Universidad Nacional Autónoma de México

Posgrado en Filosofía de la Ciencia
Facultad de Filosofía y Letras
Instituto de Investigaciones Filosóficas
Facultad de Ciencias
Dirección General de Divulgación de la Ciencia

A PHILOSOPHICAL STUDY OF EMPTY LOGICS

TESIS

que para optar por el grado de

Doctor en Filosofía de la Ciencia
Filosofía de las Matemáticas y Lógica de la Ciencia

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México, Ciudad de México
Agosto de 2024

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A PHILOSOPHICAL STUDY OF EMPTY LOGICS

Christian Romero Rodríguez

"If emptiness is empty,
how can something be borne or awaken from it?"
—Dejan Stojanovic, *The Sign and Its Children*

"I have a simple philosophy:
Fill what's empty.
Empty what's full.
Scratch where it itches."
—Alice Roosevelt Longworth

Acknowledgments

This thesis would not have been possible without my wife, Camila. These four years of doctoral studies have been the birthplace not only of new academic ideas but also of my daughter Renata. Returning home after receiving dozens of (constructive) criticisms from my colleagues and seeing Renata smiling always made me want to go on. Throughout all these years, I have had the privilege of having Camila take care of our baby so that I could bring this to a close on time. What excites me most about finishing this process is telling Camila that it is time for her to follow her dreams also and for me to give her back at least some of what she has given me.

I would like to thank Luis Estrada González for all his support and advice. He has always challenged me to push my academic boundaries. Throughout my time as a Ph.D. student, his tireless dedication and deep commitment to excellence have been a constant inspiration to me. The boss probably has more grey hair now than before I met him. I ended up getting a doctorate, and he ended up getting high blood pressure.

I want to thank my family for the moral and financial support they have given me. For not leaving me alone and for letting me know that I can count on them to support me. I would also like to thank my wife's family, who have accompanied us for a certain time in Mexico and have given us support and companionship. Thanks also to all my friends, most of whom are members of FiciFortes or the Colombian community.

I would especially like to thank Cristian Alejandro Gutierrez Ramírez, Eduardo

Alejandro Barrio, Axel Arturo Barceló Aspeitia, and Fernando Cano Jorge, who supported me throughout my doctoral process. Their advice and criticism were extremely valuable.

Finally, I would also like to thank UNAM's Institute for Philosophical Research, especially the Associated Students Program, for providing me with a place to study. This research was supported by the CONAHCyT–National Grant for doctoral studies, CONAHCyT project CBF2023-2024-55, and the UNAM-PAPIIT Projects: IN403719 "Intensionalidad hasta el final: un nuevo plan para la relevancia lógica", IG400422 "Variables proposicionales y constantes proposicionales", and IA105923 "Métodos Formales para el Razonamiento".

Chapter 0

Introduction

0.0 Introducción

En esta tesis discuto acerca de las lógicas vacías y algunos problemas que plantea su consideración como lógicas legítimas. Las lógicas vacías se caracterizan por no tener ningún argumento válido. A lo largo de esta tesis sostendré que una *lógica* es una colección de argumentos. Las lógicas que me interesan son las lógicas en donde la colección de argumentos es vacía. La cuestión de qué tipo de argumentos debemos considerar en esta colección puede ser controvertida. No obstante, voy a argumentar que la colección de argumentos de una lógica dada determina al menos tres tipos de argumentos: válidos, inválidos y antiválidos. Una manera informal de entender estos argumentos es la siguiente:

- Un argumento es *lógicamente válido* si, y sólo si, en todas las interpretaciones, si las premisas son verdaderas, la conclusión es verdadera.
- Un argumento es *lógicamente inválido* si, y sólo si, en alguna interpretación, las premisas son verdaderas y la conclusión no lo es.
- Un argumento es *lógicamente antiválido* si, y sólo si, en toda interpretación, si

las premisas son verdaderas la conclusión no lo es.

A continuación explico brevemente las razones del porqué elegir estos tres tipos de argumentos. En la literatura de la filosofía de la lógica, ha habido una serie de intentos de identificar dos lógicas aparentemente diferentes como una y la misma lógica. Tal vez la primera manera de juzgar esto fue que dos lógicas **L1** y **L2** son *idénticas* si y sólo si tienen las mismas verdades lógicas (o teoremas) (ver más en [93, p.1] y [12, p. S4995]). Esta definición es problemática porque lógicas como **LP** y la lógica clásica deberían considerarse idénticas ya que comparten las mismas verdades lógicas. Sin embargo, estas lógicas difieren en su conjunto de argumentos. Por tanto, no podemos decir que sean idénticas.

En [12], Barrio, Pailos y Szmuc han argumentado que en el debate entre pluralistas y antipluralistas lógicos se ha asumido que la identidad de conjuntos de argumentos válidos es el criterio de identidad entre lógicas. Es decir, dos lógicas **L1** y **L2** son *idénticas* si y sólo si tienen los mismos argumentos. Esto significaría que lógicas como **ST**[1] y la lógica clásica son idénticas. Sin embargo, ellos también muestran que **ST** y la lógica clásica tienen diferentes metaargumentos. Es decir, argumentos entre argumentos, en donde como premisas y como conclusión tenemos argumentos. Luego, ellos consideran que es necesario un refinamiento del criterio de identidad. Su propuesta es la siguiente: dos lógicas **L1** y **L2** son *idénticas* si y sólo si tienen los mismos argumentos y meta-argumentos en todos los niveles[2] (ver más en [12] y [8]).

En [107], Scambler probó que apelar a metaargumentos es innecesario para distinguir la lógica clásica (**CL**) de **ST**. Basta considerar los argumentos antiválidos para poder diferenciar ambas lógicas. Sean '$\vee$', '$\sim$', y '$\wedge$' una disyunción, una negación, y una conjunción, respectivamente; consideremos, por ejemplo, el argumento

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[1] Ver más sobre **ST** en el Capítulo 2.

[2] Los argumentos entre un conjuntos de fórmulas se consideran de nivel 0. Los argumentos entre argumentos se consideran de nivel 1. Los argumentos entre metaargumentos se consideran de nivel 2. Así sucesivamente.

$(A \vee \sim A) \vDash_{\mathbf{L}} (B \wedge \sim B)$. Este argumento no es antiválido en **ST** y es antiválido en **CL**. Así, si se asume que dos lógicas *idénticas* tienen los mismos argumentos antiválidos, entonces **CL** y **TS** no son la misma lógica.

Los argumentos antiválidos forman parte de la colección de argumentos inválidos. Sin embargo, una lógica puede tener la misma colección de argumentos inválidos pero una colección diferente de argumentos antiválidos. Por ejemplo, la lógica clásica y **ST** tienen los mismos argumentos válidos, los mismos inválidos, pero no los mismos antiválidos. Por otro lado, dado que la validez y la invalidez suelen ser mutuamente excluyentes y conjuntamente exhaustivas, todo lo que no es válido es inválido. Entonces, la lógica clásica y **ST** tienen los mismos argumentos inválidos. Sin embargo, la suposición de que la validez y la invalidez deben ser excluyentes y exhaustivas puede rechazarse de varias maneras. Esto nos permitiría tener algunas lógicas en las que los argumentos pueden ser simultáneamente válidos e inválidos, o ni válidos ni inválidos. Con base en esto, podemos tener lógicas con los mismos argumentos válidos, antiválidos, pero diferir en los inválidos. Por tanto, no basta con considerar los argumentos antiválidos de una lógica. También es necesario considerar los argumentos antiválidos.

Por tanto, entenderé una lógica **L** como una familia de relaciones que definen al menos tres colecciones: la colección de argumentos válidos, la colección de argumentos inválidos y la colección de argumentos antiválidos. Así, dos lógicas **L1** y **L2** son *idénticas* si y sólo si tienen los mismos argumentos válidos, argumentos inválidos y argumentos antiválidos.

Las lógicas que me interesan en esta investigación son las lógicas vacías. Las lógica sin argumentos válidos tiene muy pocos precedentes. Por ejemplo, en [91], Pailos introduce la lógica $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$. Esta lógica no tiene argumentos válidos, ni antiválidos, ni contingencias[3] de ningún tipo. Otra lógica sin argumentos válidos

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[3] En este contexto, se conserva el sentido habitual de contingencia como una fórmulas que bajo

es **TS**, que no tiene argumentos válidos, pero todos sus argumentos son inválidos; además, todos los meta-argumentos son válidos.[4] (ver más en [30]). Por último, **LKR** es una lógica no reflexiva sin argumentos válidos pero con algunos metaargumentos (pero no todos) válidos (Ver más en [57]). Todas estas lógicas tienen en común que todos sus argumentos son inválidos. Una lógica vacía puede definirse de diferentes maneras.

En primer lugar, consideremos $\mathfrak{A}$, una colección de argumentos. Podemos especificar en qué sentido A puede estar vacío:

- **V-Empty:** $\mathfrak{A}$ es *V-Empty* sii no hay argumentos válidos en él.
- **I-Empty:** $\mathfrak{A}$ es *I-Empty* sii no hay argumentos inválidos en él.
- **A-Empty:** $\mathfrak{A}$ es *A-Empty* no hay argumentos antiválidos en él.

Estas nociones dan lugar a los siguientes candidatos para definir una lógica vacía.

0.0.1. Una lógica es vacía si su colección de argumentos es (al menos) *V-Empty*.

0.0.2. Una lógica es vacía si su colección de argumentos es *V-Empty* y *I-Empty*.

0.0.3. Una lógica es vacía si su colección de argumentos es *V-Empty* y *A-Empty*.

0.0.4. Una lógica es vacía si su colección de argumentos es *V-Empty*, *I-Empty* y *A-Empty*.

Lo que todas las definiciones tienen en común, y en lo que podemos estar de acuerdo, es que lo mínimo que tiene que satisfacer una lógica para ser considerada vacía es que no tenga argumentos válidos (o que cumpla la Definición 0.0.1.).[5] Por

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alguna interpretación es verdaderas y bajo otra no lo es. Esta noción es generalizada por Pailos para aplicarse también a argumentos y metaargumentos y no sólo a fórmulas.

[4] Considerando una relación de consecuencia global de la lógica, que veremos más adelante.

[5] En el Capítulo 1 de esta tesis todas las lógicas que satisfacen la definición 0.0.1. las llamaré *mínimamente vacías*.

ejemplo, la Lógica $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ satisface la Definición 0.0.1. y 0.0.3.. Sin embargo, la Lógica **TS** y **LKR** satisfacen la Definición 0.0.1. La cuestión de cuán vacía puede ser una lógica dependerá entonces de los criterios que se consideren apropiados. A pesar de esto, lo que yo defiendo aquí es que cualquier lógica vacía debe ser (al menos) *V-Empty*.

En esta tesis respondo a la pregunta de si las lógicas vacías son lógicas. Puesto que mi hipótesis es que las lógicas vacías sí son lógicas, esto sirve de pretexto para explorar otras cuestiones, por ejemplo:

- ¿Qué es la lógica?
- ¿Cómo podemos obtener una lógica vacía?
- ¿Cuál es el papel de las interpretaciones admisibles en una lógica vacía?
- ¿Cuál es el papel de la relación de consecuencia lógica en una lógica vacía?

Considero que responder a estas preguntas aumenta nuestra comprensión de qué es la lógica y muestra uno de sus límites. Todas estas preguntas se desarrollarán en detalle en esta tesis.

0.0.1 Algunos de los problemas que se presentan

Inmediatamente después de presentar las diferentes definiciones de las lógicas vacías, surge una pregunta: ¿las lógicas vacías son realmente lógicas? Según la concepción tarskiana de la lógica (véase más en [117]), toda lógica debe satisfacer al menos las siguientes tres propiedades, donde Γ es un conjunto (posiblemente vacío) de fórmulas:

$$A \vDash_{\mathbf{L}} A \qquad \text{(Reflexividad)}$$

$$\text{Si } \Gamma \vDash_{\mathbf{L}} A \text{ entonces } \Gamma, B \vDash_{\mathbf{L}} A \qquad \text{(Monotonicidad)}$$

$$\text{Si } A \vDash_{\mathbf{L}} B \text{ y } B \vDash_{\mathbf{L}} C \text{ entonces } A \vDash_{\mathbf{L}} C \qquad \text{(Transitividad)}$$

Algunos sostienen que las lógicas en las que una de estas propiedades no es válida no son realmente lógicas (véase, por ejemplo, [59] y [37]). En [37, p. 447], Dicher dice que:

> Obsérvese que la hipótesis aquí planteada es, en efecto, que las relaciones de consecuencia subestructurales ([92]; [17]), es decir, las relaciones que no presentan todas las características especificadas en la Definición 1 [consecuencia tarskiana], no son relaciones de consecuencia lógica en absoluto.[6]

Pero esta caracterización no es inmutable. En [45], Estrada-González ha argumentado que algunas lógicas no reflexivas y no transitivas sí cumplen con los requisitos para considerarse como lógicas. Específicamente, se refiere a que estas lógicas satisfacen las propiedades deseables de una relación de consecuencia lógica presentadas por Beall y Restall en [17]: Necesidad, Normatividad y Formalidad.

Siguiendo a Beall y Restall en [17, p. 14-23], estas propiedades pueden entenderse como sigue:

- **Necesidad**: la verdad de las premisas de un argumento válido requiere la verdad de la conclusión de ese argumento.

- **Normatividad**: No es racional aceptar las premisas de un argumento válido y al mismo tiempo rechazar la conclusión.

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[6] Traducción propia de "Notice that the hypothesis here entertained is, in effect, that substructural consequence relations ([92]; [17]), i.e., relations that do not exhibit all the features specified in Definition 1 [Tarskian consequence], are not logical consequence relations at all." [37, p. 447].

- **Formalidad**: los argumentos válidos lo son en virtud de su forma lógica.

Dado que no he presentado una lógica vacía específica, no es posible definir aquí si las lógicas vacías satisfacen o no estas propiedades. Sin embargo, para satisfacer **Necesidad**, basta con que una relación de consecuencia vacía satisfaga lo siguiente: i) que la validez de los argumentos se defina en términos de preservar la verdad, y ii) que una conexión entre premisas y conclusión sea necesaria.

Tanto **Normatividad** como **Formalidad** pueden aceptarse sin ningún problema, ya que no hay argumentos válidos en la lógica vacía. Puesto que realmente no hay argumentos válidos, podemos aceptar que es irracional aceptar las premisas de un argumento válido y no la conclusión. Por tanto, nunca podremos acercarnos a esta irracionalidad de la **Normatividad**. Por otra parte, todos los argumentos son válidos, inválidos o antiválidos por su forma lógica. Éste es quizá el punto más oscuro de la propuesta de Beall y Restall, puesto que no dan ninguna pista de lo que quieren decir con 'forma lógica'. Para las lógicas no reflexivas y no transitivas (Y para la construcción de lógicas vacías) los significados de los términos aceptados en la consecuencia lógica tarskiana no están en juego, como señala Estrada-González en [45]. Por tanto, de forma analógica, la formalidad no debería verse amenazada por relaciones de consecuencia lógica vacías. No hay amenaza para el requisito de **Formalidad** si la noción original ya es formal. En una relación de consecuencia lógica vacía no hay significados de términos distintos de los aceptados por la noción de consecuencia tarskiana.

Puesto que una consecuencia vacía puede satisfacer **Necesidad**, **Normatividad** y **Formalidad**, podemos decir que son relaciones lógicas de consecuencia. Sin embargo, todavía queda un largo camino por recorrer antes de que podamos decir que las lógicas vacías son lógicas legítimas. Que sean o no lógicas dependerá de lo que entendamos por 'lógica'. El Capítulo 1 de esta investigación se ocupará de los problemas que surgen cuando queremos concluir que las lógicas vacías son lógicas.

Por otra parte, también sabemos que, puesto que una lógica vacía no tiene argumentos válidos, no es posible validar ninguna propiedad de la relación de consecuencia tarskiana. Una posición similar se encuentra también en [37], Dicher la llama *Nihilismo de Consecuencia.* Dicher cree que el Nihilismo de Consecuencia implica que la lógica estará vacía de argumentos válidos. Sin embargo, sabemos que no basta con tener una relación de consecuencia no reflexiva y no transitiva para que una lógica no tenga argumentos válidos. Lo que esto indica que es que en el estado actual de la investigación no sabemos qué es necesario para que una lógica sea vacía.

Otra pregunta que surge es, ¿cómo obtenemos una lógica sin argumentos válidos? Tenemos algunas lógicas sin argumentos válidos, como **TS** (ver más en [30]), $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ (ver más en [91]), y **LKR** (ver más en [57]). Pero, si pretendemos vaciar una lógica, también podemos preguntarnos ¿qué caminos se han explorado y cuáles quedan por explorar? Las propuestas exploradas para vaciar una lógica se desarrollan en detalle en el Capítulo 2 y en el Capítulo 3. El Capítulo 4 presenta una propuesta original para vaciar la lógica.

En [30], Cobreros, Egré, Ripley y van Rooij prueban que la relación lógica *ts*-consecuencia es vacía. De hecho, dicen literalmente: "De hecho, $\vDash_{\mathbf{ts}}$ — la relación vacía — es la relación de consecuencia más débil posible"[7] [30, p.370]. No creo que esto sea correcto. La relación de consecuencia lógica no es la única que hace el trabajo de vaciar el conjunto de argumentos válidos para una lógica dada. El conjunto de interpretaciones admisibles de una lógica dada y las condiciones de evaluación de las conectivas también desempeñan un papel crucial. En el Capítulo 5, muestro cómo podemos obtener una lógica no vacía (y casi conexiva) utilizando una relación de consecuencia lógica *ts*. Al hacerlo, muestro que no siempre se da el caso de que una

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[7] Traducción propia de "In fact, $\vDash_{\mathbf{ts}}$ — the empty relation — is the weakest possible consequence relation".

lógica con una relación de consecuencia *ts* sea la relación de consecuencia más débil posible.

El Capítulo 5 también puede verse como un capítulo bisagra para el comienzo de la investigación sobre la lógica súpercompletas. Una *lógica súpercompleta* es una lógica en la que todos sus argumentos son válidos. De forma casi dualista, los argumentos de por qué es una lógica legítima podrían valer, haciendo las modificaciones pertinentes. Sin embargo, estudiar estas lógicas en profundidad nos llevaría otra tesis doctoral. En esta tesis, por tanto, no examinaremos estas lógicas. El último capítulo sirve de bisagra para esto porque muestro cómo obtener una lógica contraclásica haciendo uso de una relación de consecuencia lógica que usualmente se relaciona con que ningún argumento vale.

0.0.2 Resumen del contenido de esta investigación

1. Lógicas vacías: ¿Qué es una lógica?

Desde el punto de vista de la filosofía de la lógica, defender la afirmación de que las lógicas vacías, que no tienen argumentos válidos, son legítimas plantea algunos retos. En este capítulo, abordaré la cuestión de si las lógicas vacías pueden considerarse legítimamente lógicas según diferentes perspectivas y enfoques tradicionales del estudio de la lógica. Argumentaré que las lógicas vacías son compatibles con todas estas perspectivas y que sólo se deslegitiman cuando se hacen varios supuestos que no son constitutivos de lo que es una lógica.

2. Interpretaciones admisibles y consecuencia lógica

En este capítulo muestro al menos dos formas de vaciar la colección de argumentos válidos de una lógica. Empiezo con argumentos clásicamente válidos para mostrar cómo un aumento en el número de interpretaciones admisibles de una lógica afecta directamente a la disminución del número de esquemas válidos. Como esto no ocurre siempre, no es trivial. Es decir, para que se dé esta relación entre interpretaciones admisibles y esquemas de argumentación inválidos hay que tener en cuenta algunos elementos del lenguaje y de la consecuencia lógica. Finalmente, reconstruyo cómo Gillian Russell introduce fórmulas dependientes del contexto para mostrar que podemos dar contraejemplos a cualquier argumento.

3. Dirección en lógicas disjuntas fuertes de Kleene

En este capítulo mostraré las características de algunos subgrupos que forman parte de algunas de las relaciones de consecuencia disjuntas definidas por Pailos en [90]. Argumento que existen tres subgrupos: a) relaciones de consecuencia disjuntas hacia adelante pero no hacia atrás, b) relaciones de consecuencia disjuntas hacia atrás pero no hacia adelante, y c) relaciones de consecuencia disjuntas en ambas direcciones. El capítulo concluye argumentando por qué es útil y fructífero considerar la dirección de la consecuencia lógica al caracterizar una lógica.

4. Validez vacía hasta el final: un camino fácil

Existe una tensión entre la definición de la lógica vacía como una lógica sin argumentos válidos ni meta-argumentos válidos, por un lado, y cómo hemos interpretado usualmente la validez de los meta-argumentos, por otro. Aquí argumentamos que una forma de eliminar la tensión es entender el "Si . . . entonces. . . " en un meta-argumento, al menos en el caso de una lógica vacía, como una transplication (también conocida como el condicional de de Finetti) en lugar de un condicional extensional o material.

5. Otra observación sobre la conexividad y la teoría de conjuntos.

En este capítulo muestro que el resultado de Wiredu en [125] no es la condena para las teorías de conjuntos conexivas, ni siquiera para aquellas basadas en lógicas similares a **CC1**. Con este propósito, presento los supuestos necesarios para la prueba de

Wiredu, haciendo algunas precisiones sobre los requisitos conexivos. Luego, presento una variante no reflexiva de **CC1** en la que la prueba de Wiredu puede ser bloqueada. Finalmente, discuto las perspectivas de una teoría de conjuntos conexiva basada en las variantes no reflexiva y no transitiva de **CC1** que, incluso asumiendo la no trivialidad, no son muy alentadoras.

0.1 Introduction

This thesis discusses empty logics and some problems with considering them as legitimate logics. Empty logics are characterized by the fact that they do not have any valid arguments at all. Throughout this thesis, it will be argued that *a logic* is a collection of arguments. In this sense, an empty logic is a logic whose collection of (valid) arguments is empty. The question of what kind of arguments we should consider in this collection may be controversial. Nevertheless, I am going to argue that the argument set of a given logic determines at least three types of arguments: valid, invalid, and anti-valid. An informal way of understanding these arguments is as follows:

- An argument is *logically valid* if and only if, in all interpretations, if the premises are true, the conclusion is true.
- An argument is *logically invalid* if and only if, in some interpretation, the premises are true and the conclusion is not true.
- An argument is *logically anti-valid* if and only, in all interpretations, if the premises are true, the conclusion is not true.

Below is a brief explanation of the reasons for the choice of these three types of argument. In the literature of the philosophy of logic, there have been a number of attempts to identify two seemingly different logics as being one and the same logic. Perhaps the first way to judge this was that two logics **L1** and **L2** are *identical* if and only if they have the same logical truths (or theorems) (see more in [93, p.1] and [12, p. S4995]). This definition is flawed because logics such as **LP** and classical logic should be considered identical since they share the same logical truths. However, these logics differ in their set of arguments. So we cannot say that they are identical.

In [12], Barrio, Pailos, and Szmuc have argued that in the debate between logical pluralists and logical anti-pluralists, it has been assumed that the identity of sets of valid arguments is the criterion of identity between logics. That is, two logics **L1** and **L2** are *identical* if and only if they have the same arguments. This would mean that logics such as **ST**[8] and classical logic are identical. However, they show that **ST** and classical logic have different meta-argument sets. That is, arguments between arguments, where we have arguments as premises and as conclusions. They then ask for a refinement of this identity in such a way that two logics **L1** and **L2** are *identical* if and only if they have the same valid arguments and meta-arguments at all levels[9] (see more in [12] and [8]).

In [107], Scambler showed that it is unnecessary to appeal to meta-arguments to distinguish classical logic (**CL**)from **ST**. To distinguish between the two logics, it is enough to consider the anti-valid arguments. Let '$\vee$', '$\sim$' and '$\wedge$' be a disjunction, a negation, and a conjunction, respectively; consider, for example, the argument $(A\vee \sim A) \vDash_{\mathbf{L}} (B\wedge \sim B)$. This argument is not anti-valid in **ST** and is anti-valid in **CL**. So if we assume that two *identical* logics have the same anti-valid arguments, then **CL** and **TS** are not the same logic.

Anti-valid arguments are part of the collection of invalid arguments. However, a logic can have the same set of invalid arguments but a different set of anti-valid arguments. For example, classical logic and **ST** have the same valid arguments, the same invalid arguments, but not the same anti-valid arguments. On the other hand, since validity and invalidity are usually mutually exclusive and jointly exhaustive, anything that is not valid is invalid. Then, classical logic and **ST** have the same invalid arguments. However, the assumption that validity and invalidity must be exclusive and exhaustive can be rejected in various ways. This would allows us to have some

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[8] See more about **ST** in Chapter 2.

[9] Arguments between sets of formulas are considered level 0. Arguments between arguments are considered level 1. Arguments between meta-arguments are considered level 2. And so on

logics in which arguments can simultaneously be valid and invalid, or neither valid nor invalid. On this basis, we can have logics with the same valid, invalid arguments, but differ in the anti-valid ones. So it is not enough to consider the anti-valid arguments of a logic. It is also necessary to consider the invalid arguments.

Therefore, I view a logic **L** as characterised by a family of relations that define at least three collections: the collection of valid arguments, the collection of invalid arguments, and the collection of anti-valid arguments. Thus, two logics **L1** and **L2** are *identical* if and only if they have the same valid arguments, invalid arguments, and anti-valid arguments.

The logics that are of interest in this research are the empty logics. There are very few precedents for logic without valid arguments. In [91], Pailos introduces the logic $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$. This logic has no valid argument, no anti-valid argument, and no contingencies[10] of any kind. We also have the logic **TS**, which has no valid arguments, but all its arguments are invalid; also, all meta-arguments are valid.[11] (see more in [30]) Finally, **LKR** is a non-reflexive logic without valid arguments but with some valid meta arguments (See more in [57]). All these logics have in common that all their arguments are invalid. An *empty logic* can be defined in different ways.

First, consider $\mathfrak{A}$, a collection of arguments. We can specify in which sense A might be empty:

- **V-Empty:** $\mathfrak{A}$ is *V-Empty* iff there are no valid arguments in it.
- **I-Empty:** $\mathfrak{A}$ is *I-Empty* iff there are no invalid arguments in it.
- **A-Empty:** $\mathfrak{A}$ is *A-Empty* iff there are no anti-valid arguments in it.

These notions give rise to the following candidates for defining an empty logic.

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[10] In this context, the usual sense of contingency is retained as a formula that is true under one interpretation and not true under another. This notion is generalised by Pailos so that it applies to arguments and meta-arguments as well, and not just to formulas.

[11] Considering a global consequence relation of logic, which we will see below.

0.1.1. A logic is empty iff its collection of arguments is (at least) *V-Empty*.

0.1.2. A logic is empty iff its collection of arguments is *V-Empty* and *I-Empty*.

0.1.3. A logic is empty iff its collection of arguments is *V-Empty* and *A-Empty*.

0.1.4. A logic is empty iff its collection of arguments is *V-Empty*, *I-Empty* and *A-Empty*.

What all the definitions have in common, and what we can agree on, is that the minimum that a logic has to satisfy to be considered empty is that it satisfies Definition 0.1.1..[12] For example, the Logic $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ satisfies, Definition 0.1.1. and 0.1.3. But, the logics **TS** and **LKR** satisfy Definition 0.1.1. These definitions will be extended when other properties are considered, such as an empty logic of contingencies or other kinds of arguments. The question of how empty a logic can be will then depend on the criteria that are considered to be appropriate. What I am arguing for here is that any empty logic must be (at least) *V-Empty*.

I answer the question of whether empty logics are logics in this thesis. Since I hypothesize that empty logics are logics, this serves as a pretext to explore other questions, such as:

- what is Logic?
- how we can get an empty logic?
- what is the role of admissible interpretations in an empty logic?
- what is the role of the logical consequence relation in an empty logic?

Answering these questions increases our understanding of logic and shows one of its limits. All of these questions are to be developed in detail in this thesis.

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[12] Hereafter, especially in Chapter 1 of this thesis, All logics that satisfy definition 0.1.1. are called *minimally empty*.

0.1.1 Some problems that arise

Immediately after presenting one of the definitions of empty logics, some questions arise: Are empty logics really logics? According to the Tarskian conception of logic (see more in [117]), every logic has to satisfy at least the following three properties, where Γ is a (possibly empty) set of formulas:

$$A \vDash_{\mathbf{L}} A \tag{Reflexivity}$$

$$\text{If } \Gamma \vDash_{\mathbf{L}} A \text{ then } \Gamma, B \vDash_{\mathbf{L}} A \tag{Monotonicity}$$

$$\text{If } A \vDash_{\mathbf{L}} B \text{ and } B \vDash_{\mathbf{L}} C \text{ then } A \vDash_{\mathbf{L}} C \tag{Transitivity}$$

Some further argue that logics where one of these properties is not valid are not logics (see for example [59] and [37]). In [37, p. 447], Dicher says that:

> Notice that the hypothesis here entertained is, in effect, that substructural consequence relations ([92]; [17]), i.e., relations that do not exhibit all the features specified in Definition 1 [Tarskian consequence], are not logical consequence relations at all.

But this characterization is not immutable. In [45], Estrada-González has argued why some non-reflexive and non-transitive logics meet the requirements of being a logic. He refers to the properties of the logical consequence relation that Beall and Restall introduced in [17]: necessity, normativity, and formality.

Following Beall and Restall in [17, p. 14-23] these properties can be understood as follows:

- **Necessity**: the truth of the premises of a valid argument necessitates the truth of the conclusion of that argument.

- **Normativity**: it is not rational to accept the premises of a valid argument while simultaneously rejecting the conclusion.
- **Formality**: valid arguments are so in virtue of their logical form

Since I have not presented a specific empty logic, it is not possible here to define whether empty logics satisfy these properties or not. However, to satisfy Necessity, it is sufficient that an empty consequence relation satisfies the following: i) that the validity of arguments is defined in terms of preserving truth, and ii) that a connection between premises and conclusion is necessary.

Both **Normativity** and **Formality** can be accepted without any problem since there are no valid arguments in empty logic. Since there really are no valid arguments, we can agree that it is irrational to accept the premises of a valid argument and not the conclusion. Therefore, we can never come close to this irrationality of **Normativity**. On the other hand, all arguments are valid, invalid or anti-valid by their logical form. This is perhaps the most obscure point in Beall and Restall's proposal. They give no hint of what they mean by *logical form*. In constructing empty logics, however, the meanings of the terms accepted in Tarskian logical consequence are not at stake, as Estrada-González notes in [45] for non-reflexive and non-transitive logics. Therefore, formality should not be threatened by empty logical consequence relations. Analogously to [45], there is no threat to **Formality** if the original notion is already formal because here, there are no meanings of terms other than those accepted by the notion of Tarskian consequence.

Since an empty consequence can satisfy **Necessity**, **Normativity**, and **Formality**, we can say that they are logical consequence relations. However, there is still a long way to go before we can say they are legitimate logics. This depends on what we mean by logic. To conclude that empty logics are logics, Chapter 1 of this investigation will deal with these problems.

On the other hand, we also know that since an empty logic does not have any valid arguments, it is not possible to validate any property of the Tarskian consequence relation. A similar position is also to be found in [37], which Dicher calls *Consequence-nihilism*. Dicher believes that Consequence-nihilism implies that a logic can be empty of valid arguments. But we know that for a logic to have no valid arguments, it is not enough to have a non-reflexive, non-transitive consequence relation. That suggests that at the present state of research, we do not know what is necessary for a logic to be empty.

Another question that arises is, how do we get a logic without valid arguments? We have some logics without valid arguments, such as **TS** (see more in [30]), $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ (see more in [91]), and **LKR** (See more in [57]). But we can also ask if we aim to empty a logic, which routes have been explored and which remain to be explored? Proposals explored for emptying a logic are developed in detail in Chapter 2 and Chapter 3. Chapter 4 presents an original proposal for emptying logic.

In [30], Cobreros, Egré, Ripley, and van Rooij show that the logical *ts*-consequence relation is empty. They say: "In fact, $\vDash_{\mathbf{ts}}$ — the empty relation — is the weakest possible consequence relation" [30, p.370]. I do not think that this is correct. The logical consequence relation is not the only one that does the job of emptying out the set of valid arguments for a given logic. The set of admissible interpretations of a given logic and the evaluation conditions of the connectives also play a crucial role. In Chapter 5, I show how we can obtain a non-empty logic (and almost connexive) using a logical *ts*-consequence relation. In doing so, I show that it is not always the case that a logic with a *ts*-consequence relation is the weakest possible consequence relation.

Chapter 5 can also be seen as a hinge for the start of research on overcomplete logic. An *overcomplete logic* is a logic in which all its arguments are valid. The arguments as to why it is a legitimate logic might hold, with appropriate modifications, in an almost dualistic way. However, studying these logics in depth would take another PhD thesis. In this thesis, therefore, we will not examine these logics. The last chapter serves as a hinge for this, because I show how to obtain a counter-classical logic by using a logical consequence relation which usually implies that does not validate any arguments

0.1.2 Overview of the contents of this investigation

1. Empty logics: What is a logic?

From the point of view of the philosophy of logic, there are challenges in defending the claim that empty logics, which have no valid arguments, are legitimate. In this chapter, I will address the question of whether or not empty logics can legitimately be regarded as logics according to different traditional perspectives and approaches to the study of logic. I will argue that empty logics are compatible with all these perspectives, and are only delegitimized when various assumptions are made that are not constitutive of what a logic is.

2. Admissible interpretations and logical consequence

In this chapter, I show at least two ways of emptying a logic's collection of valid arguments. I start with classically valid arguments to show how an increase in the number of admissible interpretations of a logic directly affects the decrease in the number of valid schemes. Since this does not happen every time, it is not trivial. That is, for this relationship between admissible interpretations and less valid argumentation schemes to be the case, some elements of language and logical consequence must be considered. Finally, I reconstruct how Gillian Russell introduces context-dependent formulas to show we can give counterexamples to any argument.

3. Direction in Strong-Kleene disjoint logics

In this chapter, I will show the characteristics of some subgroups that are part of some of the disjoint consequence relations defined by Pailos. [90]. I argue that there are three subgroups: a) forward but not backward disjoint consequence relations, b) backward but not forward disjoint consequence relations, and c) disjoint consequence relations in both directions. The chapter concludes by arguing why it is useful and fruitful to consider the direction of logical consequence when characterizing a logic.

4. Empty validity all the way up: an easy road

There is a tension between the definition of empty logic as a logic with no valid arguments and no valid meta-arguments, on the one hand, and how we have usually interpreted the validity of meta-arguments, on the other. Here we argue that one way to eliminate the tension is understanding the "If. . . then. . . " in a meta-argument, at least in the case of an empty logic, as a transplication (aka the de Finetti conditional) instead of an extensional or material conditional.

5. Another remark on connexivity and set theory

In this chapter, I show that Wiredu's result in [125] is not the doom for connexive set theories, not even for those based in logics similar to **CC1**, one of the original target logics. For this purpose, I present the necessary assumptions for Wiredu's proof, making some precisions on the connexive requirements. Then I present a non-reflexive variant of **CC1** in which Wiredu's proof can be blocked. Finally, I discuss the prospects of a connexive set theory based on both the non-reflexive and non-transitive variants of **CC1**, which, even assuming non-triviality, are not very rosy.

Contents

Chapter 1

Empty logics: What is a logic?

Abstract

From the point of view of the philosophy of logic, there are challenges in defending the claim that empty logics, which have no valid arguments, are legitimate. In this chapter, I will address the question of whether or not empty logics can legitimately be regarded as logics according to different traditional perspectives and approaches to the study of logic. I will argue that empty logics are compatible with all these perspectives, and are only delegitimized when various assumptions are made that are not constitutive of what a logic is.

Keywords: logic, empty logic, minimally empty, reasoning, logical consequence.

Introduction

A logic in which no argument is valid can be defined as an *empty logic*. Whether such logics are really logics is still disputed. In this chapter, I will look at different approaches to logic and why none of them is incompatible with empty logic. Finally, I will show that, although all approaches are compatible, in my view, the best way to understand logic is to understand logic as the study of what follows from what. This way of looking at logic does not involve questions about reasoning and some mathematical goals like other approaches to logic.

To do this, I will present four definitions of what logic is. I will also discuss some assumptions that are made when it is claimed that empty logics are not logics. In this chapter, I will concentrate on defending the idea that logic without valid arguments, invalid arguments, and anti-valid arguments is a logic, like any other known logic. Since this empty logic is the weakest of all empty logics, if I show that it is a logic, the arguments can be extended to those logics (without valid arguments) stronger than this one. Let me introduce some formal language to provide some necessary definitions before going into detail.

Let $\mathcal{L}$ be a formal language obtained in the usual way from a numerable set $PROP$ of propositional variables $p_1, \ldots, p_n$. A *interpretation* for $\mathcal{L}$ is a relation σ between $PROP$ and a set of truth values $\{1, 0\}$ in which '1' represents truth and '0' represents falsity, such that a propositional variable p_i can be related to truth values in one of the following four ways:

- p_i is true but not false, represented by '$1 \in \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1\}$
- p_i is true but also false, represented by '$1 \in \sigma(p_i)$ and $0 \in \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1, 0\}$
- p_i is neither true nor false, represented by '$1 \notin \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) =\{\ \}$
- p_i is false but not true, represented by '$0 \in \sigma(p_i)$ and $1 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{0\}$

In this thesis, and hereinafter, I understand a *logic* as a collection of arguments. Specifically, a collection of valid, invalid, and anti-valid arguments.

Definition 1.-1.1. *An argument is valid in* $\boldsymbol{L}$, $\Gamma \vDash_{\boldsymbol{L}} A$, *iff, for all interpretation* σ, *if* $1 \in \sigma(B)$, *for all* $B \in \Gamma$, *then* $1 \in \sigma(A)$.

Definition 1.-1.2. *An argument is invalid in* $\boldsymbol{L}$ *iff, for some interpretation* σ, $1 \in \sigma(B)$, *for all* $B \in \Gamma$, *and* $1 \notin \sigma(A)$.

Definition 1.-1.3. *An argument is anti-valid in* $\boldsymbol{L}$ *iff, in all interpretation* σ, *if* $1 \in \sigma(B)$, *for every* $B \in \Gamma$, *then* $1 \notin \sigma(A)$.

Consider this definition of empty logic,

Definition 1.-1.4. *A logic* $\boldsymbol{L}$ *is empty iff no argument is either valid, invalid, or anti-valid, in it.*

Whether empty logics are logics at all is still a matter of controversy. The question I want to answer in this chapter is whether empty logics are really logics. This case is interesting because their study allows us to understand what logic is. These logics have interesting properties regarding their collections of valid, invalid, and anti-valid arguments, as they are all empty. That is, empty logics is the case where all of these arguments are empty.

In [5, 6], Avron tries to draw a line between what is logic and what is not. He claims that logic must satisfy these properties:[1]

$$A \vDash_{\mathbf{L}} A \qquad \text{(Reflexivity)}$$

$$\text{If } A \vDash_{\mathbf{L}} B \text{ and } B \vDash_{\mathbf{L}} C \text{ then } A \vDash_{\mathbf{L}} C \qquad \text{(Transitivity)}$$

$$\varnothing \nvDash_{\mathbf{L}} \varnothing \qquad \text{(Consistency)}$$

---

[1] I modified the transitivity requirement for a single conclusion framework since the properties Avron requires for a logic are intended for a multiple conclusion framework.

Non-reflexive (where A $\nvDash_{\mathbf{L}} A$), non-transitive (where $A \vDash_{\mathbf{L}} B$ and $B \vDash_{\mathbf{L}} C$ but $A \nvDash_{\mathbf{L}} C$) and empty logics are excluded by these requirements that Avron asks for. In particular, the exclusion of empty logics is due to how we can understand the consistency requirement.

The consistency requirement can be understood in at least two ways. One is that $\varnothing \nvDash_{\mathbf{L}} \varnothing$ is an invalid argument and $\varnothing \nvDash_{\mathbf{L}} \varnothing$ is not valid argument. If $\varnothing \nvDash_{\mathbf{L}} \varnothing$ is an invalid argument, the invalid arguments are not empty. Therefore, empty logics could not satisfy Avron's consistency property; furthermore, an empty logic could not satisfy either reflexivity or transitivity. On the other hand, if $\varnothing \nvDash_{\mathbf{L}} \varnothing$ is not a valid argument, and the logic does not have any valid arguments, then this requirement could be compatible with the definition of an empty logic.[2]

In [28], Chemla, Egré, and Spector argue that a *respectable* relation of logical consequence must be monotonic. That is, the following argument must be valid for a logic to be respectable:

$$\text{If } \Gamma \vDash_{\mathbf{L}} A \text{ then } \Gamma, B \vDash_{\mathbf{L}} A$$

Since there are no valid arguments in an empty logic, an empty logic cannot be respectable in this sense. This uncertainty about the status of empty logics as legitimate logics is widespread. And it is common to dismiss empty logics as logics on the basis of requirements such as consistency or monotonicity. For example, in [9, p. 104], Barrio, Pailos, and Szmuc have asked in which sense is a consequence relation with no valid arguments a genuine consequence relation? They claim that 'it will be definitely non-standard to call the empty set a genuine consequence relation, let alone a logic'.

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[2] Another consequence of understanding the requirement for consistency in this way is that the set of valid arguments will not be universal. So, for Avron, an overcomplete logic (where all arguments are valid, invalid, and anti-valid.) is not a logic.

Whether a logic that has no valid arguments is a logic at all or not, remains controversial. Arguing about this leads us to question something more fundamental, and that is to ask what Logic is. While there are several different answers to the question of what is a logic, the following approaches will be the focus of my assessment. The reason for the focus on these approaches and not on others is that in the following approaches, it seems that empty logics have no place as logics:

1. **Logic is the study of how to reason.**

This position can be found in several authors, such as the following:

In [98, p.176], Priest says that:

> What is logic? Uncontroversially, logic is the study of reasoning [...] Logic does not tell us how people do reason, but how they ought to reason.

In [58, p.xiii], Goldfarb says that:

> Logic is the study of principles of reasoning. It is concerned not with how people actually reason, but rather with how people ought to reason if they wish to ensure the truth of their results.

In [27, p.161], Béziau says that:

> We can roughly define logic as the study of reasoning. Since reasoning itself can be called logic, we can say that logic is the study of logic, or better that logic is the study of Logic, using the same distinction when saying that history is the study of History.

In [75, p.1], MacFarlane says that:

> Logic is often said to provide norms for thought or reasoning. Indeed, this idea is central to the way in which logic has been demarcated as a discipline, and without it, it is hard to see how we would distinguish logic from the disciplines that crowd it on all sides: psychology, metaphysics, mathematics, and semantics. [75, p.1]

Finally, in [39, p.71], D'Ottaviano says that:

> Logic, if considered as the science of deductive reasoning, studies the consequence relation, treating of the valid inferences, that is, the inferences whose conclusion have to be true when the premises are true. [39, p.71]

2. **Logic as the study of principles and methods for distinguishing valid from invalid arguments.** (See more in [31, p.2], Orayen [89, p.16])

Some of the authors who have argued in favor of this position are the following:

In [31, p.2], Copi and Cohen say that:

> Logic is the study of the methods and principles used to distinguish correct from incorrect reasoning.

In [89, p.16], Orayen says that:

> Deductive logic is the study of the principles and methods that allow us to distinguish between valid and invalid reasoning. (The translation is mine.)

3. **Logic is the theory about what follows from what.** (See more in Priest [97, 96], Woods [128, p.32])

This is the position that Priest and Wood have argued. In [97, p.1 italics in the original], and [96, p.3209], Priest says that

> The word ‘logic’ has many meanings. Perhaps the most standard meaning amongst modern logicians is *what follows from what, and why.* That, at any rate, is how I will understand it in what follows.

> Logic, in one of the many sense of that term, is a theory about what follows from what and why.

And in [128, p.32], Woods says that:

> Logic is about what follows from what and why, period.

4. **Logic is the mathematical study of logical structures.**

This position can be found in the work of the following authors:

In [73, p.193], Lawvere says that:

> It is the study of the resulting algebra of parts of a universe of discourse and of these three transformations of parts between universes that we sometimes call “logic in the narrow sense”.

A similar position is found in [25, p.73], Brouwer says that:

> While thus mathematics is independent of logic, logic does depend upon mathematics: in the first place intuitive logical reasoning is that special kind of mathematical reasoning which remains if, considering mathematical structures, one restricts oneself to relations of whole and part.

It is easy to see why these approaches to the question of what logic is would rule out empty as logics. For example, in the first approach, an empty argument collection could not give us parameters for the way we should do reasoning. Because some consider that we use at least one valid argument in our reasoning (rejecting empty logics). In the second approach, it is not possible to distinguish valid from invalid arguments in these logics. All arguments will be empty. In the third approach, empty logics could also be excluded. 'Nothing follows' might not be considered a satisfactory answer. Finally, if logic is the mathematical study of logical structures, then empty logics are part of its research domain.

Although empty logic is not considered legitimate, this is the prevailing view. Arguments for concluding that it is not a logic require too many assumptions. However, I hypothesize that empty logics are logics like any other, according to all these traditional approaches to logic. In this sense, making these assumptions explicit delimits some prejudices about these logics. My objective is to contribute to the current knowledge of what is or is not logic and recognize empty logics as logics.

The structure of the chapter is as follows: I will present an argument to show why empty logics are logics in the first four sections. The division of the sections will be by the different approaches to the question of what a logic is. After the presentation of the argument, I will show some assumptions that one might have to have the conclusion that they are not logics, and why the consideration of these assumptions should not be part of the argument.

1.0 Logic is the study of how to reason

The position of Priest [98], Goldfarb [58], Béziau [27, 21], D'Ottaviano [39] and MacFarlane [75] can be reconstructed as follows:

- Logic is the study of how to reason.

In this way, each of the valid arguments of a logic would capture how we should reason. For example, paraconsistent logicians believe that we do not reason using arguments such as Explosion:

$$A, {\sim} A \vDash_{\mathbf{L}} B$$

We cannot claim dinosaurs are on the moon in the face of contradiction. On the other hand, relevantists consider arguments such as Disjunctive Syllogism and the Paradoxes of Material Implication (for example, Explosion) to be invalid. This is because their premises are not relevant to the conclusion. In this way, they give us some counter-examples in which we can have true premises and false conclusions. Following the position of MacFarlane in [75], I think that the dispute is, at the root, a dispute about logical validity.

If the relevant logicians are right, the logic that best tells us how we should reason is relevant logic, then paraconsistent and classical logicians would be wrong in their beliefs about what should be the way we reason. The demands we make on logic are so demanding that no logic is likely to be able to meet them because we believe in different ways that there is some connection between logical validity and the evaluation of arguments. For example, according to proscriptivism, an argument is logically valid only if the content of the conclusion is included in the content of the premises. Arguments like $A \vDash_{\mathbf{L}} A \vee B$, which are valid for most relevantists, should be logically invalid arguments based on this definition of validity (see more in [51]).

We have seen that the classical logician may think that he is right, that his logic is the right logic to tell us how we should do our reasoning. However, the paraconsistent might be able to show counter-examples to at least one of his logical laws, namely Explosion. History repeats itself, with the relevant objecting to the validity of some of the arguments of the paraconsistent and the proscriptivist objecting to at least one of the arguments of the relevantist. This could be extended to all of the arguments. That is, someone could start to give counter-examples for every logical law, such that the premises are true and the conclusion is not true.

One objection that might be raised to these counter-examples (to any logical law) is that they are not really legitimate counter-examples at all. For example, Priest argues that logic is normative. (For more, see [95, p.297]).

- A discipline is normative if and only if it is a generator of norms of behavior for agents who are capable of decisions.

Therefore, legitimate counter-examples must be informative about why we should not reason based on such a scheme of argumentation. That is, to provide a legitimate counterexample to an argument, it would not be enough to have true premises and an untrue conclusion. However, I think that this conception of a legitimate counterexample should not be taken into account because it presupposes certain approaches (e.g., normative approaches to logic) that we do not have to accept. The counterexamples that we can give to (any) argumentative schemes have the necessary structure to satisfy the definition of a logically invalid argument. Therefore, these counter-examples are just as legitimate as any other counter-examples. My point is that it is possible for any argument to have a counter-example with true premises and a non-true conclusion.

Let us suppose that our sufficient and necessary criterion for knowing which logic is right is that it captures the way in which we ought to reason. That is, a right logic captures the way we ought to reason, and any logic that captures how we ought to reason is a right logic. So far, however, we do not have a logic that does have an account of this phenomenon. This leaves at least two possibilities. i) logic is not truly normative, or ii) the right logic has not been found. Option ii) seems rather far-fetched. Given the evidence, there is no reason to believe that any logic can do this. Thus, i) logic is really not normative. On this basis, we can say that logic does not have a normative role nor an account of how we should reason. However, this situation is not exceptional for empty logics. It is the status quo for all logics studied so far, and the status of empty logic is thus the same as that of any other kind of logic.

There is, however, another sense of normativity in which the empty logic might do well. In this position, according to Wansing in [120, p.18],

Definition 1.0.1. *A discipline is normative if and only if "[it] produces theories about the meaning of [...] normative concepts. If logic is presented as a discipline that separates good arguments from bad ones, then it generates theories about normative concepts, namely the notions of a good argument and a bad argument"*

Studying logic, especially empty logic, produces an understanding of good (valid) and bad (invalid) arguments. That is, it develops a theory of what it generally means to be valid and invalid. The answer that Empty Logics would give is that there is no such thing as a good argument or a bad argument. It is important to note that if empty logic is correct, there are no valid arguments, no invalid arguments, and no anti-valid arguments. There is a break in the traditional conception of the collection of arguments of a logic. Traditionally, the collections of valid and invalid arguments have been exhaustive and exclusive. That is why it can be either valid or invalid (but not both simultaneously). As I defined above, this exhaustiveness must be broken for empty logic to be possible. This is the reason why there are no good arguments, and there are no bad arguments.

Some people may find this answer to be unsatisfactory. Nevertheless, these people would probably argue: There is at least one good argument. But this requirement is neither implicit nor explicit in characterizing logic as a study of how we reason. Empty logic does, however, provide a theory of the validity, invalidity, and anti-validity of arguments. But a person might be a normativist who wants distinctions, even if he accepts my argument above and thinks that this requirement is neither implicit nor explicit. My answer to such a person is that there are exact definitions in empty logic to distinguish a good argument (definition of validity) from a bad argument (definition of invalidity). But no argument can satisfy these definitions. This does not mean that there are no clear differences between what is a good argument and a bad argument, so there is a distinction.

So far, I have made a lot of concessions. I have conceded that logic can be normative. I have answered some of the normativist's objections to logic. However, logic is not normative. To show this, I present a more general argument, inspired by [110] and [120].

1. If the laws of logic are normative, they are laws of thought.
2. All laws of thought prescribe how we should think.
3. If we accept that logic (in this psychologistic approach) prescribes how we should think, logical laws depend on humans.
4. If logic is the science of abstract logical entities, i.e. truth-values and truth-value functions, then logical laws do not depend on humans.
5. Logic is the science of abstract logical entities, namely truth-values and truth-value functions.
6. Logical laws do not depend on humans. (By 4,5)
7. Logic does not prescribe how we should think. (By 3, 6).
8. The laws of logic are not laws of thought. (By 2, 7)
9. The laws of logic are not normative. (By 1, 8)

Assumption number 1 is based on what Shramko calls a psychologistic approach. In this approach, the laws of logic reflect the human thought process. In this sense, the laws of logic are laws of thinking, which are prescriptions for how we should think. The object of logic is a human activity in this approach. Since the object of study would not exist without human beings, Shramko considers such disciplines to be anthropological. But it seems to me that logic is a little bit more than that. In this case, it is understood as the science of abstract logical entities. For this reason, logic is not normative.

I have also shown that there is a sense of the normative that is perfectly compatible with empty logic. This version of normativity does not run the risk of being labeled *ad hoc*. Nor does it involve us in a begging-the-question fallacy. It is not *ad hoc* because it is not only empty logics that are the beneficiaries of its acceptance. This new definition of normativity is an understanding of the role of logic and its role in the study of argument. On the other hand, because it is supported by a long tradition of considering logic as a descriptive rather than a normative discipline, it is also not a fallacy of begging the question. Moreover, the normative definition does not depend on what an empty logic is or can be. Thus, I have shown that there is a meaning for the normative that is perfectly compatible with empty logic, namely definition 1.0.1. So empty logic is not incompatible with this definition of logic, whether it is normative logic or not.

1.1 Logic as the study of principles and methods for distinguishing valid from invalid arguments.

The position of Copi and Cohen, and Orayen can be reconstructed as follows:

- Logic is the study about of the distinction between valid and invalid arguments.

For our work, the following definition might be more helpful.

- Logic is the study about of the distinction between valid, invalid, and anti-valid arguments.

This approach can be easily distinguished from the psychological and sociological approaches to the reasoning process. Psychological approaches study the different forms that some people find valid or invalid. For example, Storrs McCall, in [81], conducts an empirical study with almost 90 people, novices in logic, from the University of McGill. His question is about the plausibility of the connexive theses.[3] This can be seen as an example of the study of logic from a psychological point of view. On the other hand, a sociological approach to reasoning is concerned with studying members of communities and how they use different types of reasoning to test their hypotheses. These studies look at the percentage of community members who adequately use certain forms of reasoning that are assumed to be valid.

However, logic studies the methods for distinguishing such collections of arguments. Let us take, as an example, the usual definitions of valid, invalid, and anti-valid arguments. Now, let Γ and A be a set of formulas and a formula of $\mathcal{L}$, respectively.

Definition 1.1.1. *An argument* $\Gamma \vDash_{\boldsymbol{L}} A$ *(in* $\boldsymbol{L}$*) is valid, if and only if, for every evaluation* σ*, if* $1 \in \sigma(B)$ *for every* $B \in \Gamma$, $1 \in \sigma(A)$.

Definition 1.1.2. *An argument* $\Gamma \vDash_{\boldsymbol{L}} A$ *(in* $\boldsymbol{L}$*) is invalid if and only if, there is an evaluation* σ*, such that* $1 \in \sigma(B)$ *for every* $B \in \Gamma$ *and* $1 \notin \sigma(A)$.

Definition 1.1.3. *An argument* $\Gamma \vDash_{\boldsymbol{L}} A$ *(in* $\boldsymbol{L}$*) is anti-valid iff, in all interpretation* σ*, if* $1 \in \sigma(B)$*, for every* $B \in \Gamma$*, then* $1 \notin \sigma(A)$.

We can use the definitions of validity, invalidity, and anti-validity to define different collections of arguments in a model-theoretic approach. Some people tend to use only the definition of validity and the definition of invalidity as nonvalid. In this way, it is ensured that the collections of arguments are exclusive and exhaustive. However, our

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[3] In this study, McCall evaluates the plausibility of the connexive theses. In particular, the focus is on people who do not have a background in philosophy at all. In the questionnaire, different sentences and instances of the theses of Aristotle and Boethius were written. The aim was to ask about the truth or falsity of these statements.

definition of invalidity must be appropriate to the needs of the theory in question. For example, in [70], Kapsner requires that for logic to be strongly connexive, there must be no interpretations where the connexive theses (without some negation) are satisfied. This requirement means that connexive theses must be considered logically invalid (anti-valid). Therefore, only by changing the definitions of invalidity and anti-validity can we make a logic strongly connexive.

Although anti-valid arguments are a special kind of invalid argument, distinguishing between invalid and anti-valid is possible. For example, it is impossible to distinguish Classical Logic from **ST** (except for meta-arguments) if we do not consider this collection of arguments. In [107], Scambler shows why **ST** recovers all the arguments of Classical Logic. However, it does not recover classical anti-validities.

When I refer to a collection of arguments in this research, I am considering both arguments and meta-arguments. A *meta-argument* is an argument between arguments that has the form "If $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ then $\Delta \vDash A_m$". In this approach, logicians have as their object of study the methods to distinguish at least 3 groups of arguments: I) Valid arguments and valid meta-arguments; II) Invalid arguments and invalid meta-arguments; and III) Anti-valid arguments and anti-valid meta-arguments. I will introduce two ways of evaluating meta-arguments in a model-theoretic approach to logic: by defining Global Validity and Local Validity.

Definition 1.1.4. *A meta-argument is globally valid if and only if, if the meta-premises* $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ *are valid, the meta-conclusion* $\Delta \vDash B$ *is valid as well.*

Definition 1.1.5. *A meta-argument is globally invalid if and only if, the meta-premises* $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ *are valid and the meta-conclusion* $\Delta \vDash B$ *is not valid.*

Definition 1.1.6. *A meta-argument is globally anti-valid if and only if, if the meta-premises* $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ *are valid, the meta-conclusion* $\Delta \vDash B$ *is not valid.*

To define Local Validity, it is first necessary to define *satisfaction* of an argument.

Definition 1.1.7. *An interpretation satisfies an argument if and only if, if premises are true, the conclusion is true.*

Definition 1.1.8. *A meta-argument is* locally valid *if and only if, the meta-conclusion is satisfied in every interpretation in which the arguments of the meta-premises are satisfied.*

Definition 1.1.9. *A meta-argument is* locally invalid *if and only if, there is an interpretation in which the meta-premises are satisfied and the meta-conclusion is not satisfied.*

Definition 1.1.10. *A meta-argument is* locally anti-valid *if and only if, if the meta-premises are satisfied, the meta-conclusion is not satisfied.*

In fact, in a model-theoretic approach, to affirm that a distinction can be made between these collections of arguments, it would suffice to have a precise definition of these types of arguments and meta-arguments. There is clarity about these distinctions in the case I am interested in, that of empty logic. That is to say, it is possible to define precisely these three sets of arguments. Therefore, an empty logic is a logic just like any other logic.

This answer may be unsatisfactory because some might feel that there is no real distinction being made. That is, it is not possible to distinguish the collection of valid arguments from the other collections since there are no valid, no invalid, and no anti-valid arguments. This is true, of course. But there are implicit assumptions in this kind of objection: There must be at least one valid (invalid/anti-valid) argument that is not invalid (valid/anti-valid) in every logic **L**. That is to say, the argument sets must be mutually exclusive. The acceptance of this, however, would be a departure from the object of study of logic. The focus of the study logic is not the distinction itself, but the methods which allow us to make the distinction. If studying these methods on empty logics tells us that there is no distinction between collections of arguments, the methods are doing their job, and it is our duty as logicians to study them. If logic is a theory about distinguishing between valid, invalid, and anti-valid arguments, and there is no distinction in empty logic, then empty logic remains legitimate logic. Some might think it is a pretty bad theory, but it is still a theory.

This approach differs significantly from the previous one. It does not consider issues of reasoning. The possibility of investigating how we can distinguish between types of arguments and the methods we use for this identification is characteristic of this approach. For a logic to be a logic, the reasons we have for being able to distinguish between types of argument are not essential here.

Perhaps we need to distinguish which arguments are valid or invalid for reasons proper to understanding what should be valid or invalid in the way we reason. The first approach to logic, as the study of how to reason, presupposes a stage at which we can distinguish between valid and invalid. The second approach, however, is not a prerequisite for the first. Indeed, Restall in [101], for example, insists that logic is not intended to resolve questions of epistemology or how we reason.

One of the aims of logic itself is to work out how certain features of logical consequence work and how propositions interact so that they can be good enough to prove a particular proposition. These questions are addressed by Proof Theory and Model Theory. Both approaches to logic clarify the methods and principles needed to derive theories about what should be valid, invalid, and anti-valid. In this approach to logic, if an argument is logically valid, it is not necessarily the case that it will give us normative parameters for how we should do our reasoning.

1.2 Logic is the theory about what follows from what.

The position of Priest and Wood can be reconstructed as follows

- Logic is the theory about what follows from what.

Following Barrio and Pailos in [8], logic is an abstract explanation of this connection. In particular, logic is a theory about the relation of what is a consequence of what. Different logical theories can be related through the properties of the notion of logical consequence relation.

Studying logical consequences need not be restricted to studying validity only. As I have shown, at least at the level of arguments (not meta-arguments), there are logical consequence relations of a logic (**L1**) that are coextensive with another logic (**L2**). They differ, however, at the level of the meta-arguments or the level of the anti-valid arguments. There would be no way of distinguishing **L1** from **L2** if we restricted ourselves to a definition of logic as just a collection of valid arguments. This shows that in order to establish differences and similarities between logics, it is of little use to accept this definition of logic. Then, to be committed to logic as the study of what follows from what, we must know what follows from what in each collection of arguments.

In other words, we need to be clear about which arguments are valid, which are invalid, and which are anti-valid. Clarity implies abandoning assumptions such as that valid and invalid arguments are exhaustive and exclusive. It also means abandoning the assumption that anti-valid arguments are necessarily a group of invalid arguments. Although this is the more common view, it depends on several assumptions which we do not necessarily have to hold. If we leave these assumptions aside, I can present a group of logics that satisfy the requirement of being minimally empty. Before presenting some of them, let me give a definition.

Let $\mathfrak{A}$ be a collection of arguments.

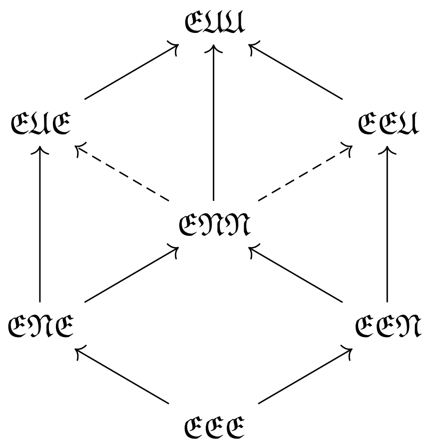
- **V-Empty:** $\mathfrak{A}$ is *V-Empty* iff there are no valid arguments in it.
- **I-Empty:** $\mathfrak{A}$ is *I-Empty* iff there are no invalid arguments in it.
- **A-Empty:** $\mathfrak{A}$ is *A-Empty* iff there are no anti-valid arguments in it.

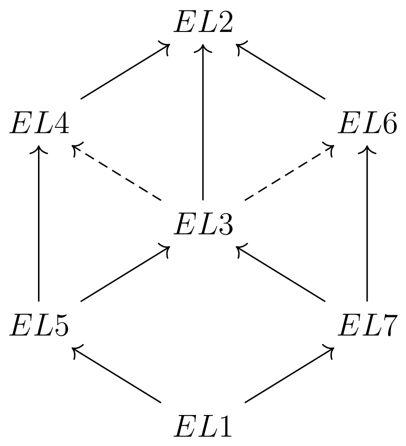
Definition 1.2.1. *A logic is minimally empty iff its collection of arguments is (at least)* V-Empty*.*

Let $\mathfrak{E}$, Let $\mathfrak{U}$, Let $\mathfrak{N}$ be an empty collection of arguments, a universal collection of arguments, and a non-empty and non-universal collection of arguments. By *universal collection* of arguments, I mean a collection of arguments where all the arguments are valid (V), invalid (I), or anti-valid (A). We can obtain the following minimally empty logics:

Logic	V	I	A
EL1	$\mathfrak{E}$	$\mathfrak{E}$	$\mathfrak{E}$
EL2	$\mathfrak{E}$	$\mathfrak{U}$	$\mathfrak{U}$
EL3	$\mathfrak{E}$	$\mathfrak{N}$	$\mathfrak{N}$
EL4	$\mathfrak{E}$	$\mathfrak{U}$	$\mathfrak{E}$
EL5	$\mathfrak{E}$	$\mathfrak{N}$	$\mathfrak{E}$
EL6	$\mathfrak{E}$	$\mathfrak{E}$	$\mathfrak{U}$
EL7	$\mathfrak{E}$	$\mathfrak{E}$	$\mathfrak{N}$

Each of these logics is characterized by the fact that it is minimally empty. However, some of the minimally empty logics will be stronger than others. Let $\mathfrak{X}\mathfrak{Y}\mathfrak{Z}$ be a logic, where $\mathfrak{X}$ represents valid arguments, $\mathfrak{Y}$ represents invalid arguments, and $\mathfrak{Z}$ represents anti-valid arguments, we can compare these logics by the following diagram:





Considering that hence there are approaches that identify logic with its argument collection (for example, [102, p.1235] and [10, p.96]), there are at least seven minimally empty logics. We can clearly distinguish each minimally empty logic because of the differences between the argument sets. The three definitions (of valid, invalid, and anti-valid arguments) determine a family of minimally empty logics that can be ordered. Sometimes we think that there is no weaker logic than a minimally empty logic. But we can have minimally empty logics that are weaker than others, as the order of these logics shows. The weakest minimally empty logic of all will be **EL1** if we consider only the arguments we have considered. However, it is possible to obtain a logic that is weaker than **EL1**. It is enough to consider other criteria, like contingencies, to have an empty logic of valid, invalid, anti-valid arguments and contingencies. This new logic is weaker than **EL1**. The important lesson to draw from this, beyond the possibility of having stronger or weaker minimally empty logics, is that if we know exactly which arguments belong to which collection, we know exactly what follows from what.

In such approaches, whether an argument is valid, invalid, or anti-valid must be determined precisely by the study of logic. It would even be alright for some arguments to be identified as both valid and invalid simultaneously or for some to be neither valid nor invalid. We would have a problem if we had some arguments that we could not classify into some of the defined sets. In such a case, we would not be clear about what follows from what, and the study of logic would not serve its purpose. One of the advantages of studying logic beyond the study of validity (considering invalidity and anti-validity) is that it allows us to distinguish different types of logic. There would be no way of distinguishing a minimally empty logic from other minimally empty logic if the study of logic were restricted to validity alone. The study of logic is, therefore, enriched by considering other sets of arguments different from the set of valid arguments. This makes it possible to evaluate not only the similarities between logics but also their differences.

From this point of view, logic is most valuable because it allows the study of minimally empty logics using proof theory. Under regular/standard considerations, it is impossible to study empty logics using proof theory. However, to make the exercise non-trivial, let us consider the minimally empty logics. Let us leave aside the logics **EL1**, **EL2**, **EL4** and **EL6**. I am going to leave these logics aside since all of their sets of arguments are either empty or universal.

For example, a logic such as **EL3**, **EL5**, or **EL7** could be characterized using a calculus for its set of invalid or anti-valid arguments. Since that in **EL3** there are invalid arguments (but not all), and there are anti-valid arguments (but not all), one could present this logic by a calculus in which all the invalid arguments can be derived, and also, jointly, by another calculus in which all the invalid arguments can be derived. The same exercise could be repeated for logics such as **EL5**, where it could be presented by a calculus of invalid arguments; and for logics such as **EL7**, where it could be presented by a calculus of anti-valid arguments. Finally, the exercise would be trivial for logics such as **EL2**, **EL4**, and **EL6**.

Pulcini and Varzi explored this idea in [99]. In this paper, Pulcini and Varzi focus on what does not follow. To do this, they first present proof systems whose theorems contain exactly all the classical contradictions (*refutation systems*). Then they present a proof system whose theorems are exactly all classical non-tautologies (*rejection systems*). The refutation system is the calculus of anti-valid arguments, and the rejection system is the calculus of invalid arguments. These two calculi, rather than just a calculus for classical theorems, could, therefore, be included in a more compact presentation of Classical Logic.

The study of logic is the study of what follows from what and what does not follow from what. The study of what follows from what need not imply what does not follow from what. It will imply it only if the distinction between validity and invalidity is exhaustive and exclusive. Otherwise, working out each approach separately will be a more general approach. On the other hand, the exhaustive study of the properties of the consequence relation and the individualization of its sets of arguments are characteristic of the study of a particular logic.

A possible objection to this treatment of empty logics in this approach is that we do not know what follows from what in an empty logic. We cannot study what follows from what because nothing follows from nothing. In other words, the argument could go like this: Logic studies what follows from what. Nothing follows from nothing in an empty logic. We cannot study what follows from what if nothing follows from nothing. So we cannot study what follows from what in an empty logic. Thus, the approach of working with empty logics may be unsuccessful.

Note that this objection only applies to empty logic (**EL1**). It does not apply to the other minimally empty logics. In the other minimally empty logics, there are indeed invalid or anti-valid arguments, so strictly speaking, there are arguments that we can individualize because they do not follow. Therefore, the conclusion of this argument does not follow for the minimally empty logics. On the other hand, it is false that we cannot study what follows from what if nothing follows from nothing.

This is because, although in an empty logic nothing follows from nothing, we do have a clear idea of what follows from what: nothing. A good theory of Pegasus includes the fact that there are none. One must not confuse a theory about an empty concept with an empty theory. Whether or not this answer is unsatisfactory or bad for some logicians is another matter. Nevertheless, it is well-known that this is what this thesis is about. I hope that by the end of this investigation, the reader will be convinced that we can study really serious and interesting philosophical problems even if nothing follows from nothing.

This approach is very different from the first, which sees logic as the study of how we should reason. In this approach, we can work more abstractly by understanding logic as the study of what follows from what. The first approach is tied to psychological issues. This restricts the study of logic to certain arguments that can account for these phenomena. However, this is not the case in this approach. Here, we seek to understand the similarities and differences between different theories and a broader understanding of what logic is. Only when we know what follows from what can we make some differences and similarities between logics explicit.

In my view, this approach (the third) is also more abstract than the second, in which logic is the study of principles and methods for distinguishing valid from invalid arguments. For at least three reasons, this approach is different. First, the principles and methods for making these distinctions are not all that logic is concerned with. It is one thing for the end of logic to be to identify what follows from what, and another for the means to identify which arguments are valid and which are invalid. The aim is to identify the consequence of a set of premises, and we can do this using different methods. However, logic is also concerned with what follows after this identification process of valid/invalid arguments. Knowing its valid and invalid arguments is insufficient to affirm that one logic is identical to another logic. When we do this kind of analysis, other factors can be decisive.

Secondly, distinguishing just between valid and invalid arguments is restrictive. This is all the more the case when invalidity is the complement of validity. Although we may be clear about what is valid or invalid, we may not be so clear about other properties that may also have logic, such as anti-valid arguments, sound arguments, contingencies, and so on. Thus the logician takes care to evaluate and study what is characteristic of a logic: what kinds of arguments (valid, invalid, antivalid, and so on) or properties formulas may have (for example, contingencies, constants, variables), in the study of what follows from what. Moreover, this approach is not limited to what is valid or invalid. It concerns any feature that can tell us what follows from what. This allows the study of what logic is to propose and consider new features.

Third, another important reason is that the above approach leaves out studying inductions, abductions, or analogies. If logic studies arguments that can be valid or invalid, and induction/abduction/analogy can be neither valid nor invalid (but can be either strong or weak), then logic does not study such arguments. In this approach, however, logic can be the study of such arguments. If we are clear about what should or should not follow from certain inductive/abductive/analogical arguments, we have a clear case of the scope of what the study of logic can be. Therefore, this approach is the most representative of the logic of the approaches under consideration.

1.3 Logic is the mathematical study of logical structures

According to Bourbaki in [23], there are three *fundamental structures*, which he calls ‘mother structures’: algebraic structures, topological structures, and order structures. These structures represent a way of thinking. In short, a structure has I) some elements that II) are related in some way. Depending on how these elements are related, we can affirm or deny certain statements of the structure.

According to Bourbaki's approach, all other structures can be constructed in a way that crosses or derives from them. Boolean structures, for example, can be obtained using algebraic (for example [61]), topological (for example [55]) or order structures (for example [68]), depending on how they are conceived. Béziau shows in his *Universal Logic* project that logical structures are another kind of fundamental structure (see more in [26]). That is, logical structures correspond to "a way of thinking" irreducible to other fundamental structures.

All logical structures have objects. Objects may or may not have different properties depending on the relation(s) they have in the structure. The only way we can know what the objects of logical structures are is through analysis of them in action within the structure. Usually, the relation of the objects is determined by the logical consequence relation, defined as $\vdash_{\mathtt{L}} \subseteq \wp(\mathtt{F}) \times \mathtt{F}$[4], where $\mathtt{F}$ can be defined as a set.

In this approach, logic is studied through mathematics. How the two are related depends on the leading author you are working with, but all approaches share the fact that logic and mathematics are closely related. There are at least three ways of understanding logic from a mathematical approach: Brouwer's approach, Lawvere's approach, and Béziau's approach.

The first is Brouwer's approach. Here, logic is retrieved from the arguments used in mathematical proofs. From Brouwer's approach, the mathematician does not restrict the inferences from his proofs following logic constraints. Otherwise, we know what may or may not be valid in a logic $\mathbf{L}$ if and only if its arguments are used in mathematical reasoning. Lawvere's approach is the second. Here, logic is extracted from mathematical structures. That is to say, every mathematical structure has an

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[4] Usually, to maintain generality, logical consequence is defined as a subset of $\wp(\mathtt{F}) \times \wp(\mathtt{F})$. That is, where we have a set of (possibly) multiple conclusions. But I preferred to leave it that way, sacrificing a bit of generality since all my work is developed with a single conclusion framework.

underlying logic that the logician can extract. Both approaches have in common that mathematics comes first. Logic is only a tool extracted from mathematics, and logic has no normative (or similar) role in mathematics. Finally, Béziau's approach. Here, logic is the study of logical structures, where the understanding of logical objects and how they relate to each other is sought.

The position of Brouwer can be reconstructed as follows:

- Logical reasoning is a type of mathematical reasoning.

Brouwer argues that mathematical language is independent of logical language. But logical language is dependent on mathematical language. He goes on to say that this can be summarised as follows.

It is sometimes thought that a mathematical proof is correct because the mathematical conclusions are justified by logic; or because they are coherent with a logic. Brouwer, however, takes a different view. For example, classical logicians often consider the Excluded Middle as a logical truth:

Let '$\vee$', '$\sim$' be a disjunction and a negation, respectively:

$$\vDash_{\mathbf{L}} A \vee \sim A$$

However, the exhaustiveness of truth values is presupposed if the Excluded Middle is to be taken as a logical truth. Following to Shapiro and Lynch in [109], consider the following recursive clause for disjunction in **BHK**(Brouwer-Heyting-Kolmogorov) semantics when it is put informally:

- A proof of a formula in the form $\sim A$ is a procedure that will transform any proof of A into a proof of absurdity (i.e., a proof of $\sim A$ is a proof that there is no proof of A).

- A proof of a formula in the form $A \vee B$ is either a proof of A or a proof of B.

That is,

- A proof of a formula in the form $A\vee \sim A$ is either a proof of A or a proof of $\sim A$.

That is, according to **BHK**, in mathematical proofs, it is valid to infer $\sim\sim A$ from A, but not the other way round. Brouwer's argument would only appeal to intuition to make the same point without appealing to **BHK**. This proposition is not true because it has no proof, but it is not false because there may be proof. Consider, for example, the Goldbach conjecture.

- Every integer that can be written as the sum of two primes can also be written as the sum of as many primes as one wishes until all terms are units.

The status of the conjecture is since this statement has not been proved in the mathematical context, it would not be legitimate to claim that its negation can, therefore, be proved because the clause for disjunction will never be met. There are propositions of this kind for which neither they nor their negation have been proved proofs. Therefore, the Excluded Middle would not hold in this kind of mathematical context. The proofs themselves give the arguments that must or must not hold.

Another way we can see the role of logic in mathematics in Brouwer's approach is as follows. Consider the following argument:

- All men are mortal.
- Socrates is a man.
- ergo: Socrates is mortal.

According to Brouwer, a mathematical system is the first mathematician's intuition that goes through a person's mind when they read such an argument. To be more precise, Brouwer is referring to a relation. That is:

- A finite set of “men” (subjects).
- A finite set of “mortals” (predicates).
- And every element of the set of “subjects” is connected to some (all, none, or one) of the elements of the set of “predicates”.

You will notice that the argument’s validity is explained in purely set-theoretic terms. In this approach, logic is not independent of mathematics, and therefore, logical validity in mathematics depends on mathematical systems.

I have two possible approaches under Brouwer’s proposal for the subject I am interested in here. First, find a case of an empty mathematical theory in the sense that it has no theorems. Or a mathematical theory in which there is no proof. We do not have this second possibility, but it may be that we do have the first one. However, immediately after it is presented, the legitimacy of empty mathematics will likely be challenged. Take a look at **Appendix 1**. We have set out what an empty mathematical theory would look like. The logic underlying the proofs that can be given in empty mathematics is empty (or minimally empty) logic.

There are no axioms in all the mathematical theories we present. And if there are no axioms, there are no proofs. If there are no proofs, no valid arguments are used in them. Logic without valid arguments is empty or minimally empty logic. So we can say that empty (or minimally empty)is the logic used in the proofs of empty mathematics. This argument could be seen as weak. However, my position on mathematics is openly pluralistic. Assuming mathematics as in **Appendix 1** and evaluating the underlying logic of its proofs should not be so problematic. An empty mathematics recovers the intuition that there can be a counterexample to every mathematical proposition. Therefore, an empty logic has to reflect the possibility that, considering all possible interpretations, it is possible to give a counter-example to every mathematical proposition. In this very special sense, we can say that the logic of the empty could be a mathematical study in the style of Brouwer.

On the other hand, the other approach is that of Lawvere. In this approach, logic is an "Algebra of parts". Following Humberstone [66, p. 17], "recall that an algebra `A` comprises a set **A** together with some operations under which A is closed. The set **A** is the universe (or 'carrier') of `A`". By *algebra of parts*, Lawvere refers to the set of all subsets of a given set (preserving certain algebraic operations).

Following [67], for example, different categories have different internal logics.

- **Regular logic**: the logical operations that can be defined are: $\top$, $\wedge$, $\exists$.
- **Coherent logic**: the logical operations that can be defined are: $\wedge$, $\vee$, $\top$, $\bot$, $\exists$.
- **Geometric logic**: the logical operations that can be defined are: $\top$, $\bot$, $\exists$, $=$, $\vee_i$ (possibly infinitary disjunction), and $\wedge_i$ (finite conjunction).

It is important to note that different kinds of logic are obtained depending on how the objects in the categories and collections of objects are structured. The organization of the categories provides internal logic. So, logic is the result of extracting how the objects are structured.

Now, I can focus on our case study: empty logics. Consider the empty set $\varnothing$. Some ways of defining it by extending and understanding it are as follows:

- $\varnothing = \{\ \}$
- $\varnothing = \{x \mid x \neq x\}$

The empty set is not foreign or strange, at least from the **ZF** perspective of sets. The empty set is considered to be a set, unlike the universal set. We have all algebraic operations when we have an empty set of objects. For example, consider the definitions of intersection, $\cap$, and union, $\cup$:

- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

All elements x that meet these definitions can belong to either $A \cap B$ or $A \cup B$. Even when operating on the empty set, these definitions are still satisfied. For example, suppose I want to work with an algebra $\langle \varnothing, \cap, \cup \rangle$, with an empty set ($\varnothing$) of elements and two operations $\cap$ and $\cup$. Under these operations, the algebra is closed. The algebra itself will give me a certain logic. I want to conjecture here that it is very likely to be an empty or minimally empty logic. One must abandon assumptions that are sometimes fundamental to the study of algebra, namely the restriction that certain sets cannot be empty, to develop such a program. If these assumptions are abandoned, perhaps an empty logic will emerge from these structures, as Lawvere thinks. Some people will think, for example, that no logic can be described in algebraic structure $\langle \varnothing, \cap, \cup \rangle$. However, it is possible to describe a logic for this kind of mathematical structure: the empty logic, if we consider that empty logics are logics. Perhaps the assumption of mathematical pluralism (which accepts as legitimate certain mathematics that some consider strange) can help in this research project. In this sense, we can affirm that empty logics are not incompatible with this approach to logics as a mathematical study.

I have shown that empty logics are not incompatible with this position on logic from Brouwer's and Lawvere's point of view. However, I think we can be dissatisfied with this conception of mathematics. From a mathematical point of view, I think that there is another sense in which it is possible to work with empty logics.

In [42, p. xi], Enderton says that:

- "Symbolic logic is a mathematical model of deductive thought".

In [26, p.73], Béziau says that:

> Universal Logic is a general study of logic in the same way as Universal Algebra is a general study of algebra. [...] Logic is then an autonomous field of mathematics, with its own intuitions and concepts and which can survive and be developed without importing specific notions from other fields of mathematics.

My point is that this approach differs from Brouwer's and Lawvere's. But it shares with them that logic can be studied from mathematics, either modeled or structured. But this approach is independent of mathematics, its proofs, or the internal organisation of the objects of the structures, at least in Beziau's approach.

Béziau's Universal Logic is a research program where a logic is a mathematical structure. Every logic is a structure that satisfies certain properties, minimally that of being a pair:

$$\langle \mathcal{S}, \mathcal{R}_{\mathcal{I}} \rangle$$

where $\mathcal{S}$ is a structure and R_I is a family of relations.

A *language* $\mathcal{L}$ can be presented as a structure (see more in [20]), specifically, an algebra:

$$\mathcal{F} = \langle \mathtt{F}, \#_1, ..., \#_n \rangle$$

where `F` is the set of formulas generated by a set of atomic formula $ATOM$ in which $\#_i$ are taken as fundamental operations. In logic, these fundamental operations are $\#$ $n-ary$ connectives of the form $\#(A_1...A_n)$. Remember that this language (an algebra) will be closed under these fundamental operations. The set `F` is closed under some fundamental operations if and only if the result of the operation with elements of `F` is also included in `F`.

On the other hand, R_I is a family of relations. I have argued sufficiently here to suggest that this family of relations must determine at least three collections: the collection of valid inferences, the collection of invalid inferences, and the collection of anti-valid inferences. This is not meant to be restrictive in any way. That is, depending on what the logician needs, this family of relations can vary. Unsurprisingly, such mathematical structures can present empty and minimally empty logics. Empty logics fulfill these properties by being clear about how the collections of the family of relations are determined, just like any other legitimate logic. Therefore, empty logics are logics in a broader approach to logic as math.

In this approach, logic has nothing to do with reasoning in general or any normative component. Perhaps in Brower's approach, it has something to do with mathematical reasoning. However, logic is a tool that allows us to use a descriptive approach to how mathematicians reason. The mathematical proofs and their reasoning are the best parameters we have about what should be valid and what should not be. So, it is not the job of logic to give us a good theory of how we should proceed with our reasoning. We already know how to do that. Just look at mathematical proofs and the patterns we use to do proofs. Indeed, Béziau in [26] considers such approaches to logic as religious stages in the history of logic.

This approach also differs from the second approach. In the second approach, we focus only on methods (almost algorithmic) where we have as input a collection of arguments. After that, we get at least two collections as output: one of the valid arguments and one of the invalid ones. In the approach of this section this is not the case. We do not only care about the method by which our input becomes an output but also what elements influence this to happen. Any computer could do logic if logic were restricted to what the second approach suggests. But this is not the case yet. I believe this is not the case because logic is much more than that. In logic, we're interested in having certain theories that validate and invalidate certain arguments but are also informative about how different elements of structures relate to each other. That is, how the language, the set of admissible interpretations, the conditions of evaluation, the logical consequence, etc., can be related. The latter approach allows us to do this.

On the other hand, in this approach, or at least in the Universal Logic approach, we have clarity about what follows from what and why certain things follow. That is to say, it is more informative than the third approach because mathematically studying logic as logical structures allows us to understand what elements are involved in logic and, above all, how these elements are related to each other. Here, different logical consequence relations (e.g., one for valid arguments, one for invalid arguments, and one for anti-valid arguments) can allow us to know how they relate to certain languages. At least the Universal Logic approach allows logic to be sufficiently general to be a way of thinking irreducible to another.

1.4 Conclusion

Empty logics are logics like any other, according to all these traditional approaches to logic. Usually, however, a large part of the academic community has discredited its status based on assumptions. We see that they are neither sufficient nor necessary in defining what a logic is if we make some of these assumptions explicit.

Logic is the study of what follows from what and why. This is a sufficiently broad and general definition of logic. On the other hand, representing a logic as a logical structure $\langle \mathcal{S}, \mathcal{R}_{\mathcal{I}} \rangle$ is compatible with this definition. The presentation of logic in this way does give us enough generality to have an understanding of what a logic is. Above all, it is an understanding that empty logics should not be strange.

In this chapter, I have only focused on different definitions of logic that seem incompatible with empty logic. But there are also different levels in the various definitions, for example:

- Logic as pure theory.
- Logic as applied theory.

Logic is a structure in the pure theory approach, similar to $\langle \mathcal{S}, \mathcal{R}_{\mathcal{I}} \rangle$. These structures can have different applications, such as electrical circuits (classical (Boolean) logic) or analyzing sentences (as in Lambek's Calculus). See [97] for more on these approaches. Approaches 2 and 3 can easily be related to the logic level as a pure logic theory. Approaches 1 and 4, conversely, can be related to the second level of logic as applied theory. Although empty logic is not entirely incompatible with the logic level as applied theory, I think the applications that various logics may have can be interesting. However, its status as a logic has nothing to do with whether or not a logic is interesting because of its applications.

Once we have established that empty logics are logics, we can take some philosophical problems more seriously. For example, how can we empty a logic, or what kinds of logical consequence relations can do this job? This kind of philosophical work can be useful in understanding what logic is. I want to emphasize that if we extend the original definition of empty logic a little with that of minimally empty logic, we get different logics that can be logically and philosophically interesting.

1.5 Appendix 1: Mathematics based on an empty logic

Joint work with Luis Estrada-González

For several authors, including scholars as diverse as da Costa and Béziau [34] or Russell [104, 105], for example, a logic has to prove its worthiness by means of at least showing its potential to support mathematics on top of it. Thus, they accept a conditional like the following one:

Logic for mathematics If a logic is worth its name, it must support at least some form of arithmetic or set theory.

In this brief note, we address the task of formulating mathematical theories based on empty logics as defined in [91]. (Empty logics has also been studied by Marcos [77, 78] and Batens [14, 15].) In particular, let us use his logic $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ that has no valid, anti-valid nor contingencies of any kind.

Case 1. Peano arithmetic. Consider the Peano axioms. Here is the full list of theorems of Peano arithmetic based on an empty logic:

And those are all. This is not so unprecedented in the history of non-classical mathematics. Ferguson [50] proved that there are no numerically inductive theories of arithmetic in any universal-identity extension of Priest's connexive logic $\mathbf{P}_N$. Let us call 'PA$_\varnothing$' this arithmetic based on $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$.

Case 2. Naive set theory. Consider the unrestricted comprehension schema and the axiom of extensionality. Here is the full list of theorems of naive set theory based on an empty logic:

And those are all. Let us call 'NST$_\varnothing$' this naive set theory based on $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$.

Case 3. ZF. Consider the Zermelo-Fraenkel axioms. (Since we are a bit skeptical about choice, we prefer not to work with ZFC, but the reader might do it if they fancy.) Here is the full list of theorems of ZF set theory based on an empty logic:

And those are all. Let us call 'ZF$_\varnothing$' this arithmetic based on $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$.

It is worth emphasizing that one can use $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ as the underlying logic of classically inconsistent axiom sets. $\mathrm{NST}_\varnothing$ was already an example of that.

Case 4. Smooth infinitesimal analysis. Consider the proper axioms of smooth infinitesimal analysis. (See [19, Chapter 8]) Here is the full list of theorems of smooth infinitesimal analysis based on an empty logic:

And those are all. Let us call '$\mathrm{SIA}_\varnothing$' this analysis based on $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$.

Mathematical theories based on an empty logic might have some meta-theoretic appeal for a classical logician, though. Let $\mathcal{M}^i_\varnothing$ be any of the empty mathematical theories described above. Classically, $\mathcal{M}^i_\varnothing$ is non-trivial: there is at least one formula A^i in the underlying language of $\mathcal{M}^i_\varnothing$ such that $A^i \notin \mathcal{M}^i_\varnothing$. Moreover, $\mathcal{M}^i_\varnothing$ is (again, classically) decidable. All the theories above are at least isomorphic, if not identical, to each other.

What is the right logic to develop meta-theoretic studies is a difficult problem. We have given some examples of what the situation looks like with a classical meta-theory. But it is well-known that some people argue that if a theory is based on a logic $\mathbf{L}$, the meta-theory should be based on $\mathbf{L}$ as well. (For a vivid instance of such a view in the case of inconsistent mathematics, see [123, Chapter 4].) Now, what if $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ is the logic for the meta-theory of $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$-based mathematical theories? That is moot. Consider the following claim:

Empty meta-theory If $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ is the logic for the meta-theory of $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$-based mathematical theories, then the meta-theory is empty.

This seems a meta-theoretical conditional claim about $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$-based mathematical theories. And if this conditional is evaluated in $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$, as per the antecedent of that very conditional, it is not valid, neither anti-valid nor invalid but not anti-valid. That is but one example of the perplexities surrounding meta-theory based on $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$.

Authors like Corfield [32] might complain that in focusing in (logical reconstructions of) arithmetic and set theory we are not dealing with any real mathematics, like higher-order algebra, but in certain parts of mathematics that have traditionally attracted the philosophers' attention. Combining this objection with the intuitions mentioned at the beginning of this piece, some people might be inclined to accept a conditional like the following one:

Logic for *real* mathematics If a logic is worth its name, it must support at least some form of a piece of real mathematics.

Point taken. Nonetheless, we are almost certain that formulating "real mathematics" on top of $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$, with the corresponding non-triviality proofs, will be no problem. We leave that work for better trained scholars, though.

As it always happens in philosophy, some people might find the results presented here suspicious. Discussing with enough detail the relationship between logic and mathematics exceeds our main focus of interest, viz. exhibiting the kind of mathematics obtained by using an empty logic. Therefore, in what follows, we only give the gist of the sort of discussion that these empty mathematics can give rise to.

Let us suppose that our reconstructions are correct. Still, non-classical mathematics always causes incredulous stares. For example, for a variety of reasons, the theories presented here might not be considered genuine pieces of mathematics, even if they are, in fact, what one obtains by employing an empty logic over some proper axioms of a mathematical theory. In that case, assuming **Logic for mathematics** (or **Logic for *real* mathematics**), we have not showed yet that $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ is worth the name 'logic'. And that would certainly fit the intuitions of most people nowadays.

Nonetheless, one can also think that $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ is worth the name 'logic', for example, by accepting something like Pailos' arguments, even if the theories it gives rise to are all like the ones presented here, that is, empty. In that case, one has at least two options: either one accepts that the theories presented here, or some others similar to them, are genuine pieces of mathematics, or else one starts doubting the acceptability of **Logic for mathematics** and **Logic for *real* mathematics**.

Since some important logics are deductively weak to support arithmetic or some meta-theory yet they seemingly deserve the name 'logic', for example, **FDE**, some authors have relaxed **Logic for mathematics** to the following:

Logic for some domain If a logic is worth its name, it must support at least some theory that could be true.
(For a recent example, see [37]. A similar stance has been adopted for different reasons, not precisely to save the logicality of some logic. See [14], [16] and [33].)

Note that empty logics challenge not only claims about the necessary applicability of logic to mathematics, but also the necessary applicability of logic to some domain. The following seems to be the case.

Case 5. ***D*.** Consider the basic background claims of any theory about a given domain D. Here is the full list of consequences of those claims if one uses an empty logic:

And those are all. Let us call '$D_\varnothing$' this theory about D based on $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$.

Summarizing: there is a common view according to which a logic must be useful to do some form of mathematics. We have given some examples of what can be done with an empty logic on that respect. Suppose that either they are pieces of genuine mathematics or they are not. In the former case, we are done. In the latter, one should ask for better examples, but seemingly there will be no better examples. In that case, either one challenges the logicality of an empty logic or the very demand that a logic must be useful for doing mathematics. Moreover, empty logics challenge the idea that a logic worth its name must be useful for doing some theory, mathematical or otherwise.

Chapter 2

Admissible interpretations and logical consequence

Abstract

In this chapter, I show at least two ways of emptying a logic's collection of valid arguments. I start with classically valid arguments to show how an increase in the number of admissible interpretations of a logic directly affects the decrease in the number of valid schemes. Since this does not happen every time, it is not trivial. That is, for this relationship between admissible interpretations and less valid argumentation schemes to be the case, some elements of language and logical consequence must be considered. Finally, I reconstruct how Russell introduces context-dependent formulas to show we can give counterexamples to any argument.

Keywords: admissible interpretations, values, logical consequence, context-dependent formulas, arguments.

Introduction

In this chapter, I answer how to empty a logic. By *empty logic* (EL), I mean that valid, invalid, and anti-valid arguments are empty or that valid arguments are (almost) empty. The guiding question of this section is relevant to the philosophy of logic because it clarifies the process by which logic gradually invalidates some argumentative schemes.

In [37], Dicher claims: A logic that does not satisfy any Tarskian property (especially reflexivity) is empty of valid arguments. Shortly thereafter, in [54], Fjellstad showed that we can have non-reflexive logic and context-dependent formulas with some valid arguments. So what the necessary elements are for a logic to be empty is not entirely clear at present. These questions have been set aside since some academics have discredited such logics as legitimate logics. These questions become of philosophical and logical interest since I have shown in my first chapter that empty logics are logics.

In [91] Pailos has shown how it is possible to create an empty logic of valid and anti-valid arguments at all inferential levels, using different standards of logical consequence. Its model-theoretic presentation uses different logical consequence relations to make the logic empty. However, this presentation leaves out the natural transition (towards an empty logic) that any logic can have when its set of admissible interpretations is expanded.

I hypothesize that increasing the set of admissible interpretations is one of the key elements to consider when one wants to empty the argument sets of a logic. That is, if you want to empty a logic, you do not need such sophisticated machinery as the one used in [91]. Note that my hypothesis goes along the same route as Gillian Russell's work in [103], [104] and [105]. If the hypothesis is correct, this will broaden our understanding of how the interaction of different logical consequence relations with sets of interpretations works. In this chapter, I will consider using the elements of the power set of truth values {1,0} to extend the set of admissible interpretations of classical logic. In addition, I will also consider interpretations that depend on whether the formula is a premise or a conclusion, on whether it is a sub-formula or not.

The plan for this chapter is as follows. First, I introduce classical logic (**CL**) to show some of its features and some valid arguments. Then, I will introduce **K3**, **LP**, and **FDE** to show how valid arguments are lost, compared to classical logic, when the set of admissible interpretations increases. Subsequently, I will show how some arguments are lost when the logical consequence relation is modified. Then, I will present how Gillian Russell proposes context-dependent formulae to further empty logics such as **FDE**.

2.0 Preliminares

Let $\mathcal{L}$ be a formal language obtained in the usual way from a numerable set $PROP$ of propositional variables $p_1, \ldots, p_n$ and a set of connectives $\{\sim, \wedge, \vee, \rightarrow\}$, which represent negation, conjunction, disjunction and conditional, respectively. A *interpretation* for $\mathcal{L}$ is a relation σ between $PROP$ and a set of truth values $\{1, 0\}$ in which '1' represents truth and '0' represents falsity, such that a propositional variable a_i can be related to truth values in one of the following three ways:

- a_i is true but not false, represented by '$1 \in i(a_i)$ and $0 \notin i(a_i)$'; more briefly, $i(a_i) = \{1\}$
- a_i is true but also false, represented by '$1 \in i(a_i)$ and $0 \in i(a_i)$'; more briefly, $i(a_i) = \{1, 0\}$

- a_i is neither true nor false, represented by '$1 \notin i(a_i)$ and $0 \notin i(a_i)$'; more briefly, $i(a_i) = \{\ \}$

- a_i is false but not true, represented by '$0 \in i(a_i)$ and $1 \notin i(a_i)$'; more briefly, $i(a_i) = \{0\}$

and that can be extended to complex formulas according to the following evaluation conditions:

$1 \in \sigma({\sim} A)$ iff $0 \in \sigma(A)$

$0 \in \sigma({\sim} A)$ iff $1 \in \sigma(A)$

$1 \in \sigma(A \wedge B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$

$0 \in \sigma(A \wedge B)$ iff $0 \in \sigma(A)$ or $0 \in \sigma(B)$

$1 \in \sigma(A \vee B)$ iff $1 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \vee B)$ iff $0 \in \sigma(A)$ and $0 \in \sigma(B)$

$1 \in \sigma(A \rightarrow B)$ iff $0 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \rightarrow B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

Some logics will have different sets of admissible interpretations, which will modify the tables resulting from the evaluation conditions of the connectives. For this reason, the corresponding tables will be presented with the presentation of the logic in question. In addition, we will work with a logical consequence relation of truth, unless otherwise noted.

An *argument* is an expression of the form $\Gamma \vDash_{\mathbf{L}} A$, where $\vDash_{\mathbf{L}}$ stands for a relation of logical consequence, and Γ is also known as a premise set and A is called 'conclusion'.

Definition 2.0.1. *An argument is valid in* $\boldsymbol{L}$, $\Gamma \vDash_{\boldsymbol{L}} A$, *iff, for all interpretation* σ, $1 \in \sigma(A)$ *if* $1 \in \sigma(B)$, *for all* $B \in \Gamma$.

A *meta-argument* is an argument between arguments that has the form "If $\Gamma_1 \vDash_{\mathbf{L}} A_1, \ldots, \Gamma_n \vDash_{\mathbf{L}} A_n$ then $\Delta \vDash_{\mathbf{L}} A_m$".

Definition 2.0.2. *A meta-argument is valid if and only if, if the meta-premises* $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ *are valid, the meta-conclusion* $\Delta \vDash B$ *is valid as well.*

Other necessary definitions are logical truth and logical falsity.

Definition 2.0.3. *A formula* A *is a logical truth iff, in every interpretation* σ, $1 \in \sigma(A)$.

Definition 2.0.4. *A formula* A *is a logical falsity iff, in every interpretation* σ, $1 \notin \sigma(\phi)$.

Since this chapter will compare some logics, it is necessary to define when a logic $\mathbf{L}_1$ is weaker than a logic $\mathbf{L}_2$:

Definition 2.0.5. *A logic* $\mathbf{L}_1$ *is weaker than a logic* $\mathbf{L}_2$ *iff* $VAL_{L_1} \subseteq VAL_{L_2}$.

Here I use mixed logic consequence relations. the definitions I will use are as follows:

Definition 2.0.6. *A logical consequence relation* $\Gamma \vDash_{\boldsymbol{L}} A$ *is mixed iff, in all interpretation* (σ), *if* $\sigma(B) \in \sigma_p$, *for every* $B \in \Gamma$, *then* $\sigma(A) \in \sigma_c$ *and* $\sigma_p \neq \sigma_c$.

Definition 2.0.7. *Attaching subscripts with the added elements indicates an expansion of* $\mathcal{L}$. *For instance,* $\mathcal{L}_{\{\circ\}}$ *denotes* $\{\circ, \wedge, \vee, \sim, \rightarrow\}$.

Throughout the chapter, I will be committed to the idea that a logic is a set of arguments. Set of arguments understood as $\{VAL, AntVAL, INVAL\}$, in which has as subsets the set of valid arguments (‘VAL’), the set of invalid arguments (‘$INVAL$’), and the set of anti-valid arguments (‘$AntVAL$’).

Assuming a classical logical consequence relation, these sets of arguments can be defined as follows:

Definition 2.0.8. *An argument* $\Gamma \vDash A$ *is anti-valid iff, in all interpretation* σ, *if* $1 \in \sigma(B)$, *for every* $B \in \Gamma$, *then* $1 \notin \sigma(A)$.

Definition 2.0.9. *An argument* $\Gamma \vDash_{\boldsymbol{L}} A$ *is invalid but not anti-valid iff, there are an interpretation* σ_j*, in which* $1 \in \sigma(B)$ *(for every* $B \in \Gamma$*) and* $1 \notin \sigma(A)$*, and other interpretation* σ_k*, in which* $1 \in \sigma(B)$ *(for every* $B \in \Gamma$*) and* $1 \in \sigma(A)$*.*

Finally, some definitions used throughout the text will be:

Definition 2.0.10. *Let '*$Sub(A)$*' be the set of proper sub-formulas of* A*. For example, if* A *is* $(p \vee \sim q) \to (r \wedge s)$*,* $Sub(A) = \{p, q, r, s, \sim q, p \vee \sim q, r \wedge s\}$*.*

Definition 2.0.11. *A conditional is extensional iff* $A > B \dashv\vDash_{\boldsymbol{L}} N - A \vee B$*, in which* N *represents a negation and* $>$ *a conditional.*

Definition 2.0.12. *A negation* N *is de Morgan iff* $\Gamma, N(A \wedge B) \vDash_{\boldsymbol{L}} NA \vee NB$*,* $\Gamma, N(A \vee B) \vDash_{\boldsymbol{L}} NA \wedge NB$*,* $\Gamma, NA \vee NB \vDash_{\boldsymbol{L}} N(A \wedge B)$ *and* $\Gamma, NA \wedge NB \vDash_{\boldsymbol{L}} N(A \vee B)$*.*

2.1 Classical logic

I'm going to start by talking about what is known as classical logic. I am doing this not because I consider it to be the most important or the most valuable, but because it is the one that has been studied the most. Classical logic can be presented in a relational semantics with two admissible interpretations:

- p_i is true but not false, represented by '$1 \in \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1\}$
- p_i is false but not true, represented by '$0 \in \sigma(p_i)$ and $1 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{0\}$

The evaluation conditions of the connectives can be represented by the following four truth tables:

$\sim A$	A
$\{0\}$	$\{1\}$
$\{1\}$	$\{0\}$

$A \wedge B$	$\{1\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$

$A \vee B$	$\{1\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$
$\{0\}$	$\{1\}$	$\{0\}$

$A \rightarrow B$	$\{1\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1\}$

The logical consequence relation of **CL** is truth-preserving.

Some logical truths of **CL**:

- $\vDash_{\mathbf{CL}} \sim (A \wedge \sim A)$
- $\vDash_{\mathbf{CL}} A \vee \sim A$
- $\vDash_{\mathbf{CL}} A \rightarrow A$

Some logical falsehoods of **CL**:

- $A \wedge \sim A \vDash_{\mathbf{CL}}$
- $\sim (A \vee \sim A) \vDash_{\mathbf{CL}}$
- $\sim (A \rightarrow A) \vDash_{\mathbf{CL}}$

The following is a set of valid arguments of **CL**. These arguments are sufficient to derive the entire set of valid arguments VAL_{LC}.

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{CL}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{CL}} A$ and $C \vDash_{\mathbf{CL}} B$, then $C \vDash_{\mathbf{CL}} A \wedge B$
- **Disjunction introduction (ID):** $A \vDash_{\mathbf{CL}} A \vee B$

- **Disjunction (DISJ):** If $A \vDash_{\mathbf{CL}} C$ and $B \vDash_{\mathbf{CL}} C$ then $A \vee B \vDash_{\mathbf{CL}} C$

- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{CL}} (A \wedge B) \vee (A \wedge C)$

- **Reflexivity of the conditional.(REFC):** $\vDash_{\mathbf{CL}} A \rightarrow A$

- **Detachment (MP):** $A, A \rightarrow B \vDash_{\mathbf{CL}} B$

- **Explosion (EXP):** $A, \sim A \vDash_{\mathbf{CL}} B$

- **Implosion (IMP) :** $B \vDash_{\mathbf{CL}} A \vee \sim A$

An in-depth study on classical logic can be found in [108]. In [126], Wisniewski, Steen, and Benzmüller write about 'classical logics' with the plural. They do so based on some logics that are presented bivalently. Here I would like to argue that there is only one classical logic. There are indeed a large number of proof-theoretic and model-theoretic presentations of classical logic. But we are discussing the same logic if all these presentations give the same set of valid, invalid, and anti-valid arguments. Speaking of classical logic in the plural is, therefore, erroneous. Further discussions of what classical logic is and what it is not at different levels can be found in [10].

2.2 Expanding the set of interpretations

In the following, I will introduce the logic **K3**, **LP**, and **FDE** to subsequently evaluate their respective sets of valid arguments. These three logics have been the subject of much study in the logic literature. If you want to learn more about **K3**, you can consult [72]. If you want to learn more about **LP**, you can consult [4] and [95]. If you want to learn more about **FDE**, you can consult [1] and [2]. See [74] or [62] for a comparative study of these and some other logics.

2.2.1 K3

K3 can be presented in a relational semantics with three admissible interpretations:

- a_i is true but not false, represented by '$1 \in i(a_i)$ and $0 \notin i(a_i)$'; more briefly, $i(a_i) = \{1\}$
- a_i is neither true nor false, represented by '$1 \notin i(a_i)$ and $0 \notin i(a_i)$'; more briefly, $i(a_i) = \{\ \}$
- a_i is false but not true, represented by '$0 \in i(a_i)$ and $1 \notin i(a_i)$'; more briefly, $i(a_i) = \{0\}$

The conditions for evaluating the connectives are the same as above. However, the tables vary as follows to reflect the expanded set of possible interpretations:

$\sim A$	A
$\{0\}$	$\{1\}$
$\{\ \}$	$\{\ \}$
$\{1\}$	$\{0\}$

$A \wedge B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

$A \vee B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{\ \}$	$\{0\}$

$A \rightarrow B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$

From the set presented in **CL**, these are some arguments that are valid in **K3**:

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{K3}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{K3}} A$ and $C \vDash_{\mathbf{K3}} B$, then $C \vDash_{\mathbf{K3}} A \wedge B$

- **Disjunction introduction (ID):** $A \vDash_{\mathbf{K3}} A \vee B$

- **Disjunction (DISJ):** If $A \vDash_{\mathbf{K3}} C$ and $B \vDash_{\mathbf{K3}} C$ then $A \vee B \vDash_{\mathbf{K3}} C$

- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{K3}} (A \wedge B) \vee (A \wedge C)$

- **Detachment (MP):** $A, A \rightarrow B \vDash_{\mathbf{K3}} B$

- **Explosion (EXP):** $A, \sim A \vDash_{\mathbf{K3}} B$

Given the properties of the set of admissible interpretations of **K3**, it is possible to state that: i) **K3** has no logical truths and ii) the set of logical falsehoods of **K3** is coextensional with the set of logical falsehoods of **CL**.

Theorem 2.2.1. *i) If* $\vDash_{CL} A$*, then* $\nvDash_{K3} A$.

Proof. Consider the following formulas, which are logical truths in **CL**: $A \vee \sim A$, $\sim (A \wedge \sim A)$, $A \rightarrow A$. **K3** has in the set of admissible interpretations to interpretation $\{\ \}$ and also a negation of de Morgan, which is characterized by the fact that the negation of a formula with the interpretation $\{\ \}$ is $\{\ \}$. Defining any logical truth in **K3** is impossible since all definable formulae will have an interpretation where $1 \notin i(A)$. Therefore, **K3** has no logical truths. □

Theorem 2.2.2. *ii)* $A \vDash_{CL}$ *iff* $A \vDash_{K3}$.

Proof. Let $\sigma_{LC} = \{\{1\}, \{0\}\}$ be the set of interpretations of **CL** and $\sigma_{K3} = \{\{1\}, \{\ \}, \{0\}\}$ the set of interpretations of **K3**. Given the extent of both sets of interpretations, $\sigma_{LC} \subseteq \sigma_{K3}$. Therefore, every logical falsehood in **K3** is also a logical falsehood of **CL**. The theorem is thus proved from right to left. In the extra interpretation that **K3** has is the case that $1 \notin i(A)$ for every formula A, for this reason, every logical falsehood in **CL** is also a logical falsehood of **K3**, proving the theorem from left to right. □

Despite their similarities, the set of valid arguments of **K3** is not coextensive with the set of valid arguments of **CL**, since **K3** does not validate arguments such as Reflexivity of the Conditional and Implosion.

2.2.2 LP

LP can be presented in a relational semantics with three admissible interpretations:

- a_i is true but not false, represented by ‘$1 \in i(a_i)$ and $0 \notin i(a_i)$’; more briefly, $i(a_i) = \{1\}$
- a_i is true but also false, represented by ‘$1 \in i(a_i)$ and $0 \in i(a_i)$’; more briefly, $i(a_i) = \{1, 0\}$
- a_i is false but not true, represented by ‘$0 \in i(a_i)$ and $1 \notin i(a_i)$’; more briefly, $i(a_i) = \{0\}$

The conditions for evaluating the connectives are the same as above. However, the tables vary as follows to reflect the expanded set of possible interpretations:

$\sim A$	A
{0}	{1}
{1,0}	{1, 0}
{1}	{0}

$A \wedge B$	$\{1\}$	$\{1,0\}$	$\{0\}$
{1}	{1}	{1,0}	{0}
{1,0}	{1,0}	{1,0}	{0}
{0}	{0}	{0}	{0}

$A \vee B$	$\{1\}$	$\{1,0\}$	$\{0\}$
{1}	{1}	{1}	{1}
{1,0}	{1}	{1,0}	{1,0}
{0}	{1}	{1,0}	{0}

$A \rightarrow B$	$\{1\}$	$\{1,0\}$	$\{0\}$
{1}	{1}	{1,0}	{0}
{1,0}	{1}	{1,0}	{1,0}
{0}	{1}	{1}	{1}

From the set presented in **CL**, these are some arguments that are valid in **LP**:

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{LP}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{LP}} A$ and $C \vDash_{\mathbf{LP}} B$, then $C \vDash_{\mathbf{LP}} A \wedge B$
- **Disjunction introduction (ID):** $A \vDash_{\mathbf{LP}} A \vee B$
- **Disjunction (DISJ):** If $A \vDash_{\mathbf{LP}} C$ and $B \vDash_{\mathbf{LP}} C$ then $A \vee B \vDash_{\mathbf{LP}} C$
- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{LP}} (A \wedge B) \vee (A \wedge C)$
- **Reflexivity of the conditional.(REFC):** $\vDash_{\mathbf{LP}} A \rightarrow A$
- **Implosion (IMP) :** $B \vDash_{\mathbf{LP}} A \vee \sim A$

Given the properties of the set of admissible interpretations of **LP**, it is possible to state that: i) **LP** has no logical falsehoods and ii) the set of logical truths of **LP** is coextensional with the set of logical truths of **CL**.

Theorem 2.2.3. *i) If A is a logical falsehood in **CL**, then* $A \nvDash_{\boldsymbol{LP}}$.

Proof. Consider the following formulas, which are logical falsehoods in **CL**: $A \wedge \sim A$, $\sim (A \vee \sim A)$, $\sim (A \rightarrow A)$. **LP** has in the set of admissible interpretations the interpretation {1,0} and also de Morgan's negation, which is characterized by the fact that the negation of a formula with the interpretation {1,0} is {1,0}, for every formula A. It is impossible to define any logical falsity in **LP** since all definable formulae will have an interpretation in which $1 \in i(A)$. Therefore, **LP** has no logical falsehoods. □

Theorem 2.2.4. *ii) A is a logical truth in textbfLC iff* $\vDash_{\boldsymbol{LP}} A$.

Proof. Let $\sigma_{LC} = \{\{1\}, \{0\}\}$ be the set of interpretations of **CL** and $\sigma_{LP} = \{\{1\}, \{1,0\}, \{0\}\}$ the set of interpretations of **LP**. Given the extent of both sets of interpretations, $\sigma_{LC} \subseteq \sigma_{LP}$. Therefore, every logical truth in **LP** is also a logical truth of **CL**. The theorem is thus proved from right to left. In the extra interpretation that **LP** has is the case that $1 \in i(A)$, for this reason, every logical truth in **CL** is also a logical truth of **LP**, proving the theorem from left to right. □

While **LP** has the same logical truths as **CL**, **LP** and **CL** differ in the set of valid arguments since **LP** does not validate arguments such as Detachment and Explosion.

2.2.3 FDE

FDE can be presented in a relational semantics with four admissible interpretations:

- a_i is true but not false, represented by '$1 \in i(a_i)$ and $0 \notin i(a_i)$'; more briefly, $i(a_i) = \{1\}$
- a_i is true but also false, represented by '$1 \in i(a_i)$ and $0 \in i(a_i)$'; more briefly, $i(a_i) = \{1,0\}$
- a_i is neither true nor false, represented by '$1 \notin i(a_i)$ and $0 \notin i(a_i)$'; more briefly, $i(a_i) =\{\ \}$
- a_i is false but not true, represented by '$0 \in i(a_i)$ and $1 \notin i(a_i)$'; more briefly, $i(a_i) = \{0\}$

The conditions for evaluating the connectives are the same as above. However, the tables vary as follows to reflect the expanded set of possible interpretations:

$\sim A$	A
{0}	{1}
{1,0}	{1, 0}
{ }	{ }
{1}	{0}

$A \wedge B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{1,0}	{ }	{0}
{1,0}	{1,0}	{1,0}	{0}	{0}
{ }	{ }	{0}	{ }	{0}
{0}	{0}	{0}	{0}	{0}

$A \vee B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{1}	{1}	{1}
{1,0}	{1}	{1,0}	{1}	{1,0}
{ }	{1}	{1}	{ }	{ }
{0}	{1}	{1,0}	{ }	{0}

$A \rightarrow B$	{1}	{1,0}	{ }	{0}
{1}	{1}	{1,0}	{ }	{0}
{1,0}	{1}	{1,0}	{1}	{1,0}
{ }	{1}	{1}	{ }	{ }
{0}	{1}	{1}	{1}	{1}

From the set presented in **CL**, these are some arguments that are valid in **FDE**:

If $C \vDash_{\mathbf{FDE}} A$ and $C \vDash_{\mathbf{FDE}} B$, then $C \vDash_{\mathbf{FDE}} A \wedge B$

If $A \vDash_{\mathbf{CL}} C$ and $B \vDash_{\mathbf{CL}} C$ then $A \vee B \vDash_{\mathbf{CL}} C$

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{FDE}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{FDE}} A$ and $C \vDash_{\mathbf{FDE}} B$, then $C \vDash_{\mathbf{FDE}} A \wedge B$
- **Disjunction introduction (ID):** $A \vDash_{\mathbf{FDE}} A \vee B$
- **Disjunction (DISJ):** If $A \vDash_{\mathbf{FDE}} C$ and $B \vDash_{\mathbf{FDE}} C$ then $A \vee B \vDash_{\mathbf{FDE}} C$
- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{FDE}} (A \wedge B) \vee (A \wedge C)$

Given the properties of this logic, **FDE** has the interpretation { }, as in **K3**, which prevents logical truths from being definable based on language formulae, and it also has the interpretation {1,0}, as in **LP**, which also prevents logical falsehoods from being definable based on language formulas. **FDE** is a logic that has neither logical truths nor logical falsities. Some interesting philosophical and logical questions about **FDE** can be found in [87].

FDE is a much weaker logic than **CL**, **LP** and **K3**. This can be graphically represented in the following diagram.

Diagram of the relationship of the logics LC, LP, K3 and FDE

$$\begin{tikzcd}
 & \mathbf{CL} & \\
\mathbf{LP} \arrow[ur] & & \mathbf{K3} \arrow[ul] \\
 & \mathbf{FDE} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

2.3 Modifying logical consequence

I have shown how the set of valid arguments decreases when the set of interpretations increases. Now, in this section, I intend to show the results of some modifications to the logical consequence relation. The proposal to have less valid arguments by changing the logical consequence relation is not similar to the proposal of Pailos in [91]. The reason is that his strategy is to define a logic with at least three different logical consequence relations. There is one logical consequence relation for valid arguments of $\mathbf{TS}_\omega$. A different logical consequence relation for the anti-valid arguments of $\mathbf{ST}_\omega$. And finally, there is another logical consequence relation for contingencies of $[\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]$. In other words, his logic will have the same valid arguments that $\mathbf{TS}_\omega$, the same anti-valid arguments of $\mathbf{ST}_\omega$ and finally, the same contingencies of $[\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]$. Hence the logic is namely $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ In this section,

I want to modify **FDE**'s logical consequence relation only to see if this can help make the logic void of valid arguments. To do this, I will introduce the logic of **ETL**, **NFL**, **ST**, and **TS**. All these logics are the result of the consideration of the following two standards in relation of consequences:

- **S**: What is sought to be preserved is truth, but not the false.
- **T**: What is sought to be preserved is truth or non-falsity.

Taking the true and the false seriously is one of the advantages of this approach. Truth is usually understood as complementing falsehood (and vice-versa). This means that one of the two concepts is dispensable. But such mixed logical consequenses approaches take more seriously what we should preserve (only truth) and what we should not preserve (only falsity).

Let XY be a logical consequence relation, where X is the set of interpretations for the premises and Y is the set of interpretations for the conclusion. This gives rise to the four logics that I will present in this section:

- **SS**
- **TT**
- **ST**
- **TS**

SS logic is also known as **ETL**, see more in [124], [71], [113] and [18]. **ETL** has a logical consequence relationship of preserving only truth (and not falsity). On the other hand, **TT** logic is also known as **NFL**, see more in [18], [111] and [113]. **NFL** has a logical consequence relation of preservation of non-falsity (It preserves everything except falsity (and not truth)). **ST** and **TS** are considered as mixed logical consequence relations. They have different standards in the premises and in the conclusion. These logics have been the subject of even more research than have **SS** and **TT**, and some notable developments have been in [30], [107], [11] and [38]. I have to warn the reader, however, that most of the research on **ST** has been done because of its closeness to classical logic and because of its non-transitive approach. As a logic without valid arguments, **TS** has gone almost unnoticed. These four logics are presented in detail below.

2.3.1 ETL

ETL can be presented in a relational semantics with four admissible interpretations (the same as **FDE**). I will present the consequence relation of the logic **ETL**.

Definition 2.3.1 (Logical consequence of ETL). *An argument is valid in* ***ETL****,*

$\Gamma \vDash_{\boldsymbol{ETL}} A$, *iff, for all interpretation* σ, $1 \in \sigma(\gamma)$ *and* $0 \notin \sigma(\Gamma)$, *for all* $\gamma \in \Gamma$, *then* $1 \in \sigma(A)$ *and* $0 \notin \sigma(A)$.

The logical consequence preserves 'exactly true' (or only true), hence the acronym 'Exactly True Logic'.

From the set presented in **CL**, these are some arguments that are valid in **ETL**:

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{ETL}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{ETL}} A$ and $C \vDash_{\mathbf{ETL}} B$, then $C \vDash_{\mathbf{ETL}} A \wedge B$
- **Disjunction introduction (ID):** $A \vDash_{\mathbf{ETL}} A \vee B$
- **Disjunction (DISJ):** If $A \vDash_{\mathbf{ETL}} C$ and $B \vDash_{\mathbf{ETL}} C$ then $A \vee B \vDash_{\mathbf{ETL}} C$
- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{ETL}} (A \wedge B) \vee (A \wedge C)$
- **Detachment (MP):** $A, A \rightarrow B \vDash_{\mathbf{ETL}} B$
- **Explosion (EXP):** $A, \sim A \vDash_{\mathbf{ETL}} B$

Given the properties of this logic, **ETL** has no logical truths. In [94], Pietz has argued that this type of logic was not worked out for a long time because some people considered that the resulting logic would be a logic like **K3** or **LP**. While at first glance it may seem that **ETL** and **K3** share the same valid arguments, this is not true since arguments like $(A \wedge \sim A) \vee (B \wedge \sim B) \vDash_{\mathbf{L}} C$ are invalid in **ETL** and valid in **K3**. The following are counterexamples to show that this argument is invalid in **ETL**: $\sigma(A) = \{1,0\}$, $\sigma(B) = \{\ \}$ and $\sigma(C) = \{0\}$, $\sigma(A) = \{\ \}$, $\sigma(B) = \{1,0\}$ and $\sigma(C) = \{0\}$, $\sigma(A) = \{1,0\}$, $\sigma(B) = \{\ \}$ and $\sigma(C) = \{\ \}$, $\sigma(A) = \{\ \}$, $\sigma(B) = \{1,0\}$ and $\sigma(C) = \{\ \}$. Since all counterexamples require the interpretation {1,0} and the model-theoretic presentation of **K3** does not have this interpretation within the set of admissible interpretations, then it is impossible to invalidate this argument in **K3**.

2.3.2 NFL

NFL can be presented in a relational semantics with four admissible interpretations (the same as **FDE**. I will present the consequence relation of the logic **NFL**.

Definition 2.3.2 (Logical consequence of NFL). *An argument is valid in* ***NFL****,* $\Gamma \vDash_{\mathbf{NFL}} A$*, iff, for all interpretation* σ*,* $1 \in \sigma(\gamma)$ *or* $0 \notin \sigma(\Gamma)$*, for all* $\gamma \in \Gamma$*, then* $1 \in \sigma(A)$ *or* $0 \notin \sigma(A)$*.*

The logical consequence preserves the interpretations 'Not only false', hence the acronym 'Non-falsity Logics'

From the set presented in **CL**, these are some arguments that are valid in **NFL**:

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{NFL}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{NFL}} A$ and $C \vDash_{\mathbf{NFL}} B$, then $C \vDash_{\mathbf{NFL}} A \wedge B$
- **Disjunction introduction (ID):** $A \vDash_{\mathbf{NFL}} A \vee B$
- **Disjunction (DISJ):** If $A \vDash_{\mathbf{NFL}} C$ and $B \vDash_{\mathbf{NFL}} C$ then $A \vee B \vDash_{\mathbf{NFL}} C$
- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{NFL}} (A \wedge B) \vee (A \wedge C)$
- **Reflexivity of the conditional.(REFC):** $\vDash_{\mathbf{NFL}} A \rightarrow A$
- **Implosion (IMP) :** $B \vDash_{\mathbf{NFL}} A \vee \sim A$

NFL is an invalid logic that invalidates Explosion, like **LP**, however, invalid arguments like $A, B, \vDash_{\mathbf{L}} A \wedge B$that are valid in **LP** but invalid in **NFL**. Counterexamples to show that this argument is invalid in **NFL** are as follows: $\sigma(A) = \{1, 0\}$, $\sigma(B) =\{\ \}$ and $\sigma(A) =\{\ \}$, $\sigma(B) = \{1, 0\}$. Since all counterexamples require the interpretation $\{\ \}$ and the model-theoretic presentation of **LP** does not have this interpretation within the set of admissible interpretations, then it is impossible to invalidate this argument in **LP**.

Both **NFL** and **ETL** are extensions of **FDE**. Although **NFL** is a paraconsistent logic and **ETL** is a paracomplete logic, they do not coincide with **LP** and **K3**, respectively.

2.3.3 $\mathbf{ST}_4$

ST_4 is a four-interpretation version of the logic **ST**. The logical consequence relation of ST_4 is mixed. I will only use **ST** to refer to ST_4 from here on. Indeed, the set of interpretations for the premises is of the interpretations only true (as in **ETL**), and the set of interpretations for the conclusion is of the interpretations not only false (as in **NFL**). Hence its mnemonic name 'Strict/tolerant logic'. In the following, I will present the definition of logical consequence in **ST**.

Definition 2.3.3 (**Logical consequence of ST**)**.** *An argument is valid in* ***ST****,* $\Gamma \vDash_{\boldsymbol{ST}} A$*, iff, for all interpretation* σ*,* $1 \in \sigma(\gamma)$ *and* $0 \notin \sigma(\Gamma)$*, for all* $\gamma \in \Gamma$*, then* $1 \in \sigma(A)$ *or* $0 \notin \sigma(A)$*.*

From the set presented in **CL**, these are some arguments that are valid in **ST**:

- **Conjunction elimination (EC):** $A \wedge B \vDash_{\mathbf{ST}} A$
- **Conjunction (CONJ):** If $C \vDash_{\mathbf{ST}} A$ and $C \vDash_{\mathbf{ST}} B$, then $C \vDash_{\mathbf{ST}} A \wedge B$
- **Disjunction introduction (ID):** $A \vDash_{\mathbf{ST}} A \vee B$
- **Disjunction (DISJ):** If $A \vDash_{\mathbf{ST}} C$ and $B \vDash_{\mathbf{ST}} C$ then $A \vee B \vDash_{\mathbf{ST}} C$
- **Distribución (DIST):** $A \wedge (B \vee C) \vDash_{\mathbf{ST}} (A \wedge B) \vee (A \wedge C)$
- **Reflexivity of the conditional.(REFC):** $\vDash_{\mathbf{ST}} A \rightarrow A$
- **Detachment (MP):** $A, A \rightarrow B \vDash_{\mathbf{ST}} B$
- **Explosion (EXP):** $A, \sim A \vDash_{\mathbf{ST}} B$

- **Implosion (IMP) :** $B \vDash_{\mathbf{ST}} A \vee \sim A$

Cobreros, Égré, Ripley, and van Rooij [30] proved that the logic **ST** has a set of valid arguments VAL_{ST} which is coextensional with VAL_{LC}. An argument in **CL** is valid if and only if it is valid in **ST**, a fact that can also be corroborated in the list of valid arguments. However, the logic **ST** and the logic **CL** are different since they differ in the set of anti-valid arguments.

Definition 2.3.4 (Anti-valid argument in **ST**)**.** *An argument is anti-valid in* ***ST*** *iff, for all interpretation* σ*, if* $1 \in \sigma(\gamma)$ *and* $0 \notin \sigma(\Gamma)$*, for all* $\gamma \in \Gamma$*, then* $0 \in \sigma(A)$ *and* $1 \notin \sigma(A)$

Consider the following invalid argument in classical logic $(A \vee \sim A) \vDash_{\mathbf{CL}} (B \wedge \sim B)$. This argument is not anti-valid in **ST** and is anti-valid in **CL**. Just consider the following counterexample: $\sigma(A) = \{1\}$ and $\sigma(B) = \{1.0\}$ or $\sigma(B) = \{\quad\}$. In this argument, the premises are only true, but the conclusion is not only false. Scambler [107] proved that the set of invalid arguments of **ST** is not coextensive with the set of invalid arguments of **CL**.

Another way of distinguishing logics like **ST** from classical logic has been put forward in [12] and [8]. **ST** differs from classical logic in the collection of meta-arguments, particularly Transitivity[1] is invalid in **ST**, whereas in classical logic it is valid. In this way of identification of logics, two logics will be identical if they have the same set of arguments and meta-arguments at all levels. However, it is unnecessary to go into this for now since Scambler has proved that **ST** differs from classical logic in its anti-valid arguments.

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[1] In [49], Ferguson and Ramírez-Cámara show that how **ST** meta-arguments have been evaluated is problematic. Classicity is lost level by level. They propose using the extensional conditional of **LP** to interpret the logical consequence for meta-arguments.

2.3.4 TS$_4$

TS_4 is a four-interpretation version of the logic **TS**. The logical consequence relation of TS_4 is mixed. I will only use **TS** to refer to TS_4. Indeed, the set of interpretations for the premises is the set of not only false interpretations (as in '**NFL**'), and the set of interpretations for the conclusion is the set of only true interpretations (as in '**ETL**'). Hence its mnemonic name 'tolerant/Strict logic'. I will present the consequence relation of **TS** in the following.

Definition 2.3.5 (Logical consequence of TS). *An argument is valid in* ***TS****,* $\Gamma \vDash_{\boldsymbol{ST}} A$*, iff, for all interpretation* σ*,* $1 \in \sigma(\gamma)$ *or* $0 \notin \sigma(\Gamma)$*, for all* $\gamma \in \Gamma$*, then* $1 \in \sigma(A)$ *and* $0 \notin \sigma(A)$*.*

From the set presented in **CL**, these are some arguments that are valid in **TS**:

- **Conjunction (CONJ):** If $C \vDash_{\mathbf{TS}} A$ and $C \vDash_{\mathbf{TS}} B$, then $C \vDash_{\mathbf{TS}} A \wedge B$
- **Disjunction (DISJ):** If $A \vDash_{\mathbf{TS}} C$ and $B \vDash_{\mathbf{TS}} C$ then $A \vee B \vDash_{\mathbf{TS}} C$

Arguments that turn out to be valid are characterized by being meta-arguments. Since I assume that meta-arguments are to be evaluated in the following way:

- A meta-argument is *valid* if and only if, if the meta-premises $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ are valid, the meta-conclusion $\Delta \vDash B$ is valid as well.

all meta-arguments are valid. Depending on the definition of the validity of meta-arguments, this characterization of the **TS** meta-argument set could vary. In another chapter of this research, however, this will be addressed.

On the other hand, **TS** logic has as one of its characteristics that the set of anti-valid arguments of **TS**, defined as $AntVAL_{TS}$, is coextensional with the set of invalid arguments of classical logic **CL**, defined as $AntVAL_{LC}$. An anti-valid argument in **TS** must satisfy the following definition:

Definition 2.3.6 (Anti-valid argument in **TS**). *An argument is anti-valid in **TS** iff, for all interpretation σ, if $1 \in \sigma(\gamma)$ or $0 \notin \sigma(\Gamma)$, for all $\gamma \in \Gamma$, then $0 \in \sigma(A)$ or $1 \notin \sigma(A)$*

TS is a special kind of logic. All of its arguments, but not its meta-arguments, are not valid. That means that all of those who argue are invalid. It could satisfy the definition of a minimally empty logic from the previous chapter if we could empty it of meta-arguments too.

2.4 Emptying a little more to FDE

It is a fact that by introducing more interpretations, it is possible to reduce the number of valid arguments, even to the point where the set of valid arguments (not meta-arguments) is empty. Secondly, it has been shown that by changing the logical consequence, as in the case of **ETL**, a logic can recover some valid arguments compared to **LP**. Third, mixed consequence relations show how it is possible to recover all the valid arguments of a logic but to differ in the anti-valid ones. I also show how a logic can be empty of valid arguments without being empty (since it has anti-valid arguments).

In [103], [104], and [106], Gillian Russell has outlined a path of context-dependent interpretations that can be used for the definition of formulas that might be useful for the emptying of a logic. This means: It is possible to empty a certain logic further, if we keep increasing the number of interpretations or change some criteria of the interpretations. Russell [103] [104] [105] wanted to prove that we can create counterexamples to any valid argument, even to arguments that seem to hold in most logics such as *Conjunction introduction* and *Reflexivity of the Conditional.*

Russell [104] presented two formulas where his interpretation is context-dependent[2]:

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[2] [69] and [65] can be seen as the predecessors of this type of context-dependent sentences.

$SOLO$ and $PREM$. $SOLO$ was presented as a counterexample to the *Conjunction introduction* and $PREM$ as a counterexample to *Reflexivity of the Conditional.* Following this route, the formulas must now be evaluated according to their context.

Some contexts to consider:

- A formula A has one interpretation when it is a premise σ_p and a different interpretation when it is a conclusion σ_c.
- A formula A has one interpretation when it is an atomic formula σ_a and a different interpretation when it is a subformula of A σ_s.

To provide some counter-examples to logical laws, such contexts are helpful. I will now go on to show how to do this.

Consider the following formulas: Now consider the following evaluating conditions:

$SOLO$[3]

The evaluation conditions of $SOLO$ are as follows:

Option 1:

- $1 \in \sigma_a(SOLO)$ iff $1 \notin \sigma_s(SOLO)$
- $0 \in \sigma_a(SOLO)$ iff $1 \in \sigma_s(SOLO)$ and $1 \notin \sigma_s(SOLO)$
- $1 \in \sigma_s(SOLO)$iff $1 \in \sigma_a(SOLO)$ and $1 \notin \sigma_a(SOLO)$
- $0 \in \sigma_s(SOLO)$ iff $0 \notin \sigma_a(SOLO)$

Option 2:

- $1 \in \sigma_a(SOLO)$ iff $1 \notin \sigma_s(SOLO)$

---

[3] Although $PREM$ could also be used to invalidate the other arguments, introducing $SOLO$ allows you to introduce another context which could invalidate other arguments.

- $0 \in \sigma_a(SOLO)$ iff $0 \in \sigma_s(SOLO)$
- $1 \in \sigma_s(SOLO)$ iff $1 \notin \sigma_a(SOLO)$ and $1 \in \sigma_a(SOLO)$
- $0 \in \sigma_s(SOLO)$ iff $1 \in \sigma_a(SOLO)$ and $0 \in \sigma_a(SOLO)$

Option 3:

- $1 \in \sigma_a(SOLO)$ iff $1 \notin \sigma_s(SOLO)$
- $0 \in \sigma_a(SOLO)$ iff $1 \notin \sigma_s(SOLO)$ and $0 \notin \sigma_s(SOLO)$
- $1 \in \sigma_s(SOLO)$ iff $1 \in \sigma_s(SOLO)$ and $1 \notin \sigma_s(SOLO)$
- $0 \in \sigma_s(SOLO)$ iff $1 \in \sigma_a(SOLO)$ and $0 \notin \sigma_a(SOLO)$

These two contexts of interpretation(σ_a and σ_s) only affect to $SOLO$. That is, for any other formula A, $\sigma(A) = \sigma_a(A) = \sigma_s(A)$. Option 1 defines $ONLY$ as a formula that is always only true when it is an atomic formula. $\sigma_p(SOLO) = \{1\}$ and only false when it is a sub-formula $\sigma_c(SOLO) = \{0\}$. Option 2 and option 3 define $ONLY$ as a formula that is always true when it is an atomic formula $1 \in \sigma_p(SOLO)$ and not true when it is a sub-formula $1 \notin \sigma_p(SOLO)$. The difference between option 1 and option 2 is that in option 1, each time that $\sigma_p(SOLO) = \{1\}$ then $\sigma_c(SOLO) = \{\ \}$ and when $\sigma_p(SOLO) = \{1, 0\}$, then $\sigma_c(SOLO) = \{0\}$, while in option 2, each time that $\sigma_p(SOLO) = \{1\}$ then $\sigma_c(SOLO) = \{0\}$ and when $\sigma_p(SOLO) = \{1, 0\}$, then $\sigma_c(SOLO) = \{\ \}$.

Counterexample to Conjunction (CONJ)

If $C \vDash_{\mathbf{L}} SOLO$ and $C \vDash_{\mathbf{L}} B$ then $C \vDash_{\mathbf{L}} SOLO \wedge B$

Based on the interpretations of B and C as follows:$\sigma(B) = \{1, 0\}$ and $\sigma(C) = \{1, 0\}$, it is possible to obtain a counterexample to Conjunction Introduction and this argument is invalidated in $\mathbf{FDE}_{SOLO}$.

Counterexample to Disjunction introduction

$$SOLO \vDash_{\mathbf{L}} SOLO \vee B$$

It is possible to obtain a counterexample to the argument *Disjunction introduction*. See the following interpretation of B: $\sigma(B) = \{0\}$. With this counterexample, the argument of *Introduction of Disjunction* is invalidated in $\mathbf{FDE}_{SOLO}$.

$$PREM$$

The evaluation conditions of $PREM$ are as follows:

Option 1:

- $1 \in \sigma_p(PREM)$ iff $1 \notin \sigma_c(PREM)$
- $0 \in \sigma_p(PREM)$ iff $1 \in \sigma_c(PREM)$ and $1 \notin \sigma_c(PREM)$
- $1 \in \sigma_c(PREM)$ iff $1 \in \sigma_c(PREM)$ and $1 \notin \sigma_c(PREM)$
- $0 \in \sigma_c(PREM)$ iff $0 \notin \sigma_p(PREM)$

Option 2:

- $1 \in \sigma_p(PREM)$ iff $1 \notin \sigma_c(PREM)$
- $0 \in \sigma_p(PREM)$ iff $0 \in \sigma_c(PREM)$
- $1 \in \sigma_c(PREM)$ iff $1 \in \sigma_p(PREM)$ and $1 \notin \sigma_p(PREM)$
- $0 \in \sigma_c(PREM)$ iff $1 \in \sigma_p(PREM)$ and $0 \in \sigma_p(PREM)$

Option 3:

- $1 \in \sigma_p(PREM)$ iff $1 \notin \sigma_c(PREM)$
- $0 \in \sigma_p(PREM)$ iff $1 \notin \sigma_c(PREM)$ and $0 \notin \sigma_c(PREM)$

- $1 \in \sigma_c(PREM)$ iff $1 \in \sigma_c(PREM)$ and $1 \notin \sigma_c(PREM)$

- $0 \in \sigma_c(PREM)$ iff $1 \in \sigma_p(PREM)$ and $0 \notin \sigma_p(PREM)$

These two contexts of interpretation(σ_a and σ_s) only affect to $PREM$. That is, for any other formula A, $\sigma(A) = \sigma_p(A) = \sigma_c(A)$.

Option 1 defines $PREM$ as a formula that is always only true when premise $\sigma_p(PREM) = \{1\}$ and only false when conclusion $\sigma_c(PREM) = \{0\}$. Option 2 and option 3 define $PREM$ as a formula that is always true when it is premise $1 \in \sigma_p(PREM)$ and not true when it is conclusion $1 \notin \sigma_p(PREM)$. The difference between option 1 and option 2 is that in option 1 whenever $\sigma_p(PREM) = \{1\}$ then $\sigma_c(PREM) = \{\ \}$ and when $\sigma_p(PREM) = \{1, 0\}$, then $\sigma_c(PREM) = \{0\}$, while in option 2, whenever$\sigma_p(PREM) = \{1\}$ then $\sigma_c(PREM) = \{0\}$ and when $\sigma_p(PREM) = \{1, 0\}$, then $\sigma_c(PREM) = \{\ \}$.

Now that I have defined $PREM$, I can show how it is useful for the definition of counter-examples to some valid arguments in **FDE**:

Counterexample to Conjunction elimination

$$PREM \wedge B \vDash_{\mathbf{L}} PREM$$

Consider the following interpretation of B: $\sigma(B) = \{1\}$. $\mathbf{FDE}_{PREM}$ invalidates the Conjunction Elimination argument.

Counterexample to Disjunction (DISJ)

$$\text{If} \sim SOLO \vDash_{\mathbf{L}} PREM \text{ and} \sim PREM \vDash_{\mathbf{L}} PREM$$

$$\text{then}$$

$$\sim SOLO \vee \sim PREM \vDash_{\mathbf{L}} PREM$$

For both $\sim SOLO \vDash_{\mathbf{L}} PREM$ and $\sim PREM \vDash_{\mathbf{L}} PREM$ to be valid, it is sufficient that $PREM$ and $SOLO$ are defined as in option 1. $\sim SOLO \vDash_{\mathbf{L}} PREM$ will be valid because the negation of $SOLO$ will only be false. And since PREM is a conclusion in $\sim SOLO \vDash_{\mathbf{L}} PREM$, it will also be false. Then this argument is valid by vacuity. On the other hand, in $\sim PREM \vDash_{\mathbf{L}} PREM$, because $PREM$ (in premises) is true, the negation is only false and $PREM$ (of the conclusion) is false, then the argument is also valid by vacuity. However, in $\sim SOLO \vee \sim PREM \vDash_{\mathbf{L}} PREM$, $SOLO$ is false (because it is in a disjunction), so its negation will be true; $PREM$ is as a premise, so its negation will be false. Therefore the disjunction of the premises of $\sim SOLO \vee \sim PREM \vDash_{\mathbf{L}} PREM$ is true. Finally, $PREM$ of the conclusion of this argument makes this argument to be logically invalid in $\mathbf{FDE}_{PREM,SOLO}$.

Counterexample to Distribution

$$PREM \wedge (B \vee C) \vDash_{\mathbf{L}} (PREM \wedge B) \vee (PREM \wedge C)$$

Based on the evaluation conditions of option 1 of $PREM$, the argument of *Distribution* will be invalid if $B = \{1\}$. C can have any interpretation. This is because in the premises, it satisfies the requirement of being true, but in the conclusion, it will always be only false because both sets of conjunctions in the conclusion will always be false. The argument of *Distribution* in $\mathbf{FDE}_{\{PREM\}}$ is invalidated by this counterexample. The same formula is used by Russell in [104] to counter the argument of *Reflexivity of the Conditional*. All valid arguments of **FDE** can be invalidated using the same strategy.

I can get the logic $\mathbf{FDE}_{\{PREM,SOLO\}}$ by defining a logic that is an expansion of **FDE** plus the two context-dependent formulas defined here. I'll call $\mathbf{FDE}_{\{PREM,SOLO\}}$ as *Context Dependent Formula Logic* (**CDFL**). We can invalidate all arguments in the versions of $PREM$ and $SOLO$ where they are not true instead of false. However,

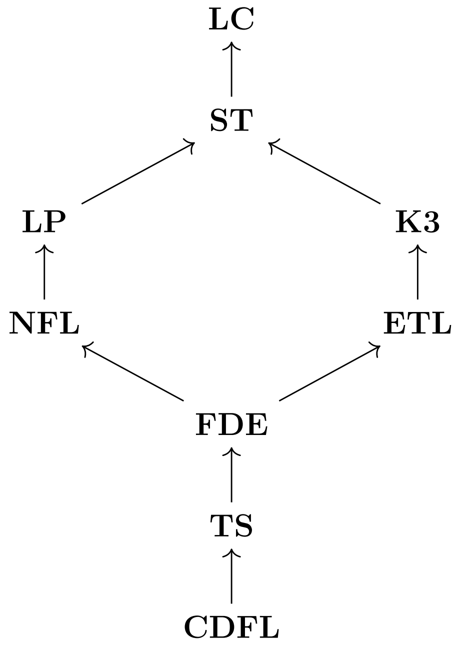
Argument	**CL**	**K3**	**LP**	**FDE**	**ETL**	**NFL**	**ST**	**TS**	**CDFL**
EC	✓	✓	✓	✓	✓	✓	✓	✗	✗
CONJ	✓	✓	✓	✓	✓	✓	✓	✓	✗
ID	✓	✓	✓	✓	✓	✓	✓	✗	✗
DISJ	✓	✓	✓	✓	✓	✓	✓	✓	✗
DIST	✓	✓	✓	✓	✓	✓	✓	✗	✗
REFLC	✓	✗	✓	✗	✗	✓	✓	✗	✗
MP	✓	✓	✗	✗	✓	✗	✓	✗	✗
EXP	✓	✓	✗	✗	✓	✗	✓	✗	✗
IMP	✓	✗	✓	✗	✗	✓	✓	✗	✗

Table 2.1: Comparative table of valid and invalid arguments

we must use a different strategy to invalidate some meta-arguments. Nevertheless, I will solve this drawback in this research to achieve at least a minimally empty logic.

2.4.1 Comparative table

Below I present a table summarising what I have shown above. Where '✓' and '✗' indicate that the argument in question is valid and that the argument in question is invalid in the logic under consideration.



2.5 Conclusion

When a particular logic is presented, one of the necessary aims is to characterize the set of valid arguments (see more in [60])), but this is not enough to be able to individualize one logic concerning another. In the following, I will show some of the reasons for this.

I showed that **LP** has the same logical truths as classical logic. However, **LP**'s set of valid arguments differs from **CL**'s set of valid arguments. Conversely, **K3** has the same falsehoods as classical logic. However, **K3** differs from the set of valid arguments of **CL**. The consequence is that the individualization of a logic based on the set of logical truths or logical falsehoods is not sufficient.

I also showed that the logic **ST** has a set of valid arguments. This set is coextensive with the valid arguments of **CL**. In the set of anti-valid arguments, however, the two logics differ. On the other hand, the logic **TS** has a set of anti-valid arguments. This set is coextensional with the anti-valid arguments of **CL**. In the set of valid arguments, however, the two logics differ.

Since a logic can have either the same set of valid arguments as Classical Logic or the same set of anti-valid arguments as Classical Logic, in [107] Scambler concludes that a good characterization of the arguments of a logic must be able to define at least these sets of arguments: valid, invalid and anti-valid arguments.

As shown in *Comparative table of valid and invalid arguments*, **K3** and **ETL** seem to have the same set of valid arguments and invalidate the same arguments. However, these two logics are not the same. Pietz [94] has proved that **K3** is different from **ETL** in the sets of anti-valid arguments. This means that the set of anti-valid arguments of **K3** is not coextensive with the set of anti-valid arguments of **ETL**. It is possible to highlight what a logic rejects by paying attention to its anti-valid arguments. In this respect, characterizing this set of arguments can be useful for individualizing a logic. To individualize a logic, it is necessary to delimit the set of valid arguments (VAL), the set of anti-valid arguments ($AntVAL$), and the set of invalid arguments ($InVAL$).

In summary, I have presented some ways in which we can empty a logic, which can be summarised in the following two points:

- Changing the logical consequence relation.
- Increasing the set of admissible interpretations.

The first possibility was also explored in [91] by Pailos with the logic $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$. This logic is empty because it combines the logical relation so that it manages to empty itself. Another similar way of emptying a logic has recently been explored in [90], also by Pailos. In this approach, logics with mixed consequence relations, especially disjoint ones, achieve some empty or minimally empty logics. That is logical consequential relations, where the intersection of the standard of the premises and the standard of the conclusion (of the logical consequential relation) is disjoint. The latter approach is developed in detail in Chapter **??**.

I have also shown how the second option seems the most natural way to empty the logic of its arguments. The dependency between the valid schemas in a logic **L** and its set of admissible interpretations can be made explicit by showing how the arguments of classical logic are emptied when the set of admissible interpretations is increased (up to $PREM$ and $SOLO$). Thus, the more admissible interpretations in a logic, the fewer valid arguments there will be.

Chapter 3

Directions in Strong-Kleene disjoint logics

Abstract

In this chapter, I will show the characteristics of some subgroups that are part of some of the disjoint consequence relations defined by Pailos. [90]. I argue that there are three subgroups: a) forward but not backward disjoint consequence relations, b) backward but not forward disjoint consequence relations, and c) disjoint consequence relations in both directions. The chapter concludes by arguing why it is useful and fruitful to consider the direction of logical consequence when characterizing a logic.

Keywords: Disjoint logics; Strong-Kleene; direction; consequence relations; backwards-preservation.

3.0 Introduction

A Strong-Kleene valuation for a logic has the following set of admissible interpretations $\{\{1\}, \{\ \}, \{0\}\}$, where $\{1\}$ represents only true, $\{\ \}$ represents neither true nor false and $\{0\}$ represents only false. Let X be a set of interpretations and $\overline{X}$ its complement. Consider the following sets of interpretations, each of which is an element of the power of $\{\{1\},\{\ \},\{0\}\}$:

- $S = \{\{1\}\}$
- $\overline{S} = \{\{\ \},\{0\}\}$
- $T = \{\{1\},\{\ \}\}$
- $\overline{T} = \{\{0\}\}$
- $N = \{\{\ \}\}$
- $\overline{N} = \{\{1\},\{0\}\}$
- $\varnothing = \{\ \}$
- $\overline{\varnothing} = \{\{1\},\{\ \},\{0\}\}$

A disjoint logical consequence is a consequence relation in which the intersection of the set of interpretations of the premises (σ_p) and the set of interpretations of the conclusions (σ_c) is empty (*cf.* [90]). Formally, it can be defined as follows:

Definition 3.0.1. $\Gamma \vDash^{disj} \delta$ *if and only if for all interpretations* σ*, if, for any* $\gamma \in \Gamma$, $\sigma(\gamma) \in \sigma_p$*, then there is a* δ[1] *such that* $\sigma(\delta) \in \sigma_c$ *and also* $\sigma_p \cap \sigma_c = \varnothing$.

Let XY be a logical consequence relation, where X is the set of interpretations for the premises and Y is the set of interpretations for the conclusion. The disjoint logical consequences based on the Strong-Kleene valuations are as follows:

(1) SN
(2) NS
(3) $\overline{T}N$
(4) $N\overline{T}$
(5) $S\overline{T}$
(6) $\overline{T}S$
(7) $S\overline{S}$
(8) $\overline{S}S$
(9) $N\overline{N}$
(10) $\overline{N}N$
(11) $\overline{T}T$
(12) $T\overline{T}$
(13) $\overline{\varnothing}\varnothing$
(14) $\varnothing\overline{\varnothing}$
(15) $\varnothing T$
(16) $\varnothing\overline{N}$
(17) $\varnothing\overline{S}$
(18) $\varnothing S$
(19) $\varnothing N$
(20) $\varnothing\overline{T}$
(21) $T\varnothing$
(22) $\overline{N}\varnothing$
(23) $\overline{S}\varnothing$
(24) $S\varnothing$
(25) $N\varnothing$
(26) $\overline{T}\varnothing$
(27) $\varnothing\varnothing$

---

[1] Moreover, one could also ask for Γ and δ to be multisets or lists.

Federico Pailos has characterized this type of logical consequence, where some cases are:

— non-trivial and non-empty logical consequences[2], like, SN, $\overline{T}N$, $S\overline{T}$, $\overline{T}S$, $\overline{S}S$, $\overline{N}N$, $\overline{T}T$, $\varnothing\overline{\varnothing}$, $\varnothing T$, $\varnothing\overline{N}$, $\varnothing\overline{S}$, $\varnothing S$, $\varnothing N$, $\varnothing\overline{T}$, $S\varnothing$, $\overline{T}\varnothing$, $\varnothing\varnothing$.

For example,

- SN, $S\overline{T}$, $S\overline{S}$ and $S\varnothing$ have at least one valid argument: $A\wedge \sim A \vDash B$.
- In $\varnothing\overline{\varnothing}$, every non-empty argument is valid — i,e,. every argument with at least one premise or at least one conclusion.
- In $\varnothing\overline{S}$, every argument without an empty set of premises and a logical falsehood in the conclusion will be valid.
- In $\varnothing T$, every argument without an empty set of premises and a logical truth in the conclusion will be valid.
- In $S\overline{S}$ every argument with a logical falsehood in premises or in conclusion be valid.
- In $\overline{T}T$ every argument with a logical truth in premises or in conclusion is valid; moreover, $\overline{T}S$ is strictly weaker than $\overline{T}T$.
- In $\overline{T}\varnothing$, $\overline{T}N$ and $\overline{T}T$ every argument with logical truths like a set of premises is valid.
- In $\varnothing\overline{T}$, $\varnothing\overline{N}$, $\varnothing T$, $\varnothing S$, $\varnothing N$ and $\varnothing\varnothing$ every argument with non-empty premises is valid.

— Empty logical consequences, like $N\overline{N}$, NS, $N\overline{T}$, $N\varnothing$, $T\varnothing$, $\overline{S}\varnothing$, $\overline{N}\varnothing$, $\overline{S}S$, $T\overline{T}$, $\overline{\varnothing}\varnothing$.

---

[2] This type of consequence relationship is characterized by having at least one valid argument and one invalid argument.

The definition of disjoint logical consequence does not take into account the *direction* of preservation of the set of interpretations. (For further discussion see [121] and [45].). Let $\models_{\overrightarrow{disj}}$ be a logical consequence defined through *forwards-preservation*: it goes from the premises to the conclusion. $\models_{\overleftarrow{disj}}$ is a logical consequence defined through *backwards-preservation*: it goes from the conclusion to the premises. In most known logical consequence relations, the truth is *forwards-preservation* and the untruth is *backwards-preservation*.

Some of Pailos' Strong-Kleene disjoint logical consequences are disjoint when defined forwards, but not when defined backwards. This motivates studying disjointness also in the other direction, but more importantly, disjointness in *both* directions. In this chapter, I present the conditions that a logical consequence, based on Strong-Kleene valuations, must meet to be $\models_{\overrightarrow{disj}}$ and $\models_{\overleftarrow{disj}}$.

The structure of the document is as follows: in Section 1, I present what a Strong-Kleene evaluation is, the set of admissible interpretations, and the set of connectives to consider. In Section 2, I present three subgroups of disjoint logical consequence relations and also show how to obtain disjoint logical consequence relations with n admissible interpretations. In Section 3, I argue why paying attention to the direction of the logical consequence relation is useful when studying disjointness and when presenting and characterizing sets of arguments in any given logic.

3.1 Strong-Kleene valuation

Let $\mathcal{L}$ be a language for **K3** with formulas built in the usual way, from a countable set of propositional variables $Var = \{p_1, \ldots, p_n\}$ with the following set of connectives $\{\neg, \wedge, \vee \rightarrow\}$. As a basis for some proofs in this text, I will assume ZFC set theory. Let X, Y, Z be *sets* and x, y and z be elements of them. I use the first capital letters of the Latin alphabet, 'A', 'B', 'C', ... as arbitrary formulas of $\mathcal{L}$.

A *Strong-Kleene valuation* or *interpretation* for a logic $\mathcal{L}$ can be presented as a relation between atomic formulas $p_1, \ldots, p_n$ and two truth values 1 (true) and 0 (false), via an assignment σ, such that an atomic formula p_i can be related to truth values in one of the following three ways:

- p_i is true but not false, represented by '$1 \in \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1\}$
- p_i is neither true nor false, represented by '$1 \notin \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) =\{\ \ \}$
- p_i is false but not true, represented by '$0 \in \sigma(p_i)$ and $1 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{0\}$

The set τ is defined as the set of connectives $\tau = \{\sim, \wedge, \vee \rightarrow\}$, which represent negation, conjunction, disjunction, and conditional. The evaluation conditions of the connectives are as follows:

$1 \in \sigma(\sim A)$ if and only if $0 \in \sigma(A)$

$0 \in \sigma(\sim A)$ if and only if $1 \in \sigma(A)$

$1 \in \sigma(A \wedge B)$ if and only if $1 \in \sigma(A)$ and $1 \in \sigma(B)$

$0 \in \sigma(A \wedge B)$ if and only if $0 \in \sigma(A)$ or $0 \in \sigma(B)$

$1 \in \sigma(A \vee B)$ if and only if $1 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \vee B)$ if and only if $0 \in \sigma(A)$ and $0 \in \sigma(B)$

$1 \in \sigma(A \rightarrow B)$ if and only if $0 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \rightarrow B)$ if and only if $1 \in \sigma(A)$ and $0 \in \sigma(B)$

The following tables correspond to the connectives defined above:

$\sim A$	A
$\{0\}$	$\{1\}$
$\{\ \}$	$\{\ \}$
$\{1\}$	$\{0\}$

$A \wedge B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

$A \vee B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{\ \}$	$\{0\}$

$A \rightarrow B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$

3.2 Directions in disjoint logics

A disjoint logical consequence relation can be defined through *forwards-preservation*:

Definition 3.2.1. $\Gamma \vDash^{\vec{d}isj} \delta$ *if and only if for all interpretations* σ*, if for any* $\gamma \in \Gamma$*,* $\sigma(\gamma) \in \sigma_p$*, then there is a* δ *such that* $\sigma(\delta) \in \sigma_c$ *and also* $\sigma_p \cap \sigma_c = \varnothing$*.*

These types of logical consequence relations were explored and characterized by Pailos, who showed some of their properties with respect to their argument sets.

On the other hand, a disjoint logical consequence relation can be defined through *backward-preservation*:

Definition 3.2.2. $\Gamma \vDash^{\overleftarrow{d}isj} \delta$ *if and only if for all interpretations* σ*, if for any* δ*s such that* $\sigma(\delta) \in \sigma_c$*, then for some* $\gamma \in \Gamma$*,* $\sigma(\gamma) \in \sigma_p$ *and also* $\overline{\sigma_c} \cap \overline{\sigma_p} = \varnothing$*.*

These logical consequence relations would preserve at least one interpretation from the conclusion to at least one of the premises, but the inverse logical consequence relation is disjoint, i.e., they do not preserve any interpretation from the premises to the conclusion.

All Strong-Kleene disjoint logical consequences defined through *backwards-preservation* are:

(1) $\overline{SN}$
(2) $\overline{NS}$
(3) $T\overline{N}$
(4) $\overline{N}T$
(5) $\overline{S}T$
(6) $T\overline{S}$
(7) $\overline{S}S$
(8) $S\overline{S}$
(9) $\overline{N}N$
(10) $N\overline{N}$
(11) $T\overline{T}$
(12) $\overline{T}T$
(13) $\overline{\varnothing}\varnothing$
(14) $\varnothing\overline{\varnothing}$
(15) $\overline{\varnothing}\overline{T}$
(16) $\overline{\varnothing}N$
(17) $\overline{\varnothing}S$
(18) $\overline{\varnothing}\overline{S}$
(19) $\overline{\varnothing}\overline{N}$
(20) $\overline{\varnothing}T$
(21) $\overline{T}\overline{\varnothing}$
(22) $N\overline{\varnothing}$
(23) $S\overline{\varnothing}$
(24) $\overline{S}\overline{\varnothing}$
(25) $\overline{N}\overline{\varnothing}$
(26) $T\overline{\varnothing}$
(27) $\overline{\varnothing\varnothing}$

For example, in the logical consequence relation X the intersection of σ_p and σ_c is empty. However, the intersection of the complements $\overline{\sigma_p}$ and $\overline{\sigma_c}$ is not empty: there is preservation of the interpretation $\{\{0\}\}$.

Also, a disjoint logical consequence relation can be defined through both *forwards* and *backward-preservation* simultaneously:

Definition 3.2.3. $\Gamma \vDash_{\overset{\leftrightarrow}{disj}} \delta$ *if and only for all interpretations* σ*, if for any* $\gamma \in \Gamma$, $\sigma(\gamma) \in \sigma_p$*, then there is a* δ *such that* $\sigma(\delta) \in \sigma_c$*, and also* $\sigma_p \cap \sigma_c = \varnothing$ *and* $\overline{\sigma_c} \cap \overline{\sigma_p} = \varnothing$*.*

This type of disjoint logical consequence does not preserve any interpretation from the premises to the conclusion nor from the conclusion to at least one of the premises. Such consequence relations are disjoint in both directions. This group is formed by the intersection of the disjoint logical consequences defined through *forwards-preservation* and *backward-preservation.*

3.2.1 Some subgroups taking into account the directions

It is possible to distinguish at least three subgroups of disjoint logical consequences that satisfy one and only one of these requirements:

1. $\sigma_p \cap \sigma_c = \varnothing$ and $\overline{\sigma_c} \cap \overline{\sigma_p} \neq \varnothing$. (Forward but not backward disjoint)

2. $\sigma_c \cap \sigma_p \neq \varnothing$ and $\overline{\sigma_c} \cap \overline{\sigma_p} = \varnothing$ (Backward but not forward disjoint)

3. $\sigma_p \cap \sigma_c = \varnothing$ and $\overline{\sigma_c} \cap \overline{\sigma_p} = \varnothing$. (Both forward and backward disjoint)

The first subgroup are disjoint logical consequences defined through *forwards-preservation* but not through *backward-preservation*; it preserves at least one interpretation from the premises to the conclusion but preserves no interpretation from the conclusion to the premises. To show this, it suffices to prove two theorems of Set Theory.

Theorem 3.2.1. *If $A \subseteq C$, $B \subseteq C$, $A \cup B = C$ and $A \neq \overline{B}$, then $A \cap B \neq \varnothing$*

Proof. Since $A \neq \overline{B}$, there is an $x \in A \wedge x \notin \overline{B}$ or $x \notin A \wedge x \in \overline{B}$. In the first case, we obtain that $x \in A$ and $x \in B$, because it is not the case that $x \notin C$ (C is our relative universal set). In the second case, we obtain that $x \notin A \cup B$. So $x \notin C$, *contradiction.* Thus, $x \in A$ and $x \in B$. Then, $A \cap B \neq \varnothing$. □

Theorem 3.2.2. *If $A \subseteq C$, $B \subseteq C$ and $A \cap B = \varnothing$, then $\overline{A \cap B} = \overline{\varnothing}$*

Proof. We know that if two sets are equal, then their complements are equal. So, if $A \cap B = \varnothing$, then $\overline{A \cap B} = \overline{\varnothing}$. □

With these theorems, I can proceed to prove the characteristics of the first subgroup where the consequence relation is disjoint through *forwards-preservation.* What I will prove is that the logical consequence relation is disjoint via *backwards-preservation* if and only if $\sigma_c \neq \overline{\sigma_p}$.

Theorem 3.2.3. *Let $\sigma_p \cap \sigma_c = \varnothing$. $\overline{\sigma_p} \cap \overline{\sigma_c} \neq \varnothing$ if and only if $\sigma_c \neq \overline{\sigma_p}$*

Proof. The proof from left to right is as follows: if $\overline{\sigma_p} \cap \overline{\sigma_c} \neq \varnothing$, then there is an x such that $x \in \overline{\sigma_p} \cap \overline{\sigma_c}$ and $x \notin \varnothing$, or, $x \in \varnothing$ and $x \notin \overline{\sigma_p} \cap \overline{\sigma_c}$. Second case is impossible, In the first case, we obtain that there is an x such that $x \in \overline{\sigma_p}$ and $x \notin \sigma_c$. Thus, $\sigma_c \neq \overline{\sigma_p}$.

Conversely, the proof from right to left is as follows: if $\sigma_c \neq \overline{\sigma_p}$, then there is an x such that $x \in \sigma_c$ and $x \notin \overline{\sigma_p}$, or, $x \in \overline{\sigma_p}$ and $x \notin \sigma_c$. In the first case, we obtain that $x \in \sigma_p \cap \sigma_c$, so $x \in \varnothing$, *impossible.* In the second case, we have that $x \in \overline{\sigma_p} \cap \overline{\sigma_c}$ and $x \notin \varnothing$. Therefore, $\overline{\sigma_p} \cap \overline{\sigma_c} \neq \varnothing$. □

First Subgroup: The Disjoint logical consequences that satisfy $\vDash^{\overrightarrow{disj}}$ but not $\vDash^{\overleftarrow{disj}}$ are:

(1) SN
(2) NS
(3) $\overline{T}N$
(4) $N\overline{T}$
(5) $S\overline{T}$
(6) $\overline{T}S$
(7) $\varnothing T$
(8) $\varnothing \overline{N}$
(9) $\varnothing \overline{S}$
(10) $\varnothing S$
(11) $\varnothing N$
(12) $\varnothing \overline{T}$
(13) $T\varnothing$
(14) $\overline{N}\varnothing$
(15) $\overline{S}\varnothing$
(16) $S\varnothing$
(17) $N\varnothing$
(18) $\overline{T}\varnothing$
(19) $\varnothing\varnothing$

The second group are disjoint logical consequences defined through *backward-preservation* but not through *forwards-preservation*; these do not preserve any interpretation from the premises to the conclusion but do preserve at least one interpretation from the conclusion to the premises. Now, let $\vDash^{\overleftarrow{disj}}$, $\vDash^{\overrightarrow{disj}}$ if and only if $\sigma_c \neq \overline{\sigma_p}$.

Theorem 3.2.4. *Let* $\overline{\sigma_p} \cap \overline{\sigma_c} = \varnothing$. $\sigma_p \cap \sigma_c \neq \varnothing$ *if and only if* $\sigma_c \neq \overline{\sigma_p}$.

Proof. The proof from left to right is as follows: if $\sigma_p \cap \sigma_c \neq \varnothing$, then there is an x such that $x \in \sigma_p \cap \sigma_c$ and $x \notin \varnothing$, or, $x \in \varnothing$ and $x \notin \sigma_p \cap \sigma_c$. The second case is impossible. In the first case, we obtain that $x \in \sigma_c$ and $x \in \sigma_p$. From $x \in \sigma_p$, we can derive that $x \notin \overline{\sigma_p}$. Thus, $x \in \sigma_c$ and $x \notin \overline{\sigma_p}$. Therefore, $\sigma_c \neq \overline{\sigma_p}$.

Conversely, the proof from right to left is as follows: if $\sigma_c \neq \overline{\sigma_p}$, then there is an x such that $x \in \sigma_c$ and $x \notin \overline{\sigma_p}$, or, $x \in \overline{\sigma_p}$ and $x \notin \sigma_c$. In the first case, we obtain that $x \in \sigma_p \cap \sigma_c$ and $x \notin \varnothing$. Then, $\sigma_p \cap \sigma_c \neq \varnothing$. In the second case, we obtain that $x \in \overline{\sigma_p} \cap \overline{\sigma_c}$, then $x \in \varnothing$, *impossible*. Thus, $\sigma_p \cap \sigma_c \neq \varnothing$. $\square$

Second Subgroup: The Disjoint logical consequences that satisfy $\vDash_{\overleftarrow{disj}}$ but not$\vDash_{\overrightarrow{disj}}$ are:

(1) $\overline{SN}$
(2) $\overline{NS}$
(3) $T\overline{N}$
(4) $\overline{N}T$
(5) $\overline{S}T$
(6) $T\overline{S}$
(7) $\overline{\varnothing}\overline{T}$
(8) $\overline{\varnothing}N$
(9) $\overline{\varnothing}S$
(10) $\overline{\varnothing}\overline{S}$
(11) $\overline{\varnothing}\overline{N}$
(12) $\overline{\varnothing}T$
(13) $\overline{T}\overline{\varnothing}$
(14) $N\overline{\varnothing}$
(15) $S\overline{\varnothing}$
(16) $\overline{S}\overline{\varnothing}$
(17) $\overline{N}\overline{\varnothing}$
(18) $T\overline{\varnothing}$
(19) $\overline{\varnothing}\overline{\varnothing}$

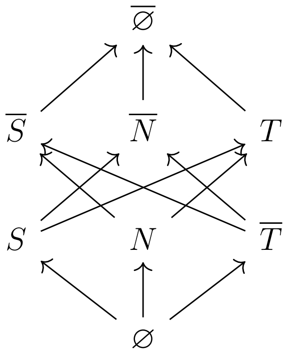
Finally, the third group are disjoint logical consequences defined through *backwards-preservation* and *forwards-preservation*, These are logical consequences that preserve no set of interpretations from the premises to the conclusion and also preserve no set of interpretations from the conclusion to the premises. This group is made up of the elements resulting from the intersection of Subgroup 1 and Subgroup 2

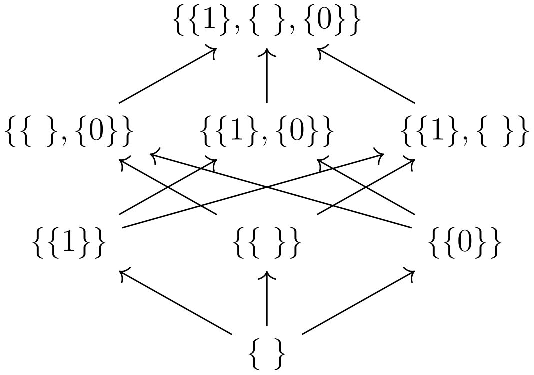
Third Subgroup: The Disjoint logical consequences that satisfy $\vDash_{\overrightarrow{disj}}$ and $\vDash_{\overleftarrow{disj}}$ are:

(1) $\overline{S}S$
(2) $S\overline{S}$
(3) $\overline{N}N$
(4) $N\overline{N}$
(5) $T\overline{T}$
(6) $\overline{T}T$
(7) $\overline{\varnothing}\varnothing$
(8) $\varnothing\overline{\varnothing}$

3.2.2 How many disjoint logical consequences can be obtained with n admissible interpretations?

Consider the following graphs of contention of the sets of interpretations in a Strong-Kleene valuation:





The fact that the intersection of the sets of interpretations is empty may be due to three reasons:

First, at the third level of the graph are the sets of interpretations S, N, and $\overline{T}$. Any combination of these sets of interpretations, whether for premises or conclusion, results in a disjoint logical consequence. Also, the intersection of a set of interpretations and the empty set is always empty. Now, when n is greater than 3 we have to take into account the relation of the unitary sets with all the subsets of the complementary set of the unitary set. The above must take into account:

a The relation of the unitary sets with the subsets of the complementary set of the unitary set.

b The relation between unitary sets.

c The relation of all the sets with the empty set.

I now proceed to prove these three points:

Theorem 3.2.5. *Let A, B, C sets, such that $A \subseteq C$ and $B \subseteq C$. $A \cap B = \varnothing$ if and only if $B \subseteq \overline{A}$*

Proof. The proof from left to right is as follows: if $A \cap B = \varnothing$ and $x \in B$, then $x \in C$. But $x \notin A$, otherwise *contradiction.* Thus, $x \in \overline{A}$. Therefore, $B \subseteq \overline{A}$. □

Theorem 3.2.6. *If A and B are different singletons, then $A \cap B = \varnothing$*

Proof. Let A the singleton of x and B the singleton of y. A and B have no more elements than x and y, respectively. Then, A and B have no elements in common. Therefore, $A \cap B = \varnothing$. □

Theorem 3.2.7. *For all A, $A \cap \varnothing = \varnothing$*

Proof. Suppose that $x \in A \cap \varnothing$, then $x \in \varnothing$. Now, suppose that $x \in \varnothing$, then *contradiction.* So, trivialy, $x \in A \cap \varnothing$. Therefore, $A \cap \varnothing = \varnothing$. □

Secondly, any set of interpretations for the conclusion (or premises), which is defined as the complement of the set of interpretations for the premises (or conclusion), will result in a disjoint logical consequence. By definition, n represents the number of admissible interpretations. The formula $\frac{2^n}{2}$ it will give information of the sets of admissible interpretations and also give information of their complementary sets. Since we seek to evaluate the relation of the sets of interpretations to each other, then we must multiply the result of $\frac{2^n}{2}$ with the result of the number of complements given by $\frac{2^n}{2}$, i.e. we must solve $2(\frac{2^n}{2})$. The formula $2(\frac{2^n}{2})$ reduces to 2^n.

Theorem 3.2.8. *For all A, $A \cap \overline{A} = \varnothing$.*

Proof. Suppose that $x \in A \cap \overline{A}$, then $x \in A$ and $x \notin A$, *contradiction.* So, trivialy, $x \in \varnothing$. Now, suppose that $x \in \varnothing$, *contradiction.* So, trivialy, $x \in A \cap \overline{A}$. Then, $A \cap \overline{A} = \varnothing$. □

Thirdly, in any process of calculating disjoint sets, it must be taken into account that the intersection between the empty set and the empty set is disjoint. Since the permutation of this relation is identical to itself, this single relation will always be summed. The sum of all these steps is equal to the number of disjoint logical consequence relations.

This makes it possible to calculate how many disjoint logical consequences (defined through *forwards-preservation* or through *backward-preservation*), have a semantics with n admissible interpretations [64]:

Let n a set of admissible interpretations for a $\mathcal{L}$, the calculation consists in:

$$\sum_{m=0}^{n} \binom{n}{m} 2^{n-m} \tag{3.1}$$

1. When $m = 0$, it is possible to calculate how many consequence relations there are between each set of interpretations and its complement.

2. When $m > 0$ and $m \neq n$, it is possible to calculate how many possible consequence relations exist between a set and the subsets of the complement set.

3. When $m = n$ it is possible to calculate the unitary interpretation when $\varnothing\varnothing$. That is, step number three.

This formula can be simplified as follows:

$$2^n \sum_{m=0}^{n} \binom{n}{m} 2^{-m} \tag{3.2}$$

$$2^n \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1}{2^m}\right) \tag{3.3}$$

$$2^n \sum_{m=0}^{n} \binom{n}{m} \left(\frac{1}{2}\right)^m \tag{3.4}$$

$$2^n \left(1 + \frac{1}{2}\right)^n \tag{3.5}$$

$$3^n \tag{3.6}$$

This formula is useful to calculate any model-theoretic presentation with n admissible interpretations when $n \in \mathbb{N}$ (See appendix for disjoint logical consequences in semantics with 4 admissible interpretations). On the other hand, in order to calculate how many disjoint logical consequence relations are defined through *forwards-preservation* and through *backwards-preservation* (at the same time), it is sufficient to solve:

$$2^n$$

There are 27 disjoint logical consequences defined through *forwards-preservation*, 27 disjoint logical consequences defined through *backward-preservation*, and 8 disjoint logical consequences defined through *forwards-preservation* and *backward-preservation* (at the same time).

3.3 The importance of logical consequence direction

In this section, I want to argue why it is useful to consider the direction of the logical consequence relation. I give two reasons:

i. If we do not take falsity and non-truth as shorthand, it allows us to construct logics where the set of arguments defined by *forwards-preservation* and *backward-preservation* are not coextensive. Item [ii.] It dispels doubts about whether such relations are, in fact, relations of logical consequence. At the end of this section, I will show how and why all the relations worked out in this text preserve a set of interpretations.

3.3.1 Non-truth and falsity

The logical consequence relation has been evaluated through *forwards-preservation*: If the premises are true, then the conclusion is true. This idea is not strange since the classical Tarskian consequence relation has this property: if the premises are true, the conclusion is also true. In this case, σ_p and σ_c are considered equal, and the preserved value is precisely intended to be the true value. In [121] Wansing and Shramko have called this t-entailment A formula A is logically true if and only if A is t-entailed by the empty set, and A is logically false if and only if A t-entails the empty set. *Logical Truth* is identified with a formula A in which, for any interpretation σ, $1 \in \sigma(A)$. When one has a logical consequence (truth-preserving) relation, where the set of interpretations is $\{1\}$ and $\{0\}$, and falsity is the complement of true, then falsity will be preserved in the other direction, from the conclusion to at least one of the premises. Wansing and Shramko have called this *f-entailment.*

When one has more than two admissible interpretations, as in the case of a Strong-Kleene valuation and other logics that preserve truth from premise to conclusion, Truth is preserved from premise to conclusion and non-truth from the conclusion to at least one of the premises. That is to say, an interpretation σ is preserved from the premises to the conclusion if and only if $\overline{\sigma}$ is preserved from the conclusion to at least one of the premises. Note that in the case of a logic, with a Strong-Kleene valuation, the preservation of the non-truth from the conclusion to at least one of the premises includes the preservation of the falsity through *backward-preservation*.

Definition 3.3.1 (non-truth-entailment in a Strong-Kleene logic ($\vDash^{\leftarrow}$))**.** *A formula* A *is a logical consequence of a set* Γ*,* $\Gamma \vDash^{\leftarrow} A$*, if and only if for all admissible interpretations* σ*,* $1 \notin \sigma(A)$*, then, there is a* $B \in \Gamma$ *such that* $1 \notin \sigma(B)$*.*

This inclusion of falsity in non-truth may make one think that the two concepts are equivalent. If the 'falsity' is not an abbreviation for 'non-truth', then f-entailment may not coincide with t-entailment. Taking an example analogous to [121], consider the logic **K3***, with a Strong-Kleene valuation, that actually uses f-entailment:

Definition 3.3.2 (t-entailment in a **K3*** ($\vDash^{\rightarrow}$))**.** *A formula* A *is a logical consequence of a set* Γ*,* $\Gamma \vDash^{\rightarrow} A$*, if and only if for all admissible interpretations* σ*, for any* $B \in \Gamma$*, if* $1 \in \sigma(B)$*, then* $1 \in \sigma(A)$*.*

Definition 3.3.3 (f-entailment in a **K3***($\vDash^{\leftarrow}$))**.** *A formula* A *is a logical consequence of a set* Γ*,* $\Gamma \vDash^{\leftarrow} A$*, if and only if for all admissible interpretations* σ*,* $0 \in \sigma(A)$ *and* $1 \notin \sigma(A)$*, then, there is a* $B \in \Gamma$ *such that* $0 \in \sigma(B)$ *and* $1 \notin \sigma(B)$*.*

It is easy to see that in **K3***, t-entailment and f-entailment do not coincide. Consider $A \wedge \sim A \vDash^{\rightarrow} B$, since it preserves the truth from premise to conclusion. However, $A \wedge \sim A \nvDash^{\leftarrow} B$ since it does not preserve falsity from the conclusion to premises when $\sigma(B) = \{0\}$ and $\sigma(A) = \{\ \}$.

The logical consequence relation of **K3*** is represented by tf-entailment:

$\Gamma \vDash^{\rightarrow,\leftarrow}_{\mathbf{K3*}} A$ if and only if for all admissible interpretations,

$\vDash^{\rightarrow}_{\mathbf{K3*}}$ For any $B \in \Gamma$, if $1 \in \sigma(B)$, then $1 \in \sigma(A)$.

$\vDash^{\leftarrow}_{\mathbf{K3*}}$ If $0 \in \sigma(A)$, then there is a $B \in \Gamma$ such that $0 \in \sigma(B)$.

Considering the logical consequence relation and giving independent treatment to non-truth and falsity allows us to construct logic where *forwards-preservation* does not coincide with *backward-preservation*. In the case of disjoint logical consequence relations, considering the logical consequence relation allows considering disjoint logics in both directions. This allows for stronger definitions of disjoint logics. I refer specifically to disjoint logics in both directions.

3.3.2 Relation of logical consequence and disjoint logics

Some definitions of logical consequence seem to entail that Pailos' disjoint logical consequence relations are not logical consequence relations. The argument is as follows:

RC1 A logical consequence relation is a relation that preserves a set of interpretations from premises to the conclusion (see more in [116] and [114]).

RC2 The disjoint logical consequence relations defined by Pailos do not preserve any set of interpretations from premises to conclusion.

RC3 The disjoint logical consequence relations defined by Pailos are not logical consequence relations.

I consider that the conclusion is false. Let me explain, the definition in RC1 does not take into account the direction of the logical consequence relation. Therefore, RC1 is false. Considering a definition of logical consequence that considers direction may clarify some doubts about the need to preserve a set of interpretations. Wansing and Shramko have defined logical consequence, taking into account the direction, as follows: "[...] a relation that (in the single conclusion case)[3] preserves membership in a certain set of algebraic values either from the premises to the conclusion of inferences, or from the conclusion to the premises" ([121, p.415]).[4] Let me call this RC1*.

---

[3] The parentheses in the quotation are from Wansing and Shramko.

[4] Here, I relate algebraic values to sets of interpretations.

Looking at this definition (RC1*), rather than RC1, allows me to conclude that Subgroup 1 and Subgroup 2 are logical consequence relations; Subgroup 1 preserve a set of interpretations from conclusion to premises and Subgroup 2 preserve a set of interpretations from premises to conclusion. But the case of Subgroup 3 is unclear. This is because these logical consequence relations do not preserve anything in both directions. Despite this last result with Subgroup 3, it is useful to consider the direction of the logical consequence relation to show why some of these relations are logical consequence relations.

However, logic is a theory about what follows from what and why. This is the position that has been argued by Priest ([97, p.1]), and ([96, p.3209]). In all disjoint logical consequence relations it is clear which arguments are valid and which are invalid. Therefore, they are legitimate logical consequence relations since they allow one to establish what follows from what and why. For those who do not find this answer plausible, I propose to answer why in Subgroup 3 there is preservation in the next session. So, they are legitimate logical consequence relations.

Preservation on another level

In this section I will show that all disjoint logics preserve a set of interpretations. This allows me to show that disjoint logics satisfy the requirement of RC1*. So far, I have assessed preservation in the three admissible interpretations: Truth Preservation, Non-Truth and Non-Falsehood Preservation, and Falsehood Preservation. In other words, the preservation of sets of values was assessed. However, this is unfair to the level of abstraction at which we have defined disjoint logics. That is, disjoint logical consequence relations have been defined in terms of sets of interpretations, not *simpliciter* interpretations.

In set theory, it is known that if a set is empty, its power set is not empty. Formally speaking:

Fact 3.3.1. *If $X = \varnothing$ then $\wp(X) \neq \varnothing$.*

Proof. Sup. $X = \varnothing$. Then, by *def.* $\wp(X)$, this set has all subsets of X as its elements. In this case, X and $\varnothing$ are the elements of $\wp(X)$. But, X and $\varnothing$ are the same set. Thus, $\wp(X)$ has at least one element. Therefore, $\wp(X) \neq \varnothing$. $\square$

That is, if we said earlier that some disjoint logics do not preserve anything, it is only because of the limitation of the set of interpretations with which I have evaluated them. Now, consider the following interpretations of the power set of admissible interpretations of Strong-Kleene logic. The names have been changed to make clear the set of interpretations preserved.

- Just just true = $\{\{1\}\}$
- Not just just true = $\{\{\ \}, \{0\}\}$
- Not just just false = $\{\{1\}, \{\ \}\}$
- Just just false = $\{\{0\}\}$
- Not Just just true and not just just false = $\{\{\ \}\}$
- Just just True and just just false = $\{\{1\}, \{0\}\}$
- Empty = $\{\ \}$
- Universal = $\{\{1\}, \{\ \}, \{0\}\}$

Let $C(\sigma) = \{\sigma_1 ..., \sigma_m\}$ be a set of interpretations, in which $0 \leq m \leq 3$ and and is defined as follows $C(\sigma)) = \sigma_p \cup \sigma_c$. Remember that σ_p and σ_c are a set of interpretations, a set of premises, and a set of conclusions, respectively. In this case, $C(\sigma)$ will represent one of the standards presented in this session. Thus, it is always the case that σ_p, $\sigma_c \subseteq C(\sigma)$. Or, to put it another way, in all disjoint logical consequence relations, one set of interpretations is always preserved. Below, you will find some examples.

Most logical disjoint logical consequences are characterized by the following features: $\sigma_c \neq \sigma_p$.

I go on to prove that there is preservation in the set $C(\sigma)$ if the consequence relation is disjoint and $\sigma_c \neq \sigma_p$.

Theorem 3.3.1. *If $X \neq Y$ and $X \cap Y = \varnothing$, then $X \cup Y \neq \varnothing$.*

Proof. Suppose $X \neq Y$ and $X \cap Y = \varnothing$. If $X \neq Y$, then there is an x such that $x \in X$ and $x \notin Y$, or, $x \in Y$ and $x \notin X$. In any case, we have that $x \in X \cup Y$. Therefore, $X \cup Y \neq \varnothing$. □

All disjoint logical consequence relations preserve a set of interpretations. Even those of Subgroup 3 would preserve the set Universal $(\{\{1\},\{\ \},\{0\}\})$ of premises to conclusion and conclusion to premises. However, some people restrict preservation to being neither empty nor universal. This new requirement would affect only relations of Subgroup 3 Subgroup. Subgroup 1 and Subgroup 2 do not preserve either the universal set nor the empty set of interpretations, as proven below:

Theorem 3.3.2. If $A \neq B$, $B \neq \overline{A}$ and $A \cap B = \varnothing$, then $A \cup B \neq \overline{\varnothing}$.

Proof. Suppose $A \neq B$, $B \neq \overline{A}$ and $A \cap B = \varnothing$. If $A \neq B$, then there is an x such that $x \in A$ and $x \notin B$, or, $x \in B$ and $x \notin A$; and, if $B \neq \overline{A}$, then there is an x such that $x \in B$ and $x \notin \overline{A}$, or, $x \in \overline{A}$ and $x \notin B$. In any case, we have an x such that $x \in \overline{\varnothing}$ but $x \notin A \cup B$. Therefore, $A \cup B \neq \overline{\varnothing}$. □

Theorem 3.3.3. If $A \neq B$, $B \neq \overline{A}$ and $A \cap B = \varnothing$, then $A \cup B \neq \overline{\varnothing}$ and $A \cup B \neq \varnothing$.

Proof. Sup. $A \neq B$, $B \neq \overline{A}$ and $A \cap B = \varnothing$. By Theorem 3.2., we know that $A \cup B \neq \overline{\varnothing}$. And, by Theorem 3.1.1., we know that $A \cup B \neq \varnothing$. Thus, $A \cup B \neq \overline{\varnothing}$ and $A \cup B \neq \varnothing$. □

Remember that $C(\sigma)$ will represent one of the standards presented in this session. I will now show how and why there will be preservation in the three sub-groups:

- For disjoint logical consequence relations of Subgroup 1: Take any relation of logical consequence XY. If $\sigma_p \cap \sigma_c = \varnothing$ and $\overline{\sigma_p} \cap \overline{\sigma_c} \neq \varnothing$, then $\sigma_p \cup \sigma_c \neq \varnothing$. If $\sigma_p \cup \sigma_c \neq \varnothing$, then the logical consequence relation preserves a set of interpretations $(C(\sigma))$ from premises to conclusion.

- For disjoint logical consequence relations of Subgroup 2: Take any relation of logical consequence XY. If $\sigma_p \cap \sigma_c \neq \varnothing$ and $\overline{\sigma_p} \cap \overline{\sigma_c} = \varnothing$, then $\overline{\sigma_p} \cup \overline{\sigma_c} \neq \varnothing$. If $\overline{\sigma_p} \cup \overline{\sigma_c} \neq \varnothing$, then the logical consequence relation preserves a set of interpretations $(C(\sigma)$ from conclusión to premises.

- For the disjoint logical consequence of Subgroup 3: Since Universal is a set of interpretations, we know the set Universal is preserved. So I can say that there is preservation.

For example, for Subgroup 1, consider the following logical consequence relation SN. This logical consequence relation belongs to Subgroup 1 because it is disjoint in *forwards-preservation* but not disjoint in *backwards-preservation.* We know that SN preserves the falsity from the conclusion to at least one of the premises. On the other hand, in the preservation from premises to conclusion, the set of interpretations for premises is equal to $\{\{1\}\}$ and the set of interpretations for the conclusion is equal to $\{\{\ \}\}$. The union of both sets of interpretations is equal to the set of interpretations $T = \{\{1\}, \{\ \}\}$ (*Not just just false*). Based on this, it is possible to affirm that there is a set of interpretations that is preserved from the premises to the conclusion: the set of the *Not just just false.* As I have shown, the argument extends for all logics in Subgroup 1.

For example, for Subgroup 2 consider the following logical consequence relation $T\overline{S}$. This logical consequence relation belongs to Subgroup 2 because it is disjoint in *backwards-preservation* but not disjoint in *forwards-preservation*. We know that $T\overline{S}$ preserves the not truth and not false from premises to conclusion. On the other hand, in the preservation from the conclusion to at least one of the premises, the set of interpretations for premises is equal to $\{\{0\}\}$ and the set of interpretations for the conclusion is equal to $\{\{1\}\}$. The union of both sets of interpretations is equal to the set of interpretations $\overline{N} = \{\{1\}, \{0\}\}$ (*Just just truth and just just false*). Based on this, it is possible to affirm that there is a set of interpretations that is preserved from the premises to the conclusion: the set of the *Just just truth and just just false*. As I have shown, the argument extends for all logics in Subgroup 2.

For example, for Subgroup 3 consider the following logical consequence relation $T\overline{T}$. This logical consequence relation belongs to Subgroup 3 because it is disjoint in *backwards-preservation* and *forwards-preservation*. In the preservation from premises to conclusion, the set of interpretations for premises is equal to $\{\{1\}, \{0\}\}$ and the set of interpretations for the conclusion is equal to $\{\{0\}\}$. The union of both sets of interpretations is equal to the set of interpretations $\overline{\varnothing} = \{\{1\}, \{\ \}, \{0\}\}$ (*Universal*). As the consequence relationship is symmetric, the same applies in the opposite direction. Based on this, it is possible to affirm that there is a set of interpretations that is preserved from the premises to the conclusion: the set of the *Universal*. As I have shown, the argument extends for all logics in Subgroup 3.

I have argued that all disjoint logical consequence relations, both Pailo's and mine, preserve a set of interpretations, either from premises to conclusion or from conclusion to premises. It does not follow that these logics would not be empty if we used these logical consequence relations in a language such as Kleene's. That is to say: Even if a logical consequence relation is truth-preserving, it does not follow that there is at least one valid argument. This is because the logical consequence relation does not do the job independently. The job of validating or invalidating arguments can also be done by language and the set of admissible interpretations of logic. That is why the results obtained in this section have to be taken abstractly. That is, logical consequence relations will preserve an interpretation like that presented in this section. But which arguments are valid will also depend on which language is used.

3.4 Conclusions

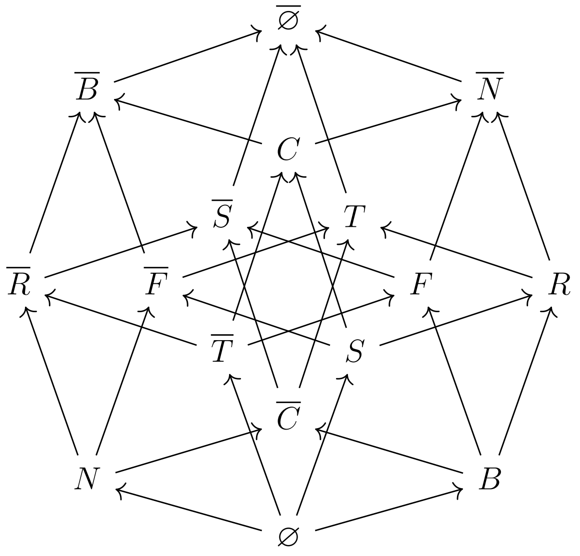
In this chapter, I have shown how it is possible to define some subgroups of disjoint consequences in a Strong-Kleene valuation: *forwards-preservation* but not *backward-preservation*, *backward-preservation* but not *forwards-preservation*, and disjoint on both sides; these subgroups allow a more detailed study of disjointness. Also, I have discussed why it is important to consider direction in logical consequence relations and how some interesting philosophical problems arise from considering this. Finally, I have defined a disjoint logical consequence equivalent to that of Pailos, which allows us to conclude that only disjoint logical consequence relations towards both sides are really disjoint.

3.5 Appendix: disjoint logical consequence relations, when $n = 4$

Consider a semantics with the following set of admissible interpretations $\{\{1\},\{1,0\},\{\ \},\{0\}\}$, where $\{1\}$ represents only true, $\{1,0\}$ represents true and false,$\{\ \}$ represents neither true nor false and $\{0\}$ represents only false (For durther discussion see [112]). Let X be a set of interpretations and $\overline{X}$ its complement. Consider the following sets of interpretations, each of which is an element of the power of $\{\{1\},\{1,0\},\{\ \},\{0\}\}$:

- $\varnothing = \{\ \}$
- $\overline{\varnothing} = \{\{1\},\{1,0\},\{\ \},\{0\}\}$
- $S = \{\{1\}\}$
- $\overline{S} = \{\{1,0\},\{\ \},\{0\}\}$
- $B = \{\{1,0\}\}$
- $\overline{B} = \{\{1\},\{\ \},\{0\}\}$
- $N = \{\{\ \}\}$
- $\overline{N} = \{\{1\},\{1,0\},\{0\}\}$
- $T = \{\{1\},\{1,0\},\{\ \}\}$
- $\overline{T} = \{0\}$
- $F = \{\{1,0\},\{0\}\}$
- $\overline{F} = \{\{1\},\{\ \}\}$
- $C = \{\{1\},\{0\}\}$
- $\overline{C} = \{\{1,0\},\{\ \}\}$
- $R = \{\{1\},\{1,0\}\}$
- $\overline{R} = \{\{\ \},\{0\}\}$

Now, consider the following graphs of contention of the sets of interpretations:



Using the same formula to calculate the number of disjoint logical consequence relations, when $n = 4$, it is obtained that there are 81 disjoint logical consequences. There are 81 disjoint logical consequences defined through *forwards-preservation*, 81 disjoint logical consequences defined through *backward-preservation*, and 16 disjoint logical consequences defined through *forwards-preservation* and *backward-preservation* (at the same time).

The disjoint logical consequences, defined through *forwards-preservation*, based on this semantic are as follows:

1. SB
2. BS
3. SN
4. NS
5. $S\overline{T}$
6. $\overline{T}S$
7. BN
8. NB
9. $B\overline{T}$
10. $\overline{T}B$
11. $N\overline{T}$
12. $\overline{T}N$
13. $S\overline{C}$
14. $\overline{C}S$
15. SF
16. FS
17. $S\overline{R}$
18. $\overline{R}S$
19. $B\overline{F}$
20. $\overline{F}B$
21. BC
22. CB
23. $B\overline{R}$
24. $\overline{R}B$
25. NF
26. FN
27. SC
28. CS
29. SR
30. RS
31. $\overline{T}R$
32. $R\overline{T}$
33. $\overline{TF}$
34. $\overline{FT}$
35. $\overline{TC}$
36. $\overline{CT}$
37. $\varnothing\overline{\varnothing}$
38. $\overline{\varnothing}\varnothing$
39. $S\overline{S}$
40. $\overline{S}S$
41. $B\overline{B}$
42. $\overline{B}B$
43. $N\overline{N}$
44. $\overline{N}N$
45. $T\overline{T}$
46. $\overline{T}T$
47. $F\overline{F}$
48. $\overline{F}F$
49. $C\overline{C}$
50. $\overline{C}C$
51. $R\overline{R}$
52. $\overline{R}R$
53. $\varnothing S$
54. $S\varnothing$
55. $\varnothing\overline{S}$
56. $\overline{S}\varnothing$
57. $\varnothing B$
58. $B\varnothing$
59. $\varnothing\overline{B}$
60. $\overline{B}\varnothing$
61. $\varnothing N$
62. $N\varnothing$
63. $\varnothing\overline{N}$
64. $\overline{N}\varnothing$
65. $\varnothing T$
66. $T\varnothing$
67. $\varnothing\overline{T}$
68. $\overline{T}\varnothing$
69. $\varnothing F$
70. $F\varnothing$
71. $\varnothing\overline{F}$
72. $\overline{F}\varnothing$
73. $\varnothing C$
74. $C\varnothing$
75. $\varnothing\overline{C}$
76. $\overline{C}\varnothing$
77. $\varnothing R$
78. $R\varnothing$
79. $\varnothing\overline{R}$
80. $\overline{R}\varnothing$
81. $\varnothing\varnothing$

The disjoint logical consequences, defined through *backward-preservation*, based on this semantic are as follows:

1. $\overline{SB}$
2. $\overline{BS}$
3. $\overline{SN}$
4. $\overline{NS}$
5. $\overline{S}T$
6. $T\overline{S}$
7. $\overline{BN}$
8. $\overline{NB}$
9. $\overline{B}T$
10. $T\overline{B}$
11. $\overline{N}T$
12. $T\overline{N}$
13. $\overline{S}C$
14. $C\overline{S}$
15. $\overline{SF}$
16. $\overline{FS}$
17. $\overline{S}R$
18. $R\overline{S}$
19. $\overline{B}F$
20. $F\overline{B}$
21. $\overline{BC}$
22. $\overline{CB}$
23. $\overline{B}R$
24. $R\overline{B}$
25. $\overline{NF}$
26. $\overline{FN}$
27. $\overline{SC}$
28. $\overline{CS}$
29. $\overline{SR}$
30. $\overline{RS}$
31. $T\overline{R}$
32. $\overline{R}T$
33. TF
34. FT
35. TC
36. CT
37. $\overline{\varnothing}\varnothing$
38. $\varnothing\overline{\varnothing}$
39. $\overline{S}S$
40. $S\overline{S}$
41. $\overline{B}B$
42. $B\overline{B}$
43. $\overline{N}N$
44. $N\overline{N}$
45. $\overline{T}T$
46. $T\overline{T}$
47. $\overline{F}F$
48. $F\overline{F}$
49. $\overline{C}C$
50. $C\overline{C}$
51. $\overline{R}R$
52. $R\overline{R}$
53. $\overline{\varnothing}\overline{S}$
54. $\overline{S}\overline{\varnothing}$
55. $\overline{\varnothing}S$
56. $S\overline{\varnothing}$
57. $\overline{\varnothing}\overline{B}$
58. $\overline{B}\overline{\varnothing}$
59. $\overline{\varnothing}B$
60. $B\overline{\varnothing}$
61. $\overline{\varnothing}\overline{N}$
62. $\overline{N}\overline{\varnothing}$
63. $\overline{\varnothing}N$
64. $N\overline{\varnothing}$
65. $\overline{\varnothing}\overline{T}$
66. $\overline{T}\overline{\varnothing}$
67. $\overline{\varnothing}T$
68. $T\overline{\varnothing}$
69. $\overline{\varnothing}\overline{F}$
70. $\overline{F}\overline{\varnothing}$
71. $\overline{\varnothing}F$
72. $F\overline{\varnothing}$
73. $\overline{\varnothing}\overline{C}$
74. $\overline{C}\overline{\varnothing}$
75. $\overline{\varnothing}C$
76. $C\overline{\varnothing}$
77. $\overline{\varnothing}\overline{R}$
78. $\overline{R}\overline{\varnothing}$
79. $\overline{\varnothing}R$
80. $R\overline{\varnothing}$
81. $\overline{\varnothing\varnothing}$

The disjoint logical consequences, defined through *forwards-preservation* and through *backward-preservation* (At the same time), based on this semantic are as follows:

1. $\varnothing\overline{\varnothing}$
2. $\overline{\varnothing}\varnothing$
3. $S\overline{S}$
4. $\overline{S}S$
5. $B\overline{B}$
6. $\overline{B}B$
7. $N\overline{N}$
8. $\overline{N}N$
9. $T\overline{T}$
10. $\overline{T}T$
11. $F\overline{F}$
12. $\overline{F}F$
13. $C\overline{C}$
14. $\overline{C}C$
15. $R\overline{R}$
16. $\overline{R}R$

Chapter 4

Empty validity all the way up: an easy road

Abstract

There is a tension between the definition of empty logic as a logic with no valid arguments and no valid meta-arguments, on the one hand, and the way in which we have usually interpreted the validity of meta-arguments, on the other. Here we argue that one way to eliminate the tension is understanding the "If…then…" in a meta-argument, at least in the case of an empty logic, as a transplication (aka the de Finetti conditional) instead of an extensional or material conditional.

Keywords: empty logics, meta-arguments, arguments, global validity, local validity

Introduction

Let Γ_i be a set of formulas of some formal language $\mathcal{L}$ and A_j a formula of that very language, with $1 \leq, j \leq n$ for some natural n. An *argument* is an expression of the form $\Gamma \vDash_{\mathbf{L}} A$, where $\vDash_{\mathbf{L}}$ stands for a relation of logical consequence, and Γ is also known as a premise set and A is called 'conclusion'. A *meta-argument* is an argument between arguments that has the form "If $\Gamma_1 \vDash A_1, \dots, \Gamma_n \vDash A_n$ then $\Delta \vDash A_m$".[1]

The logical validity of arguments is usually evaluated as truth preservation, that is, an argument $\Gamma \vDash A$ is *valid* if and only, in every interpretation, A is true if B is true for every $B \in \Gamma$. An argument $\Gamma \vDash A$ is *invalid* if and only if there is an interpretation in which B is true, for every $B \in \Gamma$, and A is not true. Such an interpretation will be considered a *countermodel* for the argument.

On the other hand, a meta-argument is *valid* if and only if, if the meta-premises $\Gamma_1 \vDash A_1, \dots, \Gamma_n \vDash A_n$ are valid, the meta-conclusion $\Delta \vDash B$ is valid as well[2].

Dicher and Paoli [38], and Barrio *et al.* [10] have called to this definition *Global validity*. This is to distinguish it from *Local validity*. The definitions of global validity and local validity work only for meta-arguments. To define local validity, it is first necessary to define *satisfaction* of an argument. An interpretation *satisfies* an argument if and only if it is not a countermodel for it. And an interpretation *unsatisfies* an argument if and only if its premises are true and the conclusion is false. A meta-argument is *locally valid* if and only if the meta-conclusion is satisfied in every interpretation in which the arguments of the meta-premises are satisfied. A meta-argument is *locally invalid* if and only if, there is an interpretation in which the meta-premises are satisfied and the meta-conclusion is unsatisfied.

For some people, like Teijeiro [118], a logic is a set of valid arguments and meta-arguments. Although this is the majority view, it is not the only one. For example, Pailos [91] says that a logic is a set of valid, anti-valid, and invalid-but-not-anti-valid

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[1] In [38], Dicher and Paoli have defined a meta-argument as a nonempty set of arguments where one of which is labeled as its conclusion. In [10], Barrio *et al.* have defined a meta-argument as an argument between a collection of arguments, and an argument.

[2] Usually, both the definition of argument and meta-argument is defined using the concept of *inference* and *meta-inference*. However, since the approach of my chapter is model-theoretical, it is not necessary to use this terminology.

arguments of all possible levels. For simplicity, in this chapter, we will assume that a logic is indeed the set of valid arguments and meta-arguments.

Thus, a logic is *empty* if and only if its sets of valid arguments and meta-arguments are empty. Since there are no valid arguments, any meta-argument with an invalid argument as a meta-premise would be globally valid. In fact, the definitions of Global and Local validity produce problems with this characterization of empty logic. For example, consider Meta-reflexivity:

$$\text{If } A \vDash A \text{ then } A \vDash A \tag{Meta-reflexivity}$$

This meta-argument would also be locally valid in an empty logic because in each interpretation in which $A \vDash A$ is satisfied, $A \vDash A$ is also satisfied. Meta-reflexivity is also globally valid because it is always the case that $A \nvDash A$.

As another example, consider Monotonicity:

$$\text{If } \Gamma \vDash A \text{ then } \Gamma \cup \Delta \vDash A \tag{Monotonicity}$$

An empty logic would be monotonic because it is always the case that $\Gamma \nvDash A$; hence, it is globally valid. In an empty logic, this meta-argument would also be locally valid. Suppose that $\Gamma \vDash A$ is valid, that is, every interpretation in which the premises in Γ are true is also an interpretation in which the conclusion is true. Suppose now that $\Gamma \cup \Delta \nvDash A$. This means that there is an interpretation in which the premises in $\Gamma \cup \Delta$ are true and A is not true. But this cannot be under the assumption that $\Gamma \vDash A$ is valid. Therefore, $\Gamma \cup \Delta \vDash A$, and hence Monotonicity is locally valid.

Thus, based on the definitions of Global and Local validity, it is impossible to have a logic that lacks valid arguments and meta-arguments. In fact, to avoid the validity of any meta-argument with invalid premises, Dicher and Paoli [38], and Barrio *et al.* [10] have suggested preferring Local validity over Global validity. More recently, Teijeiro [118] has shown that there are not enough reasons to prefer Local validity over Global validity. We show the reasons for this later.

As can be seen, the problem of finding the right notion of validity for meta-arguments is still open. In the specific case of the problems raised for empty logic, we have at least two options. Either we disregard as meaningless the notion of an empty logic as a logic without valid arguments and meta-arguments, or we modify the way we understand the validity of meta-arguments. In this chapter, we explore the possibility of keeping the working definition of empty logic by (i) preserving the definition of validity (Global validity) but (ii) understanding the logical notions in its definition in a slightly different way. Quickly said, we will argue that the "If. . . then. . . " in a meta-argument, at least in the case of an empty logic, should be understood As transplication (aka the de Finetti conditional) instead of material conditional.

A disclaimer is in order here. Giving a good definition of empty logic is already a problem. But we believe that if there is something like the right definition of 'empty logic', it should be along the lines of [91], that is, as a logic without valid, anti-valid, and invalid-but-not-invalid arguments at any level. Nonetheless, for the sake of the argument, we stick to a more conservative characterization. Part of this discussion also requires an understanding of what validity is important. As we will argue, one cannot simply take classical logic for granted at the meta-theoretical level, and much less in the very definition of validity.

The structure of the chapter is as follows: first, we present some necessary preliminary definitions. Second, we present what an empty logic is and some examples.

Third, we propose a new way we are understanding the validity of meta-arguments based on the evaluation conditions of the connective transplication. Finally, we respond to some possible replies that someone might make as to why we interpret validity in this way.

4.0 Towards empty logics

Let $\mathcal{L}$ be a formal language obtained in the usual way from a numerable set $PROP$ of propositional variables $p_1, \ldots, p_n$ and a set of connectives $\{\sim, \wedge, \vee, \rightarrow\}$, which represent negation, conjunction, disjunction and conditional, respectively. A *Strong-Kleene interpretation* for $\mathcal{L}$ is a relation σ between $PROP$ and a set of truth values $\{1, 0\}$ in which '1' represents truth and '0' represents falsity, such that a propositional variable p_i can be related to truth values in one of the following three ways:

- p_i is true but not false, represented by '$1 \in \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1\}$
- p_i is neither true nor false, represented by '$1 \notin \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) =\{\ \ \}$
- p_i is false but not true, represented by '$0 \in \sigma(p_i)$ and $1 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{0\}$

and that can be extended[3] to complex formulas according to the following evaluation conditions:

$1 \in \sigma(\sim A)$ iff $0 \in \sigma(A)$
$0 \in \sigma(\sim A)$ iff $1 \in \sigma(A)$

$1 \in \sigma(A \wedge B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$

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[3] I will abuse notation a bit and use the same symbol for both the interpretation and its extension.

$0 \in \sigma(A \wedge B)$ iff $0 \in \sigma(A)$ or $0 \in \sigma(B)$

$1 \in \sigma(A \vee B)$ iff $1 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \vee B)$ iff $0 \in \sigma(A)$ and $0 \in \sigma(B)$

$1 \in \sigma(A \to B)$ iff $0 \in \sigma(A)$ or $1 \in \sigma(B)$

$0 \in \sigma(A \to B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

$\sim A$	A
$\{0\}$	$\{1\}$
$\{\ \}$	$\{\ \}$
$\{1\}$	$\{0\}$

$A \vee B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{\ \}$	$\{0\}$

$A \wedge B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

$A \to B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$

Based on this set of connectives, their evaluation conditions, and the following definition of logical consequence, we can define the logic known as **K3**.

Definition 4.0.1. *An argument is valid in* ***K3****,* $\Gamma \vDash_{K3} A$*, iff, for all interpretation* σ, $1 \in \sigma(A)$ *if* $1 \in \sigma(B)$*, for all* $B \in \Gamma$*. In particular,* A *is a* logical truth *in* ***K3*** *iff, for every* σ, $1 \in \sigma(A)$*.*

Definition 4.0.2. *An argument is invalid in* ***K3****, iff, for some interpretation* σ, $1 \in \sigma(B)$*, for all* $B \in \Gamma$*, and* $1 \notin \sigma(A)$*.*

Since logical truths are particular cases of valid arguments, an empty logic will not have logical truths either. Although **K3** has no logical truths, it does not count as an empty logic because it does have valid arguments in some cases when the set of premises is not empty.

Lemma 4.0.1. *For any formula A of **K3**, there is an assigment $1 \notin \sigma(A)$ and $0 \notin \sigma(A)$.*

Proof. By induction on the complexity of A. If A is a proposition variable, the assignment exists. When A is of the form $\sim X$, assign $1 \notin \sigma(p)$ and $0 \notin \sigma(p)$ for all variables p of X, so $1 \notin \sigma(\sim X)$ and $0 \notin \sigma(\sim X)$. If A is of the form $X \wedge Y$, $X \vee Y$ or $X \rightarrow Y$, assign $1 \notin \sigma(p_i)$ and $0 \notin \sigma(p_i)$ and $1 \notin \sigma(p_j)$ and $0 \notin \sigma(p_j)$ for all variables p_i of X and for all variables p_j of Y, for any i, j; then $1 \notin \sigma(X \wedge Y)$ and $0 \notin \sigma(X \wedge Y)$, $1 \notin \sigma(X \vee Y)$ and $0 \notin \sigma(X \vee Y)$, and $1 \notin \sigma(X \rightarrow Y)$ and $0 \notin \sigma(X \rightarrow Y)$. □

Fact 4.0.1. ***K3** has no logical truth.*

Proof. A is a *logical truth* in **K3** iff, for every σ, $1 \in \sigma(A)$. But this is impossible by **Lemma** 4.0.1. □

Nonetheless, **K3** is useful to define a logic that is actually empty in the sense used here. To do that, consider the logic **TS**, which can be defined using the language and evaluation conditions of **K3** but using Malinowski's [76] q-consequence instead.

Definition 4.0.3. *An argument is valid in **TS**, $\Gamma \vDash_{\boldsymbol{TS}} A$, iff, for all interpretation σ, $1 \in \sigma(A)$ and $0 \notin \sigma(A)$, if $1 \in \sigma(B)$ or $0 \notin \sigma(B)$, for all $B \in \Gamma$.*

Definition 4.0.4. *An argument is invalid in **TS**, iff, for some interpretation σ, $1 \notin \sigma(A)$, and $1 \in \sigma(B)$ or $0 \notin \sigma(B)$, for all $B \in \Gamma$.*

TS is a logic without valid arguments. However, it is impossible to give a proof if we use the same logic of theory in metatheory. In order to conclude something, proofs usually require at least one valid argument. However, it is impossible to give a valid argument because there are no logically valid arguments in **TS**. Therefore, nothing can be said about empty logic: neither that it has (or does not have) valid arguments, nor that it has (or does not have) valid meta-arguments.

We cannot prove anything about a logic like **TS** using the same logic of theory in metatheory. Lets have a look at this point in detail. Let us recall the notion of validity: an argument $X \vDash_{\mathbf{L}} Y$ is valid if and only if it holds in all cases. For example, in [17], Beall and Restall show that classical logic must be used for consistent and complete cases, constructive logic must be used for possibly incomplete cases, and relevant logic must be used for inconsistent and incomplete cases. Specifically, Beall and Restall consider Tarskian models and possible worlds as classical cases, situations as relevant cases, and stages as cases of constructive logic. They also consider other kinds of variation, based on free logics and second and higher order logics, to be pluralist, though not very well developed.

If we consider all cases, such as Tarskian models, possible worlds, situations, stages, etc., no argument will be valid in all cases. So if someone asks what follows from what in an empty logic, the answer is simple: nothing follows. This approach to empty logics argues that there are no arguments holding in all cases. Following to [44], this had already been raised by Mortensen in [82] and [84], where he argued that 'there are no arguments holding in all cases' on the basis of logical developments over the last hundred years or so. For every argument considered valid, some emergent logic has a model where the argument fails.

In view of the generality of logic, we think that we have to take seriously the universal quantification of the notion of validity, which specifies 'in all cases'. 'All cases' must mean 'all cases', not just particular cases like Tarskian models, Possible Worlds, Situations, or Stages. Considering only particular would cause us to lose generality and topic-neutrality (see more in [43]). However, it is not enough to defend a logic without valid arguments to consider only the four types of cases mentioned above. It is necessary to make at least one assumption. The cases that have been presented so far are not exhaustive. All possible cases must be taken into account if we are to have a logic without valid arguments. Starting to propose and defend other kinds of cases for consideration is the work that remains to be done here. In short, assuming that the set of arguments, defined as valid in all cases, is actually empty.

But, even though we know then that there are no valid arguments, "cases may need special inferences as inferential patterns governing correct reasoning in them" ([43, p.12]). We believe that this applies to evaluating the properties of logic without valid arguments. In particular, to say something about logic without valid arguments, we need a logic that is somewhat stronger than **TS**. We need other non-empty inference patterns to assert some things in cases of logics without valid arguments. Specifically, an argument $X \vDash_{\mathbf{L}} Y$ holds in a case of **TS** if and only if, in that case, if X is non-false, then Y is true.

We believe that this is the case for evaluating the properties of a logic without valid arguments. In particular, we need a logic that is somewhat stronger than **TS**. It is nothing new to require a stronger logic in metatheory in order to be informative about a theory. For example, in order to prove that classical logic can be derived from Tarskian models, we need classical logic. One only has to have a look at the proofs of soundness and completeness of classical logic to see that if we use a weaker logic, such as intuitionistic logic, it is not possible to give constructive proof. (See more on [17, p.39]).

When defining Tarski's or semantic truth-definition, it is necessary that the metalanguage is stronger than the language. In particular, if $\mathcal{L}$ is a language, then the definition of truth is usually formulated in an $\mathcal{ML}$ metalanguage. Tarski proved that the metalanguage $\mathcal{ML}$ must be richer than the language $\mathcal{L}$ and meet at least two characteristics: "has to contain expressions of higher semantic categories than the latter

[$\mathcal{L}$] and, moreover, the metalanguage has the translations of all expressions of the object language." ([127, p.460]). The other option we have is to abandon Tarski's or semantic truth definition (And any other information about the meta-theory of **TS**). That is, the acceptance that truth is undefinable in formal languages. However, it might be better to be aware of some limiting results of formal systems, such as the Godel incompleteness theorems and the Lowenheim-Skolem

In the current state of non-classical meta-theory, there are at least three approaches. In [13], Batens has argued that the logic used in meta-theory must be classical. In [123], Weber has taken steps towards non-classical meta-theory. In this case, the logic used in the meta-theory must be the same as the logic used in the theory (Homogeneity Thesis). Finally, in [115], Tanaka and Girard have suggested that the logic used in meta-theory must be the same or weaker (Girard-Tanaka's Conjecture). However, in the case of empty logic, the situation is still not so clear. We argue that the meta-theory of an empty logic needs a non-empty logic, but not necessarily the classical one.

Then, to do a metatheory about a logic like **TS**, we have at least three options.

- Use a logic **L** weaker than **TS**.
- Use the same logic **L** (i.e. **TS**).
- Use a logic **L** stronger than **TS**.

Consider the following conjecture:

- **Girard-Tanaka's Conjecture:** The logic used in the meta-theory for a logic **L** is not stronger than **L** itself.

For example, the logic used in the meta-theory for **LP** must be **LP** itself or a weaker logic, such as **FDE**. Although you might think that it is not possible to work with a weaker logic than **TS**, it is possible. Bear in mind that **TS** is a logic with no valid arguments, where all arguments are invalid. It would suffice to have a logic where there are neither valid nor invalid arguments to have a logic weaker than **TS**. But you need valid arguments to prove (at least) that **TS** has no valid arguments. So this option is not very promising.

Now, consider the following thesis:

- **Homogeneity Thesis:** The logic used in the meta-theory for a logic **L** must be **L** itself.

For example, the logic used in the meta-theory for **LP** must be **LP** itself, but not others. As we have argued, the second option is blocked. It is not possible to do proofs in a metatheory with logic without valid arguments. In fact, every non-empty logic is stronger than an empty logic. Then, both Girard-Tanaka's Conjecture and the Homogeneity Thesis imply that the meta-theory for an empty logic must be empty. But, as we have seen, this option is problematic and not promising.

On this basis, perhaps the most promising option we have if we want to say something about the metatheoretical results of **TS** is to use a stronger logic than **TS**. In [52], Suki Finn states that the content of the basic meta-theory must include at least Modus Ponens and Universal Instantiation. We do not claim that this is necessarily the case and that it is really the only thing we need to do meta-theory of an empty logic. What we can say is that does not necessarily have to use classical logic as the underlying logic of an empty logic.

In the literature, one can find proof that **TS** is a logic with no valid arguments in [30] or [49], and also that **TS** and classical logic have the same anti-valid arguments in [53].[4] Both of these proofs use valid arguments, which **TS** does not have. The

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[4] In [53], Fitting proved that **TS** and Classical Logic (**CL**) have the same set of anti-valid arguments. An argument is *Anti-valid* in **TS**, $\Gamma \vDash^{ant}_{\mathbf{TS}} A$ iff, for all interpretation σ, $1 \notin \sigma(A)$, if $0 \notin \sigma(B)$,

argument that **TS** is a logic where all arguments are invalid can be reconstructed in the following way:

Proof. An argument is *invalid* in **TS**, $\Gamma \nvDash_{\mathbf{TS}} A$, iff, there is an interpretation σ such that $1 \notin \sigma(A)$ or $0 \in \sigma(A)$, if $1 \in \sigma(B)$ or $0 \notin \sigma(B)$, for all $B \in \Gamma$. For all arguments $\Gamma \vDash_{\mathbf{TS}} A$, there is an countermodel (guaranteed by **Lemma. ??**), specifically when $\sigma(p_i) = \{\ \}$ for all p_i occurring in all B of Γ, and when $\sigma(p_j) = \{\ \}$, for all p_j occurring in A, for any $i, j \in \mathbb{N}$. This countermodel makes all the premises not false and the conclusion not true in **TS**. Then, all arguments are invalid in **TS**. □

Using a stronger metatheory, other results also have been reported on [57]. For example, **TS** is a non-reflexive logic since $A \vDash_{\mathbf{TS}} A$ is not valid. Consider any interpretation in which $\sigma(A) = \{\ \}$ and $\sigma(B) = \{\ \}$. Under these interpretations, the argument $A \vDash_{\mathbf{L}} A$ is not valid. In particular, Reflexivity is invalid in **TS**.

Usually, the problem with non-reflexive logics is that there are too many (if not all) valid meta-arguments (see more in [57, p.120]). If we use **TS** in metatheory, we will not be able to prove absolutely anything valid about the meta-arguments. However, these claims arise only because different logics are used in theory and metatheory. For example, authors like Da Ré, Szmuc and Teijeiro conclude that in **TS** all meta-arguments are valid.

> If we take the global approach, all of the metainferences with non-empty sets of premises would be valid. This is so because any instance of the metainference schema appearing below has as a ts-counterexample the Vsk-valuation which assigns to every propositional letter the intermediate value. Thus, in some sense, the global notion makes **TS** metainferentially

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for all $B \in \Gamma$. Although **TS** is an empty logic of logical truths and arguments, both do have anti-valid arguments and logical falsehoods. In this paper, we only focus on valid arguments, so these considerations should be considered in another work.

> trivial, and so it seems like a rather useless conception of metainferential validity. [35, p.1531]

Therefore, according to this line of reasoning, **TS** is not an empty logic for judging Da Ré, Szmuc and Teijeiro. However, by using a stronger metatheory (and not precisely **TS**) for their proofs, all these problems arose. We can go even further and claim that his arguments and proofs for the assertion of this are, in fact, invalid if we are to take **TS** seriously. The right road to making a defense of **TS** as empty logics would be to invalidate every proof that there are valid meta-arguments. However, it is too late for that, and the problem has already arisen. In what follows, we show how the damage caused by the overgeneration of meta-arguments in logics with no valid arguments can be repaired and mitigated.

4.1 Global validity vs Local validity

As we said in the introduction, there are at least two ways of determining whether a meta-argument is valid, namely, Global Validity and Local validity. Let us recall the definitions:

Definition 4.1.1. ***Global Validity:*** *A meta-argument is globally valid if and only if, if the meta-premises* $\Gamma_1 \vDash A_1, \ldots, \Gamma_n \vDash A_n$ *are valid, the meta-conclusion* $(\Delta \vDash B)$ *is valid as well.*

Definition 4.1.2. ***Local validity:*** *A meta-argument is locally valid if and only if the meta-conclusion is satisfied in every interpretation in which the arguments of the meta-premises are satisfied.*

Let me show how each definition leads to different sets of valid meta-arguments. Consider the following meta-argument (built upon $\mathcal{L}$):

$$\text{If } p \vDash_{\mathbf{L}} q \text{ then } r \vDash_{\mathbf{L}} s$$

$p \vDash_{\mathbf{L}} q$ and $r \vDash_{\mathbf{L}} s$ are prototypical cases of invalid arguments in several logics. According to Global validity, this meta-argument is valid, because it cannot be that the meta-premise is valid and the meta-conclusion is not valid, simply because the meta-premise is not valid. In fact, any meta-argument with at least one invalid meta-premise is globally valid.

However, according to Local validity, the meta-argument above would be invalid, because there is an interpretation that satisfies the meta-premises but not the meta-conclusion. Consider for example any interpretation in which $1 \in \sigma(p)$, $1 \in \sigma(q)$, $1 \in \sigma(r)$ but $1 \notin \sigma(s)$.

Since different sets of arguments follow from the two definitions, it is necessary to choose one definition over the other. Barrio *et al.* gives two reasons for preferring Local validity:

- It does not overgenerate at the level of meta-argument, unlike Global validity.
- Local validity gives a unified explanation of validity for both arguments and meta-arguments (of any level).

With respect to the first item, and as we have mentioned, Global validity entails that any meta-argument with invalid meta-premises is globally valid. This seems not so bad, especially for the classicalists, since they have no qualms about things being vacuously true or valid. Nonetheless, it might create some "bad company" problems for some non-classicalists, for example, the **TS**ers. As we mentioned, **TS** is a non-reflexive logic. However, based on the global definition of validity, meta-reflexivity is globally valid.[5]

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[5] Dicher and Paoli noticed a similar problem for **ST**ers, who get Transitivity back at the meta-level.

On the other hand, with respect to the second item, Barrio *et al.* [10] consider that if we accept the definition of Global validity and apply something like that to arguments, we will need a different definition of validity for arguments, one along the following lines:

Definition 4.1.3. *An argument* $\Gamma \vDash_{\mathbf{L}} A$ *is G-valid if and only if* A *is valid if* B *is valid, for all* $B \in \Gamma$*.*

And this is so because recall that Global validity preserves validity from the meta-premises to the meta-conclusion. If we wanted Global validity also at the level of arguments, we should require preservation of validity at that level too, not merely the preservation of truth.

The overgeneration is clearer in this case. Note that $A \vDash_{\mathbf{L}} B$ would be G-valid for any atomic instances of A. (And actually for any substitution of A that is not a logical truth in **L**).

Barrio *et al.* say that both problems arising from Global validity are solved by endorsing Local validity instead. Remember that the meta-argument

$$\text{If } p \vDash_{\mathbf{L}} q \text{ then } r \vDash_{\mathbf{L}} s$$

is not locally valid. This implies that not any meta-argument with invalid meta-premises will be valid due to vacuity. According to them, this solves the problem of overgeneration in the meta-argument set.

Barrio *et al.* appeal to the usual definition of validity for arguments to settle the discussion. This definition preserves truth and not validity. That is, if we prefer the definition of validity as truth-preservation for arguments, the definition of validity for meta-arguments must also be truth-preserving. The definition of local validity meets this requirement. Therefore, if it is used for the evaluation of meta-arguments, then it is possible to provide a unified explanation of validity.

More recently, Teijeiro [118] argued that the analogy between satisfaction-preservation and truth-preservation does not quite work. Truth allows us to evaluate different instances of the same argument schema all at once, whereas with satisfiability it is required to evaluate each particular instance of a scheme. Why choosing one definition over another or whether the notion of validity for meta-arguments should be a classical notion is still an open debate. We will now delve into the specific case of **TS**.

4.2 TS and Meta-arguments

The Global validity is problematic for a logic with no valid arguments, such as **TS**. Let us see why. All arguments in **TS** are invalid. This feature makes that any meta-argument in **TS** with invalid meta-premises, would be globally valid in **TS** by vacuity. In **TS** there is an overgenerating in the set of valid meta-arguments. Therefore, the definition of empty logic, globally valid, and the logic **TS** are incompatible. In fact, Dicher and Paoli claim, in the case of **ST**, that "[...] the validity preservation requirement would vacuously vindicate all sorts of undesirables" [38, p. 291].

This overgeneration is only partially solved in **TS** by appealing to the definition of Local validity. For example, meta-reflexivity is also locally valid. Since the same interpretations that satisfy the meta-premises, will also satisfy the meta-conclusion. This is even though the argument $A \vDash_{\mathbf{TS}} A$ is not valid. Any instance of meta-reflexivity will also be locally valid in **TS**.

Dicher and Paoli have proved for **ST** that local validity implies global validity, but not vice versa. This result also holds for the meta-arguments of **TS** since the set of global meta-arguments is trivial. That is, any meta-argument locally valid is also globally valid; but there are meta-arguments globally valid (such as "if $p \vDash_{\mathbf{TS}} q$ then $r \vDash_{\mathbf{TS}} s$") that are not locally valid in **TS**.

In the definition of validity, the conditional of meta-arguments is usually interpreted by means of the conditions of evaluation of the material conditional. The evaluation conditions of the material conditional are as follows:

- $1 \in \sigma(A \supset B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$
- $0 \in \sigma(A \supset B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

Consider that truth and non-truth (at the level of arguments) are equivalent to valid and invalid, respectively (at the level of meta-arguments). These evaluation conditions of the material conditional determine which meta-argument is valid or invalid.

- **Global m-validity** A meta-argument is *globally m-valid* if and only if the meta-premises are not valid or the meta-conclusion is valid.
- **Local m-validity** A meta-argument is *locally m-valid* if and only if, for all interpretations, the meta-premises are not satisfied or the meta-conclusion is satisfied.

Here it is obvious that a logic that makes use of the definition of Global m-validity will have a more restricted set of meta-arguments. This is because Local m-validity requires meta-premises to be anti-valid arguments, while Global m-validity requires arguments to be invalid. Any anti-valid argument is invalid by definition, but not vice versa. The consequence of this is that with Global m-validity there is more possibility of having valid meta-arguments by vacuity.

Dicher and Paoli claim that "[...] for such purposes as that of assessing the classicality of **ST**, we are better served by a local notion of meta-inferential validity." And since my purpose is assessing the emptiness of **TS**, the definitions of Global and Local m-validity are not adequate —both validate meta-arguments and thus **TS** is not empty after all— and therefore it is time to propose another way of understanding validity.

But, what if we keep the working definition of empty logic by preferring the definition of validity but understanding the logical notions in its definition in a slightly different way? Consider the following conditional $\rightsquigarrow$, also known as transplication. In recent history, transplication has been extensively studied. It appeared in Reichenbach's [100] as *quasi-implication*, de Finetti's [36] conditional, and in Blamey's [22] transplication. See [40] for a more detailed study in three-valued settings and [46] for a study in four-valued settings.

The evaluation conditions of the transplication are as follows:

- $1 \in \sigma(A \rightsquigarrow B)$ iff, if $1 \in \sigma(A)$ then $1 \in \sigma(B)$
- $0 \in \sigma(A \rightsquigarrow B)$ iff $1 \in \sigma(A)$ and $0 \in \sigma(B)$

These conditions can be presented in tabular form, as follows:

$A \rightsquigarrow B$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{\ \}$
$\{0\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$

Interpreting the conditional of the definitions of validity by means of the evaluation conditions of the transplication, these definitions would be interpreted as follows:

Interpretation of Global t-validity:

- **Global t-validity:** A meta-argument is *Globally t-valid* if and only if, the meta-premises are valid and the meta-conclusion is valid.
- **Global t-Invalidity:** A meta-argument is *Globally t-invalid* if and only if, the meta-premises are valid and the meta-conclusion is invalid.

Interpretation of Local t-validity:

- **Local t-validity:** A meta-argument is *Locally t-valid* if and only, in every interpretation, the meta-conclusion is satisfied and the arguments of the meta-premises are satisfied.

- **Local t-invalidity:** A meta-argument is *Globally t-invalid* if and only if, there is an interpretation that satisfies the meta-premises but does unsatisfy the meta-conclusion

Both definitions coincide on the same set of valid meta-arguments Let me prove this:

Theorem 4.2.1. *A meta-argument is Global t-Valid iff it is local t-valid.*

Proof. First, suppose that a meta-argument is globally t-valid. A meta-argument is *globally valid* if and only if, if the meta-premises are valid, the meta-conclusion is valid as well. Interpreting this conditional by means of the evaluation conditions of transplication, then the meta-premises are t-valid and the meta-conclusion is also t-valid. Now, if the meta-premises and the meta-conclusion are t-valid, then there are no countermodels. If there are no countermodels, then in every interpretation, the meta-premises and the meta-conclusion are satisfied. Therefore, the meta-argument is locally t-valid.

Second, suppose that a meta-argument is locally t-valid. A meta-argument is *locally valid* if and only if the meta-conclusion is satisfied in every interpretation in which the arguments of the meta-premises are satisfied. Interpreting this conditional by means of the evaluation conditions of transplication, then, in every interpretation, the premises are satisfied and the meta-conclusion is satisfied. Now, if in every interpretation the meta-premises and meta-conclusion are satisfied, then there are no countermodels. Then, the meta-premises and meta-conclusion are t-valid arguments. Therefore, the meta-argument is globally t-valid. [6] ☐

Global t-invalidity coinciding with Local t-invalidity does not occur. Since there are arguments (such as an example: if $p \vDash_{\mathbf{L}} q$ then $r \vDash_{\mathbf{L}} s$) that are locally invalid but neither globally invalid nor globally valid. Let us now see how these definitions work for the case of **TS**.

Let us remember that **TS** is a logic without valid arguments. Now consider **TS** but using the interpretation of t-validity. This logic has no valid meta-arguments. Let me prove this.

Theorem 4.2.2. ***TS*** *with the interpretation of t-validity has no valid meta-arguments.*

Proof. Suppose that a meta-argument is t-valid. It follows that the meta-premises are valid and the meta-conclusion is also valid. But this is impossible since **TS** has no valid arguments. ☐

Hereinafter, we will refer to this logic as **ETS** as an abbreviation for Empty Tolerant Strict.

Fact 4.2.1. ***ETS*** *is an empty logic*

This is a good result since it allows us to address the concerns of Barrio, Dicher, and Paoli. First, special cases of validity for meta-arguments with invalid premises are no longer problematic. t-validity will not accept these special cases as valid. This implies that the interpretation of t-validity will not overgenerate the set of meta-arguments. In addition, t-validity retrieves all valid meta-arguments that are not in dispute, where the meta-premises are valid and the meta-conclusion is valid.

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[6] Since the two definitions collapse, it does not matter which definition to use to define t-valid meta-arguments. For simplicity, the Global t-validity will be used hereafter.

4.3 Possible objections

If we understand the notion of validity in terms of transplication, certain objections can be raised. In this section, we will examine four such objections and respond to each of them:

1. Transplication is really a conjunction.
2. Transplication appears nowhere in the object language.
3. The distinction between theory and meta-theory is lost in **ETS**.
4. The distinction valid/invalid is no longer exhaustive.

Let us begin with the first objection, which asks whether the transplication is a conjunction. Indeed, if we use the truth condition of the conjunction of $\mathcal{L}$ for transplication, that is

- $1 \in \sigma(A \rightsquigarrow B)$ iff, $1 \in \sigma(A)$ and $1 \in \sigma(B)$

the resulting table is exactly the same. Since this, the transplication is nothing more than a conjunction disguised as a conditional. But in our view, the tables are potentially misleading. For example, in a Dunn's semantic presentation of these logics, the tabular presentations of the connectives make it appear, for example, that the conjunction of classical logic, **K3** and **LP** are different connectives. In [85], Omori and Arenhart have presented a semantics for those logics, to show that we can agree on the classical meaning of the connectives and disagree on the logical consequence relations. That is, all these logics can have the same truth table in a Herzberger-style semantics (see more in [63]. The fact that the truth condition of the transplication and the truth condition of the conjunction result in the same table may itself be an accidental matter of semantics. This shows that inferring information from the truth tables of a given semantics is no guarantee of anything.

When you look at the evaluation conditions of transplication and its conjunctive interpretation, it seems clear that they are different connectives, even though when they result in the same table. Remember that sharing evaluation conditions is not sufficient to say that a disputed connective is a known connective. If this is the case, we must not forget that the transplication shares the condition of the falsity of the conditional. Therefore, under the same criteria, it is conditional.

If this argument is not convincing enough, let's suppose that transplication really is a conjunction. On an inferentialist approach to the meaning of connectives, it must then be the case that transplication is a co-implication of a conjunction. That is, in this perspective, inherited from Gentzen, it should be valid:

$$A \rightsquigarrow B \dashv_{\mathbf{L}} \vDash A \wedge B$$

However, this does not hold up in either direction. For $A \rightsquigarrow B \vDash_{\mathbf{ETS}} A \wedge B$, consider the following counterexample: $\sigma(A) = \{1\}$ and $\sigma(B) = \{\ \}$; or just consider any counterexample in which $1 \notin \sigma(A)$. And for $A \wedge B \vDash_{\mathbf{ETS}} A \rightsquigarrow B$, consider the following counterexample: $\sigma(A) = \{1\}$ and $\sigma(B) = \{\ \}$; or any counterexample in which $1 \in \sigma(A)$ and $1 \notin \sigma(B)$, or $0 \notin \sigma(A)$ and $1 \notin \sigma(A)$, or $0 \in \sigma(A)$ and $0 \notin \sigma(B)$. This should not be surprising because **ETS** is logic without valid arguments.

The question of whether transplication is a conditional was answered in the affirmative by Hitoshi and Arenhart in [85]. They have presented a logic, based on Herzberger's semantics, in which the transplication and the material conditional have the same conditions of truth. This allows us to see something interesting going on. Depending on the semantic presentation and the logical consequence relation, the different connectives behave differently. That is, it is easier to identify the transplication as a conditional depending on the semantic presentation. Also, connectives like transplication may or may not validate some arguments that can be considered desirable, depending on the logical consequence. For example, with the semantic presentation of **ETS** and a truth-preserving logical consequential relation (interpreted with the material conditional), the following arguments usually hold

- $A \rightsquigarrow B \vDash_{\mathbf{L}} A$
- $A \rightsquigarrow B \vDash_{\mathbf{L}} B$
- $A \rightsquigarrow B \vDash_{\mathbf{L}} B \rightsquigarrow A$
- $\nvDash_{\mathbf{L}} A \rightsquigarrow A$

These arguments are more commonly associated with a conjunction connective. Here the conjunction is separable and commutative. Therefore, some people have the view that transplication is a conjunction. However, in our approach we do not have this problem. Transplication does not validate any of the above arguments. In fact, the transplication behaves in the way that we would expect a conditional to behave for an empty logic: it does not validate anything.

We can make an even stronger claim. In these empty logics, the inferentialist criterion of the meaning of connectives becomes obsolete. For example, a conditional here does not validate arguments such as Detachment, Transitivity, or Identity; the disjunction will also not validate arguments such as Disjunction, Introduction or addition, etc. No connective validates anything, so it cannot be characterized by schemes it validates. In this case, we can either classify these types of connectives as *sui generis*, or we can determine what type of connective they are, depending on their evaluation conditions. The second disjunct gives us the best option.

The argument can even be extended further to assert then that, in empty logics, evaluation conditions do not matter either. That is, the logic will have no valid arguments even if the evaluation conditions are changed because the q-consequence is doing its work just fine. However, this is false. The logical consequence relation alone does not do the job. In [48], Estrada-González and Romero (202X) have proved that it is possible to obtain some valid arguments making use of q-consequence and the connectives of the connexive logic **CC1**. This shows that q-consequence does not do the job by itself and for the logic not to have valid arguments, the logic also needs specific evaluation conditions to make this happen.

Finally, in [41] and [47], it has recently been argued that transplication in non-transitive logics looks more like a conditional, while in Tarskian logics it looks more like a conjunction. Since empty logics are non-transitive (and non-Tarskian), we can say that transplication looks more like a conditional than a conjunction.

The second objection asks whether we have the right to use in meta-theory a connective that does not exist in the language of the object. The transplication appears nowhere in the object language. This claim is legitimate because it actually the conditional we have in $\mathcal{L}$ is called the extensional conditional. But we do not use the same conditional when we interpret the notion of validity for meta-arguments, we use transplication. Why should we use a conditional that does not appear in the language of L anywhere else? While this objection may seem more robust, let me show that if we define a logic where transplication is part of the language, this will not change the result obtained (in the case of **TS**).

Let me take $\mathcal{L}$ and expand it with the connective $\rightsquigarrow$. The evaluation conditions for these connectives have already been defined. The set of interpretations will be the same as the one worked on so far, allowing me to call this logic **TS-t**.

Lemma 4.3.1. *For any formula $A \rightsquigarrow B$ of **TS-t**, there is an assignment $1 \notin \sigma(A \rightsquigarrow B)$ and $0 \notin \sigma(A \rightsquigarrow B)$.*

Proof. By induction on the complexity of A and B. If A and B are propositional variables, the assignment exists. **Lemma.** 4.0.1 guarantees that for any formula A and B of the form $\sim X$, $X \wedge Y$, $X \vee Y$ there is an assignment $1 \notin \sigma(A)$ and $0 \notin \sigma(A)$, and $1 \notin \sigma(B)$ and $0 \notin \sigma(B)$. Then, by the table of $\rightsquigarrow$, there is always an assignment

$1 \notin \sigma(A \rightsquigarrow B)$ and $0 \notin \sigma(A \rightsquigarrow B)$. ☐

Some features of **TS-t** are:

Theorem 4.3.1. ***TS-t*** *has no logical truth.*

Proof. A is a *logical truth* in **TS-t** iff, for every σ, $1 \in \sigma(A)$. But this is impossible by **Lemma.** 4.0.1 and **Lemma** 4.3.1. ☐

Theorem 4.3.2. ***TS-t*** *has no valid arguments.*

Proof. This follows directly from **Fact. ??**, **Lemma.** 4.0.1 and **Lemma** 4.3.1, ☐

Now, if we use the interpretation of t-validity for **TS-t**, the result is as follows:

Theorem 4.3.3. ***TS-t*** *with the interpretation of t-validity has no valid meta-arguments.*

Proof. Mutatis mutandis, the proof is the same as in **Theorem.** 4.2.2. ☐

Fact 4.3.1. ***TS-t*** *with the interpretation of t-validity is a conservative extension of* ***ETS****.*

Proof. This follows directly from **Theorem.** 4.3.1 and **Theorem.** 4.3.2 and **Theorem.** 4.3.3. ☐

Since the argument set of **TS** is the same as that of **TS-t**, the result of using the t-validity in **TS-t** will be the same: **TS-t** is an empty logic. Both **TS** and **TS-t** have the same set of arguments and meta-arguments, with the interpretation of t-validity. If it is enough that two logics have exactly the same collections of valid and invalid arguments, then **TS** and **TS-t** are the same logic. It follows that it does not really matter that the transplication does not appear anywhere in the language. Conceding this, we consider that the objection was solved.

The third objection is that the distinction between theory and meta-theory is lost for this logic. The term 'meta-theory (for the logical theories)' is often used to refer to the logical study devoted to proving facts about logic, typically relating proof systems with model-theoretic structures. Usually, many meta-theoretical results are obtained by using classical logic as its underlying logic. Even though the logic of the theory is a non-classical logic. For example, the case of meta-theoretical developments of logics such as **LP** [98]. The classical logician (and even some non-classical logicians) usually interprets the validity of meta-arguments as an extensional conditional. We do not do this here and use a non-standard notion, but one that we think is the most appropriate. It does not follow, however, that the distinction between theory and metatheory is lost.

As we have already seen, transplication is not part of the language of **ETL**. Therefore, the logic that is used in theory and in meta-theory is different. This is neither positive nor negative. For our purposes, it may make sense to use this kind of interpretation of validity only in the case of meta-arguments. In short, for a logic not to have valid arguments, transplication is not necessary. But for a logic without valid arguments to have no meta-arguments is, as we have shown, a promising option.

Suppose now that we use the *t-validity* for the meta-arguments of **TS-t**. In this logic, we would also have neither valid arguments nor valid meta-arguments, as in **ETS**. In this version of **TS-t** the logic used for arguments and meta-arguments is the same. But this does not mean that the distinction between theory and metatheory is lost. We would not be able to argue anything about STD (or ST-t) in a valid way if we used an empty logic for our metatheory. Here, in order to argue (validly) that our understanding of the validity of meta-arguments is promising for logics without valid arguments, we have used a slightly weaker logic in metatheory.

Weber, Badia, Girard, and Tanaka (see more in [123], [7], [115]) have argued that we should hold non-classicality at all levels of investigations in non-classical logics. We agree with this thesis. Moreover, recently, Tanaka and Girard have argued that “when the meta-logic is stronger than the logic being (meta) theorized, bad results follow” [115, p.9]. Since every logic is stronger than an empty logic and none is weaker, the meta-theory of an empty logic must be empty. However, as we noted above, this is problematic because we would not be able to defend anything about an empty logic with valid arguments. This poses a challenge to the logical nihilist. He might have to argue that an empty logic is the only correct logic in all domains. But, in our view, taking this seriously will require another paper.

We have shown that there are those who consider that a unified explanation of logical consequence is important for arguments and meta-arguments. A logic like **ETS** that has no valid arguments and meta-arguments allows a unified explanation at both levels. For this reason, we think that our understanding of the validity of meta-arguments is appropriate for logics without valid arguments. Otherwise, all meta-arguments will be valid in a logic without valid arguments, if the validity of meta-arguments is interpreted using an extensional conditional.

Finally, the last possible objection is that, on the level of meta-arguments, the valid/invalid distinction is no longer exhaustive. This happens because meta-arguments cannot meet the interpretation of *t-validity* and *t-invalidity*. Therefore, **ETS**’s meta-arguments are neither valid nor invalid. That is, arguments can be valid, invalid, and neither invalid nor invalid. In fact, invalid arguments does not collapse with invalid arguments.

However, this should not be seen as an objection, but rather as a fact of which we can be proud. This is a consequence of taking seriously the set of interpretations we have taken. In our semantic of **ETS**, the interpretation $\{\ \}$ is a semantic value that formulas get when they are neither true nor false. Since we have collapsed the arguments and meta-arguments, some arguments are expected to be neither valid nor invalid. This should not be a problem for what we are trying to do; it should be seen as a desirable consequence.

Conclusion

In this chapter, we proposed a notion of meta-argument validity. This is intended to be compatible with an empty logic, as defined in this paper. Our definition of the validity of meta-arguments prefers the usual definition of validity, but with a slightly different understanding of the conditional used in the definition. This has allowed us to characterise the **ETS** logic as a logic with no logical truths, no arguments, and no valid meta-arguments.

In [91], Pailos asks if there is a logic without valid meta-arguments. His answer is as follows:

> Once again, the answer is positive. In fact, the path that French describes can be extended indefinitely. It starts with the Strong Kleene logic **SK**, a logic without tautologies, and follows with **TS**, and has **ST/TS** as its third step. The resulting hierarchy will be collected (in a sense we will specify below) in a logic $\mathbf{TS}_\omega$, that has no validities of any level whatsoever.

We have shown that the road does not have to be from **ST/TS** to $\mathbf{TS}_\omega$. We have obtained the empty logic **ETS**, but via an easy road compared to Pailos'. We think that understanding the causes of the overgeneration of meta-arguments (that all meta-arguments are valid) in the empty logics leads naturally to our interpretation of the validity of meta-arguments with transplication. In other words, if we understand that validity by vacuity (for meta-arguments) is the only thing that makes all meta-arguments valid, then the way seems to be to block validity by vacuity (if you want an empty logic). The result of this blocking is the interpretation of meta-arguments in the way we have proposed.

We have given some arguments for why transplication is not a conjunction and why the inferentialist criterion for the definition of connectives does not work for empty logics. That is, there seems to be no way to distinguish one connective from another in an empty logic if we define a connective $\otimes$ to be a conjunction/disjunction/negation/conditional by the arguments it validates. However, it is possible to identify these connectives with the evaluation conditions. We consider that if an intended connective shares the truth or falsity condition of a known connective, the intended connective is the known connective. Since the transplication shares the evaluation conditions of the extensional (and material) conditional, it is a conditional. On the other hand, the transplication does not share the truth condition of any conjunction, or indeed of any connective. Therefore, at least for the time being, its truth condition seems to be *sui generis*.

Finally, we think our way of interpreting the validity of meta-arguments is promising because it allows us to obtain empty logics without any further maneuvering. While Blamey used transplication for the problem of supposition, transplication has not been associated with any utility for empty logics.

Chapter 5

Another remark on connexivity and set theory

Abstract

We show that Wiredu's result in [125] is not the doom for connexive set theories, not even for those based in logics similar to **CC1**, one of the original target logics. For this purpose, we present the necessary assumptions for Wiredu's proof, making some precisions on the connexive requirements. Then we present a non-reflexive variant of **CC1** in which Wiredu's proof can be blocked. Finally, we discuss the prospects of a connexive set theory based on both the non-reflexive and non-transitive variants of **CC1**, which, even assuming non-triviality, are not very rosy.

Keywords Separation Axiom, Simplification, Conjunction Elimination, connexivity, q-consequence, p-consequence.

5.0 Introduction

Replacing the Comprehension Axiom ("Every predicate defines a set") by the Separation Axiom ("If something is a set, its definable subcollections are also sets") was Zermelo's attempt to block the rise of Russell's set and other sets that, together with some logical principles, entail the triviality of set theory. Unlike the case of the Comprehension Axiom, which requires logics that are not only paraconsistent but also non-contracting to avoid triviality, many logics seem safe to use along with Separation.

In [125], Kwasi Wiredu proved that it is possible to obtain contradictions in Zermelo's set theory based on a certain kind of logic where Aristotle's Thesis is valid. In particular, the Angell-McCall's logic **CC1** falls within the scope of the theorem, and since **CC1** is an explosive logic, a Zermelo set theory based on **CC1** is trivial. Wiredu's result is important because it shows that some connexive logics lead to triviality even when Comprehension is restricted in the way Zermelo suggested. From this, Wiredu claims that "Zermelo's kind of way out [of Russell's theorem] is not open to connexivists", and the implicature is that no set theory at all is available to connexivists.

No research on the viability of Zermelo's set theory based on a connexive logic has been carried out after Wiredu's proof. (And even before that, the only work resembling in some form a connexive set theory was McCall's connexive class theory in [80].) The aim of this chapter is to show that there is a connexive logic, built upon **CC1**, for which Wiredu's proof does not work. Such a logic is actually **CC1** but based on Malinowski's [76] q-consequence relation. Although the non-triviality of such a theory is not established here, we show that it would not face some definability problems that affect its relevant cousins, for example. For completeness, we consider also **pCC1**, that is, **CC1** based on Frankowski's [56] p-consequence, and discuss some of its properties.

The structure of the chapter is as follows. First, we reconstruct Wiredu's proof, making explicit all and only the necessary assumptions. In the third section, we present the connexive logic **CC1** at tutorial speed. In the fourth section, we make the scope of Wiredu's proof clearer, and we show that Wiredu's assessment of its own result can be called into question. Finally, we present the logics **qCC1** and **pCC1**, which are basically **CC1** with *q*-consequence and *p*-consequence, respectively. Both logics validate the usual connexive principles and coincide in many of **CC1**'s axioms. We will prove that **qCC1** is a logic where Conjunction Elimination is not a valid argument, so it is possible to block the conclusion of Wiredu's proof and yet doing some set theory in the usual shape might be tricky.

5.1 Wiredu's triviality proof

Consider a formal language with a conditional, $>$; a conjunction, $\otimes$; a negation, N; a biconditional, $<>$; a particular quantifier , $\exists$; and a universal quantifier, $\forall$. In [125], Wiredu proved that a theory **T** containing the following valid arguments is trivial:

$$\vDash_{\mathbf{T}} \forall z \exists y \forall x ((x \in y) <> ((x \in z) \otimes Fx)) \qquad \text{(Separation Axiom)}$$

$$\text{If } \vDash_{\mathbf{T}} A \text{ then } \vDash_{\mathbf{T}} A[B/C] \qquad \text{(Schema Substitution)}$$

$$\forall x B(x) \vDash_{\mathbf{T}} B(t) \qquad \text{(Universal Instantiation)}$$

$$\exists x B(x) \vDash_{\mathbf{T}} B(a) \qquad \text{(Particular Instantiation)}$$

$$A <> (B \otimes NA) \vDash_{\mathbf{T}} (A > B) \otimes (A > NA) \qquad \text{(Distribution of } > \text{ over } \otimes\text{)}$$

$$A \otimes B \vDash_{\mathbf{T}} B \qquad \text{(Conjunction Elimination)}$$

$$\vDash_{\mathbf{T}} N(A > NA) \qquad \text{(Aristotle's Thesis)}$$

$$A, NA \vDash_{\mathbf{T}} B \qquad \text{(Explosion)}$$

(The usual restrictions on the freshness of terms in Particular Instantiation also apply here.)[1]

The proof is as follows:

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[1] In fact, Wiredu considers stronger assumptions, in arrow form. Nonetheless, it does not make any harm to weaken the assumptions to rule form and still attributing the proof to Wiredu, it is essentially the same with the change just mentioned.

1. $\forall z \exists y \forall x((x \in y) <> ((x \in z) \otimes Fx))$ (Separation Axiom)
2. $\forall z \exists y \forall x((x \in y) <> ((x \in z) \otimes N(x \in x))$ (1, Substitution $Fx/\ N(x \in x)$)
3. $\exists y \forall x((x \in y) <> ((x \in z) \otimes N(x \in x)))$ (2, Universal Instantiation z/z)
4. $\forall x((x \in y) <> ((x \in z) \otimes N(x \in x)))$ (3, Particular Instantiation y/y)
5. $(y \in y) <> ((y \in z) \otimes N(y \in y))$ (4, Universal Instantiation x/y)
6. $((y \in y) > (y \in z)) \otimes ((y \in y) > N(y \in y))$ (5, Distribution of > over ⊗)
7. $(y \in y) > N(y \in y)$ (6, Conjunction Elimination)
8. $N((y \in y) > N(y \in y))$ (Aristotle's Thesis)
9. B (7, 8, Explosion)

From this, Wiredu claims "that the Zermelo type of escape route from Russell's paradox is not available in a connexively based set theory". In particular, the last four items hold in Angell-McCall's logic **CC1**. Therefore, Zermelo's set theory based on **CC1** is trivial. Yet, the very same logic can come in handy to turn Wiredu's result around. In order to make these claims more understandable, let us present **CC1** in some detail.

5.2 A crash-course on CC1

Let $\mathcal{L}$ be a language for **CC1** with formulas built in the usual way, from a countable set of propositional variables $Var = \{p_1, ..., p_n\}$ with the following set of connectives $\{\neg, \wedge_{AM}, \rightarrow_{AM}\}$. We will use the first capital letters of the Latin alphabet, 'A', 'B', 'C', ... as arbitrary formulas of $\mathcal{L}$.

McCall [79], after Angell [3], presented the logic **CC1** by means of Smiley's matrices. A *valuation* V for **CC1** is a function from propositional variables to the truth values 1, 2, 3 and 4, where $\mathcal{D} = \{1, 2\}$ is the set of designated values, and that can be extended to all formulas according to the following tables:

$\neg A$	A
1	4
2	3
3	2
4	1

$A \rightarrow_{AM} B$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	1	4	1	4
4	4	1	4	1

$A \wedge_{AM} B$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	3	4
4	4	3	4	3

Definition 5.2.1. *Let Γ be a set of formulas, A is a logical consequence of Γ in **CC1**, $\Gamma \vDash_{CC1} A$, if and only if, for every valuation V, $V(A) \in \mathcal{D}$ if $V(B) \in \mathcal{D}$ for every $B \in \Gamma$.*

The above semantics can be converted into a Dunn semantics using the method described in [86], and here we will prefer the latter presentation, which we summarize in what follows. An *interpretation* for $\mathcal{L}$ is a relation between propositional variables and truth values (1 and 0), such that they can be related in one of the following ways:

- p_i is true but not false, represented by '$1 \in \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1\}$
- p_i is true but also false, represented by '$1 \in \sigma(p_i)$ and $0 \in \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{1, 0\}$
- p_i is neither true nor false, represented by '$1 \notin \sigma(p_i)$ and $0 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{\ \}$

- p_i is false but not true, represented by '$0 \in \sigma(p_i)$ and $1 \notin \sigma(p_i)$'; more briefly, $\sigma(p_i) = \{0\}$

Now, let Γ be a set of formulas. A is a *logical consequence* of Γ in **CC1**, $\Gamma \vDash_{\mathbf{CC1}} A$, if and only if, for every evaluation σ, $1 \in \sigma(A)$, if $1 \in \sigma(B)$ for every $B \in \Gamma$. In fact, this notion of logical consequence is extended to any theory containing **CC1**, Zermelo's set theory based on **CC1**, or $Z_{\mathbf{CC1}}$ for short.

Interpretations extend to all formulas according to the following evaluation conditions:

$1 \in \sigma(\neg A)$ iff $1 \notin \sigma(A)$
$0 \in \sigma(\neg A)$ iff $0 \notin \sigma(A)$

$1 \in \sigma(A \rightarrow_{AM} B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$ and $0 \in \sigma(A)$ iff $0 \in \sigma(B)$
$0 \in \sigma(A \rightarrow_{AM} B)$ iff, $0 \in \sigma(A)$ iff $0 \notin \sigma(B)$

$1 \in \sigma(A \wedge_{AM} B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$
$0 \in \sigma(A \wedge_{AM} B)$ iff $0 \in \sigma(A)$ iff $0 \notin \sigma(B)$

The evaluation conditions above can be presented tabularly as follows:

$\neg A$	A
$\{0\}$	$\{1\}$
$\{\ \}$	$\{1,0\}$
$\{1,0\}$	$\{\ \}$
$\{1\}$	$\{0\}$

$A \rightarrow_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{\ \}$
$\{\ \}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$

$A \wedge_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{0\}$	$\{\ \}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{\ \}$	$\{0\}$	$\{\ \}$

A biconditional $A \leftrightarrow_{AM} B$ is definable as $(A \rightarrow_{AM} B) \wedge_{AM} (B \rightarrow_{AM} A)$. This biconditional $A \leftrightarrow_{AM} B$ can be presented in tabular form as follows:

$A \leftrightarrow_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$
$\{1,0\}$	$\{\ \}$	$\{1\}$	$\{\ \}$	$\{\ \}$
$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{1\}$	$\{\ \}$
$\{0\}$	$\{\ \}$	$\{\ \}$	$\{\ \}$	$\{1\}$

CC1 can be presented axiomatically, Hilbert-style, as follows:

Rules:

R1 Detachment: $A, A \rightarrow_{AM} B \vdash_{\mathbf{CC1}} B$

R2 Strict Adjunction: If $\vdash_{\mathbf{CC1}} A$ and $\vdash_{\mathbf{CC1}} B$, $\vdash_{\mathbf{CC1}} A \wedge_{AM} B$

Axioms:

1. $\vdash_{\mathbf{CC1}} (A \rightarrow_{AM} B) \rightarrow_{AM} ((B \rightarrow_{AM} C) \rightarrow_{AM} (A \rightarrow_{AM} C))$
2. $\vdash_{\mathbf{CC1}} ((A \rightarrow_{AM} A) \rightarrow_{AM} B) \rightarrow_{AM} B$
3. $\vdash_{\mathbf{CC1}} (A \rightarrow_{AM} B) \rightarrow_{AM} ((A \wedge_{AM} C) \rightarrow_{AM} (B \wedge_{AM} C))$
4. $\vdash_{\mathbf{CC1}} (A \wedge_{AM} A) \rightarrow_{AM} (B \rightarrow_{AM} B)$
5. $\vdash_{\mathbf{CC1}} (A \wedge_{AM} (B \wedge_{AM} C)) \rightarrow_{AM} (B \wedge_{AM} (A \wedge_{AM} C))$
6. $\vdash_{\mathbf{CC1}} (A \wedge_{AM} A) \rightarrow_{AM} ((A \rightarrow_{AM} A) \rightarrow_{AM} (A \wedge_{AM} A))$
7. $\vdash_{\mathbf{CC1}} A \rightarrow_{AM} (A \wedge_{AM} (A \wedge_{AM} A))$
8. $\vdash_{\mathbf{CC1}} ((A \rightarrow_{AM} \neg B) \wedge_{AM} B) \rightarrow_{AM} \neg A$
9. $\vdash_{\mathbf{CC1}} (A \wedge_{AM} \neg(A \wedge_{AM} \neg B)) \rightarrow_{AM} B$

10. $\vdash_{\mathbf{CC1}} \neg(A \wedge_{AM} \neg(A \wedge_{AM} A))$

11. $\vdash_{\mathbf{CC1}} (\neg A \vee ((A \rightarrow_{AM} A) \rightarrow_{AM} A)) \vee (((A \rightarrow_{AM} A) \vee (A \rightarrow_{AM} A)) \rightarrow_{AM} A)$

12. $\vdash_{\mathbf{CC1}} (A \rightarrow_{AM} A) \rightarrow_{AM} \neg(A \rightarrow_{AM} \neg A)$

It is known that **CC1** is sound and complete with respect to the above semantics; see [79].

CC1 is a connexive logic in the sense of McCall [79] since it validates the following argument schemas:

$$\vDash_{\mathbf{CC1}} \neg(A \rightarrow_{AM} \neg A) \qquad \text{(Aristotle's Thesis)}$$

$$\vDash_{\mathbf{CC1}} \neg(\neg A \rightarrow_{AM} A) \qquad \text{(Variant of Aristotle's Thesis)}$$

$$\vDash_{\mathbf{CC1}} (A \rightarrow_{AM} B) \rightarrow_{AM} \neg(A \rightarrow_{AM} \neg B) \qquad \text{(Boethius' Thesis)}$$

$$\vDash_{\mathbf{CC1}} (A \rightarrow_{AM} \neg B) \rightarrow_{AM} \neg(A \rightarrow_{AM} B) \qquad \text{(Variant of Boethius' Thesis)}$$

And also invalidates the following argument schema:

$$\vDash_{\mathbf{CC1}} (A \rightarrow_{AM} B) \rightarrow_{AM} (B \rightarrow_{AM} A) \qquad \text{(Symmetry of Implication)}$$

Let us prove now some useful facts about **CC1**.

Definition 5.2.2. *An inadmissible contradiction is any claim of the form "*$x \in \sigma(A)$ *and* $x \notin \sigma(A)$*", with* $x \in \{1, 0\}$.

Fact 5.2.1. *Aristotle's Thesis is never false in* ***CC1****.*

Proof. Suppose that $0 \in \sigma(\neg(A \rightarrow_{AM} \neg A))$. By the evaluation condition of negation, $0 \notin \sigma(A \rightarrow_{AM} \neg A)$. By the falsity condition of $A \rightarrow_{AM} B$, $0 \in \sigma(A)$ and $0 \in \sigma(\neg A)$ or both $0 \notin \sigma(\neg A)$ and $0 \notin \sigma(A)$. By the evaluation condition of negation, $0 \in \sigma(A)$ and $0 \notin \sigma(A)$ or both $0 \in \sigma(A)$ and $0 \notin \sigma(A)$. But neither of these disjuncts are satisfiable since each requires an inadmissible contradiction. $\square$

Fact 5.2.2. *Boethius's Thesis is never false in* ***CC1****.*

Proof. Suppose that $0 \in \sigma((A \rightarrow_{AM} B) \rightarrow_{AM} \neg(A \rightarrow_{AM} \neg B))$. By the falsity condition of $\rightarrow_{AM}$, $0 \in \sigma(A \rightarrow_{AM} B)$ iff $0 \notin \sigma\neg(A \rightarrow_{AM} \neg B)$. By the evaluation condition of negation, if $0 \notin \sigma\neg(A \rightarrow_{AM} \neg B)$ then $0 \in \sigma(A \rightarrow_{AM} \neg B)$. Therefore, $0 \in \sigma(A \rightarrow_{AM} B)$ iff $0 \in \sigma(A \rightarrow_{AM} \neg B)$. If $0 \in \sigma(A \rightarrow_{AM} B)$ then $0 \in \sigma(A)$ iff $0 \notin \sigma(B)$. And if $0 \in \sigma(A \rightarrow_{AM} \neg B)$ then $0 \in \sigma(A)$ iff $0 \notin \sigma(\neg B)$. By the evaluation condition of negation, if $0 \notin \sigma(\neg B)$ then $0 \in \sigma(B)$. But $0 \notin \sigma(B)$ and $0 \in \sigma(B)$ is an inadmissible contradiction. $\square$

The proofs for the Variants of Aristotle's and Boethius' Theses make use of similar arguments.

Fact 5.2.3. ***CC1*** *allows countermodels to Symmetry of Implication.*

Consider any interpretation in which $\sigma(A) = \{\ \}$ and $\sigma(B) = \{1\}$, or else $\sigma(A) = \{0\}$ and $\sigma(B) = \{1, 0\}$.

Remark 1. *By definition, all axioms of* ***CC1*** *are true under all interpretations. By the evaluation conditions of the connectives, they also have the property of never being false. That is, every axiom of* ***CC1*** *is true and not false under all interpretations.*

To obtain a first-order logic from **CC1**, it suffices to expand the language $\mathcal{L}$ with a universal quantifier $\forall$ and a set of variables $\{x, y, z, x_1, x_2 \ldots\}$. Let D be a collection of objects, also known as *domain.*

A pair $A = \langle D, I\rangle$ is said to be an **qCC1**-*model* structure if D is some nonempty domain of objects, and I is a function from a pair of elements a, b to $\{\{1\}, \{1, 0\}, \{\ \}, \{0\}\}$, such that for each pair a, b of elements in D, we have $I(a \in b) \in \{\{1\}, \{1, 0\}, \{\ \}, \{0\}\}$. Let any function $S : Var(\Gamma) \to D$ from the variables of Γ to the domain be called an *evaluation of the variables*. Then given some evaluation S we can assign truth values to every formula in Γ by means of a function as follows:

Let any function $S : Var(\Gamma) \to D$ from the variables of Γ to the domain be called an *evaluation of the variables*. Let $\in$ be a symbol that indicates set membership and variables $x, y, z, x_1, x_2 \ldots$. Then given some evaluation S we can assign truth values to every formula in Γ by means of a function as follows:

- $1 \in \sigma(x \in y)$ iff $\sigma(S(x) \in S(y))$
- $0 \in \sigma(x \in y)$ iff $\sigma(S(x) \notin S(y))$
- $1 \in \sigma(x = y)$ iff $1 \in \sigma(S(x) = S(y))$
- $0 \in \sigma(x = y)$ iff $0 \in \sigma(S(x) \neq S(y))$

The interpretation σ extends to quantifiers as follows:

- $1 \in \sigma(\forall x A)$ iff $1 \in \sigma(A(s(x/d))$, for all d, $d \in D$ where $s(x/d)$ is the result of sustituing x for d in s.
- $0 \in \sigma(\forall x A)$ iff $1 \notin \sigma(A(d))$, for some d, $d \in D$ where $s(x/d)$ is the result of sustituing x for d in s.

A particular quantifier $\exists x A$ is definable as $\neg \forall x \neg A$. As we have mentioned, **CC1** is a connexive logic that validates all the logical bits necessary for a set theory based on it to be trivial according to Wiredu's proof. However, **CC1** is by no means the only connexive logic, and no matter the generality of Wiredu's proof, it is not sufficient to claim, as he does, that it is not possible to have a set theory with a connexive basis.

5.3 Wiredu on connexivity and Separation, revisited

Recall that Wiredu makes a rather strong claim towards the end of his chapter, namely "that the Zermelo type of escape route from Russell's paradox is not available in a connexively based set theory". Wiredu's proof does not doom the prospects of connexive set theories, though. The proof does not go through other connexive logics, in particular, Wansing-style connexive logics (see [119], [88]), where Explosion is not valid. For simplicity, consider the logic **MC**, introduced in [119], characterized by the following evaluation conditions:

$1 \in \sigma({\sim} A)$ iff $0 \in \sigma(A)$
$0 \in \sigma({\sim} A)$ iff $1 \in \sigma(A)$

$1 \in \sigma(A \wedge B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$
$0 \in \sigma(A \wedge B)$ iff $0 \in \sigma(A)$ or $0 \in \sigma(B)$

$1 \in \sigma(A \rightarrow B)$ iff $1 \notin \sigma(A)$ or $1 \in \sigma(B)$
$0 \in \sigma(A \rightarrow B)$ iff $1 \notin \sigma(A)$ or $0 \in \sigma(B)$

$A \wedge B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\{\ \}$	$\{\ \}$	$\{0\}$	$\{\ \}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

${\sim} A$	A
$\{0\}$	$\{1\}$
$\{1,0\}$	$\{1,0\}$
$\{\ \}$	$\{\ \}$
$\{1\}$	$\{0\}$

$A \rightarrow_W B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{\ \}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

Logical consequence is defined as for **CC1**, and again a biconditional $A \leftrightarrow_W B$ can be defined as $(A \rightarrow_W B) \wedge (B \rightarrow_W A)$. It is not difficult to check that **MC** validates almost all the arguments necessary to carry Wiredu's proof on, with the only exception of Explosion: simply consider $\sigma(A) = \sigma({\sim} A) = \{1,0\}$ and $1 \notin \sigma(B)$.

Wiredu might complain that those set theories are still inconsistent, which was what one was trying to avoid. That is a moot point. One problem with Russell's inconsistency is that it leads, in classical and other logics, such as **CC1**, to the triviality of the theory. But it need not be so in other logics. Wiredu or a certain brand of "mathematicians and logicians deeply wedded to Zermelo style set theory" might insist, for a variety of reasons, that even if inconsistency and triviality do not come hand in hand, one might prefer a consistent mathematical theory over an inconsistent one. That is again a moot point, but we will not press the issue further.[2] It is enough for our purposes to note that, said against connexivists, the claim "consistent versions of Z are to be preferred to inconsistent ones" is far weaker than the original "Zermelo's kind of way out [of Russell theorem] is not open to [connexively oriented logicians]".

Wiredu's argument is not the end for $Z_{\mathbf{CC1}}$-like set theories, either. However, before proceeding to discuss that, let us address other inaccuracies in Wiredus' paper. He says [notation has been adjusted]:

> In McCall's *connexive* logic (...), this law [Simplification] is rejected principally on the ground that from $(p \land q) \to p$ substitution of $\neg p$ for q yields $(p \land \neg p) \to p$ which is unacceptable, since a contradiction cannot connexively imply anything. In the opinion of the present writer, this circumstance shows only that we must lay down a restriction on substitution so as to prevent substitutions that render contradictory any main component of a connexive proposition. It may be pointed out that McCall cannot avoid some such restriction in his connexive logic, in any case; for $p \to p$ holds in his system, and yet substitution of $p \land \neg p$ for p results in $(p \land \neg p) \to (p \land \neg p)$ which should be equally inadmissible in connexive

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[2] There are many ingenuine defenses of inconsistent mathematics. For some *loci classici*, see [83], [24] or [123].

logic.

First, as it is shown in the reconstruction above, Simplification is not needed for the triviality proof, but merely Conjunction Elimination. Wiredu himself hints at that right after the passage quoted: "if we accept the thesis that "if... then ..." is capable of both a truth-functional and a connexive significance, then it is feasible to retain 'if $(p \wedge \neg p)$ then $(p \wedge \neg p)$' in the truth-functional sense so that $(p \wedge \neg p) \supset (p \wedge \neg p)$ would stand unprejudiced. (The same remark applies, of course, to 'If $p \wedge \neg p$ then p')."

Second, the reason why Simplification fails in **CC1** is not because "a contradiction cannot connexively imply anything". There are false instances of Simplification in **CC1** whose antecedent is not a contradiction. For example, consider $\sigma(A) = \{\ \}$ and $\sigma(B) = \{0\}$. In that case, $\sigma((A \wedge B) \to A) = \{0\}$ and, in **CC1**, B is not a negation of A nor vice versa.

Third, it is highly debatable whether $(A \wedge \neg A) \to (A \wedge \neg A)$ should be "inadmissible in connexive logic". The mark of connexive logic, at least in McCall's case, is that, in a valid implication, the negation of the consequent is incompatible with the antecedent. True, it is difficult to say what compatibility is, but it can be approached in **CC1** with a connective $\circ$ defined as $A \circ B$ iff $\neg(A \to \neg B)$. Therefore, the evaluation conditions for $A \circ B$ are as follows:

$1 \in \sigma(A \circ_{AM} B)$ iff $1 \in \sigma(A)$ and $1 \in \sigma(B)$, or $0 \in \sigma(A)$ iff $0 \in \sigma(B)$.

$0 \in \sigma(A \circ_{AM} B)$ iff $0 \in \sigma(A)$ iff $0 \notin \sigma(B)$.

The evaluation condition above can be presented tabularly as follows:

$A \circ_{AM} B$	$\{1\}$	$\{1,0\}$	$\{\ \}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\{1\}$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{0\}$	$\{1\}$
$\{\ \}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$

It is clear that $\neg((A \wedge \neg A) \circ \neg(A \wedge \neg A))$, hence $(A \wedge \neg A) \to (A \wedge \neg A)$.

On the other hand, there are at least two ways of understanding the idea that contradictions are deductively inefficient, one stronger than the other:

- Contradictions cannot validly imply anything, period.
- Contradictions cannot validly imply anything that has different content than them.

As we have seen, **CC1** belongs to the latter category. Let the *content of* A, $Cont(A)$, be the set of all interpretations in which A is either false or untrue. According to the truth tables, a contradiction that is always false can validly imply only formulas that are always false as well. Suppose that there is a B such that $1 \in \sigma(B)$ and $0 \notin \sigma(B)$. By a simple inspection of the truth tables, $1 \notin \sigma((A \wedge \neg A) \to B)$. A similar reasoning applies in the case that $0 \in \sigma(B)$ and $1 \notin \sigma(B)$.[3]

Nonetheless, as we mentioned above, this implicational inefficiency of contradictions in **CC1** is not mirrored at the entailment level, since contradictions entail everything, whatever its content is. The underlying notion of logical consequence plays a crucial role in validating all the argument schemas needed for Wiredu's triviality proof. Again, they are at least two options, depending on the way one understands the inefficiency of contradictions:

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[3] Actually, this follows from the more general fact that, in **CC1**, as in many other logics, $A \to_{AM} B$ is valid only if $Cont(B) \subseteq Cont(A)$: suppose that $A \to_{AM} B$ is valid, that is, for all σ, $1 \in \sigma(A \to_{AM} B)$. This entails that if $0 \in \sigma(B)$, then $0 \in \sigma(A)$, by the truth condition of the implication. Suppose now that $1 \notin \sigma(B)$. Then, again by the truth condition, $1 \notin \sigma(A)$ as well. Therefore, in any case where $0 \in \sigma(B)$ or $1 \notin \sigma(B)$, $0 \in \sigma(A)$ or $1 \notin \sigma(A)$ as well. Hence, $Cont(B) \subseteq Cont(A)$.

- one can do without any entailment with a contradiction in the premises, including $A \wedge \neg A \vDash_{\mathbf{L}} A \wedge \neg A$;
- or one can merely do without $A \wedge \neg A \vDash_{\mathbf{L}} B$, admitting $A \wedge \neg A \vDash_{\mathbf{L}} A \wedge \neg A$, and $A \wedge \neg A \vDash_{\mathbf{L}} A$ and $A \wedge \neg A \vDash_{\mathbf{L}} \neg A$.

Although the latter seems better because it seems to draw the right conceptual distinctions, we explore the first option since it is easier. There is a well-known notion of logical consequence that can give the desired result and that, moreover, has already been studied in a context discussing connexivity and the deductive inefficiency of contradictions; see [122].

5.4 Doing without Conjunction Elimination

As we have seen, all the principles involved in Wiredu's proof are valid in **CC1**. Wiredu is not right in his claims about the failures of Simplification in connexive logic; moreover, it is irrelevant since it is not needed for his triviality proof, as Conjunction Elimination suffices. Nonetheless, there is certainly a brand of connexivity according to which Simplification and other principles in the vicinity, such as Conjunction Elimination, must fail. And there is a rather easy way of obtaining a **CC1**-like logic in which Conjunction Elimination fails, namely, by employing Malinowski's *q-consequence* [76]:

Definition 5.4.1. *Let Γ be a set of formulas. A is a logical consequence of Γ in **qCC1**, $\Gamma \vDash_{qCC1} A$, iff, for all interpretation σ, $1 \in \sigma(A)$ and $0 \notin \sigma(A)$, if $1 \in \sigma(B)$ or $0 \notin \sigma(B)$, for all $B \in \Gamma$. In particular, A is a* logical truth *in **qCC1** iff for every σ, $1 \in \sigma(A)$ and $0 \notin \sigma(A)$.*

The logic **qCC1** is non-reflexive, i.e., it invalidates the following argument schema:

$$A \vDash_{\mathbf{L}} A$$

Fact 5.4.1. *qCC1 is a non-reflexive logic.*

Consider any interpretation in which $\sigma(A) = \{1, 0\}$ or $\sigma(A) = \{\ \}$. Under these interpretations, the argument $A \vDash_{\mathbf{qCC1}} A$ is invalid.

How different is **qCC1** from **CC1**? Looking at the axiomatic base, **qCC1** does not validate Detachment, but it does validate a strict form of it: if $\vDash_{\mathbf{qCC1}} A$ and $\vDash_{\mathbf{qCC1}} A \rightarrow_{AM} B$ then $\vDash_{\mathbf{qCC1}} B$. Also, it does validate Strict Adjunction and all the implicative axioms of **CC1**.

Theorem 5.4.1. *All the implicative axioms of **CC1** are implicative axioms of **qCC1**.*

Proof. We know that an implicative axiom $A \rightarrow_{AM} B$ in **CC1** is such that $1 \in \sigma(A \rightarrow_{AM} B)$ and $0 \notin \sigma(A \rightarrow_{AM} B)$ for every interpretation σ. Then, all the implicative axioms of **CC1** are implicative axioms of **qCC1**. □

Remark 2. *qCC1 invalidates the following axioms of **CC1**:*

- $\nvDash_{qCC1} \neg(A \wedge_{AM} \neg(A \wedge_{AM} A))$. *For a countermodel, consider any interpretation in which* $\sigma(A) = \{1, 0\}$.
- $\nvDash_{qCC1} (\neg A \vee ((A \rightarrow_{AM} A) \rightarrow_{AM} A)) \vee (((A \rightarrow_{AM} A) \vee (A \rightarrow_{AM} A)) \rightarrow_{AM} A)$. *For a countermodel, consider any interpretation in which* $\sigma(A) = \{1\}$ *or* $\sigma(A) = \{\ \}$.

Theorem 5.4.2. *qCC1 validates Aristotle's Theses.*

Proof. We know that, for every interpretation σ, $1 \in \sigma(\neg(A \rightarrow_{AM} \neg A)$ and $0 \notin \sigma(\neg(A \rightarrow_{AM} \neg A)$. By the q-consequence relation, Aristotle's Thesis is valid in **qCC1**. The proof for the Variant makes use of a similar argument. □

Theorem 5.4.3. *qCC1 validates Boethius' Theses.*

Proof. Boethius' Theses are implicative schemas. By **Fact** 5.2.2, **qCC1** validates both Boethius' Thesis and its Variant. □

Remark 3. *There is an extended belief that a non-reflexive logic will be empty or near to empty, at least at the level of valid arguments. (See for example, [54] and [104].) This belief gives too much power to the consequence relation on its own. It is indeed possible that a non-reflexive logic is empty, like in the case of* ***K3*** *plus q-consequence, which is a logic without valid arguments. (See more in [29] and [91].) However, it is not only the logical consequence that is doing its job but also the connectives of the language. As one can see,* ***qCC1*** *is far from empty.*

Fact 5.4.2. ***qCC1*** *allows countermodels to Symmetry of Implication.*

Consider any interpretation in which $\sigma(A) = \{\ \}$ and $\sigma(B) = \{1\}$, or $\sigma(A) = \{0\}$ and $\sigma(B) = \{1, 0\}$.

Fact 5.4.3. ***qCC1*** *allows countermodels to Conjunction Elimination.*

5.5 Non-triviality of $Z_{\mathbf{qCC1}}$

Previously we introduced the connexive logic **qCC1** and showed that Conjunction Elimination is not a valid argument in **qCC1**. Since Conjunction Elimination is a necessary step for Wiredu's triviality proof to run, we have blocked this proof for $Z_{\mathbf{qCC1}}$. However, this does not mean that $Z_{\mathbf{qCC1}}$ is not trivial. In this section, we propose to give a proof of non-triviality.

Below we present a model in which these schemas are satisfied

Separation: $\vDash_{\mathbf{qCC1}} \forall z \exists y \forall x((x \in y) \leftrightarrow_{AM} ((x \in z) \wedge_{AM} Fx))$

All properties (Fx) that we can substitute must satisfy the property of being just true ($\sigma(Fx) = \{1\}$). That is, no property can involve a negation (or odd negations).

Extensionality: $\vDash_{\mathbf{qCC1}} \forall x \forall y(\forall z(z \in x \leftrightarrow_{AM} z \in y) \rightarrow_{AM} x = y)$

However, the following schema is not satisfied:

Foundation: $\nvDash_{\mathbf{qCC1}} \forall x(\exists y(y \in x) \rightarrow_{AM} \exists y(y \in x \wedge_{AM} \forall z \neg(z \in y \wedge_{AM} z \in x)))$

A model-structure $A = \langle D, I \rangle$, where $D = \{a_1, a_2, \ldots, a_i, \ldots, a_j\}$ and $I(a_i \in a_j) = e_{ij}$ can be illustrated in a diagram as follows:

$$\begin{array}{c|ccccc}
x \in y & a_1 & a_2 & \cdots & a_j & \cdots \\
\hline
a_1 & e_{11} & e_{12} & \cdots & e_{1j} & \cdots \\
a_2 & e_{21} & e_{22} & \cdots & e_{2j} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \\
a_i & e_{i1} & e_{i2} & \cdots & e_{ij} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \ddots
\end{array}$$

This is also known as *incidence matrix* of the model-structure A.

Consider a Universe with the following four elements: $\mathcal{U}, a, b, \varnothing$, In which $\mathcal{U}$ is the universal set, a and b are arbitrary elements, and $\varnothing$ is the empty set.

Lemma 5.5.1. *The matrixes:*

$$\begin{array}{c|cccc}
a_1 \in a_2 & \mathcal{U} & a & b & \varnothing \\
\hline
\mathcal{U} & \{1\} & \{1\} & \{1\} & \{1\} \\
a & \{1\} & \{1\} & \{1\} & \{1\} \\
b & \{1\} & \{1\} & \{1\} & \{1\} \\
\varnothing & \{1\} & \{1\} & \{1\} & \{1\}
\end{array}
\qquad
\begin{array}{c|cccc}
a_1 = a_2 & \mathcal{U} & a & b & \varnothing \\
\hline
\mathcal{U} & \{1\} & \{1\} & \{1\} & \{1\} \\
a & \{1\} & \{1\} & \{1\} & \{1\} \\
b & \{1\} & \{1\} & \{1\} & \{1\} \\
\varnothing & \{1\} & \{1\} & \{1\} & \{1\}
\end{array}$$

give a model of Separation and Extensionality and a countermodel for Foundation.

Proof. To give a model of Separation, suppose that $\sigma((x \in y) \leftrightarrow_{AM} ((y \in z) \wedge_{AM} Fx) = \{\ \}$. Remember, given the evaluation conditions of the biconditional $\leftrightarrow_{AM}$, there is no interpretation where the biconditional is JUST false ($\{0\}$). According to the evaluation conditions, there are two cases:

Case 1: $\sigma((x \in y) \neq \{1\}$.

Case 2: $\sigma((y \in z) \wedge_{AM} Fx) \neq \{1\}$.

But **Case 1** is impossible. In all cases, for any x and y, $\sigma(x \in y) = \{1\}$. And **Case 2** is impossible too, for any x and y, since $\sigma(y \in z) = \{1\}$ and $F(x) = \{1\}$, then $\sigma((y \in z) \wedge_{AM} Fx) = \{1\}$. Therefore, in all cases $\sigma((x \in y) \leftrightarrow_{AM} ((y \in z) \wedge_{AM} Fx) = \{1\}$. □

Proof. To give a model of Extensionality, suppose that $1 \notin \sigma((z \in x \leftrightarrow_{AM} z \in y) \rightarrow_{AM} x = y))$ or $0 \in \sigma((z \in x \leftrightarrow_{AM} z \in y) \rightarrow_{AM} x = y))$. Since the antecedent of Extensionality is a biconditional and the biconditional cannot be true and false ($\{1, 0\}$) or only false ($\{0\}$), then we will only consider cases where the antecedent is just true (($\{1\}$) or not true and not false ($\{\ \}$).

According to the evaluation conditions, there are five cases:

Case 1: $\sigma(z \in x \leftrightarrow_{AM} z \in y) = \{1\}$ and $\sigma(x = y) = \{1, 0\}$.

Case 2: $\sigma(z \in x \leftrightarrow_{AM} z \in y) = \{1\}$ and $\sigma(x = y) = \{\ \}$.

Case 3: $\sigma(z \in x \leftrightarrow_{AM} z \in y) = \{1\}$ and $\sigma(x = y) = \{0\}$.

Case 4: $\sigma(z \in x \leftrightarrow_{AM} z \in y) = \{\ \}$ and $\sigma(x = y) = \{1, 0\}$.

Case 5: $\sigma(z \in x \leftrightarrow_{AM} z \in y) = \{\ \}$ and $\sigma(x = y) = \{0\}$.

But all these cases are impossible. In all cases, for any x and y, $\sigma(x = y) = \{1\}$. Therefore, in all cases $\sigma((z \in x \leftrightarrow_{AM} z \in y) \rightarrow_{AM} x = y)) = \{1\}$ □

Proof. To give a countermodel for Foundation, consider the following substitutions: y/a, x/b, and z/b. It is easy to verify that $\sigma((a \in b) \wedge_{AM} \neg(b \in a \wedge b \in b)) = \{0\}$. If $\sigma(b \in a) = \{1\}$ and $\sigma(b \in b) = \{1\}$, then $\sigma(b \in a \wedge b \in b) = \{1\}$. Then $\sigma(\neg(b \in a \wedge b \in b)) = \{0\}$. If $\sigma(a \in b) = \{1\}$ and $\sigma(\neg(b \in a \wedge b \in b))) = \{0\}$ then $\sigma((a \in b \wedge_{AM} \neg(b \in a \wedge b \in b))) = \{0\}$. □

Remark 4. *Since we have a countermodel for Foundation, we have proved the non-triviality of* $Z_{\boldsymbol{qCC1}}$.

5.6 A brief look into non-transitivity

We have seen that, in general, the relation of q-consequence is not reflexive. Now, Frankowski [56] defined a related notion of logical consequence that is non-transitive instead. Thus, we also consider the logic **pCC1**, which is basically **CC1** with p-consequence, and examine its properties.

Definition 5.6.1. *Let* Γ *be a set of formulas.* A *is a logical consequence of* Γ *in* ***pCC1***, $\Gamma \vDash_{qCC1} A$, *iff, for all interpretation* σ, $1 \in \sigma(A)$ *or* $0 \notin \sigma(A)$, *if* $1 \in \sigma(B)$ *and* $0 \notin \sigma(B)$, *for all* $B \in \Gamma$. *In particular,* A *is a* logical truth *in* ***pCC1*** *iff for every* σ, $1 \in \sigma(A)$ *or* $0 \notin \sigma(A)$.

The logic **pCC1** is non-transitive, i.e., it invalidates the following argument schema:

$$\text{If } A \vDash_{\mathbf{L}} B \text{ and } B \vDash_{\mathbf{L}} C \text{ then } A \vDash_{\mathbf{L}} C$$

Fact 5.6.1. ***pCC1*** *is a non-transitive logic.*

We need two valid arguments $A \vDash_{\mathbf{pCC1}} B$ and $B \vDash_{\mathbf{pCC1}} C$, and one invalid argument $A \nvDash_{\mathbf{pCC1}} C$. The easiest way to achieve that is to have a formula A that is just true under all interpretations, a B that is neither true nor false in all interpretations, and a C that is just false under all interpretations. Fortunately, these constants are definable in the language of **pCC1** as $(p \rightarrow_{AM} p)$, $(p \rightarrow_{AM} \neg p) \wedge_{AM} (\neg p \rightarrow_{AM} p)$ and $\neg(p \rightarrow_{AM} p)$, respectively, just as in **CC1**; see [79]. Thus, **pCC1** is non-transitive.

With respect to the rules of Hilbert-style presentation of **CC1**, **pCC1** validates both Detachment and Strict Adjunction.

Theorem 5.6.1. *All the implicative axioms of* ***CC1*** *are implicative axioms of* ***pCC1***.

Proof. We know that an implicative axiom $A \to_{AM} B$ in **CC1** is such that $1 \in \sigma(A \to_{AM} B)$ and $0 \notin \sigma(A \to_{AM} B)$ for every interpretation σ. Then, all the implicative axioms of **CC1** are implicative axioms of **pCC1**. □

Theorem 5.6.2. ***pCC1*** *validates Aristotle's Theses.*

Proof. We know that, for every interpretation σ, $1 \in \sigma(\neg(A \to_{AM} \neg A))$ and $0 \notin \sigma(\neg(A \to_{AM} \neg A))$. By the *p*-consequence relation, Aristotle's Thesis is a valid argument in **pCC1**, and the proof for the Variant is similar. □

Theorem 5.6.3. ***pCC1*** *validates both Boethius' Thesis and its Variant.*

Proof. Boethius' Thesis and its Variant are implicative theorems of **CC1**. That is, in every interpretation, they are true and not false. Therefore, **pCC1** validates both. □

Theorem 5.6.4. *The symmetry of Implication is valid in* ***pCC1****.*

Proof. Suppose that $0 \in \sigma(A \to_{AM} B) \to_{AM} (B \to_{AM} A))$ and $1 \notin \sigma(A \to_{AM} B) \to_{AM} (B \to_{AM} A))$. By the evaluation conditions, there are eight possible cases:

Case 1: $\sigma(A \to_{AM} B) = \{1\}$ and $\sigma(B \to_{AM} A) = \{1, 0\}$.

Case 2: $\sigma(A \to_{AM} B) = \{1\}$ and $\sigma(B \to_{AM} A) = \{0\}$.

Case 3: $\sigma(A \to_{AM} B) = \{1, 0\}$ and $\sigma(B \to_{AM} A) = \{1\}$.

Case 4: $\sigma(A \to_{AM} B) = \{1, 0\}$ and $\sigma(B \to_{AM} A) = \{\ \}$.

Case 5: $\sigma(A \to_{AM} B) = \{\ \}$ and $\sigma(B \to_{AM} A) = \{1, 0\}$.

Case 6: $\sigma(A \to_{AM} B) = \{\ \}$ and $\sigma(B \to_{AM} A) = \{0\}$.

Case 7: $\sigma(A \to_{AM} B) = \{0\}$ and $\sigma(B \to_{AM} A) = \{1\}$.

Case 8: $\sigma(A \to_{AM} B) = \{0\}$ and $\sigma(B \to_{AM} A) = \{\ \}$.

Cases 1, 3, 4, and 5 are impossible since no implication is both true and false.

Now, for **Case 2**, since $\sigma(B \to_{AM} A) = \{0\}$, $0 \in \sigma(B)$ iff $0 \notin \sigma(A)$. But $1 \in \sigma(A \to_{AM} B)$; therefore, $0 \in \sigma(A)$ iff $0 \in \sigma(B)$. Suppose that $0 \in \sigma(A)$. Then,

$0 \in \sigma(B)$. But if $0 \in \sigma(A)$ and $0 \in \sigma(B)$, it follows, by the falsity condition of $B \rightarrow_{AM} A$ that $0 \notin \sigma(A)$, but that contradicts the assumption. Supposing that $0 \in \sigma(B)$ also leads to a contradiction by a similar reasoning.

For **Case 6**, if $\sigma(B \rightarrow_{AM} A) = \{0\}$ then $0 \in \sigma(B)$ iff $0 \notin \sigma(A)$. Suppose that $0 \in \sigma(B)$. Then $0 \notin \sigma(A)$. But if $0 \in \sigma(B)$ and $0 \notin \sigma(A)$, $\sigma(A \rightarrow_{AM} B) = \{0\}$, contradicting the assumption. Supposing that $0 \notin \sigma(A)$ also leads to a contradiction by a similar reasoning.

Cases 7 and 8 are similar to **Case 6**. □

It follows that **pCC1** is not a connexive logic, at least in the sense of McCall, and the conditional in this logic looks too similar to a biconditional.

However, one could try to say that the conditional in **pCC1** is not symmetric because it could invalidate other forms of symmetry, not necessarily the one demanded by McCall to define connexive logics. Thus, a conditional could invalidate one of the following arguments to be considered non-symmetric:

- $\vDash_{\mathbf{L}} (A > B) > (B > A)$
- $(A > B) \vDash_{\mathbf{L}} (B > A)$
- $\vDash_{\mathbf{L}} (A > B) > ((A > B) \otimes (B > A))$
- $\vDash_{\mathbf{L}} (A > B) > (NA > NB)$

Despite the attempts, the conditional of **pCC1** is symmetric also according to all these proposed forms. It follows that the conditional in **pCC1** is more biconditional-ish than one would wish. Moreover, **pCC1** validates all the arguments needed to obtain triviality in the same way as it was done with **CC1**.

One of the features of **pCC1** is that it is a non-transitive logic. This last property is assumed to be positive when one manages to recover the whole set of valid arguments of classical logic in a logic like **K3** by using a *p*-consequence. However, in this context, it is somewhat negative since it leads to validate the Symmetry of Implication and, moreover, it does not lead to invalidate Wiredu's proof. This is because with a *p*-consequence, an argument is invalid iff the premises are just true and the conclusion is just false. This feature reduces the possible countermodels to the arguments needed for the triviality result.

5.7 Conclusions

In this chapter, we reconstructed Wiredu's proof that Zermelo's set theory based on a certain kind of connexive logic is trivial. We have made explicit all and only the assumptions used by Wiredu in his proof. Based on this, we have shown that contrary to what Wiredu did, principles such as Simplification and others in arrow form are not needed; the rule forms are enough.

One of the logics in the scope of Wiredu's theorem is Angell-McCall's **CC1**. We have shown that it is possible to use Malinowski's *q*-consequence in **CC1** and, therefore, to obtain a consistent connexive logic that is not in the scope of Wiredu's proof; specifically, Conjunction Elimination is invalid in **qCC1** and thereby Wiredu's proof is blocked. What is discussed here can be extended to the identification of contradictions reinstated at higher levels and in the development of a connexive set theory. Finally, we have shown why Wiredu's proof holds in **pCC1** given its limited scope in providing countermodels for arguments.

All in all, the prospects for a sufficiently robust set theory based on **qCC1**, assuming it is non-trivial, are dime. It lacks a detachable conditional, and hence, it will surely create expressibility problems that cannot be solved by appealing to the consequence relation since it is non-reflexive. Other (consistent or inconsistent) connexive logics need to be explored as bases for a set theory.

Conclusion

In this thesis, I answer the question of whether empty logics are logics. I also explore various philosophical problems that arise from accepting this question since my answer is affirmative. I argue that a logic is a collection of valid, invalid and anti-valid arguments. In an empty logic, we know exactly the members of each collection, i.e., we have no ambiguity about which elements belong to which collection. On the other hand, I show that immediately after accepting empty logics as legitimate logics, several interesting philosophical and logical problems arise. I argue that those who claim that empty logics are not logics make several assumptions. These assumptions are alien to traditional definitions of what a logic is. These assumptions conclude that empty logics are not logics and that taking these problems seriously is no longer interesting. In Chapter 1. of this thesis, I have shown that if we question and challenge these assumptions, it is possible to admit that empty logics are logics.

Empty logics are legitimate and can be as interesting as any other logic. I have shown that empty logics does not challenge the usual definitions of logic, contrary to what some authors think. The understanding of logic as the study of what follows from what and why is perhaps the most appropriate definition of logic. The reason for this is that it does not involve us in any extralogical applications or aims. I show that it requires assumptions that we have no reason to accept to delegitimize an empty logic on this basis. Accepting empty logic like logic also broadens our conception of logic. Indeed, we may have an appreciation for many logics that have previously been thought to be too weak to be of logical and philosophical interest.

Taking empty logics seriously as logics also allows for a broader understanding of what makes a logic empty of valid arguments: We have shown that there are currently at least three ways to obtain empty logics:

1. By increasing the set of admissible interpretations, even allowing context-dependent interpretations (as in the case of Russell).
2. By modification of the logical consequence relation.
    - With mixed logical consequence relations (as in the case of the $[\mathbf{TS}_\omega, \mathbf{ST}_\omega, [\overline{\varnothing}\varnothing, \varnothing\overline{\varnothing}]]$ of Pailos).
    - With disjoint logical consequence relations (as in some of the disjoint logics worked on in Chapter 3).
3. By modification of certain connectives of the language of **TS** and their serious use in metatheory.

The first possibility was discussed in Chapter 2. To show the relationship between increasing the set of admissible interpretations and invalidating various argumentation schemes, I have presented various logics, such as classical logic, **K3**, **LP**, **FDE**, **ETL**, **NFL**, **ST**, **TS**, and **CDFL**. In Chapter 3, I examine some disjoint logics (and some philosophical problems), which may be empty logic, depending on the standards for the premises and the conclusion.

In Chapter 4, I introduced you to empty logic **ETS**. There are at least two reasons why this empty logic can be interesting. The first is that it is a logic that is empty of arguments and meta-arguments. Contrary to what it may seem, there is no need for much logical machinery to have a logic empty of valid arguments and valid meta-arguments. On the other hand, it takes into account all the necessary elements in the debate: a sufficiently robust set of interpretations (that of **K3**), a well-known mixed logical consequence relation (that of **TS**), and and it solves the problem of overgeneration of meta-arguments by interpreting validity as transplication. This leads to a collapse of the definitions of local and global validity for meta-arguments.

Finally, in Chapter 5. I show that I can obtain a contra-classical logic with logical consequence relations, which usually do not involve valid arguments. This makes it possible for us to think about two other problems. First, it is not enough that the logical consequence relation is sufficiently weak for a logic to be empty. To obtain an empty logic, the evaluation conditions of the connectives and the admissible interpretations play a crucial role. On the other hand, we can also think that the step from the absence of valid arguments to the presence of arguments in classical logic (and contra-classical arguments) is very small. That is to say, with the right ingredients, it should be possible to go from an empty logic to an overcomplete logic.

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