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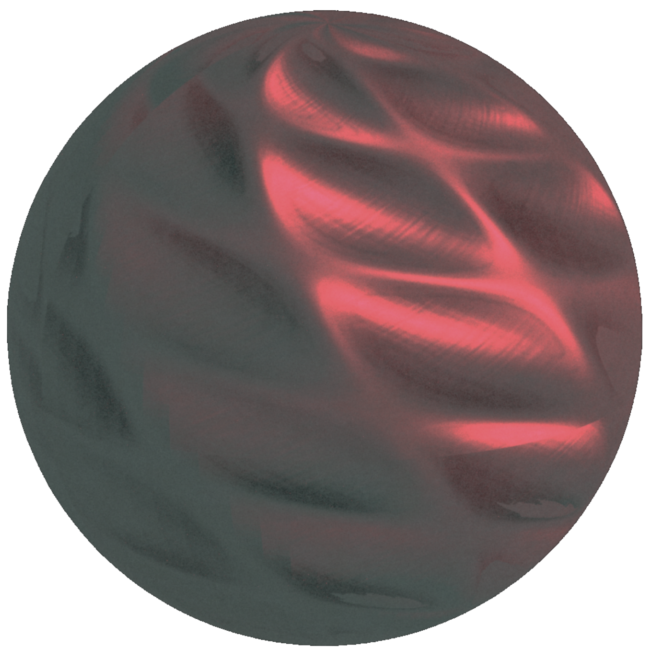
Actions on Spaces with Bounded Curvature

Andrés Ahumada Gómez



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Contents

Introducción

La geometría riemanniana genera el marco ideal para estudiar la topología de variedades usando toda la maquinaria diferenciable, métrica y algebraica. Con esta forma de pensar nos aventuramos a estudiar espacios métricos con una noción de curvatura introducida en los años 50 por A. D. Aleksandrov en el artículo [**Ale57**]. Dicha noción está inspirada por el teorema de Toponogov (ver Teorema 12.2.2, [**Pet16**]) y define cuándo un espacio métrico tiene curvatura acotada inferior o superiormente en términos de comparación de triángulos geodésicos. Además, en ese mismo trabajo Aleksandrov estudió propiedades de dichos espacios en cada caso.

Más tarde se empezó a estudiar el caso en el que se tienen ambas cotas de curvatura, surgiendo así los espacios con curvatura acotada ("spaces with bounded curvature"), una combinación de espacios de Alexandrov (cota inferior) y espacios CAT (cota superior). Estos espacios siguen siendo objeto de estudio en geometría métrica (ver por ejemplo [**Pla92, KL21**]) y ocurren, por ejemplo, como espacios CAT con una cota inferior en la curvatura de Ricci en el sentido de Lott–Sturm–Villani (ver [**KK20**]).

En 1975, V. N. Berestovskii demostró en [**Ber76**] que las cotas son muy restrictivas en la estructura topológica de los espacios con curvatura acotada:

Teorema A (Berestovskii, 1975). *Un espacio con curvatura acotada $\mathcal{M}$ es una variedad riemanniana con estructura diferenciable de clase C^1 y metrica riemanniana continua.*

Posteriormente, en 1983, I. G. Nikolaev, en [**Nik83a**] y [**Nik83b**], mejoró la regularidad de la estructura diferenciable y de la métrica riemanniana:

Teorema B (Nikolaev, 1983). *Sea $\mathcal{M}$ un espacio con curvatura acotada. Entonces, en una vecindad de cada punto $p \in \mathcal{M}$ podemos introducir un sistema de coordenadas armónicas. Las componentes g_{ij} del tensor métrico en cualquier sistema de coordenadas armónicas $\mathcal{M}$ son funciones continuas de clase $W^{2,q}(\Omega)$ para cualquier $q > 1$, donde $\Omega \subset \mathbb{R}^n$ es un dominio donde las coordenadas armónicas están definidas. Los sistemas de coordenadas armónicas en $\mathcal{M}$ forman un atlas de clase $C^{3,\alpha}$ para cualquier $0 < \alpha < 1$.*

Finalmente, en 1991 Nikolaev demostró en [**Nik91**] que podemos aproximar un espacio con curvatura acotada con variedades riemannianas suaves teniendo un control de las curvaturas:

Teorema C (Nikolaev, 1991). *Sea* $(\mathcal{M}, d(\mathsf{g}_0))$ *un espacio con curvatura acotada. Entonces, en la variedad suave* $\mathcal{M}$ *con atlas* C^∞ *que contiene el atlas armónico* $\mathfrak{h}$*, podemos definir una sucesión de métricas riemannianas suaves* $\{\mathsf{g}_m\}_{m=1}^\infty$ *que tiene las siguientes propiedades:*

(1) Los espacios métricos $(\mathcal{M}, d(\mathsf{g}_m))$ *convergen en el sentido de Lipschitz al espacio métrico* $(\mathcal{M}, d(\mathsf{g}_0))$.
(2) Las siguientes estimaciones se satisfacen para los limites de curvatura:

$$\limsup_{m\to\infty} \bar{k}_m(\mathcal{M}) \leq \bar{k}_0(\mathcal{M}) \ \ y \ \ \liminf_{m\to\infty} \underline{k}_m(\mathcal{M}) \geq \underline{k}_0(\mathcal{M}),$$

donde $\bar{k}_l(\mathcal{M})$ *y* $\underline{k}_l(\mathcal{M})$ *denotan los límites superior e inferior de la curvatura seccional de las variedades* $(\mathcal{M}, \mathsf{g}_l)$*,* $l = 0, 1, \cdots$.

A partir de este teorema, con una visión "equivariante", y tomando en cuenta que el grupo de isometrías de un espacio de curvatura acotada es un grupo de Lie (ver [**MS39, FY94**]) que es compacto si el espacio es compacto (ver [**DW28**]), nos preguntamos:

Si $(\mathcal{M}, d)$ *es compacto y tiene una acción isométrica de un grupo de Lie compacto* $\mathcal{G}$*, ¿es posible aproximar* g *mediante métricas* g_i *que también tienen una acción de* $\mathcal{G}$ *(quizás una restricción o extensión de él)?*

En esta tesis, damos respuesta a a esta pregunta en el siguiente teorema:

Teorema D. *Sean* $(\mathcal{M}, d(\mathsf{g}_0))$ *un espacio con curvatura acotada compacto y* $\mathcal{G}$ *un grupo de Lie compacto actuando por isometrías. Entonces, en la variedad suave* $\mathcal{M}$ *con atlas de clase* C^∞ *que contiene el atlas armónico* $\mathfrak{h}$ *podemos definir una sucesión de métricas riemannianas suaves* $\{\mathsf{g}_k\}_{k=1}^\infty$ *que tiene las siguientes propiedades:*

(1) El grupo de Lie $\mathcal{G}$ *actúa en* $(\mathcal{M}, d(\mathsf{g}_k))$ *por isometrías.*
(2) Los espacios métricos $(\mathcal{M}, d(\mathsf{g}_k))$ *convergen en el sentido de Lipschitz al espacio métrico* $(\mathcal{M}, d(\mathsf{g}_0))$.
(3) Las siguientes estimaciones se satisfacen para los límites de curvatura:

$$\limsup_{k\to\infty} \bar{k}_k(\mathcal{M}) \leq \bar{k}_0(\mathcal{M}) \ \ y \ \ \liminf_{k\to\infty} \underline{k}_k(\mathcal{M}) \geq \underline{k}_0(\mathcal{M}),$$

donde $\bar{k}_r(\mathcal{M})$ *y* $\underline{k}_r(\mathcal{M})$ *denotan los límites superior e inferior de la curvatura de las variedades* $(\mathcal{M}, \mathsf{g}_r)$*,* $r = 0, 1, \ldots$.

Para demostrar este resultado exploramos la teoría de corrientes y un teorema de aproximación de corrientes que De Rham estableció en [**de 84**]:

Teorema E (De Rham, 1955). *En una variedad diferenciable M podemos construir un operador lineal $\mathcal{Z}$, que depende de parámetros positivos $\varepsilon_1, \varepsilon_2, \ldots$ que pueden ser una cantidad finita o infinita de acuerdo a si M es compacto o no y que tiene las siguientes propiedades:*

(a) Si T es una m-corriente en M, entonces $\mathcal{Z}T$ es una m-corriente.
(b) El soporte de $\mathcal{Z}T$ está contenido en cualquier vecindad dada del soporte de T si tomamos los parámetros ε_i suficientemente pequeños.
(c) $\mathcal{Z}T$ es C^∞.
(d) Si cada ε_i tiende a cero, $\mathcal{Z}T$ converge débilmente a T.

En el estudio de la demostración de este resultado nos preguntamos por una versión equivariante y obtuvimos:

Teorema F. *Sea $\mathcal{G}$ un grupo de Lie compacto actuando en la variedad diferenciable $\mathcal{M}$ y sea T una m-corriente en $\mathcal{M}$. Si T es $\mathcal{G}$-invariante, podemos construir un operador $\mathcal{Z}_\mathcal{G}$ que depende de un parámetro $\varepsilon > 0$ tal que $\mathcal{Z}_\mathcal{G}T$ tiene las siguientes propiedades:.*

(a) $\mathcal{Z}_\mathcal{G}T$ es una m-corriente de clase C^∞.
(b) $\mathcal{Z}_\mathcal{G}T$ es una corriente $\mathcal{G}$-invariante.
(c) Si ε tiende a cero, $\mathcal{Z}_\mathcal{G}T$ converge débilmente a T.

Esta tesis tiene dos objetivos principales: el primero es hacer una recapitulación lo más clara y moderna posible de los fundamentos de la teoría de espacios con curvatura acotada, desde su definición y propiedades básicas, pasando por el trabajo de Berestovskii y Nikolaev, hasta el teorema de aproximacón de Nikolaev. A la fecha, este material es de difícil acceso en la literatura y la primera parte de esta tesis busca remediar esta situación (ver, por ejemplo, [**KL21**, Problema 1.12]).

El segundo objetivo es mostrar las versiones equivariantes de los teoremas de aproximación de Nikolaev (Teorema D) y de aproximación de De Rham (Teorema F).

La tesis está estructurada en 3 partes como sigue. La primera parte la forman los capítulos 1 al 4, donde se aborda la teoría de los espacios con curvatura acotada. En el capítulo 1 se introducen las definiciones y resultados básicos necesarios para entender la definición de nuestros espacios. En el segundo capítulo definimos los espacios con curvatura acotada y demostramos que son variedades riemannianas de baja regularidad estudiando el trabajo de V. Berestovskii. Continúa el capítulo 3 donde construímos un transporte paralelo a lo largo de segmentos geodésicos y a lo largo de curvas rectificables. La primera parte concluye con el capítulo 4, en el que se mejora la regularidad de las variedades riemannianas con las que estamos trabajando usando el transporte paralelo que nos da una conexión y que nos permite calcular la curvatura seccional de ellas. En los capítulos 3 y 4 exponemos trabajo hecho por I. Nikolaev.

Sigue la segunda parte con el capítulo 5; en él hacemos una introducción a la teoría de corrientes y presentamos el proceso de regularización de corrientes. Al final, incluimos el Teorema E, y como sigue el mismo esquema de demostración, el Teorema C.

Finalmente, la tercera parte incluye el capítulo 6, donde damos una introducción a la teoría de acciones de grupos de Lie y el Teorema D y el Teorema F.

Part 1

Spaces with Bounded Curvature

CHAPTER 1

Basic Concepts

In this chapter, we collect the concepts and establish the notation we will use throughout the present work. The basic references of the material presented here are [**BBI01**] and [**BH99**].

1. Length spaces and comparison triangles

The **space form** M_k^2 is the complete and simply connected Riemannian manifold of dimension 2 with constant curvature $k \in \mathbb{R}$. The diameter of M_k^2 is denoted by D_k.

A **curve** in a metric space (X, d) is continuous map $\alpha : [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is an interval. The **space of curves from** $[a, b]$ **to** X is denoted by $\mathcal{C}([a, b]; X)$. A metric space induces a length functional

$$L_d \colon \mathcal{C}([a, b]; X) \to \mathbb{R}$$

given by

$$L_d(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) \right\},$$

where we take the supremum over

$$\{\mathcal{P} = \{a = t_0 \leq t_1 \leq \cdots \leq t_n = b\}\}.$$

This functional induces a metric $\hat{d}$ on X given by

$$\hat{d}(x, y) = \inf \{L_d(\gamma) \mid \gamma \in \mathcal{C}([a, b]; X),\ \gamma(a) = x \text{ y } \gamma(b) = y\}.$$

A **length space** is a metric space such that $\hat{d} = d$.

Remark 1. A length space is not necesarily complete, for example $\mathbb{R}^2 \setminus \{0\}$ is an incomplete length space.

A **(geodesic) segment** (or a **minimizing curve**) is a curve $\gamma \colon [a, b] \to X$ such that if β is another curve joining $\gamma(a)$ with $\gamma(b)$, then $L_d(\gamma) \leq L_d(\beta)$. A segment joining x with y on X is denoted by $[x, y]$. The image of $[x, y]$ is also called a **geodesic segment**.

Any curve $\gamma\colon [a,b] \to X$ can be reparameterized over the interval $[0,1]$ using the function $\rho(t) = tb + (1-t)a$. A **rectifiable curve** (a finite length curve) has a **constant speed parameterization** if $\gamma\colon [0,1] \to X$ and $L_d(\gamma, 0, t) := L_d(\gamma\,|_{[0,t]}) = L_d(\gamma)\,t$. Moreover, we say that it is a **parameterization by arc length** if $\gamma\colon [0, L_d(\gamma)] \to X$ and $L_d(\gamma, 0, t) = t$. Unless we explicitly state the opposite, every geodesic segment is parameterized by arc length. A **geodesic** is a curve $\gamma\colon I \to X$ such that for every $t \in I$ there is an interval $[a,b]$ containing t, $[a,b] \subset I$, and $\gamma\,|_{[a,b]}$ is a minimizing segment.

A **(geodesic) triangle** in X consists of three distinct points $p, q, r \in X$ and three geodesic segments $[p,q]$, $[q,r]$ and $[r,p]$, these could be colineal. We denote such triangle by $\triangle(p,q,r)$ or $\triangle([p,q],[q,r],[r,p])$. A **comparison triangle** for $\triangle(p,q,r)$ is a triangle in M_k^2 with vertices $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ such that $d(p,q) = d(\tilde{p},\tilde{q})$, $d(q,r) = d(\tilde{q},\tilde{r})$ and $d(r,p) = d(\tilde{r},\tilde{p})$. We denote this triangle by $\triangle^k(p,q,r)$, $\triangle(\tilde{p},\tilde{q},\tilde{r})$ or $\triangle([\tilde{p},\tilde{q}],[\tilde{q},\tilde{r}],[\tilde{r},\tilde{p}])$. We illustrate these concepts in Figure 1. A **comparison point** for $x \in [p,q]$ is a point $\bar{x} \in [\bar{p},\bar{q}]$ with $d(p,x) = d(\bar{x},\bar{p})$. The angle of $\triangle(\bar{p},\bar{q},\bar{r})$ at $\bar{p}$ is called a **comparison angle** between $[p,q]$ and $[p,r]$ at p and it is denoted by $\sphericalangle^k(q,p,r)$. In particular, for the angle in the Euclidean plane, we use the notation $\sphericalangle^0(q,p,r) = \bar{\sphericalangle}(q,p,r)$.

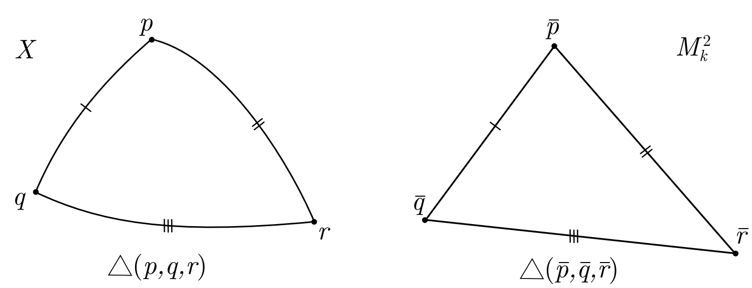

FIGURE 1. Comparison triangles

REMARK 2. Instead of denoting differently the metrics we distiguish the metric spaces involved taking notice which set the points belong to.

REMARK 3. In general comparison triangles exist and are unique up to isometry if the perimeter is less than $2D_k$ (see [**BH99**]). Thus for $k > 0$ comparison triangles always exist.

2. Known Facts

Law of cosines in M_k^2. Given a triangle in M_k^2 with length sizes a, b and c, and angle γ at the vertex opposite to the side of length c as in Figure 2, the following identities hold:

For $k = 0$,

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

For $k > 0$,

$$\cos(\sqrt{k}c) = \cos(\sqrt{k}a)\cos(\sqrt{k}b) + \sin(\sqrt{k}a)\sin(\sqrt{k}b)\cos(\gamma).$$

For $k < 0$,

$$\cosh(\sqrt{-k}c) = \cosh(\sqrt{-k}a)\cosh(\sqrt{-k}b) - \sinh(\sqrt{-k}a)\sinh(\sqrt{-k}b)\cos(\gamma).$$

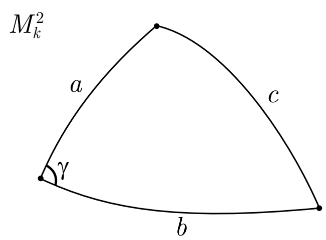

FIGURE 2. Model triangle in M_k^2

Alexandrov's lemma. Let p, q, r, x be four points in M_k^2 such that for the triangles $\triangle(p,x,q)$ and $\triangle(p,x,r)$ the points q and y lie on opposite sides of a fixed segment joining p and x. If $k > 0$ we assume that

$$d(p,q) + d(q,x) + d(x,r) + d(r,p) < 2D_k.$$

Let $\triangle(p',q',r')$ be a triangle in M_k^2 such that

$$d(p',q') = d(p,q),\ d(p',r') = d(p,r) \text{ y } d(q',r') = d(q,x) + d(x,r).$$

and let x' be a point in $[q',r']$ such that $d(q,x) = d(q',x')$.

(i) $\sphericalangle^k(q,x,p) + \sphericalangle^k(p,x,r) < \pi$ if and only if $d(p',x') < d(p,x)$. In this case illustrated in Figure 3, we have that $\sphericalangle^k(p',q',x') < \sphericalangle^k(p,q,x)$ and $\sphericalangle^k(p',r',x') < \sphericalangle^k(p,r,x)$.

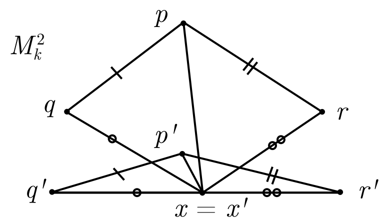

FIGURE 3

(ii) $\sphericalangle^k(q,x,p)+\sphericalangle^k(p,x,r) > \pi$ if and only if $d(p',x') > d(p,x)$. In this case illustrated in Figure 4, we have that $\sphericalangle^k(p',q',x') > \sphericalangle^k(p,q,x)$ and $\sphericalangle^k(p',r',x') > \sphericalangle^k(p,r,x)$.

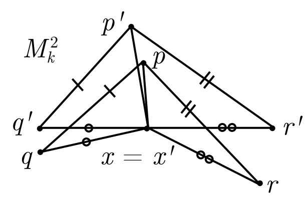

FIGURE 4

For a proof, see Lemma 4.3.3, page 115, [**BBI01**].

3. Angles

Let X be a metric space and let $c\colon [0,a] \to X$ and $c'\colon [0,a'] \to X$ be two minimizing curves with $c(0) = c'(0)$. Given $t \in (0,a]$ and $t' \in (0,a']$, we consider the comparison triangle $\bar{\triangle}(c(0),c(t),c'(t'))$ and the comparison angle $\bar{\sphericalangle}(c(t),c(0),c'(t'))$ in the plane. The **Alexandrov angle** or **upper angle** between the segments c and c' is the number $\sphericalangle(c,c') \in [0,\pi]$ defined by:

$$\sphericalangle(c,c') = \limsup_{t,t'\to 0} \bar{\sphericalangle}(c(t),c(0),c'(t')) = \lim_{\varepsilon\to 0} \sup_{0<t,t'<\varepsilon} \bar{\sphericalangle}(c(t),c(0),c'(t')).$$

If the limit

$$\lim_{t,t'\to 0} \bar{\sphericalangle}(c(t),c(0),c'(t'))$$

exists, we say that the angle exists in the strict sense and we simply call it **angle**. We denote the angle between the segments $[p,q]$ and $[p,r]$ by $\sphericalangle(q,p,r)$.

REMARK 4. We can express $\sphericalangle(c,c')$ purely in terms of the distance by noting that

$$\cos(\bar{\sphericalangle}(c(t),c(0),c'(t')) = \frac{t^2+t'^2-(d(c(t),c'(t')))^2}{2tt'}.$$

REMARK 5. By the infinitesimal character of the definition of the Alexandrov angle, we have, on the one hand that the angle only depends on the germs of the segments, i.e., if $c''\colon [0,l] \to X$ is any other segment for which there exists $\varepsilon > 0$ such that $c''\big|_{[0,\varepsilon]} = c'\big|_{[0,\varepsilon]}$, then the angle between c and c'' is the same as that between c and c'. On the other hand, we can carry out the measurement or comparison of the angle in M_k^2 instead of the plane (Proposition I.2.9, page 22, [**BH99**]).

REMARK 6. If $c\colon [a,b] \to X$ is a segment with $a < 0 < b$ and we define the segments $c'\colon [0,-a] \to X$ and $c''\colon [0,b] \to X$ by $c'(t) = c(-t)$ and $c''(t) = c(t)$, then $\sphericalangle(c', c'') = \pi$.

PROPOSITION 1.1 (Triangle inequality). *Let X be a metric space and let c, c' and c'' three minimizing curves in X starting from the same point. Then*

$$\sphericalangle(c', c'') \leq \sphericalangle(c', c) + \sphericalangle(c, c'').$$

PROOF. We proceed by contradiction. We take $\delta > 0$ such that

$$\sphericalangle(c', c'') > \sphericalangle(c', c) + \sphericalangle(c, c'') + 3\delta. \tag{1.3.1}$$

By definition of limit superior we can find $\varepsilon > 0$ in such a way that the following inequalities hold:

$$\begin{array}{rll} i) & \bar{\sphericalangle}(c(t), p, c'(t')) < \sphericalangle(c, c') + \delta & \forall t, t' < \varepsilon, \\ ii) & \bar{\sphericalangle}(c(t), p, c''(t'')) < \sphericalangle(c, c'') + \delta & \forall t, t'' < \varepsilon, \\ iii) & \bar{\sphericalangle}(c'(t'), p, c''(t'')) > \sphericalangle(c', c'') - \delta & \forall t', t'' < \varepsilon. \end{array}$$

Let t' y t'' be like in *iii*). Consider in $\mathbb{R}^2$ the triangle with vertices $\bar{0}$, x' and x'' such that $d(\bar{0}, x') = t'$, $d(\bar{0}, x'') = t''$ and the angle α at the vertex $\bar{0}$ satisfy

$$\bar{\sphericalangle}(c'(t'), p, c''(t'')) \overset{(*)}{>} \alpha \overset{(**)}{>} \sphericalangle(c', c'') - \delta.$$

In particular, $\pi > \alpha$ and the triangle $\triangle(\bar{0}, x', x'')$ is non-degenerate.

$$\begin{array}{ccc} \text{Inequality } (*) & \text{implies that} & d(x', x'') < d(c'(t'), c''(t'')), \text{ while} \\ \text{inequalities } (**) \text{ and (1.3.1)} & \text{imply that} & \alpha > \sphericalangle(c, c') + \sphericalangle(c', c'') + 2\delta. \end{array}$$

Using the last inequality we can choose a point $x \in [x', x'']$ such that the angle α' between $[\bar{0}, x]$ and $[\bar{0}, x']$ is greater than $\sphericalangle(c, c') + \delta$ and the angle α'' between $[\bar{0}, x]$ and $[\bar{0}, x'']$ is bigger than $\sphericalangle(c, c'') + \delta$. Let be $t = d(\bar{0}, x) \leq \max\{t', t''\} < \varepsilon$. Then

$$\begin{array}{rcc} \text{inequality } i) & \text{implies that} & \bar{\sphericalangle}(c(t), p, c'(t')) < \sphericalangle(c, c') + \delta < \alpha', \text{ and} \\ \text{inequality } ii) & \text{implies that} & \bar{\sphericalangle}(c(t), p, c''(t'')) < \sphericalangle(c, c'') + \delta < \alpha''. \end{array}$$

From these two inequalities we have

$$d(c(t), c'(t')) > d(x, x') \text{ and } d(c(t), c''(t'')) < d(x, x''),$$

respectively. Then we have

$$d(c'(t'), c''(t'')) > d(x', x'') = d(x', x) + d(x, x'') > d(c'(t'), c(t)) + d(c(t), c''(t'')),$$

which contradicts the triangle inequality in X. Q. E. D.

4. CAT(k) Spaces

Given $k \in \mathbb{R}$ a CAT **space** or **domain** $\mathcal{C}_k$ is a metric space X such that for every two points that are D_k-close, i.e. their distance is less than D_k, there exists a geodesic segment which joins them and such that one of the following statements holds (every triangle under consideration has perimeter less than $2D_k$):

(A) Every triangle $\triangle(p, q, r) \subset X$ and every pair of points $x, y \in \triangle(p, q, r)$ satisfy that $d(x, y) \leq d(\bar{x}, \bar{y})$ with $\bar{x}, \bar{y} \in M_k^2$ comparison points.

(B) For every triangle $\triangle([p, q], [q, r], [r, p])$ in X and $x \in [q, r]$, we have $d(p, x) \leq d(\bar{p}, \bar{x})$ for the comparison point $\bar{x} \in [\bar{q}, \bar{r}] \subset \bar{\triangle}(p, q, r) \subset M_k^2$.

(C) If $\triangle([p, q], [q, r], [r, p])$ is a triangle in X, $x \in [p, q]$ and $y \in [p, r]$ with $p \neq x$ and $p \neq y$, then the angles at the vertices corresponding to p in the comparison triangles $\bar{\triangle}(p, q, r)$ and $\bar{\triangle}(p, x, y)$ in M_k^2 satisfy $\sphericalangle^k(x, p, y) \leq \sphericalangle^k(q, p, r)$.

(D) The Alexandrov angles between the sides of a triangle in X with different vertices are not greater than the corresponding angles of a comparison triangle in M_k^2.

(E) For every triangle $\triangle([p, q], [q, r], [r, p])$ in X with $p \neq q$ and $p \neq r$, if γ denotes the Alexandrov angle at p and $\triangle(\tilde{p}, \tilde{q}, \tilde{r}) \subset M_k^2$ is a triangle with $d(\tilde{p}, \tilde{q}) = d(p, q)$, $d(\tilde{p}, \tilde{r}) = d(p, r)$ and $\sphericalangle^k(\tilde{q}, \tilde{p}, \tilde{r}) = \gamma$, then $d(q, r) \geq d(\tilde{q}, \tilde{r})$.

The first thing to do after these definitions is to prove that they are equivalent.

Proposition 1.2. *Conditions (A)-(E) are equivalent.*

Proof. The scheme of the proof is the following:

$$\begin{array}{ccccc} (A) & \rightarrow & (B) & & \\ \updownarrow & \swarrow & \uparrow & & \\ (C) & \rightarrow & (D) & \leftrightarrow & (E). \end{array}$$

(A)$\Rightarrow$(B). This implication is clear because in (A) we can take y to be the vertex p of the triangle.

(D)$\Rightarrow$(E). Consider a comparison triangle $\triangle(\bar{p}, \bar{q}, \bar{r})$ in M_k^2 for the triangle $\triangle(p, q, r)$ in X. By (D), if γ_k is the angle at $\bar{p}$, then $\gamma \leq \gamma_k$ and, therefore,

$$\cos(\gamma) \geq \cos(\gamma_k). \tag{1.4.1}$$

Using the law of cosines in M_k^2 in the triangles $\triangle(\bar{p}, \bar{q}, \bar{r})$ and $\triangle(\tilde{p}, \tilde{q}, \tilde{r})$, we obtain three cases.

Case $k = 0$:

$$\begin{aligned} d(\tilde{q},\tilde{r})^2 &= d(\tilde{p},\tilde{q})^2 + d(\tilde{p},\tilde{r})^2 - 2d(\tilde{p},\tilde{q}) \cdot d(\tilde{p},\tilde{r}) \cos(\gamma), \\ d(\bar{q},\bar{r})^2 &= d(\bar{p},\bar{q})^2 + d(\bar{p},\bar{r})^2 - 2d(\bar{p},\bar{q}) \cdot d(\bar{p},\bar{r}) \cos(\gamma_k). \end{aligned}$$

By the preceding equalities for distances, and inequality (1.4.1), we have $d(\bar{q},\bar{r})^2 = d(\tilde{q},\tilde{r})^2$. Hence we get $d(q,r) = d(\bar{q},\bar{r}) \geq d(\tilde{q},\tilde{r})$.

Case $k > 0$:

$$\begin{aligned} \cos(\sqrt{k}\, d(\tilde{q},\tilde{r})) &= \cos(\sqrt{k}\, d(\tilde{p},\tilde{q})) \cos(\sqrt{k}\, d(\tilde{p},\tilde{r})) \\ &\quad + \sin(\sqrt{k}\, d(\tilde{p},\tilde{q})) \sin(\sqrt{k}\, d(\tilde{p},\tilde{r})) \cos(\gamma), \\ \cos(\sqrt{k}\, d(\bar{q},\bar{r})) &= \cos(\sqrt{k}\, d(\bar{p},\bar{q})) \cos(\sqrt{k}\, d(\bar{p},\bar{r})) \\ &\quad + \sin(\sqrt{k}\, d(\bar{p},\bar{q})) \sin(\sqrt{k}\, d(\bar{p},\bar{r}) \cos(\gamma_k). \end{aligned}$$

Again, as a consequence of the equalities for distances, of (1.4.1), and of the monotonicity of the function cosinea and the points $\tilde{q}$ and $\tilde{r}$ lie in the sphere with sectional curvature k,

$$\cos(\sqrt{k}\, d(\tilde{q},\tilde{r})) \geq \cos(\sqrt{k}\, d(\bar{q},\bar{r})).$$

Thus, $d(\tilde{q},\tilde{r}) \leq d(\bar{q},\bar{r}) = d(q,r)$.

Case $k < 0$:

$$\begin{aligned} \cosh(\sqrt{-k}\, d(\tilde{q},\tilde{r})) &= \cosh(\sqrt{-k}\, d(\tilde{p},\tilde{q})) \cosh(\sqrt{-k}\, d(\tilde{p},\tilde{r})) \\ &\quad - \sinh(\sqrt{-k}\, d(\tilde{p},\tilde{q})) \sinh(\sqrt{-k}\, d(\tilde{p},\tilde{r})) \cos(\gamma), \\ \cosh(\sqrt{-k}\, d(\bar{q},\bar{r})) &= \cosh(\sqrt{-k}\, d(\bar{p},\bar{q})) \cosh(\sqrt{-k}\, d(\bar{p},\bar{r})) \\ &\quad - \sinh(\sqrt{-k}\, d(\bar{p},\bar{q})) \sinh(\sqrt{-k}\, d(\bar{p},\bar{r}) \cos(\gamma_k). \end{aligned}$$

Using the same arguments as before,

$$\cosh(\sqrt{-k}\, d(\tilde{q},\tilde{r})) \leq \cosh(\sqrt{-k}\, d(\bar{q},\bar{r})).$$

And, finally, $d(\tilde{q},\tilde{r}) \leq d(\bar{q},\bar{r}) = d(q,r)$.

(E)$\Rightarrow$(D). Proceeding in a similar way to the previous implication, but now with an inequality of distances instead of angles, the result follows as a consequence of the law of cosines.

(A)$\Leftrightarrow$(C). Let $\triangle([p,q],[q,r],[r,p])$ be a triangle in X and consider $x \in [p,q]$ and $y \in [p,r]$ with $x \neq p$ and $y \neq p$. Consider the comparison triangles $\bar{\triangle}(p,q,r)$ and $\bar{\triangle}(p,x,y)$ in M_k^2. We denote by $\bar{z}$ the comparison points in $\bar{\triangle}(p,q,r)$ and by $\bar{z}'$ the comparison points in $\bar{\triangle}(p, x, y)$. We have illustrated this in Figure 5. Consider also the angles $\bar{\alpha} = \sphericalangle^k(q, p, r)$ and $\bar{\alpha}' = \sphericalangle^k(x, p, y)$ in $\bar{p}$ and $\bar{p}'$, respectively. Using the laws of cosines as below, we have $d(\bar{x}, \bar{y}) \geq d(x, y) = d(\bar{x}', \bar{y}')$ if and only if $\bar{\alpha} \geq \bar{\alpha}'$.

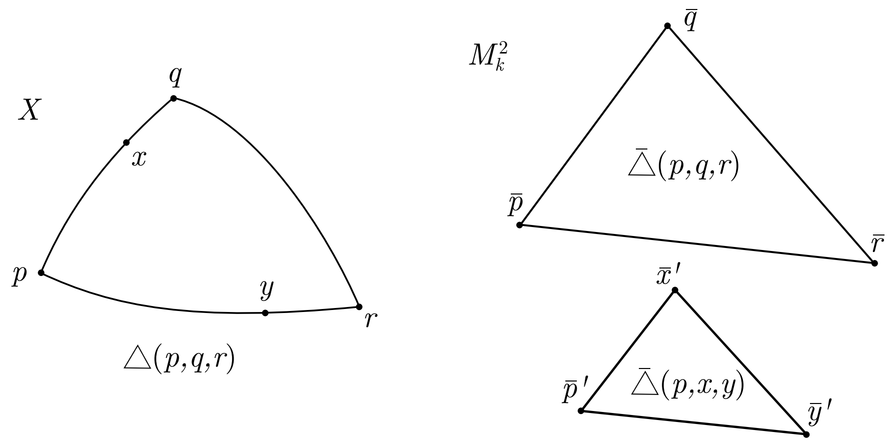

FIGURE 5

(B)⇒(C). We use the same notation as in the last part. Let $\triangle(\bar{p}'', \bar{x}'', \bar{r}'')$ be a comparison triangle for $\triangle(p, x, r)$ and $\bar{\alpha}''$ be the angle at $\bar{p}''$. By (B) we have that $d(x, y) \leq d(\bar{x}'', \bar{y}'')$, where $\bar{y}'' \in \triangle(\bar{p}'', \bar{q}'', \bar{r}'')$ is the comparison point of y. We have

$$d(\bar{x}', \bar{y}') = d(x, y) \leq d(\bar{x}'', \bar{y}'')$$

and, therefore, $\bar{\alpha}' \leq \bar{\alpha}''$. Also by (B), the inequality $d(\bar{x}'', \bar{r}'') = d(x, r) \leq d(\bar{x}, \bar{r})$ is satisfied and then $\bar{\alpha}'' \leq \bar{\alpha}$. Finally, $\sphericalangle^k(x, p, y) = \bar{\alpha}' \leq \bar{\alpha} = \sphericalangle^k(r, p, q)$.

(D)⇒(B). Let $\triangle([p, q], [q, r], [r, p])$ be a triangle in X and let $x \in [q, r]$ be different from q and r. Consider a segment $[p, x]$ in X. Let γ be the Alexandrov angle between $[p, x]$ and $[x, q] \subset [r, q]$, γ' the Alexandrov angle between $[p, x]$, and $[r, x] \subset [q, r]$ and β the Alexandrov angle between $[p, q]$ and $[q, r]$. Now consider a comparison triangle $\triangle(\bar{p}, \bar{q}, \bar{r})$ for $\triangle(p, q, r)$ in M_k^2 and $\bar{\beta}$ the angle of this triangle at $\bar{q}$. Consider also the comparison triangles $\triangle(\tilde{p}, \tilde{q}, \tilde{x})$ and $\triangle(\tilde{p}, \tilde{r}, \tilde{x})$ for $\triangle(p, q, x)$ and $\triangle(p, r, x)$, respectively. These last two are choosen such that the side $[\tilde{p}, \tilde{x}]$ common for both and $\tilde{q}$ and $\tilde{r}$ lie on the opposite sides of the line joining $\tilde{p}$ and $\tilde{x}$. Let be $\sphericalangle(\tilde{q}, \tilde{x}, \tilde{p}) = \tilde{\gamma}$, $\sphericalangle(\tilde{p}, \tilde{x}, \tilde{r}) = \tilde{\gamma}'$ and $\sphericalangle(\tilde{p}, \tilde{q}, \tilde{x}) = \tilde{\beta}$. By Remark 6 and Proposition 1.1, $\gamma + \gamma' \geq \pi$. Alexandrov's lemma implies that $\tilde{\beta} \leq \bar{\beta}$ and, using the laws of cosines, $d(p, x) = d(\tilde{p}, \tilde{x}) \leq d(\bar{p}, \bar{x})$.

Q. E. D.

REMARK 7. Another way of statting (C) is that the function

$$\theta_k(s,t) = \sphericalangle^k(x(s), p, y(t))$$

is monotone non-decreasing with s or t fixed, where $x(s)$ is the parameterization by arc length of the segment $[p,q]$ and $y(t)$ is the parameterization by arc length of $[p,r]$.

REMARK 8. By the monotonicity of θ_k, we have that for a CAT(k) the Alexandrov angles exist in the strict sense.

Properties of CAT(k) **spaces:**

PROPERTY 1. There is a unique geodesic segment joining any two points which are D_k-close and this segment varies continuously with its endpoints.

PROOF. Let $p, q \in X$ be such that $d(p,q) < D_k$. Let $[p,q]$ and $[p,q]'$ be two segments joining p and q. Let $r \in [p,q]$ and $r' \in [p,q]'$ be points satisfying that $d(p,r) = d(p,r')$. Consider $[p,r]$ and $[r,q]$ segments whose concatenation is $[p,q]$. Let $\triangle(\bar{p}, \bar{q}, \bar{r})$ be a comparison triangle for $\triangle([p,r],[r,q],[p,q]')$. By Definition (B),

$$d(r,r') \leq d(\bar{r}, \bar{r}'),$$

where $\bar{r}'$ is comparison point of r'. By Remark 6 the angle $\sphericalangle(p,r,q) = \pi$ and by (C) the comparison angle $\sphericalangle^k(p,q,r) \geq \sphericalangle(p,r,q) = \pi$. Thus, $\sphericalangle^k(p,r,q) = \pi$ and the triangle $\triangle(p,r,q)$ is degenerate. From this we have that $d(\bar{r},\bar{r}') = 0$ because the triangle $\triangle(\bar{p},\bar{q},\bar{r})$ must be degenerate by the assumed concatenation. Therefore, $d(r,r') = 0$.

Now we prove the continuous variation of the geodesic segments. Let $l < D_k$. Let $c, c' \colon [0,1] \to M_k^2$ be two minimizing curves parameterized with constant speed, length less than l, and starting from the same point, i.e., $c(0) = c'(0)$. Then there exists $C = C(l,k)$ such that

$$d(c(t), c'(t)) \leq C \cdot d(c(1), c'(1))$$

for $t \in [0,1]$. Let $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ be sequences of points such that $p_n \to p$ and $q_n \to q$. Suppose that $d(p_n,q_n)$ and $d(p,q_n)$ are smaller than $l < D_k$. Let $c, c_n, c_n' \colon [0,1] \to X$ be parameterizations with constant speed of $[p,q]$, $[p_n,q_n]$ and $[p,q_n]$, respectively. Applying Definition (A), we get

$$\begin{aligned} d(c(t), c_n(t)) &\leq d(c(t), c_n'(t)) + d(c_n'(t), c_n(t)) \\ &\leq d(\bar{c}(t), \bar{c}_n'(t)) + d(\bar{c}_n'(t), \bar{c}_n(t)) \\ &\leq C(d(\bar{q}, \bar{q}_n) + d(\bar{p}_n, \bar{p})) \\ &= C(d(q,q_n) + d(p_n,p)). \end{aligned}$$

The convergence of the sequences gives us the result. Q. E. D.

PROPERTY 2. Open balls with radius less than $D_k/2$ are convex.

PROOF. Let $x \in X$, $0 < \delta < D_k/2$, and $p, q \in B(x, \delta)$. Then the triangle $\triangle(p, q, x)$ has perimeter less than $2D_k$. In particular, by Proposition 1, $[p, q]$ is unique. Now we need to prove that $[p, q] \subset B(x, \delta)$. We take a comparison triangle $\triangle(\bar{p}, \bar{q}, \bar{x})$ in M_k^2. Let $r \in [p, q]$ and let $\bar{r} \in [\bar{p}, \bar{q}]$ be the comparison point. By Definition (B), it follows that $d(\bar{x}, \bar{r}) \geq d(x, r)$. On the other hand, $d(\bar{x}, \bar{r}) < \delta$ because the balls in M_k^2 of radius less than $D_k/2$ are convex. Hence $d(x, r) < \delta$ and $[p, q] \subset B(x, \delta)$. Q. E. D.

PROPERTY 3. Let p be a point in X. The function $(x, y) \mapsto \sphericalangle(x, p, y)$ is continuous in the ball $B(p, D_k)$.

PROOF. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences of points converging to x and y, respectively. Let c, c', c_n and c'_n be parametrizations with constant speed of the geodesic segments $[p, x]$, $[p, y]$, $[p, x_n]$ and $[p, y_n]$, respectively. For $t \in (0, 1]$, let $\alpha(t) = \sphericalangle^k(c(t), p, c'(t))$ and $\alpha_n(t) = \sphericalangle^k(c_n(t), p, c'_n(t))$. By Remark 7, these are non-decreasing functions of t. By Remark 8,

$$\alpha := \sphericalangle(x, p, y) = \lim_{t\to 0} \alpha(t) \quad \text{and} \quad \alpha_n := \sphericalangle(x_n, p, y_n) = \lim_{t\to 0} \alpha_n(t).$$

Using the Definition (D) of CAT(k) spaces:

$$\beta_n := \sphericalangle(x, p, x_n) \leq \bar{\sphericalangle}(x, p, x_n) \to 0 \text{ as } n \to \infty,$$

$$\gamma_n := \sphericalangle(y, p, y_n) \leq \bar{\sphericalangle}(y, p, y_n) \to 0 \text{ as } n \to \infty.$$

From Proposition 1.1, we get

$$|\alpha - \alpha_n| \leq \beta_n + \gamma_n.$$

Hence,

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

Q. E. D.

5. Domains $\mathcal{C}_{k',k}$

A **domain** $\mathcal{C}_{k',k}$ is a metric space X such that, for every pair of points which are D_k-close, there exists a minimizing curve joining them and every triangle $\triangle(p, q, r)$ with perimeter less than $2D_k$ satisfies

$$\begin{array}{rcccl}
\sphericalangle^{k'}(q, p, r) =: \alpha_{k'} & \leq & \alpha & \leq & \alpha_k := \sphericalangle^k(q, p, r), \\
\sphericalangle^{k'}(p, q, r) =: \beta_{k'} & \leq & \beta & \leq & \beta_k := \sphericalangle^k(p, q, r), \\
\sphericalangle^{k'}(q, r, p) =: \gamma_{k'} & \leq & \gamma & \leq & \gamma_k := \sphericalangle^k(q, r, p),
\end{array}$$

where α, β and γ are the Alexandrov angles of the triangle at p, q and r, respectively.

REMARK 9. Something important to notice about the definition is that having only the lower inequalities we can prove several equivalencies as in the CAT(k) case. A couple of them are included as Properties 4 and 5 and their proofs are very similar to what we have done before.

Properties of domains $\mathcal{C}_{k',k}$:

PROPERTY 1. A domain $\mathcal{C}_{k',k}$ is a CAT(k) space.

PROPERTY 2. For every three segments issuing from the same point, the sum of the angles between pairs of them is not greater than 2π.

PROOF. Let $[x,p]$, $[x,q]$ and $[x,r]$ be three minimizing curves issuing from the point x and sufficiently short to construct comparison triangles. Let $p(s)$, $q(s)$ and $r(s)$ be parametrizations by arc length of the segments. Let $s' \in (0, d(x,p))$ and $x' = p(s')$. Using the lower inequalities of the definition for the triangles $\triangle(x,p,q)$, $\triangle(x,p,r)$ and $\triangle(x,q,r)$, we have

$$\begin{aligned}
\sphericalangle^{k'}(p,x',q) + \sphericalangle^{k'}(p,x',r) + \sphericalangle^{k'}(r,x',q) &\leq \sphericalangle([x',p],[x',q]) + \sphericalangle([x',p],[x',r]) \\
&\quad + \sphericalangle([x',r],[x',q]) \\
&\overset{(*)}{\leq} \sphericalangle([x',p],[x',q]) + \sphericalangle([x',p],[x',r]) \\
&\quad + \sphericalangle([x',r],[x',x]) + \sphericalangle([x',x],[x',q]) \\
&= \sphericalangle([x',p],[x',q]) + \sphericalangle([x',q],[x',x]) \\
&\quad + \sphericalangle([x',r],[x',x]) + \sphericalangle([x',p],[x',r]) \\
&\overset{(**)}{=} 2\pi.
\end{aligned}$$

Inequality (*) is given by Proposition 1.1 and equality (**) is given by Remark 6. Letting $s' \to 0$, by the continuity of the angles in M_k^2, we have

$$\sphericalangle^{k'}(p(s),x,q(s)) + \sphericalangle^{k'}(p(s),x,r(s)) + \sphericalangle^{k'}(r(s),x,q(s)) \leq 2\pi.$$

Finally, taking the limit over s, we obtain the result. Q. E. D.

PROPERTY 3. The sum of adjacent angles is equal to π.

PROOF. From Proposition 1.1 we have that the sum of adjacent angles is not bigger than π and from the previous property the sum is smaller than π. Q. E. D.

PROPERTY 4. The function $\theta_{k'}(s,t)$ is monotone non-increasing if s or t are fixed.

PROPERTY 5. Let be $\triangle(p,q,r)$ a triangle in X, $x \in [p,q]$ and $y \in [p,r]$. Consider a comparison triangle $\triangle(\bar{p}',\bar{q}',\bar{r}')$ in $M_{k'}^2$, $\bar{x}' \in [\bar{p}',\bar{q}']$ and $\bar{y}' \in [\bar{p}',\bar{r}']$. Then

$$d(x,y) \geq d(\bar{x}',\bar{y}').$$

Property 6. If $[p,q] \subset [p,r]$ and $[p,q] \subset [p,x]$, then $[p,r] \subset [p,x]$ or $[p,x] \subset [p,r]$. This property is called **non-branching condition of segments**.

Proof. Without loss of generality we assume that $d(p,x) < d(p,r)$ and proceding by contradiction we suppose that $[p,x] \nsubseteq [p,r]$. Let $x' \in [p,r]$ be a point such that $d(p,x) = d(p,x')$. Then $d(x,x') \neq 0$. Consider a comparison triangle $\triangle(\bar{p},\bar{x},\bar{x}')$ of $\triangle(p,x,x')$ in $M^2_{k'}$. This comparison tirangle is isosceles and non-degenerate. But if $\bar{q}$ and $\bar{q}'$ are comparison points of q in $[p,x]$ and $[p,x']$, using Property 5 we have $d(q,q) \geq d(\bar{q},\bar{q}') > 0$, which is a contradiction to the definition of the metric. Q. E. D.

Property 7. If $[p,q]$ and $[p,r]$ are segments in X such that $\alpha = \sphericalangle([p,q],[p,r]) = 0$, then one of the segments is contained in the other.

Proof. Without loss of generality we assume that the segments are sufficiently short to construct comparison triangles and also that $d(p,q) < d(p,r)$. Consider a comparison triangle $\triangle(\bar{p},\bar{q},\bar{r})$ of $\triangle(p,q,r)$ in $M^2_{k'}$. By definition we have that $0 = \alpha \geq \sphericalangle^{k'}(q,p,r)$, which implies that $\triangle(\bar{p},\bar{q},\bar{r})$ is degenerate. From this,

$$d(p,r) = d(p,q) + d(q,r)$$

and $q \in [p,r]$. Q. E. D.

Property 8. With the same notation as the definition we have

$$\left|\alpha - \sphericalangle^0(q,p,r)\right| \leq \mu \cdot \mathrm{Area}(\triangle(\tilde{p},\tilde{q},\tilde{r}))$$

where $\triangle(\tilde{p},\tilde{q},\tilde{r})$ is a comparison triangle of $\triangle(p,q,r)$ in the plane and μ is a positive constant which depends on k' and k.

Proof. From the definition we have

$$\left|\alpha - \sphericalangle^0(q,p,r)\right| \leq \max\left\{\left|\sphericalangle^k(q,p,r) - \sphericalangle^0(q,p,r)\right|, \left|\sphericalangle^{k'}(q,p,r) - \sphericalangle^0(q,p,r)\right|\right\}.$$

If $\triangle(\bar{p},\bar{q},\bar{r})$ is a comparison triangle of $\triangle(p,q,r)$ in M^2_k,

$$\begin{aligned}
\left|\sphericalangle^k(q,p,r) - \sphericalangle^0(q,p,r)\right| &\leq \left|\sphericalangle^k(q,p,r) - \sphericalangle^0(q,p,r)\right| + \left|\sphericalangle^k(p,q,r) - \sphericalangle^0(p,q,r)\right| \\
&\quad + \left|\sphericalangle^k(q,r,p) - \sphericalangle^0(q,r,p)\right| \\
&= |k| \cdot \mathrm{Area}(\triangle(\bar{p},\bar{q},\bar{r})) \\
&= |k| \cdot \mathrm{Area}(\triangle(\tilde{p},\tilde{q},\tilde{r})) \cdot A(\triangle(\bar{p},\bar{q},\bar{r})),
\end{aligned}$$

where $A(\triangle(\bar{p},\bar{q},\bar{r})) \to 1$ if the lengths of the sides of $\triangle(p,q,r)$ tend to zero. Similarly for $|\sphericalangle^{k'}(q,p,r) - \sphericalangle^0(q,p,r)|$. Then we can take

$$\mu = \max\{|k|,|k'|\} + \varepsilon.$$

Q. E. D.

REMARK 10. Notice that we will use the upper and the lower bounds of the curvature every time we apply this last property. Esentially it tells us that our angles are not too far away from comparison angles in the plane.

CHAPTER 2

Spaces with Bounded Curvature: Riemannian Structure

1. Introduction to Spaces with Bounded Curvature

A **space with bounded curvature** is a locally compact length space $\mathcal{M}$ in which the following axioms are satisfied:

- For every point $p \in \mathcal{M}$, there exists $\rho_p > 0$ such that in the open ball $B(p, \rho_p)$ the **condition of extendibility of segments** holds: every geodesic segment $[x, y]$ with endpoints x and y in $B(p, \rho_p)$ can be extended to a segment $[x', y']$ in $\mathcal{M}$ for which x and y are internal points.
- Every $p \in \mathcal{M}$ is contained in a neighborhood U that is a domain $\mathcal{C}_{k',k}$ for some $k', k \in \mathbb{R}$ ($k' \leq k$) that depend on U.

Examples. Some examples of spaces with bounded curvature:

(1) Every smooth Riemannian manifold.
(2) The surface obtained by glueing two semispheres on the caps of a cylinder. We illustrate this example in Figure 6.

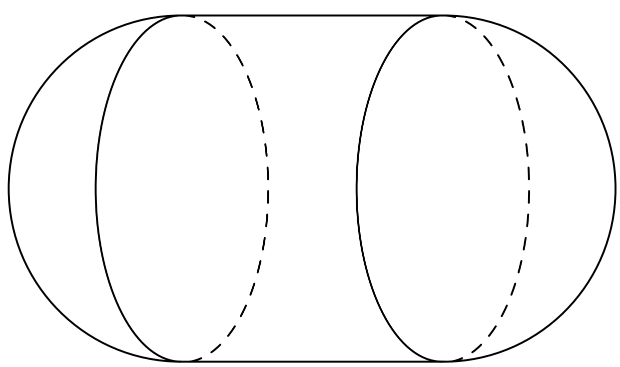

Figure 6

This surface has curvature equal to 0 on the cylinder and a positive constant on the semispheres. Thus, the curvature is between 0 and a constant.

The first goal in the treatment of spaces with bounded curvature (SBC) is to prove that these are topological manifolds. With that in mind, we need the next lemmas.

LEMMA 2.1. *Let $\mathcal{M}$ be a space with bounded curvature and let us consider geodesic segments $c_1\colon [0, L_d(c_1)] \to \mathcal{M}$, $c_2\colon [0, L_d(c_2)] \to \mathcal{M}$ and $c_3\colon [0, L_d(c_3)] \to \mathcal{M}$ issuing from the point x, i.e., $x = c_1(0) = c_2(0) = c_3(0)$. Suppose that $\sphericalangle(c_i, c_j) > 0$ for $i \neq j$ and $\sphericalangle(c_1, c_2) < \pi$. The length of these segments is denoted by a_i. We assume that $a_i < \rho_x$ and that $B(x, \rho_x)$ is a $\mathcal{C}_{k',k}$ domain. For every $t \in [0, 1]$, let w_t denote the midpoint of $[c_1(a_1 t), c_2(a_2 t)]$. Then, in the Euclidean space $\mathbb{R}^3$, there exist $\tilde{w}$, $\tilde{x}$ and $\tilde{x}_i$, $i = 1, 2, 3$, such that $d(\tilde{x}, \tilde{x}_i) = a_i$ and*

$$\tilde{w} - \tilde{x} = \frac{1}{2}((\tilde{x}_1 - \tilde{x}) + (\tilde{x}_2 - \tilde{x})), \tag{2.1.1}$$

$$\sphericalangle(c_i, c_j) = \sphericalangle(\tilde{x}_i, \tilde{x}, \tilde{x}_j), \quad i, j = 1, 2, 3, \tag{2.1.2}$$

$$\sphericalangle(\tilde{w}, \tilde{x}, \tilde{x}_i) = \lim_{t \to 0} \bar{\sphericalangle}(w_t, x, c_i(t)) \quad i = 1, 2, 3. \tag{2.1.3}$$

PROOF. From Proposition 1.1 and Property 2 of domains $\mathcal{C}_{k',k}$, we can find $\tilde{x}$ and $\tilde{x}_i$, $i = 1, 2, 3$ such that (2.1.2) is satisfied and $d(\tilde{x}_i, \tilde{x}) = a_i$. We can construct these points using spherical geometry. Now we take $\tilde{w}$ to be the point that satisfies (2.1.1). To prove (2.1.3) we are going to calculate several limits in order to use the law of cosines in the plane.

First of all

$$\lim_{t \to 0} \frac{d(c_i(a_i t), x)}{t} = \lim_{t \to 0} \frac{a_i t}{t} = a_i = d(\tilde{x}, \tilde{x}_i) \tag{2.1.4}$$

and

$$\begin{aligned}
\lim_{t \to 0} \frac{d(c_i(a_i t), c_j(a_j t))}{t} &= \lim_{t \to 0} \frac{\sqrt{\begin{array}{c} d(x, c_i(a_i t))^2 + d(x, c_j(a_j t))^2 \\ -2d(x, c_i(a_i t)) d(x, c_j(a_j t)) \cos(\bar{\sphericalangle}(c_i(a_i t), x, c_j(a_j t))) \end{array}}}{t} \\
&= \lim_{t \to 0} \sqrt{\frac{(a_i t)^2 + (a_j t)^2 - 2a_i a_j t^2 \cos(\bar{\sphericalangle}(c_i(a_i t), x, c_j(a_j t)))}{t^2}} \\
&= \sqrt{a_i^2 + a_j^2 - 2a_i a_j \cos(\lim_{t \to 0} \bar{\sphericalangle}(c_i(a_i t), x, c_j(a_j t)))} \\
&= \sqrt{a_i^2 + a_j^2 - 2a_i a_j \cos(\sphericalangle(c_i, c_j))} = d(\tilde{x}_i, \tilde{x}_j).
\end{aligned} \tag{2.1.5}$$

From this last limit and the definition of w_t, we get

$$\lim_{t \to 0} \frac{d(w_t, c_i(a_i t))}{t} = d(\tilde{w}, \tilde{x}_i) \quad i = 1, 2. \tag{2.1.6}$$

Moreover, we have

$$\sphericalangle(c_2, c_1, c_3) := \lim_{t\to 0} \bar{\sphericalangle}(c_2(a_2 t), c_1(a_1 t), c_3(a_3 t)) = \bar{\sphericalangle}(\tilde{x}_2, \tilde{x}_1, \tilde{x}_3).$$

This last angle is estimated using the law of cosines in $\mathbb{R}^2$ and the limits given by (2.1.5). From this fact and Property 8 of domains $\mathcal{C}_{k',k}$ we have the following:

$$\begin{aligned}&\lim_{t\to 0} |\bar{\sphericalangle}(c_2(a_2 t), c_1(a_1 t), c_3(a_3 t)) - \bar{\sphericalangle}(w_t, c_1(a_1 t), c_3(a_3 t))| \\ &\qquad \leq \lim_{t\to 0} \mu \operatorname{Area}(\bar{\triangle}(c_1(a_1 t), c_2(a_2 t), c_3(a_3 t))) = 0.\end{aligned}$$

Therefore,

$$\lim_{t\to 0} \bar{\sphericalangle}(w_t, c_1(a_1 t), c_3(a_3 t)) = \bar{\sphericalangle}(\tilde{w}, \tilde{x}_1, \tilde{x}_3). \tag{2.1.7}$$

Now, thanks to (2.1.5), (2.1.6) and (2.1.7)

$$\begin{aligned}\lim_{t\to 0} \frac{d(w_t, c_3(t))}{t} &= \lim_{t\to 0} \frac{\sqrt{\begin{array}{c} d(c_1(a_1 t), w_t)^2 + d(c_1(a_1 t), c_3(a_3 t))^2 \\ -2d(c_1(a_1 t), w_t)\, d(c_1(a_1 t), c_3(a_3 t)) \cdot \\ \cdot \cos(\bar{\sphericalangle}(w_t, c_1(a_1 t), c_3(a_3 t)))\end{array}}}{t} \\ &= \sqrt{d(\tilde{x}_1, \tilde{w})^2 + d(\tilde{x}_1, \tilde{x}_3)^2 - 2d(\tilde{x}_1, \tilde{w})\, d(\tilde{x}_1, \tilde{x}_3) \cos(\bar{\sphericalangle}(\tilde{x}_3, \tilde{x}, \tilde{w}))} \\ &= d(\tilde{x}_3, \tilde{w}). \end{aligned} \tag{2.1.8}$$

Using the same argument, we obtain

$$\lim_{t\to 0} \frac{d(w_t, x)}{t} = d(\tilde{x}, \tilde{w}). \tag{2.1.9}$$

Finally, by (2.1.5), (2.1.6), (2.1.8) and (2.1.9),

$$\begin{aligned}\lim_{t\to 0} \cos(\bar{\sphericalangle}(w_t, x, c_i(a_i t))) &= \lim_{t\to 0} \frac{d(w_t, x)^2 + d(x, c_i(a_i t))^2 - d(w_t, c_i(a_i t))^2}{2d(w_t, x)\, d(x, c_i(a_i t))} \\ &= \frac{d(\tilde{w}, \tilde{x})^2 + d(\tilde{x}, \tilde{x}_i)^2 - d(\tilde{w}, \tilde{x}_i)^2}{2d(\tilde{w}, \tilde{x})\, d(\tilde{x}, \tilde{x}_i)} \\ &= \cos(\bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_i)).\end{aligned}$$

Q. E. D.

Let δ be a positive number such that $B(x, \delta)$ is a domain $\mathcal{C}_{k',k}$ where the condition of extensibility of segments holds. Assume the same hypotheses of Lemma 2.1, but now with the segments inside the ball. For every $t \in (0, 1]$ let c_t be the segment of length δ that starts at x, passes through w_t and ends at x_t. By the local compactness of $\mathcal{M}$, we can assume that $B(x, \delta)$ is precompact and then there exist a point u and a sequence $\{t_n\}_{n\in\mathbb{N}}$, $0 < t_n$ with $t_n \to 0$, such that $x_{t_n} \to u$. Let c be the segment joining x with u. The next lemma describes this last segment in more detail.

LEMMA 2.2. *With the same hypotheses of Lemma 2.1, we have*

$$\sphericalangle(c, c_i) = \sphericalangle(\tilde{w}, \tilde{x}, \tilde{x}_i), \quad i = 1, 2, 3.$$

Furthermore, if $\sphericalangle(c_3, c_i) = \sphericalangle(c, c_i)$, $i = 1, 2$, *then* $\sphericalangle(c_3, c) = 0$.

PROOF. By virtue of Property 8 of domains $\mathcal{C}_{k',k}$, for t sufficiently small,

$$|\bar{\sphericalangle}(w_t, x, c_i(a_i t)) - \sphericalangle(c_i, c_t)| \leq \mu \operatorname{Area}(c_i(a_i t), \tilde{x}, \tilde{w}_t), \quad i = 1, 2, 3.$$

Using this inequality and (2.1.3), we have

$$\lim_{n\to\infty} \sphericalangle(c_{t_n}, c_i) = \bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_i), \quad i = 1, 2, 3.$$

From Property 3 of CAT(k) spaces since for s fixed $c_{t_n}(s)$ goes to $c(s)$ as $n \to \infty$ it follows that

$$\lim_{n\to\infty} \sphericalangle(c_{t_n}, c) = 0.$$

So, $\sphericalangle(c_i, c) = \bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_i)$, $i = 1, 2, 3$. Finally, if $\sphericalangle(c_3, c_i) = \sphericalangle(c, c_i)$, $i = 1, 2$, using Lemma 2.1 we have

$$\begin{aligned} \bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_1) &= \sphericalangle(c, c_1) = \bar{\sphericalangle}(\tilde{x}_3, \tilde{x}, \tilde{x}_1), \\ \bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_2) &= \sphericalangle(c, c_2) = \bar{\sphericalangle}(\tilde{x}_3, \tilde{x}, \tilde{x}_2). \end{aligned}$$

This forces $\tilde{x}_3$ to be in the same line as $\tilde{w}$ and $\sphericalangle(c_3, c) = \bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_3) = 0$. Q. E. D.

REMARK 11. The importance of these lemmas lies in the construction of the segment c whose angles with c_1 and c_2 have the same relationship of the diagonal in a parallelogram with the sides.

2. A Tangent Space to $\mathcal{M}$

Let $\mathcal{M}$ be a space with bounded curvature and $x \in \mathcal{M}$. Let $\delta > 0$ and $B(x, \delta)$ as before, i.e. $B(x, \delta)$ is a precompact domain $\mathcal{C}_{k',k}$ and the local extensibility of segments holds. We consider the set of segments joining x with $y \in B(x, \delta) \setminus \{x\}$ and we say that two of these segments are related if the angle between them is zero. This is clearly an equivalence relation, by the infinitesimal nature of Alexandrov angles. A equivalence class of segments is called a **direction** and the set Σ_x is the completion of the space of equivalence classes using the angles between directions as metric and is called **space of directions at** x. Finally, we consider the topological cone

$$T_x\mathcal{M} = \Sigma_x \times \mathbb{R}_{\geq 0} \Big/ \Sigma_x \times \{0\}$$

with metric

$$d([c, t], [c', s]) = \sqrt{t^2 + s^2 - 2ts\cos(\sphericalangle(c, c'))}.$$

The metric space is called **tangent space to** $\mathcal{M}$ **at** x.

REMARK 12. By the local extendibility condition, Properties 1 and 3 of CAT(k) spaces, and local compactness, Σ_x is compact.

REMARK 13. As a corolary to the last remark, we have that the tangent space is locally compact.

Now we define a vector space structure on $T_x\mathcal{M}$. The zero element is the vertex of the cone $[c,0]$. Let $[c,t]$ and $[c',t']$ be two elements of $T_x\mathcal{M}$. Their sum is defined by cases as follows:

(i) If t is zero, we let
$$[c,0]+[c',t']=[c',t']+[c,0]:=[c',t']$$
and viceversa if $t'=0$.

(ii) If $t,t'>0$ and $c=c'$, we let
$$[c,t]+[c',t']=[c',t']+[c,t]:=[c,t+t']=[c',t+t'].$$

(iii) If $t,t'>0$ and $\sphericalangle(c,c')=\pi$, we let
$$[c,t]+[c',t']=[c',t']+[c,t]:=\begin{cases}[c,t-t'] & \text{if } t>t',\\ [c',t'-t] & \text{if } t<t'.\end{cases}$$

(iv) If $t,t'>0$ and $\sphericalangle(c,c')\in(0,\pi)$, we let
$$[c,t]+[c',t']=[c',t']+[c,t]:=[c'',t''],$$
where $[c'',t'']$ is given by the following construction:

1. First, we take $N\in\mathbb{N}$ sufficiently large so that $t/N+t'/N<\delta$.
2. Then we choose the geodesic segments σ and σ' representing c and c' such that $L_d(\sigma)=t/N$ and $L_d(\sigma')=t'/N$.
3. Using Lemmas 2.1 and 2.2 with the segments σ and σ' we get the segment σ'' with
$$L_d(\sigma'')=\frac{\sqrt{(t)^2+(t')^2+2tt'\cos(\sphericalangle(\sigma,\sigma'))}}{N}=\frac{t''}{N},$$
$$t''\sin(\sphericalangle(\sigma'',\sigma))=t'\sin(\sphericalangle(\sigma,\sigma')),$$
$$t''\sin(\sphericalangle(\sigma'',\sigma'))=t\sin(\sphericalangle(\sigma,\sigma')).$$
4. We take the class c'' of σ'' in Σ_x.

Also, from the local extendability, if $[c,t]$ is a element of $T_x\mathcal{M}$, we can choose a unique direction $-c$ such that $\sphericalangle(c,-c)=\pi$. Then the inverse is defined by $-[c,t]:=[-c,t]$.

The product $[c,t] \in T_x\mathcal{M}$ by a real number is given by

$$\alpha[c,t] := \begin{cases} [c,\alpha t] & \text{if } \alpha \geq 0, \\ \\ -[c,-\alpha t] & \text{if } \alpha \leq 0. \end{cases}$$

THEOREM 2.3. *The tangent space $T_x\mathcal{M}$ with these operations and with scalar product*

$$\langle [c,t],[c',t'])\rangle := \begin{cases} tt'\cos(\sphericalangle(c,c')) & \text{if } t,t' > 0, \\ 0 & \text{otherwise}, \end{cases}$$

is a vector space.

PROOF. First, we prove the properties of the scalar product:

(I) $\langle [c,t],[c',t']\rangle = \langle [c',t'],[c,t]\rangle$;
(II) $\langle \alpha[c,t],[c',t']\rangle = \alpha\langle [c,t],[c',t']\rangle$;
(III) $\langle [c,t]+[c',t'],[c'',t'']\rangle = \langle [c,t],[c'',t'']\rangle + \langle [c',t'],[c'',t'']\rangle$;
(IV) $\langle [c,t],[c,t]\rangle \geq 0$ and $\langle [c,t],[c,t]\rangle = 0$ if and if only $t=0$;
(V) $\langle [c,t],[c',t']\rangle \leq tt'$; if $\langle [c,t],[c',t']\rangle = tt'$ and $t>0$, then for some $\alpha \geq 0$, $[c',t'] = \alpha[c,t]$.

Properties (I), (IV) and (V) follow directly from the definition.

Property (II): If $t=0$ or $t'=0$ or $\alpha=0$, the result follows clearly. Suppose none of t, t' oo α is zero. Then

$$\begin{aligned} \langle \alpha[c,t],[c',t']\rangle &= \begin{cases} \langle [c,\alpha t],[c',t']\rangle & \text{if } \alpha \geq 0 \\ \langle -[c,-\alpha t],[c',t']\rangle & \text{if } \alpha \leq 0 \end{cases} \\ &= \begin{cases} \alpha tt'\cos(\sphericalangle(c,c')) & \text{if } \alpha \geq 0 \\ (-\alpha t)t'\cos(\sphericalangle(-c,c')) & \text{if } \alpha \leq 0 \end{cases} \\ &= \begin{cases} \alpha tt'\cos(\sphericalangle(c,c')) & \text{if } \alpha \geq 0 \\ (-\alpha t)t'\cos(\pi - \sphericalangle(c,c')) & \text{if } \alpha \leq 0 \end{cases} \\ &= \alpha tt'\cos(\sphericalangle(c,c')) \\ &= \alpha\langle [c,t],[c',t']\rangle. \end{aligned}$$

Property (III): If $t''=0$, the result holds. If $t''>0$, we have four cases:

(i) $t'=0$. Then

$$\langle [c,t]+[c',t'],[c'',t'']\rangle = \langle [c,t],[c'',t'']\rangle = \langle [c,t],[c'',t'']\rangle + \langle [c',t'],[c'',t'']\rangle.$$

(ii) $t, t' > 0$ and $\sphericalangle(c, c') = 0$. Then $c = c'$ and thus

$$\begin{aligned}\langle [c,t] + [c',t'], [c'',t''] \rangle &= \langle [c, t+t'], [c'',t''] \rangle \\ &= (t+t')t'' \cos(\sphericalangle(c,c'')) \\ &= tt'' \cos(\sphericalangle(c,c'')) + t't'' \cos(\sphericalangle(c',c'')) \\ &= \langle [c,t],[c'',t''] \rangle + \langle [c',t'],[c'',t''] \rangle.\end{aligned}$$

(iii) $t, t' > 0$ and $\sphericalangle(c, c') = \pi$. Then

$$\begin{aligned}\langle [c,t] + [c',t'], [c'',t''] \rangle &= \begin{cases} \langle [c, t-t'], [c'',t''] \rangle & \text{if } t > t' \\ \langle [c', t'-t], [c'',t''] \rangle & \text{if } t' > t \end{cases} \\ &= \begin{cases} (t-t')t'' \cos(\sphericalangle(c,c'')) & \text{if } t > t' \\ (t'-t)t'' \cos(\sphericalangle(c',c'')) & \text{if } t' > t \end{cases} \\ &= \begin{cases} \langle [c,t],[c'',t''] \rangle - t't'' \cos(\pi - \sphericalangle(c',c'')) & \text{if } t > t' \\ \langle [c',t'],[c'',t''] \rangle - tt'' \cos(\pi - \sphericalangle(c,c'')) & \text{if } t' > t \end{cases} \\ &= \langle [c,t],[c'',t''] \rangle + \langle [c',t'],[c'',t''] \rangle.\end{aligned}$$

(iv) $t, t' > 0$ and $\sphericalangle(c, c') \in (0, \pi)$. Using Lemmas 2.1 and 2.2, taking representatives σ, σ' and σ'' of length $t/N, t'/N, t''/N < \delta$, respectively, with $N \in \mathbb{N}$, there exist $\tilde{x}$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ and $\tilde{w}$ points in $\mathbb{R}^3$ such that the following equalities hold:

$$\begin{aligned} d(\tilde{x}, \tilde{x}_1) &= t/N, \\ d(\tilde{x}, \tilde{x}_2) &= t'/N, \\ d(\tilde{x}, \tilde{x}_3) &= t''/N, \\ d(\tilde{x}, \tilde{w}) &= \frac{\sqrt{t^2 + t'^2 + 2tt' \cos(\sphericalangle(p[t], q[t']))}}{N}, \\ \bar{\sphericalangle}(\tilde{x}_1, \tilde{x}, \tilde{x}_3) &= \sphericalangle(c, c''), \\ \bar{\sphericalangle}(\tilde{x}_2, \tilde{x}, \tilde{x}_3) &= \sphericalangle(c', c'']), \\ \bar{\sphericalangle}(\tilde{w}, \tilde{x}, \tilde{x}_3) &= \sphericalangle([c,t] + [c',t'], [c'',t'']). \end{aligned}$$

Notice that we can take $\tilde{w}$ such that $\tilde{w} - \tilde{x} = (\tilde{x}_1 - \tilde{x}) + (\tilde{x}_2 - \tilde{x})$. Then we have

$$\begin{aligned}\langle [c,t] + [c',t'], [c'',t''] \rangle / N^2 &= \langle [c, t/N] + [c', t'/N], [c'', t''/N] \rangle \\ &= \langle \tilde{w} - \tilde{x}, \tilde{x}_3 - \tilde{x} \rangle_{\mathbb{R}^3} \\ &= \langle (\tilde{x}_1 - \tilde{x}) + (\tilde{x}_2 - \tilde{x}), \tilde{x}_3 - \tilde{x} \rangle_{\mathbb{R}^3} \\ &= \langle \tilde{x}_1 - \tilde{x}, \tilde{x}_3 - \tilde{x} \rangle_{\mathbb{R}^3} + \langle \tilde{x}_2 - \tilde{x}, \tilde{x}_3 - \tilde{x} \rangle_{\mathbb{R}^3} \\ &= \langle [c, t/N], [c'', t''/N] \rangle + \langle [c', t'/N], [c'', t''/N] \rangle \\ &= (\langle [c,t],[c'',t''] \rangle + \langle [c',t'],[c'',t''] \rangle)/N^2.\end{aligned}$$

It remains to show that $T_x\mathcal{M}$ is a vector space with the operations defined above. We prove the associativity of addition. The other properties are proven in a similar way. Let $[f,s]$ be an element of $T_x\mathcal{M}$. Using Property (III), we get

$$\begin{aligned}\langle [f,s],([c,t]+[c',t'])+[c'',t'']\rangle &= \langle [f,s],[c,t]+[c',t']\rangle + \langle [f,s],[c'',t'']\rangle \\ &= \langle [f,s],[c,t]\rangle + \langle [f,s],[c',t']\rangle + \langle [f,s],[c'',t'']\rangle \\ &= \langle [f,s],[c,t]\rangle + \langle [f,s],[c',t']+[c'',t'']\rangle \\ &= \langle [f,s],[c,t]+([c',t']+[c'',t''])\rangle .\end{aligned}$$

Substituting $[f,s]=([c,t]+[c',t'])+[c'',t'']$ and $[f',s']=[c,t]+([c',t']+[c'',t''])$, and using Property (V), we have

$$s^2 = \langle [f,s],[f,s]\rangle = \langle [f,s],[f',s']\rangle \geq ss'$$

and

$$(s')^2 = \langle [f',s'],[f',s']\rangle = \langle [f,s],[f',s']\rangle \geq ss'.$$

This means that $s=s'$. Again, by Property (V), we obtain that $[f,s]=[f',s']$.

As mentionated above, the remaining properties of a vector are proven following a similar argument, and hence we omit the proof. Q. E. D.

THEOREM 2.4. *The map*

$$\begin{aligned}\exp_x^{-1}\colon B(x,\delta) &\longrightarrow B([c,0],\delta), \\ p &\longmapsto [c,t],\end{aligned}$$

where c represents the unique segment joining x and p, is a homeomorphism.

PROOF. By Property 1 of CAT(k) spaces, this map is bijective and continuous, and by local compactness, it is a closed map since $B([c,0],\delta)$ is Hausdorff. Therefore, it is a homeomorphism. Q. E. D.

THEOREM 2.5. *Every space $T_x\mathcal{M}$ is finite-dimensional and they are all mutually homeomorphic.*

PROOF. By the last theorem, $T_x\mathcal{M}$ is locally compact and thus finite dimensional for every $x\in\mathcal{M}$. In addition, $\mathcal{M}$ is path-connected because it is a length space.

Let x and z be points in $\mathcal{M}$ and let $c\colon [0,1]\to\mathcal{M}$ be a curve joining these points and parameterized with constant speed. For every $c(t)=y_t$ we consider the ball $B(y_t,\delta(y_t))$ where we have the homeomorphism with $B([c,0],\delta(t))$. These balls form an open cover for $c([0,1])$ and, by the compactness of this set, we get a finite subcover. Let $\{x=y_1,y_2,\dots,y_n=z\}$ be the centers of these balls. If $B(y_i)$ and $B(y_{i+1})$ are two consecutive balls, then their intersection is non-empty and is homeomorphic to open sets of Euclidean spaces of finite dimension, $T_{y_i}\mathcal{M}$ and $T_{y_{i+1}}\mathcal{M}$. Since open sets of Euclidean spaces of different dimensions are not homeomorphic, then $T_x\mathcal{M}$ and $T_z\mathcal{M}$ have the same dimension. Q. E. D.

COROLARY 2.6. *The space $\mathcal{M}$ is a topological manifold.*

PROOF. The only thing missing is to say why $\mathcal{M}$ is second-countable. This is a consequence of local compactness, the metric structure and Theorem 2.4. (For more details, see Appendix A, page 459, [**Spi99**], Vol. 1). Q. E. D.

3. Riemannian Structure

In this section, we are going to introduce a Riemannian structure on spaces with bounded curvature. In order to do that we need the following lemmas.

LEMMA 2.7. *Let $\triangle(\tilde{x},\tilde{y},\tilde{z})$ be a triangle in $\mathbb{R}^2$. Then*

$$\mathrm{Area}(\triangle(\tilde{x},\tilde{y},\tilde{z})) = \frac{d(\tilde{y},\tilde{z})\, d(\tilde{y},\tilde{x})\ \sin(\bar{\sphericalangle}(\tilde{x},\tilde{y},\tilde{z}))}{2} \leq \frac{d(\tilde{y},\tilde{z})\, d(\tilde{y},\tilde{x})\, \bar{\sphericalangle}(\tilde{x},\tilde{y},\tilde{z})}{2}.$$

LEMMA 2.8. *Let x be a point in $\mathcal{M}$ and let $\delta(x)$ be a positive number as in Theorem 2.4. Let $y, x_1 \in B(x,\delta(x))$ and $0 < d(y,x_1) < \delta(y)$. Let $x_2 \in B(y,\delta(y))$. Then*

$$d(x_1,x_2) = d(y,x_1) - d(y,x_2)\ \cos(\sphericalangle(\exp_y^{-1}(x_1), \exp_y^{-1}(x_2))) + O((d(y,x_2))^2).$$

PROOF. We denote $\exp_y^{-1}(x_i)$ by k_i, $i = 1, 2$. Using the law of cosines, we get

$$(d(x_1,x_2))^2 = (d(y,x_1))^2 + (d(y,x_2))^2 - 2\, d(y,x_1)\, d(y,x_2)\ \cos(\bar{\sphericalangle}(x_1,y,x_2)).$$

Then

$$(d(x_1,x_2))^2 - (d(y,x_1) - d(y,x_2)\ \cos(\sphericalangle(k_1,k_2)))^2$$

$$= (d(y,x_1))^2 + (d(y,x_2))^2 - 2\, d(y,x_1)\, d(y,x_2)\ \cos(\bar{\sphericalangle}(x_1,y,x_2))$$

$$-(d(y,x_1))^2 + 2\, d(y,x_1)\, d(y,x_2)\ \cos(\sphericalangle(k_1,k_2)) - (d(y,x_2))^2 \cos^2(\sphericalangle(k_1,k_2))$$

$$= (d(y,x_2))^2(1-\cos^2(\sphericalangle(k_1,k_2)))+2\, d(y,x_1)\, d(y,x_2)(\cos(\sphericalangle(k_1,k_2))-\cos(\bar{\sphericalangle}(x_1,y,x_2))).$$

Taking absolute value we obtain

$$|d(x_1,x_2) - (d(y,x_1) - d(y,x_2)\ \cos(\sphericalangle(k_1,k_2)))|$$

$$
\begin{aligned}
&\leq \frac{(d(y,x_2))^2(1-\cos^2(\sphericalangle(k_1,k_2)))}{|d(x_1,x_2)+d(y,x_1)-d(y,x_2)\ \cos(\sphericalangle(k_1,k_2))|} \\
&\quad + \frac{2\,d(y,x_1)\,d(y,x_2)|\cos(\sphericalangle(k_1,k_2))-\cos(\bar{\sphericalangle}(x_1,y,x_2))|}{|d(x_1,x_2)+d(y,x_1)-d(y,x_2)\ \cos(\sphericalangle(k_1,k_2))|} \\
&\leq \frac{(d(y,x_2))^2+2\,d(y,x_1)\,d(y,x_2)|\cos(\sphericalangle(k_1,k_2))-\cos(\bar{\sphericalangle}(x_1,y,x_2))|}{|d(x_1,x_2)+d(y,x_1)-d(y,x_2)\ \cos(\sphericalangle(k_1,k_2))|} \\
&\overset{(*)}{\leq} \frac{(d(y,x_2))^2+2\,d(y,x_1)\,d(y,x_2)|\sphericalangle(k_1,k_2)-\bar{\sphericalangle}(x_1,y,x_2)|}{|d(x_1,x_2)+d(y,x_1)-d(y,x_2)\ \cos(\sphericalangle(k_1,k_2))|} \\
&\overset{(**)}{\leq} \frac{(d(y,x_2))^2+(d(y,x_1))^2\,(d(y,x_2))^2\mu(y)\bar{\sphericalangle}(x_1,y,x_2)}{|d(x_1,x_2)+d(y,x_1)-d(y,x_2)\ \cos(\sphericalangle(k_1,k_2))|}.
\end{aligned}
$$

The inequality $(*)$ is satisfied because $|\cos(x)-\cos(y)|<|x-y|$ and inequality $(**)$ follows combining Property 8 of $C_{k'k}$ and Lemma 2.7. The last inequality proves this lemma because we multiply both sides of the inequality by

$$|d(x_1,x_2)+d(y,x_1)-d(y,x_2)\ \cos(\sphericalangle(k_1,k_2))|.$$

Q. E. D.

THEOREM 2.9. *A space with bounded curvature $\mathcal{M}$ is a Riemannian manifold with a differentiable structure of class C^1 and a continuous Riemannian metric.*

PROOF. Let p be a point in $\mathcal{M}$. By Corolary 2.6, $\mathcal{M}$ is a topological manifold of dimension n. Let r be a positive number such that we have the homeomorphism of Theorem 2.4 over $B(p,r)$. Let $\{[c_i,\delta]\}_{i=1}^n$ be an orthogonal basis of $T_p\mathcal{M}$ with respect to the inner product defined in Theorem 2.3 and $\delta<r$.

For $i=1,\dots,n$, we define the functions

$$
\begin{aligned}
u^i\colon \mathcal{M} &\longrightarrow \mathbb{R}, \\
y &\longmapsto d(y,C_i),
\end{aligned}
$$

with $C_i=\exp_p([c_i,\delta])$. First, we claim that $u=(u^1,\cdots,u^n)$ is a coordinate chart of $\mathcal{M}$ over a neightborhood of p. To verify this claim, we define the following elements in $T_y\mathcal{M}$ with $y\in B(p,r)$ which we are going to them **vector fields** on $B(p,r)$:

$$X_i(y)=\exp_y^{-1}(C_i).$$

By Properties 1 and 3 of CAT spaces, the functions $f_{i,j}(y)=\langle X_i(y),X_j(y)\rangle$ are continuous. Thanks to the way the vectors $[c_i,\delta]$ were chosen and the continuity of $f_{i,j}(y)$ with respect to y and the fact $f_{i,j}(p)=\delta_{i,j}$, there exist $\delta_1>0$ and $0<r_1<\delta$

such that, if $d(p,y) < r_1$, then

$$\det\begin{pmatrix} \langle X_1(y), X_1(y)\rangle & \cdots & \langle X_1(y), X_n(y)\rangle \\ \vdots & \ddots & \vdots \\ \langle X_n(y), X_1(y)\rangle & \cdots & \langle X_n(y), X_n(y)\rangle \end{pmatrix} > \delta_1.$$

From this, there exists $\delta_2 > 0$ such that, if $d(p,y) < r_1$ and $[f,s] \in T_y\mathcal{M}$ with $s > 0$, then

$$\max_{i\in\{1,\ldots,n\}} |\sphericalangle([f,s], X_i(y)) - \frac{\pi}{2}| > \delta_2, \tag{2.3.1}$$

i.e. the vector $[f,s]$ is close enough to one axis.

By Property 8 of domains $\mathcal{C}_{k',k}$ and Lemma 2.7, we can choose $r_1 \geq r_2 > 0$ sufficiently small, such that, if $d(p,x_i) < r_2$, $i = 1,2$, $x_1 \neq x_2$, then:

$$|\sphericalangle(\exp_{x_1}^{-1}(x_2), \exp_{x_1}^{-1}(C_i) - \bar{\sphericalangle}(x_2,x_1,C_i)| \leq \mu(p)\,\mathrm{Area}(\triangle(\tilde{x}_2,\tilde{x}_1,\tilde{C}_i))$$

$$\leq \mu(x)\, d(x_1,C_i)\, d(x_1,x_2)\, \bar{\sphericalangle}(x_2,x_1,C_i) \leq \frac{\delta_2}{2}, \quad i \in \{1,\ldots,n\}.$$

By inequality (2.3.1), there exists $i_0 \in \{0,\cdots,n\}$ such that

$$|\sphericalangle(\exp_{x_1}^{-1}(x_2), X_{i_0}(x_1)) - \frac{\pi}{2}| > \delta_2.$$

From the last two inequalities we get that $|\bar{\sphericalangle}(x_2,x_1,C_{i_0}) - \pi/2| > \delta_2/2$ because otherwise we would obtain

$$|\sphericalangle(\exp_{x_1}^{-1}(x_2), X_{i_0}(x_1)) - \frac{\pi}{2}| \leq |\sphericalangle(\exp_{x_1}^{-1}(x_2), X_{i_0}(x_1)) - \bar{\sphericalangle}_{x_1}(x_2,C_i)|$$

$$+|\bar{\sphericalangle}(x_2,x_1,C_i) - \frac{\pi}{2}| \leq \frac{\delta_2}{2} + \frac{\delta_2}{2} = \delta_2,$$

which is a contradiction. So, we have the following two cases:

(1) If $\bar{\sphericalangle}(x_2,x_1,C_{i_0}) > \pi/2$, then $u^{i_0}(x_2) = d(x_2,C_{i_0}) > d(x_1,C_{i_0}) = u^{i_0}(x_1)$.

(2) If $\bar{\sphericalangle}(x_2,x_1,C_{i_0}) < \pi/2$, then $\bar{\sphericalangle}(x_2,x_1,C_{i_0}) < \pi/2 - \delta_2/2$. We notice that, if $y \in B(p,r_2)$, $u^{i_0}(y) = d(y,C_{i_0}) > \delta - r_2 > 0$. Putting these inequalities together we get that, if $d(x_1,x_2) < 2(\delta - r_2)\cos(\pi/2 - \delta_2/2)$,

$$2d(x_1,C_{i_0})\cos(\bar{\sphericalangle}(x_2,x_1,C_{i_0})) > 2(\delta - r_2)\cos(\bar{\sphericalangle}(x_2,x_1,C_{i_0}))$$

$$> 2(\delta - r_2)\cos(\pi/2 - \delta_2/2) > d(x_1,x_2).$$

Using this, the law of cosines implies that

$$u^{i_0}(x_2) = d(x_2,C_{i_0}) < d(x_1,C_{i_0}) = u^{i_0}(x_1),$$

because

$$d(x_2,C_{i_0})^2 = d(x_1,C_{i_0})^2 + d(x_1,x_2)(d(x_1,x_2) - 2d(x_1,C_{i_0})\cos(\bar{\sphericalangle}(x_2,x_1,C_{i_0}))).$$

Thus, if $r_3 = (\delta - r_2)\cos(\pi/2 - \delta_2/2)$, then the function

$$\begin{aligned} u\colon B(p, r_3) &\longrightarrow u(B(p, r_3)) \subset \mathbb{R}^n, \\ y &\longmapsto (u^1(y), \cdots, u^n(y)), \end{aligned}$$

is bijective because $u^{i_0}(x_2) < u^{i_0}(x_1)$.

Since $B(p, r_3)$ is open, u is continuous and injective, using Brouwer's Invariance of Domain Theorem (see [**Bro12**]), u is a homeomorphism and $u(B(p, r_3))$ is open. This proves that u is a coordinate chart and we the coordinates given by u are called **distance coordinates**.

Now, let $u = (u^1, \cdots, u^n)$ and $v = (v^1, \cdots, v^n)$ be two systems of distance coordinates over two overlapping open sets. Now we will prove that $v^i \circ u^{-1}(u^1, \cdots, u^n)$, $i \in \{1, \dots, n\}$, are continuously differentiable. For $y \in B(x, r_3)$ we define

$$\exp_y^{-1}(z) = \sum_{i=1}^{n} u_y^i(z) \frac{X_i(y)}{|X_i(y)|}, \tag{2.3.2}$$

i.e. u_y^i are coordinates of $\exp_y^{-1}(z)$ in $T_y\mathcal{M}$. Clearly we have that $u_y^i(y) = 0$ for every i. We show that $u \circ u_y^{-1}$ is differentiable at $(0, \cdots, 0)$ and its Jacobian is invertible, where $u_y\colon B(p, r_3) \subset \mathcal{M} \to \mathbb{R}^n$ given by $u_y(z) = (u_y^1(z), \cdots, u_y^n(z))$. We have that

$$u^i(u_y^{-1}(u^1, \cdots, u^n)) - u^i(u_y^{-1}(0, \cdots, 0))$$

$$= d\left(C_i, \exp_y\left(\sum_{j=1}^{n} u_y^j X_j(y)\right)\right) - d(C_i, y).$$

Since $\|(u^1, \cdots, u^n)\|$ is a small quantity of the first order in

$$\beta = d\left(y, \exp_y\left(\sum_{j=1}^{n} u_y^j X_j(y)\right)\right),$$

it follows from Lemma 2.8 and the previous equation that, up to small quantities of second order in $\|(u^1, \cdots, u^n)\|$,

$$u^i(u_y^{-1}(u^1, \cdots, u^n)) - u^i(u_y^{-1}(0, \cdots, 0))$$

$$\begin{aligned}
&= -d\left(y, \exp_y\left(\sum_{j=1}^{n} u_y^j X_j(y)\right)\right)\cos(\sphericalangle(X_i(y), \sum_{j=1}^{n} u_y^j X_j(y))) + O(\beta^2) \\
&= -\|\sum_{j=1}^{n} u_y^j X_j(y)\| \left(\frac{\sum_{j=1}^{n} u_y^j \langle X_i(y), X_j(y)\rangle}{\|\sum_{j=1}^{n} u_y^j X_j(y)\|\|X_i(y)\|}\right) + O(\beta^2) \\
&= -\sum_{j=1}^{n} u_y^j \left\langle \frac{X_i(y)}{\|X_i(y)\|}, X_j(y)\right\rangle + O(\beta^2).
\end{aligned}$$

Therefore, taking limits, $u \circ u_y^{-1}$ is differentiable at $(0, \cdots, 0)$ and its Jacobian is

$$A(y) = \begin{pmatrix} \left\langle \frac{X_1(y)}{\|X_1(y)\|}, X_1(y)\right\rangle & \cdots & \left\langle \frac{X_1(y)}{\|X_1(y)\|}, X_n(y)\right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle \frac{X_n(y)}{\|X_n(y)\|}, X_1(y)\right\rangle & \cdots & \left\langle \frac{X_n(y)}{\|X_n(y)\|}, X_n(y)\right\rangle \end{pmatrix}. \tag{2.3.3}$$

This matrix is invertible because $y \in B(p, r_3)$ and the way r_3 was chosen. Since $u \circ u_y^{-1}$ is a homeomorphism and has a differential at $(0, \cdots, 0)$, the mapping $u_y \circ u^{-1}$ has a differential at $(u^1(y), \cdots, u^n(y))$ with Jacobian $A^{-1}(y)$ due to the inverse function theorem.

Let $\{Z_j\}_{j=1}^n$ be the corresponding vector fields to v, v_y^j be the coordinates of $T_y\mathcal{M}$ corresponding to v and $B(y)$ be the Jacobian matrix of $v_y \circ v^{-1}$. If y belongs to the intersection of the domains of the distance systems u and v, then

$$v \circ u^{-1} = (v \circ v_y^{-1}) \circ (v_y \circ u_y^{-1}) \circ (u_y \circ u^{-1}),$$

where $v_y \circ u_y^{-1}$ is linear since it is a change of basis on $T_y\mathcal{M}$. Thus, $v \circ u^{-1}$ has a differential at $(u^1(y), \ldots, u^n(y))$ with Jacobian $D(y) = B(y)C(y)A^{-1}(y)$ where $C(y)$ is the associated matrix to $v_y \circ u_y^{-1}$. Since the scalar product is continuous, $D(y)$ depends continuously on y. Therefore, $v^i \circ u^{-1}$ are continuously differentiable functions of $u^1, \ldots, u^n$. This gives a C^1-atlas for $\mathcal{M}$.

It remains to introduce the Riemannian metric. Let $\alpha_1 \colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ and $\alpha_2 \colon (-\varepsilon, \varepsilon) \to \mathcal{M}$ be two curves such that have tangent vectors $V_1 = \alpha_1'(0)$ and $V_2 = \alpha_2'(0)$, respectively, and $\alpha_1(0) = \alpha_2(0) = x$. Then the paths $\exp_x^{-1}(\alpha_1(t))$ and $\exp_x^{-1}(\alpha_2(t))$ are differentiable at $t = 0$ and the Riemannian metric is defined by

$$\langle V_1, V_2\rangle := \left\langle \left(\exp_x^{-1} \circ \alpha_1\right)'(0), \left(\exp_x^{-1} \circ \alpha_2\right)'(0)\right\rangle.$$

Thanks to this definition and (2.3.3), the matrix of components of the Riemannian metric at y with distance coordinates $u = (u^1, \cdots, u^n)$ is given by

$$G(y) = (A^{-1}(y))^T \begin{pmatrix} \langle X_1(y), X_1(y)\rangle & \cdots & \langle X_1(y), X_n(y)\rangle \\ \vdots & \ddots & \vdots \\ \langle X_n(y), X_1(y)\rangle & \cdots & \langle X_n(y), X_n(y)\rangle \end{pmatrix} A^{-1}(y).$$

Multiplying the matrices, we get

$$G(y) = \begin{pmatrix} \cos(X_1(y), X_1(y)) & \cdots & \cos(X_1(y), X_n(y)) \\ \vdots & \ddots & \vdots \\ \cos(X_n(y), X_1(y)) & \cdots & \cos(X_n(y), X_n(y)) \end{pmatrix}^{-1}.$$

Hence, the Riemannian metric is continuous. Q. E. D.

Remark 14. We call the ball $B(x, r_3)$ in the preceding theorem a **normal ball**. Note that this ball has distance coordinates and is the domain of the homeomorphism in Theorem 2.4. It is important to notice that each normal ball has its curvature bounds k' and k with $k' < k$. We denote the normal ball centered at x with radius δ and curvature bounds k' and k by $B(x, \delta, k', k)$.

Remark 15. Using standard methods, a continuously differentiable curve

$$\alpha\colon [a, b] \to \mathcal{M}$$

is rectifiable and its length is given by

$$L(\alpha) = \int_a^b \|\dot{\alpha}(t)\|_G \, \mathrm{d}t.$$

CHAPTER 3

Spaces with Bounded Curvature: Parallel Transport

Through this chapter we can find the results about parallel transport defined by Nikolaev in [**Nik83a**] and inspired by the point of view of E. Cartan on the riemannian parallel transport. In the first section we introduce the parallel transport along geodesic segments and some estimates for this. And in the second section we extend it to rectifiable curves.

1. Parallel Transport along Geodesic Segments

Let x be a point of $\mathcal{M}$ and $B = B(x, \delta, k', k)$ be a normal ball around x. A **symmetry with respect to a point** $o \in B$ is a map

$$\begin{array}{rcl} \mathcal{S}_o\colon B & \longrightarrow & M, \\ p & \longmapsto & p', \end{array}$$

where p' lies on the extension of $[p, o]$ beyond o so that $d(p, o) = d(o, p')$.

Now we consider a geodesic segment $[p, q]$ in B and construct the map

$$\begin{array}{rcl} P_{p,q}\colon B([c, 0], \varepsilon) \subset T_p\mathcal{M} & \longrightarrow & B([c, 0], \varepsilon) \subset T_q\mathcal{M}, \\ \xi & \longmapsto & \exp_q^{-1}(\mathcal{S}_o(\exp_p(\xi))), \end{array}$$

where $\varepsilon > 0$ and the length h of $[p, q]$ are sufficiently small so that the three maps are defined inside B and o is the middle point of the segment $[p, q]$.

Let $[p, q]$ be an arbitrary segment in B. Let us subdivide it into 2^m equal segments by points $p = p_0, p_1, \dots, p_i, \dots, p_{2^m} = q$. We denote the map $P_{p_i,p_{i+1}}$ by $P_{i,i+1}$ and set $h_m = d(p, q)/2^m$. We define a map $P_m\colon T_p\mathcal{M} \to T_q\mathcal{M}$ as follows. Let ξ be a vector in $T_p\mathcal{M}$ and set $\xi' = h_m\xi/|\xi|$. Consider the map

$$P'_m(\xi) = P_{2^m-1,2^m} \circ \dots \circ P_{1,2} \circ P_{0,1}(\xi'),$$

which is well defined if m is sufficiently large. Then we take

$$P_m(\xi) = \begin{cases} \ \ |\xi|\, P'_m(\xi) & \text{for } m \geq 1, \\ -|\xi|\, P'_m(\xi) & \text{for } m = 0. \end{cases}$$

We will prove that $\{P_m(\xi)\}$ is convergent for every $\xi \in T_p\mathcal{M}$ and the limit will be denoted by $P(\xi)$. We call the resulting map $P\colon T_p\mathcal{M} \to T_q\mathcal{M}$ a **parallel transport along the geodesic segment** $[p, q]$. After this, we will prove that we can define this parallel transport along rectifiable curves in B and then along arbitrary rectifiable curves. As the classical parallel transport, this one will be an isometry between the tangent spaces. In order to prove all these facts, we need to make some estimates.

1.1. Estimates for the parallel transport.

LEMMA 3.1. *Let $p, q, o \in B$ and let $p' = \mathcal{S}_o(p)$ and $q' = \mathcal{S}_o(q)$, as it is illustrated in Figure 7. Then*

$$|d(p,q)^2 - d(p',q')^2| \leq c(k',k) \max\{(d(p,o))^4, (d(q,o))^4\}.$$

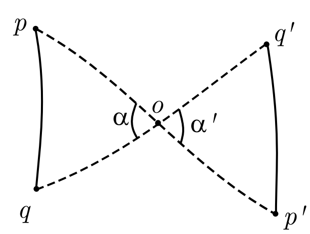

FIGURE 7

PROOF. Consider the triangles $\triangle(p,o,q)$ and $\triangle(p',o,q')$ in B and the corresponding comparison triangles in the plane $\triangle(\tilde{p},\tilde{o},\tilde{q})$ and $\triangle(\tilde{p}',\tilde{o},\tilde{q}')$. Let $\alpha = \sphericalangle(p,o,q)$ and $\alpha' = \sphericalangle(p',o,q')$. Then by Property 8 of domains $\mathcal{C}_{k',k}$:

$$\begin{aligned} |\sphericalangle^0(p,o,q) - \sphericalangle^0(p',o,q')| &\leq \mu(k',k)\left[\mathrm{Area}(\triangle(\tilde{p},\tilde{o},\tilde{q})) + \mathrm{Area}(\triangle(\tilde{p}',\tilde{o},\tilde{q}'))\right], \\ &\leq \mu(k',k)\cdot(\max\{d(o,p), d(o,q)\})^2. \end{aligned}$$

The last inequality is satisfied by Lemma 2.7. On the othe hand, using the law of cosine we get

$$(d(p,q))^2 - (d(p',q'))^2 = 2\,d(p,q)\,d(p',q')\left(\cos(\sphericalangle^0(p,o,q)) - \cos(\sphericalangle^0(p',o,q'))\right)$$

$$= -4\,d(p,q)\,d(p',q')\sin\left(\frac{\sphericalangle^0(p,o,q) - \sphericalangle^0(p',o,q')}{2}\right)\sin\left(\frac{\sphericalangle^0(p,o,q) + \sphericalangle^0(p',o,q')}{2}\right).$$

Thus

$$\begin{aligned}|(d(p,q))^2-(d(p',q'))^2| &\le 4|\sphericalangle^0(p,o,q)-\sphericalangle^0(p',o,q')| \\ &\quad\cdot\frac{d(p,q)\,d(p',q')\sin\left(\frac{\sphericalangle^0(p,o,q)+\sphericalangle^0(p',o,q')}{2}\right)}{2} \\ &\le \mu(k',k)(\max\{d(o,p),d(o,q)\})^2 \\ &\quad\cdot\frac{d(p,q)\,d(p',q')\sin\left(\frac{\sphericalangle^0(p,o,q)+\sphericalangle^0(p',o,q')}{2}\right)}{2} \\ &\le c(k',k)(\max\{d(o,p),d(o,q)\})^4.\end{aligned}$$

The last inequality is given by the fact that the fractional number is the area of a triangle and which is between Area($\triangle(\tilde{p},\tilde{o},\tilde{q})$) and Area($\triangle(\tilde{p}',\tilde{o},\tilde{q}')$) and using Lemma 2.7 as in the first inequality of the proof. Q. E. D.

LEMMA 3.2. *Let $a,o\in B$ with $a\neq o$ and $a'=\mathcal{S}_o(a)$, as it is illustrated in Figure 8. Then using the same notation as the previous lemma*

$$|\cos(\sphericalangle(p,a,q))-\cos(\sphericalangle(p',a',q'))|\le c'(k',k)M^2\left[1+\left(\frac{M}{m}\right)^2+\left(\frac{M}{m}\right)^4\right],$$

where $M=\max\{d(a,o),d(a,p),d(a,q)\}$ and $m=\min\{d(a,p),d(a,q),d(a',p'),d(a',q')\}$.

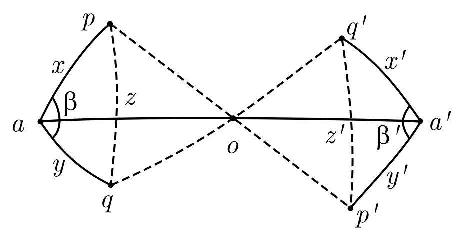

FIGURE 8

PROOF. We consider the triangles $\triangle(p,a,q)$ and $\triangle(p',a',q')$ and their comparison triangles $\triangle(\tilde{p},\tilde{a},\tilde{q})$ and $\triangle(\tilde{p}',\tilde{a}',\tilde{q})$. We introduce the following notation: $x=d(a,p)$, $y=d(a,q)$, $z=d(p,q)$, $x'=d(a',p')$, $y'=d(a',q')$, $z'=d(p',q')$, $\beta=\sphericalangle(p,a,q)$ and $\beta'=\sphericalangle(p',a',q')$.

By Property 8 of domains $\mathcal{C}_{k',k}$ and Lemma 2.7 we have

$$|\sphericalangle^0(p,a,q)-\beta|,|\sphericalangle^0(p',a',q')-\beta'|\le\tilde{\mu}(k',k)\,M^2. \tag{3.1.1}$$

Using the law of cosines on the comparison triangles we get

$$\begin{aligned}
\cos(\sphericalangle^0(p,a,q)) - \cos(\sphericalangle^0(p',a',q')) &= \frac{x^2+y^2-z^2}{2xy} - \frac{(x')^2+(y')^2-(z')^2}{2x'y'} \\
&= \frac{x^2+y^2-z^2}{2xy} - \frac{(x')^2+(y')^2-(z')^2}{2x'y'} \\
&\quad + \frac{(x')^2+(y')^2-(z')^2}{2xy} - \frac{(x')^2+(y')^2-(z')^2}{2xy} \\
&= \frac{(x^2-(x')^2)+(y^2-(y')^2)-(z^2-(z')^2)}{2xy} \\
&\quad + ((x')^2+(y')^2-(z')^2)\left(\frac{1}{2xy} - \frac{1}{2x'y'}\right).
\end{aligned}$$

The first summand of the last equation is bounded using Lemma 3.1 by

$$\frac{3}{2}\, c(k',k)\, \frac{M^4}{m^2}.$$

For the second summand we observe that

$$((x')^2+(y')^2-(z')^2)\left(\frac{1}{2xy} - \frac{1}{2x'y'}\right) =$$

$$\begin{aligned}
&= ((x')^2+(y')^2-(z')^2)\,\frac{x'y'-xy}{2x'yxy'} \\
&= ((x')^2+(y')^2-(z')^2)\,\frac{1}{2x'y}\left[\frac{x'y'-xy}{xy'}\right] \\
&= ((x')^2+(y')^2-(z')^2)\,\frac{1}{2x'y}\left[\frac{x'y'-xy'+xy'-xy}{xy'}\right] \\
&= ((x')^2+(y')^2-(z')^2)\,\frac{1}{2x'y}\left[\frac{x'-x}{x} + \frac{y'-y}{y'}\right] \\
&= ((x')^2+(y')^2-(z')^2)\,\frac{1}{2x'y}\left[\frac{(x')^2-x^2}{(x'+x)x} + \frac{(y')^2-y^2}{(y'+y)y'}\right] \\
&= ((x')^2+(y')^2-(z')^2)\cdot\left[\frac{(x')^2-x^2}{2(x')^2xy+2x'yx^2} + \frac{(y')^2-y^2}{2x'y(y')^2+2x'y^2y'}\right]
\end{aligned}$$

and using again Lemma 3.1, it is bounded by

$$3c(k',k)\,\frac{M^6}{m^4}.$$

Finally, from these two bounds and inequality (3.1.1), we have

$$\begin{aligned}
|\cos(\beta) - \cos(\beta')| &= |\cos(\beta) - \cos(\sphericalangle^0(p,a,q)) + \cos(\sphericalangle^0(p,a,q)) \\
&\quad - \cos(\sphericalangle^0(p',a',q')) + \cos(\sphericalangle^0(p',a',q')) - \cos(\beta')| \\
&\leq |\cos(\beta) - \cos(\sphericalangle^0(p,a,q))| \\
&\quad + |\cos(\sphericalangle^0(p,a,q)) - \cos(\sphericalangle^0(p',a',q'))| \\
&\quad + |\cos(\sphericalangle^0(p',a',q')) - \cos(\beta')| \\
&\leq |\beta - \sphericalangle^0(p,a,q)| + |\cos(\sphericalangle^0(p,a,q)) - \cos(\sphericalangle^0(p',a',q'))| \\
&\quad + |\sphericalangle^0(p',a',q') - \beta'| \\
&\leq \tilde{\mu}(k',k)\, M^2 + \frac{3}{2}\, c(k',k)\, \frac{M^4}{m^2} + 3c(k',k)\, \frac{M^6}{m^4} + \tilde{\mu}(k',k)\, M^2 \\
&= c'(k',k)\, M^2 \left[1 + \left(\frac{M}{m}\right)^2 + \left(\frac{M}{m}\right)^4 \right].
\end{aligned}$$

Q. E. D.

LEMMA 3.3. *Let $\triangle(a,b,c)$ be a triangle in B and let o be the middle point of the side $[b,c]$. We illustrate this configuration in Figure 9. Then*

$$|(d(a,b))^2 + (d(a,c))^2 - (2(d(a,o))^2 + 2(d(b,o))^2)| \leq c''(k',k)\, \max\{(d(a,b))^4, (d(a,c))^4\}.$$

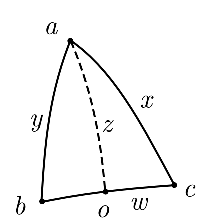

FIGURE 9

PROOF. We consider the comparison triangles in the plane $\triangle(\tilde{a},\tilde{b},\tilde{c})$ and $\triangle(\tilde{a},\tilde{b},\tilde{o})$ of the triangles $\triangle(a,b,c)$ and $\triangle(a,b,o)$, respectively. We introduce the notation: $w = d(b,c)$, $x = d(a,c)$, $y = d(a,b)$ and $z = d(a,o)$. Let $M = \max\{b,c\}$. By Property 8 of domains $\mathcal{C}_{k',k}$, we have

$$|\sphericalangle^0(a,b,o) - \sphericalangle^0(a,b,c)| \leq \mu(k',k)\, (\mathrm{Area}(\triangle(a,b,o)) + \mathrm{Area}(\triangle(a,b,c))) = O(M^2).$$

Using the law of cosines on the comparison triangles we get

$$\begin{aligned}\cos(\sphericalangle^0(a,b,o)) &= \frac{y^2+\frac{w^2}{4}-z^2}{wy},\\ x^2 &= w^2+y^2-2wy\cos(\sphericalangle^0(a,b,c)).\end{aligned}$$

By the previous inequality we have that

$$2wx\cos(\sphericalangle^0(a,b,c)) = 2wx\cos(\sphericalangle^0(a,b,o)) + O(M^4)$$

and thus,

$$\begin{aligned}x^2 &= w^2+y^2-2wy\cos(\sphericalangle^0(a,b,o))+O(M^4)\\ &= w^2+y^2-2\left(y^2+\frac{w^2}{4}-z^2\right)+O(M^4)\\ &= \frac{1}{2}w^2-y^2+2z^2+O(M^4),\\ (d(a,c))^2 &= 2(d(b,o))^2-(d(a,b))^2+2(d(a,o))^2+O(M^4).\end{aligned}$$

Q. E. D.

LEMMA 3.4. *Let λ' and λ $(0<\lambda'\leq\lambda)$ be constants. Consider p, q, a and a' distinct points in B such that*

$$\lambda' \leq \frac{d(a,p)}{d(a,a')}, \frac{d(a,q)}{d(a,a')} \leq \lambda. \tag{3.1.2}$$

Let o be the midpoint of the segment $[a,a']$ and let o_1 and o_1' be the midpoints of the segments $[o,a]$ and $[o,a']$, respectively. Set $p'=\mathcal{S}_o(p)$, $q'=\mathcal{S}_{o_1}(q)$ and $q''=\mathcal{S}_{o_1'}$. We illustrate this configuration in Figure 10.

Then

$$|\cos(\sphericalangle(q,a,p))+\cos(\sphericalangle(p',a',q''))| \leq C(k',k,\lambda',\lambda)\cdot(d(a,a'))^2.$$

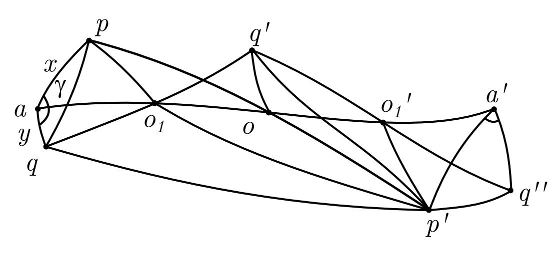

FIGURE 10

PROOF. Let us consider $x = d(a,p)$, $y = d(a,q)$, $h = d(a,a')$, $\alpha = \sphericalangle(p,a,o)$, $\beta = \sphericalangle(q,a,o)$ and $\gamma = \sphericalangle(p,a,q)$.

Using the law of cosines and Property 8 of domains $\mathcal{C}_{k',k}$ as in the proof of Lemma 3.3 on the triangles $\triangle(a,p,o)$, $\triangle(p,a,q)$ and $\triangle(q,a,o)$ along with their comparison ones we get

$$\begin{aligned}(d(p,o))^2 &= x^2 + \frac{h^2}{4} - hx\cos(\alpha) + O(h^4),\\ (d(p,q))^2 &= x^2 + y^2 - 2xy\cos(\gamma) + O(h^4),\\ (d(q,o))^2 &= y^2 + \frac{h^2}{4} - hy\cos(\beta) + O(h^4).\end{aligned}$$

With these equations and applying Lemma 3.3 to the triangle $\triangle(q,p,p')$ with o the midpoint of the side $[p,p']$, we get

$$(d(p',q))^2 = 2((d(q,o))^2 + (d(p,o))^2) - (d(p,q))^2 + O(M^4)$$

$$= x^2 + y^2 + h^2 + 2xy\cos(\gamma) - 2xh\cos(\alpha) - 2yh\cos(\beta) + O(h^4), \tag{3.1.3}$$

with $M = \max\{d(p,q), d(p',q)\}$. The last equation is satisfied by inequality (3.1.2).

By the law of cosines and Propery 8 of domains $\mathcal{C}_{k',k}$ as before applied to the triangle $\triangle(p,o_1,a)$, we have

$$(d(p,o_1))^2 = \frac{h^2}{16} + x^2 - \frac{hx\cos(\alpha)}{2} + O(h^4). \tag{3.1.4}$$

By Lemma 3.3 applied on the triangle $\triangle(p',p,o_1)$ with o the midpoint of the side $[p,p']$, the last equation and inequality (3.1.2):

$$\begin{aligned}(d(p',o_1))^2 &= 2((d(p,o))^2 + (d(o,o_1))^2) - (d(p,o_1))^2 + O((M')^4),\\ &= x^2 + \frac{9h^2}{16} - \frac{3xh\cos(\alpha)}{2} + O(h^4),\end{aligned} \tag{3.1.5}$$

where $M' = \max\{d(o_1,p), d(o_1,p')\}$.

Again, by the law of cosines and Property 8 of domains $\mathcal{C}_{k',k}$ applied to the triangle $\triangle(q,o_1,a)$, we have

$$(d(q,o_1))^2 = y^2 + \frac{h^2}{16} - \frac{hy\cos(\beta)}{2} + O(h^4). \tag{3.1.6}$$

By Lemma 3.3 applied to the triangle $\triangle(p', q', q)$, where o_1 is the midpoint of the side $[q, q']$, inequality (3.1.2), and equations (3.1.3), (3.1.6) and (3.1.5):

$$\begin{aligned}(d(p', q'))^2 &= 2((d(p', o_1))^2 + (d(q, o_1))^2) - (d(p', q))^2 + O((M'')^2) \\ &= x^2 + y^2 + \frac{h^2}{4} - 2xy\cos(\gamma) + hy\cos(\beta) - hx\cos(\alpha) + O(h^4),\end{aligned}$$

where $M'' = \max\{d(p', q'), d(p', q'')\}$. By Lemma 3.1,

$$(d(o, q'))^2 = y^2 + O(h^4). \tag{3.1.7}$$

Since the angles $\sphericalangle(a, o, q') = \sphericalangle(q', o, o_1')$ are adjacent, we have

$$\cos(\sphericalangle(q', o, o_1')) = -\cos(\sphericalangle(a, o, q')).$$

Using Lemma 3.2 with a, q, $o = \mathcal{S}_{o_1}(a)$, $q' = \mathcal{S}_{o_1}(q)$, and Property 8 of domains $\mathcal{C}_{k',k}$, we get

$$\cos(\beta) - \cos(\sphericalangle(a, o, q')) = O(h^2).$$

Thus,

$$\cos(\sphericalangle(q', o, o_1')) = -\cos(\beta) + O(h^2).$$

By equation (3.1.7) and the law of cosines on the triangle $\triangle(q', o, o_1')$, we get

$$(d(q', o_1))^2 = y^2 + \frac{h^2}{16} + \frac{hy\cos(\beta)}{2} + O(h^4).$$

By Lemma 3.1 and equation (3.1.4),

$$(d(p', o_1'))^2 = (d(p, o_1))^2 + O(h^4) = \frac{h^2}{16} + x^2 - \frac{hx\cos(\alpha)}{2} + O(h^4). \tag{3.1.8}$$

Using Lemma 3.3 as before on the triangle $\triangle(p', q', q'')$, with o_1' the middle point of the side $[q', q'']$, we get

$$\begin{aligned}(d(p', q''))^2 &= 2((d(q', o_1'))^2 + (d(p', o_1'))^2) + O((M'')^4) \\ &= x^2 + y^2 + 2xy\cos(\gamma) + O(h^4). \end{aligned} \tag{3.1.9}$$

Again, by Lemma 3.1,

$$d(p', a') = x + O(h^3) \tag{3.1.10}$$

and

$$d(a', q'') = y + O(h^3). \tag{3.1.11}$$

Finally, using Property 8 of domains $\mathcal{C}_{k',k}$, and equations (3.1.9), (3.1.10) and (3.1.11), we conclude that

$$\cos(\sphericalangle(p', a', q'')) =$$

$$
\begin{aligned}
&= \frac{(d(p',a'))^2+(d(a',q''))^2-(d(p',q''))^2}{2d(p',a')\,d(a',q'')}+O(h^2),\\
&= \frac{(x+O(h^3))^2+(y+O(h^3))^2-(x^2+y^2+2xy\cos(\gamma)+O(h^4))}{2(x+O(h^3))\,(y+O(h^3))}+O(h^2),\\
&= -\cos(\gamma)+O(h^2)
\end{aligned}
$$

and

$$|\cos(\sphericalangle(p',a',q''))+\cos(\gamma)|=O(h^2).$$

Q. E. D.

LEMMA 3.5. *Following the same notation of the beginning of this section, let*

$$\xi_l = P_{l-1,l}\circ\cdots\circ P_{1,2}\circ P_{0,1}(\xi'),\quad l=1,\ldots,2^m.$$

Then

$$||\xi_l|-h_m|\le c(k',k)h_m^2,\quad l=1,\ldots,m.$$

PROOF. By Lemma 3.1, using the definition of h_m and ξ', we have

$$||\xi_l|^2-|\xi_{l-1}|^2|\le c(k',k)h_m^2,$$

and then

$$
\begin{aligned}
||\xi_l|^2-h_m^2| &\le ||\xi_l|^2-|\xi_{l-1}|^2|+||\xi_{l-1}|^2-|\xi_{l-2}|^2|+\cdots+||\xi_1|^2-|\xi'|^2|\\
&\le l\,c(k',k)\,h_m^2. \qquad (3.1.12)
\end{aligned}
$$

Since $l\in\{1,\ldots,2^m\}$, the last term is bounded above by

$$c(k'k)\,h_m^3.$$

Thus,

$$||\xi_l|-h_m|\le\frac{c(k',k)\,h_m^3}{|\xi_l|+h_m}\le c(k',k)h_m^2.$$

Q. E. D.

LEMMA 3.6. *Let $\xi,\zeta\in\mathbb{R}^n\setminus\{0\}$ and $\varepsilon>0$. If, for each $\eta\in\mathbb{R}^n\setminus\{0\}$*

$$|\cos(\sphericalangle(\eta,\xi))-\cos(\sphericalangle(\eta,\zeta))|<\varepsilon,$$

then

$$\sphericalangle(\xi,\zeta)\le\frac{\pi\,\varepsilon}{2}.$$

PROOF. Since the angles between ξ, ζ and η are the same as the normalized ones, we can suppose that $|\xi|=|\zeta|=1$. We set

$$\eta=\frac{(\xi-\zeta)}{\|\xi-\zeta\|}.$$

Then, using the law of cosines on the plane, we have

$$\begin{aligned}|\cos(\sphericalangle(\eta,\xi))-\cos(\sphericalangle(\eta,\zeta))| &= |\langle\eta,\xi-\zeta\rangle| = |\xi-\zeta|\\ &= \sqrt{2-2\cos(\sphericalangle(\xi,\zeta))}\\ &= 2\sin\left(\frac{\sphericalangle(\xi,\zeta)}{2}\right) < \varepsilon.\end{aligned}$$

Since we know that

$$\sin(\alpha) > \frac{\alpha}{\pi},$$

if $\alpha \in [0,\pi/2]$. Thus we get using both inequalities, that

$$\sphericalangle(\xi,\zeta) \leq \frac{\pi\,\varepsilon}{2}.$$

Q. E. D.

LEMMA 3.7. *Under the same hypothesis of Lemma 3.2, we get that*

$$|\sphericalangle(p,a,q)-\sphericalangle(p',a',q')| \leq \tilde{c}\left(k',k,\frac{M}{m}\right)M^2.$$

PROOF. Let

$$\xi = \exp_a^{-1}(p), \quad \zeta = \exp_a^{-1}(q), \quad \xi' = \exp_{a'}^{-1}(p'), \quad \zeta' = \exp_{a'}^{-1}(q')$$

and

$$\xi_1 = \frac{\xi}{|\xi|}, \quad \zeta_1 = \frac{\zeta}{|\zeta|}, \quad \xi_1' = \frac{\xi'}{|\xi'|}, \quad \zeta_1 = \frac{\zeta}{|\zeta|},$$

$$\eta = \xi_1 - \zeta_1, \quad \eta' = \xi_1' - \zeta_1', \quad h = d(a,a').$$

Now we consider an onthonormal basis $\{X_i\}$ $i=1,\dots,n$, in $T_a\mathcal{M}$ and the basis $\{X_i' = -P_0(X_i)\}$, $i=1,\dots,n$ in $T_{a'}\mathcal{M}$ which is almost orthonormal up to $O(h^2)$ thanks to Lemma 3.2. That is, for $\mathsf{g}_{ij}' = \langle X_i', X_j'\rangle$, we have

$$\mathsf{g}_{ij}' - \delta_{ij} = O(h^2).$$

Also, by Lemma 3.2, we get

$$\langle\xi_1,X_i\rangle - \langle\xi_1',X_i'\rangle = O(h^2), \quad \langle\zeta_1,X_i\rangle - \langle\zeta_1',X_i'\rangle = O(h^2).$$

Thus,

$$\langle\eta,X_i\rangle = \langle\eta',X_i'\rangle = O(h^2).$$

Let $\eta = \eta^k X_k$ and $\eta' = \eta'^k X_k'$ using the Einstein summation notation. Then

$$\langle\eta,X_i\rangle = \eta^i, \quad \langle\eta',X_i'\rangle = \eta'^k g_{ik}' = \eta'^k\delta_{ik} + O(h^2) = \eta'^i + O(h^2),$$

and

$$\eta^i - \eta'^i = O(h^2).$$

This last estimate implies that

$$||\eta| - |\eta'|| \leq |\eta - \eta'| = O(h^2).$$

Noticing that the exponential map is an isometry at the origin we have that $\sphericalangle(p, a, q)$ is equal to the corresponding angle in a Euclidean triangle with the lengths of sides 1, 1 and $|\eta|$ and $\sphericalangle(p', a', q')$ is equal to the corresponding angle in a Euclidean triangle with the lengths of sides 1, 1 and $|\eta'|$ (the proof of this fact is analogous to the proof of Corolary II.1A.7, page 173, [**BH99**]). Thus, the last inequality ensures that

$$\sphericalangle(p, a, q) - \sphericalangle(p', a', q') = O(h^2).$$

Q. E. D.

1.2. Existence of parallel transport.

LEMMA 3.8. *Following the same notation in the beginning of this section,*

$$\sphericalangle(P_m(\xi), P_{m+1}(\xi)) \leq \tilde{C}(k', k)\,\frac{(d(p,q))^2}{2^m}.$$

PROOF. Let $p = p_0, p_1, \ldots, p_{2^m} = q$ be the points dividing the geodesic segment $[p, q]$ evently into 2^m segments $[p_l, p_{l+1}]$. We denote by $p_{(l+1)/2}$ the midpoint of the geodesic segment $[p_l, p_{l+1}]$, $l = 0, 1, 2, \ldots, 2^m - 1$. Also, we introduce the following notation:

$$\Phi_l = -P_{p_l, p_{l+1}}, \quad \Psi_l = P_{p_{(l+1)/2}, p_{l+1}} \circ P_{p_l, p_{(l+1)/2}}.$$

The proof is separated into two parts.

(1) Let ξ be a vector tangent to $\mathcal{M}$ at p_l such that

$$\frac{h_m}{2} \leq |\xi| \leq 2h_m. \tag{3.1.13}$$

Claim:

$$\sphericalangle(\Phi_l(\xi), \Psi_l(\xi)) \leq C'(k', k)\, h_m^2.$$

Let η be a vector tangent to $\mathcal{M}$ at p_{l+1} satisfying inequality (3.1.13). By Lemma 3.2,

$$\cos(\sphericalangle(\Phi_l^{-1}(\eta), \xi)) - \cos(\sphericalangle(\eta, \Phi_l(\xi))) = O(h_m^2).$$

By Lemma 3.1, for sufficiently large m, the vector $\Phi_l^{-1}(\eta)$ also satisfies inequality (3.1.13). Therefore, using Lemma 3.4 with $\lambda' = 1/2$ and $\lambda = 2$, we get

$$\cos(\sphericalangle(\Phi_l^{-1}(\eta), \xi)) - \cos(\sphericalangle(\eta, \Psi_l(\xi))) = O(h_m^2).$$

Thus,

$$|\cos(\sphericalangle(\eta, \Phi_l(\xi))) - \cos(\sphericalangle(\eta, \Psi_l(\xi)))| = O(h_m^2).$$

Using Lemma 3.6 and the last inequality, we obtain

$$\sphericalangle(\Phi_l(\xi), \Psi_l(\xi)) \leq C'(k', k)\, h_m^2.$$

(2) Let

$$\xi'_m = h_m \frac{\xi}{|\xi|}.$$

We define vectors $\xi_m^{(k)}, \zeta_m^{(k)} \in T_{p_k}\mathcal{M}$, $k = 0, 1, 2 \ldots, 2^m$, as follows. Let

$$\xi_m^{(0)} = \zeta_m^{(0)} = \xi'_m.$$

Then the vectors $\xi_m^{(k+1)}$ and $\zeta_m^{(k+1)}$ are defined recursively by letting

$$\xi_m^{(k+1)} = -P_{p_k, p_{k+1}}(\xi_m^{(k)}); \quad \zeta_m^{(k+1)} = P_{p_{(k+1)/2}, p_{k+1}} \circ P_{p_k, p_{(k+1)/2}}(\zeta_m^{(k)}).$$

By Lemma 3.5, for m suficiently large, the condition (3.1.13) is satified for the vectors $\xi_m^{(k)}$ and $\zeta_m^{(k)}$. Hence, we can apply the previous claim and Lemma 3.7 to these vectors. We denote the corresponding constant $\tilde{c}\left(k', k, \frac{M}{m}\right)$ ($m = h_m/2$ and $M = 2h_m$) from Lemma 3.7 by $C''(k', k)$.

Claim:

$$\sphericalangle(\xi_m^{(k)}, \zeta_m^{(k)}) \leq C(k', k)\, \frac{d(p, q)\, d(p, p_k)}{2^m}, \quad k = 0, 1, 2, \ldots, 2^m, \tag{3.1.14}$$

where $C(k', k) = \max\{C'(k', k), 2C''(k, k')\}$. We proceed by induction. Inequality (3.1.14) is obvious for $k = 0$. Suppose it holds for $k = 0, 1, \ldots, l$. Let

$$\bar{\xi}_m^{(l+1)} = P_{p_{(l+1)/2}, p_{l+1}}(\xi_m^{(l)}).$$

By the triangle inequality,

$$\sphericalangle(\xi_m^{(l+1)}, \zeta_m^{(l+1)}) \leq \sphericalangle(\bar{\xi}_m^{(l+1)}, \xi_m^{(l+1)}) + \sphericalangle(\bar{\xi}_m^{(l+1)}, \zeta_m^{(l+1)}).$$

By the previous claim,

$$\sphericalangle(\bar{\xi}_m^{(l+1)}, \xi_m^{(l+1)}) \leq C'(k', k) h_m^2.$$

Applying Lemma 3.7 twice to the maps $P_{p_l, p_{(l+1)/2}}$ and $P_{p_{(l+1)/2}, p_{l+1}}$; we get

$$\left|\sphericalangle(\bar{\xi}_m^{(l+1)}, \zeta_m^{(l+1)}) - \sphericalangle(\xi_m^{(l)}, \zeta_m^{(l)})\right| \leq 2\, C''(k', k) h_m^2.$$

By the induction hypothesis for $k = l$

$$\sphericalangle(\xi_m^{(l)}, \zeta_m^{(l)}) \leq C(k', k)\, \frac{d(p, q)\, d(p, p_l)}{2^m}.$$

Putting all together, we have

$$\begin{aligned}
\sphericalangle(\xi_m^{(l+1)}, \zeta_m^{(l+1)}) &\leq C'(k',k)h_m^2 + 2\,C''(k',k)h_m^2 + C(k',k)\,\frac{d(p,q)\,d(p,p_l)}{2^m} \\
&\leq C(k',k)\left(h_m^2 + \frac{d(p,q)\,d(p,p_l)}{2^m}\right) \\
&= C(k',k)(h_m^2 + d(p,p_l)\,h_m) \\
&= C(k',k)(d(p,p_l) + h_m)h_m \\
&= C(k',k)\frac{d(p,p_{l+1})\,d(p,q)}{2^m},
\end{aligned}$$

which proves this claim.

We finish the proof of the lemma by observing that, by definition,

$$\xi_m^{(2^m)} = P'_m(\xi), \quad \zeta_m^{(2^m)} = P'_{m+1}(\xi)$$

and

$$\sphericalangle\left(\xi_m^{(2^m)}, \zeta_m^{(2^m)}\right) = \sphericalangle(P_m(\xi), P_{m+1}(\xi)).$$

Thus, (3.1.14) for $k = 2^m$ proves this lemma. Q. E. D.

Proposition 3.9. *For any $\xi \in T_p\mathcal{M}$, the limit of $P_m(\xi)$ in $T_q\mathcal{M}$ exists, is denoted by $P(\xi)$, and*

$$\sphericalangle(P(\xi), P_m(\xi)) \leq C(k',k)\,\frac{(d(p,q)^2)}{2^m}.$$

Proof. By Lemma 3.8, we get that

$$\begin{aligned}
\sphericalangle(P_m(\xi), P_{m+k}(\xi)) &\leq \sum_{i=m}^{m+k-1} \sphericalangle(P_i(\xi), P_{i+1}(\xi)) \\
&\leq C(k',k)\,(d(p,q))^2 \sum_{i=0}^{k-1} \frac{1}{2^{m+i}} \\
&\leq C(k',k)\,\frac{(d(p,q))}{2^m}.
\end{aligned}$$

Therefore, $\{P_m(\xi)\}$ is a Cauchy sequence and, since $T_q\mathcal{M}$ is complete (it is a finite dimension vector space), there exists the limit

$$P(\xi) = \lim_{m\to\infty} P_m(\xi).$$

As $k \to \infty$ in the previous inequality we obtain the required estimate. Q. E. D.

We defined the parallel transport along sufficiently short geodesic segments. Let γ be an arbitrary geodesic segment in $\mathcal{M}$. We divide γ into short geodesic segments and define the parallel transport along γ as the composition of the parallel transport on every piece. It is clear that the definition of the parallel transport is independent of the choice of the partition of γ.

PROPOSITION 3.10. *Let $[p,q]$ be a geodesic segment. The parallel transport $P\colon T_p\mathcal{M} \to T_q\mathcal{M}$ is an isometry.*

PROOF. We assume that $[p,q]$ is a sufficiently short geodesic segment and then the procedure is the same as before. Let $\xi, \zeta \in T_p\mathcal{M}$. First, we prove the following estimate:

$$|\sphericalangle(\xi,\zeta) - \sphericalangle(P_m(\xi), P_m(\zeta))| \leq C(k',k)\,\frac{(d(p,q))^2}{2^m}. \tag{3.1.15}$$

Let

$$\xi' = \frac{\xi}{|\xi|}\,h_m, \quad \zeta' = \frac{\zeta}{|\zeta|}\,h_m$$

and

$$\Lambda_i = P_{i-1,i} \circ \ldots \circ P_{1,2} \circ P_{0,1}, \quad i = 1, \ldots, 2^m.$$

We have, by definition, that

$$\sphericalangle(P_m(\xi), P_m(\zeta)) = \sphericalangle(\Lambda_m(\xi), \Lambda_m(\zeta)).$$

By Lemmas 3.5 and 3.7,

$$|\sphericalangle(\Lambda_{i-1}(\xi'), \Lambda_{i-1}(\zeta')) - \sphericalangle(\Lambda_i(\xi'), \Lambda_i(\zeta'))| \leq C(k',k)h_m^2.$$

Therefore,

$$\begin{aligned}
|\sphericalangle(\xi,\zeta) - \sphericalangle(P_m(\xi), P_m(\zeta))| &= |\sphericalangle(\Lambda_0(\xi'), \Lambda_0(\zeta')) - \sphericalangle(\Lambda_m(\xi'), \Lambda_m(\zeta'))| \\
&\leq \sum_{i=0}^{m} |\sphericalangle(\Lambda_{i-1}(\xi'), \Lambda_{i-1}(\zeta')) - \sphericalangle(\Lambda_i(\xi'), \Lambda_i(\zeta'))| \\
&\leq 2^m\, C(k',k)\, h_m^2 \\
&= C(k',k)\,\frac{(d(p,q))^2}{2^m},
\end{aligned}$$

which proves inequality (3.1.15). Finally, since

$$|\sphericalangle(\xi,\zeta) - \sphericalangle(P(\xi), P(\zeta))| = \lim_{m\to\infty} |\sphericalangle(\xi,\zeta) - \sphericalangle(P_m(\xi), P_m(\zeta))| = 0,$$

P preserves angles and by definition of P, it preserves lengths. Thus, P is an isometry. Q. E. D.

2. Parallel Transport along Rectifiable Curves

Let $0 < \delta < \pi/2$. A triangle $\triangle(o, b, c)$ in B is called δ-**regular** if each one of its angles lies between δ and $\pi - \delta$.

Let $a, p \in B$ such that $d(a, b) = d(b, o)$ and $d(b, p) = d(o, c)$. Let o_1, o_2 and o_3 be the midpoints of the geodesic segments $[o, b]$, $[b, c]$ and $[o, c]$. We set

$$a' = \mathcal{S}_{o_1}(a), \quad p' = \mathcal{S}_{o_1}(p), \quad a'' = \mathcal{S}_{o_2}(a),$$

$$p'' = \mathcal{S}_{o_2}(p), \quad a''' = \mathcal{S}_{o_3}(a''), \quad p''' = \mathcal{S}_{o_3}(p'').$$

We illustrate this configuration in Figure 11.

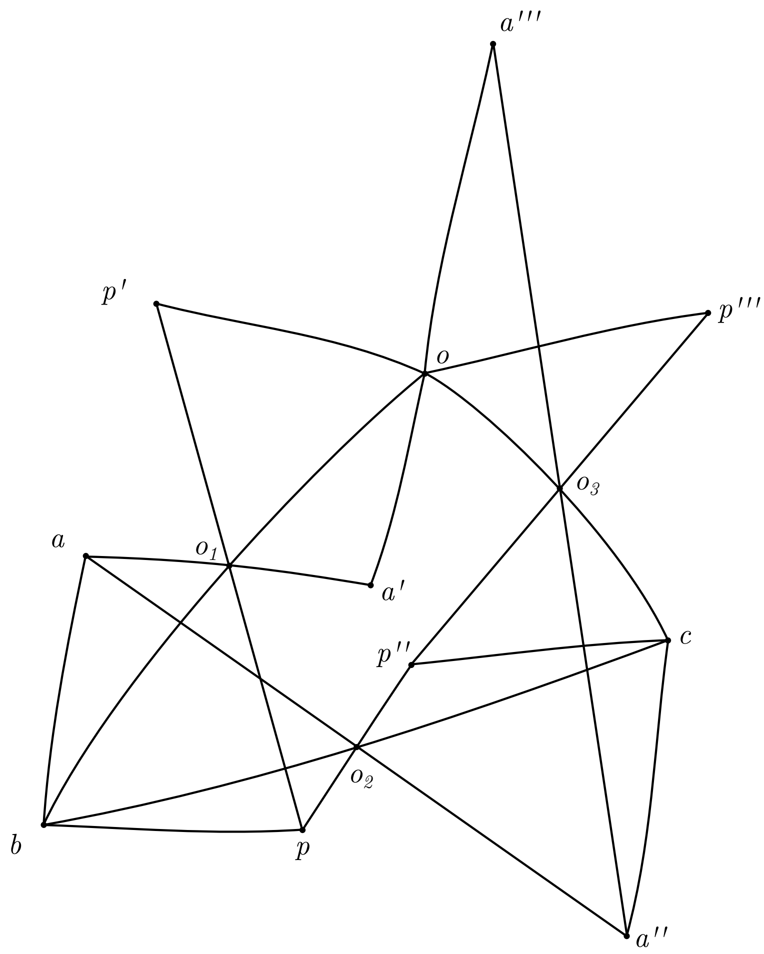

FIGURE 11

In order to prove the following lemma, we introduce the following procedure applied on the triangle $\triangle(a,b,c)$. The first step is using Property 8 of domains $\mathcal{C}_{k',k}$ as follows:

$$\begin{aligned}
|\cos(\sphericalangle^0(a,b,c)) - \cos(\sphericalangle(a,b,c))| &\leq |\sphericalangle^0(a,b,c) - \sphericalangle(a,b,c)| \\
&\leq 4\mu(k',k)\,\mathrm{Area}(\triangle^0(a,b,c)) \\
&= \frac{4\mu\, d(a,b)\, d(b,c)\sin(\sphericalangle^0(a,b,c))}{2} \\
&\leq 2\mu\, d(a,b)\, d(b,c) \leq 2\mu\, h^2.
\end{aligned}$$

For the second step, we use the previous step and the law of cosines:

$$(d(a,c))^2 = l^2 + t^2 - 2lt\cos(\sphericalangle^0(a,b,c)) = l^2 + t^2 - 2lt\cos(\sphericalangle(a,b,c)) + O(h^4).$$

We call this the **CP procedure** and say that we apply (CP) on the triangle $\triangle(a,b,c)$.

LEMMA 3.11. *If diam(B) is sufficiently small in such a way that the whole construction lies on a normal ball, then*

$$|\cos(\sphericalangle(a,b,p)) + \cos(\sphericalangle(p',o,a'''))| \leq C(k',k,\delta)\,\mathit{Area}(\triangle(o,b,c)).$$

PROOF. We introduce the following notation:

$$\begin{gathered}
l = d(b,o),\ t = d(b,c),\ h = \max\{l, t, d(o,c)\}, \\
\alpha = \sphericalangle(a,b,o),\ \beta = \sphericalangle(c,b,p)\ \gamma = \sphericalangle(a,b,p), \\
\theta = \sphericalangle(o,b,p),\ \nu = \sphericalangle(a,b,c),\ \varepsilon = \sphericalangle(o,b,c).
\end{gathered}$$

We apply (CP) on the triangle $\triangle(o,b,c)$ and get

$$(d(o,c))^2 = l^2 + t^2 - 2lt\cos(\sphericalangle^0(o,b,c)) = l^2 + t^2 - 2lt\cos(\sphericalangle(o,b,c)) + O(h^4).$$

By (CP) on $\triangle(a,b,o_2)$, we get

$$\begin{aligned}
(d(a,o_2))^2 &= (d(a,b))^2 + (d(b,o_2))^2 - 2d(a,b)\, d(b,o_2)\cos(\sphericalangle^0(a,b,o_2)) \\
&= l^2 + \left(\frac{t}{2}\right)^2 - lt\cos(\nu) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(a,b,o)$, we get

$$\begin{aligned}
(d(a,o))^2 &= (d(a,b))^2 + (d(b,o))^2 - 2d(a,b)\, d(b,o)\cos(\sphericalangle^0(a,b,o)) \\
&= 2l^2 - 2l^2\cos(\alpha) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(o,b,o_2)$, we get

$$\begin{aligned}
d(o,o_2)^2 &= l^2 + \left(\frac{t}{2}\right)^2 - lt\cos(\sphericalangle^0(o,b,c)) \\
&= l^2 + \left(\frac{t}{2}\right)^2 - lt\cos(\varepsilon) + O(h^4).
\end{aligned}$$

Using Lemma 3.3 applied to $\triangle(o,a,a'')$ and the midpoint o_2 of the segment $[a,a'']$, we obtain

$$\begin{aligned}(d(o,a''))^2 &= 2(d(a,o_2))^2+2(d(o,o_2))^2-(d(o,a))^2+O(h^4)\\ &= 2l^2+\frac{t^2}{2}-2lt\cos(\nu)+2l^2+\frac{t^2}{2}-2lt\cos(\varepsilon)\\ &\quad -2l^2+2l^2\cos(\alpha)+O(h^4)\\ &= 2l^2+t^2-2lt\cos(\nu)-2lt\cos(\varepsilon)+2l^2\cos(\alpha)+O(h^4).\end{aligned}$$

By Lemma 3.1,

$$(d(a'',c))^2=(d(a,b))^2+O(h^4)=l^2+O(h^4).$$

Using Lemma 3.3 applied to $\triangle(a'',o,c)$ and the midpoint o_3 of the segment $[o,c]$, we obtain

$$\begin{aligned}(d(a'',o_3)) &= \frac{(d(o,a''))^2}{2}+\frac{(d(a'',c))^2}{2}-\frac{(d(o,c))^2}{4}+O(h^4)\\ &= l^2+\frac{t^2}{2}-lt\cos(\nu)-lt\cos(\varepsilon)+l^2\cos(\alpha)\\ &\quad +\frac{l^2}{2}-\frac{l^2}{4}-\frac{t^2}{4}+\frac{lt\cos(\varepsilon)}{2}+O(h^4)\\ &= l^2+\frac{l^2}{4}+\frac{t^2}{4}-lt\cos(\nu)-\frac{lt\cos(\varepsilon)}{2}+l^2\cos(\alpha)+O(h^4).\end{aligned}$$

By (CP) on $\triangle(o,b,p)$ and $\triangle(b,p,o_2)$, we get

$$\begin{aligned}(d(o,p))^2 &= l^2+t^2-2lt\cos(\sphericalangle^0(o,b,p))\\ &= l^2+t^2-2lt\cos(\theta)+O(h^4)\end{aligned}$$

and

$$\begin{aligned}(d(o_2,p))^2 &= \left(\frac{t}{2}\right)^2+t^2-t^2\cos(\sphericalangle^0(c,b,p))\\ &= \frac{t^2}{4}+t^2-t^2\cos(\beta)+O(h^4).\end{aligned}$$

Using Lemma 3.3 applied to $\triangle(o,p,p'')$ and the midpoint o_2 of the segment $[p,p'']$, we obtain

$$\begin{aligned}(d(o,p''))^2 &= 2(d(o,o_2))^2+2(d(p,o_2))^2-(d(o,p))^2+O(h^4)\\ &= 2\left(l^2+\frac{t^2}{4}-lt\cos(\varepsilon)\right)+2\left(\frac{t^2}{4}+t^2-t^2\cos(\beta)\right)\\ &\quad -(l^2+t^2-2lt\cos(\theta))+O(h^4)\\ &= l^2+2t^2-2lt\cos(\varepsilon)-2t^2\cos(\beta)+2lt\cos(\theta)+O(h^4).\end{aligned}$$

By Lemma 3.1,

$$(d(p'',c))^2 = (d(b,p))^2 + O(h^4) = t^2 + O(h^4).$$

Using Lemma 3.3 applied to $\triangle(p'',c,o)$ and the midpoint o_3 of the segment $[c,o]$, we obtain

$$\begin{aligned}
(d(p'',o_3))^2 &= \frac{(d(p'',c))^2}{2} + \frac{(d(p'',o))^2}{2} - \frac{(d(o,c))^2}{4} + O(h^4)\\
&= \frac{t^2}{2} + \frac{l^2}{2} + t^2 - lt\cos(\varepsilon) - t^2\cos(\beta)\\
&\quad + lt\cos(\theta) - \frac{l^2 + t^2 - 2lt\cos(\varepsilon)}{4} + O(h^4)\\
&= t^2 + \frac{t^2 + l^2}{4} - \frac{lt\cos(\varepsilon)}{2} - t^2\cos(\beta) + lt\cos(\theta) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(a,b,p)$, we get

$$(d(a,p))^2 = l^2 + t^2 - 2lt\cos(\sphericalangle^0(a,b,p)) = l^2 + t^2 - 2lt\cos(\gamma) + O(h^4).$$

Using Lemma 3.3 applied to $\triangle(p,a,a'')$ and the midpoint o_2 of the segment $[a,a'']$, we obtain

$$\begin{aligned}
(d(p,a''))^2 &= 2(d(a,o_2))^2 + 2(d(p,o_2))^2 - (d(p,a))^2 + O(h^4)\\
&= 2\left(l^2 + \left(\frac{t}{2}\right)^2 - lt\cos(\nu)\right) + 2\left(\frac{t^2}{4} + t^2 - t^2\cos(\beta)\right)\\
&\quad - \left(l^2 + t^2 - 2lt\cos(\gamma)\right) + O(h^4)\\
&= l^2 + 2t^2 - 2lt\cos(\nu) - 2t^2\cos(\beta) + 2lt\cos(\gamma) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(o_1,b,o_2)$ and $\triangle(a,b,o_2)$, we get

$$(d(o_1,o_2))^2 = \frac{l^2}{4} + \frac{t^2}{4} - \frac{lt}{2}\cos(\sphericalangle^0(o,b,c)) = \frac{l^2}{4} + \frac{t^2}{4} - \frac{lt}{2}\cos(\varepsilon) + O(h^4)$$

and

$$(d(a,o_1))^2 = l^2 + \frac{l^2}{4} - l^2\cos(\sphericalangle^0(a,b,o)) = l^2 + \frac{l^2}{4} - l^2\cos(\alpha) + O(h^4).$$

Using Lemma 3.3 applied to $\triangle(o_1, a, a'')$ and the midpoint o_2 of the segment $[a, a'']$, we obtain

$$\begin{aligned}
(d(o_1, a''))^2 &= 2(d(o_1, o_2))^2 + 2(d(a, o_2))^2 - (d(o_1, a))^2 + O(h^4) \\
&= 2\left(\frac{l^2}{4} + \frac{t^2}{4} - \frac{lt}{2}\cos(\varepsilon)\right) + 2\left(l^2 + \left(\frac{t}{2}\right)^2 - lt\cos(\nu)\right) \\
&\quad - \left(l^2 + \frac{l^2}{4} - l^2\cos(\alpha)\right) \\
&= l^2 + \frac{l^2}{4} + t^2 - lt\cos(\varepsilon) - 2lt\cos(\nu) + l^2\cos(\alpha) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(o_1, b, p)$, we get

$$(d(p, o_1))^2 = \frac{l^2}{4} + t^2 - lt\cos(\sphericalangle^0(o, b, p)) = \frac{l^2}{4} + t^2 - lt\cos(\theta) + O(h^4).$$

Using Lemma 3.3 applied to $\triangle(a'', p', p)$ and the midpoint o_1 of the segment $[p', p]$, we obtain

$$\begin{aligned}
(d(p', a''))^2 &= 2(d(a'', o_1))^2 + 2(d(p', o_1))^2 - (d(a'', p))^2 + O(h^4) \\
&= 2\left(l^2 + \frac{l^2}{4} + t^2 - lt\cos(\varepsilon) - 2lt\cos(\nu) + l^2\cos(\alpha)\right) \\
&\quad +2\left(\frac{l^2}{4} + t^2 - lt\cos(\theta)\right) \\
&\quad - \left(l^2 + 2t^2 - 2lt\cos(\nu) - 2t^2\cos(\beta) + 2lt\cos(\gamma)\right) \\
&= 2l^2 + 2t^2 - 2lt\cos(\varepsilon) + 2l^2\cos(\alpha) - 2lt\cos(\theta) \\
&\quad +2t^2\cos(\beta) - 2lt\cos(\gamma) - 2lt\cos(\nu) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(o_2, c, o_3)$ and $\triangle(o, b, c)$, we get

$$\begin{aligned}
(d(o_2, o_3))^2 &= \frac{(d(o,c))^2 + t^2}{4} - \frac{d(o,c)\,t}{2}\cos(\sphericalangle^0(o_2, c, o_3)) \\
&= \frac{(d(o,c))^2 + t^2}{4} - \frac{d(o,c)t}{2}\cos(\sphericalangle(o, c, b)) + O(h^4)
\end{aligned}$$

and

$$\begin{aligned}
l^2 &= (d(o,c))^2 + t^2 - 2d(o,c)\,t\cos(\sphericalangle^0(o, c, b)) \\
&= (d(o,c))^2 + t^2 - 2d(o,c)\,t\cos(\sphericalangle(o, c, b)) + O(h^4).
\end{aligned}$$

These two equations imply that $(d(o_2, o_3))^2 = l^2/4 + O(h^4)$.

Using Lemma 3.3 applied to $\triangle(o_3, p, p'')$ and the midpoint o_2 of the segment $[p, p'']$, we obtain

$$\begin{aligned}
(d(p, o_3))^2 &= 2(d(o_2, o_3))^2 + 2(d(o_2, p))^2 - (d(o_3, p''))^2 + O(h^4) \\
&= 2\left(\frac{l^2}{4}\right) + 2\left(\frac{t^2}{4} + t^2 - t^2\cos(\beta)\right) \\
&\quad - \left(t^2 + \frac{t^2 + l^2}{4} - \frac{lt\cos(\varepsilon)}{2} - t^2\cos(\beta) + lt\cos(\theta)\right) \\
&= \frac{l^2 + t^2}{4} + t^2 - t^2\cos(\beta) + \frac{tl}{2}\cos(\varepsilon) - lt\cos(\theta) + O(h^4).
\end{aligned}$$

By (CP) on $\triangle(o_1, o, o_3)$ and $\triangle(o, b, c)$, we get

$$\begin{aligned}
(d(o_1, o_3))^2 &= \frac{(d(o, c))^2 + l^2}{4} - \frac{d(o, c)\, l}{2}\cos(\sphericalangle^0(o_1, o, o_3)) \\
&= \frac{(d(o, c))^2 + l^2}{4} - \frac{d(o, c)l}{2}\cos(\sphericalangle(b, o, c)) + O(h^4)
\end{aligned}$$

and

$$\begin{aligned}
t^2 &= (d(o, c))^2 + l^2 - 2d(o, c)\, l\cos(\sphericalangle^0(c, o, b)) \\
&= (d(o, c))^2 + l^2 - 2d(o, c)\, l\cos(\sphericalangle(c, o, b)) + O(h^4).
\end{aligned}$$

These two equations imply that $(d(o_1, o_3))^2 = t^2/4 + O(h^4)$.

Using Lemma 3.3 on $\triangle(o_3, p, p')$ and the midpoint o_1 of the segment $[p, p']$, we obtain

$$\begin{aligned}
(d(p', o_3))^2 &= 2(d(o_1, o_3))^2 + 2(d(p, o_1))^2 - (d(o_3, p))^2 + O(h^4) \\
&= \frac{t^2}{2} + 2\left(\frac{l^2}{4} + t^2 - lt\cos(\theta)\right) \\
&\quad - \left(\frac{l^2 + t^2}{4} + t^2 - t^2\cos(\beta) + \frac{tl}{2}\cos(\varepsilon) - lt\cos(\theta)\right) + O(h^4) \\
&= t^2 + \frac{t^2 + l^2}{4} - lt\cos(\theta) + t^2\cos(\beta) - \frac{lt}{2}\cos(\varepsilon) + O(h^4).
\end{aligned}$$

Using Lemma 3.3 on $\triangle(p', a'', a''')$ and the midpoint o_3 of the segment $[a'', a''']$, we obtain

$$\begin{aligned}
(d(p', a'''))^2 &= 2(d(p', o_3))^2 + 2(d(a'', o_3))^2 - (d(p', a''))^2 + O(h^4) \\
&= 2\left(t^2 + \frac{t^2 + l^2}{4} - lt\cos(\theta) + t^2\cos(\beta) - \frac{lt}{2}\cos(\varepsilon)\right) \\
&\quad +2\left(l^2 + \frac{l^2}{4} + \frac{t^2}{4} - lt\cos(\nu) - \frac{lt\cos(\varepsilon)}{2} + l^2\cos(\alpha)\right) \\
&\quad - \big(2l^2 + 2t^2 - 2lt\cos(\varepsilon) + 2l^2\cos(\alpha) - 2lt\cos(\theta) \\
&\quad +2t^2\cos(\beta) - 2lt\cos(\gamma) - 2lt\cos(\nu)\big) + O(h^4) \\
&= t^2 + l^2 + 2lt\cos(\gamma) + O(h^4). \qquad (3.2.1)
\end{aligned}$$

By Lemma 3.1,

$$(d(p', o))^2 = (d(b, p))^2 + O(h^4) = t^2 + O(h^4)$$

and

$$(d(a''', o))^2 = (d(a'', c))^2 + O(h^4) = (d(a, b))^2 + O(h^4) = t^2 + O(h^4).$$

Since the triangle $\triangle(o, b, c)$ is δ-regular, we have that

$$|d(p', o) - t|, |d(a''', o) - l| \leq C(k', k, \delta)\, h^3.$$

Applying (CP) on $\triangle(o, p', a''')$ we get

$$\begin{aligned}
(d(p', a'''))^2 &= (d(o, a'''))^2 + (d(o, p'))^2 - 2d(o, a''')\, d(o, p')\cos(\sphericalangle(p', o, a''')) + O(h^4) \\
&= l^2 + t^2 - 2d(o, a''')\, d(o, p')\cos(\sphericalangle(p', o, a''')) + O(h^4).
\end{aligned}$$

Thus, with equation (3.2.1) we obtain

$$2ly\cos(\gamma) + 2d(o, a''')\, d(o, p')\cos(\sphericalangle(p', o, a''')) = O(h^4).$$

And finally,

$$|\cos(\gamma) + \cos(\sphericalangle(p', o, a'''))| \leq C'(k', k, \delta)\, h^2 \leq C''(k', k, \delta)\,\mathrm{Area}(\triangle^0(o, b, c)).$$

Q. E. D.

COROLARY 3.12. *For a δ-regular triangle $T = \triangle(o, b, c) \subset B$, the increment $\Delta\xi$ of a vector $\xi \in T_o\mathcal{M}$ under parallel translation along T, i.e., the difference of the vectors in $T_b\mathcal{M}$ obtained under parallel displacement along the segment $[o, b]$ and along the polygonal line $[o, c] \cup [c, b]$, satisfies the following estimate:*

$$|\Delta\xi| \leq C(k', k)\,\mathrm{Area}(\triangle^0(o, b, c))\,|\xi|.$$

Proof. Let $\xi, \eta \in T_o\mathcal{M}$ be non-zero vectors. Let the directions of the vectors ξ and η be given by the geodesic segments $[o, a']$ and $[o, p']$, respectively, as in Lemma 3.11. Let P_{ob}, P_{bc}, P_{co}, P_{cb} and P_{oc} be the parallel translation along the geodesic segments $[o, b]$, $[b, c]$, $[c, o]$, $[c, b]$ and $[o, c]$, respectively. Let $h = \operatorname{diam}(T)$. By Lemma 3.2,

$$\cos(\sphericalangle(\xi, \eta)) - \cos(\sphericalangle(-P_{ob}(\xi), -P_{ob}(\eta))) = O(h^2).$$

Since the directions of $[b, a]$ and $[b, p]$ are given by $P_{ob}(\xi)$ and $\Phi_{ob}(\eta)$, Lemma 3.11 implies that

$$\cos(\sphericalangle(-P_{ob}(\xi), -P_{ob}(\eta))) - \cos(\sphericalangle(-P_{co} \circ P_{bc} \circ P_{ob}(\xi), \eta)) = O(h^2).$$

Therefore,

$$\cos(\sphericalangle(\xi, \eta)) - \cos(\sphericalangle(-P_{co} \circ P_{bc} \circ P_{ob}(\xi), \eta)) = O(h^2).$$

By Lemma 3.6 and the last estimate, we obtain

$$\sphericalangle(\xi, -P_{co} \circ P_{bc} \circ P_{ob}(\xi)) = O(h^2).$$

By the last estimate and Proposition 3.10 we get

$$\sphericalangle(P_{cb} \circ P_{oc}(\xi), -P_{ob}(\xi) = O(h^2).$$

Since the triangle $\triangle(b, o, c)$ is δ-regular,

$$h^2 \leq \tilde{C}(\delta) \operatorname{Area}(\triangle^0(b, o, c)).$$

From this and the previous estimate, the proof is complete. Q. E. D.

2.1. Partition into Regular Triangles. Let $T = \triangle(o, b, c)$ be an arbitrary triangle in B and fix $m \in \mathbb{N}$. We partition evenly the geodesic segment $[b, c]$ into 2^m parts by the points

$$b = b_0,\ b_1,\ b_2,\ \ldots b_{2^m-1},\ b_{2^m} = c.$$

and consider the triangles $T_i = \triangle(o, b_i, b_{i+1})$, $i = 0, 1, \ldots, 2^m - 1$ as in Figure 12.

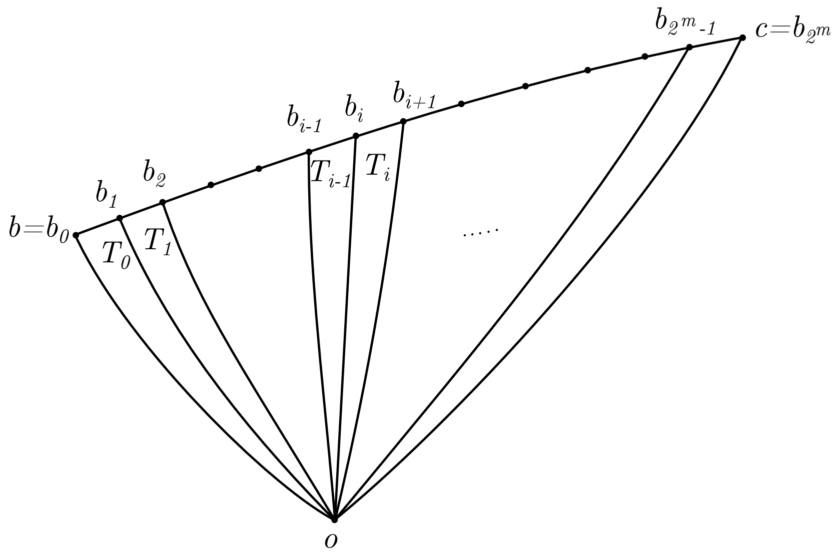

FIGURE 12

For each i, we take T_i. Suppose that $d(o,b_i) \leq d(o,b_{i+1})$. Let $b'_{i+1} \in [o,b_{i+1}]$ such that $d(o,b_i) = d(o,b'_{i+1})$ and consider the triangle $T' = \triangle(o,b_i,b_{i+1})$. We construct the following sequences of points $p_{i,l} \in [o,b_i]$ and $p_{i+1,l} \in [o,b_{i+1}]$ as follows. First we set $p_{i,0} = b_i$ and $p_{i+1,0} = b'_{i+1}$, then suppose that the points $p_{i,j} \in [o,b_i]$ and $p_{i+1,j} \in [o,b_{i+1}]$, $j = 1,\ldots,l$, have already been constructed. Finally, the points $p_{i,l+1}$ and $p_{i+1,l+1}$ are defined by

$$p_{i,l+1} \in [o,b_i],\ p_{i+1,l+1} \in [o,b_{i+1}] \text{ and } d(p_{i,l+1},p_{i,l}) = d(p_{i,l},p_{i+1,l}) = d(p_{i+1,l},p_{i+1,l+1}).$$

It is clear that

$$\lim_{l\to\infty} d(o,p_{i,l}) = \lim_{l\to\infty} d(p_{i,l},p_{i+1,l}) = 0.$$

Let $T_{i,l} = \triangle(p_{i,l},p_{i+1,l},p_{i+1,l+1})$ and $T'_{i,l} = \triangle(p_{i,l},p_{i,l+1},p_{i+1,l+1})$. We illustrate this construction in Figure 13.

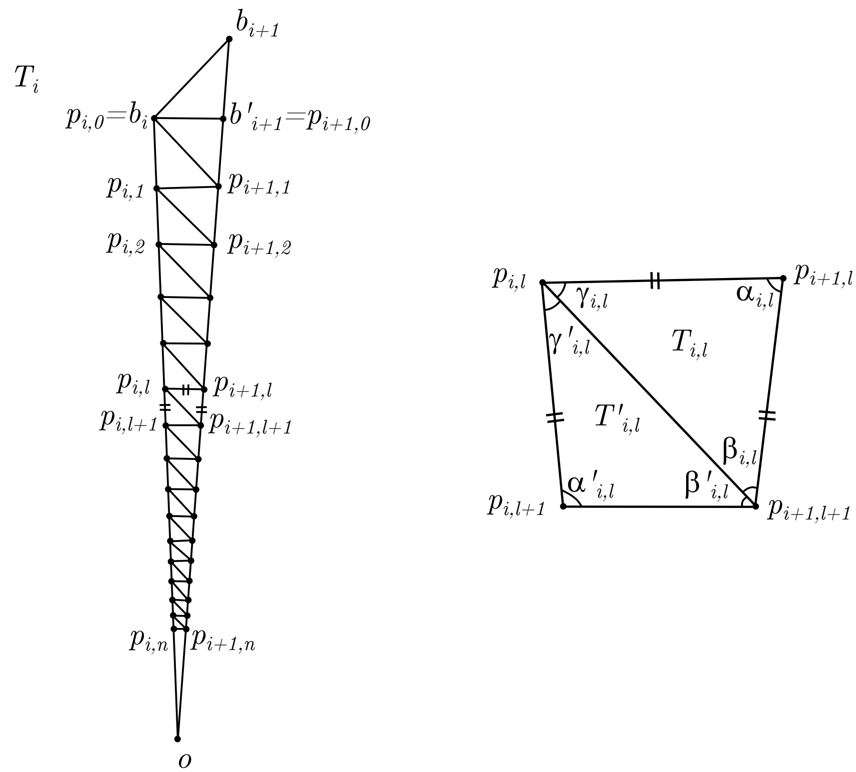

FIGURE 13

Let n be a natural number such that $d(o, p_{i,n}) \leq d(b_i, b_{i+1})$. We consider the triangles $T_{i,j}$ and $T'_{i,j}$ for $j = 0, 1, \ldots, n$. Let $\delta = \pi/6$. We will prove that these triangles are δ-regular if m is sufficiently large.

Let

$$\alpha_{i,l} = \sphericalangle(p_{i,l}, p_{i+1,l}, p_{i+1,l+1}),\ \beta_{i,l} = \sphericalangle(p_{i+1,l}, p_{i+1,l+1}, p_{i,l}),\ \gamma_{i,l} = \sphericalangle(p_{i+1,l+1}, p_{i,l}, p_{i+1,l})$$

and

$$\alpha'_{i,l} = \sphericalangle(p_{i,l}, p_{i,l+1}, p_{i+1,l+1}),\ \beta'_{i,l} = \sphericalangle(p_{i,l+1}, p_{i+1,l+1}, p_{i,l}),\ \gamma'_{i,l} = \sphericalangle(p_{i,l+1}, p_{i,l}, p_{i+1,l+1}).$$

We observe that the angle $\sphericalangle(b_i, o, b_{i+1})$ can be made as small as we wish for m large. Since the triangle $\triangle(p_{i,l}, o, p_{i+1,l})$ is isosceles and by Property 8 of domains $\mathcal{C}_{k',k}$, $\alpha_{i,l}$ is close to $\pi/2$ for m sufficiently large. Also, since the triangle $T_{i,l}$ is isosceles and Property 8 of domains $\mathcal{C}_{k',k}$ the angles $\beta_{i,l}$ and $\gamma_{i,l}$ are close to $\pi/4$ for m large. Thus, the triangles $T_{i,l}$ are at least $\pi/6$-regular.

On the other side, the limit of the angle $\sphericalangle(o, p_{i,l+1}, p_{i+1,l+1})$ as m goes to infinity is $\pi/2$. Since $\alpha'_{i,l} = \pi - \sphericalangle(o, p_{i,l+1}, p_{i+1,l+1})$, then $\alpha'_{i,l}$ is close to $\pi/2$ if m is sufficiently large. We notice too that

$$\lim_{m\to\infty} \sphericalangle(p_{i,l+1}, p_{i,l}, p_{i+1,l}) = \frac{\pi}{2}$$

and, by the triangle inequality,

$$|\gamma'_{i,l} - \gamma_{i,l}| \le \sphericalangle(p_{i,l+1}, p_{i,l}, p_{i+1,l}),$$

whence for sufficiently large m,

$$\frac{\pi}{6} \le \gamma'_{i,l} \le \frac{5\pi}{6}.$$

Analogously, we obtain that

$$\frac{\pi}{6} \le \beta'_{i,l} \le \frac{5\pi}{6}$$

and $T'_{i,l}$ is δ-regular.

Thus, we partitioned the original triangle into $\pi/6$-regular triangles $T_{i,l}$ and $T'_{i,l}$, and the triangles $T''_i = \triangle(b_i, b_{i+1}, b'_{i+1})$ and $T'''_i = \triangle(o, p_{i,n}, p_{i+1,n})$, $i = 0, 1, 2, \ldots, 2^m - 1$, $l = 0, 1, 2, \ldots, n$. Clearly, for m sufficiently large m,

$$\operatorname{diam}(T''_i), \operatorname{diam}(T'''_i) \le \frac{d(b,c)}{2^m}. \tag{3.2.2}$$

2.2. Additivity. For points $a, b, c, d, \ldots$ in B we denote by $P_{a,b,c,d,\ldots}$ the parallel translation along the polygonal line $[a,b] \cup [b,c] \cup [c,d] \cup \ldots$.

Let $\xi \in T_o\mathcal{M}$ be an arbitrary vector. We set $\xi_o = \xi$ and define the vectors $\xi_i \in T_o\mathcal{M}$, for $i = 1, 2, \ldots, 2^m$, as follows:

$$\xi_i = P_{o,b_{i-1},b_i,o}(\xi_{i-1}).$$

It is immediate that

$$\xi_{2^m} = P_{o,b,c,o}.$$

Then the increment

$$\Delta\xi = P_{o,b,c,o}(\xi) - \xi$$

of the vector ξ under the parallel translation along $[o,b] \cup [b,c] \cup [c,o]$ is given by

$$\Delta\xi = \sum_{j=0}^{2^m-1} \Delta_j,$$

where

$$\Delta_j = \xi_{j+1} - \xi_j.$$

The next step is to represent this increment as the sum of parallel translations along the triangles $T_{i,l}$, $T'_{i,l}$, T''_i and T'''_i, $i = 0, 1, 2, \ldots, 2^m - 1$, $l = 0, 1, 2, \ldots, n$. In order to do this, we set

$$\begin{aligned}
\xi_{i,0} &= \xi_i, \\
\xi_{i,1} &= P_{\mathcal{L}_{i,1}}(\xi_{i,0}) \text{ with } \mathcal{L}_{i,1} = [o, p_{i,n}] \cup [p_{i,n}, p_{i+1,n}] \cup [p_{i+1,n}, o], \\
\xi'_{i,1} &= P_{\mathcal{L}'_{i,1}}(\xi_{i,1}) \text{ with } \mathcal{L}'_{i,1} = [o, p_{i+1,n}], \\
\xi_{i,2} &= P_{\mathcal{L}_{i,2}}(\xi'_{i,1}) \text{ with } \mathcal{L}_{i,2} = [p_{i+1,n}, p_{i,n}] \cup [p_{i,n}, p_{i,n-1}] \cup \\
&\qquad\qquad\qquad\qquad \cup [p_{i,n-1}, p_{i+1,n-1}] \cup [p_{i+1,n-1}, p_{i+1,n}], \\
\xi'_{i,2} &= P_{\mathcal{L}'_{i,2}}(\xi_{i,2}) \text{ with } \mathcal{L}'_{i,2} = [p_{i+1,n}, p_{i+1,n-1}].
\end{aligned}$$

We have illustrated the configuration of the curves in Figure 14. Suppose that, for $0 \leq j \leq l$, the vectors $\xi_{i,j} \in T_{p_{i+1,n-j+2}}\mathcal{M}$ and $\xi'_{i,j} \in T_{p_{i+1,n-j+1}}\mathcal{M}$ have been constructed with $l \leq n$. Then we define the vectors $\xi_{i,l+1} \in T_{p_{i+1,n-l+1}}\mathcal{M}$ and $\xi'_{i,l+1} \in T_{p_{i+1,n-l}}\mathcal{M}$ by

$$\xi_{i,l+1} = P_{\mathcal{L}_{i,l+1}}(\xi'_{i,l})$$

with

$$\mathcal{L}_{i,l+1} = [p_{i+1,n-l+1}, p_{i,n-l+1}] \cup [p_{i,n-l+1}, p_{i,n-l}] \cup [p_{i,n-l}, p_{i+1,n-l}] \cup [p_{i+1,n-l}, p_{i+1,n-l+1}]$$

illustrated in Figure 14 and

$$\xi'_{i,l+1} = P_{\mathcal{L}'_{i,l+1}}(\xi_{i,l+1}) \text{ with } \mathcal{L}'_{i,l+1} = [p_{i+1,n-l+1}, p_{i+1,n-l}].$$

It remains to consider the translation of the vector $\xi'_{i,n+1}$ along the triangle T''_i:

$$\xi_{i,n+2} = \xi'_{i,n+2} = P_{\mathcal{L}_{i,n+1}}(\xi_{i,n+1}) \text{ with } \mathcal{L}_{i,n+1} = [p_{i+1,0}, p_{i,0}] \cup [p_{i,0}, b_{i+1}] \cup [b_{i+1}, p_{i+1,0}],$$

illustrated also in Figure 14. Finally, we also set

$$\xi''_{i,0} = \xi_{i,0} = \xi_i, \ \xi''_{i,1} = \xi_{i,1}, \ \xi''_{i,l} = P_{\mathcal{L}''_{i,l}}(\xi_{i,l})$$

with

$$\mathcal{L}''_{i,l} = [p_{i+1,n-l+2}, o], \ l = 2, 3, 4 \ldots, n+1,$$

and

$$\xi''_{n+2} = P_{\mathcal{L}''_{i,n+2}}(\xi_{i,n+2}) \text{ with } \mathcal{L}''_{i,n+2} = [b'_{i+1}, o].$$

We can represent Δ_i in the following form:

$$\Delta_i = \sum_{l=0}^{n} \Delta_{i,l},$$

where $\Delta_{i,l} = \xi''_{i,l+1} - \xi''_{i,l}$. Thus,

$$\Delta\xi = \sum_{i=0}^{2^m-1} \sum_{l=0}^{n+1} \Delta_{i,l}. \tag{3.2.3}$$

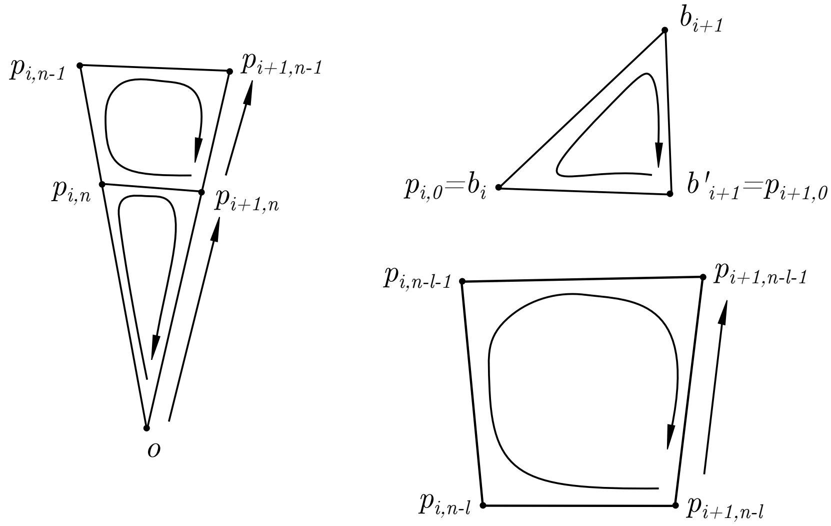

FIGURE 14

2.3. Estimates.

LEMMA 3.13.

$$|\Delta_{i,0}|, |\Delta_{i,n+1}| \le C(k', k)\frac{(d(b,c))^2}{4^m}. \tag{3.2.4}$$

PROOF. We know that the angle $\alpha_{i,0}$ is arbritrarily close to $\pi/2$ for large m. Since

$$\sphericalangle(b_i, b'_{i+1}, b_{i+1}) = \pi - \alpha_{i,0},$$

the angle $\sphericalangle(b_i, b'_{i+1}, b_{i+1})$ is arbitrarily close to $\pi/2$ for m large.

Thus, for m large,

$$\frac{\pi}{6} \le \sphericalangle(b_{i+1}, b'_{i+1}, b_i) \le \frac{5\pi}{6}.$$

There are two cases: $\sphericalangle(b'_{i+1}, b_i, b_{i+1}) \ge \pi/6$ or $\sphericalangle(b'_{i+1}, b_i, b_{i+1}) < \pi/6$. First, suppose that $\sphericalangle(b'_{i+1}, b_i, b_{i+1}) < \pi/6$. Let $q = p_{i+1,1}$. We already know that for m large the angles $\beta_{i,0}$ and $\gamma_{i,0}$ are close to $\pi/4$. Thus,

$$\begin{aligned}\frac{\pi}{12} - \varepsilon_m &= \frac{\pi}{4} - \frac{\pi}{6} - \varepsilon_m \\ &\leq \gamma_{i,0} - \sphericalangle(b'_{i+1}, b_i, b_{i+1}) \\ &\leq \sphericalangle(q, b_i, b_{i+1}) \\ &\leq \gamma_{i,0} + \sphericalangle(b'_{i+1}, b_i, b_{i+1}) \\ &\leq \frac{\pi}{4} + \frac{\pi}{6} + \varepsilon_m \\ &= \frac{5\pi}{12} + \varepsilon_m.\end{aligned}$$

For m sufficiently large we can make $\varepsilon_m = \pi/24$ and

$$\frac{\pi}{24} \leq \sphericalangle(q, b_i, b_{i+1}) \leq \frac{23\pi}{24}.$$

Therefore, the triangles $\triangle(q, b'_{i+1}, b_i)$ and $\triangle(q, b_i, b_{i+1})$ are $\pi/24$-regular. From the fact that $\sphericalangle(b'_{i+1}, b_i, b_{i+1}) < \pi/6$, we obtain that

$$\sphericalangle(b_i, b_{i+1}, b'_{i+1}) \geq \pi - \left(\frac{\pi}{2} + \frac{\pi}{6}\right) - \varepsilon_m = \frac{\pi}{3} - \frac{\pi}{24} = \frac{7\pi}{24} \leq \frac{\pi}{24}$$

for sufficiently large m.

Let $\zeta_{i,n} = P_{b'_{i+1}, b_i, b_{i+1}, b'_{i+1}}(\xi_{i,n+1})$ and observe that $\xi_{i,n+2}$ can be written as

$$\xi_{i,n+2} = P_{b'_{i+1}, b_i, q, b'_{i+1}} \circ P_{b'_{i+1}, q, b_i, b_{i+1}, b'_{i+1}}(\xi_{i,n+1}).$$

Since

$$|\xi_{i,n+2} - \xi'_{i,n+1}| \leq |\xi_{i,n+2} - \zeta_{i,n}| + |\zeta_{i,n} - \xi'_{i,n+1}|,$$

applying Corollary 3.12 and Proposition 3.10, we obtain the estimate (3.2.4) for $|\Delta_{i,n+1}|$.

Now we suppose that $\sphericalangle(b'_{i+1}, b_i, b_{i+1}) \geq \pi/6$. Using Property 8 of Domains $\mathcal{C}_{k',k}$ we get

$$\sphericalangle(b_i, b_{i+1}, b'_{i+1}) \leq \pi - (\sphericalangle(b_{i+1}, b_i, b'_{i+1}) + \sphericalangle(b_i, b'_{i+1}, b_{i+1})) + \lambda_m,$$

where λ_m is a positive constant that tends to zero as m grows. Thus,

$$\sphericalangle(b_i, b_{i+1}, b'_{i+1}) \leq \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

We extend the geodesic segment $[b_i, b'_{i+1}]$ to the geodesic segment $[b_i, x]$ such that $d(b_i, x) = d(b_{i+1}, b'_{i+1})$. Then, for sufficiently large m, the angles $\sphericalangle(b'_{i+1}, x, b_{i+1})$, $\sphericalangle(b'_{i+1}, b_{i+1}, x)$ and $\sphericalangle(q, b'_{i+1}, b_{i+1})$ are arbitrarily close to $\pi/4$, $\pi/4$ and $\pi/2$, respectively. By the triangle inequality,

$$\sphericalangle(b_i, b_{i+1}, b'_{i+1}) - \sphericalangle(b'_{i+1}, b_{i+1}, x) \leq \sphericalangle(b_i, b_{i+1}, x) \leq \sphericalangle(b_i, b_{i+1}, b'_{i+1}) + \sphericalangle(b'_{i+1}, b_{i+1}, q).$$

Thus, for m large,

$$\frac{\pi}{24} = \frac{\pi}{3} - \frac{\pi}{4} - \frac{\pi}{24} \leq \sphericalangle(b_i, b_{i+1}, x) \leq \frac{\pi}{3} + \frac{\pi}{4} + \frac{\pi}{24} = \frac{5\pi}{8} \leq \frac{23\pi}{24}.$$

Therefore, the triangles $\triangle(b_i, x, b_{i+1})$ and $\triangle(b'_{i+1}, x, b_{i+1})$ are $\pi/24$-regular. Similarly to the previous case, this implies (3.2.4) for $|\Delta_{i,n+1}|$.

Similar arguments can be applied to the triangle $\triangle(o, p_{i,n}, p_{i+1,n})$. Assuming that the diameter of this triangle is sufficiently small, extend the geodesic segment $[p_{i,n}, p_{i+1,n}]$ beyond $p_{i+1,n}$ to the geodesic segment $[p_{i,n}, y]$, so that the triangles $\triangle(p_{i,n}, o, y)$ and $\triangle(p_{i+1,n}, o, y)$ are $\pi/6$-regular. In a similar way, we represent the parallel translation of the vector $\xi_{i,0}$ as the parallel traslation along the last two $\pi/6$-regular triangles and apply (3.2.2), Corolary 3.12 and Proposition 3.10. Q. E. D.

THEOREM 3.14. *For a triangle $T = \triangle(o, b, c) \subset B$, the increment $\Delta\xi$ of a vector $\xi \in T_o\mathcal{M}$ under parallel translation along T, i.e., the difference of the vectors in $T_b\mathcal{M}$ obtained under parallel displacement along the segment $[o, b]$ and along the polygonal line $[o, c] \cup [c, b]$, satisfies the following estimate:*

$$|\Delta\xi| \leq C(k', k)\,\mathrm{Area}(\triangle^0(o, b, c))\,|\xi|.$$

PROOF. We have represented the parallel transport along the triangle $\triangle(o, b, c)$ in B as the succesive parallel translations along the triangles $T_{i,l}$, $T'_{i,l}$, T'_i, T''_i and T'''_i. Thus, by Corolary (3.12) applied to the triangles $T_{i,l}$ and $T'_{i,l}$, and (3.2.4) applied to the triangles T''_i and T'''_i combined with Proposition 3.10, we obtain

$$|\Delta\xi| \leq C(k', k)\left[\sum_{i=0}^{2^m-1}\sum_{l=0}^{n-1}\left(\mathrm{Area}(T^0_{i,l}) + \mathrm{Area}(T'^0_{i,l})\right) + \frac{(d(b, c))^2}{2^{m-2}}\right]. \tag{3.2.5}$$

Let $k^+ = \max\{0, k\}$. Let $T^{k^+}_{i,l}$, $T'^{k^+}_{i,l}$, $T''^{k^+}_i$ and $T'''^{k^+}_i$ be comparison triangles in $M^2_{k^+}$ for the triangles $T_{i,l}$, $T'_{i,l}$, T''_i and T'''_i. We glue the comparison triangles in the same order as the original ones. Thus, we obtain a polygon $\mathcal{P}$ in $M^2_{k^+}$. By the angle comparison we know by the fact that B is a CAT(k^+) space, the angles at the vertices $p_{i,l}$ and $p_{i+1,l}$ are greater than π. Thus, by rectifying the corresponding polygonal lines we arrive at the triangle T_i. Since

$$\sum_{i=0}^{2^m-1}\sum_{l=0}^{n-1}\left(\mathrm{Area}(T^0_{i,l}) + \mathrm{Area}(T'^0_{i,l})\right) \leq \sum_{i=0}^{2^m-1}\sum_{l=0}^{n-1}\Big(\mathrm{Area}(T^{k^+}_{i,l}) + \mathrm{Area}(T'^{k^+}_{i,l})\Big) \leq \mathrm{Area}(\mathcal{P}).$$

Alexandrov's lemma yields

$$\sum_{i=0}^{2^m-1}\sum_{l=0}^{n-1}\left(\mathrm{Area}(T^0_{i,l}) + \mathrm{Area}(T'^0_{i,l})\right) \leq \mathrm{Area}(T^{k^+}_i).$$

In a similar way,

$$\sum_{i=0}^{2^m-1} \text{Area}(T_i^{k^+}) \leq \text{Area}(\triangle^{k^+}(o,b,c)).$$

If $\text{diam}(\triangle(o,b,c))$ is sufficiently small, then

$$\text{Area}(\triangle^{k^+}(o,b,c)) \leq 2\text{Area}(\triangle^0(o,b,c))$$

This inequality combined with (3.2.5) completes the proof. Q. E. D.

2.4. Existence. Let $\mathcal{L} \subset B$ be an arbitrary rectifiable curve with endpoints a and b. Partition $\mathcal{L}$ evently into 2^m arcs $\widehat{a_i a_{i+1}}$, $i = 0, 1, 2, \ldots, 2^m - 1$. Let $\mathcal{L}_m$ be the polygonal line $[a, a_1] \cup [a_1, a_2], \cup \cdots \cup [a_{2m-1}, b]$. Denote by P_m the parallel translation along $\mathcal{L}_m$.

We write $P_{\mathcal{L}}(\xi) = \lim_{m\to\infty} P_m(\xi) \in T_b\mathcal{M}$ for each vetor $\xi \in T_a\mathcal{M}$ and say that the map is **the parallel translation along** $\mathcal{L}$. We define the parallel transport along an arbitrary rectifiable curve $\mathcal{L}$ in $(\mathcal{M}, d)$ as the result of succesive parallel translations along small arcs of equal length dividing $\mathcal{L}$.

Proposition 3.15. *The limit* $\lim_{m\to\infty} P_m(\xi)$ *exists for each vector* $\xi \in T_a\mathcal{M}$.

Proof. Let $a_{i+1/2}$ be the midpoint of the $\widehat{a_i a_{i+1}}$. Let $P_{\alpha,\beta}$ denote the parallel translation along the geodesic segment $[a_\alpha, a_\beta]$. We observe that

$$P_m = P_{2^m-1,2^m} \circ P_{2^m-2,2^m-1} \circ \cdots \circ P_{0,1}.$$

Let $\Phi_{i,i+1} = P_{i+1/2,i+1} \circ P_{i,i+1/2}$ and observe that

$$P_{m+1} = \Phi_{2^m-1,2^m} \circ \Phi_{2^m-2,2^m-1} \circ \cdots \circ \Phi_{0,1}.$$

Let

$$h_m = \frac{L(\mathcal{L})}{2^m}.$$

By Theorem 3.14, for each non-zero vector $\xi \in T_{a_i}\mathcal{M}$ we have

$$\sphericalangle(P_{i,i+1}(\xi), \Phi_{i,i+1}(\xi)) \leq C(k', k) h_m^2. \tag{3.2.6}$$

Define the vectors $\xi_m^{(l)}$ and $\zeta_m^{(l)}$ for $l = 0, 1, 2, \ldots, 2^m$ as follows. Set

$$\xi_m^{(0)} = \zeta_m^{(0)} = \xi.$$

Suppose that the vectors $\xi_m^{(l)}$ and $\zeta_m^{(l)}$ have been already defined and set

$$\xi_m^{(l+1)} = P_{l,l+1}(\xi_m^{(l)}) \text{ and } \zeta_m^{(l+1)} = \Phi_{l,l+1}(\zeta_m^{(l)}).$$

We claim that

$$\sphericalangle(\xi_m^{(l)}, \zeta_m^{(l)}) \leq C(k', k) L_l h_m, \tag{3.2.7}$$

where L_l is the length of the arc $\widehat{aa_l}$ of the curve $\mathcal{L}$.

Clearly inequality (3.2.7) holds for $l = 0$. Suppose that (3.2.7) holds for l. Let

$$\bar{\xi}_m^{(l+1)} = \Phi_{l,l+1}(\xi_m^{(l)}).$$

By the triangle inequality,

$$\sphericalangle(\xi_m^{(l+1)}, \zeta_m^{(l+1)}) \leq \sphericalangle(\xi_m^{(l+1)}, \bar{\xi}_m^{(l+1)}) + \sphericalangle(\bar{\xi}_m^{(l+1)}, \zeta_m^{(l+1)}).$$

By inequality (3.2.6),

$$\sphericalangle(\xi_m^{(l+1)}, \bar{\xi}_m^{(l+1)}) \leq C(k', k)h_m^2.$$

Since $\Phi_{l,l+1}$ is an isometry,

$$\sphericalangle(\bar{\xi}_m^{(l+1)}, \zeta_m^{(l+1)}) = \sphericalangle(\xi_m^{(l)}, \zeta_m^{(l)})$$

and, by inequality (3.2.7), for l we have

$$\sphericalangle(\bar{\xi}_m^{(l+1)}, \zeta_m^{(l+1)}) \leq C(k', k)L_l h_m.$$

Thus,

$$\sphericalangle(\xi_m^{(l+1)}, \zeta_m^{(l+1)}) \leq C(k', k)(h_m + L(l))h_m = C(k', k)L(l+1)h_m.$$

Observe that (3.2.7) for $l = 2^m$ yields

$$\sphericalangle(P_m(\xi), P_{m+1}(\xi)) \leq C(k', k)\frac{(L(\mathcal{L}))^2}{2^m}.$$

This last inequality implies that

$$\sphericalangle(P_m(\xi), P_{m+s}(\xi)) \leq C(k', k)\frac{(L(\mathcal{L}))^2}{2^m}, \tag{3.2.8}$$

i.e., $\{P_m(\xi)\}_{m\in\mathbb{N}}$ is a Cauchy sequence and the proof is completed as in Proposition 3.9. Q. E. D.

Corolary 3.16.

$$\sphericalangle(P_m(\xi), P_{\mathcal{L}}(\xi)) \leq C(k', k)\frac{(L(\mathcal{L}))^2}{2^m}.$$

Proof. In the inequality (3.2.8) we take the limit with respecto to s. Q. E. D.

Corolary 3.17. *$P_{\mathcal{L}}\colon T_a\mathcal{M} \to T_b\mathcal{M}$ is an isometry.*

Proof. By Proposition 3.10 P_m is an isometry. Since for each $\xi, \zeta \in T_a\mathcal{M}$,

$$P_m(\xi) \to P_{\mathcal{L}}(\xi) \text{ and } P_m(\zeta) \to P_{\mathcal{L}}(\zeta),$$

$P_{\mathcal{L}}$ is an isometry. Q. E. D.

CHAPTER 4

Spaces with Bounded Curvature: Connection and Curvature

Once we have the parallel transport in our spaces with bounded curvature, in this chapter we can define a covariant derivative and define harmonic coordinates. These coordintes allow us to improve the regularity of the spaces with bounded curvature. At the end of the chapter we compare the synthetic curvature and the sectional curvature in our spaces. This is a work done by Nikolaev in [**Nik83b**] and [**Nik91**].

1. Harmonic Coordinates

In order to define the harmonic coordinates and prove their properties, we need to introduce some concepts beforehand.

Let Ω be an nonempty connected open subset of $\mathbb{R}^n$. We denote by $L_p(\Omega)$, $p \geq 1$, the normed space of all p-integrable functions on Ω with norm

$$\|f\|_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}.$$

We define the **Sobolev space** $W^{l,p}(\Omega)$, $p \geq 1$ and $l \in \{1, 2, \cdots\}$, as the space of functions of $L_p(\Omega)$ that have in Ω all the weak derivatives with respect to x^i, $i \in \{1, \dots, n\}$, up to order l inclusive, and are p-integrable. The norm in this space is

$$\|f\|_{l,p} = \left(\int_\Omega |f|^p + \sum_{|s|=1}^{l} |D^{(s)} f|^p \right)^{\frac{1}{p}},$$

where $s = (s_1, \cdots, s_n)$, $s_i \geq 0$ are natural numbers, $|s| = s_1 + \cdots + s_n$, and

$$D^{(s)} f = \frac{\partial^{|s|} f}{(\partial x^1)^{s_1} \cdots (\partial x^n)^{s_n}}.$$

We denote by $W_o^{1,p}(\Omega)$, $p \geq 1$, the closure in the previous norm of $C_c^\infty(\Omega)$, the set of all smooth functions with compact support in Ω.

We denote by $C^{r,\alpha}(\Omega)$ the class of functions which are r times continuously differentiable in Ω, all of whose r-th partial derivatives satisfy a Hölder condition with exponent α:

$$|D^{(r)} f(x) - D^{(r)} f(y)| \leq C|x - y|^\alpha,$$

where $r \geq 0$ is an integer and $0 < \alpha < 1$.

1.1. Covariant Differentiation. Let S be an arbitrary subset of $\mathcal{M}$, J an interval and $\gamma\colon J \to \mathcal{M}$. A **vector field** X **on** S is a map that associates $p \in S$ with a vector of $T_p\mathcal{M}$ and a **vector field** Y **along** γ is a map that associates $t \in J$ with a vector $T_{\gamma(t)}\mathcal{M}$. We denote the set of vector fields on S by $\mathfrak{X}(S)$ and the set of vector fields along γ by $\mathfrak{X}(\gamma)$. Finally, $\dot{\gamma}$ denotes the vector field of tangent vectors of a differentiable curve γ.

Let $\gamma\colon J \to \mathcal{M}$ be a differentiable curve and $X \in \mathfrak{X}(\gamma)$. We denote by $(X)^t_{t_0}$ the result of applying the parallel transport P to the vector $X(t)$ along γ at the point $\gamma(t_0)$. We call the limit

$$\lim_{t\to t_0} \frac{(X)^t_{t_0} - X(t_0)}{t - t_0},$$

if it exists, **the covariant derivative of the vector field** X **at the point** $\gamma(t_0)$ **along** γ and denote it by $D_tX(t_0)$ or $\nabla_{\gamma}X(t_0)$.

PROPOSITION 4.1. *Let $\gamma\colon J \to \mathcal{M}$ be a differentiable curve, $t_0 \in J$ and $X, Y \in \mathfrak{X}(\gamma)$. The operation of covariant diferentiation D_t has the following properties:*

a) If $D_tX(t_0)$ and $D_tY(t_0)$ exist, then $D_t(X+Y)(t_0)$ also exists and

$$D_t(X+Y)(t_0) = D_tX(t_0) + D_tY(t_0).$$

b) If $\varphi\colon J \to \mathbb{R}$ is differentiable at t_0 and $D_tX(t_0)$ exists, then the covariant derivative $D_t(\varphi X)(t_0)$ also exists and

$$D_t(\varphi X)(t_0) = \varphi(t_0)\, D_tX(t_0) + \frac{\partial\varphi}{\partial t}(t_0)\, X(t_0).$$

c) If $D_tX(t_0)$ and $D_tY(t_0)$ exist, then the function $t \mapsto \langle X(t), Y(t)\rangle$ is differentiable at t_0 and

$$\frac{d}{dt}\, \langle X, Y\rangle\big|_{t_0} = \langle D_tX(t_0), Y(t_0)\rangle + \langle X(t_0), D_tY(t_0)\rangle.$$

PROOF. The proof is exactly the same as in the Riemannian case using the fact that the parallel transport is an isometry. Q. E. D.

Let B be a normal neighborhood sufficiently small such that the parallel transport can be defined there for every geodesic segment. Let $b \in B$ be a fixed point and let X_b denotes the vector field defined as follows: $X_b(q)$, $q \in B \setminus \{b\}$, is the vector in $T_q\mathcal{M}$ whose length is $d(q, b)$ and whose direction coincides with that of the geodesic segment $[q, b]$.

LEMMA 4.2. *Consider in B a triangle $\triangle(a, b, c)$ and a geodesic segment $[p, p']$ that intersects $[a, c]$ in the common middle point o. Set the angles $\sphericalangle(b, c, p) = \gamma$ and $\sphericalangle(p', a, b) = \gamma'$, and let $d(a, c) = h$, $d(c, p) = e(h)$ and $d(b, c), d(b, a) \geq \delta > 0$, where*

$e = e(h)$ denotes an infinitesimal quantity equivalent to h, i.e., a quantity such that $L_1 \le e(h)/h \le L_2$ for certain positive constants L_1 and L_2. Then

$$\cos(\gamma) + \cos(\gamma') = O(h) \cdot C(\delta),$$

with $C(\delta)$ is a constant that depends on δ.

PROOF. We consider a triangle $\triangle(\tilde{a}, \tilde{b}, \tilde{c})$ in the plane such that $d(\tilde{a}, \tilde{c}) = d(a, c)$, $d(\tilde{b}, \tilde{c}) = d(b, c)$ and $\sphericalangle(\tilde{b}, \tilde{c}, \tilde{a}) = \sphericalangle(b, c, a) = \beta$, and take a geodesic segment $[\tilde{p}, \tilde{p}']$ such that it and $[\tilde{a}, \tilde{c}]$ bisect one another at $\tilde{o}$, $d(\tilde{c}, \tilde{p}) = d(c, p)$, $\sphericalangle(\tilde{b}, \tilde{c}, \tilde{p}) = \gamma$ and $\sphericalangle(\tilde{p}, \tilde{c}, \tilde{a}) = \sphericalangle(p, c, a) = \alpha$.

Using the law of cosines we obtain:

$$\begin{aligned}
(d(\tilde{b}, \tilde{p}))^2 &= (d(\tilde{c}, \tilde{p}))^2 + (d(\tilde{b}, \tilde{c}))^2 - 2\, d(\tilde{c}, \tilde{p})\, d(\tilde{b}, \tilde{c})\, \cos(\gamma),\\
(d(\tilde{o}, \tilde{p}))^2 &= e^2 + \left(\frac{h}{2}\right)^2 - ah\cos(\alpha),\\
\cos(\sphericalangle(\tilde{c}, \tilde{a}, \tilde{p})) &= \frac{\frac{h^2}{4} + (d(\tilde{o}, \tilde{p}))^2 - e^2}{h\, d(\tilde{o}, \tilde{p})},\\
d(\tilde{a}, \tilde{p}) &= e,
\end{aligned}$$

$$\begin{aligned}
(d(\tilde{b}, \tilde{o}))^2 &= \left(\frac{h}{2}\right)^2 + (d(\tilde{b}, \tilde{c}))^2 - h\, d(\tilde{b}, \tilde{c})\, \cos(\beta),\\
\cos(\sphericalangle(\tilde{b}, \tilde{p}, \tilde{o})) &= \frac{(d(\tilde{b}, \tilde{p}))^2 + (d(\tilde{p}, \tilde{o}))^2 - (d(\tilde{b}, \tilde{o}))^2}{2d(\tilde{p}, \tilde{b})\, d(\tilde{p}, \tilde{o})},\\
(d(\tilde{p}', \tilde{b}))^2 &= (d(\tilde{p}, \tilde{b}))^2 + (2d(\tilde{p}, \tilde{o}))^2 - 4\, d(\tilde{b}, \tilde{p})\, d(\tilde{p}, \tilde{o})\, \cos 8\sphericalangle(\tilde{b}, \tilde{p}, \tilde{o}),\\
(d(\tilde{a}, \tilde{b}))^2 &= h^2 + e^2 - 2eh\cos(\beta),\\
\cos(\sphericalangle(\tilde{b}, \tilde{a}, \tilde{p}')) &= \frac{(d(\tilde{a}, \tilde{b}))^2 + e^2 - (d(\tilde{b}, \tilde{p}'))^2}{2\, d(\tilde{a}, \tilde{b})e}.
\end{aligned}$$

We set $\varepsilon = \sphericalangle(\tilde{c}, \tilde{a}, \tilde{b})$ and $\theta = \sphericalangle(\tilde{a}, \tilde{b}, \tilde{c})$. Since $\alpha + \gamma + \varepsilon + \theta = \pi$ and using again the law of cosines to find θ, we get that

$$\sphericalangle(\tilde{b}, \tilde{c}, \tilde{p}) + \sphericalangle(\tilde{b}, \tilde{a}, \tilde{p}') = \pi + C(\delta) \cdot O(h).$$

From the previous equations and the Property 8 of domains $\mathcal{C}_{k',k}$ we obtain the result. Q. E. D.

LEMMA 4.3. *Let $\gamma\colon J \to B$ be a differentiable curve and b a fixed point of B such that $d(\gamma(t), b) \ge \delta > 0$ for any $t \in J$ and some δ. Then, for any $t, t_0 \in J$, we get*

$$|(X_b)^t_{t_0} - X_b(t_0)| \le C(\delta)\, L(\gamma; t, t_0),$$

where $L(\gamma; t, t_0)$ is the length of the arc of γ corresponding to the values between t and t_0 and $C(\delta)$ is a constant depending on δ.

PROOF. Suppose that $t_0 \le t$. We put $a = \gamma(t_0)$, $c = \gamma(t)$ and let $\eta \in T_a\mathcal{M}$ be an arbitrary nonzero vector.

By Proposition 3.9 and Lemma 4.2, we obtain

$$\cos(\sphericalangle(X_b(a), \eta)) - \cos((X_b)^t_{t_0}, \eta) \le C'(\delta)\, d(a, c),$$

where $C'(\delta)$ is a constant depending on δ.

From the last inequality and Lemma 3.6, we get

$$\sphericalangle(X_b(a), (X_b)^t_{t_0}) \le C''(\delta)\, d(a, c),$$

where $C''(\delta)$ is a constant depending on δ.

Since $d(a, c) \le L(\gamma; t, t_0)$ and the fact that γ is in an annulus we obtain the result. Q. E. D.

LEMMA 4.4. *The scalar product $\varphi(y) = \langle X_b(y), X_a(y)\rangle$ satisfies a Lipschitz condition in the distance chart (u, B) with the Lipschitz constant depending on δ, where $d(a, B)$, $d(b, B) \ge \delta > 0$ and $V = u(B) \subset \mathbb{R}^n$ is a convex domain.*

PROOF. We have to prove that

$$|\varphi \circ u^{-1}(v) - \varphi \circ u^{-1}(v_0)| \le C(\delta)\, |v - v_0|,$$

where $C(\delta)$ is a constant depending on δ and $v, v_0 \in V$. We can represent $\varphi \circ u^{-1}$ in the form $(\varphi \circ u_y^{-1}) \circ (u_y \circ u^{-1})$ in any sufficiently small neighborhood of $v_y \in V$, where u_y are the coordinates introduced in the proof of Theorem 2.9. Let $x \in \mathbb{R}^n$ be a sufficiently small vector in order to apply u_y^{-1}. We put $z = u_y^{-1}(x)$, and join z and y by the geodesic segment γ_0 in B. We denote by $(X_b)^z_y$ and $(X_a)^z_y$ the vectors resulting from applying the parrallel transport of the vectors $X_b(z)$ and $X_a(z)$ at y along γ_0.

Since parallel translation is an isometry, we obtain that

$$\begin{aligned} \varphi(z) - \varphi(y) &= \langle X_b(z), X_a(z)\rangle - \langle X_b(y), X_a(y)\rangle, \\ &= \langle (X_b)^z_y, (X_a)^z_y\rangle - \langle X_b(y), X_a(y)\rangle - \langle (X_a)^z_y, X_b(y)\rangle + \langle (X_a)^z_y, X_b(y)\rangle, \\ &= \langle (X_a)^z_y, (X_b)^z_y - X_b(y)\rangle + \langle (X_a)^z_y - X_a(y), X_b(y)\rangle. \end{aligned}$$

From the last equality, the Cauchy-Schwarz inequality, and Lemma 4.3, we obtain

$$|\varphi \circ u_y^{-1}(x) - \varphi \circ u_y^{-1}(\bar{0})| \le C'(\delta)\, |x|,$$

where $C'(\delta)$ is a constant depending on δ. From the last inequality and equation (2.3.3) we obtain the desired inequality for $v - v_0$ sufficiently small. For arbitrary vectors $v, v_0 \in V$, the result follows using the previous local estimate, the convexity of V, and the compactness of the segments (the constant can be taken to be the same, taking the maximum on V). Q. E. D.

PROPOSITION 4.5. *The components of the metric tensor in distance coordinates satisfy a Lipschitz condition on a sufficiently small neighborhood $U \subset B$.*

PROOF. We apply the previous lemma to the fields $\{X_i\}$, $i \in \{1, \dots, n\}$, used to construct the distance coordinates in $\mathcal{M}$. Q. E. D.

LEMMA 4.6. *Let $\gamma_1, \gamma_2 \colon J \to B$ be differentiable curves with $\gamma_1(t_0) = \gamma_2(t_0)$ and $\dot\gamma_1(t_0) = \dot\gamma_2(t_0)$, $t_0 \in J$. If $\nabla_{\gamma_1} X_i(t_0)$ exists, then $\nabla_{\gamma_2} X_i(t_0)$ exists and*

$$\nabla_{\gamma_1} X_i(t_0) = \nabla_{\gamma_2} X_i(t_0), \quad i \in \{1, \dots, n\}.$$

PROOF. Let γ_1 and γ_2 be rectifiable curves parametrized by arc length issuing from the point $p = \gamma_1(0) = \gamma_2(0)$ and let $\alpha(x)$ be the angle in the plane triangle with sides x, x and $d(\gamma_1(x), \gamma_2(x))$ opposite to the last side.

Using the law of cosines and the series expansion of cosine, we get that $d(\gamma_1(x), \gamma_2(x))$ has order $\alpha(x) \cdot x$. We denote by γ_3 the geodesic segment joining the points $a = \gamma_1(x)$ and $b = \gamma_2(x)$. The vector $(X_i)^b_a$ is the result of the parallel translation of $X_i(b)$ along γ_3 from b to a. By Lemma 4.3,

$$(X_i)^b_a - X_i(a) = O(\alpha(x) \cdot x).$$

By Theorem 3.14,

$$((X_i)^b_a)^a_p - (X_i)^b_p = O(x^2),$$

where $((X_i)^b_a)^a_p$ is the result of applying the parallel transport of $(X_i)^b_a$ along γ_1 from a to p and $(X_i)^b_p$ is the result of applying the parallel transport of $X_i(b)$ along γ_2 from b to p. Since the parallel transport is an isometry,

$$((X_i)^b_a)^a_p - (X_i)^a_p = O(\alpha(x) \cdot x),$$

where $(X_i)^a_p$ and $((X_i)^b_a)^a_p$ are the result of parallel translation of the vectors $X_i(a)$ and $(X_i)^b_a$ along γ_1 at p. If we take the difference of the last two equations we get

$$|(X_i)^a_p - (X_i)^b_p| = \big|\big[(X_i)^a_p - X_i(p)\big] - \big[(X_i)^b_p - X_i(p)\big]\big| \le C \cdot \alpha(x) \cdot x, \tag{4.1.1}$$

where C is a constant.

If $\dot\gamma_1(t_0) = \dot\gamma_2(t_0)$, then $\alpha(x)$ and x tend to zero. Thus we obtained the result. Q. E. D.

Let $Q \subset u(B)$ be a non empty bounded connected open set. We say that $X \in \mathfrak{X}(u^{-1}(Q))$ **belongs to** $\mathfrak{X}'(Q)$ if the coordinates of X with respect to the basis U_i, $i \in \{1, \dots, n\}$, of coordinate vectors in the distance coordinates have total differential almost everywhere in Q. Similarly, we say that $X \in \mathfrak{X}(\gamma)$ **belongs to** $\mathfrak{X}'(\gamma)$ if the coordinates of X with respect to the basis X_i, $i \in \{1, \dots, n\}$, of coordinate vectors in the distance coordinates have differential almost everywhere in J.

Let $b \in B$, let $Q(\eta) \subset \mathbb{R}^n$ be a non empty convex connected open set such that $d(b, V_\eta) \ge \eta > 0$, where $V_\eta = u^{-1}(Q(\eta))$, and Q is an arbitrary non empty convex connected open set of $u(B)$. From Lemma 4.4 and Proposition 4.5 we obtain the following result,

LEMMA 4.7. *For every $\eta > 0$, $X_b \in \mathfrak{X}'(Q(\eta))$ and all the first partial derivatives of the coordinates α_b^i of the vectors X_b with respect to the basis $\{U_i\}$ are bounded by a constant depending on on η. Moreover, $X_j \in \mathfrak{X}'(Q)$ and all the first partial derivatives of the coordinates α_j^i of the vectors X_j with the basis $\{U_i\}$ are bounded by a constant.*

LEMMA 4.8. *For any differentiable curve $\gamma\colon J \to V = u^{-1}(Q)$ and any $X \in \mathfrak{X}'(\gamma)$, $\nabla_\gamma X$ exists for almost all $t \in J$.*

PROOF. Let $t_0 \in J$. We consider $f\colon J \to \mathbb{R}^n$ given by $f(t) = (X)_{t_0}^t$, i.e., it is equal to the result of displacing $X(t)$ at $\gamma(t_0)$ along γ. If $X = X_i$, then, by Lemma 4.3 and Rademacher's theorem (see Theorem 2.14, page 47, [**AFP00**]), f is differentiable almost everywhere. Since $\nabla_\gamma X$ coincides with $df/\,dt$, then the lemma is proved for $X = X_i$. Since the coordinates β_j^i of the basis vectors U_i of the distance coordinates satisfy a Lipschitz condition, by Lemma 4.4 and equation (2.3.3), the lemma is true for U_i, by a) and b) of Proposition 4.1. The assertion of the lemma follows again from a) and b) of Proposition 4.1 and the definition of $\mathfrak{X}'(\gamma)$. Q. E. D.

1.2. Lie Bracket. Let X and Y be vector fields on B whose coordinates Φ^i and Ψ^j, with $i, j = 1, \ldots, n$, in distance coordinates satisfy a Lipschitz condition with constant C. A curve $\gamma\colon J \to B$ is called **an integral curve of a vector field** $X \in \mathfrak{X}(B)$ if it is differentiable and $\dot\gamma(t) = X(\gamma(t))$.

Everywhere in this section mappings of **domains** (i.e., non empty connected open sets) of B into domains of b will be assumed to be described in distance coordinates and denoted by the same symbols, i.e, if f maps a domain of B into B, we shall denote the map $u \circ f \circ u^{-1}$of the corresponding domains of $\mathbb{R}^n$ by f and its coordinates by f^i, $i = 1, \ldots, n$.

From the definition of the integral curve γ of X that passes through x, we observe that γ is a solution of the equation

$$\gamma^i = x^i + \int_0^t \Phi^i(\gamma(s))\, ds. \tag{4.1.2}$$

Since the Φ^i satisfy a Lipschitz condition, a solution of equation (4.1.2) always exists and is unique, so we denote the integral curve of X passing through x by $\varphi_t(x)$, where t lies in a small neighborhood J of zero. As a consequence of the uniqueness of the solution, we obtain

$$\varphi_s \circ \varphi_t(x) = \varphi_{s+t}(x) \tag{4.1.3}$$

for small s and t. Thus, for small t, the φ_t are homeomorphisms (since $\varphi_0(x) = x$) and we have

$$\varphi_t^{-1}(x) = \varphi_{-t}(x). \tag{4.1.4}$$

In this way equation (4.1.2) becomes

$$\varphi_t^i(x) = x^i + \int_0^t \Phi^i(\varphi_s(x))\, ds. \tag{4.1.5}$$

In a similar way, $\psi_t(x)$ will denote an integral curve of Y and

$$\psi_t^i(x) = x^i + \int_0^t \Psi^i(\psi_s(x))\, ds.$$

LEMMA 4.9. *The maps φ_t satisfy a Lipschitz condition with a constant independent of t for sufficiently small t.*

PROOF. From equation (4.1.5) we obtain

$$\varphi_t^i(x) - \varphi_t^i(x_0) = x^i - x_0^i + \int_0^t \left(\Phi^i(\varphi_s(x)) - \Phi^i(\varphi_s(x_0))\right)\, ds.$$

If $|t| < 1/2\sqrt{n}\,C$, then

$$\begin{aligned} |\varphi_t^i(x) - \varphi_t^i(x_0)| &\leq |x^i - x_0^i| + C|t| \sup_s |\varphi_s(x) - \varphi_s(x_0)| \\ &\leq |x^i - x_0^i| + C|t|\,\sqrt{n} \sup_{s;i=1,\dots,n} |\varphi_s^i(x) - \varphi_s^i(x_0)|. \end{aligned}$$

From this inequality, we get

$$\left(1 - C\,|t|\,\sqrt{n}\right) \sup_{s;i=1,\dots,n} |\varphi_s^i(x) - \varphi_s^i(x_0)| \leq |x^i - x_0^i| \leq \max_{i\in\{1,\dots,n\}} |x^i - x_0^i| \leq |x - x_0|.$$

Thus,

$$\frac{1}{2}|\varphi_s^i(x) - \varphi_s^i(x_0)| \leq \frac{1}{2} \sup_{s;i=1,\dots,n} |\varphi_s^i(x) - \varphi_s^i(x_0)| \leq |x - x_0|. \tag{4.1.6}$$

Q. E. D.

The map $\varphi\colon Q \subset \mathbb{R}^n \to \mathbb{R}^n$ has the **N-property (N^{-1}-property)** if the image (resp. inverse image) of a set of measure zero is a set of measure zero.

LEMMA 4.10. *The maps φ_t have the N- and N^{-1}-properties for suffieciently small t. The degree of φ_t is equal to 1.*

PROOF. From equations (4.1.4) and (4.1.6) we know that φ_t and φ_t^{-1} satisfy a Lipshitz condition for small t and φ_t satisfies the N- and N^{-1}-properties. Since φ_t is homotopic to the identity for small t, then the degree of φ_t is 1. Q. E. D.

LEMMA 4.11. *If $f\colon V \to \mathbb{R}^n$, $V = u(B)$, satisfies a Lipschitz condition, then for almost all t in a sufficiently small neighborhood of zero $J \subset \mathbb{R}$, the function f has a total differential at points $\varphi_t(x)$ for almost all $x \in V$.*

Proof. By Rademacher's theorem, the map f has total differential almost everywhere on V. Then the set $O' \subset V$ of points at which f is not differentiable is of zero measure. The map φ_t is Lipschitz and, therefore, the set O''_t of non-differentiability is of zero measure. Since φ_t has the N^{-1}-property, for each $t \in [-\delta, \delta]$ the set $O'''_t = \varphi_t^{-1}(O')$ is also of zero measure. Thus, the function $g_t = f \circ \varphi_t$ is differentiable at every point of $V \setminus O_t$, where $O_t = O''_t \cup O'''_t$.

Let $A = \{(t,x) \mid t \in [-\delta,\delta],\, x \in O_t\}$ and $\chi_A(t,x)$ be the characteristic function of the set A. This function is integrable with respect to x and, for each $t \in [-\delta, \delta]$,

$$\mathrm{Measure}(O_t) = \int_V \chi_A(t,x)\,dx = 0.$$

By the Fubini–Tonelli theorem

$$0 = \int_{-\delta}^{\delta} dt \int_V \chi_A(t,x)\,dx = \iint_{[-\delta,\delta]\times V} \chi_A(t,x)\,dt\,dx = \int_V dx \int_{-\delta}^{\delta} \chi_A(t,x)\,dt.$$

Since $\chi_A(t,x) \geq 0$, for almost all $x \in V$, we have

$$\chi_A(t,x) = 0.$$

Q. E. D.

Lemma 4.12. *For sufficiently small t and almost all $x \in V = u(B)$, $\frac{\partial \varphi_t^i}{\partial x^j}(x)$ exists and satisfies the equation*

$$\frac{\partial \varphi_t^i}{\partial x^j}(x) = \delta_j^i + \int_0^t \frac{\partial \Phi^i}{\partial u^m}(\varphi_s(x)) \cdot \frac{\partial \varphi_s^m}{\partial x^j}(x)\,ds, \tag{4.1.7}$$

where δ_j^i is the Kronecker symbol and $i, j = 1, \ldots, n$.

Proof. From equation (4.1.6) and Rademacher's theorem, it follows that, for sufficiently small t and almost all $x \in V$, $\frac{\partial \varphi_t^i}{\partial x^j}(x)$ exists and is also a generalized derivative (i.e. the partial derivative exists for almost every x). Let $g\colon V \to \mathbb{R}$ be an arbitrary function of class $\mathcal{C}^1$ with compact support in V whose integral over V is equal to 1. From equation (4.1.5), Fubini–Tonelli theorem and the definition of a generalized derivative, we obtain

$$\int_V \frac{\partial \varphi_t^i}{\partial x^j}(x) \cdot g(x)\,dx = \delta_j^i - \int_0^t \int_V \Phi^i(\varphi_s(x)) \cdot \frac{\partial g}{\partial x^j}(x)\,dx\,ds. \tag{4.1.8}$$

By Lemma 4.11, for almost all $s \in J$ and almost all $x \in V$, we have

$$\frac{\partial}{\partial x^j}\left(\Phi^i(\varphi_s(x))\right) = \frac{\partial \Phi^i}{\partial u^m}(\varphi_s(x)) \cdot \frac{\partial \varphi_s^m}{\partial x^j}(x).$$

Hence, from equations (4.1.5) and (4.1.8), the Fubini-Tonelli theorem, and the definition of a generalized derivative, we obtain

$$\int_V \frac{\partial \varphi_t^i}{\partial x^j}(x) \cdot g(x)\, dx = \delta_j^i + \int_V g(x) \int_0^t \frac{\partial \Phi^i}{\partial u^m}(\varphi_s(x)) \cdot \frac{\partial \varphi_s^m}{\partial x^j}(x)\, ds\, dx,$$

and hence, since g is arbitrary, we get (4.1.7). Q. E. D.

COROLARY 4.13. *For almost all $x \in V$, the $\frac{\partial \varphi_t^i}{\partial x^j}(x)$ and $\frac{\partial (\varphi_t^{-1})^i}{\partial x^j}(x)$ satisfy a Lipschitz condition in a small neighborhood of zero with a constant independent of x.*

Let V_1, V_2 and V_0 be balls in $\mathbb{R}^n$ concentric with the ball V (which can be assumed to be a ball in $\mathbb{R}^n$), whose radii are equal to half, a quarter, and an eight of the radius of V, respectively. If t and s are sufficiently small, we always have

$$\varphi_s \circ \psi_t(V_2), \psi_s \circ \varphi_t(V_2) \subset V_1 \text{ and } \varphi_s \circ \psi_t(V), \psi_s \circ \varphi_t(V) \supset V_1. \tag{4.1.9}$$

We shall assume that J, a small neighborhood of 0, is sufficiently small such that conditions (4.1.9) and Lemmas 4.9–4.12 are satisfied for all $s, t \in J$.

We denote by ω an infinitely differentiable function in $\mathbb{R}^n$ whose integral over $\mathbb{R}^n$ is equal to 1 and whose support is contained in a ball centered at zero and radius ρ smaller than the radius of V_0. We observe that, by (4.1.9),

$$\left\{v \in \mathbb{R}^n \mid |x - (\psi_s \circ \varphi_t)^{-1}(v)| \le \rho,\, |x - (\varphi_t \circ \psi_s)^{-1}(v)| \le \rho,\, x \in V_0\right\} \subset V_1$$
$$\subsetneq \varphi_t \circ \psi_s(V), \psi_s \circ \varphi_t(V), V,$$

for $s, t \in J$.

We will denote by $J(x, f)$ the **Jacobian** of f at the point $x \in V$, where $f\colon V \to \mathbb{R}^n$ is a once differentiable map.

LEMMA 4.14. *For all $x \in V_0$, $t, h \in J$ the function*

$$f(t) = \int_0^h \int_V \omega(x-u)\Phi^l(\varphi_s(\psi_t(u)))\, du\, ds$$

is twice continuously differentiable in J.

PROOF. Using Lemma 4.11 on Φ^l, $\Phi \circ \varphi_s$, and Φ, we get the existence of their total differential. Using Lemmas 4.9 and 4.10 we obtain the existence of

$$\frac{\partial}{\partial t}\left(\Phi_l \circ \varphi_s \circ \psi_t(u)\right)$$

for almost all $s, t \in J$ and $u \in V$. And we have

$$\frac{\partial}{\partial t}\left(\Phi_l \circ \varphi_s \circ \psi_t(u)\right) = \frac{\partial \Phi^l}{\partial u^q}(\varphi_s(\psi_t(u))) \cdot \frac{\partial \varphi_s^q}{\partial u^i}(\psi_t(u)) \cdot \frac{\partial \psi_t^i}{\partial t}(u).$$

From this equation and the fact that $\Phi_l \circ \varphi_s \circ \psi_t(u)$ satisfies a Lipschitz condition in t with a constant independent of u, it follows that

$$\frac{df}{dt}(t)$$

exists for almost all $t \in J$:

$$\frac{df}{dt}(t) = \int_0^h \int_V \omega(x-u) \cdot \frac{\partial \Phi^l}{\partial u^q}(\varphi_s(\psi_t(u))) \cdot \frac{\partial \varphi_s^q}{\partial u^i}(\psi_t(u)) \cdot \Psi^i(\psi^t(u))\, du\, ds.$$

This previous expresion is defined for all t. We make a change of variable $v = \psi_t(u)$ and obtain

$$\frac{df}{dt}(t) = \int_0^h \int_{\psi_t(V)} \omega(x-\psi_t^{-1}(v)) \cdot \frac{\partial \Phi^l}{\partial u^q}(\varphi_s(v)) \cdot \frac{\partial \varphi_s^q}{\partial u^i}(v) \cdot \Psi^i(v) J(v, \psi_t^{-1})\, dv\, ds.$$

Let ε be a small number. We denote by $\mathcal{A}$ the expression analogous to

$$\frac{df}{dt}(t+\varepsilon),$$

except that instead of $J(v, \psi_{t+\varepsilon}^{-1})$ we substitute $J(v, \psi_t^{-1})$. By (4.1.9) and that the fact that ω has compact support, we have:

$$\frac{df}{dt}(t+\varepsilon) = \int_0^h \int_{\mathbb{R}^n} \chi_{\psi_{t+\varepsilon}(V)} \cdot \omega(x-\psi_{t+\varepsilon}^{-1}(v)) \cdot \frac{\partial \Phi^l}{\partial u^q}(\varphi_s(v)) \cdot \frac{\partial \varphi_s^q}{\partial u^i}(v) \cdot \Psi^i(v) J(v, \psi_{t+\varepsilon}^{-1})\, dv\, ds$$

and

$$\mathcal{A} = \int_0^h \int_{\mathbb{R}^n} \chi_{\psi_t(V)} \cdot \omega(x-\psi_{t+\varepsilon}^{-1}(v)) \cdot \frac{\partial \Phi^l}{\partial u^q}(\varphi_s(v)) \cdot \frac{\partial \varphi_s^q}{\partial u^i}(v) \cdot \Psi^i(v) J(v, \psi_t^{-1})\, dv\, ds.$$

The functions under the integral sign are defined to be zero outside their domain of definition and the domain of integration of $\mathcal{A}$ has been changed. Since all the functions under the integral sign of the previous two expressions are bounded in absolute value by a constant M, we get

$$\left| \frac{df}{dt}(t+\varepsilon) - \mathcal{A} \right| \leq M \left| \int_{\mathbb{R}^n} \chi_{\psi_{t+\varepsilon}(V)} \cdot J(v, \psi_{t+\varepsilon}^{-1})\, dv - \int_{\mathbb{R}^n} \chi_{\psi_t(V)} \cdot J(v, \psi_t^{-1}) \right|.$$

The two integrals of the last inequality are equal by the change of variables formula. Thus,

$$\frac{df}{dt}(t+\varepsilon) - \mathcal{A} = 0.$$

Using this equation, we get

$$\begin{aligned}\frac{df}{dt}(t+\varepsilon)-\frac{df}{dt}(t) &= \varepsilon\int_0^h\int_{\psi_t(V)}\frac{\partial\omega}{\partial u^i}(x-\psi_t^{-1}(v))\cdot\frac{\partial\psi_t^{-1}}{\partial t}(v)\cdot\frac{\partial\Phi^l}{\partial u^q}(\varphi_s(v))\\ &\quad\cdot\frac{\partial\varphi_s^q}{\partial u^i}(v)\cdot\Psi^i(v)J(v,\psi_t^{-1})\,dv\,ds+O(\varepsilon).\end{aligned}$$

From the last expression f is twice differentiable and

$$\frac{d^2f}{dt^2}(t)=\int_0^h\int_{\psi(V)}\frac{\partial\omega}{\partial u^i}(x-\psi_t^{-1}(v))\frac{\partial\psi_t^{-1}}{\partial t}(v)\frac{\partial\Phi^l}{\partial u^q}(\varphi_s(v))\frac{\partial\varphi_s^q}{\partial u^i}(v)\Psi^i(v)J(v,\psi_t^{-1})\,dv\,ds. \tag{4.1.10}$$

From Corollary 4.13 we get that this function is continuous in t. Q. E. D.

LEMMA 4.15. *For all $x\in V_0$ and $t\in J$ the function*

$$f(t)=\int_V\omega(x-u)\cdot\Phi^l(\varphi_t(\psi_t(u)))\,du$$

is differentiable with respect to t and its derivative satisfies a Lipshitz condition.

PROOF. As in the previous lemma, we see that

$$\frac{\partial}{\partial t}\left(\Phi_l\circ\varphi_t\circ\psi_t(u)\right)$$

exists for almost $u\in V$ and $t\in J$ and

$$\frac{\partial}{\partial t}\left(\Phi_l\circ\varphi_s\circ\psi_t(u)\right)=\frac{\partial\Phi^l}{\partial u^q}(\varphi_t(\psi_t(u)))\left[\frac{\partial\varphi_t^q}{\partial t}(\psi_t(u))+\frac{\partial\varphi_t^q}{\partial u^i}(\psi_t(u))\cdot\frac{\partial\psi_t^i}{\partial t}(u)\right].$$

In order to proceed as in the previous lemma, we needed to use the fact that $\varphi_t^q(x)$ satifies a Lipschitz condition in $J\times V$. From the last equation we see that

$$\frac{df}{dt}(t)$$

exists for almost all $t\in J$ and

$$\frac{df}{dt}(t)=\int_V\omega(x-u)\cdot\frac{\partial\Phi^l}{\partial u^q}(\varphi_t(\psi_t(u)))\left[\Phi^q(\varphi_t(\psi_t(u)))+\frac{\partial\varphi_t^q}{\partial u^i}(\psi_t(u))\cdot\Psi^i(\psi_t(u))\right]du.$$

We make the change of variable $v=\psi_t(u)$ and obtain

$$\frac{df}{dt}(t)=\int_{\psi_t(V)}\omega(x-\psi_t^{-1}(v))\cdot\frac{\partial\Phi^l}{\partial u^q}(\varphi_t(v))\left[\Phi^q(\varphi_t(v))+\frac{\partial\varphi_t^q}{\partial u^i}(v)\cdot\Psi^i(v)\right]J(v,\psi_t^{-1})\;dv.$$

Let ε be a small number. Let $\mathcal{A}_1$ denote the analogous expression to

$$\frac{df}{dt}(t+\varepsilon),$$

except that for $J(v,\psi_{t+\varepsilon}^{-1})$ we have substituted $J(v,\psi_t^{-1})$. We put

$$\mathcal{A}_2=\int_{\psi_{t+\varepsilon}(V)}\omega(x-\psi_t^{-1}(v))\cdot\frac{\partial\Phi^l}{\partial u^q}(\varphi_{t+\varepsilon}(v))\left[\Phi^q(\varphi_t(v))+\frac{\partial\varphi_t^q}{\partial u^i}(v)\cdot\Psi^i(v)\right]J(v,\psi_t^{-1})\,dv.$$

As in the previous lemma, we see that

$$\mathcal{A}_1=\frac{df}{dt}(t+\varepsilon).$$

Moreover, $|\mathcal{A}_1-\mathcal{A}_2|$ is bounded by a constant depending on the modulus of continuity of ω, φ_t^{-1} and $\Phi^q\circ\varphi_t$, and the maximum values of

$$\left|\frac{\partial(\psi_t^{-1})^j}{\partial v^i}\right|,\ \left|\frac{\partial\Phi^l}{\partial u^i}\right|,$$

$q,i,j,l\in\{1,\dots,n\}$. Thus, $\mathcal{A}_1-\mathcal{A}_2$ is $O(\varepsilon)$. By (4.1.9) and the fact that ω has compact support, the domain of integration of $\mathcal{A}_2$ for $x\in V_0$ can be replaced by $\psi_t(V)$. Since all the functions under consideration are bounded, we have

$$\begin{aligned}\left|\mathcal{A}_2-\frac{df}{dt}(t)\right| &= L\sum_{q=1}^{n}\left|\int_{\psi_t(V)}\omega(x-\psi_t^{-1}(v))\cdot\frac{\partial\Phi^l}{\partial u^q}(\varphi_{t+\varepsilon}(v))\,dv\right.\\ &\qquad\left.-\int_{\psi_t(V)}\omega(x-\psi_t^{-1}(v))\cdot\frac{\partial\Phi^l}{\partial u^q}(\varphi_t(v))\,dv\right|,\end{aligned}$$

where L is a constant. We denote the expression under the summation sign by $H_q(\varepsilon)$. We make a change of variable $z=\varphi_{t+\varepsilon}(v)$ in the first integral and $z'=\varphi_t(v)$ in the second integral in $H_q(\varepsilon)$. Then

$$\begin{aligned}H_q(\varepsilon) &= \left|\int_{\varphi_{t+\varepsilon}\circ\psi_t(V)}\omega(x-\psi_t^{-1}(\varphi_{t+\varepsilon}^{-1}(z)))\cdot\frac{\partial\Phi^l}{\partial u^q}(z)\cdot J(z,\varphi_{t+\varepsilon}^{-1})\,dz\right.\\ &\qquad\left.-\int_{\varphi_t\circ\psi_t(V)}\omega(x-\psi_t^{-1}(\varphi_t^{-1}(z')))\cdot\frac{\partial\Phi^l}{\partial u^q}(z')\cdot J(z',\varphi_t^{-1})\,dz'\right|.\end{aligned}$$

By (4.1.9) and the fact that ω has compact support, the domain of integration in both integrals can be replaced by V if $x\in V_0$, without changing their values. Finally, by Corollary 4.13, $H_q(\varepsilon)$ is $O(|\varepsilon|)$ and this completes the proof using the triangle inequality. Q. E. D.

LEMMA 4.16. *If $x\in V_0$ and $t\in J$, then the function*

$$f(t)=\int_V\omega(x-u)\cdot\Psi^l(\psi_t(u))\,du$$

is differentiable with respect to t and its derivative satisfies a Lipschitz condition.

PROOF. We just apply the previous lemma with $\varphi_t=Id$. Q. E. D.

We put $r(t,x) = \psi_t \circ \varphi_t(x) - \varphi_t \circ \psi_t(x)$. We recall that the Lie bracket in the coordinates u is written as

$$[X,Y] = \left(\Phi^q \frac{\partial \Psi^l}{\partial x^q} - \Psi^q \frac{\partial \Phi^l}{\partial x^q} \right) U_l, \tag{4.1.11}$$

where the U_l are the basis fields of the distance coordinates.

Proposition 4.17. *If the limit of $r(t,x)/t^2$ as t tends to 0 exists for almost all $x \in V_0$ and there is a neighborhood of zero on the line such that, for all t in this neighborhood and $x \in V_0$,*

$$|r(t,x)| \le C\, t^2, \tag{4.1.12}$$

where C is a constant, then, for almost all $x \in V_0$,

$$\lim_{t\to 0} \frac{r(t,x)}{t^2} = [X,Y]_x \,.$$

Proof. We set $h_1(t,u) = \varphi_t^l(\psi_t(u))$ and consider, using the Fubini–Tonelli Theorem,

$$\begin{aligned} h_1^\varepsilon(t,x) &= \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right) u^l\, du + \int_0^t \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right) \Psi^l(\psi_s(u))\, du \\ &\quad + \int_0^t \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right) \Phi^l(\varphi_s(\psi_t(u)))\, du, \end{aligned} \tag{4.1.13}$$

where $\varepsilon > 0$ and ω is an infinitely differentiable radial function whose integral over $\mathbb{R}^n$ is 1 and with support in the ball $B(\bar{0},\varepsilon) \subset \mathbb{R}^n$. This function h_1^ε is infinitely differentiable and is called the **Sobolev ε-average of** h_1 (see Section 1.4, Chapter 1, [**Nik77**]). We observe that

$$\psi_0(x) = \varphi_0(x) = x, \tag{4.1.14}$$

and

$$h_1^\varepsilon(0,x) = \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right) u^l\, du. \tag{4.1.15}$$

Differentiating, we obtain

$$\begin{aligned} \frac{\partial h_1^\varepsilon}{\partial t}(t,x) &= \frac{1}{\varepsilon^n}\int_V \Psi^l(\psi_t(u))\,\omega\left(\frac{x-u}{\varepsilon}\right)\, du \\ &\quad + \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right)\, \Phi^l(\varphi_t(\psi_t(u)))\, du \\ &\quad + \int_0^t \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right)\, \frac{\partial \Phi^l}{\partial u^q}(\varphi_s(\psi_t(u))) \\ &\quad \cdot \frac{\partial \varphi_s^q}{\partial u^i}(\psi_t(u))\Psi^i(\psi_t(u))\, du\, ds. \end{aligned} \tag{4.1.16}$$

In the last term of the previous expression we have used Lemma 4.14, for sufficiently small ε. We observe that equations (4.1.14) and (4.1.16) imply that

$$\frac{\partial h_1^{\varepsilon}}{\partial t}(0,x) = \Psi^{l,\varepsilon}(x) + \Phi^{l,\varepsilon}(x),$$

where $\Psi^{l,\varepsilon}$ and $\Phi^{l,\varepsilon}$ are the Sobolev ε-averages of Ψ^l and Φ^l, respecively. Using equation (4.1.14) and differentiating (4.1.16) with respect to t and evaluating at $t=0$, we get

$$\begin{aligned} \frac{\partial^2 h_1^{\varepsilon}}{\partial t^2}(0,x) &= \frac{1}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right)\left[\frac{\partial \Psi^l}{\partial u^i}(u)\Psi^i(u)\right. \\ &\quad \left. + \frac{\partial \Phi^l}{\partial u^i}(u)\Phi^i(u) + \frac{\partial \Phi^l}{\partial u^i}(u)\Psi^i(u)\right] du. \end{aligned} \tag{4.1.17}$$

In order to do the last calculation we have used Lemmas 4.12, 4.14, 4.15 and 4.16. We set $h_2(t,u) = \psi_t^l \circ \varphi_t(u)$ and let $h_2^{\varepsilon}(t,x)$ be its Sobolev ε-average. For this last function we obtain analogous formulas to (4.1.15), (4.1.16) and (4.1.17). From this, it follows that

$$r^{l,\varepsilon}(0,x) = 0, \qquad \frac{\partial r^{l,\varepsilon}}{\partial t}(0,t) = 0$$

and

$$\frac{\partial^2 r^{l,\varepsilon}}{\partial t^2}(0,x) = \frac{2}{\varepsilon^n}\int_V \omega\left(\frac{x-u}{\varepsilon}\right)\left[\Phi^q(u)\frac{\partial \Psi^l}{\partial x^q}(u) - \Psi^q(u)\frac{\partial \Phi^l}{\partial x^q}(u)\right] du, \tag{4.1.18}$$

where $r^{l,e} = h_2^{l,\varepsilon} - h_1^{l,\varepsilon}$. Let

$$r(x) = \lim_{t\to 0}\frac{r(t,x)}{t^2}.$$

From the approximation properties of the Sobolev ε-average (see Section 1.4, Chapter 1, [**Nik77**]), we get

$$\left\|\frac{r^{l,\varepsilon}(t,x)}{t^2} - r^{\varepsilon}(x)\right\|_{L_2(V_0)} \le \left\|\frac{r^{l,\varepsilon}(t,x)}{t^2} - r(x)\right\|_{L_2(V_0)}, \tag{4.1.19}$$

where $r^{\varepsilon}(x)$ is the Sobolev average of $r(x)$. From equations (4.1.12), (4.1.19) and Lebesgue's Theorem (on taking the limit under the integral sign), it follows that

$$r^{\varepsilon}(x) = \lim_{t\to 0}\frac{r^{l,\varepsilon}(x,t)}{t^2}$$

for almost all $x \in V_0$. From equation (4.1.18) and L'Hôpital's rule, it follows that

$$r^{\varepsilon}(x) = \frac{1}{2}\frac{\partial^2 r^{\varepsilon}}{\partial t^2}(0,x).$$

Making ε tend to zero and using (4.1.11), we obtain the result.

Q. E. D.

1.3. Simmetry of the Connection. We consider $Q \subset u(B) \subset \mathbb{R}^n$ a bounded convex domain and $V = u^{-1}(Q)$, where u are distance coordinates. We shall assume that vector fields of $\mathfrak{X}(Q)$ are defined at points $u^{-1}(x)$ for almost all $x \in Q$. Let $X, Y \in \mathfrak{X}(V)$, $p \in V$ and $X = \varphi^i X_i$. We say that the covariant derivative $\nabla_Y X|_p$ **exists at a point** p if there is a differentiable curve γ such that $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = Y(p)$, and $\nabla_\gamma X_i|_{t_0}$ and $\frac{d}{dt}(\varphi^i \circ \gamma)(t_0)$ exist. We put

$$\nabla_Y X|_p = \nabla_\gamma X|_{t_0}.$$

This expression is well defined thanks to Proposition 4.1 and Lemma 4.6.

In order to prove the symmetry of the connection, we have to prove some lemmas. Also, we introduce the following configuration. Let $x \in Q$, $a = u^{-1}(x)$ and $b, c \in B$ such that $d(b, V), d(c, V) > 0$. We consider the vector fields

$$Y_b(p) = \frac{X_b(p)}{|X_b(p)|} \text{ and } Y_c(p) = \frac{X_c(p)}{|X_c(p)|}.$$

Let φ_t and ψ_t be the integral flows of these vector fields, respectively. The applications $t \mapsto \varphi_t(a)$ and $t \mapsto \psi_t(a)$ are arc-length parametrizations of the geodesic segments joining a with $\varphi_t(a)$ and $\psi_t(a)$, respectively. Finally, we set

$$\begin{aligned} b_1 &= b_1(t) = \varphi_t(a), & b_2 &= b_2(t) = \psi_t(a), \\ c_1 &= c_1(t) = \psi_t(b_1), & c_2 &= c_2(t) = \varphi_t(b_2), \\ \xi_1 &= \xi_1(t) = \exp_a^{-1}(c_1), & \xi_2 &= \xi_2(t) = \exp_a^{-1}(c_2). \end{aligned}$$

We illustrate this configuration in the following figure.

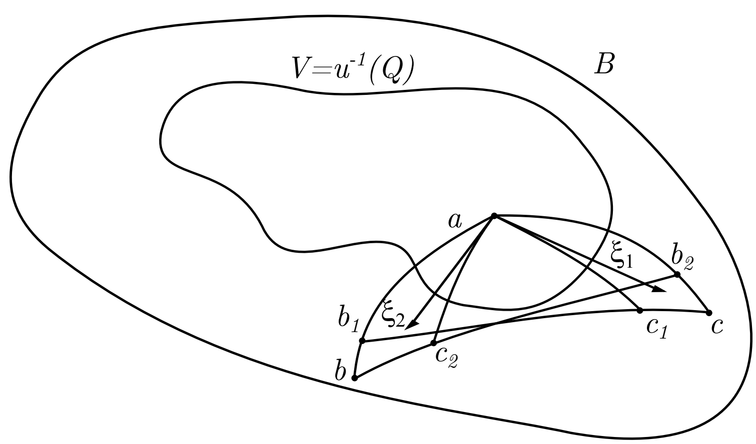

FIGURE 15

LEMMA 4.18. *Let* $\{\xi_1^l(t)\}_{l=1}^n$ *and* $\{\xi_2^l(t)\}_{l=1}^n$ *be the coordinates of the vector fields* $\xi_1(t)$ *and* $\xi_2(t)$ *with respect to the system of distance coordinates. Suppose that, for each* $l = 1, \dots, n$, *the limit*

$$\lim_{t\to 0^+} \frac{\xi_1^l(t) - \xi_2^l(t)}{t^2} \tag{4.1.20}$$

exists. Then the limit

$$\lim_{t\to 0^+} \frac{r^l(t,a)}{t^2}$$

exists and both limits coincide.

PROOF. Let ξ_{c_2} be the system of coordinates at the point c_2 given in the proof of Theorem 2.9, in equation (2.3.2). Let

$$u_1^l = \psi_t^l \circ \varphi_t(a) \text{ and } u_2^l = \varphi_t^l \circ \psi_t(a),$$

$l = 1, \dots, n$, be the distance coordinates of the points c_1 and c_2, respectively. Finally, let η^l, $l = 1, \dots, n$, be the coordinates of the point c_1 with respect to the coordinate chart ξ_{c_2}. From the definition of these coordinates, we have that the coordinates $\xi_1 - \xi_2$ are $\eta = (\eta^1, \dots, \eta^n)$.

Let $J(u \circ \xi_{c_2}^{-1}, 0)$ denote the Jacobi matrix of the map $u \circ \xi_{c_2}^{-1}$ at the point 0. We observe that

$$(\xi_1^1, \dots, \xi_1^n) - (\xi_2^1, \dots, \xi_2^n) = (\eta^1, \dots, \eta^n) \cdot J(u \circ \xi_{c_2}^{-1}, 0).$$

By equation (2.3.3), we get

$$\begin{aligned} (u_1^1, \dots, u_1^n) - (u_2^1, \dots, u_2^n) &= u \circ \xi_{c_2}^{-1}(\eta) - u \circ \xi_{c_2}^{-1}(0), \\ &= (\eta^1, \dots, \eta^n) \cdot J(u \circ \xi_{c_2}^{-1}, 0) + o(|\eta|). \end{aligned}$$

Thus,

$$(\xi_1^1, \dots, \xi_1^n) - (\xi_2^1, \dots, \xi_2^n) = (u_1^1, \dots, u_1^n) - (u_2^1, \dots, u_2^n) + o(|\eta|). \tag{4.1.21}$$

Since the limit (4.1.20) exists,

$$o(|\eta|) = o\left(\sqrt{\sum_{l=1}^n (\xi_1^l - \xi_2^l)^2}\right) = o(t^2).$$

Finally, from this equation and (4.1.21), we obtain that

$$\lim_{t\to 0^+} \frac{r^l(t,a)}{t^2} = \lim_{t\to 0^+} \frac{u_1^l - u_2^l}{t^2} = \lim_{t\to 0^+} \frac{\xi_1^l - \xi_2^l}{t^2}.$$

Q. E. D.

LEMMA 4.19. *With the same notation, let* $(Y_c)_a^{b_1}$ *and* $(Y_b)_a^{b_2}$ *be the result of the parallel translations of the vectors* $Y_c(b_1)$ *and* $Y_b(b_2)$ *along the geodesic segments* $[b_1, a]$ *and* $[b_2, a]$*, respectively. Then*

$$\begin{aligned} \sphericalangle(t \cdot (Y_c)_a^{b_1}, \xi_1 - t \cdot Y_b(a)) &= O(t^2), \\ \sphericalangle(t \cdot (Y_b)_a^{b_2}, \xi_2 - t \cdot Y_c(a)) &= O(t^2). \end{aligned}$$

PROOF. Let $\zeta \in T_a\mathcal{M}$ be a unitary vector, $q = \exp_a(t \cdot \zeta)$, o the midpoint of the geodesic segment $[a, b_1]$ and $q' = \mathcal{S}_o(q)$. We illustrate this configuration in the following figure.

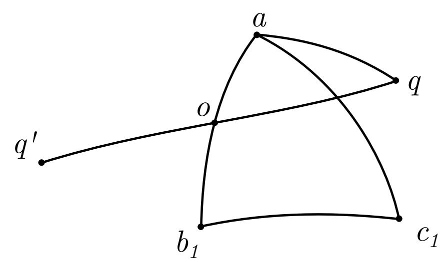

FIGURE 16

We introduce the following notation:

$$\begin{aligned} y &= d(a, c_1), \\ \alpha &= \sphericalangle(b_1, a, c_1) = \sphericalangle(Y_b(a), \xi_1), \\ \beta &= \sphericalangle(q, a, c_1) = \sphericalangle(\xi_1, \zeta), \\ \gamma &= \sphericalangle(q, a, b_1) = \sphericalangle(Y_b(a), \zeta). \end{aligned}$$

Let $\delta = \sphericalangle(\zeta, \xi_1 - t \cdot Y_b(a))$. We observe that

$$\begin{aligned} \langle \xi_1 - t \cdot Y_b(a), \zeta \rangle &= \langle \xi_1, \zeta \rangle - t \langle Y_b(a), \zeta \rangle \\ &= y \cos(\beta) - t \cos(\gamma) \end{aligned}$$

and

$$|\xi_1 - t \cdot Y_b(a)| = \sqrt{t^2 + y^2 - 2ty\cos(\alpha)}.$$

Thus,

$$\cos(\delta) = \frac{y\cos(\beta) - t\cos(\gamma)}{\sqrt{t^2 + y^2 - 2ty\cos(\alpha)}}.$$

Also we observe that $y = O(t)$. We are going to compute $\sphericalangle(q', b_1, c_1)$. We use the CP procedure introduced before Lemma 3.11 on the triangles $\triangle(q, a, c_1)$, $\triangle(q, a, o)$

and $\triangle(a,o,c_1)$ to obtain the following equalities:

$$\begin{aligned}(d(q,c_1))^2 &= t^2+y^2-2ty\cos(\beta)+O(t^4),\\ (d(q,o))^2 &= t^2+\left(\frac{t}{2}\right)^2-t^2\cos(\gamma)+O(t^4),\\ (d(o,c_1))^2 &= \left(\frac{t}{2}\right)^2+y^2-ty\cos(\alpha)+O(t^4).\end{aligned}$$

Using Lemma 3.3 applied to $\triangle(c_1,q,q')$ and the midpoint o of the segment $[q,q']$:

$$(d(q',c_1))^2 = 2t^2+y^2-2ty\cos(\alpha)-2t^2\cos(\gamma)+2ty\cos(\beta)+O(t^4).$$

By Lemma 3.1

$$(d(q',b_1))^2=(d(q,a))^2+O(t^4)=t^2+O(t^4).$$

By (CP) applied on $\triangle(b_1,c_1,a)$:

$$(d(b_1,c_1))^2=t^2+y^2-2ty\cos(\alpha)+O(t^4).$$

Thus,

$$\begin{aligned}(d(q',c_1))^2 &= (y^2+t^2-2ty\cos(\alpha))+t^2-(2t^2\cos(\gamma)-2ty\cos(\beta))+O(t^4),\\ &= (d(b_1,c_1))^2+(d(q',b_1))^2-(2t^2\cos(\gamma)-2ty\cos(\beta)).\end{aligned}$$

By (CP) applied to the triangle $\triangle(q',b_1,c_1)$ and the previous equation:

$$\begin{aligned}\cos(\sphericalangle(q',b_1,c_1)) &= \frac{(d(b_1,c_1))^2+(d(b_1,q'))^2-(d(q',c_1))^2}{2\,d(b_1,c_1)\,d(b_1,q')},\\ &= \frac{-2ty\ \cos(\beta)+2t^2\ \cos(\gamma)+O(t^4)}{2t\cdot d(b_1,c_1)}\\ &= \frac{-y\ \cos(\beta)+t\ \cos(\gamma)}{\sqrt{t^2+y^2-2ty\ \cos(\alpha)}}+O(t^2),\\ &= -\cos(\delta)+O(t^2).\end{aligned} \tag{4.1.22}$$

On the other hand, we know by Lemma 3.2 and (3.1.15) for $m=1$ that

$$\cos(\sphericalangle(q',b_1,c_1))+\cos(\sphericalangle((Y_c)_a^{b_1},\zeta))=O(t^2).$$

Combining this equation with (4.1.22), we have

$$|\cos(\sphericalangle(\zeta,t\cdot(Y_c)_a^{b_1}))-\cos(\sphericalangle(\zeta,\xi_1-t\cdot Y_b(a)))|\le Ct^2.$$

Finally, using Lemma 3.6 we get that

$$\sphericalangle(t\cdot(Y_c)_a^{b_1},\xi_1-t\cdot Y_b(a))=O(t^2).$$

The proof of the second claim is symmetric and totally similar. Q. E. D.

LEMMA 4.20. *For almost all $x \in Q$*

$$\nabla_{X_b} X_c, \quad \nabla_{X_c} X_b \quad \text{and} \quad [X_b, X_c]$$

exist at the point $a = u^{-1}(x)$, where $b, c \in B$, $d(b, V), d(c, V) > 0$. And

$$\nabla_{X_b} X_c - \nabla_{X_c} X_b = [X_b, X_c]. \tag{4.1.23}$$

PROOF. We have from Lemma 4.7 and the definition of the Lie Bracket that $[X_b, X_c]$ exists at $a = u^{-1}(x)$ for almost all $x \in Q$. From Lemmas 4.7 and 4.8 it follows that

$$\nabla_{X_b} X_c \quad \text{and} \quad \nabla_{X_c} X_b$$

exist at $a = u^{-1}(x)$ for almost all $x \in Q$.

We have

$$\begin{aligned} \xi_1 &= tY_b(a) + (\xi_1 - tY_b(a)), \\ \xi_2 &= tY_c(a) + (\xi_2 - tY_c(a)). \end{aligned}$$

Since $|\xi_i(t)| = O(t)$, $i = 1, 2$, Lemma 4.19 implies

$$\begin{aligned} \xi_1(t) &= t \cdot Y_b(a) + t \cdot (Y_c)_a^{b_1} + O(t^3), \\ \xi_2(t) &= t \cdot Y_c(a) + t \cdot (Y_b)_a^{b_2} + O(t^3). \end{aligned}$$

Thus,

$$\frac{\xi_1 - \xi_2}{t^2} = \frac{(Y_c)_a^{b_1} - Y_c(a)}{t} - \frac{(Y_b)_a^{b_2} - Y_b(a)}{t} + O(t^2). \tag{4.1.24}$$

Since $\nabla_{X_b} X_c$ and $\nabla_{X_c} X_b$ exist almost everywhere, the limit of the right-hand side of the last equality exists almost everywhere and

$$\lim_{t \to 0^+} \frac{(Y_c)_a^{b_1} - Y_c(a)}{t} - \frac{(Y_b)_a^{b_2} - Y_b(a)}{t} = \nabla_{X_b} X_c - \nabla_{X_c} X_b.$$

Thus, the limit

$$\lim_{t \to 0^+} \frac{\xi_1^l - \xi_2^l}{t^2}$$

exists almost everywhere. By Lemma 4.18

$$\lim_{t \to 0^+} \frac{\xi_1^l - \xi_2^l}{t^2} = \lim_{t \to 0^+} \frac{r^l(t, a)}{t^2}.$$

By (4.1.24)

$$\left| \frac{\xi_1^l - \xi_2^l}{t^2} \right| \le C,$$

where C is a constant, for sufficiently small positive t. From the proof of Lemma 4.18

$$\left|\frac{r^l(t,a)}{t^2}\right| \leq C.$$

We now apply Proposition 4.17 which implies that

$$\lim_{t\to 0^+} \frac{r^l(t,a)}{t^2} = [Y_b, Y_c]^l_a$$

almost everywhere. Thus, almost everywhere in V

$$\nabla_{Y_b} Y_c - \nabla_{Y_c} Y_b = [Y_b, Y_c].$$

Since $X_b(p) = d(b,p)Y_b(p)$, $X_c(p) = d(c,p)Y_c(p)$, Lemma 4.6, Proposition 4.1 and standard properties of the Lie bracket imply that almost everywhere in Q

$$\nabla_{X_b} X_c - \nabla_{X_c} X_b = [X_b, X_c].$$

Q. E. D.

We denote by $\mathfrak{X}''(Q)$ those fields $X \in \mathfrak{X}'(Q)$ for which $\nabla_X X_l$, $l \in \{1,\dots,n\}$, exist at $u^{-1}(x)$ for almost all $x \in Q$. We observe that $\mathfrak{X}''(Q) \neq \emptyset$, since, by Lemma 4.7 and Lemma 4.20, $X_l, X_b \in \mathfrak{X}''(Q)$ if $d(b,V) > 0$.

Lemma 4.21. *Let $X = \alpha^i X_i \in \mathfrak{X}''(Q)$. Then at points $u^{-1}(x)$ we have*

$$\nabla_X X_l = \alpha^i \nabla_{X_i} X_l, \quad l \in \{1,\dots,n\},$$

for almost all $x \in Q$.

Proof. We consider domains $Q_1 \subset Q_2 \subset u(B) \subset \mathbb{R}^n$ satisfying the following two properties: $Q \subset Q_1 \subset Q_2$ and $Q' = Q_2 \setminus Q_1 \neq \emptyset$. On $Q \times Q'$ we consider the function $\chi(x,y)$ to be equal to zero when

$$\nabla_{X_{u^{-1}(y)}} X_l|_{u^{-1}(x)}, \quad \nabla_{X_l} X_{u^{-1}(y)}|_{u^{-1}(x)} \quad \text{and} \quad \left[X_l, X_{u^{-1}(y)}\right]|_x$$

exist and (4.1.23) is satisfied, and $\chi(x,y) = 1$ othewise, $l \in \{1,\dots,n\}$. By Fubini's theorem, $\chi(x,y) = 0$ for almost all fixed $x \in Q$ and almost $y \in Q'$. In order to reduce the notation, we put $X_{u^{-1}(y)} = X_y$. Then for almost all $x \in Q$ there is a sequence $\{y_s\}_{s\in\mathbb{N}} \subset Q'$ such that $\chi(x,y_s) = 0$ and

$$\lim_{s\to\infty} X_{y_s}(a) = X(a), \quad a = u^{-1}(x). \tag{4.1.25}$$

From (4.1.23), Proposition 4.1 and the usual properties of the Lie bracket we see that

$$\nabla_{X_{y_s}} X_l = \alpha^i_s \nabla_{X_l} X_i + \alpha^i_s [X_l, X_i]$$

for almost all $x \in Q$ at points $u^{-1}(x)$, where α^i_s are the coordinates of X_{y_s} in distance coordinates. Again from (4.1.23) we have

$$\nabla_{X_{y_s}} X_l = \alpha^i_s \nabla_{X_i} X_l \tag{4.1.26}$$

at points $u^{-1}(x)$ for almost all $x \in Q$. From (4.1.1) we obtain

$$\lim_{s\to\infty} \nabla_{X_{y_s}} X_l|_{u^{-1}(x)} = \nabla_X X_l|_{u^{-1}(x)} \tag{4.1.27}$$

for almost all $x \in Q$. The assertion of the lemma follows from (4.1.25), (4.1.26) and (4.1.27). Q. E. D.

THEOREM 4.22. *Let v be a coordinate system in $\mathcal{M}$ such that the transition applications $u \circ v^{-1}$ from v to the distance coordinates u belong to the class $W^{2,q}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is some domain, $q > n$. Then on almost all the i-th coordinate curves of v, $i \in \{1, \dots, n\}$, the parallel displacement P is given by the usual formulas of Riemannian geometry with the help of the metric tensor.*

PROOF. Let V_i be the basis fields of the coordinates v; since

$$V^i = \frac{\partial (u \circ v^{-1})^l}{\partial v^i} U_l \tag{4.1.28}$$

and $u \circ v^{-1} \in W^{2,q}(\Omega)$, by [**VGR79**], Chapter 1, Theorem 3.2, we have that $V_i \in \mathfrak{X}'(Q)$ for some $Q \subset \mathbb{R}^n$. From Lemma 4.8 it follows that $V_i \in \mathfrak{X}''(Q)$. Applying Lemma 4.20 to V_i, we see that

$$0 = T(V_i, V_j) := \nabla_{V_i} V_j - \nabla_{V_j} V_i - [V_i, V_j]$$

almost everywhere, where T is the **torsion tensor** defined by ∇. From this and Proposition 4.1, repeating verbatim the arguments in the case of classical Riemannian geometry, we see that

$$\nabla_{V_i} V_j = \Gamma^l_{ij} V_l,$$

where Γ^l_{ij} are the Christoffel symbols defined almost everywhere, expressed by the usual rules of Riemannian geometry in terms of the components of the metric tensor.

Let H_i be the hyperplane in $\mathbb{R}^n$ passing through a point of Q' and perpendicular to V_i, and let $H_i' = H_i \cap Q'$. Since $u \circ v^{-1} \in W^{2,q}$, using Fubini's theorem, we have

$$\|\Gamma^l_{sj} \circ v^i_x\|_{L_q(I(v^i_x))} < \infty \tag{4.1.29}$$

for almost all $x \in H_i'$, where $I(v^i_x)$ is the closed domain of variation of the parameter of the i-th coordinate curve v^i_x passing through x. We set a vector field $\xi \in \mathfrak{X}(v^i_x)$ whose coordinates are given by the formula

$$\xi^r(t) = \xi^r_0 - \sum_{l=1}^n \int_0^t \Gamma^r_{il}(v^i_x(s)) \xi^l(s)\, ds, \tag{4.1.30}$$

for $r \in \{1, \dots, n\}$. Thanks to (4.1.29), a solution of (4.1.30) under the condition $\xi^r(0) = \xi^r_0$ exists and is unique (see [**Cla15**]).

Now we fix $t_0 \in I(v_x^i)$ and set the map

$$\begin{aligned} f\colon I(v_x^i) &\to T_{v_x^i(t_0)}\mathcal{M}, \\ t &\mapsto P(\xi(t)), \end{aligned} \tag{4.1.31}$$

where $P(\xi(t))$ is the parallel transport applied to the vector $\xi(t)$ at t_0 along v_x^i. Then $df(t)/\,dt$ is equal to the result of applying the parallel displacement to $\nabla_{v_x^i}\xi|_t$ at t_0 along v_x^i. From (4.1.30) and the previous lemmas, we get

$$\begin{aligned} \nabla_{v_x^i}\xi|_t &= \sum_{r=1}^{n} \nabla_{v_x^i}\xi^r V_r \\ &= \sum_{r=1}^{n} \dot{\xi}^r V_r + \xi^r \nabla_{v_x^i} V_r \\ &= \sum_{r,l=1}^{n} \xi^l \cdot \Gamma_{ij}^r(v_x^i) \cdot V_r + \xi^r \nabla_{v_x^i} V_r \\ &= \sum_{r,l=1}^{n} \xi^l \cdot \Gamma_{ij}^r(v_x^i) \cdot V_r + \xi^r \Gamma_{ir}^l(v_x^i) V_l \\ &= 0. \end{aligned}$$

All these equalities are satisfied for almost all $t \in I(v_x^i)$. Hence $df(t)/\,dt = 0$ for almost all $t \in I(v_x^i)$.

It follows from Lemma 4.3 that the map f constructed for X_l in the same way satisfies a Lipschitz condition, and so the same is true for f constructed for U_j. From (4.1.28) and the conditions of Theorem 4.22 it follows that f constructed for V_j is absolutely continuous. From this and (4.1.30) it follows that the original map f is absolutely continuous.

It follows that f is constant, which implies that P is specified on v_x^i by (4.1.30).

Q. E. D.

1.4. Harmonic Coordinates. Let (u, B) be a distance chart and $Q = u(B)$, g_{ij} the components of the metric tensor in distance coordinates and Γ_{ij}^k the Christoffel symbols, $i, j, k \in \{1, \dots, n\}$.

The Laplace operator in distance coordinates has the form

$$\Delta f = \sum_{i,j=1}^{n} \mathsf{g}^{ij} \frac{\partial^2 f}{\partial u^i \partial u^j} - \sum_{i,j,k=1}^{n} \mathsf{g}^{ij}\, \Gamma_{ij}^k \frac{\partial f}{\partial u^k}.$$

We want to find coordinate functions $\xi^l(u^1, \dots, u^n)$ that satisfy the following system of elliptic partial differential equations:

$$\sum_{i,j=1}^{n} \mathsf{g}^{ij} \frac{\partial^2 \xi^l}{\partial u^i \partial u^j} - \sum_{i,j,k=1}^{n} \mathsf{g}^{ij}\, \Gamma^k_{ij} \frac{\partial \xi^l}{\partial u^k} = 0, \tag{4.1.32}$$

$l \in \{1, \dots, n\}$, with the condition that

$$\frac{\partial \xi^l}{\partial u^i}(x_0) = \delta_{ij}, \quad x_0 \in Q, \quad i, l \in \{1, \dots, n\}. \tag{4.1.33}$$

By Proposition 4.5, g^{ij} are Lipschitz continuous and Γ^k_{ij} are bounded almost everywhere, $i, j, k \in \{1, \dots, n\}$. By Theorem 2 of [**BJS79**], Chapter 5, page 230 (the assertion of this theorem remains true when we require not continuity but merely the boundedness of the lowest coefficients of (4.1.32) (in deriving the inequality (5.28) of Lemma C in [**BJS79**], Chapter 5, page 227, we only use the boundeness of the lowest coefficients, and in the remainder of the proof the continuity of the lowest coefficients is not used)), there is always a solution ξ^l on $W^{2,q}(\Omega_0)$, where $\Omega_0 \subset Q$ is a neighborhood of x_0 and $q \geq 1$ is an arbitrary number, of the equation (4.1.32) that satisfies (4.1.33).

Since ξ^l is continuously differenciable (also by Theorem 2 of [**BJS79**], Chapter 5, page 230), in a sufficiently small neighborhood $\Omega \subset \Omega_0$ of x_0 a coordinate system is specified and is called **harmonic coordinates**.

From now on we work on harmonic coordinates $(\xi^1, \dots, \xi^n)$ with $\Omega \subset \mathbb{R}^n$ its domain of definition. g_{ij} and Γ^k_{ij} denote the components of the metric tensor in harmonic coordinates and the Christoffel symbols. Since $\xi^l \in W^{2,q}(\Omega)$ for any $q \geq 1$, we have

$$\mathsf{g}_{ij} \in W^{1,q}(\Omega) \quad \text{and} \quad \Gamma^k_{ij}, \Gamma_{ij,k} \in L_q(\Omega), \tag{4.1.34}$$

where

$$\Gamma_{ij,k} = \mathsf{g}_{il}\Gamma^l_{jk}.$$

We observe that (4.1.32) that the equality

$$\mathsf{g}^{kl}\Gamma^i_{kl} = 0, \quad i \in \{1, \dots, n\},$$

is satified in harmonic coordinates for almost all $x \in \Omega$. The last equality can be rewritten as

$$\mathsf{g}^{kl}\Gamma_{kl,i} = 0, \quad i \in \{1, \dots, n\}. \tag{4.1.35}$$

We are going to use the notation

$$D_i = \frac{\partial}{\partial \xi^i} \quad \text{and} \quad D_{ij} = \frac{\partial^2}{\partial \xi^i \partial \xi^j}.$$

Let $\eta \in W_0^{1,2}$ and we introduce the following generalized functions:

$$(\mathcal{R}_{ij}, \eta) = -\frac{1}{2}\int_\Omega \mathsf{g}^{kl}\left(D_j \mathsf{g}_{kl} D_i\eta - D_k \mathsf{g}_{il} D_j \eta + D_l \mathsf{g}_{ij} D_k \eta - D_l \mathsf{g}_{kj} D_i \eta\right) d\xi, \quad (4.1.36)$$

for $i, j \in \{1, \dots, n\}$. We observe that the right-hand side of (4.1.36) is, up to first derivatives of the metric tensor in the case of smooth Riemannian manifolds, the integral of the Ricci curvature over Ω in the direction of the i-th and j-th coordinate vectors.

If we now substitute for the $\Gamma_{kl,j}$ in (4.1.35) their expressions in terms of the g_{ij} and take the generalized derivative of the resulting identity with respect to ξ^s, we obtain

$$\int_\Omega \frac{1}{2}\mathsf{g}^{kl}(D_l \mathsf{g}_{jk} + D_k \mathsf{g}_{jl} - D_j \mathsf{g}_{kl}) D_s \eta \, d\xi = 0.$$

We denote this expression by $(\mathcal{T}_{sj})$. If we now add $\mathcal{R}_{ij}$, $\mathcal{R}_{ji}$, $\mathcal{T}_{ij}$ and $\mathcal{T}_{ji}$, we get

$$\int_\Omega \mathsf{g}^{kl} D_l \mathsf{g}_{ij} D_k \eta \, d\xi = -(\mathcal{R}_{ij} + \mathcal{R}_{ji}, \eta). \quad (4.1.37)$$

We observe that if the generalized function $(\mathcal{R}_{ij}, \eta)$ is represented in the form

$$(\mathcal{R}_{ij}, \eta) = \int_\Omega \mathcal{R}_{ij}(\xi)\eta(\xi)\, d\xi, \quad (4.1.38)$$

then g_{ij} is a generalized solution of the equation

$$\frac{\partial}{\partial \xi^k}\left(\mathsf{g}^{kl}\frac{\partial \mathsf{g}_{ij}}{\partial \xi^l}\right) = \mathcal{R}_{ij}, \quad (4.1.39)$$

(see [**LU68**], Chapter 3, Sec. 4).

LEMMA 4.23. *Let Ω_0 be a convex domain lying strictly inside the domain of definition Ω of the harmonic coordinates. We take the Sobolev average of the metric g_{ij}. Then for the resulting metric $\mathsf{g}^\varepsilon_{ij}$ we have*

$$\left\| \frac{\partial \Gamma^\varepsilon_{jl,k}}{\partial x^i} - \frac{\partial \Gamma^\varepsilon_{il,k}}{\partial x^j} \right\|_{L_q(\Omega_0)} \le C(q), \quad q \ge 1, \quad (4.1.40)$$

where $\Gamma^\varepsilon_{il,k}$ are the Christoffel symbols of the average metric and $C(q)$ is a constant depending on q and Ω.

PROOF. We consider a square in Ω whose sides are parallel to the i-th and j-th basis vectors $\{\Xi_l\}$ in Ω and have lengh $h > 0$, and the vertex $x \in \Omega_0$ of the square with least i-th and j-th coordinates We denote by $R_{h,x,i,j}$ the curve in $\mathcal{M}$ whose harmonic coordinates form the previous square in Ω as it is illustrated in Figure 17. We suppose thet $R_{h,x,i,j}$ is parametrized by the length of a side of the square, so that under a change of parameter it is first displaced along the j-th coordinate curve.

There is a small number $h_0 > 0$ such that $R_{h,x,i,j} \subset \xi^{-1}(\Omega)$, for $0 < h \leq h_0$, $x \in \Omega_0$, $i, j \in \{1, \dots, n\}$.

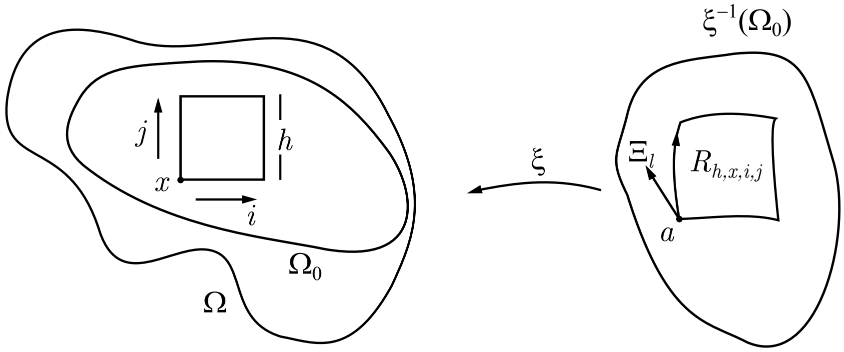

FIGURE 17

We construct a $\Delta^r_{h,ijl}(x)$ for $x \in \Omega_0$ as follows: we apply the parallel transport along $R_{h,x,i,j}$ to the vector Ξ^l. Then $\Delta^r_{h,ijl}(x)$ is the r-th coordinate of the increment of $\Xi_l(x)$. We put

$$\Delta_{h,ijl,s}(x) = \mathsf{g}_{rs}(x)\Delta^r_{h,ijl}(x) \tag{4.1.41}$$

for $i, j, l, s \in \{1, \dots, n\}$. We notice that by Theorem 4.22 the parallel displacement P along $R_{h,x,i,j}$ is specified for almost all $x \in \Omega_0$ by the usual formulas of Riemannian geometry, and by (4.1.32) we get

$$\left\| \frac{\partial \mathsf{g}_{ij}}{\partial \xi^m} \circ \gamma \right\|_{L_q([0,h_0])}, \left\| \Gamma^r_{ij} \circ \gamma \right\|_{L_q([0,h_0])}, \left\| \Gamma_{ij,s} \circ \gamma \right\|_{L_q([0,h_0])} \leq C(q), \quad q \geq 1, \tag{4.1.42}$$

where γ is an arbitrary part of $R_{h,x,i,j}$, and $C(q)$ is a constant depending only on q and Ω. Lea $a \in \mathcal{M}$ be the point corresponding to x, and b, c, d, the points on $R_{h,x,i,j}$ corresponding to the other vertices. We denote $R_{h,x,i,j}$ by $\overline{abcda}$ and the arc of this curve with its ends a, b by $\overline{ab}$. We fix the indices i, j, l, s and denote $\Delta_{h,ijl,s}$ by Δ_h. We denote by $\Xi^m_x(\lambda)$, $\lambda \in [0, 4h]$, the coordinates of the field Ξ_x of parallel vectors obtained as a result of parallel transport of Ξ_l along $\overline{abcda}$. In harmonic coordinates $\overline{abcda}$ is written:

$$\begin{aligned}
\text{on the part } \overline{ab}: &\quad (x^1, \dots, x^i, \dots, x^j + \lambda, \dots, x^n),\\
\text{on the part } \overline{bc}: &\quad (x^1, \dots, x^i + \lambda, \dots, x^j + h, \dots, x^n),\\
\text{on the part } \overline{cd}: &\quad (x^1, \dots, x^i + h, \dots, x^j + h - \lambda, \dots, x^n),\\
\text{on the part } \overline{da}: &\quad (x^1, \dots, x^i + h - \lambda, \dots, x^j, \dots, x^n),
\end{aligned}$$

where $\lambda \in [0,h]$. By (4.1.41), Δ_h is written:

$$\begin{aligned}\Delta_h(x) \;=\; & \mathsf{g}_{sr}(x)\Big[-\int_0^h \Gamma^r_{jm}(x^1,\dots,x^i,\dots,x^j+\lambda,\dots,x^n)\,\Xi^m_x(\lambda)\,d\lambda \qquad (4.1.43)\\ & -\int_0^h \Gamma^r_{jm}(x^1,\dots,x^i+\lambda,\dots,x^j+h,\dots,x^n)\,\Xi^m_x(h+\lambda)\,d\lambda\\ & +\int_0^h \Gamma^r_{jm}(x^1,\dots,x^i+h,\dots,x^j+\lambda,\dots,x^n)\,\Xi^m_x(3h-\lambda)\,d\lambda\\ & -\int_0^h \Gamma^r_{jm}(x^1,\dots,x^i+\lambda,\dots,x^j,\dots,x^n)\,\Xi^m_x(4h-\lambda)\,d\lambda\Big].\end{aligned}$$

Now we define R_h for almost all $x \in \Omega_0$ as follows:

$$\begin{aligned}R_h(x) \;=\; & \int_0^h \big(\Gamma_{jl,s}(x^1,\dots,x^i+h,\dots,x^j+\lambda,\dots,x^n) \qquad (4.1.44)\\ & -\Gamma_{jl,s}(x^1,\dots,x^i,\dots,x^j+\lambda,\dots,x^n)\big)\;d\lambda\\ & -\int_0^h \big(\Gamma_{jl,s}(x^1,\dots,x^i+\lambda,\dots,x^j+h,\dots,x^n)\\ & -\Gamma_{jl,s}(x^1,\dots,x^i+\lambda,\dots,x^j,\dots,x^n)\big)\;d\lambda.\end{aligned}$$

From the definition of Ξ^m_x and the fact that the metric is positive definite it follows that

$$|\Xi^m_x| \le \frac{|\Xi_x|}{\sqrt{\nu}} = \frac{|\Xi_l|}{\sqrt{\nu}} = \sqrt{\frac{g_{ll}}{\nu}},$$

$\nu > 0$ costant. From the formula for parallel transport we see that for almost all $x \in \Omega_0$

$$|\Xi^m_x(3h-\lambda) - \Xi^m_l| \le \Big(\frac{\mathsf{g}_{ll}}{\nu}\Big)^{1/2} \sum_{r,s=1}^{n} \int_0^\lambda |\Gamma^m_{rs}(x^1,\dots,x^i+h,\dots,x^j+t,\dots,x^n)|\,dt.$$

From this inequality, the generalized Minkowski inequality and (4.1.34) we obtain

$$\|\Xi^m_x(3h-\lambda) - \Xi^m_l(x)\|_{L_{2q}(\Omega_0)} \le C(q)\cdot\lambda,$$

From the generalized Minkowski inequality, the previous inequality, a special case of Hölder's inequality ($\|f\cdot g\|_q \le \|f\|_{2q}\cdot\|g\|_{2q}$ with $q \ge 1$) and (4.1.34) we obtain

$$\begin{aligned}&\Big\|\mathsf{g}_{rs}(x)\Big(\int_0^h \Gamma^r_{jm}(x^1,\dots,x^i+h,\dots,x^j,\dots,x^n)\Xi^m_x(3h-\lambda)\,d\lambda \qquad (4.1.45)\\ &-\int_0^h \Gamma^r_{jm}(x^1,\dots,x^i+h,\dots,x^j,\dots,x^n)\,d\lambda\Big)\Big\|_{L_q(\Omega_0)} \le C(q)h^2.\end{aligned}$$

We observe that because g_{ij} is absolutely continuous ($\mathsf{g}_{ij} \in W^{1,q}(\Omega_0)$) and Ω_0 is convex, we have for almost all $x \in \Omega_0$

$$\mathsf{g}_{rs}(x') - \mathsf{g}_{rs}(x) = h \int_0^1 \frac{\partial \mathsf{g}_{rs}}{\partial x^i}(tx' + (1-t)x)\,dt + \lambda \int_0^1 \frac{\partial \mathsf{g}_{rs}}{\partial x^j}(tx' + (1-t)x)\,dt,$$

with $x' = (x^1, \dots, x^i + h, \dots, x^j + \lambda, \dots, x^n)$. From this equality, the generalized Minkowski inequality, (4.1.34) and (4.1.42) we obtain

$$\|\mathsf{g}_{rs}(x') - \mathsf{g}_{rs}(x)\|_{L_{2q}(\Omega_0)} \le C(q) \cdot h^2,$$

From the generalized Minkowsi inequality, the special case of Hölder's inequality metioned above, the last inequality and (4.1.34) we obtain

$$\left\| \left(\Gamma_{jl,s}(x') - \mathsf{g}_{rs}(x)\Gamma^r_{jl}(x')\right)\, d\lambda \right\|_{L_q(\Omega_0)} \le C(q) \cdot h^2, \tag{4.1.46}$$

From the triangle inequality, (4.1.45) and (4.1.46) we get

$$\left\| \int_0^h \left(\Gamma_{jl,s}(x') - \mathsf{g}_{sr}(x)\Gamma^r_{jm}(x')\Xi^m_x(3h - \lambda)\right)\, d\lambda \right\|_{L_q(\Omega_0)} \le C(q) \cdot h^2.$$

Estimating the remaining differences of integrals of R_h and Δ_h in exactly the same way, we obtain

$$\|\Delta_h - R_h\|_{L_q(\Omega_0)} \le \tilde{C}(q) \cdot h^2, \quad q \ge 1, \tag{4.1.47}$$

with $\tilde{C}(q)$ is a constant depending on q and Ω.

We consider the averaged metric $\mathsf{g}^\varepsilon_{ij}$ in Ω_0. From the usual properties of averages and expressions for the Christoffel symbols it follows that

$$\Gamma^\varepsilon_{jl,s} = \int_{\mathbb{R}^n} \omega\left(\frac{x-u}{\varepsilon}\right) \Gamma_{jl,s}(u)\,du, \tag{4.1.48}$$

where ω is the averaging kernel. We now prove that if we calculate R^ε_h for the metric $\mathsf{g}^\varepsilon_{ij}$ by (4.1.44), then

$$R^\varepsilon_h(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \omega\left(\frac{x-u}{\varepsilon}\right) R_h(u)\,du, \tag{4.1.49}$$

where R_h is defined to be zero outside Ω_0. Consider the first integral of R_h:

$$I(u) = \int_0^h \Gamma_{jl,s}(u^1, \dots, u^i + h, \dots, u^j + \lambda, \dots, u^n)\,d\lambda.$$

If we average $I(u)$, we obtain by Fubini's theorem

$$I^\varepsilon(x) = \frac{1}{\varepsilon^n} \int_0^h \int_{\mathbb{R}^n} \omega\left(\frac{x-u}{\varepsilon}\right) \Gamma_{jl,s}(u^1, \dots, u^i + h, \dots, u^j + \lambda, \dots, u^n)\,du.$$

If we make the change of variables

$$x-u=v \quad \text{and} \quad u=(x^1-v^1,\dots,x^i+h-v^i,\dots,x^j+\lambda-v^j,\dots,x^n-v^n),$$

we have

$$I^\varepsilon(x)=\frac{1}{\varepsilon^n}\int_0^h\int_{\Omega_0}\omega\left(\frac{x'-u}{\varepsilon}\right)\Gamma_{jl,s}(u)\,du.$$

From (4.1.48) and the last equality we have the first term proving (4.1.49), and the other terms are proven in the same way. By properties of Sobolev average (see [**Nik77**]) we have

$$\|R_h^\varepsilon\|_{L_q(\Omega_0)}\le\|R_h\|_{L_q(\Omega_0)}, \quad q\ge 1. \tag{4.1.50}$$

We observe that from Theorem 3.14 applied on the triangles $\triangle(a,b,c)$ and $\triangle(a,c,d)$ and (4.1.41) it follows that for almost all $h\in(0,h_0]$

$$|\Delta_{h,ijl,s}(x)|\le C'\cdot h^2, \quad x\in\Omega_0, \quad i,j,l,s\in\{1,\dots,n\}, \tag{4.1.51}$$

where C' is a constant. From (4.1.47), (4.1.50) and (4.1.51) we obtain

$$\left\|\frac{R_h^\varepsilon}{h^2}\right\|\le C''(q)$$

for almost all $h\in(0,h_0]$, where $C''(q)$ is a constant depending on q and Ω. Proceeding to the limit with respect to h in the last inequality, by Fatou's lemma and the definition of R_h^ε we obtain (4.1.40). Q. E. D.

LEMMA 4.24. *The generalized function $(\mathcal{R}_{ij},\eta)$, $\eta\in W_0^{1,2}(\Omega_0)$, can be represented in the form (4.1.38) and $\mathcal{R}_{ij}(x)\in L_q(\Omega_0)$ for $q\ge 1$ and $i,j\in\{1,\dots,n\}$.*

PROOF. We consider the expression for the Ricci curvature of the average metric:

$$\mathcal{R}_{ij}^\varepsilon=\frac{1}{2}\mathsf{g}^{\varepsilon,pl}\left(D_{ij}\mathsf{g}_{pl}^\varepsilon-D_{pj}\mathsf{g}_{li}^\varepsilon+D_{pl}\mathsf{g}_{ij}^\varepsilon-D_{il}\mathsf{g}_{pj}^\varepsilon\right)+\mathsf{g}^{\varepsilon,pl}\mathsf{g}_{rs}\left(\Gamma_{pl}^{\varepsilon,r}\Gamma_{ij}^{\varepsilon,s}-\Gamma_{il}^{\varepsilon,r}\Gamma_{pj}^{\varepsilon,s}\right) \tag{4.1.52}$$

Integrating by parts, we obtain

$$-\int_{\Omega_0}\mathsf{g}^{\varepsilon,pl}D_j\mathsf{g}_{pl}^\varepsilon D_i\eta\,dx=\int_{\Omega_0}\mathsf{g}^{\varepsilon,pl}D_{ij}\mathsf{g}_{pl}^\varepsilon\eta\,dx+\int_{\Omega_0}D_i\mathsf{g}^{\varepsilon,pl}D_j\mathsf{g}_{pl}^\varepsilon\eta\,dx.$$

We denote this expression by (S_{ijpl}) and the numbering is with respect to the first term of the right-hand side. We add the equalities (S_{ijpl}), (S_{pjli}) (S_{plij}) and (S_{ilpj}), multiplying both sides of the equality by -1 if there is a minus sign in front of the corresponding term in (4.1.52). We obtain

$$-(\mathcal{R}_{ij}^\varepsilon,\eta)=\int_{\Omega_0}\mathcal{R}_{ij}^\varepsilon(x)\eta(x)\,dx+\int_{\Omega_0}f_{ij}^\varepsilon(x)\eta(x)\,dx, \tag{4.1.53}$$

where the f_{ij}^ε are expressed in terms of the metric $\mathsf{g}_{rs}^\varepsilon$ and its first derivatives, and $(\mathcal{R}_{ij}^\varepsilon,\eta)$ the generalized function given by (4.1.36). By the previous lemma, all $\mathcal{R}_{ij}^\varepsilon$ are bounded in $L_q(\Omega_0)$, $q \geq 1$, by a constant depending only on q and Ω. Hence $R^{\varepsilon}_{ij}(x)$ is also bounded. Proceeding to the limit in (4.1.53) with respect to some subsequence $\{\varepsilon_m\}$ that tends to zero, we obtain the assertion of the lemma. Q. E. D.

THEOREM 4.25. *Let $\mathcal{M}$ be a space with bounded curvature. Then in a neighborhood of each point $p \in \mathcal{M}$ we can introduce a harmonic coordinate system. The components g_{ij} of the metric tensor in any harmonic coordinate system in $\mathcal{M}$ are continuous functions of class $W^{2,q}(\Omega)$ for any $q > 1$, where $\Omega \subset \mathbb{R}^n$ is a domain in which harmonic coordinates are defined. Harmonic systems of coordinates on $\mathcal{M}$ form an atlas of class $C^{3,\alpha}$ for any $0 < \alpha < 1$.*

PROOF. By Lemma 4.24, the components g_{ij} of the metic tensor in harmonic coordinates satisfy (4.1.39), where $\mathcal{R}_{ij} \in L_q(\Omega_0)$ for every $q \geq 1$. By [**LU68**], Chapter 3, Theorem 15.1, $\mathsf{g}_{ij} \in W^{2,q}(\Omega_0) \cap C^{1,\alpha}(\Omega_0)$, $\alpha = 1 - n/q$ and $q > n$ is an arbitrary number, $i, j \in \{1, \dots, n\}$. The proof is complete using Theorem 1 of [**SS76**]. Q. E. D.

REMARK 16. From the embedding theorems of Sobolev it follows that in harmonic coordinates $\mathsf{g}_{ij} \in C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

REMARK 17. From Theorem 4.25 it follows that in harmonic coordinates g_{ij} has an ordinary second differential almost everywhere in Ω (see [**VGR79**], Chapter 1, Theorem 3.2).

2. Curvature

In this section we are going to calculate the sectional curvature of a space with bounded curvature. To do this we need to give a totally analogous definition of a space with bounded curvature.

Given a triangle $T = \triangle(p, q, r)$ in a metric space X, we define the **excess** of T as $\delta(T) := \alpha + \beta + \gamma - \pi$, where α, β and γ are the upper angles at the vertices p, q and r, respectively.

We define the **upper mean curvature** of the triangle:

$$\bar{k}(T) := \begin{cases} \frac{\delta(T)}{\sigma(T)} & \text{for } \sigma(T) \neq 0, \\ +\infty & \sigma(T) = 0 \text{ and } \delta(T) \geq 0, \\ -\infty & \sigma(T) = 0 \text{ and } \delta(T) < 0, \end{cases}$$

where $\sigma(T)$ denotes the area of a comparison triangle on the plane. Analogously, the **lower mean curvature** of the triangle:

$$\underline{k}(T) := \begin{cases} \frac{\delta(T)}{\sigma(T)} & \text{for } \sigma(T) \neq 0, \\ +\infty & \sigma(T) = 0 \text{ and } \delta(T) > 0, \\ -\infty & \sigma(T) = 0 \text{ and } \delta(T) \leq 0. \end{cases}$$

The **upper curvature** at the point $p \in X$ is defined as:

$$\bar{k}_X(p) := \limsup_{T \to p} \frac{\delta(T)}{\sigma(T)}.$$

And analogously the **lower curvature** at p is defined as:

$$\underline{k}_X(p) := \liminf_{T \to p} \frac{\delta(T)}{\sigma(T)}.$$

Both limits are taken on triangles contracting arbitrarily to the pont p. Finally the **upper limit of curvature** of X is

$$\bar{k}(X) = \sup_{p \in X}\{\bar{k}_X(p)\},$$

analogously, the **lower limit of curvature** of X is

$$\underline{k}(X) = \inf_{p \in X}\{\underline{k}_X(p)\}$$

REMARK 18. For $\bar{k}(X)$ and $\underline{k}(X)$ infinite values are admitted.

REMARK 19. For a Riemannian manifold $(\mathcal{M}, \mathtt{g})$, we have that $\bar{k}(\mathcal{M})$ coincides with $\sup\{\sec_{\mathtt{g}}(\mathcal{M})\}$ and $\underline{k}(\mathcal{M})$ with $\inf\{\sec_{\mathtt{g}}(\mathcal{M})\}$ considered over all $p \in \mathcal{M}$ and all two-dimensional sections $\sigma \subset T_p\mathcal{M}$.

Once we have these definitions, equivalently a **space with bounded curvature** is a locally compact length space $\mathcal{M}$ in which the following axioms are satisfied:

- For every point $p \in \mathcal{M}$ there exists $\rho_p > 0$ such that in the open ball $B(p, \rho_p)$ the condition of extendability of segments holds: every geodesic segment $[x, y]$ with endpoints x and y in $B(p, \rho_p)$ can be extended to a segment $[x', y']$ in $\mathcal{M}$ for which x and y are internal points.
- Every $p \in \mathcal{M}$ is contained in a neighborhood U such that for every point $p \in U$:
$$\underline{k}_{\mathcal{M}}(p) > -\infty \text{ and } \bar{k}_{\mathcal{M}}(p) < +\infty.$$

Having this new definition we need a special way of triangle convergence to calculate the sectional curvature. We say that the **sequence** $\{T_m = \triangle(p, q_m, r_m)\}$ **of triangles of** $\mathcal{M}$ **contracts to the point** p **with respect to the pair** $\{u, v\}$, where $u, v \in T_p\mathcal{M}$ are non-collinear unit vectors, if unit vectors $u_m, v_m \in T_p\mathcal{M}$ tangent to segments $[p, q_m]$ and $[p, r_m]$ converge to u and v, respectively, and $q_m \to p$ and $r_m \to p$ as $m \to \infty$. We denote this convergence by

$$T_m = \triangle(p, q_m, r_m) \xrightarrow{u,v} p$$

We shall denote by $\mathcal{O} \subset \mathcal{M}$ the set of points of two-fold differentiability of the metric tensor.

THEOREM 4.26. *Let $(\mathcal{M}, d)$ be a space with bounded curvature. Then there exists a set $\mathcal{O}_1$ ($\mathcal{O} \subset \mathcal{O}_1 \subset \mathcal{M}$) of zero n-dimensional Hausdorff measure such that the following condition holds at each point $p \in \mathcal{M} \setminus \mathcal{O}_1$:*

- *for arbitrary pairs of noncollinear unit vectors $u, v \in T_p\mathcal{M}$ one can find a sequence of triangles $\{T_m = \triangle(p, q_m, r_m)\}$ which*
$$T_m \xrightarrow{u,v} p$$
such that the following limit exists:
$$\lim_{m\to\infty} \frac{\delta(T_m)}{\sigma(T_m)}$$
and it is equal to the sectional curvature of $\mathcal{M}$ calculated formally at the point p with respect to the plane section $\sigma \subset T_p\mathcal{M}$ generated by $\{u, v\}$.

The previous theorem gives us a way to compute the curvature in a space with bounded curvature and tell us how "metric" curvature coincides with the sectional curvature. The proof of the last theorem can be found in [**Nik91**].

Part 2

Aproximation Theorems

CHAPTER 5

Currents

In this chapter we will introduce some tools in the theory of currents that are used later in the proof of Nikolaev's Approximation Theorem. Our main references are [**de 84, Fed96, Sim83, Mor16**].

1. Differential Forms

The results on differential forms in this section are standard and may be found in [**de 84, Lee13**]

Let $U \subset \mathbb{R}^n$ and $m \geq 0$. The space of **smooth differential** m**-forms** in U is denoted by

$$\mathcal{E}^m(U) = \Gamma(\Lambda^m T^* U),$$

the sections of the bundle of alternating covariant m-tensors on U.

The **support** of $\omega \in \mathcal{E}^m(U)$ is

$$\operatorname{spt}(\omega) = \overline{\{x \in U : \omega(x) \neq 0\}} \cap U$$

and we say that ω is **compactly supported** if its support is compact. The space of **compactly supported smooth differential** m**-forms** in U is denoted by

$$\mathscr{D}^m(U) \subset \mathcal{E}^m(U).$$

Notice that $\mathscr{D}^0(U) = C_c^\infty(U)$ are the space of **compactly supported smooth functions** in U.

If $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$ and $\{dx^1, \ldots, dx^n\}$ is its dual basis, every form $\omega \in \mathscr{D}^m(U)$ can be uniquely written as

$$\omega = \sum_{|I|=m} \omega_I \, dx^I,$$

where $I = (i_1, \ldots, i_m)$ is a multi-index such that $1 \leq i_1 < \ldots < i_m \leq n$, $|I| := m$, $dx^I := dx^{i_1} \wedge \ldots \wedge dx^{i_m}$, and ω_I are $C_c^\infty(U)$ functions, functions in $C^\infty(U)$ with compact support.

The **exterior derivative** of $\omega \in \mathcal{E}^m(U)$ is the form $d\omega \in \mathcal{E}^{m+1}(U)$ defined by

$$d\omega := \sum_{|I|=m} \sum_{i=1}^{n} \frac{\partial \omega_I}{\partial x_i^I} \, dx^i \wedge dx^I.$$

We have the following properties.

PROPOSITION 5.1. *If $\omega \in \mathcal{E}^m(U)$ and $\nu \in \mathcal{E}^k(U)$, then:*

(1) $d(d\omega) = 0$ and

(2) $d(\omega \wedge \nu) = (d\omega) \wedge \nu + (-1)^m \omega \wedge (d\nu)$.

Let $U \subset \mathbb{R}^n$ and $V \in \mathbb{R}^d$ be open sets and let $\varphi \colon U \to V$ be a smooth map. The **pullback** of $\omega \in \mathcal{E}^m(V)$ under φ is the differential form $\varphi^*\omega \in \mathcal{E}^m(U)$ defined by

$$(\varphi^*\omega)(x)(v_1, \ldots, v_m) := \omega(\varphi(x))(d\varphi_x(v_1), \ldots, d\varphi_x(v_m)).$$

We recall the following properties of the pullback of a form.

PROPOSITION 5.2. *If $\omega \in \mathcal{E}^m(V)$ and $\nu \in \mathcal{E}^k(V)$, then:*

(1) $\varphi^*(\omega \wedge \nu) = (\varphi^*\omega) \wedge (\varphi^*\nu)$,

(2) $d(\varphi^*\omega) = \varphi^*(d\omega)$.

2. Currents

Let $U \subset \mathbb{R}^n$ be an open set and $m \geq 0$. An m-**current** on U is a continuous linear functional on $\mathscr{D}^m(U)$, which is a topological vector space. The space of m-currents on U is denoted by $\mathscr{D}_m(U)$.

The topology on $\mathscr{D}^m$ is given as follows: we say that

$$\omega_k = \sum_{|I|=m} \omega_{k,I}\, dx^I$$

converges to

$$\omega = \sum_{|I|=m} \omega_I\, dx^I$$

if there exists a compact set $K \subset U$ such that $\mathrm{spt}(\omega_{k,I}) \subset K$ for every $k, i_1, \ldots, i_m$, $\mathrm{spt}(\omega_I) \subset K$, and

$$\lim_{k\to\infty} \mathrm{D}^\beta \omega_{k,I} = \mathrm{D}^\beta \omega_I,$$

using uniform convergence of functions and with β also a multi-index. We denote this convergence by $\omega_k \to \omega$.

EXAMPLES. Some examples of currents:

(1) If $x_0 \in \mathbb{R}^n$, then the functional

$$[[x_0]](f) := f(x_0)$$

defines a 0-current on $\mathbb{R}^n$.

(2) If we have an interval $[a, b]$, then we define a 1-current on $\mathbb{R}$ by setting

$$[[a, b]](f\, dx) := \int_a^b f(x)\, dx.$$

(3) If $\psi \in D^{n-m}(U)$, we get a m-current by letting

$$\psi(\omega) := \int_U \psi \wedge \omega.$$

We say that T is a m-**current** of class C^r with $0 \leq r \leq \infty$ on the open set U if there is a $(n-m)$-form ψ of class C^r on U such that

$$T(\phi) = \int_U \psi \wedge \phi$$

for every $\phi \in \mathscr{D}^m(U)$.

The **support** of $T \in \mathscr{D}_m(U)$ is defined by

$$\operatorname{spt}(T) = U \setminus \left(\bigcup_i V_i \right),$$

where the sets V_i are open sets such that $T(\omega) = 0$ for every $\omega \in \mathscr{D}^m(U)$ with $\operatorname{spt}(\omega) \subset V_i$.

The **boundary** of $T \in \mathscr{D}_m(U)$ is the $(m-1)$-current defined by

$$(\partial T)(\omega) = T(\,d\omega)$$

if $m \geq 1$ and $\partial T = 0$ if $m = 0$.

Now, let us consider $\varphi\colon U \to V$ a smooth map between open sets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively, and let $T \in \mathscr{D}_m(U)$. Assume that φ is proper on the support of T, that is,

$$\left(\varphi|_{\operatorname{spt}(T)} \right)^{-1}(K) = \varphi^{-1}(K) \cap \operatorname{spt}(T)$$

is compact whenever $K \subset V$ is compact. We define the **pushforward** $\varphi_*(T)$ **of** T **by** φ by

$$\varphi_*(T)(\omega) := T(\rho \cdot \varphi^*(\omega))$$

for $\omega \in \mathscr{D}^m(V)$, where $\rho \in C^\infty_c(U)$ is any function which equals 1 on a neighborhood of the compact set

$$\left(\varphi|_{\operatorname{spt}(T)} \right)^{-1}(\operatorname{spt}(\omega)) = \operatorname{spt}(T) \cap \varphi^{-1}(\operatorname{spt}(\omega)).$$

We observe that this definition does not depend on the funtion ρ and also that $\partial(\varphi_* T) = \varphi_*(\partial T)$.

We also define the **mass** of $T \in \mathscr{D}_m(U)$ by

$$\mathbb{M}(T) = \sup_{|\omega| \leq 1,\, \omega \in \mathscr{D}^m(U)} T(\omega),$$

where

$$|\omega| = \sup_{x \in U} \sqrt{\omega(x) \cdot \omega(x)},$$

working with the inner product naturally induced from $\mathbb{R}^n$. More generally, for any open set $W \subset U$, we define

$$\mathbb{M}_W(T) = \sup_{|\omega|\leq 1,\, \omega\in\mathscr{D}^m(U),\, \mathrm{spt}(\omega)\subset W} T(\omega).$$

In [**Sim83**, Lemma 2.29, Section 2, Chapter 6] we see the following two properties using the previous notation

$$\mathrm{spt}\,\varphi_* T \subset \varphi(\mathrm{spt}(T))$$

and

$$\mathbb{M}_W(\varphi_* T) \leq (\sup_{\varphi^{-1}(W)} |D\varphi|)^m \mathbb{M}_{\varphi^{-1}(W)}(T) \tag{5.2.1}$$

for every $W \subset\subset V$, i.e. W is **compactly embedded** in V, which means that $W \subset \bar{W} \subset V$ and $\bar{W}$ is compact.

3. Regularization

For a point y in $\mathbb{R}^n$ we consider the translation $\tau_y \colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\tau_y(x) = x+y$ and the homotopy $\tau \colon [0,1]\times\mathbb{R}^n \to \mathbb{R}^n$ given by $\tau(t,x) := \tau_{ty}(x) = x + ty$.

Now we choose the function $\psi \colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(t) = \begin{cases} \frac{1}{\lambda_n}\exp(\frac{t^2}{t^2-1}), & 0 \leq |t| < 1, \\ 0, & 1 \leq |t|, \end{cases}$$

where the constant λ_n is chosen so that

$$\int_{\mathbb{R}} \psi(t)\,dt = \int_{-1}^{1} \psi(t)\,dt = 1.$$

For $\varepsilon > 0$, we take $f_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}$ given by

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n}\psi\left(\frac{\|x\|}{\varepsilon}\right).$$

This is a radial nonnegative C^∞ function whose support is contained in $B(0,\varepsilon)$, the ball with center 0 and radius ε, and

$$\int_{\mathbb{R}^n} f_\varepsilon(x)\,dx = 1.$$

3.1. Representation and convolution of currents. Let T be a current in $\mathscr{D}_{m+q}(U)$ and ψ be a form in $\mathscr{D}^q(U)$. We define the **wedge product** of T and ψ as the q-current given by

$$(T \wedge \psi)(\omega) = T(\psi \wedge \omega) \quad \text{for} \quad \omega \in \mathscr{D}^q(U).$$

PROPOSITION 5.3. *If $T \in D_{n-m}(U)$, then*

$$T = \sum_{|I|=m} T_I \wedge \, dx^I$$

with $T_I = T_{i_1,\ldots,i_m} \in \mathscr{D}_n(U)$.

PROOF. We simply define these n-currents as follows:

$$T_{i_1,\ldots,i_m}(a\, dx^{i_1} \wedge \ldots \wedge dx^{i_m} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_{n-m}}) = \delta^{1,\ldots,n}_{i_1,\ldots,i_m,j_1,\ldots,j_{n-m}} T(a\, dx^{j_1} \wedge \ldots \wedge dx^{j_{n-m}}),$$

where

$$\delta^{1,\ldots,n}_{i_1,\ldots,i_m,j_1,\ldots,j_{n-m}} = \begin{cases} 1 & \text{if there exists a permutation between} \\ & \{i_1,\ldots,i_m,j_1,\ldots,j_{n-m}\} \text{ and } \{1,\ldots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Q. E. D.

REMARK 20. We have a bijection between 0-currents and n-currents, given by

$$\begin{array}{ccc} \mathscr{D}_0(U) & \to & \mathscr{D}_n(U), \\ T & \mapsto & T' \end{array}$$

where $T'(f\, dx^1 \wedge \ldots \wedge dx^n) = T(f)$. This correspondence and the previous proposition allow us to consider currents as forms whose coefficients are **distributions** (0-currents). Using this we write an m-current as

$$T = \sum_{|I|=m} T_I \; dx^I \tag{5.3.1}$$

with $T_I \in \mathscr{D}_0(U)$.

Now, if $T \in \mathscr{D}_0(\mathbb{R}^n)$ and $\varphi \in C^\infty_c(\mathbb{R}^n)$, the formula

$$(T * \varphi)(x) := T(\varphi(x - \cdot))$$

defines a function of class C^∞, called the **convolution** of T and φ. This operation has the following properties (see Chapter 7 of [**Sal16**]):

(i) The convolution conmutes with derivatives, i.e.,

$$\frac{\partial}{\partial x^j}(T * \varphi) = \left(\frac{\partial T}{\partial x^j}\right) * \varphi = T * \left(\frac{\partial \varphi}{\partial x^j}\right),$$

where the i**-th partial derivative** of the distribution T is defined by

$$\frac{\partial T}{\partial x^i}(\varphi) := -T\left(\frac{\partial \varphi}{\partial x^i}\right).$$

(ii) The support satisfies

$$\operatorname{spt}(T * \varphi) \subset \operatorname{spt}(T) + \operatorname{spt}(\varphi).$$

(iii) Using the function f_ε, then $T * f_\varepsilon \to T$ and $\operatorname{spt}(T * f_\varepsilon) \to \operatorname{spt}(T)$ in the Hausdorff sense as $\varepsilon \to 0$.

Finally, if $T \in \mathscr{D}_{n-m}(\mathbb{R}^n)$ is written as in (5.3.1) and $\varphi \in C_c^\infty(\mathbb{R}^n)$, the **convolution** of T and φ is given by

$$T * \varphi = \sum_{|I|=m} (T_I * \varphi)\, dx^I.$$

3.2. Smoothing operator. Putting everything together we can define an operator $\mathbf{Z}\colon \mathscr{D}_m(\mathbb{R}^n) \to \mathscr{D}_m(\mathbb{R}^n)$ as follows:

$$\mathbf{Z}T(\omega) := \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot (\tau_y)_*(T)(\omega)\, dy.$$

PROPOSITION 5.4. *The linear operator* $\mathbf{Z}$ *has the following properties:*

a) If T *is an* m*-dimensional current in* $\mathbb{R}^n$*, then* $\mathbf{Z}T$ *is an* m*-dimensional current.*
b) $\mathbf{Z}T$ *is* C^∞*.*
c) The support of $\mathbf{Z}T$ *is contained in the* ε*-neighborhood of the support of* T*.*
d) If ε *tends to zero,* $\mathbf{Z}T$ *converges weakly to* T*, i.e.,* $\mathbf{Z}T(\omega) \to T(\omega)$ *for every* $\omega \in \mathscr{D}^m(\mathbb{R}^n)$*.*

PROOF. The first thing we have to do is to see that the operator is well-defined, that is, to prove a). We observe first that clearly $\mathbf{Z}T$ is acting on m-forms by definition and, if ω is a m-form given by

$$\omega = \sum_{i_1<\ldots<i_m} \omega_{i_1,\ldots,i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_m},$$

then

$$\tau_y^*(\omega) = \sum_{i_1<\ldots<i_m} \omega_{i_1,\ldots,i_m} \circ \tau_y\, dx^{i_1} \wedge \ldots \wedge dx^{i_m}. \tag{5.3.2}$$

This means that $\mathbf{Z}T$ is continuous from equality (5.3.2) and the definition.

To prove b) we note that the operator can be expressed in a different way, namely,

$$
\begin{aligned}
(T * f_\varepsilon)(\omega) &= \sum_{|I|=n-m} (T_I * f_\varepsilon)\, dx^I(\omega) \\
&= \sum_{|I|=n-m} (T_I * f_\varepsilon)\, dx^I \left(\sum_{|J|=m} \omega_J\, dx^J \right) \\
&= \int_{\mathbb{R}^n} \sum_{|I|=n-m} \sum_{|J|=m} (T_I * f_\varepsilon)(x) \cdot \omega_J(x) \left(dx^I \wedge dx^J \right) \\
&= \sum_{|I|=n-m} \sum_{|J|=m} \int_{\mathbb{R}^n} T_I(f_\varepsilon(x-y)) \cdot \omega_J(x) \left(dx^I \wedge dx^J \right) \\
&= \sum_{|I|=n-m} \sum_{|J|=m} \int_{\mathbb{R}^n} T_I(\omega_J(x) \cdot f_\varepsilon(x-y)) \left(dx^I \wedge dx^J \right) \\
&= \sum_{|I|=n-m} \sum_{|J|=m} \int_{\mathbb{R}^n} T_I(\omega_J(x+y) \cdot f_\varepsilon(x)) \left(dx^I \wedge dx^J \right) \\
&= \sum_{|I|=n-m} \sum_{|J|=m} \int_{\mathbb{R}^n} f_\varepsilon(x) \cdot T_I(\omega_J(x+y)) \left(dx^I \wedge dx^J \right) \\
&= \int_{\mathbb{R}^n} \sum_{|I|=n-m} \sum_{|J|=m} f_\varepsilon(x) \cdot T_I(\omega_J(\tau_y(x))) \left(dx^I \wedge dx^J \right) \\
&= \int_{\mathbb{R}^n} f_\varepsilon(x) \sum_{|I|=n-m} T_I \left(\sum_{|J|=m} \omega_J(\tau_y(x)) \right) \left(dx^I \wedge dx^J \right) \\
&= \int_{\mathbb{R}^n} f_\varepsilon(x) \sum_{|I|=n-m} T_I \left(\tau_y^*(\omega) \right) \left(dx^I \wedge dx^J \right) \\
&= \int_{\mathbb{R}^n} f_\varepsilon(x) \cdot ((\tau_y)_* T)\,(\omega)\, dx \\
&= \mathbf{Z}T(\omega).
\end{aligned}
$$

This equation allows us to use all the properties of the convolution of a distribution with a function. In particular, since the convolution is a C^∞ function, then $\mathbf{Z}T$ is a current of class C^∞.

Finally, c) and d) follow from the previous equation and properties ii) and iii) of the convolution. Q. E. D.

3.3. De Rham's Approximation Theorem. The next step is to define a smoothing operator and to prove the analogous result to Proposition 5.4 in the case of manifolds. We need to transform the operator defined on $\mathbb{R}^n$ and this is going to be done with the aid of a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{B}^n$, where $\mathbb{B}^n$ denotes the unit open ball with center in the origin in $\mathbb{R}^n$.

Let $g\colon (0,1) \to \mathbb{R}$ be a function given by

$$g(r) = \begin{cases} r & r \in (0,1/3]\,, \\ \tilde{g}(r) & r \in [1/3,2/3]\,, \\ \exp\left(\frac{1}{(1-r)^2}\right) & r \in [2/3,1)\,, \end{cases}$$

where $\tilde{g}\colon [1/3,2/3] \to [1/3,\exp(9)]$ is such that g is C^∞ and $g'(r) > 0$. This function g is bijective and we denote by $g^{-1}\colon \mathbb{R} \to (0,1)$ its inverse. Thanks to it, we define $h\colon \mathbb{R}^n \to \mathbb{B}^n$ as follows

$$h(x) = \begin{cases} \frac{g^{-1}(\|x\|)}{\|x\|}\,x & x \neq 0, \\ \lim x \to \bar{0}\frac{g^{-1}(\|x\|)}{\|x\|}\,x & x = 0, \end{cases}$$

which is a C^∞ diffeomorphism. Then we have the diffeomorphism $s_y\colon \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞ given by

$$s_y(x) = \begin{cases} h \circ \tau_y \circ h^{-1}(x) & x \in \mathbb{B}^n, \\ x & x \in \mathbb{R}^n \setminus \mathbb{B}^n. \end{cases} \tag{5.3.3}$$

This function allow us to define an operator $\tilde{\mathbf{Z}}\colon \mathscr{D}_m(\mathbb{R}^n) \to \mathscr{D}_m(\mathbb{R}^n)$ as follows

$$\tilde{\mathbf{Z}}T(\omega) := \int_{\mathbb{R}^n} f_\varepsilon(y)\cdot (s_y)_*(T)(\omega)\,dy.$$

PROPOSITION 5.5. *The linear operator $\tilde{\mathbf{Z}}$ has the following properties:*

a) If T is an m-dimensional current in $\mathbb{R}^n$, then $\tilde{\mathbf{Z}}T$ is an m-dimensional current.
b) The support of $\tilde{\mathbf{Z}}T$ is contained in the set

$$E(T,\varepsilon) = \bigcup_{y\in\mathbb{R}^n,\,|y|<\varepsilon} (s_y)_*T.$$

c) If ε tends to zero, $\tilde{\mathbf{Z}}T$ converges weakly to T.
d) $\tilde{\mathbf{Z}}T$ is C^∞ on $\mathbb{B}^n$ and $\tilde{\mathbf{Z}}T = T$ in $\mathbb{R}^n \setminus \bar{\mathbb{B}}^n$. If T is C^r in a neighborhood of a boundary point of $\mathbb{B}^n$, then $\tilde{\mathbf{Z}}T$ is also C^r on a neighborhood of this point.

PROOF. First, we observe that a) follows from the fact that s_y is a function of class C^∞, T is a current and how the operator $\tilde{\mathbf{Z}}$ is defined.

Now, let be $\omega \in \mathscr{D}_m(\mathbb{R}^n)$ and let

$$F = \bigcup_{y \in \mathbb{R}^n,\, |y| < \varepsilon} (s_y)^* \omega$$

Using the definition of the operator $\tilde{\mathbf{Z}}$, we notice that $\tilde{\mathbf{Z}}T(\omega) = 0$ if the support of T does not meet F, which is the same as saying that the support of ω does not meet $E(T, \varepsilon)$. This means that the supprts of $\tilde{\mathbf{Z}}T$ is contained in $E(T, \varepsilon)$, proving b).

We observe that if $\{\omega_k\} \subset \mathscr{D}_m(\mathbb{R}^n)$ such that $\omega_k \to \omega$, then

$$\frac{s_y^* \omega_k - \omega_k}{|y|} \to \frac{s_y^* \omega - \omega}{|y|}.$$

Thus,

$$\frac{\int_{\mathbb{R}^n} f_\varepsilon(y) \cdot (s_y^* \omega_k)\, dy - \omega_k}{\varepsilon} \to \frac{\int_{\mathbb{R}^n} f_\varepsilon(y) \cdot (s_y^* \omega)\, dy - \omega}{\varepsilon},$$

which proves c).

The proof that $\tilde{\mathbf{Z}}T$ is C^∞ in $\mathbb{B}^n$ is totally analogous to the way in which the smoothing process of convolution is proven using that s_y is C^∞ and over $\mathbb{B}^n$ operatess as a translation like τ_y.

Clearly, from the definition of $\tilde{\mathbf{Z}}$, we have that $\tilde{\mathbf{Z}}T = T$ in $\mathbb{R}^n \setminus \mathbb{B}^n$.

If T is C^r on a neighborhood of a boundary point x_0 of $\mathbb{B}^n$, we can decompose T using bump functions into the sum $T = T_1 + T_2$ of currents where T_1 is C^r on $\mathbb{R}^n$ and where T_2 vanishes in a neighborhood of x_0. Thus, $\tilde{\mathbf{Z}}T_1$ is C^r in $\mathbb{R}^n$ and $\tilde{\mathbf{Z}}T_2$ vanishes in a neighborhood of x_0, so that $\tilde{\mathbf{Z}}T$ is C^r in a neighborhood f x_0. This proves d).

Q. E. D.

Remark 21. We can express property d) by saying that $\tilde{\mathbf{Z}}$ is **regularizing** over $\mathbb{B}^n$ and is nowhere deregularizing.

Remark 22. Thanks to the continuity of the currents and the linearity of the integration operator, the previous operators can be expressed as follows

$$\mathbf{Z}T(\omega) = T(\mathbf{Z}'\omega) \quad \text{and} \quad \tilde{\mathbf{Z}}T(\omega) = T(\tilde{\mathbf{Z}}'\omega)$$

where

$$\mathbf{Z}'\omega = \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot (\tau_y)^*(\omega)\, dy \quad \text{and} \quad \tilde{\mathbf{Z}}'\omega = \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot (s_y)^*(\omega)\, dy.$$

Theorem 5.6 (De Rham's Approximation Theorem, Theorem 12 of [**de 84**]). *On a manifold M, we can construct a linear operator $\mathcal{Z}$, depending on positive parameters $\varepsilon_1, \varepsilon_2, \dots$ which are finite or infinite in number according as M is compact or not, which have the following properties:*

(a) If T is an m dimesional current in M, then $\mathcal{Z}T$ is an m dimensional current.

(b) *The support of $\mathcal{Z}T$ is contained in any given neighborhood of the support of T provided that the parameters ε_i are sufficiently small.*
(c) *$\mathcal{Z}T$ is C^∞.*
(d) *If each ε_i tends to zero, $\mathcal{Z}T$ converges weakly to T.*

Proof. Since M is paracompact, we can find a locally finite covering $\{U_i\}$ of M such that U_i is diffeomorphic to the ball $\mathbb{B}^n$ via a diffeomorphism h_i which is C^∞ and which can be extended to neighborhoods of $\bar{U}_i$ and $\bar{\mathbb{B}}^n$. Using these diffeomorphisms, the operator $\tilde{\mathbf{Z}}$ defined on $\mathbb{R}^n$ can be transported to M. Let f be a non-negative C^∞ function which has its support in the neighborhood V_i of $\bar{U}_i$ given by the extension of h_i and which is equal to 1 on another smaller neighborhood of $\bar{U}_i$. If T is a current in M, then $T' = fT$ is a current which has its support contained in V_i and $(h_i)_*T'$ is a current which has its support contained in $h_i(V_i)$. The support of $T'' = T - T'$ does not intersect $\bar{U}_i$. By replacing the parameter ε occuring in $\tilde{\mathbf{Z}}$ and putting

$$\mathcal{Z}_i T = (h_i^{-1})_* \tilde{\mathbf{Z}} (h_i)_* T' + T'',$$

we define operators $\mathcal{Z}_i$ which possess properties in V corresponding exactly to those of the operator $\tilde{\mathbf{Z}}$ in $\mathbb{R}^n$. Here, U_i plays the role of $\mathbb{B}^n$ and ε_i replaces ε.

Put

$$\mathcal{Z}^{(l)} = \mathcal{Z}_l \circ \ldots \circ \mathcal{Z}_2 \circ \mathcal{Z}_1.$$

In the neighborhood of each compact set K, the operators $\mathcal{Z}^{(h)}$ reduce to the identity whenever h is sufficiently large because $\bar{U}_h$ does not meet K thanks to paracompacness. It follows that

$$\mathcal{Z} = \lim_{h\to\infty} \mathcal{Z}^{(h)}$$

is well defined, and there exists an integer l_0 depending only on the compact set K such that

$$\mathcal{Z}T(\omega) = \mathcal{Z}^{(l)}T(\omega)$$

for $l \geq l_0$ and for all currents T and all forms ω with support in K. This implies that $\mathcal{Z}T$ is a well defined current, proving a).

Property b) follows form the corresponding properties of each $\mathcal{Z}_i$. If the support of T is contained in an open set U, we can successively determine bounds for each of the parameters ε_i so that, as long as these bounds are not exceeded, the support of $\mathcal{Z}^{(l)}T$ remains in U.

The fact that $\mathcal{Z}T$ is C^∞ follows since $\mathcal{Z}^{(l)}T$ is C^∞ in $\cup_{j=1}^{i} U_j$ for $l \geq i$. By Proposition 5.5, $\mathcal{Z}_i$ regularizes in U_i and is not deregularizing anywhere.

To show d), we note that we have

$$\mathcal{Z}T(\omega) - T(\omega) = \sum_{l=1}^{\infty} (\mathcal{Z}^{(l)} - \mathcal{Z}^{(l-1)})T(\omega).$$

If $\omega_k \to \omega$, there exists an integer l_0 which depends only on the set L, where the convergence is given such that

$$\mathcal{Z}T(\omega) - T(\omega) = \sum_{l=1}^{l_0} (\mathcal{Z}^{(l)} - \mathcal{Z}^{(l-1)})T(\omega).$$

Thus, it is sufficient to prove that, as $\varepsilon_l \to 0$,

$$(\mathcal{Z}^{(l)} - \mathcal{Z}^{(l-1)})T(\omega_k) \to 0$$

in L and uniformly with respect to e_i $(i < l)$ which we also suppose bounded. Now, we have

$$(\mathcal{Z}^{(l)} - \mathcal{Z}^{(l-1)})T(\omega) = (\mathcal{Z}_h T - T)(\mathcal{Z}^{(h-1)'}\omega),$$

where

$$\mathcal{Z}^{(l)'}(\omega) = \mathcal{Z}'_l \circ \mathcal{Z}'_{l-1} \circ \ldots \circ \mathcal{Z}'_1(\omega)$$

and $\mathcal{Z}'_i$ is given in same way as the previous operators. Here we used Remark 22. Since

$$\mathcal{Z}'_l \circ \mathcal{Z}'_{l-1} \circ \ldots \circ \mathcal{Z}'_1(\omega_k) \to \mathcal{Z}'_l \circ \mathcal{Z}'_{l-1} \circ \ldots \circ \mathcal{Z}'_1(\omega),$$

property d) follows from property c) of Proposition 5.5. Q. E. D.

3.4. Nikolaev's Approximation Theorem. Before presenting the approximation theorem, we need to discuss how the spaces will converge.

If X and Y are two metric spaces, the **dilatation** of a Lipschitz map $f : X \to Y$ is defined by

$$\operatorname{dil}(f) = \sup_{x,x' \in X} \frac{d_Y(f(x), f(x'))}{d_X(x, x')},$$

where d_X and d_Y are the metrics of X and Y, respectively. A homeomorphism $f : X \to Y$ is called **bi-Lipschitz** if both f and f^{-1} are Lipschitz maps.

The **Lipschitz distance** d_L between two metric spaces X and Y is defined by

$$d_L(X, Y) = \inf_{f : X \to Y} \log\left(\max\left\{\operatorname{dil}(f), \operatorname{dil}(f^{-1})\right\}\right),$$

where the infimum is taken over all bi-Lipschitz homeomorphisms $f : X \to Y$.

A sequence $\{X_l\}_{l=1}^{\infty}$ of metric spaces **converges in the Lipschitz sense** to a metric space X if $d_L(X_l, X) \to 0$ as $l \to \infty$.

Remark 23. Let $(\mathcal{M}, d(\mathsf{g}_0))$ be a space with bounded curvature, where d is the metric, g_0 is the Riemannian metric, and $\mathfrak{T}$ is the $C^{3,\alpha}$-smooth structure on $\mathcal{M}$ containing the harmonic atlas $\mathfrak{h}_0$ given by Theorem 4.25. By virtue of Whitney's theorem [**Whi36**] or [**Hir76**, Theorem 2.9] in $\mathfrak{T}$ one can choose a C^{∞}-smooth atlas $\mathfrak{h}$, i.e, any chart of $\mathfrak{h}$ is a chart of $\mathfrak{T}$.

Now we can state the approximation result due to I. Nikolaev in [**Nik91**].

THEOREM 5.7 (Nikolaev's Aproximation Theorem). *Let* $(\mathcal{M}, d(\mathsf{g}_0))$ *be a space with bounded curvature. Then, on the differentiable manifold* $\mathcal{M}$ *with atlas* $\mathfrak{h}$*, one can define a sequence of infinitely differentiable Riemannian metrics* $\{\mathsf{g}_m\}_{m=1}^{\infty}$ *having the following properties:*

(1) The metric spaces $(\mathcal{M}, d(\mathsf{g}_m))$ *converge in the Lipschitz sense to the metric space* $(\mathcal{M}, d(\mathsf{g}_0))$*.*

(2) The following estimates hold for the limits of curvature:

$$\limsup_{m\to\infty} \bar{k}_m(\mathcal{M}) \leq \bar{k}_0(\mathcal{M}) \text{ and } \liminf_{m\to\infty} \underline{k}_m(\mathcal{M}) \geq \underline{k}_0(\mathcal{M}),$$

where $\bar{k}_l(\mathcal{M})$ *and* $\underline{k}_l(\mathcal{M})$ *denote the upper and lower limits of sectional curvature of the spaces* $(\mathcal{M}, \mathsf{g}_l)$*,* $l = 0, 1, \cdots$*.*

COROLARY 5.8. *If* $(\mathcal{M}, d(\mathsf{g}_0))$ *is a complete metric space, then there exists* $m_0 \in \mathbb{N}$ *such that* $(\mathcal{M}, d(\mathsf{g}_m))$ *is complete for* $m \geq m_0$*.*

REMARK 24. In the proof of the previous theorem the curvature is computed using Theorem 4.26.

REMARK 25. Although we do not include the proof of Theorem 5.7, the construction of the metrics g_m is analogous to the construction of the current $\mathcal{Z}T$ in Theorem 5.6. For the proof of Theorem 6.11 we provide a construction of such a metric.

Part 3

Equivariant Aproximation Theorems

CHAPTER 6

Equivariant Approximation Theorem

We present in this chapter the original work of this thesis. In the first section of this chapter we recall some basic results from Lie group theory we will need such as the Tubular Neighborhood Theorem and Haar measure. For further information on group actions, see [**AB15**] or [**Bre72**].

After that, in section 2 we construct a special cover to define an equivariant regularization operator on the space of currents of a riemmanian manifold with a Lie group acting on it. With this operator we prove an equivariant version of de Rham's Approximation Theorem.

As in the case of currents we also define an equivariant regularization operator on the space of riemannian metrics in a compact metric space with bounded curvature with a Lie group acting on it. Using it we prove an equivariant version of Nikolaev's Approximation Theorem.

We point out that every metric isometry of a space of bounded curvature is a Riemannian isometry (i.e., a diffeomorphism preserving the Riemannian metric) (see [**MS39**, Theorem 2]) and that the isometry group of a compact space with bounded curvature is a compact Lie group (see [**MS39, FY94, GGG13, DW28**]).

At the end of the chapter we get an equivariant sphere theorem.

1. Lie groups

In the following discussion, we consider a compact smooth Riemannian manifold $(\mathcal{M}, \mathsf{g})$ and a compact Lie group $\mathcal{G}$. A **(left) action of $\mathcal{G}$ on $\mathcal{M}$**, or a **(left) $\mathcal{G}$-action on $\mathcal{M}$**, is a smooth map $\alpha\colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ satisfying:

(i) $\alpha(e, x) = x$, for all $x \in \mathcal{M}$ and e is the identity element of the group;
(ii) $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$, for all $g_1, g_2 \in \mathcal{G}$ and $x \in \mathcal{M}$.

Whenever α is implicit, we denote the left action as $\mathcal{G} \curvearrowright \mathcal{M}$. The manifold $\mathcal{M}$ is called a **$\mathcal{G}$-manifold**, and it is common to denote $\alpha(g, x)$ by $g \cdot x$ or gx. We can define a right action in an analogous way. Although we are only considering left actions, we remark that all processes can be done in an analogous way for right actions. Given an action of $\mathcal{G}$ on $\mathcal{M}$ and a point $x \in \mathcal{M}$, the subgroup

$$\mathcal{G}_x := \{g \in \mathcal{G} : \alpha(g, x) = x\} \subset \mathcal{G}$$

is the **isotropy group** of $x \in \mathcal{M}$, and

$$\mathcal{G}(x) := \{\alpha(g,x) : g \in \mathcal{G}\} \subset \mathcal{M}$$

is called the **orbit** of $x \in \mathcal{M}$.

If $\mathcal{G}_x = \{e\}$ for all $x \in \mathcal{M}$, the action is said to be **free**.

Given an action, $g \in \mathcal{G}$ and $x \in \mathcal{M}$, we can define two auxiliary maps:

$$\begin{array}{rcl} \alpha^g \colon \mathcal{M} & \to & \mathcal{M} \\ x & \mapsto & \alpha(g,x), \end{array} \qquad \begin{array}{rcl} \alpha_x \colon \mathcal{G} & \to & \mathcal{M} \\ g & \mapsto & \alpha(g,x). \end{array}$$

Since α is smooth, the transformation $\alpha^g \colon \mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and hence $\alpha^{\mathcal{G}} := \{\alpha^g : g \in \mathcal{G}\}$ can be identified with a subgroup of the diffeomorphism group $\mathrm{Diff}(\mathcal{M})$. An orbit $\mathcal{G}(x)$ consists of all posible images $\alpha^g(x)$ for $g \in \mathcal{G}$ and the isotropy group $\mathcal{G}_x$ consist of all $g \in \mathcal{G}$ that fix x, i.e. $\alpha^g(x) = x$. An action on $(\mathcal{M}, \mathsf{g})$ is said to be by **isometries**, if α^g is an isometry of $(\mathcal{M}, \mathsf{g})$ for all $g \in \mathcal{G}$. In this case, the metric g is said to be **$\mathcal{G}$-invariant**, and $\alpha^{\mathcal{G}}$ can be identified with a subgroup of $\mathrm{Isom}(\mathcal{M}, \mathsf{g})$.

Now we consider $\mathcal{G}$-actions $\alpha_1 \colon \mathcal{G} \times \mathcal{M}_1 \to \mathcal{M}_1$ and $\alpha_2 \colon \mathcal{G} \times \mathcal{M}_2 \to \mathcal{M}_2$. A map $f \colon \mathcal{M}_1 \to \mathcal{M}_2$ is called **$\mathcal{G}$-equivariant** if $\alpha_2(g, f(x)) = f(\alpha_1(g,x))$ for all $x \in \mathcal{M}_1$ and $g \in \mathcal{G}$.

If two orbits $\mathcal{G}(x)$ and $\mathcal{G}(y)$ have nontrivial intersection, then they coincide. This means that orbits of a $\mathcal{G}$-action on $\mathcal{M}$ form a partition of $\mathcal{M}$. Hence, we can consider the set of equivalence classes

$$\mathcal{M}/\mathcal{G} := \{\mathcal{G}(x) : x \in \mathcal{M}\},$$

called the **orbit space** of the $\mathcal{G}$-action on $\mathcal{M}$. The natural projection $\pi \colon \mathcal{M} \to \mathcal{M}/\mathcal{G}$ given by $\pi(x) := \mathcal{G}(x)$, is called **projection map**, and the topology on $\mathcal{M}/\mathcal{G}$ is determined by declaring that $U \subset \mathcal{M}/\mathcal{G}$ is open if its preimage $\pi^{-1}(U) \subset \mathcal{M}$ is open. This implies that π is continuous and open.

An action $\alpha \colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ is **proper** if the map

$$\mathcal{G} \times \mathcal{M} \ni (g,x) \mapsto (\alpha(g,x), x) \in \mathcal{M} \times \mathcal{M}$$

is proper.

Proposition 6.1 (Proposition 3.19 of [**AB15**]). *All actions by compact Lie groups are proper.*

The next two results give us a description of the orbits and the orbit space.

Theorem 6.2 (Theorem 3.34 of [**AB15**]). *Let $\alpha \colon \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ be a free proper right action. Then the orbit space $\mathcal{M}/\mathcal{G}$ admits a smooth structure such that $\mathcal{G} \to \mathcal{M} \to \mathcal{M}/\mathcal{G}$ is a principal $\mathcal{G}$-bundle, where the bundle projection map $\rho \colon \mathcal{M} \to \mathcal{M}/\mathcal{G}$ is the projection map.*

Remark 26. The smooth structure on $\mathcal{M}/\mathcal{G}$ is such that ρ is smooth, and a map $h\colon \mathcal{M}/\mathcal{G} \to N$ is smooth if and only if $h \circ \rho$ is smooth. These properties uniquely characterize the smooth structure of $\mathcal{M}/\mathcal{G}$.

Proposition 6.3 (Proposition 3.41 of [**AB15**]). *Let $\alpha\colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ be a left action and define $\tilde{\alpha}_x\colon \mathcal{G}/\mathcal{G}_x \to \mathcal{M}$ by $\tilde{\alpha}_x \circ \rho = \alpha_x$ for some point $x \in \mathcal{M}$ and $\rho\colon \mathcal{G} \to \mathcal{G}/\mathcal{G}_x$ is the quotient map, i.e we have that Diagram (6.1.1) conmutes. Then $\tilde{\alpha}_x$ is a $\mathcal{G}$-equivariant injective immersion, with image $\mathcal{G}(x)$. In particular, $\mathcal{G}(x)$ is an immersed submanifold of $\mathcal{M}$ whose tangent space at $x \in \mathcal{M}$ is $T_x\mathcal{G}(x) = d(\alpha_x)_e(T_e\mathcal{G})$. In addition, if the action is proper, then $\tilde{\alpha}_x$ is an embedding and $\mathcal{G}(x)$ is an embedded submanifold of $\mathcal{M}$.*

$$\begin{tikzcd} \mathcal{G} \arrow[d, "\rho"] \arrow[dr, "\alpha_x"] & \\ \mathcal{G}/\mathcal{G}_x \arrow[r, "\tilde{\alpha}_x"] & \mathcal{M} \end{tikzcd} \tag{6.1.1}$$

With the basic definitions in hand, we now discuss two fundamental results in the theory of $\mathcal{G}$-manifolds. If $\alpha\colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ is an action, a **slice** at $x_0 \in \mathcal{M}$ is an embedded submanifold S_{x_0} containing x_0 and satisfying the following properties:

(i) $T_{x_0}\mathcal{M} = d\alpha_{x_0}(T_e\mathcal{G}) \oplus T_{x_0}S_{x_0}$ and $T_x\mathcal{M} = d\alpha_x(T_e\mathcal{G}) + T_xS_{x_0}$, for all $x \in S_{x_0}$;
(ii) S_{x_0} is invariant under $\mathcal{G}_{x_0}$, i.e., if $x \in S_{x_0}$ and $g \in \mathcal{G}_{x_0}$, then $\alpha(g, x) \in S_{x_0}$;
(iii) If $x \in S_{x_0}$ and $g \in \mathcal{G}$ are such that $\alpha(g, x) \in S_{x_0}$, then $g \in \mathcal{G}_{x_0}$.

Theorem 6.4 (Slice Theorem, Theorem 3.49 of [**AB15**]). *Let $\alpha\colon \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ be a proper action and $x_0 \in \mathcal{M}$. Then there exists a slice S_{x_0} at x_0.*

Let α be a proper $\mathcal{G}$-action on $\mathcal{M}$. Given $x_0 \in \mathcal{M}$, let S_{x_0} be a slice at x_0. We define a **tubular neighborhood** of the orbit $\mathcal{G}(x_0)$ as the image of S_{x_0} under the $\mathcal{G}$-action, that is,

$$\mathrm{Tub}(\mathcal{G}(x_0)) := \alpha(\mathcal{G}, S_{x_0}).$$

Theorem 6.5 (Tubular Neighborhood Theorem, Theorem 3.57 of [**AB15**]). *Let α be a proper action of $\mathcal{G}$ on $\mathcal{M}$. For every $x_0 \in \mathcal{M}$, there exists a $\mathcal{G}$-equivariant diffeomorphism between Tub$(\mathcal{G}(x_0))$ and the total space of the associated bundle with fiber S_{x_0},*

$$S_{x_0} \to \mathcal{G} \times_{\mathcal{G}_{x_0}} S_{x_0} \to \mathcal{G}/\mathcal{G}_{x_0},$$

to the principal $\mathcal{G}_{x_0}$-bundle $\mathcal{G}_{x_0} \to \mathcal{G} \to \mathcal{G}/\mathcal{G}_{x_0}$.

We define a method to produce $\mathcal{G}$-invariant functions. We begin by considering a function $f\colon \mathcal{G} \to \mathbb{R}$. For $h \in \mathcal{G}$, we define the functions R_hf and L_hf by $R_hf(g) = f(gh)$ and $L_hf(g) = f(h^{-1}g)$.

THEOREM 6.6 (Theorem 3.1 of [**Bre72**]). *Let $\mathcal{G}$ be a compact group. Then there is a unique function $I\colon C^0(\mathcal{G},\mathbb{R}) \to \mathbb{R}$, called **the Haar measure**, defined for continuous funtions $f\colon \mathcal{G} \to \mathbb{R}$, such that*

(a) $I(f_1 + f_2) = I(f_1) + I(f_2)$.
(b) $I(cf) = cI(f)$, where $c \in \mathbb{R}$.
(c) If $f(g) \geq 0$ for all $g \in \mathcal{G}$, then $I(f) \geq 0$.
(d) If $f(g) = 1$ for every g, then $I(f) = 1$.
(e) If $f(g) \geq 0$ and is not identically zero for all $g \in \mathcal{G}$, then $I(f) > 0$.
(f) $I(Id) = 1$.
(g) $I(R_h f) = I(f) = I(L_h f)$ for all $h \in G$.
(h) $I(f(g)) = I(f(g^{-1}))$

We use the notation

$$\int_{\mathcal{G}} f(g)\ dg$$

for $I(f)$. We call it **integral of the function** f **over the group** $\mathcal{G}$.

We have the next property.

PROPOSITION 6.7 (Proposition 3.2 of [**Bre72**]). *Let $f\colon \mathcal{G} \times A \to \mathbb{R}$ be continuous, where A is any topological space and $\mathcal{G}$ is a compact group. Then the funcion $F : A \to \mathbb{R}$ defined by*

$$F(a) = \int_{\mathcal{G}} f(g,a)\ dg$$

is continuous.

2. De Rham's Equivariant Approximation Theorem

We say that an m-current T on $\mathcal{M}$ is **$\mathcal{G}$-invariant** if $(\alpha^g)_* T = T$ for all $g \in \mathcal{G}$, where $\mathcal{G}$ is a compact Lie group acting on $\mathcal{M}$ by diffeomorphisms.

Almost as a remark we consider the special case when the manifold is $\mathbb{R}^n$ itself.

PROPOSITION 6.8. *Let $\mathcal{G}$ be a compact Lie group acting on $\mathbb{R}^n$ and let T be a m-current. The operators $\mathbf{Z}$ and $\tilde{\mathbf{Z}}$ defined in Subsections 3.2 and 3.3 are $\mathcal{G}$-invariant.*

PROOF. This result is obtained observing that the following equations are satisfied for both operators:

$$\begin{aligned}(\alpha^g)_*\mathbf{Z}T(\omega) = \mathbf{Z}T(\rho(\alpha^g)^*\omega) &= \int_{\mathbb{R}^n} f_\varepsilon(y)\cdot(\tau_y)_*(T)(\rho(\alpha^g)^*\omega)\,dy,\\ &= \int_{\mathbb{R}^n} f_\varepsilon(y)\cdot(\tau_y)_*\,(\alpha^g)_*(T)(\omega)\,dy,\\ &= \int_{\mathbb{R}^n} f_\varepsilon(y)\cdot(\tau_y)_*(T)(\omega)\,dy,\\ &= \mathbf{Z}T(\omega),\end{aligned}$$

where ρ is a C_c^∞ function that is 1 in a neighborhood of the compact set

$$\left(\alpha^g|_{\mathrm{spt}(\mathbf{Z}T)}\right)^{-1}(\mathrm{spt}(\omega)).$$

Q. E. D.

Before we consider the case of a manifold, we have the following property about invariant currents.

PROPOSITION 6.9. *Let $\mathcal{G}$ be a compact Lie group acting on a manifold $\mathcal{M}$ and let T be a $\mathcal{G}$-invariant m-current on $\mathcal{M}$. Then*

$$\left|(\alpha^g)_*\,T(\omega)\right| \le C\,|T(\omega)|$$

for every $\omega\in\mathscr{D}^m(\mathcal{M})$.

PROOF. We take $\omega\in\mathscr{D}^m(\mathcal{M})$. Then

$$\left|(\alpha^g)_*\,T(\omega)\right| = \left|T\left(\rho\,(\alpha^g)^*\,\omega\right)\right| \le \left|T\left(\rho\,(\alpha^{g_0})^*\,\omega\right)\right| \le C\,|T(\omega)|\,.$$

The first inequality works because the action is continuous and $\mathcal{G}$ is compact for some $g_0\in\mathcal{G}$. Q. E. D.

THEOREM 6.10 (Equivariant De Rham's Approximation Theorem). *Let $\mathcal{G}$ be a compact Lie group acting on a manifold $\mathcal{M}$ and let T be an m-current on $\mathcal{M}$. If T is $\mathcal{G}$-invariant, then we can construct an operator $\mathcal{Z}_{\mathcal{G}}$ such that $\mathcal{Z}_{\mathcal{G}}T$ has the following properties:*

(a) $\mathcal{Z}_{\mathcal{G}}T$ is an m-current of class C^∞.
(b) $\mathcal{Z}_{\mathcal{G}}T$ is a $\mathcal{G}$-invariant current.
(c) If ε tends to zero, $\mathcal{Z}_{\mathcal{G}}T$ converges weakly to T.

PROOF. Let α be the action of $\mathcal{G}$ on $\mathcal{M}$. The first thing we have to do is to build a special cover since we are going to use the operator $\mathcal{Z}$ defined in Theorem 5.6 which actually depends on the cover. Thanks to Theorem 6.5, given a point $p \in \mathcal{M}$ and its orbit $\mathcal{G}(p)$, the tubular neighborhood of this orbit can be seen as

$$\mathrm{Tub}(\mathcal{G}(p)) = \bigcup_{g\in\mathcal{G}} \alpha(g,V) = \bigcup_{g\in\mathcal{G}} gV$$

where V is a neighborhood of p and it is a **unit regular coordinate ball**, i.e., V is a smooth coordinate domain whose image under a smooth coordinate map φ is $\mathbb{B}^n \subset \mathbb{R}^n$ such that for some $r<1$ and $\bar{U} \subset V$:

$$\varphi(U) = B_r(0), \quad \varphi(\bar{U}) = \bar{B}_r(0) \quad \text{and} \quad \varphi(V) = \mathbb{B}^n.$$

We say that such tubular neighborhood

$$\bigcup_{g\in\mathcal{G}} gV$$

is **generated** by V. Finally, we get our special tubular cover built by the previous process:

$$\left\{\bigcup_{g\in\mathcal{G}} gV_i\right\}_{i\in\mathbb{N}}.$$

Now we want to apply the operator $\mathcal{Z}$ on one of these sets gV_i. We use the corresponding coordinate chart $\varphi_i : V_i \to \mathbb{R}^n$ and a nonnegative function h_i of class C^∞ with support in V_i that is equal to 1 in a neighborhood of $\bar{U}_i$ contained in V_i. If $T \in \mathscr{D}_m(\mathcal{M})$, then we set $T' = h_iT$ and $T'' = T - T'$. We note that T' is an m-current that is equal to T in $\bar{U}_i$ and its support is contained in V_i. Then

$$\begin{aligned}
\mathcal{Z}|_{gV_i} T(\omega) \quad &:= \quad \left(\alpha^g \circ \varphi_i^{-1}\right)_* \tilde{\mathbf{Z}} \left(\varphi_i \circ (\alpha^g)^{-1}\right)_* (\alpha^g)_* h_i T(\omega) + (\alpha^g)_* T''(\omega) \\
&= \quad \left(\alpha^g \circ \varphi_i^{-1}\right)_* \tilde{\mathbf{Z}} \left(\varphi_i \circ (\alpha^g)^{-1}\right)_* \left(h_i \circ (\alpha^g)^{-1}\right) (\alpha^g)_* T(\omega) + (\alpha^g)_* T''(\omega) \\
&= \quad \left(\alpha^g \circ \varphi_i^{-1}\right)_* \tilde{\mathbf{Z}} \left(h_i \circ (\alpha^g)^{-1}\right) \left(\varphi_i \circ (\alpha^g)^{-1}\right)^{-1} \left(\varphi_i \circ (\alpha^g)^{-1}\right)_* (\alpha^g)_* T(\omega) \\
&\quad\ + (\alpha^g)_* T''(\omega) \\
&= \quad (\alpha^g)_* \left(\varphi_i^{-1}\right)_* \tilde{\mathbf{Z}} \left(h_i \circ \varphi_i^{-1}\right) (\varphi_i)_* T(\omega) + (\alpha^g)_* T''(\omega) \\
&= \quad (\alpha^g)_* \left(\varphi_i^{-1}\right)_* \tilde{\mathbf{Z}} (\varphi_i)_* h_i T(\omega) + (\alpha^g)_* T''(\omega) \\
&= \quad (\alpha^g)_* \mathcal{Z}|_{V_i} T(\omega),
\end{aligned} \tag{6.2.1}$$

for every $\omega \in \mathscr{D}^m(\mathcal{M})$ and thanks to the following equation

$$T = (\alpha^g)_* T = (\alpha^g)_*(T' + T'') = (\alpha^g)_* T' + (\alpha^g)_* T''$$

and noticing that $(\alpha^g)_* T'$ is an m-current that is equal to T in $g\bar{U}_i$ and its support is contained in gV_i by construction.

After all the calculations in (6.2.1) we can define an operator on the tube $\bigcup_{g\in\mathcal{G}} gV_i$ using Theorem 6.6 as follows:

$$\mathcal{Z}_i^{\mathcal{G}} T(\omega) = \frac{1}{|\mathcal{G}|}\int_{\mathcal{G}} (\alpha^g)_* \, \mathcal{Z}|_{V_i} T(\omega)\; dg,$$

where $|\mathcal{G}|$ denotes the volume of $\mathcal{G}$. Finally, in the same way as in the proof of de Rham's approximation theorem we can define the operator $\mathcal{Z}^{\mathcal{G}} : \mathscr{D}_m(\mathcal{M}) \to \mathscr{D}_m(\mathcal{M})$:

$$\mathcal{Z}^{\mathcal{G}} T(\omega) = \lim_{l\to\infty} \mathcal{Z}_l^{\mathcal{G}} \circ \cdots \mathcal{Z}_2^{\mathcal{G}} \circ \mathcal{Z}_1^{\mathcal{G}} T(\omega).$$

Thanks to Proposition 6.7 the limit exists and it is invariant because of the way we constructed it.

Since the Haar integral satisfies the usual properties of theory of integration, the compactness of $\mathcal{G}$ and Proposition 6.7, we get that $\mathcal{Z}^{\mathcal{G}} T$ is a current of class C^∞.

Finally, $\mathcal{Z}^{\mathcal{G}}$ weakly converges because it does on every step of the construction and we use the properties of the integral and Proposition 6.9:

$$\begin{aligned} |\mathcal{Z}_i^{\mathcal{G}} T(\omega_k)| &= \left|\frac{1}{|\mathcal{G}|}\int_{\mathcal{G}} (\alpha^g)_* \, \mathcal{Z}|_{V_i} T(\omega_k)\; dg\right| \le \frac{1}{|\mathcal{G}|}\int_{\mathcal{G}} \left|(\alpha^g)_* \, \mathcal{Z}|_{V_i} T(\omega_k)\right|\; dg \\ &\le C \left|\mathcal{Z}|_{V_i} T(\omega_k)\right| \to C\,|T(\omega)| \text{ as } k\to\infty. \end{aligned}$$

Q. E. D.

3. Equivariant Approximation on Spaces with Bounded Curvature

We introduce some basic notation and definitions in preparation to prove the equivariant version of Nikolaev's Theorem. Our proof is based (or generalizes) Nikolaev's proof in [**Nik91**].

If $U \subset \mathbb{R}^n$, we denote by $\mathfrak{M}^{2,p}(U)$ the space of Riemannian metrics

$$\mathsf{g}(x) = \big(\mathsf{g}_{ij}(x)\big)_{i,j=1,\dots,n}$$

which are twice continuously differentiable almost everywhere on the domain U with $1 \le p \le \infty$ for which the following norm is finite:

$$|\mathsf{g}|_{\mathfrak{M}^{2,p}(U)} = \max_{i,j}\left\{|g_{ij}|_{W^{2,p}(U)}\right\}.$$

Let $\mathcal{M}$ be a differentiable manifold with fixed C^∞-atlas $\mathfrak{h} = \{(U_i,\varphi_i)\}_{i\in\mathbb{N}}$. By $\mathfrak{M}^{2,p}_{\mathfrak{h}}(\mathcal{M})$ we denote the set of continuous Riemannian metrics g on $\mathcal{M}$ for which the following seminorms are finite:

$$|\mathsf{g}|_{\mathfrak{M}^{2,p}_{\mathfrak{h}}(\mathcal{M}),i} = \left|\left(\varphi_i^{-1}\right)^* \mathsf{g}\right|_{\mathfrak{M}^{2,p}(V_i)}, \quad V_i = \varphi_i(U_i), \quad i\in\mathbb{N}.$$

One defines spaces $\mathfrak{M}^{r,\alpha}_{\mathfrak{h}}(\mathcal{M})$, $r\in\mathbb{N}$, $0<\alpha<1$, with seminorms $|\mathsf{g}|_{\mathfrak{M}^{r,\alpha}_{\mathfrak{h}}(\mathcal{M}),i}$ analogously using $C^{r,\alpha}(U)$ instead of $W^{2,p}(U)$.

We denote by $\mathfrak{M}^{\infty}_{\mathfrak{h}}(\mathcal{M})$ the **set of smooth Riemannian metrics** on $\mathcal{M}$ which are infinitely differentiable with respect to the atlas $\mathfrak{h}$.

Let $\mathsf{g} \in \mathfrak{M}^{2,p}_{\mathfrak{h}}(\mathcal{M})$. Then at almost each point $p \in \mathcal{M}$ and section $\sigma \subset T_p\mathcal{M}$ we can formally calculate the sectional curvature $K_\sigma(p)$ with respect to the Riemannian metric g, by using the Christoffel symbols and the fact that our metrc has second derivatives almost everywhere. By $\underline{K}_{\mathsf{g},\mu}$, $\bar{K}_{\mathsf{g},\mu}$, we denote the essential infimum and essential supremum, respectively, of $K_\sigma(p)$ for every point $p \in \mathcal{M}$ and every section $\sigma \subset T_p\mathcal{M}$.

Theorem 6.11 (Equivariant Nikolaev's Aproximation Theorem). *Let $(\mathcal{M}, d(\mathsf{g}_0))$ be a compact space with bounded curvature and a compact Lie group $\mathcal{G}$ acting on it by isometries. Then on the differentiable manifold $\mathcal{M}$ with atlas $\mathfrak{h}$ one can define a sequence of infinitely differentiable Riemannian metrics $\{\mathsf{g}_k\}_{k=1}^{\infty}$ having the following properties:*

(1) The Lie group $\mathcal{G}$ acts on $(\mathcal{M}, d(\mathsf{g}_k))$ by isometries.
(2) The metric spaces $(\mathcal{M}, d(\mathsf{g}_k))$ converge in the Lipschitz sense to the metric space $(\mathcal{M}, d(\mathsf{g}_0))$.
(3) The following estimates hold for the limits of the curvature:

$$\limsup_{k\to\infty} \bar{k}_k(\mathcal{M}) \le \bar{k}_0(\mathcal{M}) \ \text{ and } \ \liminf_{k\to\infty} \underline{k}_k(\mathcal{M}) \ge \underline{k}_0(\mathcal{M}),$$

where $\bar{k}_r(\mathcal{M})$ and $\underline{k}_r(\mathcal{M})$ denote the upper and lower limits of curvature of the spaces $(\mathcal{M}, \mathsf{g}_r)$, $r = 0, 1, \ldots$

Proof. We will use the same cover constructed in the proof of Theorem 6.10:

$$\left\{ \bigcup_{g\in\mathcal{G}} gV_i \right\}_{i\in\mathbb{N}} .$$

As before, we want to define the regularization operator of metrics $\mathcal{H}$ in gV_i. To do this, we use the corresponding coordinate chart $\varphi_i : V_i \to \mathbb{R}^n$ and a nonnegative function h_i of class C^∞ with support in V_i that is equal to 1 on a neighborhood of $\bar{U}_i$ contained in V_i. Then we decompose g_0 into two smooth metric tensors $\mathsf{g}_0 = \mathsf{g}_i' + \mathsf{g}_i''$, where $\mathsf{g}_i' = h_i\,\mathsf{g}_0$ and $\mathsf{g}_i'' = \mathsf{g}_0 - \mathsf{g}_i'$. We let

$$\mathcal{H}_{e,i}(\mathsf{g}_0) = \left(\alpha^e \circ \varphi_i^{-1}\right)^* \tilde{\mathbf{H}}_\varepsilon \left(\varphi_i \circ (\alpha^e)^{-1}\right)^* (\alpha^e)^* h_i\,\mathsf{g}_0 + (\alpha^e)^*\,\mathsf{g}_i'',$$

where

$$\tilde{\mathbf{H}}_\varepsilon(\mathsf{g}) = \int_{\mathbb{R}^n} f_\varepsilon(y) \cdot (s_y)^*(\mathsf{g})\,dy$$

for a Riemannian metric g on $\mathbb{R}^n$ and $e \in \mathcal{G}$ is the identity element. We note that

$$\mathsf{g}_{e,i}(x) := \mathcal{H}_{e,i}(\mathsf{g}_0)\big|_x = \mathsf{g}_0(x)$$

for $x \in \mathcal{M} \setminus V_i$ and it follows from the last two equations that $\mathrm{g}_{g,i}$ is a Riemannian metric on $\mathcal{M}$. It is the regularization process of the metric on V_i and if we want to do it on gV_i, we take the decomposition $\mathrm{g}_0 = (\alpha^g)^* \, \mathrm{g}_0 = (\alpha^g)^* \, \mathrm{g}_i' + (\alpha^g)^* \, \mathrm{g}_i''$ and then

$$\begin{aligned} \mathcal{H}_{g,i}(\mathrm{g}_0) &:= \left(\alpha^g \circ \varphi_i^{-1}\right)^* \tilde{\mathbf{H}}_\varepsilon \left(\varphi_i \circ (\alpha^g)^{-1}\right)^* (\alpha^g)^* \, h_i \, \mathrm{g}_0 + (\alpha^g)^* \, \mathrm{g}_i'' \\ &= (\alpha^g)^* \, \mathcal{H}_{e,i}(\mathrm{g}_0). \end{aligned}$$

As in the proof of Theorem 6.10, we define a metric on the tube

$$\bigcup_{g \in \mathcal{G}} gV_i$$

using Theorem 6.6 by letting

$$\mathcal{H}_i^{\mathcal{G}}(\mathrm{g}_0) = \frac{1}{|\mathcal{G}|} \int_{\mathcal{G}} (\alpha^g)^* \, \mathcal{H}_{e,i}(\mathrm{g}_0) \, dg.$$

Finally, also as in the proof of Theorem 6.10, we define the operator

$$\mathcal{H}^{\mathcal{G}} : \mathfrak{M}_{\mathfrak{h}}^{2,p}(\mathcal{M}) \to \mathfrak{M}_{\mathfrak{h}}^{\infty}(\mathcal{M})$$

by setting:

$$\mathcal{H}^{\mathcal{G}}(\mathrm{g}_0) = \lim_{s \to \infty} \mathcal{H}_s^{\mathcal{G}} \circ \cdots \circ \mathcal{H}_2^{\mathcal{G}} \circ \mathcal{H}_1^{\mathcal{G}}(\mathrm{g}_0),$$

where $\mathfrak{h}$ is the atlas defined above. Since $\mathcal{M}$ is compact, the limit exists, i.e. we finish the regularization process and defines a $\mathcal{G}$-invariant Riemannian metric on $\mathcal{M}$. To see this fact we take coordinates φ_i in gV_i for some $i \in \mathbb{N}$ and $g \in \mathcal{G}$, and coordinate vector fields $\{\partial_j\}_{j=1}^n$, then

$$\mathcal{H}^{\mathcal{G}}(\mathrm{g}_0)(\partial_l, \partial_m) = \mathcal{H}^{\mathcal{G}}(\mathrm{g}_0)_{l\,m} = \mathcal{H}_{s_i'}^{\mathcal{G}} \circ \cdots \circ \mathcal{H}_{s_i}^{\mathcal{G}}(\mathrm{g}_0)_{l\,m} = \mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m} = (\mathrm{g}_0)_{l\,m}$$

Also we notice that

$$\left|\mathcal{H}_i^{\mathcal{G}}(\mathrm{g}_0)_{l\,m}\right| = \left|\int_{\mathcal{G}} (\alpha^g)^* \, \mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m} \, dg\right| \leq \int_{\mathcal{G}} \left|(\alpha^g)^* \, \mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m}\right| \, dg$$

and since $\mathcal{G}$ is compact there exists $\tilde{g} \in \mathcal{G}$ such that

$$\int_{\mathcal{G}} \left|(\alpha^g)^* \, \mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m}\right| \, dg \leq \int_{\mathcal{G}} \left|\left(\alpha^{\tilde{g}}\right)^* \mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m}\right| \, dg.$$

Now, since $\mathcal{M}$ is compact, we get a constant C such that

$$\int_{\mathcal{G}} \left|\left(\alpha^{\tilde{g}}\right)^* \mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m}\right| \, dg \leq C \int_{\mathcal{G}} \left|\mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m}\right| \, dg.$$

Therefore,

$$\left|\mathcal{H}_i^{\mathcal{G}}(\mathrm{g}_0)_{l\,m}\right| \leq C \, |\mathcal{G}| \, \left|\mathcal{H}_{e,i}(\mathrm{g}_0)_{l\,m}\right|. \tag{6.3.1}$$

To continue with the proof we need the following couple of estimates. The first one is exactly the same as Lemma 3.1 of [**Nik91**]. The second one is similar to Lemma 3.2 also from [**Nik91**].

LEMMA 6.12 (Lemma 3.1 of [**Nik91**]). *Let $U \subset \mathbb{R}^n$ be a domain. The operator $\tilde{\mathbf{H}}_\varepsilon$ maps $\mathfrak{M}^{2,p}(U)$ into $\mathfrak{M}^{2,p}(U)$ for any $1 \le p \le \infty$ while for each positive number $\delta > 0$ we can find a $\nu_{\delta,p} > 0$ such that for all $0 < \varepsilon < \nu_{\delta,p}$ we have:*

(1) For $1 \le p \le \infty$ we have the estimate

$$\left|\tilde{\mathbf{H}}_\varepsilon(\mathsf{g}) - \mathsf{g}\right|_{\mathfrak{M}^{2,p}(U)} \le \delta. \tag{6.3.2}$$

(2) Let $K_\sigma(x)$, $K^\varepsilon_\sigma(x)$ be the sectional curvatures calculated from the metrics $\mathsf{g} \in \mathfrak{M}^{2,p}(U)$ and $\mathsf{g}_\varepsilon = \tilde{\mathbf{H}}_\varepsilon(\mathsf{g})$, respectively, where $x \in U$ and $\sigma \subset T_xU$ is a section (i.e. a 2-dimensional subspace). Then if

$$\begin{aligned} -\infty < \quad & \bar{K}_\sigma(U) = \operatorname{ess\,sup}_{x\in U}\{K_\sigma(x)\} \quad < \infty, \\ -\infty < \quad & \underline{K}_\sigma(U) = \operatorname{ess\,inf}_{x\in U}\{K_\sigma(x)\} \quad < \infty, \end{aligned}$$

then under the condition that $p > n$ the same thing is true for the corresponding quantities $\bar{K}^\varepsilon_\sigma(U)$, $\underline{K}^\varepsilon_\sigma$ calculated from the metric g_ε and for each case the following holds:

$$\left|\bar{K}_\sigma(U) - \bar{K}^\varepsilon_\sigma(U)\right|, \left|\underline{K}_\sigma(U) - \underline{K}^\varepsilon_\sigma\right| < \delta. \tag{6.3.3}$$

The proof of the previous lemma is based in an estimate of the application s_y defined in (5.3.3):

$$|s_{y\varepsilon} - \mathrm{Id}_U|_{C^3(U)} \le c_U(\varepsilon),$$

where $|y| \le 1$ and $c_U(\varepsilon) \to 0$ as $\varepsilon \to 0$, which follows from the fact that s_y is a diffeomorphism equal to 1 outside of the unitary ball and the usual way of computing the sectional curvature.

LEMMA 6.13 (Lemma 3.2 of [**Nik91**]). *Let $\mathcal{M}$ be a differentiable manifold, $\mathfrak{h}$ be a C^∞-differentiable locally finite countable atlas on $\mathcal{M}$ with the help of the operator $\mathcal{H}^\mathcal{G}$ constructed before. The the following assertions hold:*

(1) For an arbitrary uniformly bounded sequence of numbers ε_i, $i = 1, 2, \ldots$, the operator $\mathcal{H}^\mathcal{G}$ maps $\mathfrak{M}^{2,p}_\mathfrak{h}(\mathcal{M})$ into $\mathfrak{M}^\infty_\mathfrak{h}(\mathcal{M})$.

(2) For each natural number k and arbitrary sequence of positive numbers a_ν, $\nu = 1, 2 \ldots$, there exists a uniformly bounded subsequence of positive numbers ε_{k_i}, $i, k = 1, 2, \ldots$, such that for $\nu = 1, 2, \ldots$, $\mathsf{g} \in \mathfrak{M}^{2,p}_\mathfrak{h}(\mathcal{M})$ the following estimate holds:

$$\left|\mathcal{H}^\mathcal{G}(\mathsf{g}) - \mathsf{g}\right|_{\mathfrak{M}^{2,p}_\mathfrak{h}(\mathcal{M}),\nu} \le \frac{a_\nu}{k}, \tag{6.3.4}$$

now if $p > n$ then in addition we can assert the existence of an estimate for $\alpha = 1 - n/p$

$$\left|\mathcal{H}^{\mathcal{G}}(\mathrm{g}) - \mathrm{g}\right|_{\mathfrak{M}^{1,\alpha}_{\mathfrak{h}}(\mathcal{M}),\nu} \leq \frac{a_\nu}{k} \tag{6.3.5}$$

and the following relations hold:

$$\left|\bar{K}_{\mathrm{g}_k,\mu} - \bar{K}_{\mathrm{g},\mu}\right| \leq \frac{1}{k} \qquad \text{if } \bar{K}_{\mathrm{g},\mu} < \infty, \tag{6.3.6}$$

$$\bar{K}_{\mathrm{g}_k,\mu} = +\infty \qquad \text{if } \bar{K}_{\mathrm{g},\mu} = +\infty, \tag{6.3.7}$$

where $\mathrm{g}_k = \mathcal{H}(\mathrm{g})$ calculated with this sequence, and

$$\left|\underline{K}_{\mathrm{g}_k,\mu} - \underline{K}_{\mathrm{g},\mu}\right| \leq \frac{1}{k} \qquad \text{if } \underline{K}_{\mathrm{g},\mu} > -\infty, \tag{6.3.8}$$

$$\underline{K}_{\mathrm{g}_k,\mu} = -\infty \qquad \text{if } \underline{K}_{\mathrm{g},\mu} = -\infty. \tag{6.3.9}$$

The proof of this lemma follows the same structure as in the original paper. Using the construction of the operator $\mathcal{H}^{\mathcal{G}}$, the inequality (6.3.1) and (6.3.2) of Lemma 6.12 we get (6.3.4). To obtain (6.3.5) we use Rellich-Kondrashov theorem. Finally, the estimates about the curvature are the result of using (6.3.4), the convergence “almost everywhere” of the second derivative of the pullback metric on a domain in $\mathbb{R}^n$ and the estimates for the curvature of Lemma 6.12.

We continue with the proof of our theorem introducing the following notation: $\bar{\mathrm{g}}_0^\nu = (\alpha^g \circ \varphi_\nu^{-1})^* \mathrm{g}_0\big|_{gV_\nu}$ and $\bar{\mathrm{g}}_0^\nu(x) = ((\bar{\mathrm{g}}_0^\nu)_{ij}(x))$, $i, j = 1, \dots, n$, $x \in \mathbb{B}^n$, and we define a sequence of positive numbers $\{a_\nu\}$ with the help of the following equations:

$$a_\nu = \inf_{x \in \mathbb{B}^n} \left\{ \min_{|\xi| \neq 0,\, \xi \in \mathbb{R}^n} \left\{ \frac{(\bar{\mathrm{g}}_0^\nu)_{ij}(x) \xi^i \xi^j}{\delta_{ij} \xi^i \xi^j} \right\} \right\} \tag{6.3.10}$$

By $\mathrm{g}_k = \mathcal{H}^{\mathcal{G}}(\mathrm{g}_0)$ we denote the sequence of Riemannian metrics on $\mathcal{M}$ constructed in 6.13 for the sequence $\{a_\nu\}$ given by equation (6.3.10).

We also introduce the following notation: $d_k = d(\mathrm{g}_k)$, $d_0 = d(\mathrm{g}_0)$ and

$$\iota_k : (\mathcal{M}, d_0) \to (\mathcal{M}, d_k)$$

is a map for which $\iota_k(p) = p$, $p \in \mathcal{M}$ and $k = 1, \dots$. By $((\bar{\mathrm{g}}_k^\nu)_{ij}(x))$, $i, j = 1, \dots, n$ and $x \in \mathbb{B}^n$, we denote the components of the metric tensor by

$$\bar{\mathrm{g}}_k^\nu(x) = (\alpha^g \circ \varphi_\nu^{-1})^* \mathcal{H}^{\mathcal{G}}(\mathrm{g}_0)\big|_{gV_\nu}(x).$$

Let $\gamma : [0, l_0] \to \mathcal{M}$ be an arbitrary differentiable curve with respect to the atlas $\mathfrak{h}$ and parametrized by arc length in $(\mathcal{M}, d_0)$. Its length in $(\mathcal{M}, d_k)$ will be denoted by l_k.

We divide γ into a finite number of arcs γ_u each of which is contained in some $g_u V_{\nu_u}$ for $u = 1, \ldots, N$. We denote the length of γ_u in $(\mathcal{M}, d_0)$ by $l_0^{(u)}$ and its length in $(\mathcal{M}, d_k)$ by $l_k^{(u)}$. Then

$$\left| l_k^{(u)} - l_0^{(u)} \right| \leq \int_0^{l_0^{(u)}} |(\bar{\mathsf{g}}_k^{\nu_u})_{mq} - (\bar{\mathsf{g}}_0^{\nu_u})_{mq}| \, |\dot{\gamma}_u^m \dot{\gamma}_u^q| \; ds. \tag{6.3.11}$$

From the expression for a_ν it follows that

$$\max_{\substack{m,q=1,2,\ldots,n \\ 0 \leq s \leq l_0^{(u)}}} \{|\dot{\gamma}_u^m(s) \cdot \dot{\gamma}_u^q(s)|\} \leq \max_{\substack{m,q=1,2,\ldots,n \\ 0 \leq s \leq l_0^{(u)}}} \{|\delta_{mq} \dot{\gamma}_u^m(s) \cdot \dot{\gamma}_u^q(s)|\} \leq a_{\nu_u}^{-1}$$

If follows from the las inequality, inequality (6.3.11) and inequality (6.3.5) of Lemma 6.13 that

$$\left| l_k^{(u)} - l_0^{(u)} \right| \leq \frac{l_0^{(u)}}{k}.$$

Adding the last inequalities for $u = 1, \ldots N$, we get

$$|l_k - l_0| \leq \frac{l_0}{k}.$$

From this inequality, we obtain

$$\left| \frac{d_k(p,p')}{d_0(p,p')} - 1 \right| \leq \frac{1}{k} \tag{6.3.12}$$

It follows from inequality (6.3.12) that

$$\lim_{k \to \infty} \operatorname{dil} \iota_k = \lim_{k \to \infty} \operatorname{dil} \iota_k^{-1} = 1.$$

Therefore, we have assertion (2).

To prove assertion (3) we remember that for $\mathsf{g} \in \mathfrak{M}_{\mathfrak{h}}^{2,p}$ we have introduced $\underline{K}_{\mathsf{g},\mu}(\mathcal{M})$ and $\bar{K}_{\mathsf{g},\mu}(\mathcal{M})$. We denote the corresponding quantities for the Riemannian metrics g_k, $k = 1, 2 \ldots,$ by $\underline{K}_{k,\mu}(\mathcal{M})$ and $\bar{K}_{k,\mu}(\mathcal{M})$, respectively. We note that since g_k is an infinitely differentiable metric on $\mathcal{M}$ we have that $\underline{K}_{k,\mu}(\mathcal{M}) = \underline{k}_k(\mathcal{M})$ and $\bar{K}_{k,\mu}(\mathcal{M}) = \bar{k}_k(\mathcal{M})$. Considering inequalities (6.3.6)-(6.3.9) of Lemma 6.13, to prove assertion (3) it remains to note that we have:

$$\bar{K}_{0,\mu}(\mathcal{M}) \leq \bar{k}_0(\mathcal{M}) \text{ and } \underline{K}_{0,\mu}(\mathcal{M}) \geq \underline{k}_0(\mathcal{M})$$

which follow directly from Theorem 4.26 because such theorem give us a way to compute sectional curvature.

Q. E. D.

4. A Corollary of the Equivariant Nikolaev's Aproximation Theorem

A consequence of Theorem 6.11 is the following result.

THEOREM 6.14 (Equivariant Sphere Theorem). *Let $(\mathcal{M}, d)$ be a compact, simply connected space with bounded curvature of dimension $n \geq 4$ such that*

$$\frac{1}{4} < \underline{K}(\mathcal{M}) \leq \bar{K}(\mathcal{M}) \leq 1.$$

Assume that $\mathcal{G}$ is a compact Lie group and there exists a group homomorphism $\rho : \mathcal{G} \to Isom(\mathcal{M})$, i.e. $\mathcal{G}$ acts by isometries on $\mathcal{M}$. Then there exists a group homomorphism $\sigma : \mathcal{G} \to O(n+1)$ and a diffeomorphism $F : \mathcal{M} \to \mathbb{S}^n$ which is equivariant; i.e. $F \circ \rho(g) = \sigma(g) \circ F$ for all $g \in \mathcal{G}$.

PROOF. From Theorem 6.11 we get that for each $\varepsilon > 0$ we can find a Riemannian manifold $(\mathcal{M}, \mathsf{g}_\varepsilon)$ of class C^∞ that is d_L-close to the original metric space whose sectional curvatures for all point $p \in \mathcal{M}$ and section $\pi \subset T_p\mathcal{M}$ satisfy

$$\underline{K}(\mathcal{M}) - \varepsilon \leq K_\pi(p) \leq 1 + \varepsilon.$$

Also we have that $\mathcal{G}$ acts by isometries on $(\mathcal{M}, \mathsf{g}_\varepsilon)$.

We choose ε in such a way that

$$\frac{\underline{K}(\mathcal{M}) - \varepsilon}{1 + \varepsilon} > \frac{1}{4}.$$

Multiplying g_ε by a constant we can assume that

$$\frac{1}{4} < c \leq K_\pi(p) \leq 1,$$

where

$$c = \frac{\underline{K}(\mathcal{M}) - \varepsilon}{1 + \varepsilon}.$$

Therefore, by virtue of the smooth Equivariant Sphere Theorem (Theorem 2 of [**BS09**]), we obtain the result. Q. E. D.

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