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STOCHASTIC FACTORIZATIONS, DARBOUX TRANSFORMATIONS AND SPECTRAL ANALYSIS OF DISCRETE-TIME BIRTH-DEATH CHAINS

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Contents

Introducción

La conexión entre cadenas de nacimiento y muerte y polinomios ortogonales es ampliamente conocida en la comunidad matemática. Pioneros en este campo como D. G. Kendall [43], W. Ledermann y G. E. H. Reuter [45], y S. Karlin y J. McGregor [39, 40, 41], realizaron contribuciones fundamentales al encontrar representaciones espectrales de cadenas de nacimiento y muerte en la década de 1950. Dicha conexión se basa en el hecho de que la matriz de transición de probabilidades a un paso P de una cadena de nacimiento y muerte a tiempo discreto $\{X_t : t = 0, 1, \dots\}$ con espacio de estados en $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ es una matriz tridiagonal, estocástica y semi-infinita, lo que permite aplicar el teorema espectral para encontrar la correspondiente medida espectral asociada al proceso. En este contexto, con la *fórmula de representación integral de Karlin-McGregor* se calculan las probabilidades de transición a n pasos $P_{ij}^{(n)} = \mathbb{P}(X_n = j | X_0 = i)$ en términos de una medida espectral y la familia de polinomios ortogonales correspondiente. Además, esta fórmula proporciona información valiosa para comprender varios aspectos probabilísticos importantes de las cadenas de nacimiento y muerte como recurrencia, absorción o propiedades límite. A partir de los trabajos de S. Karlin y J. McGregor, numerosos autores como M. E. H. Ismail, D. Masson, G. Valent, E. A. van Doorn, H. Dette o P. Flajolet, por mencionar algunos, han contribuido a ampliar la conexión entre la teoría de polinomios ortogonales y las cadenas de nacimiento y muerte (veáse [5, 6, 9, 10, 16, 36, 44]). Referencias que contienen estos y otros resultados se pueden encontrar, por ejemplo, en [8, 47].

Esta tesis se centra en explorar diferentes tipos de factorizaciones estocásticas de la matriz de transición de probabilidades P de una cadena de nacimiento y muerte. El estudio de factorizaciones de matrices de transición de probabilidades u operadores infinitesimales de cadenas de Markov no es algo nuevo. Autores como W. K. Grassman [17], D. P. Heyman [29] o V. Vigon [49], han explorado cadenas de Markov bajo diversas condiciones y descomposiciones en factores, incluyendo factorizaciones triangulares o diagonales, y la relación con la factorización de Wiener-Hopf. La presente tesis se enfoca en factorizaciones estocásticas de tipo UL y LU de cadenas de nacimiento y muerte y se basa en los fundamentos establecidos por F.A. Grünbaum y M. D. de la Iglesia en [24]. La principal característica de las factorizaciones estocásticas de tipo UL y LU radica en la manera en que se descompone a la matriz de transición de probabilidades P en el producto de dos matrices estocásticas: P_U, una matriz bidiagonal superior, y P_L, una matriz bidiagonal inferior. Desde un punto de vista probabilístico, esta factorización se puede interpretar como la composición de dos cadenas de nacimiento y muerte: primero, un proceso de nacimiento puro asociado a P_U, seguido por un proceso de muerte puro asociado a P_L, o viceversa. Este enfoque fue usado en [24] para cadenas de nacimiento y muerte con espacio de estados en los enteros no negativos $\mathbb{Z}_{\geq 0}$, donde los autores también exploraron la transformación de

Darboux discreta, que consiste en invertir el orden de multiplicación de los factores. En el caso UL, es decir, $P = P_U P_L$, la transformación de Darboux conduce a una nueva matriz estocástica tridiagonal que se denota por $\widetilde{P} = P_L P_U$ y que describe una familia completa de cadenas de nacimiento y muerte que dependen de un *parámetro libre*. La medida espectral asociada a P y la medida espectral asociada a $\widetilde{P}$ están relacionadas a través de lo que se conoce como una *transformación de Geronimus*, que, en esta situación, consiste en dividir por x la medida espectral original y añadir una delta de Dirac en el punto $x = 0$. Por otro lado, la factorización LU, $P = \tilde{P}_L \tilde{P}_U$, es única. La medida espectral de P y la medida espectral de la transformación de Darboux $\widehat{P} = \tilde{P}_U \tilde{P}_L$ están relacionadas a través de una *transformación de Christoffel*, que, en esta situación, consiste en multiplicar por x la medida espectral original.

El objetivo principal de esta tesis es analizar diferentes tipos de factorizaciones estocásticas y transformaciones de Darboux de cadenas de nacimiento y muerte a tiempo discreto con espacio de estados en todos los números enteros $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, así como en otros espacios de estados como el *grafo araña*. Como se explicará a continuación, las principales contribuciones consistirán en obtener condiciones bajo las cuales es posible asegurar diferentes tipos de factorizaciones estocásticas para después poder explorar la relación entre las medidas espectrales de los procesos originales y las transformaciones de Darboux correspondientes. Una contribución significativa de esta investigación es la obtención de la expresión explícita de la matriz espectral en varios ejemplos. Este logro permite calcular la matriz de transición de probabilidades a n pasos de la cadena de nacimiento y muerte correspondiente utilizando la fórmula de representación integral de Karlin-McGregor. Además, se describen otras propiedades importantes como la recurrencia y la medida invariante para algunos casos específicos.

El Capítulo 1 comienza con una breve introducción a la teoría de polinomios ortogonales y abarca algunas relaciones muy útiles relacionadas con la transformada de Stieltjes, que resulta ser una herramienta muy poderosa para calcular las medidas espectrales en ejemplos posteriores. Además, se proporciona una breve introducción a la teoría de cadenas de Markov, incluido el análisis espectral de cadenas de nacimiento y muerte a tiempo discreto en $\mathbb{Z}_{\geq 0}$ y en $\mathbb{Z}$. También se introduce el concepto de *procesos cuasi de nacimiento y muerte*, que son procesos que actúan en el espacio de estados de dos dimensiones $\mathbb{Z}_{\geq 0} \times \{1, 2, \dots, N\}$ para $N \in \mathbb{Z}_{\geq 1}$, cuyas transiciones solo son posibles entre estados adyacentes de la primera componente. El análisis espectral de estos procesos se consideró en [7, 18] y está muy relacionado con la teoría de *polinomios ortogonales matriciales* de tamaño $N \times N$. Hacia el final del capítulo, se presenta una revisión de los resultados presentados en [24] sobre las factorizaciones estocásticas de tipo UL y LU y las transformaciones de Darboux para cadenas de nacimiento y muerte a tiempo discreto con espacio de estados en $\mathbb{Z}_{\geq 0}$.

En el Capítulo 2, se extienden los resultados de [24] al cambiar el espacio de estados de $\mathbb{Z}_{\geq 0}$ a $\mathbb{Z}$. Resulta interesante observar que, en este caso, tanto las factorizaciones estocásticas de tipo UL como LU dependen de un *parámetro libre*. En ambos casos, se puede asegurar la existencia de la factorización estocástica si el parámetro libre está acotado por arriba y por abajo por ciertas fracciones continuas. En el análisis espectral, se requiere una ligera modificación ya que una matriz estocástica doblemente infinita P da lugar a tres medidas espectrales soportadas en el intervalo $[-1, 1]$, denotadas por $\psi_{\alpha,\beta}$ con $\alpha, \beta = 1, 2$. Resulta que $\psi_{11}(x)$ es una medida de probabilidad, $\psi_{22}(x)$ es una medida positiva y ψ_{12} es una medida signada ($\psi_{12} = \psi_{21}$ debido a la simetría). Para capturar esta información, se define la *matriz espectral* asociada a P de la siguiente manera:

$$\mathbf{\Psi}(x) = \begin{pmatrix} \psi_{11}(x) & \psi_{12}(x) \\ \psi_{12}(x) & \psi_{22}(x) \end{pmatrix}.$$

Las matrices espectrales asociadas a las transformaciones de Darboux $\widetilde{P}$ y $\widehat{P}$ son de la forma:

$$\widetilde{\mathbf{\Psi}}(x) = \boldsymbol{S}_0(x)\mathbf{\Psi}_S(x)\boldsymbol{S}_0^*(x), \quad \text{y} \quad \widehat{\mathbf{\Psi}}(x) = \boldsymbol{T}_0(x)\mathbf{\Psi}_T(x)\boldsymbol{T}_0^*(x),$$

respectivamente, donde $\mathbf{\Psi}_S(x)$ y $\mathbf{\Psi}_T(x)$ son transformaciones de Geronimus de la matriz espectral original $\mathbf{\Psi}(x)$ asociada a P, $\boldsymbol{S}_0(x)$ y $\boldsymbol{T}_0(x)$ son ciertos polinomios matriciales de grado 1 y $\boldsymbol{S}_0^*(x)$ y $\boldsymbol{T}_0^*(x)$ denotan a las matrices transpuestas conjugadas correspondientes. Los resultados de este capítulo se publicaron en [32] y fueron la base para la tesis de maestría [38].

En el Capítulo 3 se investigan diferentes tipos de factorizaciones estocásticas. Específicamente, se examina una factorización en la que un factor representa una cadena de nacimiento y muerte en $\mathbb{Z}$ reflectante desde el estado 0, y el otro factor representa una cadena de nacimiento y muerte en $\mathbb{Z}$ absorbente hacia el estado 0. Esta factorización se denota por $P = P_R P_A$, y se conoce como una factorización *reflectante-absorbente* (RA). En este caso, para que este tipo de factorización sea posible, habrá *dos parámetros libres* que deberán estar acotados por abajo por ciertas fracciones continuas. De este análisis se derivan dos puntos importantes. Primero, en este caso, es necesario considerar a la cadena de nacimiento y muerte con espacio de estados en $\mathbb{Z}$ descrita por P como un proceso cuasi de nacimiento y muerte a tiempo discreto con espacio de estados en $\mathbb{Z}_{\geq 0} \times \{1, 2\}$. Esto se logra al reetiquetar los estados para obtener una matriz tridiagonal por bloques de tamaño 2×2 denotada por $\boldsymbol{P}$. Después de este nuevo etiquetado, la factorización RA se convierte en una factorización UL por bloques de la forma $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$. Esto nos permite extender algunos resultados utilizando técnicas de la teoría de polinomios ortogonales matriciales. Segundo, después de aplicar la transformación de Darboux, se obtiene una matriz tridiagonal por bloques de tamaño 2×2, $\widetilde{\boldsymbol{P}} = \boldsymbol{P}_A \boldsymbol{P}_R$, cuyo proceso visto como cadena de Markov en $\mathbb{Z}$ es “casi” una cadena de nacimiento y muerte, es decir, es una cadena de nacimiento y muerte regular pero además añadiendo probabilidades de transición entre los estados 1 y -1. Posteriormente, se encuentra que la relación entre las matrices espectrales de $\boldsymbol{P}$ y $\widetilde{\boldsymbol{P}}$ está dada por

$$\widetilde{\mathbf{\Psi}}(x) = \boldsymbol{S}_0 \mathbf{\Psi}_U(x) \boldsymbol{S}_0^*, \tag{1}$$

donde $\mathbf{\Psi}_U(x)$ es una transformación de Geronimus de la matriz espectral original $\mathbf{\Psi}(x)$ y $\boldsymbol{S}_0$ es la matriz constante no singular de tamaño 2×2 que corresponde con la entrada por bloques $(0, 0)$ de la matrix $\boldsymbol{P}_R$. En la Sección 3.3, se exploran las factorizaciones de tipo *absorbente-reflectante* (AR). Para aplicar la factorización AR, es necesario comenzar con una “casi” cadena de nacimiento y muerte con espacio de estados en $\mathbb{Z}$ como la descrita anteriormente. A pesar de que P no es tridiagonal, la matriz equivalente $\boldsymbol{P}$, después del reetiquetado, se convierte en una matriz tridiagonal por bloques, y la factorización correspondiente, denotada por $\boldsymbol{P} = \widetilde{\boldsymbol{P}}_A \widetilde{\boldsymbol{P}}_R$, es una factorización LU por bloques. Después de la transformación de Darboux, se obtiene una matriz tridiagonal por bloques $\widehat{\boldsymbol{P}} = \widetilde{\boldsymbol{P}}_R \widetilde{\boldsymbol{P}}_A$, cuya matriz doblemente infinita $\widehat{P}$ equivalente representa ahora una cadena de nacimiento y muerte regular en $\mathbb{Z}$. Las matrices espectrales correspondientes estarán relacionadas por una transformación de Christoffel de la forma

$$\widehat{\mathbf{\Psi}}(x) = x \widetilde{\boldsymbol{S}}_0^{-1} \mathbf{\Psi}(x) \widetilde{\boldsymbol{S}}_0^{-*},$$

donde $\widetilde{\boldsymbol{S}}_0$ es la matriz constante no singular de tamaño 2×2 que corresponde con la entrada por bloques $(0, 0)$ de la matrix $\widetilde{\boldsymbol{P}}_R$. En ambos casos (factorizaciones RA y AR), se aplican los resultados a cadenas de nacimiento y muerte con probabilidades de transición constantes. Los resultados del Capítulo 3 están incluidos en [33].

En el Capítulo 4, se analiza una cadena de nacimiento y muerte no trivial con espacio de estados en $\mathbb{Z}$, generada por los *polinomios de Jacobi asociados*. Estos polinomios pueden construirse a partir de la relación de recurrencia a tres términos que satisface la familia clásica de polinomios de Jacobi, reemplazando n por $n+t$, con $t \in \mathbb{R}$ y $n \in \mathbb{Z}$. Es importante destacar que en 1999, F. A. Grünbaum y L. Haine [23] calcularon la expresión explícita de la matriz espectral para los polinomios de Jacobi asociados con soporte en el intervalo $[0,1]$ para ciertos valores de los parámetros involucrados. Con base en estos resultados, se establecen condiciones bajo las cuales la familia de polinomios de Jacobi asociados genera una matriz estocástica tridiagonal doblemente infinita P. De hecho, la matriz P será estocástica si se elige t dentro de cierta unión numerable de intervalos reales que dependen de los parámetros de la familia clásica de polinomios de Jacobi, α y β. En este contexto, se investigan todas las posibles factorizaciones estocásticas, junto con las transformaciones de Darboux discretas y las matrices espectrales correspondientes. En este caso, las factorizaciones estocásticas de tipo UL y LU resultan únicas, mientras que la factorización estocástica RA no será posible. Aplicando los resultados del Capítulo 2, se encuentra que la matriz espectral de la transformación de Darboux $\widetilde{\mathbf{\Psi}}(x)$ para el caso UL es la misma que la matriz espectral original $\mathbf{\Psi}(x)$, pero reemplazando el parámetro α por $\alpha-1$. En cuanto al caso LU, se tiene que la matriz espectral de la transformación de Darboux $\widehat{\mathbf{\Psi}}(x)$ es de la forma

$$\widehat{\mathbf{\Psi}}(x) = \boldsymbol{T}_0(x)\frac{\mathbf{\Psi}(x)}{x}\boldsymbol{T}_0^*(x),$$

donde $\boldsymbol{T}_0(x)$ es cierto polinomio matricial de grado 1. Para concluir el capítulo, se aplican los resultados al estudio de un modelo de urnas en los enteros, correspondiente a la familia de polinomios de Jacobi asociados. Los resultados de este capítulo se encuentran publicados en [35].

Finalmente, en el Capítulo 5, se exploran las cadenas de nacimiento y muerte en una estructura conocida como *grafo araña*, que es un grafo compuesto por N líneas rectas en el plano, llamadas *piernas*, unidas en el origen, denominado *cuerpo* de la araña. Este proceso puede ser identificado como un proceso cuasi de nacimiento y muerte a tiempo discreto en el espacio de estados $\mathbb{Z}_{\geq 0} \times \{1, 2, \ldots, N\}$, representado por una matriz de transición de probabilidades tridiagonal por bloques $\boldsymbol{P}$. Se demuestra que para este proceso siempre existe una matriz espectral $\mathbf{\Psi}(x)$ de tamaño $N \times N$ soportada en el intervalo $[-1,1]$ asociada a $\boldsymbol{P}$, junto con algunos resultados para calcular la transformada de Stieltjes correspondiente. Luego, se examinan las condiciones bajo las cuales es posible obtener una factorización estocástica de tipo RA. Esta factorización puede ser vista como una factorización estocástica de tipo UL por bloques de la matriz $\boldsymbol{P}$. Dada la estructura de $\boldsymbol{P}$, se tienen N *parámetros libres*, cada uno de los cuales deberá estar acotado por debajo por cierta fracción continua. Además, se investiga la transformación de Darboux discreta, que produce nuevas familias de "casi" cadenas de nacimiento y muerte en el grafo araña, lo que implica que habrá nuevas probabilidades de transición entre los primeros estados de cada pierna. La matriz espectral asociada a la transformación de Darboux será una transformación de Geronimus de la matriz espectral original como la de que aparece en (1). Al final del capítulo, se aplican los resultados al estudio de la caminata aleatoria en el grafo araña, asumiendo probabilidades de transición constantes. Los resultados del Capítulo 5 se encuentran publicados en [34].

Introduction

The connection between birth-death chains and orthogonal polynomials is widely acknowledged in the mathematical community. Early pioneers in this field such as D. G. Kendall [43], W. Ledermann and G. E. H. Reuter [45], and S. Karlin and J. McGregor [39, 40, 41], made groundbreaking contributions by finding spectral representations of birth-death chains in the 1950s. The fundamental connection is based on the fact that the one-step transition probability matrix P of a discrete-time birth-death chain $\{X_t : t = 0, 1, \dots\}$ on $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$, is a semi-infinite tridiagonal stochastic matrix, allowing us to apply the spectral theorem to find the corresponding spectral measure associated with the process. In this context, the *Karlin-McGregor integral representation formula* computes the n-step transition probabilities $P_{ij}^{(n)} = \mathbb{P}(X_n = j | X_0 = i)$ in terms of a spectral measure and the corresponding family of orthogonal polynomials. It also provides valuable insights into the understanding of several important probability aspects of birth-death chains such as recurrence, absorption or limiting properties. Since the seminal works of S. Karlin and J. McGregor, numerous authors like M. E. H. Ismail, D. Masson, G. Valent, E. A. van Doorn, H. Dette or P. Flajolet, to mention a few, have contributed to expanding the connection between orthogonal polynomials and birth-death chains (see [5, 6, 9, 10, 16, 36, 44]). A couple of references containing these and other results can be found, for instance, in [8, 47].

This thesis focuses on exploring different types of stochastic factorizations of the one-step transition probability matrix P of a birth-death chain. The study of factorizations of transition probability matrices or infinitesimal operators of Markov chains is not new. Previous authors, like W. K. Grassman [17], D. P. Heyman [29], or V. Vigon [49], have explored Markov chains under different conditions and decompositions into factors, including triangular or diagonal factorizations and the relation with the Wiener-Hopf factorization. In this thesis, we focus on the UL and LU stochastic factorizations of birth-death chains. Our work builds upon the foundations laid by F.A. Grünbaum and M. D. de la Iglesia in [24]. The key distinction in UL and LU stochastic factorizations lies in their specific approach to decompose the matrix P into the product of two stochastic matrices: P_U, an upper bidiagonal matrix, and P_L, a lower bidiagonal matrix. From a probabilistic point of view, this factorization can be interpreted as the composition of two birth-death chains: first, a pure-birth process associated with P_U, followed by a pure-death process associated with P_L, or viceversa. This approach was used in [24] for birth-death chains with state space on the nonnegative integers $\mathbb{Z}_{\geq 0}$, where the authors also explored the discrete Darboux transformation, which involves inverting the order of multiplication of the factors. For the UL case, i.e. $P = P_U P_L$, this Darboux transformation leads to a new tridiagonal stochastic matrix denoted by $\widetilde{P} = P_L P_U$, which describes a whole family of birth-death chains depending on *one free parameter*. The spectral measure associated with P and the spectral measure associated with $\widetilde{P}$

are related through what is known as a *Geronimus transformation*, which, in this situation, consists of dividing by x the original spectral measure and adding a Dirac delta at the point $x = 0$. On the other hand, the LU factorization $P = \tilde{P}_L \tilde{P}_U$ is unique. Additionally, the spectral measure of P and the Darboux transformation $\widehat{P} = \tilde{P}_U \tilde{P}_L$ are related through a *Christoffel transformation*, which, in this situation, consists of multiplying by x the original spectral measure.

The main goal of this thesis is to analyze different types of stochastic factorizations and Darboux transformations of discrete-time birth-death chains on the whole set of integers $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, as well as other state spaces like the *spider graph*. As we will explain below, our main contributions will consist of obtaining conditions under which we can ensure different types of stochastic factorizations, and after that, exploring the relation between the spectral measures of the original processes with the corresponding Darboux transformations. A significant contribution of our research is the derivation of the explicit expression of the spectral matrix in several examples. This achievement enables us to compute the n-step transition probability matrix of the corresponding birth-death chain using the Karlin-McGregor integral representation formula. Additionally, we are able to describe other important properties such as recurrence and the invariant measure for some specific cases.

Chapter 1 begins with a concise introduction to the theory of orthogonal polynomials and covers some very useful relations concerning the Stieltjes transform, which proves to be a very powerful tool for computing the spectral measures in subsequent examples. Furthermore, we provide a brief introduction to the theory of Markov chains, including the spectral analysis of discrete-time birth-death chains on $\mathbb{Z}_{\geq 0}$ and on $\mathbb{Z}$. We also introduce the concept of *quasi-birth-and-death processes* which are processes acting on the two-dimensional state space $\mathbb{Z}_{\geq 0} \times \{1, 2, \dots, N\}$ for $N \in \mathbb{Z}_{\geq 1}$, and transitions are only possible between adjacent states of the first component. The spectral analysis of these processes was considered in [7, 18] and are very closed related to the theory of *matrix-valued orthogonal polynomials* of size $N \times N$. Towards the end of the chapter, we present a review of the results appearing in [24] about UL and LU stochastic factorizations and Darboux transformations for discrete-time birth-death chains on $\mathbb{Z}_{\geq 0}$.

In Chapter 2 we extend the results of [24] by changing the state space from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}$. It is interesting to observe that, in this case, both UL and LU stochastic factorizations depend on *one free parameter*. In both cases, the existence of the stochastic factorization can be ensured if the free parameter is bounded from above and from below by certain continued fractions. In the spectral analysis, a slight modification is necessary since a doubly infinite stochastic matrix P gives rise to three spectral measures supported on the interval $[-1, 1]$, denoted by $\psi_{\alpha,\beta}$ with $\alpha, \beta = 1, 2$. Here, $\psi_{11}(x)$ is a probability measure, $\psi_{22}(x)$ is a positive measure, and ψ_{12} is a signed measure ($\psi_{12} = \psi_{21}$ due to the symmetry). To capture this information, we define the *spectral matrix* associated with P as follows:

$$\boldsymbol{\Psi}(x) = \begin{pmatrix} \psi_{11}(x) & \psi_{12}(x) \\ \psi_{12}(x) & \psi_{22}(x) \end{pmatrix}.$$

The spectral matrices associated with the Darboux transformations $\widetilde{P}$ and $\widehat{P}$ are of the form

$$\widetilde{\boldsymbol{\Psi}}(x) = \boldsymbol{S}_0(x)\boldsymbol{\Psi}_S(x)\boldsymbol{S}_0^*(x), \quad \text{and} \quad \widehat{\boldsymbol{\Psi}}(x) = \boldsymbol{T}_0(x)\boldsymbol{\Psi}_T(x)\boldsymbol{T}_0^*(x),$$

respectively, where $\boldsymbol{\Psi}_S(x)$ and $\boldsymbol{\Psi}_T(x)$ are Geronimus transformations of the original spectral matrix $\boldsymbol{\Psi}(x)$ associated with P, $\boldsymbol{S}_0(x)$ and $\boldsymbol{T}_0(x)$ are certain matrix polynomials of degree 1 and $\boldsymbol{S}_0^*(x)$ and $\boldsymbol{T}_0^*(x)$ denote the corresponding Hermitian transposes. The results of this chapter are published in

[32] and were the basis of the master's thesis [38].

In Chapter 3 we investigate different types of stochastic factorizations. Specifically, we examine a factorization where one factor represents a reflecting birth-death chain on $\mathbb{Z}$ from the state 0, and the other factor represents an absorbing birth-death chain on $\mathbb{Z}$ to the state 0. This factorization is denoted by $P = P_R P_A$, known as a *reflecting-absorbing* (RA) factorization. In this case, in order to have this kind of factorization, there will be *two free parameters* which need to be bounded from below by certain continued fractions. Two crucial points arise from this analysis. First, we need to consider the birth-death chain on $\mathbb{Z}$ described by P as a discrete-time quasi-birth-and-death process on $\mathbb{Z}_{\geq 0} \times \{1, 2\}$. This is achieved by relabeling the states to obtain a 2×2 block tridiagonal matrix denoted by $\boldsymbol{P}$. After this relabeling, the RA factorization becomes a UL *block matrix factorization* of the form $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$. This allows us to extend some results using techniques from the theory of matrix-valued orthogonal polynomials. Second, after applying the Darboux transformation, we obtain a 2×2 block tridiagonal matrix $\widetilde{\boldsymbol{P}} = \boldsymbol{P}_A \boldsymbol{P}_R$, which process seen as a Markov chain on $\mathbb{Z}$ is an "almost" birth-death chain, i.e., a regular birth-death chain but introducing additional probability transitions between states 1 and -1. Following our procedure, we discover that the relation between the spectral matrices of $\boldsymbol{P}$ and $\widetilde{\boldsymbol{P}}$ is given by

$$\widetilde{\boldsymbol{\Psi}}(x) = \boldsymbol{S}_0 \boldsymbol{\Psi}_U(x) \boldsymbol{S}_0^*, \tag{1}$$

where $\boldsymbol{\Psi}_U(x)$ is a Geronimus transformation of the original spectral matrix $\boldsymbol{\Psi}(x)$ and $\boldsymbol{S}_0$ is the 2×2 nonsingular constant matrix located at the block entry $(0,0)$ of $\boldsymbol{P}_R$. In Section 3.3, we explore *absorbing-reflecting* (AR) factorizations. Here, in order to apply this AR factorization, we need to start with an "almost" birth-death chain on $\mathbb{Z}$ as the one described above. Despite P not being tridiagonal, the equivalent matrix $\boldsymbol{P}$ after relabeling becomes a block tridiagonal matrix, and the corresponding matrix factorization, denoted by $\boldsymbol{P} = \widetilde{\boldsymbol{P}}_A \widetilde{\boldsymbol{P}}_R$, an LU block matrix factorization. After the Darboux transformation, we arrive at a block tridiagonal matrix $\widehat{\boldsymbol{P}} = \widetilde{\boldsymbol{P}}_R \widetilde{\boldsymbol{P}}_A$, which equivalent doubly infinite matrix $\widehat{P}$ represents now a regular birth-death chain on $\mathbb{Z}$. The corresponding spectral matrices will be related by a Christoffel transformation of the form

$$\widehat{\boldsymbol{\Psi}}(x) = x \widetilde{\boldsymbol{S}}_0^{-1} \boldsymbol{\Psi}(x) \widetilde{\boldsymbol{S}}_0^{-*},$$

where $\widetilde{\boldsymbol{S}}_0$ is the 2×2 nonsingular constant matrix located at the block entry $(0,0)$ of $\widetilde{\boldsymbol{P}}_R$. In both cases, RA and AR factorizations, we apply our results to birth-death chains with constant transition probabilities. The results from Chapter 3 are included in our work [33].

In Chapter 4 we analyze a nontrivial birth-death chain on $\mathbb{Z}$ generated by the *associated Jacobi polynomials*. These polynomials can be constructed from the three-term recurrence relation satisfied by the classical family of Jacobi polynomials, but replacing n by $n+t$, with $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Notably, in 1999, F. A. Grünbaum and L. Haine [23] computed the explicit expression for the spectral matrix of the associated Jacobi polynomials supported on the interval $[0,1]$ for certain special choice of the parameters. Building upon these results we establish conditions under which the family of associated Jacobi polynomials gives rise to a doubly infinite tridiagonal stochastic matrix P. In fact, we will see that the matrix P is stochastic if we choose t inside a certain countable union of real intervals depending on the parameters of the classical family of Jacobi polynomials, α and β. In this context, we investigate all the possible stochastic factorizations, along with the discrete Darboux transformations and the corresponding spectral matrices. In this case, the UL and LU stochastic factorizations will be unique while the RA stochastic factorization will not be possible. Applying the results of Chapter 2 we find that the spectral matrix of the Darboux transformation $\widetilde{\boldsymbol{\Psi}}(x)$ for the UL case is the same as the original spectral matrix $\boldsymbol{\Psi}(x)$ but replacing the parameter α by $\alpha-1$. As for the LU case we have that the spectral matrix of the Darboux transformation $\widehat{\boldsymbol{\Psi}}(x)$ is of the form

$$\widehat{\boldsymbol{\Psi}}(x) = \boldsymbol{T}_0(x)\frac{\boldsymbol{\Psi}(x)}{x}\boldsymbol{T}_0^*(x),$$

where $\boldsymbol{T}_0(x)$ is certain matrix polynomial of degree 1. To conclude the chapter we apply our results to the study of an urn model on the integers corresponding with the family of associated Jacobi polynomials. The results of this chapter are published in [35].

Finally, in Chapter 5, we explore birth-death chains on a spider, which is a graph consisting of N discrete half lines on the plane, called *legs*, joined at the origin, called *body* of the spider. This process can be identified with a discrete-time quasi-birth-and-death process on the state space $\mathbb{Z}_{\geq 0} \times \{1, 2, \dots, N\}$, represented by a block tridiagonal transition probability matrix $\boldsymbol{P}$. We prove that for this process there always exists an $N \times N$ spectral matrix $\boldsymbol{\Psi}(x)$ supported on the interval $[-1, 1]$ associated with $\boldsymbol{P}$ along with some results to compute the corresponding Stieltjes transform. After that, we examine the conditions under which we can get an RA stochastic factorization. This factorization can be seen as a UL block stochastic factorization of the matrix $\boldsymbol{P}$. Given the structure of $\boldsymbol{P}$, we will have N *free parameters*, each of which must be bounded from below by certain continued fraction. Additionally, we investigate the Darboux transformation, which yields to new families of "almost" birth-death chains on a spider, meaning that there will be new transition probabilities between the first states of each leg. The spectral matrix associated with the Darboux transformation will be a Geronimus transformation of the original spectral matrix of the form (1). At the end of the chapter we apply our results to the random walk on a spider, assuming constant transition probabilities. The outcomes of Chapter 5 are presented in our publication [34].

CHAPTER 1

Preliminaries

In this preliminary chapter we introduce essential notions of the theory of orthogonal polynomials related to the study of discrete-time birth-death chains. Our goal is to provide a general context for the existing results in this theory and prepare the reader with the necessary tools to understand the generalizations developed in the subsequent chapters. The primary motivation for approaching the study of discrete-time birth-death chains from the perspective of spectral theory is the fact that the tridiagonal structure of the one-step transition probability matrix gives rise to a family of orthogonal polynomials in a very natural way. These polynomials will be orthogonal with respect to a specific measure known as the *spectral measure*. Once we obtain the spectral measure, it is possible to analyze the corresponding chain and to describe important probabilistic properties such as recurrence or the invariant measure.

We begin Section 1.1 by providing essential definitions about orthogonal polynomials. Additionally, we establish the connections between the spectral measure and the Stieltjes transform. Moving on to Section 1.2, we introduce the concept of a Markov chain and explore some of its properties. In Section 1.3, we establish the relation between discrete-time birth-death chains with state space on $\mathbb{Z}_{\geq 0}$ and orthogonal polynomials. In this scenario, the matrix P takes the form of a semi-infinite tridiagonal stochastic matrix. By usign this structure, we not only derive some probabilistic properties of the process but also compute the n-step transition probabilities using the Karlin-McGregor integral representation formula, first introduced in [41]. We extend this analysis to birth-death chains with state space on $\mathbb{Z}$ in Section 1.4, and we study their connection with *quasi-birth-and-death processes* in Section 1.5 . Finally, in Section 1.6, we review the UL and LU stochastic factorizations of birth-death chains with state space on $\mathbb{Z}_{\geq 0}$ and explore the example of the birth-death chain generated by the classical family of Jacobi polynomials.

1.1 Orthogonal polynomials

In this section we give some important concepts and properties about orthogonal polynomials with the intention to introduce the reader to the main tool to perform the spectral analysis of birth-death chains. We will focus on the results that will play an important role during the analysis of the different examples presented along this thesis. The theory of orthogonal polynomials has been widely studied and the bibliography on this topic is extensive. This section is mainly based on [8, Chapter 1].

Let us start by defining the concept of orthogonal polynomials. For that, we consider ψ a positive Borel measure with support on $\mathcal{S} \subset \mathbb{R}$ and assume that the *moments*

$$\mu_n = \int_{\mathcal{S}} x^n d\psi(x), \quad n \geq 0,$$

exist and are finite. Typically we normalize the measure such that $\mu_0 = 1$, i.e., ψ is a probability measure. Consider the Hilbert space $L^2_\psi(\mathcal{S})$ of all measurable functions f such that $||f||^2_\psi < \infty$ with the inner product

$$(f, g)_\psi = \int_{\mathcal{S}} f(x)g(x)d\psi(x).$$

If $\mathcal{S}$ is a countable set, we denote the Hilbert space as $\ell^2_\psi(\mathcal{S})$.

Definition 1.1.1. A sequence $(p_n(x))_{n\geq 0}$ is a *sequence of polynomials* if each element is a polynomial of degree n in the variable x.

Definition 1.1.2. A sequence of polynomials $(p_n(x))_{n\geq 0}$ is *orthogonal* with respect to the measure ψ if

$$(p_n(x), p_m(x))_\psi = \int_{\mathcal{S}} p_n(x)p_m(x)d\psi(x) = c_n^2 \delta_{nm},$$

where $c_n^2 > 0$, $c_n = ||p_n||_\psi$ is the *norm* of the polynomial p_n and δ_{nm} is the Kronecker delta.

If the norm is always equal to 1, we say that the polynomials are *orthonormal*. For this section we will denote this family as $(P_n)_{n\geq 0}$. It is very well known that every family of orthogonal polynomials satisfies a tree-term recurrence relation of the form

$$p_{-1} = 0, \quad p_0 = 1, \quad xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x), \quad n \geq 1,$$

where

$$a_n = \frac{(xp_n, p_{n+1})_\psi}{(p_{n+1}, p_{n+1})_\psi}, \quad b_n = \frac{(xp_n, p_n)_\psi}{(p_n, p_n)_\psi}, \quad c_n = \frac{(xp_n, p_{n-1})_\psi}{(p_{n-1}, p_{n-1})_\psi}.$$

The orthonormal polynomials satisfy the three-term recurrence relation

$$P_{-1} = 0, \quad P_0 = 1, \quad xP_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x), \quad n \geq 1,$$

with $a_n > 0$ and $b_n \in \mathbb{R}$. The three-term recurrence relation is one of the main characteristics of the orthogonal polynomials and it is possible to write it in matrix form. If we define $P(x) = (P_0(x), P_1(x), \dots)^T$ and

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_1 & b_1 & a_1 & 0 & \dots \\ 0 & a_2 & b_2 & a_2 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

then we have that

$$xP(x) = JP(x).$$

Note that the matrix J is a semi-infinite tridiagonal symmetric matrix also known as *Jacobi matrix*. Observe that the structure of this matrix is similar to the one-step transition probability matrix of a discrete-time birth-death chain (see Section 1.2 below). We will precisely take advantage of this fact to develop the spectral analysis of this type of Markov chains.

There is a direct relation between the Jacobi matrix J and the measure ψ. Indeed, since we have that $x^n P(x) = J^n P(x)$, then multiplying by $P(x)^T$ on the right and integrating over $\mathcal{S}$ with respect to the measure ψ, it is easy to see that

$$\int_{\mathcal{S}} x^n P_i(x) P_j(x) d\psi(x) = \sum_k \int_{\mathcal{S}} J^n_{ik} P_k(x) P_j(x) d\psi(x) = J^n_{ij}.$$

This relation gives an integral representation of the (i, j) entry of the powers of J in terms of the corresponding orthogonal polynomials. In particular, it is possible to compute the moment μ_n of the measure ψ by taking J^n_{00}. This identity can be extended to any analytic function defined on $\mathcal{S}$ of the form $f(x) = \sum_{n\geq 0} c_n x^n$ as

$$\int_{\mathcal{S}} f(x) P_i(x) P_j(x) d\psi(x) = \sum_{n\geq 0} \int_{\mathcal{S}} c_n x^n P_k(x) P_j(x) d\psi(x) = \sum_{n\geq 0} c_n J^n_{ij} = f(J)_{ij}. \tag{1.1.1}$$

As we will see below, it is possible to obtain more relations between these two objects.

Definition 1.1.3. The *Stieltjes transform* of a measure ψ is defined by

$$B(z; \psi) = \int_{\mathbb{R}} \frac{d\psi(x)}{x - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

This transform is related to the *generating function of the moments* of the measure since we have that

$$B(z; \psi) = -\frac{1}{z} \int_{\mathbb{R}} \frac{1}{1 - x/z} d\psi(x) = -\frac{1}{z} \sum_{n=0}^{\infty} \int_{\mathbb{R}} \frac{x^n}{z^n} d\psi(x) = -\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}.$$

There is a formula which allows us to calculate the measure ψ from its Stieltjes transform.

Proposition 1.1.4. *Let ψ be a probability measure with finite moments and $B(z; \psi)$ its Stieltjes transform. Then*

$$\int_a^b d\psi(x) + \frac{1}{2}\psi(\{a\}) + \frac{1}{2}\psi(\{b\}) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \int_a^b Im B(x + i\epsilon; \psi) dx, \tag{1.1.2}$$

where $\psi(\{a\}) \geq 0$ is the magnitude or size of the mass at an isolated point a. If the measure is absolutely continuous with respect to the Lebesgue measure at a then $\psi(\{a\}) = 0$.

A proof of this proposition can be found in [8, Proposition 1.1]. Equation (1.1.2) is known as the *Perron-Stieltjes inversion formula*. When the measure is absolutely continuous with respect to the Lebesgue measure, i.e., $d\psi(x) = \psi(x)dx$, we have

$$\psi(x) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} Im B(x + i\epsilon; \psi) = \lim_{\epsilon\downarrow 0} \frac{B(x + i\epsilon; \psi) - B(x - i\epsilon; \psi)}{2\pi i}.$$

For measures with an absolutely continuous part and a discrete part, we can compute the isolated points a such that $\psi(\{a\}) > 0$ since they satisfy

$$\lim_{\epsilon\downarrow 0} Im B(a + i\epsilon; \psi) = \infty,$$

and the size of the jump at $x = a$ as follows

$$\psi(\{a\}) = \lim_{\epsilon \downarrow 0} \epsilon Im B(a + i\epsilon; \psi) \geq 0. \tag{1.1.3}$$

Another important concept is the *n-th associated polynomials* generated by the matrix resulting from removing the first $n+1$ columns and rows from J. For the case where $n = 0$ we have the following important results.

If we consider the function $f(x) = (1 - zx)^{-1}$ with $z \in \mathbb{C} \setminus \mathcal{S}$, from (1.1.1) we have the following relation between the matrix J and the generating function of the measure ψ:

$$(I - Jz)^{-1}_{00} = \int_{\mathbb{R}} \frac{P_0^2(x)}{1 - xz} d\psi(x) = \int_{\mathbb{R}} \frac{d\psi(x)}{1 - xz} = \sum_{n \geq 0} \mu_n z^n.$$

where I is the identity matrix. In terms of the Stieltjes transform of ψ and J we have that

$$(I - zJ)^{-1}_{00} = -\frac{1}{z} B\left(\frac{1}{z}; \psi\right).$$

Theorem 1.1.5. *Let $J^{(0)}$ be the matrix built by removing the first column and row from J. Then we have*

$$(I - zJ)^{-1}_{00} = \frac{1}{1 - b_0 z - a_0^2 z^2 (I - zJ^{(0)})^{-1}_{00}}.$$

The proof of this theorem can be found in [8, Theorem 1.5]. If we assume that ψ is positive and there is a positive measure $\psi^{(0)}$ associated to $J^{(0)}$ then, from the previous theorem, we have

$$\int_{\mathbb{R}} \frac{d\psi(x)}{1 - xz} = \frac{1}{1 - b_0 z - a_0^2 z^2 \displaystyle\int_{\mathbb{R}} \frac{d\psi^{(0)}(x)}{1 - xz}},$$

which relates the generating functions of the moments of the measures ψ and $\psi^{(0)}$. Using the definition of the Stieltjes transform, we have

$$B(z; \psi) = \frac{1}{z - b_0 + a_0^2 B(z; \psi^{(0)})}. \tag{1.1.4}$$

Another way to generate the 0-th associated polynomials is by taking the sequence $(P_n(x))_{n \geq 0}$ and changing the initial conditions to

$$P_0^{(0)}(x) = 0, \quad P_1^{(0)} = 1/a_0.$$

Note that this change makes the degree of the polynomial $P_n^{(0)}$ equal to $n - 1$. This family has the following integral representation

$$P_n^{(0)}(x) = \int_{\mathbb{R}} \frac{P_n(x) - P_n(y)}{x - y} d\psi(y), \quad n \geq 0.$$

Finally for this section we state the *spectral theorem.* The spectral theorem is a very well known result specially in the areas of linear algebra and functional analysis. There are different versions of this theorem but the general idea is to identify a class of linear operators that can be modeled by multiplication operators. In a finite-dimensional vector space, every linear operator can be represented by a matrix. This is why in linear algebra the spectral theorem gives us conditions under which a matrix can be diagonalized, i.e., it can be represented as a diagonal matrix in some basis. In this case it is essentially enough to analyze the eigenvalues of the matrix also known as the *spectrum*.

In the infinite dimensional case the situation is more complicated. We will focus on the version for Hilbert spaces with an inner product where we can apply the spectral theorem to self-adjoint operators and find a representation in terms of projection operators. Let us consider a self-adjoint operator A defined on a Hilbert space $\mathcal{H}$. There is a measure ψ on a measurable space $\mathcal{S}$ and a unitary operator $U : \mathcal{H} \to L^2_\psi(\mathcal{S})$ such that

$$(UAU^{-1}f)(x) = F(x)f(x),$$

for some measurable and bounded real function F on $\mathcal{S}$. In the context of orthogonal polynomials, this result is known as *Favard's theorem*.

Theorem 1.1.6. *Let J be a bounded Jacobi operator. Denote by $(e_n)_{n\geq 0}$ the orthonormal canonical basis for $\ell^2(\mathbb{Z}_{\geq 0})$. Then there exists a unique probability measure ψ supported on a real compact interval such that for every polynomial P, the map $U : P(J)e_0 \to P$ extends to a unitary operator $\ell^2(\mathbb{Z}_{\geq 0}) \to L^2_\psi$ such that $UJ = MU$, where $M : L^2_\psi \to L^2_\psi$ is the multiplication operator $(Mf)(x) = xf(x)$. Moreover, the sequence $P_n = Ue_n$ is a set of orthonormal polynomials with respect to ψ. Therefore, the operator J can be diagonalized in the following way:*

$$(UJU^{-1}f)(x) = (Mf)(x) = xf(x), \quad f \in L^2_\psi.$$

If we start from the symmetric tridiagonal Jacobi matrix J which is self-adjoint in the Hilbert space $\ell^2(\mathbb{Z}_{\geq 0})$, the previous theorem provides us with a measure ψ and a complete orthonormal basis in $L^2_\psi(\mathcal{S})$ formed by the sequence of orthonormal polynomials P_n. Two different ways to prove the spectral theorem can be found in [8, Section 1.3].

1.2 Discrete-time Markov chains

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let us consider a discrete-time stochastic process $\{X_n : n = 0, 1, \dots\}$ with state space $\mathcal{S}$ satisfying the following property for all $n \geq 0$ and any states $i_0, i_1, \dots, i_{n-1}, i, j \in \mathcal{S}$

$$\mathbb{P}(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i) = \mathbb{P}(X_{n+1} = j | X_n = i) = P_{ij}^{n,n+1}.$$

This last relation is known as the *Markov property* and $P_{ij}^{n,n+1}$ for $i, j \in \mathcal{S}$ are the *one-step transition probabilities*. A stochastic process with the previous property is called a *discrete-time Markov chain* or simply a Markov chain. The state space $\mathcal{S}$ can be a finite collection of discrete points, the space of nonnegative integers $\mathbb{Z}_{\geq 0}$ or the whole set of integers $\mathbb{Z}$. We will assume $\mathcal{S} = \mathbb{Z}_{\geq 0}$ in this section.

In general, there is no restriction for the transition probabilities. However, in this thesis we will work with *homogeneous* Markov chains, i.e., the one-step transition probabilities does not depend on n and can be written simply as P_{ij}. Note that there is a one-step transition probability for each pair of states in $\mathcal{S}$, so it is usual to represent these probabilities as the following matrix:

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ P_{20} & P_{21} & P_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

also known as the *one-step transition probability matrix* of the Markov chain and it is a stochastic matrix, i.e., it satisfies the following properties

$$P_{ij} \geq 0, \quad \text{for all } i, j \in \mathcal{S} \quad \text{and} \quad \sum_{j=0}^{\infty} P_{ij} = 1, \quad \text{for all } i \in \mathcal{S}.$$

Definition 1.2.1. We say that the state i is *absorbing* if once the Markov chain reaches it, it becomes impossible to transition to any other state, i.e., $P_{ii} = 1$ and $P_{ij} = 0$ for all $j \neq i$.

A Markov chain is characterized by a state space, a transition probability matrix and an initial distribution across the state space. We also can define the *n-step transition probabilities* as

$$P_{i,j}^{(n)} = \mathbb{P}(X_n = j | X_0 = i), \quad \text{for all } i, j \in \mathcal{S}.$$

These probabilities form the following matrix

$$P^{(n)} = \begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots \\ P_{20}^{(n)} & P_{21}^{(n)} & P_{22}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

called the *n-step transition probability matrix*. For each pair of states, we can compute the n-step transition probabilities using the *Chapman-Kolmogorov equations*:

$$P_{ij}^{(n)} = \sum_{k \in \mathcal{S}} P_{ik}^{(r)} P_{kj}^{(n-r)}, \quad i, j \in \mathcal{S}, \quad 0 \leq r \leq n.$$

As a consequence of the last equation we have that

$$P^{(n)} = P^n.$$

Therefore, the n-step transition probability matrix satisfies the following property:

$$P^0 = I, \quad P^{(1)} = P, \quad P^{(n+1)} = P^{(n)} P = P P^{(n)}.$$

One can describe different properties of the Markov chain through the transition probabilities. For example, there is a classification of the states depending on the possible transitions. We say that the state $j \in \mathcal{S}$ is *accesible* from state $i \in \mathcal{S}$ if there is an $n \in \mathbb{Z}_{\geq 0}$ such that $P_{ij}^{(n)} > 0$. We denote this property as $i \to j$. If we have that $i \to j$ and $j \to i$, then we say that i and j are communicated and we denote this as $i \leftrightarrow j$. The property of communication is, in fact, an *equivalence relation* that induces a partition of the state space $\mathcal{S}$ which *equivalence clases* are called *communication clases*. We denote the communication class of $i \in \mathcal{S}$ by $C(i)$. Therefore, $i \leftrightarrow j$ if and only if $C(i) = C(j)$.

Definition 1.2.2. A Markov chain is *irreducible* if the equivalence relation induces only a single communication class, i.e., $C(i) = \mathcal{S}$ for all $i \in \mathcal{S}$. This means that all states in $\mathcal{S}$ communicate with each other.

Definition 1.2.3. The *period* of a state i, denoted by $d(i)$ is defined as the greatest common divisor of all positive integers such that $P_{ii}^{(n)} > 0$, that is,

$$d(i) = \gcd\{n \geq 1 : P_{ii}^{(n)} > 0\}.$$

If $d(i) = 1$ then we say that the state i is *aperiodic*. If $d(i) = k \geq 2$, we say that the state i has period k. If all the states of a Markov chain are aperiodic, i.e., $d(i) = 1$ for all $i \in \mathcal{S}$, we say that the Markov chain is aperiodic.

Definition 1.2.4. The *first passage time probability* $f_{ij}^{(n)}$ is the probability that, starting in state i, the Markov chain reaches the state j *for the first time* in exactly n steps, that is

$$f_{ij}^{(n)} = \mathbb{P}(X_n = j, X_m \neq i, m = 1, 2, \ldots, n-1 | X_0 = i).$$

Note that $f_{ii}^{(n)}$ denotes the probability of returning to the state i for the first time. We can compute the probability of an eventual visit from the state i to the state j as

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}.$$

Definition 1.2.5. The state $i \in \mathcal{S}$ is *recurrent* if $f_{ii} = 1$. The state $i \in \mathcal{S}$ is *transient* if $f_{ii} < 1$.

Definition 1.2.6. The *first passage time to the state* j is given by

$$T_j = \min\{n \geq 1 | X_n = j\}.$$

Then the first passage time probability and the first passage time are related by $f_{ij}^{(n)} = \mathbb{P}(T_j = n | X_0 = i)$, and these probabilities are related to the probabilities $P_{ij}^{(n)}$ by the formula

$$P_{ij}^{(n)} = \sum_{k=1}^{n} f_{ij}^{(k)} P_{jj}^{(n-k)}, \quad n \geq 1.$$

Definition 1.2.7. The *generating functions* associated with $P_{ij}^{(n)}$ and $f_{ij}^{(n)}$ are given by

$$P_{ij}(s) = \sum_{n=0}^{\infty} P_{ij}^{(n)} s^n, \quad F_{ij}(s) = \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n, \quad |s| < 1.$$

These generating functions are related by

$$\begin{aligned} P_{ij}(s) &= F_{ij}(s) P_{jj}(s), \quad i \neq j, \\ P_{ii}(s) &= 1 + F_{ii}(s) P_{ii}(s). \end{aligned}$$

Therefore, from the previous relations we can also say that the state i is recurrent if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ and transient otherwise. Note that if i is recurrent and $i \leftrightarrow j$, then j is recurrent. The same occurs with the transience property.

Definition 1.2.8. The *first passage time from the state* i *to the state* j is given by

$$T_{ij} = \min\{n \geq 0 | X_n = j, X_0 = i\},$$

while the *mean recurrence time* is given by

$$\mu_{ij} = \mathbb{E}[T_{ij}].$$

If $i = j$, then we simply write μ_i.

Definition 1.2.9. If $\mu_i = \infty$, the state i is called *null recurrent*. If $\mu_i < \infty$, the state i is called *positive recurrent* or *ergodic*.

To state some limiting properties of the Markov chains, we need the following definition.

Definition 1.2.10. The vector $\pi = (\pi_i)_{i\in\mathcal{S}}$ is an *invariant* or *stationary vector* for the Markov chain if

$$\pi_i \geq 0, \quad \text{for all } i \in \mathcal{S} \quad \text{and} \quad \pi P = \pi.$$

If we can normalize this vector in such a way that it is a probability distribution, i.e., $\sum_{i\in\mathcal{S}} \pi_i = 1$, then we call it an *invariant* or *stationary distribution*.

Finally, we are going to state two very well known results on the convergence of Markov chains.

Theorem 1.2.11. *If a Markov chain is irreducible and aperiodic, then the invariant distribution exists and it is given by*

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j, \quad \textit{for all } i, j \in \mathcal{S}.$$

Theorem 1.2.12. *If a Markov chain is irreducible, positive recurrent and aperiodic, then the invariant distribution exists and it is given by*

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{\mu_j},$$

and it is the only solution to the system $\pi P = \pi$ subject to $\sum_{i\in\mathcal{S}} \pi_i = 1$ and $\pi_i \geq 0$ for all $i \in \mathcal{S}$.

The theory of Markov chains has been widely developed for both, discrete and continuous time as well as for countable and continuous or general state spaces. For more results on this topic we refer the reader to [42, 46].

1.3 Discrete-time birth-death chains on $\mathbb{Z}_{\geq 0}$

In the following chapters we are going to study a special case of discrete-time Markov chains called birth-death chains or random walks. Therefore, in this section we will state some general results about this particular process. These results will prove to be very useful later. From now on we will use the term discrete-time birth-death chain, or simply birth-death chain, since random walks are usually understood as discrete-time birth-death chains with constant transition probabilities.

Roughly speaking, a discrete-time birth-death chain is a Markov chain with countable state space $\mathcal{S}$ that at each step can move $+1$ or -1, i.e., the process can move from state i to state j only if $|i - j| \leq 1$. Let us consider $\{X_t : t = 0, 1, \dots\}$ an irreducible birth-death chain with the state space on $\mathbb{Z}_{\geq 0}$. Then the transition probability matrix P is given by the following semi-infinite tridiagonal matrix

$$P = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ c_1 & b_1 & a_1 & 0 & \cdots \\ 0 & c_2 & b_2 & a_2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.3.1}$$

where $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1$ for all $n \geq 1$ and $a_n, c_{n+1} > 0, b_n \geq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Observe that P is a tridiagonal or Jacobi matrix (see Section 1.1). A diagram of the birth-death chain described by P is given in Figure 1.1.

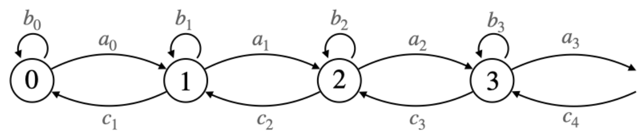

Figure 1.1: Diagram for a discrete-time birth-death chain with state space on $\mathbb{Z}_{\geq 0}$.

Definition 1.3.1. Let $(\pi_n)_{n\geq 0}$ be a solution of the symmetry equation $\pi P = \pi$ for P defined in (1.3.1). Then $(\pi_n)_{n\geq 0}$ is called the *sequence of potential coefficients* of P and it is given by

$$\pi_0 = 1, \quad \pi_n = \frac{a_0 a_1 \cdots a_{n-1}}{c_1 c_2 \cdots c_n}, \quad n \geq 1.$$

Notice that $(\pi_n)_{n\geq 0}$ is an invariant vector of the birth-death chain and it will be a distribution if $\sum_{n=0}^{\infty} \pi_n < \infty$.

Now, there is an important representation formula concerning the *spectral measure* associated to P. To derive it, we need to define the polynomials $Q(x) = (Q_0(x), Q_1(x), \dots)^T$ as a solution of the eigenvalue equation $xQ(x) = PQ(x)$. This polynomial family is given by

$$\begin{aligned} Q_0(x) =& 1, \quad Q_{-1}(x) = 0, \\ xQ_0(x) =& a_0 Q_1(x) + b_0 Q_0(x), \\ xQ_n(x) =& a_n Q_{n+1}(x) + b_n Q_n(x) + c_n Q_{n-1}(x), \quad n \geq 1. \end{aligned} \tag{1.3.2}$$

If we consider the matrix P as an operator acting in the Hilbert space $\ell^2_\pi(\mathbb{Z}_{\geq 0}) = \{(\alpha_n)_{n\in\mathbb{Z}_{\geq 0}} : \sum_{n=0}^{\infty} |\alpha_n|^2 \pi_n < \infty\}$, as follows:

$$(Pf)_n = a_n f_{n+1} + b_n f_n + c_n f_{n-1}, \quad f \in \ell^2_\pi(\mathbb{Z}_{\geq 0}),$$

where the sequence $(\pi_n)_{n\geq 0}$ is given by Definition 1.3.1, then, P gives rise to a self-adjoint operator of norm ≤ 1 that we are going to call also P (see [8, Lemma 2.4]). Note that in the following four equations we are abusing the notation since P represents a self-adjoint operator. Let us define the vector $e^{(i)}, i \geq 0$, as the vector with entries $e^{(i)}_j = \delta_{ij}/\pi_i$, where δ_{ij} is the Kronecker delta. Then we have

$$Pe^{(i)} = 1/\pi_i(0, \dots, a_{i-1}, b_i, c_{i+1}, 0, \dots)^T \quad \text{and} \quad P^n e^{(j)} = \frac{1}{\pi_j}(P^n_{0j}, P^n_{1j}, \dots), \quad n > 1.$$

Then

$$(P^n e^{(j)}, e^{(i)})_\pi = \frac{1}{\pi_j} \sum_{k=0}^{\infty} P^n_{kj} \frac{\delta_{ik}}{\pi_i} \pi_k = \frac{1}{\pi_j} P^{(n)}_{ij}. \tag{1.3.3}$$

Also, using the three-term recurrence formula (1.3.2) and induction, it is possible to prove that $Q_n(P)e^{(0)} = e^{(n)}$. Therefore we have

$$P^{(n)}_{ij} = \pi_j (P^n e^{(j)}, e^{(i)})_\pi = \pi_j (P^n Q_j(P) e^{(0)}, Q_i(P) e^{(0)})_\pi = \pi_j (P^n Q_j(P) Q_i(P) e^{(0)}, e^{(0)})_\pi.$$

Using the spectral theorem (Theorem 1.1.6) we know that there exists a probability measure ψ supported on the interval $[-1,1]$ such that

$$\int_{-1}^{1} f(x)d\psi(x) = (f(P)e^{(0)}, e^{(0)})_\pi.$$

The support of ψ is a consequence of the Perron-Frobenius theorem (see [30, Theorem 8.4.4]) since P is self-adjoint in $\ell^2_\pi(\mathbb{Z}_{\geq 0})$ and all eigenvalues of P are contained inside the unit circle.

Therefore we obtain the so-called *Karlin–McGregor representation formula* for birth-death chains on $\mathbb{Z}_{\geq 0}$:

$$P_{ij}^{(n)} = \pi_j \int_{-1}^{1} x^n Q_i(x)Q_j(x)d\psi(x). \tag{1.3.4}$$

Let us highlight that if we take $n=0$ in the previous equation we get

$$\int_{-1}^{1} Q_i(x)Q_j(x)d\psi(x) = \frac{\delta_{ij}}{\pi_j}, \tag{1.3.5}$$

where δ_{ij} is the Kronecker delta.

Therefore, the polynomials defined by (1.3.2) are orthogonal with respect to the measure ψ. We know the family $(Q_n(x))_{n\geq 0}$ as the family of *orthogonal polynomials associated with* P and ψ as the *spectral measure associated with* P. For more details on this topic see [8, Section 2.2].

As a matter of fact, in the 1950s, S. Karlin and J. McGregor [39, 40, 41] studied continuous-time birth-death processes and subsequently, the case of discrete-time birth-death chains. This series of papers represented the beginning of the study of the relation between orthogonal polynomials and stochastic processes. S. Karlin and J. McGregor derived equation (1.3.4) for the n-step transition probabilities in [41] and showed its usefulness in the study of properties such as recurrence and transience in the following theorem.

Theorem 1.3.2. *Consider* $\{X_t : t = 0, 1, \dots\}$ *a discrete-time birth-death chain with one-step transition probability matrix* P *given by* (1.3.1) *where* $a_0 + b_0 = 1$ *(i.e., 0 is a reflecting state). Then the following are equivalent:*

- *The birth-death chain is recurrent.*
- $\displaystyle\int_{-1}^{1} \frac{d\psi(x)}{1-x} = \infty.$
- $\displaystyle\sum_{n=0}^{\infty} \frac{1}{a_n \pi_n} = \infty.$

In addition, the following are equivalent

- *The birth-death chain is positive recurrent.*
- *The measure* ψ *has a finite jump at* $x = 1$ *of size* $\displaystyle\left(\sum_{n=0}^{\infty} \pi_n\right)^{-1}$.

- $\displaystyle\sum_{n=0}^{\infty} \pi_n < \infty.$

Remark 1.3.3. In a similar way to the context of orthonormal polynomials, in the context of birth-death chains we have the concept of the *k-th discrete-time birth-death chain* which transition probability matrix is generated by eliminating the first $k+1$ rows and columns of P given by (1.3.1). The case with $k=0$ is particularly useful as it will help us to compute the spectral measure for later examples. Let us denote $\psi^{(0)}$ the spectral measure associated with the 0-th birth-death chain. Then we have

$$B(z;\psi) = -\frac{1}{z - b_0 + a_0 c_1 B(z;\psi^{(0)})}.$$

Note that this equation is very similar to (1.1.4) with the difference that P here is not necessarily a symmetric matrix.

1.4 Discrete-time birth-death chains on $\mathbb{Z}$

In this section we will describe a discrete-time birth-death chain in a similar way that in the previous section but now considering as state space the whole set of integers $\mathbb{Z}$. These processes are also known as *bilateral birth-death chains*. We will follow the last section of [8, Chapter 2] and [41]. Note that in this section we are going to use the same notation for the same concepts as in the previous section but with indices in $\mathbb{Z}$ instead of $\mathbb{Z}_{\geq 0}$.

Let us consider $\{X_t : t = 0, 1, \dots\}$ an irreducible discrete-time birth-death chain with state space on the integers $\mathbb{Z}$ with transition probability matrix P given by

$$P = \left(\begin{array}{ccc|cccc} \ddots & \ddots & \ddots & & & & \\ & c_{-1} & b_{-1} & a_{-1} & & & \\ \hline & & c_0 & b_0 & a_0 & & \\ & & & c_1 & b_1 & a_1 & \\ & & & & \ddots & \ddots & \ddots \end{array}\right). \tag{1.4.1}$$

Note that, for this case, the matrix P is a doubly infinite tridiagonal stochastic matrix. Then all entries are nonnegative and

$$c_n + b_n + a_n = 1, \quad n \in \mathbb{Z}.$$

Since the process is irreducible, then we have that $0 < a_n, c_n < 1, n \in \mathbb{Z}$. A diagram of the birth-death chain described by P is given by Figure 1.2

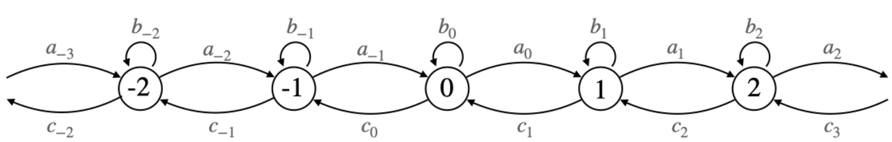

Figure 1.2: Diagram for a discrete-time bilateral birth-death chain.

The behavior of this birth-death chain is characterized by the potential coefficients.

Definition 1.4.1. Let $(\pi_n)_{n\in\mathbb{Z}}$ be a solution of the symmetry equation $\pi P = \pi$ for P defined in (1.4.1). Therefore $(\pi_n)_{n\in\mathbb{Z}}$ is called the *sequence of potential coefficients* of P and it is given by

$$\pi_0 = 1, \quad \pi_n = \frac{a_0 a_1 \cdots a_{n-1}}{c_1 c_2 \cdots c_n}, \quad \pi_{-n} = \frac{c_0 c_{-1} \cdots c_{-n+1}}{a_{-1} a_{-2} \cdots a_{-n}}, \quad n \geq 1.$$

In other words, $(\pi_n)_{n\in\mathbb{Z}}$ is an invariant vector of P. Now we consider the eigenvalue equation

$$xQ(x) = PQ(x),$$

where $Q(x) = (\ldots, Q_{-1}(x), Q_0(x), Q_1(x), \ldots)^T$. It turns out that for each x real or complex there are two polynomial families of linearly independent solutions, so we have $Q_n^\alpha(x)$ for $\alpha = 1, 2, n \in \mathbb{Z}$. Each of the solutions depends on the initial values at $n = -1$ and $n = 0$ and they are given by

$$\begin{aligned} Q_0^1(x) &= 1, \quad Q_0^2(x) = 0, \\ Q_{-1}^1(x) &= 0, \quad Q_{-1}^2(x) = 1, \\ xQ_n^\alpha(x) &= a_n Q_{n+1}^\alpha(x) + b_n Q_n^\alpha(x) + c_n Q_{n-1}^\alpha(x), \quad n \in \mathbb{Z}, \quad \alpha = 1, 2. \end{aligned} \tag{1.4.2}$$

It is easy to see that the degree of the previous polynomials depends on n as follows

$$\begin{gathered} \deg(Q_n^1(x)) = n, \quad n \geq 0, \quad \deg(Q_n^2(x)) = n - 1, \quad n \geq 1, \\ \deg(Q_{-n-1}^1(x)) = n - 1, \quad n \geq 1, \quad \deg(Q_{-n-1}^2(x)) = n, \quad n \geq 0. \end{gathered}$$

Also, it is possible to compute the leading coefficients of the polynomials $(Q_n^\alpha(x))_{n\in\mathbb{Z}}, \alpha = 1, 2$. First, from the three-term recurrence relation (1.4.2) we have

$$\begin{aligned} Q_{n+1}^\alpha(x) = &\frac{x - b_n}{a_n} Q_n^\alpha(x) - \frac{c_n}{a_n} Q_{n-1}^\alpha(x), \\ Q_{n-1}^\alpha(x) = &\frac{x - b_n}{c_n} Q_n^\alpha(x) - \frac{a_n}{c_n} Q_{n+1}^\alpha(x). \end{aligned}$$

Now we define R_n^1 and L_n^1 as the leading coefficient of $Q_n^1(x)$ and $Q_{-n-1}^1(x)$, respectively. Then we get

$$\begin{aligned} R_0^1 = 1, \quad R_n^1 &= \frac{1}{a_0 a_1 \cdots a_{n-1}}, \quad n \geq 1, \\ L_{-1}^1 = 0, \quad L_{n-1}^1 &= -\frac{a_{-1}}{c_{-1} c_{-2} \cdots c_{-n}}, \quad n \geq 1. \end{aligned}$$

In the same way, if R_n^2 and L_n^2 are the leading coefficients of $Q_n^2(x)$ and $Q_{-n-1}^2(x)$, respectively, we get

$$\begin{aligned} L_0^2 = 1, \quad L_{n-1}^2 &= \frac{L_{n-2}^2}{c_{-n}} = \frac{1}{c_{-1} c_{-2} \cdots c_{-n}}, \quad n \geq 1, \\ R_{-1}^2 = 0, \quad R_{n-1}^2 &= \frac{R_{n-2}^2}{a_{n-1}} = -\frac{c_0}{a_0 a_1 \cdots a_{n-1}}, \quad n \geq 1. \end{aligned}$$

Then, for $n \geq 1$, it is possible to get an expression of the two polynomial families using the leading coefficients as follows

$$\begin{aligned} Q_n^1(x) &= R_n^1 x^n + \mathcal{O}(x^{n-1}), \\ Q_{-n-1}^1(x) &= L_{n-1}^1 x^{n-1} + \mathcal{O}(x^{n-2}), \end{aligned} \tag{1.4.3}$$

and

$$
\begin{aligned}
Q_n^2(x) &= R_{n-1}^2 x^{n-1} + \mathcal{O}(x^{n-2}),\\
Q_{-n-1}^2(x) &= L_n^2 x^n + \mathcal{O}(x^{n-1}).
\end{aligned} \tag{1.4.4}
$$

Let us consider the Hilbert space $\ell_\pi^2(\mathbb{Z}) = \{(\alpha_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |\alpha_n|^2 \pi_n < \infty\}$. Here the matrix P gives rise to a self-adjoint operator of norm ≤ 1, which we will also denote by P. In this space, if we define the vectors $e^{(i)}$ with entries $e_j^{(i)} = \frac{\delta_{ij}}{\pi_j}$, we obtain the same expression as in equation (1.3.3). Now, the main difference is that we have two linearly independent polynomial families. Then, using the three term recurrence relation (1.4.2) and induction, we can prove that

$$
Q_i^1(P)e^{(0)} + Q_i^2(P)e^{(-1)} = e^{(i)}, \quad i \in \mathbb{Z}.
$$

Therefore

$$
\begin{aligned}
P_{ij}^{(n)} &= \pi_j (P^n e^{(j)}, e^{(i)})_\pi = \pi_j (P^n [Q_j^1(P)e^{(0)} + Q_j^2(P)e^{(-1)}], Q_i^1(P)e^{(0)} + Q_i^2(P)e^{(-1)})_\pi\\
&= \pi_j [(P^n Q_j^1(P) Q_i^1(P) e^{(0)}, e^{(0)})_\pi + (P^n Q_j^1(P) Q_i^2(P) e^{(0)}, e^{(-1)})_\pi\\
&\quad + (P^n Q_j^2(P) Q_i^1(P) e^{(-1)}, e^{(0)})_\pi + (P^n Q_j^2(P) Q_i^2(P) e^{(-1)}, e^{(-1)})_\pi].
\end{aligned}
$$

Here, we can apply the spectral theorem *three times* from where we get the existence of three unique measures $\psi_{11}(x), \psi_{22}(x)$ and $\psi_{12}(x)$ supported on the interval $[-1,1]$ such that

$$
\begin{aligned}
\int_{-1}^1 f(x)\psi_{11}(x)dx &= (f(P)e^{(0)}, e^{(0)})_\pi,\\
\int_{-1}^1 f(x)\psi_{12}(x)dx &= (f(P)e^{(0)}, e^{(-1)})_\pi,\\
\int_{-1}^1 f(x)\psi_{22}(x)dx &= (f(P)e^{(-1)}, e^{(-1)})_\pi,
\end{aligned}
$$

where f is a real bounded function on the interval $[-1,1]$. Note that, in fact, there are four measures but we have $\psi_{12}(x) = \psi_{21}(x)$ as a consequence of P being self-adjoint and the symmetry of the inner product. Then we have the *Karlin-McGregor* integral representation formula for birth-death chains with state space on $\mathbb{Z}$ as follows

$$
P_{ij}^{(n)} = \pi_j \int_{-1}^1 x^n \sum_{\alpha,\beta=1}^2 Q_i^\alpha(x) Q_j^\beta(x) d\psi_{\alpha\beta}(x), \quad i,j \in \mathbb{Z}.
$$

If we take $n = 0$ in the previous equation, together with equations (1.4.2) for $i,j = 0$, we have

$$
P_{00}^0 = \pi_0 \int_{-1}^1 Q_0^1(x) Q_0^1(x) d\psi_{11}(x) = \int_{-1}^1 d\psi_{11}(x),
$$

but it is clear that $P_{00}^0 = 1$. Therefore ψ_{11} is a probability measure. If we take $i,j = -1$ we have

$$
P_{-1-1}^0 = \pi_{-1} \int_{-1}^1 Q_{-1}^2(x) Q_{-1}^2(x) d\psi_{22}(x) = \pi_{-1} \int_{-1}^1 d\psi_{22}(x).
$$

Therefore $\pi_{-1}\psi_{22}$ is a probability measure. If we take $i=-1$ and $j=0$ we have

$$P_{-11}^{0}=\pi_0\int_{-1}^{1}Q_{-1}^2(x)Q_0^1(x)d\psi_{12}(x)=\int_{-1}^{1}d\psi_{12}(x).$$

Therefore ψ_{12} is a signed measure satisfying $\int_{-1}^{1}d\psi_{12}(x)=0$. Taking $n=0$, for any i,j in the state space, we also get the following orthogonality relation

$$\sum_{\alpha,\beta=1}^{2}\int_{-1}^{1}Q_i^{\alpha}(x)Q_j^{\beta}(x)d\psi_{\alpha\beta}(x)=\frac{\delta_{ij}}{\pi_j},\quad i,j\in\mathbb{Z}, \tag{1.4.5}$$

with δ_{ij} the Kronecker delta. For simplicity let us assume that the three measures are continuously differentiable with respect to the Lebesgue measure, i.e., abusing the notation we have $d\psi_{\alpha\beta}(x)=\psi_{\alpha\beta}(x)dx, \alpha,\beta=1,2$. Now, let us define the *spectral matrix* associated with P as

$$\mathbf{\Psi}(x)=\begin{pmatrix}\psi_{11}(x) & \psi_{12}(x)\\ \psi_{12}(x) & \psi_{22}(x)\end{pmatrix}. \tag{1.4.6}$$

Then the orthogonality relations (1.4.5) can be written in matrix form as

$$\int_{-1}^{1}\left(Q_i^1(x),Q_i^2(x)\right)\mathbf{\Psi}(x)\begin{pmatrix}Q_j^1(x)\\ Q_j^2(x)\end{pmatrix}dx=\frac{\delta_{ij}}{\pi_j},\quad i,j\in\mathbb{Z}, \tag{1.4.7}$$

where δ_{ij} is the Kronecker delta and the Karlin-McGregor integral representation formula can be written as follows

$$P_{ij}^{(n)}=\pi_j\int_{-1}^{1}x^n\left(Q_i^1(x),Q_i^2(x)\right)\mathbf{\Psi}(x)\begin{pmatrix}Q_j^1(x)\\ Q_j^2(x)\end{pmatrix}dx,\quad i,j\in\mathbb{Z}. \tag{1.4.8}$$

Remark 1.4.2. Observe that the invariant measure of P is given by the potential coefficients given in (1.4.1), which can also be computed using the inverse of the norms of the corresponding orthogonal polynomials (see (1.4.7)). This invariant measure will be a probability distribution if and only if the process is positive recurrent, or, in other words, if $\sum_{n\in\mathbb{Z}}\pi_n<\infty$.

In general, the computation of the spectral matrix for birth-death chains on $\mathbb{Z}$ is not an easy subject. If we consider the irreducible birth-death chain $\{X_t : t=0,1,\dots\}$ with transition probability matrix P given by (1.4.1), we can reduce the computation of the spectral matrix $\mathbf{\Psi}(x)$ to the study of two birth-death chains corresponding to the two directions to infinity. Let ψ^+ be the measure associated with the birth-death chain with state space $\{0,1,2,\dots\}$ whose probability transition matrix P^+ is given by $P_{ij}^+=P_{ij}$, $i,j\geq 0$, i.e.

$$P^+=\begin{pmatrix}b_0 & a_0 & 0 & 0 & \cdots\\ c_1 & b_1 & a_1 & 0 & \cdots\\ 0 & c_2 & b_2 & a_2 & \cdots\\ \vdots & \vdots & \ddots & \ddots & \ddots\end{pmatrix}, \tag{1.4.9}$$

with probability of absorption to the state -1 given by $c_0>0$. And let ψ^- be the measure associated with the birth-death chain with state space $\{-1,-2,\dots\}$ whose probability transition matrix P^- is given by $P^-_{ij} = P_{ij}$, $i,j \leq -1$, i.e.

$$P^- = \begin{pmatrix} b_{-1} & c_{-1} & 0 & 0 & \cdots \\ a_{-2} & b_{-2} & c_{-2} & 0 & \cdots \\ 0 & a_{-3} & b_{-3} & c_{-3} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.4.10}$$

with probability of absorption to the state 0 given by $a_{-1} > 0$. Then, using the corresponding generating functions and the definition of the Stieltjes transform of both birth-death chains, it is possible to compute the Stieltjes transforms of $\psi_{\alpha,\beta}$, $\alpha,\beta = 1,2$ given in (1.4.6) with the following relations:

$$\begin{aligned} B(z;\psi_{11}) &= \frac{B(z;\psi^+)}{1 - a_{-1}c_0 B(z;\psi^+)B(z;\psi^-)}, \\ \frac{c_0}{a_{-1}} B(z;\psi_{22}) &= \frac{B(z;\psi^-)}{1 - a_{-1}c_0 B(z;\psi^+)B(z;\psi^-)}, \\ B(z;\psi_{12}) &= \frac{-a_{-1}B(z;\psi^+)B(z;\psi^-)}{1 - a_{-1}c_0 B(z;\psi^+)B(z;\psi^-)}. \end{aligned} \tag{1.4.11}$$

For details about the derivation of these relations we refer the reader to [41] (see also [8, Section 2.6]). Finally, to get the spectral matrix associated with P, we can use the Perron-Stieltjes inversion formula given by (1.1.2).

In a similar way to the spectral measure of the previous section, the computation of the spectral matrix for birth-death chains on $\mathbb{Z}$ is very useful when we want to study recurrence or the invariant measure of the process described by P. Following [8, Section 2.6], we have that the birth-death chain is recurrent if and only if

$$\int_{-1}^{1} \frac{\psi_{11}(x)}{1-x} dx = \infty \quad \text{or} \quad \pi_{-1} \int_{-1}^{1} \frac{\psi_{22}(x)}{1-x} dx = \infty,$$

and it is positive recurrent if and only if $\psi_{11}(x)$ and/or $\pi_{-1}\psi_{22}(x)$ has a jump at the point 1.

1.5 Quasi-birth-and-death processes

Let us consider the homogeneous discrete-time Markov chain $\{Z_t : t = 0,1,\dots\}$ with two-dimensional state space given by $\mathbb{Z}_{\geq 0} \times \{1,2,\dots N\}$, where $N \in \mathbb{Z}_{\geq 1}$. The process is called a *quasi-birth-and-death process* if it satisfies the following property:

$$\mathbb{P}[Z_1 = (i,j) | Z_0 = (i_0,j_0)] = 0, \quad \text{if} \quad |i - i_0| > 1. \tag{1.5.1}$$

This means that the one-step transition probability matrix of the process can be given by a block tridiagonal matrix of the form

$$\boldsymbol{P} = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.5.2}$$

where A_n, B_n, and C_n, are $N \times N$ matrices containing the probabilities of the one-step transitions.

Note that if $N = 1$ we go back to a regular discrete-time birth-death chain. In general, the first component of the state space is called *level* while the second is the *phase*. Given the property (1.5.1), the main characteristic of these processes is that they allow transitions between all *adjacent* levels and *all* phases. The matrices C_n represent the probabilities of all the different ways of moving down 1 level while going from any phase to any other phase, starting at level n. We can interpret A_n and B_n similarly with the probabilities of moving up 1 level and staying at the same level, respectively. A diagram of a quasi-birth-and-death process with 2 phases is represented in Figure 1.3.

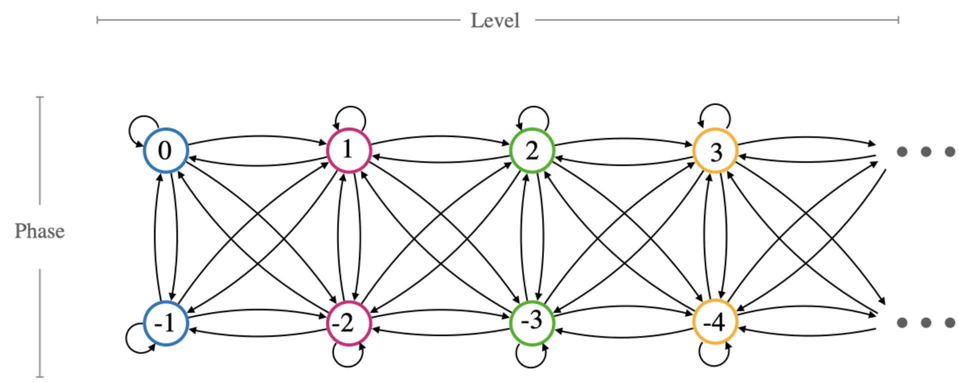

Figure 1.3: Diagram for the discrete-time quasi-birth-and-death process described by $\boldsymbol{P}$ with $N = 2$.

There are some interesting results about quasi-birth-and-death processes concerning the spectral analysis of the one-step transition probability matrix $\boldsymbol{P}$ (1.5.2). For instance, in [7] (see also [18]), the authors study the spectral matrix associated to quasi-birth-and-death processes. With this in mind, let us define the sequence of $N \times N$ *matrix-valued polynomials* $(\boldsymbol{Q}_n(x))_{n\geq 0}$ satisfying the following three-term recurrence relation

$$\begin{aligned} x\boldsymbol{Q}_0(x) &= A_0\boldsymbol{Q}_1(x) + B_0\boldsymbol{Q}_0(x), \quad \boldsymbol{Q}_0(x) = \boldsymbol{I}_N, \\ x\boldsymbol{Q}_n(x) &= A_n\boldsymbol{Q}_{n+1}(x) + B_n\boldsymbol{Q}_n(x) + C_n\boldsymbol{Q}_{n-1}(x), \quad n \geq 1, \end{aligned} \tag{1.5.3}$$

where $\boldsymbol{I}_N$ denotes the $N \times N$ identity matrix. Then, it is possible to characterize the existence of a spectral matrix $\boldsymbol{\Psi}$ such that the polynomials $(\boldsymbol{Q}_n(x))_{n\geq 0}$ are orthogonal with respect to $d\boldsymbol{\Psi}(x)$ in terms of the matrix-valued inner product

$$\int_{\mathbb{R}} \boldsymbol{Q}_i(x) d\boldsymbol{\Psi}(x) \boldsymbol{Q}_j^*(x) = \boldsymbol{0}_N, \quad \text{for all} \quad i \neq j,$$

where $\boldsymbol{0}_N$ is the null matrix of size $N \times N$ and A^* is the Hermitian transpose of a matrix A.

Then, in [7, Section 2] we can find the following results.

Theorem 1.5.1. *Assume that the matrices A_n and C_n in the one step block tridiagonal transition matrix* (1.5.2) *are nonsingular. There exists a $N \times N$ spectral matrix $\boldsymbol{\Psi}(x)$ supported on the real line such that the polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ defined by* (1.5.3) *are orthogonal with respect to $d\boldsymbol{\Psi}(x)$ if and only if there exists a sequence of nonsingular matrices $(R_n)_{n\geq 0}$ such that the following relations are satisfied:*

$$R_n B_n R_n^{-1} \quad \textit{is Hermitian for all} \quad n \geq 0,$$

$$R_n^* R_n = (C_1^* \cdots C_n^*)^{-1}(R_0^* R_0) A_0 \cdots A_{n-1}, \quad n \geq 1.$$

Theorem 1.5.2. *Assume that the conditions of Theorem 1.5.1 are satisfied and define the block diagonal matrix $R = diag(R_0, R_1, R_2, \dots)$. If the matrix R is Hermitian and the matrix $R^* \boldsymbol{P} R^{-1}$ has nonnegative entries, then the spectral matrix $\boldsymbol{\Psi}$ corresponding to the polynomials given by* (1.5.3) *is supported on the interval* $[-1, 1]$.

Therefore we have the Karlin-McGregor representation formula for quasi-birth-and-death processes.

Theorem 1.5.3. *If the assumptions of Theorems 1.5.1 and 1.5.2 are satisfied, then the (i, j) block $\boldsymbol{P}_{ij}^n$ of the n-step block transition probability matrix $\boldsymbol{P}^n$ can be represented by*

$$\boldsymbol{P}_{ij}^n = \left(\int_{-1}^{1} x^n \boldsymbol{Q}_i(x) d\boldsymbol{\Psi}(x) \boldsymbol{Q}_j^*(x) \right) \Pi_j,$$

where $\boldsymbol{\Psi}$ is the spectral matrix associated with the one step block transition probability matrix $\boldsymbol{P}$ (1.5.2) *and*

$$\Pi_j = (||\boldsymbol{Q}_j(x)||_{\boldsymbol{\Psi}}^2)^{-1} = \left(\int_{-1}^{1} \boldsymbol{Q}_j(x) d\boldsymbol{\Psi}(x) \boldsymbol{Q}_j^*(x) \right)^{-1}.$$

Using the matrix notation we have an alternative way of writing Π_n as follows (see [31])

$$\Pi_n = (C_1^* \cdots C_n^*)^{-1} \Pi_0 A_0 \cdots A_{n-1}, \quad n \geq 1. \tag{1.5.4}$$

Besides, if we assume conditions of Theorem 1.5.1 and that the spectral matrix is supported on the interval $[-1, 1]$, we have the concept of recurrence for quasi-birth-and-death processes. Let us denote by $e_k^T = (0, \dots, 0, 1, 0, \dots, 0)^T$ the k-th canonical vector in $\mathbb{R}^N$. If the quasi-birth-and-death process is *irreducible*, it is recurrent if and only if the condition

$$e_j^T \left(\int_{-1}^{1} \frac{d\boldsymbol{\Psi}(x)}{1-x} \right) \Pi_0^{-1} e_j = \infty, \tag{1.5.5}$$

is satisfied for some $j \in \{1, \dots, N\}$. The quasi-birth-and-death process is *positive recurrent* if and only if one of the measures

$$e_j^T d\boldsymbol{\Psi}(x) \Pi_0^{-1} e_j, \quad j \in \{1, \dots, N\}, \tag{1.5.6}$$

has a jump at the point $x = 1$. For more details on this topic we refer the reader to [7, Section 4.1] and [18].

Now, let us present some results similar to those given in Section 1.1 for a scalar measure to relate the Jacobi matrix, the spectral matrix and the corresponding Stieltjes transform. These results can be found in [4] where the author shows a generalization of the Karlin-McGregor representation formula for quasi-birth-and-death processes.

Let us consider the Jacobi matrix $\boldsymbol{P}$ given by (1.5.2). Here, the 0-th associated process is generated by removing the first block column and the first block row, that is

$$\boldsymbol{P}_0 = \begin{pmatrix} B_1 & A_1 & & & \\ C_2 & B_2 & A_2 & & \\ & C_3 & B_3 & A_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The matrix-valued Stieltjes transform is given by

$$B(z;\mathbf{\Psi}) = \int_{\mathbb{R}} \frac{d\mathbf{\Psi}(x)}{x-z}, \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{1.5.7}$$

The relation between the matrix-valued Stieltjes transforms associated to $\boldsymbol{P}$ and $\boldsymbol{P}_0$ is given by

$$B(z;\mathbf{\Psi})\Pi_{\mathbf{\Psi}} = \left[z\boldsymbol{I}_N - B_0 - A_0 B(z;\mathbf{\Psi}_0)\Pi_{\mathbf{\Psi}_0}C_1\right]^{-1}, \tag{1.5.8}$$

where $\Pi_{\mathbf{\Psi}} = (\int_{\mathbb{R}} d\mathbf{\Psi}(x))^{-1}$ and $\Pi_{\mathbf{\Psi}_0} = (\int_{\mathbb{R}} d\mathbf{\Psi}_0(x))^{-1}$.

Remark 1.5.4. In [4], the definition of the matrix-valued Stieltjes transform and $\Pi_{\mathbf{\Psi}}$ depends on the value of $\boldsymbol{Q}_0(x)$ that, in this case, is the identity matrix.

To end this section, we will highlight the important observation that a birth-death chain on $\mathbb{Z}$ can be interpreted as a quasi-birth-death chain with 2 phases. Indeed, by relabeling the states in $\mathbb{Z}$ in the following way

$$\{0,1,2,\ldots\} \to \{0,2,4,\ldots\}, \quad \text{and} \quad \{-1,-2,-3,\ldots\} \to \{1,3,5,\ldots\}, \tag{1.5.9}$$

we have that the doubly infinite tridiagonal matrix P given in (1.4.1) is equivalent to the semi-infinite block tridiagonal matrix $\boldsymbol{P}$ given in (1.5.2) whith blocks given by

$$B_0 = \begin{pmatrix} b_0 & c_0 \\ a_{-1} & b_{-1} \end{pmatrix}, \quad B_n = \begin{pmatrix} b_n & 0 \\ 0 & b_{-n-1} \end{pmatrix}, \quad n \geq 1,$$
$$A_n = \begin{pmatrix} a_n & 0 \\ 0 & c_{-n-1} \end{pmatrix}, \quad n \geq 0, \quad C_n = \begin{pmatrix} c_n & 0 \\ 0 & a_{-n-1} \end{pmatrix}, \quad n \geq 1.$$

The matrix-valued polynomials associated to $\boldsymbol{P}$ are given by

$$\boldsymbol{Q}_n(x) = \begin{pmatrix} Q_n^1(x) & Q_n^2(x) \\ Q_{-n-1}^1(x) & Q_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0, \tag{1.5.10}$$

where $(Q_n^\alpha)_{n\in\mathbb{Z}}$, $\alpha = 1, 2$ are given by (1.4.2). A diagram of this process looks like Figure 1.4.

Note that by equations (1.4.3) and (1.4.4), $\deg(\boldsymbol{Q}_n) = n$ and the leading coefficient is a nonsingular matrix. Using Theorem 1.5.1 with the sequence of nonsingular matrices given by

$$R_n = \begin{pmatrix} \sqrt{\pi_n} & 0 \\ 0 & \sqrt{\pi_{-n-1}} \end{pmatrix}, \quad n \geq 0,$$

we can ensure the existence of the spectral matrix $\mathbf{\Psi}$ and we can write the orthogonality relation defined in terms of the matrix-valued inner product

$$\int_{-1}^{1} \boldsymbol{Q}_n(x)\mathbf{\Psi}(x)\boldsymbol{Q}_m^*(x)dx = \Pi_n\delta_{nm}, \tag{1.5.11}$$

where δ_{nm} is the Kronecker delta and

$$\Pi_n = \begin{pmatrix} \pi_n & 0 \\ 0 & \pi_{-n-1} \end{pmatrix}, \quad n \geq 0, \tag{1.5.12}$$

and $\pi = (\pi_n)_{n\in\mathbb{Z}}$ are given by Definition 1.4.1.

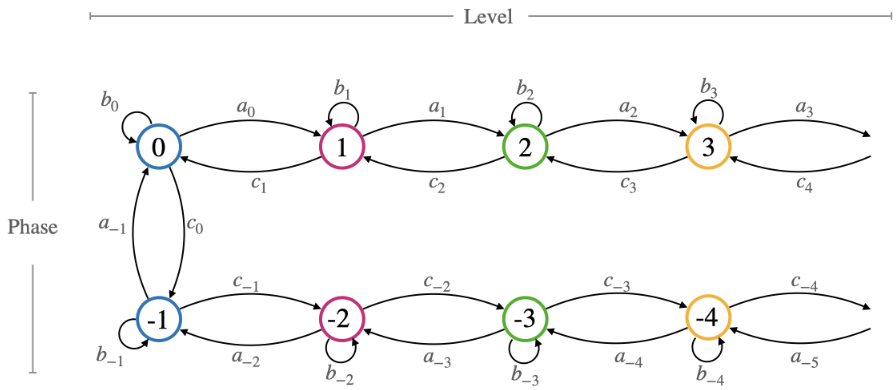

Figure 1.4: Diagram for the discrete-time quasi-birth-and-death process described by $\boldsymbol{P}$ equivalent to the discrete-time birth-death chain described by P on the integers.

In this form, the Karlin-McGregor integral representation formula for the 2×2 block entry (i,j) is given by

$$\boldsymbol{P}_{ij}^{(n)} = \left(\int_{-1}^{1} x^n \boldsymbol{Q}_i(x) \boldsymbol{\Psi}(x) \boldsymbol{Q}_j^*(x) dx \right) \Pi_j, \quad i,j \in \mathbb{Z}_{\geq 0}, \tag{1.5.13}$$

where $\boldsymbol{\Psi}(x)$ is given by (1.4.6).

Remark 1.5.5. In the same lines as mentioned in Remark 1.4.2 we have that the invariant measure of the quasi-birth-and-death process generated by $\boldsymbol{P}$, using (1.5.12) and [31, Theorem 3.1], is given by the vector

$$\boldsymbol{\pi} = (\pi_0, \pi_{-1}; \pi_1, \pi_{-2}; \cdots),$$

and clearly, this invariant measure will be a probability distribution if and only if the process is positive recurrent, or, in other words, if $\sum_{n \in \mathbb{Z}} \pi_n < \infty$.

1.6 Stochastic UL and LU factorizations on $\mathbb{Z}_{\geq 0}$ and Darboux transformations

The main goal of this section is to describe the UL and LU stochastic factorizations and the Darboux transformation of the one-step transition probability matrix P associated with a discrete-time birth-death chain with state space on the nonnegative integers $\mathbb{Z}_{\geq 0}$. This section is based on [24] (see also [38, Chapter 1]).

Let us consider $\{X_t : t = 0, 1, \dots\}$ an irreducible discrete-time birth-death chain with state space on $\mathbb{Z}_{\geq 0}$ with transition probability matrix P given by (1.3.1). The UL stochastic factorization is given by

$$P = \begin{pmatrix} y_0 & x_0 & 0 & 0 & \dots \\ 0 & y_1 & x_1 & 0 & \dots \\ 0 & 0 & y_2 & x_2 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} s_0 & 0 & 0 & 0 & \dots \\ r_1 & s_1 & 0 & 0 & \dots \\ 0 & r_2 & s_2 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = P_U P_L,$$

where P_U and P_L are also stochastic matrices. Note that P_U is an upper bidiagonal semi-infinite matrix describing a pure-birth process while P_L is a lower bidiagonal semi-infinite matrix describing a pure-death process. This means that

$$\begin{aligned} x_n + y_n =&1, \quad n \geq 0, \\ s_0 = 1, \quad r_n + s_n =&1, \quad n \geq 1, \end{aligned}$$

$0 \leq x_n, y_n \leq 1$ for all $n \geq 0$ and $0 \leq r_n, s_n \leq 1$ for all $n \geq 1$. The system of equations generated by the UL factorization has one *free parameter*, say y_0. This means that we have a whole family of factorizations depending on the choice of the free parameter $0 \leq y_0 < 1$.

On the other hand, the LU stochastic factorization is given by

$$P = \begin{pmatrix} \tilde{s}_0 & 0 & 0 & 0 & \dots \\ \tilde{r}_1 & \tilde{s}_1 & 0 & 0 & \dots \\ 0 & \tilde{r}_2 & \tilde{s}_2 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \tilde{y}_0 & \tilde{x}_0 & 0 & 0 & \dots \\ 0 & \tilde{y}_1 & \tilde{x}_1 & 0 & \dots \\ 0 & 0 & \tilde{y}_2 & \tilde{x}_2 & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = \widetilde{P}_L \widetilde{P}_U,$$

where $\widetilde{P}_L$ and $\widetilde{P}_U$ are also stochastic matrices, i.e., all entries are nonnegative and

$$\begin{aligned} \tilde{x}_n + \tilde{y}_n =&1, \quad n \geq 0, \\ \tilde{s}_0 = 1, \quad \tilde{r}_n + \tilde{s}_n =&1, \quad n \geq 1. \end{aligned}$$

In this case there is no free parameter. This means that, if it exists, the stochastic factorization will be unique.

From a probabilistic point of view, these factorizations involve dividing the probabilistic model associated with the birth-death chain into two different and simpler experiments, and then combine them together to obtain a simpler description of the original probabilistic model. Note that although we can solve the equation system for the respective factorizations, nothing ensures that the two matrices are stochastic. For that reason we have to introduce some basic elements of the theory of continued fractions.

Let us define H as the *continued fraction* generated by alternatively choosing a_n and c_n as follows

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \dots}}}. \tag{1.6.1}$$

The sequence $a_0, c_1, a_1, c_2, \dots$ is called the sequence of *partial numerators* of the continued fraction H. A different notation for H is the following:

$$H = 1 - \frac{a_0 |}{| 1} - \frac{c_1 |}{| 1} - \frac{a_1 |}{| 1} - \frac{c_2 |}{| \cdots}.$$

Now let us consider the sequence of *convergents* $(h_n)_{n \geq 0}$ given by

$$h_{2n} =1 - \frac{a_0 |}{| 1} - \frac{c_1 |}{| 1} - \frac{a_1 |}{| 1} - \dots \frac{c_n |}{| 1}, \quad h_{2n+1} = 1 - \frac{a_0 |}{| 1} - \frac{c_1 |}{| 1} - \frac{a_1 |}{| 1} - \dots \frac{a_n |}{| 1},$$

for $n \geq 0$. An important property of the convergents is that we can write them as the following quotients

$$h_n = \frac{N_n}{D_n}, \quad n \geq 0,$$

where the sequences $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ can be computed recursively. For basic concepts on the theory of continued fractions we refer the reader to [3, 50].

Theorem 1.6.1. *Let H be the continued fraction defined by* (1.6.1) *with convergents* $(h_n)_{n\geq 0}$. *If we assume that*

$$0 < N_n < D_n, \quad n \geq 1,$$

then H is convergent. Moreover P_U and P_L are stochastic matrices if and only if

$$0 \leq y_0 \leq H.$$

The complete proof of this theorem can be found in [24] where we can also find the following equivalent theorem for the LU stochastic factorization.

Theorem 1.6.2. *Consider the continued fraction*

$$\tilde{H} = 1 - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|1} - \cdots,$$

with convergents $\tilde{h}_n = \tilde{N}_n/\tilde{D}_n$. If we have

$$0 < \tilde{N}_n < \tilde{D}_n, \quad \textit{for all} \quad n \geq 1,$$

then $\tilde{H}$ is convergent. Moreover, for the LU factorization we have that both matrices $\tilde{P}_L$ and $\tilde{P}_U$ are stochastic if and only if

$$0 \leq a_0 \leq \tilde{H}.$$

Once one can ensure that the UL and LU stochastic factorizations are possible, the authors of [24] proceed to study the so-called *discrete Darboux transformation* which implies to invert the order of multiplication of the factors. In the context of urn models, with the stochastic factorization, it is possible to interpret the birth-death chain described by P as the consecutive performance of two simpler experiments, A and B. The Darboux transformation can be interpreted as exchanging the order of the two experiments, i.e., perform experiment B first and then experiment A.

Note that, as both matrices are stochastic, the product is again stochastic. In fact, this new matrix describes a birth-death chain which intuitively, in some way, must be related to the original one. This relation will be studied through the spectrum of both processes.

For the UL case we denote the Darboux transformation by $\tilde{P} = P_L P_U$. The matrix $\tilde{P}$ describes a new birth-death chain and it generates a family of polynomials $(\tilde{Q}(x))_{n\geq 0}$ defined by

$$\tilde{Q} = P_L Q, \tag{1.6.2}$$

so that it satisfies the eigenvalue equation $x\tilde{Q} = \tilde{P}\tilde{Q}$, with initial condition $\tilde{Q}_0 = 1$ and $\deg(\tilde{Q}_n) = n$. The relation between the process described by $\tilde{P}$ and the original process described by P is given in the following theorem.

Theorem 1.6.3. *Consider the family of polynomials* $(Q_n)_{n\geq 0}$*, defined by* (1.3.2)*, associated to* $P = P_U P_L$ *which are orthogonal with respect to* ψ *in the sense of equation* (1.3.5)*. Let* $(\tilde{Q}_n)_{n\geq 0}$ *be the family of polynomials associated to* $\tilde{P} = P_L P_U$ *and generated by* (1.6.2) *and the Geronimus transformation of* ψ *given by*

$$\tilde{\psi}(x) = y_0 \frac{\psi(x)}{x} + M\delta_0, \quad M = 1 - y_0 \mu_{-1},$$

where y_0 *is the free parameter for the stochastic factorization,* $\delta_0(x)$ *is the Dirac delta and* $\mu_{-1} = \int_{-1}^{1} x^{-1} d\psi(x)$ *is well defined. Then*

$$\int_{-1}^{1} \tilde{Q}_i(x)\tilde{Q}_j(x) d\tilde{\psi}(x) = \frac{\delta_{ij}}{\tilde{\pi}_j},$$

where δ_{ij} *is the Kronecker delta and* $(\tilde{\pi}_n)_{n\geq 0}$ *is the sequence of potential coefficients of* $\tilde{P}$ *given by*

$$\tilde{\pi}_0 = 1, \quad \tilde{\pi}_n = \frac{\tilde{a}_0, \dots, \tilde{a}_{n-1}}{\tilde{c}_1, \dots, \tilde{c}_n}, \quad n \geq 1.$$

On the other hand, for the Darboux transformation of the LU stochastic factorization $\hat{P} = \tilde{P}_U \tilde{P}_L$, following the same idea, we can define the family of polynomials

$$\bar{Q} = \tilde{P}_U Q.$$

However this family does not satisfy the same initial conditions, so we have to make a slight adjustment by defining the family $(\hat{Q}_n)_{n\geq 0}$ through the equation

$$\bar{Q}_n(x) = x\hat{Q}_n(x),$$

that satisfies $\hat{Q}_0(x) = 1$ and $\deg(\hat{Q}_n) = n$. In [38, Lemma 1.5] we can find the proof that this family is indeed a family of polynomials and the following results as part of [38, Theorem 1.6].

Theorem 1.6.4. *Let* $(\hat{Q}_n)_{n\geq 0}$ *be the family of polynomials associated to* $\hat{P} = \tilde{P}_U \tilde{P}_L$ *and consider the Christoffel transformation of* ψ *given by*

$$\hat{\psi}(x) = \frac{x\psi(x)}{\tilde{y}_0}.$$

Therefore

$$\int_{-1}^{1} \hat{Q}_i(x)\hat{Q}_j(x) d\hat{\psi}(x) = \frac{\delta_{ij}}{\hat{\pi}_j},$$

where δ_{ij} *is the Kronecker delta and* $(\hat{\pi}_n)_{n\geq 0}$ *is the sequence of potential coefficients of* $\hat{P}$ *given by*

$$\hat{\pi}_0 = 1, \quad \hat{\pi}_n = \frac{\hat{a}_0, \dots, \hat{a}_{n-1}}{\hat{c}_1, \dots, \hat{c}_n}, \quad n \geq 1.$$

Now, let us highlight the importance of having the spectral measures of the processes described in the previous paragraphs. In addition to being able to describe some probabilistic properties of the associated process and having the relationship between P and $\tilde{P}$ or $\hat{P}$, we have the Karlin-McGregor integral representation formula for the n-step transition probabilities as the one given for P in (1.3.4).

1.6.1 Birth-death chain generated by the Jacobi polynomials

The family of Jacobi polynomials is one of the *classical families of orthogonal polynomials of a continuous variable* (see [3, 48]). In a general context, this family depends on two parameters α and β, and is orthogonal with respect to $\omega(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1$ in the interval $\mathcal{S} = (-1, 1)$. These polynomials, denoted by $P_n^{(\alpha,\beta)}(x)$, can be generated through the following three-term recurrence relation

$$\begin{aligned}
&P_{-1}^{(\alpha,\beta)}(x) = 0, \quad P_0^{(\alpha,\beta)}(x) = 1,\\
&xP_n^{(\alpha,\beta)}(x) = \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x)\\
&\qquad + \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}P_n^{(\alpha,\beta)}(x)\\
&\qquad + \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x), \quad n \geq 0,
\end{aligned}$$

and it is possible to see that

$$||P_n^{(\alpha,\beta)}||_\omega^2 = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}, \quad n \geq 0.$$

The simplicity of this family has made it one of the best known and used in different contexts. There are some special cases of Jacobi polynomials for which formulas simplify considerably. For instance, for $\alpha = \beta = 0$ we get the *Legendre* polynomials

$$P_n(x) = P_n^{(0,0)}(x), \quad n \geq 0.$$

For $\alpha = \beta = -1/2$ we get the *Chebychev polynomials of the first kind*

$$T_n(x) = 2^{2n}\binom{2n}{n}^{-1}P_n^{(-1/2,-1/2)}(x), \quad n \geq 0.$$

For $\alpha = \beta = 1/2$ we get the *Chebychev polynomials of the second kind*

$$U_n(x) = 2^{2n}\binom{2n+1}{n+1}^{-1}P_n^{(1/2,1/2)}(x), \quad n \geq 0.$$

For $\alpha = \beta$ we get the *Gegenbauer* polynomials also known as *ultraspherical* polynomials

$$P_n^{(\lambda)}(x) = \binom{2\alpha}{\alpha}^{-1}\binom{n+2\alpha}{\alpha}P_n^{(\alpha,\alpha)}(x), \quad n \geq 0,$$

with $\alpha = \lambda - 1/2 \neq -1/2$. In fact some of these polynomials appeared chronologically before than the Jacobi polynomials (see [3] and [8, Section 1.4]).

Jacobi polynomials are usually defined on the interval $(-1, 1)$, but in fact, they can always be defined on any other interval (a, b) by the change of variables $y = \frac{a+b}{2} + x\frac{b-a}{2}$. In particular in this thesis, we will use the Jacobi polynomials $(Q_n^{(\alpha,\beta)})_{n\geq 0}$ defined on the interval $(0, 1)$ by the three-term recurrence relation

$$\begin{aligned}
&Q_{-1}^{(\alpha,\beta)}(x) = 0, \quad Q_0^{(\alpha,\beta)}(x) = 1,\\
&xQ_n^{(\alpha,\beta)}(x) = a_nQ_{n+1}^{(\alpha,\beta)}(x) + b_nQ_n^{(\alpha,\beta)}(x) + c_nQ_{n-1}^{(\alpha,\beta)}(x), \quad n \geq 0,
\end{aligned} \tag{1.6.3}$$

with

$$
\begin{aligned}
a_n &= \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0, \\
b_n &= \frac{(n+\beta+1)(n+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} + \frac{(n+\alpha)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad n \geq 0, \\
c_n &= \frac{n(n+\alpha)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad n \geq 1,
\end{aligned}
\tag{1.6.4}
$$

and are orthogonal with respect to the normalized weight

$$\psi(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^\alpha (1-x)^\beta, \quad x \in [0,1].$$

Observe that $a_n > 0$, $n \geq 0$, $b_n \geq 0$, $n \geq 0$, $c_n > 0$, $n \geq 1$, and $a_0 + b_0 = 1$, $a_n + b_n + c_n = 1$, $n \geq 1$. Therefore the Jacobi matrix generated by these coefficients is stochastic. To verify if the UL and LU stochastic factorizations are possible, we have to compute the continued fractions H and $\tilde{H}$ from Theorems 1.6.1 and 1.6.2 respectively. In general, this is not an easy matter but, in this case, as it was done in [24], we can use some results related to *chain sequences*.

Definition 1.6.5. A sequence $(n_i)_{i\geq 1}$ is a chain sequence if there is a sequence $(m_j)_{j\geq 0}$ such that $0 \leq m_0 < 1$, $0 < m_j < 1$ for all $j \geq 1$ and we can write

$$n_i = (1 - m_{i-1}) m_i, \quad i \geq 1.$$

We call $(m_j)_{j\geq 0}$ the *parameter sequence* and m_0 the *initial parameter*.

We have the following result related to the convergence of continued fractions.

Theorem 1.6.6. *Consider the continued fraction*

$$C = 1 - \frac{n_1|}{|1} - \frac{n_2|}{|1} - \frac{n_3|}{|1} - \cdots.$$

Assume that $(n_i)_{i\geq 1}$ is a chain sequence. Then

$$C = m_0 + \frac{1-m_0}{1+L},$$

where $(m_j)_{j\geq 0}$ is the the parameter sequence of $(n_i)_{i\geq 1}$ and

$$L = \sum_{j=1}^{\infty} \frac{m_1 m_1 \cdots m_j}{(1-m_1)(1-m_2)\cdots(1-m_j)}.$$

Moreover, if $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ are the convergents of C, we have

$$N_n = \prod_{k=1}^{n}(1-m_k) + m_0 \left(\sum_{k=1}^{n-1} m_1 \cdots m_k (1-m_{k+1}) \cdots (1-m_n) + \prod_{k=1}^{n} m_k \right),$$

$$D_n = \prod_{k=1}^{n}(1-m_k) + \sum_{k=1}^{n-1} m_1 \cdots m_k (1-m_{k+1}) \cdots (1-m_n) + \prod_{k=1}^{n} m_k.$$

Therefore $0 < N_n < D_n$, for all $n \geq 1$.

The previous result and more about chain sequences can be found in [3].

Going back to the example of the birth-death chain generated by the Jacobi polynomials that appears in [24], we can obtain the conditions under which the stochastic factorizations exist. For the family of polynomials defined by (1.6.3) and the continued fraction H, the sequence of partial numerators $a_0, c_1, a_1, c_2, \dots$ is a chain sequence with parameter sequence given by

$$m_{2n+1} = \frac{n+\beta+1}{2n+\alpha+\beta+2}, \quad m_{2n} = \frac{n}{2n+\alpha+\beta+1}, \quad n \geq 0.$$

Note that $m_0 = 0$. Then H converges to $\frac{1}{1+L}$ with $L = \frac{\beta+1}{\alpha}$. Therefore, following Theorem 1.6.1, if we take

$$0 \leq y_0 \leq \frac{\alpha}{\alpha+\beta+1},$$

the UL stochastic factorization is possible.

On the other hand, for the continued fraction $\tilde{H}$, the sequence of partial numerators $c_1, a_1, c_2, \dots$ is a chain sequence with parameter sequence given by

$$m_{2n+1} = \frac{n+1}{2n+\alpha+\beta+3}, \quad m_{2n} = \frac{n+\beta+1}{2n+\alpha+\beta+2}, \quad n \geq 0.$$

Then $\tilde{H}$ converges to $m_0 + \frac{1-m_0}{1+L}$ with $L = \frac{1}{\alpha}$. Therefore $\tilde{H}$ converges to $\frac{\alpha+\beta+1}{\alpha+\beta+2}$.

Recall that for this case there is no free parameter, but it is clear that

$$\frac{\beta+1}{\alpha+\beta+2} = a_0 \leq \tilde{H} = \frac{\alpha+\beta+1}{\alpha+\beta+2},$$

where the first equality follows directly from equation (1.6.4). Therefore, following Theorem 1.6.2, the LU stochastic factorization is always possible.

About the Darboux transformation, the first step we need is to get the moment μ_{-1}. For this case it is easy to see that

$$\begin{aligned}\mu_{-1} &= \int_0^1 \frac{\psi(x)}{x} dx = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 x^{\alpha-1}(1-x)^{\beta} dx \\ &= \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left(\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \right) = \frac{\alpha+\beta+1}{\alpha}.\end{aligned}$$

Therefore, following Theorem 1.6.3 the spectral measure associated to the Darboux transformation $\tilde{P} = P_L P_U$ is given by

$$\tilde{\psi}(x) = y_0 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} x^{\alpha-1}(1-x)^{\beta} + \left(1 - y_0 \frac{\alpha+\beta+1}{\alpha}\right) \delta_0(x), \quad x \in [0,1].$$

We have to be careful with some details here. The measure $\tilde{\psi}$ is integrable as long as $\alpha > 0$ and $\beta > -1$. Also, if y_0 is in the range for the existence of the stochastic factorization, the mass at 0 is always nonnegative, and vanishes if $y_0 = \frac{\alpha}{\alpha+\beta+1}$. The case $y_0 = 0$ gives us a *degenerate measure* since $\tilde{\psi}(x) = \delta_0(x)$ (see [24, Remark 3.1]).

On the other hand, following Theorem 1.6.4, the spectral measure associated to the Darboux transformation $\hat{P} = \tilde{P}_U \tilde{P}_L$ is given by

$$\hat{\psi}(x) = \frac{\Gamma(\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(\beta+1)} x^{\alpha+1}(1-x)^{\beta}, \quad x \in [0,1].$$

Finally, following Theorem 1.3.2, we can see that if $-1 < \beta \leq 0$ the integral $\int_{-1}^{1} \frac{\psi(x)}{1-x} dx$ is equal to ∞. Therefore the birth-death chain will be null recurrent since there is no mass at $x = 1$. If $\beta > 0$ the birth-death chain will be transient. This example can be found in [24, Section 5] along with a probability interpretation through an urn model (see also [21]).

CHAPTER 2

Stochastic UL factorizations on $\mathbb{Z}$

In this chapter we extend the results given in [24] but applied to bilateral birth-death chains. We start Section 2.1 with the structure of the UL (LU) factorization that we are going to consider. Recall that, from a probabilistic point of view, these factorizations involve dividing the birth-death chain into two separate and simpler experiments. The first experiment is a pure-birth chain, described by P_U, followed by a pure-death chain, described by P_L. The main result of this section will consist of getting the necessary conditions for the existence of the UL and LU stochastic factorizations. We will see that in both cases the condition is given in terms of continued fractions. An important difference with birth-death chains on $\mathbb{Z}_{\geq 0}$ is that now, for the LU factorization, we also have one free parameter.

In Section 2.2 we study the Darboux transformation which implies reversing the order of multiplication of the factors. The main motivation of Section 2.2 is to derive an explicit relation between the original process and the one resulting from the Darboux transformation. This relation will be given by a Geronimus transformation for *both* UL and LU stochastic factorizations. This situation is different from the case of birth-death chains on $\mathbb{Z}_{\geq 0}$ where the spectral measure associated with the LU factorization is given by a Christoffel transformation.

Towards the end of this chapter, we apply our results to a couple of examples. The first one has constant transition probabilities for all the states and the second one has also constant transition probabilities but inverted probabilities for the negative states. All the results presented in this chapter were published in [32] which was the basis of the master's thesis [38].

2.1 Stochastic UL and LU factorizations on the integers

Let us consider $\{X_t : t = 0, 1, \dots\}$ an irreducible discrete-time birth-death chain with state space on the integers $\mathbb{Z}$ with transition probability matrix P given by (1.4.1) in Section 1.4. Following the same idea of [24], the UL factorization of P is given by

$$P = \left(\begin{array}{ccc|cccc} \ddots & \ddots & & & & & \\ 0 & y_{-1} & & x_{-1} & & & \\ \hline & 0 & & y_0 & x_0 & & \\ & & & 0 & y_1 & x_1 & \\ & & & & & \ddots & \ddots \end{array}\right) \left(\begin{array}{ccc|ccc} \ddots & \ddots & & & & \\ & r_{-1} & s_{-1} & 0 & & \\ \hline & & r_0 & s_0 & 0 & \\ & & & r_1 & s_1 & 0 \\ & & & & \ddots & \ddots \end{array}\right) = P_U P_L, \tag{2.1.1}$$

with the condition that P_U and P_L are also stochastic matrices. Notice that it is important to have the zero state as a reference so that the structure of these two doubly bidiagonal matrices is clear. In probabilistic terms, this factorization gives us a decomposition of the Markov chain described by P into two independent processes: The first one is a pure-birth chain described by P_U illustrated in Fig. 2.1, while the second one is a pure-death chain described by P_L illustrated in Fig. 2.2, both with state space in $\mathbb{Z}$.

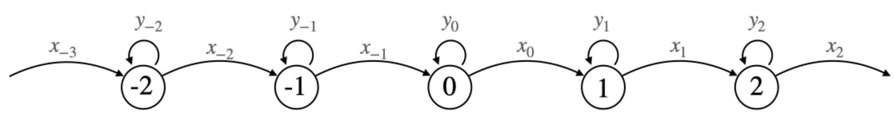

Figure 2.1: Diagram for the pure-birth process described by P_U.

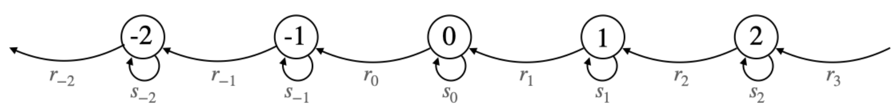

Figure 2.2: Diagram for the pure-death process described by P_L.

Since the factorization is stochastic, all entries of P_U and P_L are nonnegative and the sums of all rows are equal to one, that is

$$x_n + y_n = 1, \quad s_n + r_n = 1, \quad n \in \mathbb{Z}. \tag{2.1.2}$$

Equation (2.1.1) is equivalent to

$$\begin{aligned} a_n &= x_n s_{n+1}, \\ b_n &= x_n r_{n+1} + y_n s_n, \quad n \in \mathbb{Z}, \\ c_n &= y_n r_n. \end{aligned} \tag{2.1.3}$$

Also, as a direct consequence of the irreducibility conditions of P, we have that

$$0 < x_n, y_n, s_n, r_n < 1, \quad n \in \mathbb{Z}.$$

Notice that, if we fix x_0 or y_0, it is possible to compute all entries of P_U and P_L recursively. For y_0 fixed and nonnegative values of the indices we can compute $x_0, s_1, r_1, y_1, x_1, s_2, r_2, y_2, \dots$ recursively using (2.1.2) and (2.1.3). Similarly, for negative values of the indices we can compute r_0, s_0, x_{-1}, y_{-1}, r_{-1}, s_{-1}, x_{-2}, y_{-2},... recursively using again (2.1.2) and (2.1.3). This property of UL factorization is similar to the case of birth-death chains on $\mathbb{Z}_{\geq 0}$ where the UL factorization gives rise to an entire family of factorizations of the original transition probability matrix P depending on the choice of y_0 (see [24]). Although the previous procedure describes how to obtain the coefficients of the UL factorization it remains to obtain a formal proof to ensure that this factorization is possible.

Lemma 2.1.1. *Let $\{X_t : t = 0, 1, 2, \dots\}$ be an irreducible birth-death chain with state space on $\mathbb{Z}$ and transition probability matrix P given by* (1.4.1) *and let P_U and P_L be the factors in* (2.1.1) *with $0 < y_0 < 1$ fixed. Then $x_n + y_n = 1$ for all $n \in \mathbb{Z}$ if and only if $s_n + r_n = 1$ for all $n \in \mathbb{Z}$.*

Proof. Let P be the stochastic matrix given by (1.4.1) and let us assume that $x_n + y_n = 1$ for all $n \in \mathbb{Z}$. Using equations (2.1.3), if $s_0 + y_0 = 1$, we have

$$s_1 + r_1 = \frac{a_0}{x_0} + \frac{b_0 - y_0 s_0}{x_0} = \frac{1 - c_0 - y_0(1 - r_0)}{x_0} = \frac{x_0}{x_0} = 1.$$

Now, if we assume that $s_n + r_n = 1$ then, using equations (2.1.3) we have

$$s_{n+1} + r_{n+1} = \frac{a_n}{x_n} + \frac{b_n - r_n s_n}{x_n} = \frac{1 - c_n - y_n(1 - r_n)}{x_n} = \frac{x_n}{x_n} = 1.$$

The proof for the negative indices is similar. Now, let us assume that $x_n + y_n = 1$ for all $n \in \mathbb{Z}$ and $s_0 + r_0 = 1$, then

$$s_{-1} + r_{-1} = \frac{b_{-1} - r_0 x_{-1}}{y_{-1}} + \frac{c_{-1}}{y_{-1}} = \frac{1 - a_{-1} - x_{-1}(1 - s_0)}{y_{-1}} = \frac{1 - a_{-1} + a_{-1} - x_{-1}}{y_{-1}} = 1.$$

Then, if we assume that $s_{-n} + r_{-n} = 1$, we have

$$\begin{aligned} s_{-n-1} + r_{-n-1} = &\frac{b_{-n-1} - r_{-n} x_{-n-1}}{y_{-n-1}} + \frac{c_{-n-1}}{y_{-n-1}} \\ =&\frac{1 - a_{-n-1} - x_{-n-1}(1 - s_{-n})}{y_{-n-1}} = \frac{1 - a_{-n-1} + a_{-n-1} - x_{-n-1}}{y_{-n-1}} = 1. \end{aligned}$$

Therefore we can conclude that $s_n + r_n = 1$ for all $n \in \mathbb{Z}$. For the second part of the proof if we assume that $s_n + r_n = 1$, using the relations (2.1.3), we have

$$\begin{aligned} b_n =& y_n s_n + r_{n+1} x_n = y_n(1 - r_n) + x_n(1 - s_{n+1}) \\ =& y_n - y_n r_n + x_n - s_{n+1} x_n = y_n - c_n + x_n - a_n. \end{aligned}$$

This means that $x_n + y_n = a_n + b_n + c_n = 1$ for all $n \in \mathbb{Z}$ since P is stochastic. □

Before introducing the main theorem of this section, let us highlight the fact that (2.1.2) does not imply that P_U and P_L are stochastic matrices. It remains to prove that all coefficients are positive. To this end, the proof of Theorem 2.1 of [24] for birth-death chains on $\mathbb{Z}_{\geq 0}$ provides the main keys for the case of birth-death chains on $\mathbb{Z}$. There, the authors showed that matrices P_U and P_L are stochastic if and only if $0 \leq y_0 \leq H$, where H is the continued fraction (2.1.4) below (see Section 1.6). The difference now is that the free parameter y_0 will be also bounded from below by another continued fraction H'. A somewhat rough argument of why this happens is that here the birth-death chain is bilateral, i.e., it has positive and also negative states, so in addition to the upper bound we must have a lower bound generated by the probabilities of the negative states of the birth-death chain.

A first key for the proof is to realize that we can compute all coefficients $(y_n)_{n\in\mathbb{Z}}$ and $(s_n)_{n\in\mathbb{Z}}$ by using the following equations

$$y_n = \frac{c_n}{1-s_n}, \quad s_{n+1} = \frac{a_n}{1-y_n}, \quad n \in \mathbb{Z},$$

and then use equation (2.1.2) to compute all coefficients $(x_n)_{n\in\mathbb{Z}}$ and $(r_n)_{n\in\mathbb{Z}}$. A second important key is to define H and H' as the continued fractions generated by alternatively choosing a_n and c_n in different directions, that is

$$H = 1 - \cfrac{a_0}{1 - \cfrac{c_1}{1 - \cfrac{a_1}{1 - \cfrac{c_2}{1-\dots}}}} \quad \text{and} \quad H' = \cfrac{c_0}{1 - \cfrac{a_{-1}}{1 - \cfrac{c_{-1}}{1 - \cfrac{a_{-2}}{1-\dots}}}}. \tag{2.1.4}$$

The sequence $a_0, c_1, a_1, c_2, \dots$ is the sequence of partial numerators of the continued fraction H while the sequence $c_0, a_{-1}, c_{-1}, a_{-2}, \dots$ is the sequence of partial numerators for H'. Alternatively H and H' can be denoted by

$$H = 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|\cdots} \quad \text{and} \quad H' = \frac{c_0|}{|1} - \frac{a_{-1}|}{|1} - \frac{c_{-1}|}{|1} - \frac{a_{-2}|}{|\cdots}.$$

For H consider the sequence of convergents $(h_n)_{n\geq 0}$ given by

$$h_{2n} = 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \dots \frac{c_n|}{|1}, \quad h_{2n+1} = 1 - \frac{a_0|}{|1} - \frac{c_1|}{|1} - \frac{a_1|}{|1} - \dots \frac{a_n|}{|1},$$

for $n \geq 0$, while for H' consider the sequence of convergents $(h'_{-n})_{n\geq 0}$ given by

$$h'_{-2n} = \frac{c_0|}{|1} - \frac{a_{-1}|}{|1} - \dots \frac{a_{-n}|}{|1}, \quad h'_{-2n-1} = \frac{c_0|}{|1} - \frac{a_{-1}|}{|1} - \dots \frac{c_{-n}|}{|1}.$$

An important property of these continued fractions is that we can write them as the following quotients

$$h_n = \frac{N_n}{D_n} \quad \text{and} \quad h'_{-n} = \frac{N'_{-n}}{D'_{-n}}, \quad \text{for all} \quad n \geq 0, \tag{2.1.5}$$

where the sequences $(N_n)_{n\geq 0}$, $(D_n)_{n\geq 0}$, $(N'_{-n})_{n\geq 0}$ and $(D'_{-n})_{n\geq 0}$ can be computed from the following system of equations

$$\begin{aligned} &N_{2n} = N_{2n-1} - c_n N_{2n-2}, \quad n \geq 1, \quad N_{2n+1} = N_{2n} - a_n N_{2n-1}, \quad n \geq 0, \quad N_{-1} = 1, \quad N_0 = 1, \\ &D_{2n} = D_{2n-1} - c_n D_{2n-2}, \quad n \geq 1, \quad D_{2n+1} = D_{2n} - a_n D_{2n-1}, \quad n \geq 0, \quad D_{-1} = 0, \quad D_0 = 1, \end{aligned} \tag{2.1.6}$$

for the positive indices and

$$\begin{aligned} &N'_{-2n} = N'_{-2n+1} - a_{-n} N'_{-2n+2}, n \geq 1, \quad N'_{-2n-1} = N'_{-2n} - c_{-n} N'_{-2n+1}, n \geq 0, \quad N'_1 = -1, N'_0 = 0, \\ &D'_{-2n} = D'_{-2n+1} - a_{-n} D'_{-2n+2}, n \geq 1, \quad D'_{-2n-1} = D'_{-2n} - c_{-n} D'_{-2n+1}, n \geq 0, \quad D'_1 = 0, D'_0 = 1, \end{aligned}$$

for the negative indices. For basic concepts on the theory of continued fractions we refer the reader to [3, 50].

Based on the above system of equations for $(N_n)_{n\geq 0}$ and $(D_n)_{n\geq 0}$ with the corresponding initial conditions, it is easy to see that

$$\begin{aligned} N_1D_0 - D1N_0 &= -a_1, \\ N_2D_1 - D_2N_1 &= -a_1c_1. \end{aligned}$$

Let us assume that for $n \in \mathbb{Z}_{\geq 0}$ we have that

$$\begin{aligned} N_{2n-2}D_{2n-3} - N_{2n-3}D_{2n-2} &= -a_0c_1a_1c_2 \cdots a_{n-2}c_{n-1}, \\ N_{2n-1}D_{2n-2} - N_{2n-2}D_{2n-2} &= -a_0c_1 \cdots c_{n-1}a_{n-1}. \end{aligned}$$

Using (2.1.6) for $n = n+1$ we get

$$\begin{aligned} N_{2n}D_{2n-1} - N_{2n-1}D_{2n} &= [N_{2n-1} - c_nN_{2n-2}]D_{2n-1} - N_{2n-1}[D_{2n-1} - c_nD_{2n-2}] \\ &= N_{2n-1}D_{2n-1} - c_nN_{2n-2}D_{2n-1} - N_{2n-1}D_{2n-1} + c_nN_{2n-1}D_{2n-2} \\ &= c_n[N_{2n-1}D_{2n-2} - N_{2n-2}D_{2n-1}] = -a_0c_1 \cdots c_{n-1}a_{n-1}c_n. \end{aligned}$$

Then we have

$$\begin{aligned} N_{2n+1}D_{2n} - N_{2n}D_{2n+1} &= [N_{2n} - a_nN_{2n-1}]D_{2n} - N_{2n}[D_{2n} - a_nD_{2n-1}] \\ &= N_{2n}D_{2n} - a_nN_{2n-1}D_{2n} + N_{2n}D_{2n} + a_nN_{2n}D_{2n-1} \\ &= a_n[N_{2n}D_{2n-1} - N_{2n-1}D_{2n}] = -a_0c_1 \cdots a_{n-1}c_na_n. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} N_{2n}D_{2n-1} - N_{2n-1}D_{2n} &= -a_oc_1a_1c_2 \cdots a_{n-1}c_n, \quad n \geq 1, \\ N_{2n+1}D_{2n} - N_{2n}D_{2n+1} &= -a_0c_1a_1 \cdots c_na_n, \quad n \geq 0. \end{aligned}$$

Following a similar procedure for negative indices, it can also be proved that

$$\begin{aligned} N'_{-2n}D'_{-2n-1} - N'_{-2n-1}D'_{-2n} &= -c_0a_{-1}c_{-1} \cdots a_{-n}c_{-n}, \quad n \geq 0, \\ N'_{-2n-1}D'_{-2n-2} - N'_{-2n-2}D'_{-2n-1} &= -c_0a_{-1}c_{-1} \cdots c_{-n}a_{-n-1}, \quad n \geq 0. \end{aligned} \tag{2.1.7}$$

With the previous considerations we can state the main theorem of this section.

Theorem 2.1.2. *Let H and H' be the continued fractions defined by* (2.1.4) *and the corresponding convergents $(h_n)_{n\geq 0}$ and $(h_{-n})_{n\geq 0}$ defined by* (2.1.5)*. Assume that*

$$0 < N_n < D_n, \quad \text{and} \quad 0 < N'_{-n} < D'_{-n}, \quad n \geq 1. \tag{2.1.8}$$

Then both H and H' are convergent. Moreover, let $P = P_UP_L$ be as in (2.1.1)*. Assume that $H' \leq H$. Then, both P_U and P_L are stochastic matrices if and only if we choose y_0 in the following range*

$$H' \leq y_0 \leq H. \tag{2.1.9}$$

Proof. The proof of the convergence of H and the upper bound for y_0 can be found in [24, Theorem 2.1]. For the convergence of H' and the lower bound for y_0, if we consider equation (2.1.7) and using the assumptions (2.1.8) we have that

$$\begin{aligned} h'_{-2n-2} - h'_{-2n-1} &= \frac{N'_{-2n-2}}{D'_{-2n-2}} - \frac{N'_{-2n-1}}{D'_{-2n-1}} = \frac{c_0 a_{-1} c_{-1} \cdots c_{-n} a_{-n-1}}{D'_{-2n-1} D'_{-2n-2}} > 0, \quad n \geq 0, \\ h'_{-2n-1} - h'_{-2n} &= \frac{N'_{-2n-1}}{D'_{-2n-1}} - \frac{N'_{-2n}}{D'_{-2n}} = \frac{c_0 a_{-1} c_{-1} \cdots a_{-n} c_{-n}}{D'_{-2n} D'_{-2n-1}} > 0, \quad n \geq 0. \end{aligned}$$

Therefore we have the inequalities

$$0 = h'_0 < h'_{-1} < h'_{-2} < \cdots < h'_{-2n} < h'_{-2n-1} < h'_{-2n-2} < \cdots < 1,$$

that is, $(h'_{-n})_{n\geq 0}$ is a strictly increasing and bounded sequence, so it is convergent to H'.

For the lower bound for y_0, assume first that both P_U and P_L are stochastic matrices, so that $0 < x_n, y_n, s_n, r_n < 1, n \in \mathbb{Z}$. Then we have, using (2.1.3), that

$$s_0 = 1 - \frac{c_0}{y_0} > 0 \Leftrightarrow \frac{c_0}{y_0} < 1 \Leftrightarrow c_0 < y_0 \Leftrightarrow h'_{-1} < y_0,$$

and

$$\begin{aligned} y_{-1} &= 1 - \frac{a_{-1}}{s_0} > 0 \Leftrightarrow \frac{a_{-1}}{s_0} < 1 \Leftrightarrow a_{-1} < s_0 \Leftrightarrow a_{-1} < 1 - \frac{c_0}{y_0} \\ &\Leftrightarrow 1 - a_{-1} > \frac{c_0}{y_0} \Leftrightarrow \frac{1}{1 - a_{-1}} < \frac{y_0}{c_0} \Leftrightarrow \frac{c_0}{1 - a_{-1}} < y_0 \Leftrightarrow h'_{-2} < y_0. \end{aligned}$$

Following the same argument we have

$$\begin{aligned} y_{-n} = 1 - \frac{a_{-n}}{s_{-n+1}} > 0 &\Leftrightarrow a_{-n} < s_{-n+1} \Leftrightarrow 1 - a_{-n} > \frac{c_{-n+1}}{y_{-n+1}} \Leftrightarrow \frac{c_{-n+1}}{1 - a_{-n}} < y_{-n+1} \\ &\Leftrightarrow \frac{c_{-n+1}}{1 - a_{-n}} < 1 - \frac{a_{-n+1}}{s_{-n+2}} \Leftrightarrow \frac{a_{-n+1}}{1 - \frac{c_{-n+1}}{1 - a_{-n}}} < s_{-n+2} \\ &\cdots \Leftrightarrow \frac{a_{-n+n-1}\mid}{\mid 1} - \frac{c_{-1}\mid}{\mid 1} - \frac{a_{-2}\mid}{\mid 1} \cdots - \frac{a_{-n}\mid}{\mid 1} < s_{-n+n} \\ &\Leftrightarrow \frac{a_{-1}\mid}{\mid 1} - \frac{c_{-1}\mid}{\mid 1} - \frac{a_{-2}\mid}{\mid 1} \cdots - \frac{a_{-n}\mid}{\mid 1} < 1 - \frac{c_0}{y_0} \\ &\Leftrightarrow \frac{c_0}{1} - \frac{a_{-1}\mid}{\mid 1} - \frac{c_{-1}\mid}{\mid 1} - \frac{a_{-2}\mid}{\mid 1} \cdots - \frac{a_{-n}\mid}{\mid 1} < y_0 \\ &\Leftrightarrow h'_{-2n} < y_0. \end{aligned}$$

Therefore, for all $n \geq 0$, we have

$$0 = h'_0 < h'_{-n} < H' \leq y_0,$$

and we get the lower bound for y_0. On the contrary, if (2.1.9) holds, in particular we have that $h_{-n} < H' \leq y_0 \leq H < h_n$ for every $n \geq 0$. Following [24, Theorem 2.1] and the same steps as before together with an argument of strong induction, we can conclude that both P_U and P_L are stochastic matrices with the conditions that $0 < x_n, y_n, s_n, r_n < 1, n \in \mathbb{Z}$. □

For the rest of this section we focus our attention on the LU factorization. Here we consider the same discrete-time birth-death chain with state space on $\mathbb{Z}$ described by P given in (1.4.1) but now we are interested in the study of a LU factorization of the form $P = \widetilde{P}_L \widetilde{P}_U$ where $\widetilde{P}_L$ is a lower bidiagonal matrix describing a pure-death process and $\widetilde{P}_U$ is an upper bidiagonal matrix describing a pure-birth process. Notice that we use a tilde to differentiate the matrices of this factorization from the UL case. The LU factorization of the stochastic matrix P can be written as

$$P = \left(\begin{array}{cccc|cccc} \ddots & \ddots & & & & & & \\ & \tilde{r}_{-1} & \tilde{s}_{-1} & & 0 & & & \\ \hline & & \tilde{r}_0 & & \tilde{s}_0 & 0 & & \\ & & & & \tilde{r}_1 & \tilde{s}_1 & 0 & \\ & & & & & \ddots & \ddots & \end{array}\right) \left(\begin{array}{ccc|cccc} \ddots & \ddots & & & & & \\ 0 & \tilde{y}_{-1} & & \tilde{x}_{-1} & & & \\ \hline & 0 & & \tilde{y}_0 & \tilde{x}_0 & & \\ & & & 0 & \tilde{y}_1 & \tilde{x}_1 & \\ & & & & & \ddots & \ddots \end{array}\right) = \widetilde{P}_L \widetilde{P}_U, \qquad (2.1.10)$$

where again we have the condition that $\widetilde{P}_L$ and $\widetilde{P}_U$ are stochastic matrices, i.e., all entries of $\tilde{P}_L$ and $\tilde{P}_U$ are nonnegative and the sum of all rows is equal to 1, that is

$$\tilde{r}_n + \tilde{s}_n = 1, \quad \tilde{y}_n + \tilde{x}_n = 1, \quad n \in \mathbb{Z}. \qquad (2.1.11)$$

Now, (2.1.10) is equivalent to the system of equations

$$\begin{aligned} a_n &= \tilde{s}_n \tilde{x}_n, \\ b_n &= \tilde{r}_n \tilde{x}_{n-1} + \tilde{s}_n \tilde{y}_n, \quad n \in \mathbb{Z}, \\ c_n &= \tilde{r}_n \tilde{y}_{n-1}. \end{aligned} \qquad (2.1.12)$$

Thanks to the irreducibility condition we have that $0 < \tilde{r}_n,\ \tilde{s}_n,\ \tilde{y}_n,\ \tilde{x}_n < 1$, $n \in \mathbb{Z}$. Again we can choose $\tilde{s}_0$ or $\tilde{r}_0$ in certain interval and compute all entries of $\widetilde{P}_L$ and $\widetilde{P}_U$ recursively. Let us consider $\tilde{r}_0$ as *free parameter*. For the nonnegative values of the indices we can compute $\tilde{s}_0$, $\tilde{x}_0$, $\tilde{y}_0$, $\tilde{r}_1$, $\tilde{s}_1$, $\tilde{x}_1$, $\tilde{y}_1$, $\tilde{r}_2$, $\ldots$ recursively from (2.1.11) and (2.1.12), while for the negative values of the indices we can compute $\tilde{y}_{-1}$, $\tilde{x}_{-1}$, $\tilde{s}_{-1}$, $\tilde{r}_{-1}$, $\tilde{y}_{-2}$, $\tilde{x}_{-2}$, $\tilde{s}_{-2}$, $\tilde{r}_{-2}$, $\ldots$ recursively again from (2.1.11) and (2.1.12).

Before moving on, let us highlight that there is an important difference between LU factorizations when we consider a birth-death chain on $\mathbb{Z}_{\geq 0}$ and the case where the state space is $\mathbb{Z}$. The first case was studied in [24] where if the stochastic factorization exists, then it is unique. However, for the case on $\mathbb{Z}$, as we saw in the previous paragraph, the solution depends on a free parameter within a certain range, which means that if the stochastic factorization exists, then there will be a one-parameter family of LU factorizations. Next we study the existence of the stochastic factorization.

Lemma 2.1.3. *Let $\{X_t : t = 0, 1, 2, \ldots\}$ be an irreducible birth-death chain with state space in $\mathbb{Z}$ and transition probability matrix P given by* (1.4.1) *and let $\widetilde{P}_L$ and $\widetilde{P}_U$ be the factors described in* (2.1.10) *with $0 < \tilde{r}_0 < 1$ fixed. Then $\tilde{r}_n + \tilde{s}_n = 1$ for all $n \in \mathbb{Z}$ if and only if $\tilde{y}_n + \tilde{x}_n = 1$ for all $n \in \mathbb{Z}$.*

Proof. The proof of this Lemma follows the same steps as the proof of [24, Lemma 2.1] (see also Lemma 2.1.1). □

Before stating the final result of this section, note that we can compute all coefficients of $(\tilde{r}_n)_{n\in\mathbb{Z}}$ and $(\tilde{x}_n)_{n\in\mathbb{Z}}$ by using the following equations

$$\tilde{r}_n = \frac{c_n}{1 - \tilde{x}_{n-1}}, \quad \tilde{x}_n = \frac{a_n}{1 - \tilde{r}_n}, \quad n \in \mathbb{Z},$$

together with equation (2.1.11).

At this point, we can compute all coefficients $\tilde{r}_n$, $\tilde{s}_n$, $\tilde{y}_n$ and $\tilde{x}_n$ in terms of one free parameter $\tilde{r}_0$. Nevertheless, we can not infer anything about the positivity of these coefficients yet. A natural way to study this factorization is with a similar argument as in Theorem 2.1.2 but now taking the free parameter $\tilde{r}_0$ and, surprisingly, the same continued fractions as upper and lower bounds.

Theorem 2.1.4. *Let H and H' be the continued fractions given by* (2.1.4) *and the corresponding convergents $(h_n)_{n\geq 0}$ and $(h'_{-n})_{n\geq 0}$ defined by* (2.1.5) *and assume conditions* (2.1.8)*. Then both H and H' are convergent. Moreover, let $P = \widetilde{P}_L\widetilde{P}_U$ be as in* (2.1.10)*. Assume that $H' \leq H$. Then, both $\widetilde{P}_L$ and $\widetilde{P}_U$ are stochastic matrices if and only if we choose $\tilde{r}_0$ in the following range*

$$H' \leq \tilde{r}_0 \leq H.$$

Proof. The proof is similar to the proof of Theorem 2.1.2 but using (2.1.12) instead of (2.1.3). $\square$

Now that we know the conditions under which we have UL and LU stochastic factorizations for the transition probability matrix P given by (1.4.1), let us continue to the next point of interest, the discrete Darboux transformation and the relation between the corresponding spectral matrices.

2.2 Stochastic Darboux transformations and spectral matrices

This section is devoted to the study of the *discrete Darboux transformation* and is closely related to [24, Section 3]. If we consider the tridiagonal stochastic matrix P as in (1.4.1) and the UL factorization described in (2.1.1) then, in our context, the discrete Darboux transformation consists of inverting the order of multiplication of the factors so we have $\widetilde{P} = P_L P_U$. The importance of this transformation is clear when we note that the Darboux transformation $\widetilde{P}$ will be again a stochastic matrix. One of the most important contributions of this section is that we get a relation between the spectral matrix of P and the spectral matrix of $\widetilde{P}$ and with this, a way to study certain probabilistic properties of the process described by P and the process described by the Darboux transformation $\widetilde{P}$. Of course, we can perform this transformation in both UL and LU stochastic factorizations.

We begin by considering $P = P_U P_L$ as in (2.1.1). Then, by inverting the order of multiplication of the factors, we obtain another tridiagonal matrix of the form

$$\widetilde{P} = P_L P_U = \left(\begin{array}{ccc|ccc} \ddots & \ddots & & & & \\ & r_{-1} & s_{-1} & 0 & & \\ \hline & & r_0 & s_0 & 0 & \\ & & & r_1 & s_1 & 0 \\ & & & & \ddots & \ddots \end{array}\right)\left(\begin{array}{ccc|ccc} \ddots & \ddots & & & & \\ 0 & y_{-1} & & x_{-1} & & \\ \hline & 0 & & y_0 & x_0 & \\ & & & 0 & y_1 & x_1 \\ & & & & \ddots & \ddots \end{array}\right) \tag{2.2.1}$$

$$= \left(\begin{array}{ccc|cccc} \ddots & \ddots & & & & & \\ 0 & \tilde{c}_{-1} & \tilde{b}_{-1} & \tilde{a}_{-1} & & & \\ \hline & 0 & \tilde{c}_0 & \tilde{b}_0 & \tilde{a}_0 & & \\ & & 0 & \tilde{c}_1 & \tilde{b}_1 & \tilde{a}_1 & \\ & & & & \ddots & \ddots & \ddots \end{array}\right).$$

From this, it is clear that the new coefficients of the matrix $\widetilde{P}$ are given by

$$\begin{aligned}
\tilde{a}_n &= s_n x_n, \\
\tilde{b}_n &= r_n x_{n-1} + s_n y_n, \quad n \in \mathbb{Z}, \\
\tilde{c}_n &= r_n y_{n-1}.
\end{aligned} \tag{2.2.2}$$

If we assume that the factorization is stochastic, then the matrix $\widetilde{P}$ is again stochastic since the product of stochastic matrices is again stochastic. Now, as the factorization depends on the free parameter y_0, the Darboux transformation gives rise to an entire *family* of new discrete-time birth-death chains $\{\widetilde{X}_t : t = 0, 1, \ldots\}$ on the integers $\mathbb{Z}$ with coefficients $(\tilde{a}_n)_{n\in\mathbb{Z}}$, $(\tilde{b}_n)_{n\in\mathbb{Z}}$ and $(\tilde{c}_n)_{n\in\mathbb{Z}}$ depending on y_0.

On the other hand, we can do the same for the LU factorization (2.1.10) of the form $P = \widetilde{P}_L \widetilde{P}_U$. For this case, after the Darboux transformation, the new tridiagonal matrix is given by

$$\widehat{P} = \widetilde{P}_U \widetilde{P}_L = \left(\begin{array}{cc|cccc}
\ddots & \ddots & & & & \\
0 & \tilde{y}_{-1} & \tilde{x}_{-1} & & & \\
\hline
 & 0 & \tilde{y}_0 & \tilde{x}_0 & & \\
 & & 0 & \tilde{y}_1 & \tilde{x}_1 & \\
 & & & & \ddots & \ddots
\end{array}\right)
\left(\begin{array}{ccc|ccc}
\ddots & \ddots & & & & \\
 & \tilde{r}_{-1} & \tilde{s}_{-1} & 0 & & \\
\hline
 & & \tilde{r}_0 & \tilde{s}_0 & 0 & \\
 & & & \tilde{r}_1 & \tilde{s}_1 & 0 \\
 & & & & \ddots & \ddots
\end{array}\right) \tag{2.2.3}$$

$$= \left(\begin{array}{ccc|cccc}
\ddots & \ddots & & & & & \\
0 & \hat{c}_{-1} & \hat{b}_{-1} & \hat{a}_{-1} & & & \\
\hline
 & 0 & \hat{c}_0 & \hat{b}_0 & \hat{a}_0 & & \\
 & & 0 & \hat{c}_1 & \hat{b}_1 & \hat{a}_1 & \\
 & & & & \ddots & \ddots & \ddots
\end{array}\right), \tag{2.2.4}$$

where the new coefficients are given by

$$\begin{aligned}
\hat{a}_n &= \tilde{x}_n \tilde{s}_{n+1}, \\
\hat{b}_n &= \tilde{x}_n \tilde{r}_{n+1} + \tilde{y}_n \tilde{s}_n, \quad n \in \mathbb{Z}, \\
\hat{c}_n &= \tilde{y}_n \tilde{r}_n.
\end{aligned} \tag{2.2.5}$$

In the previous section we saw that the stochastic LU factorization is not unique, so again, the matrix $\widehat{P}$ is stochastic and it gives rise to an entire one-parameter family of new discrete-time birth-death chains $\{\widehat{X}_t : t = 0, 1, \ldots\}$ on the integers $\mathbb{Z}$ with coefficients $(\hat{a}_n)_{n\in\mathbb{Z}}$, $(\hat{b}_n)_{n\in\mathbb{Z}}$ and $(\hat{c}_n)_{n\in\mathbb{Z}}$ depending on one free parameter $\tilde{r}_0$.

So far in this section we are assuming that all factors of the UL and LU factorizations are stochastic matrices. Therefore, the corresponding Darboux transformations are also stochastic matrices and they describe new processes. From a probabilistic perspective, we can think of the birth-death chain P as a model driven by urn experiments. In this context, both factorizations may be thought as two urn experiments, Experiment 1 and Experiment 2, respectively. We first perform the Experiment 1 and with the result we immediately perform the Experiment 2. The model for the Darboux transformation will be reversing the order of both experiments. Some examples of urn models of this type can found in [24, 25]. In this thesis we will also see an urn model for the example in Chapter 4.

In Chapter 1 we saw that the eigenvalue equation for the matrix P generates two families of linearly independent polynomials $(Q_n^\alpha)_{n\in\mathbb{Z}}$, $\alpha = 1, 2$. Through the use of the spectral theorem, we ensured the existence of three spectral measures that, in matrix form, we called the spectral matrix $\mathbf{\Psi}(x)$ associated with P. An important characteristic of the spectral matrix is that it allows us to write the orthogonality relation in matrix form as in (1.4.7).

In this section we will return to these concepts since our goal is to give an explicit expression for the relation between the spectral matrix associated with P and the spectral matrix associated with the corresponding Darboux transformation. With this in mind, we present the following result that provides us with a characterization of the orthogonality for the vector-valued polynomials $(Q_n^1(x), Q_n^2(x))$ associated with P in terms of monomials. This result will simplify some computations later in the main theorem of this section.

Lemma 2.2.1. *Let $(Q_n^\alpha)_{n\in\mathbb{Z}}$ be the polynomials defined by* (1.4.2)*. Then the vector-valued polynomials $\big(Q_n^1(x), Q_n^2(x)\big)$, $n \in \mathbb{Z}$, are orthogonal in the sense of* (1.4.7) *if and only if for $n \geq 0$ we have*

$$\int_{-1}^{1} \big(Q_n^1(x), Q_n^2(x)\big)\,\mathbf{\Psi}(x) x^j\,dx = \begin{cases} (0,0), & \textit{for} \quad j = 0,1,\dots,n-1, \\ (\alpha_n, 0), \alpha_n \neq 0, & \textit{for} \quad j = n, \end{cases} \tag{2.2.6}$$

and

$$\int_{-1}^{1} \big(Q_{-n-1}^1(x), Q_{-n-1}^2(x)\big)\,\mathbf{\Psi}(x) x^j\,dx = \begin{cases} (0,0), & \textit{for} \quad j = 0,1,\dots,n-1, \\ (0, \beta_n), \beta_n \neq 0, & \textit{for} \quad j = n. \end{cases} \tag{2.2.7}$$

Moreover $\alpha_0 = 1, \alpha_n = c_1 \cdots c_n, n \geq 1$ and $\beta_n = c_0^{-1} a_{-1} \cdots a_{-n-1}, n \geq 0$.

Proof. First of all, it is clear that the orthogonality conditions (1.4.7) are equivalent to the matrix orthogonality (1.5.11). Since $\boldsymbol{Q}_n(x)$ in (1.5.10) is a matrix-valued polynomial of degree n with nonsingular leading coefficient, the orthogonality is equivalent to

$$\int_{-1}^{1} \boldsymbol{Q}_n(x)\mathbf{\Psi}(x)x^j\,dx = \mathbf{0}_2 \quad \text{ for } j = 0,1,\dots,n-1,$$

where $\mathbf{0}_2$ denotes the 2×2 null matrix, and $\int_{-1}^1 \boldsymbol{Q}_n(x)\mathbf{\Psi}(x)x^n\,dx$ is a nonsingular and diagonal matrix. The rows of this expression are the same as equations (2.2.6) and (2.2.7), so it only remains to compute the values of α_n and β_n using (1.4.7) as follows

$$\begin{aligned} \frac{1}{\pi_n} &= \int_{-1}^{1} (Q_n^1(x), Q_n^2(x))\mathbf{\Psi}(x) \begin{pmatrix} Q_n^1(x) \\ Q_n^2(x) \end{pmatrix} dx \\ &= \int_{-1}^{1} (Q_n^1(x), Q_n^2(x))\mathbf{\Psi}(x) \begin{pmatrix} R_n^1 x^n + \mathcal{O}(x^{n-1}) \\ R_{n-1}^2 x^{n-1} + \mathcal{O}(x^{n-2}) \end{pmatrix} dx \\ &= \int_{-1}^{1} (Q_n^1(x), Q_n^2(x))\mathbf{\Psi}(x) \begin{pmatrix} R_n^1 x^n \\ R_{n-1}^2 x^{n-1} \end{pmatrix} dx \\ &= R_n^1 \int_{-1}^{1} (Q_n^1(x), Q_n^2(x))\mathbf{\Psi}(x) \begin{pmatrix} x^n \\ 0 \end{pmatrix} dx, \end{aligned}$$

so we have the value of α_n as follows

$$\int_{-1}^{1} (Q_n^1(x), Q_n^2(x))\mathbf{\Psi}(x) \begin{pmatrix} x^n \\ 0 \end{pmatrix} dx = \frac{R_n^1}{\pi_n} = c_1 c_2 \cdots c_n.$$

Following the same steps we can compute the value of β_n as

$$\int_{-1}^{1}(Q^1_{-n-1}(x),Q^2_{-n-1}(x))\boldsymbol{\Psi}(x)\begin{pmatrix}x^n\\0\end{pmatrix}dx=\frac{a_{-1}a_{-2}\cdots a_{-n-1}}{c_0}.$$

$\square$

The previous result will be very useful whenever one tries to prove orthogonality results on polynomials related to $(Q^\alpha_n(x))_{n\in\mathbb{Z}}$, $\alpha=1,2$. We will see an application of this lemma in the next two subsections.

2.2.1 Darboux transformation for the UL case

Now it is time to study the spectral matrix associated with the Darboux transformation for the UL factorization of the matrix P given in (1.4.1). Let us consider the discrete Darboux transformation $\widetilde{P}$ described in (2.2.1) with probability coefficients $(\tilde{a}_n)_{n\in\mathbb{Z}}$, $(\tilde{b}_n)_{n\in\mathbb{Z}}$ and $(\tilde{c}_n)_{n\in\mathbb{Z}}$ given by (2.2.2). In order to study the spectral matrix associated with $\widetilde{P}$ we have to find the family of polynomials that solves the eigenvalue equation. With this in mind, let us introduce an auxiliary family of polynomials $S^\alpha_n(x)$ given by the relation

$$s^\alpha(x)=P_Lq^\alpha(x),$$

where $q^\alpha(x)=(\cdots,Q^\alpha_{-1}(x),Q^\alpha_0(x),Q^\alpha_1(x),\cdots)^T$, and $s^\alpha(x)=(\cdots,S^\alpha_{-1}(x),S^\alpha_0(x),S^\alpha_1(x),\cdots)^T$, for $\alpha=1,2$. Using the coefficients of P_L we have

$$S^\alpha_n(x)=s_nQ^\alpha_n(x)+r_nQ^\alpha_{n-1}(x),\quad n\in\mathbb{Z},\tag{2.2.8}$$

for $\alpha=1,2$. From the expression of the UL factorization we also have

$$P_Us^\alpha(x)=P_UP_Lq^\alpha(x)=Pq^\alpha(x)=xq^\alpha(x),$$

so we get

$$xQ^\alpha_n(x)=x_nS^\alpha_{n+1}(x)+y_nS^\alpha_n(x),\quad n\in\mathbb{Z},\quad \alpha=1,2.\tag{2.2.9}$$

Evaluating (2.2.9) at $x=0$ and after some computations we get

$$\begin{aligned}S^\alpha_n(0)&=(-1)^n\frac{y_0\cdots y_{n-1}}{x_0\dots x_{n-1}}S^\alpha_0(0),\quad n\geq 1,\\ S^\alpha_{-n-1}(0)&=(-1)^{n+1}\frac{x_{-1}\cdots x_{-n-1}}{y_{-1}\cdots y_{-n-1}}S^\alpha_0(0),\quad n\geq 0,\end{aligned}\tag{2.2.10}$$

where

$$S^\alpha_0(0)=\begin{cases}s_0, & \text{if}\quad \alpha=1,\\ r_0, & \text{if}\quad \alpha=2.\end{cases}$$

This last equation establishes a direct relation between the polynomials $(S^\alpha_n)_{n\in\mathbb{Z}}$ for $\alpha=1,2$, given by

$$s_0S^2_n(0)=r_0S^1_n(0),\quad n\in\mathbb{Z}.\tag{2.2.11}$$

Another relation that eventually will prove to be very useful arises from equations (2.2.9) and (2.2.10) which gives the polynomials $(S_n^\alpha)_{n\in\mathbb{Z}}$ in terms of $(Q_n^\alpha)_{n\in\mathbb{Z}}$. Indeed, for $n \geq 0$,

$$\begin{aligned} S_{n+1}^\alpha(x) &= \frac{x}{x_n} Q_n^\alpha(x) - \frac{y_n}{x_n} S_n^\alpha(x) = \frac{x}{x_n} Q_n^\alpha(x) + \frac{S_{n+1}^\alpha(0)}{S_n^\alpha(0)} \left[\frac{x}{x_{n-1}} Q_{n-1}^\alpha(x) + \frac{S_n^\alpha(0)}{S_{n-1}^\alpha(0)} S_{n-1}^\alpha(x) \right] \\ &= x \left[\frac{Q_n^\alpha(x)}{x_n} + \frac{S_{n+1}^\alpha(0)}{S_n^\alpha(0)} \frac{Q_{n-1}^\alpha(x)}{x_{n-1}} \right] + \frac{S_{n+1}^\alpha(0)}{S_{n-1}^\alpha(0)} S_{n-1}^\alpha(x) = \cdots \\ &= x \sum_{j=0}^{n} \frac{S_{n+1}^\alpha(0)}{S_{j+1}^\alpha(0)} \frac{Q_j^\alpha(x)}{x_j} + S_{n+1}^\alpha(0), \end{aligned}$$

since $S_0^\alpha(x)$ is constant. Therefore

$$S_n^\alpha(x) = S_n^\alpha(0) \left[1 + x \sum_{j=0}^{n-1} \frac{Q_j^\alpha(x)}{S_{j+1}^\alpha(0) x_j} \right], \quad n \geq 1. \tag{2.2.12}$$

Following the same steps as before, we have

$$S_{-n-1}^\alpha(x) = S_{-n-1}^\alpha(0) \left[1 + x \sum_{j=0}^{n} \frac{Q_{-j-1}^\alpha(x)}{S_{-j-1}^\alpha(0) y_{-j-1}} \right], \quad n \geq 0. \tag{2.2.13}$$

Observe that this auxiliary family $(S_n^\alpha)_{n\in\mathbb{Z}}$ does not satisfy the same initial conditions as the family $(Q_n^\alpha)_{n\in\mathbb{Z}}$ since by (2.2.8) we have

$$\begin{aligned} S_0^1(x) &= s_0, & S_0^2(x) &= r_0, \\ S_{-1}^1(x) &= -\frac{x_{-1}s_0}{y_{-1}}, & S_{-1}^2(x) &= \frac{x - x_{-1}r_0}{y_{-1}}. \end{aligned}$$

Also, the degrees of the polynomials $(S_n^\alpha)_{n\in\mathbb{Z}}$ are not the same as the degrees of the polynomials $(Q_n^\alpha)_{n\in\mathbb{Z}}$, since

$$\begin{aligned} \deg(S_n^1) &= n, \quad n \geq 0, \quad \deg(S_n^2) = n, \quad n \geq 0, \\ \deg(S_{-n-1}^1) &= n, \quad n \geq 0, \quad \deg(S_{-n-1}^2) = n+1, \quad n \geq 0. \end{aligned} \tag{2.2.14}$$

In order to obtain the family of orthogonal polynomials associated with $\widetilde{P}$, we need a polynomial family satisfying the same initial conditions and degrees as $(Q_n^\alpha)_{n\in\mathbb{Z}}$. In matrix form these conditions are given by $\boldsymbol{Q}_0(x) = \boldsymbol{I}_2$ and $\deg(\boldsymbol{Q}_n(x)) = n$. It is possible to obtain this family of matrix-valued polynomials by taking

$$\boldsymbol{S}_n(x) = \begin{pmatrix} S_n^1(x) & S_n^2(x) \\ S_{-n-1}^1(x) & S_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0,$$

which has degree $n+1$ and singular leading coefficient and defining

$$\widetilde{\boldsymbol{Q}}_n(x) = \boldsymbol{S}_n(x) \boldsymbol{S}_0^{-1}(x), \quad n \geq 0, \tag{2.2.15}$$

where

$$\boldsymbol{S}_0(x) = \begin{pmatrix} s_0 & r_0 \\ -\dfrac{x_{-1}s_0}{y_{-1}} & \dfrac{x - x_{-1}r_0}{y_{-1}} \end{pmatrix}. \tag{2.2.16}$$

Following the same representation as in (1.5.10) we can define the functions $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1, 2$, but first we have to show that we are dealing with polynomials.

Proposition 2.2.2. *Let $(\widetilde{\boldsymbol{Q}}_n(x))_{n\geq 0}$, be the sequence of matrix functions defined by* (2.2.15). *Then, for $n \geq 0$, $\widetilde{\boldsymbol{Q}}_n(x)$ is a matrix polynomial of degree exactly n with nonsingular leading coefficient and $\widetilde{\boldsymbol{Q}}_0(x) = \boldsymbol{I}_2$.*

Proof. First we compute the inverse of $\boldsymbol{S}_0(x)$, given by

$$\boldsymbol{S}_0^{-1}(x) = \frac{1}{x}\begin{pmatrix} \dfrac{x - x_{-1}r_0}{s_0} & -\dfrac{y_{-1}r_0}{s_0} \\ x_{-1} & y_{-1} \end{pmatrix}.$$

Observe that $|\boldsymbol{S}_0(x)| = \frac{xs_0}{y_{-1}}$, so the inverse is not well-defined at $x = 0$. We will show that we can avoid this problem using the properties of the polynomials $(S_n^\alpha)_{n\in\mathbb{Z}}$. Indeed, from (2.2.15), we have

$$\begin{aligned} \widetilde{Q}_n^1(x) &= \frac{S_n^1(x)}{s_0} + \frac{x_{-1}}{xs_0}\left(s_0 S_n^2(x) - r_0 S_n^1(x)\right), \quad n \in \mathbb{Z}, \\ \widetilde{Q}_n^2(x) &= \frac{y_{-1}}{xs_0}\left(s_0 S_n^2(x) - r_0 S_n^1(x)\right), \quad n \in \mathbb{Z}. \end{aligned}$$

A straightforward computation using (2.2.12), (2.2.13) and (2.2.11) gives

$$\begin{aligned} s_0 S_0^2(x) - r_0 S_0^1(x) &= 0, \\ s_0 S_n^2(x) - r_0 S_n^1(x) &= x\sum_{j=0}^{n-1}\frac{1}{x_j}\left(s_0\frac{S_n^2(0)Q_j^2(x)}{S_{j+1}^2(0)} - r_0\frac{S_n^1(0)Q_j^1(x)}{S_{j+1}^1(0)}\right), \quad n \geq 1, \\ s_0 S_{-n-1}^2(x) - r_0 S_{-n-1}^1(x) &= x\sum_{j=0}^{n}\frac{1}{y_{-j-1}}\left(s_0\frac{S_{-n-1}^2(0)Q_{-j-1}^2(x)}{S_{-j-1}^2(0)} - r_0\frac{S_{-n-1}^1(0)Q_{-j-1}^1(x)}{S_{-j-1}^1(0)}\right), \quad n \geq 0. \end{aligned}$$

Therefore from these relations we can see that $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1, 2$ are indeed polynomials. Also, from equation (2.2.14) we have

$$\begin{aligned} \deg\left((s_0 S_n^2(x) - r_0 S_n^1(x))/x\right) &= n - 1, \quad \text{for} \quad n \geq 1, \quad \text{and} \\ \deg\left((s_0 S_{-n-1}^2(x) - r_0 S_{-n-1}^1(x))/x\right) &= n, \quad \text{for } n \geq 0. \end{aligned}$$

Then, from (2.2.14) we get

$$\begin{aligned} \deg(\widetilde{Q}_n^1) &= n, \quad n \geq 0 \quad \deg(\widetilde{Q}_n^2) = n - 1, \quad n \geq 1, \\ \deg(\widetilde{Q}_{-n-1}^2) &= n, \quad n \geq 0. \end{aligned}$$

Finally, $\widetilde{Q}_{-n-1}^1$ is in principle a polynomial of degree n, but if we call Λ_n the coefficient of x^n in $\widetilde{Q}_{-n-1}^1$, then, using (1.4.3), (1.4.4), (2.2.8) and (2.1.3), we have

$$\Lambda_n = \frac{1}{s_0}\left(-\frac{a_{-1}r_{-n-1}}{c_{-1}\cdots c_{-n-1}} + \frac{x_{-1}s_0}{y_{-n-1}c_{-1}\cdots c_{-n}}\right) = \frac{1}{s_0 c_{-1}\cdots c_{-n}}\left(-\frac{a_{-1}r_{-n-1}}{c_{-n-1}} + \frac{x_{-1}s_0}{y_{-n-1}}\right) = 0.$$

Therefore $\deg(\widetilde{Q}_{-n-1}^1) = n - 1, n \geq 1$. The fact that $\widetilde{\boldsymbol{Q}}_0(x) = \boldsymbol{I}_2$ comes from the definition. □

Now, we know that $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, is a family of polynomials satisfying the same initial and degree conditions than the original polynomials $(Q_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$. They also satisfy the three-term recurrence relation

$$\begin{aligned}
\widetilde{Q}_0^1(x) &= 1, \quad \widetilde{Q}_0^2(x) = 0,\\
\widetilde{Q}_{-1}^1(x) &= 0, \quad \widetilde{Q}_{-1}^2(x) = 1,\\
x\widetilde{Q}_n^\alpha(x) &= \tilde{a}_n\widetilde{Q}_{n+1}^\alpha(x) + \tilde{b}_n\widetilde{Q}_n^\alpha(x) + \tilde{c}_n\widetilde{Q}_{n-1}^\alpha(x), \quad n\in\mathbb{Z}, \quad \alpha = 1,2,
\end{aligned} \tag{2.2.17}$$

where the coefficients $(\tilde{a}_n)_{n\in\mathbb{Z}}$, $(\tilde{b}_n)_{n\in\mathbb{Z}}$ and $(\tilde{c}_n)_{n\in\mathbb{Z}}$ are defined by (2.2.2).

We will continue with the main result of this section that establishes a relation between the spectral matrix associated with the Darboux transformation $\widetilde{P}$ and the one associated with P. We will also prove that $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, are the corresponding orthogonal polynomials associated with $\widetilde{P}$. For that, let us define the potential coefficients associated with $\widetilde{P}$ in a similar way as for P, i.e.,

$$\tilde{\pi}_0 = 1, \quad \tilde{\pi}_n = \frac{\tilde{a}_0\cdots\tilde{a}_{n-1}}{\tilde{c}_1\cdots\tilde{c}_n}, \quad \tilde{\pi}_{-n} = \frac{\tilde{c}_0\cdots\tilde{c}_{-n+1}}{\tilde{a}_{-1}\cdots\tilde{a}_{-n}}, \quad n\geq 1. \tag{2.2.18}$$

Theorem 2.2.3. *Let $\{X_t : t = 0,1,\dots\}$ be the bilateral birth-death chain on $\mathbb{Z}$ with transition probability matrix P given by* (1.4.1) *and $\{\widetilde{X}_t : t = 0,1,\dots\}$ the bilateral birth-death chain on $\mathbb{Z}$ with transition probability matrix $\widetilde{P}$ given by* (2.2.1)*. Assume that*

$$\boldsymbol{M}_{-1} = \int_{-1}^{1} \frac{\boldsymbol{\Psi}(x)}{x}dx \tag{2.2.19}$$

is well-defined entry by entry, where $\boldsymbol{\Psi}(x)$ is the original spectral matrix given by (1.4.6)*. Then the polynomials $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, defined by* (2.2.17) *are orthogonal with respect to the following spectral matrix*

$$\widetilde{\boldsymbol{\Psi}}(x) = \boldsymbol{S}_0(x)\boldsymbol{\Psi}_S(x)\boldsymbol{S}_0^*(x), \tag{2.2.20}$$

where $\boldsymbol{S}_0(x)$ is defined by (2.2.16) *and*

$$\boldsymbol{\Psi}_S(x) = \frac{y_0}{s_0}\frac{\boldsymbol{\Psi}(x)}{x} + \left[\begin{pmatrix} 1/s_0 & 0\\ 0 & 1/r_0\end{pmatrix} - \frac{y_0}{s_0}\boldsymbol{M}_{-1}\right]\delta_0(x), \tag{2.2.21}$$

with $\delta_0(x)$ the Dirac delta at $x=0$. Moreover, we have

$$\int_{-1}^{1} \widetilde{\boldsymbol{Q}}_n(x)\widetilde{\boldsymbol{\Psi}}(x)\widetilde{\boldsymbol{Q}}_m^*(x)dx = \begin{pmatrix} 1/\tilde{\pi}_n & 0\\ 0 & 1/\tilde{\pi}_{-n-1}\end{pmatrix}\delta_{nm}, \tag{2.2.22}$$

where δ_{nm} is the Kronecker delta and $(\tilde{\pi}_n)_{n\in\mathbb{Z}}$ are the potential coefficients defined by (2.2.18)*.*

Proof. Let $(Q_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, be the polynomials defined by (1.4.2), which are orthogonal with respect to the original spectral matrix $\boldsymbol{\Psi}(x)$. By Lemma 2.2.1 we have the orthogonality conditions (2.2.6) and (2.2.7). Since $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, satisfies the same initial and degree conditions than $(Q_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, we will use Lemma 2.2.1 to prove that $(\widetilde{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1,2$, are orthogonal with respect to $\widetilde{\boldsymbol{\Psi}}(x)$ in (2.2.20).

Assume first that $n \geq 1$. Then we have, using (2.2.20), (2.2.15) and (2.2.21), that

$$\begin{aligned}
\int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) \widetilde{\mathbf{\Psi}}(x) x^j dx &= \int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) x \widetilde{\mathbf{\Psi}}(x) x^{j-1} dx \\
&= \int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) \boldsymbol{S}_0(x) x \mathbf{\Psi}_S(x) \boldsymbol{S}_0^*(x) x^{j-1} dx \\
&= \frac{y_0}{s_0} \int_{-1}^{1} \left(S_n^1(x), S_n^2(x)\right) \boldsymbol{S}_0^{-1}(x) \boldsymbol{S}_0(x) \mathbf{\Psi}(x) \boldsymbol{S}_0^*(x) x^{j-1} dx \\
&= \frac{y_0}{s_0} \int_{-1}^{1} \left(S_n^1(x), S_n^2(x)\right) \mathbf{\Psi}(x) \boldsymbol{S}_0^*(x) x^{j-1} dx.
\end{aligned}$$

Now, using (2.2.8), the above expression can be written as

$$\begin{aligned}
\int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) \widetilde{\mathbf{\Psi}}(x) x^j dx = &\frac{s_n y_0}{s_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \boldsymbol{S}_0^*(x) x^{j-1} dx \\
&+ \frac{r_n y_0}{s_0} \int_{-1}^{1} \left(Q_{n-1}^1(x), Q_{n-1}^2(x)\right) \mathbf{\Psi}(x) \boldsymbol{S}_0^*(x) x^{j-1} dx.
\end{aligned}$$

Writing $\boldsymbol{S}_0(x) = A + xB$, where A and B are given by

$$A = \begin{pmatrix} s_0 & r_0 \\ -\dfrac{x_{-1} s_0}{y_{-1}} & -\dfrac{x_{-1} r_0}{y_{-1}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{y_{-1}} \end{pmatrix}, \tag{2.2.23}$$

the above expression can be written as

$$\begin{aligned}
&\int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) \widetilde{\mathbf{\Psi}}(x) x^j dx = \\
&\quad \frac{s_n y_0}{s_0} \left[\int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) A^* x^{j-1} dx + \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) B^* x^j dx\right] \\
&\quad + \frac{r_n y_0}{s_0} \left[\int_{-1}^{1} \left(Q_{n-1}^1(x), Q_{n-1}^2(x)\right) \mathbf{\Psi}(x) A^* x^{j-1} dx + \int_{-1}^{1} \left(Q_{n-1}^1(x), Q_{n-1}^2(x)\right) \mathbf{\Psi}(x) B^* x^j dx\right].
\end{aligned}$$

Using (2.2.6) we have that the first term of the sum vanishes for $j = 1, \ldots, n$, the second term vanishes for $j = 0, \ldots, n-1$, the third term vanishes for $j = 1, \ldots, n-1$ and the fourth term vanishes for $j = 0, \ldots, n-2$. Therefore the above expression vanishes for $j = 1, \ldots, n-2$. For $j = n-1$ the only term that does not vanish is the fourth one. But in this case we have, using (2.2.6) and (2.2.23), that

$$\begin{aligned}
\int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) \widetilde{\mathbf{\Psi}}(x) x^{n-1} dx &= \frac{r_n y_0}{s_0} \int_{-1}^{1} \left(Q_{n-1}^1(x), Q_{n-1}^2(x)\right) \mathbf{\Psi}(x) B^* x^{n-1} dx \\
&= \frac{r_n y_0}{s_0} (\alpha_{n-1}, 0) \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{y_{-1}} \end{pmatrix} = (0, 0).
\end{aligned} \tag{2.2.24}$$

For $j=0$ we have, using (2.2.15) and (2.2.20), that

$$\int_{-1}^{1}\left(\widetilde{Q}_n^1(x),\widetilde{Q}_n^2(x)\right)\widetilde{\boldsymbol{\Psi}}(x)dx=\int_{-1}^{1}\left(S_n^1(x),S_n^2(x)\right)\boldsymbol{S}_0^{-1}(x)\boldsymbol{S}_0(x)\boldsymbol{\Psi}_S(x)\boldsymbol{S}_0^*(x)dx$$
$$=\left[\int_{-1}^{1}\left(S_n^1(x),S_n^2(x)\right)\boldsymbol{\Psi}_S(x)dx\right]A^*+\left[\int_{-1}^{1}\left(S_n^1(x),S_n^2(x)\right)x\boldsymbol{\Psi}_S(x)dx\right]B^*.$$

The second term of the sum of the above expression vanishes as a consequence of (2.2.21), (2.2.8) and (2.2.6). Indeed, for $n\geq 2$, we have

$$\left[\int_{-1}^{1}\left(S_n^1(x),S_n^2(x)\right)x\boldsymbol{\Psi}_S(x)dx\right]B^*=\left[\frac{y_0}{s_0}\int_{-1}^{1}\left(S_n^1(x),S_n^2(x)\right)\boldsymbol{\Psi}(x)dx\right]B^*$$
$$=\left[\frac{s_ny_0}{s_0}\int_{-1}^{1}\left(Q_n^1(x),Q_n^2(x)\right)\boldsymbol{\Psi}(x)dx+\frac{r_ny_0}{s_0}\int_{-1}^{1}\left(Q_{n-1}^1(x),Q_{n-1}^2(x)\right)\boldsymbol{\Psi}(x)dx\right]B^*=(0,0).$$

For $n=1$ we can use the same argument as in (2.2.24) and get again $(0,0)$. Now, using (2.2.12), we can write

$$\left(S_n^1(x),S_n^2(x)\right)=x\sum_{j=0}^{n-1}\frac{1}{x_j}\left(\frac{S_n^1(0)}{S_{j+1}^1(0)}Q_j^1(x),\frac{S_n^2(0)}{S_{j+1}^2(0)}Q_j^2(x)\right)+\left(S_n^1(0),S_n^2(0)\right).$$

Substituting this in the remaining integral we get

$$\int_{-1}^{1}\left(\widetilde{Q}_n^1(x),\widetilde{Q}_n^2(x)\right)\widetilde{\boldsymbol{\Psi}}(x)dx=\left[\int_{-1}^{1}\left(S_n^1(x),S_n^2(x)\right)\boldsymbol{\Psi}_S(x)dx\right]A^*$$
$$=\left(S_n^1(0),S_n^2(0)\right)\left[\begin{pmatrix}1/s_0 & 0\\ 0 & 1/r_0\end{pmatrix}-\frac{y_0}{s_0}\boldsymbol{M}_{-1}\right]A^*$$
$$+\frac{y_0}{s_0}\sum_{j=0}^{n-1}\frac{1}{x_j}\int_{-1}^{1}\left(\frac{S_n^1(0)}{S_{j+1}^1(0)}Q_j^1(x),\frac{S_n^2(0)}{S_{j+1}^2(0)}Q_j^2(x)\right)\boldsymbol{\Psi}(x)A^*dx+\frac{y_0}{s_0}\left(S_n^1(0),S_n^2(0)\right)\boldsymbol{M}_{-1}A^*$$
$$=\left(S_n^1(0),S_n^2(0)\right)\begin{pmatrix}1/s_0 & 0\\ 0 & 1/r_0\end{pmatrix}A^*+\frac{y_0}{s_0x_0}\frac{S_n^1(0)}{S_1^1(0)}\int_{-1}^{1}\left(Q_0^1(x),Q_0^2(x)\right)\boldsymbol{\Psi}(x)A^*dx.$$

The third step is a consequence of $\dfrac{S_n^1(0)}{S_{j+1}^1(0)}=\dfrac{S_n^2(0)}{S_{j+1}^2(0)}$ using (2.2.11) and the orthogonality properties.

Since $\displaystyle\int_{-1}^{1}\left(Q_0^1(x),Q_0^2(x)\right)\boldsymbol{\Psi}(x)dx=(1,0)$, $S_1^1(0)=-s_0y_0/x_0$, and A is given by (2.2.23) then we have

$$\int_{-1}^{1}\left(\widetilde{Q}_n^1(x),\widetilde{Q}_n^2(x)\right)\widetilde{\boldsymbol{\Psi}}(x)dx=\left(S_n^1(0),S_n^2(0)\right)\begin{pmatrix}1/s_0 & 0\\ 0 & 1/r_0\end{pmatrix}A^*-\frac{1}{s_0^2}\left(S_n^1(0),0\right)A^*$$
$$=\left((1-1/s_0)\frac{S_n^1(0)}{s_0},\frac{S_n^2(0)}{r_0}\right)\begin{pmatrix}s_0 & -\dfrac{x_{-1}s_0}{y_{-1}}\\ r_0 & -\dfrac{x_{-1}r_0}{y_{-1}}\end{pmatrix}$$
$$=\left((-r_0/s_0)S_n^1(0)+S_n^2(0),-\frac{x_{-1}}{y_{-1}}\left((-r_0/s_0)S_n^1(0)+S_n^2(0)\right)\right)$$
$$=(0,0),$$

as a consequence of (2.2.11).

For $n \leq -1$ the proof is similar but now using (2.2.7) and (2.2.13). Therefore we have proved (2.2.22) for $n \neq m$. Observe that this implies in particular that the family of vector-valued polynomials $\left(S_n^1(x), S_n^2(x)\right), n \in \mathbb{Z}$, is also orthogonal for $n \neq m$, with respect to the weight matrix $\mathbf{\Psi}_S(x)$ in (2.2.21). For $n = m$, using this fact and (2.2.15), (2.2.9), (2.2.8), (2.2.20) and (2.2.21), we have that

$$
\begin{aligned}
&\int_{-1}^{1} \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right) \widetilde{\mathbf{\Psi}}(x) \left(\widetilde{Q}_n^1(x), \widetilde{Q}_n^2(x)\right)^* dx = \int_{-1}^{1} \left(S_n^1(x), S_n^2(x)\right) \mathbf{\Psi}_S(x) \left(S_n^1(x), S_n^2(x)\right)^* dx \\
&\quad = \int_{-1}^{1} \left[\frac{x}{y_n} \left(Q_n^1(x), Q_n^2(x)\right) - \frac{x_n}{y_n} \left(S_{n+1}^1(x), S_{n+1}^2(x)\right)\right] \mathbf{\Psi}_S(x) \left(S_n^1(x), S_n^2(x)\right)^* dx \\
&\quad = \frac{1}{y_n} \int_{-1}^{1} x \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}_S(x) \left(S_n^1(x), S_n^2(x)\right)^* dx - \frac{x_n}{y_n} \int_{-1}^{1} \left(S_{n+1}^1(x), S_{n+1}^2(x)\right) \mathbf{\Psi}_S(x) \left(S_n^1(x), S_n^2(x)\right)^* dx \\
&\quad = \frac{y_0}{y_n s_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \left[s_n \left(Q_n^1(x), Q_n^2(x)\right)^* + r_n \left(Q_{n-1}^1(x), Q_{n-1}^2(x)\right)^*\right] \\
&\quad = \frac{s_n y_0}{y_n s_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \left(Q_n^1(x), Q_n^2(x)\right)^* dx + \frac{r_n y_0}{y_n s_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \left(Q_{n-1}^1(x), Q_{n-1}^2(x)\right)^* dx \\
&\quad = \frac{s_n y_0}{y_n s_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \left(Q_n^1(x), Q_n^2(x)\right)^* dx = \frac{s_n y_0}{y_n s_0} \frac{1}{\pi_n} = \frac{1}{\tilde{\pi}_n}.
\end{aligned}
$$

The last step follows using (2.1.3), (2.2.2) and π_n given in Definition 1.4.1 and $\tilde{\pi}_n$ given in (2.2.18). $\square$

Remark 2.2.4. The assumption of $\boldsymbol{M}_{-1}$ being well-defined entry by entry in (2.2.19) is actually too restrictive. It is enough to assume the weaker condition that $\boldsymbol{S}_0(0)\boldsymbol{M}_{-1}\boldsymbol{S}_0^*(0)$ is well defined entry by entry in order to have the same result.

Notice that the spectral matrix $\widetilde{\mathbf{\Psi}}(x)$ associated with the Darboux transformation given in the previous theorem is a *conjugation* by a matrix-valued polynomial $\boldsymbol{S}_0(x)$ of degree 1 of a *Geronimus transformation* of the original spectral matrix $\mathbf{\Psi}(x)$. This behavior is consistent, except for the conjugation, with the case of the Darboux transformation of transition probability matrices on $\mathbb{Z}_{\geq 0}$ for the UL factorization appearing in [24] (see also Theorem 1.6.3).

2.2.2 Darboux transformation for the LU case

Now we turn our attention to the LU case. For this, we will follow the same procedure as the last subsection. Consider now the discrete Darboux transformation $\widehat{P}$ in (2.2.3) with probability coefficients $(\hat{a}_n)_{n\in\mathbb{Z}}$, $(\hat{b}_n)_{n\in\mathbb{Z}}$ and $(\hat{c}_n)_{n\in\mathbb{Z}}$ given by (2.2.5). Since the main results are similar to the ones in the previous subsection, we will only give the important and necessary formulas for the proofs.

First we need to introduce the auxiliary family of polynomials $(T_n^\alpha(x))_{n\in\mathbb{Z}}$, $\alpha, \beta = 1, 2$, given by the relation

$$t^\alpha(x) = \widetilde{P}_U q^\alpha(x),$$

where $q^\alpha(x) = (\cdots, Q_{-1}^\alpha(x), Q_0^\alpha(x), Q_1^\alpha(x), \cdots)^T$, and $t^\alpha(x) = (\cdots, T_{-1}^\alpha(x), T_0^\alpha(x), T_1^\alpha(x), \cdots)^T$ for $\alpha = 1, 2$. Using the coefficients of $\widetilde{P}_U$ we get

$$T_n^\alpha(x) = \tilde{y}_n Q_n^\alpha(x) + \tilde{x}_n Q_{n+1}^\alpha(x), \quad n \in \mathbb{Z}, \quad \alpha = 1, 2. \tag{2.2.25}$$

From the LU factorization we also have $\widetilde{P}_L t^{\alpha}(x) = xq^{\alpha}(x)$, that is

$$xQ_n^{\alpha}(x) = \tilde{r}_n T_{n-1}^{\alpha}(x) + \tilde{s}_n T_n^{\alpha}(x), \quad n \in \mathbb{Z}, \quad \alpha = 1, 2. \tag{2.2.26}$$

Evaluating (2.2.26) at $x = 0$ we get recursively

$$\begin{aligned} T_n^{\alpha}(0) &= (-1)^{n+1} \frac{\tilde{r}_0 \dots \tilde{r}_n}{\tilde{s}_0 \dots \tilde{s}_n} T_{-1}^{\alpha}(0), \quad n \geq 1, \\ T_{-n-1}^{\alpha}(0) &= (-1)^{n} \frac{\tilde{s}_{-1} \dots \tilde{s}_{-n}}{\tilde{r}_{-1} \dots \tilde{r}_{-n}} T_{-1}^{\alpha}(0), \quad n \geq 0, \end{aligned} \tag{2.2.27}$$

where

$$T_{-1}^{\alpha}(0) = \begin{cases} \tilde{x}_{-1}, & \text{if} \quad \alpha = 1, \\ \tilde{y}_{-1}, & \text{if} \quad \alpha = 2. \end{cases}$$

The equations (2.2.27) establish a direct relation between the polynomials $(T_n^{\alpha})_{n \in \mathbb{Z}}, \alpha = 1, 2$, given by

$$\tilde{x}_{-1} T_n^2(0) = \tilde{y}_{-1} T_n^1(0), \quad n \in \mathbb{Z}. \tag{2.2.28}$$

Again, the polynomials $(T_n^{\alpha})_{n \in \mathbb{Z}}$ can be written in terms of the polynomials $(Q_n^{\alpha})_{n \in \mathbb{Z}}$ with the following computations

$$\begin{aligned} T_n^{\alpha}(x) &= \frac{x}{\tilde{s}_n} Q_n^{\alpha}(x) - \frac{\tilde{r}_n}{\tilde{s}_n} T_{n-1}^{\alpha}(x) \\ &= \frac{x}{\tilde{s}_n} Q_n^{\alpha}(x) + \frac{T_n^{\alpha}(0)}{T_{n-1}^{\alpha}(0)} \left[\frac{x}{\tilde{s}_{n-1}} Q_{n-1}^{\alpha}(x) + \frac{T_{n-1}^{\alpha}(0)}{T_{n-2}^{\alpha}(0)} T_{n-2}^{\alpha}(x) \right] \\ &= x \left[\frac{Q_n^{\alpha}(x)}{\tilde{s}_n} + \frac{T_n^{\alpha}(0) Q_{n-1}^{\alpha}(x)}{T_{n-1}^{\alpha}(0) \tilde{s}_{n-1}} \right] + \frac{T_n^{\alpha}(0)}{T_{n-2}^{\alpha}(0)} T_{n-2}^{\alpha}(x) \\ &= \dots = x \sum_{j=0}^{n} \frac{T_n^{\alpha}(0)}{T_j^{\alpha}(0) \tilde{s}_j} Q_j^{\alpha}(x) + T_n^{\alpha}(0), \end{aligned}$$

so we have

$$T_n^{\alpha}(x) = T_n^{\alpha}(0) \left[1 + x \sum_{j=0}^{n} \frac{Q_j^{\alpha}(x)}{T_j^{\alpha}(0) \tilde{s}_j} \right], \quad n \geq 1. \tag{2.2.29}$$

In the same way as before, we have

$$T_{-n-1}^{\alpha}(x) = T_{-n-1}^{\alpha}(0) \left[1 + x \sum_{j=0}^{n-1} \frac{Q_{-j-1}^{\alpha}(x)}{T_{-j-2}^{\alpha}(0) \tilde{r}_{-j-1}} \right], \quad n \geq 0.$$

This auxiliary family $(T_n^{\alpha})_{n \in \mathbb{Z}}$ does not satisfy the same initial conditions as the family $(Q_n^{\alpha})_{n \in \mathbb{Z}}$. In fact we have

$$\begin{aligned} T_0^1(x) = \frac{x - \tilde{r}_0 \tilde{x}_{-1}}{\tilde{s}_0}, \quad & T_0^2(x) = -\frac{\tilde{r}_0 \tilde{y}_{-1}}{\tilde{s}_0}, \\ T_{-1}^1(x) = \tilde{x}_{-1}, \quad & T_{-1}^2(x) = \tilde{y}_{-1}. \end{aligned}$$

The degrees of the polynomials $(T_n^\alpha)_{n\in\mathbb{Z}}$ are

$$\begin{aligned}\deg(T_n^1) &= n+1, \quad n\geq 0, \quad \deg(T_n^2)=n, \quad n\geq 0,\\ \deg(T_{-n-1}^1) &= n-1, \quad n\geq 0, \quad \deg(T_{-n-1}^2)=n, \quad n\geq 0.\end{aligned}$$

Again, it is possible to obtain a family of polynomials that satisfy the same initial conditions as $(Q_n^\alpha(x))_{n\in\mathbb{Z}}$. In matrix form, we take

$$\boldsymbol{T}_n(x) = \begin{pmatrix} T_n^1(x) & T_n^2(x) \\ T_{-n-1}^1(x) & T_{-n-1}^2(x)\end{pmatrix}, \quad n\geq 0,$$

that has degree $n+1$ and singular leading coefficient. Now we will define a new family of polynomials which will turn out to be the associated family of the Darboux transformation $\widehat{P}$. For $n\geq 0$ define

$$\widehat{\boldsymbol{Q}}_n(x) = \boldsymbol{T}_n(x)\boldsymbol{T}_0^{-1}(x), \quad n\geq 0, \tag{2.2.30}$$

where

$$\boldsymbol{T}_0(x) = \begin{pmatrix} \dfrac{x-\tilde{r}_0\tilde{x}_{-1}}{\tilde{s}_0} & -\dfrac{\tilde{r}_0\tilde{y}_{-1}}{\tilde{s}_0} \\ \tilde{x}_{-1} & \tilde{y}_{-1}\end{pmatrix}. \tag{2.2.31}$$

Following the same representation as in (1.5.10) we can define the functions $(\widehat{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha=1,2$, which turn out to be polynomials, as the following proposition shows.

Proposition 2.2.5. *Let $(\widehat{\boldsymbol{Q}}_n(x))_{n\geq 0}$, be the sequence of matrix functions defined by* (2.2.30)*. Then, for $n\geq 0$, $\widehat{\boldsymbol{Q}}_n(x)$ is a matrix polynomial of degree exactly n with nonsingular leading coefficient and $\widehat{\boldsymbol{Q}}_0(x)=\boldsymbol{I}_2$.*

Proof. Now, since

$$\boldsymbol{T}_0^{-1}(x) = \frac{1}{x}\begin{pmatrix} \tilde{s}_0 & \tilde{r}_0 \\ -\dfrac{\tilde{s}_0\tilde{x}_{-1}}{\tilde{y}_{-1}} & \dfrac{x-\tilde{r}_0\tilde{x}_{-1}}{\tilde{y}_{-1}}\end{pmatrix},$$

we have from (2.2.30)

$$\begin{aligned}\widehat{Q}_n^1(x) &= \frac{\tilde{s}_0}{x\tilde{y}_{-1}}\left(\tilde{y}_{-1}T_n^1(x) - \tilde{x}_{-1}T_n^2(x)\right), \quad n\in\mathbb{Z},\\ \widehat{Q}_n^2(x) &= \frac{T_n^2(x)}{\tilde{y}_{-1}} + \frac{\tilde{r}_0}{x\tilde{y}_{-1}}\left(\tilde{y}_{-1}T_n^1(x) - \tilde{x}_{-1}T_n^2(x)\right), \quad n\in\mathbb{Z}.\end{aligned}$$

From here the proof follows the same lines as the proof of Proposition 2.2.2 but now using (2.2.28) and (2.2.29). □

Now, let us prove the analogue of Theorem 2.2.3 for the spectral matrix for the polynomials $(\widehat{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha=1,2$. We define first the potential coefficients associated with $\widehat{P}$ given by

$$\hat{\pi}_0 = 1, \quad \hat{\pi}_n = \frac{\hat{a}_0\cdots\hat{a}_{n-1}}{\hat{c}_1\cdots\hat{c}_n}, \quad \hat{\pi}_{-n} = \frac{\hat{c}_0\cdots\hat{c}_{-n+1}}{\hat{a}_{-1}\cdots\hat{a}_{-n}}, \quad n\geq 1. \tag{2.2.32}$$

Theorem 2.2.6. *Let $\{X_t : t = 0, 1, \dots\}$ be the bilateral birth-death chain on $\mathbb{Z}$ with transition probability matrix P given by (1.4.1) and $\{\widehat{X}_t : t = 0, 1, \dots\}$ the bilateral birth-death chain on $\mathbb{Z}$ with transition probability matrix $\widehat{P}$ given by (2.2.3). Assume that*

$$\boldsymbol{M}_{-1} = \int_{-1}^{1} \frac{\boldsymbol{\Psi}(x)}{x} dx$$

is well-defined entry by entry, where $\boldsymbol{\Psi}(x)$ is the original spectral matrix given by (1.4.6). Then the polynomials $(\widehat{Q}_n^\alpha)_{n\in\mathbb{Z}}, \alpha = 1, 2,$ defined by (2.2.30) are orthogonal with respect to the following spectral matrix

$$\widehat{\boldsymbol{\Psi}}(x) = \boldsymbol{T}_0(x)\boldsymbol{\Psi}_T(x)\boldsymbol{T}_0^*(x), \tag{2.2.33}$$

where $\boldsymbol{T}_0(x)$ is defined by (2.2.31) and

$$\boldsymbol{\Psi}_T(x) = \frac{\tilde{s}_0}{\tilde{y}_0}\frac{\boldsymbol{\Psi}(x)}{x} + \left[\frac{\hat{a}_{-1}}{\hat{c}_0}\begin{pmatrix} 1/\tilde{x}_{-1} & 0 \\ 0 & 1/\tilde{y}_{-1}\end{pmatrix} - \frac{\tilde{s}_0}{\tilde{y}_0}\boldsymbol{M}_{-1}\right]\delta_0(x), \tag{2.2.34}$$

with $\delta_0(x)$ the Dirac delta at $x = 0$. Moreover, we have

$$\int_{-1}^{1} \widehat{\boldsymbol{Q}}_n(x)\widehat{\boldsymbol{\Psi}}(x)\widehat{\boldsymbol{Q}}_m^*(x)dx = \begin{pmatrix} 1/\hat{\pi}_n & 0 \\ 0 & 1/\hat{\pi}_{-n-1}\end{pmatrix}\delta_{nm},$$

where δ_{nm} is the Kronecker delta and $(\hat{\pi}_n)_{n\in\mathbb{Z}}$ are the potential coefficients defined by (2.2.32).

Proof. The proof follows the same lines as the proof of Theorem 2.2.3. For $n \geq 1$ and $j = 1, \dots, n-1$ we have, using (2.2.33), (2.2.30), (2.2.34) and (2.2.25), that

$$\begin{aligned}\int_{-1}^{1}\left(\widehat{Q}_n^1(x), \widehat{Q}_n^2(x)\right)\widehat{\boldsymbol{\Psi}}(x)x^j dx = \frac{\tilde{y}_n\tilde{s}_0}{\tilde{y}_0}\left[\int_{-1}^{1}\left(Q_n^1(x), Q_n^2(x)\right)\boldsymbol{\Psi}(x)A^*x^{j-1}dx + \int_{-1}^{1}\left(Q_n^1(x), Q_n^2(x)\right)\boldsymbol{\Psi}(x)B^*x^j dx\right] \\ + \frac{\tilde{x}_n\tilde{s}_0}{\tilde{y}_0}\left[\int_{-1}^{1}\left(Q_{n+1}^1(x), Q_{n+1}^2(x)\right)\boldsymbol{\Psi}(x)A^*x^{j-1}dx + \int_{-1}^{1}\left(Q_{n+1}^1(x), Q_{n+1}^2(x)\right)\boldsymbol{\Psi}(x)B^*x^j dx\right],\end{aligned}$$

where A and B, from $\boldsymbol{T}_0(x) = A + xB$, are given now by

$$A = \begin{pmatrix} -\dfrac{\tilde{r}_0\tilde{x}_{-1}}{\tilde{s}_0} & -\dfrac{\tilde{r}_0\tilde{y}_{-1}}{\tilde{s}_0} \\ \tilde{x}_{-1} & \tilde{y}_{-1}\end{pmatrix}, \quad B = \begin{pmatrix} \dfrac{1}{\tilde{s}_0} & 0 \\ 0 & 0\end{pmatrix}. \tag{2.2.35}$$

Using (2.2.6) we have that the first term of the sum vanishes for $j = 1, \dots, n$, the second term vanishes for $j = 0, \dots, n-1$, the third term vanishes for $j = 1, \dots, n+1$ and the fourth term vanishes for $j = 0, \dots, n$. Therefore the above expression vanishes for $j = 1, \dots, n-1$.

For $j = 0$ we have, using (2.2.30) and (2.2.33), that

$$\begin{aligned}\int_{-1}^{1}\left(\widehat{Q}_n^1(x), \widehat{Q}_n^2(x)\right)\widehat{\boldsymbol{\Psi}}(x)dx &= \int_{-1}^{1}\left(T_n^1(x), T_n^2(x)\right)\boldsymbol{T}_0^{-1}(x)\boldsymbol{T}_0(x)\boldsymbol{\Psi}_T(x)\boldsymbol{T}_0^*(x)dx \\ &= \left[\int_{-1}^{1}\left(T_n^1(x), T_n^2(x)\right)\boldsymbol{\Psi}_T(x)dx\right]A^* + \left[\int_{-1}^{1}\left(T_n^1(x), T_n^2(x)\right)x\boldsymbol{\Psi}_T(x)dx\right]B^*.\end{aligned}$$

As before, the second term of the sum of the above expression vanishes as a consequence of (2.2.34), (2.2.25) and (2.2.6). Now, using (2.2.29), we can write

$$\left(T_n^1(x), T_n^2(x)\right) = x \sum_{j=0}^{n} \frac{1}{\tilde{s}_j} \left(\frac{T_n^1(0)}{T_j^1(0)} Q_j^1(x), \frac{T_n^2(0)}{T_j^2(0)} Q_j^2(x) \right) + \left(T_n^1(0), T_n^2(0)\right).$$

Substituting this in the remaining integral we get

$$\begin{aligned}
\int_{-1}^{1} \left(\widehat{Q}_n^1(x), \widehat{Q}_n^2(x)\right) \widehat{\mathbf{\Psi}}(x) dx &= \left[\int_{-1}^{1} \left(T_n^1(x), T_n^2(x)\right) \mathbf{\Psi}_T(x) dx \right] A^* \\
&= \left(T_n^1(0), T_n^2(0)\right) \left[\frac{\hat{a}_{-1}}{\hat{c}_0} \begin{pmatrix} 1/\tilde{x}_{-1} & 0 \\ 0 & 1/\tilde{y}_{-1} \end{pmatrix} - \frac{\tilde{s}_0}{\tilde{y}_0} \boldsymbol{M}_{-1} \right] A^* \\
&\quad + \frac{\tilde{s}_0}{\tilde{y}_0} \sum_{j=0}^{n} \frac{1}{\tilde{s}_j} \int_{-1}^{1} \left(\frac{T_n^1(0)}{T_j^1(0)} Q_j^1(x), \frac{T_n^2(0)}{T_j^2(0)} Q_j^2(x) \right) \mathbf{\Psi}(x) A^* dx + \frac{\tilde{s}_0}{\tilde{y}_0} \left(T_n^1(0), T_n^2(0)\right) \boldsymbol{M}_{-1} A^* \\
&= \frac{\hat{a}_{-1}}{\hat{c}_0} \left(T_n^1(0), T_n^2(0)\right) \begin{pmatrix} 1/\tilde{x}_{-1} & 0 \\ 0 & 1/\tilde{y}_{-1} \end{pmatrix} A^* + \frac{1}{\tilde{y}_0} \frac{T_n^1(0)}{T_0^1(0)} \int_{-1}^{1} \left(Q_0^1(x), Q_0^2(x)\right) \mathbf{\Psi}(x) A^* dx.
\end{aligned}$$

The third step is a consequence of $\dfrac{T_n^1(0)}{T_j^1(0)} = \dfrac{T_n^2(0)}{T_j^2(0)}$ using (2.2.28) and the orthogonality properties. Since $\int_{-1}^{1} \left(Q_0^1(x), Q_0^2(x)\right) \mathbf{\Psi}(x) dx = (1, 0)$, $T_0^1(0) = -\tilde{r}_0 \tilde{x}_{-1}/\tilde{s}_0$, $\hat{a}_{-1}/\hat{c}_0 = \tilde{x}_{-1}\tilde{s}_0/\tilde{y}_0\tilde{r}_0$ and A is given by (2.2.35) then we have

$$\begin{aligned}
\int_{-1}^{1} \left(\widehat{Q}_n^1(x), \widehat{Q}_n^2(x)\right) \widehat{\mathbf{\Psi}}(x) dx &= \frac{\tilde{x}_{-1}\tilde{s}_0}{\tilde{y}_0\tilde{r}_0} \left(T_n^1(0), T_n^2(0)\right) \begin{pmatrix} 1/\tilde{x}_{-1} & 0 \\ 0 & 1/\tilde{y}_{-1} \end{pmatrix} A^* - \frac{\tilde{s}_0}{\tilde{y}_0\tilde{r}_0\tilde{x}_{-1}} \left(T_n^1(0), 0\right) A^* \\
&= \left(\frac{\tilde{s}_0}{\tilde{y}_0\tilde{r}_0} (1 - 1/\tilde{x}_{-1}) T_n^1(0), \frac{\tilde{x}_{-1}\tilde{s}_0}{\tilde{r}_0\tilde{y}_0\tilde{y}_{-1}} T_n^2(0) \right) \begin{pmatrix} -\dfrac{\tilde{r}_0\tilde{x}_{-1}}{\tilde{s}_0} & \tilde{x}_{-1} \\ -\dfrac{\tilde{r}_0\tilde{y}_{-1}}{\tilde{s}_0} & \tilde{y}_{-1} \end{pmatrix} \\
&= \left(\frac{1}{\tilde{y}_0} (\tilde{y}_{-1} T_n^1(0) - \tilde{x}_{-1} T_n^2(0)), -\frac{\tilde{s}_0}{\tilde{y}_0\tilde{r}_0} (\tilde{y}_{-1} T_n^1(0) - \tilde{x}_{-1} T_n^2(0)) \right) \\
&= (0, 0),
\end{aligned}$$

as a consequence of (2.2.28). For $n \leq -1$ the proof is similar but now using (2.2.7) and (2.2.29).

Finally, using (2.2.30), (2.2.26), (2.2.25), (2.2.33) and (2.2.34), we have that

$$\begin{aligned}
\int_{-1}^{1} \left(\widehat{Q}_n^1(x), \widehat{Q}_n^2(x)\right) \widehat{\mathbf{\Psi}}(x) \left(\widehat{Q}_n^1(x), \widehat{Q}_n^2(x)\right)^* dx &= \int_{-1}^{1} \left(T_n^1(x), T_n^2(x)\right) \mathbf{\Psi}_T(x) \left(T_n^1(x), T_n^2(x)\right)^* dx \\
&= \int_{-1}^{1} \left[\frac{x}{\tilde{s}_n}\left(Q_n^1(x), Q_n^2(x)\right) - \frac{\tilde{r}_n}{\tilde{s}_n}\left(T_{n-1}^1(x), T_{n-1}^2(x)\right)\right] \mathbf{\Psi}_T(x) \left(T_n^1(x), T_n^2(x)\right)^* dx \\
&= \frac{1}{\tilde{s}_n} \int_{-1}^{1} x\left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}_T(x) \left(T_n^1(x), T_n^2(x)\right)^* dx \\
&= \frac{\tilde{s}_0}{\tilde{s}_n \tilde{y}_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \left[\tilde{y}_n \left(Q_n^1(x), Q_n^2(x)\right)^* + \tilde{x}_n \left(Q_{n+1}^1(x), Q_{n+1}^2(x)\right)^*\right] \\
&= \frac{\tilde{y}_n \tilde{s}_0}{\tilde{s}_n \tilde{y}_0} \int_{-1}^{1} \left(Q_n^1(x), Q_n^2(x)\right) \mathbf{\Psi}(x) \left(Q_n^1(x), Q_n^2(x)\right)^* dx \\
&= \frac{\tilde{y}_n \tilde{s}_0}{\tilde{s}_n \tilde{y}_0} \frac{1}{\pi_n} \\
&= \frac{1}{\hat{\pi}_n}.
\end{aligned}$$

The last step follows using (2.1.3), (2.2.5) and the definition of π_n and $\hat{\pi}_n$ in (1.4.1) and (2.2.32), respectively. □

Remark 2.2.7. As before (see Remark 2.2.4), it is enough to assume that $\boldsymbol{T}_0(0)\boldsymbol{M}_{-1}\boldsymbol{T}_0^*(0)$ is well-defined entry by entry in order to obtain the same results as in Theorem 2.2.6.

As a final remark, we will mention that the spectral matrix $\widehat{\mathbf{\Psi}}(x)$ associated with the Darboux transformation given in the previous theorem is, again, a conjugation by a matrix polynomial $\boldsymbol{T}_0(x)$ of degree 1 of a *Geronimus transformation* of the original spectral matrix $\mathbf{\Psi}(x)$. This phenomenon is different from a Darboux transformation of a transition probability matrix on $\mathbb{Z}_{\geq 0}$ for the LU factorization studied in [24] (see also Theorem 1.6.4), where the associated spectral measure is given by a *Christoffel transformation*, i.e., multiplying the original measure by the polynomial x.

2.3 Examples

In this section we will apply all the results shown in this chapter to two examples. The first one is the bilateral birth-death chain with constant transition probabilities, also known as random walk on $\mathbb{Z}$, and the second one has also constant transition probabilities but inverted probabilities for negative states.

2.3.1 Random walk on $\mathbb{Z}$

For this example we will consider a discrete-time birth-death chain with one-step transition probability matrix P as in (1.4.1) with constant coefficients given by

$$a_n = a, \quad b_n = b, \quad c_n = c, \quad n \in \mathbb{Z}, \quad a + b + c = 1, \quad a, c > 0, \quad b \geq 0.$$

The first step is to analyze the existence of UL and LU stochastic factorizations. With this in mind, we have to analyze the continued fractions given by (2.1.4). It is easy to see that both H and H' can be computed explicitly as follows. For H we have

$$H = 1 - \frac{a|}{|1} - \frac{c|}{|1} - \frac{a|}{|1} - \cdots = 1 - \frac{a|}{|1} - \frac{c|}{|H}.$$

Then H must solve the following equation

$$H^2 + H(a - c - 1) + c = 0.$$

Indeed, we have that

$$H = \frac{1}{2}\left(1 + c - a + \sqrt{(1 + c - a)^2 - 4c}\right), \tag{2.3.1}$$

with $a \leq (1 - \sqrt{c})^2$ to ensure convergence. Also we have

$$H' = \frac{c}{H},$$

therefore the value for H' is given by

$$H' = \frac{1}{2}\left(1 + c - a - \sqrt{(1 + c - a)^2 - 4c}\right).$$

It is also possible to see, under the conditions on the parameters, that $0 \leq H \leq H' \leq 1$. Then, by Theorem 2.1.2, if we choose $H' \leq y_0 \leq H$ we have a stochastic UL factorization where both factors are stochastic matrices. On the other hand, following Theorem 2.1.4, we have a stochastic LU factorization if and only if we choose the free parameter $\tilde{r}_0$ in the range $H' \leq \tilde{r}_0 \leq H$.

The next step is to perform a discrete Darboux transformation and to analyze the relation between the spectrum of the respective matrices. A great advantage of this example is that, thanks to its simplicity, we can compute the explicit expression of the spectral matrix associated with P. This spectral matrix appeared for the first time in the last section of [41] for the case of $b = 0$, together with the method to compute the spectral matrix using Stieltjes transforms and the spectral measures associated with the positive and negative states given in Section 1.4.

For this case, let ψ^+ be the spectral measure associated with the birth-death chain on $\mathbb{Z}_{\geq 0}$ with transition probability matrix P^+ given by (1.4.9), with probability of absorption to the state -1 given by c. Also, let ψ^- be the spectral measure associated with the birth-death chain on $\mathbb{Z}_{\leq -1}$ and transition probability matrix P^- given by (1.4.10) with probability of absorption to state 0 given by a. The relations between the Stieltjes transforms given in (1.4.11) are the following:

$$\begin{aligned} B(z;\psi_{11}) &= \frac{B(x;\psi^+)}{1 - acB(z;\psi^+)B(z;\psi^-)}, \\ \frac{c}{a}B(z;\psi_{22}) &= \frac{B(x;\psi^-)}{1 - acB(z;\psi^+)B(z;\psi^-)}, \\ B(z;\psi_{12}) &= \frac{-aB(x;\psi^-)B(x;\psi^+)}{1 - acB(z;\psi^+)B(z;\psi^-)}. \end{aligned} \tag{2.3.2}$$

Note that if we remove the first row and the first column of P^+ we get the same matrix. Then, following Remark 1.3.3, the Stieltjes transform $B(z;\psi^+)$ associated with ψ^+ satisfy the following equation

$$acB(z;\psi^+) + (z - b)B(z;\psi^+) + 1 = 0.$$

Therefore we have the expression

$$B(z;\psi^+) = \frac{b - z \pm \sqrt{(z-\sigma_-)(z-\sigma_+)}}{2ac}, \quad \sigma_\pm = 1 - (\sqrt{a} \mp \sqrt{c})^2,$$

from which we choose the positive solution so that $B(z;\psi^+)$ is bounded when $z \to \infty$. Given the structure of P^- it is easy to see that $B(z;\psi^+) = B(z;\psi^-)$. Then, using (2.3.2), after some computations we have

$$B(z;\psi_{11}) = \frac{B(z;\psi^+)}{1 - acB(z;\psi^+)^2} = \frac{B(z;\psi^+)}{2 + (z-b)B(z;\psi^+)} = -\frac{1}{\sqrt{(z-\sigma_-)(z-\sigma_+)}}, \quad \sigma_\pm = 1 - (\sqrt{a} \mp \sqrt{c})^2.$$

Using the Perron–Stieltjes inversion formula (1.1.2) we have

$$\psi_{11}(x) = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im} B(x + i\epsilon; \psi_{11}) = \frac{1}{\pi\sqrt{(x-\sigma_-)(\sigma_+ - x)}}, \quad \sigma_\pm = 1 - (\sqrt{a} \mp \sqrt{c})^2.$$

Following the same procedure we can compute ψ_{22} and ψ_{12}. Therefore the spectral matrix of the discrete-time birth-death chain described by P is given by

$$\mathbf{\Psi}(x) = \frac{1}{\pi\sqrt{(x-\sigma_-)(\sigma_+ - x)}} \begin{pmatrix} 1 & \dfrac{x-b}{2c} \\ \dfrac{x-b}{2c} & a/c \end{pmatrix}, \quad x \in [\sigma_-, \sigma_+], \quad \sigma_\pm = 1 - \left(\sqrt{a} \mp \sqrt{c}\right)^2. \tag{2.3.3}$$

Remark 2.3.1. Another way to identify the spectral measure without using the Stieltjes transform is to identify the polynomials generated from te recursion formula $xQ(x) = PQ(x)$ which in this case is given by

$$\begin{aligned}
&Q_0^1(x) = 1, \quad Q_0^2(x) = 0,\\
&Q_{-1}^1(x) = 0, \quad Q_{-1}^2(x) = 1,\\
&xQ_n^\alpha(x) = cQ_{n-1}^\alpha(x) + bQ_n^\alpha(x) + aQ_{n+1}^\alpha(x), \quad n \in \mathbb{Z}, \quad \alpha = 1, 2.
\end{aligned}$$

It is possible to see that these families of polynomials can be written in terms of the Chebyshev polynomials of the second kind $(U_n)_{n\geq 0}$ (defined at the end of Section 1.6) as follows:

$$Q_n^1(x) = \left(\frac{c}{a}\right)^{n/2} U_n\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n \geq 0, \quad Q_{-n-1}^1(x) = -\left(\frac{a}{c}\right)^{(n+1)/2} U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n \geq 1,$$

and

$$Q_n^2(x) = -\left(\frac{c}{a}\right)^{(n+1)/2} U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n \geq 1, \qquad Q_{-n-1}^2(x) = \left(\frac{a}{c}\right)^{n/2} U_n\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n \geq 0.$$

If we analyze this spectral measure, following Theorem 1.3.2, it is easy to see that the process is always transient except for the case $a = c$. The birth-death chain is never positive recurrent since the spectral matrix does not have a jump at the point 1, then for the case $a = c$ the random walk is null recurrent, as expected. The moment $\boldsymbol{M}_{-1}$ of $\mathbf{\Psi}(x)$ is given by following expression

$$\boldsymbol{M}_{-1} = \begin{pmatrix} \dfrac{1}{\sqrt{\sigma_-\sigma_+}} & \dfrac{1}{2c}\left(1 - \dfrac{b}{\sqrt{\sigma_-\sigma_+}}\right) \\ \dfrac{1}{2c}\left(1 - \dfrac{b}{\sqrt{\sigma_-\sigma_+}}\right) & \dfrac{a}{c\sqrt{\sigma_-\sigma_+}} \end{pmatrix}. \tag{2.3.4}$$

In order for $\boldsymbol{M}_{-1}$ to be well-defined we need to assume that $\sigma_- > 0$, i.e. $\sqrt{a}+\sqrt{c}<1$, or equivalent, $a<(1-\sqrt{c})^2$, which is also the condition for convergence of the continued fractions H and H'.

Regarding the Darboux transformations, for the UL factorization we are interested in the spectral matrix associated with $\widetilde{P}=P_LP_U$. Using Theorem 2.2.3, the spectral matrix is given by

$$\widetilde{\boldsymbol{\Psi}}(x)=\boldsymbol{S}_0(x)\boldsymbol{\Psi}_S(x)\boldsymbol{S}_0^*(x),$$

where

$$\boldsymbol{S}_0(x)=\begin{pmatrix} s_0 & r_0 \\ -\dfrac{x_{-1}s_0}{y_{-1}} & \dfrac{x-x_{-1}r_0}{y_{-1}}\end{pmatrix}=\begin{pmatrix} \dfrac{y_0-c}{y_0} & \dfrac{c}{y_0} \\ -\dfrac{a(y_0-c)}{y_0(1-a)-c} & \dfrac{x(y_0-c)-ac}{y_0(1-a)-c}\end{pmatrix},$$

$$\boldsymbol{\Psi}_S(x)=\frac{y_0}{y_0-c}\left(y_0\frac{\boldsymbol{\Psi}(x)}{x}+\left[\begin{pmatrix}1 & 0\\ 0 & \dfrac{y_0-c}{c}\end{pmatrix}-y_0\boldsymbol{M}_{-1}\right]\delta_0(x)\right),$$

where $\delta_0(x)$ is the Dirac delta at $x=0$, $\boldsymbol{\Psi}(x)$ and $\boldsymbol{M}_{-1}$ are defined by (2.3.3) and (2.3.4), respectively, and y_0 is the free parameter. With these equations we can compute explicitly the spectral matrix

$$\widetilde{\boldsymbol{\Psi}}(x)=\frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}\left[\widetilde{A}+\widetilde{B}x+\widetilde{C}x^2\right]+\widetilde{\boldsymbol{M}}\delta_0(x),$$

where

$$\widetilde{A}=\frac{(H'-y_0)(H-y_0)}{s_0y_0}\begin{pmatrix}1 & -x_{-1}/y_{-1}\\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2\end{pmatrix},$$

$$\widetilde{B}=\begin{pmatrix}1 & -\dfrac{by_0}{2cy_{-1}}\\ -\dfrac{by_0}{2cy_{-1}} & \dfrac{(y_0b-c(1-c))x_{-1}^2}{acy_{-1}^2}\end{pmatrix},\quad \widetilde{C}=\frac{y_0}{2cy_{-1}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\widetilde{\boldsymbol{M}}=\frac{(y_0-H')(H-y_0)}{s_0y_0\sqrt{\sigma_-\sigma_+}}\begin{pmatrix}1 & -x_{-1}/y_{-1}\\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2\end{pmatrix}.$$

So we can conclude that if we choose the free parameter y_0 in the range $H'\leq y_0\leq H$, then $\widetilde{\boldsymbol{M}}$ is a positive semidefinite matrix, so $\widetilde{\boldsymbol{\Psi}}(x)$ is a proper *weight matrix*.

If we observe the last expressions, we see some interesting cases. When we choose either $y_0=H$ or $y_0=H'$, some elements of $\widetilde{\boldsymbol{\Psi}}(x)$ vanishes. In fact, we have

$$\widetilde{A}=\boldsymbol{0}_2,\quad \widetilde{\boldsymbol{M}}=\boldsymbol{0}_2,\quad \widetilde{B}=\begin{pmatrix}1 & -\dfrac{b}{2c}\\ -\dfrac{b}{2c} & a/c\end{pmatrix},\quad \widetilde{C}=\frac{1}{2c}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

With all simplifications we recover the original weight matrix (2.3.3), i.e., the Darboux transformation is invariant. This phenomenon is not possible for Darboux transformations of Jacobi matrices on $\mathbb{Z}_{\geq 0}$. Finally, as we already know, from the spectral matrix $\widetilde{\boldsymbol{\Psi}}(x)$ we get the Karlin-McGregor formula for the n-step transition probabilities of $\widetilde{P}$ using equation (1.4.8) and, thanks to the properties of the Geronimus transformation, recurrence of the process is not affected by the transformation.

On the other hand, for the LU case, using Theorem 2.2.6 and after some computations, we can also compute the explicit expression for the spectral matrix associated to the Darboux transformation $\widehat{P} = \widetilde{P}_U \widetilde{P}_L$ as follows

$$\widehat{\mathbf{\Psi}}(x) = \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+ - x)}}\left[\widehat{A} + \widehat{B}x + \widehat{C}x^2\right] + \widehat{\boldsymbol{M}}\delta_0(x),$$

where $\delta_0(x)$ is the Dirac delta at $x=0$ and

$$\widehat{A} = \frac{\tilde{r}_0(H'-\tilde{r}_0)(H-\tilde{r}_0)}{\tilde{x}_{-1}\tilde{s}_0^2}\begin{pmatrix} 1 & -\tilde{s}_0/\tilde{r}_0 \\ -\tilde{s}_0/\tilde{r}_0 & (\tilde{s}_0/\tilde{r}_0)^2 \end{pmatrix},$$

$$\widehat{B} = \frac{\tilde{r}_0\tilde{y}_0}{\tilde{x}_{-1}\tilde{s}_0}\begin{pmatrix} 1 & -\dfrac{b}{2\tilde{y}_0\tilde{r}_0} \\ -\dfrac{b}{2\tilde{y}_0\tilde{r}_0} & \dfrac{\tilde{s}_0\ \tilde{x}_{-1}}{\tilde{y}_0\tilde{r}_0} \end{pmatrix}, \quad \widehat{C} = \frac{1}{2\tilde{s}_0\tilde{x}_{-1}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\widehat{\boldsymbol{M}} = \frac{\tilde{r}_0(\tilde{r}_0-H')(H-\tilde{r}_0)}{\tilde{s}_0^2\tilde{x}_{-1}\sqrt{\sigma_-\sigma_+}}\begin{pmatrix} 1 & -\tilde{s}_0/\tilde{r}_0 \\ -\tilde{s}_0/\tilde{r}_0 & (\tilde{s}_0/\tilde{r}_0)^2 \end{pmatrix}.$$

Again we clearly see that if we choose $\tilde{r}_0$ in the range $H' \leq \tilde{r}_0 \leq H$, then $\widehat{\boldsymbol{M}}$ is a positive semidefinite matrix, so $\widehat{\mathbf{\Psi}}(x)$ is a proper *weight matrix*.

For this spectral matrix we can also observe some special cases. If we choose either $\tilde{r}_0 = H$ or $\tilde{r}_0 = H'$, then we recover the original weight matrix (2.3.3). In addition, from the spectral matrix $\widehat{\mathbf{\Psi}}(x)$ we get the Karlin-McGregor formula (1.4.8) for the n-step transition probabilities of the process described by $\widehat{P}$ and the recurrence is not affected by the transformation.

2.3.2 Birth-death chain on $\mathbb{Z}$ with constant attractive or repulsive forces

The second example is very similar to the Example 2.3.1 but in this case we exchange probabilities a and c for the negative states. The effect of this change is that if $a < c$, we have a discrete-time birth-death chain where the origin is an attractive state. On the contrary, if $a > c$, then the origin is a repulsive state. Then, let us consider P as in (1.4.1) with

$$a_n = a, \quad c_n = c, \quad n \geq 0, \quad a_{-n} = c, \quad c_{-n} = a, \quad n \geq 1, \quad b_n = b, \quad n \in \mathbb{Z},$$

and, as before, $a+b+c=1, a, c > 0, b \geq 0$. Diagram for this example looks like follows

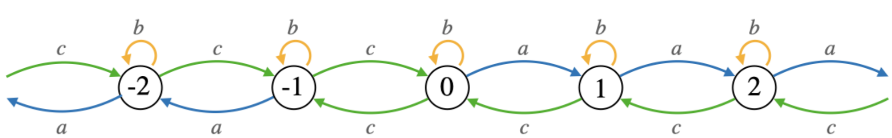

Figure 2.3: Diagram for Example 2.3.1.

Since the probabilities are constants, the continued fractions given in (2.1.4) can be computed explicitly. As the positive states are the same as in Example 2.3.1, then H has the same value, i.e.

$$H = \frac{1}{2}\left(1+c-a+\sqrt{(1+c-a)^2-4c}\right), \tag{2.3.5}$$

as long as $a \leq (1-\sqrt{c})^2$, so H is convergent. On the other hand, we have

$$H' = \frac{c}{1 - \dfrac{c}{H}},$$

and rationalizing we get

$$H' = \frac{c}{2a}\left(1 + a - c - \sqrt{(1+c-a)^2 - 4c}\right). \tag{2.3.6}$$

Here we have that $H' > 0$ if and only if $a > 0$, but we have to be very careful because now it is not true that $H' \leq H$ for all values of the parameters a and c such that $a \leq (1-\sqrt{c})^2$. The analysis of the inequality $H' \leq H$ shows that a must be in the following range

$$\begin{cases} 0 < a \leq (1-\sqrt{c})^2, & \text{if} \quad 0 < c \leq 1/4, \\ 0 < a \leq \dfrac{1-2c}{2}, & \text{if} \quad 1/4 \leq c < 1. \end{cases}$$

Here there is a graph of the range of the previous equations.

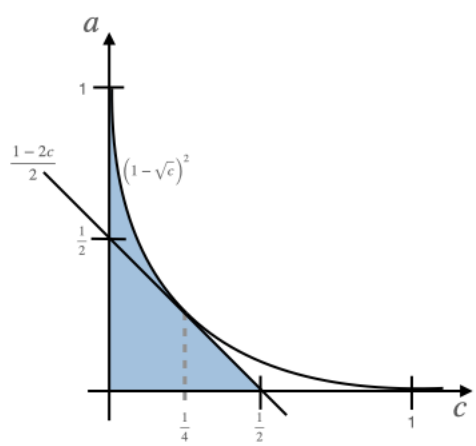

Figure 2.4: Range where $H' \leq H$ in blue.

As before, by Theorem 2.1.2, if we choose $H' \leq y_0 \leq H$ we have a stochastic UL factorization where both factors are stochastic matrices. On the other hand, following Theorem 2.1.4, we have a stochastic LU factorization if and only if we choose the free parameter $\tilde{r}_0$ in the range $H' \leq \tilde{r}_0 \leq H$.

The spectral matrix associated with this example appeared for the first time in Section 6 of [19] for the case of $b = 0$. We will use the same procedure that we used in previous example described in Section 1.4. Note that, for this case, the matrices P^+ and P^- given in (1.4.9) and (1.4.10), associated with the birth-death chains on $\mathbb{Z}_{\geq 0}$ and on $\mathbb{Z}_{\leq -1}$ respectively, are the same. Therefore $\psi^+ = \psi^-$ and

$B(z;\psi^+) = B(z;\psi^-) = B(z)$ and equations (1.4.11) simplify to

$$B(z;\psi_{11}) = B(z;\psi_{22}) = \frac{B(z)}{1-c^2B(z)^2},$$
$$B(z;\psi_{12}) = \frac{-cB(z)^2}{1-c^2B(z)^2}.$$

Also, we have that $acB(z)^2 - (z-b)B(z) + 1 = 0$. Then it is easy to see that

$$c^2B(z)^2 = c\frac{z-b}{a}B(z) - \frac{c}{a}.$$

After some computations we have that

$$B(z;\psi_{11}) = \frac{aB(z)}{a+c+c(z-b)B(z)} = \frac{(c-a)(z-b)+(1-b)\sqrt{(z-\sigma_-)(z-\sigma_+))}}{2c\left[(1-z)(z-2b+1)\right]}, \quad \sigma_\pm = 1-(\sqrt{a}\mp\sqrt{c})^2.$$

Using the Perron–Stieltjes inversion formula (1.1.2) we have

$$\psi_{11}(x) = \frac{1}{\pi}\lim_{\epsilon\to 0^+}\mathrm{Im}B(x+i\epsilon;\psi_{11}) = \frac{(a+c)\sqrt{(x-\sigma_-)(\sigma_+-x)}}{2\pi c(1-x)(x-2b+1)}, \quad \sigma_\pm = 1-(\sqrt{a}\mp\sqrt{c})^2.$$

Observe that the Stieltjes transform has two isolated poles at $z=1$ and $z=2b-1$. Therefore there will be two jumps at those points for certain values of a. We have that the size of the jump at $x=1$ (see (1.1.3)) is given by

$$\psi_{11}(\{1\}) = \lim_{\epsilon\to 0^+}\epsilon\mathrm{Im}B(x+i\epsilon;\psi_{11}) = \frac{c-a+|c-a|}{4c} = \left(\frac{c-a}{2c}\right)\chi_{\{c>a\}},$$

where $\chi_{\{A\}}$ is the indicator function. Following the same procedure we can compute the size of the jump at $x=2b-1$ and ψ_{12} with its respective jumps. This means that the spectral matrix associated to P is given by an absolutely continuous $\mathbf{\Psi}_c(x)$ and a discrete part $\mathbf{\Psi}_d(x)$. We can write the spectral matrix as $\mathbf{\Psi}(x) = \mathbf{\Psi}_c(x) + \mathbf{\Psi}_d(x)$ where the absolutely continuous part is given by

$$\mathbf{\Psi}_c(x) = \frac{(a+c)\sqrt{(x-\sigma_-)(\sigma_+-x)}}{2\pi c(1-x)(x-2b+1)}\begin{pmatrix} 1 & \dfrac{x-b}{a+c} \\ \dfrac{x-b}{a+c} & 1 \end{pmatrix}, \quad x\in[\sigma_-,\sigma_+],$$

where $\sigma_\pm = 1-(\sqrt{a}\mp\sqrt{c})^2$ and the discrete part is given by

$$\mathbf{\Psi}_d(x) = \frac{c-a}{2c}\left[\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\delta_{2b-1}(x) + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\delta_1(x)\right]\chi_{\{c>a\}},$$

where $\delta_w(x)$ is the Dirac delta at $x=w$. With this expression and the Karlin-McGregor formula (1.4.8) we can get the n-step transition probabilities of the process associated with P and we know that this birth-death chain is always recurrent, in particular it is positive recurrent if $c>a$ since for that case the spectral matrix has a jump at the point 1.

Remark 2.3.2. Following Remark 2.3.1, it is possible to see that the families of polynomials generated by P can be written in terms of the Chebyshev polynomials of the second kind $(U_n)_{n\geq 0}$ (see Section 1.6) as follows:

$$Q_n^1(x) = \left(\frac{c}{a}\right)^{n/2} U_n\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n\geq 0, \quad Q_{-n-1}^1(x) = -\left(\frac{c}{a}\right)^{(n+1)/2} U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n\geq 1,$$

and

$$Q_n^2(x) = -\left(\frac{c}{a}\right)^{(n+1)/2} U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n\geq 1, \qquad Q_{-n-1}^2(x) = \left(\frac{c}{a}\right)^{n/2} U_n\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n\geq 0.$$

Now the computation of the moment $\boldsymbol{M}_{-1}$ of $\boldsymbol{\Psi}(x)$ is more complicated since there is an absolutely continuous and a discrete part. Also one of the Dirac deltas of the discrete part can be located at $x=0$ if $b=1/2$, so that $\boldsymbol{M}_{-1}$ may have a different expression in that case. Nevertheless, after some computations, we obtain

$$\boldsymbol{M}_{-1} = \begin{pmatrix} \mu_{-1} & \dfrac{\gamma - b\mu_{-1}}{a+c} \\ \dfrac{\gamma - b\mu_{-1}}{a+c} & \mu_{-1} \end{pmatrix} + \frac{c-a}{c(2b-1)} \begin{pmatrix} b & -(a+c) \\ -(a+c) & b \end{pmatrix} \chi_{\{c>a\}},$$

where

$$\mu_{-1} = \frac{1}{2c(2b-1)}\left((a+c)\sqrt{\sigma_-\sigma_+} - b|a-c|\right), \quad \gamma = \begin{cases} 1, & \text{if} \quad c\leq a, \\ a/c, & \text{if} \quad c>a. \end{cases}$$

In order for $\boldsymbol{M}_{-1}$ to be well-defined we need to assume that $\sigma_- > 0$, i.e. $\sqrt{a}+\sqrt{c}<1$, or equivalently, $a<(1-\sqrt{c})^2$. For the case of $b=1/2$ we obtain

$$\boldsymbol{M}_{-1} = \begin{pmatrix} \dfrac{4a}{|4a-1|} & 2\left(\gamma - \dfrac{2a}{|4a-1|}\right) \\ 2\left(\gamma - \dfrac{2a}{|4a-1|}\right) & \dfrac{4a}{|4a-1|} \end{pmatrix} + \frac{1-4a}{1-2a}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\chi_{\{a<1/4\}}. \tag{2.3.7}$$

Now we proceed to compute the spectral matrix for the Darboux transformation for the UL case. Using Theorem 2.2.3 and after some computations we get the following expression

$$\widetilde{\boldsymbol{\Psi}}(x) = \frac{\sqrt{(x-\sigma_-)(\sigma_+-x)}}{2\pi c x(1-x)(x-2b+1)}\left[\widetilde{A} + \widetilde{B}x + \widetilde{C}x^2\right] + \widetilde{\boldsymbol{M}}_0\delta_0(x) + \widetilde{\boldsymbol{M}}_{2b-1}\delta_{2b-1}(x) + \widetilde{\boldsymbol{M}}_1\delta_1(x),$$

where

$$\widetilde{A} = \frac{(a+c)(\alpha_+-y_0)(\alpha_--y_0)}{s_0y_0}\begin{pmatrix} 1 & -x_{-1}/y_{-1} \\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2 \end{pmatrix}, \quad \alpha_\pm = \frac{c}{a+c}\left(1\pm\sqrt{1-2a-2c}\right),$$

$$\widetilde{B} = \begin{pmatrix} 2c & \dfrac{(c(1-2c)-by_0)x_{-1}}{cy_{-1}} \\ \dfrac{(c(1-2c)-by_0)x_{-1}}{cy_{-1}} & \dfrac{2(by_0-c(1-c))x_{-1}^2}{cy_{-1}^2} \end{pmatrix}, \quad \widetilde{C} = \frac{y_0s_0x_{-1}}{cy_{-1}}\begin{pmatrix} 0 & 1 \\ 1 & \dfrac{(a-c)x_{-1}}{cy_{-1}} \end{pmatrix},$$

$$\widetilde{\boldsymbol{M}}_0 = \frac{a(\bar{H}'-\bar{H})(y_0-H')(H-y_0)}{c(2b-1)s_0y_0}\begin{pmatrix} 1 & -x_{-1}/y_{-1} \\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2 \end{pmatrix},$$

$$\widetilde{\boldsymbol{M}}_{2b-1} = \frac{c-a}{2c(2b-1)}\begin{pmatrix} (s_0-r_0)^2 & -(s_0-r_0)(s_{-1}-r_{-1}) \\ -(s_0-r_0)(s_{-1}-r_{-1}) & (s_{-1}-r_{-1})^2 \end{pmatrix}\chi_{\{c>a\}}, \quad \widetilde{\boldsymbol{M}}_1 = \frac{c-a}{2c}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\chi_{\{c>a\}}.$$

Notice that we are implicitly assuming that $b \neq 1/2$. If we analyze this expression, we get some special cases. For the case where $b = 1/2$, the Geronimus transformation is not well-defined for the Dirac delta at $x = 0$. However, it is possible to define the spectral matrix in terms of the *derivative* of the Dirac delta at $x = 0$. Therefore, the spectral matrix for the Darboux transformation is given by

$$\widetilde{\mathbf{\Psi}}(x) = \frac{\sqrt{(x-\sigma_-)(\sigma_+ - x)}}{2\pi c x^2(1-x)} \left[\widetilde{A} + \widetilde{B}x + \widetilde{C}x^2\right] + \widetilde{\boldsymbol{M}}_0\delta_0(x) - \widetilde{\boldsymbol{M}}_0'\delta_0'(x) + \widetilde{\boldsymbol{M}}_1\delta_1(x), \tag{2.3.8}$$

where $\widetilde{A}, \widetilde{B}, \widetilde{C}$ and $\widetilde{\boldsymbol{M}}_1$ are the same as before writing $b = 1/2$ and $c = 1/2 - a$ and

$$\widetilde{\boldsymbol{M}}_0' = \lim_{b\to 1/2}(2b-1)\widetilde{\boldsymbol{M}}_{2b-1},$$

$$\widetilde{\boldsymbol{M}}_0 = \eta\frac{(y_0 - H')(H - y_0)}{s_0 y_0}\begin{pmatrix} 1 & -x_{-1}/y_{-1} \\ -x_{-1}/y_{-1} & (x_{-1}/y_{-1})^2 \end{pmatrix}, \quad \eta = \begin{cases} \dfrac{1}{2(1-2a)(1-4a)}, & \text{if} \quad a < 1/4, \\ \dfrac{a}{4a-1}, & \text{if} \quad a > 1/4. \end{cases}$$

For the case where $a = 1/4$ then the moment (2.3.7) is not well-defined, but in this case we are in the situation of the previous example and, from the spectral matrix $\widetilde{\mathbf{\Psi}}(x)$ we get the Karlin-McGregor formula (1.4.8) for the n-step transition probabilities and we know that the recurrence of the Darboux transformation is not affected.

If we fix $y_0 = H$, we have a similar behavior to the previous example since we observe that the positive states of the original process remain invariant under the Darboux transformation (2.2.2), while if $y_0 = H'$, then the negative states of the original random walk remain invariant.

Finally, we proceed to analyze the LU case. Using Theorem 2.2.6 and after some computations, the expression for the spectral matrix of $\widehat{P} = \widetilde{P}_U\widetilde{P}_L$ is the following

$$\widehat{\mathbf{\Psi}}(x) = \frac{\sqrt{(x-\sigma_-)(\sigma_+ - x)}}{2\pi c x(1-x)(x-2b+1)} \left[\widehat{A} + \widehat{B}x + \widehat{C}x^2\right] + \widehat{\boldsymbol{M}}_0\delta_0(x) + \widehat{\boldsymbol{M}}_{2b-1}\delta_{2b-1}(x) + \widehat{\boldsymbol{M}}_1\delta_1(x),$$

where

$$\widehat{A} = \frac{(a+c)(\beta_+ - \tilde{s}_0)(\beta_- - \tilde{s}_0)}{\tilde{s}_0\tilde{y}_0}\begin{pmatrix} 1 & -\tilde{s}_0/\tilde{r}_0 \\ -\tilde{s}_0/\tilde{r}_0 & (\tilde{s}_0/\tilde{r}_0)^2 \end{pmatrix}, \quad \beta_\pm = \frac{a + c\sqrt{2b-1}}{a+c},$$

$$\widehat{B} = \frac{1}{\tilde{y}_0\tilde{r}_0}\begin{pmatrix} \dfrac{2\tilde{r}_0}{\tilde{s}_0}(c(1-c) - p\tilde{r}_0) & (a-c)\tilde{r}_0 - c(1-2c) \\ (a-c)\tilde{r}_0 - c(1-2c) & 2c\tilde{s}_0\tilde{x}_{-1} \end{pmatrix}, \quad \widehat{C} = \frac{\tilde{y}_{-1}}{\tilde{y}_0}\begin{pmatrix} \dfrac{a-c}{\tilde{y}_{-1}} & 1 \\ 1 & 0 \end{pmatrix},$$

$$\widetilde{\boldsymbol{M}}_0 = \frac{a(\bar{H}' - \bar{H})(\tilde{r}_0 - H')(H - \tilde{r}_0)}{c(2b-1)\tilde{s}_0\tilde{y}_0}\begin{pmatrix} 1 & -\tilde{s}_0/\tilde{r}_0 \\ -\tilde{s}_0/\tilde{r}_0 & (\tilde{s}_0/\tilde{r}_0)^2 \end{pmatrix}, \quad \widehat{\boldsymbol{M}}_1 = \frac{c-a}{2c}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\chi_{\{c>a\}},$$

$$\widehat{\boldsymbol{M}}_{2b-1} = \frac{c-a}{2c(2b-1)}\begin{pmatrix} (\tilde{x}_0 - \tilde{y}_0)^2 & -(\tilde{x}_0 - \tilde{y}_0)(\tilde{x}_{-1} - \tilde{y}_{-1}) \\ -(\tilde{x}_0 - \tilde{y}_0)(\tilde{x}_{-1} - \tilde{y}_{-1}) & (\tilde{x}_{-1} - \tilde{y}_{-1})^2 \end{pmatrix}\chi_{\{c>a\}}.$$

As before, from the spectral matrix $\widehat{\mathbf{\Psi}}(x)$ we get the Karlin-McGregor formula (1.4.8) for the n-step transition probabilities and the recurrence of the Darboux transformation is not affected. We have a similar behavior if we assume either $\tilde{r}_0 = H$ or $\tilde{r}_0 = H'$ for the factorization.

Before finishing this section we would like to comment on some special cases that occur in both factorizations, UL and LU. First, if we assume $a = c$ then we recover the previous example and the Darboux transformation is invariant if we choose the free parameter $y_0 = H = \frac{1}{2}(1 + \sqrt{1-4a})$ or $y_0 = H' = \frac{1}{2}(1 - \sqrt{1-4a})$.

Aditionally, there are some values of the parameters where the Darboux transformation is *almost invariant*. For instance, for $0 < a < 1/2$ consider $c = 1/2 - a$, then $b = 1/2$ and the values of the continued fractions (2.3.5) and (2.3.6) depend on the value of a. We have two situations:

- If $0 < a \leq 1/4$, then $H = H' = 1 - 2a$. Therefore the only choice of the parameter y_0 in order to have a stochastic factorization is $y_0 = 1 - 2a$. For this value we always have

$$\begin{aligned} &y_n = 1 - 2a, \quad n \geq 0, \quad y_{-n} = 2a, \quad n \geq 1, \\ &s_n = r_n = 1/2, \quad x_n = 1 - y_n, \quad n \in \mathbb{Z}. \end{aligned} \tag{2.3.9}$$

  Therefore the transition probabilities of the Darboux transformation are exactly the same as the original case except for the state 0, where we have

$$\tilde{c}_0 = a, \quad \tilde{a}_0 = a, \quad \tilde{b}_0 = 1 - 2a. \tag{2.3.10}$$

  The birth-death chain generated by $\widetilde{\boldsymbol{P}}$ is almost the same as the original one except for the state 0. The spectral matrix is given in this case by

$$\widetilde{\boldsymbol{\Psi}}(x) = \frac{\sqrt{(x - \sigma_-)(\sigma_+ - x)}}{2\pi c x(1 - x)} \left[\widetilde{B} + \widetilde{C}x\right] + \widetilde{\boldsymbol{M}}_1 \delta_1(x), \quad \sigma_\pm = 1/2 \pm \sqrt{2a(1 - 2a)}, \tag{2.3.11}$$

  where

$$\widetilde{B} = (1 - 2a) \begin{pmatrix} 1 & -\dfrac{1 - 2a}{2a} \\ -\dfrac{1 - 2a}{2a} & \dfrac{(1 - 2a)^2}{4a^2} \end{pmatrix}, \quad \widetilde{C} = \frac{1 - 2a}{2a} \begin{pmatrix} 0 & 1 \\ 1 & -\dfrac{1 - 4a}{2a} \end{pmatrix}, \quad \widetilde{\boldsymbol{M}}_1 = \frac{1 - 4a}{2(1 - 2a)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

  Similar results hold for the LU case.

- If $1/4 < a < 1/2$, then $H = 1/2$ and $H' = (1 - 2a)/4a$. Therefore the parameter y_0 can be chosen in the range

$$(1 - 2a)/4a \leq y_0 \leq 1/2.$$

  In particular, if take $y_0 = 1 - 2a$, then we are in the same situation of the previous case, i.e. we have (2.3.9) for the sequences x_n, y_n, s_n, r_n and the transition probabilities of the Darboux transformation are exactly the same as the original case except for the state 0, where we have again (2.3.10). The spectral matrix is then given by (2.3.11). If $y_0 = 1/2$ then we get invariance on the positive states of the Darboux transformation but not on the negative states nor state 0. On the contrary, if $y_0 = (1-2a)/4a$, then we get invariance on the negative states of the Darboux transformation but not on the nonnegative states. The spectral matrix can be computed from (2.3.8). Similar results hold for the LU case.

Finally, let us highlight that, despite the fact that both examples are considered *simple* since all probabilities are constant, both processes were considered by previous authors (at least for the case $b = 0$). Besides, considerable effort was necessary to get the explicit expression of the spectral matrices and our analysis led to very interesting results such as the cases where there is an invariance property or an "almost" invariance property. For these reasons, we will return to these examples in the next chapters. Nonetheless we will also study a nontrivial example in Chapter 4, given by the so-called associated Jacobi polynomials.

CHAPTER 3

Stochastic Absorbing-Reflecting factorizations

In Chapter 2, we explored UL and LU stochastic factorizations of the one-step transition probability matrix P, which describes an irreducible birth-death chain on $\mathbb{Z}$. These factorizations represent either pure-birth or pure-death chains. Given that our study goes around the one-step transition probability matrix P and begins with a stochastic factorization, we seek decompositions that are meaningful and interesting from a theoretical point of view. In this context, examining absorbing-reflecting (AR) factorizations presents a promising path for further investigation. The RA factorization consists of treating the first factor, P_R, as a reflecting birth-death chain on $\mathbb{Z}$ from the state 0, and the second factor, P_A, as an absorbing birth-death chain on $\mathbb{Z}$ to the state 0. While there is no difference between UL (LU) stochastic factorizations and RA (AR) factorizations on $\mathbb{Z}_{\geq 0}$, these factorizations represent different chains when the state space is $\mathbb{Z}$.

The first part of this chapter is dedicated to the RA factorization. Section 3.1 provides the fundamental definitions and necessary conditions for the existence of a stochastic RA factorization. These conditions are expressed in terms of the continued fractions presented in Chapter 2 (see (2.1.4)). In Section 3.2, we perform the discrete Darboux transformation and conduct the spectral analysis. In this case, after the Darboux transformation we get an "almost" birth-death chain on $\mathbb{Z}$ since the resulting matrix $\widetilde{P}$ has transition probabilities between states 1 and -1. To achieve this, we use the relabeling method given by (1.5.9) to obtain an equivalent representation of $\widetilde{P}$ as a 2×2 block tridiagonal matrix, denoted by $\widetilde{\boldsymbol{P}}$ (see (1.5.2)). This block structure proves to be very useful in certain proofs, as it allows us to apply techniques from the theory of matrix-valued orthogonal polynomials. This analysis offers crucial insights that will also be employed in Chapter 5. The reason for not using the 2×2 block structure in Chapter 2 was that it did not preserve the UL or LU block structure of the original factorization of P. Following the main result, we revisit Example 2.3.1 and apply our results concerning the RA factorization.

The second part of this chapter explores AR factorizations. In Section 3.3, we explain why considering P as in (1.4.1) is not particularly interesting. Instead, we introduce the simplest process applicable to this factorization: An "almost" birth-death chain similar to the one described by $\widetilde{P}$, i.e., we add transition probabilities between states 1 and -1. We then provide conditions under which the stochastic

AR factorization exists. Following the same steps as before, Section 3.4 investigates the Darboux transformation and the relation between the corresponding spectral matrices. It is interesting to see that, after the Darboux transformation, we recover the tridiagonal structure of an usual birth-death chain. This chapter concludes with an example, closely resembling Example 2.3.1, but with the modifications described in Section 3.3. All results presented in this chapter are included in [33].

3.1 Stochastic RA factorization on the integers

Let $\{X_t : t = 0, 1, \ldots\}$ be an irreducible discrete-time birth-death chain on the integers $\mathbb{Z}$ with transition probability matrix P given by (1.4.1). First, we are going to study the RA factorization, that is, $P = P_R P_A$ where P_R represents a reflecting birth-death chain on $\mathbb{Z}$ from the state 0 (lower bidiagonal for the negative states and upper bidiagonal for the positive states) and P_A represents an absorbing birth-death chain on $\mathbb{Z}$ to the state 0 (upper bidiagonal for the negative states and lower bidiagonal for the positive states). In other words

$$P_R = \left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & x_{-2} & y_{-2} & 0 & & & & \\ & & x_{-1} & y_{-1} & 0 & & & \\ \hline & & & \alpha & y_0 & x_0 & & \\ & & & & 0 & y_1 & x_1 & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right), \tag{3.1.1}$$

and

$$P_A = \left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & 0 & s_{-2} & r_{-2} & & & & \\ & & 0 & s_{-1} & r_{-1} & & & \\ \hline & & & 0 & 1 & 0 & & \\ & & & & r_1 & s_1 & 0 & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right). \tag{3.1.2}$$

Here we have to make two important observations. First, in P_R we have to add a probability α of going from state 0 to -1 in order to connect the chain from positive to negative states. Second, the state 0 of P_A is an absorbing state. Therefore $s_0 = 1$ and $r_0 = 0$. Below we show diagrams for these two processes.

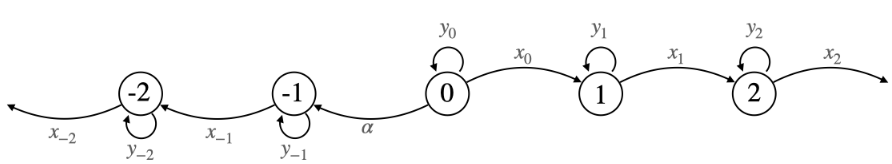

Figure 3.1: Diagram for the discrete-time bilateral birth-death chain described by P_R.

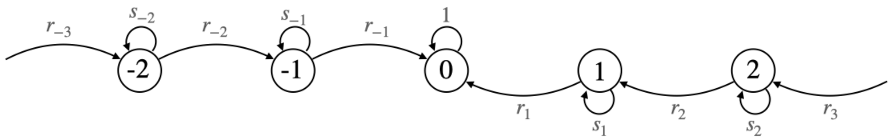

Figure 3.2: Diagram for the discrete-time bilateral birth-death chain described by P_A.

Since we are looking for a stochastic factorization we have the following conditions:

$$\alpha + x_0 + y_0 = 1, \quad x_n + y_n = 1, \quad s_n + r_n = 1, \quad n \in \mathbb{Z} \setminus \{0\}. \tag{3.1.3}$$

Now, using equation $P = P_R P_A$ entry by entry, it is easy to see that

$$a_n = x_n s_{n+1}, \quad n \geq 0, \quad a_{-n} = y_{-n} r_{-n}, \quad n \geq 1, \tag{3.1.4}$$
$$b_n = y_n s_n + x_n r_{n+1}, \quad n \geq 1, \quad b_0 = y_0 + x_0 r_1 + \alpha r_{-1}, \quad b_{-n} = y_{-n} s_{-n} + x_{-n} r_{-n-1}, \quad n \geq 1,$$
$$c_n = y_n r_n, \quad n \geq 1, \quad c_0 = \alpha s_{-1}, \quad c_{-n} = x_{-n} s_{-n-1}, \quad n \geq 1. \tag{3.1.5}$$

We are going to use this system of equations for the first part of our analysis to derive conditions for the existence of the stochastic RA factorization. As in the previous chapter, we have a free parameter α from which we can compute $s_{-1}, r_{-1}, y_{-1}, x_{-1}, s_{-2}, \ldots$ recursively using (3.1.5), (3.1.3) and (3.1.4). In this case, for positive values of the indices, we need to fix a second parameter x_0, in order to obtain recursively s_1, r_1, y_1, x_1, $s_2, \ldots$ using again (3.1.4), (3.1.3) and (3.1.5). Therefore we have *two* free parameters.

Proposition 3.1.1. *Let H and H' be the continued fractions defined by* (2.1.4) *with condition* (2.1.8). *Then both H and H' are convergent. Moreover, let $P = P_R P_A$ and assume that $H' \leq H$. Then, both P_R and P_A are stochastic matrices if and only if we choose α and x_0 in the following ranges*

$$\alpha \geq H', \quad x_0 \geq 1 - H. \tag{3.1.6}$$

Proof. The first part of the theorem was proved in Theorem 2.1.2. For the second part, assume $H' \leq H$, P_R and P_A stochastic matrices and let $(h_n)_{n\geq 0}$ and $(h'_{-n})_{n\geq 0}$ be the sequences of convergents for H and H' respectively. Then it is clear that

$$x_0 > 0 = 1 - h_0,$$

and then

$$s_1 = \frac{a_0}{x_0} < 1 \Leftrightarrow x_0 > a_0 = 1 - h_1.$$

Similarly

$$y_1 = \frac{c_1}{r_1} \Leftrightarrow c_1 < r_1 \Leftrightarrow s_1 < 1 - c_1 \Leftrightarrow \frac{a_0}{x_0} < 1 - c_1 \Leftrightarrow x_0 > \frac{a_0}{1 - c_1} = \frac{a_0 \rfloor}{\lceil 1} - \frac{c_1 \rfloor}{\lceil 1} = 1 - h_2.$$

Using the same argument recursively, for even indices we have

$$
\begin{aligned}
y_{n+1} = \frac{c_{n+1}}{r_{n+1}} < 1 \Leftrightarrow s_{n+1} < 1 - c_{n+1} \Leftrightarrow \frac{a_n}{x_n} < 1 - c_{n+1} \Leftrightarrow 1 - y_n > \frac{a_n}{1 - c_{n+1}} \\
\Leftrightarrow y_n < 1 - \frac{a_n\rfloor}{\lceil 1} - \frac{c_{n+1}\rfloor}{\lceil 1} \Leftrightarrow \cdots \Leftrightarrow y_1 < 1 - \frac{c_1\rfloor}{\lceil 1} - \frac{a_1\rfloor}{\lceil 1} - \frac{c_2\rfloor}{\lceil 1} - \cdots - \frac{c_{n+1}\rfloor}{\lceil 1} \\
\Leftrightarrow \frac{a_0}{x_0} < 1 - \frac{c_1\rfloor}{\lceil 1} - \frac{a_1\rfloor}{\lceil 1} - \frac{c_2\rfloor}{\lceil 1} - \cdots - \frac{c_{n+1}\rfloor}{\lceil 1} \\
\Leftrightarrow x_0 > \frac{a_0\rfloor}{\lceil 1} - \frac{c_1\rfloor}{\lceil 1} - \frac{a_1\rfloor}{\lceil 1} - \frac{c_2\rfloor}{\lceil 1} - \cdots - \frac{c_{n+1}\rfloor}{\lceil 1} = 1 - h_{2n+2},
\end{aligned}
$$

while for odd indices, we have

$$
\begin{aligned}
s_{n+1} = \frac{a_n}{x_n} < 1 \Leftrightarrow y_n < 1 - a_n \Leftrightarrow \frac{c_n}{r_n} < 1 - a_n \Leftrightarrow 1 - s_n > \frac{c_n}{1 - a_n} \\
\Leftrightarrow s_n < 1 - \frac{c_n\rfloor}{\lceil 1} - \frac{a_n\rfloor}{\lceil 1} \Leftrightarrow \cdots \Leftrightarrow s_1 < 1 - \frac{c_1\rfloor}{\lceil 1} - \frac{a_1\rfloor}{\lceil 1} - \frac{c_2\rfloor}{\lceil 1} - \cdots - \frac{a_n\rfloor}{\lceil 1} \\
\Leftrightarrow \frac{a_0}{x_0} < 1 - \frac{c_1\rfloor}{\lceil 1} - \frac{a_1\rfloor}{\lceil 1} - \frac{c_2\rfloor}{\lceil 1} - \cdots - \frac{a_n\rfloor}{\lceil 1} \\
\Leftrightarrow x_0 > \frac{a_0\rfloor}{\lceil 1} - \frac{c_1\rfloor}{\lceil 1} - \frac{a_1\rfloor}{\lceil 1} - \frac{c_2\rfloor}{\lceil 1} - \cdots - \frac{a_n\rfloor}{\lceil 1} = 1 - h_{2n+1}.
\end{aligned}
$$

Therefore, since in Theorem 2.1.2 we proved that $(h_n)_{n\geq 0}$ is a decreasing sequence, we have

$$0 = 1 - h_0 < 1 - h_n < 1 - H \leq x_0.$$

Following the same lines as before for $(h'_{-n})_{n\geq 0}$ we get

$$0 = h'_0 < h'_{-n} < H' \leq \alpha.$$

On the contrary, if (3.1.6) holds, in particular we have that $1 - h_n < 1 - H \leq x_0$ and $h'_{-n} < H' \leq \alpha$ for every $n \geq 0$. Following the same steps as before, using an argument of strong induction, will lead us to the fact that both P_R and P_L are stochastic matrices. □

Next we will proceed to the study of the discrete Darboux transformation which consists of inverting the order of multiplication of the matrices in the stochastic factorization of the previous theorem, as well as the spectral analysis.

3.2 Stochastic Darboux transformations and spectral matrices for the RA case

Now it is time to turn our attention to the discrete Darboux transformation. As we know, if we invert the order of multiplication of the factors in the stochastic factorization $P = P_R P_A$, we obtain a new stochastic matrix, say $\widetilde{P} = P_A P_R$. This matrix $\widetilde{P}$ is not tridiagonal but *pentadiagonal*, so we do not obtain again a birth-death chain on $\mathbb{Z}$, as in the case of UL (or LU) factorization in Chapter 2. However we obtain a two-parameter family of new Markov chains on $\mathbb{Z}$, which we denote by $\{\widetilde{X}_t : t = 0, 1, \dots\}$, which is “almost” a birth-death chain. The only difference is that we have new transition probabilities between the state 1 to the state -1 and vice versa. The tridiagonal coefficients of $\widetilde{P}$ are given by

$$
\begin{aligned}
&\tilde{a}_n = s_n x_n, \quad \tilde{a}_{-n-1} = y_{-n} r_{-n-1}, \quad n \geq 0,\\
&\tilde{b}_0 = y_0, \quad \tilde{b}_{-1} = r_{-1}\alpha + s_{-1}y_{-1}, \quad \tilde{b}_n = r_n x_{n-1} + s_n y_n, \quad \tilde{b}_{-n} = r_{-n}x_{-n+1} + s_{-n}y_{-n}, \quad n \geq 1,\\
&\tilde{c}_0 = \alpha, \quad \tilde{c}_n = r_n y_{n-1}, \quad \tilde{c}_{-n} = s_{-n}x_{-n}, \quad n \geq 1,
\end{aligned}
\tag{3.2.1}
$$

while the probability transitions between the states 1 and -1 are given by

$$
\tilde{d}_1 = \mathbb{P}(\widetilde{X}_1 = -1 | \widetilde{X}_0 = 1) = r_1\alpha, \quad \tilde{d}_{-1} = \mathbb{P}(\widetilde{X}_1 = 1 | \widetilde{X}_0 = -1) = r_{-1}x_0. \tag{3.2.2}
$$

A diagram for this process looks as follows

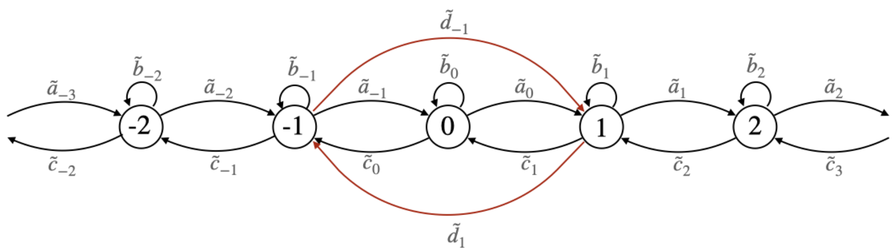

Figure 3.3: Diagram for the Markov chain generated by the Darboux transformation $\widetilde{P}$.

As we anticipated, we will consider the equivalent discrete-time *quasi-birth-and-death process* described by $\boldsymbol{P}$ as in (1.5.2). In this context we have to look for the same structure on the factors P_R and P_A. This will be very convenient in our computations and also for the spectral analysis that we will see later in this chapter. If we perform the relabeling (1.5.9), we get the following 2×2 block matrices

$$
\boldsymbol{P}_R = \left(\begin{array}{cc|cc|cc|c}
y_0 & \alpha & x_0 & 0 & & & \\
0 & y_{-1} & 0 & x_{-1} & & & \\
\hline
 & & y_1 & 0 & x_1 & 0 & \\
 & & 0 & y_{-2} & 0 & x_{-2} & \\
\hline
 & & & & \ddots & & \ddots
\end{array}\right),
$$

$$
\boldsymbol{P}_A = \left(\begin{array}{cc|cc|c}
1 & 0 & & & \\
r_{-1} & s_{-1} & & & \\
\hline
r_1 & 0 & s_1 & 0 & \\
0 & r_{-2} & 0 & s_{-2} & \\
\hline
 & & \ddots & & \ddots
\end{array}\right).
$$

Observe that $\boldsymbol{P}_R$ is a semi-infinite 2×2 *upper* block matrix while $\boldsymbol{P}_A$ is a semi-infinite 2×2 *lower* block matrix. If we call

$$
\begin{aligned}
&Y_0 = \begin{pmatrix} y_0 & \alpha \\ 0 & y_{-1} \end{pmatrix}, \quad Y_n = \begin{pmatrix} y_n & 0 \\ 0 & y_{-n-1} \end{pmatrix}, \quad n \geq 1, \quad X_n = \begin{pmatrix} x_n & 0 \\ 0 & x_{-n-1} \end{pmatrix}, \quad n \geq 0,\\
&S_0 = \begin{pmatrix} 1 & 0 \\ r_{-1} & s_{-1} \end{pmatrix}, \quad S_n = \begin{pmatrix} s_n & 0 \\ 0 & s_{-n-1} \end{pmatrix}, \quad n \geq 1, \quad R_n = \begin{pmatrix} r_n & 0 \\ 0 & r_{-n-1} \end{pmatrix}, \quad n \geq 0,
\end{aligned}
\tag{3.2.3}
$$

then $\boldsymbol{P}_R$ and $\boldsymbol{P}_A$ can be written as

$$\boldsymbol{P}_R = \begin{pmatrix} Y_0 & X_0 & & & \\ & Y_1 & X_1 & & \\ & & Y_2 & X_2 & \\ & & & \ddots & \ddots \end{pmatrix}, \quad \boldsymbol{P}_A = \begin{pmatrix} S_0 & & & & \\ R_1 & S_1 & & & \\ & R_2 & S_2 & & \\ & & \ddots & \ddots & \end{pmatrix}.$$

With this notation, a direct computation from $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$ shows that

$$\begin{aligned} A_n &= X_n S_{n+1}, \quad n \geq 0, \\ B_n &= Y_n S_n + X_n R_{n+1}, \quad n \geq 0, \\ C_n &= Y_n R_n, \quad n \geq 1. \end{aligned} \tag{3.2.4}$$

This change plays a very important role in this case because although $P = P_R P_A$ *is not* a UL factorization, the block matrix factorization $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$ *is* a UL block matrix factorization similar to the one we studied in Chapter 2. Now, if we perform a discrete Darboux transformation we get the equivalent family of discrete-time quasi-birth-and-death processes $\widetilde{\boldsymbol{P}}$ as follows

$$\widetilde{\boldsymbol{P}} = \begin{pmatrix} \widetilde{B}_0 & \widetilde{A}_0 & & & & \\ \widetilde{C}_1 & \widetilde{B}_1 & \widetilde{A}_1 & & & \\ & \widetilde{C}_2 & \widetilde{B}_2 & \widetilde{A}_2 & & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} S_0 & & & & \\ R_1 & S_1 & & & \\ & R_2 & S_2 & & \\ & & \ddots & \ddots & \end{pmatrix} \begin{pmatrix} Y_0 & X_0 & & & \\ & Y_1 & X_1 & & \\ & & Y_2 & X_2 & \\ & & & \ddots & \ddots \end{pmatrix}. \tag{3.2.5}$$

And the relations (3.2.1) are equivalent to the matrix relations

$$\begin{aligned} \widetilde{A}_n &= S_n X_n, \quad n \geq 0, \\ \widetilde{B}_0 &= S_0 Y_0, \quad \widetilde{B}_n = R_n X_{n-1} + S_n Y_n, \quad n \geq 1, \\ \widetilde{C}_n &= R_n Y_{n-1}, \quad n \geq 0. \end{aligned} \tag{3.2.6}$$

To continue with our analysis, let us consider the spectral matrix given by (1.4.6) associated with $\boldsymbol{P}$ and the corresponding matrix-valued orthogonal polynomials in (1.5.10). Notice that the orthogonality relation is given by (1.5.11), the Karlin-McGregor representation formula is given by (1.5.13) and we can compute the invariant measure following Remark 1.5.5.

For the main result of this section we will follow similar steps as in Chapter 2. Let us consider the family of matrix-valued polynomials generated by $\boldsymbol{U} = \boldsymbol{P}_A \boldsymbol{Q}$ with $\boldsymbol{U} = (\boldsymbol{U}_0^T, \boldsymbol{U}_1^T, \cdots)^T$. This family is given by

$$\begin{aligned} \boldsymbol{U}_0(x) &= S_0 \boldsymbol{Q}_0(x) = S_0, \\ \boldsymbol{U}_n(x) &= R_n \boldsymbol{Q}_{n-1}(x) + S_n \boldsymbol{Q}_n(x), \quad n \geq 1, \end{aligned} \tag{3.2.7}$$

where $(S_n)_{n\geq 0}$ and $(R_n)_{n\geq 1}$ defined in (3.2.3). Using the RA factorization of $\boldsymbol{P}$, we have

$$\boldsymbol{P}_R \boldsymbol{U} = \boldsymbol{P}_R \boldsymbol{P}_A \boldsymbol{Q} = \boldsymbol{P}\boldsymbol{Q} = x\boldsymbol{Q},$$

or, in other words,

$$x\boldsymbol{Q}_n(x) = Y_n \boldsymbol{U}_n(x) + X_n \boldsymbol{U}_{n+1}(x), \quad n \geq 0. \tag{3.2.8}$$

If we evaluate the last equation at $x=0$, it is easy to see that

$$\boldsymbol{U}_n(0)=(-1)^n X_{n-1}^{-1}Y_{n-1}\cdots X_0^{-1}Y_0S_0, \tag{3.2.9}$$

and we also have

$$\begin{aligned}\boldsymbol{U}_n(0)\boldsymbol{U}_{n-k}^{-1}(0)&=\left[(-1)^n X_{n-1}^{-1}Y_{n-1}\cdots X_0^{-1}Y_0S_0\right]\left[(-1)^{n-k}X_{n-k-1}^{-1}Y_{n-k-1}X_{n-k-2}^{-1}Y_{n-k-2}\cdots X_0^{-1}Y_0S_0\right]^{-1}\\&=(-1)^k X_{n-1}^{-1}Y_{n-1}\cdots X_{n-k}^{-1}Y_{n-k}, \quad k=1,\dots,n.\end{aligned} \tag{3.2.10}$$

This last equation is very useful when we try to solve (3.2.8) recursively, because we have

$$\begin{aligned}\boldsymbol{U}_n(x)=&X_{n-1}^{-1}\left(x\boldsymbol{Q}_{n-1}(x)-Y_{n-1}\boldsymbol{U}_{n-1}(x)\right)\\=&xX_{n-1}^{-1}\boldsymbol{Q}_{n-1}(x)-X_{n-1}^{-1}Y_{n-1}\boldsymbol{U}_{n-1}(x)\\=&xX_{n-1}^{-1}\boldsymbol{Q}_{n-1}(x)-X_{n-1}^{-1}Y_{n-1}\left[X_{n-2}^{-1}\left(x\boldsymbol{Q}_{n-2}(x)-Y_{n-2}\boldsymbol{U}_{n-2}(x)\right)\right].\end{aligned}$$

Here, using (3.2.10) we can write

$$\boldsymbol{U}_n(x)=\boldsymbol{U}_n(0)\left[\boldsymbol{I}_2+x\sum_{k=0}^{n-1}\boldsymbol{U}_{k+1}^{-1}(0)X_k^{-1}\boldsymbol{Q}_k(x)\right]. \tag{3.2.11}$$

As a matter of fact, through the Darboux transformation we have

$$\widetilde{\boldsymbol{P}}\boldsymbol{U}=\boldsymbol{P}_A\boldsymbol{P}_R\boldsymbol{P}_A\boldsymbol{Q}=x\boldsymbol{P}_A\boldsymbol{Q}=x\boldsymbol{U}.$$

Hence the family of matrix-valued polynomials $\boldsymbol{U}_n(x)$ solves the eigenvalue equation for $\widetilde{\boldsymbol{P}}$ and from (3.2.7) we have that $\deg(\boldsymbol{U}_n(x))=n, n\geq 0$. Unfortunately this family does not satisfy the initial conditions since $\boldsymbol{U}_0(x)=S_0\neq\boldsymbol{I}_2$. In view of this fact, we will be interested in a new family of matrix-valued polynomials $(\widetilde{\boldsymbol{Q}}_n)_{n\geq 0}$ where $\widetilde{\boldsymbol{Q}}_0=\boldsymbol{I}_2$. Since S_0 is a nonsingular constant matrix, this new family can be defined as

$$\widetilde{\boldsymbol{Q}}_n(x)=\boldsymbol{U}_n(x)S_0^{-1}. \tag{3.2.12}$$

Before the main result, let us define the matrix-valued potential coefficients $(\widetilde{\Pi}_n)_{n\geq 0}$ as the solution of the symmetry equations for $\widetilde{\boldsymbol{P}}$, given by

$$\widetilde{\Pi}_n=(\widetilde{C}_1^T\cdots\widetilde{C}_n^T)^{-1}\widetilde{\Pi}_0\widetilde{A}_0\cdots\widetilde{A}_{n-1}, \quad n\geq 1.$$

Then, using (3.2.4), (3.2.6) and the previous equation, we get

$$\begin{aligned}\widetilde{\Pi}_n&=(R_nY_{n-1})^{-T}(R_{n-1}Y_{n-2})^{-T}\cdots(R_2Y_1)^{-T}(R_1Y_0)^{-T}\widetilde{\Pi}_0(S_0X_0)(S_1X_1)\cdots(S_{n-1}X_{n-1})\\&=R_n^{-T}Y_{n-1}^{-T}R_{n-1}^{-T}Y_{n-2}^{-T}\cdots R_2^{-T}Y_1^{-T}R_1^{-T}Y_0^{-T}\widetilde{\Pi}_0S_0X_0S_1X_1\cdots S_{n-1}X_{n-1}\\&=Y_n^{-T}R_n^{-T}Y_{n-1}^{-T}R_{n-1}^{-T}Y_{n-2}^{-T}\cdots R_2^{-T}Y_1^{-T}R_1^{-T}Y_0^{-T}\widetilde{\Pi}_0S_0X_0S_1X_1\cdots S_{n-1}X_{n-1}S_n^{-1}\\&=Y_n^{-T}(C_1^TC_2^T\cdots C_n^T)^{-1}Y_0^{-T}\widetilde{\Pi}_0S_0A_0\cdots A_{n-1}S_n^{-1}.\end{aligned}$$

Then, if $\widetilde{\Pi}_0=Y_0^T\Pi_0S_0^{-1}$ we obtain

$$\widetilde{\Pi}_n=Y_n^T\Pi_nS_n^{-1}, \quad n\geq 0. \tag{3.2.13}$$

Computing $\widetilde{\Pi}_0$ using (3.1.4) and (3.1.5), we get

$$\widetilde{\Pi}_0 = \begin{pmatrix} y_0 & 0 \\ 0 & \alpha/r_{-1} \end{pmatrix}.$$

And, of course, $(\widetilde{\Pi}_n)_{n\geq 0}$ are always *diagonal matrices*. We are now ready to state the main result of this section.

Theorem 3.2.1. *Let $\{X_t : t = 0, 1, \dots\}$ be the birth-death chain on $\mathbb{Z}$ with transition probability matrix P given by (1.4.1) and $\{\widetilde{X}_t : t = 0, 1, \dots\}$ be the Markov chain generated by the Darboux transformation of $P = P_R P_A$ with transition probabilities given by (3.2.1) and (3.2.2). Assume that*

$$\boldsymbol{M}_{-1} = \int_{-1}^{1} x^{-1} \boldsymbol{\Psi}(x) dx$$

is well-defined entry by entry, where $\boldsymbol{\Psi}(x)$ is the original spectral matrix (1.4.6). Then the matrix-valued polynomials $(\widetilde{\boldsymbol{Q}}_n)_{n\geq 0}$ defined by (3.2.12) are orthogonal with respect to the following spectral matrix

$$\widetilde{\boldsymbol{\Psi}}(x) = S_0 \boldsymbol{\Psi}_{\boldsymbol{U}}(x) S_0^T, \tag{3.2.14}$$

where the constant matrix S_0 is defined by (3.2.3) and

$$\boldsymbol{\Psi}_{\boldsymbol{U}}(x) = \frac{\boldsymbol{\Psi}(x)}{x} + \left[\frac{1}{y_0} \begin{pmatrix} 1 & -r_{-1}/s_{-1} \\ -r_{-1}/s_{-1} & (r_{-1}/s_{-1})^2 + y_0 r_{-1}/\alpha s_{-1}^2 \end{pmatrix} - \boldsymbol{M}_{-1} \right] \delta_0(x), \tag{3.2.15}$$

where $\delta_0(x)$ is the Dirac delta at $x = 0$. Moreover, we have

$$\int_{-1}^{1} \widetilde{\boldsymbol{Q}}_n(x) \widetilde{\boldsymbol{\Psi}}(x) \widetilde{\boldsymbol{Q}}_m^T(x) dx = \widetilde{\Pi}_n^{-1} \delta_{nm},$$

where δ_{nm} is the Kronecker delta and $(\widetilde{\Pi}_n)_{n\geq 0}$ are defined by (3.2.13).

Proof. For $n \geq 1$ and $j = 1, \dots, n-1$, we have

$$\begin{aligned} \int_{-1}^{1} \widetilde{\boldsymbol{Q}}_n(x) \widetilde{\boldsymbol{\Psi}}(x) x^j dx &= \int_{-1}^{1} \boldsymbol{U}_n(x) \boldsymbol{\Psi}_{\boldsymbol{U}}(x) x^j S_0^T dx = \int_{-1}^{1} [R_n \boldsymbol{Q}_{n-1}(x) + S_n \boldsymbol{Q}_n(x)] \boldsymbol{\Psi}(x) x^{j-1} S_0^T dx \\ &= \int_{-1}^{1} R_n \boldsymbol{Q}_{n-1}(x) \boldsymbol{\Psi}(x) x^{j-1} S_0^T dx + \int_{-1}^{1} S_n \boldsymbol{Q}_n(x) \boldsymbol{\Psi}(x) x^{j-1} S_0^T dx = \boldsymbol{0}_2, \end{aligned}$$

where for the first equality we have used (3.2.12) and (3.2.14), for the second equality we have used (3.2.7) and (3.2.15), and finally we have used the orthogonality of the family $(\boldsymbol{Q}_n)_{n\geq 0}$. Now, for $n \geq 1$, we have, using (3.2.11), that

$$\begin{aligned} \int_{-1}^{1} \widetilde{\boldsymbol{Q}}_n(x) \widetilde{\boldsymbol{\Psi}}(x) dx &= \int_{-1}^{1} \boldsymbol{U}_n(x) \boldsymbol{\Psi}_{\boldsymbol{U}}(x) S_0^T dx = \int_{-1}^{1} \boldsymbol{U}_n(0) \left[\boldsymbol{I}_2 + x \sum_{k=0}^{n-1} \boldsymbol{U}_{k+1}^{-1}(0) X_k^{-1} \boldsymbol{Q}_k(x) \right] \boldsymbol{\Psi}_{\boldsymbol{U}}(x) S_0^T dx \\ &= \boldsymbol{U}_n(0) \left[\int_{-1}^{1} \boldsymbol{\Psi}_{\boldsymbol{U}}(x) dx + \sum_{k=0}^{n-1} \boldsymbol{U}_{k+1}^{-1}(0) X_k^{-1} \int_{-1}^{1} \boldsymbol{Q}_k(x) \boldsymbol{\Psi}(x) dx \right] S_0^T. \end{aligned}$$

The second part of the previous sum vanishes for $k=1,\dots,n-1$. Therefore the only nonzero term is for $k=0$, i.e., $\boldsymbol{U}_1^{-1}(0)X_0^{-1}\int_{-1}^1 \boldsymbol{\Psi}(x)dx = \boldsymbol{U}_1^{-1}(0)X_0^{-1}\Pi_0^{-1}$. A direct computation using (3.2.9), (3.2.3), (1.5.12) and Definition 1.4.1 shows that

$$\boldsymbol{U}_1^{-1}(0)X_0^{-1}\Pi_0^{-1} = -S_0^{-1}Y_0^{-1}\Pi_0^{-1} = -\frac{1}{y_0}\begin{pmatrix} 1 & -r_{-1}/s_{-1} \\ -r_{-1}/s_{-1} & (r_{-1}/s_{-1})^2 + y_0 r_{-1}/\alpha s_{-1}^2 \end{pmatrix}.$$

From the definition of $\boldsymbol{\Psi_U}(x)$ in (3.2.15) we obtain that $\int_{-1}^1 \widetilde{\boldsymbol{Q}}_n(x)\widetilde{\boldsymbol{\Psi}}(x)dx = \boldsymbol{0}_2$.

Finally, for $n \geq 0$, and using (3.2.12), (3.2.14), (3.2.8), (3.2.7) and the orthogonality properties, we have

$$\begin{aligned}
\int_{-1}^1 \widetilde{\boldsymbol{Q}}_n(x)\widetilde{\boldsymbol{\Psi}}(x)\widetilde{\boldsymbol{Q}}_n^T(x)dx &= \int_{-1}^1 \boldsymbol{U}_n(x)S_0^{-1}S_0\boldsymbol{\Psi_U}(x)S_0^T S_0^{-T}\boldsymbol{U}_n^T(x)dx \\
&= \int_{-1}^1 Y_n^{-1}\left[x\boldsymbol{Q}_n(x) - X_n\boldsymbol{U}_{n+1}(x)\right]\boldsymbol{\Psi_U}(x)\boldsymbol{U}_n^T(x)dx \\
&= Y_n^{-1}\int_{-1}^1 x\boldsymbol{Q}_n(x)\boldsymbol{\Psi_U}(x)\boldsymbol{U}_n^T(x)dx \\
&= Y_n^{-1}\int_{-1}^1 \boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)\left[R_n\boldsymbol{Q}_{n-1}(x) + S_n\boldsymbol{Q}_n(x)\right]^T dx \\
&= Y_n^{-1}\left[\int_{-1}^1 \boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)\boldsymbol{Q}_n^T(x)dx\right]S_n^T \\
&= Y_n^{-1}\Pi_n^{-1}S_n^T = (S_n^{-T}\Pi_n Y_n)^{-1} = \widetilde{\Pi}_n^{-T} = \widetilde{\Pi}_n^{-1},
\end{aligned}$$

where in the last steps we have used the formula (3.2.13) and the fact that $(\widetilde{\Pi}_n)_{n\geq 0}$ are diagonal matrices. □

Let us highlight that the previous theorem states that the spectral matrix associated with the Darboux transformation $\widetilde{\boldsymbol{P}}$ considering the stochastic factorization $\boldsymbol{P} = \boldsymbol{P_R P_A}$ is a Geronimus transformation of the original spectral matrix $\boldsymbol{\Psi}(x)$. More notably, the result is not restricted to 2×2 block tridiagonal matrices. We can follow the same steps for any block tridiagonal matrix $\boldsymbol{P}$ with $N\times N$ blocks, if we are able to find a factorization of the form $\boldsymbol{P} = \boldsymbol{P}_R\boldsymbol{P}_A$ where $\boldsymbol{P}_R$ and $\boldsymbol{P}_A$ are $N\times N$ block bidiagonal matrices as in (3.2). The spectral matrix associated with the Darboux transformation $\widetilde{\boldsymbol{P}} = \boldsymbol{P}_A\boldsymbol{P}_R$ will be given by

$$\widetilde{\boldsymbol{\Psi}}(x) = S_0\left(\frac{\boldsymbol{\Psi}(x)}{x} + \left[(\Pi_0 Y_0 S_0)^{-1} - \boldsymbol{M}_{-1}\right]\delta_0(x)\right)S_0^T. \tag{3.2.16}$$

as long as $\boldsymbol{M}_{-1} = \int_{-1}^1 x^{-1}\boldsymbol{\Psi}(x)dx$ is well-defined entry by entry and we can compute the spectral matrix $\boldsymbol{\Psi}(x)$. This last condition is perhaps the most difficult to fulfill because in this case is not always possible. This idea opens the door to the study of more general structures, as we will see in Chapter 5.

Certainly, talking about bilateral birth-death chains, one of the most important consequences of this result is related to the probabilistic properties of the process described by $\widetilde{\boldsymbol{P}}$. As the spectral matrix after the discrete Darboux transformation $\widetilde{\boldsymbol{\Psi}}(x)$ is a Geronimus transformation of the original spectral matrix, the recurrence properties will be preserved for any values of the free parameters α and x_0 satisfying the conditions (3.1.6) in view of the fact that the Geronimus transformation does not affect the behavior of the transformed spectral matrix at the point 1. Actually, the Geronimus transformation affects only the point 0, since it consists of dividing by x and adding a Dirac delta at 0. This means that the original process is recurrent if and only if the transformed process is recurrent. The same can be inferred for null or positive recurrence or transience properties. Finally, the invariant measure of the transformed process can be computed through the invariant measure of the original process as follows

$$\widetilde{\boldsymbol{\pi}} = \left(y_0, \frac{\alpha}{r_{-1}}; \frac{y_1\pi_1}{s_1}, \frac{y_{-2}\pi_{-2}}{s_{-2}}; \frac{y_2\pi_2}{s_2}, \frac{y_{-3}\pi_{-3}}{s_{-3}}; \cdots \right). \tag{3.2.17}$$

Note that this expression comes from the fact that the invariant measure is given by the diagonal entries of $\widetilde{\Pi}_n$ and it corresponds to an invariant measure of a quasi-birth-and-death process.

3.2.1 RA study of the random walk on $\mathbb{Z}$

Let $\{X_t : t = 0, 1, \dots\}$ be an irreducible birth-death chain on $\mathbb{Z}$ with transition probability matrix P as in (1.4.1) with constant transition probabilities, say,

$$a_n = a, \quad b_n = b, \quad c_n = c, \quad n \in \mathbb{Z}, \quad a+b+c = 1, \quad a, c > 0, \quad b \geq 0.$$

Then, we can compute explicitly the continued fractions as in Example 2.3.1. These fractions are given by

$$H = \frac{1}{2}\left(1 + c - 1 + \sqrt{(1+c-1)^2 - 4c}\right) \quad \text{and} \quad H' = \frac{1}{2}\left(1 + c - 1 - \sqrt{(1+c-1)^2 - 4c}\right).$$

Following condition (3.1.6) we have that the RA stochastic factorization is possible if and only if we take $\alpha \geq H'$ and $x_0 \geq 1 - H$, with $\alpha + x_0 \leq 1$.

Again, we have some interesting cases. If we choose $\alpha = H'$ and $x_0 = 1 - H$, then, using (3.1.5), (2.3.1), (3.1.3) and (3.1.4) we get

$$\begin{aligned} s_{-1} = \frac{c}{H'} &= \frac{2c}{1+c-a-\sqrt{(1+c-a)^2-4c}} = \frac{2c(1+c-a+\sqrt{(1+c-a)^2-4c})}{(1+c-a)^2-(1+c-a)^2+4c} \\ &= \frac{1}{2}\left(1+c-a+\sqrt{(1+c-a)^2-4c}\right) = H. \end{aligned}$$

Then $r_{-1} = 1 - H$ and

$$y_{-1} = \frac{a}{1-H} = 1 - H'.$$

Note that this last equality holds using the expressions of H and H' and that $H = 1 - \frac{a}{1-H'}$. Therefore $x_{-1} = H'$. Following similar steps for the other indices, we have

$$\begin{aligned} s_{-n} = H, \quad r_{-n} = 1 - H, \quad y_{-n} = 1 - H', \quad x_{-n} = H', \quad n \geq 1, \\ s_n = 1 - H', \quad r_n = H', \quad y_n = H, \quad x_n = 1 - H, \quad n \geq 1, \end{aligned} \tag{3.2.18}$$

and $y_0 = H - H'$, $s_0 = 1$ and $r_0 = 0$. Now, if we compute the coefficients of the Darboux transformation using equations (3.2.1), we get

$$
\begin{aligned}
\tilde{a}_{-1} &= (H-H')(1-H), \quad \tilde{a}_0 = 1-H, \quad \tilde{a}_n = 1, \quad n \in \mathbb{Z}\setminus\{-1,0\},\\
\tilde{b}_{-1} &= \tilde{b}_1 = H'(1-H) + H(1-H'), \quad \tilde{b}_0 = H-H', \quad \tilde{b}_n = b, \quad n \in \mathbb{Z}\setminus\{-1,0,1\},\\
\tilde{c}_0 &= H', \quad \tilde{c}_1 = H'(H-H'), \quad \tilde{c}_n = c, \quad n \in \mathbb{Z}\setminus\{0,1\},\\
\tilde{d}_{-1} &= (1-H)^2, \quad \tilde{d}_1 = (H')^2.
\end{aligned}
$$

This means that the coefficients of the Darboux transformation remain "almost" invariant.

From (2.3.3) we have an explicit expression of the weight matrix $\boldsymbol{\Psi}(x)$ associated with $\boldsymbol{P}$. The moment $\boldsymbol{M}_{-1}$ of $\boldsymbol{\Psi}(x)$ is given by (2.3.4) where we assume that $a < (1-\sqrt{c})^2$, which is the same condition for the convergence of the continued fractions H and H'. With this information it is possible to compute the spectral matrix associated with the Darboux transformation $\widetilde{P} = P_A P_R$, which we recall it is an "almost" birth-death chain except for the states 1 and -1 and two free parameters, α and x_0. In this case we have, by equation (3.2.3), that

$$
S_0 = \begin{pmatrix} 1 & 0 \\ 1-c/\alpha & c/\alpha \end{pmatrix}.
$$

Now, following Theorem 3.2.1, to get the spectral matrix associated with $\widetilde{P}$ we have to perform a series of simple computations. First, for the absolutely continuous part we have

$$
\begin{aligned}
S_0\left(\frac{\boldsymbol{\Psi}(x)}{x}\right)S_0^T &= \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}\begin{pmatrix} 1 & 0 \\ 1-c/\alpha & c/\alpha \end{pmatrix}\begin{pmatrix} 1 & \dfrac{x-b}{2c} \\ \dfrac{x-b}{2c} & a/c \end{pmatrix}\begin{pmatrix} 1 & 1-c/\alpha \\ 0 & c/\alpha \end{pmatrix}\\
&= \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}\begin{pmatrix} 1 & 1-\dfrac{x-b-c}{2\alpha} \\ 1-\dfrac{x-b-c}{2\alpha} & 1+\dfrac{x-b-2c}{\alpha}+\dfrac{c(x+a-b)+c^2}{\alpha^2} \end{pmatrix}.
\end{aligned}
$$

Using the fact that we can write $a = (1-H)(1-H')$ and $c = HH'$ we have

$$
S_0\left(\frac{\boldsymbol{\Psi}(x)}{x}\right)S_0^T = \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}\begin{pmatrix} 1 & \dfrac{2\alpha-H-H'+x}{2\alpha} \\ \dfrac{2\alpha-H-H'+x}{2\alpha} & \dfrac{(\alpha-H)(\alpha-H')+x(\alpha-HH')}{\alpha^2} \end{pmatrix}.
$$

For the value at point 0, which will lead us to calculate $\widetilde{\boldsymbol{M}}_{-1} = \int_{-1}^{1} x^{-1}\widetilde{\boldsymbol{\Psi}}(x)$, first we note that

$$
\sqrt{\sigma_-\sigma_+} = \sqrt{(H+H'-2HH')^2 - 4(1-H)(1-H')HH'} = H-H'.
$$

Then we have to compute

$$
S_0\begin{pmatrix} \dfrac{1}{y_0}-\dfrac{1}{H-H'} & \dfrac{c-\alpha}{y_0c}-\dfrac{1}{2c}\left(1-\dfrac{b}{H-H'}\right) \\ \dfrac{c-\alpha}{y_0c}-\dfrac{1}{2c}\left(1-\dfrac{b}{H-H'}\right) & \dfrac{(\alpha-c)^2-y_0(\alpha-c)}{y_0c^2}-\dfrac{a}{c(H-H')} \end{pmatrix}S_0^T.
$$

Using the expression of a and c we get

$$\widetilde{\boldsymbol{M}}_{-1} = \begin{pmatrix} \dfrac{H-H'-y_0}{y_0(H-H')} & \dfrac{H'-\alpha}{\alpha(H-H')} \\ \dfrac{H'-\alpha}{\alpha(H-H')} & \dfrac{(\alpha-H')(H-\alpha)}{\alpha^2(H-H')} \end{pmatrix}.$$

Therefore the final expression for the spectral matrix is given by

$$\widetilde{\boldsymbol{\Psi}}(x) = \frac{1}{\pi x\sqrt{(x-\sigma_-)(\sigma_+-x)}}[\widetilde{\boldsymbol{A}}+\widetilde{\boldsymbol{B}}x] + \widetilde{\boldsymbol{M}}_{-1}\delta_0,$$

where

$$\widetilde{\boldsymbol{A}} = \begin{pmatrix} 1 & \dfrac{2\alpha-H-H'}{2\alpha} \\ \dfrac{2\alpha-H-H'}{2\alpha} & \dfrac{(\alpha-H')(\alpha-H)}{\alpha^2} \end{pmatrix}, \quad \widetilde{\boldsymbol{B}} = \frac{1}{2\alpha}\begin{pmatrix} 0 & 1 \\ 1 & \dfrac{2(\alpha-HH')}{\alpha} \end{pmatrix},$$

and $\delta_0(x)$ the Dirac delta at $x=0$. About the probabilistic properties, as we mentioned in Example 2.3.1, it is easy to see that the original random walk is always transient, except for $a=c$, where it is null recurrent. The transformed process will have the same recurrent behavior. In addition, the invariant measure of the original process is given by the following simple expression

$$\boldsymbol{\pi} = \left(1, \frac{c}{a}; \frac{a}{c}, \frac{c^2}{a^2}; \frac{a^2}{c^2}, \frac{c^3}{a^3}; \cdots\right), \tag{3.2.19}$$

while the invariant measure of the transformed process will be a complicated two-parameter family of vectors. Nevertheless, for the special choice of $\alpha = H'$ and $x_0 = H$ we have, using (3.2.17) and (3.2.18), that the invariant measure is given by following vector

$$\widetilde{\boldsymbol{\pi}} = \left(H-H', \frac{H'}{1-H}; \frac{aH}{c(1-H')}, \frac{c^2(1-H')}{a^2H}; \frac{a^2H}{c^2(1-H')}, \frac{c^3(1-H')}{a^3H}; \cdots\right).$$

3.3 Stochastic AR factorization on the integers

Now it is time to study the second case, where we are looking for an absorbing-reflecting (AR) factorization. First of all note that if we start from a birth-death chain P as in equation (1.4.1) and consider an AR factorization of the form $P = \widetilde{P}_A\widetilde{P}_R$, we will end with both $\widetilde{P}_A$ and $\widetilde{P}_R$ two separated birth-death chains at the state 0, i.e., there will not be a probability that connects positive and negative indexes. To fix this detail, we notice that the multiplication of matrices of the form $\widetilde{P}_A\widetilde{P}_R$, with $\widetilde{P}_A$ as in (3.1.2) and $\widetilde{P}_R$ as in (3.1.1), gives rise to a Markov chain which is "almost" a birth-death chain, except for the states 1 and -1. Therefore, for our purposes, it will make more sense to start with an irreducible Markov chain $\{X_t : t = 0,1,\ldots\}$ on $\mathbb{Z}$ with transition probability matrix given by

$$P = \left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & c_{-2} & b_{-2} & a_{-2} & & & & \\ & & c_{-1} & b_{-1} & a_{-1} & d_{-1} & & \\ \hline & & & c_0 & b_0 & a_0 & & \\ & & & d_1 & c_1 & b_1 & a_1 & \\ & & & & & c_2 & b_2 & a_2 \\ & & & & & & \ddots & \ddots & \ddots \end{array}\right), \tag{3.3.1}$$

similar to the one illustrated in Figure 3.3.

As before, to simplify some computations, we will work with the equivalent quasi-birth-and-death process. After the relabeling (1.5.9) we get the following semi-infinite 2×2 block tridiagonal matrix

$$
\boldsymbol{P} = \left(\begin{array}{cc|cc|cc|cc}
b_0 & c_0 & a_0 & 0 & & & & \\
a_{-1} & b_{-1} & d_{-1} & c_{-1} & & & & \\
\hline
c_1 & d_1 & b_1 & 0 & a_1 & 0 & & \\
0 & a_{-2} & 0 & b_{-2} & 0 & c_{-2} & & \\
\hline
 & & c_2 & 0 & b_2 & 0 & a_2 & 0 \\
 & & 0 & a_{-3} & 0 & b_{-3} & 0 & c_{-3} \\
\hline
 & & & & \ddots & & \ddots & & \ddots
\end{array}\right), \tag{3.3.2}
$$

where the only difference with the coefficients in (1.5.2) is the triangular shape of the blocks A_0 and C_1, now given by

$$
A_0 = \begin{pmatrix} a_0 & 0 \\ d_{-1} & c_{-1} \end{pmatrix}, \quad C_1 = \begin{pmatrix} c_1 & d_1 \\ 0 & a_{-2} \end{pmatrix}.
$$

Observe that the block matrix factorization $\boldsymbol{P} = \widetilde{\boldsymbol{P}}_A \widetilde{\boldsymbol{P}}_R$ corresponds to an LU 2×2 block matrix factorization. A direct computation shows that

$$
\begin{aligned}
A_n &= \widetilde{S}_n \widetilde{X}_n, \quad n \geq 0, \\
B_n &= \widetilde{S}_n \widetilde{Y}_n + \widetilde{R}_n \widetilde{X}_{n-1}, \quad n \geq 1, \quad B_0 = \widetilde{S}_0 \widetilde{Y}_0, \\
C_n &= \widetilde{R}_n \widetilde{Y}_{n-1}, \quad n \geq 1,
\end{aligned} \tag{3.3.3}
$$

or equivalently, using $P = \widetilde{P}_A \widetilde{P}_R$, we obtain

$$
a_n = \tilde{x}_n \tilde{s}_n, \quad n \geq 1, \quad a_0 = \tilde{x}_0, \quad a_{-n} = \tilde{y}_{-n+1} \tilde{r}_{-n}, \quad n \geq 1, \tag{3.3.4}
$$

$$
b_n = \tilde{y}_n \tilde{s}_n + \tilde{x}_{n-1} \tilde{r}_n, \quad n \geq 1, \quad b_0 = \tilde{y}_0, \quad b_{-n} = \tilde{y}_{-n} \tilde{s}_{-n} + \tilde{x}_{-n+1} \tilde{r}_{-n}, \quad n \geq 1, \tag{3.3.5}
$$

$$
c_n = \tilde{y}_{n-1} \tilde{r}_n, \quad n \geq 1, \quad c_0 = \tilde{\alpha}, \quad c_{-n} = \tilde{x}_{-n} \tilde{s}_{-n}, \quad n \geq 1, \tag{3.3.6}
$$

$$
d_{-1} = \tilde{r}_{-1} \tilde{x}_0, \quad d_1 = \tilde{r}_1 \tilde{\alpha}, \tag{3.3.7}
$$

with $\widetilde{P}_A$ and $\widetilde{P}_R$ stochastic matrices, i.e., all entries are nonnegative and

$$
\tilde{\alpha} + \tilde{x}_0 + \tilde{y}_0 = 1, \quad \tilde{x}_n + \tilde{y}_n = 1, \quad \tilde{s}_n + \tilde{r}_n = 1, \quad n \in \mathbb{Z} \setminus \{0\}. \tag{3.3.8}
$$

Observe that, from (3.3.4) - (3.3.7), the values of the probability transitions between states 1 and -1 are given by

$$
d_{-1} = \frac{a_{-1} a_0}{b_0}, \quad d_1 = \frac{c_0 c_1}{b_0}. \tag{3.3.9}
$$

This means that the factorization is restricted to condition (3.3.9). Additionally we need to have that $0 < d_{-1}, d_1 < 1$, so these coefficients are probabilities. Using again equation (3.3.9), we have to assume that

$$
b_0 > \max\{a_{-1} a_0, c_0 c_1\}.
$$

With the previous restrictions it is possible to get the coefficients $\tilde{y}_0$, $\tilde{r}_1$, $\tilde{s}_1$, $\tilde{x}_1$, $\tilde{y}_1, \dots$ recursively using (3.3.5), (3.3.6) and (3.3.8) in that order. We also can get $\tilde{y}_0$, $\tilde{r}_{-1}$, $\tilde{s}_{-1}$, $\tilde{x}_{-1}$, $\tilde{y}_{-1}$, $\tilde{r}_{-2}, \dots$ recursively using (3.3.5), (3.3.4) and (3.3.8) in that order. This means that *there is no free parameter* and the factorization is unique.

Now, we proceed to analyze conditions under which the factorization is stochastic. As we have a slightly different expression of P, for this case, the continued fractions $\widetilde{H}$ and $\widetilde{H}'$ are given by

$$\widetilde{H} = \frac{c_1|}{|1} - \frac{a_1|}{|1} - \frac{c_2|}{|1} - \frac{a_2|}{|1} - \cdots, \quad \widetilde{H}' = \frac{a_{-1}|}{|1} - \frac{c_{-1}|}{|1} - \frac{a_{-2}|}{|1} - \frac{c_{-2}|}{|1} - \cdots. \tag{3.3.10}$$

For each continued fraction, consider the corresponding sequence of convergents $(\tilde{h}_n)_{n\geq 0}$ and $(\tilde{h}'_{-n})_{n\geq 0}$, given by

$$\tilde{h}_n = \frac{\tilde{N}_n}{\tilde{D}_n}, \quad \tilde{h}'_{-n} = \frac{\tilde{N}'_{-n}}{\tilde{D}'_{-n}}. \tag{3.3.11}$$

With these fractions and the convergents defined we are ready to establish conditions for the existence of the stochastic factorization as follows.

Proposition 3.3.1. *Let $\widetilde{H}$ and $\widetilde{H}'$ be the continued fractions given by* (3.3.10) *and the corresponding convergents $(\tilde{h}_n)_{n\geq 0}$ and $(\tilde{h}'_{-n})_{n\geq 0}$ defined by* (3.3.11)*. Assume that*

$$0 < \tilde{N}_n < \tilde{D}_n, \quad \textit{and} \quad 0 < \tilde{N}'_{-n} < \tilde{D}'_{-n}, \quad n \geq 1.$$

Then both $\widetilde{H}$ and $\widetilde{H}'$ are convergent. Moreover, let $P = \widetilde{P}_A\widetilde{P}_R$. Then, both $\widetilde{P}_A$ and $\widetilde{P}_R$ are stochastic matrices if and only if

$$b_0 \geq \max\{\widetilde{H}, \widetilde{H}'\}. \tag{3.3.12}$$

Proof. Following exactly the same lines as the proof of Proposition 3.1.1, replacing the formulas by the tilde notation, one can prove that $(\tilde{h}_n)_{n\geq 0}$ and $(\tilde{h}'_{-n})_{n\geq 0}$ converge to $\widetilde{H}$ and $\widetilde{H}'$, respectively. After that, using a similar argument about equivalences as in Proposition 3.1.1, one can prove that $0 = \tilde{h}_0 < \tilde{h}_n < \widetilde{H} \leq \tilde{y}_0$, and $0 = \tilde{h}'_0 < \tilde{h}'_{-n} < \widetilde{H}' \leq \tilde{y}_0$. Therefore we get (3.3.12) taking in mind that $\tilde{y}_0 = b_0$ in this case (see (3.3.5)). $\square$

Observe that if $b_0 \geq \widetilde{H}$, then in particular $b_0 > c_1$ and since we know that $0 < c_0 < 1$, then we have $b_0 > c_1 > c_0c_1$. In the same way, if $b_0 \geq \widetilde{H}'$, in particular we have $b_0 > a_{-1}$ and since $0 < a_0 < 1$, then we get $b_0 > a_{-1} > a_0a_{-1}$. Therefore it can be concluded that if $b_0 \geq \max\{\widetilde{H}, \widetilde{H}'\}$, then $b_0 > \max\{a_{-1}a_0, c_0c_1\}$ from which we conclude that $0 < d_1, d_{-1} < 1$. Observe that this does not mean that the factorization is always possible since we need to have (3.3.9).

3.4 Stochastic Darboux transformations and spectral matrices for the AR case

Now that we have $P = \widetilde{P}_A\widetilde{P}_R$, or equivalently $\boldsymbol{P} = \widetilde{\boldsymbol{P}}_A\widetilde{\boldsymbol{P}}_R$, we can perform a discrete Darboux transformation, which we will denote by a hat superscript, $\widehat{P} = \widetilde{P}_R\widetilde{P}_A$ or equivalently $\widehat{\boldsymbol{P}} = \widetilde{\boldsymbol{P}}_R\widetilde{\boldsymbol{P}}_A$. In block matrix form we have

$$\widehat{\boldsymbol{P}} = \begin{pmatrix} \widehat{B}_0 & \widehat{A}_0 & & & \\ \widehat{C}_1 & \widehat{B}_1 & \widehat{A}_1 & & \\ & \widehat{C}_2 & \widehat{B}_2 & \widehat{A}_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \widetilde{Y}_0 & \widetilde{X}_0 & & & \\ & \widetilde{Y}_1 & \widetilde{X}_1 & & \\ & & \widetilde{Y}_2 & \widetilde{X}_2 & \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \widetilde{S}_0 & & & & \\ \widetilde{R}_1 & \widetilde{S}_1 & & & \\ & \widetilde{R}_2 & \widetilde{S}_2 & & \\ & & & \ddots & \ddots \end{pmatrix}. \tag{3.4.1}$$

A direct computation shows

$$\begin{aligned}\widehat{A}_n &= \widetilde{X}_n\widetilde{S}_{n+1}, \quad n\geq 0,\\ \widehat{B}_n &= \widetilde{X}_n\widetilde{R}_{n+1}+\widetilde{Y}_n\widetilde{S}_n, \quad n\geq 0,\\ \widehat{C}_n &= \widetilde{Y}_n\widetilde{R}_n, \quad n\geq 1.\end{aligned} \tag{3.4.2}$$

Here, after the Darboux transformation, we get that $\widehat{P}$ is now a tridiagonal matrix, i.e., it describes a discrete-time birth-death chain on $\mathbb{Z}$.

An important difference now is that we can not guarantee the existence of a weight matrix associated with $\boldsymbol{P}$. But we know, following Theorem 1.5.1, that there exists a spectral matrix $\boldsymbol{\Psi}(x)$ such that the polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ defined by the three-term recurrence relation (1.5.3) are orthogonal with respect to the spectral matrix $d\boldsymbol{\Psi}(x)$. The sequence of nonsingular matrices $(R_n)_{n\geq 0}$ in that theorem is given by

$$R_n=\begin{pmatrix}\sqrt{\pi_n} & 0\\ 0 & \sqrt{\pi_{-n-1}}\end{pmatrix}, \quad n\geq 0,$$

where $\pi=(\pi_n)_{n\in\mathbb{Z}}$ are the potential coefficients given by Definition 1.4.1. Using (3.3.9) we can see that $R_nB_nR_n^{-1}, n\geq 0$, are always symmetric matrices and that

$$R_n^TR_n=(C_1^T\cdots C_n^T)^{-1}R_0^TR_0A_0\cdots A_{n-1}, \quad n\geq 1.$$

Therefore we have $R_n^TR_n=\Pi_n$ where Π_n is defined by (1.5.12).

Now we have to define the matrix-valued polynomials that we are going to use for the main theorem of this section. As in Chapter 2, we have an auxiliary family that satisfies the eigenvalue equation but not the initial conditions. First remember that, in this case, $\boldsymbol{P}$ is almost tridiagonal, so the family of matrix-valued polynomials we are looking for is very similar to the family given in (1.5.3). Let us consider the polynomial vector $\bar{\boldsymbol{Q}}=(\bar{\boldsymbol{Q}}_0^T,\bar{\boldsymbol{Q}}_1^T,\cdots)^T$ where entries are given by $\bar{\boldsymbol{Q}}=\widetilde{\boldsymbol{P}}_R\boldsymbol{Q}$. Therefore this family can be generated by

$$\bar{\boldsymbol{Q}}_n(x)=\widetilde{Y}_n\boldsymbol{Q}_n(x)+\widetilde{X}_n\boldsymbol{Q}_{n+1}(x), \quad n\geq 0,$$

where $(\widetilde{Y}_n)_{n\geq 0}$ and $(\widetilde{X}_n)_{n\geq 0}$ are defined by equation (3.2.3). Now, from the AR factorization of $\boldsymbol{P}$, we get

$$\widetilde{\boldsymbol{P}}_{\boldsymbol{A}}\bar{\boldsymbol{Q}}=\widetilde{\boldsymbol{P}}_{\boldsymbol{A}}\widetilde{\boldsymbol{P}}_{\boldsymbol{R}}\boldsymbol{Q}=\boldsymbol{P}\boldsymbol{Q}=x\boldsymbol{Q},$$

or the equivalent system

$$\begin{aligned}x\boldsymbol{Q}_0(x)&=\widetilde{S}_0\bar{\boldsymbol{Q}}_0(x),\\ x\boldsymbol{Q}_n(x)&=\widetilde{R}_n\bar{\boldsymbol{Q}}_{n-1}(x)+\widetilde{S}_n\bar{\boldsymbol{Q}}_n(x), \quad n\geq 1.\end{aligned}$$

From the previous equation we have that

$$\bar{\boldsymbol{Q}}_0(x)=x\widetilde{S}_0^{-1}\boldsymbol{Q}_0(x)=x\widetilde{S}_0^{-1},$$

and $\bar{\boldsymbol{Q}}_n(x)=x\boldsymbol{T}_n(x)$, where $(\boldsymbol{T}_n)_{n\geq 0}$ is a family of matrix-valued polynomials with $\deg(\boldsymbol{T}_n(x))=n, n\geq 0$, and nonsingular leading coefficient. We can rewrite the previous two formulas in terms of the polynomials $(\boldsymbol{T}_n)_{n\geq 0}$. Indeed,

$$x\boldsymbol{T}_n(x)=\widetilde{Y}_n\boldsymbol{Q}_n(x)+\widetilde{X}_n\boldsymbol{Q}_{n+1}(x), \quad n\geq 0, \tag{3.4.3}$$

and in terms of the polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ as follows

$$\begin{aligned}\boldsymbol{Q}_0(x) &= \widetilde{S}_0\boldsymbol{T}_0(x),\\ \boldsymbol{Q}_n(x) &= \widetilde{R}_n\boldsymbol{T}_{n-1}(x) + \widetilde{S}_n\boldsymbol{T}_n(x), \quad n\geq 1.\end{aligned} \tag{3.4.4}$$

As expected, we have $\boldsymbol{T}_0(x) = \widetilde{S}_0^{-1} \neq \boldsymbol{I}_2$, so we have to define a new family $(\widehat{\boldsymbol{Q}}_n)_{n\geq 0}$ that satisfies the initial condition $\widehat{\boldsymbol{Q}}_0 = \boldsymbol{I}_2$. In this case it is easy to see that this family is given by

$$\widehat{\boldsymbol{Q}}_n(x) = \boldsymbol{T}_n(x)\widetilde{S}_0. \tag{3.4.5}$$

Finally, the last element we need to define is the sequence of potential coefficients given by the solution of the symmetry equations for $\widehat{\boldsymbol{P}}$ in (3.4.1). Therefore the potential coefficients are given by

$$\widehat{\Pi}_n = (\widehat{C}_1^T\cdots\widehat{C}_n^T)^{-1}\widehat{\Pi}_0\widehat{A}_0\cdots\widehat{A}_{n-1}, \quad n\geq 1,$$

where, using (3.3.4) and (3.3.6), we have

$$\widehat{\Pi}_0 = \frac{1}{\widetilde{y}_0}\begin{pmatrix}1 & 0\\ 0 & \frac{\tilde{\alpha}\tilde{s}_{-1}}{\tilde{y}_{-1}\tilde{r}_{-1}}\end{pmatrix}.$$

Note that using equations (3.3.3), (3.4.2) and the previous equation, we get

$$\widehat{\Pi}_n = \widetilde{Y}_n^{-T}\Pi_n\widetilde{S}_n, \quad n\geq 0. \tag{3.4.6}$$

This means that, as in the RA factorization, $(\widehat{\Pi}_n)_{n\geq 0}$ are always *diagonal matrices.*

Theorem 3.4.1. *Let $\{X_t : t = 0, 1, \dots\}$ be the Markov chain on $\mathbb{Z}$ with transition probability matrix P given by* (3.3.1) *and $\{\widetilde{X}_t : t = 0, 1, \dots\}$ the birth-death chain generated by the Darboux transformation of $P = \widetilde{P}_A\widetilde{P}_R$. Then the matrix-valued polynomials $(\widehat{\boldsymbol{Q}}_n)_{n\geq 0}$ defined by* (3.4.5) *are orthogonal with respect to the following spectral matrix*

$$\widehat{\boldsymbol{\Psi}}(x) = x\widetilde{S}_0^{-1}\boldsymbol{\Psi}(x)\widetilde{S}_0^{-T}, \tag{3.4.7}$$

where the constant matrix $\widetilde{S}_0$ is defined by (3.2.3) *and $\boldsymbol{\Psi}(x)$ is the original spectral matrix associated with P. Moreover, we have*

$$\int_{-1}^{1}\widehat{\boldsymbol{Q}}_n(x)\widehat{\boldsymbol{\Psi}}(x)\widehat{\boldsymbol{Q}}_m^T(x)dx = \widehat{\Pi}_n^{-1}\delta_{nm},$$

where δ_{nm} is the Kronecker delta and $(\widehat{\Pi}_n)_{n\geq 0}$ are defined by (3.4.6).

Proof. For $n\geq 1$ and $j = 0,\dots,n-1$, we have

$$\begin{aligned}\int_{-1}^{1}\widehat{\boldsymbol{Q}}_n(x)\widehat{\boldsymbol{\Psi}}(x)x^j dx &= \int_{-1}^{1} x\boldsymbol{T}_n(x)\boldsymbol{\Psi}(x)x^j\widetilde{S}_0^{-T}dx\\ &= \int_{-1}^{1}[\widetilde{Y}_n\boldsymbol{Q}_n(x) + \widetilde{X}_n\boldsymbol{Q}_{n+1}(x)]\boldsymbol{\Psi}(x)x^j\widetilde{S}_0^{-T}dx\\ &= \widetilde{Y}_n\int_{-1}^{1}\boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)x^j\widetilde{S}_0^{-T}dx + \widetilde{X}_n\int_{-1}^{1}\boldsymbol{Q}_{n+1}(x)\boldsymbol{\Psi}(x)x^j\widetilde{S}_0^{-T}dx\\ &= \boldsymbol{0}_2,\end{aligned}$$

where for the first equality we have used (3.4.5) and (3.4.7), for the second equality we have used (3.4.3), and finally we have used the orthogonality of the family $(\boldsymbol{Q}_n)_{n\geq 0}$. Finally, for $n \geq 0$, and using (3.4.3), (3.4.4), (3.4.6) and the orthogonality properties, we have

$$\begin{aligned}
\int_{-1}^{1} \widehat{\boldsymbol{Q}}_n(x)\widehat{\boldsymbol{\Psi}}(x)\widehat{\boldsymbol{Q}}_n^T(x)dx &= \int_{-1}^{1} \boldsymbol{T}_n(x)x\boldsymbol{\Psi}(x)\boldsymbol{T}_n^T(x)dx \\
&= \int_{-1}^{1} [\widetilde{Y}_n\boldsymbol{Q}_n(x) + \widetilde{X}_n\boldsymbol{Q}_{n+1}(x)]\boldsymbol{\Psi}(x)\boldsymbol{T}_n^T(x)dx \\
&= \widetilde{Y}_n \int_{-1}^{1} \boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)\boldsymbol{T}_n^T(x)dx \\
&= \widetilde{Y}_n \int_{-1}^{1} \boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)[\widetilde{S}_n^{-1}\boldsymbol{Q}_n(x) - \widetilde{S}_n^{-1}\widetilde{R}_n\boldsymbol{T}_{n-1}(x)]^T dx \\
&= \widetilde{Y}_n \int_{-1}^{1} \boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)\boldsymbol{Q}_n^T(x)\widetilde{S}_n^{-T} dx \\
&= \widetilde{Y}_n\Pi_n^{-1}\widetilde{S}_n^{-T} = (\widetilde{S}_n^T\Pi_n\widetilde{Y}_n^{-1})^{-1} = \widehat{\Pi}_n^{-T} = \widehat{\Pi}_n^{-1},
\end{aligned}$$

where in the final step we have used the fact that $(\widehat{\Pi}_n)_{n\geq 0}$ are diagonal matrices. $\square$

With this final result, let us comment that, after the Darboux transformation, we get a Christoffel transformation of the original spectral matrix, i.e., there is a multiplication by x. Equally important, we can conclude that the transformed process is unique. Although we have an explicit expression of the n-step transition probability matrix $\widehat{\boldsymbol{P}}_{ij}^{(n)}$ using the Karlin-McGregor formula (1.5.13) and, since $\widehat{\boldsymbol{\Psi}}(x)$ is a Christoffel transformation of the original spectral matrix, we have that the recurrence properties will always be preserved. Finally, we know that the invariant measure of the transformed process is given by the diagonal entries of the matrices $\widehat{\Pi}_n$ in (3.4.6). Therefore we have the explicit expression as follows

$$\widehat{\boldsymbol{\pi}} = \left(\frac{1}{\tilde{y}_0}, \frac{\tilde{s}_{-1}\tilde{\pi}_{-1}}{\tilde{y}_{-1}}; \frac{\tilde{s}_1\pi_1}{\tilde{y}_1}, \frac{\tilde{s}_{-2}\pi_{-2}}{\tilde{y}_{-2}}; \frac{\tilde{s}_2\pi_2}{\tilde{y}_2}, \frac{\tilde{s}_{-3}\pi_{-3}}{\tilde{y}_{-3}}; \cdots\right).$$

We finish this section with some brief remarks about the connection between RA and AR factorization. Now that we understand what RA and AR factorizations are about, we can infer that we are in the presence of dual problems just as the UL and LU factorizations. It is clear that if we start with an RA factorization of P as in (1.4.1), i.e., $P = P_R P_A$ and consider the Darboux transformation $\widetilde{P} = P_A P_R$, then, if we consider $\widetilde{P}$ with an AR factorization and perform the Darboux transformation, we will get the original matrix P. Although this fact is clear from the structure of the one step transition probability matrices, one can also explore this fact through the spectral matrices as follows. First note that, after the second Darboux transformation we will have the spectral matrix given in (3.4.7) where the matrix $\widetilde{S}_0$ appears. Second, note that if we apply the AR factorization to $\widetilde{P}$ we have that $\widetilde{S}_0 = S_0$, where S_0 is the matrix in (3.2.14). Hence the final spectral matrix turns out to be the following

$$\widehat{\boldsymbol{\Psi}}(x) = x\widetilde{S}_0^{-1}\widetilde{\boldsymbol{\Psi}}(x)\widetilde{S}_0^{-T} = x\widetilde{S}_0^{-1}S_0\boldsymbol{\Psi}_{\boldsymbol{U}}(x)S_0^T\widetilde{S}_0^{-T} = x\left[\frac{\boldsymbol{\Psi}(x)}{x} + C\delta_0(x)\right] = \boldsymbol{\Psi}(x),$$

where C is the constant 2×2 matrix appearing in front of the Dirac delta $\delta_0(x)$ in (3.2.15). It is possible to follow the same argument for the case where we start with an AR factorization of P as in (3.3.1).

To finish this section we will revisit Example 2.3.1 with a small change to meet the conditions required by the AR factorization.

3.4.1 AR study of the random walk on $\mathbb{Z}$ with transitions between the states 1 and -1

As an example for this section we will analyze the random walk with constant transition probabilities as in Examples 2.3.1 and 3.2.1, but we will make a small change to have transition probabilities between the states 1 and -1. Note that, by equation (3.3.9), there is one way to choose these transition probabilities so that the factorization is possible. Let us consider an irreducible Markov chain on $\mathbb{Z}$ with transition probability matrix P as in (3.3.1) where

$$a_n = a, \quad n \in \mathbb{Z} \setminus \{-1\}, \quad b_n = b, \quad n \in \mathbb{Z}, \quad c_n = c, \quad n \in \mathbb{Z} \setminus \{1\},$$

and as usual $a + b + c = 1, a, c > 0, b \geq 0$. From (3.3.9) the values of a_{-1}, d_{-1}, c_1, d_1 must be given in terms of a, b, c. Indeed,

$$a_{-1} = \frac{ab}{1-c}, \quad d_{-1} = \frac{a^2}{1-c}, \quad c_1 = \frac{bc}{1-a}, \quad d_1 = \frac{c^2}{1-a}.$$

Since $1 - c > 0$ and $1 - a > 0$, this implies that $a_{-1}, d_{-1}, c_1, d_1 > 0$. Now, observe that $a + c \leq 1$ and $ab < a$ since $0 \leq b < 1$. Then we have $ab + c < a + c \leq 1$, which implies that $ab < 1 - c$ and therefore $a_{-1} < 1$. In the same way but using $a^2 < a$, $bc < c$ and $c^2 < c$ we get $d_{-1} < 1$, $c_1 < 1$ and $d_1 < 1$, respectively. Therefore, independently of the choice of a, b and c, P is always a stochastic matrix. A diagram of this process looks like Figure 3.4.

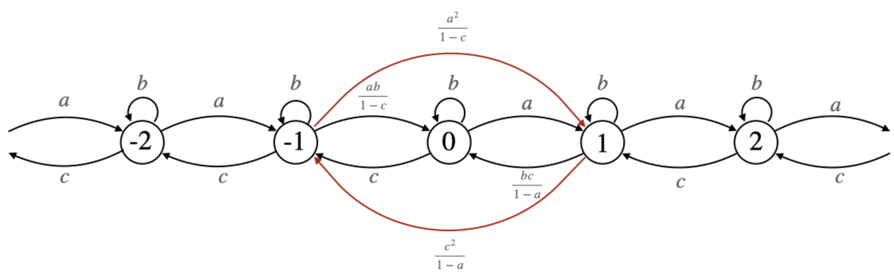

Figure 3.4: Diagram for the Markov chain of Example 3.4.1.

In this case, the continued fractions in (3.3.10) can be explicitly computed as follows. First notice that

$$\widetilde{H} = \frac{c_1}{J} = \frac{bc}{J(1-a)}, \quad \widetilde{H}' = \frac{a_{-1}}{J'} = \frac{ab}{J'(1-c)},$$

where

$$J = 1 - \cfrac{a}{1 - \cfrac{c}{J}}, \quad J' = 1 - \cfrac{c}{1 - \cfrac{a}{J}},$$

or in other words, J and J' are solutions of the quadratic equations

$$J^2 + J(-1 - c + a) + c = 0, \quad J'^2 + J'(-1 + c - 1) + a = 0,$$

respectively. Using the fact that $(1 + a - c)^2 - 4a = (1 + c - a)^2 - 4c$ and after rationalizing, we have the following expressions for the continued fractions

$$\widetilde{H} = \frac{b}{2(1-a)} \left(1 + c - a + \sqrt{(1 + c - a)^2 - 4c}\right), \quad \widetilde{H}' = \frac{b}{2(1-c)} \left(1 + a - c + \sqrt{(1 + c - a)^2 - 4c}\right),$$

with $a \leq (1-\sqrt{c})^2$. It is clear that with this condition we have that $b > \widetilde{H}$, $b > \widetilde{H}'$ and then

$$b > \max\{\widetilde{H}, \widetilde{H}'\}.$$

Therefore, according to Proposition 3.3.1, we have that the AR stochastic factorization of P is always possible.

Now that P is not tridiagonal and we do not have a birth-death chain on the integers $\mathbb{Z}$, we can not apply the same methodology as we did in the Chapter 2. However we can consider the block tridiagonal structure of P given by $\boldsymbol{P}$ in (3.3.2) and use the theory of matrix-valued orthogonal polynomials to compute the spectral matrix $\boldsymbol{\Psi}(x)$. This procedure is similar to the one used in Example 2.3.1 where we used the relation between the Stieltjes transform of the original process described by P in (1.4.1) and the Stieltjes transform of the 0-th associated process constructed by removing the first row and column of P. In this example we use the same results but with the block matrices. To start, after relabeling (1.5.9), we get that $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \left(\begin{array}{cc|cc|cc|cc} b & c & a & 0 & & & & \\ \frac{ab}{1-c} & b & \frac{a^2}{1-c} & c & & & & \\ \hline \frac{bc}{1-a} & \frac{c^2}{1-a} & b & 0 & a & 0 & & \\ 0 & a & 0 & b & 0 & c & & \\ \hline & & c & 0 & b & 0 & a & 0 \\ & & 0 & a & 0 & b & 0 & c \\ \hline & & & & \ddots & & \ddots & & \ddots \end{array}\right).$$

Then, we study the relation between the Stieltjes transform of the spectral matrix $\boldsymbol{\Psi}(x)$ and the Stieltjes transform of the spectral matrix $\boldsymbol{\Psi}_0(x)$ of the 0-th associated process $\boldsymbol{P}_0$ defined in Remark 1.3.3.

Using equation (1.5.7) we can compute the corresponding Stieltjes transform $B(z;\boldsymbol{\Psi}_0)$, given by

$$B(z;\boldsymbol{\Psi}_0) = \begin{pmatrix} \dfrac{z-b\pm\sqrt{(z-\sigma_+)(z-\sigma_-)}}{2ac} & 0 \\ 0 & \dfrac{z-b\pm\sqrt{(z-\sigma_+)(z-\sigma_-)}}{2c^2} \end{pmatrix}, \quad z \in \mathbb{C}\setminus[\sigma_-,\sigma_+],$$

where $\sigma_\pm = 1-(\sqrt{a}\mp\sqrt{c})^2$. Now using equation (1.5.8) we have that the Stieltjes transform $B(z;\boldsymbol{\Psi})$ of $\boldsymbol{P}$ satisfies the algebraic equation

$$B(z;\boldsymbol{\Psi})\Pi_{\boldsymbol{\Psi}}\left[z\boldsymbol{I}_2 - B_0 - A_0 B(z;\boldsymbol{\Psi}_0)\Pi_{\boldsymbol{\Psi}_0} C_1\right] = \boldsymbol{I}_2, \tag{3.4.8}$$

where

$$A_0 = \begin{pmatrix} a & 0 \\ \dfrac{a^2}{1-c} & c \end{pmatrix}, \quad B_0 = \begin{pmatrix} b & c \\ \dfrac{ab}{1-c} & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} \dfrac{bc}{1-a} & \dfrac{c^2}{1-a} \\ 0 & a \end{pmatrix},$$

and $\Pi_{\boldsymbol{\Psi}}$ and $\Pi_{\boldsymbol{\Psi}_0}$ are given in this case by

$$\Pi_{\boldsymbol{\Psi}} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{c(1-c)}{ab} \end{pmatrix}, \quad \Pi_{\boldsymbol{\Psi}_0} = \begin{pmatrix} 1 & 0 \\ 0 & c/a \end{pmatrix}.$$

Solving (3.4.8) and after some straightforward computations we get

$$B(z;\psi_{ij}) = \frac{p_{ij}(z) + q_{ij}(z)\sqrt{(z-\sigma_+)(z-\sigma_-)}}{r_{ij}(z)}, \quad z \in \mathbb{C} \setminus [\sigma_-, \sigma_+],$$

where $p_{ij}(z), q_{ij}(z), r_{ij}(z)$ are polynomials given by

$$\begin{aligned}
p_{11}(z) &= 2(1-a)(1-c)z^3 - 4b(1-a)(1-c)z^2 + \gamma_{11}z - b^2((a-c)^2 - a - c),\\
q_{11}(z) &= b\left[-2(1-a)(1-c)z - a(1-a) - c(1-c)\right],\\
r_{11}(z) &= 2(1-a)(1-c)z^4 - 4b(1-a)(1-c)z^3 + (\gamma_{11} - b^2(2ac-a-c))z^2 + 4ab^2cz - 2ab^2c,\\
p_{12}(z) &= b\left[-(1-a)(1-c)z^3 + b(1-a)(2-3c)z^2 + \gamma_{12}z - bc((1-c)^2 - a(1+c))\right]\\
q_{12}(z) &= b\left[-(1-a)(1-c)z^2 + (1-a)(1+2c^2-a-3c)z + bc(1-c)\right]\\
r_{12}(z) &= cr_{11}(z),\\
p_{22}(z) &= b^2\left[-(1-a)z^3 + (1-a)(b+2(1-c))z^2 + \gamma_{22}z - b(ac-c^2+a+2c-1)\right]\\
q_{22}(z) &= b\left[-(1-a)(1-c+a)z^2 + 2b(1-a)(1-c)z - b^2(1-c)\right]\\
r_{22}(z) &= cr_{11}(z),
\end{aligned} \tag{3.4.9}$$

where

$$\begin{aligned}
\gamma_{11} &= 2(1-a)^3 + 2(1-c)^3 - 2 + 2ac(2+a+c-4ac) + b^2(2ac-a-c),\\
\gamma_{12} &= a^3 + a^2(2c^2+2c-3) + a(1-c)(2c^2-4c+3) - (1-c)^2(1-3c),\\
\gamma_{22} &= -2a^2(1+c) + a(2c^2-5c+5) - 3(1-c)^2.
\end{aligned}$$

Therefore the Stieltjes transform $B(z;\mathbf{\Psi})$ can be written as

$$B(z;\mathbf{\Psi}) = \frac{\sqrt{(z-\sigma_+)(z-\sigma_-)}}{cr_{11}(z)}\begin{pmatrix} cq_{11}(z) & q_{12}(z) \\ q_{12}(z) & q_{22}(z)\end{pmatrix} + \frac{1}{cr_{11}(z)}\begin{pmatrix} cp_{11}(z) & p_{12}(z) \\ p_{12}(z) & p_{22}(z)\end{pmatrix}.$$

We can see that $r_{11}(z)$ is a polynomial of degree 4, so we know that the Stieltjes transform may have at most 4 real poles. Nevertheless, we have not been able to compute an explicit expression of these zeros. Instead we will assume that $c = a$ and get a significant simplification.

If $c = a$, then $b = 1 - 2a$ and the polynomial $r_{11}(z)$ has now a simpler expression:

$$r_{11}(z) = 2(1-z)(z(1-a)+a)[(a-1)z^2 + b^2z - ab^2].$$

The zeros of $r_{11}(z)$ are given by

$$1, \quad -\frac{a}{1-a}, \quad \frac{b(b \pm \sqrt{2b^2-1})}{1+b}. \tag{3.4.10}$$

If $\sqrt{2}/2 < b < 1$, i.e., $0 < a < (2-\sqrt{2})/4$, there can be at most 4 different real zeros. This means that the spectral matrix will consist of an absolutely continuous density plus possibly some Dirac delta masses located at these zeros with certain weights. Let us write the spectral matrix as $\mathbf{\Psi}(x) = \mathbf{\Psi}_c(x) + \mathbf{\Psi}_d(x)$. Using the Stieltjes-Perron inversion formula (see Proposition 1.1.4) the continuous part of the spectral matrix is given by

$$\mathbf{\Psi}_c(x) = \frac{\sqrt{(\sigma_+ - x)(x-\sigma_-)}}{c\pi r_{11}(x)}\begin{pmatrix} cq_{11}(x) & q_{12}(x) \\ q_{12}(x) & q_{22}(x)\end{pmatrix}, \quad x \in [\sigma_-, \sigma_+] = [1-4a, 1], \tag{3.4.11}$$

where

$$
\begin{aligned}
q_{11}(x) &= -2(1-2a)(1-a)(x(1-a)+a),\\
q_{12}(x) &= -(1-2a)(1-a)(x(1-a)+a)(x-1+2a),\\
q_{22}(x) &= -(1-2a)(1-a)(x^2-2(1-a)(1-2a)x+(1-2a)^2).
\end{aligned}
$$

The discrete masses come from the residues at the simple poles of $B(z;\boldsymbol{\Psi})$, given by (3.4.10). After some computations it is possible to see that all discrete masses are identically $\mathbf{0}_2$. Therefore the spectral matrix only have an absolutely continuous part given by (3.4.11).

Now that we have an explicit expression for the spectral matrix, let us consider the discrete Darboux transformation. In block matrix form, the Darboux transformation is given by $\widehat{\boldsymbol{P}}$ which equivalent matrix $\widehat{P}$ is now a discrete-time birth-death chain on $\mathbb{Z}$. Using (3.4.2) together with the following continued fraction

$$
J = \frac{c\rfloor}{\lceil 1} - \frac{a\rfloor}{\lceil 1} - \frac{c\rfloor}{\lceil 1} - \frac{a\rfloor}{\lceil 1} - \cdots
$$

and the corresponding convergents

$$
j_n = \frac{\alpha_n}{\beta_n}, \quad n \geq 0,
$$

it is possible to find an expression for the entries of $\widehat{P}$. The sequences $(\alpha_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 0}$ of the convergents $(j_n)_{n\geq 0}$ can be computed recursively using the following relations:

$$
\begin{aligned}
&\alpha_{2n} = \alpha_{2n-1} - a\alpha_{2n-2}, \quad n \geq 1, \quad \alpha_{2n+1} = \alpha_{2n} - c\alpha_{2n-1}, \quad n \geq 0, \quad \alpha_{-1} = 1, \quad \alpha_0 = 1,\\
&\beta_{2n} = \beta_{2n-1} - a\beta_{2n-2}, \quad n \geq 1, \quad \beta_{2n+1} = \beta_{2n} - c\beta_{2n-1}, \quad n \geq 0, \quad \beta_{-1} = 0, \quad \beta_0 = 1.
\end{aligned}
$$

The first few convergents are given by

$$
j_0 = 0, \quad j_1 = c, \quad j_2 = \frac{c}{1-a}, \quad j_3 = \frac{c(1-c)}{b}, \quad j_4 = \frac{cb}{b-a(1-a)}, \quad j_5 = \frac{c(b-c(1-c))}{b(1-c)-a(1-a)}, \quad \ldots.
$$

First note that using (3.3.4) and (3.3.6) shows that the coefficients $\tilde{x}_n, \tilde{y}_n, \tilde{r}_n, \tilde{s}_n, n \in \mathbb{Z}$, can be written in terms of the convergents j_n as follows

$$
\begin{aligned}
&\tilde{\alpha} = c, \quad \tilde{x}_n = \frac{a}{1-j_{2n}}, \; n \geq 0, \quad \tilde{x}_{-n} = j_{2n+1}, \quad n \geq 1,\\
&\tilde{y}_0 = b, \quad \tilde{y}_n = 1 - \tilde{x}_n, \quad \tilde{y}_{-n} = 1 - j_{2n+1}, \; n \geq 1,\\
&\tilde{r}_n = j_{2n}, \; n \geq 0, \quad \tilde{r}_{-n} = \frac{a}{1-j_{2n-1}}, \; n \geq 1,\\
&\tilde{s}_n = 1 - j_{2n}, \; n \geq 0, \quad \tilde{s}_{-n} = 1 - \tilde{r}_{-n}, \; n \geq 1.
\end{aligned}
\tag{3.4.12}
$$

Therefore, the coefficients $\hat{a}_n, \hat{b}_n, \hat{c}_n, n \in \mathbb{Z}$, of the birth-death chain $\widehat{P}$ are given by

$$
\begin{aligned}
\hat{a}_n &= \frac{a(1-j_{2n+2})}{1-j_{2n}}, \quad n \geq 0, \quad \hat{a}_{-n} = \frac{a(1-j_{2n+1})}{1-j_{2n-1}}, \quad n \geq 1,\\
\hat{c}_n &= \frac{j_{2n}(1-a-j_{2n})}{1-j_{2n}}, \quad n \geq 0, \quad \hat{c}_{-n} = \frac{j_{2n+1}(1-a-j_{2n+1})}{1-j_{2n+3}}, \quad n \geq 1,\\
\hat{b}_n &= 1 - \hat{a}_n - \hat{c}_n, \quad n \in \mathbb{Z}.
\end{aligned}
$$

To finish the example, it is possible to compute the spectral matrix $\widehat{\mathbf{\Psi}}(x)$ of the birth-death chain $\widehat{P}$ using Theorem 3.2.1. Indeed, the spectral matrix is given by

$$\widehat{\mathbf{\Psi}}(x) = x\widetilde{S}_0^{-1}\mathbf{\Psi}(x)\widetilde{S}_0^{-T},$$

where $\mathbf{\Psi}(x)$ is given by (3.4.11) and

$$\widetilde{S}_0^{-1} = \begin{pmatrix} 1 & 0 \\ -a/b & (1-c)/b \end{pmatrix}.$$

After some computations we get

$$\widehat{\mathbf{\Psi}}(x) = \frac{x\sqrt{(\sigma_+ - x)(x-\sigma_-)}}{c\pi r_{11}(x)} \begin{pmatrix} cq_{11}(x) & -\frac{ac}{b}q_{11}(x) + \frac{1-c}{b}q_{12}(x) \\ -\frac{ac}{b}q_{11}(x) + \frac{1-c}{b}q_{12}(x) & \frac{a^2c}{b^2}q_{11}(x) - \frac{2a(1-c)}{b^2}q_{12}(x) + \frac{(1-c)^2}{b^2}q_{22}(x) \end{pmatrix},$$

where $q_{ij}(x)$ are given by (3.4.9).

Finally, it is possible to see that the original random walk is always transient, except for $a = c$, where it is null recurrent. Therefore, as we mentioned before, the transformed process will have the same recurrent behavior. The invariant measure of the original process is given by (3.2.19). Using (3.4) and (3.4.12), we have that the invariant measure of the transformed process is given by

$$\widehat{\boldsymbol{\pi}} = \left(\frac{1}{b}, \frac{c}{a-c(1-c)}; \frac{ab^2}{c(1-a)((1-a)^2-c)}, \frac{c^2((1-a)^2+(1-c)^2-1+ac)^2}{a^2((1-c)^2+a)((1-c)^3+(1-a)^2-1+2ac)}; \cdots\right).$$

The rest of the coefficients can be computed using the convergents $(j_n)_{n\geq 0}$.

CHAPTER 4

An example of a nontrivial birth-death chain on $\mathbb{Z}$: the associated Jacobi polynomials

In previous chapters, the main object of study was the one-step transition probability matrix P given in (1.4.1) which is a doubly infinite tridiagonal stochastic matrix. In each chapter, we considered a different type of stochastic factorization, some of them subject to free parameters. Then we gave conditions in terms of certain continued fractions such that the stochastic factorization is always possible. Afterwards, performing a Darboux factorization, we got new families of birth-death chains on the integers. We identified the spectral matrices associated with these Darboux transformations which, in general, are conjugations of a Geronimus or a Christoffel transformation of the original spectral matrix. At the end of the analysis, in each chapter, we presented an example in which it is possible to perform the stochastic factorization and apply our results. However, as we already emphasized, the methods for obtaining the spectral matrices are not so simple, even for the simplest variations of random walks on $\mathbb{Z}$. We also used as an example a version of the birth-death chain with constant transition probabilities but adding transitions between states 1 and -1 in which case we managed to compute the spectral matrix using methods from the theory of matrix-valued orthogonal polynomials.

The main motivation of this chapter is to study a nontrivial example of birth-death chain on $\mathbb{Z}$, given by the so-called *associated Jacobi polynomials*. The transition probabilities will be given by rational functions depending on the state of the process instead of constants. In Section 1.6 we studied the example of the birth-death chain corresponding with the classical family of Jacobi polynomials. For our purposes, we considered the family restricted to the interval $[0, 1]$ and the coefficients a_n, b_n and c_n in such a way that the Jacobi matrix is stochastic. In Section 4.1 we will provide a general context about the families of *associated polynomials*, with a specific focus on the family of associated Jacobi polynomials. We will explore some of its key characteristics, such as the Jacobi matrix and the spectral matrix. Moving on to Section 4.2, we will derive conditions under which the Jacobi matrix associated to this polynomial family is a stochastic matrix. This will allow us to interpret this matrix as the one-step transition probability matrix of a non-trivial birth-death chain on $\mathbb{Z}$, enabling us to study the UL and LU stochastic factorizations by the end of this section. Continuing our investigation, we will explore Darboux transformations in Section 4.3, and finally, we will conclude with the study of an urn model corresponding to the family of associated Jacobi polynomials in Section 4.4.

4.1 Associated Jacobi polynomials

The associated orthogonal polynomials have been widely studied before. As far as we know, the concept of associated polynomials was considered in [2] and then in a paper by D. Askey and J. Wimp [1], where the spectral measures for associated Laguerre and Hermite polynomials were computed. The case of associated Jacobi polynomials was considered later by many authors as in [52, 37]. These references are a great precedent in our subject since we are interested in spectral measures and, as we know, there are only a handful of cases where the spectral matrix has been explicitly computed.

The families of *associated polynomials* are generated by considering polynomials satisfying the three-term recurrence relation for the classical families of Hermite, Laguerre, Jacobi or Bessel polynomials, but replacing n by $n+t$, where t is an arbitrary real parameter. F.A. Grünbaum and L. Haine in [22], extended this idea to the whole set of integers $n \in \mathbb{Z}$, in which case we have a doubly infinite tridiagonal matrix. The authors gave a solution of the *bispectral problem*, also called, Bochner's problem, for the doubly infinite tridiagonal matrices corresponding to the associated Hermite, Laguerre, Jacobi and Bessel polynomials, and the corresponding differential operator has order two. Although a complete classification of the bispectral problem for this extension of the Bochner's problem was given in [22], very little is known about the spectral matrices for these associated polynomials. However, in [23, Theorem 1], F.A. Grünbaum and L. Haine managed to compute the explicit expression of the spectral matrix for the associated Jacobi polynomials supported on $[0,1]$ for some special choice of the parameters involved. The main purpose of this section is to present the associated Jacobi polynomials on $\mathbb{Z}$ and the expression for the corresponding spectral matrix.

First, for every $n \in \mathbb{Z}$ let

$$\begin{aligned} d_n &= \frac{(n+t)(n+t+\alpha)(n+t+\beta)(n+t+\alpha+\beta)}{(2n+2t+\alpha+\beta-1)(2n+2t+\alpha+\beta)^2(2n+2t+\alpha+\beta+1)}, \\ e_n &= \frac{1}{2}\left(1+\frac{\alpha^2-\beta^2}{(2n+2t+\alpha+\beta-2)(2n+2t+\alpha+\beta)}\right), \end{aligned} \tag{4.1.1}$$

with $d_n > 0$ for all $n \in \mathbb{Z}$. Let us define J as the doubly infinite matrix given by

$$J = \left(\begin{array}{cccc|cccc} \ddots & \ddots & \ddots & & & & & \\ & \sqrt{d_{-2}} & e_{-1} & \sqrt{d_{-1}} & & & & \\ & & \sqrt{d_{-1}} & e_0 & \sqrt{d_0} & & & \\ \hline & & & \sqrt{d_0} & e_1 & \sqrt{d_1} & & \\ & & & & \sqrt{d_1} & e_2 & \sqrt{d_2} & \\ & & & & & \ddots & \ddots & \ddots \end{array}\right), \tag{4.1.2}$$

and we consider the eigenvalue equation $xP(x) = JP(x)$, with the vector of polynomials $P(x) = (\cdots, P_{-1}(x), P_0(x), P_1(x), \cdots)^T$. In fact, we have two linearly independent solutions depending on the initial conditions. These solutions define two families of linearly independent polynomials $(P_n^\eta)_{n\in\mathbb{Z}}, \eta =$

1, 2, satisfying the following three-term recurrence relation

$$
\begin{aligned}
&P_0^1(x) = 1, \quad P_0^2(x) = 0, \\
&P_{-1}^1(x) = 0, \quad P_{-1}^2(x) = 1, \\
&xP_n^\eta(x) = \sqrt{d_{n+1}}P_{n+1}^\eta(x) + e_{n+1}P_n^\eta(x) + \sqrt{d_n}P_{n-1}^\eta(x), \quad n \in \mathbb{Z}, \quad \eta = 1, 2.
\end{aligned}
$$

Now, the coefficients of the three-term recurrence relation for the classical Jacobi orthonormal polynomials on the interval $[0, 1]$ are given by

$$
\begin{aligned}
\tilde{a}_n &= \frac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \\
\tilde{b}_n &= \frac{1}{2}\left(\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta-2)(2n+\alpha+\beta)} + 1\right),
\end{aligned}
$$

for $n \geq 1$. Therefore there is a simple equivalence with coefficients in equation (4.1.1) given by

$$
\begin{aligned}
d_n &= \tilde{a}_{n+t}, \\
e_n &= \tilde{b}_{n+t},
\end{aligned}
$$

for $n \in \mathbb{Z}$. Consequently we call $(P_n^\eta)_{n\in\mathbb{Z}}$ the *associated Jacobi polynomials* and J is the *associated Jacobi matrix*. In [23], the authors specify that if we assume that α and β are inside the square $-1 < \alpha, \beta < 1$, then it is possible to find some t such that $d_n > 0$, $n \in \mathbb{Z}$. Outside of this square it is possible to see that there is no value of t that makes all d_n positive.

The polynomials $(P_n^\eta)_{n\in\mathbb{Z}}, \eta = 1, 2$ are orthonormal in the classical sense with respect to certain spectral matrix $\boldsymbol{W}$ supported on the interval $[0, 1]$. Additionally, if the coefficients d_n are all positive, the spectral matrix has only an absolutely continuous part given by the following expression

$$
\boldsymbol{W}(x) = x^\alpha(1-x)^\beta \begin{pmatrix} \Sigma_{11}(x) & \Sigma_{12}(x) \\ \Sigma_{12}(x) & \Sigma_{22}(x) \end{pmatrix}, \quad x \in [0, 1], \tag{4.1.3}
$$

where

$$
\begin{aligned}
\Sigma_{11}(x) &= \gamma L\left(G_1^2(x) - \mu K^2 x^{-2\alpha} G_2^2(x)\right), \\
\Sigma_{12}(x) &= -L\left(G_1(x)G_3(x) - \mu\nu K^2 x^{-2\alpha} G_2(x)G_4(x)\right), \\
\Sigma_{22}(x) &= \frac{L}{\gamma}\left(G_3^2(x) - \mu\nu^2 K^2 x^{-2\alpha} G_4^2(x)\right),
\end{aligned} \tag{4.1.4}
$$

and

$$
\begin{aligned}
G_1(x) &= {}_2F_1\left(\alpha+\beta+t+1, -t; \alpha+1; x\right), \quad G_2(x) = {}_2F_1\left(\beta+t+1, -\alpha-t; 1-\alpha; x\right), \\
G_3(x) &= {}_2F_1\left(\alpha+\beta+t, 1-t; \alpha+1; x\right), \quad G_4(x) = {}_2F_1\left(\beta+t, 1-t-\alpha; 1-\alpha; x\right), \\
\mu &= \frac{\sin(\pi t)\sin(\pi(\beta+t))}{\sin(\pi(\alpha+\beta+t))\sin(\pi(\alpha+t))}, \quad \nu = \frac{(\alpha+t)(\alpha+\beta+t)}{t(\beta+t)}, \\
K &= -\frac{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(t+1)\Gamma(-\alpha-\beta-t)\sin(\pi\alpha)\sin(\pi(\alpha+\beta+t))}{\pi\Gamma(\alpha+t+1)\Gamma(-\beta-t)\sin(\pi(\beta+t))}, \\
L &= \frac{t(\beta+t)\sin(\pi\alpha)}{\pi\sqrt{d_0}(\alpha+\beta+2t)\alpha(\mu-1)K}, \quad \gamma = \frac{(\alpha+t)(\alpha+\beta+t)}{\sqrt{d_0}(\alpha+\beta+2t-1)(\alpha+\beta+2t)},
\end{aligned}
$$

where ${}_2F_1$ denotes the standard Gauss hypergeometric function.

Now, with the relabeling given in (1.5.9), we transform the doubly infinite Jacobi matrix given in equation (4.1.2) into the following semi-infinite 2×2 block Jacobi matrix

$$\boldsymbol{J} = \begin{pmatrix} E_1 & D_1 & & & \\ D_1 & E_2 & D_2 & & \\ & D_2 & E_3 & D_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$E_1 = \begin{pmatrix} e_1 & \sqrt{d_0} \\ \sqrt{d_0} & e_0 \end{pmatrix}, \quad E_{n+1} = \begin{pmatrix} e_{n+1} & 0 \\ 0 & e_{-n} \end{pmatrix}, \quad n \geq 1, \quad D_{n+1} = \begin{pmatrix} \sqrt{d_{n+1}} & 0 \\ 0 & \sqrt{d_{-n-1}} \end{pmatrix}, \quad n \geq 0,$$

and we define the matrix-valued polynomials

$$\boldsymbol{P}_n(x) = \begin{pmatrix} P_n^1(x) & P_n^2(x) \\ P_{-n-1}^1(x) & P_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0. \tag{4.1.5}$$

These matrix-valued polynomials satisfy the following three-term recurrence relation

$$\begin{aligned} x\boldsymbol{P}_0(x) &= D_1\boldsymbol{P}_1(x) + E_1\boldsymbol{P}_0(x), \quad \boldsymbol{P}_0(x) = \boldsymbol{I}_2, \\ x\boldsymbol{P}_n(x) &= D_{n+1}\boldsymbol{P}_{n+1}(x) + E_{n+1}\boldsymbol{P}_n(x) + D_n\boldsymbol{P}_{n-1}(x), \quad n \geq 1, \end{aligned}$$

where $\boldsymbol{I}_2$ denotes the 2×2 identity matrix. Therefore the orthonormality is defined in terms of the following entry by entry integral

$$\int_0^1 \boldsymbol{P}_n(x)\boldsymbol{W}(x)\boldsymbol{P}_m^T(x)dx = \boldsymbol{I}_2\delta_{nm},$$

where δ_{nm} is the Kronecker delta.

All these results were developed in [23] with the difference that the authors used parameters a, b, c, according to the standard notation of the Gauss hypergeometric equation. For this chapter we decided to use a notation more related with the classical Jacobi polynomials. Our notation and the one used in [23] are related by

$$\begin{aligned} a &= \alpha + \beta + t + 1, \\ b &= -t, \\ c &= \alpha + 1. \end{aligned}$$

Remark 4.1.1. Observe that in [23] the authors use a different representation of the matrix-valued orthonormal polynomials $(\boldsymbol{P}_n(x))_{n\geq 0}$ given in (4.1.5). It is easy to see that both notations are connected by a conjugation of the σ_1 Pauli matrix (of size 2×2). Thus, the spectral matrix given in (4.1.3) has been modified according to our notation.

Remark 4.1.2. In [23] the authors gave a complete solution of the discrete-continuous version of the following bispectral problem: Describe all families of functions $f_n(z)$, $n \in \mathbb{Z}$, $z \in \mathbb{C}$, that satisfy

$$(Jf)_n(z) = zf_n(z), \text{ and } Ef_n(z) = \lambda_n f_n(z), \text{ for all } z \text{ and } n,$$

where J is a doubly infinite matrix like in (4.1.2) and E is a second-order differential operator with coefficients independent of n. In the case of associated Jacobi polynomials the functions $f_n(z)$ can be given in terms of an arbitrary solution of Gauss' hypergeometric equation (see (2.1)–(2.6) of [23]). The corresponding second order differential operator E and eigenvalue λ_n are given by

$$E = z(1-z)\frac{d^2}{dz^2} + (\alpha+1+(\alpha+\beta+2)z)\frac{d}{dz}, \quad \lambda_n = -(n+t)(n+t+\alpha+\beta+1).$$

4.2 The stochastic associated Jacobi matrix

In this section we consider a normalization of the associated Jacobi polynomials in such a way that we get a stochastic Jacobi matrix. For all $n \in \mathbb{Z}$, let us define

$$\begin{aligned} a_n &= \frac{(n+t+\beta+1)(n+t+\alpha+\beta+1)}{(2n+2t+\alpha+\beta+1)(2n+2t+\alpha+\beta+2)}, \\ b_n &= \frac{(n+t+\beta+1)(n+t+1)}{(2n+2t+\alpha+\beta+1)(2n+2t+\alpha+\beta+2)} + \frac{(n+t+\alpha)(n+t+\alpha+\beta)}{(2n+2t+\alpha+\beta)(2n+2t+\alpha+\beta+1)}, \\ c_n &= \frac{(n+t)(n+t+\alpha)}{(2n+2t+\alpha+\beta)(2n+2t+\alpha+\beta+1)}. \end{aligned} \tag{4.2.1}$$

Observe that

$$a_n + b_n + c_n = 1, \quad n \in \mathbb{Z}.$$

Again, there is a relation between these coefficients and the classical family of Jacobi polynomials. If we denote by $\tilde{a}_n, \tilde{b}_n, \tilde{c}_n$ the coefficients of the three-term recurrence relation for the classical Jacobi polynomials on $[0,1]$ such that the corresponding Jacobi matrix is stochastic given by (1.6.4), then we have

$$\begin{aligned} a_n &= \tilde{a}_{n+t}, \\ b_n &= \tilde{b}_{n+t}, \\ c_n &= \tilde{c}_{n+t}, \end{aligned}$$

for $n \in \mathbb{Z}$. The corresponding doubly infinite Jacobi matrix associated with the coefficients (4.2.1) can be written as P in equation (1.4.1). If we compute explicitly the sequence of potential coefficients given by Definition 1.4.1, we get the following expression

$$\pi_n = \frac{(\alpha+\beta+t+1)_n(-t)_{-n}(2n+2t+\alpha+\beta+1)}{(\alpha+t+1)_n(-\beta-t)_{-n}(2t+\alpha+\beta+1)}, \quad n \in \mathbb{Z}, \tag{4.2.2}$$

where $(a)_n$ denotes the Pochhammer symbol, i.e.,

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), n \geq 1,$$

and

$$(a)_{-n} = \frac{1}{(a-n)_n}, \quad n \geq 1.$$

Now, let us define

$$\Pi = \left(\begin{array}{ccc|ccc} \ddots & \ddots & \ddots & & & \\ & & \sqrt{\pi_{-1}} & & & \\ \hline & & & \sqrt{\pi_0} & & \\ & & & & \sqrt{\pi_1} & \\ & & & \ddots & \ddots & \ddots \end{array}\right) = \operatorname{diag}\left(\cdots, \sqrt{\pi_{-1}}, \sqrt{\pi_0}, \sqrt{\pi_1}, \cdots\right).$$

Then, the relation between the Jacobi matrices J and P given by (4.1.2) and (1.4.1), respectively, is

$$J\Pi = \Pi P.$$

Under the conditions described above, we know that the sum of all rows of the matrix P is always equal to one. To ensure that P is stochastic it is also necessary that all entries are positive. With this in mind, let us state the following proposition. Recall from the previous section that α and β are inside the square $-1 < \alpha, \beta < 1$.

Proposition 4.2.1. *If $0 \leq \beta < 1$ then it is not possible to find t such that $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$. Therefore assume that $-1 < \alpha < 1$ and $-1 < \beta < 0$. Then the coefficients a_n, b_n, c_n defined by (4.2.1) are all positive if we choose t according to one of the following 8 regions (see Figure 4.1):*

$\boldsymbol{A}_1 = \{\beta - \alpha + 1 > 0, \alpha > -\beta, \beta < 0\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n, n-\beta) \cup (n-\alpha, n-\alpha-\beta). \tag{4.2.3}$$

$\boldsymbol{A}_2 = \{\beta - \alpha + 1 > 0, \alpha < -\beta, \alpha > 0\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n-\alpha, n) \cup (n-\alpha-\beta, n-\beta).$$

$\boldsymbol{B}_1 = \{\beta - \alpha + 1 < 0, \alpha > -\beta, \alpha < 1\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n, n-\alpha+1) \cup (n-\beta, n-\alpha-\beta+1).$$

$\boldsymbol{B}_2 = \{\beta - \alpha + 1 < 0, \alpha < -\beta, \beta > -1\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n-\beta-1, n) \cup (n-\alpha-\beta, n-\alpha+1).$$

$\boldsymbol{C}_1 = \{\beta + \alpha + 1 > 0, \alpha > \beta, \alpha < 0\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n, n-\alpha) \cup (n-\beta, n-\alpha-\beta).$$

$\boldsymbol{C}_2 = \{\beta + \alpha + 1 > 0, \alpha < \beta, \beta < 0\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n, n-\beta) \cup (n-\alpha, n-\alpha-\beta).$$

$\boldsymbol{D}_1$= $\{\beta+\alpha+1<0, \alpha>\beta, \beta>-1\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n-\beta-1, n) \cup (n-\alpha-\beta, n-\alpha+1).$$

$\boldsymbol{D}_2$= $\{\beta+\alpha+1<0, \alpha<\beta, \alpha>-1\}$. *Then t must be chosen in the following real set:*

$$t \in \bigcup_{n\in\mathbb{Z}} (n-\alpha-1, n) \cup (n-\alpha-\beta, n-\beta+1).$$

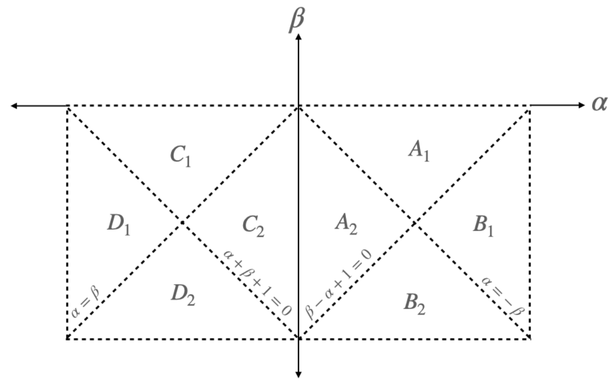

Figure 4.1: Representation of the regions in Proposition 4.2.1.

Proof. We will prove first that in region $0 \leq \beta < 1$ there is no t such that all coefficients $a_n, b_n, c_n > 0$. Let us write the coefficients given in (4.2.1) as follows

$$a_n = x_n s_{n+1}, \quad b_n = x_n r_{n+1} + y_n s_n, \quad c_n = r_n y_n, \quad n \in \mathbb{Z}, \tag{4.2.4}$$

where

$$x_n = \frac{n+t+\beta+1}{2n+2t+\alpha+\beta+1},\ y_n = \frac{n+t+\alpha}{2n+2t+\alpha+\beta+1},\ s_n = \frac{n+t+\alpha+\beta}{2n+2t+\alpha+\beta},\ r_n = \frac{n+t}{2n+2t+\alpha+\beta}. \tag{4.2.5}$$

This representation is inspired by the coefficients of the UL factorization of the doubly infinite matrix P. We will prove the first part of the proposition in the region $R = \{\alpha > \beta \geq 0, \beta < 1-\alpha\}$. The rest of cases are similar. In the region R, if we take $n+t$ as a variable, the values of the zeros of each factor in (4.2.5) are located as follows

$$-2 < -1-\alpha-\beta < -1-\frac{\alpha+\beta}{2} < -1-\beta < -1 < -\frac{\alpha+\beta+1}{2} < -\alpha-\beta < -\alpha < -\frac{\alpha+\beta}{2} < 0. \tag{4.2.6}$$

For the proof, it is enough to see what happens in the interval $[-1,0]$, since $n+t$ can be moved to any other interval of this size for some integer n. Now, we are going to analyze the sign of a_n, b_n, c_n in (4.2.4) through the sign of x_n, y_n, s_n, r_n in (4.2.5). According to the position of $n+t$, we have the following possibilities:

1. If $-1 < n+t < -\frac{\alpha+\beta+1}{2}$, then we have

$$x_n = \frac{\text{Sign}(+)}{\text{Sign}(-)} \quad \text{and} \quad s_{n+1} = \frac{\text{Sign}(+)}{\text{Sign}(+)}.$$

   Therefore $a_n = x_n s_{n+1} < 0$.

2. If $-\frac{\alpha+\beta+1}{2} < n+t < -\alpha$, then we have

$$r_n = \frac{\text{Sign}(-)}{\text{Sign}(-)} \quad \text{and} \quad y_n = \frac{\text{Sign}(-)}{\text{Sign}(+)}.$$

   Therefore $c_n = r_n y_n < 0$.

3. If $-\alpha < n+t < -\frac{\alpha+\beta}{2}$, then we have

$$y_n = \frac{\text{Sign}(+)}{\text{Sign}(+)} \quad \text{and} \quad s_n = \frac{\text{Sign}(+)}{\text{Sign}(-)}.$$

   Therefore $y_n s_n < 0$ and the second summand in b_n of (4.2.4) is negative.

4. If $-\frac{\alpha+\beta}{2} < n+t < 0$, then we have

$$r_n = \frac{\text{Sign}(-)}{\text{Sign}(+)} \quad \text{and} \quad y_n = \frac{\text{Sign}(+)}{\text{Sign}(+)}.$$

   Therefore $c_n = r_n y_n < 0$.

In all the above cases it is not possible to find t such that $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$. Finally there are another two possibilities:

5. If $n+t > 0$ then all x_n, y_n, s_n, r_n are positive and $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$. This means that we have to choose t such that $t > -n$. As $|n| \to \infty$ it will not be possible to find a finite t such that $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$.

6. If $n+t < -1-\alpha-\beta$, then all x_n, y_n, s_n, r_n are negative and $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$. But then we have to choose t such that $t < -1-\alpha-\beta-n$. Again, as $|n| \to \infty$ it will not be possible to find a finite t such that $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$.

The rectangle $-1 < \alpha < 1, 0 \leq \beta < 1$ can be divided in a similar way as in Figure 4.1. The region R is just one of these triangles. The proof for the rest of the regions is similar, only changing the position of the values of the zeros in equation (4.2.6).

For the second part of the proposition we will focus on the region $\boldsymbol{A}_1$. The rest of the cases are similar. As before, we have that the values of the zeros of each of the factors in (4.2.5), for $n+t$ as a variable, are located as follows

$$-2 < -1-\alpha-\beta < -1-\frac{\alpha+\beta}{2} < -1 < -1-\beta < -\frac{\alpha+\beta+1}{2} < -\alpha < -\alpha-\beta < -\frac{\alpha+\beta}{2} < 0, \quad (4.2.7)$$

so it is enough to see what happens in the interval $[-1, 0]$. We have the following possibilities according to the position of $n+t$:

1. If $-1 < n+t < -1-\beta$, then we have
$$x_n = \frac{\text{Sign}(-)}{\text{Sign}(-)}, s_{n+1} = \frac{\text{Sign}(+)}{\text{Sign}(+)}, r_{n+1} = \frac{\text{Sign}(+)}{\text{Sign}(+)}, y_n = \frac{\text{Sign}(-)}{\text{Sign}(-)}, s_n = \frac{\text{Sign}(-)}{\text{Sign}(-)} \text{ and } r_n = \frac{\text{Sign}(-)}{\text{Sign}(-)}.$$
Therefore $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$.

2. If $-1-\beta < n+t < -\frac{\alpha+\beta+1}{2}$, then we have
$$x_n = \frac{\text{Sign}(+)}{\text{Sign}(-)} \text{ and } s_{n+1} = \frac{\text{Sign}(+)}{\text{Sign}(+)}.$$
Therefore $a_n = x_n s_{n+1} < 0$.

3. If $-\frac{\alpha+\beta+1}{2} < n+t < -\alpha$, then we have
$$r_n = \frac{\text{Sign}(-)}{\text{Sign}(-)} \text{ and } y_n = \frac{\text{Sign}(-)}{\text{Sign}(+)}.$$
Therefore $c_n = r_n y_n < 0$.

4. If $-\alpha < n+t < -\alpha-\beta$, then we have
$$x_n = \frac{\text{Sign}(+)}{\text{Sign}(+)}, s_{n+1} = \frac{\text{Sign}(+)}{\text{Sign}(+)}, r_{n+1} = \frac{\text{Sign}(+)}{\text{Sign}(+)}, y_n = \frac{\text{Sign}(+)}{\text{Sign}(+)}, s_n = \frac{\text{Sign}(-)}{\text{Sign}(-)} \text{ and } r_n = \frac{\text{Sign}(-)}{\text{Sign}(-)}.$$
Therefore $a_n, b_n, c_n > 0$ for all $n \in \mathbb{Z}$.

5. If $-\alpha-\beta < n+t < -\frac{\alpha+\beta}{2}$, then we have
$$y_n = \frac{\text{Sign}(-)}{\text{Sign}(-)} \text{ and } s_n = \frac{\text{Sign}(+)}{\text{Sign}(-)}.$$
Therefore $y_n s_n < 0$ and the second summand in b_n of (4.2.4) is negative.

6. If $-\frac{\alpha+\beta}{2} < n+t < 0$, then we have
$$r_n = \frac{\text{Sign}(-)}{\text{Sign}(+)} \text{ and } y_n = \frac{\text{Sign}(+)}{\text{Sign}(+)}.$$
Therefore $c_n = r_n y_n < 0$.

Cases (1) and (4) correspond to what we wanted to prove in (4.2.3) for region $\boldsymbol{A}_1$. The proof for the rest of the regions is similar, only changing the position of the values in equation (4.2.7). □

As a consequence of the previous proposition we can ensure that the Jacobi matrix P in (1.4.1) with coefficients (4.2.1) is stochastic. Therefore it can be interpreted as the one-step transition probability matrix of a nontrivial bilateral birth-death chain $\{Z_t : t = 0, 1, \dots\}$ on $\mathbb{Z}$ depending on three parameters α, β and t.

Let us consider $(Q_n^\alpha(x))_{n\in\mathbb{Z}}, \alpha = 1, 2$, the two sets of linearly independent polynomials associated to P which are generated by the eigenvalue equation. In a similar way to what we have done before, we proceed to perform the relabeling (1.5.9), procedure that allows us to collect all the information of P in a semi-infinite 2×2 block tridiagonal matrix given by (1.5.2) which describes a quasi-birth-and-death process. If we define the matrix-valued polynomials as in (1.5.10),

$$\boldsymbol{Q}_n(x) = \begin{pmatrix} Q_n^1(x) & Q_n^2(x) \\ Q_{-n-1}^1(x) & Q_{-n-1}^2(x) \end{pmatrix}, \quad n \geq 0,$$

the three-term recurrence relation is given by (1.5.3) and the orthogonality relation is given by

$$\int_0^1 \boldsymbol{Q}_n(x)\boldsymbol{\Psi}(x)\boldsymbol{Q}_m^*(x)dx = \begin{pmatrix} 1/\pi_n & 0 \\ 0 & 1/\pi_{-n-1} \end{pmatrix} \delta_{nm},$$

with δ_{nm} the Kronecker delta.

Notice that this relation is equivalent to equation (1.5.11) but with the difference that here the interval is restricted to $[0,1]$. In this case, the spectral measure is given by

$$\boldsymbol{\Psi}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{\pi_{-1}} \end{pmatrix} \boldsymbol{W}(x) \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{\pi_{-1}} \end{pmatrix} = x^{\alpha}(1-x)^{\beta} \begin{pmatrix} \Sigma_{11}(x) & \dfrac{1}{\sqrt{\pi_{-1}}}\Sigma_{12}(x) \\ \dfrac{1}{\sqrt{\pi_{-1}}}\Sigma_{12}(x) & \dfrac{1}{\pi_{-1}}\Sigma_{22}(x) \end{pmatrix}, \quad x \in [0,1], \tag{4.2.8}$$

where Σ_{ij} are given by (4.1.4). The n-step transition probabilities of the bilateral birth-death chain can be computed using the Karlin-McGregor integral representation formula (1.5.13).

Finally, it is possible to see that, under the conditions of Proposition 5.1.1, the functions inside the matrix in the expression of $\boldsymbol{\Psi}(x)$ are bounded at the point $x = 1$. Therefore the divergence of the integral

$$\int_0^1 \frac{\boldsymbol{\Psi}(x)}{1-x}dx,$$

only depends on the part $(1-x)^{\beta}$ from the spectral measure. Besides, in Proposition 4.2.1 we stated that $-1 < \beta < 0$, thus we always have that all entries of the integral are divergent. Therefore the bilateral birth-death chain is always recurrent. To complete the description, since the spectral measure has only an absolutely continuous part, there are no jumps at the point 1. Then the birth-death chain is always null recurrent.

Now we turn our attention to the study of UL and LU stochastic factorizations described in Chapter 2. We divide this analysis into two brief subsections below.

Remark 4.2.2. The matrix-valued orthogonal polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ are bispectral. Not only they are eigenfuntions of the block tridiagonal Jacobi operator $\boldsymbol{P}$ like in (1.5.2), but they are also eigenfunctions of the following matrix-valued second-order differential operator:

$$\boldsymbol{B} = x(1-x)\frac{d^2}{dx^2} + (C - xU)\frac{d}{dx}, \tag{4.2.9}$$

that is, $\boldsymbol{Q}_n(x)\boldsymbol{B} = \Lambda_n \boldsymbol{Q}_n(x)$, where

$$C = \begin{pmatrix} \alpha + 1 + \frac{2t(\beta+t)}{\alpha+\beta+2t} & 2t - \frac{2t(\beta+t)}{\alpha+\beta+2t} \\ -2(\beta+t) + \frac{2t(\alpha+t)}{\alpha+\beta+2t} & 1 - \alpha - \frac{2t(\beta+t)}{\alpha+\beta+2t} \end{pmatrix}, \quad U = \begin{pmatrix} \alpha+\beta+2t+2 & 0 \\ 0 & -\alpha-\beta-2t+2 \end{pmatrix},$$
$$V = \begin{pmatrix} -\alpha-\beta-2t & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_n = \begin{pmatrix} -(n+1)(n+\alpha+\beta+2t) & 0 \\ 0 & -n(n-\alpha-\beta-2t+1) \end{pmatrix}, \quad n \geq 0. \tag{4.2.10}$$

Observe that the coefficients of $\boldsymbol{B}$, independent of n, are multiplied on the right while the eigenvalue Λ_n is multiplied on the left. This is consistent with the theory of matrix-valued orthogonal polynomials satisfying second-order differential equations initiated by A. J. Durán, F.A. Grünbaum, I. Pacharoni, and J.A. Tirao (see [11, 27]). In terms of the two linearly independent families of polynomials $(Q_n^\eta(x))_{n\in\mathbb{Z}}$, $\eta = 1, 2$, we have two coupled second-order differential equations of the form

$$\begin{aligned}
x(1-x)(Q_n^\eta(x))'' + \left(1+\epsilon\left(\alpha+\frac{2t(\beta+t)}{\alpha+\beta+2t}\right) - x(2+\epsilon(\alpha+\beta+2t))\right)(Q_n^\eta(x))' \\
+ \left(-\beta(1+\epsilon)+2\epsilon t\left(-1+\frac{\beta+t}{\alpha+\beta+2t}\right)\right)(Q_n^{\eta+\epsilon}(x))' \\
- \frac{1}{2}(1+\epsilon)(\alpha+\beta+2t)Q_n^\eta(x) \\
= \lambda_n^{\pm} Q_n^\eta(x), \quad n\in\mathbb{Z},
\end{aligned}$$

where

$$\epsilon = \begin{cases} 1, & \text{if} \quad \eta = 1 \\ -1, & \text{if} \quad \eta = 2 \end{cases}, \quad \lambda_n^{\pm} = \begin{cases} -(n+1)(n+\alpha+\beta+2t), & \text{if} \quad n\geq 0 \\ -n(n-\alpha-\beta-2t+1), & \text{if} \quad n<0 \end{cases}.$$

As a final comment J. Wimp [52] found a fourth-order differential equation with coefficients depending on n for the family of polynomials $(Q_n^1(x))_{n\geq 0}$.

4.2.1 Stochastic UL factorization

According to Proposition 2.1.2, the UL stochastic factorization is possible if and only if we choose y_0 in the following range

$$H' \leq y_0 \leq H,$$

where H and H' are the continued fractions given by (2.1.4).

Theorem 4.2.3. *Assume that $\alpha > 0$. Then we have that*

$$H = H' = \frac{\alpha+t}{\alpha+\beta+2t+1}. \tag{4.2.11}$$

Therefore there exists only one value of the parameter y_0 ($y_0 = H$) such that we obtain a stochastic UL factorization of the form (2.1.1) *and the coefficients of each of the factors P_U and P_L are given by*

$$\begin{aligned}
y_n &= \frac{n+t+\alpha}{2n+2t+\alpha+\beta+1}, & x_n &= \frac{n+t+\beta+1}{2n+2t+\alpha+\beta+1}, \\
s_n &= \frac{n+t+\alpha+\beta}{2n+2t+\alpha+\beta}, & r_n &= \frac{n+t}{2n+2t+\alpha+\beta},
\end{aligned} \qquad n\in\mathbb{Z}. \tag{4.2.12}$$

Proof. We will follow the same ideas as the proof of [24, Proposition 5.1]. First, for H, we have that the sequence of alternating numbers $a_0, c_1, a_1, c_2, \ldots$ is a chain sequence (see Definition 1.6.5). Let us call $(\alpha_n)_{n\geq 1}$, the sequence of partial numerators. Then $\alpha_n = (1-m_{n-1})m_n$, with parameter sequence given by

$$m_{2n} = \frac{n+t}{2n+2t+\alpha+\beta+1}, \quad m_{2n+1} = \frac{n+t+\beta+1}{2n+2t+\alpha+\beta+2}.$$

According to Theorem 1.6.6, we have that

$$H = m_0 + \frac{1 - m_0}{1 + L}, \quad L = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{m_k}{1 - m_k}.$$

It is possible to show that L is convergent as long as $\alpha > 0$, in which case we have

$$L = \frac{\beta + t + 1}{\alpha}.$$

Using the previous considerations and the value of m_0 we have

$$H = \frac{\alpha + m_0(\beta + t + 1)}{\alpha + \beta + t + 1} = \frac{\alpha(2t + \alpha + \beta + 1) + t(\beta + t + 1)}{(2t + \alpha + \beta + 1)(\alpha + \beta + t + 1)} = \frac{\alpha + t}{2t + \alpha + \beta + 1}.$$

On the other hand, for H', we have again that the sequence $\alpha'_n = (1 - m'_{n-1})m'_n, n \geq 1$ of alternating numbers $c_0, a_{-1}, c_{-1}, a_{-2}, \ldots$ is a chain sequence where

$$m'_{2n} = \frac{-n + t + \alpha + \beta + 1}{-2n + 2t + \alpha + \beta + 1}, \quad m'_{2n+1} = \frac{-n + t + \alpha}{-2n + 2t + \alpha + \beta}.$$

Therefore

$$1 - H' = m'_0 + \frac{1 - m'_0}{1 + L'}, \quad L' = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{m'_k}{1 - m'_k}.$$

It is possible to show that L' is convergent as long as $\alpha > 0$, in which case we have

$$L' = -\frac{\alpha + t}{\alpha}.$$

Therefore

$$H' = \frac{\alpha + t - m_0(\alpha + t)}{t} = \frac{(\alpha + t)(2t + \alpha + \beta + 1) - (\alpha + t)(t + \alpha + \beta + 1)}{t(2t + \alpha + \beta + 1)} = \frac{\alpha + t}{2t + \alpha + \beta + 1}.$$

Finally, a direct computation using (4.2.12) and (2.1.3) gives the coefficients (4.2.1). □

4.2.2 Stochastic LU factorization

Let us now consider the LU stochastic factorization of the matrix P. Now, we have to apply Proposition 2.1.4 which establishes that we need to take the free parameter $\tilde{r}_0$ in following range

$$H' \leq \tilde{r}_0 \leq H,$$

in order to have a stochastic LU factorization, where H and H' are defined by (2.1.4).

As before, there is only one value that satisfies the condition and we have following theorem.

Theorem 4.2.4. *Assume that $\alpha > 0$. Then $H = H'$ is convergent to* (4.2.11) *and there exists only one value of the parameter $\tilde{r}_0$ ($\tilde{r}_0 = H$) such that we obtain a stochastic LU factorization of the form* (2.1.10) *and the coefficients of each of the factors $\widetilde{P}_L$ and $\widetilde{P}_U$ are given by*

$$\tilde{y}_n = r_{n+1}, \quad \tilde{x}_n = s_{n+1}, \quad \tilde{s}_n = x_n, \quad \tilde{r}_n = y_n \quad n \in \mathbb{Z}, \tag{4.2.13}$$

where x_n, y_n, s_n, r_n are defined by (4.2.12).

Proof. Identical to the proof of Theorem 5.2.1 but using (2.1.12). □

Theorems 4.2.3 and 4.2.4 prove that there is a unique UL, or LU, stochastic factorization. Nonetheless, following Proposition 4.2.1, the factorization depends on our choice of t. For the regions $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_1$ and $\boldsymbol{B}_2$ we always have that $0 < H < 1$ and a stochastic UL or LU factorization will always be possible. In contrast, in regions $\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{D}_1, \boldsymbol{D}_2$ it is not possible to give a stochastic UL or LU factorization since $\alpha < 0$ and the convergence of H and/or H' is not guaranteed.

Following our procedure, next we perform the discrete Darboux transformation.

4.3 Stochastic Darboux transformations and the associated spectral matrices

First, for the UL factorization, we have that the Darboux transformation is given by $\widetilde{P} = P_L P_U$. In fact, using (4.2.12), we have that $\tilde{a}_n, \tilde{b}_n$, and $\tilde{c}_n$ are the coefficients a_n, b_n, and c_n in (4.2.1) replacing α by $\alpha - 1$. In other words,

$$\tilde{a}_n = a_n\big|_{\alpha=\alpha-1}, \quad \tilde{b}_n = b_n\big|_{\alpha=\alpha-1}, \quad \tilde{c}_n = c_n\big|_{\alpha=\alpha-1}.$$

Therefore the new discrete-time birth-death chain $\{\widetilde{Z}_t : t = 0, 1, \ldots\}$ on the integers $\mathbb{Z}$ with coefficients $(\tilde{a}_n)_{n\in\mathbb{Z}}$, $(\tilde{b}_n)_{n\in\mathbb{Z}}$ and $(\tilde{c}_n)_{n\in\mathbb{Z}}$ is the same as the original birth-death chain Z_t but replacing the parameter α by $\alpha - 1$. The spectral matrix $\widetilde{\boldsymbol{\Psi}}(x)$ associated with $\widetilde{P}$ is then given by

$$\widetilde{\boldsymbol{\Psi}}(x) = \boldsymbol{\Psi}(x)\big|_{\alpha=\alpha-1},$$

where $\boldsymbol{\Psi}(x)$ is the spectral matrix (4.2.8). Since we are assuming that $\alpha > 0$, we have that $\widetilde{\boldsymbol{\Psi}}(x)$ is well-defined on $[0, 1]$. Also $\widetilde{\boldsymbol{\Psi}}(x)$ does not have a discrete part, just as $\boldsymbol{\Psi}(x)$. We can also derive the expression for $\widetilde{\boldsymbol{\Psi}}(x)$ from Theorem 2.2.3, given in terms of a Geronimus transformation of the spectral matrix $\boldsymbol{\Psi}(x)$. This representation is given by equation (2.2.20) where

$$\begin{aligned}\boldsymbol{S}_0(x) &= \begin{pmatrix} s_0 & r_0 \\ -\dfrac{x_{-1}s_0}{y_{-1}} & \dfrac{x - x_{-1}r_0}{y_{-1}} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{t+\alpha+\beta}{2t+\alpha+\beta} & \dfrac{t}{2t+\alpha+\beta} \\ -\dfrac{(t+\beta)(t+\alpha+\beta)}{(2t+\alpha+\beta)(t+\alpha-1)} & \dfrac{2t+\alpha+\beta-1}{t+\alpha-1}x - \dfrac{t(t+\beta)}{(2t+\alpha+\beta)(t+\alpha-1)} \end{pmatrix},\end{aligned}$$

and

$$\boldsymbol{\Psi}_S(x) = \frac{y_0}{s_0}\frac{\boldsymbol{\Psi}(x)}{x} + \left[\begin{pmatrix} 1/s_0 & 0 \\ 0 & 1/r_0 \end{pmatrix} - \frac{y_0}{s_0}\boldsymbol{M}_{-1}\right]\delta_0(x),$$

where $\delta_0(x)$ is the Dirac delta at $x = 0$ and $\boldsymbol{M}_{-1} = \int_0^1 x^{-1}\boldsymbol{\Psi}(x)dx$. We know that $\widetilde{\boldsymbol{\Psi}}(x)$ does not have discrete spectrum. That means that the matrix in front of $\delta_0(x)$ should be the null matrix. In Proposition 2.2.3 it is assumed that $\boldsymbol{M}_{-1}$ is well-defined entry by entry, but for $\alpha > 0$ it is not. According to Remark 2.2.4, this assumption is too restrictive. However it is enough to assume that $\boldsymbol{S}_0(0)\boldsymbol{M}_{-1}\boldsymbol{S}_0^*(0)$ is well-defined. $\boldsymbol{S}_0(0)$ is a singular matrix and after simplifications it turns out that the integral of $x^{-1}\boldsymbol{S}_0(0)\boldsymbol{\Psi}(x)\boldsymbol{S}_0^*(0)$ over $[0,1]$ is always well-defined. In fact we have, after a simple computation that

$$\frac{y_0}{s_0}\boldsymbol{S}_0(0)\boldsymbol{M}_{-1}\boldsymbol{S}_0^*(0)=\begin{pmatrix}1 & -\dfrac{t+\beta}{t+\alpha-1}\\ -\dfrac{t+\beta}{t+\alpha-1} & \left(\dfrac{t+\beta}{t+\alpha-1}\right)^2\end{pmatrix}=\boldsymbol{S}_0(0)\begin{pmatrix}1/s_0 & 0\\ 0 & 1/r_0\end{pmatrix}\boldsymbol{S}_0^*(0),$$

so we have that the matrix in front of $\delta_0(x)$ in $\widetilde{\boldsymbol{\Psi}}(x)$ is the null matrix. Therefore we can conclude that a second representation of the spectral matrix $\widetilde{\boldsymbol{\Psi}}(x)$ is given by

$$\widetilde{\boldsymbol{\Psi}}(x)=\frac{y_0}{s_0}\boldsymbol{S}_0(x)\frac{\boldsymbol{\Psi}(x)}{x}\boldsymbol{S}_0^*(x).$$

Finally, if we construct the matrix-valued polynomials $(\widetilde{\boldsymbol{Q}}_n)_{n\geq 0}$, associated to P, we have that

$$\int_0^1\widetilde{\boldsymbol{Q}}_n(x)\widetilde{\boldsymbol{\Psi}}(x)\widetilde{\boldsymbol{Q}}_m^*(x)dx=\begin{pmatrix}1/\tilde{\pi}_n & 0\\ 0 & 1/\tilde{\pi}_{-n-1}\end{pmatrix}\delta_{nm},$$

where $(\tilde{\pi}_n)_{n\in\mathbb{Z}}$ are the potential coefficients defined by (4.2.2) replacing α by $\alpha-1$ and δ_{nm} is the Kronecker delta. The bilateral birth-death chain $\{\widetilde{Z}_t : t=0,1,\ldots\}$ associated with $\widetilde{P}$ is always null recurrent as the original one.

On the other hand, for the LU factorization given in equation (2.1.10) of the form $P=\widetilde{P}_L\widetilde{P}_U$, the Darboux transformation is given by $\widehat{P}=\widetilde{P}_U\widetilde{P}_L$. As before, a simple computation, using (4.2.13) and (4.2.12), gives

$$\hat{a}_n=a_{n+1}\big|_{\alpha=\alpha-1},\quad \hat{b}_n=b_{n+1}\big|_{\alpha=\alpha-1},\quad \hat{c}_n=c_{n+1}\big|_{\alpha=\alpha-1}.$$

The previous shifted coefficients give a spectral matrix which is not as easily identifiable as the case of the UL factorization. However we can still apply Theorem 2.2.6 to compute $\widehat{\boldsymbol{\Psi}}(x)$ in terms of a Geronimus transformation of the original spectral matrix $\boldsymbol{\Psi}(x)$. The new spectral matrix is given by

$$\widehat{\boldsymbol{\Psi}}(x)=\boldsymbol{T}_0(x)\boldsymbol{\Psi}_T(x)\boldsymbol{T}_0^*(x),$$

where

$$\begin{aligned}\boldsymbol{T}_0(x)&=\begin{pmatrix}\dfrac{x-s_0y_0}{x_0} & -\dfrac{y_0r_0}{x_0}\\ s_0 & r_0\end{pmatrix}\\ &=\begin{pmatrix}\dfrac{2t+\alpha+\beta+1}{t+\beta+1}x-\dfrac{(t+\alpha)(t+\alpha+\beta)}{(2t+\alpha+\beta)(t+\beta+1)} & -\dfrac{t(t+\alpha)}{(2t+\alpha+\beta)(t+\beta+1)}\\ \dfrac{t+\alpha+\beta}{2t+\alpha+\beta} & \dfrac{t}{2t+\alpha+\beta}\end{pmatrix},\end{aligned}$$

and

$$\boldsymbol{\Psi}_T(x)=\frac{x_0}{r_1}\frac{\boldsymbol{\Psi}(x)}{x}+\left[\frac{x_0}{y_0r_1}\begin{pmatrix}1 & 0\\ 0 & s_0/r_0\end{pmatrix}-\frac{x_0}{r_1}\boldsymbol{M}_{-1}\right]\delta_0(x).$$

The spectrum of $\widehat{P}$ is the same as the spectrum of P, since we are only shifting the coefficients one step forward and replacing α by $\alpha-1$. Therefore we should expect, as in the case of UL factorization, that the matrix in front of $\delta_0(x)$ in $\widehat{\mathbf{\Psi}}(x)$ is the null matrix. Proceeding as before we have

$$\frac{x_0}{r_1}\boldsymbol{T}_0(0)\boldsymbol{M}_{-1}\boldsymbol{T}_0^*(0)=\frac{(2t+\alpha+\beta+2)(t+\alpha+\beta)(t+\alpha)}{(t+1)(2t+\alpha+\beta)(t+\beta+1)}\begin{pmatrix}1 & -\dfrac{t+\beta+1}{t+\alpha}\\ -\dfrac{t+\beta+1}{t+\alpha} & \left(\dfrac{t+\beta+1}{t+\alpha}\right)^2\end{pmatrix},$$

which is the same matrix as

$$\frac{x_0}{y_0t_1}\boldsymbol{T}_0(0)\begin{pmatrix}1 & 0\\ 0 & s_0/t_0\end{pmatrix}\boldsymbol{T}_0^*(0).$$

Therefore the matrix in front of $\delta_0(x)$ in $\widehat{\mathbf{\Psi}}(x)$ is the null matrix. As a consequence we have that the spectral matrix $\widehat{\mathbf{\Psi}}(x)$ is given by

$$\widehat{\mathbf{\Psi}}(x)=\frac{x_0}{r_1}\boldsymbol{T}_0(x)\frac{\mathbf{\Psi}(x)}{x}\boldsymbol{T}_0^*(x).$$

Finally, if we construct the matrix-valued polynomials associated to $\widehat{P}$, we have that

$$\int_0^1\widehat{\boldsymbol{Q}}_n(x)\widehat{\mathbf{\Psi}}(x)\widehat{\boldsymbol{Q}}_m^*(x)dx=\begin{pmatrix}1/\hat{\pi}_n & 0\\ 0 & 1/\hat{\pi}_{-n-1}\end{pmatrix}\delta_{nm},$$

where now

$$\hat{\pi}_n=\frac{(\alpha+\beta+t+1)_n(-t-1)_{-n}(2n+2t+\alpha+\beta+2)}{(\alpha+t+1)_n(-\beta-t-1)_{-n}(2t+\alpha+\beta+2)},\quad n\in\mathbb{Z}.$$

The birth-death chain $\{\widehat{Z}_t : t=0,1,\ldots\}$ associated with $\widehat{P}$ will have a similar Karlin-McGregor representation formula as in (1.5.13) and again it is always null recurrent.

As a final remark for this section, we would like to explain the reason why we do not consider an RA stochastic factorization. Recall that in that case there will be 2 free parameters, namely α and x_0. As we stated in equation (3.1.6), to guarantee a stochastic RA factorization we need $\alpha\geq H'$ and $x_0\geq 1-H$. But since in this case $\alpha+x_0=1$, then $y_0=1$, so the RA stochastic factorization will not be possible.

Remark 4.3.1. The families of matrix-valued orthogonal polynomials $(\widetilde{\boldsymbol{Q}}_n)_{n\geq 0}$ and $(\widehat{\boldsymbol{Q}}_n)_{n\geq 0}$ constructed from the UL and LU Darboux transformations are also eigenfunctions of a matrix-valued second-order differential operator of the form (4.2.9). The coefficients and eigenvalues of the differential operator for the first family $(\widetilde{\boldsymbol{Q}}_n)_{n\geq 0}$ are given by (4.2.10) replacing α by $\alpha-1$. On the other hand, the coefficients and eigenvalues of the differential operator for the second family $(\widehat{\boldsymbol{Q}}_n)_{n\geq 0}$ are given by (4.2.10) replacing α by $\alpha-1$ and t by $t+1$. In particular both families are *bispectral*. In the scalar case, and choosing special values of the parameters involved, the order of the differential operator after a Darboux transformation is always higher than 2. In the matrix case we have that after one step of the Darboux transformation, the order of the differential operator can be the same as the original one. This phenomenon is not new and appeared for the first time in [12] using a method different than the Darboux transformation. For other examples of the bispectral property following a Darboux transformation see [25].

4.4 An urn model for the associated Jacobi polynomials

Finally we give an urn model for the associated Jacobi polynomials. For simplicity, we will restrict the parameters α and β to the region $\boldsymbol{A}_1=\{\beta-\alpha+1>0,\alpha>-\beta,\beta<0\}$ given in Proposition 5.1.1.

For the rest of regions we can proceed in a similar way. In order to have a stochastic matrix P, we need to choose the parameter t in the real set (4.2.3). Since $-1 < \alpha < 1, -1 < \beta < 0$ and t is a real parameter, in order to find an urn model including numbers of nonnegative blue or red balls, in this section we will assume that

$$\alpha = \frac{1}{A}, \quad \beta = -\frac{1}{B}, \quad t = \frac{1}{T} + K, \quad A, B, T \in \mathbb{Z}_{\geq 2}, \quad K \in \mathbb{Z}_{\geq 0}. \tag{4.4.1}$$

On one side, the restriction on the region $\boldsymbol{A}_1$ is equivalent to $AB > A + B$ and $A < B$. On the other side, the restriction (4.2.3) gives two possibilities:

- $n < t < n - \beta, n \in \mathbb{Z}$. Substituting (4.4.1) in the previous inequalities we get that we need $n < 1/T + K < n + 1/B, n \in \mathbb{Z}$. It turns out that, since $A, B, T \geq 2$ and $K \geq 0$ are nonnegative integers, the only choice of $n \in \mathbb{Z}$ such that the previous both inequalities hold is for $n = K$, in which case we need that $T > B$. If $n < K$ then the first inequality is not possible and if $n > K$ then the second inequality does not hold.

- $n - \alpha < t < n - \alpha - \beta, n \in \mathbb{Z}$. Substituting (4.4.1) in the previous inequalities we get that we need $n - 1/A < 1/T + K < n - 1/A + 1/B, n \in \mathbb{Z}$. Again, it turns out that, since $A, B, T \geq 2$ and $K \geq 0$ are nonnegative integers, the only choice of $n \in \mathbb{Z}$ such that the previous both inequalities hold is for $n = K$, in which case we need that $B < AT/(A+T)$. If $n < K$ then the first inequality is not possible and if $n > K$ then the second inequality does not hold.

We will choose T according to the first possibility (for the second we can proceed in a similar way). In summary, our nonnegative parameters A, B, T, K will be restricted to

$$A, B, T \geq 2, \quad K \geq 0, \quad AB > A + B, \quad \text{and} \quad A < B < T. \tag{4.4.2}$$

We focus now on the case of the UL stochastic factorization $P = P_U P_L$ with coefficients $x_n, y_n, s_n, r_n, n \in \mathbb{Z}$ given by (4.2.12). Substituting (4.4.1) in these coefficients we obtain

$$\begin{aligned} y_n &= \frac{B(AT(n+K) + A + T)}{ABT(2n + 2K + 1) + 2AB + T(B - A)}, \quad & x_n &= \frac{A(BT(n+K+1) + B - T)}{ABT(2n + 2K + 1) + 2AB + T(B - A)}, \\ s_n &= \frac{ABT(n+K) + AB + T(B - A)}{2ABT(n+K) + 2AB + T(B - A)}, \quad & r_n &= \frac{AB(T(n+K) + 1)}{2ABT(n+K) + 2AB + T(B - A)}, \end{aligned} \quad n \in \mathbb{Z}. \tag{4.4.3}$$

To simplify the notation, let us call

$$\begin{aligned} Y_n &= B(AT(n+K) + A + T), \quad & X_n &= A(BT(n+K+1) + B - T), \\ S_n &= ABT(n+K) + AB + T(B - A), \quad & R_n &= AB(T(n+K) + 1), \end{aligned} \quad n \in \mathbb{Z}, \tag{4.4.4}$$

so that we have

$$y_n = \frac{Y_n}{X_n + Y_n}, \quad x_n = \frac{X_n}{X_n + Y_n}, \quad s_n = \frac{S_n}{S_n + R_n}, \quad r_n = \frac{R_n}{S_n + R_n}, \quad n \in \mathbb{Z}.$$

Lemma 4.4.1. *Assume that we have that A, B, T, K are nonnegative integers satisfying (4.4.2). If $n + K \geq 0$ then $Y_n, X_n, S_n, R_n \geq 0$ for all $n \in \mathbb{Z}$ and if $n + K < 0$ then $Y_n, X_n, S_n, R_n < 0$ for all $n \in \mathbb{Z}$.*

Proof. From (4.4.4) we have that $Y_n, X_n, S_n, R_n \geq 0$ if

$$n + K \geq \begin{cases} -\dfrac{1}{T} - \dfrac{1}{A}, & \text{for } Y_n, \\[2ex] -1 - \dfrac{1}{T} + \dfrac{1}{B}, & \text{for } X_n, \\[2ex] -\dfrac{1}{T} - \dfrac{1}{A} + \dfrac{1}{B}, & \text{for } S_n, \\[2ex] -\dfrac{1}{T} & \text{for } R_n. \end{cases}$$

A straightforward computation using (4.4.2) shows that

$$-1 < -1 - \frac{1}{T} + \frac{1}{B} < -\frac{1}{T} - \frac{1}{A} + \frac{1}{B} < -\frac{1}{T} - \frac{1}{A} < -\frac{1}{T} < 0.$$

From the previous inequalities it is now easy to see that if $n + K \geq 0$ then $Y_n, X_n, S_n, R_n \geq 0$, while if $n + K < 0$ then $Y_n, X_n, S_n, R_n < 0$ for all $n \in \mathbb{Z}$. $\square$

Let $\{Z_t : t = 0, 1, \ldots\}$ be the bilateral birth-death chain associated with the transition probability matrix P. Consider the UL stochastic factorization $P = P_U P_L$. We have that each one of the matrices P_U and P_L will give rise to an experiment in terms of an urn model, which we call Experiment 1 and Experiment 2, respectively. Let us call $\{Z_t^{(i)} : t = 0, 1, \ldots\}, i = 1, 2$, the chains associated with P_U and P_L, respectively. At times $t = 0, 1, \ldots,$ the state $n \in \mathbb{Z}$ in each of these chains will be given by the *number of blue balls minus the number of red balls.* Therefore we may have nonnegative and negative integer states. We finally assume that the urn sits in a bath consisting of an infinite number of blue and red balls.

At the beginning of every experiment for $t = 0, 1, \ldots,$ we have to decide how many blue and red balls are going to be in the urn. This will depend on the state $n \in \mathbb{Z}$ according to the following rule:

1. If the initial state n satisfies $n \geq -K$, then we initially put in the urn $n + K$ blue balls and K red balls.

2. If the initial state n satisfies $n < -K$, then we initially put in the urn K blue balls and $-n + K$ red balls.

Experiment 1, for P_U, will give a pure-birth chain on $\mathbb{Z}$. Initially, $Z_0^{(1)} = n$. On one hand, if $n \geq -K$, then we place $n + K$ blue balls and K red balls in the urn. After that, we add or remove balls until we have X_n blue balls and Y_n red balls. Note that both are nonnegative integers by Lemma 4.4.1. Draw one ball from the urn at random with the uniform distribution. We have two possibilities:

- If we get a blue ball (with probability x_n in (4.4.3)) then we remove/add balls until we have $n + K + 1$ blue balls and K red balls in the urn and start over. Then we have $Z_1^{(1)} = n + 1$.

- If we get a red ball (with probability y_n in (4.4.3)) then we remove/add balls until we have $n+K$ blue balls and K red balls in the urn and start over. Then we have $Z_1^{(1)} = n$.

On the other hand, if $n < -K$, then we place K blue balls and $-n+K$ red balls in the urn. After that, we add or remove balls until we have $-X_n$ blue balls and $-Y_n$ red balls. Again, both are nonnegative integers by Lemma 4.4.1. Draw one ball from the urn at random with the uniform distribution. We have two possibilities:

- If we get a blue ball (with probability x_n in (4.4.3)) then we remove/add balls until we have $K+1$ blue balls and $-n+K$ red balls in the urn and start over. Then we have $Z_1^{(1)} = n+1$.
- If we get a red ball (with probability y_n in (4.4.3)) then we remove/add balls until we have K blue balls and $-n+K$ red balls in the urn and start over. Then we have $Z_1^{(1)} = n$.

Experiment 2 (for P_L) will give a pure-death chain on $\mathbb{Z}$. Initially, $Z_0^{(2)} = n$. On one hand, if $n \geq -K$, then we place $n+K$ blue balls and K red balls in the urn. After that, we add or remove balls until we have S_n blue balls and R_n red balls. Both are nonnegative integers by Lemma 4.4.1. Draw one ball from the urn at random with the uniform distribution. We have two possibilities:

- If we get a blue ball (with probability s_n in (4.4.3)) then we remove/add balls until we have $n+K$ blue balls and K red balls in the urn and start over. Then we have $Z_1^{(2)} = n$.
- If we get a red ball (with probability r_n in (4.4.3)) then we remove/add balls until we have $n+K$ blue balls and $K+1$ red balls in the urn and start over. Then we have $Z_1^{(2)} = n-1$.

On the other hand, if $n < -K$, then we place K blue balls and $-n+K$ red balls in the urn. After that, we add or remove balls until we have $-S_n$ blue balls and $-R_n$ red balls. Again, both are nonnegative integers by Lemma 4.4.1. Draw one ball from the urn at random with the uniform distribution. We have two possibilities:

- If we get a blue ball (with probability s_n in (4.4.3)) then we remove/add balls until we have K blue balls and $-n+K$ red balls in the urn and start over. Then we have $Z_1^{(2)} = n$.
- If we get a red ball (with probability r_n in (4.4.3)) then we remove/add balls until we have K blue balls and $-n+K+1$ red balls in the urn and start over. Then we have $Z_1^{(2)} = n-1$.

The urn model for P with state space on $\mathbb{Z}$ is obtained by repeatedly alternating Experiments 1 and 2 in that order. The following figures explain the 3 possible situations that we can find, given the value of the initial state $n \in \mathbb{Z}$. Figures 4.2 and 4.3 are the diagrams for the cases where $n \geq -K$ and $n < -K-1$, respectively, where the number of blue and red balls at the beginning of Experiment 2 does not change. Figure 4.4, for $n = -K-1$ is the only case where at the beginning of Experiment 2, when the state is $-K$, we need to place a different number of blue and red balls. In the figures, the boxed regions represent the number of blue balls B_b and red balls R_r contained within the urn, so that the state of the system is $n = b - r$. The event that a ball is drawn from an urn is indicated by B_1^{Draw} or R_1^{Draw}.

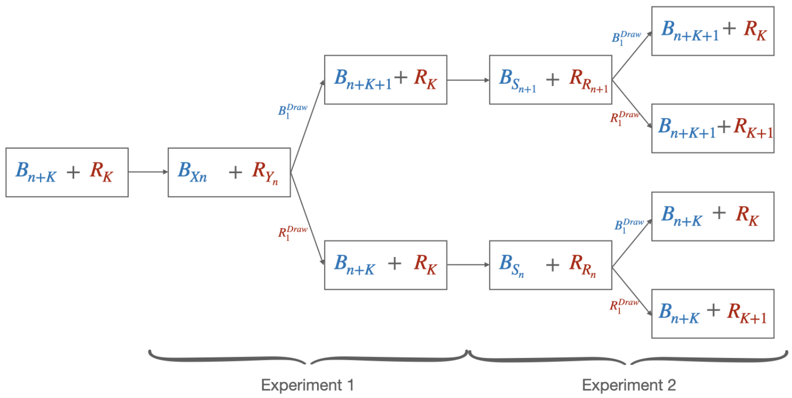

Figure 4.2: Experiments 1 and 2 when $n \geq -K$.

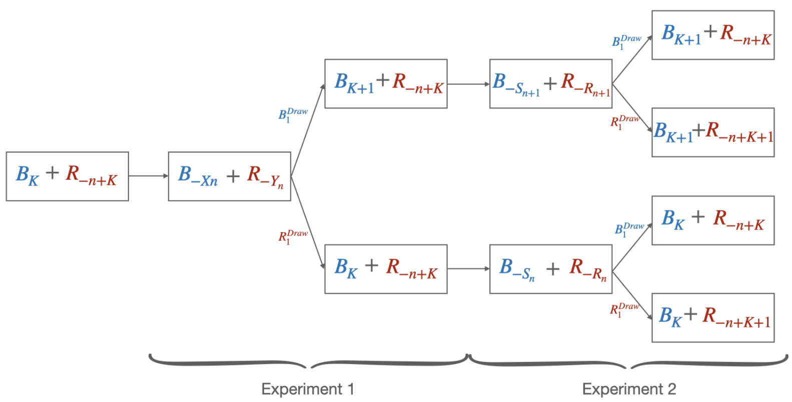

Figure 4.3: Experiments 1 and 2 when $n < -K - 1$.

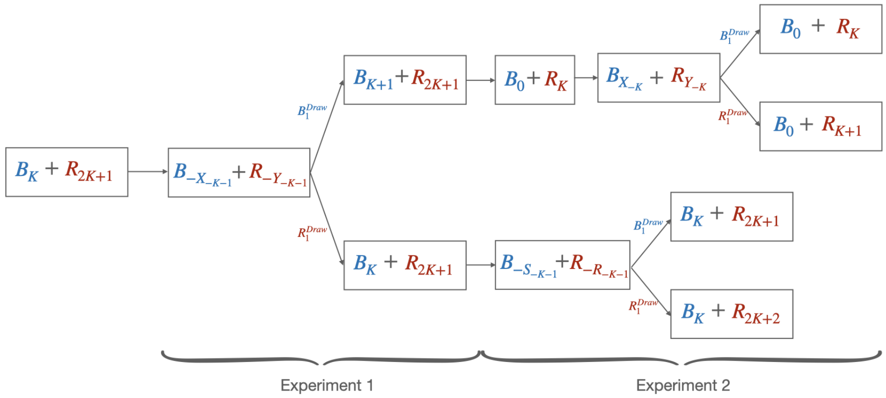

Figure 4.4: Experiments 1 and 2 when $n = -K - 1$.

Similar urn models can be derived for the LU factorization with small modifications.

CHAPTER 5

Birth-death chains on a spider

So far we have studied different types of stochastic factorizations for bilateral birth-death chains and we have performed a detailed and thorough analysis of the relation between the spectral matrix associated to the different processes. The main object to perform this study is the doubly infinite tridiagonal matrix P describing the process. In Chapter 3 we emphasized that the use of the theory of matrix-valued orthogonal polynomials through the quasi-birth-and-death processes allowed us to have more general results in the sense that we can extrapolate some ideas for tridiagonal block matrices of size $N \times N$.

This chapter is dedicated to the study of stochastic factorizations and the spectral analysis of processes with a more general structure than a simple birth-death chain. In particular we consider discrete-time birth-death chains on a *spider* which is a graph consisting of N discrete half lines on the plane, called *legs*, that are joined at the origin, called the *body of the spider*. This chain behaves like a regular discrete-time birth-death chain in each of the legs, but once it reaches the body of the spider it continues towards any of the N legs with a given probability. This process can be identified with a discrete-time quasi-birth-and-death process on the state space $\mathbb{Z}_{\geq 0} \times \{1, 2, \dots, N\}$, represented by a block tridiagonal transition probability matrix, precisely, with blocks of size $N \times N$.

In Section 5.1 we give a definition of this process and perform the spectral analysis to derive the corresponding spectral matrix. In Section 5.2 we consider a reflecting-absorbing (RA) stochastic factorization of the birth-death chain on a spider. The difference with our results in Chapter 3 is that now we will have N free parameters, one for each leg, and we will show that each of these parameters must be bounded from below by certain continued fraction built from the transition probabilities of each leg. After that, we consider a discrete Darboux transformation which describes an "almost" birth-death chain on a spider (now there will be extra transition probabilities between the first states of each leg). Finally, in Section 5.3, we apply our results to the birth-death chain on a spider with constant transition probabilities (or random walk on a spider). This example was considered in [7, Example 3.5] where an explicit expression of the corresponding spectral matrix $\mathbf{\Psi}$ was computed, and it was improved later by F. A. Grünbaum in [20]. We derive another simplified explicit expression of the spectral matrix $\mathbf{\Psi}$ (as a 2×2 block matrix), which is different from the ones given in [7, 20]. The content of this chapter has been published in [34].

5.1 Spectral analysis of birth-death chains on a spider

The study of the so-called *Walsh's spider* dates back to 1978 when J. B. Walsh [51] characterized a Brownian motion with excursions around zero in random directions on the plane which takes values in $[0, 2\pi)$. More recently S. N. Evans and R. B. Sowers [14] considered the same construction but with different methods and used the name *Walsh's spider*. Roughly speaking, the process behaves like a regular Brownian motion on each leg and once it reaches the origin it continues on any of the N legs with a given probability. If we replace the Brownian motions with simple symmetric random walks on the legs we get a discrete version of the Walsh's spider, also called *random walk on a spider*. H. Hajri [28] studied this discrete version as an approximation of the Walsh's spider. In fact, the author studied the solutions to the Tanaka's differential equation related to the Walsh's spider (called Walsh Brownian motion) as limits of discrete models.

define an extension of the Tanaka's equation called Tanaka's stochastic differential equation related to the skew brownian motion and there are only one Wiener solution and only one flow of mappings solving this equation.

The main purpose of this section is to give the definition of the process that we will study together with the one-step transition probability matrix that describes it. We also perform the spectral analysis to derive the sequence of orthogonal polynomials and the spectral matrix that we need.

For $N \in \mathbb{N}$ consider the spider graph given by

$$\mathbb{S}_N := \{v_N(k,m),\ k \in \mathbb{N}_0,\ m = 1, \dots, N\},$$

where

$$v_N(k,m) = k \exp\left(\frac{2\pi i(m-1)}{N}\right), \quad i = \sqrt{-1}.$$

The number N is the number of legs of the spider $\mathbb{S}_N$. If $N = 1$ then we go back to regular birth-death chains on $\mathbb{Z}_{\geq 0}$, while if $N = 2$ we have a birth-death chain on $\mathbb{Z}$ or a bilateral birth-death chain. The point $v_N(0) := v_N(0,m), m = 1, \dots, N$, will be called the body of the spider.

Consider an homogeneous birth-death chain $\{Z_n, n = 0, 1, \dots\}$ on the spider $\mathbb{S}_N$. The transition probabilities are given by

$$\mathbb{P}\left[Z_{n+1} = v_N(0) \,|\, Z_n = v_N(0)\right] = \alpha_0, \quad \mathbb{P}\left[Z_{n+1} = v_N(1,m) \,|\, Z_n = v_N(0)\right] = \alpha_m, \quad m = 1, \dots, N,$$

where $\sum_{m=0}^{N} \alpha_m = 1$, and

$$\begin{aligned}
\mathbb{P}\left[Z_{n+1} = v_N(k+1,m) \,|\, Z_n = v_N(k,m)\right] &= a_{k,m}, \\
\mathbb{P}\left[Z_{n+1} = v_N(k,m) \,|\, Z_n = v_N(k,m)\right] &= b_{k,m}, \\
\mathbb{P}\left[Z_{n+1} = v_N(k-1,m) \,|\, Z_n = v_N(k,m)\right] &= c_{k,m},
\end{aligned}$$

where $a_{k,m} + b_{k,m} + c_{k,m} = 1$ and $0 < a_{k,m}, c_{k,m} < 1$ for all $k \geq 1$ and $m = 1, \dots, N$.

This whole process can be seen as a quasi-birth-and-death process on the state space $\mathbb{Z}_{\geq 0} \times \{1, 2, \dots, N\}$. The labeling follows putting $v_N(0)$ as the origin 0. Then the first N nodes on the first circle as $1, \dots, N$, in a counter-clock wise fashion. The second circle with $N+1, \dots, 2N$, and so on. The transition probability matrix of the birth-death chain $\{Z_n, n = 0, 1, \dots\}$, seen as a quasi-birth-and-death process, is

$$\boldsymbol{P} = \begin{pmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{5.1.1}$$

where the blocks are given by

$$B_0 = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\ c_{1,1} & b_{1,1} & 0 & \cdots & 0 \\ c_{1,2} & 0 & b_{1,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1,N-1} & 0 & 0 & \cdots & b_{1,N-1} \end{pmatrix}, \quad A_0 = \begin{pmatrix} \alpha_N & & & \\ & a_{1,1} & & \\ & & \ddots & \\ & & & a_{1,N-1} \end{pmatrix}, \tag{5.1.2}$$

and

$$A_n = \begin{pmatrix} a_{n,N} & & & \\ & a_{n+1,1} & & \\ & & \ddots & \\ & & & a_{n+1,N-1} \end{pmatrix}, \quad n \geq 1, \tag{5.1.3}$$

$$B_n = \begin{pmatrix} b_{n,N} & & & \\ & b_{n+1,1} & & \\ & & \ddots & \\ & & & b_{n+1,N-1} \end{pmatrix}, \quad n \geq 1, \tag{5.1.4}$$

$$C_n = \begin{pmatrix} c_{n,N} & & & \\ & c_{n+1,1} & & \\ & & \ddots & \\ & & & c_{n+1,N-1} \end{pmatrix}, \quad n \geq 1. \tag{5.1.5}$$

A diagram of the probability transitions between the states of this process is given in Figure 5.1.

In the same fashion as previous chapters, we consider the matrix-valued polynomials generated by the three-term recurrence relation

$$\begin{aligned} &x\boldsymbol{Q}_n(x) = A_n\boldsymbol{Q}_{n+1}(x) + B_n\boldsymbol{Q}_n(x) + C_n\boldsymbol{Q}_{n-1}(x), \quad n \geq 0, \\ &\boldsymbol{Q}_0(x) = \boldsymbol{I}_N, \quad \boldsymbol{Q}_{-1}(x) = \boldsymbol{0}_N, \end{aligned}$$

where $\boldsymbol{I}_N$ and $\boldsymbol{0}_N$ denote the identity and the null matrix of dimension $N \times N$, respectively (in this section whenever we write $\boldsymbol{0}$ we will mean the null vector or matrix which dimension will be determined by the context). This family of matrix-valued polynomials can be written as

$$\boldsymbol{Q}_n(x) = \begin{pmatrix} Q_{n,N}(x) & \alpha_1 Q_{n,N}^{(0)}(x) & \alpha_2 Q_{n,N}^{(0)}(x) & \cdots & \alpha_{N-1} Q_{n,N}^{(0)}(x) \\ Q_{n,1}^{(0)}(x) & Q_{n,1}(x) & 0 & \cdots & 0 \\ Q_{n,2}^{(0)}(x) & 0 & Q_{n,2}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{n,N-1}^{(0)}(x) & 0 & 0 & \cdots & Q_{n,N-1}(x) \end{pmatrix}, \quad n \geq 0, \tag{5.1.6}$$

where $Q_{n,N}(x)$ satisfies the scalar-valued three-term recurrence relation

$$\begin{aligned} &xQ_{n,N}(x) = a_{n,N}Q_{n+1,N}(x) + b_{n,N}Q_{n,N}(x) + c_{n,N}Q_{n-1,N}(x), \quad n \geq 0, \\ &Q_{0,N}(x) = 1, \quad Q_{-1,N}(x) = 0, \end{aligned} \tag{5.1.7}$$

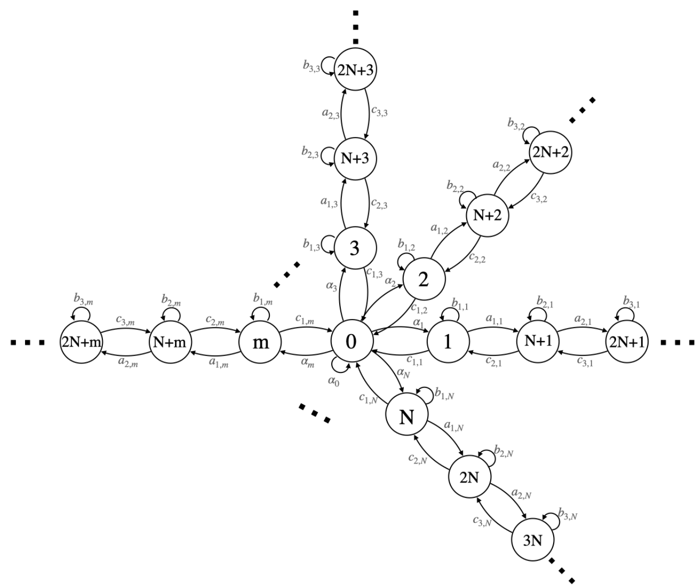

Figure 5.1: Diagram for the N homogeneous birth-death chains on the Walsh's spider.

where $a_{0,N} = \alpha_N$, $b_{0,N} = \alpha_0$ and $Q_{n,N}^{(0)}$ will denote the corresponding 0-th associated polynomials (see Remark 1.3.3). These are polynomials satisfying the same three-term recurrence relation (5.1.7) but with initial conditions

$$Q_{0,N}^{(0)} = 0, \quad Q_{1,N}^{(0)} = -1/\alpha_N.$$

Also $Q_{n,k}(x), k = 1, \ldots, N-1$, satisfy the scalar-valued three-term recurrence relations

$$\begin{aligned} xQ_{n,k}(x) &= a_{n+1,k}Q_{n+1,k}(x) + b_{n+1,N}Q_{n,k}(x) + c_{n+1,N}Q_{n-1,k}(x), \quad n \geq 0, \\ Q_{0,k}(x) &= 1, \quad Q_{-1,k}(x) = 0, \end{aligned} \tag{5.1.8}$$

and $Q_{n,k}^{(0)}(x), k = 1, \ldots, N-1$, will denote the corresponding associated polynomials with initial conditions $Q_{0,k}^{(0)} = 0, Q_{1,k}^{(0)} = -c_{1,k}/a_{1,k}, k = 1, \ldots, N-1$.

As the associated polynomials have degree $n-1$, the matrix-valued polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ in equation (5.1.6) satisfy $\deg(\boldsymbol{Q}_n) = n$ and have nonsingular leading coefficient. This means that this is the family of orthogonal polynomials associated to $\boldsymbol{P}$ in (5.1.1).

Proposition 5.1.1. *Let $\{Z_n, n = 0, 1, \ldots\}$ be a birth-death chain on the spider $\mathbb{S}_N$ with transition probability matrix $\boldsymbol{P}$ (5.1.1). Then there exists a weight matrix $\boldsymbol{\Psi}$ supported on the interval $[-1, 1]$ such that the polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ defined by (5.1.6) are orthogonal with respect to $\boldsymbol{\Psi}$ in the following sense*

$$\int_{-1}^{1} \boldsymbol{Q}_n(x) d\boldsymbol{\Psi}(x) \boldsymbol{Q}_m^T(x) = \|\boldsymbol{Q}_n\|_{\boldsymbol{\Psi}}^2 \delta_{nm},$$

where $\|\boldsymbol{Q}_n\|_{\boldsymbol{\Psi}}^2 := \int_{-1}^{1} \boldsymbol{Q}_n(x) d\boldsymbol{\Psi}(x) \boldsymbol{Q}_n^T(x)$ is the matrix-valued norm of the polynomial $\boldsymbol{Q}_n(x)$ and δ_{nm} is the Kronecker delta.

Proof. For the existence and orthogonality we apply Theorem 1.5.1. We need to define a sequence of nonsingular matrices $(T_n)_{n\geq 0}$ such that

$$
\begin{aligned}
T_n T_n^T B_n &= B_n^T T_n T_n^T, \quad n \geq 0,\\
T_n T_n^T A_n &= C_{n+1}^T T_{n+1} T_{n+1}^T, \quad n \geq 0,
\end{aligned}
$$

where the coefficients $(A_n)_{n\geq 0}, (B_n)_{n\geq 0}$ and $(C_n)_{n\geq 1}$ are defined by (5.1.2), (5.1.3), (5.1.4) and (5.1.5), respectively. Let us define the following sequences

$$
\begin{aligned}
\pi_{0,N} &= 1, \quad \pi_{n,N} = \alpha_N \frac{a_{1,N}\cdots a_{n-1,N}}{c_{1,N}\cdots c_{n,N}}, \quad n \geq 1,\\
\pi_{1,k} &= \frac{\alpha_k}{c_{1,k}}, \quad \pi_{n+1,k} = \alpha_k \frac{a_{1,k}\cdots a_{n,k}}{c_{1,k}\cdots c_{n+1,k}}, \quad n \geq 1, \quad k = 1,\ldots,N-1.
\end{aligned}
$$

Then a straightforward computation shows that the diagonal matrix

$$
T_n = \begin{pmatrix} \sqrt{\pi_{n,N}} & & & \\ & \sqrt{\pi_{n+1,1}} & & \\ & & \ddots & \\ & & & \sqrt{\pi_{n+1,N-1}} \end{pmatrix},
$$

satisfies the conditions (5.1). Finally the weight matrix $\boldsymbol{\Psi}$ is supported on the interval $[-1,1]$ as a consequence of Theorem 1.5.2. $\square$

With the elements defined in the proof of the previous proposition we can define the sequence of potential coefficients for the birth-death chain on the spider $\mathbb{S}_N$ as

$$
\Pi_n = T_n T_n^T = \begin{pmatrix} \pi_{n,N} & & & \\ & \pi_{n+1,1} & & \\ & & \ddots & \\ & & & \pi_{n+1,N-1} \end{pmatrix}, \quad n \geq 0, \tag{5.1.9}
$$

where

$$
\begin{aligned}
\pi_{0,N} &= 1, \quad \pi_{n,N} = \alpha_N \frac{a_{1,N}\cdots a_{n-1,N}}{c_{1,N}\cdots c_{n,N}}, \quad n \geq 1,\\
\pi_{1,k} &= \frac{\alpha_k}{c_{1,k}}, \quad \pi_{n+1,k} = \alpha_k \frac{a_{1,k}\cdots a_{n,k}}{c_{1,k}\cdots c_{n+1,k}}, \quad n \geq 1, \quad k = 1,\ldots,N-1,
\end{aligned}
$$

and

$$
T_n = \begin{pmatrix} \sqrt{\pi_{n,N}} & & & \\ & \sqrt{\pi_{n+1,1}} & & \\ & & \ddots & \\ & & & \sqrt{\pi_{n+1,N-1}} \end{pmatrix}.
$$

This sequence can be identified with the inverse of the norms of the matrix-valued orthogonal polynomials $(\boldsymbol{Q}_n)_{n\geq 0}$ defined by (5.1.6), i.e.

$$
\Pi_n = \left(\|\boldsymbol{Q}_n\|_{\boldsymbol{\Psi}}^2\right)^{-1}, \quad n \geq 0.
$$

Before moving on the analysis of the stochastic factorization, let us remark that the existence of a weight matrix $\mathbf{\Psi}(x)$ for the birth-death chain $\{Z_n, n = 0, 1, \dots\}$ on the spider $\mathbb{S}_N$ gives one way of computing the (i, j)-block of the n-step transition probability matrix $\boldsymbol{P}^n$ by the Karlin-McGregor formula given in Theorem 1.5.3. We also can study the recurrence of the process following equations (1.5.5) and (1.5.6).

Next we will analyze the Stieltjes transform of the birth-death chain on a spider. These series of results will be very useful to compute the spectral matrix associated with the process. This computation will not always be possible since, at this point, the process is much more complex than those seen in previous chapters. However we will be able to derive this expression for the example at the end of this chapter.

First of all, let us recall that the family of polynomials $(Q_{n,k})_{n\geq 0}$, $k = 1, \dots, N$, and the corresponding associated polynomials $(Q_{n,k}^{(0)})_{n\geq 0}$, $k = 1, \dots, N$, are defined in terms of regular three-term recurrence relations. This means that we can use the spectral theorem for orthogonal polynomials (Theorem 1.1.6) and ensure the existence of spectral measures supported on the interval $[-1, 1]$ such that each of these two families of polynomials are orthogonal.

For $k = 1, \dots, N$, let us denote ω_k and $\omega_k^{(0)}$ the spectral probability measures associated with the polynomials $(Q_{n,k})_{n\geq 0}$ and $(Q_{n,k}^{(0)})_{n\geq 0}$, respectively. Now we can use the well-known connection between the Stieltjes transforms of ω_k and $\omega_k^{(0)}$ (see Section 1.1), in this case given by

$$
\begin{aligned}
B(z;\omega_N) &= -\frac{1}{z - \alpha_0 + \alpha_N c_{1,N} B(z;\omega_N^{(0)})},\\
B(z;\omega_k) &= -\frac{1}{z - b_{1,k} + a_{1,k} c_{2,k} B(z;\omega_k^{(0)})}, \qquad k = 1, \dots, N-1.
\end{aligned}
\tag{5.1.10}
$$

From Proposition 5.1.1 we know that any birth-death chain $\{Z_n, n = 0, 1, \dots\}$ on the spider $\mathbb{S}_N$ can be identified with some weight matrix $\mathbf{\Psi}(x)$. It is possible to compute the Stieltjes transform of $\mathbf{\Psi}(x)$, entry by entry, in terms of the Stieltjes transforms of the measures ω_k, $k = 1, \dots, N$. For that we will need the following notation

$$
\vec{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_{N-1})^T, \quad \boldsymbol{\alpha}_D = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_{N-1} \end{pmatrix}, \tag{5.1.11}
$$

$$
\vec{\boldsymbol{c}} = (c_{1,1}, \dots, c_{1,N-1})^T, \quad \boldsymbol{c}_D = \begin{pmatrix} c_{1,1} & & & \\ & c_{1,2} & & \\ & & \ddots & \\ & & & c_{1,N-1} \end{pmatrix},
$$

$$
\vec{\boldsymbol{\omega}}(x) = (\omega_1(x), \dots, \omega_{N-1}(x))^T, \quad \boldsymbol{\omega}_D(x) = \begin{pmatrix} \omega_1(x) & & & \\ & \omega_2(x) & & \\ & & \ddots & \\ & & & \omega_{N-1}(x) \end{pmatrix}.
$$

From now on, if we write $B(z;\vec{\boldsymbol{\omega}})$ or $B(z;\boldsymbol{\omega}_D)$ we mean that we are taking the Stieltjes transform on each entry or component.

Proposition 5.1.2. *Let $\{Z_n, n = 0, 1, \ldots\}$ be a birth-death chain on the spider $\mathbb{S}_N$ with transition probability matrix $\boldsymbol{P}$ (5.1.1). The Stieltjes transform of the weight matrix $\boldsymbol{\Psi}$ obtained in Proposition 5.1.1 can be written as*

$$B(z;\boldsymbol{\Psi}) = \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & -B(z;\boldsymbol{\omega}_D)\boldsymbol{c}_D\boldsymbol{\alpha}_D^{-1} \end{array}\right) + \mathfrak{b}(z)\left(\begin{array}{c|c} 1 & -\vec{\boldsymbol{c}}^T B(z;\boldsymbol{\omega}_D) \\ \hline -B(z;\boldsymbol{\omega}_D)\vec{\boldsymbol{c}} & B(z;\boldsymbol{\omega}_D)\vec{\boldsymbol{c}}\vec{\boldsymbol{c}}^T B(z;\boldsymbol{\omega}_D) \end{array}\right), \quad (5.1.12)$$

where

$$\mathfrak{b}(z) = \frac{1}{\dfrac{1}{B(z;\omega_N)} - \vec{\boldsymbol{\alpha}}^T B(z;\boldsymbol{\omega}_D)\vec{\boldsymbol{c}}}. \quad (5.1.13)$$

Proof. As we know (see (1.5.8)), the relation between the Stieltjes transform of $\boldsymbol{\Psi}$ and the Stieltjes transform of the spectral matrix $\boldsymbol{\Psi}^{(0)}$ of the 0-th associated process is given by

$$B(z;\boldsymbol{\Psi})\Pi_0 = -\left[z\boldsymbol{I}_N - B_0 + A_0 B(z;\boldsymbol{\Psi}^{(0)})\Pi_0^{(0)}C_1\right]^{-1},$$

where $\Pi_0^{(0)} = \boldsymbol{I}_N$ and

$$\Pi_0 = \left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{\alpha}_D\boldsymbol{c}_D^{-1} \end{array}\right). \quad (5.1.14)$$

The rest of the proof is reduced to identify the corresponding matrices and perform the computations. Since $A_n, B_n, C_{n+1}, n \geq 1$, are diagonal matrices we have that $B(z;\boldsymbol{\Psi}^{(0)})$ is also a diagonal matrix given by

$$B(z;\boldsymbol{\Psi}^{(0)}) = \begin{pmatrix} B(z;\omega_N^{(0)}) & & & \\ & B(z;\omega_1^{(0)}) & & \\ & & \ddots & \\ & & & B(z;\omega_{N-1}^{(0)}) \end{pmatrix}.$$

Using the definition of B_0, A_0 and C_1 in (5.1.2) and (5.1.5) we can write the Stieltjes transform of $\boldsymbol{\Psi}$ in a 2×2 block matrix expression

$$B(z;\boldsymbol{\Psi}) = -\left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right)^{-1}\left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{c}_D\boldsymbol{\alpha}_D^{-1} \end{array}\right),$$

where

$$M_{11} = z - \alpha_0 + \alpha_N c_{1,N} B(z;\omega_N^{(0)}) = -\frac{1}{B(z;\omega_N)}, \quad M_{12} = -\vec{\boldsymbol{\alpha}}^T, \quad M_{21} = -\vec{\boldsymbol{c}},$$

$$M_{22} = \begin{pmatrix} z - b_{1,1} + a_{1,1}c_{2,1}B(z;\omega_1^{(0)}) & & \\ & \ddots & \\ & & z - b_{1,N-1} + a_{1,N-1}c_{2,N-1}B(z;\omega_{N-1}^{(0)}) \end{pmatrix} = -B(z;\boldsymbol{\omega}_D)^{-1}.$$

Using the well-known formula for the inverse of a 2×2 block matrix

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)^{-1} = \left(\begin{array}{c|c} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ \hline -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1} \end{array}\right),$$

and using the fact that

$$\vec{\boldsymbol{\alpha}}^T\boldsymbol{c}_D\boldsymbol{\alpha}_D^{-1} = \vec{\boldsymbol{c}}^T,$$

we get (5.1.12). $\square$

The previous proposition provides us with a very clear method for the calculation of the Stieltjes transformation of the spectral matrix which, in turn, will allow the calculation of the spectral matrix associated to $\boldsymbol{P}$. The importance of this result will be clear when we analyze the example in Section 5.3. However we will also use it to obtain the spectral matrix of the Darboux transformation. With this in mind, we proceed to study the existence of the RA stochastic factorization to later study the Darboux transformation.

5.2 RA factorization for birth-death chains on a spider

In this section we will analyze the RA stochastic factorization of the birth-death chain $\{Z_n, n = 0, 1, \ldots\}$ on the spider $\mathbb{S}_N$ described by $\boldsymbol{P}$ (5.1.1). Let us recall that this factorization is a *block* UL factorization. Therefore, in block structure, we are looking for a factorization of the form $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$, where

$$\boldsymbol{P}_R = \begin{pmatrix} Y_0 & X_0 & & & \\ & Y_1 & X_1 & & \\ & & Y_2 & X_2 & \\ & & & \ddots & \ddots \end{pmatrix}, \quad \boldsymbol{P}_A = \begin{pmatrix} S_0 & & & \\ R_1 & S_1 & & \\ & R_2 & S_2 & \\ & & \ddots & \ddots \end{pmatrix}, \tag{5.2.1}$$

with blocks given by

$$Y_0 = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{N-1} \\ 0 & y_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & y_{1,2} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & y_{1,N-1} \end{pmatrix}, \quad Y_n = \begin{pmatrix} y_{n,N} & & & \\ & y_{n+1,1} & & \\ & & \ddots & \\ & & & y_{n+1,N-1} \end{pmatrix}, \quad n \geq 1, \tag{5.2.2}$$

$$X_0 = \begin{pmatrix} \beta_N & 0 & 0 & \cdots & 0 \\ 0 & x_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & x_{1,2} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & x_{1,N-1} \end{pmatrix}, \quad X_n = \begin{pmatrix} x_{n,N} & & & \\ & x_{n+1,1} & & \\ & & \ddots & \\ & & & x_{n+1,N-1} \end{pmatrix}, \quad n \geq 1,$$

$$S_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ r_{1,1} & s_{1,1} & 0 & \cdots & 0 \\ r_{1,2} & 0 & s_{1,2} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ r_{1,N-1} & 0 & \cdots & & s_{1,N-1} \end{pmatrix}, \quad S_n = \begin{pmatrix} s_{n,N} & & & \\ & s_{n+1,1} & & \\ & & \ddots & \\ & & & s_{n+1,N-1} \end{pmatrix}, \quad n \geq 1, \tag{5.2.3}$$

$$R_n = \begin{pmatrix} r_{n,N} & & & \\ & r_{n+1,1} & & \\ & & \ddots & \\ & & & r_{n+1,N-1} \end{pmatrix}, \quad n \geq 1, \tag{5.2.4}$$

and we have to add the conditions that all these matrices are stochastic, i.e.,

$$\sum_{k=0}^{N} \beta_k = 1,$$

$$x_{n,m} + y_{n,m} = 1, \quad n \geq 1, \quad m = 1, 2, \ldots, N, \tag{5.2.5}$$

$$r_{n,m} + s_{n,m} = 1, \quad n \geq 1, \quad m = 1, 2. \ldots, N. \tag{5.2.6}$$

Diagrams of the possible transitions between the states of both birth-death chains are given in Figure 5.2.

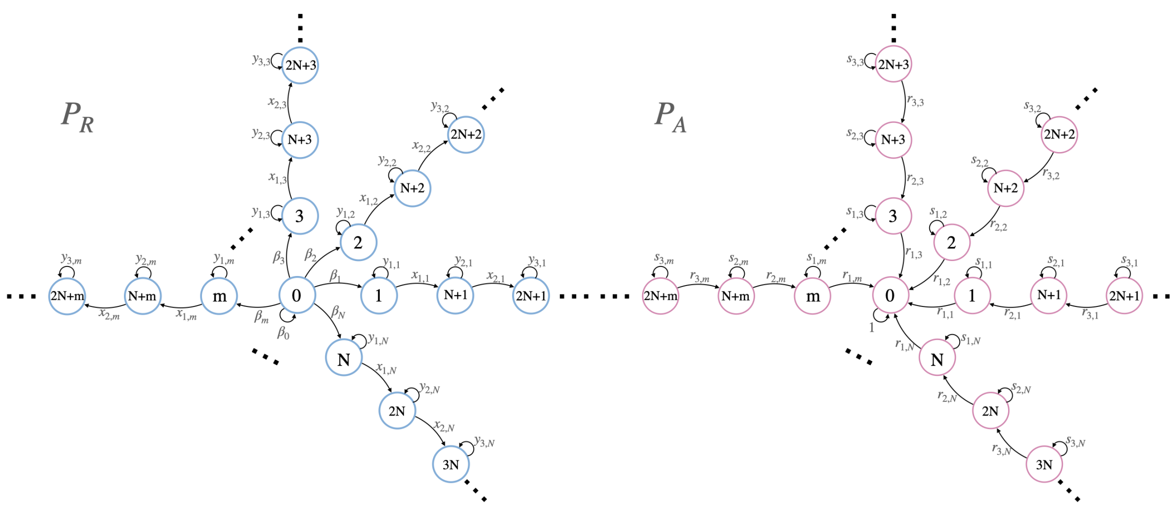

Figure 5.2: Diagrams of the possible transitions between the states of the reflecting process given by $\boldsymbol{P}_R$ and the absorbing process given by $\boldsymbol{P}_A$.

The factorization gives the following relations

$$A_n = X_n S_{n+1}, \quad n \geq 0,$$
$$B_n = X_n R_{n+1} + Y_n S_n, \quad n \geq 0,$$
$$C_n = Y_n R_n, \quad n \geq 1.$$

Or equivalently, entry by entry, we have

$$\alpha_0 = \beta_0 + \sum_{k=1}^{N} \beta_k r_{1,k},$$

$$\alpha_m = \beta_m s_{1,m}, \quad m = 1, 2, \ldots, N, \tag{5.2.7}$$

$$a_{n,m} = x_{n,m} s_{n+1,m}, \quad n \geq 1, \quad m = 1, 2, \ldots, N, \tag{5.2.8}$$

$$b_{n,m} = y_{n,m} s_{n,m} + x_{n,m} r_{n+1,m}, \quad n \geq 1, \quad m = 1, 2, \ldots, N,$$

$$c_{n,m} = y_{n,m} r_{n,m}, \quad n \geq 1, \quad m = 1, 2, \ldots, N. \tag{5.2.9}$$

As in previous cases, we can compute all the coefficients $x_{n,m}, y_{n,m}, r_{n,m}, s_{n,m}$, in terms of N *free parameters* $\beta_1, \dots, \beta_N$, one for each leg. Indeed, if we fix β_m for $m = 1, 2, \dots, N$, we get $s_{1,m}$, for $m = 1, 2, \dots, N$, from equation (5.2.7) and $r_{1,m}$, for $m = 1, 2, \dots, N$, from equation (5.2.6). After this we get $y_{1,m}$, for $m = 1, 2, \dots, N$, from equation (5.2.9) and $x_{1,m}$, for $m = 1, 2, \dots, N$, from equation (5.2.5). Then we get $s_{2,m}$, for $m = 1, 2, \dots, N$ from equation (5.2.8) and so on using the same equations.

In a similar way to the previous chapters, we will define the continued fractions that will be the key to derive conditions to guarantee the stochastic RA factorization. Notice that for this case we will have one restriction for each leg. Let

$$H_m = \frac{\alpha_m|}{|1} - \frac{c_{1,m}|}{|1} - \frac{a_{1,m}|}{|1} - \frac{c_{2,m}|}{|1} - \cdots, \quad m = 1, 2, \dots, N, \tag{5.2.10}$$

be the continued fraction with sequence of convergents given by

$$h_{n,m} = \frac{N_{n,m}}{D_{n,m}}, \quad n \geq 0, \quad m = 1, \dots, N. \tag{5.2.11}$$

The sequences $(N_{n,m})_{n\geq 0}$ and $(D_{n,m})_{n\geq 0}$ for every $m = 1, \dots, N$, can be recursively obtained using the formulas

$$\begin{aligned}
&N_{2n,m} = N_{2n-1,m} - c_{n,m} N_{2n-2,m}, \quad n \geq 1, \quad N_{2n+1,m} = N_{2n,m} - a_{n,m} N_{2n-1,m}, \quad n \geq 0,\\
&N_{-1,m} = -1, \quad N_{0,m} = 0,
\end{aligned}$$

$$\begin{aligned}
&D_{2n,m} = D_{2n-1,m} - c_{n,m} D_{2n-2,m}, \quad n \geq 1, \quad D_{2n+1,m} = D_{2n,m} - a_{n,m} D_{2n-1,m}, \quad n \geq 0,\\
&D_{-1,m} = 0, \quad D_{0,m} = 1,
\end{aligned}$$

where $a_{0,m} = \alpha_m$.

Theorem 5.2.1. *Let H_m, $m = 1, 2, \dots, N$, be the continued fractions defined by* (5.2.10) *with their corresponding sequences of convergents defined by* (5.2.11). *Assume that*

$$0 < N_{n,m} < D_{n,m}, \quad n \geq 0, \quad m = 1, \dots, N.$$

Then the continued fractions H_m, $m = 1, 2, \dots, N$, are all convergent. Additionally, let $\boldsymbol{P} = \boldsymbol{P}_R \boldsymbol{P}_A$ and assume that $\sum_{m=1}^{N} H_m < 1$. Then $\boldsymbol{P}_R$ and $\boldsymbol{P}_A$ are stochastic matrices if and only if

$$\beta_m \geq H_m, \quad m = 1, 2, \dots, N. \tag{5.2.12}$$

Proof. For $m = 1, 2, \dots, N$, following same steps as Theorem 2.1.2, it is not hard to prove that

$$\begin{aligned}
N_{2n,m} D_{2n+1,m} - D_{2n,n} N_{2n+1,m} &= -\alpha_m c_{1,m} a_{1,m} \cdots a_{n,m}, \quad n \geq 0,\\
N_{2n+1,m} D_{2n+2,m} - D_{2n+1,m} N_{2n+2,m} &= -\alpha_m c_{1,m} a_{1,m} \cdots c_{n+1,m}, \quad n \geq 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
h_{2n,m} - h_{2n+1,m} &= \frac{N_{2n,m}}{D_{2n,m}} - \frac{N_{2n+1,m}}{D_{2n+1,m}} = -\frac{\alpha_m c_{1,m} a_{1,m} \cdots a_{n,m}}{D_{2n,m} D_{2n+1,m}} < 0, \quad n \geq 0,\\
h_{2n+1,m} - h_{2n+2,m} &= \frac{N_{2n+1,m}}{D_{2n+1,m}} - \frac{N_{2n+2,m}}{D_{2n+2,m}} = -\frac{\alpha_m c_{1,m} a_{1,m} \cdots c_{n+1,m}}{D_{2n+2,m} D_{2n+1,m}} < 0, \quad n \geq 0.
\end{aligned}$$

Then we have following inequality

$$0 = h_{0,m} < \cdots < h_{2n,m} < h_{2n+1,m} < h_{2n+2,m} < \cdots < 1,$$

and then the sequences $(h_{n,m})_{n\geq 0}$ are all bounded and strictly increasing, so they converge to H_m for every $m = 1, \ldots, N$. Now assume that $\sum_{m=1}^{N} H_m < 1$ and $\boldsymbol{P}_R$ and $\boldsymbol{P}_A$ are stochastic matrices. Then it is clear that

$$\beta_m > 0 = h_{0,m},$$

and using equation (5.2.7) we have

$$s_{1,m} = \frac{\alpha_m}{\beta_m} < 1 \Leftrightarrow \beta_m > \alpha_m = h_{1,m}.$$

Using now equations (5.2.9), (5.2.6) and (5.2.7) we have

$$y_{1,m} = \frac{c_{1,m}}{r_{1,m}} \Leftrightarrow 1 - s_{1,m} > c_{1,m} \Leftrightarrow \beta_m > \frac{\alpha_m}{1 - c_{1,m}} = h_{2,m},$$

and in general it can be shown that

$$\beta_m > h_{n,m}.$$

Therefore $0 = h_{0,m} < h_{n,m} < H_m \leq \beta_m$.

On the contrary, if (5.2.12) holds, in particular we have that $\beta_m > h_{n,m}$ for every $n \geq 0, m = 1, \ldots, N$. Following the same steps as before, using an argument of strong induction, will lead us to the fact that both $\boldsymbol{P}_R$ and $\boldsymbol{P}_L$ are stochastic matrices. □

Remark 5.2.2. The reflecting-absorbing factorization $\boldsymbol{P} = \boldsymbol{P}_R\boldsymbol{P}_A$ is just one type of a stochastic block UL factorization of $\boldsymbol{P}$, but there can be more possibilities. Also we could have considered a stochastic block LU factorization of $\boldsymbol{P}$. As it was pointed out in [25], the different stochastic block factorizations of $\boldsymbol{P}$ may come with many degrees of freedom, and the analysis is more complicated than the case of classical birth-death chains.

5.2.1 Stochastic Darboux transformation and the associated spectral matrix

Once we have the conditions under we can perform a stochastic RA factorization, it is possible to compute the discrete Darboux transformation. If $\boldsymbol{P} = \boldsymbol{P}_R\boldsymbol{P}_A$ as in (5.2.1), then, by inverting the order of multiplication of the factors, we obtain another stochastic matrix of the form $\widetilde{\boldsymbol{P}} = \boldsymbol{P}_A\boldsymbol{P}_R$. This new matrix preserves the block tridiagonal structure. In a very similar way that the RA stochastic factorization studied in Chapter 3, for this case the matrix $\widetilde{\boldsymbol{P}}$ will describe a whole family, depending on N free parameters $\beta_1, \ldots, \beta_N$, of Markov chains $\{\widetilde{Z}_n, n = 0, 1, \ldots\}$ on the spider $\mathbb{S}_N$ which is a family of “almost” birth-death chains. The only difference is that we will have extra transitions between the first states of each leg or, in other words, between the states $1, 2, \ldots, N$.

If we call $\widetilde{B}_n, \widetilde{A}_n, \widetilde{C}_{n+1}, n \geq 0$, the new coefficients of the block tridiagonal matrix $\widetilde{\boldsymbol{P}}$, a direct computations shows

$$\begin{aligned}
&\widetilde{A}_n = S_n X_n, \quad n \geq 0,\\
&\widetilde{B}_0 = S_0 Y_0, \quad \widetilde{B}_n = R_n X_{n-1} + S_n Y_n, \quad n \geq 1,\\
&\widetilde{C}_n = R_n Y_{n-1}, \quad n \geq 1.
\end{aligned}$$

If we compute values entry by entry, these relations can also be written as

$$
\begin{aligned}
\tilde{\alpha}_m &= \beta_m, \quad m = 0, 1, \ldots, N, \\
\tilde{\alpha}_{n,m} &= s_{n,m} x_{n,m}, \quad n \geq 1, \quad m = 1, \ldots, N, \\
\tilde{b}_{1,m} &= r_{1,m} \beta_m + s_{1,m} y_{1,m}, \quad m = 1, \ldots, N, \\
\tilde{b}_{n,m} &= r_{n,m} x_{n-1,m} + s_{n,m} y_{n,m}, \quad n \geq 1, \quad m = 1, \ldots, N, \\
\tilde{c}_{1,m} &= r_{1,m} \beta_m, \quad m = 0, 1, \ldots, N, \\
\tilde{c}_{n,m} &= r_{n,m} y_{n-1,m}, \quad n \geq 1, \quad m = 1, \ldots, N,
\end{aligned}
$$

and we also have transition probabilities between the first states of each leg denoted by $\tilde{d}_{i,j}$. These probabilities are given by

$$\tilde{d}_{i,j} = \beta_j r_{1,i}, \quad i, j = 1, \ldots, N.$$

A diagram of this process is similar to the one for the process described by $\boldsymbol{P}$ in Figure 5.1 but now we have to add probabilities between the first states of each leg. For instance, for $N = 4$ we have the diagram in Figure 5.3. In general we have to add $N(N-1)$ extra transition probabilities between the first states of each leg.

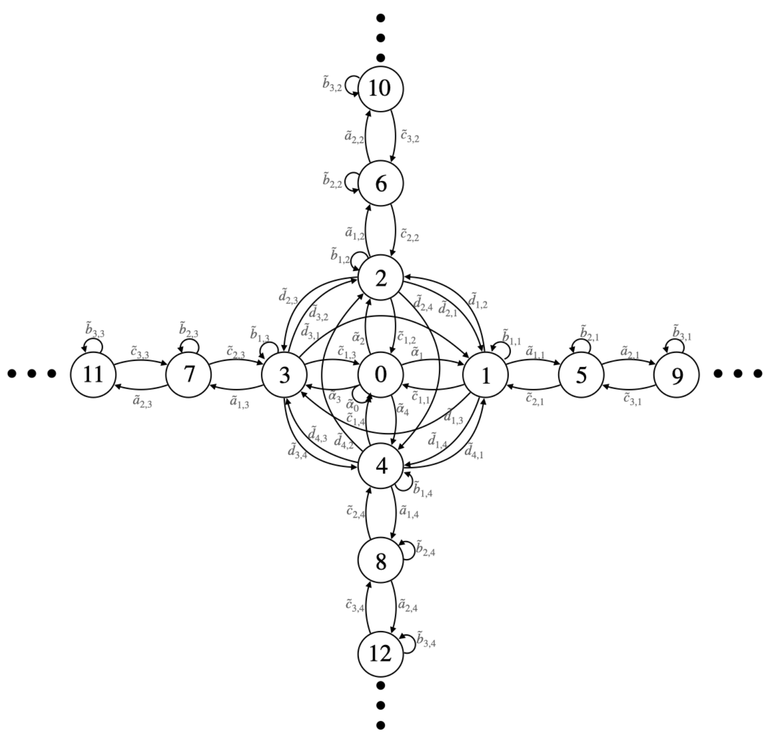

Figure 5.3: Diagram for the Darboux transformation $\widetilde{\boldsymbol{P}}$ with $N = 4$.

Now, as we stated at the end of Section 3.2, the spectral matrix is given by equation (3.2.16) where

S_0, Y_0 are given by (5.2.3) and (5.2.2), and Π_0 is given by (5.1.9). In fact, we can compute

$$\Pi_0 Y_0 S_0 = \begin{pmatrix} \alpha_0+\alpha_N-\beta_N & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\ \alpha_1 & \dfrac{\alpha_1^2}{\beta_1-\alpha_1} & 0 & \cdots & 0 \\ \alpha_2 & 0 & \dfrac{\alpha_2^2}{\beta_2-\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{N-1} & 0 & 0 & \cdots & \dfrac{\alpha_{N-1}^2}{\beta_{N-1}-\alpha_{N-1}} \end{pmatrix}.$$

If we call $\boldsymbol{X} = (\Pi_0 Y_0 S_0)^{-1}$, a straightforward computation shows that the entries of $\boldsymbol{X} = (X_{ij})$ for $i \le j$ are given by

$$X_{ij} = \frac{1}{\beta_0}\begin{cases} 1, & \text{if} \quad i=j=1, \\ 1-\dfrac{\beta_{j-1}}{\alpha_{j-1}}, & \text{if} \quad i=1, j>1, \\ \left(1-\dfrac{\beta_{i-1}}{\alpha_{i-1}}\right)\left(1-\dfrac{\beta_{j-1}}{\alpha_{j-1}}\right), & \text{if} \quad i>1, j>i, \\ \left(1-\dfrac{\beta_{i-1}}{\alpha_{i-1}}\right)\left(1-\dfrac{\beta_0}{\alpha_{i-1}}-\dfrac{\beta_{i-1}}{\alpha_{i-1}}\right), & \text{if} \quad i>1, i=j. \end{cases} \tag{5.2.13}$$

For entries where $i \ge j$ we have to change i by j since $\boldsymbol{X}$ is always symmetric.

One of the main challenges here is the computation of the moment $\boldsymbol{M}_{-1}$. However, using Proposition 5.1.2 we may have a way to compute explicitly $B(z;\boldsymbol{\Psi})$ and then use the fact that

$$\boldsymbol{M}_{-1} = B(0;\boldsymbol{\Psi}).$$

We will follow this procedure for the example of the next section. Finally, we can also compute the matrix-valued orthogonal polynomials $(\widetilde{\boldsymbol{Q}}_n)_{n\ge0}$ associated with $\widetilde{\boldsymbol{\Psi}}(x)$ using Theorem 3.2.1. Since we have an explicit expression of the polynomials $(\boldsymbol{Q}_n)_{n\ge0}$ in (5.1.6) and

$$S_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\dfrac{r_{1,1}}{s_{1,1}} & \dfrac{1}{s_{1,1}} & 0 & \cdots & 0 \\ -\dfrac{r_{1,2}}{s_{1,2}} & 0 & \dfrac{1}{s_{1,2}} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -\dfrac{r_{1,N-1}}{s_{1,N-1}} & 0 & \cdots & & \dfrac{1}{s_{1,N-1}} \end{pmatrix}, \tag{5.2.14}$$

then we have that

$$\widetilde{\boldsymbol{Q}}_n(x) = \begin{pmatrix} V_{n,N}(x) & \dfrac{\alpha_1}{s_{1,1}}V_{n,N}^{(0)}(x) & \dfrac{\alpha_2}{s_{1,2}}V_{n,N}^{(0)}(x) & \cdots & \dfrac{\alpha_{N-1}}{s_{1,N-1}}V_{n,N}^{(0)}(x) \\ V_{n,1}^{(0)}(x) & V_{n,1}(x) & 0 & \cdots & 0 \\ V_{n,2}^{(0)}(x) & 0 & V_{n,2}(x) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_{n,N-1}^{(0)}(x) & 0 & 0 & \cdots & V_{n,N-1}(x) \end{pmatrix}, \quad n \ge 0, \tag{5.2.15}$$

where

$$
\begin{aligned}
V_{n,N}(x) =& s_{n,N}Q_{n,N}(x) + r_{n,N}Q_{n-1,N}(x) - \left(s_{n,N}Q_{n,N}^{(0)}(x) + r_{n,N}Q_{n-1,N}^{(0)}(x)\right)\sum_{k=1}^{N-1}\frac{r_{1,k}\alpha_k}{s_{1,k}},\\
V_{n,N}^{(0)}(x) =& s_{n,N}Q_{n,N}^{(0)}(x) + r_{n,N}Q_{n-1,N}^{(0)}(x),\\
V_{n,k}(x) =& \frac{1}{s_{1,k}}\left(s_{n+1,k}Q_{n,k}(x) + r_{n+1,k}Q_{n-1,k}(x)\right), \quad k = 1,\dots,N-1,\\
V_{n,k}^{(0)}(x) =& s_{n+1,k}Q_{n,k}^{(0)}(x) + r_{n+1,k}Q_{n-1,k}^{(0)}(x) - \frac{r_{1,k}}{s_{1,k}}\left(s_{n+1,k}Q_{n,k}(x) + r_{n+1,k}Q_{n-1,k}(x)\right).
\end{aligned}
$$

As for the potential coefficients for the Markov chain on the spider generated by $\widetilde{\boldsymbol{P}}$, we have, using formula (3.2.13) that

$$\widetilde{\Pi}_n = Y_n^T\Pi_n S_n^{-1}, \quad n \geq 0,$$

and together with (5.2.2), (5.2.14), (5.2.7) and (5.2.9) we have that

$$\widetilde{\Pi}_0 = \begin{pmatrix} \beta_0 & & & \\ & \dfrac{\beta_1}{r_{1,1}} & & \\ & & \ddots & \\ & & & \dfrac{\beta_{N-1}}{r_{1,N-1}} \end{pmatrix} = \begin{pmatrix} \beta_0 & & & \\ & \dfrac{\beta_1^2}{\beta_1-\alpha_1} & & \\ & & \ddots & \\ & & & \dfrac{\beta_{N-1}^2}{\beta_{N-1}-\alpha_{N-1}} \end{pmatrix}.$$

Therefore $(\widetilde{\Pi}_n)_{n\geq 0}$ are always diagonal matrices. As a consequence we obtain the norms of the matrix-valued orthogonal polynomials

$$\widetilde{\Pi}_n = \left(\|\widetilde{\boldsymbol{Q}}_n\|_{\widetilde{\boldsymbol{\Psi}}}^2\right)^{-1} = \left(\int_{-1}^{1}\widetilde{\boldsymbol{Q}}_n(x)d\widetilde{\boldsymbol{\Psi}}(x)\widetilde{\boldsymbol{Q}}_n^T(x)\right)^{-1},$$

and the orthogonality relations

$$\int_{-1}^{1}\widetilde{\boldsymbol{Q}}_n(x)d\widetilde{\boldsymbol{\Psi}}(x)\widetilde{\boldsymbol{Q}}_m^T(x) = \widetilde{\Pi}_n^{-1}\delta_{nm},$$

with δ_{nm} the Kronecker delta

5.3 Random walk on a spider

In this example we will consider a set of N simple birth-death chains with constant transition probabilities and state space on $\mathbb{Z}_{\geq 0}$ all linked in the state 0. Of course we have to add transition probabilities in the body of this spider. Let us consider the block tridiagonal transition probability matrix (5.1.1) with constant transition probability coefficients, i.e.,

$$B_0 = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\ c & b & 0 & \cdots & 0 \\ c & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & 0 & 0 & \cdots & b \end{pmatrix}, \quad A_0 = \begin{pmatrix} \alpha_N & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{pmatrix},$$

and

$$A_n = a\boldsymbol{I}_N, \quad B_n = b\boldsymbol{I}_N, \quad C_n = c\boldsymbol{I}_N, \quad n \geq 1,$$

where

$$\sum_{k=0}^{N} \alpha_k = 1, \quad a + b + c = 1.$$

The first step for this example is to derive the spectral matrix. As we anticipated, for the spectral analysis we will use the Stieltjes transform. For $\omega_k^{(0)}, k = 1, \ldots, N$, the Stieltjes transform does not depend on k and it is given by

$$B(z;\omega_k^{(0)}) = \frac{b - z + \sqrt{(z-\sigma_+)(z-\sigma_-)}}{2ac}, \quad \sigma_\pm = 1 - (\sqrt{a} \mp \sqrt{c})^2. \tag{5.3.1}$$

Therefore, using the second formula in (5.1.10), as $b_{1,k} = b, a_{1,k} = a, c_{2,k} = c$, we have

$$B(z;\omega_k) = -\frac{2}{z - b + \sqrt{(z-\sigma_+)(z-\sigma_-)}} = \frac{b - z + \sqrt{(z-\sigma_+)(z-\sigma_-)}}{2ac}, \quad k = 1, \ldots, N-1.$$

As a consequence

$$B(z;\boldsymbol{\omega}_D) = \frac{b - z + \sqrt{(z-\sigma_+)(z-\sigma_-)}}{2ac}\boldsymbol{I}_{N-1}.$$

On the other hand, using the first formula in (5.1.10) and (5.3.1), we obtain

$$\frac{1}{B(z;\omega_N)} = -\left[z - \alpha_0 + \alpha_N c B(z;\omega_N^{(0)})\right] = -\left[z - \alpha_0 + \frac{\alpha_N}{2a}\left(b - z + \sqrt{(z-\sigma_+)(z-\sigma_-)}\right)\right].$$

Using the previous two formulas in (5.1.13) we get

$$\mathfrak{b}(z) = \frac{1}{-\left[z - \alpha_0 + \frac{\alpha_N}{2a}\left(b - z + \sqrt{(z-\sigma_-)(z-\sigma_+)}\right)\right] - \left(\sum_{k=1}^{N-1}\alpha_k\right)\left(\frac{b - z + \sqrt{(z-\sigma_-)(z-\sigma_+)}}{2a}\right)}.$$

Since $\alpha_1 + \cdots + \alpha_{N-1} = 1 - \alpha_0 - \alpha_N$, after some computations we have

$$\mathfrak{b}(z) = \frac{1}{\alpha_0 - \frac{b(1-\alpha_0)}{2a} - \left(1 - \frac{1-\alpha_0}{2a}\right)z - \frac{1-\alpha_0}{2a}\sqrt{(z-\sigma_+)(z-\sigma_-)}},$$

and after rationalizing we get

$$\mathfrak{b}(z) = \frac{(1-2a-\alpha_0)z - b + \alpha_0(1+a-c) + (1-\alpha_0)\sqrt{(z-\sigma_+)(z-\sigma_-)}}{2(1-z)\left[(1-a-\alpha_0)z + c - \alpha_0(1-a+c-\alpha_0)\right]}.$$

Therefore we have all the necessary functions to compute the Stieltjes transform of $\boldsymbol{\Psi}$, given by (5.1.12) of Proposition 5.1.2. After some computations we can write $B(z;\boldsymbol{\Psi})$ as

$$B(z;\boldsymbol{\Psi}) = \left(\begin{array}{c|c} B_{11}(z;\boldsymbol{\Psi}) & B_{12}(z;\boldsymbol{\Psi})\vec{\boldsymbol{e}}_{N-1}^T \\ \hline B_{12}(z;\boldsymbol{\Psi})\vec{\boldsymbol{e}}_{N-1} & B_{22}(z;\boldsymbol{\Psi}) \end{array}\right), \tag{5.3.2}$$

where

$$
\begin{aligned}
B_{11}(z;\boldsymbol{\Psi}) &= \mathfrak{b}(z),\\
B_{12}(z;\boldsymbol{\Psi}) &= \frac{2c-\alpha_0(1-a+c)+(b+\alpha_0)z-z^2+(z-\alpha_0)\sqrt{(z-\sigma_+)(z-\sigma_-)}}{2(1-z)\left[(1-a-\alpha_0)z+c-\alpha_0(1-a+c-\alpha_0)\right]},\\
B_{22}(z;\boldsymbol{\Psi}) &= \frac{b-z+\sqrt{(z-\sigma_+)(z-\sigma_-)}}{2a}\boldsymbol{\alpha}_D^{-1}\\
&\quad + \frac{p(z)+r(z)\sqrt{(z-\sigma_+)(z-\sigma_-)}}{2a(1-z)\left[(1-a-\alpha_0)z+c-\alpha_0(1-a+c-\alpha_0)\right]}\vec{\boldsymbol{e}}_{N-1}\vec{\boldsymbol{e}}_{N-1}^T,
\end{aligned}
$$

and

$$
\begin{aligned}
p(z) &= -z^3+(\alpha_0+2b)z^2-(\alpha_0(2-2a-c)+b^2-2ac-c)z-bc+\alpha_0(b-a(1-a+c)),\\
r(z) &= z^2-(\alpha_0+b)z-c+\alpha_0(1-a).
\end{aligned}
$$

Then, the weight matrix $\boldsymbol{\Psi}$ will consist in the addition of an absolutely continuous part $\boldsymbol{\Psi}_c$ and a discrete part $\boldsymbol{\Psi}_d$, i.e., $\boldsymbol{\Psi}=\boldsymbol{\Psi}_c+\boldsymbol{\Psi}_d$. First, using the Perron-Stieltjes inversion formula (1.1.2), $\boldsymbol{\Psi}_c$ is given by

$$
\boldsymbol{\Psi}_c(x)=\left(\begin{array}{c|c}\Psi_{11}(x) & \Psi_{12}(x)\vec{\boldsymbol{e}}_{N-1}^T\\ \hline \Psi_{12}(x)\vec{\boldsymbol{e}}_{N-1} & \Psi_{22}(x)\end{array}\right),\quad x\in[\sigma_-,\sigma_+], \tag{5.3.3}
$$

where

$$
\begin{aligned}
\Psi_{11}(x) &= \frac{(1-\alpha_0)\sqrt{(\sigma_+-x)(x-\sigma_-)}}{2\pi(1-x)\left[(1-a-\alpha_0)x+c-\alpha_0(1-a+c-\alpha_0)\right]},\\
\Psi_{12}(x) &= \frac{(x-\alpha_0)\sqrt{(\sigma_+-x)(x-\sigma_-)}}{2\pi(1-x)\left[(1-a-\alpha_0)x+c-\alpha_0(1-a+c-\alpha_0)\right]},\\
\Psi_{22}(x) &= \frac{\sqrt{(\sigma_+-x)(x-\sigma_-)}}{2\pi a}\boldsymbol{\alpha}_D^{-1}+\frac{r(x)\sqrt{(\sigma_+-x)(x-\sigma_-)}}{2\pi a(1-x)\left[(1-a-\alpha_0)x+c-\alpha_0(1-a+c-\alpha_0)\right]}\boldsymbol{e}_{N-1}\boldsymbol{e}_{N-1}^T.
\end{aligned}
$$

On the other hand, the discrete part $\boldsymbol{\Psi}_d$ will be given by the behavior of the Stieltjes transform $B(z;\boldsymbol{\Psi})$ (5.3.2) at its poles, given in this case by

$$
z_1=1,\quad z_2=\frac{\alpha_0(1-a+c-\alpha_0)-c}{1-a-\alpha_0}.
$$

The expression for the discrete part is given by

$$
\boldsymbol{\Psi}_d(x)=\frac{c-a}{c-a+1-\alpha_0}\delta_1(x)\vec{\boldsymbol{e}}_N\vec{\boldsymbol{e}}_N^T\chi_{\{c>a\}}+\frac{(1-\alpha_0-a)^2-ac}{(1-\alpha_0)(1-\alpha_0-a+c)}\delta_{z_2}(x)\vec{\boldsymbol{u}}_N\vec{\boldsymbol{u}}_N^T\chi_{\{(1-\alpha_0-a)^2>ac\}}, \tag{5.3.4}
$$

where χ_A is the indicator function and

$$
\vec{\boldsymbol{u}}_N=\left(-1,\frac{q}{1-\alpha_0-a},\ldots,\frac{q}{1-\alpha_0-a}\right)^T.
$$

As always, one of the advantages of having an explicit expression for the spectral matrix is that we can describe some probability properties of the process. We have three cases:

- If $a > c$, then we have that $[\sigma_-, \sigma_+] \subsetneq [-1, 1]$. Therefore all integrals in (1.5.5) are bounded and the random walk on a spider is transient.
- If $a = c$, then we have $[\sigma_-, \sigma_+] = [1 - 4a, 1]$. Therefore all integrals in (1.5.5) are divergent and the random walk on a spider is null recurrent (since there is no jump at the point 1).
- If $a < c$, there will always be a jump at the point 1 (see (5.3.4)). Therefore the random walk on a spider is positive recurrent.

Finally, it is possible to see, by looking at the three-term recurrence relations (5.1.7) and (5.1.8), that the entries of the matrix-valued polynomials $\boldsymbol{Q}_n(x)$ in (5.1.6) are given by

$$
\begin{aligned}
Q_{n,N}(x) &= \frac{1}{\alpha_N}\left(\frac{c}{a}\right)^{n/2}\left[2(\alpha_N - a)T_n\left(\frac{x-b}{2\sqrt{ac}}\right) + (2a - \alpha_N)U_n\left(\frac{x-b}{2\sqrt{ac}}\right) + \sqrt{\frac{a}{c}}(b-\alpha_0)U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right)\right],\\
Q_{n,N}^{(0)}(x) &= -\frac{1}{\alpha_N}\left(\frac{c}{a}\right)^{(n-1)/2} U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right),\\
Q_{n,k}(x) &= \left(\frac{c}{a}\right)^{n/2} U_n\left(\frac{x-b}{2\sqrt{ac}}\right), \quad Q_{n,k}^{(0)}(x) = -\left(\frac{c}{a}\right)^{(n+1)/2} U_{n-1}\left(\frac{x-b}{2\sqrt{ac}}\right), \quad k = 1, \ldots, N-1,
\end{aligned}
\tag{5.3.5}
$$

where $(T_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ are the Chebychev polynomials of the first and second kind, respectively (see Section 1.6).

Remark 5.3.1. The polynomials $(Q_{n,N})_{n\geq 0}$ in (5.3.5) can be written in terms of *perturbed Chebychev polynomials* (see [3, pp. 204-205]). Indeed, these perturbed Chebychev polynomials $(P_n)_{n\geq 0}$ are defined in terms of the three-term recurrence relation

$$
P_0(x) = 1, \quad P_1(x) = \delta x - \gamma, \quad xP_n(x) = \frac{1}{2}P_{n+1}(x) + \frac{1}{2}P_{n-1}(x), \quad n \geq 1, \quad \delta \neq 0.
$$

These polynomials can also be written in terms of Chebychev polynomials of the first and second kind (see Section 1.6). If we use the well-known relation $U_{n-2}(x) = U_n(x) - 2T_n(x)$, we obtain

$$
P_n(x) = (2-\delta)T_n(x) - (1-\delta)U_n(x) - \gamma U_{n-1}(x), \quad n \geq 0.
$$

A direct identification with the expression of the polynomials $(Q_{n,N})_{n\geq 0}$ in (5.3.5) shows that we need to choose

$$
\delta = \frac{2a}{\alpha_N}, \quad \gamma = \sqrt{\frac{a}{c}}\frac{\alpha_0 - b}{\alpha_N},
$$

in order to relate $(Q_{n,N})_{n\geq 0}$ with the perturbed Chebychev polynomials $(P_n)_{n\geq 0}$. Therefore we obtain

$$
Q_{n,N}(x) = \left(\frac{c}{a}\right)^{n/2} P_n\left(\frac{x-b}{2\sqrt{ac}}\right), \quad n \geq 0.
$$

Let us now apply Theorem 5.2.1 and see under what conditions we can perform a stochastic RA factorization of the form $\boldsymbol{P} = \boldsymbol{P}_R\boldsymbol{P}_A$. As we have constant transition probabilities, the continued fractions in (5.2.10) can be written as $H_m = \alpha_m/H$, where

$$
H = 1 - \frac{c\rfloor}{\lceil 1} - \frac{a\rfloor}{\lceil 1} - \frac{c\rfloor}{\lceil 1} - \frac{a\rfloor}{\lceil 1} - \cdots,
$$

so we can compute the explicit value given by

$$H=\frac{1}{2}\left(1+a-c+\sqrt{(1+c-a)^2-4c}\right),$$

as long as $a\leq(1-\sqrt{c})^2$. Therefore, after rationalizing, we have

$$H_m=\frac{\alpha_m}{2a}\left(1+a-c-\sqrt{(1+c-a)^2-4c}\right),\quad m=1,\ldots,N.$$

According to Theorem 5.2.1, the stochastic reflecting-absorbing factorization will be possible if and only if

$$\beta_m\geq\frac{\alpha_m}{2a}\left(1+a-c-\sqrt{(1+c-a)^2-4c}\right),\quad m=1,\ldots,N.$$

As $\sum_{m=1}^N H_m<1$ we need to have

$$\alpha_0>\frac{1}{2}\left(1-a+c-\sqrt{(1+c-a)^2-4c}\right).$$

After performing the discrete Darboux transformation given by $\widetilde{\boldsymbol{P}}=\boldsymbol{P}_A\boldsymbol{P}_R$ we get a new family of Markov chains (depending on N free parameters $\beta_1,\ldots,\beta_N$) $\{\widetilde{Z}_n,n=0,1,\ldots\}$ on the spider $\mathbb{S}_N$. As we mentioned before, the matrix $\widetilde{\boldsymbol{P}}$ describes an "almost" birth-death chain on a spider. The new coefficients $\widetilde{B}_0$ and $\widetilde{A}_0$, which give the extra transitions between the first states of each leg are given by

$$\widetilde{B}_0=\begin{pmatrix}\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{N-1}\\ \beta_0\left(1-\dfrac{\alpha_1}{\beta_1}\right) & \beta_1-\alpha_1+\dfrac{c\alpha_1}{\beta_1-\alpha_1} & \beta_2\left(1-\dfrac{\alpha_1}{\beta_1}\right) & \cdots & \beta_{N-1}\left(1-\dfrac{\alpha_1}{\beta_1}\right)\\ \beta_0\left(1-\dfrac{\alpha_2}{\beta_2}\right) & \beta_1\left(1-\dfrac{\alpha_2}{\beta_2}\right) & \beta_2-\alpha_2+\dfrac{c\alpha_2}{\beta_2-\alpha_2} & \cdots & \beta_{N-1}\left(1-\dfrac{\alpha_2}{\beta_2}\right)\\ \vdots & \vdots & & \ddots & \vdots\\ \beta_0\left(1-\dfrac{\alpha_{N-1}}{\beta_{N-1}}\right) & \beta_1\left(1-\dfrac{\alpha_{N-1}}{\beta_{N-1}}\right) & \cdots & & \beta_{N-1}-\alpha_{N-1}+\dfrac{c\alpha_{N-1}}{\beta_{N-1}-\alpha_{N-1}}\end{pmatrix},$$

$$\widetilde{A}_0=\begin{pmatrix}\beta_N & 0 & 0 & \cdots & 0\\ \beta_N\left(1-\dfrac{\alpha_1}{\beta_1}\right) & \dfrac{\alpha_1}{\beta_1}-\dfrac{c\alpha_1}{\beta_1-\alpha_1} & 0 & \cdots & 0\\ \beta_N\left(1-\dfrac{\alpha_2}{\beta_2}\right) & 0 & \dfrac{\alpha_2}{\beta_2}-\dfrac{c\alpha_2}{\beta_2-\alpha_2} & \cdots & 0\\ \vdots & \vdots & & \ddots & \vdots\\ \beta_N\left(1-\dfrac{\alpha_{N-1}}{\beta_{N-1}}\right) & 0 & \cdots & & \dfrac{\alpha_{N-1}}{\beta_{N-1}}-\dfrac{c\alpha_{N-1}}{\beta_{N-1}-\alpha_{N-1}}\end{pmatrix}.$$

Finally, the weight matrix $\widetilde{\boldsymbol{\Psi}}$ associated with $\widetilde{\boldsymbol{P}}$ is given by

$$\widetilde{\boldsymbol{\Psi}}(x)=S_0\left(\frac{\boldsymbol{\Psi}(x)}{x}+[\boldsymbol{X}-\boldsymbol{M}_{-1}]\,\delta_0(x)\right)S_0^T,$$

where

$$S_0=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 1-\dfrac{\alpha_1}{\beta_1} & \dfrac{\alpha_1}{\beta_1} & 0 & \cdots & 0\\ 1-\dfrac{\alpha_2}{\beta_2} & 0 & \dfrac{\alpha_2}{\beta_2} & \cdots & 0\\ \vdots & \vdots & & \ddots & \vdots\\ 1-\dfrac{\alpha_{N-1}}{\beta_{N-1}} & 0 & \cdots & & \dfrac{\alpha_{N-1}}{\beta_{N-1}} \end{pmatrix},$$

the weight matrix $\boldsymbol{\Psi}=\boldsymbol{\Psi}_c+\boldsymbol{\Psi}_d$ is given by (5.3.3) and (5.3.4), $\boldsymbol{X}$ is the symmetric matrix given by (5.2.13) and from (5.3.2) we have

$$\boldsymbol{M}_{-1}=\left(\begin{array}{c|c} \mu_{11} & \mu_{12}\vec{\boldsymbol{e}}^T_{N-1}\\ \hline \mu_{12}\vec{\boldsymbol{e}}_{N-1} & \boldsymbol{\mu}_{22}\end{array}\right),$$

where

$$\mu_{11}=\frac{\alpha_0(1+a-c)-b-(1-\alpha_0)\sqrt{\sigma_+\sigma_-}}{2\left[c-\alpha_0(1-a+c-\alpha_0)\right]},\quad \mu_{12}=\frac{2c-\alpha_0(1-a+c)+\alpha_0\sqrt{\sigma_+\sigma_-}}{2\left[c-\alpha_0(1-a+c-\alpha_0)\right]},$$
$$\boldsymbol{\mu}_{22}=\frac{b-\sqrt{\sigma_+\sigma_-}}{2a}\boldsymbol{\alpha}_D^{-1}+\frac{\alpha_0(b-a(1-a+c))-bc-(\alpha_0(1-a)-c)\sqrt{\sigma_+\sigma_-}}{2a\left[c-\alpha_0(1-a+c-\alpha_0)\right]}\vec{\boldsymbol{e}}_{N-1}\vec{\boldsymbol{e}}^T_{N-1},$$

and $\boldsymbol{\alpha}_D$ defined by (5.1.11). With this expression we have the n-step transition probability matrix using the Karlin-McGregor formula and the recurrence of the process is not affected by the Darboux transformation.

The matrix-valued polynomials $(\widetilde{\boldsymbol{Q}}_n)_{n\geq 0}$ orthogonal with respect to $\widetilde{\boldsymbol{\Psi}}$ can be computed from (5.2.15) and can also be written as combinations of Chebychev polynomials of the first and second kind (see (5.3.5)).

Conclusion and final comments

This work has been dedicated to the exploration of the spectral properties of birth-death chains and their relation with Darboux transformations. The main goal was to generalize the results published by F. A. Grünbaum and M. D. de la Iglesia in [24], which were summarized in Section 1.6. Building upon this seminal work, we successfully extended the study to discrete-time birth-death chains with a state space on $\mathbb{Z}$, an achievement presented in the author's master's thesis [38] and detailed in Chapter 2. Chapter 3 explored different kinds of stochastic factorizations like RA or AR, different from the ones presented in Chapter 2. Chapter 4 presented a non-trivial example related to the family of associated Jacobi polynomials. At the end of the chapter, an urn model for the associated Jacobi polynomials family is described, providing a valuable application of the theoretical results. In Chapter 5 we generalized our results to the discrete version of the Walsh's spider, a more complex process. Let us highlight that the explicit expressions for the spectral matrices and the Karlin-McGregor representation formulas enable the computation of n-step transition probabilities and offer valuable insights into fundamental properties of these birth-death chains. Each chapter in this thesis is directly linked to a published paper, namely [32], [33], [34], and [35].

In summary, this work has provided novel perspectives and tools for understanding and analyzing birth-death chains. By exploring diverse factorizations and transformations, we have expanded our knowledge and enriched the field with new contributions. The application of the theoretical results to the urn model for the associated Jacobi polynomials highlights the practical relevance of our research. Looking ahead, there are several avenues for future research. For instance, the stochastic factorization methods could be applied to other extensions of birth-death chains models related to multiple orthogonal polynomials, orthogonal polynomials in several variables or Krall-type orthogonal polynomials (see [13, 26, 15]). Another promising direction is the extension of our findings to continuous-time scenarios. Exploring deeper into the physical and probabilistic interpretations of stochastic factorizations and spectral properties may lead to new insights in different fields.

Bibliography

[1] R. Askey, and J. Wimp, (1984), *Associated Laguerre polynomials*, Proceedings of the Royal Society of Edinburgh, 96, 15–37.

[2] P. Barrucand, and D. Dickinson, (1968), *On the associated Legendre polynomials*, In: Orthogonal Expansions and their Continual Analogues (Proc. Conf., Edwardsville, Ill., 1967), 43–50.

[3] T. S. Chihara, (1978), *An Introduction to Orthogonal Polynomials*, New York: Gordon and Breach.

[4] A. Clayton, (2010), *Quasi-birth-and-death processes and matrix-valued orthogonal polynomials*, SIAM Journal on Matrix Analysis and Applications, 31(5), 2239–2260.

[5] P. Coolen-Schrijner and E. A. van Doorn, (1998), *Analysis of random walks using orthogonal polynomials*, Journal of Computational and Applied Mathematics, 99(1-2), 387–399.

[6] H. Dette, (2000), *First return probabilities of birth and death chains and associated orthogonal polynomials*, Proceedings of the American Mathematical Society, 129(6), 1805–1815.

[7] H. Dette, B. Reuther, W. Studden and M. Zygmunt, (2006), *Matrix measures and random walks with a block tridiagonal transition matrix*, SIAM Journal on Matrix Analysis and Applications, 29(1), 117–142.

[8] M. Domínguez de la Iglesia, (2022), *Orthogonal Polynomials in the Spectral Analysis of Markov Processes. Birth-Death Models and Diffusion*, Encyclopedia of Mathematics and its Applications 181, Cambridge University Press.

[9] E. A. van Doorn, (1981), *Stochastic Monotonicity and Queueing Applications of Birth-Death Processes*, Lecture Notes in Statistics, 4, Springer, New York.

[10] E. A. van Doorn and P. Schrijner, (1993), *Random walk polynomials and random walk measures*, Journal of Computational and Applied Mathematics, 49(1-3), 289–296.

[11] A. J. Durán and F. A. Grünbaum, (2004), *Orthogonal matrix polynomials satisfying second order differential equations*. International Mathematics Research Notices, 2004(10), 461–484.

[12] A. J. Durán and M. D. de la Iglesia, (2008), *Second order differential operators having several families of orthogonal matrix polynomials as eigenfunctions*, International Mathematics Research Notices, 2008, rnn084.

[13] A. J. Durán and M. D. de la Iglesia, (2015), *Constructing Bispectral Orthogonal Polynomials from the Classical Discrete Families of Charlier, Meixner and Krawtchouk*, Constructive Approximation, 41, 49–91.

[14] S. N. Evans and R. B. Sowers, (2003), *Pinching and twisting Markov Processes*, The Annals of Probability, 31(1), 486–527.

[15] L. Fernández and M. D. de la Iglesia, (2021), *Quasi-birth-and-death processes and multivariate orthogonal polynomials*, Journal of Mathematical Analysis and Applications, 499(1), 125029.

[16] P. Flajolet and F. Guillemin, (2000), *The formal theory of birth-and-death processes, lattice path combinatorics and continued fractions*, Advances in Applied Probability, 32(3), 750–778.

[17] W. K. Grassmann, (1993), *Means and variances in Markov reward systems*, Meyer C. D. and Plemmons, R.J. (Ed.) Linear Algebra, Markov Chains and Queueing Models. The IMA Volumes in Mathematics and its Applications. New York, NY: Springer, 48, 193–204.

[18] F. A. Grünbaum, (2007), *Random walks and orthogonal polynomials: some challenges*, Probability, Geometry and Integrable Systems, 55, 241–260.

[19] F. A. Grünbaum, (2008), *QBD processes and matrix orthogonal polynomials: some new explicit examples*, Numerical Methods for Structured Markov Chains, Eds. D. Bini, B. Meini, V. Ramaswami, M.A. Remiche and P. Taylor, Dagstuhl Seminar Proceedings.

[20] F. A. Grünbaum, (2010), *A spectral weight matrix for a discrete version of Walsh's spider*, Topics in Operator theory: Advances and Applications, 202, 253–264.

[21] F. A. Grünbaum, (2010), *An urn model associated with Jacobi polynomials*, Communications on Applied Mathematics and Computation Sciences, 5(1), 55–63.

[22] F. A. Grünbaum and L. Haine, (1997), *A theorem of Bochner, revisited*, in: Algebraic Aspects of Integrable Systems (A.S. Fokas and I.M.Gelfand, eds.), Progr. Nonlinear Differential Equations Appl., 26, 143–172.

[23] F. A. Grünbaum and L. Haine, (1999), *Associated polynomials, spectral matrices and the bispectral problem*, Methods and Applications of Analysis, 6(2), 209–224.

[24] F. A. Grünbaum and M. D. de la Iglesia, (2018), *Stochastic LU factorizations, Darboux transformations and urn models*, Journal of Applied Probability, 55(3), 862–886.

[25] F. A. Grünbaum and M. D. de la Iglesia, (2019), *Stochastic Darboux transformations for quasi-birth-and-death processes and urn models*, Journal of Mathematical Analysis and Applications, 478(2), 634–654.

[26] F. A. Grünbaum and M. D. de la Iglesia, (2022), *An urn model for the Jacobi-Piñeiro polynomials*, Proceedings of the American Mathematical Society, 150, 3613-3625.

[27] F. A. Grünbaum, I. Pacharoni and J. A. Tirao, (2002), *Matrix-valued spherical functions associated to the complex projective plane*, Journal of Functional Analysis, 188(2), 350–441.

[28] H. Hajri, (2012), *Discrete approximations to solution flows of Tanaka's SDE related to Walsh Brownian motion*, in: Donati-Martin C., Lejay A., Rouault A. (eds. Séminaire de Probabilités XLIV. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2046, 167–190.

[29] D. P. Heyman, (1995), *A decomposition theorem for infinite stochastic matrices*, Journal of Applied Probability, 32(4), 893–903.

[30] R. A. Horn and C. R. Johnson, (1990), *Matrix Analysis*, Cambridge University Press, Cambridge.

[31] M. D. de la Iglesia, (2011), *A note on the invariant distribution of a quasi-birth-and-death process*, Journal of Physics A: Mathematical and Theoretical, 44, 135201.

[32] M. D. de la Iglesia and C. Juarez, (2020), *The spectral matrices associated with the stochastic Darboux transformations of random walks on the integers*, Journal of Approximation Theory, 258, 105458.

[33] M. D. de la Iglesia and C. Juarez, (2021), *Absorbing–reflecting factorizations for birth–death chains on the integers and their Darboux transformations*, Journal of Approximation Theory, 266, 105583.

[34] M. D. de la Iglesia and C. Juarez, (2023), *Birth-death chains on a spider: Spectral analysis and reflecting-absorbing factorization*, Journal of Mathematical Analysis and Applications, 517(2), 126624.

[35] M. D. de la Iglesia and C. Juarez, (2023), *The bilateral birth-death chain generated by the associated Jacobi polynomials*, Studies in Applied Mathematics, 151(2), 616–642.

[36] M.E.H. Ismail, J. Letessier, D. Masson and G. Valent, (1990), *Birth and death processes and orthogonal polynomials*, in: Orthogonal Polynomials, P. Nevai (ed. Kluwer Acad. Publishers), 229–255.

[37] M.E.H. Ismail and D. Masson, (1990), *Two families of orthogonal polynomials related to Jacobi polynomials*, Rocky Mountain Journal of Mathematics, 21(1), 359–375.

[38] C. Juarez, (2019), *Transformaciones de Darboux estocásticas para caminatas aleatorias en $\mathbb{N}_0$ y $\mathbb{Z}$*, Masters's Thesis, Universidad Nacional Autónoma de México. TESIUNAM.

[39] S. Karlin and J. McGregor, (1957), *The differential equations of birth and death processes, and the Stieltjes moment problem*, Transactions of the American Mathematical Society, 85(2), 489–546.

[40] S. Karlin and J. McGregor, (1957), *The classification of birth-and-death processes*, Transactions of the American Mathematical Society, 86(2), 366–400.

[41] S. Karlin and J. McGregor, (1959), *Random walks*, Illinois Journal of Mathematics, 3(1), 66–81.

[42] S. Karlin and H. Taylor, (1975), *A First Course in Stochastic Processes*, Academic Press, New York.

[43] D. G. Kendall, (1959), *Integral representations for Markov transition probabilities*, Bulletin of the American Mathematical Society, 64(6), 358–362.

[44] M. Kijima, M. G. Nair, P. K. Pollett, and E. A. van Doorn, (1997), *Limiting conditional distributions for birth-death processes*, Advances in Applied Probability, 29(1), 185–204.

[45] W. Ledermann and G. E. H. Reuter, (1954), *Spectral theory for the differential equations of simple birth and death processes*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences. 246(914), 321–369.

[46] J. R. Norris, (1997), *Markov Chains*, Cambridge University Press.

[47] W. Schoutens, (2000), *Stochastic Processes and Orthogonal Polynomials*, Springer New York.

[48] G. Szegö, (1975), *Orthogonal Polynomials*, 4th ed., American Mathematical Society Colloquium Publications: Providence, Rhode Island, USA, 23.

[49] V. Vigon, (2013), *LU factorization versus Wiener-Hopf factorization for Markov chains*, Acta Applicandae Mathematicae, 128, 1–37.

[50] H. S. Wall, (1948), *Analytic theory of continued fractions*, D. Van Nostrand Company Inc. London.

[51] J. B. Walsh, (1978), *A diffusion with discontinuous local time*, Astérisque, 52-53, 37–45.

[52] J. Wimp, (1987), *Explicit formulas for the associated Jacobi polynomials and some applications*, Canadian Journal of Mathematics, 39(4), 983–1000.