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PRODUCTION OF PRIMORDIAL GRAVITATIONAL WAVES IN TELEPARALLEL GRAVITY

T E S I S

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Scientific publications

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Introduction

Teleparallel gravity is a gauge theory of translations locally invariant under the Lorentz group where the gravitational field is a manifestation of the non-trivial geometry given by the torsion and not the curvature. In Sec. 1 we revisit the foundations of general relativity and teleparallel gravity from a geometric point of view, we study the teleparallel equivalent to general relativity formulation and why we need to look for cosmological viable models in teleparallel gravity. In this latter idea, we explore the field equations of $f(T, B)$ and $f(T)$ gravity.

In Sec. 2 we explore the so called cosmological principle and the associated FLRW geometry in order to explore the Friedman equation in both general relativity and teleparallel gravity. We also recall the basics of inflationary cosmology in the slow roll approximation, briefly discussing how it solves some of the issues of the standard cosmology scenario, the presence of the scalar field as the responsible mechanism of inflation and finally introduce the slow roll parameters.

In Sec. 3 we study the basics of linear perturbation theory. We begin by focusing in the cosmological case in general relativity by writing the most general metric and identify background and perturbed quantities in order to fully decompose the perturbed quantities into scalar, vector and tensor perturbations using the SVT decomposition. We briefly discuss the gauge problem, obtain the gauge transformations of the perturbations in general relativity and construct the gauge-invariant potentials associated to each type of perturbation. Finally, we discuss linear perturbation theory in teleparallel gravity, focusing only in the tensor perturbation case, showing the relation between the perturbation in teleparallel gravity and general relativity and at last obtaining the field equations associated to the tensor perturbations, which are the equation we will use to study the production of primordial gravitational waves in the next sections.

In Sec. 4 we explore the production of gravitational waves from vacuum fluctuations. Vacuum fluctuations are understood as fluctuations of the scalar field responsible for inflation, and such scalar field perturbations do not produce tensor anisotropic stress. Since the fluctuations are in the inflaton field, we consider two backgrounds compatible with an exponentially accelerated expansion as that required by inflation, namely, a de Sitter and quasi de Sitter expansions. We proceed to study the gravitational waves generated both general relativity and teleparallel gravity from these fluctuation within both backgrounds. We found that there is not differences in the de Sitter background case in comparison to general relativity and really strong differences in the value of the tensor spectral index on both theories. We also discuss the density energy spectrum of such waves in the context of both theories and finally, briefly discuss what are the implications on the power spectrum if we consider any other quantum state for the universe different from the vacuum one.

In Sec. 5 we explore the power spectrum of primordial gravitational waves, but this time we also include any possible source of tensor anisotropic stress. We use the method of Green's function to obtain the most general solution of the gravitational waves equation in teleparallel gravity with tensor anisotropic stress. We compute the most general contribution to the peaks of the power spectrum coming from the anisotropic stress and briefly discuss some particular cases, namely, local thermal fluctuations, first order cosmological phase transitions and primordial magnetic fields, which are well known sources of tensor anisotropic stress in general relativity.

In Sec. 6 we present the conclusions of this thesis. We discuss the main observational predictions of teleparallel gravity compared to those of general relativity. We discuss how teleparallel gravity extended models allow for a high value of the tensor spectral index whereas general relativity only a small value. We also analyse the observational implications on the peaks of the power spectrum in the $\epsilon \to 0$, limit which allows us to directly compare the peaks on teleparallel gravity with those of general relativity. We also discuss the difference between the energy density of gravitational waves in teleparallel gravity and general relativity.

Finally, in appendix A with discuss about the spin connection in teleparallel gravity and in appendix B the gauge problem of the cosmological linear perturbation in teleparallel gravity. Such topics are closely related with the topic of the thesis, they are not needed to read the thesis and understand the results but are worth reading as supplementary information about the subtleties of the topics on this thesis.

Table of Contents

1. Theoretical Foundations

1.1. General Relativity

1.1.1. Geometrical setup

General Relativity (GR) is the most successful theory of gravity nowadays [1]. GR describes the spacetime structure as differentiable 4-dimensional manifold $\mathcal{M}$ with a symmetric non-degenerate bilinear form $g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ in each tangent space of the manifold such that, $\forall x \in \mathcal{M}$ it satisfies

$$\begin{aligned} g_x : T_x\mathcal{M} \times T_x\mathcal{M} &\to \mathbb{R} \\ (v_x, w_x) &\mapsto g_x(v_x, w_x) = g_{\mu\nu}(x) v_x^\mu w_x^\nu, \end{aligned} \tag{1}$$

where the positive-definite conditions is not required, .i.e, the metric tensor is a Lorentz metric [2]. The manifold is also endowed with a linear connection ∇ which is compatible with the metric $\nabla g = 0$ and symmetric $\nabla_{[\mu}\nabla_{\nu]} f = 0$, $\forall f : \mathcal{M} \to \mathbb{R}$ such that [3]

$$\nabla : \mathcal{X}(\mathcal{M}) \times \mathcal{X}(\mathcal{M}) \to \mathcal{X}(\mathcal{M}) \tag{2}$$
$$(V, W) \mapsto \nabla_V W, \tag{3}$$

where $\mathcal{X}(\mathcal{M})$ is the space of vector fields on $\mathcal{M}$, such that, if $X, Y, Z \in \mathcal{X}(\mathcal{M})$ then $\forall f : \mathcal{M} \to \mathbb{R}$ and $a \in \mathbb{R}$, ∇ satisfies:

1. $\nabla_X(aY + Z) = a\nabla_X Y + \nabla_X Z$,
2. $\nabla_X(fY) = f\nabla_X Y + (X[f])Y$,
3. $\nabla_{fX+aY} Z = f\nabla_X Z + a\nabla_Y Z$.

$\nabla_V W$ is called *the covariant derivative of* W *w.r.t* V [3].
The symmetric condition along with the metric compatibility condition over the linear connection define a unique form of the connection coefficients $\nabla_\mu e_\nu = \Gamma^\rho_{\nu\mu} e_\rho$, the so called *Christoffel Symbols*

$$\Gamma^\lambda_{\alpha\nu} = \frac{1}{2} g^{\lambda\mu} \left(g_{\mu\alpha,\nu} + g_{\nu\mu,\alpha} - g_{\alpha\nu,\mu} \right). \tag{4}$$

The connection associated with the Christoffel Symbols is called *the Levi-Civita connection*[4].

In this geometrical setup, six axioms are imposed on the theory. To be more specific, any point-like test particle follow a trajectory defined by the geodesic equations, which is a geometrical form of the *Weak Equivalence Principle*. The existence of *Riemann normal coordinates* guarantees that locally, the laws of physics are those of the Special Relativity, which is the *Einstein Equivalence Principle*.

The four final axioms are related to the field equations that govern the physics in this geometrical setup and the source of the gravitational field. The *principle of general covariance* states that the metric tensor $g_{\mu\nu}$ is the unique fundamental mathematical object to appear in the field equations and, in general, all laws of physics. The other axioms states that the field equations must be of second order, linear on the second derivatives of the metric tensor, and that the matter, modelled by its energy-momentum tensor, is source for gravitational interaction. These conditions are achieve by the *Einstein Field equations* which are going to be discussed in the following.

1.1.2. Field equations

The standard way of deducing the Einstein Field Equations is through a variational principle (See Appendix B of [5] for a brief discussion on the topic) which consists of varying and minimizing an action w.r.t the dynamical object of the theory. In the case of GR, the action is the *Einstein-Hilbert* action plus a matter action minimally coupled to gravity, and the variation is w.r.t. the metric tensor.

The Einstein-Hilbert action is given by

$$S_{EH}[g_{\mu\nu}] = \frac{1}{2k}\int R\sqrt{-g}d^4x, \tag{5}$$

with R the Ricci scalar, g the determinant of the metric and $\kappa = 8\pi G$. On the other hand, the matter action is given by

$$S_{\text{matt}} = \int \mathcal{L}_M\sqrt{-g}d^4x \quad \text{with} \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\mathcal{L}_M\sqrt{-g})}{\partial g^{\mu\nu}}. \tag{6}$$

Hence, the action of GR is given by

$$S[g_{\mu\nu},\boldsymbol{\psi}] = \frac{1}{2\kappa}\int R\sqrt{-g}d^4x + S_{\text{matt}}[g_{\mu\nu},\boldsymbol{\psi}], \tag{7}$$

such that, by varying w.r.t $g_{\mu\nu}$ and minimizing we get

$$0 = \delta S = \int\left(\frac{1}{2\kappa}\left[R_{\mu\nu}\sqrt{-g} - \frac{1}{2}g_{\mu\nu}R\sqrt{-g}\right] + \frac{\partial(\mathcal{L}_M\sqrt{-g})}{\partial g^{\mu\nu}}\right)\delta g^{\mu\nu}d^4x, \tag{8}$$

which lead us to the Einstein-Field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \tag{9}$$

The deduction of the Einstein Field Equations (9) seems to be highly dependent on the Einstein-Hilbert action. However, this is not the case, since there exists a more general result obtained by Lovelock [6][7], which states that the most general rank 2 tensor $A^{\mu\nu}$ which is concomitant of the metric tensor $g_{\mu\nu}$ and its first two derivatives, divergence free, symmetric and linear in the second derivatives of $g_{\mu\nu}$ in 4 dimensions is

$$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu}, \tag{10}$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ is the Einstein tensor. Hence, the only possible second-order Euler-Lagrange equations in 4 dimensions coming from a scalar density $\mathcal{L} = \mathcal{L}(g_{\mu\nu})$ are the Einstein Field Equations with a possible cosmological constant, provided one does not add extra fields or vary the number of dimensions [1]. Thus, in order to obtain field equations which are not the

Einstein Field Equations, relaxing the previous conditions are necessary, namely, allow more fundamental fields besides the metric tensor, accept derivatives of the metric tensor with order greater than 2, work on a manifolds with dimension different from 4, etc.

Several models of modified and extended gravity rely on relaxing these conditions. In this work we will explore the so called *Teleparallel Gravity* and a brief introduction on the topic is presented in the next subsection.

1.2. Teleparallel Gravity

Teleparallel Gravity (TG) is one of the possibilities to solve some of the observational issues present in GR. The fundamental concept of TG is not the curvature of spacetime, as in GR, but rather the curvature of spacetime's torsion. The mathematical foundation is a vector fiber bundle where the tangent vector spaces are identified with the Minkowski space, compared to the coordinate spaces of GR. In this sense, the dynamical fields are the tetrad fields and spin connection, compared to the metric and linear connection in GR. The success of TG consist of being a possible solution of the Dark Energy problem through extended models of the Teleparallel Equivalent to General Relativity, see [8]. Let us dive into the details of this theory.

1.2.1. Geometrical setup and gauge theory

TG has quite a different geometrical setup than GR, as mentioned previously. In TG we work on a 4-dimensional vector bundle (cf. refs [9][10]) $\xi = (\mathcal{M}, E, \pi)$ with $\mathcal{M}$ the space-time manifold, $E = \mathbb{R}^{1,3}$ the Minkowski space-time and $\pi : E \to \mathcal{M}$ a smooth map such that $E_p = \pi^{-1}(p)$ is a vector space isomorphic to E.

Analogous to GR, we have two geometrical objects that play important roles on the theory [11].The first one is the *tetrad* $\mathbf{e} \in \Omega^1(\mathcal{M}, \mathbb{R}^{1,3})$, i.e., a set of differential 1-forms on $\mathcal{M}$ assuming values on the Minkowski space.

Upon the introduction of local coordinates, the tetrad fields $\mathbf{e} = \{e^A\}_{A=0}^3$ read as $e^A = e^A{}_\mu dx^\mu$.

The tetrad fields and their dual vectors $e^A = e^A{}_\mu dx^\mu$ and $E_A = E_A{}^\mu \partial_\mu$ constitute non-coordinate basis of [1] $\Gamma(T^*U)$ and $\Gamma(TU)$ at each local coordinate system $U \subset \mathcal{M}$, respectively [12][13], and satisfy orthonormality conditions

$$e^A{}_\mu E_A{}^\nu = \delta^\nu_\mu \quad \text{and} \quad e^A{}_\mu E_B{}^\mu = \delta^B_A, \tag{11}$$

i.e., $E_A{}^\mu = (e^A{}_\mu)^{-1}$.

Since $e^A{}_\mu$ is a 1-form on $\mathcal{M}$ assuming values on $\mathbb{R}^{1,3}$, under diffeomorphism over $\mathcal{M}$ and local Lorentz transformation $\Lambda^A{}_B(\mathbf{x}) \in \mathrm{SO}^+(1,3)$, the components of the tetrad fields transform as

$$e^A{}_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}} e^A{}_\nu \quad \text{and} \quad e^A{}_\mu = \Lambda^A{}_B e^B{}_\mu. \tag{12}$$

[1] $\Gamma(TU)$ are the sections over the tangent bundle and $\Gamma(T^*U)$ the sections over the cotangent bundle.

Among all the possible tetrad fields, a particular set of tetrad fields exist such that, in every point of the manifold $p \in \mathcal{M}$, the metric is diagonalized

$$g_{AB}(p) = \eta_{AB}(p) = g_{\mu\nu}(p) E^{\mu}_{A}(p) E^{\nu}_{B}(p), \tag{13}$$

or consequently

$$g_{\mu\nu}(p) = e^{A}{}_{\mu}(p) e^{B}{}_{\nu}(p) \eta_{AB}(p). \tag{14}$$

Such tetrad are called *Orthonormal Basis* or *Orthonormal Frames* [14], which are the type of tetrad field we will be working with throughout the entire text.

The second one is the *spin connection* $\boldsymbol{\omega} \in \Omega^1(\mathcal{M}, \mathfrak{so}(1,3))$, i.e., is a set of differential 1-forms assuming values on $\mathfrak{so}(1,3)$, the Lie Algebra of the Lorentz Group [15]. To be more specific, the spin connection, in a local coordinate system U, is defined as [13]

$$\boldsymbol{\omega} = \frac{1}{2} \omega^{AB}{}_{\mu} dx^{\mu} \otimes S^{AB} \in \Gamma\left(T^{*}U \otimes \mathfrak{so}(1,3)\right), \quad x \mapsto (x, \boldsymbol{\omega}_x) \tag{15}$$

where $\boldsymbol{\omega}_x = \frac{1}{2} \omega^{AB}{}_{\mu}(x) dx^{\mu} \otimes S^{AB} \in T^{*}U \otimes \mathfrak{so}(1,3)$. The components of the spin connection must be antisymmetric w.r.t the Lorentzian indices.

With the spin connection and the tetrad fields, it is possible to induce a linear connection on the base manifold $\mathcal{M}$ [15] whose coefficients are given by

$$\Gamma^{\rho}_{\nu\mu} \equiv E^{\rho}_{A} \partial_{\mu} e^{A}_{\nu} + E^{\rho}_{A} \omega^{A}{}_{B\mu} e^{B}_{\nu} = E^{\rho}_{A} \mathscr{D}_{\mu} e^{A}_{\nu}, \tag{16}$$

where $\mathscr{D}_{\mu}$ is the Fock-Ivanenko derivative.

From these properties it is possible to show that the covariant derivative, whose coefficient of connection are defined in (16), of the tetrad field is zero and that, due to the antisymmetry of the spin connection, the metric is compatible with this linear connection.

Nevertheless, similar to GR, we require some axioms imposed over the geometry. TG requires that the curvature 2-form vanishes (cf. ref [15])

$$\mathbf{R} = \frac{1}{4} R^{A}{}_{B\nu\mu} S^{B}_{A} dx^{\nu} \wedge dx^{\mu} \quad \text{with} \quad R^{A}{}_{B\nu\mu} = \partial_{\nu} \omega^{A}{}_{B\mu} - \partial_{\mu} \omega^{A}{}_{B\nu} + \omega^{A}{}_{D\nu} \omega^{D}{}_{B\mu} - \omega^{A}{}_{D\mu} \omega^{D}{}_{B\nu} \equiv 0. \tag{17}$$

This condition can only be achieved through the so called *purely inertial* spin connection [15][11]

$$\omega^{A}{}_{B\mu} = \Lambda^{A}{}_{C}(x) \partial_{\mu} \Lambda^{C}{}_{B}(x). \tag{18}$$

On the other hand, TG also requires the torsion 2-form to be non-vanishing

$$\mathbf{T} = \frac{1}{2} T^{A}_{\nu\mu} P_{A} dx^{\nu} \wedge dx^{\mu} \quad \text{with} \quad T^{A}_{\nu\mu} = \partial_{\nu} e^{A}_{\mu} - \partial_{\mu} e^{A}_{\nu} + \omega^{A}{}_{C\nu} e^{C}_{\mu} - \omega^{A}{}_{C\mu} e^{C}_{\nu} \neq 0, \tag{19}$$

where P_A are the generators of the translational group. The latter condition is achieved by imposing covariance w.r.t an infinitesimal gauge translation, associating a non-trivial translational potential to the construction of the covariant derivative (cf. ref [15]). Hence, the theory arises as a Gauge Theory of the translational Group and invariant under local Lorentz Transformation due to condition (18) over the spin connection.

These are the main theoretical differences and theoretical strengths of the theory. Similar to all other theories of the fundamental forces, TG arises as Gauge Theory which is also locally Lorentz invariant. In TG, the space-time manifold is globally flat but with non-trivial geometry given by the torsion, hence, the gravitational interaction is mediated through torsion and not curvature.

Since the torsion is different from zero, we can see that

$$-2\Gamma^{\rho}_{[\nu\mu]} = T^{\rho}_{\nu\mu} = e^{\rho}_{A} T^{A}_{\nu\mu} \neq 0, \tag{20}$$

hence, the linear connection on the base manifold is not symmetric and thus a different connection than the Levi-Civita connection, receiving the name of *Teleparallel Connection*.

According to the definition of the contortion tensor $K^{\rho}_{\mu\nu}$ [16], we can relate the Teleparallel Connection with the Levi-Civita connection [2] as

$$\Gamma^{\rho}_{\mu\nu} = \mathring{\Gamma}^{\rho}_{\mu\nu} + K^{\rho}_{\mu\nu}, \tag{21}$$

and being able to compute the curvature tensor as [17]

$$R^{\alpha}{}_{\beta\mu\nu} = \mathring{R}^{\alpha}{}_{\beta\mu\nu} + \mathring{\nabla}_{\mu} K^{\alpha}{}_{\nu\beta} - \mathring{\nabla}_{\nu} K^{\alpha}{}_{\mu\beta} + K^{\alpha}{}_{\mu\rho} K^{\rho}{}_{\nu\beta} - K^{\alpha}{}_{\nu\rho} K^{\rho}{}_{\mu\beta}. \tag{22}$$

Consequently, the Ricci scalar is split as

$$R = \mathring{R} + T - B, \tag{23}$$

where

$$T = T^{\alpha}{}_{\sigma\rho} S_{\alpha}{}^{\sigma\rho} = \frac{1}{4} T^{\mu\nu\lambda} T_{\mu\nu\lambda} + \frac{1}{2} T^{\mu\nu\lambda} T_{\nu\mu\lambda} - T^{\mu} T_{\mu}, \tag{24}$$

$$S_{\alpha}{}^{\sigma\rho} = \frac{1}{4} (T_{\alpha}{}^{\sigma\rho} + T^{\rho\sigma}{}_{\alpha} - T^{\sigma\rho}{}_{\alpha} - 2T^{\lambda\sigma}{}_{\lambda} \delta^{\rho}_{\alpha} + 2T^{\lambda\rho}{}_{\lambda} \delta^{\sigma}_{\alpha}) = \frac{1}{2} \left(K^{\sigma\rho}{}_{\alpha} + T^{\sigma} \delta^{\rho}_{\alpha} - T^{\rho} \delta^{\sigma}_{\alpha} \right), \tag{25}$$

$$B = \frac{2}{e} \partial_{\mu} \left(e T^{\mu} \right) = 2 \nabla_{\mu} T^{\mu}, \tag{26}$$

are the *Torsion Scalar*, *Superpotential Tensor* and *Boundary Term*, respectively, with $T_{\mu} = T^{\lambda}{}_{\lambda\mu}$. The determinant of the tetrad is given by $e = \det(e^{a}_{\lambda}) = \sqrt{-g}$.

However, since no curvature is present in TG, $R = 0$ and then

$$\mathring{R} = -T + B, \tag{27}$$

which will allow us to discuss the field equations down below.

[2] Henceforward, when discussing TG, we will refer every quantity computed with the Levi-Civita connection with a circle over it, thus, $\mathring{\Gamma}^{\rho}_{\mu\nu}$ refers to the Levi-Civita connection itself.

1.2.2. TEGR field equations

Consider the GR action (7), and substitute $\sqrt{-g} = e$ and $\mathring{R} = -T + B$ into it

$$S_{\text{TEGR}} = \frac{1}{2\kappa}\int (-T+B)\, ed^4x + S_{\text{matt}}[g_{\mu\nu}, \psi]. \tag{28}$$

Since B is a boundary term

$$\int Bed^x = 0, \tag{29}$$

and then we arrive to the *Teleparallel Equivalent to General Relativity* (TEGR) action [17]

$$S_{\text{TEGR}} = -\frac{1}{2\kappa}\int Ted^4x + S_{\text{matt}}[g_{\mu\nu}, \psi]. \tag{30}$$

Hence, TEGR and GR Lagrangian density differ only by a boundary term and as a consequence, the field action are completely equivalent in TEGR and GR, providing the same physical effect. That is where TEGR acquires its name. Therefore, whether the gravitational effects are due to curvature or torsion is a matter of interpretation [18].

In TG it is usual to work within a particular Gauge called *Weitzenböck Gauge*, which consists of working with zero spin connection $\omega^A{}_{B\mu} = 0$. Within this Gauge, the Teleparallel connection is called the Weitzenböck connection, and then our entire theory depends exclusively on the tetrad.

Taking variation of (30) w.r.t the tetrad and minimizing leads to [5]

$$\frac{1}{2\kappa}\int d^4x\,[e\delta T + T\delta e] = \delta S_{\text{matt}}, \tag{31}$$

where

$$\delta e = e e^\lambda_A \delta e^A_\lambda, \quad \delta T = -4T^\alpha{}_{\mu A} S_\alpha{}^{\mu\lambda}\delta e^A_\lambda + 4S_A{}^{\mu\lambda}\partial_\mu \delta e^A_\lambda. \tag{32}$$

Therefore, neglecting total derivatives, the TEGR field equations are

$$-4T^\alpha{}_{\mu A} S_\alpha{}^{\mu\lambda} - \frac{4}{e}\partial_\mu(eS_A{}^{\mu\lambda}) + Te^\lambda_A = 2\kappa\theta^\lambda_A, \tag{33}$$

such that

$$\theta^\lambda_A = \frac{1}{e}\frac{\delta(e\mathcal{L}_{\text{matt}})}{\delta e^A_\lambda}. \tag{34}$$

These equations can be recast in a more familiar way as

$$\mathring{G}^\lambda_\nu \equiv \kappa\mathcal{T}^\lambda_\nu = \frac{2}{e}e^A_\nu\partial_\mu(eS_A{}^{\lambda\mu}) - 2T^\alpha{}_{\mu\nu}S_\alpha{}^{\mu\lambda} + \frac{1}{2}T\delta^\lambda_\nu, \tag{35}$$

with $\mathring{G}^\lambda_\nu$ the Einstein tensor computed with the Levi-Civita connection and $\mathcal{T}^\lambda_\nu = e^A_\nu\theta^\lambda_A$ the energy-momentum tensor [3].

[3] We changed the notation of the energy-momentum tensor from T to $\mathcal{T}$ in order to avoid confusion with the torsion.

1.2.3. $f(T)$ and $f(T,B)$ gravity

Since TEGR and GR are equivalent, the field equation (35) suffer from the same advantages and disadvantages phenomenologically speaking, i.e. at the level of the classical equations of motion. Therefore, we will look for extensions of TEGR in the context of TG. A way to extend TEGR is considering, instead of $-T+B$, a general function of the torsion scalar and the boundary term $f(T,B)$ in the action (30) as

$$S_{f(T,B)} = \frac{1}{2\kappa}\int ef(T,B)d^4x + S_{\text{matt}}, \tag{36}$$

which is known as $f(T,B)$ gravity.

By varying and minimizing the action w.r.t to the tetrad, we obtain [5]

$$\delta S_{f(T,B)} = \int \frac{1}{2\kappa}\left[f\delta e + ef_B\delta B + ef_T\delta T + 2\kappa\delta(e\mathcal{L}_{\text{matt}})\right] d^4x, \tag{37}$$

where, neglecting total derivatives, we obtain

$$ef_T\delta T = 4\left[-f_T\partial_\mu(eS_A^{\ \mu\lambda}) - \partial_\mu(f_T)eS_A^{\mu\lambda} + ef_T T^\alpha_{\ \mu A}S_\alpha^{\ \lambda\mu}\right]\delta e^A_\lambda, \tag{38}$$

$$f\delta e = efe^\lambda_a\delta e^A_\lambda, \tag{39}$$

$$ef_B\delta B = -ef_BBe^\lambda_A - 2ee^\lambda_A\mathring{\Box}f_B + 2ee^\nu_A\mathring{\nabla}^\lambda\mathring{\nabla}_\nu f_B + 4e(\partial_\mu f_B)S_A^{\ \lambda\mu}. \tag{40}$$

Hence, the field equations of $f(T,B)$ gravity are

$$\begin{aligned}-ef_BBe^\lambda_A - 2ee^\lambda_A\mathring{\Box}f_B + &2ee^\nu_A\mathring{\nabla}^\lambda\mathring{\nabla}_\nu f_B - 4e(\partial_\mu f_B)S_A^{\ \mu\lambda} \\ &- 4\partial_\mu(f_T)eS_A^{\ \mu\lambda} - 4f_T\partial_\mu\left(eS_A^{\mu\lambda}\right) + 4ef_T T^\alpha_{\ \mu A}S_\alpha^{\ \lambda\mu} + efe^\lambda_A = -2e\kappa\theta^\lambda_A,\end{aligned} \tag{41}$$

that can be recast in term of the Einstein tensor in the Levi-Civita connection as

$$\begin{aligned}\mathring{G}^\lambda_\nu \equiv \kappa\mathcal{T}^\lambda_\nu =& \delta^\lambda_\nu\mathring{\Box}f_B - \mathring{\nabla}^\lambda\mathring{\nabla}_\nu f_B + \frac{1}{2}f_BB\delta^\lambda_\nu + 2\left[\partial_\mu f_B + \partial_\mu f_T\right]S_\nu^{\ \mu\lambda} + \frac{2}{e}e^A_{\ \nu}f_T\partial_\mu(eS_A^{\ \mu\lambda}) \\ &- 2f_T T^\alpha_{\ \mu\nu}S_\alpha^{\ \lambda\mu} - \frac{1}{2}f\delta^\lambda_\nu.\end{aligned} \tag{42}$$

If we suppose that there is no dependence on the boundary term B of the functional $f(T,B) = f(T)$, which implies that the functional cannot longer imitate $f(\mathring{R})$ gravity [8], the equations (42) reduce to

$$\mathring{G}^\lambda_\nu \equiv \kappa\mathcal{T}^\lambda_\nu = 2(\partial_\mu f_T)S_\nu^{\ \mu\lambda} + \frac{2}{e}e^A_{\ \nu}f_T\partial_\mu(eS_A^{\ \mu\lambda}) - 2f_T T^\alpha_{\ \mu\nu}S_\alpha^{\ \lambda\mu} - \frac{1}{2}f\delta^\lambda_\nu, \tag{43}$$

which are the field equations of $f(T)$ gravity. If we choose $f(T) = -T$, equations (43) recover equations (35) which is expected.

It is in $f(T)$ and $f(T,B)$ gravity where we are going to analyse the quantum production of Gravitational Waves (GW), since TEGR is completely equivalent to GR and thus the production of GW is the same. However, before engaging in such endeavour we need to discuss the background cosmology and linear perturbation theory around that scheme.

2. Background Cosmology

2.1. FLRW metric

The FLRW metric relies on the *cosmological principle*, which states that the universe is spatially isotropic and homogeneous and has been shown to be successful in describing the universe on large scales [19][20].
The mathematical description of this is to consider the space-time manifold to be foliated with maximally symmetric spacelike slices $\mathcal{M} = \mathbb{R} \times \Sigma$ [21], where $\mathbb{R}$ represents the time direction and $\Sigma = \{\Sigma_t\}_{t\in\mathbb{R}}$ such that $\forall t \in \mathbb{R}$ and $\forall$ points $p, q \in \Sigma_t$ there exists an isometry Φ of $g_{\mu\nu}$ [2]

$$\Phi : M \to M, \quad (\Phi^* g)_{\mu\nu} = g_{\mu\nu}, \tag{44}$$

such that $\Phi(p) = q$ [4], that accounts for homogeneity.
For isotropy, we state that the spacetime is spatially isotropic if $\forall p \in \Sigma_t$ and any two vectors V and W in $T_p\Sigma_t$, there is an isometry of the spacetime such that $\Phi_* W$[5] is parallel to V [22].

The relationship between homogeneity and isotropy is not necessary, the space can be homogeneous but anisotropic and vice versa, nevertheless, if a space is isotropic in every point, then homogeneity is a consequence of such scheme [22].

The geometry of the entire manifold is encoded in the metric tensor $g_{\mu\nu}$, and for a spacetime spatially homogeneous and isotropic, the metric tensor is given by

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad d\Omega^2 = d\theta^2 + \sin^2(\theta) d\varphi^2, \tag{45}$$

which is the *Friedman-Lemaître-Robertson-Walker* (FLRW) metric, where the components of the metric are given explicitly as

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a(t)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & -a(t)^2 r^2 \sin^2(\theta) \end{pmatrix}. \tag{46}$$

The FLRW metric is given in spatial *comoving coordinates* (r, θ, ϕ) and *cosmic time*. $a(t)$ is called the *scale factor* and $k = +1, 0, -1$ the spatial curvature.

The evolution of the scale factor $a(t)$ will be given by the solution of the field equations of the theory.

It is possible to work with a different, yet useful, time measurement called *conformal time* [23]

$$a d\eta = dt \quad \to \quad \eta - \eta_i = \int_{t_i}^{t} \frac{dt'}{a(t')}, \tag{47}$$

[4] Φ^* is the *pull-back* of the mapping.
[5] Φ_* is the *pushforward* of the mapping.

where $c(\eta - \eta_i)$ represents the comoving distance travelled by a photon between times t_i and t [23]. Using the conformal time, the FLRW metric can be recast as

$$ds^2 = a(\eta)^2 \left(d\eta^2 - \frac{dr^2}{1 - kr^2} - r^2 d\Omega^2 \right). \tag{48}$$

From now on we will be working with the spatially flat case $k = 0$, which is the scenario supported by observations [24], and with Cartesian coordinates (x, y, z).

2.2. Basics of inflationary cosmology

The standard cosmological model have several achievements like, to name a few, explaining the thermal history of the universe, the CMB, the Large-Scale-Structure and expansion of the Universe [25][26] but also have shortcomings. For discussion of cosmological inflation, we will describe three of these issues.

1. *The flatness problem.* Observational data shows that the spatial curvature of the universe is approximately flat $\Omega_k \approx 0$ [24]. The flatness problem states exactly that: why do we observe today this value of the curvature density? Since this condition requires a fine tuning of $|\Omega - 1|$ in the early universe in order to achieve this current value of the curvature density [27].

2. *The horizon problem.* This problems relies on the validity of the cosmological principle on large scales. Why casually disconnected regions of the universe appear to be in thermal equilibrium $T_0 = 2.7255$ K? Since it appears that even going backwards on time, the distance between those zones were large enough to be casually disconnected and then no possible mechanism would exist to bring thermal equilibrium [27][28].

3. *The monopole problem.* This is a problem that does not purely rely on cosmology but also in particle physics. The problem is basically the lack of observation of magnetic monopoles in the present day of the universe since GUT models predict a high density number of magnetic monopoles [27][28].

These problems are solved by the proposal of an early period of the universe where the space-time expanded exponentially from a tiny size to a really big one. This period is known as *cosmological inflation* or simply inflation.

In this way, inflation solves the flatness problem by predicting that $|1 - \Omega|$ decays exponentially with time, decaying to zero in the present day, regardless of the initial conditions on the curvature of space [28]. The horizon problem is solved by predicting a really small size right before inflation $d_{\text{hor}} \sim 10^{-28}$ m, hence, the regions of the universe are casually connected and are able to get into thermal equilibrium, and then inflation expands the universe to $d_{\text{hor}} \sim 10^{16}$ m where some regions become casually disconnected as observed today [28]. Finally, it solves the monopole problem by predicting that the density number of magnetic monopoles decreases exponentially with time, hence, being approximately zero today and then practically undetectable [28].

There are several ways to model the inflationary era, the standard way is by including a scalar field, as a perfect fluid, to the gravitational theory in the energy-momentum tensor of the field equations.

It begins by defining the matter action of the scalar field as [29]

$$S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{49}$$

where we can derive the energy-momentum tensor as

$$\mathcal{T}_\phi^{\mu\nu} = g^{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right] - g^{\mu\rho}g^{\nu\sigma}\partial_\rho\phi\partial_\sigma\phi, \tag{50}$$

which has the same form as a perfect fluid with energy density, pressure and four velocity given by

$$\rho = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi), \tag{51}$$

$$P = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{52}$$

$$u^\mu = \left[-g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right]^{-1/2} g^{\mu\tau}\partial_\tau\phi, \tag{53}$$

respectively.

For a spatially homogeneous scalar field, only the pressure and energy density are non-vanishing and are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{54}$$

where the conservation equation of the energy momentum tensor implies that

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{55}$$

in a FLRW metric (see subsection 2.1).

In the case of a FLRW background, the acceleration equation for a spatially homogeneous scalar field reads as

$$\dot{H} = -4\pi G\dot{\phi}^2. \tag{56}$$

Hence, if an exponential acceleration like an inflationary era is required, the only dimensionless combination possible with $\dot{H}$ requires [23]

$$\frac{|\dot{H}|}{H^2} \ll 1, \tag{57}$$

or in terms of the scalar field

$$\dot{\phi}^2 \ll V(\phi). \tag{58}$$

This is the first *slow-roll* condition, and encourage the definition o the slow-roll parameter ϵ as

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \tag{59}$$

then the first slow-roll condition is achieved by $|\epsilon| \ll 1$.

It is also necessary that the exponential expansion lasts for long enough, which is realised by defining the second slow-roll parameter

$$\eta_\phi = -\frac{1}{H}\frac{\ddot{\phi}}{\dot{\phi}}, \tag{60}$$

and imposing $\eta \ll 1$ [23]. The conditions $|\epsilon|, \eta_\phi \ll 1$ are enough to guarantee an exponential phase acceleration that lasts for long enough to produce the desired results of an inflationary era.

2.3. FLRW in General Relativity

2.3.1. Friedman equations

Using the FLRW metric in cosmic time (45), the Ricci tensor can be computed as

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2}, \quad R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k), \\ R_{33} = r^2\sin^2(\theta)(a\ddot{a} + 2\dot{a}^2 + 2k), \tag{61}$$

and then the Ricci scalar becomes

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]. \tag{62}$$

On the other hand, we will model the matter content as a perfect fluid whose energy-momentum tensors is

$$\mathcal{T}_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu}, \tag{63}$$

where on comoving coordinates $U_\mu = (1, 0, 0, 0)$, and $\mathcal{T} = \mathcal{T}^\mu_\mu = \rho - 3P$. The conservation equation of the perfect fluid is given by

$$\dot{\rho} + 3H(\rho + P) = 0. \tag{64}$$

Finally, the field equations of FLRW in GRn called *Friedman equations* are given by

$$H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}, \quad H^2 + \dot{H} = \frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P). \tag{65}$$

It seems that the conservation equation (64) along with the Friedman equations (65) would give us enough information to obtain the time evolution of the three quantities involved in this system of equations $(a(t), \rho(t), P(t))$. Nonetheless, this is not the situation, since combining the conservation equation with the first Friedman equation will result in the second Friedman equation. Hence, instead of 3 equations for three variables we haver got two, and an extra condition is needed to solve completely the system.

The extra condition is usually achieved through the *barotropic equations of state*

$$P = \omega\rho, \tag{66}$$

where in general $\omega = \omega(t)$ [28]. The case where $\omega =$ const is of great importance, since it can model radiation $\omega = 1/3$, dust (baryons or CDM) $\omega = 0$, or the cosmological constant $\omega = -1$ [4]. By the introduction of the critical density

$$\rho_c = \frac{3H^2}{8\pi}, \tag{67}$$

the first Friedman equation is rewritten as

$$\frac{k}{H^2a^2} = \frac{\rho}{\rho_c} - 1 \equiv \Omega - 1, \tag{68}$$

with $\Omega = \rho/\rho_c$ is called the *density parameter*.

2.4. FLRW in Teleparallel Gravity

2.4.1. Isotropic and homogeneous tetrad

For the discussion of isotropic and homogeneous tetrad, we will summarize briefly the discussion found in [11] for the flat case $k = 0$.

The idea is to recover the concept of symmetries from Cartan geometry and then use a similar approach in TG. With this in mind, the definition of symmetry in TG is as follows

Definition 2.1. " A *symmetry* of a TG geometry $(\mathcal{M}, \mathbf{e}, \boldsymbol{\omega})$ is a group action $\varphi : G \times \mathcal{M} \to M$ of a Lie group G such that the metric (14) and the linear connection induced on the base manifold (16) are invariant, i.e., $\varphi_u^* g = g$ and $\varphi_u^*\Gamma = \Gamma$, $\forall u \in G$. The teleparallel geometry is then called *symmetric* under the group action φ."

Nevertheless, in physics we are usually interested in working with infinitesimal symmetries. With this in mind, it is possible to arrive at the following conditions

$$(\mathcal{L}_{X_\xi} e)^A{}_\mu = -\lambda_\xi^A{}_B e^B{}_\mu, \quad (\mathcal{L}_{X_\xi}\omega)^A{}_{B\mu} = D_\mu \lambda_\xi^A{}_B, \tag{69}$$

where

$$(\mathcal{L}_{X_\xi} e)^A{}_\mu = X_\xi^\nu \partial_\nu e^A{}_\mu + \partial_\mu X_\xi^\nu e^A{}_\nu, \quad (\mathcal{L}_{X_\xi}\omega)^A{}_{B\mu} = X_\xi^\nu \partial_\nu \omega^A{}_{B\mu} + \partial_\mu X_\xi^\nu \omega^A{}_{B\nu}, \tag{70}$$

are the Lie derivatives of the tetrad and the spin connection, X_ξ^ν are the generator of the symmetry,

$$D_\mu \lambda_\xi^A{}_B = \partial_\mu \lambda_\xi^A{}_B + \omega^A{}_{C\mu}\lambda_\xi^C{}_B - \omega^C{}_{B\mu}\lambda_\xi^A{}_C, \tag{71}$$

and $\lambda : \mathfrak{g} \times M \to \mathfrak{so}(1,3)$ is the local Lie algebra homomorphism.

When working on the Weitzenböck gauge, (69) reduce to

$$(\mathcal{L}_{X_\xi} e)^A{}_\mu = -\lambda_\xi^A{}_B e^B{}_\mu, \quad 0 \equiv (\mathcal{L}_{X_\xi} \omega)^A{}_{B\mu} = \partial_\mu \lambda_\xi^A{}_B, \tag{72}$$

giving a system of equation, in terms of the generators of the symmetry, for the tetrad and then giving the sufficient conditions to find the tetrad that exhibits the desired symmetry in the Weitzenböck gauge.

With this in consideration, the tetrad that exhibits the cosmological symmetry in Cartesian coordinate sis given by

$$e^A{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t) & 0 & 0 \\ 0 & 0 & a(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}, \tag{73}$$

with a constant lapse function $n(t) = 1$ in order to have the FLRW metric as (45).
In conformal time the tetrad becomes

$$e^A{}_\mu = a(\eta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{74}$$

2.4.2. FLRW equations: TEGR, $f(T)$ and $f(T,B)$

Using the tetrad (73) in TEGR, we arrive at the same Friedmann equation in GR (65) with spatially flat curvature $k = 0$. Now, in order to obtain the Friedmann equation in $f(T)$ and $f(T,B)$, we need to compute the torsion and superpotential tensor along with the torsion scalar and boundary term.
The non-vanishing components of the torsion and superpotential tensor are [30]

$$T^i{}_{0j} = H\delta^i_j, \quad S_i{}^{0j} = -H\delta^i_j, \tag{75}$$

whereas the torsion scalar and boundary term are given by

$$T = -6H^2 \quad \text{and} \quad B = -6(3H^2 + \dot{H}). \tag{76}$$

Hence, the field equation of $f(T)$ gravity (43) are given by [8]

$$-6H^2 f_T - \frac{1}{2} f = \kappa\rho, \tag{77}$$

$$-2f_T(3H^2 + \dot{H}) - 2H\dot{f}_T - \frac{1}{2} f = -\kappa P, \tag{78}$$

and the field equations for $f(T,B)$ gravity are

$$3H\dot{f}_B - 3H^2(3f_B + 2f_T) - 3f_B\dot{H} - \frac{1}{2} f(T,B) = \kappa\rho, \tag{79}$$

$$-(3H^2 + \dot{H})(2f_T + 3f_B) - 2H\dot{f}_T + \ddot{f}_B - \frac{1}{2} f(T,B) = -\kappa P. \tag{80}$$

3. Linear Perturbation Theory

3.1. Perturbation around FLRW

For this section, we will follow principally the discussion found in ref. [23], but also use ref. [31] and ref. [29] when needed and summarize what we need for this wok.
The idea of linear perturbation theory is to work with small deviation $\delta g_{\mu\nu}$ from a background manifold

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{81}$$

where $\bar{g}_{\mu\nu}$ is the metric of the background manifold, in our case, the FLRW metric in conformal time (48). The deviation form the background metric are considered to be small, so any non-linear term involving the deviation must be vanishing

$$|\delta g_{\mu\nu}| \ll 1 \quad \rightarrow \quad (\delta g_{\mu\nu})^n = 0, n = 2, 3, \ldots . \tag{82}$$

In the following, we will first discuss linear perturbation theory in GR and later on, discuss in detail linear perturbation theory in TG.

Using the decomposition of the metric (81), we can compute the Christoffel Symbols as

$$\Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\mu\nu}, \tag{83}$$

where $\bar{\Gamma}^{\rho}_{\mu\nu}$ are the background Christoffel Symbols given by (4) in terms of the background metric [6], and the perturbed Christoffel Symbols are given by

$$\delta\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\bar{g}^{\mu\sigma}\left(\delta g_{\sigma\nu,\rho} + \delta g_{\sigma\rho,\nu} - \delta g_{\nu\rho,\sigma} - 2\delta g_{\sigma\alpha}\bar{\Gamma}^{\alpha}_{\nu\rho}\right). \tag{84}$$

Analogously, the Ricci tensor is given by

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} , \tag{85}$$

where the perturbation of the Ricci tensor is

$$\delta R_{\mu\nu} = \delta\Gamma^{\rho}_{\mu\nu,\rho} - \delta\Gamma^{\rho}_{\mu\rho,\nu} + \bar{\Gamma}^{\rho}_{\mu\nu}\delta\Gamma^{\sigma}_{\rho\sigma} + \delta\Gamma^{\rho}_{\mu\nu}\bar{\Gamma}^{\sigma}_{\rho\sigma} - \bar{\Gamma}^{\rho}_{\mu\sigma}\delta\Gamma^{\sigma}_{\nu\rho} - \delta\Gamma^{\rho}_{\mu\sigma}\bar{\Gamma}^{\sigma}_{\nu\rho} . \tag{86}$$

Finally, the Einstein tensor is also given by

$$G^{\mu}{}_{\nu} = \bar{G}^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu}, \tag{87}$$

with the perturbation given by

$$\delta G^{\mu}{}_{\nu} = \bar{g}^{\mu\rho}\delta R_{\rho\nu} + \delta g^{\mu\rho}\bar{R}_{\rho\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\delta R \quad \text{with} \quad \delta g^{\mu\nu} = -\bar{g}^{\mu\rho}\delta g_{\rho\sigma}\bar{g}^{\nu\sigma}. \tag{88}$$

On the other hand, we also need to compute the perturbations at first order of the energy-momentum tensor.

[6] Any mathematical symbol with a bar indicates that it was computed with the background metric.

Recall the definition of the energy-momentum tensor (63) for the background cosmology

$$\bar{\mathcal{T}}_{\mu\nu} = (\bar{\rho} + \bar{P})\bar{u}_\mu \bar{u}_\nu - \bar{P}\bar{g}_{\mu\nu}. \tag{89}$$

With the introduction of a projector onto the hypersurface orthogonal to $\bar{u}_\mu$

$$\bar{\theta}_{\mu\nu} = \bar{g}_{\mu\nu} - \bar{u}_\mu \bar{u}_\nu \quad \text{where} \quad \bar{\theta}_{\mu\nu}\bar{u}^\mu = 0, \tag{90}$$

the energy-momentum of the perfect fluid can be rewritten as

$$\bar{\mathcal{T}}_{\mu\nu} = \bar{\rho}\bar{u}_\mu \bar{u}_\nu - \bar{P}\bar{\theta}_{\mu\nu}. \tag{91}$$

In ref. [32], based on a covariant split of any two-rank tensor onto the four-velocity, it was obtained that the energy-momentum tensor for an *almost* perfect fluid, which includes dissipation effects, is given by

$$\mathcal{T}_{\alpha\beta} = \rho u_\alpha u_\beta - (P + \pi)\theta_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha + \pi_{\alpha\beta}, \tag{92}$$

where the dissipative terms q_α, $\pi_{\alpha\beta}$ and π (called the *heat transfer* contribution, *anisotropic stress* and *bulk viscosity* respectively), satisfy

$$q_\mu u^\mu = 0, \quad \pi_{\alpha\beta}u^\beta = \pi_{[\alpha\beta]} = \pi^\alpha{}_\alpha = 0 \tag{93}$$

and π is the trace of the anisotropic stress. Observe that we can work with $\pi = 0$ by redefining the pressure term as $P + \pi \to P$, which is the convention we will follow.

By comparing (91) with (92), we can observe that q_α and $\pi_{\alpha\beta}$ are already first order contribution to the perturbation of the energy-momentum tensor of the perfect fluid, whereas the perturbation of the energy density, pressure and four-velocity proceed as usual

$$\rho = \bar{\rho} + \delta\rho(\eta, \boldsymbol{x}), \quad P = \bar{P} + \delta P(\eta, \boldsymbol{x}), \quad u^\mu = \bar{u}^\mu + \delta u^\mu(\eta, \boldsymbol{x}). \tag{94}$$

The splitting of the energy-momentum tensor into a zero and a first order perturbation reads as

$$\mathcal{T}_{\mu\nu} = \bar{\mathcal{T}}_{\mu\nu} + \delta\mathcal{T}_{\mu\nu}, \tag{95}$$

such that the first order perturbation of the energy-momentum tensor is

$$\delta\mathcal{T}_{\mu\nu} = \delta\rho\bar{u}_\mu\bar{u}_\nu + \bar{\rho}\delta u_\mu\bar{u}_\nu + \bar{\rho}\bar{u}_\mu\delta u_\nu + q_\mu\bar{u}_\nu + q_\nu\bar{u}_\mu - \bar{\theta}_{\mu\nu}(\delta P + \pi) - \bar{P}\delta\theta_{\mu\nu} + \pi_{\mu\nu}\ , \tag{96}$$

with

$$\delta\theta_{\mu\nu} = \delta g_{\mu\nu} - \delta u_\mu\bar{u}_\nu - \bar{u}_\mu\delta u_\nu\ . \tag{97}$$

The perturbation for the energy-momentum tensor with mixed components is

$$\delta\mathcal{T}^\mu{}_\nu = \bar{g}^{\mu\rho}\delta\mathcal{T}_{\rho\nu} + \delta g^{\mu\rho}\bar{\mathcal{T}}_{\rho\nu}\ . \tag{98}$$

Considering that, the Einstein field equations are given by

$$G^\mu{}_\nu = \kappa\mathcal{T}^\mu{}_\nu \tag{99}$$

with $G^{\mu}{}_{\nu} = \bar{G}^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu}$ and $\mathcal{T}^{\mu}{}_{\nu} = \bar{\mathcal{T}}^{\mu}{}_{\nu} + \delta\mathcal{T}^{\mu}{}_{\nu}$, the perturbed Einstein field equations are

$$\delta G^{\mu}{}_{\nu} = \kappa \delta \mathcal{T}^{\mu}{}_{\nu}. \tag{100}$$

The previous discussion did not make use of the FLRW background and any coordinate system, however, we can make this explicit by writing the metric as

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} [1 + 2\psi(\eta, \boldsymbol{x})] & -w_i(\eta, \boldsymbol{x}) \\ & \\ -w_i(\eta, \boldsymbol{x}) & -\delta_{ij}[1 - 2\phi(\eta, \boldsymbol{x})] - \chi_{ij}(\eta, \boldsymbol{x}) \end{pmatrix}, \tag{101}$$

where $\delta^{ij}\chi_{ij} = 0$. Observe that, if $\psi = \phi = w_i = \chi_{ij} = 0$, we recover the components of the FLRW metric tensor in conformal time (48), hence, our linear perturbation $\delta g_{\mu\nu}$ is encoded in terms of the scalar ψ, ϕ, vector w_i and tensor χ_{ij} perturbation.

3.2. The Gauge Problem

We started the previous discussion by stating the small deviation from the background as

$$g_{\mu\nu}(\eta, \boldsymbol{x}) = \bar{g}_{\mu\nu}(\eta, \boldsymbol{x}) + \delta g_{\mu\nu}(\eta, \boldsymbol{x}), \tag{102}$$

which can be recast as

$$\delta g_{\mu\nu}(\eta, \boldsymbol{x}) = g_{\mu\nu}(\eta, \boldsymbol{x}) - \bar{g}_{\mu\nu}(\eta, \boldsymbol{x}). \tag{103}$$

This difference is an *"ill-posed statement"* [23], since $\bar{g}_{\mu\nu}$ is a tensor on the background manifold and $g_{\mu\nu}$ is a tensor on the *physical manifold*, hence, we are comparing two different tensors on two different manifolds. Thus, a mapping that allows us to identify points on the background manifold with those of the physical manifold is required. Such mapping is called the *gauge*, and will allow us to use the conformal time and comoving coordinates of the background manifold on the physical manifold.

If we change the coordinates system, we can change the value of the perturbation variables and even introduce fictitious perturbations [31]. It is then mandatory to build the so-called *gauge-invariant potentials*, which are expressions that will remain unchanged under a change of coordinates and will serve as the physical perturbations of the theory.

3.2.1. Gauge transformations

We begin by an infinitesimal change of coordinates on the background coordinates

$$x^{\mu} \to \tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}(\eta, \boldsymbol{x}). \tag{104}$$

From the tensor transformation rules, at first order on ξ, the metric tensor transform as

$$\begin{aligned} g_{\mu\nu}(x) &= \hat{g}_{\mu\nu}(x) + \partial_{\alpha} g_{\mu\nu}(x)\xi^{\alpha} + \partial_{\mu}\xi^{\rho} g_{\rho\nu}(x) + \partial_{\nu}\xi^{\rho} g_{\rho\mu}(x) \\ &= \hat{g}_{\mu\nu}(x) + \nabla_{\nu}\xi_{\mu} + \nabla_{\mu}\xi_{\nu}. \end{aligned} \tag{105}$$

Therefore, at first order, the perturbed quantities transform as

$$\hat{\psi} = \psi - \mathcal{H}\xi^0 - \xi^{0\prime}, \tag{106}$$
$$\hat{w}_i = w_i - \zeta_i' + \partial_i \xi^0, \tag{107}$$
$$\hat{\phi} = \phi - \mathcal{H}\xi^0 - \frac{1}{3}\partial_l \xi^l, \tag{108}$$
$$\hat{\chi}_{ij} = \chi_{ij} - \partial_j \zeta_i - \partial_i \zeta_j + \frac{2}{3}\delta_{ij}\partial_l \xi^l, \tag{109}$$

where the prime denotes the derivative w.r.t the conformal time, $\mathcal{H} = \frac{a'}{a}$ and $\zeta_i \equiv \delta_{il}\xi^l$.

Moreover, the transformation of the energy-momentum tensor w.r.t this change of coordinates is

$$\hat{\mathcal{T}}_{\mu\nu}(x) = \mathcal{T}_{\mu\nu}(x) - \partial_\alpha \mathcal{T}_{\mu\nu}(x)\xi^\alpha - \partial_\mu \xi^\rho \mathcal{T}_{\rho\nu}(x) - \partial_\nu \xi^\rho \mathcal{T}_{\rho\mu}(x) \ , \tag{110}$$

and the, the perturbed quantities transform as

$$\hat{\delta\rho} = \delta\rho - \bar{\rho}'\xi^0, \qquad \hat{v}_i = v_i + \partial_i \xi^0, \qquad \hat{q}_i = q_i, \tag{111}$$
$$\delta\hat{P} = \delta P - \bar{P}'\xi^0 \qquad \hat{\pi}_{ij} = \pi_{ij}, \tag{112}$$

where $\delta u_i = a v_i$.

The fact that the heat transfer contribution and anisotropic stress is zero is just a manifestation of the Stewart-Walker lemma [33], which states that a linear perturbation δQ of a background quantity Q is gauge-invariant if and only if one of the following conditions hold

1. $\bar{Q} = 0$.
2. $\bar{Q}$ is a constant scalar.
3. $\bar{Q}$ is a linear combination of products of Kronecker deltas.

3.2.2. SVT decomposition

The Scalar-Vector-Tensor (SVT) decomposition consists in fully identify the scalar, vector and tensor parts of the perturbation. We have already seen that our perturbations have scalar parts ϕ, ψ, vector part w_i and tensor part χ_{ij}, however, it is still possible to obtain other two scalar contributions from w_i and χ_{ij}, and another vector one from χ_{ij}.

This can be accomplished by the use of Helmholtz Theorem [34], which states that, if we consider a vector function $\mathbf{F}(\mathbf{r})$, and define its divergence and curl as

$$D(\mathbf{r}) \equiv \nabla \cdot \mathbf{F}(\mathbf{r}), \quad \mathbf{C}(\mathbf{r}) \equiv \nabla \times \mathbf{F}(\mathbf{r}), \tag{113}$$

the vector field can be rewritten as

$$\mathbf{F}(\mathbf{r}) = -\nabla U + \nabla \times \mathbf{W}, \quad \nabla \times (\nabla U) = 0 \quad \text{and} \quad \nabla \cdot \mathbf{C}(\mathbf{r}) = 0, \tag{114}$$

where

$$U(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{D(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}', \quad \mathbf{W}(\mathbf{r}) = \frac{1}{4\pi}\int \frac{\mathbf{C}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}', \tag{115}$$

i.e., the vector field can be decomposed as a divergenceless plus an irrotational part.

Based on this, we decompose our vector perturbation as a divergenceless plus an irrotational part

$$w_i = w_i^{||} + w_i^{\perp}, \quad \text{with} \quad \epsilon^{ijk}\partial_j w_i^{||} = 0, \quad \partial^k w_k^{\perp} = 0. \tag{116}$$

By comparing (116) and (114), we observe that the irrotational part is the gradient of a scalar $w_i^{||} = \partial_i w$, and we define the divergenceless part as $w_i^{\perp} = S_i$, and then our vector perturbation read as

$$w_i = \partial_i w + S_i. \tag{117}$$

Analogously, the tensor part cam be split into three parts, a longitudinal part $\chi_{ij}^{||}$, an orthogonal part $\chi_{ij}^{\perp}$, and the transverse contribution χ_{ij}^{T} as follows

$$\chi_{ij} = \chi_{ij}^{||} + \chi_{ij}^{\perp} + \chi_{ij}^{T} \ , \tag{118}$$

where every part satisfy

$$\epsilon^{ijk}\partial^l\partial_j\chi_{lk}^{||} = 0 \ , \quad \partial^i\partial^j\chi_{ij}^{\perp} = 0 \quad \text{and} \quad \partial^j\chi_{ij}^{T} = 0 \ . \tag{119}$$

The longitudinal and orthogonal part are built from the divergence of χ_{ij} and apply Helmholtz Theorem to it. Hence, we can rewrite each part as

$$\chi_{ij}^{||} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)2\mu \ , \qquad \chi_{ij}^{\perp} = \partial_j A_i + \partial_i A_j \ , \qquad \partial^i A_i = 0 \ , \quad \chi_{ij}^{T} = h_{ij}, \tag{120}$$

and then the tensor perturbation becomes

$$\chi_{ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)2\mu + \partial_j A_i + \partial_i A_j + h_{ij}. \tag{121}$$

Therefore, the 10 degrees of freedom of the metric have been decomposed into 4 scalar modes ϕ, ψ, w, μ, 4 divergenceless vector modes S_i, A_i, and two tensor degrees of freedom h_{ij}.

The same idea can be applied to the gauge fields ξ^μ of the infinitesimal transformation (104) as follows

$$\xi^0 \equiv \alpha, \quad \zeta_i = \delta_{il}\xi^l = \partial_i\beta + \varepsilon_i, \tag{122}$$

where ε_i is divergenceless $\partial^l\varepsilon_l = 0$.
Using the SVT decomposition of the perturbation and the gauge fields, and using the gauge transformation rules of the perturbation quantities (106), we can compute the gauge transformation rules for the scalar ϕ, ψ, w, μ, vector w_i, S_i and tensor part h_{ij}, which turn out to be

$$\hat{\psi} = \psi - \mathcal{H}\alpha - \alpha', \tag{123}$$

$$\hat{\phi} = \phi + \mathcal{H}\alpha + \frac{1}{3}\nabla^2\beta, \tag{124}$$

$$\hat{w} = w - \beta' + \alpha, \tag{125}$$

$$\hat{\mu} = \mu - \beta, \tag{126}$$

for the scalar part,

$$\hat{S}_i = S_i - \epsilon_i' \tag{127}$$
$$\hat{A}_i = A_i - \epsilon_i, \tag{128}$$

for the vector part, and

$$\hat{\chi}_{ij} = \chi_{ij}, \tag{129}$$

i.e., the tensor part is already gauge-invariant.

By combining equation (123)-(126), we obtain the so called *Bardeen Potentials*

$$\Psi = \psi + \frac{1}{a}\left[(w - \mu')\, a\right]', \quad \Phi = \phi - \mathcal{H}\,(w - \mu') + \frac{1}{3}\nabla^2\mu, \tag{130}$$

which are the scalar gauge-invariant potentials.

Likewise, by combing (127) and (128) we arrive at the vector gauge-invariant potential

$$W_i = S_i - A_i'. \tag{131}$$

Since we are working in linear perturbation theory, any interaction between modes is a second order perturbation and the negligible, thus the scalar, vector and tensor modes evolve independently.

The same procedure can be done for the matter perturbations, however, this topic lies beyond the scope of this thesis and will not be performed here.

3.2.3. Gauge fixing

Since we have already constructed the gauge-independent potential for every mode, we can exploit the gauge freedom to simplify the equations, Let us discuss some examples.

Considering only scalar perturbations (ϕ, ψ, w, μ), we can select a gauge, called the *Newtonian gauge*, such that

$$\hat{w} = \hat{\mu} = 0, \tag{132}$$

which from equations (123)-(126) determine the sacalar gauge transformation as

$$\beta = \mu, \quad \alpha = \mu' - w, \tag{133}$$

and the Bardeen potentials become

$$\Psi = \hat{\psi}, \quad \Phi = \hat{\phi}, \tag{134}$$

hence, our scalar perturbation depends exclusively on the Bardeen potentials.

In this gauge, the metric tensor reads as

$$ds^2 = a^2(\eta)[(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j], \tag{135}$$

and the perturbation theory have several consequences.

One of the most important consequences, in GR, is that Φ and Ψ are not physically independent fields but differ only by the scalar part of the anisotropic stress of the energy-momentum tensor and, in absence of scalar anisotropic stress $\pi^S = 0$, they are the same field $\Psi = \Phi$ and the metric resembles the weak field limit of GR about a Minkowski spacetime with Φ playing the role of the gravitational potential [31].

Other choices are also possible, for instance, by choosing

$$\hat{\psi} = 0 \quad \text{and} \quad \hat{w} = 0, \tag{136}$$

we arrive at the *Synchronous gauge* whose metric tensor is

$$ds^2 = a^2(\eta)\left[d\eta^2 - \left((1-2\phi)\delta_{ij} + \left\{\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right\}2\mu\right)dx^i dx^j\right]. \tag{137}$$

This gauge is useful when dealing with fluctuation of the inflation field or when dealing with a universe dominated by cold dark matter [31][29]. There exist different gauges and gauge transformations that will not be covered here, some of these can be found in ref. [35].

3.3. Gravitational Waves in General Relativity

When considering a pure tensor perturbation

$$\delta g_{00} = 0, \quad \delta g_{0i} = 0, \quad \delta g_{ij} = h_{ij}, \tag{138}$$

with $\delta^{ij}h_{ij} = \partial^i h_{ij} = 0$ as discussed before, the non-zero components of the perturbed Einstein tensor (88) is

$$-2a^2\delta G^i{}_j = h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}. \tag{139}$$

If we consider the Fourier transform of a generic field $f(\eta, \boldsymbol{x})$ to be

$$\tilde{f}(\eta, \boldsymbol{k}) = \int d^3x f(\eta, \boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad f(\eta, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3}\tilde{f}(\eta, \boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{140}$$

and apply it to (139), then the Einstein field equations in Fourier Space are

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = -16\pi G a^2 \pi^T_{ij}, \tag{141}$$

where π^T_{ij} is the tensor part of the anisotropic stress. Equations (141) constitute the equations for Gravitational Waves (GW) from a FLRW background.

Generally speaking, since h_{ij} is a symmetric 3-dimensional tensor, it has six independent components, however the transverse condition $\partial^i h_{ij} = 0$ impose 3 constrictions, and the traceless condition $\delta^{ij}h_{i,j} = 0$ another one, hence, reducing the number of independent components from six to only two, which are the so called h_+ and $h_\times$ polarizations, such that, if the GW propagates in the $\hat{z}$ direction $\hat{k} = \hat{z}$, the tensor perturbation is

$$h_{ij}(k\hat{z}) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{142}$$

or rather compact as

$$h_{ij}(k\hat{z}) = h_+ \pm ih_\times. \tag{143}$$

If a rotation about the z-axis is performed, the tensor perturbation transforms as [23]

$$\bar{h}_+ \pm i\bar{h}_\times = e^{\mp 2i\theta}(h_+ \pm ih_\times), \tag{144}$$

and it is said that the GW has helicity 2 or is a Spin 2 field when it is quantized.

For a general direction, we can write the tensor field in terms of the *polarization tensors* ϵ^λ_{ij} as [36]

$$h_{ij}(\eta.\boldsymbol{k}) = \sum_{\lambda=\pm 2} \epsilon^\lambda_{ij}(\hat{k})h_\lambda(\eta,\boldsymbol{k}), \quad \sum_{\lambda=\pm 2} \epsilon^\lambda_{ij}\epsilon^{\lambda' *}_{ij} = \delta^{\lambda\lambda'}, \tag{145}$$

where the polarization tensors and vectors are given by

$$e_{ij}(\hat{k},\pm 2) = \sqrt{2}e_{\pm,i}e_{\pm,j} \ , \quad e_{\pm,i}(\hat{k}) \equiv \frac{(e_1 \pm ie_2)_{,i}}{\sqrt{2}} \ , \tag{146}$$

with $\{e_a, e_b\}$ two dimensional vectors orthogonal to a vector $\hat{k}$ that results of going to Fourier space with the transverse condition

$$\partial^i h_{ij} = 0 \quad \rightarrow \quad \hat{k}^i h_{ij} = 0. \tag{147}$$

In this scenario, the $+, \times$ polarizations are given by

$$h_+(\eta,\boldsymbol{k}) = \frac{1}{\sqrt{2}}h_{+2}(\eta,\boldsymbol{k}) + \frac{1}{\sqrt{2}}h_{-2}(\eta,\boldsymbol{k}) \ , \tag{148}$$

$$h_\times(\eta,\boldsymbol{k}) = \frac{i}{\sqrt{2}}h_{+2}(\eta,\boldsymbol{k}) - \frac{i}{\sqrt{2}}h_{-2}(\eta,\boldsymbol{k}) \ . \tag{149}$$

From the previous equation, we can see that $h_\lambda(\eta,\boldsymbol{k})$ satisfy the GW equation (141), and $h_{+,\times}$ also satisfy it.

3.4. Linear Perturbation in Teleparallel Gravity

3.4.1. Tetrad perturbation

Now it is time to discuss linear perturbation theory in TG. Since this is one the central topics of this work, we will discuss it in detail.

We begin by considering small deviation from the isotropic and homogeneous tetrad $\bar{e}^A{}_\mu$ as follows

$$e^A{}_\mu = \bar{e}^A{}_\mu + \delta e^A{}_\mu. \tag{150}$$

From the relation between the tertad and the metric (14), we observe that

$$\bar{g}_{\mu\nu} + \delta g_{\mu\nu} = \left(\bar{e}^A{}_\mu + \delta e^A{}_\mu\right)\left(\bar{e}^B{}_\nu + \delta e^B{}_\nu\right)\eta_{AB}, \tag{151}$$

from where, neglecting second order perturbations, we have that

$$\delta g_{\mu\nu} = \bar{e}^{A}{}_{\mu}\delta e^{B}{}_{\nu}\eta_{AB} + \bar{e}^{B}{}_{\nu}\delta e^{A}{}_{\mu}\eta_{AB}, \tag{152}$$

or in a simplified manner [37]

$$\delta g_{\mu\nu} = 2\tau_{(\mu\nu)} \quad \text{where} \quad \tau_{\mu\nu} = \bar{e}^{A}{}_{\mu}\delta e^{B}{}_{\nu}\eta_{AB}. \tag{153}$$

From eq. (84) and eq. (153), we obtain the perturbed Chrystoffel symbols in TG

$$\delta\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\overset{\circ}{\nabla}}_{\mu}\delta g_{\sigma\nu} + \bar{\overset{\circ}{\nabla}}_{\nu}\delta g_{\mu\sigma} - \bar{\overset{\circ}{\nabla}}_{\sigma}\delta g_{\mu\nu}\right) = \bar{g}^{\rho\sigma}\left(\bar{\overset{\circ}{\nabla}}_{\mu}\tau_{(\sigma\nu)} + \bar{\overset{\circ}{\nabla}}_{\nu}\tau_{(\mu\sigma)} - \bar{\overset{\circ}{\nabla}}_{\sigma}\tau_{(\mu\nu)}\right) \tag{154}$$

$$= \bar{g}^{\rho\sigma}\left(\tau_{(\sigma\mu),\nu} + \tau_{(\sigma\nu),\mu} - \tau_{(\mu\nu),\sigma} - 2\tau_{(\sigma\alpha)}\bar{\Gamma}^{\alpha}{}_{\mu\nu}\right). \tag{155}$$

On the other hand, since we are working on linear perturbation of the gravitational sector, the Weitzenböck gauge is still employed throughout this discussion, and no extra degrees of freedom are added to the theory. Thus, from Weitzenböck connection (16) we have

$$\bar{\Gamma}^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu} = (\bar{E}^{\rho}{}_{A} + \delta E^{\rho}{}_{A})\partial_{\mu}(\bar{e}^{A}{}_{\nu} + \delta e^{A}{}_{\nu}), \tag{156}$$

from where the perturbed Weitzenböck connection is

$$\delta\Gamma^{\rho}_{\nu\mu} = \bar{E}^{\rho}{}_{A}\partial_{\mu}\delta e^{A}{}_{\nu} + \delta E^{\rho}{}_{A}\partial_{\mu}\bar{e}^{A}{}_{\nu}, \tag{157}$$

where the perturbed inverse tetrad is

$$\delta E^{\nu}{}_{A} = -\bar{E}^{\mu}{}_{A}\bar{E}^{\nu}{}_{B}\delta e^{B}{}_{\mu}. \tag{158}$$

Consequently, from the definition of the contortion tensor (21), we obtain the perturbed contortion tensor as

$$\delta K^{\rho}_{\mu\nu} = \delta\Gamma^{\rho}_{\mu\nu} - \delta\overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu}, \tag{159}$$

and also from (20), we can obtain the perturbed torsion tensor as

$$\delta T^{\rho}_{\nu\mu} = \delta\Gamma^{\rho}_{\mu\nu} - \delta\Gamma^{\rho}_{\nu\mu}. \tag{160}$$

Now, for the superpotential tensor (25), its linear perturbation is

$$\delta S_{\alpha}{}^{\sigma\rho} = \frac{1}{2}\left(\bar{g}^{\rho\beta}\delta K^{\sigma}_{\beta\alpha} + \bar{g}^{\sigma\beta}\delta T_{\beta}\delta^{\rho}_{\alpha} - \bar{g}^{\rho\beta}\delta T_{\beta}\delta^{\sigma}_{\alpha} + \delta g^{\rho\beta}\bar{K}^{\sigma}_{\beta\alpha} + \delta g^{\sigma\beta}\bar{T}_{\beta}\delta^{\rho}_{\alpha} - \delta g^{\rho\beta}\bar{T}_{\beta}\delta^{\sigma}_{\alpha}\right), \tag{161}$$

and then, the perturbation of the torsion scalar is

$$\delta T = \bar{T}^{\sigma}_{\mu\nu}\delta S_{\sigma}{}^{\mu\nu} + \delta T^{\sigma}_{\mu\nu}\bar{S}_{\sigma}{}^{\mu\nu}. \tag{162}$$

Finally, for the boundary term we have (26)

$$B = 2\nabla_{\mu}T^{\mu} = 2g^{\mu\alpha}\nabla_{\mu}T^{\lambda}{}_{\lambda\alpha} = 2g^{\mu\alpha}\left(\partial_{\mu}T_{\alpha} + \Gamma^{\lambda}_{\rho\mu}T^{\rho}{}_{\lambda\alpha} - \Gamma^{\rho}_{\lambda\mu}T^{\lambda}{}_{\rho\alpha} - \Gamma^{\rho}_{\alpha\mu}T_{\rho}\right),$$

and consequently, the linear perturbation of the boundary term is

$$\delta B = 2\left(\delta g^{\mu\alpha}\left[\partial_{\mu}\bar{T}_{\alpha} + \bar{\Gamma}^{\lambda}_{\rho\mu}\bar{T}^{\rho}{}_{\lambda\alpha} - \bar{\Gamma}^{\rho}_{\lambda\mu}\bar{T}^{\lambda}{}_{\rho\alpha} - \bar{\Gamma}^{\rho}_{\alpha\mu}\bar{T}_{\rho}\right]\right. \tag{163}$$

$$\left. + \bar{g}^{\mu\alpha}\left[\partial_{\mu}\delta T_{\alpha} + \delta\Gamma^{\lambda}_{\rho\mu}\bar{T}^{\rho}_{\lambda\alpha} + \bar{\Gamma}^{\lambda}_{\rho\mu}\delta T^{\rho}_{\lambda\alpha} - \delta\Gamma^{\rho}_{\lambda\mu}\bar{T}^{\lambda}{}_{\rho\alpha} - \bar{\Gamma}^{\rho}_{\lambda\mu}\delta T^{\lambda}{}_{\rho\alpha} - \delta\Gamma^{\rho}_{\alpha\mu}\bar{T}_{\rho} - \bar{\Gamma}^{\rho}_{\alpha\mu}\delta T_{\rho}\right]\right).$$

Now we are in position of computing the linear perturbation of the field equations in TG.

3.4.2. Perturbed equations

In this case, we will focus our attention into the $f(T,B)$ gravity scenario (42), since the $f(T)$ and TEGR scenarios are achieved by imposing the conditions $\bar{f}_{\bar{B}} = 0$ and $\bar{f}(\bar{T}) = -\bar{T}$ in the $f(T,B)$ scenario.

Consider the most general functional of T and B as $f(T,B)$, with the first order partial derivatives f_T and f_B. The linear perturbation of these functional are

$$f(T,B) = f(\bar{T}+\delta T, \bar{B}+\delta B) = \bar{f} + \bar{f}_{\bar{T}}\delta T + \bar{f}_{\bar{B}}\delta B \equiv \bar{f} + \delta f, \tag{164}$$

where we have defined $\bar{f} = f(\bar{T},\bar{B})$ and $\delta f = \bar{f}_{\bar{T}}\delta T + \bar{f}_{\bar{B}}\delta B$.

For f_T and f_B we have

$$f_T(T,B) = \bar{f}_{\bar{T}} + \delta f_T, \quad f_B(T,B) = \bar{f}_{\bar{B}} + \delta f_B \tag{165}$$

where

$$\delta f_T = \bar{f}_{\bar{T}\bar{T}}\delta T + \bar{f}_{\bar{T}\bar{B}}\delta B \quad \text{and} \quad \delta f_B = \bar{f}_{\bar{B}\bar{T}}\delta T + \bar{f}_{\bar{B}\bar{B}}\delta B. \tag{166}$$

From (42), recall the Einstein tensor in TG as

$$\begin{aligned}\mathring{G}^{\lambda}_{\nu} =& \delta^{\lambda}_{\nu}\mathring{\Box} f_B - \mathring{\nabla}^{\lambda}\mathring{\nabla}_{\nu} f_B + \frac{1}{2} f_B B \delta^{\lambda}_{\nu} + 2\left[\partial_\mu f_B + \partial_\mu f_T\right] S_{\nu}{}^{\mu\lambda} + \frac{2}{e} e^{A}_{\nu} f_T \partial_\mu (e S_a{}^{\mu\lambda}) \\ &- 2 f_T T^{\alpha}_{\mu\nu} S_{\alpha}{}^{\lambda\mu} - \frac{1}{2} f \delta^{\lambda}_{\nu},\end{aligned} \tag{167}$$

and define the following expressions

$$A = \mathring{\Box} f_B, \tag{168}$$
$$B^{\lambda}{}_{\nu} = \mathring{\nabla}^{\lambda}\mathring{\nabla}_{\nu} f_B, \tag{169}$$
$$C = f_B B, \tag{170}$$
$$D^{\lambda}{}_{\nu} = \left[\partial_\mu f_B + \partial_\mu f_T\right] S_{\nu}{}^{\mu\lambda}, \tag{171}$$
$$F^{\lambda}{}_{\nu} = \frac{1}{e} e^{A}{}_{\nu} f_T \partial_\mu (e S_a{}^{\mu\lambda}), \tag{172}$$
$$H^{\lambda}{}_{\nu} = f_T T^{\alpha}_{\mu\nu} S_{\alpha}{}^{\lambda\mu}, \tag{173}$$

such that the Einstein tensor is just

$$\mathring{G}^{\lambda}_{\nu} = \delta^{\lambda}_{\nu} A - B^{\lambda}{}_{\nu} + \frac{1}{2}\delta^{\lambda}_{\nu} C + 2D^{\lambda}{}_{\nu} + 2F^{\lambda}{}_{\nu} - 2H^{\lambda}{}_{\nu} - \frac{1}{2} f \delta^{\lambda}_{\nu}. \tag{174}$$

Now, we follow to compute the perturbation of the quantities defined in equations (168)-(173) in order to obtain the perturbation of the Einstein tensor.

From (168), it can be seen that

$$\begin{aligned} A = \mathring{\Box} f_B &= \mathring{\nabla}_\mu (\partial^\mu f_B) = g^{\mu\beta}\mathring{\nabla}_\mu(\partial_\beta f_B) \\ &= g^{\mu\beta}\left(\partial_\mu \partial_\beta f_B - \mathring{\Gamma}^{\rho}_{\beta\mu}\partial_\rho f_B\right),\end{aligned} \tag{175}$$

and thus, its linear perturbation is

$$\boxed{\delta A = \delta g^{\mu\beta}\left(\partial_\mu\partial_\beta \bar{f}_{\bar{B}} - \bar{\mathring{\Gamma}}^{\rho}_{\beta\mu}\partial_\rho \bar{f}_{\bar{B}}\right) + \bar{g}^{\mu\beta}\left(\partial_\mu\partial_\beta \delta f_B - \delta\mathring{\Gamma}^{\rho}_{\beta\mu}\partial_\rho \bar{f}_{\bar{B}} - \bar{\mathring{\Gamma}}^{\rho}_{\beta\mu}\partial_\rho \delta f_B\right)}. \tag{176}$$

From (169) we have that

$$B^\lambda{}_\nu = \mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B = g^{\lambda\beta}\left(\partial_\beta\partial_\nu f_B - \mathring{\Gamma}^{\rho}_{\nu\beta}\partial_\rho f_B\right), \tag{177}$$

where the linear perturbation is

$$\delta B^\lambda{}_\nu = \delta g^{\lambda\beta}\left(\partial_\beta\partial_\nu \bar{f}_{\bar{B}} - \bar{\mathring{\Gamma}}^{\rho}_{\nu\beta}\partial_\rho \bar{f}_{\bar{B}}\right) + \bar{g}^{\lambda\beta}\left(\partial_\beta\partial_\nu \delta f_B - \delta\mathring{\Gamma}^{\rho}_{\nu\beta}\partial_\rho \bar{f}_{\bar{B}} - \bar{\mathring{\Gamma}}^{\rho}_{\nu\beta}\partial_\rho \delta f_B\right), \tag{178}$$

or succinctly as

$$\boxed{\delta B^\lambda{}_\nu = \delta g^{\lambda\beta}\bar{\mathring{\nabla}}_\beta \bar{\mathring{\nabla}}_\nu \bar{f}_{\bar{B}} + \bar{g}^{\lambda\beta}\left(\bar{\mathring{\nabla}}_\beta \bar{\mathring{\nabla}}_\nu \delta f_B - \delta\mathring{\Gamma}^{\rho}_{\nu\beta}\partial\rho \bar{f}_{\bar{B}}\right)}. \tag{179}$$

From equation (170) we have that

$$C = f_B B \to \bar{C} + \delta C = (\bar{f}_{\bar{B}} + \delta f_B)(\bar{B} + \delta B), \tag{180}$$

from where

$$\boxed{\delta C = \bar{f}_{\bar{B}}\delta B + \delta f_B \bar{B}}. \tag{181}$$

From equation (171) we have

$$D^\lambda{}_\nu = \partial_\mu \left(f_B + f_T\right) S_\nu{}^{\mu\lambda}, \tag{182}$$

where it is straightforward to see that

$$\boxed{\delta D^\lambda{}_\nu = \partial_\mu \left(\delta f_B + \delta f_T\right) \bar{S}_\nu{}^{\mu\lambda} + \partial_\mu \left(\bar{f}_{\bar{B}} + \bar{f}_{\bar{T}}\right) \delta S_\nu{}^{\mu\lambda}}. \tag{183}$$

In the equation (172) the determinant of the tetrad appears, hence, we need first to compute the linear perturbation of the determinant.

Let us begin by considering

$$e = \det(e^A{}_\lambda), \tag{184}$$

and expand the determinant in a first order Taylor series about $\bar{e}$ as follows

$$\det(e^A{}_\lambda) = \det(\bar{e}^A{}_\lambda + \delta e^A{}_\lambda) = \det(\bar{e}^A{}_\lambda) + \delta e^B{}_\mu \left.\frac{\partial \det(e^A{}_\lambda)}{\partial e^B{}_\mu}\right|_{e=\bar{e}}, \tag{185}$$

and from the *Jacobi's formula* [38] we have that

$$\frac{\partial \det(e^A{}_\lambda)}{\partial e^B{}_\mu} = \mathrm{adj}(e^A{}_\lambda)^\mu{}_B, \tag{186}$$

however, since the tetrad is non-singular

$$\mathrm{adj}(e^A{}_\lambda)^\mu{}_B = \det(e^A{}_\lambda)E^\mu{}_B, \tag{187}$$

and therefore

$$\left.\frac{\partial \det(e^A{}_\lambda)}{\partial e^B{}_\mu}\right|_{e=\bar{e}} = \det(\bar{e}^A{}_\lambda)\bar{E}^\mu{}_B = \bar{e}\bar{E}^\mu{}_B. \tag{188}$$

Thus, the linear perturbation of the tetrad is

$$\boxed{\delta e = \bar{e}\bar{E}^\mu{}_B \delta e^B{}_\mu}. \tag{189}$$

Hence, from equation (172) we have that

$$\begin{aligned}\bar{F}^\lambda{}_\nu + \delta F^\lambda{}_\nu &= \frac{1}{\bar{e}+\delta e}\left(\bar{e}^A{}_\nu + \delta e^A{}_\nu\right)\left(\bar{f}_{\bar{T}} + \delta f_T\right)\partial_\mu\left(\left[\bar{e}+\delta e\right]\left[\bar{S}_A{}^{\mu\lambda} + \delta S_A{}^{\mu\lambda}\right]\right) \\ &= \left(\frac{1}{\bar{e}} - \frac{\delta e}{\bar{e}^2}\right)\left(\bar{e}^A{}_\nu + \delta e^A{}_\nu\right)\left(\bar{f}_{\bar{T}} + \delta f_T\right)\partial_\mu\left(\left[\bar{e}+\delta e\right]\left[\bar{S}_A{}^{\mu\lambda} + \delta S_A{}^{\mu\lambda}\right]\right),\end{aligned} \tag{190}$$

from where we obtain the linear perturbation to be

$$\boxed{\delta F^\lambda{}_\nu = \frac{1}{\bar{e}}\bar{e}^A{}_\nu \bar{f}_{\bar{T}}\partial_\mu\left(\bar{e}\delta S_A{}^{\mu\lambda} + \delta e \bar{S}_A{}^{\mu\lambda}\right) + \left[\frac{1}{\bar{e}}\bar{e}^A{}_\nu \delta f_T + \left(\frac{1}{\bar{e}}\delta e^A{}_\nu - \frac{\delta e}{\bar{e}^2}\bar{e}^A{}_\nu\right)\bar{f}_{\bar{T}}\right]\partial_\mu\left(\bar{e}\bar{S}_A{}^{\mu\lambda}\right)}, \tag{191}$$

with

$$\bar{S}_A{}^{\mu\lambda} = \bar{E}^\alpha{}_A \bar{S}_\alpha{}^{\mu\lambda} \quad \text{and} \quad \delta S_A{}^{\mu\lambda} = \bar{E}^\alpha{}_A \delta S_\alpha{}^{\mu\lambda} + \delta E^\alpha{}_A \bar{S}_\alpha{}^{\mu\lambda}. \tag{192}$$

Finally, from (173) we have

$$\bar{H}^\lambda{}_\nu + \delta H^\lambda{}_\nu = \left(\bar{f}_{\bar{T}} + \delta f_T\right)\left(\bar{T}^\alpha_{\mu\nu} + \delta T^\alpha_{\mu\nu}\right)\left(\bar{S}_\alpha{}^{\lambda\mu} + \delta S_\alpha{}^{\lambda\mu}\right), \tag{193}$$

and the its linear perturbation is

$$\delta H^\lambda{}_\nu = \bar{S}_\alpha{}^{\lambda\mu}\left[\bar{f}_{\bar{T}}\delta T^\alpha_{\mu\nu} + \delta f_T \bar{T}^\alpha_{\mu\nu}\right] + \bar{f}_{\bar{T}}\bar{T}^\alpha_{\mu\nu}\delta S_\alpha{}^{\lambda\mu}. \tag{194}$$

Therefore, the perturbed Einstein tensor is simply

$$\delta \mathring{G}^\lambda_\nu = \delta^\lambda_\nu \delta A - \delta B^\lambda{}_\nu + \frac{1}{2}\delta^\lambda_\nu \delta C + 2\delta D^\lambda{}_\nu + 2\delta F^\lambda{}_\nu - 2\delta H^\lambda{}_\nu - \frac{1}{2}\delta f \delta^\lambda_\nu. \tag{195}$$

3.4.3. Gravitational Waves in Teleparallel Gravity

Consider the background tetrad in conformal time (74) and a transverse and traceless tensor perturbation

$$\delta e^A{}_\mu = \frac{a(\eta)}{2}\begin{pmatrix} 0 & 0 \\ 0 & h_{ij}\end{pmatrix}, \tag{196}$$

where from (153), the nature of the tensor perturbation in TG is the same as the GR scenario. Using (196), the linear perturbation of the torsion tensor and the superpotential tensor are [30]

$$\delta T^{i}{}_{0j} = \frac{1}{2}h'_{ij}, \quad \delta T^{i}{}_{jk} = \frac{1}{2}(\partial_j h_{ik} - \partial_k h_{ij}), \delta S_0{}^{0i} = 0, \quad \delta S_i{}^{0j} = \frac{1}{4a^2}h'_{ij}, \quad \delta S_i{}^{jk} = -\frac{1}{4a^2}(\partial_j h_{ik} - \partial_k h_{ij}). \tag{197}$$

Analogously, using the equations (162) and (163), a direct computation of the perturbation of the torsion scalar (162) and the boundary term (163), from the perturbed tetrad (196), gives

$$\delta T = \delta B = 0.$$

From the previous result, we have

$$\delta f = \delta f_T = \delta f_B = 0 \rightarrow f(T,B) = \bar{f}(\bar{T},\bar{B}), \tag{198}$$
$$f_T(T,B) = \bar{f}_{\bar{T}}(\bar{T},\bar{B}), \quad f_B(T,B) = \bar{f}_{\bar{B}}(\bar{T},\bar{B}), \tag{199}$$

hence, we will drop the bar notation on the functional $f(T,B)$ and its derivatives when discussing tensor perturbation.
Finally, the perturbed Einstein tensor (195) is given by

$$\delta G^i_j = \frac{f_T}{2a^2}\delta^{ik}\left(h''_{kj} + [2+\nu]\mathcal{H}h'_{kj} - \nabla^2 h_{kj}\right), \tag{200}$$

where the parameter ν

$$\nu = \frac{1}{\mathcal{H}}\frac{f'_T}{f_T}, \tag{201}$$

called the *Planck mass run rate*, encodes all the information about the extension of TG, since $\nu = 0$ recovers the equation of GR. The Planck mass run is the friction term caused by the expansion of the universe [39].
By combining the perturbation of the Einstein tensor and the energy-momentum tensor, and working in the Fourier space, we arrive to [7]

$$h''_{kj} + [2+\nu]\mathcal{H}h'_{kj} + k^2 h_{kj} = \frac{16\pi G a^2}{f_T}\pi^T_{ij}. \tag{202}$$

If $f_T = -1$ we recover the GW equation in GR (141). The equation (141) corresponds to a tensor wave propagating at the speed of light [8], in agreement to measurements of GW [40].

Observe that in the case $f(T,B) = f(T)$ the same equation holds, and in the case of $f(T,B) = f(-T+B) = f(\mathring{R})$, with $f_T \rightarrow f_R$, then $\nu = \frac{1}{\mathcal{H}}\frac{f'_R}{f_R}$ which is found in the literature [30][41].

Finally, if we consider (202) in vacuum

$$h''_{kj} + [2+\nu]\mathcal{H}h'_{kj} + k^2 h_{kj} = 0, \tag{203}$$

which can be derived from an action of the type (cf. [42])

$$S = \int d^3\boldsymbol{x}\, d\eta(-a^2 f_T)\left[h'_{ij}h'^{ij} - \partial_k h_{ij}\partial^k h^{ij}\right], \tag{204}$$

[7] Observe that π^T_{ij} in the $(+,-,-,-)$ is $-\pi^T_{ij}$ in the signature $(-,+,+,+)$, with $\pi^T_{ij} = \delta_{ik}\pi_j^{k\,T}$.

and the going to Fourier space, which requires $f_T < 0$ in order to preserve the positive sign o the Lagrangian density as in the GR limit $f_T = -1$ [8].
Therefore, we have obtained that the main differences on the GW coming from GR and TG are the dissipation term ν and the extra factor of f_T over the right hand side of the equations. GW in TG are not constrained by present observations, since tensor waves propagate at speed of light and also no constraints exist for the Planck run rate [30], hence. we can explore its production in the early universe and track its Power Spectrum which could be detected on future GW detectors like LISA.

4. Production of Gravitational Waves: Vacuum case

Let us begin by considering the production of GW coming directly from inflation. As we discussed in Subsec. 2.2, inflation is usually modelled by an scalar field through the energy-momentum tensor in the form of a perfect fluid. Therefore, when making perturbation of the energy-momentum tensor of the scalar field (50), we only deal with $\delta\rho$, δP and δu^μ perturbations, hence, no tensor anisotropic stress is generated during inflation

$$\pi^T_{ij} = 0, \tag{205}$$

thus, the GW equations on GR and TG are given by

$$h''_{ij} + 2\mathcal{H}h'_{ij} + k^2 h_{ij} = 0 \quad \text{and} \quad h''_{ij} + [2+\nu]\mathcal{H}h'_{ij} + k^2 h_{ij} = 0, \tag{206}$$

with ν the parameter defined in (201).
Since we are interested in studying the production during inflation, which occurs at the very beginning of the universe, we propose a quantized tensor field

$$h_{ij}(\eta, \mathbf{x}) = \sum_{\lambda=\pm2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[h(\eta, k) e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{k}, \lambda) e_{ij}(\hat{k}, \lambda) + h^*(\eta, k) e^{-i\mathbf{k}\cdot\mathbf{x}} a^\dagger(\mathbf{k}, \lambda) e^*_{ij}(\hat{k}, \lambda) \right] , \tag{207}$$

where the creator and annihilation operators satisfy the commutation relations

$$[a(\mathbf{k}, \lambda), a(\mathbf{k}', \lambda')] = 0 \, , \qquad \left[a(\mathbf{k}, \lambda), a^\dagger(\mathbf{k}', \lambda')\right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')\delta_{\lambda\lambda'} \, . \tag{208}$$

We will suppose that the universe is in a vacuum quantum state $|0\rangle$ such that

$$a(\mathbf{k}, \lambda) \left|0\right\rangle = 0, \tag{209}$$

and such that, in the infinite past $\eta \to -\infty$, the modes resemble the quantisation of a free field in Minkowski space [23][43]. Such vacuum state is called the *Bunch-Davies* vacuum.
Thus, the Primordial Power Spectrum will be defined from the correlator, the vacuum value expectation, of the gravitational wave

$$\langle 0 | h_{ij}(\eta, \mathbf{x}) h_{lm}(\eta, \mathbf{x}') | 0 \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} |h(\eta, k)|^2 e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \Pi_{ij,lm}(\hat{k}) \, , \tag{210}$$

where

$$\Pi_{ij,lm}(\hat{k}) \equiv \sum_{\lambda=\pm2} e_{ij}(\hat{k}, \lambda) e^*_{lm}(\hat{k}, \lambda) \, . \tag{211}$$

But, what is exactly the Power Spectrum? defining $h(\eta, \mathbf{k}, \lambda) = h(\eta, k) a(\mathbf{k}, \lambda)$, we can rewrite the value expectation as

$$\langle 0 | h_{ij}(\eta, \mathbf{x}) h_{lm}(\eta, \mathbf{x}') | 0 \rangle = \sum_{\lambda=\pm2} \int \frac{d^3\mathbf{k} d^3\mathbf{k}'}{(2\pi)^6} \langle 0 | h(\eta, \mathbf{k}, \lambda) h^\dagger(\eta, \mathbf{k}, \lambda) | 0 \rangle e^{i(\mathbf{k}\cdot\mathbf{x} - \mathbf{k}'\cdot\mathbf{x}')} e_{ij}(\hat{k}, \lambda) e^*_{lm}(\hat{k}, \lambda). \tag{212}$$

Outside the horizon $k|\eta| \to 0$ the field loses its quantum nature and then we identify the expectation value with the ensemble average, the expectation value of a random field among all the possible values that it can assume, of the classical random fields[8]

$$\langle 0|h(\eta,\mathbf{k},\lambda)h^{\dagger}(\eta,\mathbf{k},\lambda)|0\rangle \underset{k|\eta|\to 0}{\longrightarrow} \langle h(\eta,\mathbf{k},\lambda)h^{*}(\eta,\mathbf{k},\lambda)\rangle = (2\pi)^3\delta_{\rm D}(\mathbf{k}-\mathbf{k}')P_h(\eta,k), \tag{213}$$

therefore

$$\langle 0|h_{ij}(\eta,\mathbf{x})h_{lm}(\eta,\mathbf{x}')|0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}P_h(\eta,k)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\Pi_{ij,lm}(\hat{k}) \;, \tag{214}$$

Where the equality must be understood outside the horizon. From (210) and (214) we conclude that

$$P_h(\eta,k) = |h(\eta,k)|^2 \tag{215}$$

is the Primordial Power Spectrum (PPS) of gravitational waves coming from inflation.

We will study the PPS of GW in GR and TG for two different backgrounds, the first one a perfect de Sitter space $\dot{H}=0$ and the second one a quasi-de Sitter space $\dot{H}=-\epsilon H_\Lambda^2$.

4.1. De Sitter background: General Relativity and Teleparallel Gravity

Let us consider a perfectly de-Sitter background

$$H = H_\Lambda = \text{const} \to \dot{H} = 0, \tag{216}$$

and recall that for tensor perturbation

$$f(T,B) = \bar{f}(\bar{T},\bar{B}), \tag{217}$$

i.e., the functional is just the background functional, and since $\bar{T}=-6H^2$ and $\bar{B}=-6(3H^2+\dot{H})$, then

$$f_T' = \frac{dt}{d\eta}\dot{f}_T = \frac{dt}{d\eta}\left(f_{TT}\dot{\bar{T}} + f_{TB}\dot{\bar{B}}\right) = 0, \tag{218}$$

and then $\nu = 0$. This implies that for a perfect de Sitter background, GW waves satisfy the same equation on GR and TG

$$h_{ij}'' + 2\mathcal{H}h_{ij}' + k^2h_{ij} = 0. \tag{219}$$

If we insert (207) into the latter equation, we observe that $h(\eta,k)$ and its conjugate satisfy

$$h'' + 2\mathcal{H}h' + k^2h = 0, \tag{220}$$

with $h = h_{+,\times}$ accounting for the two degrees of freedom, then we only need to solve this equation to obtain the PPS.

[8] We will assume that the GW field is a Gaussian random field, i.e., its statistics obeys isotropy and homogeneity.

The condition $H = H_\Lambda = \text{const}$ in conformal time reads as

$$a(\eta) = -\frac{1}{H_\Lambda \eta} \quad \text{or} \quad \mathcal{H} = -\frac{1}{\eta}, \tag{221}$$

with $\eta < 0$. Then the equation for GW becomes

$$h'' - \frac{2}{\eta}h' + k^2 h = 0. \tag{222}$$

Now, we will work canonically normalized tensor modes [23][36]

$$\hat{h} = \frac{h}{\sqrt{32\pi G}}, \tag{223}$$

and then perform the following change of variable

$$g = a\hat{h} = \frac{ah}{\sqrt{32\pi G}} \equiv \frac{M_{\rm Pl}}{2}ah. \tag{224}$$

The normalization factor is in order to properly compare tensor and scalar perturbation, and to give units of L^3 or M^{-3} to the power spectrum [23]. With this change of variable , the equation to solve is

$$g'' + \left(k^2 - \frac{2}{\eta^2}\right)g = 0, \tag{225}$$

with the initial condition on the infinite past to be the

$$g = \frac{1}{\sqrt{2k}}e^{-ik\eta}. \tag{226}$$

Generally speaking, the power spectrum will depend on the conformal time, so, at what time it should be analysed? Since the tensor mode is a constant outside the horizon $k/a \ll H$ [29], it provides a initial condition when the scale re enters the horizon during the radiation-dominated era, hence it should be at a time such that $k|\eta| \to 0$, i.e., for an asymptotic behaviour at large scales.

The formal solutions of (225) are the complex functions

$$g \subset \left\{ C(k)k\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}, \quad C^*(k)k\left(1 + \frac{i}{k\eta}\right)e^{ik\eta} \right\}, \tag{227}$$

where, in order to satisfy the initial condition in the infinite past, we need to choose the first solution

$$g(\eta, k) = C(k)k\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}, \tag{228}$$

such that, for a time η_i as the infinite past limit $k|\eta_i| \to \infty$, or $k|\eta_i| \gg 1$, we have

$$g(\eta_i, k) = C(k)ke^{-ik\eta_i} = \frac{1}{\sqrt{2k}}e^{-ik\eta_i} \to C(k)k = \frac{1}{\sqrt{2k}}. \tag{229}$$

In this case, the solution is just

$$g(\eta, k) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right). \tag{230}$$

Returning to the original field using (224)

$$h = \sqrt{32\pi G}\frac{1}{a}\frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right), \tag{231}$$

the power spectrum is

$$P_h(\eta, k) = \frac{16\pi G}{k}H_\Lambda^2\eta^2\left(1 + \frac{1}{k^2\eta^2}\right), \tag{232}$$

and then the dimensionless power spectrum, which is defined for an arbitrary random field G as [23]

$$\Delta_G^2(k) \equiv \frac{k^3 P_G(k)}{2\pi^2}, \tag{233}$$

for the tensor perturbation, considering that there are two polarization states, is [42]

$$\Delta_h^2(\eta, k) \equiv \frac{d\,\langle 0|\,\hat{h}_{ik}^2\,|0\rangle}{d\ln k} = 64\pi G\frac{k^3}{2\pi^2}|\hat{h}(\eta, k)|^2 = \frac{2H_\Lambda^2}{\pi^2 M_{\rm pl}^2}[1 + k^2\eta^2]. \tag{234}$$

In the limit $k|\eta| \to 0$, the scale-invariant dimensionless power spectrum is recovered

$$\Delta_h^2(k) = \frac{2}{\pi^2}\frac{H_\Lambda^2}{M_{\rm pl}^2}, \tag{235}$$

which is an important prediction from inflation [31].

4.2. Quasi de-Sitter background: General Relativity

4.2.1. Power Spectrum and Tensor Spectral Index

The case of a perfect de Sitter expansion is not realistic, hence, our approach here consists in working with a quasi-de Sitter expansion of the type

$$\dot{H} = -\epsilon H_\Lambda^2, \tag{236}$$

with ϵ the first slow roll parameter $|\epsilon| \ll 1$, hence, all calculations are done at first order in ϵ. In this case, GR and TG obey different GW equations, hence we split the discussion into two parts.

In this case, the scale factor is

$$a(\eta) = \frac{1}{H_\Lambda}\frac{1}{|\eta|^{1+\varepsilon}}, \quad \text{where} \quad |\eta| = -\eta, \tag{237}$$

and the conformal Hubble factor is

$$\mathcal{H} = -\frac{1+\epsilon}{\eta}. \tag{238}$$

Hence, using the same change of variable (224), the equation of GW in GR with a quasi-de Sitter background becomes

$$g'' + \left(k^2 - \frac{2+3\varepsilon}{\eta^2}\right) g = 0. \tag{239}$$

The solution to the equation (239) is in term of Hankel functions of the first and second kind

$$g(\eta, k) \subset \left\{C(k)\sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|), \quad C^*(k)\sqrt{|\eta|}H_\alpha^{(2)}(k|\eta|)\right\}, \quad \alpha = \frac{3}{2} + \epsilon, \tag{240}$$

with α given at first order in ϵ.

Using the asymptotic expansion of Hankel functions for large $|z|$ and α real [44]

$$H_\alpha^{(1)} \sim \sqrt{\frac{2}{\pi z}} e^{i\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)}, \quad H_\alpha^{(2)} \sim \sqrt{\frac{2}{\pi z}} e^{-i\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right)}, \tag{241}$$

we can choose the $H_\alpha^{(1)}$ solution in order to satisfy the initial condition

$$g(\eta, k) = C(k)\sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|). \tag{242}$$

Therefore, using (241) for the infinite past $k|\eta| \to \infty$ and matching with equation (226), the integration constant is

$$C(k) = \frac{\sqrt{\pi}}{2} e^{i\alpha\frac{\pi}{2} + i\frac{\pi}{4}}, \tag{243}$$

and then the solution is simply

$$g(\eta, k) = \frac{\sqrt{\pi}}{2} e^{i\frac{\alpha\pi}{2} + i\frac{\pi}{4}} \sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|). \tag{244}$$

Returning to the original field h, the solution is

$$h = \sqrt{32\pi G}\frac{1}{a}\frac{\sqrt{\pi}}{2} e^{i\frac{\alpha\pi}{2} + i\frac{\pi}{4}} \sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|), \tag{245}$$

and then the power spectrum is

$$P_h(\eta, k) = \frac{\pi|\eta|}{M_{\rm pl}^2 a^2} \left|H_\alpha^{(1)}(k|\eta|)\right|^2. \tag{246}$$

In the limit outside the horizon $k|\eta| \to 0$, we can use that

$$H_\alpha^{(1)}(x) \underset{x\to 0}{\sim} -i\frac{\Gamma(\alpha)}{\pi}\left(\frac{x}{2}\right)^{-\alpha}, \tag{247}$$

with $\Gamma(z)$ the gamma function, to obtain the dimensionless power spectrum as

$$\Delta_h^2(\eta,k) = \frac{k^3|\eta|\Gamma(\alpha)^2}{\pi^3 M_{\rm pl}^2 a^2}\left(\frac{k|\eta|}{2}\right)^{-2\alpha}, \quad \alpha = \frac{3}{2}+\epsilon, \tag{248}$$

where have taken into account the two polarizations. Considering ϵ at first order in the scale dependence and zero elsewhere, we obtain that

$$\Delta_h^2(\eta,k) = \frac{2}{\pi^2 M_{\rm pl}^2 a^2|\eta|^2}(k|\eta|)^{-2\epsilon} = \frac{2}{\pi^2}\frac{H_\Lambda^2}{M_{\rm pl}^2}k^{-2\epsilon}, \tag{249}$$

due to (237). The power spectrum is no longer scale-invariant, but depends on k as a power law whose exponent is

$$n_T = -2\epsilon, \tag{250}$$

which is the *tensor spectral index* and is expected to be small due to the smallness of ϵ.

4.2.2. The energy spectrum of GW

The energy spectrum, i.e. the gravitational-wave energy density per logarithmic wave number interval is defined as [42]

$$\Omega_{\rm GW}(\eta,k) = \frac{1}{\rho_{\rm crit}(\eta)}\frac{\langle 0|\,\rho_{\rm GW}(\eta)\,|0\rangle}{d\ln k}, \tag{251}$$

where the critical density is given by

$$\rho_{\rm crit}(\eta) = \frac{3H^2(\eta)}{8\pi G}. \tag{252}$$

In order to determine $\rho_{\rm GW}$, we need to work with the action of the gravitational field as a fluid in a FLRW background and compute its energy-momentum tensor (6). In the case of GR, the action for GW is

$$S = \int d^3\boldsymbol{x}\,d\eta\sqrt{-\bar{g}}\left[\bar{g}^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij}\right], \tag{253}$$

from where we can identify the Lagrangian density as

$$\mathcal{L} = \bar{g}^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij}. \tag{254}$$

Since the energy-momentum tensor, considering the signature $(+,-,-,-)$, is given by

$$\mathcal{T}_{\mu\nu} = \frac{2}{\sqrt{-\bar{g}}}\frac{\partial(\mathcal{L}\sqrt{-\bar{g}})}{\partial\bar{g}^{\mu\nu}} = 2\frac{\partial\mathcal{L}}{\partial\bar{g}^{\mu\nu}} - \bar{g}_{\mu\nu}\mathcal{L}, \tag{255}$$

we obtain the energy-momentum tensor to be

$$\mathcal{T}_{\mu\nu} = 2\partial_\mu h_{ij}\partial_\nu h_{ij} - \bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\partial_\alpha h_{ij}\partial_\beta h_{ij}, \tag{256}$$

from where the gravitational-wave energy density is

$$\rho_{\rm GW} = \mathcal{T}^0_{\ 0} = \frac{(h'_{ij})^2 + (\nabla h_{ij})^2}{a(\eta)^2}, \tag{257}$$

where its vacuum expectation value is

$$\langle 0| \rho_{\rm GW} |0\rangle = \int_0^\infty \frac{k^3}{2\pi^2} \frac{|h'|^2 + k^2|h|^2}{a^2} \frac{dk}{k}, \tag{258}$$

such that the energy spectrum is just

$$\Omega_{\rm GW}(\eta, k) = \frac{8\pi G}{3H(\eta)^2} \frac{k^3}{2\pi^2} \frac{|h'|^2 + k^2|h|^2}{a^2}. \tag{259}$$

According to [42], when the k-mode re-enters the horizon after inflation, it satisfies

$$|h'(\eta, k)|^2 = k^2 |h(\eta, k)|, \tag{260}$$

and then, the energy spectrum is simply

$$\Omega_{\rm GW}(\eta, k) = \frac{1}{12} \frac{k^2 \Delta_h^2(\eta, k)}{a^2(\eta) H^2(\eta)}, \tag{261}$$

considering the canonically normalized tensor fields.

This energy spectrum can be computed at different times so we can analyse the possible measurements at different experiments, for instance, at today's time we can analyse the possible density that LIGO or LISA may detect, or at a time of recombination such that CMB experiments may provide information. However, in order to do such analysis, we need the transfer function for the power spectrum, since equation (249) can only be detected at a time η_* when the k-mode re-enters the horizon after inflation, and then it evolves after that time η_*. However, such analysis lies beyond the scope of this work, since our main purpose is to study the power spectrum of GW in TG and the physics after inflation on this theory has been barely studied [8][45].

4.3. Quasi-de Sitter background: Teleparallel Gravity

4.3.1. Power Spectrum and Tensor Spectral Index for several models

As stated before, in the quasi-de Sitter scenario, GW in TG satisfy a different equation because of the presence of the parameter ν. Since the gravitational wave equation depends on the functional proposed, we will consider the following models that have been shown to be compatible with late time cosmological observations [46][8].

1. *Power law models.* We will consider two models in $f(T)$ and $f(T, B)$ in the form of a power law model [9]

$$f(T) = -T + f_0(-T)^m, \quad f(T, B) = -T + f_0(-T)^m + f_1(-B)^m. \tag{262}$$

[9] Remember that $\delta T = \delta B = 0$, and hence $T = \bar{T}$ and $\bar{B} = B$.

Both cases provide the same derivative w.r.t the torsion scalar

$$f_T = -1 - m f_0(-T)^{m-1}, \tag{263}$$

and hence produce the same equation for GW. Observe that the stability condition $f_T < 0$ is achieved if $\text{sign}(f_0) = \text{sign}(m)$ [10] which from observations [47][46] imply $m, f_0 > 0$, $m \neq 1$.

2. *Mixed power law model.* In this case, the $f(T,B)$ is

$$f(T,B) = -T + f_0(-T)^m(-B)^n, \tag{264}$$

such that

$$f_T = -1 - m f_0(-T)^{m-1}(-B)^n, \tag{265}$$

and the stability condition $f_T < 0$ is achieved by $m, f_0 > 0$ according to observations [46].

3. *Exponential model.* In this case, the $f(T)$ model to consider is

$$f(T) = -T + \beta T_\Lambda(1 - e^{-qT/T_\Lambda}), \tag{266}$$

where we have defined $T_\Lambda = -6H_\Lambda^2$. This model recovers GR in the $q \to \infty$ limit [48]. The derivative w.r.t the torsion scalar is

$$f_T = -1 + \beta q e^{-qT/T_\Lambda}, \tag{267}$$

where the stability condition $f_T < 0$ requires $\beta q < 0$ which is fulfilled, since observations requires $q > 0$ and $\beta < 0$ [49].

These models have been studied in literature from various points of views ranging from Noether symmetries to precision cosmology and inflation [50][51][52], showing that they are good candidate to explain the current accelerated expansion era [53][54].

Now, we are working in a quasi-de Sitter expansion, where all computations are done at first order in ϵ, which allow us to unify the discussion of all the models into a single discussion by computing the ν parameter as follows.
For the power law models of $f(T)$ and $f(T,B)$ we have that

$$\nu = -\frac{f_0 2^{m+1} 3^m (m-1) m\epsilon \left((\epsilon+1)^2 H_\Lambda^2 (-\eta)^{2\epsilon}\right)^m}{f_0 6^m m(\epsilon+1)\left((\epsilon+1)^2 H_\Lambda^2(-\eta)^{2\epsilon}\right)^m + 6(\epsilon+1)^3 H_\Lambda^2 (-\eta)^{2\epsilon}}, \tag{268}$$

which at first order in ϵ gives

$$\nu = -\frac{f_0 2^{m+1} 3^m (m-1) m\epsilon \left(H_\Lambda^2\right)^m}{f_0 6^m m \left(H_\Lambda^2\right)^m + 6H_\Lambda^2} = 2\frac{f_0 m(1-m)|T_\Lambda|^{m-1}}{1 + m f_0 |T_\Lambda|^{m-1}}\epsilon, \tag{269}$$

where we have defined $T_\Lambda = -6H_\Lambda^2$ and $B_\Lambda = -18H_\Lambda^2$. If $m = 0, 1$ we obtain $\nu = 0$ as expected since those cases are just TEGR.

[10] A change on the signature would imply $\text{sign}(f_0) = -\text{sign}(m)$.

For the mixed power law model of $f(T,B)$ we have

$$\nu = -\frac{f_0 m\epsilon 2^{n+m}3^{n+m-1}\left(\frac{3n}{3(\epsilon+1)^2(-\eta)^{2\epsilon}-\epsilon}+\frac{(m-1)(-\eta)^{-2\epsilon}}{(\epsilon+1)^2}\right)\left(H_\Lambda^2\left(3(\epsilon+1)^2(-\eta)^{2\epsilon}-\epsilon\right)\right)^n\left((\epsilon+1)^2H_\Lambda^2(-\eta)^{2\epsilon}\right)^m}{(\epsilon+1)H_\Lambda^2\left(f_0 m 6^{n+m-1}\left(H_\Lambda^2\left(3(\epsilon+1)^2(-\eta)^{2\epsilon}-\epsilon\right)\right)^n\left((\epsilon+1)^2H_\Lambda^2(-\eta)^{2\epsilon}\right)^{m-1}+1\right)}, \tag{270}$$

which at first order in ϵ is

$$\nu = -\frac{f_0 m\epsilon 2^{n+m+1}3^{2n+m}(n+m-1)\left(H_\Lambda^2\right)^{n+m}}{f_0 m 2^{n+m}3^{2n+m}\left(H_\Lambda^2\right)^{n+m}+6H_\Lambda^2} = 2\frac{f_0 m(1-m-n)|T_\Lambda|^{m-1}|B_\Lambda|^n}{1+mf_0|T_\Lambda|^{m-1}|B_\Lambda|^n}\epsilon. \tag{271}$$

Finally, for the exponential $f(T)$ model we have

$$\nu = \frac{2\beta q^2\epsilon(\epsilon+1)(-\eta)^{2\epsilon}}{\beta q - e^{q(\epsilon+1)^2(-\eta)^{2\epsilon}}}, \tag{272}$$

which at first order in ϵ gives

$$\nu = \frac{2\beta q^2\epsilon}{\beta q - e^q}. \tag{273}$$

We observe that all the studied models recover the following form of the ν parameter

$$\nu = 2\gamma\epsilon, \tag{274}$$

where the form of γ depends on the model. Hence, our discussion can be performed with $\nu = 2\gamma\epsilon$ taking into account that γ takes the following form for each model

$$\gamma = \begin{cases} \frac{f_0 m(1-m)|T_\Lambda|^{m-1}}{1+mf_0|T_\Lambda|^{m-1}}, & \text{for the power law models,} \\ \frac{f_0 m(1-m-n)|T_\Lambda|^{m-1}|B_\Lambda|^n}{1+mf_0|T_\Lambda|^{m-1}|B_\Lambda|^n}, & \text{for the mixed power law model,} \\ \frac{\beta q^2}{\beta q - e^q}, & \text{for the exponential model.} \end{cases} \tag{275}$$

It is possible to find γ for an arbitrary functional $f(T,B)$, we will show this later on, however we would like to point out that these particular models not only guarantee that $f_T < 0$, but the also provide $f_T < -1$ which is something that will affect the peaks of the power spectrum coming from tensor anisotropic stress sources.

Consider the ν parameter as a function of ϵ and expand at first order in Taylor

$$\nu(\epsilon) = \nu(\epsilon = 0) + \epsilon\partial_\epsilon\nu|_{\epsilon=0}, \tag{276}$$

where we have already argued in (218) that at $\epsilon = 0$, $f_T' = 0$ and then $\nu(\epsilon = 0) = 0$.

Under the same argument, we have

$$\partial_\epsilon\nu|_{\epsilon=0} = -\frac{\eta}{f_{T_\Lambda}}\partial_\epsilon f_T'|_{\epsilon=0}, \tag{277}$$

where we introduce the notation of subscript Λ to indicate that the derivative is evaluated at a pure de Sitter background $T_\Lambda = -6H_\Lambda^2$ and $B_\Lambda = 3T_\Lambda$. A direct calculation using the chain rule shows that

$$\partial_\epsilon \nu|_{\epsilon=0} = 2\left(\frac{f_{T_\Lambda T_\Lambda}|T_\Lambda| + f_{T_\Lambda B_\Lambda}|B_\Lambda|}{f_{T_\Lambda}}\right), \tag{278}$$

therefore, the γ parameter for a general $f(T,B)$ functional is

$$\gamma = \frac{f_{T_\Lambda T_\Lambda}|T_\Lambda| + f_{T_\Lambda B_\Lambda}|B_\Lambda|}{f_{T_\Lambda}}. \tag{279}$$

However, in spite of the general form of γ for any functional has been obtained, the condition $f_T < 0$ needs to be fulfilled and only some particular models accomplish this according to observational data, that is why we considered such particular models at the beginning of the discussion.

With this in mind, the computation of the power spectrum goes as follows.

As in the case of GR, we need a change of variable that takes us from the equation (202) in vacuum to an equation of an harmonic oscillator without damping effect, however, for TG, the transformation (224) does not give the desired result, hence, we need to find the proper change of variable for TG.

We propose the following ansatz

$$h(\eta,k) = \frac{2}{M_{\text{pl}}} f(\eta) g(\eta,k), \tag{280}$$

thus, the GW equation in vacuum is

$$g'' f + g(f'' + [2+\nu]\mathcal{H} f' + k^2 f) + g'(2f' + [2+\nu]\mathcal{H} f) = 0. \tag{281}$$

Hence, in order to remove the damping term, the $f(\eta)$ function, must satisfy

$$2f' + [2+\nu]\mathcal{H} f = 0, \tag{282}$$

whose solution is

$$f(\eta) = \frac{1}{a(\eta)} \text{Exp}\left(-\frac{1}{2}\int \mathcal{H}\nu d\eta\right). \tag{283}$$

We observe that the transformation does not depend on the model and if $\nu = 0$ we recover the transformation in GR (224).

The GW equation is then given by

$$g'' + g\left(k^2 + \frac{f''}{f} + [2+\nu]\mathcal{H}\frac{f'}{f}\right) = 0, \tag{284}$$

such that, for the quasi-de Sitter background and $\nu = 2\gamma\epsilon$, we obtain at first order that

$$g'' + g\left(k^2 - \frac{2 + 3(1+\gamma)\epsilon}{\eta^2}\right) = 0. \tag{285}$$

We observe that the solutions is the same as the GR case but with a different α parameter

$$g(\eta,k)\subset\left\{C(k)\sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|),\quad C^*(k)\sqrt{|\eta|}H_\alpha^{(2)}(k|\eta|)\right\},\quad \alpha=\frac{3}{2}+\epsilon(1+\gamma), \tag{286}$$

and the same integration constant. Thus the solution is the same form as GR

$$g(\eta,k)=\frac{\sqrt{\pi}}{2}e^{i\frac{\alpha\pi}{2}+i\frac{\pi}{4}}\sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|). \tag{287}$$

When returning the the original field we have a different situation than GR, since

$$h=\frac{2}{M_{\rm pl}}f(\eta)g(\eta,k), \tag{288}$$

and

$$f(\eta)=\frac{1}{a(\eta)}{\rm Exp}\left(-\frac{1}{2}\int\mathcal{H}\nu d\eta\right)=a(\eta)^{-(1+\epsilon\gamma)}, \tag{289}$$

then the original field is

$$h=\sqrt{32\pi G}a^{-(1+\epsilon\gamma)}\frac{\sqrt{\pi}}{2}e^{i\frac{\alpha\pi}{2}+i\frac{\pi}{4}}\sqrt{|\eta|}H_\alpha^{(1)}(k|\eta|). \tag{290}$$

The power spectrum is then

$$P_h(\eta,k)=32\pi Ga^{-2(1+\epsilon\gamma)}\frac{\pi}{4}|\eta||H_\alpha^{(1)}(k|\eta|)|^2, \tag{291}$$

with the dimensionless power spectrum given by

$$\Delta_h^2(\eta,k)=\frac{k^3|\eta|}{\pi M_{\rm pl}^2}a^{-2(1+\epsilon\gamma)}|H_\alpha^{(1)}(k|\eta|)|^2. \tag{292}$$

At large scales $k|\eta|\to 0$ the dimensionless power spectrum is

$$\Delta_h^2(\eta,k)=\frac{k^3|\eta|\Gamma(\alpha)^2}{\pi^3M_{\rm pl}^2}a^{-2(1+\epsilon\gamma)}\left(\frac{k|\eta|}{2}\right)^{-2\alpha},\quad \alpha=\frac{3}{2}+\epsilon(1+\gamma) \tag{293}$$

such that using (237), it simplifies to

$$\Delta_h^2(\eta,k)=\frac{k^3|\eta|\Gamma(\alpha)^2}{\pi^3M_{\rm pl}^2}H_\Lambda^{2(1+\gamma\epsilon)}|\eta|^{2+2\epsilon(1+\gamma)}\left(\frac{k|\eta|}{2}\right)^{-3-2\epsilon(1+\gamma)}=\frac{\Gamma(\alpha)^2 2^{3+2\epsilon(1+\gamma)}}{\pi^3M_{\rm pl}^2}H_\Lambda^{2(1+\gamma\epsilon)}k^{-2\epsilon(1+\gamma)}, \tag{294}$$

and then, taking the k-dependence at first order in ϵ and zero order everywhere else, we obtain

$$\Delta_h^2(\eta,k)=\frac{2H_\Lambda^2}{\pi^2M_{\rm pl}^2}k^{-2\epsilon(1+\gamma)}, \tag{295}$$

then, the tensor spectral index is

$$n_T=-2\epsilon(1+\gamma), \tag{296}$$

with γ given by (275).

This is an important result since the tensor spectral index of TG allows a large dependency on the scale for the tensor waves, and hence, if future detectors like LISA are able to measure the primordial GW background and find a non-small tensor spectral index, it would directly suggest that modification of gravity are needed. Current estimations of the tensor spectral index suggest it to be very small, however it is an inferred value from the tensor-to-scalar ratio which highly depends on GR [24][55].

4.3.2. Energy density of GW in Teleparallel Gravity

In this case we proceed in a similar manner as GR, where we need to find ρ_{GW} in TG in order to compute (251). We begin by considering the action of GW in TG

$$S = \int d^3\boldsymbol{x} d\eta (-a^2 f_T) \left[h'_{ij} h'^{ij} - \partial_k h_{ij} \partial^k h^{ij} \right], \tag{297}$$

that can be written in terms of the background tetrad as

$$S = \int d^3\boldsymbol{x} d\eta \bar{e} (-f_T) \left[\bar{E}^{\mu}{}_{A} \bar{E}^{\nu}{}_{B} \eta^{AB} \partial_\mu h_{ij} \partial_\nu h_{ij} \right], \tag{298}$$

where the Lagrangian density is

$$\mathcal{L} = (-f_T) \left[\bar{E}^{\mu}{}_{A} \bar{E}^{\nu}{}_{B} \eta^{AB} \partial_\mu h_{ij} \partial_\nu h_{ij} \right]. \tag{299}$$

Then, the energy momentum tensor of GW is given by

$$\mathcal{T}^{\lambda}_{\nu} = \bar{e}^{A}_{\nu} \theta^{\lambda}_{A}, \quad \theta^{\lambda}_{A} = \frac{1}{\bar{e}} \frac{\delta(\bar{e}\mathcal{L})}{\delta \bar{e}^{A}_{\lambda}}. \tag{300}$$

In order to compute the energy-momentum tensor of GW in TG, we need to use the following variational identity [5][8]

$$\frac{\delta \bar{E}^{\mu}{}_{C}}{\delta \bar{e}^{A}_{\lambda}} = -\bar{E}^{\lambda}{}_{C} \bar{E}^{\mu}{}_{A}, \tag{301}$$

and remember that total derivatives vanish under the integral of the action. Hence, taking into consideration the signature, we obtain

$$\mathcal{T}^{\lambda}_{\nu} = 2(-f_T) \bar{E}^{\lambda}{}_{A} \bar{E}^{\mu}{}_{B} \eta^{AB} \partial_\mu h_{ij} \partial_\nu h_{ij} + \left[\bar{E}^{\mu}{}_{C} \bar{E}^{\alpha}{}_{B} \eta^{CB} \partial_\mu h_{ij} \partial_\alpha h_{ij} \right] \bar{e}^{A}{}_{\nu} \frac{1}{\bar{e}} \frac{\delta(\bar{e} f_T)}{\delta \bar{e}^{A}_{\lambda}}. \tag{302}$$

For the calculation of $\frac{\delta(\bar{e} f_T)}{\delta \bar{e}^{A}_{\lambda}}$, we will use the field equation of $f(T,B)$. Observe that, neglecting total derivatives [5]

$$\begin{aligned} W^{\lambda}{}_{A}[f,e] \equiv \frac{\delta(ef(T,B))}{\delta e^{A}{}_{\lambda}} = &- e f_B B E^{\lambda}_{A} - 2e E^{\lambda}_{A} \mathring{\Box} f_B + 2e E^{\nu}_{A} \mathring{\nabla}^{\lambda} \mathring{\nabla}_{\nu} f_B - 4e(\partial_\mu f_B) S_{A}^{\ \mu\lambda} \\ &- 4\partial_\mu (f_T) e S_{A}^{\ \mu\lambda} - 4 f_T \partial_\mu \left(e S_{A}^{\ \mu\lambda} \right) + 4 e f_T T^{\alpha}_{\ \mu A} S_{\alpha}^{\ \lambda\mu} + e f E^{\lambda}_{A}, \end{aligned} \tag{303}$$

from where we can observe that

$$W^{\lambda}{}_{A}[f_T,\bar{e}] \equiv \frac{\delta(\bar{e}f_T(T,B))}{\delta\bar{e}^{A}{}_{\lambda}} = -\,\bar{e}f_{TB}\bar{B}\bar{E}_A^{\lambda} - 2\bar{e}\bar{E}_A^{\lambda}\mathring{\Box}f_{TB} + 2\bar{e}\bar{E}_A^{\nu}\mathring{\bar{\nabla}}^{\lambda}\mathring{\bar{\nabla}}_{\nu}f_{TB} - 4\bar{e}(\partial_\mu f_{TB})\bar{S}_A{}^{\mu\lambda}$$
$$- 4\partial_\mu(f_{TT})\bar{e}\bar{S}_A{}^{\mu\lambda} - 4f_{TT}\partial_\mu\left(\bar{e}\bar{S}_A^{\mu\lambda}\right) + 4\bar{e}f_{TT}\bar{T}^{\alpha}{}_{\mu A}\bar{S}_{\alpha}{}^{\lambda\mu} + \bar{e}f_T\bar{E}_A^{\lambda}, \tag{304}$$

therefore, the energy momentum tensor is

$$\mathcal{T}^{\lambda}_{\ \nu} = 2(-f_T)\bar{E}^{\lambda}{}_{A}\bar{E}^{\mu}{}_{B}\eta^{AB}\partial_\mu h_{ij}\partial_\nu h_{ij} + \left[\bar{E}^{\mu}{}_{C}\bar{E}^{\alpha}{}_{B}\eta^{CB}\partial_\mu h_{ij}\partial_\alpha h_{ij}\right]\bar{e}^{A}{}_{\nu}\frac{1}{\bar{e}}W^{\lambda}{}_{A}[f_T,\bar{e}]. \tag{305}$$

Observe that in the TEGR limit $f_T = -1$ we recover the same energy-momentum tensor of GR (256) with mixed indices. Hence, for $\lambda = \nu = 0$, we observe that

$$\bar{e}^{A}{}_{0}\frac{1}{\bar{e}}W^{0}{}_{A}[f_T,\bar{e}] = -2\kappa\bar{\rho}_{f_T}, \tag{306}$$

where we introduce the notation

$$a^2(\eta)\kappa\bar{\rho}_{f_T} \equiv 3\mathcal{H}f'_{TB} - 6\mathcal{H}^2(f_{TB}+f_{TT}) - 3f_{TB}\mathcal{H}' - \frac{a^2(\eta)}{2}f_T, \tag{307}$$

to define the background energy density given by the background field equations in $f(T,B)$ gravity (79), in conformal time, if the functional $f(T,B)$ is replaced by $f_T(T,B)$. Thus, the energy density of GW in TG is

$$\rho_{\rm GW}^{\rm TG} = \frac{2}{a^2(\eta)}\left[-(h'_{ij})^2(f_T+\kappa\bar{\rho}_{f_T}) + \kappa\bar{\rho}_{f_T}(\nabla h_{ij})^2\right]. \tag{308}$$

Again, if $f_T = -1$, then $\kappa\bar{\rho}_{f_T} = \frac{1}{2}$ and the energy density is the same found in GR (257). Analogously as the GR case, we can obtain the energy density by computing the vacuum expectation of the energy density and imposing the same condition when k-mode re-enters the horizon, from where we obtain

$$\Omega(\eta,k) = \frac{1}{12}\frac{k^2\Delta_h^2(\eta,k)}{a^2(\eta)H^2(\eta)}(-f_T), \tag{309}$$

where it is trivial to see that $f_T = -1$ recovers the same result as GR.

4.4. From the Bunch-Davies vacuum state to Quantum Coherent States

4.4.1. About the polarization of primordial GW

Generally speaking, when GW are discussed in GR, the weak field limit is employed

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{310}$$

and then arriving to the simple wave equation

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{311}$$

by defining the trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2}\eta_{\mu\nu} \to h = -\bar{h}. \tag{312}$$

The transverse and traceless condition is imposed on both $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$, leaving only two propagating degrees of freedom (DoF) $+, \times$ [56]. However, this is not the case for a general metric theory of gravity, where there exists six possible polarizations given by the six electric components of the Riemann tensor R_{i0j0} [57] that, in the notation of Newman-Penrose, are encoded into four functions

$$\Psi_2 = -\frac{1}{6}R_{z0z0}, \quad \Psi_3 = \frac{1}{2}(-R_{x0z0} + iR_{y0z0}), \quad \Psi_4 = R_{y0y0} - R_{x0x0} + 2iR_{x0y0}, \tag{313}$$

$$\Phi_{22} = -(R_{x0x0} + R_{y0y0}), \tag{314}$$

where each function has an associated helicity s, namely, Ψ_2 and Φ_{22} have s=0 (scalar DoF), Ψ_3 with $s = \pm 1$ (vector DoF) and Ψ_4 has $s = \pm 2$ (tensor DoF). In the case of TG we have two scenarios. If $f(T)$ gravity is the theory, then we still only have two polarizations $+, \times$ [58]. If $f(T, B)$ gravity is the theory, we have a similar situation to $f(\mathring{R})$ gravity, where the trace-reversed perturbation is defined by

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} + \frac{f_{BB}(0,0)}{f_T(0,0)}\eta_{\mu\nu}R^{(1)}. \tag{315}$$

We observe that both traces h and $\bar{h}$ are not the same due to the presence of $\frac{f_{BB}(0,0)}{f_T(0,0)}R^{(1)}$. Although the transverse and traceless condition can be imposed over $\bar{h}_{\mu\nu}$ in order to obtain plane wave solutions, they cannot be imposed over $h_{\mu\nu}$ at the same time [56][59], hence, more degrees of freedom appears in the perturbation, namely, the longitudinal and breathing modes appear.

However, when discussing tensor perturbations around a FLRW background, we employ the SVT decomposition and we consider our primordial GW to be the transverse and traceless part of this decomposition, imposing exactly four constriction leaving two degrees of freedom, the $+, \times$ and it is independent of the theory and only dependent on the SVT decomposition. And since the pertrubation in GR is proportional to the symmetric part of the perturbation in TG, the primordial GW have the same polarization in GR and TG, and that is what was meant by having the same nature as GR, the tensor perturbation in both GR and TG is a massless helicity-2 field with polarizations $+, \times$.

4.4.2. Expectation value and Power Spectrum

In the previous section we consider the state of the universe to be the Bunch-Davies vacuum state, however, since the physics before inflation is unknown, we can assume a different quantum state for the universe. We will summarize the discussion found in [60] where the universe is supposed to be in a quantum coherent state.

We begin by assuming that the initial state is a Schrödinger type coherent state as a consequence of the pre-inflationary dynamics, where some excitation is assumed in the beginning of the universe. For both polarization modes we define their creation and annihilation operators as

$$b_{\oplus}\left|n,m\right\rangle = \sqrt{n}\left|n-1,m\right\rangle, \quad b_{\oplus}^{\dagger}\left|n,m\right\rangle = \sqrt{n+1}\left|n+1,m\right\rangle, \tag{316}$$

$$b_{\otimes}\left|n,m\right\rangle = \sqrt{m}\left|n,m-1\right\rangle, \quad b_{\otimes}^{\dagger}\left|n,m\right\rangle = \sqrt{m+1}\left|n,m+1\right\rangle, \tag{317}$$

where $|n,m\rangle$ represents n gravitons with $+$ polarization and m with $\times$ polarization. The total number of gravitons is $\mathcal{N} = n+m$. It is also possible to define operators that create and annihilate both polarizations as

$$A^{\dagger}_{\alpha,\beta} = \alpha b^{\dagger}_{\otimes} + \beta b^{\dagger}_{\otimes}, \tag{318}$$

$$A\alpha,\beta = \alpha^{*} b_{\otimes} + \beta^{*} b_{\otimes}, \tag{319}$$

such that $|\alpha|^2$ and $|\beta|^2$ represent the probability the create or annihilate a polarization mode, hence, $|\beta|^2 + |\alpha|^2 = 1$. The quantum state of a graviton of number $\mathcal{N}$ that consists of any polarization mode is

$$|\mathcal{N},\alpha,\beta\rangle = \frac{1}{\sqrt{\nu!}}\left(A^{\dagger}_{\alpha,\beta}\right)^{\nu}|0\rangle, \tag{320}$$

where the vacuum state is $|0,\alpha,\beta\rangle = |0\rangle$. There is a degeneracy of $\nu+1$ gravitons in the graviton state. The operator $A^{\dagger}_{\alpha,\beta}$ increase by unity the number of gravitons as

$$A^{\dagger}_{\alpha,\beta}\left|\mathcal{N},\alpha,\beta\right\rangle = \sqrt{\mathcal{N}+1}\left|\mathcal{N}+1,\alpha,\beta\right\rangle, \quad A_{\alpha,\beta}\left|\mathcal{N},\alpha,\beta\right\rangle = \sqrt{\mathcal{N}}\left|\mathcal{N}-1,\alpha,\beta\right\rangle. \tag{321}$$

Then, we define the state of the universe to be a Schrödinger type coherent state as

$$A_{\alpha,\beta}\left|\Psi,\alpha,\beta\right\rangle = \Psi\left|\Psi,\alpha,\beta\right\rangle, \tag{322}$$

where $\Psi = |\Psi|e^{i\theta}$ is a complex eigenvalue. The state $|\Psi,\alpha,\beta\rangle$ can be written in the $|\mathcal{N},\alpha,\beta\rangle$ basis as

$$\begin{aligned} |\Psi,\alpha,\beta\rangle &= e^{-\frac{|\Psi|^2}{2}}\sum_{\mathcal{N}=0}^{\infty}\frac{\Psi^{\mathcal{N}}}{\sqrt{\mathcal{N}!}}\left|\mathcal{N},\alpha,\beta\right\rangle = e^{-\frac{|\Psi|^2}{2}+\Psi A^{\dagger}_{\alpha\beta}}\left|0\right\rangle \\ &= \exp\left(\Psi A^{\dagger}_{\alpha\beta} - \Psi^{*}A_{\alpha\beta}\right)|0\rangle \equiv D(\Psi)\left|0\right\rangle, \end{aligned} \tag{323}$$

where $D(\Psi)$ is the displacement operator which satisfies

$$D(\Psi) = \exp\left(\alpha\Psi b^{\dagger}_{\oplus} - \alpha^{*}\Psi^{*}b_{\otimes} + \beta\Psi b^{\dagger}_{\otimes} - \beta^{*}\Psi^{*}b_{\otimes}\right) \equiv D_{\oplus}(\alpha\Psi)D_{\otimes}(\beta\Psi). \tag{324}$$

Equation (323) implies that

$$b_{\oplus}\left|\Psi,\alpha,\beta\right\rangle = \alpha\Psi\left|\Psi,\alpha,\beta\right\rangle, \quad b_{\otimes}\left|\Psi,\alpha,\beta\right\rangle = \beta\Psi\left|\Psi,\alpha,\beta\right\rangle. \tag{325}$$

This relations are needed in order to compute the power spectrum. If we decompose the tensor field as (207) where the creation and annihilation operators, we obtain each operator for each polarization as

$$h^{k}_{\oplus} = b_{\oplus}h(k,\eta) + b^{\dagger}_{\oplus}h^{*}(k,\eta), \tag{326}$$

$$h^{k}_{\otimes} = b_{\otimes}h(k,\eta) + b^{\dagger}_{\otimes}h^{*}(k,\eta), \tag{327}$$

with $h(k,\eta)$ the solution of the GW equation either in GR or TG. Hence, the power spectrum for the plus mode is

$$\langle\Psi,\alpha,\eta|\, h^k_\oplus h^k_\oplus \,|\Psi,\alpha,\eta\rangle = |h(\eta,k)|^2 + 2|\Psi|^2|\alpha|^2|h(\eta,k)|^2 + \Psi^2\alpha^{*2}h(\eta,k)^2 + \Psi^{*2}\alpha^2 h^*(\eta,k)^2, \tag{328}$$

for the cross mode is

$$\langle\Psi,\alpha,\eta|\, h^k_\otimes h^k_\otimes \,|\Psi,\alpha,\eta\rangle = |h(\eta,k)|^2 + 2|\Psi|^2|\beta|^2|h(\eta,k)|^2 + \Psi^2\beta^{*2}h(\eta,k)^2 + \Psi^{*2}\beta^2 h^*(\eta,k)^2, \tag{329}$$

and the power spectrum between both modes is

$$\langle\Psi,\alpha,\eta|\, h^k_\oplus h^k_\otimes \,|\Psi,\alpha,\eta\rangle = |\Psi|^2|h(\eta,k)|^2(\alpha^*\beta + \beta^*\alpha) + \Psi^2 h(\eta,k)^2\alpha\beta + \Psi^{*2}h^*(\eta,k)^2\alpha^*\beta^*. \tag{330}$$

Observe that this result is independent of the theory, and the distinction between theories appear in the $h(\eta,k)$ solutions. This results is based on the supposition of an excited quantum state of the universe, however the mechanism for such excitation is not discussed, then to formally obtain a power spectrum like this detected in an experiment we must specify the concrete mechanism for such excitation, and then being able to specify the parameters α,β,Ψ. It is possible to perform a similar analysis for different states, but this lays outside the purpose of this work.

5. Production of Gravitational Waves: Including tensor anisotropic effects

5.1. Stochastic derivation of Gravitational Waves

In this section, we will discuss another way of dealing with perturbation that will allow us to introduce effects of tensor anisotropic stress, like those coming from thermal fluctuations, into the power spectrum. We will begin by discussing the GR case in a perfect de Sitter background, we will follow the discussion in [36] and [61] for such endeavour.

Consider the tensor field h to be divided into two different perturbations, a short wavelength much more smaller than the horizon $h_<$, and a long-wavelength part $h_>$, such that

$$h(\eta, \boldsymbol{x}) = h_<(\eta, \boldsymbol{x}) + h_>(\eta, \boldsymbol{x}). \tag{331}$$

The short wavelength mode summarizes all vacuum fluctuations and its physical effects appear through the phenomenology of the long wavelength perturbation, in this case, it will appear through the power spectrum. The short wavelength mode represents high momentum modes of the field $k > \mathcal{H}$, i.e., wavelength much smaller than the horizon, this is achieved through a filter or window function , hence, our short wavelength mode is written as

$$h_<(\eta, \boldsymbol{x}) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}} W(\eta, k) \left[h(\eta, k) e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{k}, \lambda) + h^*(\eta, k) e^{-i\mathbf{k}\cdot\mathbf{x}} a^\dagger(\mathbf{k}, \lambda) \right], \tag{332}$$

where the window function is

$$W(\eta, k) = \theta(k - \varepsilon\mathcal{H}), \tag{333}$$

and ε is a suitable parameter that allows to achieve the purpose of $h_<$ to be a short wavelength contribution.

In the case of GR, the equation of gravitational waves in vacuum for a perfect de Sitter background is

$$h'' - \frac{2}{\eta} h' - \nabla^2 h = 0, \tag{334}$$

working in the physical space. By introducing the separation of the field (331) we obtain that

$$h''_> - \frac{2}{\eta} h'_> - \nabla^2 h_> = \varrho_Q, \quad \varrho_Q = -\left(h''_< - \frac{2}{\eta} h'_< - \nabla^2 h_< \right), \tag{335}$$

with ϱ_Q called the *quantum noise*. Let us compute the quantum noise explicitly. Observe that the only quantities that depend on the conformal time are $W(\eta, k)$ and $h(\eta, k)$, hence

$$\varrho_Q = -\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}} \left[\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2 \right) (W(\eta, k) h(\eta, k)) e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{k}, \lambda) \right. \tag{336}$$

$$\left. + \left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2 \right) (W(\eta, k) h^*(\eta, k)) e^{-i\mathbf{k}\cdot\mathbf{x}} a^\dagger(\mathbf{k}, \lambda) \right], \tag{337}$$

and observe that

$$\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2\right) W(\eta, k)h(\eta, k) = \left(W'' - \frac{2}{\eta}W'\right) h + 2W'h' + W\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2\right) h, \quad (338)$$

but we require $h(\eta, k)$ to satisfy the GW equation

$$\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2\right) h = 0, \quad (339)$$

and then

$$\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2\right) W(\eta, k)h(\eta, k) = \left(W'' - \frac{2}{\eta}W'\right) h + 2W'h'. \quad (340)$$

Thus, defining

$$f_k(\eta) = \left(W'' - \frac{2}{\eta}W'\right) h + 2W'h', \quad (341)$$

the quantum noise becomes

$$\varrho_Q(\eta, \boldsymbol{x}) = -\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}} \left[a(\mathbf{k}, \lambda) f_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}, \lambda) f_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \quad (342)$$

With this obtained, the vacuum expectation value of the quantum noise is

$$\langle 0| \varrho_Q(\eta_1, \boldsymbol{x}_1)\varrho_Q(\eta_2, \boldsymbol{x}_2) |0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)} f_k(\eta_1) f_k^*(\eta_2), \quad (343)$$

or in Fourier space is

$$\langle 0| \varrho_Q(\eta_1, \boldsymbol{k})\varrho_Q(\eta_2, \boldsymbol{q}) |0\rangle = \delta_D(\mathbf{k} - \mathbf{q}) f_k(\eta_1) f_k^*(\eta_2). \quad (344)$$

In Fourier space, the equation (335) becomes

$$\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2\right) h_> = \varrho_Q(\eta, \mathbf{k}), \quad (345)$$

which is a inhomogeneous version of the equation studied previously. In order to solve this equation we will use the method of Green's function. The idea of Green's function is that we have a differential operator $\mathcal{L}$ given by

$$\mathcal{L} = \alpha(x)\frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(x), \quad (346)$$

such that we require to solve the inhomogeneous equation

$$\mathcal{L}y(x) = f(x), \quad (347)$$

under certain boundary conditions [62]. Green's function is then a function $G(x; \xi)$ that satisfies

$$\mathcal{L}G(x; \xi) = \delta_D(x - \xi), \quad (348)$$

from where the solution of the inhomogeneous equation reads as

$$y(x) = \int G(x;\xi) f(\xi). \tag{349}$$

The condition (348) implies that the Green's function must satisfy the following continuity conditions

$$G(x;\xi)\big|_{x=\xi^+} = G(x;\xi)\big|_{x=\xi^-}, \quad \frac{\partial G}{\partial x}\bigg|_{x=\xi^+} - \frac{\partial G}{\partial x}\bigg|_{x=\xi^-} = \frac{1}{\alpha(\xi)}. \tag{350}$$

In our case, we are interested in a retarded Green's function whose boundary condition reads as $G(x;\xi) = 0$ if $x < \xi$. In order to construct such Green's function with the desired boundary conditions, we propose that the Green's function is just a Heaviside step function times a linear combination of the independent solutions of the homogeneous equation $\mathcal{L}y_1 = \mathcal{L}y_2 = 0$ as

$$G(x;\xi) = \theta(x-\xi)\left[A(\xi)y_1(x) + B(\xi)y_2(\xi)\right] \equiv \theta(x-\xi)L(x,\xi), \tag{351}$$

such that $\mathcal{L}L(x,y) = 0$. The continuity conditions (350) imply that

$$L(\xi,\xi) = 0, \quad L'(\xi,\xi) = \frac{1}{\alpha(\xi)}. \tag{352}$$

Hence, we can see explicitly that

$$\mathcal{L}G = \alpha(x)\delta_D'(x-\xi)L(x,\xi) + 2\alpha(x)\delta_D(x-\xi)L'(x,\xi) + \beta(x)L(x,\xi). \tag{353}$$

Now, in order to see that $\mathcal{L}G = \delta_D(x-\xi)$, we need to show that

$$\int \mathcal{L}G f(x)dx = f(\xi), \tag{354}$$

since $\delta_D(x-\xi)$ only makes sense as a distribution. Using the following property of the Dirac delta

$$\int \delta'(x)\phi(x)dx = -\int \delta(x)\phi'(x)dx, \tag{355}$$

such that, using the conditions (352) we observe

$$\begin{aligned}\int \mathcal{L}G f(x)dx &= \int \alpha(x)\delta_D'(x-\xi)L(x,\xi)f(x)dx + 2\int \alpha(x)\delta_D(x-\xi)L'(x,\xi)f(x)dx \\ &+ \int \beta(x)L(x,\xi)f(x)dx = -\int \delta_D(x-\xi)(\alpha(x)L(x,\xi)f(x))'dx \\ &+ 2\int \alpha(x)\delta_D(x-\xi)L'(x,\xi)f(x)dx + \int \beta(x)L(x,\xi)f(x)dx \\ &= -L(\xi,\xi)(\alpha(x)f(x))'_{x=\xi} - L'(\xi,\xi)\alpha(\xi)f(\xi) + 2\alpha(\xi)L'(\xi,\xi)f(\xi) \\ &+ \beta(\xi)L(\xi,\xi)f(\xi) = \alpha(\xi)L'(\xi,\xi)f(\xi) = f(\xi),\end{aligned}$$

which shows that $G(x;\xi)$ is indeed the required Green's function. Applying the boundary conditions, we can find the coefficients $A(\xi)$ and $B(\xi)$ to be

$$A(\xi) = \frac{1}{\alpha(\xi)}\frac{y_2(x)}{y_2(x)y_1'(x) - y_1(x)y_2'(x)}\bigg|_{x=\xi}, \quad B(\xi) = -\frac{1}{\alpha(\xi)}\frac{y_1(x)}{y_2(x)y_1'(x) - y_1(x)y_2'(x)}\bigg|_{x=\xi}. \tag{356}$$

In the case of GW, we have the differential equation for GW

$$\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + k^2\right) G^R(\eta, \eta_i, k) = \delta(\eta - \eta_i), \tag{357}$$

we now take advantage of the solutions for the vacuum case where the solutions of the homogeneous equation have been obtained (227), however, we first need to perform the change of variable (224) for $h_>$, which leave us with the following equation

$$\left(\partial_\eta^2 - \frac{2}{\eta^2} + k^2\right) g_> = a(\eta)\hat{\varrho}_Q(\eta, \mathbf{k}), \tag{358}$$

where $\hat{\varrho}_Q$ is the quantum noise computed with the canonically quantized tensor fields. The, the retarded Green's function satisfy

$$\left(\partial_\eta^2 - \frac{2}{\eta^2} + k^2\right) G^R(\eta, \eta_i, k) = \delta_D(\eta - \eta_i), \tag{359}$$

and hence, from the previous discussion of Green's functions and using the solutions of the vacuum case (227), we obtain the Green's function to be

$$G^R(\eta, \eta_i, k) = \frac{\theta(\eta - \eta_i)}{\eta\eta_i k^3}\mathrm{Im}\left[e^{ik(\eta-\eta_i)}(1 - ik\eta)(1 + ik\eta_i)\right], \tag{360}$$

and then the solution is simply

$$g_>(\eta, \mathbf{k}) = \int_{-\infty}^{\eta} G^R(\eta, \eta_i, k)a(\eta_i)\hat{\varrho}_Q(\eta_i, \mathbf{k})d\eta_i, \tag{361}$$

such that returning to the original field

$$\begin{aligned} h_>(\eta, \mathbf{k}) &= \frac{1}{a(\eta)}\int_{-\infty}^{\eta} G^R(\eta, \eta_i, k)a(\eta_i)\varrho_Q(\eta_i, \mathbf{k})d\eta_i \\ &= \int_{-\infty}^{\eta} G_R(\eta, \eta_i, k)\varrho_Q(\eta_i, \mathbf{k})d\eta_i, \end{aligned} \tag{362}$$

where by using that $a(\eta) = -\frac{1}{H_\Lambda \eta}$, we can recover

$$G_R(\eta, \eta_i, k) = \frac{a(\eta_i)}{a(\eta)}G^R(\eta, \eta_i, k) = \frac{\theta(\eta - \eta_i)}{\eta_i^2 k^3}\mathrm{Im}\left[e^{ik(\eta-\eta_i)}(1 - ik\eta)(1 + ik\eta_i)\right], \tag{363}$$

which is the Green's function found in [36].
Since the solution for the field is given in terms of the Green's function

$$h_>(\eta, \mathbf{k}) = \int_{-\infty}^{\eta} G_R(\eta, \eta_i, k)\varrho_Q(\eta_i, \mathbf{k})d\eta_i, \tag{364}$$

taking the equal time correlator, we get

$$\langle h_>(\eta, \mathbf{k})h_>(\eta, \mathbf{q})\rangle = \delta_D(\mathbf{k} + \mathbf{q})\left|\int_{-\infty}^{\eta} G_R(\eta, \eta_i, k)f_k(\eta_i)d\eta_i\right|^2, \tag{365}$$

where we have used the expectation value of the quantum noise in Fourier space (344). An integration by parts shows that

$$\int_{-\infty}^{\eta} d\eta_i G_R(\eta,\eta_i,k) f_k(\eta_i) = \frac{iH_\Lambda}{\sqrt{2k^3}}(1+ik\eta), \tag{366}$$

and hence, considering the canonically normalized fields, we have

$$\left\langle \hat{h}_>(\eta,\mathbf{k})\hat{h}_>(\eta,\mathbf{q})\right\rangle = P_h(k) = \frac{16\pi G H_\Lambda^2}{k^3}(1+k^2\eta^2), \tag{367}$$

the same power spectrum obtained in the previous section. Therefore, this stochastic derivation is completely analogous to the previously discussed method, however, the introduction of the Green's function gives an enormous advantage of being able to introduce effects of possible tensor anisotropic stress. In the following we will discuss the stochastic derivation in TG with a general tensor anisotropic stress contribution to later on analyse different sources of this anisotropic stress.

5.2. Stochastic derivation in Teleparallel Gravity

Now, we will follow the stochastic derivation in TG with a quasi-de Sitter background, but we will consider also the effects of tensor anisotropic stress into the calculation. Let us consider the equation for Gw in TG with canonically normalized fields

$$\hat{h}''_{kj} + [2+\nu]\mathcal{H}\hat{h}'_{kj} + k^2\hat{h}_{kj} = \frac{16\pi G a^2}{f_T}\pi^T_{ij}, \tag{368}$$

and multiply both sides by $\epsilon^{\lambda *}_{ij}$ and sum over the spatial indices

$$\left(\partial_\eta^2 + [2+\nu]\mathcal{H}\partial_\eta + k^2\right)\sum_{ij}\epsilon^{\lambda *}_{ij}\hat{h}_{ij} = \frac{16\pi G a^2}{f_T}\sum_{ij}\epsilon^{\lambda *}_{ij}\pi^T_{ij}, \tag{369}$$

and now, observe that

$$\sum_{ij}\epsilon^{\lambda *}_{ij}\hat{h}_{ij} = \sum_{ij}\epsilon^{\lambda *}_{ij}\sum_{\lambda'}\epsilon^{\lambda'}_{ij}\hat{h}^{\lambda'} = \sum_{\lambda'}\left(\sum_{ij}\epsilon^{\lambda'}_{ij}\epsilon^{\lambda *}_{ij}\right)\hat{h}^{\lambda'} = \sum_{\lambda'}\delta^{\lambda'\lambda}\hat{h}^{\lambda'} = \hat{h}^{\lambda},$$

therefore, the equation for GW becomes

$$\left(\partial_\eta^2 + [2+\nu]\mathcal{H}\partial_\eta + k^2\right)\hat{h}^{\lambda}(\eta,\mathbf{k}) = \frac{16\pi G a^2}{f_T}\sum_{ij}\epsilon^{\lambda *}_{ij}\pi^T_{ij} \equiv -\frac{1}{f_T}\varrho^{\lambda}_T(\eta,\mathbf{k}), \tag{370}$$

$\varrho^{\lambda}_T(\eta,\mathbf{k})$ is the tensor anisotropic noise.
We now perform the split into short and long wavelength modes (331) from where we obtain

$$\left(\partial_\eta^2 + [2+\nu]\mathcal{H}\partial_\eta + k^2\right)\hat{h}^{\lambda}_>(\eta,\mathbf{k}) = \hat{\varrho}^{\lambda}_Q(\eta,\mathbf{k}) - \frac{1}{f_T}\varrho^{\lambda}_T(\eta,\mathbf{k}), \tag{371}$$

and then proceed to make the change of variable

$$\hat{h}^{\lambda}_>(\eta,\mathbf{k}) = f(\eta)g^{\lambda}_>(\eta,\mathbf{k}) \quad \text{with} \quad f(\eta) = a(\eta)^{-(1+\gamma\epsilon)}, \tag{372}$$

the equation becomes

$$\left(\partial_\eta^2 + k^2 - \frac{2+3\epsilon(1+\gamma)}{\eta^2}\right) g_>^\lambda(\eta,\mathbf{k}) = a(\eta)^{1+\gamma\epsilon}\left(\hat{\varrho}_Q^\lambda(\eta,\mathbf{k}) - \frac{1}{f_T}\varrho_T^\lambda(\eta,\mathbf{k})\right). \tag{373}$$

Observe that in our case, the quantum noise is different from GR as

$$\varrho_Q^\lambda(\eta,\boldsymbol{x}) = -\int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}\left[a(\mathbf{k},\lambda) f_k^{\lambda,\mathrm{TG}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k},\lambda) f_k^{\lambda,\mathrm{TG}*}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{374}$$

with

$$f_k^{\lambda,\mathrm{TG}}(\eta) = \left(W'' - \frac{2}{\eta}\left[1+\epsilon(1+\gamma)\right]W'\right)h^\lambda + 2W'h^{\lambda\prime}. \tag{375}$$

Hence, in order to solve (373) we need to compute the retarded Green's function for TG in a quasi-de Sitter background. We take advantage of the solutions in vacuum (286) together with the previous discussion of retarded Green's function (356) to obtain the retarded Green's function in TG given by

$$G^R_{\mathrm{TG}}(\eta,\eta_i,k) = \theta(\eta-\eta_i)\frac{\pi}{2}\sqrt{|\eta|}\sqrt{|\eta_i|}\ \mathrm{Im}\left[H_\alpha^{(1)}(k|\eta_i|)H_\alpha^{(2)}(k|\eta|)\right], \quad \alpha = \frac{3}{2} + \epsilon(1+\gamma), \tag{376}$$

then the solution of (373) is

$$g_>^\lambda(\eta,\boldsymbol{k}) = \int_{-\infty}^{\eta} d\eta_i a(\eta_i)^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\hat{\varrho}_Q^\lambda(\eta_i,\boldsymbol{k}) - \int_{-\infty}^{\eta} d\eta_i \frac{a(\eta_i)^{1+\gamma\epsilon}}{f_T(\eta_i)} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\varrho_T^\lambda(\eta_i,\boldsymbol{k}). \tag{377}$$

Therefore, returning to the $\hat{h}_>^\lambda$ field, we have

$$\begin{aligned}\hat{h}_>^\lambda(\eta,\boldsymbol{k}) = &\int_{-\infty}^{\eta} d\eta_i \left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\hat{\varrho}_Q^\lambda(\eta_i,\boldsymbol{k}) \\ &- \int_{-\infty}^{\eta} d\eta_i \frac{1}{f_T(\eta_i)}\left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\varrho_T^\lambda(\eta_i,\boldsymbol{k}).\end{aligned} \tag{378}$$

This the general solution for the GW field in TG in a quasi-de Sitter background with the effects of tensor anisotropic stress included. Observe that, if $f_T = -1$, then $\gamma = 0$ and then we have also obtained the general solution in GR for a quasi-de Sitter background. Now, we proceed to compute the equal time correlator of this field in order to obtain the power spectrum.

5.3. The contribution to the power spectrum from the anisotropic stress

5.3.1. General Case

Taking the equal time correlator of the general solution (378) we have

$$\left\langle \hat{h}^{\lambda}_{>}(\eta,\boldsymbol{k})\hat{h}^{\lambda'}_{>}(\eta,\boldsymbol{q})\right\rangle = \tag{379}$$
$$\int_{-\infty}^{\eta} d\eta_1 \left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_1,k)\int_{-\infty}^{\eta} d\eta_2 \left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_2,k)\langle 0|\,\hat{\varrho}^{\lambda}_{Q}(\eta_1,\boldsymbol{k})\hat{\varrho}^{\lambda'}_{Q}(\eta_2,\boldsymbol{q})\,|0\rangle$$
$$+\int_{-\infty}^{\eta} d\eta_1 \frac{1}{f_T(\eta_1)}\left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_1,k)\int_{-\infty}^{\eta} d\eta_2 \frac{1}{f_T(\eta_2)}\left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_2,k)$$
$$\langle 0|\,\varrho^{\lambda}_{T}(\eta_1,\boldsymbol{k})\varrho^{\lambda'}_{T}(\eta_2,\boldsymbol{q})\,|0\rangle\,,$$

and we recall the form of the autocorrelator of the quantum noise (344)

$$\langle 0|\,\hat{\varrho}^{\lambda}_{Q}(\eta_1,\boldsymbol{k})\hat{\varrho}^{\lambda'}_{Q}(\eta_2,\boldsymbol{q})\,|0\rangle = 32\pi G\delta_D(\mathbf{k}+\mathbf{q})\delta^{\lambda\lambda'} f^{\rm TG}_k(\eta_1)f^{\rm TG*}_k(\eta_2), \tag{380}$$

where the factor $32\pi G$ is due to the canonical normalization. Therefore, taking into account that there are two polarization states, the equal time correlator is

$$\left\langle \hat{h}_{>}(\eta,\boldsymbol{k})\hat{h}_{>}(\eta,\boldsymbol{q})\right\rangle = 64\pi G\delta_D(\mathbf{k}+\mathbf{q})\left|\int_{-\infty}^{\eta} d\eta_i \left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_i,k)f^{\rm TG}_k(\eta_i)\right|^2 \tag{381}$$
$$+\int_{-\infty}^{\eta} d\eta_1 \frac{1}{f_T(\eta_1)}\left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_1,k)\int_{-\infty}^{\eta} d\eta_2 \frac{1}{f_T(\eta_2)}\left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_2,k)$$
$$\sum_{\lambda}\langle 0|\,\varrho^{\lambda}_{T}(\eta_1,\boldsymbol{k})\varrho^{\lambda}_{T}(\eta_2,\boldsymbol{q})\,|0\rangle\,.$$

By construction, the first term is just the power spectrum from vacuum fluctuation [11]

$$P_h(\eta,k) = 64\pi G\left|\int_{-\infty}^{\eta} d\eta_i \left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_i,k)f^{\rm TG}_k(\eta_i)\right|^2 = \frac{2\pi|\eta|}{M^2_{\rm pl}}a^{-2(1+\epsilon\gamma)}|H^{(1)}_{\alpha}(k|\eta|)|^2, \tag{382}$$

which has been previously computed, whereas we define the second term as

$$\delta_D(\mathbf{k}-\mathbf{q})P_T(\eta,k) = \int_{-\infty}^{\eta} d\eta_1 \frac{1}{f_T(\eta_1)}\left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_1,k)\int_{-\infty}^{\eta} d\eta_2 \frac{1}{f_T(\eta_2)}\left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\rm TG}(\eta,\eta_2,k) \tag{383}$$
$$\sum_{\lambda}\langle 0|\,\varrho^{\lambda}_{T}(\eta_1,\boldsymbol{k})\varrho^{\lambda}_{T}(\eta_2,\boldsymbol{q})\,|0\rangle\,,$$

as the contribution from tensor anisotropic stress to the power spectrum, such that the total power spectrum is simply

$$P(\eta,k) = P_h(\eta,k) + P_T(\eta,k). \tag{384}$$

For the sake of completeness, we now proceed to discuss the contribution $P_T(\eta,k)$ from different sources.

[11] This might not be clear at first sight, however a direct calculation shows this result explicitly taking into account that $|\eta| \leq |\eta_*|$ with η_* the time where the mode k re-enters the horizon.

5.3.2. From Thermal Fluctuations

From equation (381), we observe that the tensor anisotropic stress contribution only affects the power spectrum of the long wavelength tensor field. In this case, we will consider non-zero values of the tensor anisotropic stress coming from thermal fluctuations. Hence, the use of hydrodynamics as a long wavelength description of fluids is required [63]. The energy-momentum tensor for a fluid in perfect equilibrium (thermal, chemical and mechanical) is just the energy-momentum tensor of a perfect fluid [63].

$$\bar{\mathcal{T}}^{\mu\nu} = (\bar{\rho} + \bar{P})\bar{u}^{\mu}\bar{u}^{\nu} - \bar{P}\bar{g}^{\mu\nu}, \tag{385}$$

with $\bar{\rho} + \bar{P} = \bar{T}\bar{s} + \mu_B n_B$ the local enthalpy density. We will consider that the temperature and four velocities are not constant but vary slowly, thus non-equilibrium correction $\delta\mathcal{T}^{\mu\nu}$ and occasional long-wavelength thermal fluctuations must be considered [64]

$$\begin{aligned} \mathcal{T}^{\mu\nu} &= \bar{\mathcal{T}}^{\mu\nu} + \delta\mathcal{T}^{\mu\nu} + S^{\mu\nu} \\ &= \bar{\mathcal{T}}^{\mu\nu} - \eta_v \theta^{\rho\gamma}\theta^{\nu\sigma}W_{\gamma\sigma} - \chi\left(\theta^{\mu\gamma}u^{\nu} + \theta^{\nu\gamma}u^{\mu}\right)Q_{\gamma} - \zeta\theta^{\mu\nu}u^{\gamma}_{;\gamma} + S^{\mu\nu}, \end{aligned} \tag{386}$$

where we have defined the shear tensor $W_{\alpha\beta}$ and the heat-flow vector Q_α as [65]

$$W_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha} - \frac{2}{3}g_{\mu\nu}u^{\gamma}_{;\gamma}, \quad Q_\alpha = T_{;\alpha} + Tu_{\alpha;\beta}u^{\beta}, \tag{387}$$

with η_v the shear viscosity, ζ the bulk viscosity and χ the heat conduction. The background term $\bar{\mathcal{T}}^{\mu\nu}$ contains the average values of the temperature and four velocity ($T = \bar{T}, \bar{v}_i = 0$) with $\delta u_i = av_i$, the non-equilibrium correction $\delta\mathcal{T}^{\mu\nu}$ the first order correction ($T = \delta, v_i = \delta v_i$), and $S^{\mu\nu}$ local thermal fluctuations. The non-equilibrium correction have no projection onto tensor modes [36], only the local thermal fluctuations will contribute to the tensor anisotropic stress. It was found that [63][66] the correlator for the local thermal fluctuations is [36][64]

$$\left\langle S^{ij}(X)S^{mn}(Y)\right\rangle = 2T\left[\eta_v\left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) + \left(\zeta - \frac{2\eta_v}{3}\right)\delta_{ij}\delta_{mn}\right]\frac{\delta_D(X-Y)}{\sqrt{-\bar{g}}}, \tag{388}$$

where the convention of raising and lowering spatial indices with the Kronecker delta has been used. Hence, identifying the tensor anisotropic stress with the thermal fluctuations, we have

$$\begin{aligned} \sum_{\lambda}\langle 0|\, \varrho^{\lambda}_{T}(\eta_1, \boldsymbol{k})\varrho^{\lambda}_{T}(\eta_2, \boldsymbol{q})\,|0\rangle &= (16\pi G)^2 a^4 \sum_{\lambda}\epsilon^{\lambda}_{ij}\epsilon^{\lambda *}_{mn}\left\langle S^{ij}(\eta_1, \mathbf{k})S^{mn}(\eta_2, \mathbf{q})\right\rangle \\ &= (16\pi G)^2 a^4 \Pi_{ij,mn}\left\langle S^{ij}(\eta_1, \mathbf{k})S^{mn}(\eta_2, \mathbf{q})\right\rangle, \end{aligned} \tag{389}$$

where $\Pi_{ij,mn}$ is the sum over helicities [23][29] given by

$$\begin{aligned} \Pi_{ij,mn}(\hat{k}) = {} & \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm} - \delta_{ij}\delta_{mn} + \delta_{ij}\hat{k}_m\hat{k}_n + \delta_{mn}\hat{k}_i\hat{k}_j - \delta_{im}\hat{k}_j\hat{k}_n - \delta_{jn}\hat{k}_i\hat{k}_m - \delta_{in}\hat{k}_j\hat{k}_m - \delta_{jm}\hat{k}_i\hat{k}_n \\ & + \hat{k}_i\hat{k}_j\hat{k}_m\hat{k}_n\,, \end{aligned} \tag{390}$$

and $\hat{k}_i = k_i/k$ the normalized wave vector. Therefore, considering that $\delta_{im}\delta_{im} = \delta_{ii} = 3$, it is straightforward to see that

$$\Pi_{ij,mn}\left\langle S^{ij}(\eta_1, \mathbf{k})S^{mn}(\eta_2, \mathbf{q})\right\rangle = \frac{8T\eta_v\delta_D(\mathbf{k}-\mathbf{q})\delta_D(\eta_1-\eta_2)}{a^4}, \tag{391}$$

from where we have

$$\int_{-\infty}^{\eta} d\eta_1 \frac{1}{f_T(\eta_1)} \left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_1,k) \int_{-\infty}^{\eta} d\eta_2 \frac{1}{f_T(\eta_2)} \left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_2,k)$$
$$\sum_\lambda \langle 0|\, \varrho^\lambda_T(\eta_1,\boldsymbol{k})\varrho^\lambda_T(\eta_2,\boldsymbol{q})\,|0\rangle = \int_{-\infty}^{\eta} d\eta_1 \frac{1}{f_T(\eta_1)} \left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_1,k) \int_{-\infty}^{\eta} d\eta_2 \frac{1}{f_T(\eta_2)} \left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon}$$
$$G^R_{\mathrm{TG}}(\eta,\eta_2,k)(16\pi G)^2 a^4 \frac{8T\eta_v \delta_D(\mathbf{k}-\mathbf{q})\delta_D(\eta_1-\eta_2)}{a^4}$$
$$= \delta_D(\mathbf{k}-\mathbf{q}) 64\pi G \int_{-\infty}^{\eta} d\eta_i 32\pi G \left(\frac{1}{f_T(\eta_i)} \left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\right)^2 T(\eta_i)\eta_\nu(\eta_i),$$

and thus the thermal contribution to the power spectrum in TG is

$$P_T(\eta,k) = 64\pi G \int_{-\infty}^{\eta} d\eta_i 32\pi G \left(\frac{1}{f_T(\eta_i)} \left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\right)^2 T(\eta_i)\eta_\nu(\eta_i). \tag{392}$$

Then the dimensionless power spectrum is

$$\Delta^2_T(\eta,k) = \frac{64\pi G k^3}{2\pi^2} \int_{-\infty}^{\eta} d\eta_i 32\pi G \left(\frac{1}{f_T(\eta_i)} \left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\right)^2 T(\eta_i)\eta_\nu(\eta_i). \tag{393}$$

Observe that there is a lot of physics to be yet discussed. To be more specific, the value of the shear viscosity has to be studied in the context of TG, since in the case of GR this value depends on the damping coefficient of a heat bath in thermal equilibrium with the inflaton field [36][67]. Such effects has not yet been studied in the context of TG, so a more exact prediction of the power spectrum will depend on physics of inflation in TG and the possibility to generate such thermal fluctuations. However, if such thermal effects can be fully studied in TG, equation (393) will account its contribution to the power spectrum of primordial GW.

5.3.3. From cosmological phase transitions

Another possible source of GW is a first order phase transition of the relativistic matter in the early universe [68]. Formally speaking, a first order phase transition is a discontinuity on a phase boundary, loci of separate regions of analyticity, on the partial derivatives of the bulk free energy per unit volume w.r.t the axes of the phase diagram [69]. Roughly speaking, it is an abrupt change of a thermodynamic variable when a parameter of control is changed, like in the liquid-gas transition of water when temperature is increased. If a first order phase transition occurred in the early universe, the mechanism is through bubble nucleation that generates inhomogeneities and turbulence in the cosmic plasma sourcing the background of GW [70]. Let us discuss the generation of GW in TG from a general phase transition with an associated anisotropic stress, which is generated by colliding bubbles and turbulence in the cosmic plasma [71].

Consider a tensor anisotropic stress split into two uncorrelated helicity modes

$$\pi_{ij}(\eta,\mathbf{k}) = \epsilon^+_{ij}\pi_+(\eta,\mathbf{k}) + \epsilon^\times_{ij}\pi_\times(\eta,\mathbf{k}), \tag{394}$$

such that

$$\langle \pi_+(\eta, \mathbf{k})\pi_+(\eta', \mathbf{q})\rangle = \langle \pi_\times(\eta, \mathbf{k})\pi_\times(\eta', \mathbf{q})\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{q})\rho_X^2 P_\pi(\eta, \eta', k), \tag{395}$$

$$\langle \pi_+(\eta, \mathbf{k})\pi_\times(\eta', \mathbf{q})\rangle = 0. \tag{396}$$

Thus, the contribution to the power spectrum is

$$P_T(\eta, k) = (64\pi G)^2\pi^3 \int_{-\infty}^{\eta} d\eta_1 \frac{1}{f_T(\eta_1)}\left[\frac{a(\eta_1)}{a(\eta)}\right]^{1+\gamma\epsilon} a^2(\eta_1)\rho_X(\eta_1)G^R_{\mathrm{TG}}(\eta, \eta_1, k) \int_{-\infty}^{\eta} d\eta_2 \frac{1}{f_T(\eta_2)}\left[\frac{a(\eta_2)}{a(\eta)}\right]^{1+\gamma\epsilon} a^2(\eta_2)\rho_X(\eta_2)G^R_{\mathrm{TG}}(\eta, \eta_2, k)P_\pi(\eta_1, \eta_2, k). \tag{397}$$

There exist different possibilities of the unequal time correlator of the the anisotropic stress $P_\pi(\eta_1, \eta_2, k)$, for instance, in the case of turbulence, it is possible to have a totally coherent source such that

$$P_\pi(\eta_1, \eta_2, k) = \sqrt{P_\pi(\eta_1, \eta_1, k)P_\pi(\eta_2, \eta_2, k)}, \tag{398}$$

it is also possible to have a source with finite coherence time with

$$P_\pi(\eta_1, \eta_2, k) = \sqrt{P_\pi(\eta_1, \eta_1, k)P_\pi(\eta_2, \eta_2, k)}\theta(x_c - |\eta_| - \eta_2|k), \quad x_c \sim 1, \tag{399}$$

or even a totally incoherent source

$$P_\pi(\eta_1, \eta_2, k) = P_\pi(\eta_2, \eta_2, k)\Delta\eta_*\delta(\eta_1 - \eta_2), \tag{400}$$

with $\Delta\eta_*$ the time duration of the phase transition at $T = T_*$. It is also possible to determine the unequal correlator in the case of bubble collision in terms of the bubble nucleation rate and the number of bubbles [72]. The details of these cases lie beyond the scope of this work. Among all the possibilities, a phase transition due to electroweak symmetry breaking is a highly possible source of a first order phase transition, hence, the search for such symmetry breaking and its contributions to the physics of the early universe in the context of TG has to be studied, since in GR it appears as a peak on the spectrum of GW, in TG we should analyse in detail the peak due to this symmetry breaking in order to distinguish both models phenomenologically. In the context of a totally incoherent source, we have

$$P_T = (64\pi G)^2\pi^3\Delta\eta_* \int_{-\infty}^{\eta}\left(d\eta_i \frac{1}{f_T(\eta_i)}\left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} a^2(\eta_i)\rho_X(\eta_i)G^R_{\mathrm{TG}}(\eta, \eta_i, k)P_\pi(\eta_i, \eta_i, k)\right)^2, \tag{401}$$

this will let us to easily discuss in the conclusion section the principal differences with the vacuum and thermal fluctuation contributions.

5.3.4. From magnetic fields

For this case, we will follow the discussion found in ref. [73]. Similar to turbulence and bubble nucleation, primordial magnetic fields can be generated from phase transitions and even from second order phase transitions. The idea is to define a correlation of the magnetic field over a certain scale. The most general way to achieve that is by introducing the following correlator of the divergenceless magnetic field

$$\langle B_i(\mathbf{k})B_j(\mathbf{q})\rangle = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{q})P_B(k), \tag{402}$$

with

$$P_B(k) = \begin{cases} C_B k^2, & 0 < k < L^{-1} \\ C_B L^{\alpha-2}k^{\alpha}, & L^{-1} < k < \lambda^{-1} \\ 0, & \text{otherwise,} \end{cases} \tag{403}$$

with $\alpha < -3$, L the correlation scale, λ the dissipation scale and C_B an undetermined constant. The idea is that once the magnetic field is generated, it is dissipated slowly unlike turbulence. The correlation scale is time dependent as

$$L(\eta) = L_* \left(\frac{\eta}{\eta_*}\right)^{\gamma}, \tag{404}$$

with $0 < \gamma < 1$, L_* is the correlation length at η_* when the magnetic field is created. The dissipation scale also depends on time, however the exact form is irrelevant for the calculation. Now, similar to the Loitsyansky's integral that measures the angular momentum of the fluid and is a constant in time, the following conserved quantity is imposed

$$\left\langle B^2\right\rangle L^5 = \text{constant}, \tag{405}$$

which is equivalent to C_B being constant [74]. Then, the comoving magnetic energy is given by

$$\left\langle B^2\right\rangle(\eta) = \left\langle B^2\right\rangle(\eta_*)\left(\frac{\eta_*}{\eta}\right)^{5\gamma}. \tag{406}$$

Finally, the correlator of the comoving anisotropic stress for magnetic fields is the one of a coherent source [75]

$$\langle \pi_{ij}(\eta_1,\mathbf{k})\pi_{mn}(\eta_2,\mathbf{q})\rangle = \sqrt{\pi_B(\eta_1,k)}\sqrt{\pi_B(\eta_2,k)}\delta_D(\mathbf{k}-\mathbf{q})\Pi_{ij,mn}, \tag{407}$$

with

$$\pi_B \approx C_B^2 \begin{cases} A_1^2 L^{-7}, & 0 < k < L^{-1} \\ A_2^2 L^{\alpha-7}k^{\alpha}, & L^{-1} < k < \lambda^{-1} \\ 0, & \text{otherwise,} \end{cases} \tag{408}$$

and $A_1^2 = \frac{2\alpha-4}{7(2\alpha+3)}$, $A_2^2 = \frac{\alpha-2}{5(2\alpha+3)}$. Therefore, we obtain

$$\sum_{\lambda} \langle 0|\, \varrho_T^{\lambda}(\eta_1,\boldsymbol{k})\varrho_T^{\lambda}(\eta_2,\boldsymbol{q})\,|0\rangle = 8a^{-4}(16\pi G)^2\sqrt{\pi_B(\eta_1,k)}\sqrt{\pi_B(\eta_2,k)}\delta_D(\mathbf{k}-\mathbf{q}), \tag{409}$$

and then the contribution to the power spectrum is

$$P_T(\eta,k) =(64\pi G)(32\pi G)\left|\int_{-\infty}^{\eta} d\eta_i \frac{a^{-2}(\eta_i)}{f_T(\eta_i)}\left[\frac{a(\eta_i)}{a(\eta)}\right]^{1+\gamma\epsilon} G^R_{\mathrm{TG}}(\eta,\eta_i,k)\sqrt{\pi_B(\eta_i,k)}\right|^2. \tag{410}$$

Therefore, we can explore a pletora of possible sources of anisotropic stress in the study of primordial GW in TG with the provided solution, as we have seen through the last examples.

6. Conclusion

The results found in this work have several implications. Let us begin by considering the vacuum fluctuations in a perfect de Sitter background. In this case we have that the propagation equations in GR and TG are exactly the same and then their associated GW fields are the same. This has the implication of having the same scale-invariant power spectrum (235) which means that if the primordial GW background is directly measured with an almost zero scale dependence, both GR and TG survive as a possible theory to describe such phenomenology. If the background is a quasi de Sitter expansion driven by the first slow roll parameter, the propagation equations for both theories are different which has a direct observational implication on the tensor spectral indices. In the case of GR the tensor spectral index is $n_T = -2\epsilon$ whereas in TG is $n_T = -2\epsilon(1+\gamma)$ with γ given by (275). The slow roll parameter is considered to be small, this implies that in GR the tensor spectral index should be small, however, the γ parameter appearing in the TG case allows for a high value of the tensor index. This implies that, although a value near to zero of the tensor spectral index will not discard TG as a possible gravitational theory, a high value of the tensor spectral index will strongly suggest the need of an extensions of gravity, with TG being a potential explanation of such value as we can see from the results of this thesis. Notice that in order to perform these observations, we need a direct measurement of the tensor spectral index and not an inferred value from the tensor-to scalar-ratio as done nowadays.

When tensor anisotropic stress is included, we have found the most general solution for GW in TG, which serves a cornerstone for the TG community since this will allow for future studies of primordial GW with any desired tensor anisotropic stress. From this solution, the associated power spectrum from such anisotropies was obtained in (384), however, it is not possible to easily obtain physics from that result but we can take the $\epsilon \to 0$ to see the physics behind the solution, since, even if in the vacuum case GR and TG are indistinguishable, when tensor anisotropies are present it is not the case. In this limit, the Green's function of TG found in (376) recovers that of GR [67]

$$\frac{a(\eta_i)}{a(\eta)} G^R(\eta,\eta_i,k) = \frac{\theta(\eta-\eta_i)}{\eta_i^2 k^3} \mathrm{Im}\left[e^{ik(\eta-\eta_i)}(1-ik\eta)(1+ik\eta_i)\right] \equiv G_R(\eta,\eta_i,k)\,, \tag{411}$$

and then the power spectrum becomes

$$\delta_D(\boldsymbol{k}-\boldsymbol{q})P_T(\eta,k) = \frac{1}{f_T^2}\int_{-\infty}^{\eta} d\eta_i G_R^2(\eta,\eta_i,k)\sum_{\lambda}\langle 0|\,\varrho_T^{\lambda}(\eta_i,\boldsymbol{k})\varrho_T^{\lambda}(\eta_i,\boldsymbol{q})\,|0\rangle \equiv \frac{1}{f_T^2}\delta_D(\boldsymbol{k}-\boldsymbol{q})P_T^{GR}(\eta,k)\,, \tag{412}$$

where we can identify the contribution in GR as

$$\delta_D(\boldsymbol{k}-\boldsymbol{q})P_T^{GR}(\eta,k) = \int_{-\infty}^{\eta} d\eta_i G_R^2(\eta,\eta_i,k)\sum_{\lambda}\langle 0|\,\varrho_T^{\lambda}(\eta_i,\boldsymbol{k})\varrho_T^{\lambda}(\eta_i,\boldsymbol{q})\,|0\rangle\,. \tag{413}$$

This is an import result on the cosmological phenomenology of TG. This result tells us that the peaks of the power spectrum coming from tensor anisotropies in TG are suppressed, with a suppression factor given by $1/f_T^2$ where the cosmological extended models in TG satisfy $f_T < -1$, compared with the same peaks in GR. We discussed some of these peaks in the thesis and their features are well known in GR, hence, if a measurement of the power spectrum of GW and the peaks due to tensor anisotropies are obtained and they turn out to be suppressed with respect to the amplitude expected in GR the need of extensions of gravity will be manifest and TG extended models provide an explanation for this suppression and would pose as the possible successful gravitational theory. If the peaks are amplified, TG is also an explanation for that but we need to look for new cosmological viable models in TG that satisfying $-1 < f_T < 0$ compared to those studied in the literature and discussed in subsection 4.3. Finally, working with the same $\epsilon \to 0$ limit, the energy density of GW that was found in (309), and as we have discussed, the cosmological viable extended models in TG, at least the ones discussed in subsection 4.3, are in the form $f_T = -1 + F_T$ with $F_T < 0$, from where it is possible to notice that

$$\Omega(\eta, k) = \frac{1}{12}\frac{k^2\Delta_h^2(\eta, k)}{a^2(\eta)H^2(\eta)} - F_T\frac{1}{12}\frac{k^2\Delta_h^2(\eta, k)}{a^2(\eta)H^2(\eta)} \equiv \Omega_{\rm GR} + \Omega_{F_T}, \tag{414}$$

from where we have defined the energy density of GR and from the extensions as

$$\Omega_{\rm GR} \equiv \frac{1}{12}\frac{k^2\Delta_h^2(\eta, k)}{a^2(\eta)H^2(\eta)}, \quad \Omega_{F_T} = -F_T\frac{1}{12}\frac{k^2\Delta_h^2(\eta, k)}{a^2(\eta)H^2(\eta)}, \tag{415}$$

and since $F_T < 0$, we have that

$$\Omega(\eta, k) = \Omega_{\rm GR} + \Omega_{F_T} > \Omega_{GR}. \tag{416}$$

The latter inequality tells us the the amount of energy density of GW that we can expect from TG extended models is greater than that of GR providing another important prediction for extended models of TG.

Therefore, we found that Teleparallel Gravity in its TEGR formulation has the same strength and shortcoming as GR, hence, has the same accuracy in explaining the current data as GR. For extended models, it has been shown that those models are capable of solving some of the issues of GR about the late universe. However, few studies have been done in the context of the early universe, that is why the focus of this thesis is to explore the primordial universe. In the latter context, we have found so far important observational prediction of the primordial background of GW in TG and we have discussed how a direct measurement of such background can help us to see if TG is a better gravitational theory than GR in explaining the phenomena of the early universe, particularly, by looking into the power spectrum and the energy density. Finally, these results open the TG community to explore the early universe using GW in the light of future GW detectors, like LISA or the Einstein Telescope and providing the general solution of such waves.

A. Spin connection and inertial effects

When discussing the geometrical setup of TG, we introduced a geometrical object called the spin connection, which is a differential 1-form assuming values on the Lie Algebra of the Lorentz Group. The idea of introducing such connection is to construct a theory which is covariant under the translation group and locally invariant under Lorentz. Let us begin by considering an infinitesimal transformation of coordinates

$$x^A \longrightarrow x^A = x'^A + \epsilon^A(x^\mu), \tag{417}$$

from where a scalar field ϕ transforms as

$$\phi(x^A) \longrightarrow \phi(x'^A) = \phi(x^A - \epsilon^A), \tag{418}$$

and the infinitesimal difference is

$$\delta\phi \equiv \phi(x^A) - \phi(x'^A) = \epsilon^A \partial_A \phi. \tag{419}$$

However, we should expect a similar transformation for any Lorentz scalar field, particularly for the partial derivative $\partial_\mu \phi$, but this is not the case, since the infinitesimal difference is given by

$$\delta(\partial_\mu \phi) = \epsilon^A \partial_A \partial_\mu \phi + (\partial_A \phi)(\partial_\mu \epsilon^A). \tag{420}$$

This justifies the introduction of a translational potential

$$B_\mu = B^A_\mu P_A, \tag{421}$$

which is a 1-form assuming values on the Lie algebra of the translation group, such that its infinitesimal difference is

$$\delta B^A_\mu = -\partial_\mu \epsilon^A, \tag{422}$$

and the derivative is changed by

$$\partial_\mu \to e_\mu = \partial_\mu + B^A_\mu \partial_A. \tag{423}$$

This new derivative transform covariantly w.r.t to translational change of coordinates

$$\delta(e_\mu \phi) = \epsilon^A \partial_A e_\mu \phi. \tag{424}$$

The presence of this covariant derivative defines a non-trivial tetrad field given by

$$e_\mu = e^A{}_\mu \partial_A \quad \to \quad e^A{}_\mu = \partial_\mu x^A + B^A{}_\mu, \tag{425}$$

where $B^A{}_\mu \neq \partial_\mu \epsilon^A$ accounts for the non-triviality of the tetrad.

Now, we require the theory to be invariant under local Lorentz transformations, hence, we perform a local Lorentz transformation on the Minkowski coordinates and consequently also on the tetrad field

$$x^A \longrightarrow \Lambda^A{}_B x^B, \quad e^A_\mu = \Lambda^A{}_B e^B_\mu, \tag{426}$$

however, from eq. (425) we see that this transformation implies

$$\Lambda^A_{\ B} e^B_{\ \mu} = \partial_\mu (\Lambda^A_{\ B} x^B) + \Lambda^A_{\ B} B^B_{\ \mu}, \tag{427}$$

from where isolating the tetrad field we obtain

$$e^A_{\ \mu} = \partial_\mu x^A + B^A_{\ \mu} + \omega^A_{\ B\mu} x^B, \tag{428}$$

from where

$$\omega^A_{\ B\mu} = \Lambda^A_{\ D} \partial_\mu \Lambda^D_{\ B}, \tag{429}$$

is the purely inertial spin connection introduced previously. This is a very particular spin connection, to see exactly what type of connection it is consider the general transformation of the spin connection w.r.t to a Lorentz transformation [15]

$$\omega^A_{\ B\mu}{}'(\boldsymbol{x}) = \Lambda^A_{\ C}(\boldsymbol{x}) \omega^C_{\ D\mu}(\boldsymbol{x}) \Lambda_B^{\ D}(\boldsymbol{x}) + \Lambda^A_{\ D}(\boldsymbol{x}) \partial_\mu \Lambda^D_{\ B}(\boldsymbol{x}), \tag{430}$$

hence, the purely inertial spin connection corresponds to a zero spin connection in a general inertial frame, thus the spin connection represents all the inertial effects on the arbitrary Lorentz frame. The Weitzenböck gauge corresponds to working in the particular Lorentz frame $\Lambda = 0$. The presence of the spin connection causes TG to be completely equivalent to GR even in the presence of spinor field, in the sense that any particle, spinless or not, follows the same trajectories as in GR [76][77]. In the absence of the spin connection, GR and TG are only equivalent for spinless particles. Hence, the spin connection and the formulation of TG allows us to successfully include spinors to a gravitational theory.

The presence of the spin connection not only affects the trajectories of test particles but also the field equations themselves. When computing the field equations, we fixed the spin connection to zero and made variations of the action w.r.t the tetrad field, but, if we consider a non-zero purely inertial spin connection we need to also perform variations of the action w.r.t the spin connection. Performing variations w.r.t to the purely inertial spin connection, parametrised by the Lorentz transformation Λ does not impose any new field equations [17], since such variation is a surface term on the action. Hence, whether or not a purely spin connection is explicitly included in TEGR is a matter of preference in the context of field equations.

If we go beyond TEGR and work with extended models, $f(T)$ or $f(T, B)$ gravity in our case, the presence of the spin connection a total different story. When performing variation of the action w.r.t the spin connection it will not act as a surface term inside the action but reduces to the antisymmetric part of the field equations [78][79]. However, the antisymmetric part of the field equations do not provide new dynamics but rather a constriction of the spin connection, hence, if we manage to find a spin connection that makes the antisymmetric part of field equations to vanish we are only left with equations coming from variations w.r.t to the tetrad. When the this issue first arose our understanding of the problem was to try to find a pair tetrad-spin connection that make the antisymmetric part of the field equation zero, leaving an entire discussion of *good and bad tetrads* when the spin connection was chosen in Weitzenböck gauge [80]. Nowadays the discussion is quite different, the choose of a vanishing or non-vanishing spin connection is up to preference since both symmetric and antisymmetric parts of field equations need to be solved simultaneously even if the background symmetry is displayed for both the tetrad and spin connection field. For instance, in the case of spherical symmetry, even if we choose a tetrad and a non-vanishing spin connection that exhibit spherical symmetry, we will always have two non-vanishing antisymmetric part of the field equations [11]. The case of cosmological symmetry implies a general result, which states that any rank-2 tensor with cosmological symmetry, generated by the vector fields generator of translations or rotations, vanishes, particularly the antisymmetric part of the field equations. Then, independently of the theory, the antisymmetric part of the field equations of a theory with cosmological symmetry will vanish, hence, we can always work with the Weitzenböck gauge and only focus to solve the symmetric part of the field equations when working in a background with cosmological symmetry.

If we break the cosmological symmetry, as in the case of linear perturbation theory, we only need to perturb the tetrad field and keep the spin connection as in the background scenario since we still want to include inertial effects through the spin connection. Nevertheless, we will require to solve both the symmetric and antisymmetric part of the field equation, since they will not vanish due to the breaking of the cosmological symmetry. In the case of a transverse and traceless tensor perturbation, shown in eq. (200), the antisymmetric part of the perturbed field equation vanishes due to the symmetry of h_{ij}, however, the same result does not hold for scalar perturbations, as we show in the next section.

B. Gauge problem on Teleparallel Gravity

In this appendix we will discuss the gauge problem in the context of TG. We will follow a similar discussion as in the GR case. Let us consider an infinitesimal change of the background coordinates

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{\xi}, \tag{431}$$

such that the tetrad fields will transform as

$$e^A_{\ \mu}(\boldsymbol{x}) = \frac{\partial \tilde{x}^\sigma}{\partial x^\mu} \tilde{e}^A_{\ \sigma}(\tilde{\boldsymbol{x}}) = \left(\delta^\sigma_\mu + \partial_\mu \xi^\sigma\right)\left(\tilde{e}^A_{\ \sigma}(\boldsymbol{x}) + \xi^\alpha \partial_\alpha \tilde{e}^A_{\ \sigma}(\boldsymbol{x})\right), \tag{432}$$

from where

$$e^A_{\ \mu}(\boldsymbol{x}) = \tilde{e}^A_{\ \mu}(\boldsymbol{x}) + \xi^\alpha \partial_\alpha \bar{e}^A_{\ \mu}(\boldsymbol{x}) + \left(\partial_\mu \xi^\sigma\right) \bar{e}^A_{\ \sigma}(\boldsymbol{x}), \tag{433}$$

where we have already considered only first order contributions. Now, we will consider the most general tetrad already in the SVT decomposition [30][81]

$$e^A_{\ \mu} = \bar{e}^A_{\ \mu} + \delta e^A_{\ \mu}, \tag{434}$$

with the background tetrad $\bar{e}^A_{\ \mu}$ given by eq. (74) and the perturbed tetrad is

$$\delta e^A_{\ \mu} = a(\eta) \begin{pmatrix} \psi & \partial_i w + w_i \\ \partial_i \omega + \omega_i & -\phi \delta_{ij} + \partial_{\langle i}\partial_{j\rangle} h + \epsilon_{ijk}(\partial^k \sigma + \sigma^k) + \partial_i E_j + \frac{1}{2} h_{ij}, \end{pmatrix} \tag{435}$$

with

$$\partial_{\langle i}\partial_{j\rangle} h = \left(\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) h, \tag{436}$$

and from where the metric perturbation is

$$\delta g_{\mu\nu} = 2\tau_{(\mu\nu)} = a^2(\eta) \begin{pmatrix} 2\psi & \partial_i(w-\omega) + w_i - \omega_i \\ \partial_i(w-\omega) + w_i - \omega_i & -2\left[-\phi\delta_{ij} + \partial_{\langle i}\partial_{j\rangle} h + \partial_{(i}E_{j)} + \frac{1}{2}h_{ij}\right] \end{pmatrix} \tag{437}$$

which is the same perturbation as in eq. (101) in the SVT decomposition [31]. From this, considering the SVT transformation of the gauge field (122) the gauge transformation of the perturbed quantities is

$$\psi \to \tilde{\psi} = \psi - \mathcal{H}\alpha - \alpha', \quad w \to \tilde{w} = w - \alpha, \quad \tilde{w}_i = w_i, \quad \omega \to \tilde{\omega} = \omega - \beta', \tag{438}$$

$$\omega_i \to \tilde{\omega}_i = \omega_i - \epsilon_i{}', \quad \phi \to \tilde{\phi} = \phi + \mathcal{H}\alpha + \frac{1}{3}\nabla^2\beta, \quad h \to \tilde{h} = h - \beta, \tag{439}$$

$$E_j \to \tilde{E}_j = E_j - \epsilon_j, \quad \tilde{h}_{ij} = h_{ij}, \quad \tilde{\sigma} = \sigma, \quad \tilde{\sigma}^k = \sigma^k. \tag{440}$$

Observe that these transformations are the same as in the GR case. Fro, these transformations, we obtain the same Bardeen potentials as in the GR case

$$\Psi = \psi + \frac{1}{a}\left[(\omega - w - h')a\right]', \quad \Phi = \phi - \mathcal{H}(\omega - w - h') + \frac{1}{3}\nabla^2 h, \tag{441}$$

the same vector potential

$$W_i = \omega_i - w_i - E_i', \tag{442}$$

and the tensor perturbation are also gauge invariant. Once the gauge invariant potentials have been introduced in TG, we are able to choose a particular gauge to work the scalar and vector perturbations.

In the case of scalar perturbations, we can work with the Newtonian gauge such that $w = \omega$ and $h = 0$, and then, we obtain the components of the perturbed Einstein tensor in $f(T)$ gravity for scalar perturbations given by

$$\frac{\delta G_0^0}{2} = -a^{-2} f_T \nabla^2 \psi - 12 a^{-4} \mathcal{H}^2 f_{TT} \nabla^2 \zeta + 3a^{-2}\mathcal{H}(f_T - 12a^{-2}\mathcal{H}^2 f_{TT})\left(\psi' + \mathcal{H}\phi\right), \tag{443}$$

$$\frac{\delta G_i^0}{2} = -a^{-1}(f_T - 12a^{-2} f_{TT}\mathcal{H}^2)(\partial_i \psi' + \mathcal{H}\partial_i \phi) + 4F_{TT}\mathcal{H}a^{-3}\partial_i \nabla^2 \zeta \tag{444}$$

$$\frac{\delta G_0^i}{2} = a^{-3} f_T \left[\partial_i(\psi' + \mathcal{H}\phi)\right] - 12a^{-4} f_{TT}\mathcal{H}\left[a^{-1}\mathcal{H}\right]' \partial_i \psi, \tag{445}$$

$$\frac{\delta G_j^{i\;(i=j)}}{2} = a^{-2} f_T \left[\phi(2a(a^{-1}\mathcal{H})' + 3\mathcal{H}^2) + \frac{1}{2}(\nabla^2 - \partial_j^2)(\phi - \psi) + \mathcal{H}\phi' + 3\mathcal{H}\psi' + a(a^{-1}\psi')'\right] \tag{446}$$

$$\begin{aligned} &+ a^{-4} f_{TT}\left[-\phi(36\mathcal{H}^4 + 60a(a^{-1}\mathcal{H})'\mathcal{H}^2) - 12\mathcal{H}^3\phi' - 36\mathcal{H}(a(a^{-1}\mathcal{H})' + \mathcal{H}^2)\psi' - 12a\mathcal{H}^2(a^{-1}\psi')' \right. \\ &\left. -10a(a^{-1}\mathcal{H})'\nabla^2\zeta - 4\mathcal{H}^2\nabla^2\zeta + 6a(a^{-1}\mathcal{H})'\partial_j^2\zeta - 4\mathcal{H}\nabla^2\zeta'\right] \\ &+ 12a^{-5} f_{TTT}\mathcal{H}^2(a^{-1}\mathcal{H})'\left[12\mathcal{H}(\psi' + \mathcal{H}\phi) - 4\nabla^2\zeta\right], \end{aligned} \tag{447}$$

$$\begin{aligned} \frac{1}{3}\mathrm{Tr}\left(\frac{\delta G_j^i}{2}\right) &= a^{-2} f_T \left[\phi(2a(a^{-1}\mathcal{H})' + 3\mathcal{H}^2) + \frac{1}{3}\nabla^2(\phi - \psi) + \mathcal{H}\phi' + 3\mathcal{H}\psi' + a(a^{-1}\psi)'\right] \\ &+ f_{TT}a^{-4}\left[-\phi(36\mathcal{H}^4 + 60a(a^{-1}\mathcal{H})'\mathcal{H}^2) - 12\mathcal{H}^3\phi' - 36\mathcal{H}(a(a^{-1}\mathcal{H})' + \mathcal{H}^2)\psi' - 12a\mathcal{H}^2(a^{-1}\psi')' \right. \\ &\left. -(8a(a^{-1}\mathcal{H})' + 4\mathcal{H}^2)\nabla^2\zeta - 4\mathcal{H}\nabla^2\zeta'\right] + 12a^{-5} f_{TTT}(a^{-1}\mathcal{H})'\mathcal{H}^2\left[12\mathcal{H}(\psi' + \mathcal{H}\phi) - 4\nabla^2\zeta\right], \end{aligned} \tag{448}$$

$$\frac{\delta G_j^{i\;(i\neq j)}}{2} = a^{-2}\left[12 f_{TT}\dot{H}\partial_i\partial_j\zeta + f_T\partial_i\partial_j(\psi - \phi)\right], \tag{449}$$

where we have defined the parameter $\zeta = \mathcal{H}\omega$. To obtain the perturbed field equations, we just need to match these perturbations with perturbation of the energy-momentum tensor, which result in the same equations reported in ref. [82].

In the case of $f(T, B)$ gravity, we have the following perturbed Einstein tensor for scalar perturbations

$$\delta G^0{}_0 = -\frac{1}{2} f_T \delta T + \left(\frac{\bar{B}}{2} - \frac{\nabla^2}{a^2}\right) \delta f_B + \frac{3\mathcal{H}\delta f_B{}'}{a^2} - \frac{6\mathcal{H}^2 \delta f_T}{a^2} + \frac{2\mathcal{H} f_T}{a^2}\nabla^2\omega + \frac{\psi'}{a^2}\left(12\mathcal{H} f_T - 3 f_B'\right) \tag{450}$$
$$- \frac{2}{a^2} f_T \nabla^2 \psi + \frac{6\mathcal{H}\phi}{a^2}\left(2\mathcal{H} f_T - f_B'\right),$$

$$\delta G^0{}_i = \frac{1}{a^2}\left[\mathcal{H}\partial_i(3\delta f_B + 2\delta f_T) - 2 f_T \partial_i(\mathcal{H}\phi + \psi') + \partial_i(f_B'\phi - \delta f_B')\right], \tag{451}$$

$$\delta G^i{}_0 = \frac{1}{a^2}\left[-\mathcal{H}\partial_i \delta f_B + 2 f_T \partial_i(\mathcal{H}\phi + \psi') + \partial_i(-f_B'\phi + 2(f_B' + f_T')\partial_i\psi + \delta f_B')\right], \tag{452}$$

$$\delta G^i{}_j{}^{(i\neq j)} = \frac{1}{a^2}\partial_i\partial_j\left[f_T(\psi - \phi) - \omega(f_B' + f_T') + \delta f_B\right], \tag{453}$$

$$\frac{1}{3}\mathrm{Tr}\left[\delta G^i{}_j{}^{(i=j)}\right] = -\frac{1}{2} f_T \delta T + \delta f_B\left(\frac{\bar{B}}{2} - \frac{2}{3a^2}\nabla^2\right) - 2\frac{\delta f_T}{a^2}\left(\mathcal{H}' + 2\mathcal{H}^2\right) + \frac{1}{a^2}\delta f_B{}'' - \frac{2}{a^2}\mathcal{H}\delta f_T{}' - \frac{1}{a^2}\mathcal{H} f_B' \tag{454}$$
$$+ \frac{1}{a^2}\left[f_t\left(4(2\mathcal{H}^2 + \mathcal{H}') + \frac{2}{3}\nabla^2\right) + 2\mathcal{H}(f_B' + 2 f_T') - 2 f_B''\right]\phi + \frac{2\nabla^2\omega}{3a^2}(3 f_T \mathcal{H} + f_T' + f_B')$$
$$- \frac{2 f_T}{3a^2}\nabla^2\psi + \frac{\phi'}{a^2}(2\mathcal{H} f_T - f_B') + \frac{2\psi'}{a^2}(5\mathcal{H} f_T + f_T') + \frac{2 f_T}{a^2}\psi'',$$

with

$$\delta f_T = f_{BT}\delta T + f_{TB}\delta B, \quad \delta f_B = f_{BT}\delta T + f_{BB}\delta B, \tag{455}$$

and the perturbation of the boundary term and torsion scalar are

$$\delta B = \frac{2}{a^2}\left[6\phi(2\mathcal{H}^2 + \mathcal{H}') + 2\mathcal{H}\nabla^2\omega + 3\mathcal{H}\phi' + 15\mathcal{H}\psi' + \nabla^2(\phi - 2\psi)\right], \quad \delta T = \frac{4\mathcal{H}}{a^2}\left(3\mathcal{H}\phi + \nabla^2\omega + 3\psi'\right), \tag{456}$$

respectively. These equations are completely equivalent to those found in ref. [30]. We can see that the antisymmetric part of the field equations odes not vanish as in the tensor case, hence, both the symmetric and antisymmetric parts need to be solved simultaneously. We will not discuss the case of vector perturbation, however, it is worth mentioning that vector perturbations require the same stability condition $f_T < 0$ as in the case of GW, and is a similar result in the case of $f(R)$ where the vector perturbation are not propagating [30].

References

[1] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified gravity and cosmology. *Physics Reports*, 513(1-3):1–189, March 2012. doi:10.1016/j.physrep.2012.01.001. URL https://doi.org/10.1016/j.physrep.2012.01.001.

[2] Robert M. Wald. *General Relativity*. University of Chicago Pr., 1984. ISBN 0226870332.

[3] Eduardo Nahmad-Achar. *Differential Topology and Geometry with Applications to Physics*. Institute of Physics Publishing, 2018.

[4] S. Ramos-Sánchez. *Relatividad para futuros físicos*. Textos contemporáneos. Copit-Arxives, 2018. ISBN 9781938128189.

[5] Geovanny A. Rave-Franco. Análisis de Sistemas Dinámicos en teorías teleparalelas a la gravedad. Bachelor Thesis, 2021. URL http://132.248.9.195/ptd2021/mayo/0811921/Index.html.

[6] David Lovelock. The einstein tensor and its generalizations. *Journal of Mathematical Physics*, 12(3):498–501, March 1971. doi:10.1063/1.1665613. URL https://doi.org/10.1063/1.1665613.

[7] David Lovelock. The four-dimensionality of space and the einstein tensor. *Journal of Mathematical Physics*, 13(6):874–876, June 1972. doi:10.1063/1.1666069. URL https://doi.org/10.1063/1.1666069.

[8] Sebastian Bahamonde, Konstantinos F. Dialektopoulos, Celia Escamilla-Rivera, Gabriel Farrugia, Viktor Gakis, Martin Hendry, Manuel Hohmann, Jackson Levi Said, Jurgen Mifsud, and Eleonora Di Valentino. Teleparallel gravity: From theory to cosmology, 2021. URL https://arxiv.org/abs/2106.13793.

[9] Alexei Kovalev. Vector bundles. Lecture Notes, 2004. URL https://www.dpmms.cam.ac.uk/~agk22/vb.pdf. Part III: Differential geometry (Michaelmas 2004).

[10] Aleksey Zinger. Notes on Vector Bundles, 2018. URL http://www.math.stonybrook.edu/~azinger/mat566-spr18/vectorbundles.pdf.

[11] Manuel Hohmann, Laur Järv, Martin Krššák, and Christian Pfeifer. Modified teleparallel theories of gravity in symmetric spacetimes. *Physical Review D*, 100(8), October 2019. doi:10.1103/physrevd.100.084002. URL https://doi.org/10.1103/physrevd.100.084002.

[12] Miguel Socolovsky. Fiber bundles, connections, general relativity, and the einstein-cartan theory – part i. *Advances in Applied Clifford Algebras*, 22(3):837–872, July 2012. doi:10.1007/s00006-012-0365-1. URL https://doi.org/10.1007/s00006-012-0365-1.

[13] Miguel Socolovsky. Fiber bundles, connections, general relativity, and the einstein-cartan theory – part II. *Advances in Applied Clifford Algebras*, 22(3):873–909, July 2012. doi:10.1007/s00006-012-0366-0. URL https://doi.org/10.1007/s00006-012-0366-0.

[14] Matthias Blau. Lecture Notes on General Relativity. Technical report, Albert Einstein Center for Fundamental Physics, University of Bern, 2011. URL http://www.blau.itp.unibe.ch/newlecturesGR.pdf.

[15] Ruben Aldrovandi and José Geraldo Pereira. *Teleparallel Gravity*. Springer Netherlands, 2013. doi:10.1007/978-94-007-5143-9.

[16] Mikio Nakahara. *Geometry, Topology and Physics*. CRC Press, October 2018. doi:10.1201/9781315275826. URL https://doi.org/10.1201/9781315275826.

[17] Alexey Golovnev, Tomi Koivisto, and Marit Sandstad. On the covariance of teleparallel gravity theories. *Classical and Quantum Gravity*, 34(14):145013, June 2017. doi:10.1088/1361-6382/aa7830. URL https://doi.org/10.1088/1361-6382/aa7830.

[18] Sebastián Bahamonde. *Modified Teleparallel theories of gravity*. PhD thesis, University College London, 2018.

[19] Ofer Lahav. Testing homogeneity on large scales. *ASP Conf. Ser.*, 201:377, 2000.

[20] Ofer Lahav. Observational tests for the cosmological principle and world models. In *Structure Formation in the Universe*, pages 131–142. Springer Netherlands, 2001. doi:10.1007/978-94-010-0540-1_7.

[21] S. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Addison Wesley, 2004. ISBN 9780805387322.

[22] Sean M. Carroll. Lecture notes on general relativity, 1997. URL https://arxiv.org/abs/gr-qc/9712019.

[23] Oliver Piattella. *Lecture Notes in Cosmology*. Springer International Publishing, 2018. doi:10.1007/978-3-319-95570-4. URL https://doi.org/10.1007/978-3-319-95570-4.

[24] Planck Collaboration. Planck 2018 results. *Astronomy & Astrophysics*, 641:A6, September 2020. doi:10.1051/0004-6361/201833910. URL https://doi.org/10.1051/0004-6361/201833910.

[25] Joel R. Primack. Cosmological structure formation, 2015. URL https://arxiv.org/abs/1505.02821.

[26] George Ellis. The standard cosmological model: Achievements and issues. *Foundations of Physics*, 48(10):1226–1245, June 2018. doi:10.1007/s10701-018-0176-x. URL https://doi.org/10.1007/s10701-018-0176-x.

[27] J. Alberto Vazquez Gonzalez, Luis E. Padilla, and Tonatiuh Matos. Inflationary cosmology: from theory to observations. *Revista Mexicana de Física E*, 17(1 Jan-Jun):73–91, January 2020. doi:10.31349/revmexfise.17.73. URL https://doi.org/10.31349/revmexfise.17.73.

[28] B. Ryden. *Introduction to Cosmology*. Cambridge University Press, 2017. ISBN 9781107154834.

[29] S. Weinberg. *Cosmology*. Cosmology. OUP Oxford, 2008. ISBN 9780191523601. URL https://books.google.com.mt/books?id=nqQZdg020fsC.

[30] Sebastian Bahamonde, Viktor Gakis, Stella Kiorpelidi, Tomi Koivisto, Jackson Levi Said, and Emmanuel N. Saridakis. Cosmological perturbations in modified teleparallel gravity models: boundary term extension. *The European Physical Journal C*, 81(1), January 2021. doi:10.1140/epjc/s10052-021-08833-2. URL https://doi.org/10.1140/epjc/s10052-021-08833-2.

[31] Daniel Baumann. *Cosmology*. Cambridge University Press, June 2022. doi:10.1017/9781108937092. URL https://doi.org/10.1017/9781108937092.

[32] Roy Maartens. Causal thermodynamics in relativity, 1996. URL https://arxiv.org/abs/astro-ph/9609119.

[33] J. M. Stewart and M. Walker. Perturbations of space-times in general relativity. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 341(1624):49–74, October 1974. doi:10.1098/rspa.1974.0172. URL https://doi.org/10.1098/rspa.1974.0172.

[34] David J Griffiths. *Introduction to Electrodynamics*. Pearson, Upper Saddle River, NJ, 4 edition, September 2012.

[35] Timothy Clifton, Christopher S. Gallagher, Sophia Goldberg, and Karim A. Malik. Viable gauge choices in cosmologies with nonlinear structures. *Physical Review D*, 101 (6), March 2020. doi:10.1103/physrevd.101.063530. URL https://doi.org/10.1103/physrevd.101.063530.

[36] P. Klose, M. Laine, and S. Procacci. Gravitational wave background from vacuum and thermal fluctuations during axion-like inflation. *Journal of Cosmology and Astroparticle Physics*, 2022(12):020, December 2022. doi:10.1088/1475-7516/2022/12/020. URL https://doi.org/10.1088/1475-7516/2022/12/020.

[37] Manuel Hohmann. General cosmological perturbations in teleparallel gravity. *The European Physical Journal Plus*, 136(1), January 2021. doi:10.1140/epjp/s13360-020-00969-6. URL https://doi.org/10.1140/epjp/s13360-020-00969-6.

[38] Jan R Magnus and Heinz Neudecker. *Matrix differential calculus with applications in statistics and econometrics*. Wiley Series in Probability and Statistics: Texts and References Section. John Wiley & Sons, Chichester, England, 2 edition, February 1999.

[39] Jose María Ezquiaga and Miguel Zumalacárregui. Dark energy after GW170817: Dead ends and the road ahead. *Physical Review Letters*, 119(25), December 2017. doi:10.1103/physrevlett.119.251304. URL https://doi.org/10.1103/physrevlett.119.251304.

[40] The Virgo Collaboration The LIGO Scientific Collaboration. GW170817: Observation of gravitational waves from a binary neutron star inspiral. *Physical Review Letters*, 119 (16), October 2017. doi:10.1103/physrevlett.119.161101. URL https://doi.org/10.1103/physrevlett.119.161101.

[41] Antonio De Felice and Shinji Tsujikawa. f(r) theories. *Living Reviews in Relativity*, 13(1), June 2010. doi:10.12942/lrr-2010-3. URL https://doi.org/10.12942/lrr-2010-3.

[42] Latham A. Boyle and Paul J. Steinhardt. Probing the early universe with inflationary gravitational waves. *Physical Review D*, 77(6), March 2008. doi:10.1103/physrevd.77.063504. URL https://doi.org/10.1103/physrevd.77.063504.

[43] Sandipan Kundu. Inflation with general initial conditions for scalar perturbations. *Journal of Cosmology and Astroparticle Physics*, 2012(02):005–005, February 2012. doi:10.1088/1475-7516/2012/02/005. URL https://doi.org/10.1088/1475-7516/2012/02/005.

[44] Encyclopedia of Mathematics. Hankel functions. URL http://encyclopediaofmath.org/index.php?title=Hankel_functions&oldid=31410.

[45] Rafael C. Nunes, Supriya Pan, and Emmanuel N. Saridakis. New observational constraints on $f(T)$ gravity through gravitational-wave astronomy. *Physical Review D*, 98(10), November 2018. doi:10.1103/physrevd.98.104055. URL https://doi.org/10.1103/physrevd.98.104055.

[46] Celia Escamilla-Rivera and Jackson Levi Said. Cosmological viable models in $f(t, b)$ theory as solutions to the h_0 tension. *Classical and Quantum Gravity*, 37(16):165002, July 2020. doi:10.1088/1361-6382/ab939c. URL https://doi.org/10.1088/1361-6382/ab939c.

[47] F. B. M. dos Santos, J. E. Gonzalez, and R. Silva. Observational constraints on f(t) gravity from model-independent data. *The European Physical Journal C*, 82(9), September 2022. doi:10.1140/epjc/s10052-022-10784-1. URL https://doi.org/10.1140/epjc/s10052-022-10784-1.

[48] Chunlong Li, Yong Cai, Yi-Fu Cai, and Emmanuel N. Saridakis. The effective field theory approach of teleparallel gravity, $f(t)$ gravity and beyond. *Journal of Cosmology and Astroparticle Physics*, 2018(10):001–001, October 2018. doi:10.1088/1475-7516/2018/10/001. URL https://doi.org/10.1088/1475-7516/2018/10/001.

[49] S. Nesseris, S. Basilakos, E. N. Saridakis, and L. Perivolaropoulos. Viable $f(t)$ models are practically indistinguishable from lcdm. *Physical Review D*, 88(10), November 2013. doi:10.1103/physrevd.88.103010. URL https://doi.org/10.1103/physrevd.88.103010.

[50] Sebastian Bahamonde and Salvatore Capozziello. Noether symmetry approach in f(t, b) teleparallel cosmology. *The European Physical Journal C*, 77(2), February 2017. doi:10.1140/epjc/s10052-017-4677-0. URL https://doi.org/10.1140/epjc/s10052-017-4677-0.

[51] Sebastian Bahamonde, Ugur Camci, and Salvatore Capozziello. Noether symmetries and boundary terms in extended teleparallel gravity cosmology. *Classical and Quantum Gravity*, 36(6):065013, February 2019. doi:10.1088/1361-6382/ab0510. URL https://doi.org/10.1088/1361-6382/ab0510.

[52] K. Rezazadeh, A. Abdolmaleki, and K. Karami. Power-law and intermediate inflationary models in f(t)-gravity. *Journal of High Energy Physics*, 2016(1), January 2016. doi:10.1007/jhep01(2016)131. URL https://doi.org/10.1007/jhep01(2016)131.

[53] Geovanny A. Rave-Franco, Celia Escamilla-Rivera, and Jackson Levi Said. Dynamical complexity of the teleparallel gravity cosmology. *Physical Review D*, 103(8), April 2021. doi:10.1103/physrevd.103.084017. URL https://doi.org/10.1103/physrevd.103.084017.

[54] Geovanny A. Rave Franco, Celia Escamilla-Rivera, and Jackson Levi Said. Stability analysis for cosmological models in f(t, b) gravity. *The European Physical Journal C*, 80(7), July 2020. doi:10.1140/epjc/s10052-020-8253-7. URL https://doi.org/10.1140/epjc/s10052-020-8253-7.

[55] M. Tristram, A. J. Banday, K. M. Górski, R. Keskitalo, C. R. Lawrence, K. J. Andersen, R. B. Barreiro, J. Borrill, H. K. Eriksen, R. Fernandez-Cobos, T. S. Kisner, E. Martínez-González, B. Partridge, D. Scott, T. L. Svalheim, H. Thommesen, and I. K. Wehus. Planck constraints on the tensor-to-scalar ratio. *Astronomy & Astrophysics*, 647:A128, March 2021. doi:10.1051/0004-6361/202039585. URL https://doi.org/10.1051/0004-6361/202039585.

[56] H. Rizwana Kausar, Lionel Philippoz, and Philippe Jetzer. Gravitational wave polarization modes in $f(r)$ theories. *Physical Review D*, 93(12), June 2016. doi:10.1103/physrevd.93.124071. URL https://doi.org/10.1103/physrevd.93.124071.

[57] Douglas M. Eardley, David L. Lee, Alan P. Lightman, Robert V. Wagoner, and Clifford M. Will. Gravitational-wave observations as a tool for testing relativistic gravity. *Physical Review Letters*, 30(18):884–886, April 1973. doi:10.1103/physrevlett.30.884. URL https://doi.org/10.1103/physrevlett.30.884.

[58] Gabriel Farrugia, Jackson Levi Said, Viktor Gakis, and Emmanuel N. Saridakis. Gravitational waves in modified teleparallel theories. *Physical Review D*, 97(12), June 2018. doi:10.1103/physrevd.97.124064. URL https://doi.org/10.1103/physrevd.97.124064.

[59] Yun Soo Myung. Propagating degrees of freedom in $f(r)$ gravity. *Advances in High Energy Physics*, 2016:1–6, 2016. doi:10.1155/2016/3901734. URL https://doi.org/10.1155/2016/3901734.

[60] Sugumi Kanno and Jiro Soda. Polarized initial states of primordial gravitational waves. *Symmetry*, 12(4):672, April 2020. doi:10.3390/sym12040672. URL https://doi.org/10.3390/sym12040672.

[61] Rudnei O Ramos and L.A. da Silva. Power spectrum for inflation models with quantum and thermal noises. *Journal of Cosmology and Astroparticle Physics*, 2013(03):032–032, March 2013. doi:10.1088/1475-7516/2013/03/032. URL https://doi.org/10.1088/1475-7516/2013/03/032.

[62] David Skinner. Mathematical Methods. University of Cambridge Part IB Mathematical Tripos, 2015. URL https://www.damtp.cam.ac.uk/user/dbs26/1BMethods/All.pdf.

[63] J. I. Kapusta, B. Müller, and M. Stephanov. Relativistic theory of hydrodynamic fluctuations with applications to heavy-ion collisions. *Physical Review C*, 85(5), May 2012. doi:10.1103/physrevc.85.054906. URL https://doi.org/10.1103/physrevc.85.054906.

[64] G. Jackson and M. Laine. Hydrodynamic fluctuations from a weakly coupled scalar field. *The European Physical Journal C*, 78(4), April 2018. doi:10.1140/epjc/s10052-018-5791-3. URL https://doi.org/10.1140/epjc/s10052-018-5791-3.

[65] Steven Weinberg. *Gravitation and Cosmology*. John Wiley & Sons, 1972. ISBN 0471925675.

[66] L D Landau and E M Lifshits. *Statistical physics: Theory of the condensed state pt. 2*. Course of Theoretical Physics S. Pergamon, 3 edition, January 1980.

[67] P. Klose, M. Laine, and S. Procacci. Gravitational wave background from non-abelian reheating after axion-like inflation. *Journal of Cosmology and Astroparticle Physics*, 2022 (05):021, May 2022. doi:10.1088/1475-7516/2022/05/021. URL https://doi.org/10.1088/1475-7516/2022/05/021.

[68] C. J. Hogan. Gravitational radiation from cosmological phase transitions. *Monthly Notices of the Royal Astronomical Society*, 218(4):629–636, February 1986. doi:10.1093/mnras/218.4.629. URL https://doi.org/10.1093/mnras/218.4.629.

[69] Nigel Goldenfeld. *Lectures on Phase Transitions and the Renormalization Group*. Addison Wesley, London, England, December 1991.

[70] Ruth Durrer. Gravitational waves from cosmological phase transitions. *Journal of Physics: Conference Series*, 222:012021, April 2010. doi:10.1088/1742-6596/222/1/012021. URL https://doi.org/10.1088/1742-6596/222/1/012021.

[71] Christophe Grojean and Géraldine Servant. Gravitational waves from phase transitions at the electroweak scale and beyond. *Physical Review D*, 75(4), February 2007. doi:10.1103/physrevd.75.043507. URL https://doi.org/10.1103/physrevd.75.043507.

[72] Ryusuke Jinno and Masahiro Takimoto. Gravitational waves from bubble collisions: An analytic derivation. *Physical Review D*, 95(2), January 2017. doi:10.1103/physrevd.95.024009. URL https://doi.org/10.1103/physrevd.95.024009.

[73] Chiara Caprini and Ruth Durrer. Gravitational waves from stochastic relativistic sources: Primordial turbulence and magnetic fields. *Physical Review D*, 74(6), September 2006. doi:10.1103/physrevd.74.063521. URL https://doi.org/10.1103/physrevd.74.063521.

[74] Robi Banerjee and Karsten Jedamzik. Evolution of cosmic magnetic fields: From the very early universe, to recombination, to the present. *Physical Review D*, 70(12), December 2004. doi:10.1103/physrevd.70.123003. URL https://doi.org/10.1103/physrevd.70.123003.

[75] R. Durrer, P. G. Ferreira, and T. Kahniashvili. Tensor microwave anisotropies from a stochastic magnetic field. *Physical Review D*, 61(4), January 2000. doi:10.1103/physrevd.61.043001. URL https://doi.org/10.1103/physrevd.61.043001.

[76] V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira. Teleparallel spin connection. *Physical Review D*, 64(2), June 2001. doi:10.1103/physrevd.64.027502. URL https://doi.org/10.1103/physrevd.64.027502.

[77] V. C. de Andrade and J. G. Pereira. Gravitational lorentz force and the description of the gravitational interaction. *Physical Review D*, 56(8):4689–4695, October 1997. doi:10.1103/physrevd.56.4689. URL https://doi.org/10.1103/physrevd.56.4689.

[78] Manuel Hohmann, Laur Järv, Martin Krššák, and Christian Pfeifer. Teleparallel theories of gravity as analogue of nonlinear electrodynamics. *Physical Review D*, 97(10), May 2018. doi:10.1103/physrevd.97.104042. URL https://doi.org/10.1103/physrevd.97.104042.

[79] Alexey Golovnev and María-José Guzmán. Foundational issues in f(t) gravity theory. *International Journal of Geometric Methods in Modern Physics*, 18(supp01): 2140007, April 2021. doi:10.1142/s0219887821400077. URL https://doi.org/10.1142/s0219887821400077.

[80] Nicola Tamanini and Christian G. Böhmer. Good and bad tetrads in $f(T)$ gravity. *Physical Review D*, 86(4), August 2012. doi:10.1103/physrevd.86.044009. URL https://doi.org/10.1103/physrevd.86.044009.

[81] Alexey Golovnev and Tomi Koivisto. Cosmological perturbations in modified teleparallel gravity models. *Journal of Cosmology and Astroparticle Physics*, 2018(11):012–012, November 2018. doi:10.1088/1475-7516/2018/11/012. URL https://doi.org/10.1088/1475-7516/2018/11/012.

[82] Rui Zheng and Qing-Guo Huang. Growth factor in $f(T)$ gravity. *Journal of Cosmology and Astroparticle Physics*, 2011(03):002–002, March 2011. doi:10.1088/1475-7516/2011/03/002. URL https://doi.org/10.1088/1475-7516/2011/03/002.

[83] Geovanny A. Rave-Franco, Celia Escamilla-Rivera, and Jackson Levi Said. Production of Primordial Gravitational Waves in Teleparallel Gravity. 4 2023.

[84] Ji-Xiang Zhao, Xiao-Hui Liu, and Nan Li. Primordial black holes and scalar-induced gravitational waves from the perturbations on the inflaton potential in peak theory. *Physical Review D*, 107(4), February 2023. doi:10.1103/physrevd.107.043515. URL https://doi.org/10.1103/physrevd.107.043515.