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PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y
DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

OCCUPATIONAL LOCAL TIMES OF STABLE PROCESSES VIA FRACTIONAL CALCULUS AND APPLICATIONS TO DRIFTLESS SDEs.

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Chapter 1

Introduction

Stable processes are ubiquitous in the theory of stochastic processes as they belong to the wider classes of Lévy, Markov, and self-similar processes. Brownian motion is by far the most famous and studied example of a stable process; furthermore, it is the only one with continuous paths. Other well-known examples correspond to stable subordinators, the Cauchy process and symmetric stable processes.

The objective of this thesis is to study stable processes whose infinitesimal generator can be related to some adequate fractional derivatives, in order to establish new results via fractional calculus.

Symmetric stable processes have been the main focus of the existing literature. Perhaps, this could be due to the resemblance to the Brownian motion or the significant number of applications they have in physics. Moreover, many properties get messier when we consider asymmetric stable processes and the standard techniques seem to fail in many cases. Our approach, using fractional calculus, considers the infinitesimal generator as a linear combination of fractional derivatives, which allows us to generalize the fractional Laplacian (the infinitesimal generator of a symmetric stable process) in such a way that it accounts for possible asymmetry of the associated stable process.

Even in the study of fractional calculus there has been a bias toward completely asymmetric and symmetric combinations of fractional operators, so that the general asymmetric case has not been sufficiently developed. We focus on this gap to prove some interesting results which later will be useful in the study of stable processes.

Let us briefly explain the main contributions of this work in the following sections.

1.1 Fractional operators

Fractional operators have been studied through many points of view: operator theory, fractional calculus or their relation to the infinitesimal generator of stable processes. In most of these approaches, they only describe completely asymmetric (only left or right fractional operators) or symmetric fractional operators (fractional Laplacian and Riez potential). The main reason for this is that general linear combinations of right and left operators (cf. Definition 3.2.1) are more complicated to work when we consider their compositions. Another reason, which is stressed throughout this work, is that an adequate domain of definition for these operators is necessary in order to even define their compositions.

As we will see in Chapter 3, the natural domain of definition is the Schwartz space of rapidly decreasing functions. However, this space is not invariant for fractional operators; this means that once we apply a fractional operator to functions in this space, we can no longer assure the result stays in the Schwartz space.

The way we address this problem is by considering a subspace of Schwartz which remains invariant under fractional operators action, namely the Lizorkin space (cf. Definition 3.3.1). In this space we can consider the composition of two (or any finite number) fractional operators, since they stay in Lizorkin space. This will lead us to our first problem:

Problem 1. ***How to compute a crossed (right/left with left/right) composition of fractional operators?***

This problem arises when we try to compute the composition of linear combinations of fractional operators. First, we can argue that the composition of the same side (left/right) fractional operators of order α and β in an invariant space are well-defined and are the fractional operators of order $\alpha+\beta$. Nevertheless, the compositions of left/right with right/left, at least well defined in Lizorkin space, are not a common calculation in the standard literature. We compute this in full generality, that is, for the case of fractional derivatives, integrals, or mixed compositions. This result is stated in Proposition 4.2.2.

Given that we compute these kinds of compositions, it remains to compute the crossed composition of derivatives and integrals of the same order. The Definition 3.2.1 suggests that the composition of fractional derivatives and integrals of the same side gives us the identity operator. However, the crossed case is not well defined as it diverges when the sum of the orders is zero (cf. Proposition 4.2.2).

Problem 2. ***How to compute the crossed compositions of fractional derivatives and integrals of the same order?***

Through an adequate limit, we were able to compute the sum of compositions of crossed fractional derivatives and integrals of the same order and found an interesting result, which is stated in Lemma 4.2.1. This result gives us the key ingredient to compute the inverse operator of the infinitesimal generator of a stable process.

Problem 3. ***What is the inverse operator of the infinitesimal generator of a stable process?***

It is known that the infinitesimal generator of a stable process can be identified with a linear combination of Riemann-Liouville fractional derivatives, with the same order as the stability index of the underlying process. The solution for this question uses the solution of problem 2. As a consequence, we were able to prove that the inverse corresponds to a linear combination of Riemann-Liouville fractional integrals with the same index of stability (cf. Theorem 4.2.1). The constants that appear in this linear combination of integrals are explicitly shown in terms of those appearing in the infinitesimal generator.

These three problems constitute our contribution in terms of fractional calculus. They are interesting not only for the stable processes study's sake, but also in the realm of fractional calculus.

These results can be stated for all non-integer orders $\alpha \in \mathbb{R}$, not only the intervals $\alpha \in (0,2) \setminus \{1\}$ as the stable process requires. Moreover, with the adequate definition of multidimensional Riemann-Liouville fractional operators (cf. [MS12]), n-dimensional results could be tackled having in mind the methods used for the one-dimensional case in this work.

1.2 Stable processes via fractional calculus

The solutions to problems 1, 2 and 3 provide the right tools to state and solve the following problems related to stable processes.

Problem 4. ***Which kind of functions are appropriate for an occupational Meyer-Itô theorem for stable processes?***

In the case of Brownian motion, or even continuous semimartingales, the biggest class which remains invariant through the Itô-Meyer theorem is the class of (differences of) convex functions. This class can be obtained by noticing that the fundamental solution of the Laplacian is the absolute value function. In a similar fashion, we find the fundamental solution of the infinitesimal generator of a stable process by using the inverse operator of Problem 3.

This class is defined in Definition 4.3.1. Since stable processes have poorer integrability conditions than Brownian motion, functions in this class have some additional constraints in order to be well-defined.

Problem 5. ***How can we define an occupational Meyer-Itô theorem?***

Meyer-Itô theorem for stable processes was already stated, for example in [Pro04]. However, the local time that appears in this formulation is the semimartingale local time, which is defined through the quadratic variation, and since we are working with pure jump processes this local time is zero.

However, for the recurrent stable processes ($\alpha \in (1,2)$) there exists an occupational local time, defined by the classic occupation formula. So this Problem refers to finding a Meyer-Itô theorem where an occupational local time appears. In Theorem 4.3.1 we provide some conditions on functions belonging to the class of Problem 4 in order to have this occupational version of Meyer-Itô theorem.

Finally, we provide some applications of this theorem. For instance, the Tanaka formula comes as a particular case; and we are able to generalize a result from Engelbert and Kurenok [EK19] regarding a submartingale decomposition of power functions of symmetric stable process. We find that in the general, asymmetric, case this decomposition is not always a submartingale and we state the conditions where you have it.

Problem 6. ***Is it possible to extend these results to solutions of a driftless stochastic differential equation driven by stable processes?***

Given the result by Rosínski and Woyczy [RW86], and Kallenberg [Kal92], we can identify the solution to these kind of SDEs as a time-changed stable process, allowing us to extend the previous results from the stable case to stochastic integrals with respect to a stable process.

1.3 Summary of results

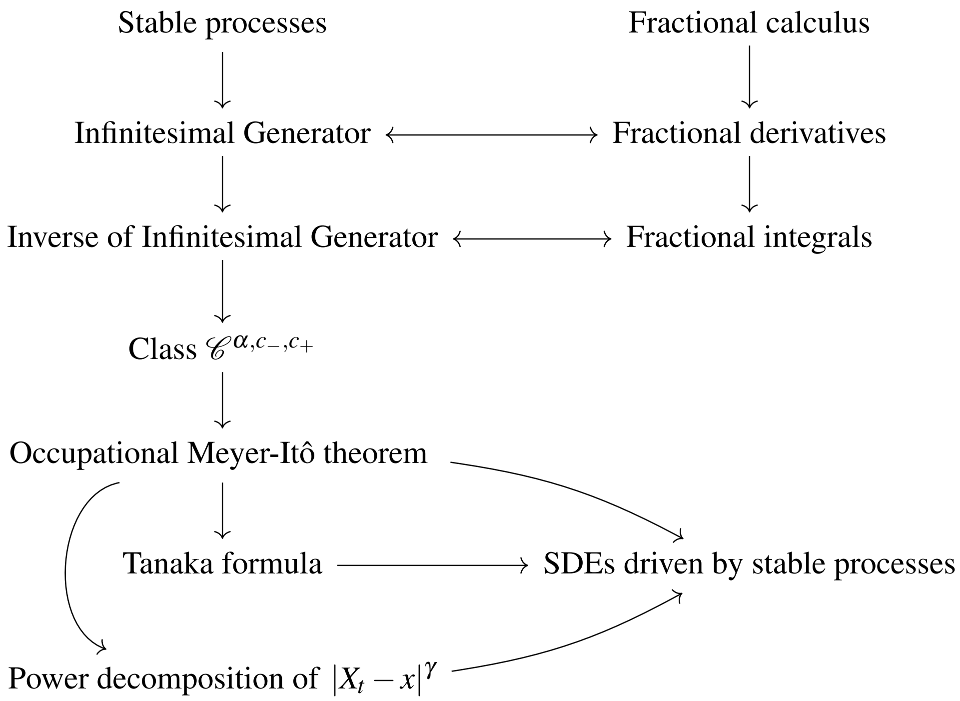
In Chapters 2 and 3 we give the preliminary definitions and some results concerning stable processes and fractional calculus respectively, which allow us to prove the main results of this work in Chapters 4 and 5. The results given in Chapters 2, 3 and 4 are part of the article "A Meyer-Itô formula for stable processes via fractional calculus" [SU23]. The results of Chapter 5 are part of a working paper.

The results obtained in this work can be summarized by:

- Crossed composition of fractional operators (cf. Proposition 4.2.2).
- Crossed composition of fractional derivatives and integrals of the same order (cf. Lemma 4.2.1).
- Inverse of the infinitesimal generator (cf. Theorem 4.2.1).
- Definition of Class $\mathscr{C}^{\alpha,c_-,c_+}$ (cf. Definition 4.3.1).
- Meyer-Itô theorem with occupational local time (cf. Theorem 4.3.1).
- Semimartingale or Doob-Meyer decomposition of the process $|X_t - x|^\gamma$ (cf. Theorem 4.4.1).
- Extension of the previous results from stable processes to solutions of stochastic differential equations driven by stable processes via change of time (cf. Section 5.3).

The main idea is to use the natural relationship between fractional calculus and infinitesimal generators of stable processes, prove new results on the fractional calculus domain and use them in the stable processes realm.

In the next diagram, we briefly show the path we have followed throughout this work.



Chapter 2

Stable processes

In this chapter, we will state definitions and results concerning stable processes that we use to prove the main results of this work. The main references are: Bertoin [Ber98], Sato [Sat99], Applebaum [App09].

2.1 Definition and characterization of stable processes

There are many ways to define stable processes, since they belong to more general classes of stochastic processes. We will take the Lévy process definition as our base and defer the other ones as properties for later on.

Lévy processes are characterized by the distribution of the processes at a given time, in fact, their distributions belong to the class of infinitely divisible distributions. Let us start with the definition of a strictly stable random variable as a member of the infinitely divisible distributions.

Definition 2.1.1 (Strictly stable random variable)**.** *We will say that a real valued random variable Y is strictly stable with index of stability $\alpha \in (0,2]$ if for any $n \geq 1$ we have:*

$$Y \stackrel{d}{=} \frac{Y_1 + \cdots + Y_n}{n^{1/\alpha}},$$

where $Y_1, \ldots, Y_n$ are i.i.d. copies of Y.

The *strictly* part means that we do not need to add an extra constant (or shift) to the RHS in order to achieve the distribution equality. On the sequel, we omit the term *strictly* when we talk about stable random variables or processes.

The cases where α is 1 and 2, which correspond to Cauchy and Gaussian random variables respectively, are the most well-known examples of stable r.v. Nevertheless, the fractional calculus technique we will emphasize later, exclude both cases. So, hereafter we will assume that $\alpha \in (0,2) \setminus \{1\}$.

Before we define a stable process, let us recall the definition of a Lévy process.

Definition 2.1.2 (Lévy Process)**.** *Let $\{X_t\}_{t \geq 0}$ be a continuous time stochastic process defined in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We will say it is a Lévy process if the following conditions hold:*

- $X_0 = 0$ $\mathbb{P}$-*a.s.*
- *X has independent and stationary increments.*
- *X has* $\mathbb{P}$-*a.s. càdlàg paths.*

The fundamental examples of Lévy processes are the Compound Poisson process and the Brownian motion. In fact, they play a central role in characterizing Lévy processes as we could see in the following theorems. Following Applebaum [App09] (Theorem 1.2.14 and 2.4.16), we state the Lévy-Khintchine formula and the Lévy-Itô decomposition for Lévy processes.

Theorem 2.1.1 (Lévy-Khintchine). *Let* $b \in \mathbb{R}$, $\sigma \in \mathbb{R}$ *and* ν *a measure concentrated in* $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ *such that* $\int_{\mathbb{R}_0} (1 \wedge x^2)\nu(dx) < \infty$. *Given* (b, σ, ν), *for each* $u \in \mathbb{R}$ *define*

$$\phi(u) = ibu + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}_0} \left(1 - e^{iux} + iux\, \mathbb{1}_{|x|<1}\right) \nu(dx).$$

Then, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, *on which a Lévy process* $\{X_t\}_{t \geq 0}$, *such that* $\mathbb{E}\left(e^{iuX_t}\right) = e^{-t\phi(u)}$, *is defined.*

The function ϕ is called the characteristic exponent of X and the vector (b, σ, ν) is its characteristic triplet. Furthermore, the term b is called the drift component, σ the Gaussian component and ν is the measure of the jumps of X since for any $A \in \mathscr{B}(\mathbb{R}_0)$:

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1] : X_t - X_{t-} \in A\}\right].$$

Despite the fact that (b, σ, ν) is unique for a given Lévy process, one may find in the literature other representations (b_g, σ, ν) where they use a regularizing function $g(x)$, and the characteristic exponent takes the form:

$$\phi(u) = ib_g u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}_0} \left(1 - e^{iux} + iug(x)\right) \nu(dx),$$

the drift b_g component depends on the function $g(x)$.

These name conventions can be well understood in sight of the following theorem.

Theorem 2.1.2 (Lévy-Itô decomposition). *Let* $\{X_t\}_{t \geq 0}$ *a Lévy process with characteristic triplet given by* (b, σ, ν). *Then, for each* $t \geq 0$ *we have the following decomposition:*

$$X_t = bt + \sigma B_t + \int_0^t \int_{|x|<1} x\tilde{N}(ds, dx) + \int_0^t \int_{|x|>1} xN(ds, dx),$$

where B is a standard Brownian motion and N an independent Poisson random measure[1] *defined in* $\mathbb{R}_+ \times \mathbb{R}_0$, *whose intensity is* $ds\nu(dx)$, *and* $\tilde{N}$ *is the compensated Poisson random measure given by*

$$\tilde{N}(ds, dx) := N(ds, dx) - ds\nu(dx).$$

[1] See appendix C for its definition and main properties.

We are ready to define a stable process as a Lévy process, state its characteristic triplet and its Lévy-Itô decomposition. The form that the characteristic exponent takes, depends on α being in $(0,1)$ or $(1,2)$. For the first case, since ν is integrable near zero, we could take the regularizing function $g(h)=0$ and this implies that $b=-\int_{|h|\leq 1} h\nu(dh)$. For the second case, since ν integrates h^2 near zero, we could take $g(h)=h$ and the drift becomes $b=\int_{|h|>1} h\nu(dh)$.

Definition 2.1.3 (Strictly stable process)**.** *A Lévy process $\{X_t\}_{t\geq 0}$ with characteristic triplet $(b_\alpha, 0, \nu)$, is called a strictly stable process with index of stability $\alpha \in (0,2)\setminus\{1\}$ if*

$$\begin{aligned} b_\alpha &= \begin{cases} -\displaystyle\int_{|h|\leq 1} h\nu(dh) & \alpha \in (0,1), \\ \displaystyle\int_{|h|>1} h\nu(dh) & \alpha \in (1,2), \end{cases} \qquad \textit{and} \\ \nu(dh) &= \left(c_- \mathbb{1}_{\{h<0\}} + c_+ \mathbb{1}_{\{h>0\}}\right) \frac{dh}{|h|^{\alpha+1}}, \end{aligned}$$

where $c_-, c_+ \geq 0$ not both zero.

In fact, for each $t \geq 0$ we have that $X_t = t^{1/\alpha} X_1$, and X_1 is an α-stable random variable. We will write $X \sim S_\alpha(c_-, c_+)$ when we refer to a strictly stable process with such parameters.

Let us give the Lévy-Khintchine formula and Lévy-Itô decomposition as a corollary from the general Lévy case and considering the drift b that depends on α as mentioned above.

Corollary 2.1.1 (of Lévy-Khintchine)**.** *Let $X \sim S_\alpha(c_-, c_+)$, if $\alpha \in (0,1)$, then the characteristic exponent can be written as*

$$\phi(u) = \int_{\mathbb{R}_0} \left(1 - e^{iuh}\right) \nu(dh),$$

and if $\alpha \in (1,2)$,

$$\phi(u) = \int_{\mathbb{R}_0} \left(1 - e^{iuh} + iuh\right) \nu(dh).$$

Moreover, it can be proved (cf. Applebaum [App09] Theorem 1.2.21) that in the case $\alpha \in (0,2)\setminus\{1\}$ the characteristic exponent of a stable process X of index α is equal to:

$$\phi(u) = \exp\left[-\sigma|u|^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right)\right]. \tag{2.1}$$

Here we have another parametrization of a stable process in terms of the skewness β and scale σ. In this case the stable process is denoted by $X \sim S_\alpha(\beta, \sigma)$. We can recover the (c_-, c_+) parametrization solving:

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}, \qquad \sigma = -(c_+ + c_-)\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right).$$

There are many equivalent parametrizations for stable processes, as shown by Zolotarev in [Zol86]. Besides the ones mentioned before, we can consider the positivity parameter, denoted by $\rho = \mathbb{P}(X_t \geq 0)$. It can be proved that

$$\rho = \frac{1+\theta}{2}, \qquad \text{where } \theta = \begin{cases} \beta & \text{if } \alpha \in (0,1), \\ \beta\left(\frac{\alpha-2}{\alpha}\right) & \text{if } \alpha \in (1,2). \end{cases}$$

The values that θ and ρ can take are in $[-1,1]$ and $[0,1]$ when $\alpha \in (0,1)$. In the other case, where $\alpha \in (1,2)$, we get $\alpha\rho \leq 1$, $\alpha(1-\rho) \leq 1$ and $\theta \leq 2/\alpha - 1$. For instance, it is quite common to find $c_{\pm}$ in terms of α, ρ and $\hat{\rho} := 1-\rho$ in the following way:

$$c_{-} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \quad \text{and} \quad c_{+} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}.$$

With this parametrization, we will say that a stable process X has parameters α and ρ and we denote it by $X \sim S(\alpha,\rho)$.

Remark 2.1.1. *The characteristic exponent of a stable process is intrinsically related to the Fourier transform of the Riemann-Liouville fractional derivatives (cf. Remark 3.3.1).*

Corollary 2.1.2 (of Levy-Itô decomposition)**.** *Let $X \sim S_{\alpha}(c_{-},c_{+})$. If $\alpha \in (0,1)$, there are no finite moments, hence no martingale representation, but the Levy-Itô decomposition can be written as:*

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}_0} hN(ds,dh).$$

If $\alpha \in (1,2)$, it has finite first moment, and the Lévy-Itô decomposition can be written as:

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}_0} h\tilde{N}(ds,dh).$$

Where N is a Poisson random measure with intensity $ds\nu(dh)$ and $\tilde{N}$ is the compensated Poisson random measure. In fact, note that in the recurrent case, $\alpha \in (1,2)$, we get an integrable martingale.

Example 2.1.1 (Stable subordinator)**.**
Let $\alpha = 0.95$ and $\beta = 1$. Consider the process $X \sim S(\alpha = 0.95, \rho = 1)$, then an example of a sample path of X and the jumps associated to this path are:

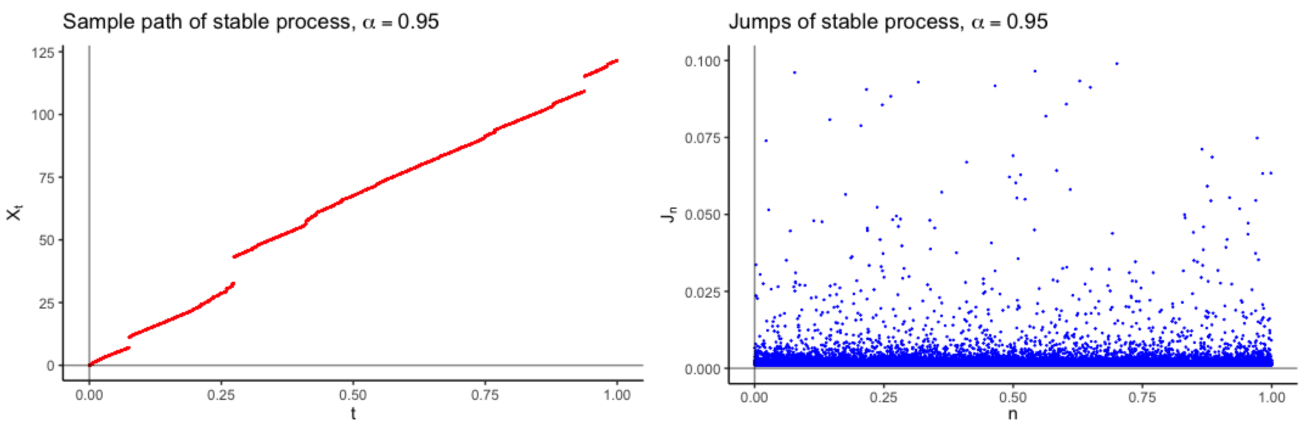


Figure 2.1: Stable subordinator and its associated jumps.

2.2 Some properties of stable processes

The behavior and further properties of the process X differ substantially whether α is in $(0,1)$ or $(1,2)$ as we have seen in the Lévy-Khintchine and Lévy-Itô theorems. For instance, we may consider the transient/recurrent dichotomy, the polar/non-polar character of zero, or the bounded/unbounded variation of the sample paths.

A useful characterization of transience and recurrence is given by the finiteness or divergence of the so-called potential measures. Following [Ber98] we consider the family of linear operators U^q for $q > 0$ called the q-resolvents. These operators are given for every measurable positive function f by

$$U^q f(x) = \mathbb{E}_x \left(\int_0^\infty e^{-qt} f(X_t) dt \right).$$

The family of finite measures $(U^q(x,dy))$ for any $x \in \mathbb{R}$ such that

$$U^q f(x) = \int_{-\infty}^\infty f(y) U^q(x,dy),$$

is known as the resolvent kernel.

The potential measures $U(x,\cdot)$ for any $x \in \mathbb{R}$ correspond to the limit case of the q-resolvent kernel as $q \to 0$. That is, for every $x \in \mathbb{R}$ and $A \in \mathscr{B}(\mathbb{R})$

$$U(x,A) = \mathbb{E}_x \left(\int_0^\infty \mathbb{1}_{X_t \in A} \, dt \right) = \int_0^\infty \mathbb{P}_x(X_t \in A) dt.$$

The definition of transience and recurrence in terms of the potential measures is as follows.

Definition 2.2.1. *A Lévy process is transient if the potential measures satisfy that for any compact set K, we have that $U(x,K) < \infty$ for any $x \in \mathbb{R}$. On the contrary, we say that a Lévy process is recurrent if $U(0,B) = \infty$ for every open ball B centered at the origin.*

According to this definition, it can be proved that for $\alpha \in (0,1)$ a stable process is transient and for $\alpha \in (1,2)$ it is recurrent.

Another important property of a stable process is whether or not the process reaches single points. According to [RY13] we have the following definition.

Definition 2.2.2. *Consider a Markov process $\{X_t\}_{t\geq 0}$ with state space $\mathbb{R}$, a set $A \in \mathscr{B}(\mathbb{R})$ is called polar if*

$$\mathbb{P}_x(X_t \in A \text{ for some } t > 0) = 0, \quad \text{for any } x \in \mathbb{R}.$$

If the set A consists of a single point, say a, a polar process X does not reach the point a. Otherwise, this probability is 1 and it is said that the process is non-polar.

In the case of stable processes, it can be proved that for $\alpha \in (0,1)$ a stable process is polar, and for $\alpha \in (1,2)$ it is non-polar.

In terms of the sample paths, there is also an important characterization. Since there is no Gaussian component in the characteristic triplet of our stable processes, the bounded or unbounded variation of the sample paths is completely described by the finiteness or divergence of the sum of the absolute value of the jumps.

We have that $\sum_{0\leq s\leq t}|\Delta X_s|$ converges for every $t>0$ a.s. if and only if $\int_{\mathbb{R}_0}(1\wedge|x|)\nu(dx)<\infty$. In this case, we say that X has bounded variation and unbounded variation otherwise.

Since $\nu(dx)$ is of the order $|x|^{-1-\alpha}$ we have that for $\alpha\in(0,1)$ a stable process has bounded variation sample paths; whereas for $\alpha\in(1,2)$ it has unbounded variation sample paths.

The following result concerns the finiteness of moments for stable processes. Since they have poorer integrability conditions than Brownian motion, we will have to be careful when we consider Itô formula, for each component to be integrable. For the first part, the proof can be consulted in [Tsu19] and the second one in [Ber98].

Proposition 2.2.1. *Let $\alpha\in(0,2)\setminus\{1\}$, $c_-,c_+\geq 0$, not both zero, and consider a strictly stable process $X\sim S_\alpha(c_-,c_+)$. Then, the following bounds are satisfied:*
1. For all $t>0$, $x\in\mathbb{R}$ and $0<\gamma<1$,

$$\mathbb{E}\left[|X_t-x|^{-\gamma}\right]\leq S(\alpha,\gamma)t^{-\gamma/\alpha},$$

where $S(\alpha,\gamma)$ is a constant which depends on α and γ. Note that this bound does not depend on x.
2. For all $t>0$ and $0<\gamma<\alpha$,

$$\mathbb{E}\left[|X_t|^{\gamma}\right]<\infty,$$

and if $\gamma\geq\alpha$ it is infinite.

2.3 Some stochastic calculus results for stable processes

Stochastic calculus is a very useful set of techniques that allows us to study more properties of the stochastic processes involved. Stochastic calculus for Brownian motion is by far the most used and well-known, and it has been successfully extended to the case of continuous semimartingales and even to Lévy processes. We are considering strictly stable processes, which are pure jump processes; however, as they belong to the Lévy processes class we also have in hand a lot of results, which we will use accordingly.

One of the most important results in stochastic calculus is the Itô formula, which gives us the dynamics of the process $f(X_t)$ for any $f\in C^2$ and X_t a Lévy process, in terms of a stochastic differential equation (SDE). The main feature of this result is the fact that if X is a semimartingale, then $f(X)$ is also a semimartingale.

In order to define the infinitesimal generator of a stable process we state Itô's formula for stable processes. The following proposition is a version of Itô's formula (termed predictable in [SY05]) for stable processes. The statement, and the useful notation $C^2_{1+,b}$ for the functional space of twice continuously differentiable functions whose derivatives or order greater than 1 are bounded, are taken from [Tsu19]. In contrast to the standard Itô's formula, the semimartingale decomposition of $f(X)$

in this version clearly features the infinitesimal generator and big and small jumps are compensated. Hence the need to restrict the class of C^2 functions to $C^2_{1+,b}$.

Proposition 2.3.1 (Itô's formula)**.** *Let $X \sim S_\alpha(c_-,c_+)$ with $c_-,c_+ \geq 0$, not both zero and $f \in C^2_{1+,b}$. If $\alpha \in (0,1)$, then for any $t \geq 0$, with probability 1 we have*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \int_{\mathbb{R}_0} [f(X_{s-}+h) - f(X_{s-})]\tilde{N}(ds,dh) \\ &+ \int_0^t \int_{\mathbb{R}_0} [f(X_{s-}+h) - f(X_{s-})]\,\nu(dh)ds. \end{aligned}$$

If $\alpha \in (1,2)$, then for any $t \geq 0$, with probability 1 we have

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t \int_{\mathbb{R}_0} [f(X_{s-}+h) - f(X_{s-})]\tilde{N}(ds,dh) \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[f(X_{s-}+h) - f(X_{s-}) - hf'(X_{s-})\right]\nu(dh)ds. \end{aligned}$$

The infinitesimal generator $\mathscr{L}$ for a Feller process X can be defined as the time derivative at zero of the semigroup associated with the process. For instance, let $f \in \mathscr{S}$, the Schwartz space of rapidly decreasing functions, then we have:

$$\mathscr{L}f(x) := \frac{\partial}{\partial t}\mathbb{E}_x(f(X_t))\bigg|_{t=0},$$

in other words, the infinitesimal generator describes the change of X_t in an infinitesimal period of time.

Following the book of Sato [Sat99], we are going to use the following equivalent expression of the infinitesimal generator associated with a stable process.

Definition 2.3.1 (Infinitesimal generator)**.** *Let $X \sim S_\alpha(c_-,c_+)$, then its infinitesimal generator $\mathscr{L}$ is the operator defined by*

$$\mathscr{L}f(x) = \begin{cases} \int_{\mathbb{R}_0} [f(x+h) - f(x)]\,\nu(dh), & \text{if } \alpha \in (0,1) \\ \int_{\mathbb{R}_0} \left[f(x+h) - f(x) - hf'(x)\right]\nu(dh), & \text{if } \alpha \in (1,2), \end{cases}$$

for $f \in \mathscr{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.

Considering the stable process X as a semimartingale, Itô theorem implies that the family of functions C^2 is an invariant transformation in the class of semimartingales. In fact, there is a well-known extension of this result, called the Meyer-Itô theorem. Following [Pro04](Theorem 70) we have the following theorem.

Theorem 2.3.1. *Let f be the difference of two convex functions, let f' be its left derivative, and let μ be the signed measure (when restricted to compacts) which is the second derivative of f in the generalized function sense. Then the following equation holds:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{0<s\leq t}\left[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\right] + \frac{1}{2}\int_{-\infty}^{\infty} L_t^a(X)\mu(da),$$

where X is a semimartingale and L_t^a is its semimartingale local time at a up to time t.

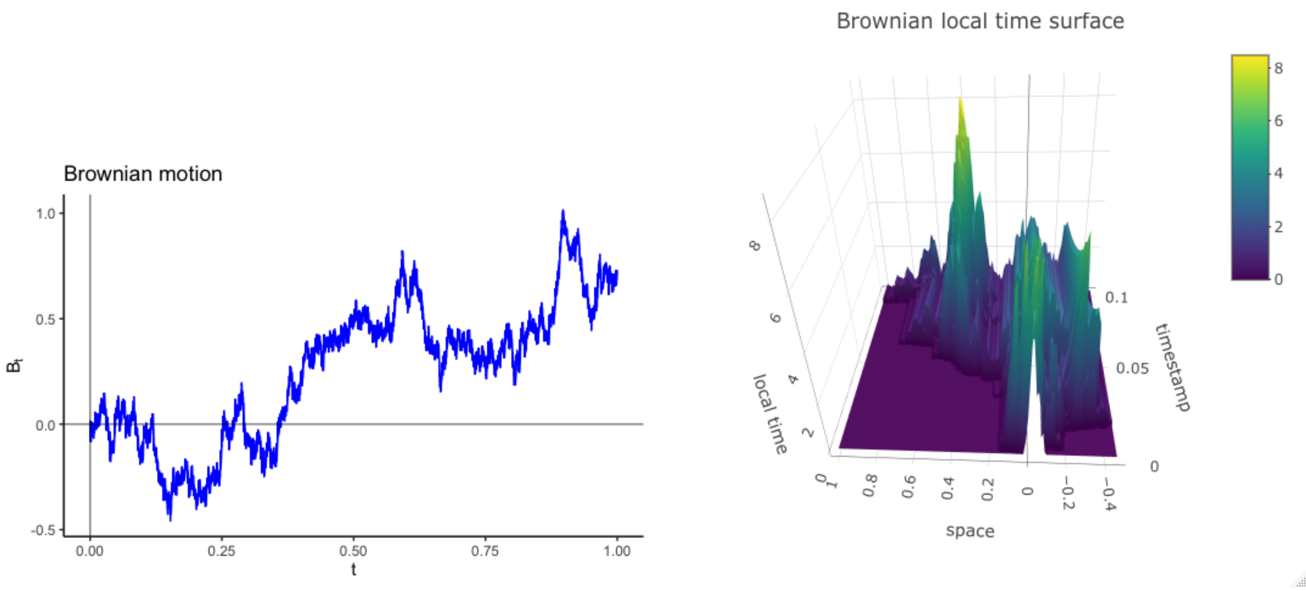


Figure 2.2: Brownian motion sample path and its local time surface.

The semimartingale local time of X satisfies the following property (cf. [Pro04] Corollary 1).

Proposition 2.3.2. *Let X be a semimartingale with semimartingale local time $(L^a(X))_{a\in\mathbb{R}}$. Let g be a bounded Borel measurable function. Then a.s.*

$$\int_0^t g(X_s)d[X,X]_s^c = \int_{-\infty}^{\infty} g(a)L_t^a(X)da,$$

where $[X,X]^c$ denotes the path by path continuous part of the quadratic variation, with $[X,X]_0^c = 0$.

In the case of Brownian motion, since it is a continuous semimartingale, this local time is not zero since $d[B,B]_s^c = ds$. Unfortunately, our stable process has no continuous martingale part. Then, for a stable process X we have that the semimartingale local time is identically equal to zero.

Local times can be defined in several ways, for example: semimartingle, Markov or occupational local times. The latter is more convenient in our case. Recall that in the case of recurrent stable processes, that is for $\alpha \in (1,2)$, it is non-polar for every Borel set, in particular for any single point set. The definition of an occupational local time is as follows.

Definition 2.3.2 (Occupational local time)**.** *Consider a family of random variables with two indices, $\{L_t^a(X) : a \in \mathbb{R}, t \geq 0\}$. We will call it an occupational local time of a process X if, the occupation time formula is satisfied for any positive Borel measurable function $f : \mathbb{R} \to [0,\infty)$:*

$$\int_0^t f(X_s)\,ds = \int_{-\infty}^{\infty} f(a)\,L_t^a(X)da \quad a.s.$$

The fact that this local time exists for recurrent stable processes, as well as being jointly continuous in time and space, was studied by Boylan [Boy64] and Barlow [Bar88]. From now on, when we talk about a local time for stable processes, we will refer to the occupational one.

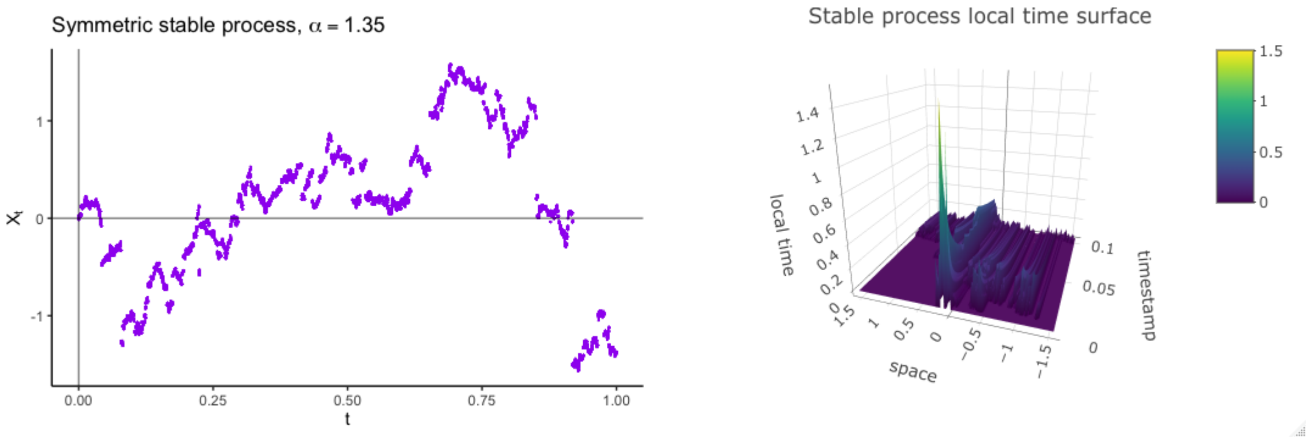


Figure 2.3: Symmetric stable process sample path and its local time surface.

In the case of Brownian motion, the local time appears in the Doob-Meyer decomposition of the process $|B_t|$ as the increasing process part. This decomposition is well known as the Tanaka formula for Brownian motion; given any $x \in \mathbb{R}$ we have that:

$$|B_t - x| = |B_0 - x| + \int_0^t \operatorname{sgn}(B_s - x)dB_s + L_t^x(B). \tag{2.2}$$

Given that we can define a local time for stable processes, one natural question is if we can also achieve its corresponding Tanaka formula. An affirmative answer was given by Tsukada in [Tsu19].

The following definition corresponds to the function that appears in [Tsu19], but within our notation.

Definition 2.3.3. *For every fixed $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, we define the function:*

$$F^{\alpha,c_-,c_+}(x) := K(\alpha, c_-, c_+)\left(1 - \left(\frac{c_+ - c_-}{c_+ + c_-}\right)\operatorname{sgn}(x)\right)|x|^{\alpha-1}, \tag{2.3}$$

where

$$K(\alpha, c_-, c_+) = \frac{c_- + c_+}{2\Gamma(-\alpha)\Gamma(\alpha)\left(c_-^2 + c_+^2 + 2c_-c_+\cos(\pi\alpha)\right)}$$

By means of Fourier and stochastic calculus arguments, Tsukada proved in [Tsu19] that $F^{\alpha,c_-,c_+}(x)$ satisfies the Tanaka formula for the stable process $X \sim S_\alpha(c_-, c_+)$. Given any $a \in \mathbb{R}$ we have:

$$F^{\alpha,c_-,c_+}(X_t - a) = F^{\alpha,c_-,c_+}(X_0 - a) + M_t^a(X) + L_t^a(X),$$

where $L_t^a(X)$ is the occupational local time at a up to time t of X and $M_t^a(X)$ is a martingale given by

$$M_t^a(X) = \int_0^t \int_{\mathbb{R}_0} \left[F^{\alpha,c_-,c_+}(X_{s-} - a + h) - F^{\alpha,c_-,c_+}(X_{s-} - a)\right]\tilde{N}(ds, dh).$$

Finally, we will be interested in stochastic differential equations driven by stable processes; that is, for any $X \sim S_\alpha(c_-, c_+)$ and some measurable functions b and σ we will consider the following SDE:

$$dZ_t = b(Z_t)dt + \sigma(Z_{s-})dX_t.$$

The existence and uniqueness of solutions for this type of equation (considering asymmetric stable processes) have been studied by Fournier in [Fou13]. He provided some properties on b and σ such that the solution Z_t is pathwise unique.

After defining an occupational local time for stochastic integrals with respect to stable processes, we could follow the method Le Gall stated in [LG83] for the Brownian motion, to prove pathwise uniqueness of driftless SDEs driven by stable processes. In a nutshell, Le Gall uses the Meyer-Itô theorem for continuous semimartingales and give conditions to ensure the pathwise uniqueness of solutions to the following SDE:

$$dZ_t = b(Z_t)dt + \sigma(Z_s)dB_t, \tag{2.4}$$

where B is a Brownian motion and b, σ some measurable functions. Given the Tanaka formula (2.2) the classic Itô formula for continuous semimartingales can be extended from C^2 functions to the family of the difference of convex functions. If f is the difference of convex functions then:

$$f(Z_t) = f(Z_0) + \int_0^t f'(Z_s)dZ_s + \int_{-\infty}^{\infty} L_t^a(Z)f''(da), \tag{2.5}$$

and the local time $L_t^a(Z)$ satisfies the occupational time formula for any positive Borel measurable function $g : \mathbb{R} \to [0,\infty)$:

$$\int_0^t g(X_s)\, d < Z >_s = \int_{-\infty}^{\infty} g(a)\, L_t^a(Z)da \quad \text{a.s.},$$

where $< Z >_t$ stands for the quadratic variation of Z at time t.

If we take b to be Lipschitz and σ such that there exists an increasing function $\rho : [0,\infty) \to [0,\infty)$ such that

$$\int_{0+} \frac{du}{\rho(u)} = +\infty \quad \text{and } (\sigma(x) - \sigma(y))^2 \leq \rho(|x-y|) \quad \forall x, y.$$

Then, if we consider two solutions, Z^1 and Z^2, for the SDE (2.4) with the same initial value, we have that:

$$\forall t \geq 0, \quad L_t^0(Z^1 - Z^2) = 0.$$

Then by the Meyer-Itô formula (2.5) applied to $Z^1 - Z^2$ with the function $f(x) = |x|$, we have that:

$$|Z_t^1 - Z_t^2| = \int_0^t \operatorname{sgn}(Z_s^1 - Z_s^2)\left(b(Z_s^1) - b(Z_s^2)\right) ds + \int_0^t \operatorname{sgn}(Z_s^1 - Z_s^2)\left(\sigma(Z_s^1) - \sigma(Z_s^2)\right) dB_s + L_t^0(Z^1 - Z^2).$$

Since $L_t^0(Z^1 - Z^2) = 0$ and b is Lipschitz, taking expectations in both sides, by Gronwall lemma we have:

$$\mathbb{E}\left(|Z_t^1 - Z_t^2|\right) \leq \mathbb{E}\left(\int_0^t \left|b(Z_s^1) - b(Z_s^2)\right| ds\right) \leq K \int_0^t \mathbb{E}\left(|Z_s^1 - Z_s^2|\right) ds.$$

where K is the Lipschitz constant for b. So we have that $\forall t \geq 0$, $\mathbb{E}\left(|Z_t^1 - Z_t^2|\right) = 0$ and we get pathwise uniqueness of the solution to the SDE (2.4).

Note that the key property is that $L_t^0(Z^1 - Z^2) = 0$ for all $t \geq 0$. The work of Le Gall [LG83] gives us the condition on σ in order to achieve this property, as well as the condition on b to conclude the pathwise uniqueness.

Following the same reasoning, we could prove pathwise uniqueness for driftless SDEs driven by strictly stable processes; however, this problem is out of the scope of this work, because the local time we will define in the last chapter does not consider the case where σ can take both signs.

Chapter 3

Fractional calculus

Fractional calculus has been studied almost since the invention of calculus. One of the most famous applications is the solution to the tautochrone problem by Abel (cf. [PMT17]). Even though many mathematicians have contributed to the formalization of the field; it was Marcel Riesz who systematized several results in terms of non-local operator theory. The book of Samko, Kilbas and Marichev [SKM93] will be our main reference for the theory of fractional calculus in what follows. We will focus on the results that will be useful to prove the inversion theorem 4.2.1; as we will see, the connection between fractional calculus and stable processes will appear very natural by means of their infinitesimal generator.

3.1 Brief introduction to fractional calculus

The simplest way to introduce fractional operators is through fractional integrals. First, recalling the Cauchy formula for the n-fold integral (right and left) of an integrable function in the interval $[a,b]$. If $f \in L^1(a,b)$, then for any $a \leq x \leq b$ we have:

$$
\begin{aligned}
I^n_{a+}f(x) &:= \int_a^x \int_a^{x_1} \cdots \int_a^{x_{n-1}} f(x_n)dx_n \cdots dx_2 dx_1 = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt, \\
I^n_{b-}f(x) &:= \int_x^b \int_{x_1}^{b} \cdots \int_{x_{n-1}}^{b} f(x_n)dx_n \cdots dx_2 dx_1 = \frac{1}{(n-1)!}\int_x^b (t-x)^{n-1} f(t)dt,
\end{aligned}
$$

Given these representations, the generalization for any $\alpha > 0$ is straightforward. Using the fact that $\Gamma(n) = (n-1)!$ we can write the left and right fractional integral as:

$$
\begin{aligned}
I^\alpha_{a+}f(x) &:= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \\
I^\alpha_{b-}f(x) &:= \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt.
\end{aligned}
$$

The heuristic of how these operators can be considered as nice interpolations between the corresponding n-fold integration formulas is quite simple. However, this is not the case for fractional derivatives. There are several properties of derivatives that their fractional counterparts will preserve or lose, depending on the definition we adopt. Let us present the two most common definitions of fractional derivatives, the Riemann-Liouville and the Caputo definitions.

If $n-1<\alpha\leq n$, the Riemann-Liouville fractional derivatives are defined by:

$$D_{a+}^{\alpha}f(x) := \left(\frac{d}{dx}\right)^n I_{a+}^{n-\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1}f(t)dt,$$
$$D_{b-}^{\alpha}f(x) := \left(-\frac{d}{dx}\right)^n I_{b-}^{n-\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (t-x)^{n-\alpha-1}f(t)dt.$$

These definitions seem convenient, since for any $\alpha = n \in \mathbb{N}$ we have that

$$D_{a+}^{n}f(x) = \left(\frac{d}{dx}\right)^n f(x) \quad \text{and} \quad D_{b-}^{n}f(x) = (-1)^n\left(\frac{d}{dx}\right)^n f(x),$$

so that, in other words, these fractional derivatives also nicely interpolate the standard derivatives. To make our point, on where these definitions could cause troubles, let us calculate the fractional derivative of a power function.

Example 3.1.1. *Let $p>0$ and consider the function $f(x)=x^p\,\mathbb{1}_{x\geq 0}$. Let us calculate the right fractional derivative of $f(x)$ of order $0<\alpha<1$:*

$$\begin{aligned} D_{0+}^{\alpha}x^p &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[\int_0^x (x-t)^{-\alpha}t^p dt\right] \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left[\frac{\Gamma(p+1)\Gamma(1-\alpha)}{\Gamma(p+2-\alpha)}x^{p+1-\alpha}\right] \\ &= \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)}x^{p-\alpha}. \end{aligned}$$

As $p\to 0$ we have that $f(x)\to 1$ if $x\geq 0$, but the fractional derivative $D_{0+}^{\alpha}x^p \to \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$, which is not zero as we would like it to be. In other words, the Riemann-Liouville fractional derivatives of a constant are not necessarily zero, which is an important property of standard derivatives.

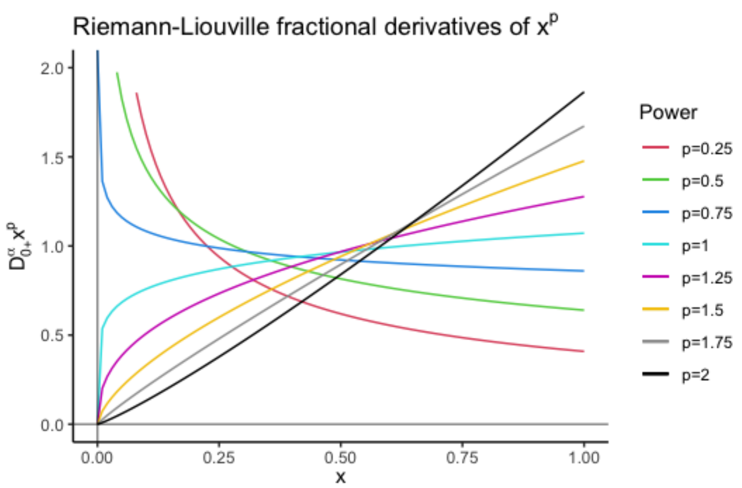


Figure 3.1: Fractional derivatives of order $\alpha = 0.85$ of the function $f(x)=x^p$.

If $n-1<\alpha\leq n$, the Caputo fractional derivatives are defined by:

$$\begin{aligned} D^{\alpha}_{a+*}f(x) &:= I^{n-\alpha}_{a+}\left(\frac{d^n}{dx^n}f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\frac{d^n}{dt^n}f(t)dt,\\ D^{\alpha}_{b-*}f(x) &:= I^{n-\alpha}_{b-}\left(\frac{d^n}{dx^n}f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1}\frac{d^n}{dt^n}f(t)dt. \end{aligned}$$

It can be shown that these definitions also lead to the standard derivatives when $\alpha=n\in\mathbb{N}$. The main difference is that in the Caputo case, we have that the fractional derivative of a constant is zero. From this property, is immediate that the operators $\frac{d^n}{dx^n}$ and $I^{n-\alpha}$ do not commute. In fact, we can express one in terms of the other; since we are focusing on the cases $\alpha\in(0,1)$ and $\alpha\in(1,2)$, let us write the relationship between both definitions (c.f. [Kol15]).

Proposition 3.1.1. *Let $\alpha\in(0,1)$, then for any $a<x$ we have:*

$$D^{\alpha}_{a+*}f(x)=D^{\alpha}_{a+}f(x)-\frac{f(a)}{\Gamma(1-\alpha)(x-a)^{\alpha}}.$$

Let $\alpha\in(1,2)$, then for any $a<x$ we have:

$$D^{\alpha}_{a+*}f(x)=D^{\alpha}_{a+}f(x)-\frac{f(a)}{\Gamma(1-\alpha)(x-a)^{\alpha}}-\frac{f'(a)}{\Gamma(2-\alpha)(x-a)^{\alpha-1}}.$$

We can see that the difference between the fractional derivatives comes from the initial conditions in the extreme point a. There is an analogous relation between D^{α}_{b-*} and D^{α}_{b-}.

3.2 Riemann-Liouville fractional operators

The fractional operators we will use are the Riemann-Liouville's, with $a=-\infty$ and $b=\infty$, for the left and right operators respectively.[1] These definitions and further properties can be consulted in [SKM93](Section 2.3). In the sequel we will denote by $\mathscr{S}(\mathbb{R})$ the Schwartz space of real rapidly decreasing functions (see Appendix B).

Definition 3.2.1 (Riemann-Liouville fractional operators)**.** *Let $\alpha\geq 0$ and $\varphi\in\mathscr{S}(\mathbb{R})$. Then, the left and right Riemann-Liouville fractional operators of order α applied to φ are defined in three cases:*

- *For $\alpha=0$ we get the identity operator*

$$W^{\alpha}_{-}\varphi(x)=W^{\alpha}_{+}\varphi(x):=\varphi(x).$$

- *For $\alpha>0$, the Riemann-Liouville fractional integrals are given by*

$$\begin{aligned} W^{\alpha}_{-}\varphi(x) &:= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}\varphi(t)\,dt,\\ W^{\alpha}_{+}\varphi(x) &:= \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}\varphi(t)\,dt. \end{aligned}$$

[1] In fact, in this case, the R-L derivatives coincide with the Caputo's, since the initial conditions will be zero in the extreme points.

- *For $n-1<\alpha<n$, with $n\in\mathbb{N}$, the Riemann-Liouville fractional derivatives are given by*

$$\begin{aligned} W_-^{-\alpha}\varphi(x) &:= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^{x}(x-t)^{n-\alpha-1}\varphi(t)\,dt,\\ W_+^{-\alpha}\varphi(x) &:= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{x}^{\infty}(t-x)^{n-\alpha-1}\varphi(t)\,dt. \end{aligned}$$

Remark 3.2.1.

- *Fractional operators can be defined for absolutely continuous functions with some differentiability conditions, but we stick with the Schwartz space so that their Fourier transforms are well-defined.*
- *If $\alpha>0$, we will use the following notation for the fractional integrals and derivatives:*

$$\begin{aligned} I_\mp^\alpha &:= W_\mp^\alpha,\\ D_\mp^\alpha &:= W_\mp^{-\alpha}. \end{aligned}$$

- *If $\alpha\in\mathbb{N}$, then the left fractional operators I_-^α and D_-^α, are the iterated integral and classical differential operators of order α.*
- *On an adequate domain (the so-called Lizorkin space, to be introduced), they satisfy the group property for $\alpha,\beta\in\mathbb{R}$:*

$$\begin{aligned} W_-^\alpha\circ W_-^\beta &= W_-^{\alpha+\beta},\\ W_+^\alpha\circ W_+^\beta &= W_+^{\alpha+\beta}. \end{aligned}$$

- *On Schwartz space of rapidly decreasing smooth functions, fractional derivatives satisfy the semigroup property $D_\pm^\alpha\circ D_\pm^\beta=D_\pm^{\alpha+\beta}$ for $\alpha,\beta\geq 0$.*

In the next proposition, we rewrite the definition of fractional derivative depending on the index α, this representation is called the generator form. The fact that they are equivalent can be found in the book of Meerschaert and Sikorskii [MS12] and the article of Kolokoltsov [Kol15].

Proposition 3.2.1 (Generator form)**.** *Let $f\in\mathscr{S}(\mathbb{R})$ and $\alpha\in(0,2)\setminus\{1\}$. Then the generator form of the left and right fractional derivatives are as follows:*

$$\begin{aligned} D_-^\alpha f(x) &= \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^\infty \frac{f(x-h)-f(x)}{h^{1+\alpha}}dh, & \text{if } \alpha\in(0,1)\\ \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^\infty \frac{f(x-h)-f(x)+hf'(x)}{h^{1+\alpha}}dh, & \text{if } \alpha\in(1,2) \end{cases}\\ D_+^\alpha f(x) &= \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^\infty \frac{f(x+h)-f(x)}{h^{1+\alpha}}dh, & \text{if } \alpha\in(0,1)\\ \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^\infty \frac{f(x+h)-f(x)-hf'(x)}{h^{1+\alpha}}dh, & \text{if } \alpha\in(1,2) \end{cases} \end{aligned}$$

This generator form will be the key ingredient to state the fractional derivative weighted sum representation for the infinitesimal generator of strictly stable processes. Now we will focus on the further properties of the fractional operators that will lead us to the proof of the inversion theorem.

3.3 Lizorkin space

The main problem working in the fractional calculus framework is the domain of definition of these operators; Schwartz space is not invariant under fractional operators (cf. Samko, Kilbas and Marichev [SKM93], section 8.2). Since we are seeking for the inverse of the infinitesimal generator, it would be useful to have a space that remains invariant under the action of the Riemann-Liouville fractional operators. This kind of space has been thoroughly studied by Lizorkin [Liz68, Liz71], Samko, Kilbas and Marichev [SKM93] and Rubin [Rub96, Rub15].

The Lizorkin space of test functions Φ (which is a subspace of the Schwartz space of rapidly decreasing functions), is a space that remains invariant with respect to fractional integration and differentiation. The intuition behind this relies on the Fourier transform of these operators.

Proposition 3.3.1 (Fourier transform of fractional operators)**.** *Let $f \in \Phi$ and $\alpha \geq 0$, then the Fourier transforms of the Riemann-Liouville fractional operators of index α, considering the principal branch of the logarithm, satisfy the following identities:*

$$\begin{aligned}\mathscr{F}\left[D_{-}^{\alpha} f\right](u) &= (-iu)^{\alpha}\mathscr{F}[f](u),\\ \mathscr{F}\left[D_{+}^{\alpha} f\right](u) &= (iu)^{\alpha}\mathscr{F}[f](u),\\ \mathscr{F}\left[I_{-}^{\alpha} f\right](u) &= (-iu)^{-\alpha}\mathscr{F}[f](u),\\ \mathscr{F}\left[I_{+}^{\alpha} f\right](u) &= (iu)^{-\alpha}\mathscr{F}[f](u).\end{aligned}$$

Where the Fourier transform of an element $f \in \mathscr{S}$ is defined by:

$$\mathscr{F}[f](u) = \int_{\mathbb{R}} f(x)e^{iux}dx.$$

The proof of this proposition can be found in the book of Samko, Kilbas and Marichev [SKM93] Lemma 8.1.

Remark 3.3.1. *If we take the principal branch of the logarithm, we have*

$$\begin{aligned}(\pm iu)^{\alpha} &= |u|^{\alpha}e^{\pm i\,\mathrm{sgn}(u)\alpha\pi/2}\\ &= |u|^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right) \pm i\,\mathrm{sgn}(u)\sin\left(\frac{\alpha\pi}{2}\right)\right),\end{aligned}$$

for all $u, \alpha \in \mathbb{R}$. These are precisely the characteristic functions of the one-sided stable processes, see equation (2.1) with $\sigma = 1$ and $\beta = \pm 1$.

Thus, we have to consider functions in $\mathscr{S}$, such that their Fourier transform is well-behaved in the singularity of the multiplier.

Definition 3.3.1 (Lizorkin space)**.** *Consider the space of functions that vanish at zero together with all their derivatives:*

$$\Psi = \left\{\psi \in \mathscr{S}(\mathbb{R}) \,\middle|\, \psi^{(j)}(0) = 0, j \in \{0,1,2,\dots\}\right\}.$$

Then, the space of functions whose Fourier transforms are in Ψ is called the Lizorkin space and is defined by

$$\Phi = \{\phi \in \mathscr{S}(\mathbb{R})\,|\,\mathscr{F}[\phi] \in \Psi\}.$$

In the article [Liz71], Lizorkin defines the fractional derivatives of functions in $L^p(\mathbb{R})$, $p > 1$, using functions in the Lizorkin space in the weak and strong sense. Considering the following function

$$\tilde{\kappa}_\beta(u) = e^{-\beta^2\left(u^2+\frac{1}{u^2}\right)}, \qquad \beta > 0,$$

it can be proved that $\tilde{\kappa}_\beta \in \Psi$. Since it is an odd function its Fourier transform can be written as

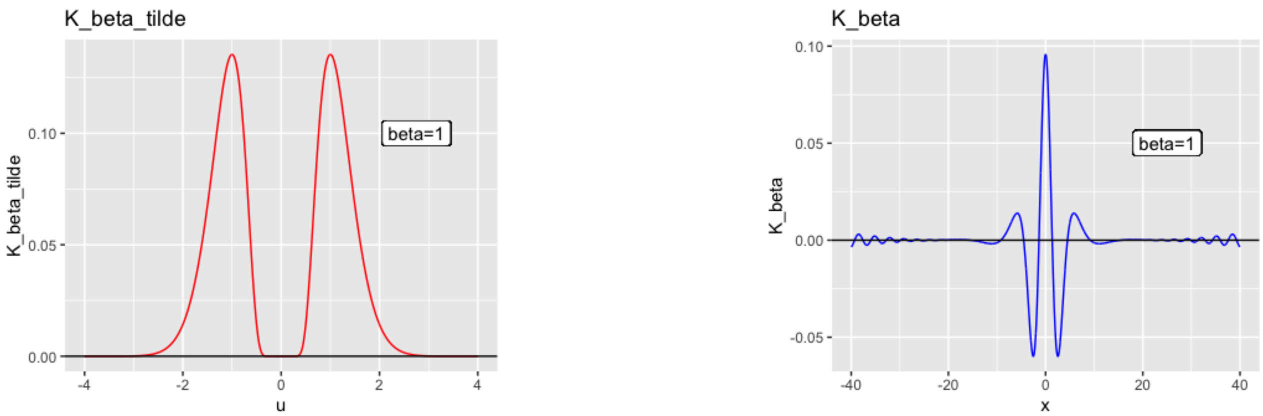
$$\kappa_\beta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\beta^2\left(u^2+\frac{1}{u^2}\right)} \cos(ux)du.$$

So that $\kappa_\beta \in \Phi$. Since $\tilde{\kappa}_\beta \to 1$ as $\beta \to 0$, we would expect that $\kappa_\beta/\sqrt{2\pi}$ converge to the Dirac delta distribution in zero as $\beta \to 0$ as well. The proofs of these properties of κ_β and its Fourier transform can be consulted in [Liz71, Ch. II§3].

Given a function $f \in L^p(\mathbb{R})$, $p > 1$, Lizorkin defines infinite differentiable approximations, called *completely balanced averages* of f, by means of the κ_β so that as $\beta \to 0$ we have:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \kappa_\beta(x-y)f(y)dy \xrightarrow{L^p(\mathbb{R})} f(x).$$

It allows us to characterize the fractional differentiation of f through its completely balanced averages in the limit and provide the required tools to prove Lemma 4.3.2.



(a) $\tilde{\kappa}_1(u)$ (b) $\kappa_1(x)$

Figure 3.2

By definition of these spaces, we have that for any $k \in \mathbb{N}_0$:

$$0 = \frac{d^k}{du^k}\mathscr{F}[\phi]\bigg|_{u=0} = \int_{-\infty}^{\infty} x^k e^{i\cdot 0\cdot x}\phi(x)dx.$$

So that the Lizorkin space can be characterized as the subspace of $\mathscr{S}$ whose elements are orthogonal to all polynomials, since:

$$\int_{-\infty}^{\infty} x^k \phi(x)dx = 0, \quad k \in \mathbb{N}_0.$$

This property implies that Lizorkin space does not contain any infinitely differentiable function of compact support other than the zero function. In order to extend results from Φ to more general space functions such as L^p or C_0^∞, we could not rely on the classic denseness of, for example, C_0^∞ in $\mathscr{S}$ or L^p; and here is where the completely balanced averages come in handy.

Another interesting property is that Φ does not contain real valued functions everywhere different from zero, since if $\phi \in \Phi$, it must satisfy

$$\int_{-\infty}^{\infty} \phi(x)dx = 0.$$

Space Φ may be considered as a topological vector space with the topology of the space $\mathscr{S}$. The latter is generated by the countable family of seminorms

$$\sup_x \left(1+x^2\right)^{\frac{m}{2}} \left|\phi^{(k)}(x)\right|,$$

which are finite for any $k \in \mathbb{N}_0$. Moreover, the spaces Φ and Ψ are closed in $\mathscr{S}$.

One may define a topology in Ψ which embraces the behavior of functions $\psi(x)$ not only at infinity, but for $x \to 0$ as well, namely by means of a countable number of seminorms

$$\sup_x \left(1+x^2\right)^{\frac{m}{2}} |x|^{-p} \left|\phi^{(k)}(x)\right|,$$

which are finite for any $k \in \mathbb{N}_0$. In fact, this topology coincides with that of $\mathscr{S}$ for functions $\psi \in \Psi$ (cf. [Rub96] Chapter 1.3).

The space of linear continuous functionals[2] on Φ will be denoted by Φ' as is usual. Let us compare Φ' with $\mathscr{S}'$ by comparing first Ψ' with $\mathscr{S}'$. Since Ψ is closed in $\mathscr{S}$, we may identify Ψ' with the quotient space of the Schwartzian space $\mathscr{S}'$ modulo the subspace Ψ'_0 of functionals in $\mathscr{S}'$ having Ψ as a null space, i.e.

$$\Psi' = \mathscr{S}'/\Psi'_0,$$

where $\Psi'_0 = \{f \in \mathscr{S}' \,|(f,\psi) = 0, \psi \in \Psi\}$.

This is a particular case of the following more general result: if M is a closed subspace in a linear topological space E, then $M' = E'/M^\perp$, where $M^\perp$ is the space of all functionals in E', which are orthogonal to M.

In our case, the space Ψ consists of functions that vanish at zero together with all their derivatives, so that Ψ'_0 consist of functions that are concentrated at the point $x = 0$, and these are the linear combinations of delta distributions and their derivatives. Moreover, it is known that $\delta^{(k)}(u)$ is the Fourier transform of the power function, that is

$$\mathscr{F}[(-ix)^k](u) = 2\pi\delta^{(k)}(u),$$

the Fourier transform is understood in the sense of generalized functions as:

$$(\mathscr{F}[f],\phi) = (f,\mathscr{F}[\phi]),$$

[2] See Appendix B for an elementary compilation of results on distribution theory.

where $\phi \in \mathscr{S}$ or $\phi \in \Phi$. Consequently, we have the representation

$$\Phi' = \mathscr{S}'/\Phi'_0,$$

where $\Phi'_0 = \{f \in \mathscr{S}' \,|\, (f,\phi) = 0, \phi \in \Phi\}$, consisting of polynomials.

In other words, Φ' can be obtained from $\mathscr{S}'$ in a way such that whenever an element is of the form $f + P \in \mathscr{S}'$ with P a polynomial, we eliminate the P part, i.e., two functionals in $\mathscr{S}'$ differing by a polynomial are indistinguishable as elements of Φ'.

If we consider two locally integrable functions f and g which coincide in the Φ' sense, i.e. $(f,\varphi) = (g,\varphi)$ for all $\varphi \in \Phi$. Then, we could ask ourselves, when does $f(x) = g(x)$ pointwise for almost all $x \in \mathbb{R}$. The following results are sufficient conditions to achieve the latter.

Consider the following function, which characterizes the "size" of $|f(x)|$:

$$\lambda_f(\gamma) = \text{Leb}\left(\{x \in \mathbb{R} : |f(x)| > \gamma\}\right), \quad \gamma > 0.$$

Note that $\lambda_f(\gamma)$ is a non-increasing function.

Lemma 3.3.1. *Let $f,g \in L^1_{loc} \cap \mathscr{S}'$ which coincide in the Φ' sense. If the corresponding $\lambda_f(\gamma)$ and $\lambda_g(\gamma)$ are finite for all $\gamma > 0$, then $f(x) = g(x)$ a.e. in $\mathbb{R}$.*

Proof. Since $f,g \in \mathscr{S}'$ and $f = g$ in Φ' sense, then $f(x) = g(x) + P(x)$, where $P(x)$ is a polynomial. Then, we have $\lambda_P(2\gamma) \leq \lambda_f(\gamma) + \lambda_g(\gamma) < \infty$ for all $\gamma > 0$, but this is possible only if $P(x) = 0$. □

The fact that $\lambda_f(\gamma) < \infty$ means that the function f is bounded, but most examples of distributions in Φ' are (slowly) increasing. In order to conclude the a.e. equality of functions with these characteristics, we have the following corollary.

Corollary 3.3.1. *Let $f \in L^r$, $1 \leq r < \infty$, and $f \in L^p$, $1 \leq p < \infty$. If $f = g$ in Φ' sense, then $f = g$ a.e. in $\mathbb{R}$.*

Proof. For all $\gamma > 0$ we have that

$$\|f\|_r^r \geq \int_{|f(x)| \geq \gamma} |f(x)|^r dx \geq \gamma^r \lambda_f(\gamma),$$

so that $\lambda_f(\gamma) \leq \|f\|_r^r/\gamma^r < \infty$. Similarly for g, we have that $\lambda_g(\gamma) \leq \|g\|_p^p/\gamma^p < \infty$. Then by lemma 3.3.1, $f = g$ a.e. in $\mathbb{R}$. □

It will be useful to define the fractional operators for distributions in Φ'. One way to do this is by duality in Φ. Let $\alpha \in \mathbb{R}$, then

$$\left(W^\alpha_\pm f, \phi\right) = \left(f, W^\alpha_\mp \phi\right), \quad \phi \in \Phi. \tag{3.1}$$

Moreover, if we consider the following identity

$$\left(f, W^\alpha_\mp \phi\right) = \left(\mathscr{F}[f], \mathscr{F}\left[W^\alpha_\mp \phi\right]\right) = \left(\mathscr{F}[f], (\mp iu)^{-\alpha} \mathscr{F}[\phi](u)\right),$$

and since $\mathscr{F}[\phi](u) \in \Psi$ and multiplication by $(\mp iu)^{-\alpha}$ is a continuous operation in Ψ, we have that $\left(f, W^\alpha_\mp \phi\right)$ is a continuous functional on Φ.

Remark 3.3.2. *Equation 3.1 may serve as a definition of the fractional integral for a function $f(x) \in L^p(\mathbb{R})$ with $p \geq 1/\alpha$ when integrals $W^\alpha_\pm f$ do not exist in the usual sense, being divergent at infinity. In this case $W^\alpha_\pm f$ is a generalized function.*

The same results can be obtained if we define fractional integrals and derivatives of distributions $f \in \Phi'$ not by duality, but by using the notion of convolution. The latter approach coincides with the former, but on many occasions it is preferable.

Let $\alpha > 0$, consider

$$h^\alpha_\pm = \frac{x^{\alpha-1}_\pm}{\Gamma(\alpha)}, \quad x^{\alpha-1}_\pm = \begin{cases} |x|^{\alpha-1} & \text{if } \pm x > 0, \\ 0 & \text{if } \pm x < 0 \end{cases}.$$

The functions $h^\alpha_\pm$ can be regarded as Φ' distributions and if we take $\varphi \in \Phi$, their convolution coincides with the corresponding fractional integrals, that is

$$\left(h^\alpha_\pm * \varphi\right)(x) = I^\alpha_\pm \varphi(x).$$

In a similar way, fractional derivatives are defined for $\alpha < 0$ by

$$\left(h^\alpha_\pm * \varphi\right)(x) = D^{-\alpha}_\pm \varphi(x).$$

In the Lizorkin space, compositions of fractional operators are well-defined. In fact, we will recall in the following section that in this space left (right) fractional derivative and left (right) fractional integral are inverses of each other. But, there is one more issue, the case of a crossed composition is not immediate, in fact, it is not even mentioned in the literature. We defer this discussion to the next chapter.

Chapter 4

Stable processes via fractional calculus

4.1 Introduction

The connection between fractional calculus and stable processes has not been completely developed, even though it is common knowledge that fractional Laplacians are related to the infinitesimal generator of symmetric stable processes (cf. [MS12, LPGea20] with Remark 4.2.1). The oldest references that relate fractional calculus and stable random variables, are the seminal work of Feller [Fel52], which uses fractional calculus to compute a series for stable densities, and the articles of Gorenflo and Mainardi (cf. [GM98, MPG07]), which identify a correspondence between stable characteristic function and the Fourier transform of fractional derivatives. More recent references are the book of Meerschaert and Sikorskii [MS12] and the article of Kolokoltsov [Kol15], where they identify the infinitesimal generator of a stable process with fractional derivatives.

This chapter contains new results concerning both fractional calculus and stable processes. The main goal is to unveil new connections between both areas and to prove interesting results for the less studied case of asymmetric stable processes.

Considering a space of functions that remains invariant under fractional operators' action, we identify fractional integrals as inverses of fractional derivatives. Nevertheless, for a general one-dimensional strictly stable process, the generator corresponds to a linear combination of left and right fractional derivatives, which requires a consideration of crossed compositions of left and right fractional derivatives and integrals.

4.2 Infinitesimal generator of stable processes

After some algebraic manipulations, the infinitesimal generator of a strictly stable process can be written in terms of the Riemann-Liouville fractional derivatives of order α. For a detailed proof see for example the article of Kolokoltsov [Kol15] or the book of Meerschaert and Sikorskii [MS12].

Proposition 4.2.1 (Infinitesimal generator)**.** *Let $\alpha \in (0,2)\setminus\{1\}$, $c_-, c_+ \geq 0$, not both zero. If $X \sim S_\alpha(c_-, c_+)$ and $\varphi \in \mathscr{S}(\mathbb{R})$, then, its infinitesimal generator $\mathscr{L}$ takes the following form*

$$\begin{aligned} \mathscr{L}\varphi(x) &= c_-\Gamma(-\alpha)D_-^\alpha\varphi(x) + c_+\Gamma(-\alpha)D_+^\alpha\varphi(x) \\ &= M_-D_-^\alpha\varphi(x) + M_+D_+^\alpha\varphi(x), \end{aligned}$$

where $M_{\pm} = c_{\pm}\Gamma(-\alpha)$.

Remark 4.2.1. *This representation is consistent in the case* $\alpha = 2$ *and* $c_- = c_+$*, which corresponds to the Brownian motion, and its infinitesimal generator is the Laplacian* Δ*. In the case* $\alpha \in (0,2)\setminus\{1\}$ *and* $c_- = c_+$*, corresponding to a symmetric strictly* α*-stable process, the infinitesimal generator is given by the fractional Laplacian* $-(-\Delta)^{\alpha/2}$.

In the Lizorkin space, left (right) fractional derivative and left (right) fractional integral are inverses of each other. However, the crossed compositions are not immediate.

Our first result concerns a characterization of crossed compositions as a linear combination of right and left fractional operators. The result is stated without proof for fractional integrals in the article of Feller [Fel52].

Proposition 4.2.2. *Let* $\lambda,\mu \in \mathbb{R}$ *with* $\lambda+\mu \notin \mathbb{Z}$ *and* $\phi \in \Phi$*, then the crossed composition of Riemann-Liouville operators satisfy:*

$$W_+^{\lambda}W_-^{\mu}\phi(x) = \frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)}W_-^{\lambda+\mu}\phi(x) + \frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)}W_+^{\lambda+\mu}\phi(x). \tag{4.1}$$

Proof. Since the Fourier transform characterizes a function $\phi\in\Phi$, we will prove that the Fourier transforms of both sides of the statement are the same. First, using the Fourier transform of fractional operators with polar representation of the multipliers we have for the LHS:

$$\begin{aligned}\mathscr{F}\left[W_+^{\lambda}W_-^{\mu}\phi\right](u) &= |u|^{\lambda}e^{i\frac{\pi}{2}\operatorname{sgn}(u)\lambda}|u|^{\mu}e^{-i\frac{\pi}{2}\operatorname{sgn}(u)\mu}\mathscr{F}[\phi](u)\\ &= |u|^{\lambda+\mu}e^{i\frac{\pi}{2}\operatorname{sgn}(u)(\lambda-\mu)}\mathscr{F}[\phi](u).\end{aligned}$$

For the RHS we have

$$\begin{aligned}&\frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)}\mathscr{F}\left[W_-^{\lambda+\mu}\phi\right](u) + \frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)}\mathscr{F}\left[W_+^{\lambda+\mu}\phi\right](u)\\ =\;&\frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)}|u|^{\lambda+\mu}e^{-i\frac{\pi}{2}\operatorname{sgn}(u)(\lambda+\mu)}\mathscr{F}[\phi](u)\\ &+\frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)}|u|^{\lambda+\mu}e^{i\frac{\pi}{2}\operatorname{sgn}(u)(\lambda+\mu)}\mathscr{F}[\phi](u).\end{aligned}$$

After canceling out the common factors, it suffices to prove that:

$$e^{i\frac{\pi}{2}\operatorname{sgn}(u)(\lambda-\mu)} = \frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)}e^{-i\frac{\pi}{2}\operatorname{sgn}(u)(\lambda+\mu)} + \frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)}e^{i\frac{\pi}{2}\operatorname{sgn}(u)(\lambda+\mu)}.$$

This is equivalent to the real and imaginary parts agreeing. We refer to the lemma A.0.1 in Appendix A, which gives us:

$$\begin{aligned}\cos\left((\lambda-\mu)\frac{\pi}{2}\right) &= \frac{\cos\left((\lambda+\mu)\frac{\pi}{2}\right)[\sin(\mu\pi)+\sin(\lambda\pi)]}{\cos((\lambda+\mu)\pi)},\\ \sin\left((\lambda-\mu)\operatorname{sgn}(u)\frac{\pi}{2}\right) &= \frac{\sin\left((\lambda+\mu)\operatorname{sgn}(u)\frac{\pi}{2}\right)[\sin(\lambda\pi)-\sin(\mu\pi)]}{\sin((\lambda+\mu)\pi)},\end{aligned}$$

finishing the proof. □

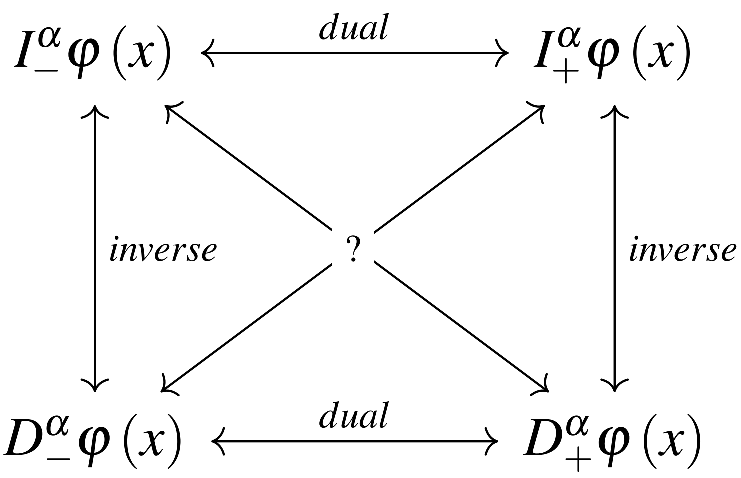


Figure 4.1: Relationships between Riemann-Liouville fractional operators.

Before the proof of the Inversion Theorem 4.2.1, we will prove one more lemma regarding the composition of fractional derivatives and integrals. Since we are taking functions in the Lizorkin space, these compositions are well-defined.

In figure 4.1 we show how the Riemann-Liouville integrals and derivatives relate to each other. For any $\phi_1, \phi_2 \in \Phi$ the right/left and left/right integrals satisfy:

$$\int_{-\infty}^{\infty} I^{\alpha}_{\pm}(\phi_1(x))\,\phi_2(x)dx = \int_{-\infty}^{\infty} \phi_1(x) I^{\alpha}_{\mp}(\phi_2(x))\,dx,$$

that is, they are dual operators acting on Φ, seen as a Hilbert space with inner product given by $(\phi_1, \phi_2) = \int \phi_1(x)\phi_2(x)dx$ for any $\phi_1, \phi_2 \in \Phi$.

On the other hand, if we consider the right/left operators, we have that for any $\phi \in \Phi$:

$$I^{\alpha}_{\pm} D^{\alpha}_{\pm}(\phi(x)) = \phi(x),$$

that is, they are inverse operators.

Finally, it remains to see how the crossed compositions are defined. Note that we can't just evaluate $\lambda = -\alpha$ and $\mu = \alpha$ in equation (4.1) since in that case $\lambda + \mu = 0 \in \mathbb{Z}$ and the crossed composition becomes indeterminate. The next result gives us a way to understand the crossed compositions in a useful manner.

Lemma 4.2.1 (Fractional compositions)**.** *Let $\phi \in \Phi$ and $\alpha > 0$ with $\alpha \notin \mathbb{N}$, then the compositions of fractional derivatives and integrals of order α satisfy:*

$$\begin{aligned} D^{\alpha}_{-} I^{\alpha}_{-}\phi(x) &= \phi(x), \\ D^{\alpha}_{+} I^{\alpha}_{+}\phi(x) &= \phi(x), \\ D^{\alpha}_{-} I^{\alpha}_{+}\phi(x) + D^{\alpha}_{+} I^{\alpha}_{-}\phi(x) &= 2\cos(\alpha\pi)\phi(x). \end{aligned}$$

Proof. The first two equations as well as the fact that all the compositions of fractional operators commute follow from Proposition 3.3.1. For the last equation, we use Proposition 4.2.2 with $\lambda = \alpha$ and $\mu \to -\alpha$, to get the result. This last limit can be taken since the semigroup associated is strongly continuous in the parameter μ ([SKM93] Section 2.7).

Using equation (4.1) twice[1] to obtain both crosses we have:

$$\begin{aligned} W_-^{\lambda} W_+^{\mu} \phi(x) \quad + \quad & W_+^{\lambda} W_-^{\mu} \phi(x) \\ = \quad & \frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)} W_-^{\lambda+\mu} \phi(x) + \frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)} W_+^{\lambda+\mu} \phi(x) \\ & + \frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)} W_-^{\lambda+\mu} \phi(x) + \frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)} W_+^{\lambda+\mu} \phi(x) \\ = \quad & \left[\frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)} + \frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)} \right] W_-^{\lambda+\mu} \phi(x) \\ & + \left[\frac{\sin(\mu\pi)}{\sin((\lambda+\mu)\pi)} + \frac{\sin(\lambda\pi)}{\sin((\lambda+\mu)\pi)} \right] W_+^{\lambda+\mu} \phi(x). \end{aligned} \tag{4.2}$$

Moreover, we have that the following limit is indeterminate, so using L'Hôspital's rule we have:

$$\begin{aligned} \lim_{\mu\to-\alpha} \left(\frac{\sin(\mu\pi) + \sin(\alpha\pi)}{\sin((\alpha+\mu)\pi)} \right) \quad &= \quad \lim_{\mu\to-\alpha} \left(\frac{\pi\cos(\mu\pi)}{\pi\cos((\alpha+\mu)\pi)} \right) \\ &= \quad \cos(\alpha\pi). \end{aligned}$$

Finally, with $\lambda = \alpha$ and taking the limit $\mu \to -\alpha$ in equation (4.2) we have:

$$\begin{aligned} D_-^{\alpha} I_+^{\alpha} \phi(x) \quad + \quad & D_+^{\alpha} I_-^{\alpha} \phi(x) \\ & = \lim_{\mu\to-\alpha} W_-^{\alpha} W_+^{\mu} \phi(x) + W_+^{\alpha} W_-^{\mu} \phi(x) \\ & = \lim_{\mu\to-\alpha} \left[\frac{\sin(\alpha\pi)}{\sin((\alpha+\mu)\pi)} + \frac{\sin(\mu\pi)}{\sin((\alpha+\mu)\pi)} \right] W_-^{\alpha+\mu} \phi(x) \\ & \quad + \lim_{\mu\to-\alpha} \left[\frac{\sin(\mu\pi)}{\sin((\alpha+\mu)\pi)} + \frac{\sin(\alpha\pi)}{\sin((\alpha+\mu)\pi)} \right] W_+^{\alpha+\mu} \phi(x) \\ & = 2\cos(\alpha\pi)\phi(x). \end{aligned}$$

Where we used that W^0 is the identity operator as in Definition 3.2.1. □

Working in the Lizorkin space and using the last result we can compute the inverse of the infinitesimal generator and prove our main theorem.

Theorem 4.2.1 (Inverse of the Infinitesimal Generator)**.** *Let $\alpha \in (0,2) \setminus \{1\}$, $c_-, c_+ \geq 0$, not both zero. Consider $X \sim S_\alpha(c_-, c_+)$ with infinitesimal generator $\mathscr{L}$. Then, $\mathscr{L}$ is invertible in Φ and for every $\phi \in \Phi$*

$$\mathscr{L}^{-1}\phi(x) = K_- I_-^{\alpha} \phi(x) + K_+ I_+^{\alpha} \phi(x),$$

where

$$K_\pm = \frac{M_\pm}{M_-^2 + M_+^2 + 2M_- M_+ \cos(\pi\alpha)},$$

and the constants $M_\pm$ as defined in Proposition 4.2.1.

[1] Note that $W_-^{\lambda} W_+^{\mu} = W_+^{\mu} W_-^{\lambda}$ since $W_\pm$ commute for functions in Lizorkin space.

Proof. Define the operator $\mathscr{G}$ as

$$\mathscr{G}\phi\left(x\right)=K_{-}I_{-}^{\alpha}\phi\left(x\right)+K_{+}I_{+}^{\alpha}\phi\left(x\right),$$

we will prove that $\mathscr{G}\left(\mathscr{L}\phi\right)=\mathscr{L}\left(\mathscr{G}\phi\right)=\phi$, so that $\mathscr{L}$ is invertible and $\mathscr{L}^{-1}=\mathscr{G}$. By our definition of $\mathscr{G}$, we have:

$$\begin{aligned}\mathscr{G}\left(\mathscr{L}\phi\left(x\right)\right) &= \mathscr{G}\left(M_{-}D_{-}^{\alpha}\phi\left(x\right)+M_{+}D_{+}^{\alpha}\phi\left(x\right)\right)\\ &= K_{-}M_{-}I_{-}^{\alpha}\left(D_{-}^{\alpha}\phi\left(x\right)\right)+K_{-}M_{+}I_{-}^{\alpha}\left(D_{+}^{\alpha}\phi\left(x\right)\right)\\ &+ K_{+}M_{-}I_{+}^{\alpha}\left(D_{-}^{\alpha}\phi\left(x\right)\right)+K_{+}M_{+}I_{+}^{\alpha}\left(D_{+}^{\alpha}\phi\left(x\right)\right)\end{aligned}$$

Substituting the values of K_{-} and K_{+}, and defining $M=M_{-}^{2}+M_{+}^{2}+2M_{-}M_{+}\cos(\alpha\pi)$ to temporarily ease notation, and using Lemma 4.2.1, we get

$$\begin{aligned}\mathscr{G}\left(\mathscr{L}\phi\left(x\right)\right) &= \frac{M_{-}^{2}}{M}\phi\left(x\right)+\frac{M_{+}^{2}}{M}\phi\left(x\right)+\frac{M_{-}M_{+}}{M}I_{-}^{\alpha}D_{+}^{\alpha}\phi\left(x\right)+\frac{M_{+}M_{-}}{M}I_{+}^{\alpha}D_{-}^{\alpha}\phi\left(x\right)\\ &= \frac{M_{-}^{2}+M_{+}^{2}+2M_{-}M_{+}\cos(\alpha\pi)}{M}\phi\left(x\right)\\ &= \phi\left(x\right).\end{aligned}$$

We conclude that $(\mathscr{G}\circ\mathscr{L})\phi=\phi$ and analogous computations prove that $(\mathscr{L}\circ\mathscr{G})\phi=\phi$. ☐

We will be interested in the distributions that are generated by the space of Lizorkin test functions. For the definition of the action of Riemann-Liouville operators on distributions we refer to [SKM93](Section 8.1).

Remark 4.2.2. *We are going to define the fractional operators for distributions taken in the Lizorkin's dual space Φ' by duality, in the same way the Fourier transform are defined for distributions using test functions in the Schwartz space.*

For instance, let $f\in\Phi'$ and consider the Riemann-Liouville operators W_{-}^{α} and W_{+}^{α}, then for any $\phi\in\Phi$ we define the distributions $W_{-}^{\alpha}f$ and $W_{+}^{\alpha}f$ by means of:

$$\begin{aligned}\left(W_{-}^{\alpha}f,\phi\right) &= \left(f,W_{+}^{\alpha}\phi\right),\\ \left(W_{+}^{\alpha}f,\phi\right) &= \left(f,W_{-}^{\alpha}\phi\right),\end{aligned}$$

where the $(\cdot,\cdot)$ is the inner product with Lebesgue measure.

Note that, if we consider the infinitesimal generator $\mathscr{L}$ of $X\sim S_{\alpha}\left(c_{-},c_{+}\right)$, then its dual $\tilde{\mathscr{L}}$ is the infinitesimal generator of $\tilde{X}\sim S_{\alpha}(c_{+},c_{-})$, the dual process of X. This corresponds to the notion that the left Riemann-Liouville operator is dual to the right one.

The main theorem 4.2.1 provides a stronger link between fractional calculus, stable processes and potential theory. The following known results can be recovered as an application of this theorem:

1. For the case $\alpha\in(0,1)$, the Lévy process X is transient. Therefore, its potential corresponds to the inverse of the negative of the infinitesimal generator, $(-\mathscr{L})^{-1}$. The above theorem can recover the expression given by Sato [Sat72].

2. For the case $\alpha \in (1,2)$, the Lévy process X is recurrent and its classical potential is infinite. Nevertheless, Port [Por67] defined the recurrent potential for stable processes (by an appropriate compensated kernel) and computed it explicitly. As Sato [Sat72] notes, for a wide class of Lévy processes, including the stable processes, $(-\mathscr{L})^{-1}$ corresponds to a potential and coincides with that defined by Port. Since we have explicitly this inverse, Port's computation can be recovered with the aid of the above theorem.

3. It provides an heuristic explanation of the function involved in the Tanaka formula for strictly stable processes given by Tsukada in [Tsu19]: it corresponds to applying the inverse of the infinitesimal generator to the Dirac δ distribution. Note that the Itô formula for Lévy processes tells us that for any Schwartz function f, writing $g = \mathscr{L}f$, we have

$$f(x+X_t) = f(x) + M_t^f + \int_0^t g(x+X_s)ds,$$

where M^f is a martingale whose explicit expression is only needed later. Formally, if g equals the Dirac δ distribution, the last term equals the time that X spends at x on $[0,t]$, which is one guiding principle behind the construction of the local time of X at x. Hence, if $\mathscr{L}F = \delta$ then the local time should equal $F(x+X) - M^F$. Our formula for $\mathscr{L}^{-1}$ allows us to guess the solution to $\mathscr{L}F = \delta$ in terms of $K_{\mp}x_{\pm}^{\alpha-1}$, which can be combined to obtain Tsukada's formula. That $K_- \neq K_+$ in general is a manifestation of the asymmetry in the jumps of X.

4.3 Class $\mathscr{C}^{\alpha,c_-,c_+}$ and the Meyer-Itô formula for stable processes

The objective of this section is to prove the occupational Meyer-Itô theorem, with a non-zero local time component, for stable processes. The special class of functions which satisfy this theorem will be defined through the inverse of the infinitesimal generator. The preliminary results we will prove before, mainly concern how to pass from Lizorkin space to a more general space which allows us to generalize the Tanaka formula given by Tsukada [Tsu19].

Recalling Definition 2.3.3, the function that appears in the Tanaka formula for a stable process $X \sim S_\alpha(c_-,c_+)$, with $\alpha \in (1,2)$, $c_-,c_+ \geq 0$, not both zero, given by Tsukada in [Tsu19] is:

$$F^{\alpha,c_-,c_+}(x) := K(\alpha,c_-,c_+)\left(1 - \left(\frac{c_+ - c_-}{c_+ + c_-}\right)\operatorname{sgn}(x)\right)|x|^{\alpha-1}, \tag{4.3}$$

where

$$K(\alpha,c_-,c_+) = \frac{c_- + c_+}{2\Gamma(-\alpha)\Gamma(\alpha)\left(c_-^2 + c_+^2 + 2c_-c_+\cos(\pi\alpha)\right)}$$

As we have remarked and will prove in Lemma 4.3.1, $F^{\alpha,c_+,c_-} = \mathscr{L}^{-1}\delta$ in the sense of distributions. This is useful to define a class of functions f such that $\mathscr{L}^{-1}f$ coincides with a Radon measure (in the sense of distributions).

Definition 4.3.1. *For every fixed $\alpha \in (1,2)$, $c_-,c_+ \geq 0$, not both zero, we define the class of real functions*

$$\mathscr{C}^{\alpha,c_-,c_+} = \left\{ f = F^{\alpha,c_-,c_+} * \mu \,\middle|\, \mu \text{ is a signed Radon measure such that } \int |x|^{\alpha-1}\mu(dx) < \infty \right\}.$$

The condition on the $(\alpha-1)$-moment is to ensure the convolution is well defined. In the symmetric case with $\alpha=2$ (i.e. Brownian motion), such class corresponds to the difference of convex functions (cf. [RY13] discussion before Theorem VI.1.5).

Considering the following generalized functions in the dual space Φ', we will prove an important relationship between the Dirac delta distribution and the power functions, which are strongly related to the strictly stable processes. Moreover, the following lemma could be regarded as the key result to obtain Tanaka formula.

Lemma 4.3.1. *If $\lambda>0$, then, the (generalized) functions $f_+^{\lambda}(x):=x^{\lambda}\,1\!\!1_{\{x>0\}}$ and $f_-^{\lambda}(x):=|x|^{\lambda}\,1\!\!1_{\{x<0\}}$ belong to Φ' and*

$$\begin{aligned} f_+^{\lambda}(x) &= \Gamma(\lambda+1)I_-^{\lambda+1}\delta(x), \\ f_-^{\lambda}(x) &= \Gamma(\lambda+1)I_+^{\lambda+1}\delta(x). \end{aligned}$$

Therefore,

$$\mathscr{L}^{-1}(\delta)=F^{\alpha,c_-,c_+}.$$

Proof. Since the Dirac delta distribution is a linear functional contained in Φ', and using the duality that was pointed out in equation 3.1, we have:

$$\begin{aligned} \left(I_-^{\lambda+1}\delta,\phi\right) &= \left(\delta,I_+^{\lambda+1}\phi\right) \\ &= \left(I_+^{\lambda+1}\phi\right)\Big|_{x=0} \\ &= \frac{1}{\Gamma(\lambda+1)}\int_0^{\infty}t^{\lambda}\phi(t)\,dt \\ &= \left(\frac{1}{\Gamma(\lambda+1)}t^{\lambda}\,1\!\!1_{\{t\geq 0\}},\phi\right) \\ &= \left(\frac{1}{\Gamma(\lambda+1)}f_+^{\lambda},\phi\right). \end{aligned}$$

We can proceed similarly to prove the second identity in the lemma.

Finally, it follows from equation (3.1), the Inversion Theorem 4.2.1 that:

$$\begin{aligned} \mathscr{L}^{-1}(\delta) &= K_-I_-^{\alpha}\delta(x)+K_+I_+^{\alpha}\delta(x) \\ &= \frac{K_-}{\Gamma(\alpha)}f_+^{\alpha-1}(x)+\frac{K_+}{\Gamma(\alpha)}f_-^{\alpha-1}(x). \end{aligned}$$

If we substitute the values of K_- and K_+ in terms of α, c_- and c_+ we will get that $\mathscr{L}^{-1}(\delta)=F^{\alpha,c_-,c_+}$ in the sense of Φ' distributions. □

Thus, the Inverse Theorem 4.2.1 provides an insight to the function that satisfies the Tanaka formula.

The class of convolutions $f=F^{\alpha,c_-,c_+}*\mu$ in Definition 4.3.1, is defined in such a way that the distribution induced by the measure μ coincides with $\mathscr{L}f$, in the sense of Φ' distributions. As a consequence, μ can be considered as the extension of $\mathscr{L}f$ from the Lizorkin space to the class C_c of continuous functions with compact support. A precise version of this is contained in the following lemma. It is here that the *completely balanced averages* of Lizorkin play a fundamental rôle: they constitute a way to approximate δ and other distributions from within Lizorkin space.

Lemma 4.3.2. *Let $f \in \mathscr{C}^{\alpha,c_-,c_+}$ be given by $f = F^{\alpha,c_-,c_+} * \mu$. Then, $\mathscr{L}f = \mu$ in the Φ' sense; that is, for every $\phi \in \Phi$:*

$$(\mathscr{L}f, \phi) = (\mu, \phi).$$

Finally, if μ is a finite measure with compact support, then $\phi \mapsto (\mathscr{L}f, \phi)$ extends by continuity to $\phi \mapsto (\mu, \phi)$ from Φ to C_c with the topology of uniform convergence.

Proof. Let $\phi \in \Phi$. Since $\alpha - 1 \in (0,1)$, then $x \mapsto x^{\alpha-1}$ is subadditive on $[0,\infty)$. Hence,

$$\begin{aligned}\int |f(x)\phi(x)|\,dx &\leq \int\int [|x|^{\alpha-1} + |a|^{\alpha-1}]|\phi(x)|\,|\mu|(da)\,dx \\ &\leq |\mu|(\mathbb{R})\int |x|^{\alpha-1}|\phi(x)|\,dx + \|\phi\|_1 \int |a|^{\alpha-1}|\mu|(da) < \infty.\end{aligned}$$

From equation (3.1) and Fubini's theorem:

$$\begin{aligned}(\mathscr{L}f,\phi) &= \int_{-\infty}^{\infty} f(x)\tilde{\mathscr{L}}\phi(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{\alpha,c_-,c_+}(x-a)\mu(da)\tilde{\mathscr{L}}\phi(x)dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F^{\alpha,c_-,c_+}(x-a)\tilde{\mathscr{L}}\phi(x)dx\mu(da) = \int_{-\infty}^{\infty}\left(\mathscr{L}^{-1}\delta_a, \tilde{\mathscr{L}}\phi\right)\mu(da) \\ &= \int_{-\infty}^{\infty}\left(\delta_a, \tilde{\mathscr{L}^{-1}}\tilde{\mathscr{L}}\phi\right)\mu(da) = \int_{-\infty}^{\infty}(\delta_a,\phi)\,\mu(da) = \int_{-\infty}^{\infty}\phi(a)\mu(da),\end{aligned}$$

where the operator $\tilde{\mathscr{L}}\phi$ is the dual operator of $\mathscr{L}$ (cf. Remark 4.2.2); yielding that $\mathscr{L}f = \mu$ on Φ'.

Lizorkin, in [Liz71] (cf. after Definition 3.3.1), gives an approximation of δ in Φ' by means of a collection of functions $\kappa_\beta \in \Phi$ with the following property. If $\phi \in C_c$, then $\phi_\beta := \kappa_\beta * \phi \to \phi$ uniformly on compact sets; note that $\phi_\beta \in \Phi$. Indeed, Lizorkin writes $\kappa_\beta = \kappa_\beta^1 - \kappa_\beta^2$ where κ_β^1 is a centered Gaussian density of variance $2\beta^2$. Hence $\kappa_\beta^1 * \phi \to \phi$ uniformly if $\phi \in C_c$. On the other hand, the proof of Theorem 1 [Liz71, Ch. II§4] tells us that $\kappa_\beta^2 * \phi \to 0$ uniformly on compact sets since ϕ is integrable. Hence, Φ is dense in C_c. If μ is finite and of compact support then it also has a finite moment of order $\alpha - 1$ and so, by the previous paragraph, $\mathscr{L}(F * \mu) = \mu$ in Φ'. The bounded linear functional $\phi \mapsto (\mu, \phi)$ on C_c coincides with $\phi \mapsto (\mathscr{L}f, \phi)$ on $\mathscr{L}$, so that, by denseness, the latter extends uniquely by continuity to C_c. □

An important example of the result in Lemma 4.3.2, is the Brownian motion case, where $\mathscr{L}f(x) = \frac{1}{2}\Delta f(x)$, so that μ is the second derivative of f in the sense of distributions and in fact, the class $\mathscr{C}^{2,c,c}$ corresponds to the class of difference of convex functions.

Recall that the Meyer-Itô theorem for semimartingales, for example from [Pro04](Theorem 70), gives a semimartingale decomposition for $|X|$ which contains a semimartingale local time term. However, the latter is zero for a strictly stable process. For functions in the class $\mathscr{C}^{\alpha,c_-,c_+}$ we prove the following occupational Meyer-Itô theorem, with a non-zero local time term.

Theorem 4.3.1 (Occupational Meyer-Itô formula). *Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, and consider a strictly stable process $X \sim S_\alpha(c_-, c_+)$. Let $f = F * \mu \in \mathscr{C}^{\alpha, c_-, c_+}$ and furthermore assume that μ is finite and compactly supported. Then,*

$$f(X_t) = f(X_0) + M_t + \int_{-\infty}^{\infty} L_t^a(X)\,\mu(da), \tag{4.4}$$

where

$$M_t = \int_0^t \int_{\mathbb{R}_0} [f(X_{s-} + h) - f(X_{s-})]\,\tilde{N}(ds, dh),$$

is a martingale and $L_t^a(X)$ is the occupational local time at a up to time t of X.

Remark 4.3.1.

- *In the limit case $\alpha = 2$, we can recover $F(x) = \frac{1}{2}|x|$ and the corresponding class $\mathscr{C}$ can be identified with the convex functions as in Revuz-Yor [RY13].*
- *For recurrent symmetric stable process, that is $\alpha \in (1,2)$ and $c_- = c_+ = c > 0$, we have $F^{\alpha,c,c}(x) = K_{\alpha,c}|x|^{\alpha-1}$ for some constant $K_{\alpha,c}$. This particular case was obtained by Salminen and Yor in [SY05].*
- *The Tanaka formula of Tsukada [Tsu19], corresponds to the case $\mu = \delta$.*
- *The compact support hypothesis of μ is sufficient to ensure the integrability of all the terms in (4.4). Since strictly stable processes have finite κ-moments for $\kappa \in (-1, \alpha)$, we have to be careful with the growth of $f = F * \mu$.*

The novel part of this result is the representation of the semimartingale in terms of an occupational local time. Before we prove this theorem let us state some lemmas which will be useful in the proof.

The following results are inspired by the work of Tsukada [Tsu19], which we will generalize relying on a well-known procedure to construct approximations of a function, smoothing it with mollifiers (cf. [KS91], Theorem 6.22), allowing us to use Itô formula (2.3.1).

A positive real function $\rho \in C_c^\infty$, with support in $[-1,1]$ and integral equal to one, is said to be a mollifier. Then, if we consider a sequence of functions given by $\rho_n(x) = n\rho(nx)$ for all $n \in \mathbb{N}$, this sequence converges weakly to the Dirac delta distribution in the sense of Schwartz distributions, that is

$$\left| \int_{-\infty}^{\infty} \rho_n(x)\phi(x)dx - \phi(0) \right| \longrightarrow 0, \quad \text{as } n \to \infty,$$

for all $\phi \in \mathscr{S}$.

Let $C_{1+,b}^\infty$ be the family of infinitely differentiable continuous functions with bounded derivatives of any order greater than or equal to one. We are going to use some bounds for the function F^{α,c_-,c_+} as well as of its increments, for a proof of the following results we refer to [Tsu19](Theorem 3.1 and Lemma 3.1). For fixed α, c_- and c_+, to ease the notation, we are going to write F instead of F^{α,c_-,c_+} when there is no confusion with the parameters.

Lemma 4.3.3. *Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, and consider a strictly stable process $X \sim S_\alpha(c_-, c_+)$. Consider the function $F = F^{\alpha, c_-, c_+}$ in equation (2.3), then the following results are satisfied:*
1. Let $(\rho_n)_{n \geq 1}$ as above, then $F_n := F^{\alpha, c_-, c_+} * \rho_n \in C^\infty_{1+,b}$ for all $n \in \mathbb{N}$ and $F_n \to F$, uniformly in compact sets as $n \to \infty$.
2. Let $|h| \leq 1$, $a \in \mathbb{R}$, $s > 0$ and $\varepsilon_0 \leq (\alpha - 1) \wedge (2 - \alpha)$, then we have:

$$\mathbb{E}\left[\left|F(X_{s_-} - a + h) - F(X_{s_-} - a)\right|^2\right] \leq c_1 S(\alpha, 2 + \varepsilon_0 - \alpha) s^{(\alpha - 2 - \varepsilon_0)/\alpha} |h|^{\alpha + \varepsilon_0},$$

where $c_1 = 20K(\alpha, c_-, c_+)^2$ and the constant $S(\cdot, \cdot)$ as in Proposition 2.2.1, and the same bound holds if we replace F by F_n.
Moreover, this bound satisfies:

$$\int_0^t \int_{|h| \leq 1} s^{(\alpha - 2 - \varepsilon_0)/\alpha} |h|^{\alpha + \varepsilon_0} \nu(dh) ds = \left(\frac{c_+ + c_-}{\varepsilon_0}\right)\left(\frac{\alpha}{2\alpha - \varepsilon_0 - 2}\right) t^{(2\alpha - \varepsilon_0 - 2)/\alpha} < \infty.$$

3. Let $|h| > 1$, $a \in \mathbb{R}$ and $s > 0$, then we have:

$$\mathbb{E}\left[\left|F(X_s - a + h) - F(X_s - a)\right|\right] \leq c_2 |h|^{\alpha - 1},$$

where $c_2 = 4K(\alpha, c_-, c_+)$ and the same bound holds if we replace F by F_n.
Moreover, this bound satisfies:

$$\int_0^t \int_{|h| > 1} |h|^{\alpha - 1} \nu(dh) ds = (c_+ + c_-) t < \infty.$$

In the proof of Meyer-Itô we will need the same kind of bounds, but for $f(x) = (F * \mu)(x)$. In order to prove these bounds, we will rely on a result similar to Jensen's inequality. In general finite measure spaces this inequality does not hold; however, for the special case $f(x) = x^2$, we are able to get a similar result.

Lemma 4.3.4. *Let μ be a compactly supported Radon measure, with support in K and mass $\mu(K) < \infty$, and $g : \mathbb{R} \to \mathbb{R}$. Then, the following inequality holds:*

$$\left(\int_K g(x) \mu(dx)\right)^2 \leq \mu(K) \int_K g^2(x) \mu(dx).$$

Proof. Define $\mu_p(dx) = \mu(dx)/\mu(K)$, so that μ_p is a probability measure, then by using Jensen inequality we have:

$$\begin{aligned} \left(\int_K g(x)\mu(dx)\right)^2 &= \left(\int_K g(x)\mu(K)\frac{\mu(dx)}{\mu(K)}\right)^2 \\ &= (\mu(K))^2 \left(\int_K g(x)\mu_p(dx)\right)^2 \\ &\leq (\mu(K))^2 \int_K g^2(x)\mu_p(dx) \\ &= \mu(K) \int_K g^2(x)\mu(dx). \qquad \square \end{aligned}$$

The following result is a corollary of Lemma 4.3.3 and it will be useful in several steps of the Meyer-Itô theorem's proof.

Corollary 4.3.1. *Under the assumptions of Lemma 4.3.3, let $f \in \mathscr{C}^{\alpha,c_-,c_+}$, such that $f = F * \mu$ with μ a finite Radon measure and consider $f_n = f * \rho_n$ for $n \in \mathbb{N}$. Then we have:*

$$\mathbb{E}\left[\left|f(X_{s_-}+h)-f(X_{s_-})\right|^2\right] \leq (\mu(\mathbb{R}))^2 c_1 S(\alpha, 2+\varepsilon_0-\alpha) s^{(\alpha-2-\varepsilon_0)/\alpha}|h|^{\alpha+\varepsilon_0}, \quad |h| \leq 1,$$
$$\mathbb{E}\left[\left|f(X_{s_-}+h)-f(X_{s_-})\right|\right] \leq \mu(\mathbb{R}) c_2 |h|^{\alpha-1}, \quad |h| > 1,$$

and the same bounds are satisfied if we replace f with f_n.
These bounds are elements of $L^1\left((0,t)\times A, \mathscr{B}((0,t)\times A), \lambda\otimes\nu\right)$, with $A = [-1,1]\setminus\{0\}$ and $A = [-1,1]^c$ respectively.

These results follow from the Lemma 4.3.3 and Lemma 4.3.4.

The proof of the Occupational Meyer-Itô theorem essentially consists in smoothing the function f using mollifiers, enabling the use of Itô's formula. Then, provided that all its terms are well defined as elements of $L^1(\mathbb{P})$ we have to prove that they converge to the desired result.

Proof. (Occupational Meyer-Itô 4.3.1)
Without loss of generality, we assume that μ is actually a positive measure, which was assumed to be finite with compact support and, therefore, with moments of order α and $2(\alpha-1)$. Then, we have the representation:

$$f(x) \quad = \quad \int_{-\infty}^{\infty} F(x-a)\,\mu(da).$$

Consider the sequences $F_n = F * \rho_n$ and $f_n = f * \rho_n = F * \rho_n * \mu$ as the infinitely differentiable approximations of F and f by the sequence $\{\rho_n\}_{n\geq 0}$, with $n \in \mathbb{N}$, and we have that $f_n \to f$ uniformly on compact sets ([EG15] Theorem 4.1: Properties of mollifiers).

Since $f_n \in C^{\infty}_{1+,b} \subset C_2$, using Itô's formula (Proposition 2.3.1) we have:

$$f_n(X_t) = f_n(X_0) + M_t^n + V_t^n, \tag{4.5}$$

where the last two terms are

$$M_t^n \quad = \quad \int_0^t \int_{\mathbb{R}_0} \left[f_n(X_{s-}+h) - f_n(X_{s-})\right]\tilde{N}(ds,dh),$$
$$V_t^n \quad = \quad \int_0^t \mathscr{L} f_n(X_s)ds.$$

Moreover, since the behavior of M_t^n is different depending on the size of the jumps, we will consider $M_t^n = M_t^{1,n} + M_t^{2,n}$, where

$$M_t^{1,n} \quad = \quad \int_0^t \int_{h\leq 1} \left[f_n(X_{s-}+h) - f_n(X_{s-})\right]\tilde{N}(ds,dh),$$
$$M_t^{2,n} \quad = \quad \int_0^t \int_{h>1} \left[f_n(X_{s-}+h) - f_n(X_{s-})\right]\tilde{N}(ds,dh).$$

In a similar fashion, we define $M_t = M_t^1 + M_t^2$, by replacing f_n with f.

The proof consists in establishing the following steps:

Step 1 $f(X_t)$ and $f_n(X_t)$ are in $L^1(\mathbb{P})$ and $f_n(X_t) \to f(X_t)$ in L^1.

Step 2 M^1 and $M^{1,n}$ are square integrable martingales and $M_t^{1,n} \to M_t^1$ in L^2.

Step 3 M^2 and $M^{2,n}$ are integrable martingales and $M_t^{2,n} \to M_t^2$ in L^1.

Step 4 $V_t^n \to \int L_t^a \, \mu(da)$ in L^1.

Let us begin with **Step 1**. First, we provide a bound for $f(x)$ and $f_n(x)$ in terms of x and which does not depend on n. Using that $\alpha - 1 \in (0,1)$, we have that $x \mapsto x^{\alpha-1}$ is subadditive on $[0,\infty)$, so that

$$
\begin{aligned}
0 \quad &\leq \quad f_n(x) = \int_{-\infty}^{\infty} f(x-y)\rho_n(y)\,dy \\
&= \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x-a-y)\rho_n(y)\,dy\,\mu(da) \\
&\leq \quad \int_{-\infty}^{\infty}\int_{-1/n}^{1/n} 2K(|x|^{\alpha-1}+|a|^{\alpha-1}+|y|^{\alpha-1})\rho_n(y)\,dy\,\mu(da) \\
&\leq \quad 2K\int_{-\infty}^{\infty}(|x|^{\alpha-1}+|a|^{\alpha-1}+1)\mu(da),
\end{aligned}
$$

which is finite for any $x \in \mathbb{R}$ by the assumptions on μ and does not depend on n.

By similar arguments, we have that

$$
0 \leq f(x) \leq 2K\int_{-\infty}^{\infty}(|x|^{\alpha-1}+|a|^{\alpha-1})\mu(da). \tag{4.6}
$$

For the squared difference, using a Jensen-like inequality for finite measures, we have,

$$
\begin{aligned}
&|f_n(x)-f(x)|^2 \leq 2|f_n(x)|^2+2|f(x)|^2 \\
\leq \quad & 16K^2\left(\int_{-\infty}^{\infty}(|x|^{\alpha-1}+|a|^{\alpha-1}+1)\,\mu(da)\right)^2 \\
\leq \quad & 16K^2\mu(\mathbb{R})\int_{-\infty}^{\infty}\left((|x|^{\alpha-1}+|a|^{\alpha-1}+1)\right)^2\mu(da) \\
\leq \quad & 48K^2\mu(\mathbb{R})\int_{-\infty}^{\infty}(|x|^{2\alpha-2}+|a|^{2\alpha-2}+1)\,\mu(da)
\end{aligned} \tag{4.7}
$$

Then, similar arguments give

$$
\begin{aligned}
|f_n(X_t)|^2 \quad &\leq \quad 12K^2\mu(\mathbb{R})\int_{-\infty}^{\infty}(|X_t|^{2\alpha-2}+|a|^{2\alpha-2}+1)\mu(da) \quad \text{and} \\
|f(X_t)|^2 \quad &\leq \quad 8K^2\mu(\mathbb{R})\int_{-\infty}^{\infty}(|X_t|^{2\alpha-2}+|a|^{2\alpha-2})\mu(da),
\end{aligned}
$$

and these bounds are independent of n and belong to $L^1(\mathbb{P})$ since $0 < 2\alpha - 2 < \alpha$ and μ is a finite measure with a moment of order $2\alpha - 2$. We can conclude that $f_n(X_t)$ and $f(X_t)$ are elements of $L^2(\mathbb{P})$. Moreover, by dominated convergence, we get

$$
\lim_{n\to\infty}\mathbb{E}\left[|f_n(X_t)-f(X_t)|^2\right] = \mathbb{E}\left[\lim_{n\to\infty}|f_n(X_t)-f(X_t)|^2\right] = 0, \tag{4.8}
$$

so that $f_n(X_t) \to f(X_t)$ in $L^2(\mathbb{P})$, which implies **Step 1**'s assertions.

Let us move to **Step 2**. In this case, we are considering the jumps smaller than one, i.e. $h \leq 1$. To prove that $M^{1,n}$ is a square integrable martingale, according to Ikeda and Watanabe ([IW89] section II.3), we need to show that:

$$m_t^{1,n} := \mathbb{E}\left[\int_0^t \int_{|h|\leq 1} |f_n(X_{s-}+h) - f_n(X_{s-})|^2 \nu(dh)ds\right] < \infty.$$

Since the integrand is positive and $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$-measurable with $\mathscr{X} = (\Omega \times [-1,1] \setminus \{0\} \times [0,t])$, by the Fubini theorem (cf. [Kal02] Theorem 1.27), it suffices to prove the finiteness in any order of integration. Then, using the bound in Corollary 4.3.1 for $|h| \leq 1$ and Lemma 4.3.3, we have:

$$\begin{aligned}
m_t^{1,n} &= \int_0^t \int_{|h|\leq 1} \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-})|^2\right] \nu(dh)ds \\
&\leq \int_0^t \int_{|h|\leq 1} (\mu(\mathbb{R}))^2 c_1 S(\alpha, 2+\varepsilon_0-\alpha) s^{(\alpha-2-\varepsilon_0)/\alpha} |h|^{\alpha+\varepsilon_0} \nu(dh)ds \\
&\leq (\mu(\mathbb{R}))^2 c_1 S(\alpha, 2+\varepsilon_0-\alpha) \int_0^t \int_{|h|\leq 1} s^{(\alpha-2-\varepsilon_0)/\alpha} |h|^{\alpha+\varepsilon_0} \nu(dh)ds \\
&< \infty.
\end{aligned}$$

The result for m_t^1 follows from Corollary 4.3.1 in a similar fashion. Hence, M^1 is also a square integrable martingale.

In order to prove the convergence of $M_t^{1,n} \to M_t^1$ in $L^2(\mathbb{P})$, first note that according to Corollary 4.3.1 we have

$$\begin{aligned}
\mathscr{M}_n^1 &:= \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-}) - (f(X_{s-}+h) + f(X_{s-}))|^2\right] \\
&\leq 2\mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-})|^2\right] + 2\mathbb{E}\left[|f(X_{s-}+h) - f(X_{s-})|^2\right] \\
&\leq 4(\mu(\mathbb{R}))^2 c_1 S(\alpha, 2+\varepsilon_0-\alpha) s^{(\alpha-2-\varepsilon_0)/\alpha} |h|^{\alpha+\varepsilon_0},
\end{aligned}$$

Thus, $(\mathscr{M}_n^1)_{n\geq 1}$ is dominated in $L^1\left((0,t)\times[-1,1]\setminus\{0\}, \mathscr{B}((0,t)\times[-1,1]\setminus\{0\}), \mathrm{Leb}\otimes\nu\right)$.

We know that $(M_t^{1,n} - M_t^1)$ is a square integrable martingale for any $n \in \mathbb{N}$, then using Itô's isometry ([App09] p. 223) and dominated convergence theorem for the sequence $(\mathscr{M}_n^1)_{n\geq 1}$ we have:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\left|M_t^{1,n} - M_t^1\right|^2\right] \\
&= \lim_{n\to\infty} \int_0^t \int_{|h|\leq 1} \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-}) - (f(X_{s-}+h) - f(X_{s-}))|^2\right] \nu(dh)ds \\
&= \int_0^t \int_{|h|\leq 1} \lim_{n\to\infty} \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-}) - (f(X_{s-}+h) - f(X_{s-}))|^2\right] \nu(dh)ds \\
&= 0.
\end{aligned}$$

The convergence to zero of the last equation is a consequence of equation (4.8) in **Step 1**. So that $M_t^{1,n} \to M_t^1$ in $L^2(\mathbb{P})$, ending with **Step 2**.

For **Step 3**, we are considering the jumps greater than one, i.e. $h > 1$. To prove that $M_t^{2,n}$ is a martingale, following Ikeda and Watanabe ([IW89] section II.3) we must show:

$$m_t^{2,n} := \mathbb{E}\left[\int_0^t \int_{|h|>1} |f_n(X_{s-}+h) - f_n(X_{s-})| \, \nu(dh)ds\right] < \infty.$$

Since the integrand is positive and $(\mathscr{X}, \mathscr{B}(\mathscr{X}))$-measurable with $\mathscr{X} = (\Omega \times [-1,1]^c \times [0,t])$, by the Fubini theorem it suffices to prove the finiteness in any order of integration. Then, using the bound in Corollary 4.3.1 for $|h| > 1$ and Lemma 4.3.3, we have:

$$\begin{aligned} m_t^{2,n} &= \int_0^t \int_{|h|>1} \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-})|\right] \nu(dh)ds \\ &\leq \mu(\mathbb{R}) \int_0^t \int_{|h|>1} c_2 |h|^{\alpha-1} \nu(dh)ds \\ &< \infty. \end{aligned}$$

The result for m_t^2 follows by the same bounds in Corollary 4.3.1, so that M_t^2 is also a martingale. As in the previous step, to prove the convergence of $M^{2,n} \to M^2$ in $L^1(\mathbb{P})$, first note that according to Corollary 4.3.1 we have

$$\begin{aligned} \mathscr{M}_n^2 &:= \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-}) - (f(X_{s-}+h) + f(X_{s-}))|\right] \\ &\leq \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-})|\right] + \mathbb{E}\left[|f(X_{s-}+h) - f(X_{s-})|\right] \\ &\leq 2\mu(\mathbb{R}) c_2 |h|^{\alpha-1}. \end{aligned}$$

Thus, $(\mathscr{M}_n^2)_{n\geq 1}$ is dominated in $L^1((0,t) \times [-1,1]^c, \mathscr{B}((0,t) \times [-1,1]^c), \mathrm{Leb} \otimes \nu))$.

We know that $(M_t^{2,n} - M_t^2)$ is a stochastic integral with respect to a Poisson random measure for any $n \in \mathbb{N}$, then using Campbell's theorem ([Kin93] section 3.2) and dominated convergence theorem for the sequence $(\mathscr{M}_n^2)_{n\geq 1}$ we have:

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left[\left|M_t^{2,n} - M_t^2\right|\right] \leq \\ &\int_0^t \int_{|h|>1} \lim_{n\to\infty} \mathbb{E}\left[|f_n(X_{s-}+h) - f_n(X_{s-}) - (f(X_{s-}+h) - f(X_{s-}))|\right] \nu(dh)ds. \\ &= 0. \end{aligned}$$

The convergence to zero of the last equation is a consequence of equation (4.8) in **Step 1**. So that $M_t^{2,n} \to M_t^2$ in $L^1(\mathbb{P})$, ending with **Step 3**.

By **Step 2** and **Step 3** we conclude that M^n and M in equation (4.5) are martingales and $M_t^n \to M_t$ in L^1.

Finally, for **Step 4**, we have from equation (4.5) that:

$$V_t^n = f_n(X_t) - f_n(X_0) - M_t^n \quad \overset{L^1(\mathbb{P})}{\longrightarrow} \quad f(X_t) - f(X_0) - M_t,$$

as $n \to \infty$, so that the limit $\lim_{n\to\infty} V_t^n(X_t) \in L^1(\mathbb{P})$. We just need to verify that this limit coincides with the one stated in the theorem.

We know that $f_n = F * (\rho_n * \mu) \in C^{\infty}_{1+,b} \cap C^{\alpha,c_-,c_+}$ is positive and measurable and that $\rho_n * \mu$ is a finite measure with compact support.

Then $\mathscr{L} f_n$ is well-defined, positive and measurable as well. So, by the occupation formula, we have:

$$V_t^n = \int_0^t \mathscr{L} f_n(X_s) ds = \int_{-\infty}^{\infty} L_t^a \mathscr{L} f_n(a) da.$$

Since $L_t^a(\omega) \in C_c$ for almost all $\omega \in \Omega$, Lemma 4.3.2 tells us that

$$V_t^n = \int_{-\infty}^{\infty} L_t^a (\mu * \rho_n)(da),$$

and since $\rho_n \to \delta$ weakly as $n \to \infty$, then $(\mu * \rho_n) \to \mu$ weakly as $n \to \infty$ as well. Hence,

$$\left| \int_{-\infty}^{\infty} L_t^a(\mu * \rho_n)(da) - \int_{-\infty}^{\infty} L_t^a \mu(da) \right| \to 0, \quad \text{as } n \to \infty.$$

Steps 1-4 finish the proof of Theorem 4.3.1. □

For the first application, we have the Tanaka formula for asymmetric strictly stable processes.

Corollary 4.3.2 (Tanaka formula)**.** *Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, and consider a strictly stable process $X \sim S_\alpha(c_-, c_+)$. Then, the Tanaka formula is satisfied:*

$$F^{\alpha,c_-,c_+}(X_t - a) = F^{\alpha,c_-,c_+}(X_0 - a) + M_t^a(X) + L_t^a(X), \tag{4.9}$$

where $L_t^a(X)$ is the occupational local time at a up to time t of X and $M_t^a(X)$ is a square integrable martingale given by

$$M_t^a(X) = \int_0^t \int_{\mathbb{R}_0} [F^{\alpha,c_-,c_+}(X_{s-} - a + h) - F^{\alpha,c_-,c_+}(X_{s-} - a)] \tilde{N}(ds, dh).$$

Proof. Consider the unitary measure concentrated in a, that is $\delta_a(E) = 1$ if $a \in E$ and zero otherwise, with $f(x) = (F^{\alpha,c_-,c_+} * \delta_a)(x) = F^{\alpha,c_-,c_+}(x - a)$, using the occupational Meyer-Itô theorem we have:

$$\begin{aligned} F^{\alpha,c_-,c_+}(X_t - a) &= F^{\alpha,c_-,c_+}(X_0 - a) \\ &+ \int_0^t \int_{\mathbb{R}_0} [F^{\alpha,c_-,c_+}(X_{s-} - a + h) - F^{\alpha,c_-,c_+}(X_{s-} - a)] \tilde{N}(ds, dh) \\ &+ \int_{-\infty}^{\infty} L_t^x(X) \delta_a(dx), \\ &= F^{\alpha,c_-,c_+}(X_0 - a) + M_t^a + L_t^a(X). \end{aligned}$$

□

Even though the subclass of functions $\mathscr{C}^{\alpha,c_-,c_+}$ is enough to prove important results such as as the Meyer-Itô and the Tanaka formula, we are limited to functions with a compactly supported measure in their decomposition. However, in the next section we are going to combine these techniques and the fact that we can find explicitly the infinitesimal generator of power functions to give a decomposition of power functions of the type $|X_t - x|^\gamma$.

4.4 Power decomposition of $|X_t - x|^\gamma$

The objective of this section is to generalize the work of Salminen and Yor [SY05] and of Engelbert and Kurenok [EK19], regarding the Doob-Meyer decomposition of absolute powers of symmetric stable processes to the asymmetric case.

Our first step will be to explicitly compute the infinitesimal generator of the power functions in Lemma 4.3.1. Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$ not both zero and $\alpha - 1 < \gamma < \alpha$. From Lemma 4.3.1 we know that $f_\pm^\gamma$ belong to Φ' and can be identified with the following fractional integrals:

$$\begin{aligned} f_+^\gamma(x) &= \Gamma(\gamma+1) I_-^{\gamma+1}\delta(x), \\ f_-^\gamma(x) &= \Gamma(\gamma+1) I_+^{\gamma+1}\delta(x). \end{aligned}$$

Consider the infinitesimal generator evaluated at $f_+^\gamma(x)$: with the constants $M_\pm$ as defined in Proposition 4.2.1, we have

$$\begin{aligned} \mathscr{L} f_+^\gamma(x) &= M_- D_-^\alpha f_+^\gamma(x) + M_+ D_+^\alpha f_+^\gamma(x) \\ &= \Gamma(\gamma+1) M_- D_-^\alpha I_-^{\gamma+1}\delta(x) + \Gamma(\gamma+1) M_+ D_+^\alpha I_-^{\gamma+1}\delta(x) \end{aligned}$$

Using the fractional composition formulas in Lemma 4.2.1, we get

$$\begin{aligned} \mathscr{L} f_+^\gamma(x) &= \Gamma(\gamma+1) M_- I_-^{\gamma-\alpha+1}\delta(x) + \Gamma(\gamma+1) M_+ \frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)} I_-^{\gamma-\alpha+1}\delta(x) \\ &\quad + \Gamma(\gamma+1) M_+ \frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)} I_+^{\gamma-\alpha+1}\delta(x) \\ &= \frac{\Gamma(\gamma+1) M_-}{\Gamma(\gamma-\alpha+1)} f_+^{\gamma-\alpha}(x) + \frac{\Gamma(\gamma+1) M_+}{\Gamma(\gamma-\alpha+1)} \frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)} f_+^{\gamma-\alpha}(x) \\ &\quad + \frac{\Gamma(\gamma+1) M_+}{\Gamma(\gamma-\alpha+1)} \frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)} f_-^{\gamma-\alpha}(x). \end{aligned}$$

For the function $f_-^\gamma(x)$, we can proceed similarly to get

$$\begin{aligned} \mathscr{L} f_-^\gamma(x) &= \Gamma(\gamma+1) M_- \frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)} I_-^{\gamma-\alpha+1}\delta(x) \\ &\quad + \Gamma(\gamma+1) M_- \frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)} I_+^{\gamma-\alpha+1}\delta(x) + \Gamma(\gamma+1) M_+ I_+^{\gamma-\alpha+1}\delta(x) \\ &= \frac{\Gamma(\gamma+1) M_-}{\Gamma(\gamma-\alpha+1)} \frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)} f_+^{\gamma-\alpha}(x) \\ &\quad + \frac{\Gamma(\gamma+1) M_-}{\Gamma(\gamma-\alpha+1)} \frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)} f_-^{\gamma-\alpha}(x) + \frac{\Gamma(\gamma+1) M_+}{\Gamma(\gamma-\alpha+1)} f_-^{\gamma-\alpha}(x). \end{aligned}$$

Before we prove Theorem 4.4.1, we need to understand the constants $k_\pm(\alpha, \gamma, c_-, c_+)$ that are used there. They play an important role in the bounded variation part of the power decomposition (4.10), because in order to be an increasing process, both need to be positive. The following lemma states the critical exponent γ from which both $k_\pm(\alpha, \gamma, c_-, c_+)$ are positive. Recall the definition of c in Corollary 4.4.1.

Lemma 4.4.1. *Let $\alpha \in (1,2)$, $\gamma \in (\alpha-1,\alpha)$ and $k_{\pm}(\alpha,\gamma,c_-,c_+)$ as in Theorem 4.4.1. Define*

$$\beta(a,c) := \frac{1}{\pi}\arccos\left(\frac{c^2(1-a^2)-(1+ac)^2}{c^2(1-a^2)+(1+ac)^2}\right) \in (\alpha-1,1),$$

where $a = \cos(\alpha\pi)$ and $c = \frac{\min(c_-,c_+)}{\max(c_-,c_+)}$. Then, if $c_- < c_+$ we have that $k_-(\alpha,\gamma,c_-,c_+)$ is positive for all $\gamma \in (\alpha-1,\alpha)$ while $k_+(\alpha,\gamma,c_-,c_+)$ is negative if $\gamma \in (\alpha-1,\beta(a,c))$ and positive if $\gamma \in (\beta(a,c),1)$. The same conclusion follows for $c_+ < c_-$ after switching the roles of the $k_{\pm}(\alpha,\gamma,c_-,c_+)$.

Proof. Assume that $c_- < c_+$. First, we prove $k_-(\alpha,\gamma,c_-,c_+) > 0$ for all $\gamma \in (\alpha-1,\alpha)$. Note that:

$$\begin{aligned}
k_-(\alpha,\gamma,c_-,c_+) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\left[M_+\frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)}+M_-\frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)}+M_+\right]\\
&= \frac{\Gamma(\gamma+1)M_+}{\Gamma(\gamma-\alpha+1)\sin((\gamma-\alpha+1)\pi)}\left[\sin(-\alpha\pi)+c\sin((\gamma+1)\pi)+\sin((\gamma-\alpha+1)\pi)\right]\\
&= \frac{\Gamma(\gamma+1)c_+\Gamma(-\alpha)}{\Gamma(\gamma-\alpha+1)\sin((\gamma-\alpha+1)\pi)}\left[\sin(-\alpha\pi)-c\sin(\gamma\pi)-\sin((\gamma-\alpha)\pi)\right].
\end{aligned}$$

Since we have

$$\frac{\Gamma(\gamma+1)c_+\Gamma(-\alpha)}{\Gamma(\gamma-\alpha+1)\sin((\gamma-\alpha+1)\pi)} > 0,$$

for all $\alpha \in (1,2)$ and $\gamma \in (\alpha-1,\alpha)$, then $k_-(\alpha,\gamma,c_-,c_+) > 0$ is equivalent to:

$$h_-(\gamma) := \sin(-\alpha\pi) - c\sin(\gamma\pi) - \sin((\gamma-\alpha)\pi) > 0,$$

for all $\gamma \in (\alpha-1,\alpha)$. Lemma A.0.2 tells us that $h_{\pm}$ are 2-periodic. Moreover, we have that:

$$\begin{aligned}
h_-(0) &= \sin(-\alpha\pi) - \sin(-\alpha\pi) = 0,\\
h_-(\alpha-1) &= \sin(-\alpha\pi) - c\sin((\alpha-1)\pi)\\
&= \sin(-\alpha\pi)(1-c)\\
&> 0,
\end{aligned}$$

because $c < 1$ and $\alpha \in (1,2)$. This means that $h_-(\gamma)$ has just one zero in $(0,2)$ and it is before $\alpha-1$, so that $h(\gamma) > 0$ for all $\gamma \in (\alpha-1,\alpha)$, as well as $k_-(\alpha,\gamma,c_-,c_+) > 0$ in the same interval.

We will prove in a similar way the change of signs of $k_+(\alpha,\gamma,c_-,c_+)$. Note that, as in the previous case, we just need to analyze the change of signs of the function:

$$h_+(\gamma) := c\sin(-\alpha\pi) - \sin(\gamma\pi) - c\sin((\gamma-\alpha)\pi).$$

Since

$$h_+(0) := c\sin(-\alpha\pi) - c\sin(-\alpha\pi) = 0,$$

there must be just one zero in $(0,2\pi)$, this zero is precisely $\gamma = \beta(a,c)$ ([Fou13] Lemma 9). But, by definition $\beta(a,c) \in (\alpha-1,1)$, this means that:

$$\begin{aligned}
h_+(\gamma) &< 0, \quad \text{if } \gamma \in (\alpha-1,\beta(a,c)) \text{ and}\\
h_+(\gamma) &\geq 0, \quad \text{if } \gamma \in [\beta(a,c),1).
\end{aligned}$$

Finally, when $c_+ < c_-$, just note that since $k_+(\alpha,\gamma,c_-,c_+) = k_-(\alpha,\gamma,c_+,c_-)$ we can use the same proof. □

We are ready to prove the power decomposition theorem. These results are a generalization of the works of Salminen and Yor [SY05] and of Engelbert and Kurenok [EK19]. The proof of the decomposition uses the Tanaka formula for asymmetric stable processes (4.9) and relies on the representation of the infinitesimal generator of a power function given in Lemma 4.3.1. Note that in [SY05] it was easy to find the measure which could recover the power decomposition in the symmetric case and for the generalization we made direct use of fractional calculus to find the relevant measure needed for the asymmetric case.

Theorem 4.4.1 (Power decomposition). *Let $\alpha \in (1,2)$ and $c_-, c_+ \geq 0$, not both zero, and consider a strictly stable process $X \sim S_\alpha(c_-, c_+)$. Then for all $x \in \mathbb{R}$ and $\gamma \in (\alpha-1, \alpha)$ we have the decomposition*

$$\begin{aligned} |X_t - x|^\gamma &= |X_0 - x|^\gamma + \int_0^t \int_{\mathbb{R}_0} \left[|X_{s-} - x + h|^\gamma - |X_{s-} - x|^\gamma \right] \tilde{N}(ds, dh) \\ &+ \int_0^t |X_s - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{X_s > x\}} + k_+ \mathbb{1}_{\{X_s < x\}} \right] ds, \end{aligned} \tag{4.10}$$

where $k_\pm := k_\pm(\alpha, \gamma, c_-, c_+)$ are given by

$$\begin{aligned} k_- &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} \left[M_+ \frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)} + M_- \frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)} + M_+ \right] \quad \textit{and} \\ k_+ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} \left[M_- \frac{\sin(-\alpha\pi)}{\sin((\gamma-\alpha+1)\pi)} + M_+ \frac{\sin((\gamma+1)\pi)}{\sin((\gamma-\alpha+1)\pi)} + M_- \right]. \end{aligned}$$

Note that the last integral in (4.10) could be written in terms of the local time as

$$\int_{-\infty}^{\infty} |a - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{a > x\}} + k_+ \mathbb{1}_{\{a < x\}} \right] L_t^a da.$$

The main result of Engelbert and Kurenok [EK19] is that this decomposition corresponds to a submartingale, thus providing the Doob-Meyer decomposition for $|X_t - x|^\gamma$, when X is a symmetric stable process. However, if asymmetry in the jumps of the stable process is allowed, this decomposition will not be in general a submartingale. By direct inspection, the last term of the decomposition will correspond to an increasing process if and only if $k_\pm \geq 0$.

The constants $k_\pm$ have been found and used by Fournier [Fou13] by other means and in a different context. Fournier proved pathwise uniqueness for SDEs driven by an asymmetric strictly stable process and, in order to use the Gronwall inequality, he defined a constant $\beta(a,c) \in (\alpha-1, 1)$, where $a = \cos(\pi\alpha)$ and $c = c_-/c_+$, assuming $0 < c_- < c_+$. Then, he proved that $k_+ = 0$ for $\gamma = \beta(a,c)$.

We will prove in Lemma 4.4.1 that, in fact, both $k_\pm$ are nonnegative for all $\gamma \geq \beta(a,c)$ and otherwise one of them is negative.

Proof. (of Theorem 4.4.1) Recall that from Lemma 4.3.1 we know that for $f(y) = |y|^\gamma$, the infinitesimal generator associated to f is given by:

$$\mu(dy) = \left(k_- |y|^{\gamma-\alpha} \mathbb{1}_{\{y>0\}} + k_+ |y|^{\gamma-\alpha} \mathbb{1}_{\{0<y\}} \right) dy.$$

Taking the Tanaka formula (4.9) at the level a and integrating both sides by $\mu^x(da)$ (the measure μ translated by x) we have:

$$\int_{-\infty}^{\infty} F(X_t - a) \mu^x(da) = \int_{-\infty}^{\infty} F(X_0 - a) \mu^x(da) + \int_{-\infty}^{\infty} M_t^a(X) \mu^x(da) + \int_{-\infty}^{\infty} L_t^a(X) \mu^x(da).$$

Note that the representation of f as a member of the Class $\mathscr{C}^{\alpha,c_-,c_+}$ is precisely $F * \mu$. We will now use a version of Fubini's theorem for compensated Poisson random measures and apply it to the small jumps of $M^a(X)$ above. See [MR15, Lemma A.1.2]. We need to verify some integrability assumptions to apply it, which are (4.11) and (4.12) below. Applying the Fubini theorem, we get

$$\begin{aligned}
|X_t - x|^\gamma &= |X_0 - x|^\gamma + \int_0^t \int_{\mathbb{R}_0} \left[|X_{s-} - x + h|^\gamma - |X_{s-} - x|^\gamma\right] \tilde{N}(ds, dh) \\
&+ \int_{-\infty}^{\infty} |a - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{a>x\}} + k_+ \mathbb{1}_{\{a<x\}}\right] L_t^a da.
\end{aligned}$$

Using the occupational formula for the local time, the last integral is equivalent to

$$\int_0^t |X_s - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{X_s>x\}} + k_+ \mathbb{1}_{\{X_s<x\}}\right] ds.$$

This finishes the proof modulo showing that the first integral is a martingale and the applicability of Fubini's theorem. The proof of the martingale character will follow the ideas of [EK19, Section 3]. Incidentally, the same argument will justify the application of Fubini's theorem above. We can identify two cases depending on the size of the jump:

$$\begin{aligned}
M_t^\gamma &= \int_0^t \int_{\mathbb{R}_0} \left[|X_{s-} - x + h|^\gamma - |X_{s-} - x|^\gamma\right] \tilde{N}(ds, dh) \\
&= M_t^{\gamma,1} + M_t^{\gamma,2} \\
&:= \int_0^t \int_{|h| \le |X_{s-} - x|} \left[|X_{s-} - x + h|^\gamma - |X_{s-} - x|^\gamma\right] \tilde{N}(ds, dh) \\
&\quad + \int_0^t \int_{|h| > |X_{s-} - x|} \left[|X_{s-} - x + h|^\gamma - |X_{s-} - x|^\gamma\right] \tilde{N}(ds, dh).
\end{aligned}$$

In order to prove that $M^{1,\gamma}$ is a square integrable martingale, according to Ikeda and Watanabe ([IW89] section II.3) we need to show that:

$$m_t^{1,\gamma} := \mathbb{E}\left[\int_0^t \int_{|h| \le |X_{s-} - x|} \left||X_{s-} - x + h|^\gamma - |X_{s-} - x|^\gamma\right|^2 \nu(dh) ds\right] < \infty. \tag{4.11}$$

Take $\bar{c} = c_- \vee c_+$, then the intensity measure $\nu_{\bar{c}}(dh) = \bar{c}|h|^{-\alpha-1}dh$ is greater than the intensity measure $\nu(dh)$, corresponding to X_t, and if we consider the change of variable $h = (X_{s-} - x)u$ we have:

$$\begin{aligned}
m_t^{1,\gamma} &\le \mathbb{E}\left[\int_0^t \int_{|(X_{s-} - x)u| \le |X_{s-} - x|} \frac{\bar{c}\left||X_{s-} - x + (X_{s-} - x)u|^\gamma - |X_{s-} - x|^\gamma\right|^2}{|X_{s-} - x|^\alpha |u|^{\alpha+1}} duds\right] \\
&= \mathbb{E}\left[\int_0^t \int_{|u| \le 1} |X_{s-} - x|^{2\gamma} \left(|1+u|^\gamma - 1\right)^2 \frac{\bar{c}}{|X_{s-} - x|^\alpha |u|^{\alpha+1}} duds\right] \\
&= \mathbb{E}\left[\int_0^t |X_{s-} - x|^{2\gamma-\alpha} ds\right] \int_{|u| \le 1} \left(|1+u|^\gamma - 1\right)^2 \frac{\bar{c}}{|u|^{\alpha+1}} du.
\end{aligned}$$

Since $-1 < \alpha - 2 < 2\gamma - \alpha < \alpha$, the integral $\mathbb{E}\left[\int_0^t |X_{s-} - x|^{2\gamma-\alpha} ds\right]$ is finite for all $t \ge 0$. It remains to check that the second integral is finite. Consider the auxiliary function $g(u) = |1+u|^\gamma$, and note that for any $u \in (-1,1)$ we have that $g(u) = (1+u)^{\gamma}$, which is differentiable. By the mean value theorem we can choose $u_* \in (-1,0)$ and $u^* \in (0,1)$ such that:

$$f(u) - f(0) = \begin{cases} f'(u_*)u & -1 < u < 0, \\ f'(u^*)u & 0 < u < 1, \end{cases}$$

This corresponds to:

$$(1+u)^{\gamma} - 1 = \begin{cases} \gamma(1+u_*)^{\gamma-1}u & -1 < u < 0, \\ \gamma(1+u^*)^{\gamma-1}u & 0 < u < 1. \end{cases}$$

We get the following bound for any $u \in (-1,1)$:

$$|(1+u)^{\gamma} - 1| \leq \gamma c_1(\gamma)|u|,$$

where $c_1(\gamma) = \max((1+u_*)^{\gamma-1}, (1+u^*)^{\gamma-1})$. Then, we have that

$$\begin{aligned} \int_{|u|\leq 1} \left(|1+u|^{\gamma} - 1\right)^2 \frac{\bar{c}}{|u|^{\alpha+1}} du &\leq \gamma^2 c_1^2(\gamma) \int_{|u|\leq 1} \bar{c}|u|^{1-\alpha} du \\ &\leq \gamma^2 c_1^2(\gamma) \frac{2\bar{c}}{2-\alpha} \\ &< \infty. \end{aligned}$$

So that $m_t^{1,\gamma}$ for any $t \geq 0$ and $M^{1,\gamma}$ is a square integrable martingale.

Now, to prove that $M^{2,\gamma}$ is a martingale, according to Ikeda and Watanabe ([IW89] section II.3) we need to show that:

$$m_t^{2,\gamma} := \mathbb{E}\left[\int_0^t \int_{|h|>|X_{s-}-x|} \left||X_{s-} - x + h|^{\gamma} - |X_{s-} - x|^{\gamma}\right| \nu(dh)ds\right] < \infty. \tag{4.12}$$

Similarly, we have:

$$\begin{aligned} m_t^{2,\gamma} &\leq \mathbb{E}\left[\int_0^t \int_{|(X_{s-}-x)u|>|X_{s-}-x|} \frac{c\left||X_{s-} - x + (X_{s-}-x)u|^{\gamma} - |X_{s-}-x|^{\gamma}\right|}{|X_{s-}-x|^{\alpha}|u|^{\alpha+1}} duds\right] \\ &= \mathbb{E}\left[\int_0^t \int_{|u|>1} |X_{s-}-x|^{\gamma}\left||1+u|^{\gamma} - 1\right| \frac{c}{|X_{s-}-x|^{\alpha}|u|^{\alpha+1}} duds\right] \\ &= \mathbb{E}\left[\int_0^t |X_{s-}-x|^{\gamma-\alpha} ds\right] \int_{|u|>1} \left||1+u|^{\gamma} - 1\right| \frac{c}{|u|^{\alpha+1}} du \\ &< \infty, \end{aligned}$$

since $\gamma - \alpha \in (-1,0)$ and this moment of X_t is finite for any $t \geq 0$, the expectation is finite. To see the last integral is finite, just note that $\left||1+u|^{\gamma} - 1\right|$ behaves like $|u|^{\gamma}$ as $|u| \to \infty$. Then, we have that $m_t^{2,\gamma}$ is finite for any $t \geq 0$ and we can conclude that $M^{2,\gamma}$ is a martingale. This allow us to conclude that $M^{\gamma} = M^{1,\gamma} + M^{2,\gamma}$ is a martingale. □

Finally, we state when this power decomposition is a submartingale or just a semimartingale.

Corollary 4.4.1. *Let $a = \cos(\pi\alpha)$ and $c = (c_- \wedge c_+)/(c_- \vee c_+)$. Then the power decomposition in Theorem 4.4.1 for the process $|X_t - x|^{\gamma}$ is a submartingale if $\gamma \in [\beta(a,c), \alpha)$; whereas, for $\gamma \in (\alpha - 1, \beta(a,c))$ it is a semimartingale, whose finite variation part is not monotone.*

Proof. From the Lemma 4.4.1 and Theorem 4.4.1 the last integral is a non decreasing process if and only if $\gamma \in [\beta(a,c), \alpha)$, by Lemma 4.4.1, so that we get a Doob-Meyer decomposition for $|X_t - x|^\gamma$. In the other case, $\gamma \in (\alpha - 1, \beta(a,c))$, this results in a semimartingale instead of a submartingale. □

Chapter 5

SDEs driven by stable processes

The results given for strictly stable processes can be extended to driftless stochastic differential equations driven by strictly stable processes. This extension is more or less straightforward due to a beautiful result of Kallenberg in [Kal92], which states conditions in order to identify a stochastic integral with respect to a stable process as a time-changed stable process. In the case of symmetric stable processes, this result is due to Rosinski and Woyczynski [RW86].

In this sense, by appropriately changing the time we will be able to prove the analogous results for driftless SDEs driven by stable processes. In the case where the integrand is non-negative, we provide a definition of an occupational local time for the corresponding stochastic integral with respect to a stable process; moreover, we provide a proof for the Tanaka, Meyer-Itô and the power decomposition for these integrals.

Finally, we provide some insights for the case where the integrand can take positive and negative values; and how it could be helpful to prove pathwise uniqueness of solutions to driftless SDEs driven by stable process.

5.1 Stochastic integrals with respect to stable processes

As we have pointed out before, the case of Brownian motion is the most studied stable process. For instance, we have the Dambis-Dubins-Schwarz theorem, which asserts that a continuous local martingale M can be written as a Brownian motion in a suitable random time change. In fact, $M = B \circ [M]$ a.s., where $[M]$ is the quadratic variation of M, with $M_0 = 0$ and B a Brownian motion possibly defined in some extension of the original probability space.

A similar result is proved in the case of Poisson processes. If $\{X_t\}_{t\geq 0}$ is a non-homogeneous Poisson process with parameter $\lambda(t)$ and intensity function given by $\Lambda(t) = \int_0^t \lambda(s)ds$. If we consider the function

$$\Lambda^{-1}(t) = \inf\{u \geq 0 | \Lambda(u) = t\}.$$

Then the process $\left\{X_{\Lambda^{-1}(t)}\right\}_{t\geq 0}$ is an homogeneous Poisson process with parameter $\lambda = 1$.

In a more general setting, Papangelou [Pap72] and Meyer [Mey71] proved that any simple and quasi-left continuous point Process N becomes a Poisson point process after using the compensator of N as a time change. That is, considering N with compensator $\tilde{N}$, there exists a Poisson point process π such that $N = \pi \circ \tilde{N}$ a.s.

Finally, in the case of stable processes, Kallenberg showed in [Kal92] that stochastic integration followed by a suitable random time change transform the marked point process of jump times and sizes for the original stable process into a Poisson point process on the new time scale with the same distribution and concluding that time-changed integral processes are again stable processes.

Through all the following sections we are considering a strictly stable process $X \sim S_\alpha(c_-, c_+)$ with $\alpha \in (1,2)$ and some $c_-, c_+ \geq 0$ not both zero. Moreover, taking a strictly positive bounded measurable function σ, then we can consider the process Z given by:

$$Z_t = \int_0^t \sigma(Z_{s-}) dX_s, \quad t \geq 0.$$

Given these conditions, following Kallenberg [Kal92] (who works in the more general setting of predictable integrands), there exists a $X' \stackrel{d}{=} X$ such that with the following random change of time

$$A_t = \int_0^t \sigma^\alpha(Z_{u-}) du,$$

provided this integral is meaningful, i.e. $\sigma \circ Z^-$ is locally L^α-integrable, we have that a.s.

$$\int_0^t \sigma(Z_{s-}) dX_s = X'_{A_t}, \quad t \geq 0. \tag{5.1}$$

This is a remarkable result, since through this random time change we will be able to extend our principal results from strictly stable processes to stochastic integrals driven by them in a simple way. Namely, we can state a Tanaka Formula, Occupational Meyer-Itô and the power decomposition we previously proved. Note that the integrability condition will be satisfied trivially if σ is bounded.

We can see this as an extension of the self-similarity property, if we consider $X \sim S_\alpha(c_-, c_+)$ and $c > 0$ then $cX_t \stackrel{d}{=} X_{c^\alpha t}$.

5.2 Driftless SDEs driven by stable processes

We are going to consider the following SDE:

$$Z_t = z_0 + \int_0^t \sigma(Z_{s-}) dX_s. \tag{5.2}$$

A priori, the above equation makes sense whenever $\sigma \circ Z^-$ is locally in L^α. However, a negative σ will change the signs of the jumps of Z requiring two stable processes in Kallenberg's time-change representation of stochastic integrals. It will therefore be more convenient to consider only non-negative functions σ. In the symmetric case, such considerations are unnecessary.

In the case of continuous martingales, i.e. when X is a Brownian motion, the notion of the local time of Brownian stochastic integrals is well defined through their quadratic variation $d\langle Z \rangle_t =$

$\sigma^2(Z_s)ds$. Nevertheless, since a stochastic integral with respect to a stable process has zero continuous component, the semimartingale local time is zero. Instead, we will provide an occupational local time for stochastic integrals of the form:

$$Z_t = z_0 + \int_0^t H_s dX_s, \tag{5.3}$$

where H_s is a predictable strictly positive process, whose paths lie locally in L^α a.s.

Definition 5.2.1. *The family of random variables $\{L_t^a(Z)\}$ for $a \in \mathbb{R}$ and $t \geq 0$ will be called the occupational local time of* (5.3) *if for any bounded Borel measurable function $f : \mathbb{R} \to [0,\infty)$ and $t \geq 0$ and $z_0 \in \mathbb{R}$, the following occupation formula is satisfied:*

$$\int_0^t f(Z_s) H_s^\alpha ds = \int_{-\infty}^{\infty} f(a) L_t^{a+z_0}(Z) da.$$

As a matter of fact, due to Kallenberg's identification of stochastic integrals driven by stable processes as a time-changed stable process, we can prove this occupational local time for Z is well defined and satisfies all the nice properties of the local time for X.

Proposition 5.2.1. *Consider the process Z as in equation* (5.3)*, where H is predictable, strictly positive and with paths locally in L^α. Then, Z admits a jointly continuous occupational local time.*

Proof. First, let us consider the case $z_0 = 0$ and recall Kallenberg's random time change is given by $A_t := \int_0^t H_s^\alpha ds$, which gives us $Z = X' \circ A$, where $X' \stackrel{d}{=} X$. We can consider the (right-continuous) inverse $(T_s)_{s\geq 0}$ of A. Since $H(u) > 0$, then $A_{T_s} = s$ and $T_{A_t} = t$; moreover, we have $dA_t = H_t^\alpha dt$.

Using the random time change equivalence between Z and X, and the occupational formula for the local time of X' we have:

$$\begin{aligned} \int_0^t f(Z_s) H_s^\alpha ds &= \int_0^t f(X'_{A_s}) dA_s \\ &= \int_0^{A_t} f(X'_u) du \\ &= \int_{-\infty}^{\infty} f(a) L_{A_t}^a(X') da. \end{aligned}$$

This means that $L_{A_t}^a(X')$ satisfies the occupational formula for Z, so that defining $L_t^a(Z)$ by means of

$$L_t^a(Z) = L_{A_t}^a(X'), \quad a.s.,$$

we obtain a jointly continuous version of the occupational local time of Z, since $L_t^a(X)$ is jointly continuous (cf. Boylan [Boy64] and Barlow [Bar88]). Moreover, its straightforward to check that when $z_0 \neq 0$ we have that $L_t^{a+z_0}(Z) = L_{A_t}^a(X')$. □

Note that when $\alpha \to 2$, the definition of the local time $L(Z)$ is consistent with the local time of a stochastic integral with respect to a Brownian motion. Also, if we take $H \equiv \sigma$, a constant function, we recover the definition of the local time of a stable process times this constant.

5.3 Tanaka formula, power decomposition and Meyer-Itô formula

In this section we will generalize Tanaka formula, the power decomposition and the Meyer-Itô formula from the stable case to the SDE driven by a stable process given by (5.2). We leave the precise description of the local martingale component in such formulae to future work. The description we are looking for would generalize that obtained for the stable process.

Theorem 5.3.1. *Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, consider $X \sim S_\alpha(c_-, c_+)$ and let $F(x)$ as in equation (4.3). Consider the process Z as equation (5.2) with σ a strictly positive bounded measurable function. Then*

$$F(Z_t - a) = F(z_0 - a) + M_t^F + L_t^a(Z), \tag{5.4}$$

where M_t^F is a martingale and $L_t^a(Z)$ is the local time of Z at level a up to time t.

Proof. Consider $X' \stackrel{d}{=} X$ given in the stochastic integral representation of Z as in equation (5.1). The proof relies on the Tanaka formula for X' at level $x = a - z_0$ after an evaluation at time $A_t = \int_0^t \sigma^\alpha(Z_{s-})ds$. First, by the Tanaka formula, we have:

$$\begin{aligned} F(X'_t - (a - z_0)) &= F(X'_0 - (a - z_0)) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \left[F(X'_{u-} - (a - z_0) + w) - F(X'_{u-} - (a - z_0))\right] \tilde{N}'(du, dw) \\ &\quad + L_t^{a - z_0}(X'), \end{aligned}$$

where $\tilde{N}'(du, dw) = N'(du, dw) - \nu(dw)du$ is the compensated Poisson random measure associated to X'.

Now, by evaluating at time A_t:

$$\begin{aligned} F(X'_{A_t} - (a - z_0)) &= F(X'_{A_0} - (a - z_0)) \\ &\quad + \int_0^{A_t} \int_{\mathbb{R}_0} \left[F(X'_{u-} - (a - z_0) + w) - F(X'_{u-} - (a - z_0))\right] \tilde{N}'(du, dw) \\ &\quad + L_{A_t}^{a - z_0}(X'). \end{aligned}$$

Finally, using the equivalence between X' and Z after the change of time we have:

$$F(Z_t - a) = F(z_0 - a) + M_t^F + L_t^a(Z),$$

where

$$M_t^F := \int_0^{A_t} \int_{\mathbb{R}_0} \left[F(X'_{u-} - (a - z_0) + w) - F(X'_{u-} - (a - z_0))\right] \tilde{N}'(du, dw)$$

In order to proof that M_t^F is a martingale, note that it is an integral with respect to a Poisson random measure and by Ikeda and Watanabe [IW89] we just need to show that:

$$\mathbb{E}\left[\int_0^{A_t} \int_{|w|>1} \left|F(X'_{u-} + z_0 - a + w) - F(X'_{u-} + z_0 - a)\right| \nu(dw)du\right] < \infty, \quad \text{and}$$
$$\mathbb{E}\left[\int_0^{A_t} \int_{|w|\leq 1} \left|F(X'_{u-} + z_0 - a + w) - F(X'_{u-} + z_0 - a)\right|^2 \nu(dw)du\right] < \infty.$$

Since σ is bounded, we have that $A_t \leq \|\sigma\|_\infty t$ and from the bounds in Lemma 4.3.3 we get the result. □

Following the same ideas we are going to state a power decomposition for the process Z given by equation (5.2).

Theorem 5.3.2. *Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, consider $X \sim S_\alpha(c_-, c_+)$. Let the process Z as equation (5.2) with σ a strictly positive bounded measurable function. Then for all $\gamma \in (\alpha-1, \alpha)$ and $x \in \mathbb{R}$ we have the decomposition:*

$$|Z_t - x|^\gamma = |z_0 - x|^\gamma + M_t^\gamma \tag{5.5}$$

$$+ \int_0^t |Z_s - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{Z_s > x\}} + k_+ \mathbb{1}_{\{Z_s < x\}}\right] \sigma^\alpha(Z_{s-}) ds, \tag{5.6}$$

where $k_\pm := k_\pm(\alpha, \gamma, c_-, c_+)$ are given in Theorem 4.4.1. Note that the last integral can be expressed in terms of the local time of Z as:

$$\int_{-\infty}^{\infty} |a - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{a > x\}} + k_+ \mathbb{1}_{\{a < x\}}\right] L_t^a(Z) da.$$

Proof. Consider $X' \stackrel{d}{=} X$ given in the stochastic integral representation of Z as in equation (5.1). The proof uses the power decomposition of $|X - (x - z_0)|^\gamma$ at level $x - z_0$ after an evaluation at time $A_t = \int_0^t \sigma^\alpha(Z_{s-}) ds$. Then, by the power decomposition, we have:

$$\begin{aligned} |X'_t - (x - z_0)|^\gamma &= |z_0 - x|^\gamma \\ &+ \int_0^t \int_{\mathbb{R}_0} \left[\left|X'_{u-} - (x - z_0) + w\right|^\gamma - \left|X'_{u-} - (x - z_0)\right|^\gamma\right] \tilde{N}'(du, dw) \\ &+ \int_0^t \left|X'_u - (x - z_0)\right|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{X'_u + z_0 > x\}} + k_+ \mathbb{1}_{\{X'_u + z_0 < x\}}\right] du, \end{aligned}$$

where $\tilde{N}'(du, dw) = N'(du, dw) - \nu(dw) du$ is the compensated Poisson random measure associated to X'.

Now, by evaluating at time A_t:

$$\begin{aligned} |X'_{A_t} + z_0 - x|^\gamma &= |X'_{A_0} + z_0 - x|^\gamma \\ &+ \int_0^{A_t} \int_{\mathbb{R}_0} \left[|X'_{u-} + z_0 - x + w|^\gamma - |X'_{u-} + z_0 - x|^\gamma\right] \tilde{N}'(du, dw) \\ &+ \int_0^{A_t} \left|X'_u + z_0 - x\right|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{X'_u + z_0 > x\}} + k_+ \mathbb{1}_{\{X'_u + z_0 < x\}}\right] ds. \end{aligned}$$

Using the equivalence between X' and Z after the change of time we have:

$$\begin{aligned} |Z_t - x|^\gamma &= |z_0 - x|^\gamma \\ &+ \int_0^{A_t} \int_{\mathbb{R}_0} \left[|X'_{u-} + z_0 - x + w|^\gamma - |X'_{u-} + z_0 - x|^\gamma\right] \tilde{N}'(du, dw) \\ &+ \int_0^{A_t} \left|X'_u + z_0 - x\right|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{X'_u + z_0 > x\}} + k_+ \mathbb{1}_{\{X'_u + z_0 < x\}}\right] ds. \end{aligned}$$

To get the same representation as in equation (5.6) in the theorem, let us consider the following calculations. Let us define the double integral as

$$M_t^{\gamma} := \int_0^{A_t} \int_{\mathbb{R}_0} \left[|X'_{u-} + z_0 - x + w|^{\gamma} - |X'_{u-} + z_0 - x|^{\gamma} \right] \tilde{N}'(du, dw).$$

As in Tanaka formula, to prove that M_t^{γ} is a martingale, we just need to show that:

$$\mathbb{E}\left[\int_0^{A_t} \int_{|w|>1} \left| |X'_{u-} + z_0 - x + w|^{\gamma} - |X'_{u-} + z_0 - x|^{\gamma} \right| \nu(dw) du \right] < \infty, \quad \text{and}$$
$$\mathbb{E}\left[\int_0^{A_t} \int_{|w|\leq 1} \left| |X'_{u-} + z_0 - x + w|^{\gamma} - |X'_{u-} + z_0 - x|^{\gamma} \right|^2 \nu(dw) du \right] < \infty.$$

Since σ is bounded, we have that $A_t \leq \|\sigma\|_{\infty} t$ and the result follows from the proof of Theorem 4.4.1.

For the last term, using the change of variable $u = A_s$ and changing $X_{A_s} + z_0$ by Z_s, we have:

$$\begin{aligned} &\int_0^{A_t} \left| X'_u + z_0 - x \right|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{X'_u + z_0 > x\}} + k_+ \mathbb{1}_{\{X'_u + z_0 < x\}} \right] du \\ = &\int_0^t |Z_s - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{Z_s > x\}} + k_+ \mathbb{1}_{\{Z_s < x\}} \right] \sigma^{\alpha}(Z_{s-}) ds \end{aligned}$$

As it was noted at the end of the theorem, we can use the occupational formula for Z from Definition 5.2.1 to write this last integral as:

$$\int_{-\infty}^{\infty} |a - x|^{\gamma-\alpha} \left[k_- \mathbb{1}_{\{a > x\}} + k_+ \mathbb{1}_{\{a < x\}} \right] L_t^a(Z) da.$$

Finishing the proof. □

Given the proof techniques for the Tanaka and Power decomposition for the solutions of the SDE (5.2) we can also provide a generalization of the Meyer-Itô formula in this context.

Theorem 5.3.3 (Occupational Meyer-Itô formula). *Let $\alpha \in (1,2)$, $c_-, c_+ \geq 0$, not both zero, and consider a strictly stable process $X \sim S_{\alpha}(c_-, c_+)$. Let the process Z as equation (5.2) with σ a strictly positive bounded measurable function. Let $f = F * \mu \in \mathscr{C}^{\alpha, c_-, c_+}$ and furthermore assume that μ is finite and compactly supported. Then,*

$$f(Z_t) = f(Z_0) + M_t^f + \int_{-\infty}^{\infty} L_t^a(Z) \mu(da), \tag{5.7}$$

where M_t^f is a martingale and $L_t^a(Z)$ is the occupational local time at a up to time t of Z.

Proof. Consider $X' \stackrel{d}{=} X$ given in the stochastic integral representation of Z as in equation (5.1). The proof is an application of the Occupational Meyer-Itô theorem for X' after an evaluation at time $A_t = \int_0^t \sigma^{\alpha}(Z_{s-}) ds$. By the Occupational Meyer-Itô theorem, we have:

$$\begin{aligned} f\left(X'_t + z_0\right) &= f\left(X'_0 + z_0\right) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \left[f(X'_{u-} + z_0 + w) - f(X'_{u-} + z_0) \right] \tilde{N}'(du, dw) \\ &\quad + \int_{-\infty}^{\infty} L_t^a\left(X'\right) \mu(da). \end{aligned}$$

Now, by evaluating at time A_t:

$$\begin{aligned} f(X'_{A_t}+z_0) &= f(X'_{A_0}+z_0) \\ &+\int_0^{A_t}\int_{\mathbb{R}_0}\left[f(X'_{u-}+z_0+w)-f(X'_{u-}+z_0)\right]\tilde{N}'(du,dw) \\ &+\int_{-\infty}^{\infty} L_{A_t}^{a+z_0}\left(X'\right)\mu\left(da\right), \end{aligned}$$

where $\tilde{N}'(du,dw)=N'(du,dw)-\nu(dw)du$ is the compensated Poisson random measure associated to X'.

Using the equivalence between X' and Z after the change of time we have:

$$f(Z_t)=f(z_0)+M_t^f+L_t^a(Z),$$

where

$$M_t^f:=\int_0^{A_t}\int_{\mathbb{R}_0}\left[f(X'_{u-}+z_0+w)-f(X'_{u-}+z_0)\right]\tilde{N}'(du,dw).$$

In order to proof that M_t^f is a martingale, note that it is an integral with respect to a Poisson random measure and by Ikeda and Watanabe [IW89] we just need to show that:

$$\begin{aligned} &\mathbb{E}\left[\int_0^{A_t}\int_{|w|>1}\left|f(X'_{u-}+z_0-a+w)-f(X'_{u-}+z_0-a)\right|\nu(dw)du\right]<\infty, \quad \text{and} \\ &\mathbb{E}\left[\int_0^{A_t}\int_{|w|\le 1}\left|f(X'_{u-}+z_0-a+w)-f(X'_{u-}+z_0-a)\right|^2\nu(dw)du\right]<\infty. \end{aligned}$$

Since σ is bounded, we have that $A_t\le\|\sigma\|_\infty t$ and from the bounds in Lemma 4.3.3 together with the proof of Theorem 4.3.1 we get the result.

□

5.4 Stochastic integral with changing signs integrand

The objective of this last section is to provide some ideas toward the definition of an occupational local time when the integrand can change signs.

Consider a predictable process H with path locally in L^α and X an α-stable process, then:

$$\int_0^t H_s dX_s = X'_{A_t^+}-X''_{A_t^-},$$

where $X'\stackrel{d}{=}X''\stackrel{d}{=}X$ and

$$A_t^+=\int_0^t H_s^+dX_s, \qquad A_t^-=\int_0^t H_s^-dX_s.$$

As Kallenberg noted in [Kal92], since the positive a negative part of H have disjoint supports, we can consider X' and X'' to be independent.

In-between sign changes of H, we can study the stochastic integral as in the previous sections. However, these sort of excursions of H between sign changes seem somewhat difficult to tackle directly, and a definition of an occupational local time for this process appears to be complicated.

Given the appropriate way to deal with the sign changes and the local time, it should be easy to extend the previous results in a similar fashion. Furthermore, this could be useful to emulate the proof of pathwise uniqueness from Le Gall [LG83]: first by proving a Tanaka formula for the solution of this general SDE, state conditions in order to have a zero local time for these stochastic integrals and finally combining both of them.

Appendix A

Trigonometric results

The following trigonometric result is used in the proof of the composition of crossed fractional operators.

Lemma A.0.1. *Let $\lambda, \mu \in \mathbb{R}$, then the following identity holds*

$$\cos\left((\lambda-\mu)\frac{\pi}{2}\right)\sin\left((\lambda+\mu)\frac{\pi}{2}\right) = \sin\left(\mu\frac{\pi}{2}\right)\cos\left(\mu\frac{\pi}{2}\right)+\sin\left(\lambda\frac{\pi}{2}\right)\cos\left(\lambda\frac{\pi}{2}\right).$$

Proof. Using the trigonometric identities for the sum of angles we start from the LHS:

$$\begin{aligned}
&\left[\cos\left(\lambda\frac{\pi}{2}\right)\cos\left(\mu\frac{\pi}{2}\right)+\sin\left(\lambda\frac{\pi}{2}\right)\sin\left(\mu\frac{\pi}{2}\right)\right]\left[\sin\left(\lambda\frac{\pi}{2}\right)\cos\left(\mu\frac{\pi}{2}\right)+\cos\left(\lambda\frac{\pi}{2}\right)\sin\left(\mu\frac{\pi}{2}\right)\right]\\
&=\cos\left(\lambda\frac{\pi}{2}\right)\sin\left(\lambda\frac{\pi}{2}\right)\cos^2\left(\mu\frac{\pi}{2}\right)+\cos\left(\mu\frac{\pi}{2}\right)\sin\left(\mu\frac{\pi}{2}\right)\cos^2\left(\lambda\frac{\pi}{2}\right)\\
&+\sin\left(\mu\frac{\pi}{2}\right)\cos\left(\mu\frac{\pi}{2}\right)\sin^2\left(\lambda\frac{\pi}{2}\right)+\sin\left(\lambda\frac{\pi}{2}\right)\cos\left(\lambda\frac{\pi}{2}\right)\sin^2\left(\mu\frac{\pi}{2}\right)\\
&=\sin\left(\mu\frac{\pi}{2}\right)\cos\left(\mu\frac{\pi}{2}\right)\left[\cos^2\left(\lambda\frac{\pi}{2}\right)+\sin^2\left(\lambda\frac{\pi}{2}\right)\right]\\
&+\sin\left(\lambda\frac{\pi}{2}\right)\cos\left(\lambda\frac{\pi}{2}\right)\left[\cos^2\left(\mu\frac{\pi}{2}\right)+\sin^2\left(\mu\frac{\pi}{2}\right)\right]\\
&=\sin\left(\mu\frac{\pi}{2}\right)\cos\left(\mu\frac{\pi}{2}\right)+\sin\left(\lambda\frac{\pi}{2}\right)\cos\left(\lambda\frac{\pi}{2}\right).
\end{aligned}$$

□

The following lemma is used to analyze the constant $\beta(\alpha,c)$ in Theorem 4.4.1.

Lemma A.0.2. *The functions $h_\pm$ of Lemma 4.4.1 have minimum period* 2.

Proof. Let $f_\pm(x) = h_\pm(x/2\pi)$, so that we now wish to prove that the minimum period of $f_\pm$ is 2π. First, note that $f_\pm$ is a solution to $f''+f=0$. Second, all solutions to the above ODE are given by $a\cos+b\sin$. Finally, we assert that the minimum period of the above linear combination is 2π as long as a and b are not both zero. Let us assume that $a\neq 0$. If $\tilde{f}_\pm(x) = f_\pm(x+p)$ for some p, by equating initial conditions at zero, we obtain

$$a = a\cos p + b\sin p \quad \text{and} \quad b = -a\sin p + b\cos p.$$

By substituting the value for b obtained in the second equation in the first and cancelling a, since it is non-zero, we get

$$1-\cos^2 p = \sin^2(p) = (1-\cos p)^2.$$

Expanding the square, we get

$$\cos p = \cos^2 p$$

from which $p = 2k\pi$. The case when $b \neq 0$ is handled similarly. □

Appendix B

Distribution theory

In this chapter we are going to give the relevant definitions which serve as preliminaries needed for Lizorkin space, following [Rub96].

B.1 Distributions

Since most of this work relies on distribution theory and in particular with the Lizorkin space of test functions, we are going to state the results which serve our purpose.

B.1.1 Schwartz space

We begin by considering the Schwartz space of test functions, $\mathscr{S} := \mathscr{S}(\mathbb{R})$, which consists of infinitely differentiable real valued functions $\varphi(x)$ such that the norms

$$\|\varphi\|_{(m)} = \max_x (1+|x|)^m \sum_{j=0}^{m} \left|\varphi^{(j)}(x)\right| \tag{B.1}$$

are finite for all $m \in \mathbb{N}_0$. The space $\mathscr{S}$ is a locally convex topological vector space that is metrizable and complete, equipped with its natural topology generated by the sequence of norms in (B.1).

This space has several properties which are important, for instance:

- For any $j \in \mathbb{N}$ the map $\varphi \mapsto \varphi^{(j)}$ is continuous in the topology of $\mathscr{S}$.
- If $a(x) \in C^\infty$ increases at infinity with all derivatives no faster than a polynomial,

  $$\left|a^{(j)}(x)\right| \leq c_j (1+|x|)^{m_j} \quad \forall j \in \mathbb{N},$$

  then $a(x)$ is a multiplier in $\mathscr{S}$, i.e. that the mapping $\varphi(x) \to \varphi(x)a(x)$ is continuous in the topology of $\mathscr{S}$ and such that $\varphi(x)a(x) \in \mathscr{S}$.
- The following inclusions have a dense image: $C_c^\infty \subset \mathscr{S}$, $\mathscr{S} \subset L^p$ for $1 \leq p < \infty$ and $\mathscr{S} \subset C_0$.
- The Fourier transform $\mathscr{F}$ is a topological isomorphism of $\mathscr{S}$ onto itself.
- If $\varphi, \psi \in \mathscr{S}$ then the convolution $\varphi * \psi \in \mathscr{S}$. Moreover, $\mathscr{F}[\varphi * \psi](u) = \mathscr{F}[\varphi](u)\mathscr{F}[\psi](u)$.

B.1.2 Tempered distributions

Working with special spaces of test functions leads us inevitably to consider distributions, which are continuous linear functionals on these spaces.

For now, let us stick with the Schwartz space of test functions $\mathscr{S}$. Consider a functional f on $\mathscr{S}$, we will write (f,φ) for the value of f at a test function $\varphi \in \mathscr{S}$. We will say that f is a distribution if it is:

- Linear: if for any $\varphi_1, \varphi_2 \in \mathscr{S}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$,

$$(f, \alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 (f, \varphi_1) + \alpha_2 (f, \varphi_2).$$

- Continuous: if for any convergent sequence $\varphi_n(x) \to \varphi(x)$ in $\mathscr{S}$, we have that $(f,\varphi_n) \to (f,\varphi)$ as $n \to \infty$.

The space of all distributions will be denoted by $\mathscr{S}'(\mathbb{R}) := \mathscr{S}'$. We say that a sequence $f_n \in \mathscr{S}'$ is convergent to $f \in \mathscr{S}'$ if $(f_n,\varphi) \to (f,\varphi)$ $\forall \varphi \in \mathscr{S}$, and the space $\mathscr{S}'$ is complete under such convergence.

Let $\Omega \subset \mathbb{R}$ be an open domain and $f \in \mathscr{S}'$. The distribution f is equal to zero, $f = 0$, in Ω if $(f,\varphi) = 0$ for any $\varphi \in C_c^\infty(\Omega)$. The largest open set Ω_{max} on which $f = 0$ is said to be a zero set for f, and is complement Ω_{max}^c is called the support of f, denoted by supp f.

Let $f(x)$ be a locally integrable function, which also satisfies

$$\int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^m} dx < \infty,$$

for some $m > 0$, so that $f(x)$ is a slowly increasing function. Then, a distribution $f \in \mathscr{S}'$ can be identified with $f(x)$ by the formula

$$(f,\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx.$$

Distributions that can be identified with locally integrable functions are called regular distributions. In fact, there is a one to one correspondence between regular distributions and locally integrable slowly increasing functions. For instance, we have that $L^p \subset \mathscr{S}'$ for $1 \le p \le \infty$.

Another class of distributions, that will be very useful in this work, which contains the regular distributions is generated by Radon measures. We consider measures defined in $\mathscr{B}(\mathbb{R})$ such that $\mu(K)$ is finite for every compact set K. If the measure satisfies

$$\int_{\mathbb{R}} \frac{1}{(1+|x|)^m} \mu(dx) < \infty,$$

for some $m > 0$, we say it is a tempered measure and we can consider a distribution in $\mathscr{S}'$ defined by

$$(\mu,\phi); = \int_{\mathbb{R}} \varphi(x)\mu(dx), \quad \varphi \in \mathscr{S}.$$

The simplest example of a distribution defined by a tempered measure is the Dirac delta distribution. As a measure is defined by $\delta(E) = 1$ if $0 \in E$ and zero otherwise, and as a distribution, this leads to $(\delta, \varphi) = \varphi(0)$. This is also an example of a distribution that is not regular, in such case, we will say they are singular distributions.

Derivatives of distributions are defined through the space of test functions. The derivative D^jF, $j \in \mathbb{N}$, of the distribution $f \in \mathscr{S}'$ is defined by the relation

$$\left(D^j f, \varphi\right) = (-1)^j \left(f, D^j \varphi\right), \quad \varphi \in \mathscr{S}.$$

In this sense, distributions have derivatives of all orders and the mapping $f \to D^jF$ is continuous in $\mathscr{S}'$, with supp $D^j \subset$ supp f.

If $f \in \mathscr{S}'$ and $a(x)$ is an infinitely differentiable slowly increasing function, then the product $a(x)f$ is a distribution defined by the relation $(af, \varphi) = (f, a\varphi)$ $\forall \varphi \in \mathscr{S}$, and the mapping $f \to af$ is continuous in $\mathscr{S}'$. A function $a(x)$ as above, is called a multiplier in $\mathscr{S}'$. Moreover, translations and scaling of distributions are defined by:

- If $f \in \mathscr{S}'$ and $x_0 \in \mathbb{R}$, then $f(x - x_0)$ is defined by
$$(f(x - x_0), \varphi(x)) = (f(x), a\varphi(x + x_0)) \quad \forall \varphi \in \mathscr{S}.$$
- If $f \in \mathscr{S}'$ and $\lambda \neq 0$, then $f(\lambda x)$ is defined by
$$(f(\lambda x), \varphi(x)) = |\lambda|^{-n}(f(x), \varphi(x/\lambda)) \quad \forall \varphi \in \mathscr{S}.$$

Another useful definitions regards the convolution of a distribution $f \in \mathscr{S}'$ with a test function $\phi \in \mathscr{S}$, which is defined by

$$(f * \varphi)(x) = (f(y), \varphi(x - y)),$$

where $\varphi(x - y)$ is a function of y, with x fixed. In this case, the function $(f * \varphi)(x)$ is infinitely differentiable and slowly increasing with all its derivatives. Moreover, if f has compact support then $f * \varphi \in \mathscr{S}$ and $\varphi \to f * \varphi$ is continuous in $\mathscr{S}$.

The convolution of two distributions is defined as follows. For any $f \in \mathscr{S}'$ and any compactly supported distribution $g \in \mathscr{S}'$, the convolution $g * f$ is defined by

$$(g * f, \varphi) = (f, g_1 * \varphi), \quad \varphi \in \mathscr{S},$$

where g_1 is a distribution such that $(g_1, \varphi) = (g, \varphi(-x))$ for every $\varphi \in \mathscr{S}$. For such f and g, the Fourier transform of the convolution satisfies $\mathscr{F}[g * f] = \mathscr{F}[g]\mathscr{F}[f]$.

In fact, the definition of convolution can be extended to more general classes of functions. Let $\Phi := \Phi(\mathbb{R})$ be a closed linear subspace of $\mathscr{S}$. The space Φ can be regarded as a linear topological space with the topology induced by that of $\mathscr{S}$. If we consider its Fourier image $\Psi = \mathscr{F}[\Phi]$, since $\mathscr{F}$ is a topological isomorphism of the space $\mathscr{S}$, the space Ψ is a linear topological space which is isomorphic to Φ under the action of $\mathscr{F}$. Denoting by Φ' and Ψ' the spaces of linear continuous functionals on Φ and Ψ respectively, then $\mathscr{S}' \subset \Phi'$ and $\mathscr{S}' \subset \Psi'$.

In this context, we can consider Φ and Ψ as test function spaces and Φ' and Ψ' as their corresponding spaces of distributions. As before, if some operation is admissible and continuous in the spaces Φ and Ψ, then a dual operation can be defined for their corresponding distributions as well.

Appendix C

Poisson random measures

Poisson random measures are an essential part of stable processes as its Levy-Itô decomposition showed. In this chapter we are going to give the relevant definitions on this topic, following [Kyp14] and [Kin93].

Definition C.0.1 (Poisson random measure)**.** *Let μ be a σ-finite measure on $(S,\mathscr{S})$ and $(\Omega,\mathscr{F},\mathbb{P})$ a probability space. Let $N:\Omega\times S\to\mathbb{N}\cup\{0,\infty\}$ such that the family $\{N(\cdot,A),A\in\mathscr{S}\}$ are random variables defined on $(\Omega,\mathscr{F},\mathbb{P})$, and for convenience the dependence on ω is usually omitted. Then N is called a Poisson random measure on S with intensity measure μ if it satisfies that:*

1. *If $A_1,\dots,A_n\in\mathscr{B}(\mathbb{R})$ are mutually disjoint, the random variables $N(A_1),\dots,N(A_n)$ are independent.*
2. *For each $A\in\mathscr{B}(\mathbb{R})$, $N(A)\sim Poi(\mu(A))$.*
3. *$N(\cdot)$ is a measure $\mathbb{P}$-a.s.*

Note that when $\mu(A)=0$, then $\mathbb{P}[N(A)=0]=1$ and when $\mu(A)=\infty$, then $\mathbb{P}[N(A)=\infty]=1$.

We will be interested in integral with respect to Poisson random measures, as the Lévy-Itô decomposition of stable processes makes clear. If we consider a Poisson random measure $N(\cdot)$ defined on measure space $(S,\mathscr{S},\mu)$, since it is a measure $\mathbb{P}$-a.s., by classical measure theory we can consider the following kind of integrals

$$\int_S f(x)N(dx),$$

as a well-defined random variable, for any measurable function f, such that the integral with respect to N of either f^+ or f^- is finite.

The main properties of this kind of random variable consist of integrability and moments. The following theorem and its proof can be consulted in [Kyp14] Theorem 2.7.

Theorem C.0.1. *Let $N(\cdot)$ be a Poisson random measure defined on measure space $(S,\mathscr{S},\mu)$ and $f:S\to\mathbb{R}$ a measurable function. Then*

$$X=\int_S f(x)N(dx),$$

is a.s. absolutely convergent if and only if

$$\int_S (1 \wedge |f(x)|)\, \mu(dx) < \infty.$$

And regarding its first two moments, we have:

$$\mathbb{E}(X) = \int_S f(x)\mu(dx), \quad \textit{if } \int_S |f(x)|\mu(dx) < \infty$$

and

$$\mathbb{E}(X^2) = \int_S f^2(x)\mu(dx) + \left(\int_S f(x)\mu(dx)\right)^2,$$

if

$$\int_S f^2(x)\mu(dx) < \infty \quad \textit{and} \quad \int_S |f(x)|\mu(dx) < \infty.$$

In the context of this thesis, we will be interested in the case where the Poisson random measure N is defined on $([0,\infty)\times\mathbb{R}, \mathscr{B}[0,\infty)\times\mathscr{B}(\mathbb{R}), dt\times\nu(dh))$, where ν is the measure concentrated in $\mathbb{R}_0$ associated to a stable process, that is

$$\nu(dh) = \left(c_- \mathbb{1}_{\{h<0\}} + c_+ \mathbb{1}_{\{h>0\}}\right) \frac{dh}{|h|^{\alpha+1}},$$

for some non negative constants c_- and c_+, not both zero. So, in this case, we are interested in stochastic processes with the following integral form:

$$X_t := \int_0^t \int_B hN(ds,dh),$$

with $B \in \mathscr{B}(\mathbb{R})$.

When $0 < \nu(B) < \infty$ we can identify X_t with a compound Poisson process as the following proposition states (cf. [Kyp14] Lemma 2.8).

Proposition C.0.1. *Let N a Poisson random measure on $([0,\infty)\times\mathbb{R}, \mathscr{B}[0,\infty)\times\mathscr{B}(\mathbb{R}), dt\times\nu(dh))$, and $B \in \mathscr{B}(\mathbb{R})$ such that $0 < \nu(B) < \infty$. Then*

$$X_t = \int_0^t \int_B hN(ds,dh), \quad t \geq 0,$$

is a compound Poisson process with arrival rate $\nu(B)$ and jump distribution $\nu(B)^{-1}\,\nu(dh)|_B$.

Moreover, if we consider $\{\mathscr{F}_t\}_{t\geq 0}$ the natural filtration associated with the process $\{X_t\}_{t\geq 0}$, we have the following result (cf. [Kyp14] Lemma 2.9).

Proposition C.0.2. *Let N as in the previous proposition and consider $B \in \mathscr{B}(\mathbb{R})$ such that $\int_B |h|\nu(dh) < \infty$. Then,*

$$M_t := \int_0^t \int_B hN(ds,dh) - t\int_B h\nu(dh), \quad t \geq 0,$$

is a $\mathscr{F}_t$-martingale. And, if we have $\int_B h^2\nu(dh) < \infty$, then it is a square integrable martingale.

The last propositions assume that $\int_B |h|\nu(dh) < \infty$; however, for our measure ν this is not always the case. For example, if we take the set $\tilde{B} = (-\varepsilon, 0) \cup (0, \varepsilon)$, then $\int_{\tilde{B}} |h|\nu(dh) = \infty$ for any $\varepsilon > 0$; nevertheless, we have that $\int_{\tilde{B}} h^2\nu(dh) < \infty$. Thus, we need to understand what happens to the martingale in the last proposition, with $B_\varepsilon = (-1, -\varepsilon) \cup (\varepsilon, 1)$, in the limit as $\varepsilon \downarrow 0$. The following result can be consulted in [Kyp14] Theorem 2.10.

Theorem C.0.2. *Let N as in the previous proposition and note that* $\int_{B_\varepsilon} h^2\nu(dh) < \infty$. *For each* $\varepsilon \in (0,1)$ *define the following martingale*

$$M_t^\varepsilon = \int_0^t \int_{B_\varepsilon} hN(ds,dh) - t\int_{B_\varepsilon} h\nu(dh), \quad t \geq 0.$$

Then there exists a $\mathscr{F}_t$*-martingale* $M = \{M_t\}_{t\geq 0}$ *such that:*

1. *For each* $T > 0$*, there exists a non random sequence* $\left(\varepsilon_n^T\right)_{n\in\mathbb{N}}$ *with* $\varepsilon_n^T \downarrow 0$ *and such that*

$$\lim_{n\uparrow\infty} \sup_{0\leq s\leq T} \left(M_s^{\varepsilon_n^T} - M_s\right)^2 = 0 \quad a.s.$$

2. *it is càdlàg a.s.*

3. *it has, at most a countable number of discontinuities on* $[0,T]$ *a.s.*

4. *it has stationary and independent increments.*

In other words, for any fixed $T > 0$*, the sequence of martingales* (M_t^ε) *converges uniformly on* $[0,T]$ *with probability one along a subsequence in* ε *which may depend on* T.

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