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On the behavior of the Hartshorne-Rao module in the Hilbert scheme of curves in $\mathbb{P}^3$

Tesis
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Muchos geómetras han buscado durante mucho tiempo (como otros persiguen a las ballenas blancas) curvas "generales", vagamente míticas, que serían, en cada esquema de Hilbert, más bellas que las demás.

-M. Martin-Deschamps

-D. Perrin

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Contents

Introducción

Dos curvas en espacio proyectivo de dimensión 3 están geométricamente enlazadas si su unión como esquemas es igual a la intersección de dos superficies (ver Definición 1.1.1). Esta definición puede ser extendida para definir una relación de equivalencia sobre el conjunto de todas las curvas localmente Cohen-Macauay en $\mathbb{P}^3$, y es conocida como *equivalencia de enlaces*. Prabhakar Rao probó que dada una curva C en $\mathbb{P}^3$, su clase de enlace está determinada por la clase de isomorfismo, salvo traslación, del módulo de Hartshorne-Rao (H-R) de la curva (ver Definición 1.1.2), [Rao78]. Por otro lado, Federico Gaeta probó que la propiedad de ser una curva aritméticamente Cohen-Macaulay (ACM) se preserva bajo enlaces, [Gae48]. La clase de enlace de las curvas ACM está determinada por el módulo trivial. Una pregunta natural es:

Pregunta. *¿Cómo se comporta el módulo de Hartshorne-Rao en familias de curvas? En particular, ¿cómo se comporta en la cerradura de las curvas ACM?*

El propósito de esta tesis es estudiar esta pregunta en algunos casos en particular. Nosotros consideramos la cerradura de la familia de curvas localmente Cohen-Macaulay (lcm) en el esquema de Hilbert de curvas de grado $d_r = \frac{r(r+1)}{2}$ y género aritmético $g_r = \frac{r(r+1)(2r-5)}{6} + 1$ en el espacio proyectivo $\mathbb{P}^3$, denotado por $\mathscr{H}_r$. En estos casos existe una única componente que contiene curvas ACM. Denotamos a esta componente por $\overline{\mathscr{C}_r} \subseteq \mathscr{H}_r$.

Dada una curva $C_r \in \mathscr{H}_r$, sea $C_{r-1} \in \mathscr{H}_{r-1}$ una curva enlazada a C_r por la intersección completa de dos superficies X, X' de grado r. Si X es suave y $m \geqslant r-3$ calculamos la siguiente ecuación:

$$\begin{aligned} h^0(\mathbb{P}^3, \mathscr{I}_{C_r}(m)) &= h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(m)) - h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(2r-m-4)) \\ &\quad + \frac{r(r-m-2)(r-m-1)}{2} + \binom{m-r+3}{3}. \end{aligned} \tag{$\star$}$$

Esta ecuación nos fue utíl, ya que nos permitio calcular el grado mínimo de una superficie que contiene una curva en términos del grado mínimo de una superficie que contiene a una curva directamente enlazada a esta.

Por otro lado, si nos enfocamos en el caso $r = 3$, es decir, curvas de grado 6 y género 3 entonces podemos considerar dos familias especiales dentro de la componente $\overline{\mathscr{C}_3}$: la familia de curvas hiperelípticas de grado 6 denotada por $\mathscr{C}^h$ y la familia que parametriza curvas que son la unión de una cuártica plana y dos lineas alabeadas que la intersectan, esta familia es denotada por $\mathscr{A}$. La cerradura de estas familias son divisores de $\overline{\mathscr{C}_3}$ y los elementos en estas dos familias tienen módulo H-R de rango uno. Más aún, son los únicos divisores con esta propiedad, [Amr98]. Sin embargo, con las tecnicas usuales no es posible demostrar que la componente es normal. Por lo que tomamos una subvariedad de $\overline{\mathscr{C}_3}$, denotada por $\mathscr{B}$, que contiene a todas las curvas localmente Cohen-Macaulay de $\overline{\mathscr{C}_3}$, en particular las familias $\overline{\mathscr{C}^h}$ y $\overline{\mathscr{A}}$. En este caso, si denotamos por $N^1(\mathscr{B})$ al espacio de divisores en $\mathscr{B}$ módulo equivalencia númerica, demostramos el siguiente resultado:

Teorema A (Teorema 4.0.5). *Las clases de* $\overline{\mathscr{A}}$ *y* $\overline{\mathscr{C}^h}$ *en* $N^1(\mathscr{B})$ *son linealmente independientes y cada uno genera un rayo extremal del cono efectivo*

$$\mathit{Eff}(\mathscr{B}) \subseteq N^1(\mathscr{B}).$$

El Teorema A responde a la pregunta (en el mundo de las curvas localmente Cohen-Macaulay) para $\overline{\mathscr{C}_3}$, donde al incrementar el rango del módulo de H-R aparece un divisor. Nosotros esperamos que ocurra un comportamiento similiar para cualquier r. En particular, esperamos que la subvariedad de $\overline{\mathscr{C}_r}$ que parametriza curvas con módulo H-R de rango uno defina un divisor reducible.

Siendo más específicos, siguiendo la notación de la Definición 1.1.5, consideramos familias $\mathscr{L}^{r-3}\mathscr{C}^h$ y $\mathscr{L}^{r-3}\mathscr{A}$ en $\mathscr{H}_r$ en la misma clase de enlace que las familias $\mathscr{C}^h$ y $\mathscr{A}$ respectivamente, los elementos de estas familias tienen módulo H-R de rango uno. Más aún, estas familias tienen la dimensión correcta para ser divisores en la componente $\overline{\mathscr{C}_r}$. El principal obstáculo es probar que estas familias efectivamente están contenidas en la componente $\overline{\mathscr{C}_r}$. En el siguiente teorema presentamos los casos en los que obtuvimos resultados concretos:

Teorema B (Corolario 2.4.4). *Si* r *es un numéro impar, entonces la familia* $\mathscr{L}^{r-3}\mathscr{A}$ *está contenida en* $\overline{\mathscr{C}_r}$. *Si* r *es un numéro par, entonces la familia* $\mathscr{L}^{r-3}\mathscr{C}^h$ *está contenida en* $\overline{\mathscr{C}_r}$.

El teorema anterior responde, en particular, la pregunta en estos esquemas, es decir:

Corolario 1. *La condición de tener módulo H-R de rango uno, siempre produce un divisor de* $\overline{\mathscr{C}_r}$.

Esta tesis se organiza de la siguiente manera:

Empezamos con el Capítulo 1 donde presentamos los preliminares de la teoría de Enlaces y definiciones de los objetos que estudiaremos a lo largo de todo este trabajo. El principal resultado de este capítulo es el siguiente:

Teorema C (Teorema 1.2.3). *El esquema de Hilbert* $\mathscr{H}_r$ *tiene una única componente cuyo elemento genérico es una curva ACM.*

Más aún, dado que las curvas ACM son puntos suaves en su esquema de Hilbert (Corolario 8.10 en [Har10]), esta componente es genéricamente suave. Con frecuencia, nos enfocamos en describir familias dentro de esta componente. En los capítulos posteriores calcularemos la dimensión de una familia en la misma clase de enlace. Para esto deducimos una formula nos permite comparar las secciones globales de la gavilla de ideales de una curva con las secciones globales de la gavilla de ideales de una curva ligada a esta.

En el Capítulo 2 nos enfocamos en la componente de curvas ACM. En este capítulo daremos algunas propiedades de las familias $\mathscr{C}^h$ y $\mathscr{A}$. Probamos que, en algunos casos, las correspondientes familias en $\mathscr{H}_r$ están contendidas en la componente principal.

El resultado principal en este capítulo es el Teorema B, el cual probamos dando una descripción cohomologica conveniente de algunas de estas familias. En los casos especiales que consideramos podemos verificar que están contenidos en la cerradura de las curas ACM usando la geometría de las superficies en las que están contenidas. En general, el obstáculo es encontrar un criterio que no dependa de la descripción de la familia.

El Capítulo 3 está enfocado en curvas de grado 6 y género 3. Se sabe que $\mathscr{H}_3$ es un esquema reducible y conexo por [Amr00]. Para describir las componentes de este esquema construimos varias familias parametrizadas por la unión de curvas suaves e irreducibles de grado estrictamente menor y con intersecciones transversales. Con ayuda de Macaulay2, [M2], calculamos ideales iniciales, tablas de Betti y módulos H-R de elementos genéricos de cada una de estas familias. Esta información nos permite identificar a que componente corresponde cada familia. En general, la información que obtuvimos de estas familias nos permite tener una imagen parcial del esquema de Hilbert de curvas de grado 6 y género 3 en el espacio proyectivo. El código que usamos en todo este capítulo puede ser consultado en los apéndices de este trabajo.

El Capítulo 4 está dedicado a la única componente irreducible de curvas de grado 6 y género 3 con elementos suaves, a menudo llamada *componente principal*, que coincide con la componente de las curvas ACM, es decir, la variedad $\overline{\mathscr{C}_3}$. Nos interesa estudiar el espacio de clases de divisores $N^1(\overline{\mathscr{C}_3})$. Dado que no conocemos el comportamiento de $\overline{\mathscr{C}_3}$ fuera de la familia de curvas localmente Cohen-Macaulay, no podemos afirmar que la dimensión de $N^1(\overline{\mathscr{C}_3})$ es finita. Sin embargo, consideramos el esquema normal $\mathscr{B} := \overline{(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm}} \cup \mathscr{C}_3$ como una compactificación parcial de $\mathscr{C}_3$. En este caso $N^1(\mathscr{B})$ tiene dimensión finita por lo que podemos demostrar el Teorema A y, como consecuencia, obtuvimos el siguiente resultado:

Corolario 2 (Corolario 4.0.7)**.** *La dimensión del espacio vectorial* $N^1(\mathscr{B})$ *es* 3.

Introduction

Two curves in projective space of dimension 3 are *geometrically directly linked* if their union (scheme theoretically) is equal to the complete intersection of two surfaces (see Definition 1.1.1). This definition can be extended to define an equivalence relation, called *linkage equivalence*. Prabhakar Rao proved that given a curve C in $\mathbb{P}^3$, its linkage class is determinate by the isomorphism class, up to shifting of degrees, the Harthsorne-Rao module (H-R) of the curve (see Definition 1.1.2), [Rao78]. On the other hand, Federico Gaeta proved that the property of being arithmetically Cohen-Macaulay (ACM) is preserved by links, [Gae48]. Since the linkage class of the ACM curves is labeled by the trivial module, a natural question is:

Question 1. How does the Hartshorne-Rao module behave in families of curves? In particular, in the closure of the locus of ACM curves?

The purpose of this thesis is to study this question in some particular cases. We consider the closure of the family of curves locally Cohen-Macaulay (lcm) in the Hilbert scheme of curves of degree $d_r = \frac{r(r+1)}{2}$ and genus $g_r = \frac{r(r+1)(2r-5)}{6} + 1$ in the projective space, denoted by $\mathscr{H}_r$. In these cases, there is a unique component that contains ACM curves. We denote this component by $\overline{\mathscr{C}_r} \subseteq \mathscr{H}_r$.

Given a curve $C_r \in \mathscr{H}_r$ let $C_{r-1} \in \mathscr{H}_{r-1}$ be a curve linked to C_r by the complete intersection of two surfaces X, X' of degree r. If X is smooth and $m \geqslant r-3$ we compute the following formula:

$$\begin{aligned} h^0(\mathbb{P}^3, \mathscr{I}_{C_r}(m)) &= h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(m)) - h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(2r-m-4)) \\ &\quad + \frac{r(r-m-2)(r-m-1)}{2} + \binom{m-r+3}{3}. \end{aligned} \qquad (\star)$$

This formula is useful. If we know the minimal degree of the surfaces that contain a curve. Then, this equality gives us the minimal degree of a surface that contains a curve directly linked to it.

On the other hand, we focus on the case $r = 3$, i.e., curves of degree 6 and genus 3, then we get two families inside $\overline{\mathscr{C}_3}$: the family of hyperelliptic curves of degree 6, denoted by $\mathscr{C}^h$, and the family of curves that parameterize the union of a plane quartic with two incident skew lines denoted by $\mathscr{A}$. The closure of these two families are divisors of $\overline{\mathscr{C}_3}$ and the elements in these two families have H-R module of rank one. Moreover, these are the only divisors with this property. Nevertheless, with the information that we have, is not possible to prove that the main component is normal, thus, we have to take a normal subvariety of $\overline{\mathscr{C}_3}$, denoted by $\mathscr{B}$, that contains all the locally Cohen-Macaulay curves, in particular, the families $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$. In this case, if $N^1(\mathscr{B})$ is the space of divisors of $\mathscr{B}$ modulo numerical equivalence, then we prove:

Theorem A (Theorem 4.0.5)**.** *The classes of $\overline{\mathscr{A}}$ and $\overline{\mathscr{C}^h}$ in $N^1(\mathscr{B})$ are linearly independent and each of them spans an extremal ray of the effective cone*

$$\mathit{Eff}(\mathscr{B}) \subseteq N^1(\mathscr{B}).$$

Theorem A answers Question 1 for $\overline{\mathscr{C}_3}$ where appears a subvariety of codimension 1 when the rank of the H-R modulo increase by one. We expect that similar behavior occurs for every r. In particular, we expect that the subvariety of $\overline{\mathscr{C}_r}$ that parameterizes curves with H-R module of rank one defines a reducible divisor. Let us elaborate on this.

Following the notation of the Definition 1.1.5, we consider families $\mathscr{L}^{r-3}\mathscr{C}^h$ and $\mathscr{L}^{r-3}\mathscr{A}$ in $\mathscr{H}_r$ in the same linkage class of the families $\mathscr{C}^h$ and $\mathscr{A}$ respectively, the elements in these families have H-R module of rank one. Furthermore, we verify that these families have the correct dimension to be divisors of the component $\overline{\mathscr{C}_r}$. The main obstacle in showing they are indeed divisors is to prove that these families are contained in the component $\overline{\mathscr{C}_r}$. Here is what we prove:

Theorem B (Corollary 2.4.4)**.** *If r is an odd number then the family $\mathscr{L}^{r-3}\mathscr{A}$ is contained in $\overline{\mathscr{C}_r}$. If r is an even number, then the family $\mathscr{L}^{r-3}\mathscr{C}^h$ is contained in $\overline{\mathscr{C}_r}$.*

Consequently, we can give an answer to question 1, in these cases, with the following corollary:

Corollary 1. *The condition of having H-R of rank one induces a divisor on $\overline{\mathscr{C}_r}$ for every r.*

The organization of the thesis is as follows.

We start in Chapter 1 with preliminaries of Liaison Theory and the definitions of the objects that we study throughout. The main result of this Chapter is the following:

Theorem C (Theorem 1.2.3)**.** *The Hilbert scheme $\mathscr{H}_r$ has only one component whose generic element is an ACM curve.*

Furthermore, since the curves ACM are smooth points in Hilbert scheme (Corollary 8.10 in [Har10]), this component is generically smooth. Frequently, we focus on describing families inside of this component. At the end of this Chapter, in order to compute the dimension of a family in the same linkage class, we compute a formula that allows us to compare the global sections of the ideal sheaf of a curve and the global sections of the ideal sheaf of a curve linked to it.

In Chapter 2 we focus on the component of ACM curves. We give properties of the families $\mathscr{C}^h$ and $\mathscr{A}$. We prove that, in some cases, the corresponding linked families in $\mathscr{H}_r$ are contained in the main component. The main result of this chapter is Theorem B.

We prove this Theorem by giving a convenient cohomological description of these families. In some particular cases, we can verify the contention using the special geometry of the family, but in general, the obstacle is to find criteria that do not depend on the description of the family.

Chapter 3 focuses on curves of degree 6 and genus 3. It is known that this Hilbert scheme is reducible and connected by [Amr00]. In order to understand the components of this scheme, we construct several families. Each family parametrizes curves that are the union of smooth irreducible curves of strictly minor degree, intersecting transversely. For each of these families we compute, using the software Macaulay2, the initial ideal, the Betti table and the Hartshorne-Rao module of the generic element of each in these families. This information helps us to organize these families in the three components of this Hilbert scheme. In general, the information that we obtain from these families gives us a partial image of the Hilbert scheme of curves of degree 6 and genus 3. The code that we use is in the Appendix.

Chapter 4 is devoted to the unique irreducible component of curves of degree 6 and genus 3 with smooth elements, often called the *main component*. This component coincides with the component of ACM curves, that is, the variety $\overline{\mathscr{C}_3}$. We are interested in studying the space of classes of divisors $N^1(\overline{\mathscr{C}_3})$. Nevertheless, with the information that we have, we do not know the behavior of $\overline{\mathscr{C}_3}$ outside of the family of locally Cohen-Macaulay curves, thus we can not claim that the dimension of $N^1(\overline{\mathscr{C}_3})$ is finite. For this reason we consider the normal scheme $\mathscr{B} := \overline{(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm}} \cup \mathscr{C}_3$ as a partial compactification of $\mathscr{C}_3$. In this case $N^1(\mathscr{B})$ has finite dimension. For this subvariety we proved Theorem A and as a consequence, we are able to prove the following result:

Corollary 2 (Corollary 4.0.7)**.** *The dimension of the vector space* $N^1(\mathscr{B})$ *is* 3*.*

Chapter 1

Preliminaries

This chapter contains the background needed for the following chapters. We review prerequisites from Liaison Theory (for more details see [Mig98]) that we frequently use and establish the notation.

In this thesis, k always denotes an algebraically closed field of characteristic zero. All varieties and subschemes will be assumed to be projective. We shall denote by S the homogeneous polynomial ring $k[x, y, z, w]$ and we let $\mathbb{P}^3 := \mathbb{P}^3_k = Proj(S)$ stand for the projective 3-space. In this work, by *curve* we mean a one-dimensional closed subscheme of $\mathbb{P}^3$ that is locally Cohen-Macaulay (lcm). These are closed subschemes of dimension one that may be reducible and non-reduced but that have no isolated or embedded points.

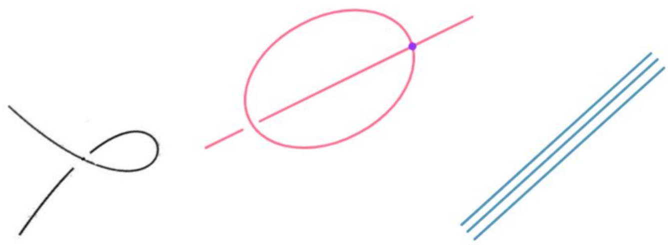

Figure 1.1: In the picture we represent curves of degree 3 and genus 0 (twisted cubic, conic union line, and triple line) all of them are locally Cohen-Macaulay curves.

1.1 Liaison Theory preliminaries

Liaison theory began with Apery and Gaeta in the 1940's (cf. [Apé45] and [Gae48]). They show that a smooth curve C in $\mathbb{P}^3$ is in the linkage class of a complete intersection if and only if it is arithmetically Cohen-Macaulay (ACM). This result was extended to arbitrary codimension two subschemes of projective space by Peskine and Szpiro [Pes74], and they put the whole theory of liaison into the framework of modern scheme theory. The next major contribution to the theory of liaison was made by Rao in [Rao78]. Rao finds a necessary and sufficient condition for two curves of projective 3-space to be linked.

Definition 1.1.1. Two curves C and C' in $\mathbb{P}^3$ without components in common are *directly geometrically linked* (or simply directly linked) by the complete intersection of two surfaces X and X' if $C \cup C' = X \cap X'$ scheme theoretically; that is $\mathscr{I}_C \cap \mathscr{I}_{C'} = \mathscr{I}_X + \mathscr{I}_{X'}$. The curves C, C' are *linked* if there exist a finite number of curves $C_1, \ldots, C_m$ in $\mathbb{P}^3$ such that C_i are directly linked to C_{i+1} for all i, with $C = C_1$ and $C' = C_m$.

Example 1. If X is the union of two general planes in $\mathbb{P}^3$ and X' is a transversal plane to X, then $X \cap X'$ is the union of two lines C and C' that intersect at a point (see figure 1.2).

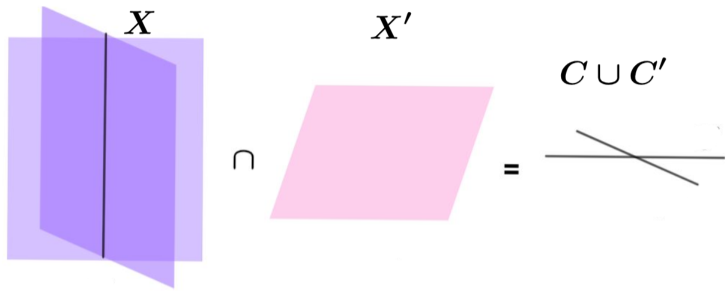

Figure 1.2: Two lines linked by the complete intersection of X and X'

The previous definition induces an equivalence relation called *linkage equivalence*. The linkage classes can be labeled via the Hartshorne-Rao module.

Definition 1.1.2. Let C be a closed subscheme in $\mathbb{P}^3$. The *Hartshorne-Rao module* (H-R module) of C is defined by:

$$M(C) := \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathscr{I}_C(n)).$$

If C is a curve, in particular lcm, this module is a graded S-module of finite length. For example, the H-R module of two skew lines is isomorphic to k.

Definition 1.1.3. A curve C is arithmetically Cohen-Macaulay (ACM) if its coordinate ring is a Cohen-Macaulay ring.

A necessary and sufficient condition to be an ACM curve is to have a trivial H-R module.

Example 2.

- A curve C of bidegree (a, b) on a nonsingular quadric surface in $\mathbb{P}^3$ is ACM if and only if $|a - b| \leq 1$.
- A curve C on a nonsingular cubic surface in $\mathbb{P}^3$ is ACM if and only if it is linearly equivalent to $B + mH$, where H is a hyperplane section, $m \geq 0$ is an integer, and B is either a line, a conic, a twisted cubic, or a hyperplane section.

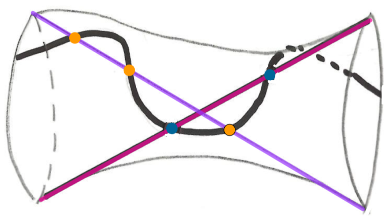

Figure 1.3: An ACM curve of bidegree $(2, 3)$

Two curves in the projective space lie in the same linkage class if and only if their H-R modules are isomorphic. Here is the precise statement of this claim.

Theorem 1.1.4. [Rao78, Thm. 2.6]

- *Two curves C and C' are in the same linkage class if and only if their Hartshorne-Rao-modules are isomorphic (except for a degree translation).*
- *For every S-module of finite length M, there exists a non singular irreducible curve $C \subseteq \mathbb{P}^3$ with Hartshorne-Rao module isomorphic to M (except for a degree translation).*

In order to extend Theorem A to other families of curves in the same linkage class, we are interested in studying families of curves linked to a fixed family $\mathscr{C}$. For this reason, we set notation to refer to this type of families.

Definition 1.1.5. Given a family $\mathscr{C}$ in a Hilbert scheme of curves in $\mathbb{P}^3$ we denote by $\mathscr{L}_s\mathscr{C}$ the family of curves linked to elements of $\mathscr{C}$ by the complete intersection of two surfaces of degree s, we called to the family $\mathscr{L}\mathscr{C}$ a *family linked* to the family $\mathscr{C}$. In many cases, we iterate this construction and write $\mathscr{L}_s^{n+1}\mathscr{C}$ instead of $\mathscr{L}_s(\mathscr{L}_{s-1}^n\mathscr{C})$.

Remark 1.1.6. Observe that, if a curve C in the family $\mathscr{C}$ has degree d and genus g, then the curves C' in the family $\mathscr{L}_s\mathscr{C}$ linked to C have degree $d' = s^2 - d$ and genus $g' = \frac{(d'-d)(2s-4)}{2} + g$. Furthermore, we have that $H^1(\mathbb{P}^3, \mathscr{I}_{C'}(n))$ is isomorphic to the dual of $H^1(\mathbb{P}^3, \mathscr{I}_C(2s-4-n))$ for each n. This implies that $M(C') = 0$ if $M(C) = 0$.

1.2 Triangular curves

In this section, we present the Hilbert schemes we will be working on, and prove that each of such schemes has only one component whose generic element is an ACM curve.

Let us start with the most simple curve in $\mathbb{P}^3$, the line. Given two different quadrics that contain a fixed line, their intersection is the union of the line with a curve C. We know the degree and genus of the line. Since C is linked to it by the complete intersection of two surfaces of degree 2, we can use Remark 1.1.6 to conclude that C has degree 3 and genus 0. That means, C is a twisted cubic. If we take two different cubic surfaces that contain this twisted cubic and intersect them, one obtains a residual curve of degree 6 and genus 3. We can continue iterating this process to produce an infinite family of curves and all these curves are linked to the line. By the previous observation, we have that for every r, we obtain a curve of degree $d_r = \frac{r(r+1)}{2}$ and genus $g_r = \frac{r(r+1)(2r-5)}{6} + 1$. We are interested in studying the Hilbert scheme of each of these curves. Thus, for any positive integer r, we set the numbers:

$$d_r = \frac{r(r+1)}{2} \quad \text{and} \quad g_r = \frac{r(r+1)(2r-5)}{6} + 1.$$

Let us consider the Hilbert scheme $Hilb_{p_r(t),3}$ of one-dimensional closed subschemes with Hilbert polynomial $p_r(t) = d_r t + (1 - g_r)$ in $\mathbb{P}^3$. The set of locally Cohen-Macaulay curves of degree d_r and arithmetic genus g_r is denoted by $\mathscr{H}_r^{lcm}$.

Definition 1.2.1. Let $\mathscr{H}_r$ be the closure of $\mathscr{H}_r^{lcm}$ in $Hilb_{p_r(t),3}$. We refer to the curves in $\mathscr{H}_r^{lcm}$ as *triangular curves*.

Triangular curves satisfy:

1. Their degrees are triangular numbers,
2. We have that $\mathscr{L}_{r+1}\mathscr{H}_r^{lcm} \subseteq \mathscr{H}_{r+1}^{lcm}$ for all r.

Example 3.

a) The Hilbert scheme $\mathscr{H}_1$ is the Grassmannian of lines in $\mathbb{P}^3$, it is smooth and irreducible of dimension 4.

b) The scheme $\mathscr{H}_2$ is the component of the Hilbert scheme of curves of degree 3 and genus 0 whose generic element is a twisted cubic. This scheme is smooth and irreducible of dimension 12 (cf. [Che08], [HSS21] and [PS85]).

c) The Hilbert scheme $\mathscr{H}_3$ is reducible and connected [Amr00, Thm.2]. In order to better understand the geometry of the curves in $\mathscr{H}_3$, we give a description of many families inside of the components of $\mathscr{H}_3$ in Chapter 3.

On the other hand, there exists a family in $\mathscr{H}_r$ that parametrizes curves C with minimal free resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-(r+1))^r \xrightarrow{M_r} \mathcal{O}_{\mathbb{P}^3}(-r)^{r+1} \longrightarrow \mathscr{I}_C \longrightarrow 0 \quad (1.1)$$

where $\mathscr{I}_C$ is the ideal sheaf of C. It follows from this resolution that the degree of an element in this family is d_r and the genus is g_r.

Definition 1.2.2. We denote by $\mathscr{C}_r$ the family of curves in $\mathscr{H}_r$ with minimal free resolution as (1.1).

In fact, observe that the elements in $\mathscr{C}_1$ are lines in $\mathbb{P}^3$, the elements in $\mathscr{C}_2$ are twisted cubics, etc. In particular, from the previous discussion we have that $\mathscr{L}_{r+1}\mathscr{C}_r \subseteq \mathscr{C}_{r+1}$.

For curves $C \in \mathscr{C}_r$, the H-R module $M(C)$ is trivial, which means that the curves in $\mathscr{C}_r$ are ACM curves. We also know that these curves are smooth points in $\mathscr{H}_r$ by a result of [Har10, Cor.8.10]. We claim that these families form the unique component of ACM curves in $\mathscr{H}_r$ for each $r > 0$.

Theorem 1.2.3. *The family $\overline{\mathscr{C}_r}$ is the only irreducible component in $\mathscr{H}_r$ of ACM curves.*

Proof. By the Ellingsrud-Hilbert-Bursch Theorem the associated ideal of an ACM curve C in $\mathbb{P}^3$ is minimally generated by the $r \times r$-minors of an $r \times (r+1)$ matrix of homogeneous elements of $k[x, y, z, w]$. Therefore the minimal free resolution of the ideal $\mathscr{I}_C$ of an ACM curve C has the following form:

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^3}(-b_i) \to \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{P}^3}(-a_i) \to \mathscr{I}_C \to 0.$$

In the case of the curves in $\mathscr{C}_r$, we have that $b_i = r+1$ and $a_i = r$ for all i. The h-vector of this family is $\{(1, 2, \dots, r)\}$, which corresponds with an open set of an irreducible component of $\mathscr{H}_r$. We verify that this is the only possible h-vector in $\mathscr{H}_r$ of ACM curves. Then, by [Mig98], $\mathscr{C}_r$ is the only irreducible component in $\mathscr{H}_r$ of ACM curves.

□

Remark 1.2.4. For $r > 0$, we can consider the incidence variety W_r defined by:

$$W_r := \{(C_r, C_{r+1}) \in \mathscr{C}_r \times \mathscr{C}_{r+1} | C_r \cup C_{r+1} = X \cap X'\}$$

$$\mathscr{C}_r \xleftarrow{\pi_0} W_r \xrightarrow{\pi_1} \mathscr{C}_{r+1}$$

If $C_r \in \mathscr{C}_r$, then C_{r+1} denotes a curve linked to C_r by the complete intersection of two surfaces X and X' of degree $r+1$; that means, $C_{r+1} \in \mathscr{L}_{r+1}\{C_r\}$. A consequence of Theorem 1.2.3 is that $\mathscr{C}_{r+1} = \mathscr{L}_{r+1}\mathscr{C}_r$ and therefore, using the incidence correspondence W_r, we can compute the dimension of the family $\mathscr{C}_r$ in terms of the family $\mathscr{C}_{r-1}$ and conclude that:

Lemma 1.2.5. *For all $r \geq 1$ we have that dim $\mathscr{C}_r = 4d_r = 2r(r+1)$.*

Observe that if $r \geq 3$ the Hilbert scheme $\mathscr{H}_r$ is reducible. These Hilbert schemes have two different components: the generically smooth component $\overline{\mathscr{C}_r}$ of ACM curves and a generically nonreduced component of dimension $\frac{3}{2}d_r(d_r - 3) + 9 - 2g_r$ whose general element is an extremal curve, [MDP96, Thm.4.3]. That means, curves with H-R module of maximal rank. The existence of more than one component complicates verifying the contention $\mathscr{L}\mathscr{B} \subseteq \overline{\mathscr{C}_r}$ for a family $\mathscr{B}$ in the component $\overline{\mathscr{C}_{r-1}}$. An important part of this work is to verify these contentions.

1.3 A Formula relating linked curves

In this section, we investigate the relation between the ideal sheaves of two linked curves in order to compute the dimension of a linked family $\mathscr{L}\mathscr{C}$ in terms of the dimension of the family $\mathscr{C}$. For this, we start with a curve C_r in $\mathscr{H}_r$, and suppose that there exists a curve C_{r-1} in $\mathscr{H}_{r-1}$ and surfaces X and X' of degree r such that C_r and C_{r-1} are linked by the complete intersection of X and X'. Assume that X is smooth. Then, we can consider C_r as a divisor in X with class $X \cap X' - C_{r-1} = rL - C_{r-1}$, where L is a hyperplane section of X.

Lemma 1.3.1. *Under the above hypotheses, the following equation is satisfied:*

$$\begin{aligned} h^0(X, \mathcal{O}_X(-C_r)(m)) - h^1(X, \mathcal{O}_X(-C_r)(m)) \\ = -h^0(X, \mathcal{O}_X(-C_{r-1})(2r-m-4)) + \frac{r(r-m-2)(r-m-1)}{2}. \end{aligned} \tag{1.2}$$

Proof. Since X is a smooth surface in $\mathbb{P}^3$ of degree r, we have that $\chi(\mathcal{O}_X) = \frac{r(r^2-6r+11)}{6}$ and $K_X = (r-4)L$. On the other hand, since C_r is a curve of degree d_r and genus g_r in X, we have that $C_r^2 = \frac{r(r+1)(r+2)}{2}$ and $C_r \cdot K_X = \frac{(r-4)r(r+1)}{2}$. Thus, the formula follows from Riemann-Roch and Serre duality on the surface X.

□

We want to compute the cohomology of the ideal of the curve over the projective space, thus let us write the cohomology of the sheaf $\mathcal{O}_X(-C_r)(m) = \mathcal{O}_X(mL - A))$, over the surface X, in terms of the cohomology of the sheaf $\mathscr{I}_{C_r}(m)$ over $\mathbb{P}^3$.

Using the exact sequence of the surface X of degree r in $\mathbb{P}^3$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(m-r) \to \mathcal{O}_{\mathbb{P}^3}(m) \to \mathcal{O}_X(m) \to 0,$$

we have that

$$H^1(X, \mathcal{O}_X(m)) \cong H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0 \quad \text{for all } m, \tag{1.3}$$

and

$$h^0(X, \mathcal{O}_X(m)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m-r)) \qquad \text{for all } m \geqslant r-3 \tag{1.4}$$

$$= \binom{m+3}{3} - \binom{m-r+3}{3} \qquad \text{for all } m \geqslant r-3. \tag{1.5}$$

Using the exact sequence of a curve $C \subseteq \mathbb{P}^3$,

$$0 \to \mathscr{I}_C(m) \to \mathcal{O}_{\mathbb{P}^3}(m) \to \mathcal{O}_C(m) \to 0$$

we have the next exact sequence for all m:

$$0 \to H^0(\mathbb{P}^3, \mathscr{I}_C(m)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(C, \mathcal{O}_C(m)) \to H^1(\mathbb{P}^3, \mathscr{I}_C(m)) \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) = 0$$

This gives us the next equation:

$$h^0(C, \mathcal{O}_C(m)) = -h^0(\mathbb{P}^3, \mathscr{I}_C(m)) + h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) + h^1(\mathbb{P}^3, \mathscr{I}_C(m)) \quad \text{for all } m. \tag{1.6}$$

For any curve $C \subseteq X$, we can consider the structure exact sequence of C in X:

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and from isomorphism (1.3), we obtain the next exact sequence in cohomology for all m:

$$0 \to H^0(X, \mathcal{O}_X(-C)(m)) \to H^0(X, \mathcal{O}_X(m)) \to H^0(C, \mathcal{O}_C(m)) \to H^1(X, \mathcal{O}_X(-C)(m)) \to H^1(X, \mathcal{O}_X(m)) = 0.$$

That implies the next equation:

$$h^0(C, \mathcal{O}_C(m)) = -h^0(X, \mathcal{O}_X(-C)(m)) + h^0(X, \mathcal{O}_X(m)) + h^1(X, \mathcal{O}_X(-C)(m)) \text{ for all } m. \quad (1.7)$$

Thus, we can use equations (1.6) and (1.7) and combine them with equation (1.5) to obtain:

$$\begin{aligned} &h^0(X, \mathcal{O}_X(-C)(m)) - h^1(X, \mathcal{O}_X(-C)(m)) \\ &\quad = -\binom{m-r+3}{3} + h^0(\mathbb{P}^3, \mathscr{I}_C(m)) - h^1(\mathbb{P}^3, \mathscr{I}_C(m)) \quad \text{for all } m \geqslant r-3. \end{aligned} \quad (1.8)$$

Lemma 1.3.2. *Let C be a curve contained in a surface X of degree r in $\mathbb{P}^3$, then:*

$$h^0(X, \mathcal{O}_X(mL - C)) = h^0(\mathbb{P}^3, \mathscr{I}_C(m)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m-r)).$$

Proof. From the last resolutions we have the following inclusions:

$$u : H^0(X, \mathcal{O}_X(mL - C)) \hookrightarrow H^0(X, \mathcal{O}_X(mL))$$

and

$$v : H^0(\mathbb{P}^3, \mathscr{I}_C(m)) \hookrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)).$$

On the other hand, we can consider the restriction morphism:

$$f : H^0(\mathbb{P}^3, \mathscr{I}_C(m)) \to H^0(X, \mathcal{O}_X(mL - C)).$$

Furthermore, if T is the equation that defines the surface X, then we have a morphism

$$g : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m-r)) \to H^0(\mathbb{P}^3, \mathscr{I}_C(m))$$

which maps a section s in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m-r))$ to the element $T \cdot s$ that vanishes on C. These morphisms, and the short exact sequence in cohomology induced by the inclusion of the surface X in $\mathbb{P}^3$, imply the next commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m-r)) & \xrightarrow{g} & H^0(\mathbb{P}^3, \mathscr{I}_C(m)) & \xrightarrow{f} & H^0(X, \mathcal{O}_X(mL-C)) & & \\ & & \| & & \downarrow v & & \downarrow u & & \\ 0 & \to & H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m-r)) & \xrightarrow{G} & H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) & \xrightarrow{F} & H^0(X, \mathcal{O}_X(mL)) & \to & 0 \end{array}$$

From the last commutative diagram is enough to prove that f is a surjective morphism and g is injective.

Let $a \in H^0(X, \mathcal{O}_X(mL - C))$ then $u(a) \in H^0(X, \mathcal{O}_X(mL))$. Since F is a surjective morphism, there exists an element $b \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m))$ such that $F(b) = u(a)$. The last equation implies that b is a section that vanishes on C, which implies that $b \in H^0(\mathbb{P}^3, \mathscr{I}_C(m))$ Since the diagram is commutative, $f(b) = a$ and f is surjective.

If $g(a) = g(b)$ then $G(a) = v(g(a)) = v(g(b)) = G(b)$. Since G is injective, $a = b$ and g is injective.

□

We find a formula that allows us to compare the global sections of the ideal sheaf of a curve to those of a curve linked to it. This formula is useful since it allows us to know the minimal degree of a surface that contains a fixed curve if we know the minimal degree of the surfaces that contain a curve directly linked to it.

Proposition 1.3.3. *Let $C_r \in \mathscr{H}_r$ be a curve and $C_{r-1} \in \mathscr{H}_{r-1}$ be a curve linked to C_r by the complete intersection of two surfaces X, X' of degree r. Suppose that X is smooth and $m \geqslant r-3$ then:*

$$\begin{aligned} h^0(\mathbb{P}^3, \mathscr{I}_{C_r}(m)) &= h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(m)) - h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(2r-m-4)) \\ &\quad + \frac{r(r-m-2)(r-m-1)}{2} + \binom{m-r+3}{3}. \end{aligned} \tag{$\star$}$$

Proof. It follows from 1.2, 1.8 and Lemma 1.3.2.

□

Remark 1.3.4. The preliminaries and more details of liaison theory that we present in this section can be found in [Mig98]. To my knowledge, the novelty is the uniqueness of the ACM component in $\mathscr{H}_r$ and the Proposition 1.3.3.

Chapter 2

Families in the closure of ACM curves

The goal of this Chapter is to construct families in $\mathscr{H}_r$ from known families using Liaison Theory. The purpose is to verify that the property of being a divisor can be preserved by liaison. In particular, we consider two families on $\mathscr{H}_3$ of curves with H-R module of rank one that are divisors of $\overline{\mathscr{C}_3}$ and conjecture that the linked families are divisors in their respective spaces $\overline{\mathscr{C}_r}$. In the end of this chapter, we prove that this conjecture is true for an infinite of cases.

2.1 Hyperelliptic curves

In this section, we consider families of curves linked to hyperelliptic curves of degree 6 and genus 3 and give some properties of such families.

Since a generic ACM curve C on $\mathscr{H}_3$ has $h^1(\mathbb{P}^3, \mathscr{I}_C(2)) = 0$, then it is not hyperelliptic. In fact, the hyperelliptic curves C' satisfy $h^1(\mathbb{P}^3, \mathscr{I}_{C'}(2)) = 1$ and in consequence are contained in a quadric surface [Har10, Ex.8.8, pp. 70-71]. Thus, let $\mathscr{C}^h$ be the family of curves of bidegree $(2,4)$ in a smooth quadric surface, see figure 2.1, the elements in this family are hyperelliptic curves of degree 6 and genus 3. This means that $\mathscr{C}^h$ is a subset of $\mathscr{H}_3 - \mathscr{C}_3$. The dimension of the family $\mathscr{C}^h$ is 23 ([Har10, Ex.1.2 a)]) and the ideal of a generic element in it has minimal free resolution:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-6) \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-5)^4 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-4)^3 \bigoplus \mathscr{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathscr{I}_3^h \longrightarrow 0.$$

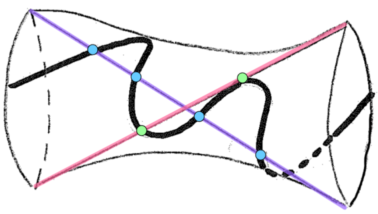

Figure 2.1: A curve of bidegree $(2,4)$ in a quadric.

Definition 2.1.1. Let $\mathscr{L}\mathscr{C}^h$ be the family of curves in $\mathscr{H}_4$ that are linked to the elements of $\mathscr{C}^h$ by the complete intersection of two surfaces of degree 4. We define recursively for all $r \geq 4$, the family $\mathscr{L}^{r-3}\mathscr{C}^h$ in $\mathscr{H}_r$ as the curves linked to curves in the family $\mathscr{L}^{r-4}\mathscr{C}^h$ by the complete intersection of two surfaces of degree r.

Given an element in $\mathscr{C}^h$, one can link it by a complete intersection of surfaces of degree 2 and 4 to a curve of degree 2 and genus -1 according to Remark 1.1.6. This must be the union of two skew lines or a double line of genus -1. In any case, its H-R module has rank one. In fact, we know that their H-R module is isomorphic to k in degree 2 and 0 in all other degrees. This allows us to know the H-R modules for all r.

Lemma 2.1.2. *Let C be a curve in $\mathscr{L}^{r-3}\mathscr{C}^h$ then:*

- *If r is odd, $H^1(\mathbb{P}^3, \mathscr{I}_C(r-1)) \cong k$ and $H^1(\mathbb{P}^3, \mathscr{I}_C(n)) = 0$ for all $n \neq r-1$.*
- *If r is even, $H^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) \cong k$ and $H^1(\mathbb{P}^3, \mathscr{I}_C(n)) = 0$ for all $n \neq r-2$.*

Proof. By definition, the curve C is linked to a curve $C' \in \mathscr{L}^{(r+1)-3}\mathscr{C}^h$ by the complete intersection of two surfaces of degree $r+1$. By [Har10, Ex. 8.4 c)] we have that:

$$H^1(\mathbb{P}^3, \mathscr{I}_{C'}(n)) \cong H^1(\mathbb{P}^3, \mathscr{I}_C(2r-2-n))^{\vee}. \tag{2.1}$$

We know the H-R modulo of an element in $\mathscr{C}^h$. Thus inductively with the equation 2.1 we have the result.

□

We used [M2] to compute the minimal free resolution of curves in $\mathscr{L}^{r-3}\mathscr{C}^h$ for small values of r. We conjecture the following:

Conjecture 2.1.3. The truncated minimal free resolution of the ideal of a generic element C in $\mathscr{L}^{r-3}\mathscr{C}^h$ is:
for r an odd number,

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-(r+3)) \to \mathcal{O}_{\mathbb{P}^3}(-(r+2))^4 \oplus \mathcal{O}_{\mathbb{P}^3}(-(r+1))^{r-6} \to \mathcal{O}_{\mathbb{P}^3}(-r)^{r-3} \oplus \mathcal{O}_{\mathbb{P}^3}(-(r-1)) \text{ and}$$

for r an even number,

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-(r+2)) \to \mathcal{O}_{\mathbb{P}^3}(-(r+2)) \oplus \mathcal{O}_{\mathbb{P}^3}(-(r+1))^r \to \mathcal{O}_{\mathbb{P}^3}(-r)^{r+1}.$$

Using the Proposition 1.3.3 we are able to compute the global sections of the sheaf $\mathscr{I}_C(m)$ for all positive m and all $r \geq 3$ for a generic element C of $\mathscr{L}^{r-3}\mathscr{C}^h$. This provides evidence that conjecture 2.1.3 may hold.

Corollary 2.1.4. *Let C a generic curve in $\mathscr{L}^{r-3}\mathscr{C}^h$, then we have that:*

a) $h^0(\mathbb{P}^3, \mathscr{I}_C(r-3)) = 0$

b) $h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = \begin{cases} 0 & \text{if } r \text{ is even} \\ 1 & \text{if } r \text{ is odd} \end{cases}$

c) $h^0(\mathbb{P}^3, \mathscr{I}_C(r)) = r+1$

d) $h^0(\mathbb{P}^3, \mathscr{I}_C(r+1)) = 3r+4$*; in general we have:*

$$h^0(\mathbb{P}^3, \mathscr{I}_C(r+a)) = \frac{(a+1)(a+2)}{2}r + \frac{(a+1)(a+2)(a+3)}{6} \quad \text{for all } a \geqslant 1.$$

Proof. *a*) We use induction over r: for the case $r = 3$ we know that $h^0(\mathbb{P}^3, \mathscr{I}_{C_3}(0)) = 0$. Suppose that is it true for all $s \leqslant r-1$. Let $C_r \in \mathscr{L}^{r-3}\mathscr{C}^h$. By definition there exists a curve $C_{r-1} \in \mathscr{L}^{(r-1)-3}\mathscr{C}^h$ linked to C_r by the complete intersection of two surfaces of degree r. Applying the formula ($\star$) we have that:

$$h^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r-3)) = h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(r-3)) - h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(r-1)) + r.$$

Now, we know that $h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(r-3)) = h^1(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(r-1)) = 0$ and we can use the formula ($\star$) with C_{r-1} and other curve $C_{r-2} \in \mathscr{L}^{(r-2)-3}\mathscr{C}^h$ linked to C_{r-1} to obtain:

$$h^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r-3)) = -(h^1(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(r-1)) - h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-2}}(r-1)) + r) + r = h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-2}}(r-1)).$$

The last term is zero by induction.

c), *d*) Follow from *a*) and Proposition 1.3.3.

b) Let $C_r \in \mathscr{L}^{r-3}\mathscr{C}^h$. We consider a curve $C_{r-1} \in \mathscr{L}^{(r-1)-3}\mathscr{C}^h$ linked to C_r as before. From ($\star$) we have that:

$$h^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r-1)) = h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(r-1)) - h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(r-3)) + 0.$$

From *a*), we have that $h^0(\mathbb{P}^3, \mathscr{I}_{C_{r-1}}(r-3)) = 0$ for all r and $h^1(\mathbb{P}^3, \mathscr{I}_{C_r}(r-1))$ is one if r is odd and zero if r is even by Lemma 2.1.2.

□

Now we can prove that the families $\mathscr{L}^{r-3}\mathscr{C}^h$ are irreducible and compute their dimension.

Proposition 2.1.5. *Let $r \geq 3$, then the family $\mathscr{L}^{r-3}\mathscr{C}^h$ is irreducible of dimension $2r(r+1)-1$.*

Proof. We use induction over r. For the case $r = 3$, we know that $\mathscr{C}^h$ is irreducible of dimension 23. Suppose that $\mathscr{L}^{r-3}\mathscr{C}^h$ is an irreducible family of dimension $2r(r+1)-1$.
For every $r \geq 3$ we consider the incidence variety

$$W_r^h := \{(C_r, C_{r+1}) \in \mathscr{L}^{r-3}\mathscr{C}^h \times \mathscr{L}^{(r+1)-3}\mathscr{C}^h | C_r \cup C_{r+1} = X \cap X'\}$$

with X and X' surfaces of degree $r+1$. That is, the elements of W_r^h are pairs of curves in $\mathscr{L}^{r-3}\mathscr{C}^h \times \mathscr{L}^{(r+1)-3}\mathscr{C}^h$ that are linked by the complete intersection of two surfaces of degree $r+1$. This subvariety comes with projection morphisms to each factor:

$$\pi_0^r : W_r^h \to \mathscr{L}_r^{r-3}\mathscr{C}^h \qquad \text{and} \qquad \pi_1^r : W_r^h \to \mathscr{L}^{(r+1)-3}\mathscr{C}^h$$

Using these projection morphisms, we have that:

$$\begin{aligned} dim\ \mathscr{L}^{(r+1)-3}\mathscr{C}^h &= dim\ W_r^h - dim\ (\pi_1^r)^{-1} \\ &= dim\ \mathscr{L}^{r-3}\mathscr{C}^h + dim\ (\pi_0^r)^{-1} - dim\ (\pi_1^r)^{-1}. \end{aligned}$$

For a generic element C_r in $\mathscr{L}^{r-3}\mathscr{C}^h$ the fibre of π_0^r is the Grassmannian of planes in the vector space $h^0(\mathbb{P}^3, \mathscr{I}_{C_r}^h(r+1))$ and Corollary 2.1.4 implies:

$$dim\ (\pi_0^r)^{-1} = dim\ G(2, h^0(\mathscr{I}_r^h(r+1))) = dim\ G(2, 3r+4) = 6r+6.$$

The fibre of π_1^r for a generic element C_{r+1} in $\mathscr{L}^{(r+1)-3}\mathscr{C}^h$ is the Grassmannian of 2-planes in the vector space $h^0(\mathbb{P}^3, \mathscr{I}_{C_{r+1}}^h(r+1))$ and again Corollary 2.1.4 implies:

$$dim\ (\pi_1^r)^{-1} = dim\ G(2, h^0(\mathscr{I}_{r+1}^h(r+1))) = dim\ G(2, r+2) = 2r+2.$$

Thus, we obtain:

$$\begin{aligned} dim\ \mathscr{L}^{(r+1)-3}\mathscr{C}^h &= dim\ \mathscr{L}^{r-3}\mathscr{C}^h + dim\ (\pi_0^r)^{-1} - dim\ (\pi_1^r)^{-1} \\ &= 2r(r+1) - 1 + (6r+6) - (2r+2) \\ &= 2r^2 + 6r + 3 \\ &= 2(r+1)(r+2) - 1. \end{aligned}$$

On the other hand, the family $\mathscr{C}^h$ is irreducible and the Grassmannian is irreducible. Therefore, the fibres of π_0^3 and π_1^3 are irreducible, in particular that implies that $\mathscr{L}\mathscr{C}^h$ is irreducible and inductively is follows that all families $\mathscr{L}^{r-3}\mathscr{C}^h$ are irreducible.

□

2.2 Reducible curves in $\overline{\mathscr{C}_3}$

This section has a similar structure to the previous section. In this case, we study the family that parametrizes reducible curves in $\mathscr{H}_3$. Indeed, we consider curves defined by a plane quartic union two incident skew lines.

Let us start with a geometric description of a generic element in this family. Consider a plane H and two skew lines L_1, L_2 in $\mathbb{P}^3$ that intersect H in two different points p_1 and p_2 respectively. If Q is a plane quartic on H that passes through the points p_1 and p_2, then the curve $C = L_1 \cup L_2 \cup Q$ (see Figure 2.2) has degree 6 and genus 3, thus $C \in \mathscr{H}_3$. Let $\mathscr{A}$ be the family of these curves.

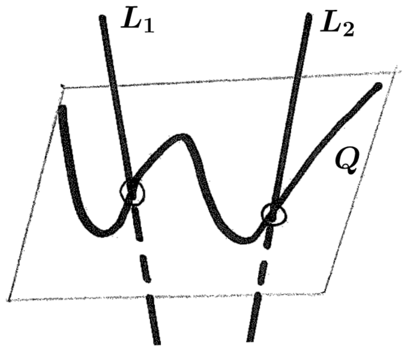

Figure 2.2: $C = L_1 \cup L_2 \cup Q$ is a generic element of $\mathscr{A}$

Proposition 2.2.1. *The family $\mathscr{A}$ is irreducible of dimension* 23*. Furthermore, it is contained in the closure of $\mathscr{C}_3$.*

Proof. The dimension and irreducibility follows from the previous construction. On the other hand, by [Nai02, Thm.1] the only smooth curves in $\mathscr{H}_3$ are ACM curves or a generic element in $\mathscr{C}^h$. By [HH06, Cor.4.3], the generic element of $\mathscr{A}$ is smoothable. This implies that the family $\mathscr{A}$ is contained in the closure of $\mathscr{C}_3$ or the closure of $\mathscr{C}^h$, but the latter case is not possible because both families have the same dimension. Therefore, $\mathscr{A}$ is contained in $\overline{\mathscr{C}_3}$.

□

The minimal free resolution of the ideal of an element C of $\mathscr{A}$ is:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-5) \bigoplus \mathcal{O}_{\mathbb{P}^3}(-4)^4 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \bigoplus \mathcal{O}_{\mathbb{P}^3}(-3)^4 \longrightarrow \mathscr{I}_C \longrightarrow 0.$$

We define families in $\mathscr{H}_r$ as before:

Definition 2.2.2. Let $\mathscr{L}\mathscr{A}$ be the family of curves in $\mathscr{H}_4$ that are linked to the elements of $\mathscr{A}$ by the complete intersection of two surfaces of degree 4. Similarly, the family $\mathscr{L}^{r-3}\mathscr{A}$ in $\mathscr{H}_r$ parametrizes curves linked to curves in the family $\mathscr{L}^{r-4}\mathscr{A}$ by the complete intersection of two surfaces of degree r.

The curves in $\mathscr{A}$ are linked to the union of two skew lines, hence their H-R modules have rank one. In fact, it is isomorphic to k in degree 1 and 0 in all other degree. Therefore, they are in the same linkage class as the elements of $\mathscr{C}^h$. Furthermore, the same arguments that go into the proof of the Lemma 2.1.2 can be used in this case.

Lemma 2.2.3. *Let C be a curve in $\mathscr{L}^{r-3}\mathscr{A}$ then:*

- *If r is even, $H^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) \cong k$ and $H^1(\mathbb{P}^3, \mathscr{I}_C(n)) = 0$ for all $n \neq r-2$.*
- *If r is odd, $H^1(\mathbb{P}^3, \mathscr{I}_C(r-1)) \cong k$ and $H^1(\mathbb{P}^3, \mathscr{I}_C(n)) = 0$ for all $n \neq r-1$.*

Proof. The proof is similar that Lemma 2.1.2.

□

In order to compare the families $\mathscr{L}^{r-3}\mathscr{C}^h$ and $\mathscr{L}^{r-3}\mathscr{A}$, we reproduce the proof of Corollary 2.1.4 and Lemma 2.1.5 and obtain the following:

Corollary 2.2.4. *1. Let C be a generic curve in $\mathscr{L}^{r-3}\mathscr{A}$, then:*

a) $h^0(\mathbb{P}^3, \mathscr{I}_C(r-3)) = 0$

b) $h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 1 & \text{if } r \text{ is even} \end{cases}$

c) $h^0(\mathbb{P}^3, \mathscr{I}_C(r)) = r+1$

d) $h^0(\mathbb{P}^3, \mathscr{I}_C(r+1)) = 3r+4$*; in general we have:*

$$h^0(\mathbb{P}^3, \mathscr{I}_C(r+a)) = \frac{(a+1)(a+2)}{2}r + \frac{(a+1)(a+2)(a+3)}{6} \quad \textit{for all } a \geqslant 1.$$

2. The family $\mathscr{L}^{r-3}\mathscr{A}$ is irreducible of dimension $2r(r+1)-1$.

As in the case of the families $\mathscr{L}^{r-3}\mathscr{C}^h$ we used [M2] to compute the minimal free resolution of elements on $\mathscr{L}^{r-3}\mathscr{A}$ in low cases and expect that the resolution of their elements are

Conjecture 2.2.5. The truncated minimal free resolution of the ideal of a generic element in $\mathscr{L}^{r-3}\mathscr{A}$ is:
for r an even number,

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-(r+3)) \to \mathscr{O}_{\mathbb{P}^3}(-(r+2))^4 \oplus \mathscr{O}_{\mathbb{P}^3}(-(r+1))^{r-6} \to \mathscr{O}_{\mathbb{P}^3}(-r)^{r-3} \oplus \mathscr{O}_{\mathbb{P}^3}(-(r-1)) \text{ and}$$

for r an odd number,

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-(r+2)) \to \mathscr{O}_{\mathbb{P}^3}(-(r+2)) \oplus \mathscr{O}_{\mathbb{P}^3}(-(r+1))^r \to \mathscr{O}_{\mathbb{P}^3}(-r)^{r+1}.$$

2.3 The first cases

The families $\mathscr{C}^h$ and $\mathscr{A}$ have many similarities. Indeed, both are irreducible families of dimension 23, and the generic element in both families is linked to the union of two skew lines. By Proposition 2.2.1 we know that the family $\mathscr{A}$ lies in the closure of the family $\mathscr{C}_3$ of ACM curves. One of the purposes of this section is to verify that the family $\mathscr{C}^h$ lies in the closure of the family $\mathscr{C}_3$. Thus, both families $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$ are different divisors of $\overline{\mathscr{C}_3}$. The second goal of this section is to prove that the closure of the linked families $\mathscr{L}\mathscr{C}^h$ and $\mathscr{L}\mathscr{A}$ are divisors in $\overline{\mathscr{C}_4}$.

Showing that a family lies along the closure of another family is in general difficult. Thus, let us start with a useful proposition that helps us to prove that the families $\mathscr{L}\mathscr{C}^h$ and $\mathscr{L}\mathscr{A}$ are contained in the closure of $\mathscr{C}_4$.

Proposition 2.3.1. *Let $r < 9$. If $\{C_t\}_{t\in T}$ is a family of curves in $\mathbb{P}^3$ such that:*

- *$deg(C_t) = \frac{r(r+1)}{2}$ and $g(C_t) = \frac{r(r+1)(2r-5)}{6} + 1$ for all $t \in T$.*
- *C_0 is contained in a surface of degree $r-1$*
- *$h^1(\mathbb{P}^3, \mathscr{I}_{C_0}(n)) = 0$ for all $n \neq r-1$*
- *The generic element C_t is not contained in a surface of degree $r-1$,*

then the generic element of C_t is ACM.

Proof. For all $n \geq 0$ and t generic we have by semi-continuity that:

$$0 \leq h^1(\mathbb{P}^3, \mathscr{I}_{C_t}(n)) \leq h^1(\mathbb{P}^3, \mathscr{I}_{C_0}(n)).$$

Since $h^1(C_0, \mathscr{I}_{C_0}(n)) = 0$ for all $n \neq r-1$ then $h^1(C_t, \mathscr{I}_{C_t}(n)) = 0$ for all $n \neq r-1$ and t generic. By hypothesis $h^0(\mathbb{P}^3, \mathscr{I}_{C_t}(r-1)) = 0$ thus:

$$\begin{aligned} h^2(\mathbb{P}^3, \mathscr{I}_{C_t}(r-1)) - h^1(\mathbb{P}^3, \mathscr{I}_{C_t}(r-1)) &= \chi(\mathscr{I}_{C_t}(r-1)) \\ &= \chi(\mathscr{O}_{\mathbb{P}^3}(r-1)) - \chi(\mathscr{O}_{C_t}(r-1)) \\ &= \binom{r+2}{3} - (1 - g(C_t) + (r-1)(deg(C_t)) \\ &= 0. \end{aligned}$$

We have that $H^2(\mathbb{P}^3, \mathscr{I}_{C_t}(r-1)) \cong H^1(C_t, \mathscr{O}_{C_t}(r-1))$ and the last cohomology group is isomorphic to $H^0(C_t, K_{C_t} - (r-1)H_t)$ here H_t is the hyperplane section of C_t. Since $r < 9$, then $2g(C_t) - 2 < (r-1)deg(C_t)$. Then the degree of the divisor $K_{C_t} - (r-1)H_t$ is negative which implies $H^0(C_t, K_{C_t} - (r-1)H_t) = 0$ and therefore $H^2(\mathbb{P}^3, \mathscr{I}_{C_t}(r-1)) = 0$.

We conclude that $h^1(C_t, \mathscr{I}_{C_t}(r-1)) = 0$ for generic t; then the Hartshorne-Rao module of a generic element of the family C_t is the trivial module. This is equivalent to being ACM. □

In the proof of the previous proposition, we can see that when $r \geqslant 9$ the degree of the divisor $K_{C_t} - (r-1)H_t$ is positive. However, the degree of this divisor is less than the genus of the curve C_t thus there exists the possibility that it does not have sections.

Lemma 2.3.2. *The family $\overline{\mathscr{C}^h}$ is contained in $\overline{\mathscr{C}_3}$.*

Proof. This result is an exercise of [Har10, Ex. 8.8 d)] and follows from the Proposition 2.3.1. Indeed, since $\overline{\mathscr{C}^h}$ is irreducible, this family must be contained in only one component or in the intersection of two o more components of $\mathscr{H}_3$. On the other hand, using the program [M2] we obtain explicit examples of elements of $\overline{\mathscr{C}^h}$ that are smooth curves and are smooth points in $\mathscr{H}_3$. Their tangent spaces have dimension 24. Then the family $\overline{\mathscr{C}^h}$ has to be contained in a component of dimension 24. Suppose that $\overline{\mathscr{C}^h} \not\subseteq \overline{\mathscr{C}_3}$. Then there exists another component U of dimension 24 that contains $\overline{\mathscr{C}^h}$, since the generic element of U cannot be an ACM curve, the Proposition 2.3.1 implies that the generic element of U is contained in a quadric surface. This implies that $\overline{\mathscr{C}^h} = U$, hence U has dimension 23 that is a contradiction thus $\overline{\mathscr{C}^h}$ is contained in the closure of $\mathscr{C}_3$.

□

Now we are going to study the Hilbert scheme $\mathscr{H}_4$. We will focus on the linked families $\mathscr{L}\mathscr{C}^h$ and $\mathscr{L}\mathscr{A}$ of dimension 39 contained in the Hilbert scheme $\mathscr{H}_4$ by Proposition 2.1.5 and Corollary 2.2.4. We have to verify that they are contained in the component $\overline{\mathscr{C}_4}$ that has dimension 40 to prove that these families are divisors on it. This is not obvious since the Hilbert scheme $\mathscr{H}_4$ is reducible and the next Theorem describes three components in it.

Theorem 2.3.3. *The Hilbert scheme of curves of degree* 10 *and genus* 11 *is reducible and has at least the following three components:*

1. *The component of ACM curves, denoted by $\overline{\mathscr{C}_4}$, has dimension* 40.
2. *The component of extremal curves, denoted by $\mathscr{E}_4$, has dimension* 92.
3. *A component $\mathscr{R}$ of dimension at least* 46 *that contains a family that parametrizes the union of two disjoint plane quintic curves.*

Proof. The family of ACM curves is a component of dimension 40 by Theorem 1.2.3. The existence of the component $\mathscr{E}_4$ is a result from [MDP96, Thm.4.3].

Set Q the family that parametrizes the union of two disjoint plane quintic curves. This family is irreducible of dimension 46.

The family Q is not contained in $\overline{\mathscr{C}_4}$ since the dimension of $\overline{\mathscr{C}_4}$ is 40. On the other hand, the H-R module of an element of Q has rank 25 and the H-R module of an extremal curve has rank 425. Thus, by semi-continuity, Q cannot be contained in the component $\mathscr{E}_4$.

Therefore, there exists a component $\mathscr{R}$ different from $\overline{\mathscr{C}_4}$ and $\mathscr{E}_4$ that contains Q.

□

We are now going to prove that the linked family $\mathscr{L}\mathscr{A}$ is in the closure of $\overline{\mathscr{C}_4}$.

Proposition 2.3.4. *The closure of the family of curves linked to a plane quartic union two skew lines incident to it are contained in the closure of the ACM curves, that means*

$$\overline{\mathscr{L}\mathscr{A}} \subseteq \overline{\mathscr{C}_4}.$$

Proof. First, we prove that the family of curves in $\mathscr{H}_4$ that are contained in a cubic surface has dimension 39. Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree 3. We use the notation of [Har77, Not. 4.7.3, pag. 401]. If C is a curve in X of degree 10 and genus 11, then its class is one of the following:

$$8l - 3e_1 - 3e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6 = B_1$$
$$9l - 4e_1 - 3e_2 - 3e_3 - 3e_4 - 2e_5 - 2e_6 = B_2$$
$$10l - 4E_1 - 4e_2 - 4e_3 - 3e_4 - 3e_5 - 2e_6 = B_3$$

$$10l - 5E_1 - 3e_2 - 3e_3 - 3e_4 - 3e_5 - 3e_6 = B_4$$

$$11l - 5E_1 - 4e_2 - 4e_3 - 4e_4 - 3e_5 - 3e_6 = B_5$$

$$12l - 5E_1 - 5e_2 - 4e_3 - 4e_4 - 4e_5 - 4e_6 = B_6$$

Let U_i be the family of all curves of degree 10 and genus 11 that are contained in a linear system $|B_i|$ of a smooth cubic surface X. Cubic surfaces form an irreducible family of dimension 19. Thus U_i is irreducible. Since $Pic(X)$ is discrete, the maximal families of curves inside X are complete linear systems. Therefore, the dimension of U_i is $19+dim|B_i|$. On the other hand, there exists a smooth irreducible curve in $|B_i|$ for each i by [Har77, Cor. 4.13, pag. 406], thus $U_i \neq \emptyset$. We compute the dimension for the case U_1, but a similar arguments holds for all cases. Let $C \in |B_1|$ be a smooth irreducible curve. Since the degree of C is 10, there exist a unique surface of degree 3 that contains C. From the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$$

in cohomology we have that:

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(C)) \to H^0(X, \mathcal{O}_C(C)) \to H^1(X, \mathcal{O}_X) = 0.$$

Thus

$$h^0(X, \mathcal{O}_C(C)) = h^0(X, \mathcal{O}_X(C)) - h^0(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(C)) - 1 = dim|C|.$$

The sheaf $\mathcal{O}_C(C)$ has degree $C^2 = 30$ which implies that the linear system is not special. By the Riemann-Roch Theorem, we have that:

$$h^0(C, \mathcal{O}_C(C)) = 30 - 10 = 20.$$

Therefore, the dimension of U_i is $19 + 20 = 39$ as we expected.
Now, by Corollary 2.2.4 the family $\overline{\mathscr{L}\mathscr{A}}$ is irreducible, thus it must be contained in only one component or in the intersection of two or more components of $\mathscr{H}_4$. Using [M2] we obtain explicit examples of elements in $\mathscr{L}\mathscr{A}$ that are smooth curves and are smooth points in $\mathscr{H}_4$ with tangent space of dimension 40. Thus $\overline{\mathscr{L}\mathscr{A}}$ has to be contained in a component of dimension 40. If $\mathscr{L}\mathscr{A}$ is not contained in $\overline{\mathscr{C}_4}$ then Proposition 2.3.1 implies that $\mathscr{L}\mathscr{A}$ is contained in a component of curves that are contained in a cubic surface. But as we saw, such component has dimension 39. That is a contradiction. □

Observe that in the proof of Lemma 2.3.2 and Proposition 2.3.4 we used the geometry of the surfaces in which the curves are contained, thus we can not generalize the proof for other cases.

In the next section, we prove that the linked families to $\mathscr{C}^h$ and $\mathscr{A}$ are contained in their respective $\overline{\mathscr{C}_r}$. In particular, the family $\mathscr{L}\mathscr{C}^h$ is contained in the closure $\overline{\mathscr{C}_4}$. Since the component $\overline{\mathscr{C}_3}$ has two divisors $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$, then we prove that the linked component $\overline{\mathscr{L}\mathscr{C}_3} = \overline{\mathscr{C}_4}$ has two divisors that correspond to the closure of the linked families $\mathscr{L}\mathscr{C}^h$ and $\mathscr{L}\mathscr{A}$.

We expected that the previous behavior is true in general; that means, the families $\overline{\mathscr{L}^{r-3}\mathscr{C}^h}$ and $\overline{\mathscr{L}^{r-3}\mathscr{A}}$ are divisors of the component $\overline{\mathscr{C}_r}$. We have proved that the families $\overline{\mathscr{L}^{r-3}\mathscr{C}^h}$ and $\overline{\mathscr{L}^{r-3}\mathscr{A}}$ have dimension $dim\overline{\mathscr{C}_r} - 1$. Nevertheless, since the spaces $\mathscr{H}_r$ are reducible for all $r \geq 3$ it is not obvious that these families are contained in the closure of $\mathscr{C}_r$. We are able to prove the last claim in many cases; which is what follows.

2.4 Almost canonical curves in $\overline{\mathscr{C}_r}$

In this section, we will describe two families in each $\mathscr{H}_r$ that allow us to verify that the families $\mathscr{L}^{r-3}\mathscr{C}^h$ and $\mathscr{L}^{r-3}\mathscr{A}$ have the same geometric description when they have different parity. That means, the families $\mathscr{L}^{r-3}\mathscr{C}^h$ and $\mathscr{L}^{r-2}\mathscr{A}$ have the same description for all r. One of such families has a geometric description and the other a cohomological one. The cohomological description helps us to prove that infinitely many of families $\mathscr{L}^{r-3}\mathscr{C}^h$ and $\mathscr{L}^{r-3}\mathscr{A}$ are contained in the component of ACM curves.

Definition 2.4.1. We denote the open set of all smooth curves on $\mathscr{H}_r$ by $\mathscr{H}_r^{smooth}$.

- Let $\mathscr{D}_{r-1}$ be the closure of the set of all smooth curves in $\mathscr{H}_r$ that lie in a surface of degree $r-1$. That means:

$$\mathscr{D}_{r-1} := \overline{\{C \in \mathscr{H}_r^{smooth} | h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = 1\}} \subseteq \mathscr{H}_r.$$

- Let $\mathscr{M}_r$ be the closure in $\mathscr{H}_r$ of the set of all smooth curves such that the canonical divisor minus $r-2$ times the hyperplane section H has one section and the rank of their H-R module is 1. That means:

$$\mathscr{M}_r := \overline{\{C \in \mathscr{H}_r^{smooth} | h^0(C, K_C - (r-2)H) = 1 \quad \text{and} \quad rkM(C) = 1\}} \subseteq \mathscr{H}_r.$$

The relation between these two families and the families $\mathscr{L}^{r-3}\mathscr{C}^h$ and $\mathscr{L}^{r-3}\mathscr{A}$ defined in 2.1.1 and 2.2.2 is stated in the next Theorem:

Theorem 2.4.2.

1. *If r is an odd number, then the closure of the family $\mathscr{L}^{r-3}\mathscr{C}^h$ is equal to the family $\mathscr{D}_{r-1}$ and the closure of the family $\mathscr{L}^{r-3}\mathscr{A}$ is an irreducible component of the family $\mathscr{M}_r$.*

2. *If r is an even number, then the closure of the family $\mathscr{L}^{r-3}\mathscr{A}$ is equal to the family $\mathscr{D}_{r-1}$ and the closure of the family $\mathscr{L}^{r-3}\mathscr{C}^h$ is an irreducible component of the family $\mathscr{M}_r$.*

Proof. Suppose that r is an odd number.

First we prove that $\overline{\mathscr{L}^{r-3}\mathscr{C}^h} = \mathscr{D}_{r-1}$. The set $\mathscr{L}^{r-3}\mathscr{C}^h \cap \mathscr{H}_r^{smooth}$ is dense in $\mathscr{L}^{r-3}\mathscr{C}^h$ and by Corollary 2.1.4 we have that $\mathscr{L}^{r-3}\mathscr{C}^h \cap \mathscr{H}_r^{smooth} \subseteq \mathscr{D}_{r-1}$. To prove the other containment we use induction over s, with $r = 2s+1$. The base case holds by definition. Let us assume it holds for $r-2 = 2(s-1)+1$.

Let $C \in \mathscr{H}_r^{smooth} \cap \mathscr{D}_{r-1}$, that means, $h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = 1$. We can then find two surfaces without common components X and Y of degree r that contains C. By Bertini's Theorem we can assume that X is smooth. Then we have a curve $C' \in \mathscr{H}_{r-1}$ linked to C by the complete intersection of X and Y and by the Proposition 1.3.3 we have that:

$$h^0(\mathbb{P}^3, \mathscr{I}^h_{C'}(r-1)) = h^1(\mathbb{P}^3 \mathscr{I}_C(r-3)) - h^0(\mathbb{P}^3, \mathscr{I}_C(r-3)) + r.$$

Since $h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = 1$, we have that $h^0(\mathbb{P}^3, \mathscr{I}_C(r-3)) = 0$, thus $h^0(\mathbb{P}^3, \mathscr{I}^h_{C'}(r-1)) > r-1$. Then we can link C' to a curve $C'' \in \mathscr{H}_{r-2}$ by the complete intersection of two surfaces without common components X' and Y' of degree $r-1$.

Again by Proposition 1.3.3, we have that:

$$h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = h^1(\mathbb{P}^3, \mathscr{I}_C(r-1)) - h^0(\mathbb{P}^3, \mathscr{I}_{C'}(r-3)) + h^0(\mathbb{P}^3, \mathscr{I}_{C''}(r-3)).$$

By construction C and C' are linked by the complete intersection of surfaces of degree r, which implies $H^1(\mathbb{P}^3, \mathscr{I}_{C'}(r-3)) \cong H^1(\mathbb{P}^3, \mathscr{I}_C(r-1))^\vee$. Therefore $h^0(\mathbb{P}^3, \mathscr{I}_{C''}(r-3)) = h^0(\mathbb{P}^3, \mathscr{I}_C(r-1)) = 1$. Then, by induction $C'' \in \mathscr{D}_{(r-2)-1} = \mathscr{L}^{(r-2)-3}\mathscr{C}^h$ and the definition of our families it follows that $C' \in \mathscr{L}^{(r-1)-3}\mathscr{C}^h$ and thus $C \in \mathscr{L}^{r-3}\mathscr{C}^h$.

Now we prove that $\overline{\mathscr{L}^{r-3}\mathscr{A}}$ is a component of $\mathscr{M}_r$. Let us start with a generic element C in $\mathscr{L}^{r-3}\mathscr{A}$. It follows that $h^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) = 1$ and $h^1(\mathbb{P}^3, \mathscr{I}_C(n)) = 0$ for all $n \neq r-2$ by Lemma 2.2.3. Hence the H-R module of C has rank one. On the other hand, by the exact sequence:

$$0 \to \mathscr{I}_C(r-2) \to \mathcal{O}_{\mathbb{P}^3}(r-2) \to \mathcal{O}_C(r-2) \to 0,$$

we have the next exact sequence in cohomology:

$$\begin{array}{l} 0 \longrightarrow H^0(\mathbb{P}^3, \mathscr{I}_C(r-2)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r-2)) \longrightarrow H^0(C, \mathcal{O}_C(r-2)) \\ \longrightarrow H^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r-2)) \longrightarrow H^1(C, \mathcal{O}_C(r-2)) \\ \longrightarrow H^2(\mathbb{P}^3, \mathscr{I}_C(r-2)) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r-2)) \longrightarrow H^2(C, \mathcal{O}_C(r-2)) \longrightarrow \dots \end{array} \tag{2.2}$$

By Serre duality $H^1(C, \mathcal{O}_C(r-2)) \cong H^0(C, \omega_C \otimes (\mathcal{O}_C(r-2)^\vee))$, thus using Riemann-Roch we have that:

$$\chi(\mathcal{O}_C(r-2)) + 1 = (r-2)d_r + 1 - g_r + 1 = \binom{r+1}{3} + 1 \tag{2.3}$$

On the other hand, $h^0(\mathbb{P}^3, \mathscr{I}_C(r-2)) = 0$ by Corollary 2.2.4, thus from the sequence 2.2 it follows that:

$$h^0(C, \mathcal{O}_C(r-2)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r-2)) + h^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) = \binom{r-2+3}{3} + 1. \tag{2.4}$$

Then the equations (2.3) and (2.4) imply that $h^1(C, \mathcal{O}_C(r-2)) = 1$; therefore $C \in \mathscr{M}_r$.

Since $\mathscr{L}^{r-3}\mathscr{A}$ is irreducible, there exists an irreducible component of $\mathscr{M}$ that contains it. We denote this component by $\mathscr{M}'$. Let C be a generic element of $\mathscr{M}'$. Observe that we have an exact sequence as 2.2 for the ideal of C. By Serre duality we have that $h^1(C, \mathcal{O}_C(r-2)) = h^0(C, K_C - (r-2)H) = 1$. Thus

$$\begin{aligned} h^0(C, \mathcal{O}_C(r-2)) - 1 &= h^0(C, \mathcal{O}_C(r-2)) - h^1(C, \mathcal{O}_C(r-2)) = \chi\mathcal{O}_C(r-2) \\ &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r-2)) = h^0(C, \mathcal{O}_C(r-2)) - h^1(\mathbb{P}^3, \mathscr{I}_C(r-2)). \end{aligned}$$

Therefore $h^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) = 1$ but $rkM(C) = 1$ implies that $C \in \mathscr{L}^{r-3}\mathscr{A}$.

The proof in which r is even is analogous.

□

We denote by $\mathscr{M}_r'$ the component $\mathscr{M}_r$ that is equal to $\mathscr{L}^{r-3}\mathscr{C}^h$ in the odd case and to $\mathscr{L}^{r-3}\mathscr{A}$ in the even case. In particular, Theorem 2.4.2 tells us that $\mathscr{D}_r = \mathscr{L}_{r+1}\mathscr{M}_r'$ and $\mathscr{M}_r' = \mathscr{L}_{r+1}\mathscr{D}_{r-1}$ for all $r \geq 4$.

The next Proposition is inspired in Proposition 2.3.1 and the proof follows from similar arguments.

Lemma 2.4.3. *If $\{C_t\}_{t\in T}$ is a family of curves in $\mathbb{P}^3$ such that:*

- *C_t is smooth for a generic element.*
- *$deg(C_t) = \frac{r(r+1)}{2}$ and $g(C_t) = \frac{r(r+1)(2r-5)}{6} + 1$*
- *C_0 is a smooth element of $\mathscr{M}_r'$ and*

- *the sheaf* $\omega_{C_t} \otimes (\mathcal{O}_{C_t}(r-2)^\vee)$ *has not global sections for* t *generic,*

then the generic element of $\{C_t\}_{t\in T}$ *is ACM.*

Proof. For all $n \geq 0$ and t generic we have by semi-continuity that:

$$0 \leq h^1(\mathbb{P}^3, \mathscr{I}_{C_t}(n)) \leq h^1(\mathbb{P}^3, \mathscr{I}_{C_0}(n)).$$

Since $C_0 \in \mathscr{M}'_r$, by Theorem 2.4.2, C_0 is contained in $\overline{\mathscr{L}^{r-3}\mathscr{C}^h}$ if r is even or in $\overline{\mathscr{L}^{r-3}\mathscr{A}}$ if r is odd. In both cases we have that $h^1(C_0, \mathscr{I}_{C_0}(n)) = 0$ for all $n \neq r-2$. Then $h^1(C_t, \mathscr{I}_{C_t}(n)) = 0$ for all $n \neq r-2$ and t generic.

By hypothesis $h^0(C_t, \omega_{C_t} \otimes (\mathcal{O}_{C_t}(r-2)^\vee)) = h^1(\mathbb{P}^3, \mathcal{O}_{C_t}(r-2)) = 0$ and we have an exact sequence as (2.2). Therefore

$$\begin{aligned} h^1(\mathbb{P}^3, \mathscr{I}_C(r-2)) &= h^0(\mathbb{P}^3, \mathcal{O}_{C_t}(r-2)) - h^0(\mathbb{P}^3, \mathcal{O}_{C_t}(r-2)) \\ &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(r-2)) - (\chi\mathcal{O}_{C_t} - h^1(\mathbb{P}^3, \mathcal{O}_{C_t}(r-2))) = 0. \end{aligned}$$

We conclude that $h^1(C_t, \mathscr{I}_{C_t}(r-2)) = 0$ for t generic. Then the Hartshorne-Rao module of a generic element of the family C_t is the trivial module. That is equivalent to being ACM.

□

A consequence of this Lemma, Theorem 2.4.2 and Theorem 1.2.3 is the following Corollary that is the Theorem B of the introduction:

Corollary 2.4.4. *If* r *is an odd number, then the family* $\mathscr{L}^{r-3}\mathscr{A}$ *is contained in the closure* $\overline{\mathscr{C}_r}$. *If* r *is an even number, then the family* $\mathscr{L}^{r-3}\mathscr{C}^h$ *is contained in* $\overline{\mathscr{C}_r}$.

Proof. By Theorem 2.4.2 when r is an odd number, the family $\mathscr{L}^{r-3}\mathscr{A}$ coincides with the family $\mathscr{M}'_r$ and if r is an even number, the family $\mathscr{L}^{r-3}\mathscr{C}^h$ is equal to $\mathscr{M}'_r$. Lemma 2.4.3 proves that the family $\mathscr{M}'_r$ is contained in an ACM component but by Theorem 1.2.3 the only ACM component is $\overline{\mathscr{C}_r}$.

□

Chapter 3

Curves of degree 6 and genus 3

This is a technical Chapter and it is completely dedicated to the case when r is equal to 3, namely the Hilbert scheme of curves locally Cohen-Macaulay in $\mathbb{P}^3$ of degree 6 and genus 3. In [Amr98, Thm.7] it is proved that this scheme is reducible and has 3 components. Also, it gives a description of modules that appear as H-R modules of locally Cohen-Macaulay curves of degree 6 and genus 3. We use this work to describe some families in the components of this scheme that we construct with [M2]. For this section, we denote by $M(n)$ the graded module M shifted by n.

The three components of the Hilbert scheme $\mathscr{H}_3$ are the following:

1. **The main component:** This component is the closure of the ACM curves whose dimension is 24 and coincides with the closure of the family of smooth curves of degree 6 and genus 3. The generic element has the following minimal free resolution:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^4 \longrightarrow \mathscr{I}_C \longrightarrow 0.$$

   In this component, the generic element has trivial H-R module. Nevertheless, there exist two families of codimension one with H-R module of rank one:

   - **Hyperelliptic curves:** We denote this family $\mathscr{C}^h$, as in the Chapter 2. The dimension of $\mathscr{C}^h$ is 23, the elements in this family have H-R module isomorphic to $k(-2)$ and the ideal of a generic element in it has the following minimal free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-6) \to \mathcal{O}_{\mathbb{P}^3}(-5)^4 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathscr{I}_3^h \to 0.$$

   - **"Antenitas" family:** As in Chapter 2 this family is denoted by $\mathscr{A}$. The dimension of this family is 23, the elements in $\mathscr{A}$ have H-R module isomorphic to $k(-1)$ and the ideal of a generic element in it has minimal free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-5) \to \mathcal{O}_{\mathbb{P}^3}(-5) \oplus \mathcal{O}_{\mathbb{P}^3}(-4)^4 \to \mathcal{O}_{\mathbb{P}^3}(-4) \oplus \mathcal{O}_{\mathbb{P}^3}(-3)^4 \to \mathscr{I}_C \to 0.$$

2. **The component of reducible curves:** We denote this component by $\mathscr{R}_3$, the generic element is the union of a plane quartic with a conic that intersects it at a point. The dimension of this component is 24 and the H-R module that appears for the elements of this component is $M = k[x, y, z, w](-1)/\langle x, y, z, w^3\rangle$ by [Amr98, Thm.4]. The minimal free resolution of the ideal of an element in $\mathscr{R}_3$ is:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-7) \to \mathcal{O}_{\mathbb{P}^3}(-6)^3 \oplus \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3}(-5)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-3) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathscr{I}_C \to 0.$$

3. **The extremal component:** By [MDP96, Thm.4.3] there exists a component $\mathscr{E}_3$ generically non reduced of dimension 30, whose generic element is an extremal curve. The H-R

module of a generic element in this component is $M = k[x,y,z,w](2)/\langle x,y,F,G\rangle$ with F, G polynomials of degree 3 and 7 respectively. The minimal free resolution of the ideal of an element in $\mathscr{E}_3$ is:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-10) \to \mathcal{O}_{\mathbb{P}^3}(-9)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-7) \to \mathcal{O}_{\mathbb{P}^3}(-8) \oplus \mathcal{O}_{\mathbb{P}^3}(-6) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^2 \to \mathscr{I}_C \to 0.$$

For any sum of positive integers $\sum_{i=1}^{n} a_i$ equals to six, we construct curves C in $\mathscr{H}_3$ that are the union $C_1 \cup \ldots \cup C_n$ of smooth irreducible curves C_i of degree $deg(C_i) = a_i$ and considering all possible arithmetic genus for every C_i. The code in [M2] of all these families is found in the Appendix. We separate all these families into four lists that correspond to the different Betti tables that appear in $\mathscr{H}_3$. At the end of any list, we add a diagram that represents the degeneration between these families.

Betti tables. Given a curve C in $\mathbb{P}^3$, (which we assume non-degenerate) the Betti table of C presents its graded Betti numbers. With this information is possible to recover the minimal free resolution of the ideal associated to C. If C has the following Betti table:

0	1	2	…	n
1	$\beta_{1,1}$	$\beta_{1,1}$	$\cdots$	$\beta_{n,1}$
2	$\beta_{1,2}$	$\beta_{2,2}$	$\cdots$	$\beta_{n,2}$
$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$
m	$\beta_{1,m}$	$\beta_{2,m}$	$\cdots$	$\beta_{n,m}$

then the associated ideal of C has the next minimal free resolution:

$$0 \to \oplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(-(n+i))^{\beta_{n,j}} \to \cdots \to \oplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^3}(-(1+i))^{\beta_{1,j}} \to \mathscr{I}_C \to 0$$

with $\beta_{i,j}$ a natural number for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. We write "$-$" instead of 0 in a Betti table.

Example 4. The ACM curves of degree 6 and genus 3 have minimal free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4)^3 \to \mathcal{O}_{\mathbb{P}^3}(-3)^4 \to \mathscr{I}_C \to 0.$$

and the Betti table is:

0	1	2
1	-	-
2	4	3

3.1 The ACM component

We call the component $\overline{\mathscr{C}_3}$ the main component since by [Nai02, Thm.1] this is the only component where there are smooth curves.

We have found 3 different Betti tables corresponding to curves in $\overline{\mathscr{C}_3}$ and for each table, we have some families that we list below:

- Families $Ai \subseteq \mathscr{C}_3$ with $i \in \{0, \ldots, 14\}$; the elements in these families are smooth points in the Hilbert scheme, have H-R module trivial, and the next Betti table:

0	1	2
1	-	-
2	4	3

- A0 The generic point of $\mathscr{C}_3$ is a smooth ACM curve. The family of all ACM smooth curves is an open set of $\overline{\mathscr{C}_3}$ (dimension 24).
- A1 Line + quintic of genus 2: Take a curve C of bidegree $(2, 3)$ in a smooth quadric and a line L that intersect C in two points (dimension 22).
- A2 Line + quintic of genus 1: The elements in this family are a quintic of genus one union a trisecant line (dimension 21).
- A3 Twisted cubic + plane cubic: For three generic points in $\mathbb{P}^3$ take a twisted cubic through the points and in the plane defined by the points take a plane cubic through to the points (dimension 21).
- A4 Conic + quartic of genus 1: For three points p, q, s in $\mathbb{P}^3$ take two quadrics that contain the points, the intersection of these quadrics is a quartic of genus one, and add a conic through the three points (dimension 21).
- A5 Two lines + quartic of genus 1: For four points p, q, r, s in $\mathbb{P}^3$ take two quadrics that contain the points, the intersection of these quadrics is a quartic of genus one, and add the line through the points p and q and the line through the points r and s (dimension 20).
- A6 Line + rational quintic: Let L a line in $\mathbb{P}^3$ and choose two distinct points p, q in L. Take a curve $C_0 \in \mathscr{C}_3$ and link to a curve $C_1 \in \mathscr{C}_4$ by the complete intersection of two quartics that contain the curve C_0 and the points p, q. Now take two quartics that contain the curve C_1 and the line L, the complete intersection of these quartics liked C_1 with a curve $C \in \mathscr{C}_3$ that is the union of L with a rational quintic. (dimension 20).
- A7 Conic + rational quartic: Take a rational quartic and in the four points of the intersection of a general plane with the quartic put a conic through the four points (dimension 20).
- A8 Two twisted cubic: For four generic points in $\mathbb{P}^3$ take two different twisted cubic that passes through the four points. The union of these cubics is a curve in $\mathscr{C}_3$ (dimension 20).
- A9 Two lines + rational quartic: Take a rational quartic and put two lines over two pairs of points in the intersection of the quartic with a general plane (dimension 19).
- A10 Line + conic + twisted cubic: For three generic points in $\mathbb{P}^3$ take a twisted cubic that passes through the three points and a conic in the plane determined by these that through for the three points. Now take a line in the same plane and the union of the twisted cubic, the conic and the line is a curve in $\mathscr{C}_3$ (dimension 19).
- A11 Three conics: The elements in this family are three conics in three different planes that intersect two by two in two points (dimension 18).
- A12 Three lines + Twisted cubic: The elements in this family are twisted cubics union three secant lines (dimension 18).
- A13 Four lines + conic: Let $p_1, \ldots, p_4$ four points in a plane H and $p_5 \notin H$. Take Q a conic in H through $p_1, \ldots, p_4$, L_1 the line between p_1 and p_5 and L_2 the line between p_2 and p_5. Let $p_6 \in L_1 - \{p_1, p_5\}$ and L_3 the line between p_3 and p_6. Lastly take a point $p_7 \in L_3 - \{p_3, p_6\}$ and the line L_4 between p_4 and p_7. The curves in this family be $Q \cup L_1 \cup L_2 \cup L_3 \cup L_4$ (dimension 17).
- A14 Six lines: Given four generic points $p_1, \ldots, p_4 \in \mathbb{P}^3$ we call the tetrahedron T determinate by $p_1, \ldots, p_4$ to the union of the six lines that contain pairs of these points. The family of all tetrahedra is an irreducible subfamily of $\mathscr{C}_3$ (dimension 12).

Remark 3.1.1. All these families have as generic initial ideal the ideal of the triple line $\langle x^3, x^2y, xy^2, y^3\rangle$.

In Figure 3.1 we add a diagram that describes the following:

- The pointed lines separated the families by dimension.
- We draw an arrow from a family A to a family B if there exists a flat limit from an element in A to an element of B. The code of these limits is in Appendix C.
- In the shaded part is the families $\mathscr{C}^h$ and $\mathscr{A}$ that are not ACM but we include them since there are some degenerations to some subfamilies of these that we will describe later.
- We do not write arrows where do not exist plane limits. The way to check that do not exist a flat limit from an element C_a in a family A to an element C_b of a family B is to assume that it exists and follows all components of the curve C_a under such limit and find a contradiction. For example, since a quartic of genus 1 cannot degenerate to a rational quartic, there is not exist a flat limit from elements of the family A_4 to the family A_7.
- Of course there exist flat limits from a generic element in $\mathscr{C}_3$ to any family A_i and from any family, A_i to the triple line $\langle x^3, x^2y, xy^2, y^3\rangle$. But we do not draw these lines so as not to overload the diagram

• Families $Bi \subseteq \mathscr{C}^h$ with $i \in \{0, \dots, 7\}$: the elements in this family are smooth points in the Hilbert scheme, has H-R module $k(-2)$ and the next Betti table:

0	1	2	3
1	1	-	-
2	-	-	-
3	3	4	1

B0 The generic element in $\mathscr{C}^h$ is a smooth irreducible curve of bidegree $(2,4)$ in a smooth quadric surface (dimension 23).

B1 Two plane cubics: The elements in this family are two plane cubics in two different planes that intersect at two points (dimension 22).

B2 Line + quintic of genus 2: The elements in this family be a quintic of bidegree $(2,3)$ and a line of bidegree $(0,1)$ in a smooth quadric surface (dimension 20).

B3 Line + conic + plane cubic: The elements in this family be a line and a conic in the same plane and a plane cubic in another plane that does not intersect with the line and intersects with the conic in two points (dimension 20).

B4 Conic + rational quartic: The elements in this family be a quartic of bidegree $(3,1)$ and a conic of bidegree $(1,1)$ in a smooth quadric surface(dimension 19).

B5 Line + rational quintic: It is the same construction that in the case ACM but now take $C_0 \in \mathscr{C}^h$ (dimension 18).

B6 Two lines + two conics: In two different planes take a line and a conic so that the conics intersect at two points (dimension 18).

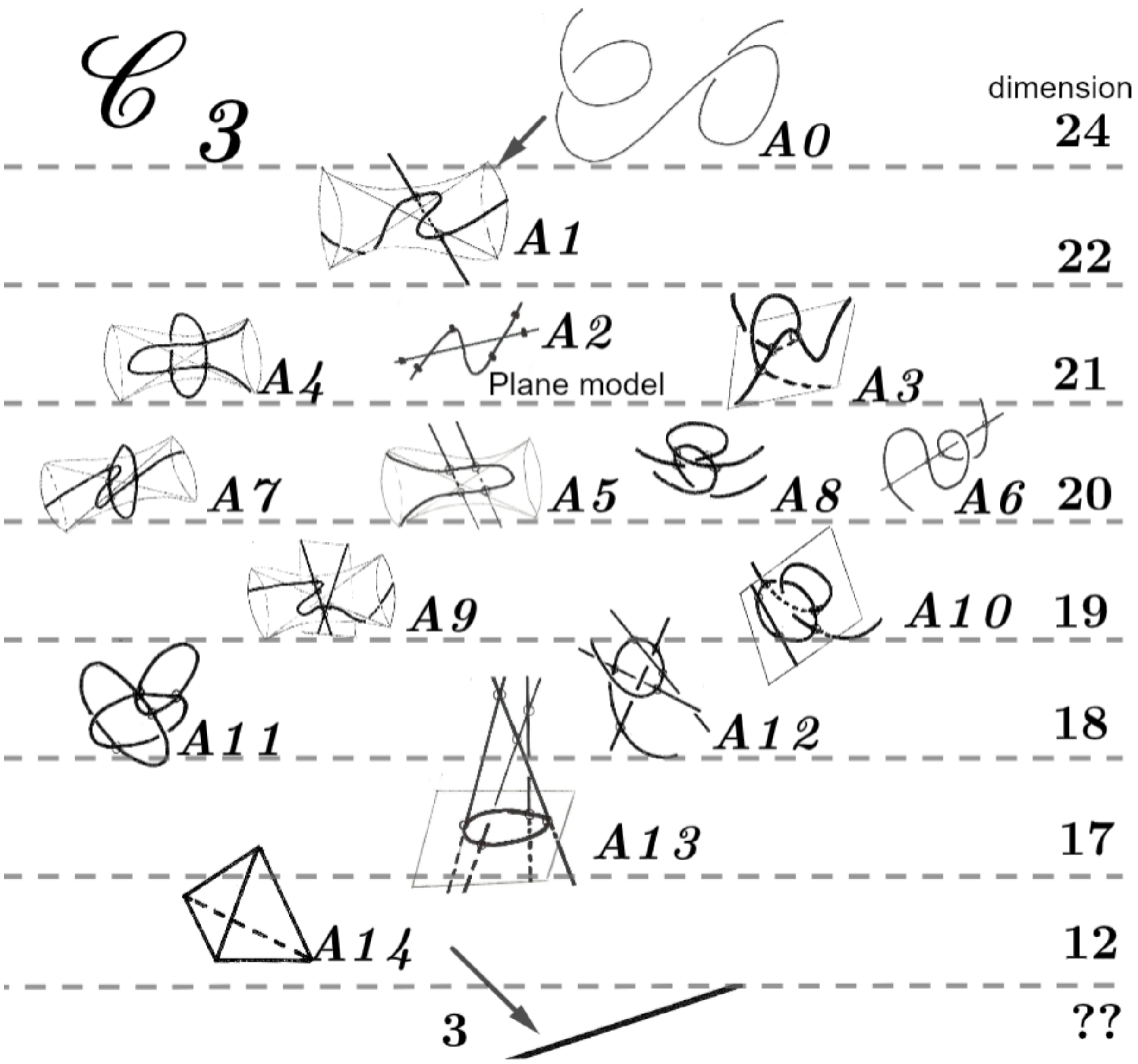

Figure 3.1: ACM curves

B7 Six lines: A triangle is a curve determined by the union of all lines between three points in general position. The elements in this family are the union of two triangles that intersect at two points (dimension 16).

Remark 3.1.2. All these families have as generic initial ideal the ideal $\langle x^2, xy^3, y^4, xy^2z\rangle$ that decompose as $\langle x^2, xy^2, y^4\rangle$ and $\langle x^2, y^3, z\rangle$ (an embedding point of degree 6).

As before we include Figure 3.2 to represent the flat limits inside of $\mathscr{C}^h$.

- Families $Ci \subseteq \overline{\mathscr{A}}$ with $i \in \{0, \ldots, 3\}$: the elements in this family are smooth points in the Hilbert scheme, has H-R module $k(-1)$ and the next Betti table:

0	1	2	3
1	-	-	-
2	4	4	1
3	1	1	-

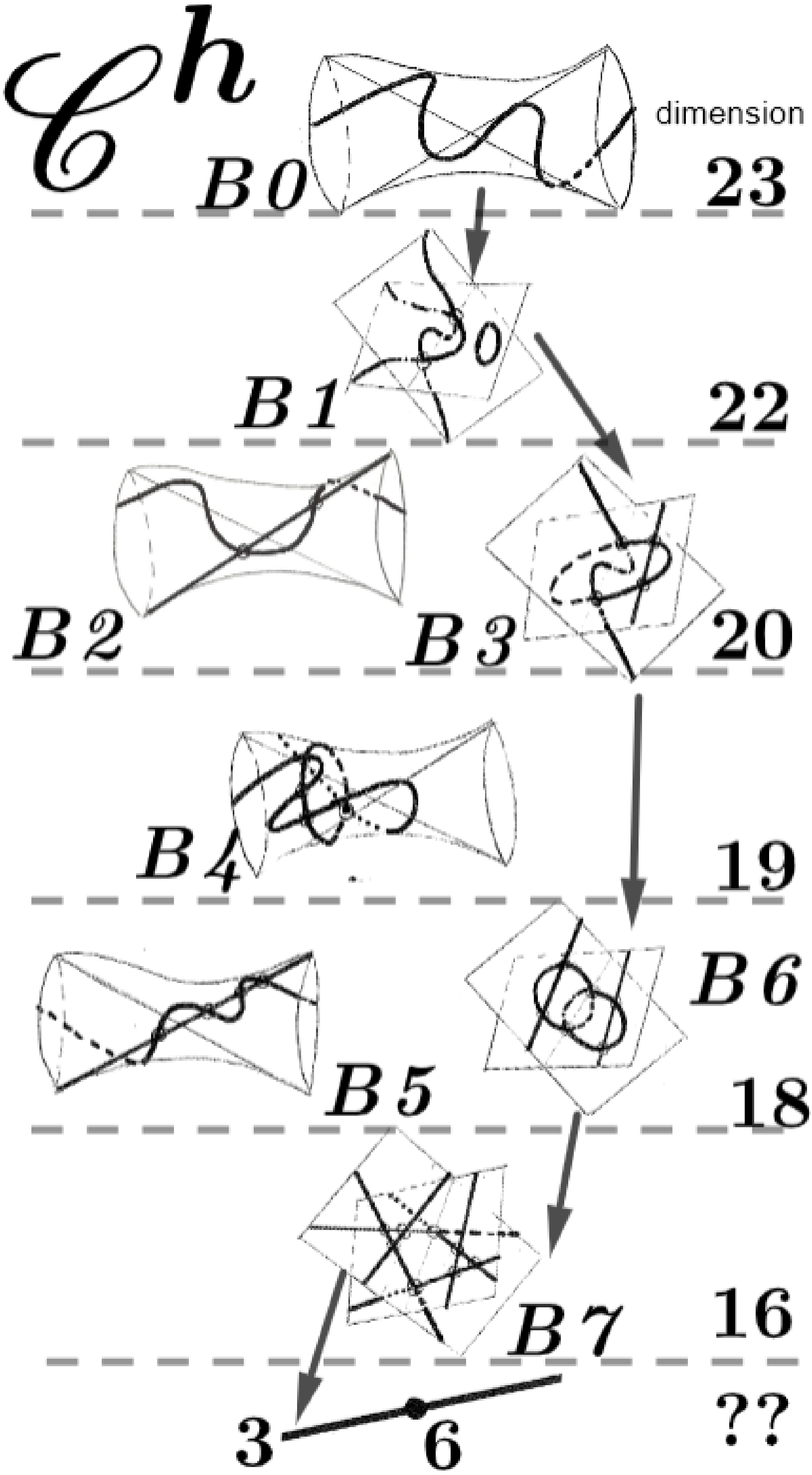

Figure 3.2: Hyperelliptic curves

C0 The elements in $\mathscr{A}$ are a plane quartic with two skew lines that intersect the quartic at two different points (dimension 23).

C1 Three lines + plane cubic: The family consists of two skew lines each of them intersect a plane cubic in one point and other line in the plane of the cubic (dimension 20).

C2 Two lines + two conics: The family consists of two conics in the same plane H and two lines outside of H and every line intersects with some conic (dimension 19).

C3 Four lines + conic: In a plane H take a conic and two lines, the other two lines are outside of H and through two different points of the conic (dimension 18).

Remark 3.1.3. All these families have as generic initial ideal the ideal $\langle x^3, x^2y, xy^2, x^2z\rangle$ that decompose as $\langle x^2, xy^2, y^4\rangle$ and $\langle x^3, y, z\rangle$ (an embedding point of degree 3).

In Figure 3.3 we represent the flat limits on $\mathscr{A}$.

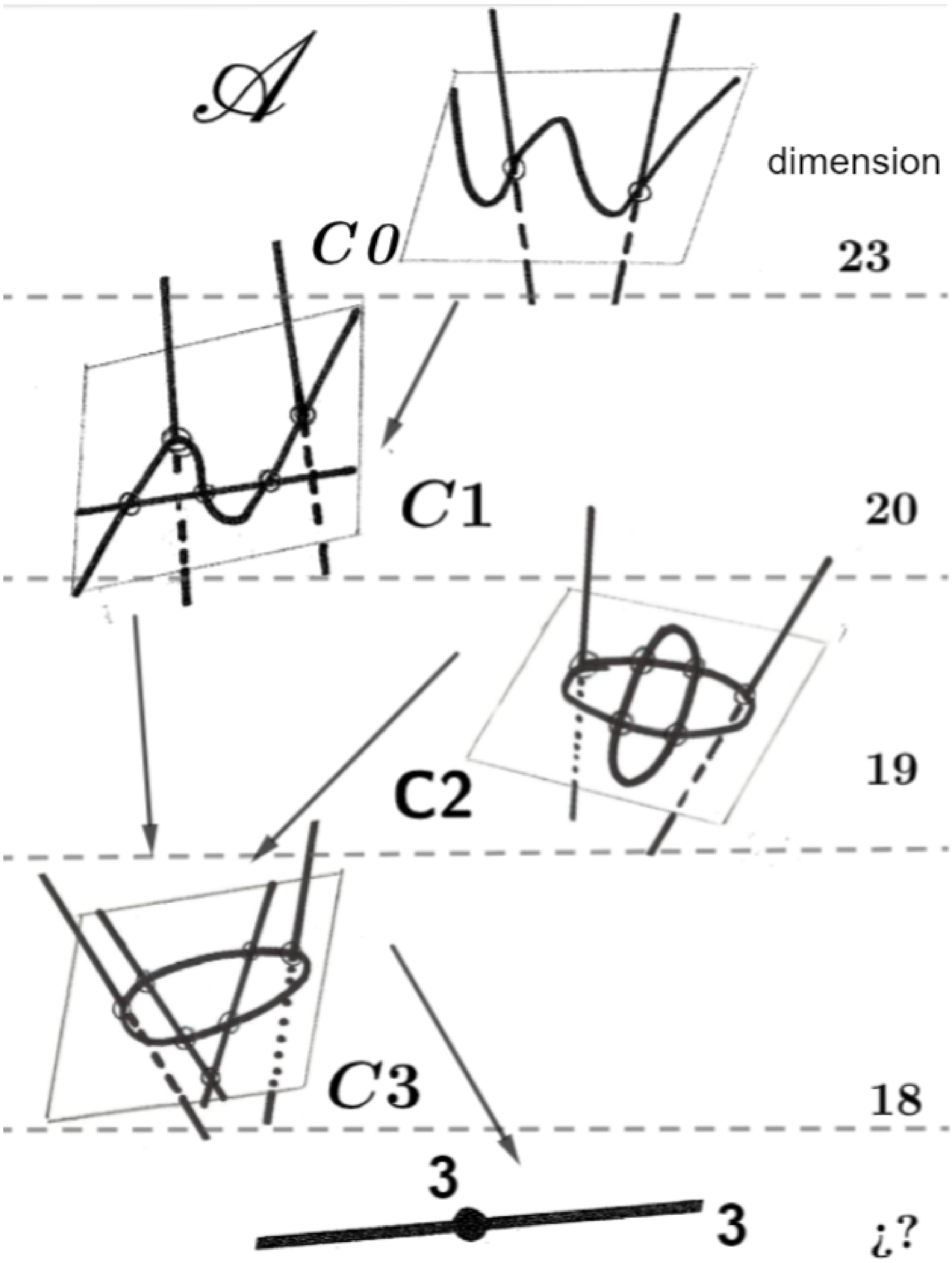

Figure 3.3: Reducible curves in $\overline{\mathscr{C}_3}$

3.2 The component of reducible curves

In this component there are no smooth curves, the generic element is reducible and the H-R module that appears for the elements of this component is $M = k[x, y, z, w](-1)/\langle x, y, z, w^3\rangle$.

Here there is only one Betti table and we list some families below:

0	1	2	3
1	1	-	-
2	1	1	-
3	-	-	-
4	2	3	1

D0 The elements in $\mathscr{R}_3$ as we saw before are the union of a plane quartic and a conic that intersect in one point. The generic elements in this family are smooth points in the Hilbert scheme (dimension 24).

D1 Two lines + plane quartic (I): The elements of this family are a plane quartic, a line that intersects the conic in one point, and another line that does not intersect the quartic and intersects the other line in one point (dimension 23).

D2 Three conics: Take two conics in the same plane and the third conic in another plane that intersects with another conic at one point (dimension 23).

D3 Line + conic + plane cubic: In a plane take a cubic and a line and for one point in the cubic (or the line) take a conic through the point in another plane (dimension 21).

D4 Two lines + two conics: Take two lines and a conic in the same plane and the second conic in another plane that intersects the other conic (or some line) at one point (dimension 19).

- In this component we find two special families whose elements have the same Betti table but are singular points of the Hilbert scheme:

D5 Two lines + plane quartic (II): The elements of this family are a plane quartic and two lines that intersects the conic in the same point (dimension 22).

D6 Four lines + conic: In a plane H takes a conic and two lines, the other two lines are outside of H and through the same point in the intersection of the other lines with the conic. (dimension 16).

Remark 3.2.1. All these families have as generic initial ideal the ideal $\langle x^2, xy^2, y^5, y^4z\rangle$ that decompose as $\langle x^2, xy^2, y^4\rangle$ and $\langle x, y^5, z\rangle$ (an embedding point of degree 5).

As in the previous section in Figure 3.4 we have a diagram with the flat limits in $\mathscr{R}_3$.

3.3 The extremal component

By [MDP96, Thm.4.3] there exists a component $\mathscr{E}_3$ generically non-reduced of dimension 30, whose generic element is an extremal curve. By [Amr00, Thm.4.3] the H-R module of a generic element in this component is $M = k[x, y, z, w](2)/\langle x, y, F, G\rangle$ with F, G polynomials of degree 3 and 7 respectively.

We only identify a family in this component. This family consist of curves with ideal associated $\mathscr{I}_E = \langle x^2, xy, y^6, xG - Fy^5\rangle$ with $F, G \in k[z, w]$ of degree 3 and 7 respectively. The Betti table of an element in this family be:

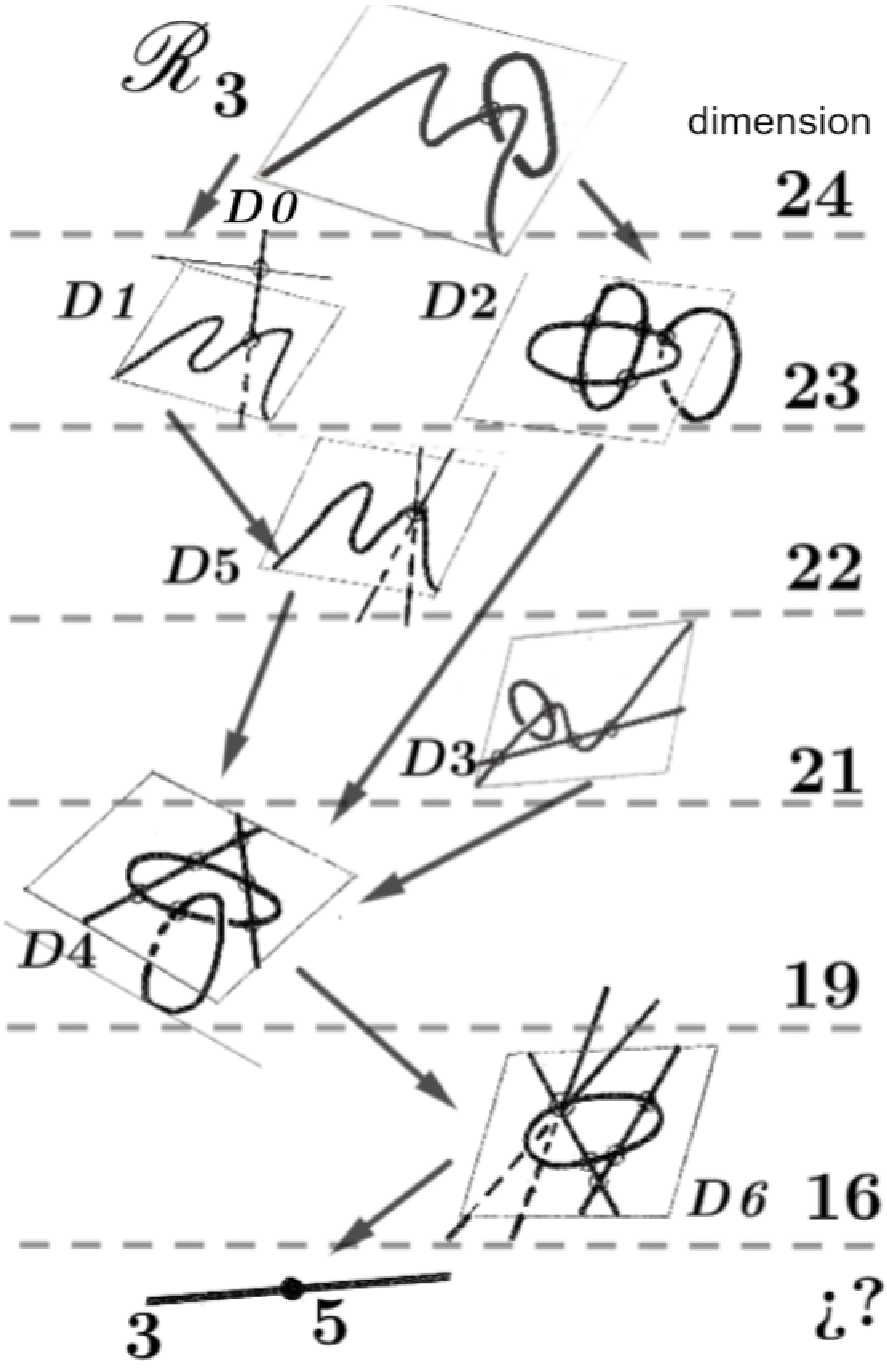

Figure 3.4: Curves in $\mathscr{R}_3$

0	1	2	3
1	2	1	-
2	-	-	-
3	-	-	-
4	-	-	-
5	1	1	-
6	-	-	-
7	1	2	1

Chapter 4

The main component of curves of degree 6 and genus 3

In this chapter, we focus on the main component $\overline{\mathscr{C}_3}$. Note that in the case when r is 2 we have that $\mathscr{H}_2 = \overline{\mathscr{C}_2}$ and the only lcm curves in this space are the ACM curves. This changes in the case $r = 3$. As we saw in the last chapter, the closure of the ACM curves is only one of three components of $\mathscr{H}_3$. Inside this component, there are exactly two families of codimension one in $(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm}$ as the following Lemma shows:

Lemma 4.0.1. *The set of locally Cohen-Macaulay curves in $\overline{\mathscr{C}_3} - \mathscr{C}_3$ is equal to the set of locally Cohen-Macaulay curves in the union $\overline{\mathscr{A}} \cup \overline{\mathscr{C}^h}$. In other words,*

$$(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm} = (\overline{\mathscr{A}} \cup \overline{\mathscr{C}^h})^{lcm}.$$

Moreover, taking closure we have that

$$\overline{(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm}} = \overline{\mathscr{A}} \cup \overline{\mathscr{C}^h}. \tag{4.1}$$

Proof. The union $(\overline{\mathscr{A}} \cup \overline{\mathscr{C}^h})^{lcm}$ is contained in $(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm}$ since the elements in $\overline{\mathscr{C}_3} - \mathscr{C}_3$ are curves with non-trivial H-R-module. By [Amr00, Thm.1.9] any curve in $(\overline{\mathscr{C}_3})^{lcm}$ with non-trivial H-R-module is contained in $(\overline{\mathscr{A}})^{lcm}$ or $(\overline{\mathscr{C}^h})^{lcm}$. Then $(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm} \subseteq (\overline{\mathscr{A}} \cup \overline{\mathscr{C}^h})^{lcm}$. □

Let $N^1_{\mathbb{Z}}(\overline{\mathscr{C}_3})$ be the quotient of the Cartier divisors on $\overline{\mathscr{C}_3}$ modulo numerical equivalence and $N^1(\overline{\mathscr{C}_3})$ be the tensor product of $N^1_{\mathbb{Z}}(\overline{\mathscr{C}_3})$ with the field of the real numbers $\mathbb{R}$. We do not know the behavior of $\overline{\mathscr{C}_3}$ outside of the family of locally Cohen-Macaulay curves, thus we can not claim that the dimension of $N^1(\overline{\mathscr{C}_3})$ is finite. But the equality (4.1) allows us to consider the normal scheme $\mathscr{B} := \overline{(\overline{\mathscr{C}_3} - \mathscr{C}_3)^{lcm}} \cup \mathscr{C}_3$ as a partial compactification of $\mathscr{C}_3$. In this case $N^1(\mathscr{B})$ has finite dimension.

Furthermore, we claim that the classes of the families $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$ are linearly independent in $N^1(\mathscr{B})$. To show this, we begin by defining a birational map: for a generic element in $\mathscr{C}_3$, its ideal is generated by exactly 4 linearly independent cubics. These four cubics define a linear subspace of dimension 3 in the space of cubics $\mathbb{P}^{19}$. Thus we can define the map:

$$h : \overline{\mathscr{C}_3} \dashrightarrow \mathbb{G}(3, 19)$$

$$C \longmapsto \mathbb{P}(H^0(\mathbb{P}^3, \mathscr{I}_C(3))) .$$

The map h is not a morphism: by [Amr00, Prop. 3.13 and 3.15], the intersection of $\overline{\mathscr{C}^h}$ with $\mathscr{R}_3$ is non-empty and the general element in this intersection has five linearly independent cubics. Similarly, the intersection of $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$ with $\mathscr{E}_3$ is not empty and the ideals of elements in this intersection have 7 linearly independent cubics. In particular, h is not well defined even in $\mathscr{B}$. Nevertheless, this occurs outside of $\mathscr{C}_3 \cup \mathscr{C}^h \cup \mathscr{A}$ which implies that h is well defined over $\mathscr{B}$ in codimension one.

On the other hand, observe that for a generic C in $\mathscr{C}_3$, the four defining cubics determine C scheme-theoretically. This implies that the map h is birational.

We aim to show that this map contracts the divisors $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$. In order to prove this we need the following two Lemmas:

Lemma 4.0.2. *The map h contracts $\mathscr{C}^h$ to a family of dimension* 9.

Proof. Given a quadric in $\mathbb{P}^3_{[x,y,z,w]}$ we can obtain an element of the Grasmannian of 3-planes in the space $\mathbb{P}^{19}$ of cubics in $\mathbb{P}^3_{[x,y,z,w]}$ by taking the subspace generated by the quadric multiplied by the four variables x, y, z, w. Thus we can define a morphism:

$$f : \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))) \to \mathbb{G}(3, 19)$$
$$q \longmapsto V_q := \langle q \cdot x, q \cdot y, q \cdot z, q \cdot w \rangle.$$

Set $Q := Im\ f$. Observe that f is an embedding, thus Q is a subvariety of $\mathbb{G}(3, 19)$ of dimension 9.

The cubics that contain a curve C of $\mathscr{C}^h$ are multiples of the smooth quadric q_0 that contains C. In particular, the elements $q_0 \cdot x, q_0 \cdot y, q_0 \cdot z$ and $q_0 \cdot w$ generate the space $H^0(\mathbb{P}^3, \mathscr{I}_C(3))$. Thus, by the definition of h, we have that $h(\mathscr{C}^h) \subseteq Q$. Let us argue that in fact h is dominant onto Q. Let q be a smooth quadric in $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)))$. Let L and L' be two lines in the same ruling of q and p an irreducible quartic that contains them. The residual curve to $L \cup L'$ in the intersection of q and p is a curve of bidegree $(2, 4)$ in q and therefore an element of $\mathscr{C}^h$, thus $f(q) \in h(\mathscr{C}^h)$. That implies that $\overline{h(\mathscr{C}^h)} = Q$.

□

Lemma 4.0.3. *The map h contracts $\mathscr{A}$ to a family of dimension* 11.

Proof. To prove this we give an auxiliary set-theoretic map that helps us understand the image under h of the family $\mathscr{A}$: given two lines in general position L and L' in $\mathbb{P}^3$, there exist exactly four linearly independent quadric surfaces that contain them. Thus for any linear equation F of $\mathbb{P}^3$ we can multiply F by these quadrics and obtain a 3-space in the space of cubics in $\mathbb{P}^3$. Therefore, we have the following map:

$$g : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \times \mathbb{G}(1, 3)^2 \dashrightarrow \mathbb{G}(3, 19)$$
$$(F, L, L') \longmapsto \mathbb{P}(F \cdot H^0(\mathbb{P}^3, \mathscr{I}_{L \cup L'}(2)))$$

where $F \cdot H^0(\mathbb{P}^3, \mathscr{I}_{L \cup L'}(2))$ denotes the subspace generated by the four quadrics that contain the lines L and L' multiplied by the lineal equation F. The map g is generically injective, then the subvariety $U := \overline{Im\ g}$ has dimension 11.
Since a generic curve C in $\mathscr{A}$ is the union of a plane quartic q_0 and two skew lines, each of them intersecting q_0 at a point, then the cubics in $H^0(\mathbb{P}^3, \mathscr{I}_C(3))$ must contain the plane F that contains q_0. Consequently, any of such cubics have to be $F \cdot Q$ for some Q of degree 2 that contains the two lines of C. This implies $h(\mathscr{A}) \subseteq U$. Additionally, given a generic element $g(F, L, L') \in U$, let q be a general plane quartic in F that passes through the points of intersection of L and L' with F. Then q union the lines L and L' is an element of $\mathscr{A}$. That implies $\overline{g(\mathscr{A})} = U$.

□

Remark 4.0.4. A birational map $\varphi : X - \to Y$ between normal projective varieties is called contracting if the inverse map φ^{-1} does not contract any divisors. Let $ex(\varphi)$ be the subcone of Eff(X) spanned by the φ-exceptional effective divisors. By [Oka16, Lem.2.7] the extremal rays of this cone are generated by the φ-exceptional prime divisors, then by [HK00, Lem.1.6] $ex(\varphi)$ is a extremal face of the effective cone and therefore the φ-exceptional prime divisors are extremal rays of $Eff(X)$. On the other hand, since two different prime divisors have different support, the second part of [Oka16, Lem.2.7] proved that any two φ-exceptional prime divisors are linearly independent.

Now we are able to prove the main result of this section:

Theorem 4.0.5. *The classes of $\overline{\mathscr{A}}$ and $\overline{\mathscr{C}^h}$ in $N^1(\mathscr{B})$ are linearly independent and each of them spans an extremal ray of the effective cone*

$$Eff(\overline{\mathscr{C}_3}) \subseteq N^1(\mathscr{B}).$$

Proof. Lemmas 4.0.2 and 4.0.3 prove that the birational map h is a divisorial contraction that contracts the divisors $\overline{\mathscr{A}}$ and $\overline{\mathscr{C}^h}$ onto the subscheme $Q \cup U$. This implies that the divisors $\overline{\mathscr{A}}$ and $\overline{\mathscr{C}^h}$ are h-exceptionals and by remark 4.0.4 the theorem follows. □

Observe that for each $r \geq 3$ we can define the map

$$h_r : \overline{\mathscr{C}_r} \dashrightarrow \mathbb{G}(r, \tbinom{r+3}{3})$$

$$C \longmapsto \mathbb{P}(H^0(\mathbb{P}^3, \mathscr{I}_C(r))) .$$

Nevertheless, we cannot repeat the argument of Theorem 4.0.5. In general, we do not know if the map h_r is well defined in codimension 2 and which divisors are contracted by h_r.

The definition of $\mathscr{B}$ and Theorem 4.0.5 tell us that $N^1(\mathscr{B})$ is generated by the classes of $\overline{\mathscr{A}}$, $\overline{\mathscr{C}^h}$ and generators of $N^1(\mathscr{C}_3)$. In order to compute the dimension of the space $N^1(\mathscr{C}_3)$, we consider the incidence variety W_r defined in remark 1.2.4. For a generic element C_{r+1} in $\mathscr{C}_{r+1}$, we can identify any element (C_r, C_{r+1}) of the fiber $\pi_1^{-1}(C_{r+1})$ with the 2-plane of surfaces of degree $r+1$ in $H^0(\mathbb{P}^3, \mathscr{I}_{C_{r+1}}(r+1))$ that determines the union $C_r \cup C_{r+1}$. Since the curves in $\mathscr{C}_{r+1}$ are not contained in surfaces of degree r, these surfaces do not have common components. Thus we have that

$$\pi_1^{-1}(C_{r+1}) \cong G(2, H^0(\mathbb{P}^3, \mathscr{I}_{C_{r+1}}(r+1))),$$

hence $dim_{\mathbb{R}} N^1(\mathscr{C}_{r+1}) + 1 = dim_{\mathbb{R}} N^1(W_r)$, [EH16, pag. 346].

On the other hand, for a generic element C_r of $\mathscr{C}_r$ we can identify any element (C_r, C_{r+1}) of the fiber $\pi_0^{-1}(C_r)$ with the 2-plane of surfaces of degree $r+1$ in $H^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r+1))$ that determine the union $C_r \cup C_{r+1}$. However, there exist elements in $G(2, H^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r+1)))$ whose surfaces have common components. Thus we can identify the fiber with an open set of $G(2, H^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r+1)))$. Therefore, W_r is an open set of an incidence correspondence $\widetilde{W_r} \xrightarrow{\pi} \mathscr{C}_r$ that has as fiber the Grasmannian $G(2, H^0(\mathbb{P}^3, \mathscr{I}_{C_r}(r+1)))$ and such that $\pi|_{W_r} = \pi_0$. Then $dim_{\mathbb{R}} N^1(W_r) \leq dim_{\mathbb{R}} N^1(\widetilde{W_r}) = dim_{\mathbb{R}} N^1(\mathscr{C}_r) + 1$.

These inequalities allow us to write the following:

$$dim_{\mathbb{R}} N^1(\mathscr{C}_{r+1}) \leq dim_{\mathbb{R}} N^1(\mathscr{C}_r). \tag{4.2}$$

Proposition 4.0.6. *For all r we have that $dim_{\mathbb{R}}\ N^1(\mathscr{C}_r) = 1$.*

Proof. We know that $\mathscr{C}_1 = \overline{\mathscr{C}_1} = \mathbb{G}(1,3)$ and using the equation (4.2) we have that for all r

$$dim_{\mathbb{R}} N^1(\mathscr{C}_r) \leq dim_{\mathbb{R}} N^1(\mathscr{C}_1) = dim_{\mathbb{R}} N^1(\mathbb{G}(1,\mathbb{P}^3)) = 1.$$

□

With the previous Proposition, we can prove the following Corollary.

Corollary 4.0.7. *The dimension of the vector space* $N^1(\mathscr{B})$ *is* 3.

Proof. By Proposition 4.0.6 and Lemma 4.0.1 we have that $dim_{\mathbb{R}} N^1(\mathscr{B}) \leq 3$. Therefore, it is enough to find three linearly independent divisors in $N^1(\mathscr{B})$. Consider H the locus of all curves in $\mathscr{B}$ that intersect a fixed-line. This is a divisor of $\mathscr{B}$; we denote its class in $N^1(\mathscr{B})$ by $[H]$. Let $[\overline{\mathscr{C}^h}]$ and $[\overline{\mathscr{A}}]$ be the classes of the divisors $\overline{\mathscr{C}^h}$ and $\overline{\mathscr{A}}$ in $N^1(\mathscr{B})$, respectively. We aim to show that $[H], [\overline{\mathscr{C}^h}]$ and $[\overline{\mathscr{A}}]$ are linearly independent.

Since $[\overline{\mathscr{C}^h}]$ and $[\overline{\mathscr{A}}]$ are linearly independent in $N^1(\overline{\mathscr{C}_3})$ by Theorem 4.0.5, then they are linearly independent in $N^1(\mathscr{B})$, thus there exist two classes α and β in $N_1(\mathscr{B})$ such that $[\overline{\mathscr{A}}] \cdot \alpha \neq 0$, $[\overline{\mathscr{C}^h}] \cdot \alpha = 0$, $[\overline{\mathscr{A}}] \cdot \beta = 0$ and $[\overline{\mathscr{C}^h}] \cdot \beta \neq 0$.

We know that the curves in $\mathscr{C}_3$ are flexible curves, that means, for every twelve general points in $\mathbb{P}^3$ there exists a curve in $\mathscr{C}_3$ passing through them [Per86, Prop. 5.6, Cor. 5.7, Prop. 5.11.bis]. We use this property to construct a curve in $\mathscr{C}_3$. Let Z be a set of 11 points in general position on $\mathbb{P}^3$ and L such that $Z \cap L = \emptyset$. For every $t \in L$ we consider a curve δ_t on $\mathscr{C}_3$ that contains the set of points $Z \cup \{t\}$. This construction gives us a curve δ in $\mathscr{C}_3$ which does not intersect $\overline{\mathscr{A}} \cup \overline{\mathscr{C}^h}$. On the other hand, the intersection of δ with $[H]$ is positive. Let γ be the class of δ in $N_1(\mathscr{B})$.

The following table summarizes the intersection numbers of curves and divisors we have discussed:

$\cdot$	$[\overline{\mathscr{A}}]$	$[\overline{\mathscr{C}^h}]$	$[H]$
α	$\neq 0$	0	?
β	0	$\neq 0$	?
γ	0	0	> 0

Therefore, the classes $[\overline{\mathscr{A}}]$, $[\overline{\mathscr{C}^h}]$ and $[H]$ are linearly independents in $N^1(\mathscr{B})$ and hence form a base of this space.

□

Initially, we expect to compute the dimension of $N^1(\overline{\mathscr{C}_3})$ but we do not know how big $\overline{\mathscr{C}_3} - \mathscr{B}$ is or even if $\overline{\mathscr{C}_3}$ is normal. We know that there exists a component G of the Hilbert scheme $Hilb_{p(t),3}$ of dimension 27 that intersects $\overline{\mathscr{C}_3}$, but we can not prove that this is the only component that intersects it. Thus we do not know how singular $\overline{\mathscr{C}_3}$ is outside of $\mathscr{B}$.

Appendix A

Classes of curves in a smooth cubic surface

Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree 3. Let us denote a line and the six (-1)-curves that generate $Pic(X)$ as follows $l, E_1, \ldots, E_6$. The following code in [M2] lists all possible classes of a curve of degree d and genus g in X. The class $C = al - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4 - b_5E_5 - b_6E_6$ in [M2] is denoted by

```
{a,{b_{1},b_{2},b_{3},b_{4},b_{5},b_{6}}}
```

Example: All possible classes of curves of degree 10 and genus 11 as in the proof of Proposition 2.3.4 are:

```
 L = classOnCubic(10, 11)
```

returns

```
{{8, {3, 3, 2, 2, 2, 2}}, {9, {4, 3, 3, 3, 2, 2}},
{10, {4, 4, 4, 3, 3, 2}}, {10, {5, 3, 3, 3, 3, 3}},
{11, {5, 4, 4, 4, 3, 3}},{12, {5, 5, 4, 4, 4, 4}}}
```

```
classOnCubic = (d, g) ->
{
    L := {};
aMax := floor( sqrt( ( 2*d^2+6*(2-2*g) )/3 ) + d );
for a in (ceiling( (d+1)/3 ) )..aMax do
for b1 in (ceiling((3*a-d)/6))..(3*a-d) do
for b2 in (ceiling((3*a-d-b1)/5))..min(3*a-d-b1,b1) do
for b3 in (ceiling((3*a-d-b1-b2)/4))..min(3*a-d-b1-b2,b2) do
for b4 in (ceiling((3*a-d-b1-b2-b3)/3))..min(3*a-d-b1-b2-b3,b3) do
for b5 in (ceiling((3*a-d-b1-b2-b3-b4)/2))..min(3*a-d-b1-b2-b3-b4,b4) do
{
b6 := 3*a-d-b1-b2-b3-b4-b5;
if(b6 < 0 or b6 > b5) then continue;
if(b1+b2+b3+b4+b5+b6 == 3*a-d and 2*g == 2+a^2-d-b1^2-b2^2-b3^2-b4^2-b5^2-b6^2) then
L = join(L, {{a, {b1, b2, b3, b4, b5, b6}}});
};
return L;
}
```

Appendix B

Families of degree 6 and genus 3

In this appendix, we present code in [M2] to construct examples of the curves given in Chapter 3. All codes of the appendix B and C start with the following base code:

```
R=ZZ/2011[x,y,z,w]
------R=QQ[x,y,z,w]
randomElement = (d, I) ->
{ randomElementR := I;
    randomElementI := I;
    if(toString class I == "PolynomialRing") then
    { randomElementR = I;
        randomElementI = ideal vars randomElementR;
    }
    else if(toString class I == "Ideal") then
    {
        randomElementR = ring I;
        randomElementI = I;
};
    randomElementF := sub(0, randomElementR);
   for p in flatten entries gens randomElementI do
{
randomElementF = randomElementF + p * (random(randomElementR^{d- (degree p)_0},
randomElementR^{0}))_0_0;};
return randomElementF;
}
```

B.1 ACM curves

Generic element

```
M = random(R^{-3,-3,-3,-3}, R^{-4,-4,-4});
C=fittingIdeal(1,coker M);
```

A1

```
m1=randomElement(1,ideal(vars R));
m2=randomElement(1,ideal(vars R));
n1=randomElement(1,ideal(vars R));
n2=randomElement(1,ideal(vars R));
L=ideal(m1,m2);
Lred=ideal(n1,n2);
q=randomElement(2,Lred);
Z=saturate(ideal(q)+L);
c=randomElement(3,saturate(Z*Lred));
Q=saturate(ideal(c,q),Lred);
A=saturate (Q*L);
C= radical A;
```

A2

```
R=ZZ/2011[x,y,z]
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
```

```
p2=saturate (ideal(m12,m22));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
p3=saturate (ideal(m13,m23));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
p4=saturate (ideal(m14,m24));
n11=randomElement(1,ideal(vars R))
n21=randomElement(1,ideal(vars R))
q1=saturate (ideal(n11,n21));
n12=randomElement(1,ideal(vars R))
n22=randomElement(1,ideal(vars R))
q2=saturate (ideal(n12,n22));
P=saturate(p1*p2*p3*p4);
A=saturate(P*q1*q2);
q=randomElement(3, P);
l=randomElement(1, saturate(q1*q2));
H=ideal(l);
Q=ideal(q);
d= #flatten entries gens image basis(3,A)
S=ZZ/2011[a_0 .. a_(d-1)]
k=map(R,S,flatten entries gens image basis(3,A));
C1=preimage_k Q;
L=preimage_k H;
C=saturate(C1*L);
C=radical C;
```

A3

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate(ideal(m11,m21,m31));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
p2=saturate(ideal(m12,m22,m32));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
p3=saturate(ideal(m13,m23,m33));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
L=saturate(ideal(m14,m24));
a11=randomElement(2,saturate(L*p2*p1*p3));
a12=randomElement(2,saturate(L*p2*p1*p3));
S=ideal(a11,a12);
T=saturate(S,L);
h=randomElement(1,saturate(p2*p1*p3));
b=randomElement(3,saturate(p2*p1*p3));
a=ideal(b,h);
A=saturate (T*a);
C= radical trim A;
```

A4

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p2=saturate (ideal(m12,m22,m31));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
p3=saturate (ideal(m13,m23,m31));
q=randomElement(2, saturate(p1*p2*p3));
T=ideal(q,m31);
q1=randomElement(2, saturate(p1*p2*p3));
q2=randomElement(2,  saturate(p1*p2*p3));
Q=ideal(q1,q2)
C=saturate(Q*T);
C=radical trim C;
```

A5

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
p2=saturate (ideal(m12,m22,m32));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
p3=saturate (ideal(m13,m23,m33));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
m34=randomElement(1,ideal(vars R));
p4=saturate (ideal(m14,m24,m34));
l121=randomElement(1,saturate(p1*p2));
l122=randomElement(1,saturate(p1*p2));
l341=randomElement(1,saturate(p3*p4));
l342=randomElement(1,saturate(p3*p4));
L12=ideal(l121,l122);
L34=ideal(l341,l342);
q1=randomElement(2, saturate(p1*p2*p3*p4));
q2=randomElement(2, saturate(p1*p2*p3*p4));
Q=ideal(q1,q2);
C=  saturate(Q*L12*L34);
C=radical trim C;
```

A6

```
M = random(R^{-3,-3,-3,-3}, R^{-4,-4,-4})
C3=fittingIdeal(1,coker M);
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
p2=saturate (ideal(m12,m22,m32));
l1=randomElement(1,saturate(p1*p2));
l2=randomElement(1,saturate(p1*p2));
L=ideal(l1,l2);
a1=randomElement(4,saturate(C3*p1*p2));
a2=randomElement(4,saturate(C3*p1*p2));
C4=saturate(ideal(a1,a2),C3);
b1=randomElement(4,saturate((C4*L),C4+L));
b2=randomElement(4,saturate((C4*L),C4+L));
C=saturate(ideal(b1,b2),C4);
```

A7

```
l11=randomElement(1, ideal vars R);
l12=randomElement(1, ideal vars R);
l21=randomElement(1, ideal vars R);
l22=randomElement(1, ideal vars R);
L1=ideal(l11,l12);
L2=ideal(l21,l22);
q=randomElement(2, saturate(L1*L2));
c=randomElement(3, saturate(L1*L2));
C=saturate(ideal(q,c), saturate(L1*L2));
h=randomElement(1, ideal vars R);
t=randomElement(2, saturate(ideal(h)+C));
A=saturate(C*ideal(t,h));
C=radical trim A;
```

A8

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
```

```
m24=randomElement(1,ideal(vars R));
m34=randomElement(1,ideal(vars R));
n11=randomElement(1,ideal(vars R));
n12=randomElement(1,ideal(vars R));
n21=randomElement(1,ideal(vars R));
n22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m32));
p3=saturate (ideal(m13,m23,m33));
p4=saturate (ideal(m13,m24,m34));
L1=saturate (ideal(n11,n12));
L2=saturate (ideal(n21,n22));
a11=randomElement(2,saturate(L1*p2*p1*p3*p4));
a12=randomElement(2,saturate(L1*p2*p1*p3*p4));
b11=randomElement(2,saturate(L2*p2*p1*p3*p4));
b12=randomElement(2,saturate(L2*p2*p1*p3*p4));
S1=ideal(a11,a12);
S2=ideal(b11,b12);
T1=saturate(S1,L1);
T2=saturate(S2,L2);
A=saturate (T1*T2);
C= radical trim A;
```

A9

```
m11=randomElement(1,ideal(vars R))
m21=randomElement(1,ideal(vars R))
m31=randomElement(1,ideal(vars R))
p1=saturate (ideal(m11,m21,m31))
m12=randomElement(1,ideal(vars R))
m22=randomElement(1,ideal(vars R))
m32=randomElement(1,ideal(vars R))
p2=saturate (ideal(m12,m22,m32));
l11=randomElement(1, ideal vars R);
l12=randomElement(1, ideal vars R);
l21=randomElement(1, ideal vars R);
l22=randomElement(1, ideal vars R);
L1=ideal(l11,l12);
L2=ideal(l21,l22);
q=randomElement(2, saturate(L1*L2*p1*p2));
c=randomElement(3, saturate(L1*L2*p1*p2));
C=saturate(ideal(q,c), saturate(L1*L2));
h=randomElement(1, saturate(p1*p2));
t=saturate(ideal(h)+C,saturate(p1*p2));
l1=randomElement(1, saturate(p1*p2));
l2=randomElement(1, saturate(p1*p2));
h1=randomElement(1, t);
h2=randomElement(1, t);
H1=ideal(l1,l2);
H2=ideal(h1,h2);
A=saturate(C*H1*H2);
C=radical trim A;
```

A10

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m32));
p3=saturate (ideal(m13,m23,m33));
L=saturate (ideal(m14,m24));
a11=randomElement(2,saturate(L*p2*p1*p3));
a12=randomElement(2,saturate(L*p2*p1*p3));
S=ideal(a11,a12);
T=saturate(S,L);
h1=randomElement(1,ideal(vars R));
h=randomElement(1,saturate(p2*p1*p3));
b=randomElement(2,saturate(p2*p1*p3));
l=ideal(h1,h);
c=ideal(b,h);
A=saturate (T*c*l);
C= radical trim A;
```

A11

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
p3=saturate (ideal(m13,m23,m31));
h2=randomElement(1,saturate(p1*p2));
n11=randomElement(1,ideal(vars R));
n12=randomElement(1,ideal(vars R));
n21=randomElement(1,ideal(vars R));
n22=randomElement(1,ideal(vars R));
p4=saturate(ideal(h2,n11,n12));
p5=saturate(ideal(h2,n21,n22));
l1=randomElement(1,saturate(p5*p3*p4));
a=randomElement(2,saturate(p1*p2*p3));
b=randomElement(2,saturate(p1*p2*p4*p5));
l=randomElement(2,saturate(p3*p4*p5));
Q1=ideal(m31,a);
Q2=ideal(h2,b);
Q3=ideal(l1,l);
A=saturate (Q1*Q2*Q3);
C= radical trim A;
```

A12

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
m34=randomElement(1,ideal(vars R));
m15=randomElement(1,ideal(vars R));
m25=randomElement(1,ideal(vars R));
m35=randomElement(1,ideal(vars R));
n1=randomElement(1,ideal(vars R));
n2=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m32));
p3=saturate (ideal(m13,m23,m33));
p4=saturate (ideal(m14,m24,m34));
p5=saturate (ideal(m15,m25,m35));
L=saturate (ideal(n1,n2));
a11=randomElement(2,saturate(L*p2*p1*p3*p4*p5));
a12=randomElement(2,saturate(L*p2*p1*p3*p4*p5));
S=ideal(a11,a12);
T=saturate(S,L);
s=randomElement(2,saturate(p2*p1*p3*p4*p5));
Z=T+ideal(s);
p6=saturate(Z,saturate(p2*p1*p3*p4*p5));
h1=randomElement(1,saturate(p2*p1));
b1=randomElement(1,saturate(p2*p1));
h2=randomElement(1,saturate(p3*p4));
b2=randomElement(1,saturate(p3*p4));
h3=randomElement(1,saturate(p5*p6));
b3=randomElement(1,saturate(p5*p6));
l1=ideal(b1,h1);
l2=ideal(b2,h2);
l3=ideal(b3,h3);
A=saturate (T*l1*l2*l3);
C= radical trim A;
```

A13

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
p3=saturate (ideal(m13,m23,m31));
p4=saturate (ideal(m14,m24,m31));
q=randomElement(2,saturate(p1*p2*p3*p4));
Q=ideal(q,m31);
n11=randomElement(1,ideal(vars R));
n21=randomElement(1,ideal(vars R));
n31=randomElement(1,ideal(vars R));
p5=saturate (ideal(n11,n21,n31));
b1=randomElement(1,saturate(p1*p5));
b2=randomElement(1,saturate(p1*p5));
L1=ideal(b2,b1);
n12=randomElement(1,ideal(vars R));
p6=saturate (L1+ideal(n12));
b1=randomElement(1,saturate(p1*p5));
b2=randomElement(1,saturate(p1*p5));
c1=randomElement(1,saturate(p2*p5));
c2=randomElement(1,saturate(p2*p5));
d1=randomElement(1,saturate(p4*p6));
d2=randomElement(1,saturate(p4*p6));
L1=ideal(b2,b1);
L2=ideal(c1,c2);
L3=ideal(d1,d2);
s12=randomElement(1,ideal(vars R));
p7=saturate (L3+ideal(s12));
a1=randomElement(1,saturate(p3*p7));
a2=randomElement(1,saturate(p3*p7));
L4=ideal(a1,a2),
A=saturate (L1*L2*L3*L4*Q);
C= radical trim A;
```

A14

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
m34=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m32));
p3=saturate (ideal(m13,m23,m33));
p4=saturate (ideal(m14,m24,m34));
h112=randomElement(1,saturate(p1*p2));
h212=randomElement(1,saturate(p1*p2));
h123=randomElement(1,saturate(p2*p3));
h223=randomElement(1,saturate(p2*p3));
h113=randomElement(1,saturate(p1*p3));
h213=randomElement(1,saturate(p1*p3));
h114=randomElement(1,saturate(p1*p4));
h214=randomElement(1,saturate(p1*p4));
h124=randomElement(1,saturate(p4*p2));
h224=randomElement(1,saturate(p4*p2));
h134=randomElement(1,saturate(p3*p4));
h234=randomElement(1,saturate(p3*p4));
L12=ideal(h112,h212);
L23=ideal(h123,h223);
L13=ideal(h113,h213);
L14=ideal(h114,h214);
L24=ideal(h124,h224);
L34=ideal(h134,h234);
L=saturate(L12*L23);
L=saturate(L*L13);
L=saturate(L*L14);
L=saturate(L*L24);
C=radical saturate(L*L34);
```

B.2 Family $\mathscr{C}^h$

Generic element

```
l11=randomElement(1,ideal(vars R));
l12=randomElement(1,ideal(vars R));
L1=ideal(l11,l12);
l21=randomElement(1,ideal(vars R));
l22=randomElement(1,ideal(vars R));
L2=ideal(l21,l22);
M=saturate(L1*L2);
q=randomElement(2,M);
Q=ideal(q);
A=M+Q;
f= randomElement(4, A);
C=saturate(ideal(f)+Q,M);
```

B1

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
m1=randomElement(1,saturate(p1*p2));
a=randomElement(3,saturate(p1*p2));
b=randomElement(3,saturate(p1*p2));
q1=ideal(m31,a);
q2=ideal(m1,b);
A=saturate (q1*q2);
C= radical trim A;
```

B2

```
n1=randomElement(1,ideal(vars R));
n2=randomElement(1,ideal(vars R));
Lred=ideal(n1,n2);
q=randomElement(2,Lred);
c=randomElement(3,saturate(Lred));
h=randomElement(1,Lred);
L=saturate(ideal(h,q),Lred);
Q=saturate(ideal(c,q),Lred);
A=saturate (Q*L);
C= radical A;
```

B3

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
p3=saturate (ideal(m13,m23,m31));
m1=randomElement(1,saturate(p1*p2));
m2=randomElement(1,saturate(p3));
a=randomElement(2,saturate(p1*p2*p3));
b=randomElement(3,saturate(p1*p2));
Q1=ideal(m31,a);
Q2=ideal(m1,b);
L=ideal(m31,m2);
A=saturate (Q1*Q2*L);
C= radical trim A;
```

B4

```
l11=randomElement(1, ideal vars R);
l12=randomElement(1, ideal vars R);
l21=randomElement(1, ideal vars R);
l22=randomElement(1, ideal vars R);
L1=ideal(l11,l12);
L2=ideal(l21,l22);
q=randomElement(2, saturate(L1*L2));
c=randomElement(3, saturate(L1*L2));
C1=saturate(ideal(q,c), saturate(L1*L2));
h=randomElement(1, ideal vars R);
T=ideal(q,h);
A=saturate(C1*T);
C=radical trim A;
```

B5

```
l11=randomElement(1,ideal(vars R));
l12=randomElement(1,ideal(vars R));
L1=ideal(l11,l12);
l21=randomElement(1,ideal(vars R));
l22=randomElement(1,ideal(vars R));
L2=ideal(l21,l22);
M=saturate(L1*L2);
q=randomElement(2,M);
Q=ideal(q);
A=M+Q;
f= randomElement(4, A);
C3=saturate(ideal(f)+Q,M);
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
p2=saturate (ideal(m12,m22,m32));
l1=randomElement(1,saturate(p1*p2));
l2=randomElement(1,saturate(p1*p2));
L=ideal(l1,l2);
a1=randomElement(4,saturate(C3*p1*p2));
a2=randomElement(4,saturate(C3*p1*p2));
C4=saturate(ideal(a1,a2),C3);
b1=randomElement(4,saturate((C4*L),C4+L));
b2=randomElement(4,saturate((C4*L),C4+L));
C=saturate(ideal(b1,b2),C4);
```

B6

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q1=randomElement(2,saturate(p1*p2));
q2=randomElement(2,saturate(p1*p2));
h1=randomElement(1,saturate(p1*p2));
h2=randomElement(1,saturate(p1*p2));
b1=randomElement(1,ideal(vars R));
b2=randomElement(1,ideal(vars R));
c1=ideal(q1,h1);
c2=ideal(q2,h2);
l1=ideal(b1,h1);
l2=ideal(b2,h2);
A=saturate (c1*l1*c2*l2);
C= radical trim A;
```

B7

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m33=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
m34=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m32));
p3=saturate (ideal(m13,m23,m33));
p8=saturate (ideal(m14,m24,m34));
a11=randomElement(1,saturate(p1*p3));
a12=randomElement(1,saturate(p1*p3));
a21=randomElement(1,saturate(p1*p2));
a22=randomElement(1,saturate(p1*p2));
a31=randomElement(1,saturate(p3*p2));
a32=randomElement(1,saturate(p3*p2));
L1=ideal(a11,a12);
L2=ideal(a21,a22);
L3=ideal(a31,a32);
m15=randomElement(1,ideal(vars R));
m16=randomElement(1,ideal(vars R));
p4=saturate (L3+ideal(m15));
p5=saturate (L1+ideal(m16));
```

```
a41=randomElement(1,saturate(p8*p5));
a42=randomElement(1,saturate(p8*p5));
a51=randomElement(1,saturate(p4*p8));
a52=randomElement(1,saturate(p4*p8));
L4=ideal(a41,a42);
L5=ideal(a51,a52);
m17=randomElement(1,ideal(vars R));
m18=randomElement(1,ideal(vars R));
p6=saturate (L5+ideal(m17));
p7=saturate (L4+ideal(m18));
a61=randomElement(1,saturate(p7*p6));
a62=randomElement(1,saturate(p7*p6)),
L6=ideal(a61,a62);
A=saturate (L1*L2*L3*L4*L5*L6);
C= radical trim A;
```

B.3 Family $\mathscr{A}$

Generic element

```
m1=randomElement(1,ideal(vars R));
m2=randomElement(1,ideal(vars R));
m3=randomElement(1,ideal(vars R));
p1=ideal(m1,m2,m3);
s1=randomElement(1,ideal(vars R));
s2=randomElement(1,ideal(vars R));
s3=randomElement(1,ideal(vars R));
p2=ideal(s1,s2,s3);
h=randomElement(1,saturate(p1*p2));
c=randomElement(4,saturate(p1*p2));
n11=randomElement(1,p1);
n12=randomElement(1,p1);
n21=randomElement(1,p2);
n22=randomElement(1,p2);
L1=ideal(n11,n12);
L2=ideal(n21,n22);
Q=saturate(ideal(h,c)),
A=saturate (L1*L2*Q);
C= radical trim A;
```

C1

```
l11=randomElement(1,ideal vars R);
l12=randomElement(1,ideal vars R);
l13=randomElement(1,ideal vars R);
l21=randomElement(1,ideal vars R);
l22=randomElement(1,ideal vars R);
l23=randomElement(1,ideal vars R);
p1=ideal(l11,l12,l13);
p2=ideal(l21,l22,l23);
H=ideal(randomElement(1,saturate(p1*p2)));
Q=H+ideal(randomElement(3,saturate(p1*p2)));
m11=randomElement(1,p1);
m12=randomElement(1,p1);
m21=randomElement(1,p2);
m22=randomElement(1,p2);
L1=ideal(m11,m12);
L2=ideal(m21,m22);
L=H+ideal(randomElement(1,ideal vars R));
A=saturate(Q*L1*L2*L);
C=radical A;
```

C2

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q1=randomElement(2,saturate(p1));
q2=randomElement(2,saturate(p2));
h1=randomElement(1,saturate(p2));
h2=randomElement(1,saturate(p2));
b1=randomElement(1,saturate(p1));
b2=randomElement(1,saturate(p1));
c1=ideal(q1,m31);
c2=ideal(q2,m31);
l1=ideal(b1,b2);
l2=ideal(h1,h2);
A=saturate (c1*l1*c2*l2);
C= radical trim A;
```

C3

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
p3=saturate (ideal(m13,m23,m31));
p4=saturate (ideal(m14,m24,m31));
q=randomElement(2,saturate(p1*p2*p3*p4));
Q=ideal(q,m31);
b1=randomElement(1,saturate(p1));
b2=randomElement(1,saturate(p2));
c1=randomElement(1,saturate(p3));
c2=randomElement(1,saturate(p3));
d1=randomElement(1,saturate(p4));
d2=randomElement(1,saturate(p4));
L1=ideal(m31,b1);
L2=ideal(m31,b2);
L3=ideal(d1,d2);
L4=ideal(c1,c2);
A=saturate (L1*L2*L3*L4*Q);
C= radical trim A;
```

B.3.1 A non-reduced family

Double conic union two skew lines

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m32=randomElement(1,ideal(vars R));
p2=saturate (ideal(m12,m22,m32));
P=saturate(p1*p2);
L1=saturate (ideal(m11,m21));
L2=saturate (ideal(m12,m22));
q=randomElement(2,P);
h=randomElement(1,P);
Q=saturate(ideal(h,q*q));
C=saturate(L1*L2*Q,P);
```

B.4 The component $\mathscr{R}$

Generic element

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m1=randomElement(1,p1);
a=randomElement(2,p1);
b=randomElement(4,p1);
q1=ideal(m31,a);
q2=ideal(m1,b);
A=saturate (q1*q2);
C= radical trim A;
```

D1 (I)

```
m1=randomElement(1,ideal(vars R));
m2=randomElement(1,ideal(vars R));
m3=randomElement(1,ideal(vars R));
p1=ideal(m1,m2,m3);
s1=randomElement(1,ideal(vars R));
s2=randomElement(1,ideal(vars R));
s3=randomElement(1,ideal(vars R));
p2=ideal(s1,s2,s3);
n11=randomElement(1,p1);
n12=randomElement(1,p1);
n21=randomElement(1,saturate(p1*p2));
n22=randomElement(1,saturate(p1*p2));
L1=ideal(n11,n12);
L2=ideal(n21,n22);
h=randomElement(1,p2);
c=randomElement(4,p2);
Q=saturate(ideal(h,c));
A=saturate (L1*L2*Q);
C= radical trim A;
```

D2

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
h1=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,h1));
h2=randomElement(1,p1);
a=randomElement(2,p1);
b=randomElement(2,ideal(vars R));
l=randomElement(2,p1);
Q1=ideal(h1,a);
Q2=ideal(h1,b);
Q3=ideal(h2,l);
A=saturate (Q1*Q2*Q3);
C= radical trim A;
```

D3

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
h1=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,h1));
h2=randomElement(1,p1);
a=randomElement(3,p1);
b=randomElement(2,p1);
l=randomElement(1,ideal(vars R));
Q1=ideal(h2,a);
Q2=ideal(h1,b);
L=ideal(h2,l);
A=saturate (Q1*Q2*L);
C= radical trim A;
```

D4

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
q1=randomElement(2,saturate(p1));
q2=randomElement(2,saturate(p1));
h1=randomElement(1,saturate(p1));
h2=randomElement(1,saturate(p1));
b1=randomElement(1,ideal(vars R));
b2=randomElement(1,ideal(vars R));
c1=ideal(q1,h1);
c2=ideal(q2,h2);
l1=ideal(b1,h1);
l2=ideal(b2,h1);
A=saturate (c1*l1*c2*l2);
C= radical trim A;
```

D5 (II)

```
m1=randomElement(1,ideal(vars R));
```

```
m2=randomElement(1,ideal(vars R));
m3=randomElement(1,ideal(vars R));
p1=ideal(m1,m2,m3);
h=randomElement(1,saturate(p1));
c=randomElement(4,saturate(p1));
n11=randomElement(1,p1);
n12=randomElement(1,p1);
n21=randomElement(1,p1);
n22=randomElement(1,p1);
L1=ideal(n11,n12);
L2=ideal(n21,n22);
Q=saturate(ideal(h,c));
A=saturate (L1*L2*Q),
C= radical trim A;
```

D6

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
m13=randomElement(1,ideal(vars R));
m23=randomElement(1,ideal(vars R));
m14=randomElement(1,ideal(vars R));
m24=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
p3=saturate (ideal(m13,m23,m31));
p4=saturate (ideal(m14,m24,m31));
q=randomElement(2,saturate(p1*p2*p3*p4));
Q=ideal(q,m31);
b1=randomElement(1,saturate(p1));
b2=randomElement(1,saturate(p2));
c1=randomElement(1,saturate(p4));
c2=randomElement(1,saturate(p4));
d1=randomElement(1,saturate(p4));
d2=randomElement(1,saturate(p4));
L1=ideal(m31,b1);
L2=ideal(m31,b2);
L3=ideal(d1,d2);
L4=ideal(c1,c2);
A=saturate (L1*L2*L3*L4*Q);
C= radical trim A;
```

B.5 The extremal component

```
F=randomElement(3,ideal(z,w));
G=randomElement(7,ideal(z,w));
C=ideal(x^2 ,x*y,y^6 ,x*G-F*y^5);
```

Appendix C

Flat limits

In this appendix are the codes in [M2] of the flat limits that we claim that there exist in chapter 3. All codes of this appendix start with the following base code:

```
R=ZZ/2011[x,y,z,w]
------R=QQ[x,y,z,w]
randomElement = (d, I) ->
{randomElementR = ring I;
randomElementF = sub(0, randomElementR);
for p in flatten entries gens I do (randomElementF = randomElementF +
p * (random(randomElementR^{d- (degree p)_0}, randomElementR^{0}))_0_0);
return randomElementF}
```

C.1 Limits in $\mathscr{C}^h$

- Flat limit from B1 to B3.

```
    m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q=randomElement(3,saturate(p1*p2));
c=randomElement(2,saturate(p1*p2));
l=randomElement(1,ideal(vars R));
h1=randomElement(1,saturate(p1*p2));
Q=randomElement(3,saturate(p1*p2));

Rt=R[t]

I1=saturate(ideal(t*sub(q,Rt)+(1-t)*sub(c*l,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(Q,Rt),sub(h1,Rt)),sub(p1*p2,Rt));
I=saturate(I1*I2,sub(p1*p2,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

- Flat limit from B3 to B6.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
```

```
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q=randomElement(3,saturate(p1*p2));
c=randomElement(2,saturate(p1*p2));
l1=randomElement(1,ideal(vars R));
h1=randomElement(1,saturate(p1*p2));
l2=randomElement(1,ideal(vars R));
Q=randomElement(2,saturate(p1*p2));

Rt=R[t]

I1=saturate(ideal(t*sub(q,Rt)+(1-t)*sub(c*l1,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(Q*l2,Rt),sub(h1,Rt)),sub(p1*p2,Rt));
I=saturate(I1*I2,sub(p1*p2,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

- Flat limit from B6 to B7.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q1=randomElement(2,saturate(p1*p2));
l11=randomElement(1,ideal(vars R));
c1=randomElement(1,ideal vars R);
l12=randomElement(1,p1);
l13=randomElement(1,p2);
q2=randomElement(2,saturate(p1*p2));
l21=randomElement(1,saturate(p1*p2));
c2=randomElement(1,ideal vars R);
l22=randomElement(1,p1);
l23=randomElement(1,p2);
l24=randomElement(1,ideal(vars R));

Rt=R[t]

I1=saturate(ideal(t*sub(q1*l11,Rt)+(1-t)*sub(c1*l12*l13,Rt),sub(m31,Rt)));
I2=saturate(ideal(t*sub(q2*l24,Rt)+(1-t)*sub(c2*l22*l23,Rt),sub(l21,Rt)));
I=saturate(I1*I2,sub(p1*p2,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

C.2 Limits in $\mathscr{A}$

- Flat limit from C1 to C3.

```
l11=randomElement(1,ideal vars R);
l12=randomElement(1,ideal vars R);
l13=randomElement(1,ideal vars R);
l21=randomElement(1,ideal vars R);
l22=randomElement(1,ideal vars R);
l23=randomElement(1,ideal vars R);
p1=ideal(l11,l12,l13);
p2=ideal(l21,l22,l23);
```

```
h=randomElement(1,saturate(p1*p2));
H=ideal(h);
n11=randomElement(1,ideal vars R);
n12=randomElement(1,ideal vars R);
q1=ideal(n11,n12,h);
n21=randomElement(1,ideal vars R);
n22=randomElement(1,ideal vars R);
q2=ideal(n21,n22,h);
Q=randomElement(3,saturate(p1*p2*q1*q2));
m11=randomElement(1,p1);
m12=randomElement(1,p1);
m21=randomElement(1,p2);
m22=randomElement(1,p2);
L1=ideal(m11,m12);
L2=ideal(m21,m22);
L=H+ideal(randomElement(1,q1));
E=randomElement(2,saturate(p1*p2*q1*q2));
Le=randomElement(1,q1);

Rt=R[t]
I1=saturate(ideal(t*sub(Q,Rt)+(1-t)*sub(E*Le,Rt),sub(h,Rt)));
I=saturate(I1,saturate(sub(p1,Rt)*sub(p2,Rt)*sub(q1,Rt)*sub(q2,Rt)));

At=saturate(I,t)
A0=sub(At,{t=>0})
A=radical saturate(A0*L1*L2*L);
```

- Flat limit from C0 to C2.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q1=randomElement(2,ideal vars R);
q2=randomElement(2,saturate(p1*p2));
b1=randomElement(1,saturate(p1));
b2=randomElement(1,saturate(p1));
c1=randomElement(1,saturate(p2));
c2=randomElement(1,saturate(p2));
d=randomElement(4,saturate(p1*p2));

Rt=R[t]

I1=saturate(ideal(t*sub(d,Rt)+(1-t)*sub(q1*q2, Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(b1,Rt),sub(b2,Rt))*ideal(sub(c1,Rt),sub(c2,Rt)),sub(p1*p2,Rt));
I=saturate(I1*I2,sub(p1*p2,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

- Flat limit from C2 to C3.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
```

```
p2=saturate (ideal(m12,m22,m31));
q1=randomElement(2,ideal vars R);
q2=randomElement(2,saturate(p1*p2));
b1=randomElement(1,saturate(p1));
b2=randomElement(1,saturate(p1));
c1=randomElement(1,saturate(p2));
c2=randomElement(1,saturate(p2));
d1=randomElement(1,ideal vars R);
d2=randomElement(1,ideal vars R);

Rt=R[t]

I1=saturate(ideal(t*sub(q1*q2,Rt)+(1-t)*sub(q2*d1*d2,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(b1,Rt),sub(b2,Rt))*ideal(sub(c1,Rt),sub(c2,Rt)),sub(p1*p2,Rt));
I=saturate(I1*I2,sub(p1*p2,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

C.3 Limits in $\mathscr{R}_3$

- Flat limit from D0 to D1.

```
m11=randomElement(1,ideal(vars R))
m21=randomElement(1,ideal(vars R))
m31=randomElement(1,ideal(vars R))
p1=saturate (ideal(m11,m21,m31))
m1=randomElement(1,p1)
n1=randomElement(1,p1)
n2=randomElement(1,p1)
a=randomElement(2,p1)
b=randomElement(4,p1)

Rt=R[t]
I1=saturate(ideal(t*sub(a,Rt)+(1-t)*sub(n1*n2,Rt),sub(m31,Rt)))
I2=ideal(sub(b,Rt),sub(m1,Rt))
I=saturate(I1*I2,sub(p1,Rt))

At=saturate(I,t)
A0=sub(At,{t=>0})
```

- Flat limit from D0 to D2.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m1=randomElement(1,p1);
n1=randomElement(1,ideal vars R);
n2=randomElement(1,ideal vars R);
a=randomElement(2,p1);
b=randomElement(2,p1);
c=randomElement(4,p1);

Rt=R[t]

I1=saturate(ideal(t*sub(c,Rt)+(1-t)*sub(n1*b*n2,Rt),sub(m31,Rt)));
I2=ideal(sub(a,Rt),sub(m1,Rt));
I=saturate(I1*I2,sub(p1,Rt));
```

```
At=saturate(I,t);

A0=sub(At,{t=>0});
```

- Flat limit from D0 to D3.

```
m11=randomElement(1,ideal(vars R))
m21=randomElement(1,ideal(vars R))
m31=randomElement(1,ideal(vars R))
p1=saturate (ideal(m11,m21,m31))
m1=randomElement(1,p1)
n1=randomElement(1,p1)
n2=randomElement(1,ideal(vars R))
a=randomElement(2,p1)
b=randomElement(4,p1)

Rt=R[t]

I1=saturate(ideal(t*sub(a,Rt)+(1-t)*sub(n1*n2,Rt),sub(m31,Rt)))
I2=ideal(sub(b,Rt),sub(m1,Rt))
I=saturate(I1*I2,sub(p1,Rt))

At=saturate(I,t)
A0=sub(At,{t=>0})
```

- Flat limit from D2 to D5.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m1=randomElement(1,p1);
c=randomElement(2,p1);
n1=randomElement(1,ideal vars R);
n2=randomElement(1,ideal vars R);
a=randomElement(2,p1);
b1=randomElement(2,p1);
b2=randomElement(2,ideal vars R);

Rt=R[t]

I1=saturate(ideal(t*sub(b1*b2,Rt)+(1-t)*sub(n1,Rt)*sub(a,Rt)*sub(n2,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(c,Rt),sub(m1,Rt)),sub(p1,Rt));
I=saturate(I1*I2,sub(p1,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

- Flat limit from D4 to D5.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m1=randomElement(1,p1);
c=randomElement(2,p1);
n1=randomElement(1,ideal vars R);
n2=randomElement(1,ideal vars R);
a=randomElement(2,p1);
b1=randomElement(1,p1);
b2=randomElement(3,ideal vars R);
```

```
Rt=R[t]

I1=saturate(ideal(t*sub(b1*b2,Rt)+(1-t)*sub(n1,Rt)*sub(a,Rt)*sub(n2,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(c,Rt),sub(m1,Rt)),sub(p1,Rt));
I=saturate(I1*I2,sub(p1,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

- Flat limit from D5 to D6.

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
m1=randomElement(1,p1);
c1=randomElement(2,p1);
m11=randomElement(1, ideal vars R);
m12=randomElement(1, ideal vars R);
b1=randomElement(2,p1);
n1=randomElement(1, p1);
n2=randomElement(1, p1);

Rt=R[t]

I1=saturate(ideal(t*sub(b1,Rt)+(1-t)*sub(n1,Rt)*sub(n2,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(c1*m11*m12,Rt),sub(m1,Rt)),sub(p1,Rt));
I=saturate(I1*I2,sub(p1,Rt));

At=saturate(I,t);
A0=sub(At,{t=>0});
```

C.4 Families outside of $\mathscr{H}_3^{lcm}$.

Taking plane limits we find two new families, one of them is outside of $\mathscr{H}_3^{lcm}$.

- **Four lines union a conic with a embedded point**

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q=randomElement(1,saturate(p1*p2));
c=randomElement(2,saturate(p1*p2));
l1=randomElement(1,ideal(vars R));
l3=randomElement(1,ideal(vars R));
h1=randomElement(1,saturate(p1*p2));
l2=randomElement(1,ideal(vars R));
Q=randomElement(2,saturate(p1*p2));
Rt=R[t]
I1=saturate(ideal(t*sub(c*l1,Rt)+(1-t)*sub(q*l1*l3,Rt),sub(m31,Rt)));
I2=saturate(ideal(sub(Q*l2,Rt),sub(h1,Rt)),sub(p1*p2,Rt));
I=saturate(I1*I2,sub(p1*p2,Rt));
At=saturate(I,t);
```

```
A0=sub(At,{t=>0});
```

- **A double line union four lines**

```
m11=randomElement(1,ideal(vars R));
m21=randomElement(1,ideal(vars R));
m31=randomElement(1,ideal(vars R));
m12=randomElement(1,ideal(vars R));
m22=randomElement(1,ideal(vars R));
p1=saturate (ideal(m11,m21,m31));
p2=saturate (ideal(m12,m22,m31));
q1=randomElement(2,saturate(p1*p2));
l11=randomElement(1,ideal(vars R));
c1=randomElement(1,saturate(p1*p2));
l12=randomElement(1,ideal(vars R));
l13=randomElement(1,ideal(vars R));
q2=randomElement(2,saturate(p1*p2));
l21=randomElement(1,saturate(p1*p2));
c2=randomElement(1,saturate(p1*p2));
l22=randomElement(1,ideal(vars R));
l23=randomElement(1,ideal(vars R));
l24=randomElement(1,ideal(vars R));
Rt=R[t]
I1=saturate(ideal(t*sub(q1*l11,Rt)+(1-t)*sub(c1*l12*l13,Rt),sub(m31,Rt)));
I2=saturate(ideal(t*sub(q2*l24,Rt)+(1-t)*sub(c2*l22*l23,Rt),sub(l21,Rt)));
I=saturate(I1*I2,sub(p1*p2,Rt));
At=saturate(I,t);
A0=sub(At,{t=>0});
```

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