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Minimal obstructions for some matrix partitions on selected graph classes

T E S I S

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Contents

Preface

Many problems in Graph Theory are about deciding whether the vertex set of an input graph admits a partition satisfying some adjacency constraints between the parts of the partition. Matrix partitions describe problems as above when the adjacency constraints are either two parts are completely adjacent, they are completely nonadjacent, or there is no restriction between the parts.

For each fixed matrix partition M, the class of graphs that admit an M-partition can be characterized by the family of all minimal graphs with the property of not admitting such partition, which are called its minimal obstructions. This characterization is interesting since a certifying recognition algorithm for the graphs with an M-partition can potentially use M-partitions as yes-certificates and minimal M-obstructions as no-certificates.

Nevertheless, complete lists of minimal obstructions usually are difficult to obtain for general graphs, so it is common to study these problems in restricted graph classes. That is what we do in this thesis. For some well-behaved graph classes, we study a kind of matrix partition problems called (s,k)-polar partitions as well as other related vertex-partition problems encompassed under the name of polarity. The selected graph classes that we study are divided in two: graph classes that, in some sens, have just a few induced paths of length three, and graph classes whose members have certain initial vertex-partition. Additionally, just for some small matrix partitions we give complete lists of minimal obstructions for general graphs.

It is worth mentioning that this work is a natural continuation of the thesis that the author of this document wrote to obtain his master's degree. In such document was given a complete recursive characterization of minimal $(s,1)$-polar obstructions for graphs without induced paths on four vertices (called cographs), as well as the complete family of four minimal obstructions for $(\infty,1)$-polarity when restricted to cographs.

The present document is organized in four parts, each of them subdivided in two chapters. The first chapter contains the basic terminology needed to read the rest of the document, including elementary concepts of Graph Theory and Computational Complexity Theory, while the second chapter serves to introduce graph classes with just a few induced P_4s, as well as matrix partitions.

Both chapters in the second part are about characterizing properties related to polarity when restricted to P_4-free, P_4-sparse, and P_4-extendible graphs. The main result of Chapter 3 is a partial recursive characterization for the minimal (∞, k)-polar obstructions when restricted to cographs, and the complete lists of cograph minimal (∞, k)-polar obstructions for the cases $k = 2$ and $k = 3$. In Chapter 4, we generalize to P_4-sparse and P_4-extendible graphs almost all known results about characterizations of polar properties in cographs, and we develop linear time algorithms to decide whether a graph in the mentioned classes has said properties. In that same chapter, it is proven that any hereditary property, when restricted to either P_4-sparse or P_4-extendible graphs, has only a finite number of minimal obstructions.

The third part of this document is also about polar properties, but this time restricted to $\mathcal{H}$-split graphs, which are families of graphs that generalize the so-called split graphs. Due to their properties, we focus in two families of $\mathcal{H}$-split graphs, namely the pseudo-split graphs and the $2K_2$-split graphs. Polarity on pseudo-split graphs is treated in Chapter 5, where we provided finite lists of minimal obstructions for the main polar properties, and give linear-time algorithms to recognize such properties on pseudo-split graphs from their degree sequences; at the end of the chapter are studied the (k, ℓ)-colorings of pseudo-split graphs. Results about polarity in $2K_2$-split graphs that are analogous to those given in Chapter 5 for pseudo-split graphs are developed in Chapter 6. We show that, among other differences, $2K_2$-split graphs that are (s, k)-polar cannot be recognized from their degree sequence as pseudo-split (s, k)-polar graphs do, but they are still efficiently recognizable. Since C_4-split graphs are the complements of $2K_2$-split graphs, analogous results are deduced for these graphs.

Finally, in the fourth and last part, we give complete lists of minimal obstructions for matrix partitions with three parts where every pair of different parts are either completely adjacent or completely nonadjacent, except for at most one of such pairs, which has no adjacency restriction. There are essentially seven of such matrices; in Chapter 7 we characterize all of them but one, the most difficult one, to which the final chapter is devoted.

Open problems and conjectures are provided throughout all the document, at the end of each chapter. Conclusions and future lines of work follows Chapter 8, as well as the bibliography, a glossary of symbols, and an alphabetical index of concepts.

Part I

Introduction

Chapter 1

Elementary concepts

Throughout this chapter we set the necessary concepts to read the rest of the document. In general, we follow the terminology of [2], although some definitions may differ a bit.

1.1 Basics of Graph Theory

A **graph** G is an ordered pair (V_G, E_G) such that V_G is a finite set whose elements are called **vertices**, and E_G is a set whose elements, called **edges**, are 2-subsets of V_G. The **order** and **size** of the graph G are $|V_G|$ and $|E_G|$, respectively. The subscripts of the vertex and edge sets of a graph are usually omitted when we work with a single graph or there is not risk of confusion. To simplify the notation, we write uv to denote the edge $\{u, v\}$. Two vertices, u and v, are said to be **adjacent** if uv is an edge; in such a case u and v are said to be the **ends** of uv. Graphs are usually represented by drawing a small circle for each of its vertices and joining any pair of adjacent vertices with a line.

The (open) **neighborhood** of a vertex u in a graph G, denoted by $N_G(u)$, or simply $N(u)$ when no confusion is possible, is the set of all vertices of G that are adjacent to u; the **closed neighborhood** of u is the set $N[u] = N(u) \cup \{u\}$. Two vertices are called **neighbors** if one of them belongs to the neighborhood of the other. The **degree** of a vertex u, denoted $\mathrm{d}(u)$, is the cardinality of $N(u)$. The non-increasing sequence of the vertex degrees of a graph G is called the **degree sequence** of G.

Given two graphs, G and H, an **isomorphism** from G to H is a bijection $\theta\colon V_G \to V_H$ such that, for any vertices u and v of G, $uv \in E_G$ if and only if

$\theta(u)\theta(v) \in E_H$. If there is an isomorphism from G to H, we say that G and H are **isomorphic**, and we denoted it by $G \cong H$.

A **subgraph** of a graph G is a graph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$. An **induced subgraph** of G is a subgraph H of G such that two vertices are adjacent in H if and only if they are adjacent in G. The induced subgraph of G whose vertex set is V' is denoted by $G[V']$. The graph $G[V \setminus V']$ is also denoted by $G - V'$ and, if V' is the single set $\{v\}$, we write $G - v$ instead of $G - \{v\}$. The graphs $G - v$ are referred as the **vertex-deleted subgraphs** of G. We use $H \leq G$ to indicate that G has an induced subgraph isomorphic to H. We say that G is an H**-free** graph when $H \nleq G$; for a family of graphs $\mathcal{H}$, we say that G is $\mathcal{H}$-free if it is H-free for every graph $H \in \mathcal{H}$.

A class of graphs is **hereditary** if it is closed under induced subgraphs. A property $\mathcal{P}$ of graphs is said to be **hereditary** if the class of graphs having property $\mathcal{P}$ is hereditary. For a property of graphs $\mathcal{P}$, a graph G is called a $\mathcal{P}$**-obstruction** if G does not have the property $\mathcal{P}$. A $\mathcal{P}$-obstruction such that any proper induced subgraph of G has the property $\mathcal{P}$ is said to be a **minimal** $\mathcal{P}$**-obstruction**. Notice that, if $\mathcal{P}$ is a hereditary property of graphs, then a $\mathcal{P}$-obstruction G is minimal if and only if any vertex-deleted subgraph of G has the property $\mathcal{P}$. For a hereditary class of graphs $\mathcal{G}$, $\mathcal{G}$-obstructions and minimal $\mathcal{G}$-obstructions are respectively defined as the $\mathcal{P}$-obstructions and minimal $\mathcal{P}$-obstructions, where $\mathcal{P}$ is the property of being a graph in $\mathcal{G}$. The following folklore characterization of hereditary properties implies that any hereditary class of graphs is suitable to be characterized by its set of minimal obstructions.

Theorem 1.1. *Let $\mathcal{P}$ be a property of graphs, and let $\mathcal{O}_{\mathcal{P}}$ be the family of all minimal $\mathcal{P}$-obstructions. Then, $\mathcal{P}$ is hereditary if and only if the following assertions are equivalent for any graph G.*

1. *G has the property $\mathcal{P}$.*

2. *G is an $\mathcal{O}_{\mathcal{P}}$-free graph.*

A **walk** of length k in a graph G is a sequence of vertices $W = (v_0, v_1, \dots, v_k)$ such that, for any $i \in \{1, \dots, k\}$, v_{i-1} is adjacent to v_i; in this case v_0 and v_k are called the **ends** of W. A **path** is a walk that does not repeat vertices; up to isomorphisms, for each positive integer k, there exists one and only one path of order k, which is denoted by P_k. A **cycle** is a walk $W = (v_0, v_1, \dots, v_k)$ of length at least three such that, for any i and j with $0 \leq i < j \leq k$, $v_i = v_j$ if and only if $i = 0$ and $j = k$. Up to isomorphisms, for any integer k with $k \geq 3$, there is only one cycle of order k, which we denote by C_k. A **hole** is a cycle of length at least 5. A cycle C_k is said to be odd or even accordingly to the parity of k.

A graph G is **connected** if, for any two vertices u and v of G, there is a path in G whose ends are u and v. A graph that is not connected is said to be **disconnected**. The **connected componets** of a graph G, sometimes simply called the **components** of G, are the induced subgraphs of G that are maximum with the property of being connected.

The **complement** of a graph G, denoted $\overline{G}$, is the graph with vertex set V_G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A class of graphs $\mathcal{F}$ is called **self-complementary** if it is closed under graph complements. A graph G is self-complementary if the family $\{G\}$ is, i.e. if $G \cong \overline{G}$. Given vertex disjoint graphs, G and H, we define the **disjoint union** of G and H, denoted by $G + H$, as the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$. Congruently, we denote by nG the graph with n connected components, each of them isomorphic to G. The graph $G \oplus H$, called the **join** of G and H, is the graph obtained from $G + H$ by adding every edge with one of its ends in V_G and the other in V_H. It is a simple observation that $\overline{G \oplus H} = \overline{G} + \overline{H}$. We will use W_n to denote the **wheel** graph $C_n \oplus K_1$.

A **complete graph** is a graph such that any two of its vertices are adjacent; up to isomorphisms, for each positive integer n, there exist only one complete graph of order n, which is denoted by K_n. We say that a graph is **trivial** if it is isomorphic to K_1, and **nontrivial** otherwise. The complement of a complete graph is an **empty graph**. A **cluster** is a graph whose connected components are complete graphs. A **k-cluster** is a cluster with at most k connected components. A graph whose complement is a cluster (respectively a k-cluster) is called a **complete multipartite graph** (respectively, a **complete k-partite graph**). It is easy to verify that clusters are characterized as the P_3-free graphs, while k-clusters are precisely the $\{P_3, (k+1)K_1\}$-free graphs. Complementarily, complete multipartite graphs are precisely the $\overline{P_3}$-free graphs, and complete k-partite graphs are exactly the $\{\overline{P_3}, K_{k+1}\}$-free graphs.

Two no-necessarily distinct vertex subsets of a graph G, V_1 and V_2, are said to be **completely adjacent** if every vertex $u \in V_1$ is adjacent to any vertex of $V_2 \setminus \{u\}$. Similarly, V_1 is **completely nonadjacent** to V_2 if no vertex in V_1 is adjacent to a vertex in V_2. A vertex subset V' of a graph G is called a **clique** if it is completely adjacent to itself, or equivalently, if $G[V']$ is a complete graph.

The vertex subset V' is **independent** or **stable** if it is completely nonadjacent to itself, i.e., if $G[V']$ is an empty graph. A vertex subset U is said to be **homogeneous** if $V \setminus U$ admits a partition[1] (A, B) such that U is completely adjacent to A and completely nonadjacent to B.

The largest cardinality among all cliques of G is called the **clique number** of G, and it is denoted by $\omega(G)$. Analogously, the largest cardinality among all independent sets of G is called the **independence number** of G, and it is denoted by $\alpha(G)$. Given nonnegative integers k and ℓ, a (k, ℓ)**-coloring** of G is a partition $(V_1, V_2, \ldots, V_{k+\ell})$ of V_G such that V_i is an independent set for any $i \in \{1, \ldots, k\}$, and V_j is a clique for each $j \in \{k + 1, \ldots, k + \ell\}$; a graph G that admits a (k, ℓ)-coloring is said to be (k, ℓ)**-colorable** and sometimes it is referred as a (k, ℓ)**-graph**. A $(k, 0)$-coloring of G is called a (proper) k**-coloring** of G, and G is said to be k**-colorable** if it admits a k-coloring. The 2-colorable graphs receive the special name of **bipartite graphs**, and it is well known that they can be characterized as the graphs that do not have odd-cycles as induced subgraphs. The minimum integer k such that G admits a k-coloring is the **chromatic number** of G, and it is denoted $\chi(G)$. The minimum integer ℓ for which G has a $(0, \ell)$-coloring is denoted by $\theta(G)$, and it is called the **clique covering number** of G. A z**-cocoloring** of G is any (k, ℓ)-coloring of G such that $k + \ell = z$. We use $\chi^c(G)$ to denote the **cochromatic number** of G, which is the minimum integer z for which G admits a z-cocoloring. A graph G is said to be z**-bicolorable** if, for any integers k and ℓ such that $k + \ell = z$, G is (k, ℓ)-colorable. The **bichromatic number** of G, denoted $\chi^b(G)$, is the minimum integer z such that G is z-bicolorable. Notice that, for any graph G,

$$\chi^c(G) \leq \min\{\chi(G), \theta(G)\} \leq \max\{\chi(G), \theta(G)\} \leq \chi^b(G).$$

Clearly, the chromatic number of any graph is at least as big as its clique number. A graph G is called **perfect** if, for each induced subgraph H of G, $\chi(H) = \omega(H)$. The Strong Perfect Graph Theorem establish that a graph G is perfect if and only if neither G or its complement has induced odd holes [13].

1.2 Algorithms: complexity and certificates

In this section, we present some ways to measure the *quality* of an algorithm and the *difficulty* of a problem. With that purpose, we give a brief explanation

[1] As is usual in graph theory and combinatorics, we do not require nonempty parts in a partition.

of the basic terminology of Complexity Theory, although some concepts such as *problem* or *algorithm* are only treated in an informal way because an intuitive idea of them is enough to understand the rest of the text. Most of the content presented in this section was taken from [53].

By an *instance* of a problem, we refer to the input data for the problem in a prescribed array; for example, in the problem of determining if a given graph is connected, an instance of the problem is a fixed graph, as can be C_4, $\overline{P_3}$, or any other. Intuitively, an *algorithm* is a set of steps that can be used to solve a problem for each of its instances. An algorithm is commonly described as a set of rules that precisely define a sequence of operations in such a way that it allows us to obtain an *output* answer from the specific data of an *input* instance in a finite number of steps.

The time an algorithm takes to solve a given problem can be used to compare different algorithms to solve the same problem. Concretely, the **time complexity** of an algorithm A is the function f such that $f(n)$ is the maximum number of *steps* that A needs to solve any problem instance of size n. Nevertheless, in most cases it is impossible to accurately calculate the complexity of an algorithm if it is defined in this way, so we must settle for an asymptotic estimate of how fast it grows. To establish such an estimate, we introduce the so-called **big O notation**.

Let f and g be two functions from $\mathbb{N}$ to $\mathbb{R}^+$. We say that f has at most the rate of growth of g if there is a positive constant c such that $f(n) \leq cg(n)$ for all sufficiently large n. The class of all functions that has at most the rate of growth of $g(n)$ is denoted by $O(g(n))$. We say that an algorithm with time complexity f has complexity $O(g(n))$ if $f \in O(g(n))$. If there is an algorithm to solve a problem P whose complexity is $O(f(n))$, we say that P has complexity at most $O(f(n))$.

The following sequence on growth rates is well known and can be found in [11]; it indicates that a function with complexity $O(n!)$ requires a longer computation time than a function with complexity $O(2^n)$, that a function with complexity $O(2^n)$ requires a longer computation time than a function with complexity $O(n^3)$, and so on.

$$O(1) \subsetneq O(\log n) \subsetneq O(n \log n) \subsetneq O(n^2) \subsetneq O(n^3) \subsetneq O(2^n) \subsetneq O(n!).$$

Algorithms of complexity $O(n^k)$, which are naturally called **polynomial algorithms**, have demonstrated to be the most useful ones, for such a reason, these algorithms are also called **efficient** or **good**. Problems for which a polynomial algorithm exists are called **easy**, whereas problems for which no polynomial algorithm can exist are called **intractable** or **hard**.

1.2.1 NP-complete problems

In the restricted class of problems whose solution is either *yes* or *no*, which are naturally called **decision problems**, we distinguish two subclasses of great importance: the class of polynomial decision problems, denote by P (for *polynomial*), and the class of decision problems such that each positive answer can be verified in polynomial time, which is denoted by NP (for *non-deterministic polynomial*).

Since the output of a polynomial algorithm can be verified in polynomial time by running again the algorithm, it follows that P ⊆ NP, but it remains unclear whether P ≠ NP [42]. A decision problem is said to be **NP-hard** if the polynomial solvability of such problem imply that any problem in NP is solvable in polynomial time, that is, P = NP. An NP-hard problem in NP is called **NP-complete**.

In 1971, Stephen Cook [22] showed the existence of NP-problems by proving that *boolean satisfiability problem* (SAT-problem) is one of them. Observe that, once a problem P has been identified as NP-complete, to show that an NP problem P' is NP-complete it is enough to prove that there is a **polynomial-time reduction** from P to P', that is, a polynomial algorithm that assigns to each instance of P an instance of P' in such a way that an instance of P has solution "yes" if and only if its corresponding output also has solution "yes".

It is worth mentioning that, although particular techniques were required to prove that SAT-problem is NP-complete, just a year after Cook proved the existence of NP-complete problems, Karp [54] published a list of 21 new NP-complete problems by using the technique of polynomial-time reductions and taking as base the result of Cook.

1.2.2 Certifying algorithms

When executing an algorithm, one of three things can happen: either it produces a correct output (the desired case), or it is detected a bug in the algorithm (which is undesired, of course, but is generally preferable to continue without detecting the bug), or the algorithm fails in a way that masks bugs and prevents it from being detected (which is completely undesired). To avoid the latter case occur, it is desirable to have a **certifying algorithm**, which is an algorithm that outputs, together with the solution to the problem it solves, a *certificate* that the given solution is correct. In this way, the implementation of a certifying algorithm (including a checker for the certificates) may be considered to be more reliable than non-certifying algorithms. Notice that the checkers for the certifies produced by a certifying algorithm should be, in some sense, faster than the algorithm itself, otherwise any algorithm could be considered certifying (with its output being verified by running the algorithm again). Usually this is formalized by requiring that a verification of the proof take less time than the original algorithm.

Chapter 2

Some graph classes

In this section we introduce some families of graphs that we will study in the following chapters. We start describing some classes of graphs with the property of having, in some sens, just a few induced paths on four vertices, and we continue with graph families defined from partitions of their vertex sets.

2.1 Graphs with few induces P_4s

Complement reducible graphs, or **cographs** for short, were introduced in 1981 based on the following recursive conditions: K_1 is a cograph; if G is a cograph, then its complement $\overline{G}$ is also a cograph; if G and H are cographs, so is $G + H$. Additionally, in [23] was proved that some graph classes defined in a wide variety of ways since the 1970s coincide with the class of cographs. In the next theorem, we highlight some cograph characterizations that are particularly interesting for this work.

Theorem 2.1 ([23]). *Let G be a graph. The following statements are equivalent.*

1. *G is a cograph.*

2. *G is a P_4-free graph.*

3. *G can be constructed from trivial graphs by means of join and disjoint union operations.*

4. *For any nontrivial induced subgraph H of G, either H or $\overline{H}$ is disconnected.*

From the theorem above, it is clear that cographs constitute a hereditary and self-complementary class of graphs. Additionally, each cograph G can be uniquely represented by a rooted labeled tree taking as base Item 3 of the previous theorem: the leaf vertices are associated with the vertices of G, and each internal node is labeled 0 or 1 indicating the operation, join or disjoint union, performed on the cographs associated with their children, respectively. Such a tree is called the **cotree** associated with G. Remarkably, cographs can be recognized and their cotree can be constructed in $O(|V| + |E|)$-time by a certifying LexBFS algorithm [10]. Also, it follows from the uniqueness of the cotree representation that many algorithmic problems that are difficult for general graphs can be efficiently solved on cographs using bottom-up algorithms on their cotrees [23].

Cographs possess many desirable structural properties, and they are particularly interesting since some real-life applications involve graph models where paths of length four are unlikely to appear [24]. From this point of view, cographs are the most restrictive class (P_4-free), so a natural question is whether some cograph superclass with weaker restrictions on the amount of induced P_4s has similar properties, i.e., it allows us to develop efficient algorithms for problems that are difficult in general graphs. For the above reasons, the study of cographs was naturally followed by the introduction of many cograph superclasses having both, few induced P_4s and a unique tree representation.

Below, we introduce some graph classes with few induced paths of length three, which have the property of having a constructive characterization from simple primitive graphs and using simple graph operations. Such characterizations imply that these graph classes can be recognized in linear time and a tree representation (similar to the cotree) can be efficiently computed. Before introducing such graph families, we give some necessary definitions.

A **split partition** of a graph G is a partition (S, K) of V_G such that S and K are an independent set and a clique, respectively. The graphs admitting a split partition are the **split graphs** and they are characterized as the $\{2K_2, C_4, C_5\}$-free graphs [41]. A graph with a split partition (S, K) such that S is completely adjacent to K is called a **complete split graph**; these graphs are characterized as the $\{C_4, \overline{P_3}\}$-free graphs. The $\{2K_2, C_4\}$-free graphs are known as **pseudo-split graphs**. A graph G of order at least four is said to be a **headless spider** if there exists a split partition (S, K) of V and a bijection $f\colon S \to K$ such that either $N(s) = \{f(s)\}$ for any $s \in S$, or

$N(s) = K \setminus \{f(s)\}$ for every $s \in S$. Given a graph G and an induced path P of length three, a **partner** of P is a vertex v of G such that $V_P \cup \{v\}$ induces some of C_5, P_5, P, F or its complements (see Figure 2.1). Now, we introduce some cograph generalizations.

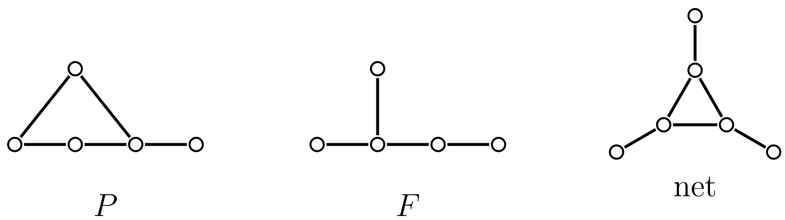


Figure 2.1: P, F, and the net graph.

A graph G is said to be:

1. P_4**-reducible** if any vertex belongs to at most one induced P_4.

2. A (q,t)**-graph** if no set of at most q vertices induces more than t distinct P_4s. The $(5,1)$-graphs are called P_4**-sparse graphs**.

3. **Extended** P_4**-reducible** if both, G and $\overline{G}$, are $\{P_5, F, P, \text{net}\}$-free graphs.

4. **Extended** P_4**-sparse** if both, G and $\overline{G}$, are $\{P_5, F, P\}$-free graphs.

5. P_4**-lite** if every induced subgraph of order at most six is either isomorphic to a headless spider, or it contains at most two induced P_4s.

6. P_4**-extendible** if for any vertex subset W inducing a P_4, there exists at most one vertex $v \notin W$ that belongs to a P_4 sharing vertices with W.

7. P_4**-tidy** if any induced P_4 has at most one partner.

8. P_4**-laden** if any induced subgraph of G of order at most six either is a split graph or it contains at most two induced P_4s.

9. **Extended** P_4**-laden** if any induced subgraph of G of order at most six either is a pseudo-split graph or it contains at most two induced P_4s.

It is worth emphasizing that we are not really interested in general (q,t)-graphs, but in $(q,q-4)$-graphs. This is due to, for any fixed q, $(q,q-4)$-graphs are known to have simple enough tree representations, while it remains unknown whether (q,t)-graphs have such representations for arbitrary values of q and t.

Giakoumakis and Vanherpe [43] observed that P_4-reducible graphs are the graphs that are both P_4-sparse and P_4-extendible graphs. Some other relations between the graph classes introduced above can be established from their definitions or using diverse characterizations for them. We represent the containment relationships between these classes in Figure 2.2, where an arc from a class $\mathcal{G}$ to a class $\mathcal{H}$ means that $\mathcal{H} \subseteq \mathcal{G}$. We remark that any graph class represented in Figure 2.2 can be recognized, and a tree representation can be obtained, in polynomial time. Moreover, in most cases this can be done in linear time.

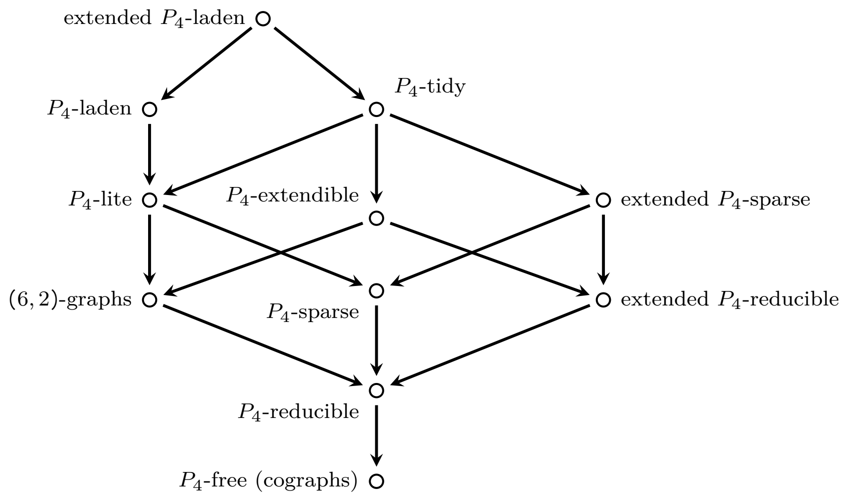


Figure 2.2: Relations between some graph classes with just a few induced paths on four vertices.

From the graph classes defined above, in this work we mainly focus on cographs and two of its proper superclasses, namely P_4-sparse and P_4-extendible graphs, for which we give some useful characterizations in Section 4.1.

2.2 Matrix Partitions

Many graph problems are about deciding whether the vertex set of a graph admits a partition with some constraints *within* and *between* the parts of the partition. Moreover, in many of these problems the inner conditions are simply being an independent set or a clique, while the external conditions correspond to some parts are completely adjacent or completely nonadjacent. Conveniently, these partition problems can be easily described with the concept of *matrix partitions*.

Given a symmetric matrix M of size $n \times n$ with entries on the set $\{0, 1, *\}$, which we call a **pattern** (of size n) for simplicity, an **M-partition** of a graph G is a partition $(V_1, V_2, \dots, V_n)$ of V_G such that, V_i is completely adjacent to V_j whenever $M_{i,j} = 1$, and V_i is completely nonadjacent to V_j whenever $M_{i,j} = 0$. Note that, if $M_{i,j} = *$, then there are not restrictions on the adjacencies between vertices of V_i and V_j.

Given a pattern M, the *M-partition problem* is the problem of deciding whether an input graph admits an M-partition. Observe that, if $M_{i,i} = *$ for any i, then every graph G admits an M-partition, because we can simply make $V_i = V_G$. Thus, this simple version of the M-partition problem is studied only for patterns without entries $*$ on the main diagonal. It is a straightforward observation that these kind of patterns can always be transformed by means of simultaneous row and column permutations into an equivalent pattern of the form

$$\left(\begin{array}{c|c} A & C \\ \hline C^T & B \end{array}\right),$$

where A is a block matrix with only entries 0 on its main diagonal and B is a block matrix with only entries 1 on its main diagonal. For the rest of this document we assume that any pattern has the form of (A, B, C)-blocks described before. Given $a, b, c \in \{0, 1, *\}$ an (a, b, c)-constant pattern is a pattern M such that its block A (respectively, B) has only entries a (resp. b) off the main diagonal, and whose block C has only entries c.

Some well-studied partition problems of graphs are generalized by matrix partitions. For instance, (k, ℓ)-colorings, and hence k-colorings, are particular cases of M-partitions: a (k, ℓ)-coloring corresponds to a $(*, *, *)$-constant pattern whose blocks A and B have size $k \times k$ and $\ell \times \ell$, respectively.

Another example of problems generalized by matrix partitions is provided by graph homomorphisms. Given two graphs, G and H, a **homomorphism** from G to H, also known as an *H-homomorphism* or an *H-coloring* of G, is a function $\phi\colon V_G \to V_H$ that preserve adjacencies, that is to say, that $\phi(u)\phi(v) \in E_H$ whenever $uv \in E_G$. It is straightforward to verify that, for a fixed graph H, the problem of determining whether a graph G admits an H-coloring is precisely the M-partition problem associated to the pattern M obtained from the adjacency matrix of H by interchanging 1 entries by $*$ entries. Some other matrix partitions that are particularly relevant in this work are going to be introduced in Section 2.3.

Notice that, for any fixed pattern M, the graphs admitting an M-partition constitute a hereditary class of graphs, $\mathcal{G}_M$. Hence, by Theorem 1.1, $\mathcal{G}_M$ can be characterized by the family of minimal $\mathcal{G}_M$-obstructions. For brevity, in this context $\mathcal{G}_M$-obstructions and minimal $\mathcal{G}_M$-obstructions are simply going to be called **M-obstructions** and **minimal M-obstructions**.

Observe that the set of all minimal obstructions of a hereditary class of graphs is not necessarily finite, for instance, the class of all bipartite graphs is hereditary but its set of minimal obstructions is the infinite family of all odd cycles. Notice that, if a pattern M has a finite set of minimal obstructions, then there exists a brute force algorithm that solves the M-partition problem in polynomial time. Otherwise, if the family of minimal M-obstructions is not finite, the M-partition problem can be NP-complete as in the well-known case of 3-coloring, or it can be efficiently solved as in the case of bipartite graphs that can be recognized in $O(|V| + |E|)$-time.

Knowing the minimal obstructions of a hereditary class of graphs also posses a great computational relevance, for example, by designing certifying algorithms for the problem of deciding whether an arbitrary graph belongs to such family. Going back to the example of bipartite graphs, a modified version of the Breadth First Search algorithm can be used as a certifying algorithm for the decision problem associated, deciding if a graph G is bipartite in time $O(|V| + |E|)$, using 2-colorings as yes-certificates and returning odd cycles as no-certificates. Observe that the yes-certificates can be checked in time $O(|E|)$, whilst the no-certificates take only $O(|V|)$-time to be checked. As we observed before, the relevance of certifying algorithms lies on the fact that, once the algorithm has been implemented, we have a warranty that such implementation is correct. From this point of view is of great computational interest to know the minimal obstructions of a hereditary property $\mathcal{P}$ because, as we exemplify with the bipartite graphs, such minimal obstructions are natural candidates to be no-certificates in a certifying algorithm for the decision problem associated to $\mathcal{P}$.

2.3 Polarity

Given two nonnegative integers, s and k, a graph G is called **(s, k)-polar** if the vertex set of G admits a partition (A, B) such that A induces a complete s-partite graph and B induces a k-cluster; such a partition is called an **(s, k)-polar partition** of G. A (k, k)-polar partition is simply referred as a **k-polar partition**, and a graph that admits such partition is a **k-polar graph**.

We use ∞ instead of s, k, or both, to indicate that the number of parts in the multipartite graph, or the number of components in the cluster is unbounded. Hence, we say that a graph G is an (s, ∞)-polar graph if its vertex set admits a partition (A, B) where A is a complete s-partite graph, and B is a cluster; such partition is an (s, ∞)-polar partition. The concepts of (∞, k)- and (∞, ∞)-polar graphs and partitions are analogously defined. The (∞, ∞)-polar graphs are commonly called **polar graphs**, whilst a $(1, \infty)$-polar graph is referred as a **monopolar graph**. A graph with a polar partition (A, B) such that A induces a clique is called a **unipolar graph**. Unipolar and monopolar graphs are particularly interesting because many recognition algorithms for polar graphs on specific graph classes first check whether the input graph is either unipolar or monopolar.

In [39] it was shown that, for any pattern M without $*$ entries in its blocks A or B, but such that C has only $*$ entries, the set of minimal M-obstructions is finite. Remarkably, for any nonnegative integers s and k, an (s, k)-polar partition correspond to an M-partition with the $(1, 0, *)$-constant pattern whose blocks A and B have size $s \times s$ and $k \times k$, respectively. Hence, for any pair of fixed nonnegative integers, s and k, there is only a finite number of minimal (s, k)-polar obstructions. Nevertheless, the complete lists of minimal (s, k)-polar obstructions are known just for a few pairs of values of s and k. A graph is $(0, k)$-polar if and only if it is a k-cluster, hence a $\{P_3, (k + 1)K_1\}$-free graph; it is $(s, 0)$-polar if and only if it is a complete s-partite graph, hence a $\{\overline{P_3}, K_{s+1}\}$-free graph; it is 1-polar if and only if it is a split graph, hence a $\{2K_2, C_4, C_5\}$-free graph [41]. In contrast, it is known that the families of minimal obstructions for polarity, monopolarity, and unipolarity, are all of them infinite, but structural descriptions of such families of forbidden subgraphs remain unknown.

It is worth mentioning that polar graphs were introduced as a generalization of split graphs, which are precisely the $(1, 1)$-polar graphs. In addition, split graphs are not the only interesting subclass of polar graphs, for instance, k-clusters and complete k-partite graphs are precisely the $(0, k)$- and $(k, 0)$-

polar graphs, and, in consequence, clusters are the $(0, \infty)$-polar graphs, while complete multipartite graphs are the $(\infty, 0)$-polar graphs. Moreover, for any bipartite graph $G[X, Y]$, a polar partition for both G and $\overline{G}$ is given by (X, Y), so polar graphs also generalize bipartite graphs and co-bipartite graphs. Nevertheless, although bipartite, co-bipartite, and split graphs can be recognized in linear time [44, 52], polar graphs are not easy to recognize. In fact, the problems of deciding whether an arbitrary graph is polar or monopolar are NP-complete [12, 36]. In contrast, unipolar graphs was proven to be recognizable in polynomial time [17, 35], and an explicit polynomial-time algorithm to solve the problem of deciding whether an arbitrary input graph admits an (s, k)-polar partition when s and k are fixed nonnegative integers was given in [38].

Since the problems of recognizing polar and monopolar graphs turned out to be NP-complete, they have been studied in its restricted version to many graph classes. Table 2.1 shows some results on complexity of polarity and monopolarity recognition in specific graph classes. It does not pretend to be an exhaustive compilation of all the known results, but rather to present the most representative ones.

We want to point out some remarks on known results on polarity. The polarity partition problem can be expressed in monadic second order logic without edge quantification, where it follows from the results in [1, 25] and [26] that polar graphs on bounded tree-width or bounded clique-width can be recognized in polynomial time. In this sense, results about polarity on chordal and permutation graphs are remarkable, since such graph families do not have bounded tree-width or clique-width.

An interesting result relating polarity and monopolarity complexities is given in [17], where the authors proved that, if it is known a polar partition of a graph G, then it can be decided in $O(|V|^2 \cdot |E|)$ time whether G is monopolar. Then, if polarity recognition is polynomial when we restrict it to a graph class, monopolarity will be too. Moreover, as we shown in Table 2.1, there are examples of graph classes in which polarity remains NP-complete while monopolarity recognition has an efficient algorithm. Related to above discussion, in [55] was formulated the question whether there exists a graph class for which polarity is easier than monopolarity. A partial result for this question was given by Yolov [60], who proved that polarity is not easier than monopolarity for graph classes closed under disjoint union.

The results above are all concerned to the complexity problem of polarity and monopolarity. Additionally of them, for a few classes the problem of

Graph class	**Monopolarity**	**Polarity**
general graphs	NP-c [36]	NP-c [12]
chordal	$O(n + m)$ [32]	P [32]
line graphs	$O(n)$ [14]	$O(n)$ [14]
permutation	$O(nm)$ [31]	$O(nm^2)$ [31]
maximal planar	$O(n^4)$ [56]	P [56]
co-comparability	P [17]	NP-c [60]
P_4-free (cographs)	$O(n)$ [34]	$O(n)$ [34]
claw-free	$O(n^3)$ [16]	NP-c [16, 56]
P_5-free	$O(n^4)$ [56]	NP-c [56]
triangle-free planar	NP-c [15, 56]	NP-c [15, 56]
claw-free planar	$O(n^3)$ [16]	P [56]
3-colorable comparability	NP-c [60]	NP-c [60]
comparability co-comparability	P [30]	P [30]

Table 2.1: Parameters n and m stands for the order and the size of a graph, respectively. P and NP-c means that the complexity status of the corresponding problem is polynomial or NP-complete, respectively.

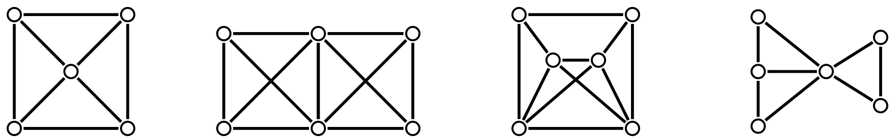
determining the minimal polar obstructions was treated. In [32] was shown that the family of minimal obstructions for polar and monopolar chordal graphs has infinitely many elements; nevertheless in [59] was given a simple recursive procedure to obtain all the chordal minimal monopolar obstructions. Also, in [48] was proven a forbidden subgraph characterization of line-polar bipartite graphs by several infinite families on minimal obstructions, which contrast with the linear algorithm for recognizing general line-polar graphs given in [14]. In the next section we will mention the work that has been done about determining minimal obstructions for polarity partitions on cographs, which will be important in the development of the results of Chapters 3 and 4.

2.3.1 Preliminary results on polar cographs

In [37] was demonstrated that, for any fixed nonnegative integers s and k, every cograph minimal (s, k)-polar obstruction has at most $(s + 1)(k + 1)$ vertices. Ekim, Mahadev and de Werra were pioneers giving explicit lists of cograph minimal (s, k)-polar obstructions; they exhibited the only eight cograph minimal polar obstructions, as well as the complete list of cograph minimal (s, k)-polar obstructions when $\min\{s, k\} = 1$ [33, 34]. In the last years, the study of cograph minimal (s, k) -polar obstructions has continued with the following main results. The exhaustive list of nine cograph minimal $(2, 1)$-polar obstructions was found by Bravo, Nogueira, Protti and Vianna [9] in 2016. Then, in 2019, Hell, Hernández-Cruz and Linhares-Sales [46] provided a full characterization of cograph minimal 2-polar obstructions and, in 2020, Contreras-Mendoza and Hernández-Cruz [19] proved a simple recursive characterization for the cograph minimal $(s, 1)$-polar obstructions for any arbitrary integer s, as well as the complete list of cograph minimal monopolar obstructions.

In order to this document be self-contained, we quote below some of the results referred in the previous paragraph concerning to cograph minimal obstructions for polarity, monopolarity, and $(1, s)$-polarity for a fixed positive integer s. Such results will be helpful in the development of some of the original results in this document.

Theorem 2.2 ([19]). *A graph G is a cograph minimal $(1, \infty)$-polar obstruction if and only if G is isomorphic to one of the graphs depicted in Figure 2.3.*



$K_1 \oplus C_4$ $\quad$ $K_2 \oplus 2K_2$ $\quad$ $\overline{2P_3}$ $\quad$ $K_1 \oplus (K_2 + P_3)$

Figure 2.3: Cograph minimal $(1, \infty)$-polar obstructions.

Theorem 2.3 ([34])**.** *A graph G is a cograph minimal polar obstruction if and only if G or its complement is isomorphic to $P_3 + H$, where H is any cograph minimal $(1, \infty)$-polar obstruction.*

Theorem 2.4 ([19])**.** *Let s be an integer, $s \geq 2$.*

1. *The graph G is a connected cograph minimal $(1, s)$-polar obstruction if and only if G is either a cograph minimal $(1, \infty)$-polar obstruction or it is isomorphic to $K_{s+1,s+1}$, $\overline{K_2} \oplus (K_2 + sK_1)$, or $K_1 \oplus (2K_2 + (s-1)K_1)$.*

2. *The graph G is a disconnected cograph minimal $(1, s)$-polar obstruction if and only there exists a positive integer t and nonnegative integers $s_0, s_1, \dots, s_t$ such that $G = G_0 + \cdots + G_t$, where G_i is a connected cograph minimal $(1, s_i)$-polar obstruction that is not a cograph minimal $(1, \infty)$-polar obstruction, and $s = t + \sum_{i=0}^{t} s_i$.*

Part II

Polarity on cograph superclasses

Chapter 3

Cograph minimal (∞, k)-polar obstructions

Exact lists of cograph minimal (∞, k)-polar obstructions are known only for $k \leq 1$. Throughout this section, we provide a partial recursive characterization for cograph minimal (∞, k)-polar obstructions. Our results allow us to give explicit lists for cograph minimal (∞, k)-polar obstructions for the cases $k = 2$ and $k = 3$. Notice that, by taking complements, analogous results can be trivially obtained for cograph minimal (s, ∞)-polar obstructions. All the results on this chapter can be found published in [18].

We start with two propositions exhibiting remarkable properties of cograph minimal (∞, k)-polar obstructions. Note that, if G is a cluster such that either G has at most $k + 1$ components or G has at most k nontrivial components, then G is an (∞, k)-polar graph. Hence, every cograph minimal (∞, k)-polar obstruction that is a cluster, has at least $k + 2$ components and at least $k + 1$ of them are nontrivial. In consequence, we have the following useful observation.

Remark 3.1. *Let k be an integer. Up to isomorphism, the graph $K_1 + (k+1)K_2$ is the only cograph minimal (∞, k)-polar obstruction that is a cluster.*

The following lemma is a slight modification of Lemma 1 in [46]; the proof is very similar, and thus it is omitted.

Lemma 3.2. *Let k be a nonnegative integer, and let G be a cograph minimal (∞, k)-polar obstruction. Then*

1. *G has at most $k + 2$ connected components,*

2. *G has at least one nontrivial component,*

3. *G has at most $k + 1$ trivial components,*

4. *if G has at least one trivial component, then G has at most one non-complete component,*

5. *every complete component of G has order one or two.*

3.1 Connected obstructions

The $(\infty, 0)$-polar cographs are precisely the complete multipartite graphs, and it is well known that the only cograph minimal $(\infty, 0)$-polar obstruction is $\overline{P_3}$. Furthermore, from Theorem 2.2, each cograph minimal $(\infty, 1)$-polar obstruction is disconnected. Thus, for $k \leq 1$, there exist no connected cograph minimal (∞, k)-polar obstructions. Next, we characterize the connected cograph minimal (∞, k)-polar obstructions for every integer k such that $k \geq 2$. In contrast with the case $k \leq 1$, it results that for $k \geq 2$ there exist connected cograph minimal (∞, k)-polar obstructions, and there is a fixed number of them as we show below.

Theorem 3.3. *Let k be an integer, $k \geq 2$, and let G be a connected cograph. Then, G is a minimal (∞, k)-polar obstruction if and only if G is a minimal polar obstruction.*

Proof. Let H be a disconnected cograph minimal (k, ∞)-polar obstruction. Note that, by the minimality of H, if K is a complete component of H, then $H - K$ is a (k, ∞)-polar graph, and thus also is H, which is absurd. Hence, every component of H is non-complete. Moreover, since H is not a $(1, \infty)$-polar graph, then H contains a cograph minimal $(1, \infty)$-polar obstruction H' as an induced subgraph. From Theorem 2.2, H' is connected, so it is completely contained in a single component of H. Thus, since H has no complete components, H contains $H' + P_3$ as an induced subgraph, but by Theorem 2.3, $H' + P_3$ is a not (k, ∞)-polar, so $H = H' + P_3$, which proves that H is a cograph minimal polar obstruction. The converse implication follows easily from Theorem 2.2 and Theorem 2.3. The result follows since a graph G is a connected cograph minimal (s, k)-polar obstruction if and only if its complement, $\overline{G}$, is a disconnected cograph minimal (k, s)-polar obstruction. $\square$

3.2 Disconnected obstructions

Since we have already characterized the connected cograph minimal (∞, k)-polar obstructions, we are now concerned only within the disconnected obstructions. We found useful the following notation to classify disconnected cograph minimal (∞, k)-polar obstructions with similar properties. Let c and i be integers such that $0 \leq i < c$. We say that a graph has **type** (c, i) if it has exactly c connected components and precisely i of them are trivial. We divide the study of cograph minimal (∞, k)-polar obstructions on three classes depending on their type, as follows: type $(c, 0)$, type $(c, c-1)$, and the rest of the types.

3.2.1 Type $(c, 0)$ obstructions

We begin our study of cograph minimal (∞, k)-polar obstructions without isolated vertices by noticing some restrictions on their connected components.

Lemma 3.4. *Let k be an integer, $k \geq 2$, and let G be a disconnected cograph minimal (∞, k)-polar obstruction without isolated vertices. Then, G has at least two non-complete components.*

Proof. Let G be as in the hypothesis. From Remark 3.1, we have that G is not a cluster, so G has at least one non-complete component. Aiming for a contradiction, suppose that G has precisely one non-complete component. Then, by Lemma 3.2, for some integer $j \in \{1, \ldots, k+1\}$, $G \cong jK_2 + H$, where H is a connected non-complete graph. Note that since G is not an (∞, k)-polar graph, H is not an $(\infty, k-j)$-polar graph.

Let v be a vertex of H, and suppose that $H - v$ is not a cluster. Thus, for every (∞, k)-polar partition (A, B) of $G - v$, $A \cap V(H - v) \neq \varnothing$, which implies that $(A \cap V(H - v), B \cap V(H - v))$ is an $(\infty, k-j)$-polar partition of $H - v$. Hence, for each vertex v of H, $H - v$ is either a cluster or an $(\infty, k-j)$-polar graph.

Since H is not an $(\infty, k-j)$-polar graph, H contains a cograph minimal $(\infty, k-j)$-polar obstruction H' as an induced subgraph. Nevertheless, by Theorem 3.3, if H' is connected, then it is a cograph minimal (∞, k)-polar obstruction, in contradiction with the minimality of G. Thus H' is a disconnected induced subgraph of the connected cograph H. Let v be a vertex of $H - H'$. Since H' is an induced subgraph of $H - v$, we have that $H - v$ is not an $(\infty, k-j)$-polar graph, which implies that $H - v$ and H' are clusters. However, from Remark 3.1, H' is isomorphic to $K_1 + (k - j + 1)K_2$, but in such a case G properly contains $K_1 + (k + 1)K_2$ as an induced subgraph, contradicting its minimality. The contradiction arose from supposing that G has no more than one non-complete component, so G must have at least two non-complete components. □

The following lemma characterizes a family of graphs with some properties that are common to all cograph minimal (∞, k)-polar obstructions without isolated vertices. It will be very useful to give recursive constructions of such graphs.

Lemma 3.5. *Let k be a positive integer, and let H be a cograph. Then, H is such that*

1. *H is not a cluster,*
2. *H is $(1, k)$-polar, but not $(1, k-1)$-polar, and*
3. *for each vertex v of H, the graph $H - v$ is either a $(1, k-1)$-polar graph or a cluster,*

if and only if exactly one of the following statements is satisfied:

a. *H is a cograph minimal $(1, k-1)$-polar obstruction, that is neither a cograph minimal $(1, \infty)$-polar obstruction nor isomorphic to kK_2.*

b. *$H \cong P_3 + (k-1)K_2$.*

c. *$k \geq 2$, and $H \cong (k-2)K_2 + (K_1 \oplus 2K_2)$.*

Proof. Let H be a cograph that satisfies Items 1 to 3. Since H is a cograph, we have from Item 2 that H contains a cograph minimal $(1, k-1)$-polar obstruction H'. Observe that, since H is not a cluster but it is $(1, k)$-polar, if $H = H'$ then it satisfies item *a*.

Let K be a complete component of H (if any). We claim that K has order two. From item 3, for every vertex w of K, $H - w$ admits a $(1, k-1)$-polar partition (A, B). If K is a trivial graph, then $(A \cup \{w\}, B)$ is a $(1, k-1)$-polar partition of H, which is impossible. Else, if K has order at least three, then $V(K - w) \cap B \neq \varnothing$ (otherwise $A = V(K - w)$ and B covers $H - K$, which cannot occur since $H - K$ is not a cluster), and then $(A, B \cup \{w\})$ is a $(1, k-1)$-polar partition of H, a contradiction. Therefore, every complete component of H is isomorphic to K_2.

Suppose that H properly contains a cograph minimal $(1, k-1)$-polar obstruction as an induced subgraph. It implies that there exists a vertex v of H such that $H - v$ is not a $(1, k-1)$-polar graph, and from item 3, $H - v$ is a cluster. Note that from item 1, H has a subgraph P isomorphic to P_3, and that v is necessarily a vertex of P, or $H - v$ would not be a cluster.

Let v and P be as described above, then we have two cases: either $d_P(v) = 1$ or $d_P(v) = 2$. Suppose first that $d_P(v) = 1$. Since H is a cograph and $d_P(v) = 1$, v is adjacent to exactly one component of the cluster $H - v$, and therefore $H \cong jK_2 + (K_a \oplus (v + K_b))$ for some positive integers a and b and some nonnegative integer j. Moreover, since H is not a $(1, k-1)$-polar graph, $j \geq k-1$, but if $j > k-1$ then H contains $(k+1)K_2$ as a proper induced subgraph, and then it is not a $(1, k)$-polar graph, contradicting our assumptions. Thus, $j = k-1$, and $H \cong (k-1)K_2 + (K_a \oplus (v + K_b))$. Observe that H contains $H' \cong P_3 + (k-1)K_2$ as an induced subgraph, and H' is neither $(1, k-1)$-polar nor a cluster. Therefore, by item 3, $H \cong P_3 + (k-1)K_2$, that is, H satisfies item b.

For the second case, suppose that $d_P(v) = 2$. Note that since v is adjacent to at least two components of the cluster $H - v$, then v is completely adjacent or completely not-adjacent to each component of $H - v$, and therefore, it is completely adjacent to at least two components of $H - v$. Let K be a component of $H - v$ that is completely adjacent to v, and suppose to reach a contradiction that K has more than two vertices: if w is a vertex of K, then $H - w$ is not a cluster, so it admits a $(1, k-1)$-polar partition (A, B) and therefore $(A, B \cup \{w\})$ is a $(1, k)$-polar partition of H, a contradiction. Hence, every component of $H - v$ that is completely adjacent to v has at most two vertices, and in consequence H is isomorphic to $qK_2 + (v \oplus (\ell K_2 + mK_1))$ for some nonnegative integers ℓ, m and q such that $\ell + m \geq 2$.

Observe that if $\ell + q \geq k + 1$ then H contains $(k+1)K_2$ as an induced subgraph, and then H is not a $(1, k)$-polar cograph, contradicting our hypothesis. Therefore, $\ell + q \leq k$. Furthermore, since $H - v$ is a cluster that is not a $(1, k-1)$-polar graph, it contains a cograph minimal $(1, k-1)$-polar obstruction H' that is a cluster as an induced subgraph. Nevertheless, the only cograph minimal $(1, k-1)$-polar obstruction that is a cluster is $H' \cong kK_2$. The above observation implies that $\ell + q \geq k$, so we have that $\ell + q = k$.

It is straightforward to show that if $\ell \leq 1$, then H has $P_3 + (k-1)K_2$ as a proper induced subgraph, which is impossible as we have noted when proving the case $d_P(v) = 1$. Thus, $\ell \geq 2$. Furthermore, note that the component of

H that contains v is a $(1, \ell + m)$-polar graph that is not a $(1, \ell + m - 1)$-polar graph, which implies that H is a $(1, k + m)$-polar graph that admits no $(1, k + m - 1)$-polar partitions. However, by hypothesis H is a $(1, k)$-polar graph that is not a $(1, k - 1)$-polar graph, so we have that $m = 0$, and then $H \cong (k - \ell)K_2 + (v \oplus \ell K_2)$. Aiming for a contradiction, suppose that $\ell \geq 3$, and let w be a vertex of H adjacent to v. Since $H - w$ is not a cluster, it is a $(1, k - 1)$-polar graph, and therefore the component of $H - w$ that contains v is $(1, \ell - 1)$-polar, but this is impossible since such component is isomorphic to $K_1 \oplus ((\ell - 1)K_2 + K_1)$, which contains the cograph minimal $(1, \ell - 1)$-polar obstruction $K_1 \oplus (2K_2 + (\ell - 2)K_1)$ as an induced subgraph. Hence, $\ell = 2$ and $H \cong (k - 2)K_2 + (K_1 \oplus 2K_2)$, so item *c.* is satisfied.

To prove that the graphs described in items *b.* and *c.* satisfy the statements of items 1, 2, and 3 is a simple routine work. The analogous result for graphs described in item *a.* follows from Theorem 2.4. □

It also will be useful to know when do the graphs described in the above lemma posses some specific properties. The following remark identifies some interesting cases. The proof is straightforward and thus omitted.

Remark 3.6. *Let k be an integer and, let H be a cograph.*

1. *Suppose that H is a cograph minimal $(1, k - 1)$-polar obstruction that is neither a cograph minimal $(1, \infty)$-polar obstruction nor isomorphic to kK_2. Then, H is an $(\infty, k - 1)$-polar graph if and only if H has precisely one component non-isomorphic to K_2.*

2. *The graph H, with $H \cong P_3 + (k - 1)K_2$ is a $(2, k - 1)$-polar graph, and for each vertex v of H, $H - v$ is either a $(1, k - 1)$-polar graph or it is isomorphic to kK_2.*

3. *The graph H, with $H \cong (k - 2)K_2 + (K_1 \oplus 2K_2)$ is a $(3, k - 1)$-polar graph, and for each vertex v of H, $H - v$ is either a $(1, k - 1)$-polar graph or it is isomorphic to kK_2.*

It results convenient to divide the study of disconnected cograph minimal (∞, k)-polar obstructions without isolated vertices into two cases, depending on whether some component is isomorphic to P_3. We start by treating the case in which the graphs have not components isomorphic to P_3.

Lemma 3.7. *Let k be a nonnegative integer, and let G be a graph without components isomorphic to P_3. Then, G is a disconnected cograph minimal (∞, k)-polar obstruction without isolated vertices if and only if there exist positive integers k_1 and k_2, and cographs H_1 and H_2 such that $G = H_1 + H_2$, and for $i \in \{1, 2\}$, the following statements are satisfied:*

1. *H_i is not a cluster,*
2. *H_i is a $(1, k_i)$-polar graph that admits no $(1, k_i - 1)$-polar partitions,*
3. *for each vertex v of H_i, the graph $H_i - v$ is either a $(1, k_i - 1)$-polar graph or a cluster,*
4. *for $j \in \{1, 2\}$ such that $j \neq i$, if H_i is not a cograph minimal $(1, k_i - 1)$-polar obstruction, then H_j is an $(\infty, k_j - 1)$-polar graph, and*
5. *$k = k_1 + k_2 - 1$.*

Proof. Suppose that G is a disconnected cograph minimal (∞, k)-polar obstruction without isolated vertices. From Lemma 3.4, G has at least two non-complete components, say G_1 and G_2. Let $H_1 = G_1$ and $H_2 = G - G_1$. Clearly, $G = H_1 + H_2$ and both, H_1 and H_2, are cographs that are not clusters.

Let $\{i, j\} = \{1, 2\}$. Observe that, since G_j is a non-complete component of G, and G has no components isomorphic to P_3, there exists a vertex v of H_j such that $H_j - v$ is not a cluster. In addition, by the minimality of G, $G - v$ admits an (∞, k)-polar partition (A, B), but $G - v = H_i + (H_j - v)$, so $G - v$ has at least two non-complete components, and therefore (A, B) is a $(1, k)$-polar partition. Furthermore, since H_i contains P_3 as an induced subgraph, it is not a $(1, 0)$-polar graph. The above observations imply that there exists an integer $k_i \in \{1, \dots, k - 1\}$ such that H_i is a $(1, k_i)$-polar graph that is not $(1, k_i - 1)$-polar. Note that G is a $(1, k_1 + k_2)$-polar graph that is not (∞, k)-polar, which implies that $k \leq k_1 + k_2 - 1$.

Let v be a vertex of H_i, and let (A, B) be an (∞, k)-polar partition of $G - v$. If $H_i - v$ is not a cluster, and given that H_j is neither, (A, B) is a $(1, k)$-polar partition, and since H_j is not a $(1, k_j - 1)$-polar graph, then $(A \cap V(H_i - v), B \cap V(H_i - v))$ is a $(1, k - k_j)$-polar partition, which implies that $H_i - v$ is a $(1, k_i - 1)$-polar graph, because $k - k_j \leq k_i - 1$. Therefore, for each vertex v of H_i, the graph $H_i - v$ is either a cluster or a $(1, k_i - 1)$-polar graph.

Suppose that H_i is not a cograph minimal $(1, k_i - 1)$-polar obstruction. Since H_i is not a $(1, k_i - 1)$-polar graph, it follows from Lemma 3.5 and Remark 3.6 that there exists a vertex v of H_i for which $H_i - v \cong k_i K_2$. Let (A, B) be an (∞, k)-polar partition of $G - v$. The graph H_j is not a cluster, so we have that $A \cap V(H_j) \neq \varnothing$, and then $(A \cap V(H_j), B \cap V(H_j))$ is an $(\infty, k - k_i)$-polar partition of H_j, and therefore H_j is an $(\infty, k_j - 1)$-polar graph, because $k - k_i \leq k_j - 1$. Hence, if H_i is not a cograph minimal $(1, k_i - 1)$-polar obstruction, then H_j is an $(\infty, k_j - 1)$-polar graph.

So far, we have only shown that $k \leq k_1 + k_2 - 1$. To prove the equality, we will show that G is not a cograph minimal (∞, j)-polar obstruction for $j \leq k_1 + k_2 - 2$, which implies that $k \geq k_1 + k_2 - 1$.

It follows from Lemma 3.5 that $k_i \geq 2$, and by construction we have that $k_i \leq k - 1$. The above observations imply that if $k \leq 2$, then there exist no (∞, k)-polar obstructions without isolated vertices or components isomorphic to P_3, so we can assume that $k \geq 3$. Aiming for a contradiction, suppose that $k < k_1 + k_2 - 1$, in which case at least one of k_1 and k_2 is greater than or equal to three. Let us assume without loss of generality that $k_1 \geq 3$.

Since $k_1 \geq 3$ we have from Theorem 2.4 and Lemma 3.5 that H_1 contains, as a proper induced subgraph, a cograph H_1' which is not a cluster and such that it is a $(1, k_1 - 1)$-polar graph but it is not $(1, k_1 - 2)$-polar. Observe that the cograph $G' = H_1' + H_2$ is not an $(\infty, k_1 + k_2 - 2)$-polar graph, because since neither H_1' nor H_2 are clusters, every $(\infty, k_1 + k_2 - 2)$-polar partition of G' is a $(1, k_1 + k_2 - 2)$-polar partition, which is impossible since H_1' is not $(1, k_1 - 2)$-polar and H_2 is not $(1, k_2 - 1)$-polar. Therefore G has a cograph $(\infty, k_1 + k_2 - 2)$-polar obstruction as a proper induced subgraph, and then G is not a cograph minimal (∞, j)-polar obstruction for $j < k_1 + k_2 - 2$. As we have mentioned, it proves that $k = k_1 + k_2 - 1$, which is absurd since we are supposing that $k < k_1 + k_2 - 1$. Thus, $k = k_1 + k_2 - 1$ as we intended. This finalizes the proof of the first implication of the proposition.

For the converse implication let us suppose that $G = H_1 + H_2$ is a cograph without components isomorphic to P_3 such that, for some positive integers k_1 and k_2 and any election of $i, j \in \{1, 2\}$, $i \neq j$, the graphs H_i and H_j satisfy the enumerated items of this lemma's statement.

Aiming for a contradiction, suppose that G admits an (∞, k)-polar partition (A, B). Since H_1 and H_2 are not clusters and $k = k_1 + k_2 - 1$, (A, B) is a $(1, k_1 + k_2 - 1)$-polar partition of G, but this is impossible since for hypothesis H_i is not a $(1, k_i - 1)$-polar cograph for any $i \in \{1, 2\}$. Thus G is not an

(∞, k)-polar graph.

Let v be a vertex of G, let us suppose without loss of generality that $v \in V(H_1)$. If $H_1 - v$ admits a $(1, k_1 - 1)$-polar partition (A_1, B_1), then, for any $(1, k_2)$-polar partition (A_2, B_2) of H_2, $(A_1 \cup A_2, B_1 \cup B_2)$ is a $(1, k)$-polar partition of $G - v$. Otherwise, if $H_1 - v$ is not a $(1, k_1 - 1)$-polar graph, by item 3 we have that $H_1 - v$ is a cluster, and by Lemma 3.5 and Remark 3.6 it has exactly k_1 components. In addition, by item 4, H_2 is an $(\infty, k_2 - 1)$-polar graph. Thus, if (A_1, B_1) is a $(0, k_1)$-polar partition of $H_1 - v$ and (A_2, B_2) is an $(\infty, k_2 - 1)$-polar partition of H_2, then $(A_1 \cup A_2, B_1 \cup B_2)$ is an (∞, k)-polar partition of $G - v$. Hence, G is a cograph minimal (∞, k)-polar obstruction. Clearly, G is a disconnected graph, and it follows from Lemma 3.5 that G has no isolated vertices. □

Based on Lemmas 3.5 and 3.7 and Remark 3.6 it is straightforward to deduce the following recursive construction of cograph minimal (∞, k)-polar obstructions without isolated vertices nor components isomorphic to P_3.

Theorem 3.8. *Let k be a positive integer, and let G be a graph without components isomorphic to P_3. Then, G is a disconnected cograph minimal (∞, k)-polar obstruction without isolated vertices if and only for some positive integers k_1 and k_2, and some cographs H_1 and H_2,*

1. *$G = H_1 + H_2$,*
2. *$k = k_1 + k_2 - 1$,*
3. *for $i \in \{1, 2\}$, H_i is either a cograph minimal $(1, k_i - 1)$-polar obstruction that is neither a cograph minimal $(1, \infty)$-polar obstruction nor isomorphic to $k_i K_2$, or $k_i \geq 2$ and $H \cong (k_i - 2)K_2 + (K_1 \oplus 2K_2)$, and*
4. *if $H_i \cong (k_i - 2)K_2 + (K_1 \oplus 2K_2)$ and $G - H_i$ is a cograph minimal $(1, k_i - 1)$-polar obstruction, then $G - H_i$ has exactly one component non-isomorphic to K_2.*

We now turn our attention to cograph minimal (∞, k)-polar obstructions without isolated vertices that have some component isomorphic to P_3. We begin with a technical characterization of such family of graphs, followed by two lemmas that treat with specific subcases, and finalize with a recursive construction for these obstructions. It is worth noticing that, by Lemma 3.9, any cograph minimal (∞, k)-polar obstruction with a component isomorphic to P_3 has no isolated vertices.

Lemma 3.9. *Let k be a positive integer, and let G be a graph with at least one component isomorphic to P_3. Then G is a cograph minimal (∞, k)-polar obstruction if and only if $G \cong P_3 + H$, where H is a cograph that satisfies the following statements:*

1. *H is not a $(1, k-1)$-polar graph,*
2. *H is not a cluster,*
3. *H is an $(\infty, k-1)$-polar graph,*
4. *H is either a $(1, k)$-polar graph or an $(\infty, k-2)$-polar graph, and*
5. *for each vertex v of H, the graph $H - v$ is either a $(1, k-1)$-polar graph or a k-cluster.*

Proof. Suppose that G is a cograph minimal (∞, k)-polar obstruction with a component isomorphic to P_3, and let H be such that $G \cong P_3 + H$. Note that H cannot be a $(1, k-1)$-polar graph, because P_3 is a $(1, 1)$-polar graph, and then G would be a $(1, k)$-polar graph.

To prove that H is not a cluster we will first prove by means of a contradiction that H has no isolated vertices. If H has at least one isolated vertex, we have from Lemma 3.2 that for some positive integers p and q, G is isomorphic to $pK_1 + qK_2 + P_3$, but in such a case G is a $(1, q+1)$-polar graph, which implies that $k \leq q$. Furthermore, for each integer $j \in \{2, \ldots, q\}$, G contains the cograph minimal (∞, j)-polar obstruction $K_1 + (j+1)K_2$ as a proper induced subgraph, which implies that $k \leq 1$. But it is impossible, since the cograph minimal (∞, k)-polar obstructions for $k \leq 1$ have no components isomorphic to P_3. Hence, H has no isolated vertices, and if H is a cluster, then for some positive integer q, $G \cong qK_2 + P_3$. We have that $q \geq k+1$ because G is not an (∞, k)-polar graph, but then G contains the cograph minimal (∞, k)-polar obstruction $K_1 + (k+1)K_2$ as a proper induced subgraph, in contradiction with the minimality of G. This contradiction arose from supposing that H is a cluster, so it is not.

Items 3 to 5 can be easily proved by considering (∞, k)-polar partitions of $G - v$ when v is either a vertex of $G - H$, or a vertex of H. We have used similar arguments before, so the details of these arguments are omitted. Also, the proof of the converse implication is very similar to the proof of the converse of Lemma 3.7, so it will be also omitted. $\square$

Lemma 3.10. *Let k be a positive integer, and let $G \cong P_3 + H$ be a cograph minimal (∞, k)-polar obstruction. If H is not a $(1, k)$-polar graph, then H is a connected non-complete graph.*

Proof. Let k, G and H be as in the hypothesis. By Lemma 3.9, H is not a cluster, so H has at least one non-complete component. Moreover, it also follows from Lemma 3.9 that H is an $(\infty, k-1)$-polar graph that is not $(1, k-1)$-polar, which implies that H cannot have more than one non-complete component. Therefore, H has precisely one non-complete component. In addition, since H is an induced subgraph of G, it follows from Lemma 3.2 that every non-complete component of H is isomorphic to K_1 or K_2. In addition, it also follows from Lemma 3.2 that H has no isolated vertices, otherwise G would have at most one non-complete component, which is not the case. Hence, for some nonnegative integer ℓ, $H \cong \ell K_2 + H'$, where H' is a connected non-complete graph.

Suppose that $\ell \geq 1$, and let $v \in V(H - H')$. Note that since $H' \leq H - v$, we have that $H - v$ is not a cluster. Hence, by Lemma 3.9, we have that $H - v$ admits a $(1, k-1)$-polar partition (A, B). But in such case, $(A, B \cup \{v\})$ is a $(1, k)$-polar partition of H, which is impossible from our original hypotheses. The contradiction arose from supposing that $\ell \geq 1$, so $\ell = 0$ and then $H = H'$, which proves that H is a connected non-complete graph. □

The next trivial observation will be helpful in some of the following results. It is immediate from the cotree representation of cographs.

Remark 3.11. *Let H be a connected cograph, and let H' be a disconnected induced subgraph of H. Then $K_1 \oplus H'$ is also an induced subgraph of H.*

Lemma 3.12. *Let k be a positive integer, and let G be a graph with at least one component isomorphic to P_3.*

1. *If $k = 2$ then, G is a cograph minimal (∞, k)-polar obstruction of type $(2, 0)$ if and only if $G \cong P_3 + C_4$ or $G \cong P_3 + (K_1 \oplus 2K_2)$.*

2. *If $k \geq 3$ then, G is a cograph minimal (∞, k)-polar obstruction of type $(2, 0)$ if and only if $G \cong P_3 + H$, where H is any connected cograph minimal $(1, k-1)$-polar obstruction.*

Proof. We prove only the second statement, the case $k = 2$ can be treated in a very similar way. Suppose that H is any connected cograph minimal

$(1, k-1)$-polar obstruction. Since $k \geq 3$, we have from Theorems 2.2 and 2.4 that H is isomorphic to some cograph in the set

$$\{K_1 \oplus C_4,\ K_2 \oplus 2K_2,\ \overline{2P_3},\ K_1 \oplus (K_2 + P_3),\ K_{k,k},\\ \overline{K_2} \oplus (K_2 + (k-1)K_1),\ K_1 \oplus (2K_2 + (k-2)K_1)\}.$$

It is straightforward to check that, in any case, H satisfies the items enumerated in Lemma 3.9, which implies that $P_3 + H$ is a cograph minimal (∞, k)-polar obstruction of type $(2, 0)$.

Conversely, let us suppose that G is a cograph minimal (∞, k)-polar obstruction of type $(2, 0)$. From Lemma 3.9, $G \cong P_3 + H$ where H is a connected non-complete cograph that contains a cograph minimal $(1, k-1)$-polar obstruction H' as an induced subgraph. As we just mentioned, if H' is connected, then $P_3 + H'$ is a cograph minimal (∞, k)-polar obstruction, so $H = H'$. Otherwise, if H' is disconnected, it follows from Remark 3.11 that H contains $K_1 \oplus H'$ as an induced subgraph. Nevertheless, from Theorem 2.4 and Lemma 3.9, $H' \cong kK_2$, but in this case H contains properly the connected cograph minimal $(1, k-1)$-polar obstruction $K_1 \oplus (2K_2 + (k-2)K_1)$ as an induced subgraph, which contradicts the minimality of G. □

Theorem 3.13. *Let k be an integer, $k \geq 2$, and let G be a graph with at least one component isomorphic to P_3. Then, G is a cograph minimal (∞, k)-polar obstruction if and only if $G \cong P_3 + H$ and one of the following statements is satisfied:*

1. *$H \cong P_3 + (k-1)K_2$.*
2. *$H \cong (k-2)K_2 + (K_1 \oplus 2K_2)$.*
3. *for some integer $j \in \{1, \ldots, k-1\}$, $H \cong (k-j-1)K_2 + H_j$, where H_j is a connected cograph minimal $(1, j)$-polar obstruction that is not a cograph minimal $(1, \infty)$-polar obstruction.*
4. *$k \geq 3$, and H is any cograph minimal $(1, \infty)$-polar obstruction.*

Proof. Suppose that $G \cong P_3 + H$ is a cograph minimal (∞, k)-polar obstruction. From Lemmas 3.5 and 3.9 and Remark 3.6, we have that if H is a $(1, k)$-polar graph, then H satisfies one of the following statements:

a. $H \cong P_3 + (k-1)K_2$.

b. $k \geq 2$, and $H \cong (k-2)K_2 + (K_1 \oplus 2K_2)$.

c. H is a cograph minimal $(1, k-1)$-polar obstruction, that is neither a cograph minimal $(1, \infty)$-polar obstruction nor isomorphic to kK_2, and such that exactly one of its components is not isomorphic to K_2.

Furthermore, from Theorem 2.4 we have that the graphs described in item c are precisely the graphs H such that, for some integer $j \in \{1, \ldots, k-1\}$, $H \cong (k-j-1)K_2 + H_j$, where H_j is a connected cograph minimal $(1, j)$-polar obstruction that is not a cograph minimal $(1, \infty)$-polar obstruction.

Suppose then that H is not a $(1, k)$-polar graph. It follows from Lemma 3.10 that H is a connected non-complete graph, and then G is a cograph minimal (∞, k)-polar obstruction of type $(2, 0)$, and then it follows from Lemma 3.12 that H satisfies item 3 or item 4 of the theorem statement.

The converse follows easily from Lemmas 3.5, 3.9 and 3.12 and Remark 3.6. □

3.2.2 Obstructions of type $(c, c-1)$

So far, we have obtained a recursive characterization of cograph minimal (∞, k)-polar obstructions without isolated vertices. Next, we focus on the special case of cograph minimal (∞, k)-polar obstructions such that all its components, except one, are trivial. We begin giving a technical characterization of these obstructions. The proof of this result is omitted since it is very similar to that of Lemma 3.9.

Lemma 3.14. *Let j and k be integers such that $0 \leq j+1 \leq k$. Then, G is a cograph minimal (∞, k)-polar obstruction of type $(k-j+1, k-j)$ if and only if $G \cong (k-j)K_1 + H$, where H is a connected non-complete cograph that satisfies the following statements:*

1. *H is not a $(1, k)$-polar graph,*
2. *H is not an (∞, j)-polar graph,*
3. *H is an $(\infty, j+1)$-polar graph,*
4. *for each vertex v of H, $H - v$ is either $(1, k)$-polar or (∞, j)-polar.*

The following result provides a pleasant recursive characterization of cograph minimal (∞, k)-polar obstructions of type $(2, 1)$.

Theorem 3.15. *Let k be an integer, $k \geq 2$, and let G be a graph with precisely two connected components, one of them trivial. Then G is a cograph minimal (∞, k)-polar obstruction if and only if $G \cong K_1 + (K_1 \oplus H')$, where H' is a disconnected cograph minimal $(\infty, k-1)$-polar obstruction that is $(1, k)$-polar.*

Proof. Let $H = K_1 \oplus H'$, where H' is a disconnected cograph minimal $(\infty, k-1)$-polar obstruction that is $(1, k)$-polar, and let $G = K_1 + H$. Observe that by the election of H', H is an (∞, k)-polar graph that is not $(1, k)$-polar, and for each vertex v of H, the graph $H - v$ is either $(1, k)$- or $(\infty, k-1)$-polar.

Let us suppose, to reach a contradiction, that G admits an (∞, k)-polar partition (A, B). Since H is not $(1, k)$-polar, $G[A]$ must be a nontrivial connected graph. Thus, since H is not complete, $A \subseteq V(H)$ and H is an $(\infty, k-1)$-polar graph, which is impossible. Therefore G is not an (∞, k)-polar graph.

Let v be a vertex of G. If v is the only isolated vertex of G, then $G - v = H$, and H is an (∞, k)-polar graph, so $G - v$ is. Otherwise $v \in V(H)$ and, as we have noted above, $H - v$ is is either $(1, k)$- or $(\infty, k-1)$-polar, so $G - v$ is an (∞, k)-polar graph. Hence, $G \cong K_1 + (K_1 \oplus H')$ is a cograph minimal (∞, k)-polar obstruction whenever H' is a cograph minimal $(\infty, k-1)$-polar obstruction that is $(1, k)$-polar.

Conversely, suppose that G is a cograph minimal (∞, k)-polar obstruction with precisely two connected components, one of them trivial. By Lemma 3.14, $G \cong K_1 + H$ for some connected cograph H that is not $(\infty, k-1)$-polar, and such that for every vertex v of H, $H - v$ is either $(1, k)$- or $(\infty, k-1)$-polar. Note that H contains a cograph minimal $(\infty, k-1)$-polar obstruction H' as an induced subgraph, but from Theorem 3.3, H' cannot be connected, or H' would be a proper induced subgraph of G that is not an (∞, k)-polar graph, an absurd. Thus, H' must be disconnected, and from Remark 3.11, $K_1 \oplus H' \leq H$. But in such a case G contains the cograph minimal (∞, k)-polar obstruction $K_1 + (K_1 \oplus H')$ as an induced subgraph, so $G \cong K_1 + (K_1 \oplus H')$. □

Notice that, by Lemma 3.14, a graph G is a cograph minimal (∞, k)-polar obstruction of type $(k+2, k+1)$ if and only if $G \cong (k+1)K_1 + H$, where H is a connected non-complete cograph minimal $(1, k)$-polar obstruction that is a complete multipartite graph. Moreover, from Theorem 2.4, for any integer k, $k \geq 2$, the only cograph minimal $(1, k)$-polar obstructions that are complete multipartite graphs are $K_{k+1,k+1}$ and $K_1 \oplus C_4$. Thus, the following result follows immediately from Lemma 3.14.

Theorem 3.16. *Let k be an integer, $k \geq 2$. Thus, G is a cograph minimal (∞, k)-polar obstruction of type $(k+2, k+1)$ if and only of $G \cong (k+1)K_1 + H$, where H is isomorphic to $K_{k+1,k+1}$ or to $K_1 \oplus C_4$.*

Unfortunately, obtaining explicit lists of disconnected cograph minimal (∞, k)-polar obstructions with precisely one nontrivial component is a very difficult task. As we have shown above, a simple recursive construction of $(2, 1)$- and $(k + 2, k + 1)$-type obstructions is possible, but as we show below, it is not enough for covering the general case.

The following three lemmas are auxiliary results that will be the cornerstone for obtaining explicit lists of cograph minimal (∞, k)-polar obstructions of types $(k + 1, k)$ and $(k, k - 1)$. The three of them are based on the same proof technique: we consider all the distinct ways in which a connected cograph can be generated from another connected cograph whose cotree has specific characteristics. For the sake of length, we only sketch the proof of the first proposition, the other two are very similar.

Lemma 3.17. *Let k be an integer, $k \geq 2$, and let H be a complete multipartite graph with at least two parts and such that each part has at least $k + 1$ vertices. If H' is a connected cograph of order $|V(H)| + 1$ that contains H as an induced subgraph, then exactly one of the following statements is satisfied:*

1. *H' is a complete multipartite graph with at least two parts and such that each part has at least $k + 1$ vertices.*

2. *H' contains $\overline{K_2} \oplus (K_2 + kK_1)$ or $K_1 \oplus C_4$ as an induced subgraph.*

Proof. Let k, H and H' be as in the hypothesis. Note that the cotree of H is a rooted tree (T, r) of height two, such that r is labeled 1, r has at least two children, and each of them is the parent of at least $k + 1$ leaves. Then, by the properties of cotrees, the cotree of H' is a rooted tree (T', r) with exactly one more leaf than T, that contains T as an induced tree.

It can be verified that a tree T' as described above is necessarily the result of one of the following modifications on T: (a) adding a new leaf x as a child of r, (b) adding a new leaf x as a child of a child of r, (c) for a child c of r, deleting a child ℓ of c, adding a child c' to c, and adding to c' the leave ℓ and a new leaf x, (d) for a child c of r with t children, and $s \in \{2, \dots, t-1\}$, deleting the children $\ell_1, \dots, \ell_s$ of c, adding a child c' to c, adding a new leaf x as a child of c', adding a child c'' to c', and adding the leaves $\ell_1, \dots, \ell_s$ as children of c'', or (e) supposing r has t children, for an integer $s \in \{2, \dots, t - 1\}$, deleting the children $c_1, \ldots, c_s$ of r (each with its own children), adding a child c' to r, adding a new leaf x to c', adding a child c'' to c, and adding the vertices $c_1, \ldots, c_s$ (with their children) as children of c''.

It is straightforward to corroborate that such modifications on T correspond to the following modifications on H:

a. Add a universal vertex to H.

b. Add a false twin to a vertex of H.

c. Add a true twin to a vertex of H.

d. Add a vertex v to H in such a way that v is completely adjacent to every part of H, except for a part P, and v is adjacent to at least two vertices in P but it is not adjacent to every vertex of P.

e. Add a vertex v to H in such a way that v is completely nonadjacent to at least two parts of H, and it is completely adjacent to at least one part of H.

Then, if H' corresponds to the operation described in c, then H has $\overline{K_2} \oplus (K_2 + kK_1)$ as a proper induced subgraph, while if H corresponds to an operation described in items a, d, or e, then H' contains $K_1 \oplus C_4$ as an induced subgraph, and if H' is obtained from the operation described in item b, then H is a complete multipartite graph with at least two parts and such that each part contains at least $k + 1$ vertices. □

Lemma 3.18. *Let H be a complete multipartite graph with at least three parts and such that at least two of them have more than one vertex. If H' is a connected cograph obtained by adding a new vertex to H, then exactly one of the following conditions is satisfied:*

1. H' is a complete multipartite graph.

2. H' contains, as an induced subgraph, at least one of the following cographs: $K_1 \oplus (K_1 + C_4), \overline{K_2} \oplus (K_1 + P_3)$, or $K_1 \oplus (\overline{P_3 + K_2})$.

Lemma 3.19. *Let k be an integer, $k \geq 3$, and let H be a connected $(1, k)$-polar cograph that contains $K_1 \oplus (2K_2 + K_1)$ as an induced subgraph. If H' is a connected cograph obtained by adding a new vertex to H, then some of the following statements is satisfied:*

1. *H' is a $(1,k)$-polar cograph.*

2. *H' contains some of the following cographs as an induced subgraph: $K_1 \oplus (2K_2 + (k-1)K_1)$, $K_2 \oplus (2K_2 + K_1)$, $K_1 \oplus (P_3 + \overline{P_3})$, $K_1 \oplus (K_2 + \overline{K_1 + P_3})$, or $K_1 \oplus (K_1 + (K_1 \oplus 2K_2))$.*

Now, we are ready to give explicit lists of cograph minimal (∞,k)-polar obstructions of types $(k+1,k)$ and $(k,k-1)$. As we have mentioned above, these lists are directly based on the previous lemmas.

Corollary 3.20. *Let k be an integer, $k \geq 2$. Then, G is a cograph minimal (∞,k)-polar obstruction of type $(k+1,k)$ if and only if $G \cong kK_1 + H$, where H is isomorphic to some cograph of the set:*

$$\{\overline{2P_3},\ (P_3+K_2)\oplus K_1,\ 2K_2\oplus K_2,\ K_1\oplus(C_4+K_1),\\ K_1\oplus\overline{P_3+K_2},\ \overline{K_2}\oplus(P_3+K_1),\ \overline{K_2}\oplus(K_2+kK_1)\}.$$

Proof. By Lemma 3.14, it is routine to verify that, if H is isomorphic to some of the listed graphs, then $kK_1 + H$ is a cograph minimal (∞,k)-polar obstruction. For the converse, let us consider G, a cograph minimal (∞,k)-polar obstruction of type $(k+1,k)$. By Lemma 3.14 we have that $G \cong kK_1 + H$, where H is a connected non-complete cograph that contains a cograph minimal $(1,k)$-polar obstruction H' as an induced subgraph, and such that for each vertex $v \in V(H)$, $H - v$ is either a $(1,k)$-polar graph or a complete multipartite graph. In addition, by Theorem 2.4 we know that every disconnected cograph minimal $(1,k)$-polar obstruction is not a complete multipartite graph, which implies from Remark 3.11 that H' cannot be disconnected. Then, since $k \geq 2$, we have that H' is either isomorphic to some graph of the set $\{K_{k+1,k+1}, K_1 \oplus (2K_2 + (k-1)K_1), \overline{K_2} \oplus (K_2 + kK_1)\}$, or it is isomorphic to some $(1,\infty)$-polar obstruction, that is, to some graph of the set $\{K_1 \oplus C_4, K_2 \oplus 2K_2, \overline{2P_3}, K_1 \oplus (P_3 + K_2)\}$.

We observed at the beginning of this proof that if H' is isomorphic to $K_2 \oplus 2K_2, \overline{2P_3}, K_1 \oplus (P_3 + K_2)$ or $\overline{K_2} \oplus (K_2 + kK_1)$, then $kK_1 + H'$ is a cograph minimal (∞,k)-polar obstruction, so in this cases $H = H'$. Furthermore, since $k \geq 2$, Lemma 3.14 implies that $G' \cong (k-1)K_1 + K_1 \oplus (2K_2 + (k-1)K_1)$ is a cograph minimal (∞,k)-polar obstruction. In consequence, $H' \ncong K_1 \oplus (2K_2 + (k-1)K_1)$, or G would contain G' as a proper induced subgraph, a contradiction.

Thus, we have only two remaining cases, $H' \cong K_1 \oplus C_4$, or $H' \cong K_{k+1,k+1}$. Note that, in both cases, H' is a complete multipartite graph, and by Lemma 3.14 H is not a complete multipartite graph, so H' must be a proper induced subgraph of H. Furthermore, by Theorem 3.16, in both cases, $G' \cong (k+1)K_1 + H'$ is a cograph minimal (∞, k)-polar obstruction, which implies that for each vertex $v \in V(H - H')$, v is adjacent to some vertex of H'.

Suppose that $H' \cong K_1 \oplus C_4$. As we have mentioned before, it is straightforward to show that $kK_1 + K_1 \oplus (C_4 + K_1)$, $kK_1 + K_1 \oplus \overline{P_3 + K_2}$ and $kK_1 + \overline{K_2} \oplus (P_3 + K_1)$ are all cograph minimal (∞, k)-polar obstructions, so, if H contains as an induced subgraph a graph H^* that is isomorphic to either $K_1 \oplus (C_4 + K_1), K_1 \oplus \overline{P_3 + K_2}$, or $\overline{K_2} \oplus (P_3 + K_1)$, then $kK_1 + H^*$ is a cograph minimal (∞, k)-polar obstruction contained as an induced subgraph in G, and then $G \cong kK_1 + H^*$ and $H \cong H^*$. Moreover, from Lemma 3.18, H must contain as an induced subgraph a graph H^* as described before, or H would be a complete multipartite graph, a contradiction.

For the last case, suppose that $H' \cong K_{k+1,k+1}$. Then, since H is not a complete multipartite graph, we have from Lemma 3.17 that H either contains $K_1 \oplus C_4$ as an induced subgraph, or it contains $\overline{K_2} \oplus (K_2 + kK_1)$ as a proper induced subgraph. Since we have already treated both cases before, we conclude that the only cograph minimal (∞, k)-polar obstructions of type $(k+1, k)$ are the listed one in the statement of the corollary. □

Corollary 3.21. *Let k be an integer, $k \geq 3$. The graph G is a cograph minimal (∞, k)-polar obstruction of type $(k, k-1)$ if and only if $G \cong (k-1)K_1 + H$, where H is isomorphic to some cograph of the set*

$$\{K_1 \oplus (C_4 + 2K_1),\ K_1 \oplus 2P_3,\ K_1 \oplus (K_1 + \overline{P_3 + K_2}),\\ K_1 \oplus (K_2 + \overline{P_3 + K_1}),\ K_2 \oplus (K_1 + 2K_2),\ K_1 \oplus (K_1 + K_2 + P_3),\\ K_1 \oplus (K_1 + (K_1 \oplus 2K_2)),\ K_1 \oplus ((k-1)K_1 + 2K_2)\}$$

Proof. Based on Lemma 3.14, it is routine to verify that, if H is isomorphic to some of the listed graphs, then $(k-1)K_1 + H$ is a cograph minimal (∞, k)-polar obstruction.

Conversely, let G be a cograph minimal (∞, k)-polar obstruction of type $(k, k-1)$. By Lemma 3.14 we have that $G \cong (k-1)K_1 + H$, where H is a connected cograph that contains a cograph minimal $(\infty, 1)$-polar obstruction H' as an induced subgraph. Thus, from Theorem 2.2 and Remark 3.11, there exists a cograph H' isomorphic to some graph in the set

$$\{K_1 \oplus (K_1 + 2K_2),\ K_1 \oplus (2K_1 + C_4),\ K_1 \oplus 2P_3,\ K_1 \oplus (K_1 + \overline{K_2 + P_3})\}$$

contained as an induced subgraph of H. As we have observed at the start of this proof, if H' is isomorphic to either $K_1 \oplus (2K_1 + C_4), K_1 \oplus 2P_3$ or $K_1 \oplus (K_1 + \overline{K_2 + P_3})$, then $(k-1)K_1 + H'$ is a cograph minimal (∞, k)-polar obstruction, and then $H = H'$. Suppose then that $H' \cong K_1 \oplus (K_1 + 2K_2)$. It follows from Lemma 3.14 that H is not a $(1,k)$-polar graph, which implies from Lemma 3.19 that H contains a graph H^* in the set

$$\{K_1 \oplus (2K_2 + (k-1)K_1),\ K_2 \oplus (2K_2 + K_1),\ K_1 \oplus (P_3 + \overline{P_3}),$$
$$K_1 \oplus (K_2 + \overline{K_1 + P_3}),\ K_1 \oplus (K_1 + (K_1 \oplus 2K_2))\}$$

as an induced subgraph. Since we have proved that in every such case $(k-1)K_1 + H^*$ is a cograph minimal (∞, k)-polar obstruction, we have that $H = H^*$, which finishes the proof. □

3.2.3 The remaining types

In contrast with the obstructions with precisely one nontrivial component, we show in the following proposition that cograph minimal (∞, k)-polar obstructions that have at least one trivial component, and at least one complete nontrivial component, can be nicely obtained from the cograph minimal $(\infty, k-1)$-polar obstructions with at least one isolated vertex.

Theorem 3.22. *Let j, k and p be nonnegative integers such that $1 \leq p \leq k-j$. The graph G is a cograph minimal (∞, k)-polar obstruction of type $(k-j+2, p)$ if and only if $G \cong K_2 + G'$ where G' is a cograph minimal $(\infty, k-1)$-polar obstruction of type $(k-j+1, p)$ that is a $(1,k)$-polar graph.*

Proof. Suppose that G' is a cograph minimal $(\infty, k-1)$-polar obstruction that is a $(1,k)$-polar graph, and let $G = K_2 + G'$. Note that since G' is not an $(\infty, k-1)$-polar graph, G is not an (∞, k)-polar graph. Moreover, for $v \in V(G - G')$, since G' is a $(1,k)$-polar graph, $G - v$ is also a $(1,k)$-polar graph, while for $w \in G'$, since $G' - w$ is an $(\infty, k-1)$-polar graph, we have that $G - w$ is an (∞, k)-polar graph. Thus, G is a cograph minimal (∞, k)-polar obstruction.

Conversely, let G be a cograph minimal (∞, k)-polar obstruction of type $(k-j+2, p)$, so $G \cong pK_1 + (k-j-p+1)K_2 + H$, where H is a connected nontrivial graph. Thus, for $G' = pK_1 + (k-j-p)K_2 + H$, we have that $G \cong K_2 + G'$. Observe that, since G is not an (∞, k)-polar graph, G' is not an $(\infty, k-1)$-polar graph.

Let $v \in V(G - G')$, and let w be the only neighbor of v in G. Let (A, B) be an (∞, k)-polar partition of $G - v$. Note that w must belong to A, or G would be an (∞, k)-polar graph. Thus (A, B) is a $(1, k)$-polar partition of $G - v$, and then G' is a $(1, k)$-polar graph. Hence, since G' is not $(\infty, k-1)$-polar but it is $(1, k)$-polar, G' contains a cograph minimal $(\infty, k-1)$-polar obstruction G^* that is $(1, k)$-polar as an induced subgraph, but we have shown at the beginning of the proof that in such a case $K_2 + G^*$ is a cograph minimal (∞, k)-polar obstruction, so we have that $G' = G^*$, which finishes the proof. □

A somewhat surprising consequence of the previous results is that for $c \in \{k, k+1, k+2\}$ and $i \in \{1, \dots, c-2\}$, there exists exactly one cograph minimal (∞, k)-polar obstruction of type (c, i). In the following proposition we specify the known cases.

Corollary 3.23. *Let p and k be nonnegative integers.*

1. *If $1 \leq p \leq k+1$, then the graph $pK_1 + (k-p+1)K_2 + K_{p,p}$ is a cograph minimal (∞, k)-polar obstruction. Moreover, for $p \leq k$, up to isomorphism, this is the only cograph minimal (∞, k)-polar obstruction of type $(k+2, p)$.*

2. *If $1 \leq p \leq k$, the graph $pK_1 + (k-p)K_2 + (\overline{K_2} \oplus (K_2 + pK_1))$ is a cograph minimal (∞, k)-polar obstruction. Moreover, for $p \leq k-1$, up to isomorphism, this is the only cograph minimal (∞, k)-polar obstruction of type $(k+1, p)$.*

3. *If $1 \leq p \leq k-1$, the graph $pK_1 + (k-p-1)K_2 + (K_1 \oplus (2K_2 + pK_1)$ is a cograph minimal (∞, k)-polar obstruction. Moreover, for $p \leq k-2$, up to isomorphism, this is the only cograph minimal (∞, k)-polar obstruction of type (k, p).*

Proof. Let k and p be nonnegative integers such that $1 \leq p \leq k+1$. It is routine to show that $pK_1 + (k-p+1)K_2 + K_{p,p}$ is a cograph minimal (∞, k)-polar obstruction that admits a $(1, k+1)$-polar partition.

Suppose that $p \leq k$. We proceed by mathematical induction on k to show that the only cograph minimal (∞, k)-polar obstruction of type $(k+2, p)$ is isomorphic to $pK_1 + (k-p+1)K_2 + K_{p,p}$.

The base case, $k = 1$, follows from Theorem 2.2. For the inductive step, suppose that $k \geq 2$, and let G be a cograph minimal (∞, k)-polar obstruction of type $(k+2, p)$. Theorem 3.22 implies that $G \cong K_2 + G'$, where G' is a cograph minimal $(\infty, k-1)$-polar obstruction of type $(k+1, p)$ that admits a $(1, k)$-polar partition. If $p \leq k-1$, the induction hypothesis implies that $G' \cong pK_1 + (k-p)K_2 + K_{p,p}$, where the result is immediate. In a similar way, if $p = k$, Theorem 3.16 implies that $G' \cong pK_1 + (k-p)K_2 + K_{p,p}$, which ends the proof of the first item. The proof of items 2 and 3 are analogous to the proof of item 1, but using Corollaries 3.20 and 3.21 instead of Theorem 3.16. □

3.3 Cases $k = 2$ and $k = 3$

The exhaustive list of cograph minimal $(\infty, 1)$-polar obstructions was given in [19] (see Theorem 2.2). As the next two theorems show, the results exposed along this section allow us to give complete lists of cograph minimal (∞, k)-polar obstructions for the cases $k = 2$ and $k = 3$.

Theorem 3.24. *Let G be a cograph minimal $(\infty, 2)$-polar obstruction. Then*

1. *G is connected if and only if $\overline{G}$ is isomorphic to $P_3 + (K_1 \oplus C_4)$, $P_3 + \overline{2P_3}$, $P_3 + (K_2 \oplus 2K_2)$, or $P_3 + (K_1 \oplus (K_2 + P_3))$,*

2. *G is disconnected and has no isolated vertices if and only if G is isomorphic to $P_3 + C_4$, $P_3 + (K_1 \oplus 2K_2)$, or $2P_3 + K_2$,*

3. *G has exactly 4 connected components if and only if G is isomorphic to $3K_1 + (K_1 \oplus C_4)$, or $pK_1 + (3-p)K_2 + K_{p,p}$ for some integer p with $p \in \{1, 2, 3\}$,*

4. *G has exactly 3 connected components and at least one isolated vertex if and only if G is isomorphic to $K_1 + \overline{2P_3}$, $K_1 + (K_1 \oplus (P_3 + K_2))$, $K_1 + (K_2 \oplus 2K_2)$, $K_1 + (K_1 \oplus (C_4 + K_1))$, $K_1 + (K_1 \oplus \overline{P_3 + K_2})$, $K_1 + (\overline{K_2} \oplus (P_3 + K_1))$, or $pK_1 + (2-p)K_2 + (2K_1 \oplus (K_2 + pK_1))$ for some integer p with $p \in \{1, 2\}$,*

5. *G has exactly 2 connected components and one isolated vertex if and only if G is isomorphic to $K_1 + (K_1 \oplus (K_1 + 2K_2))$, $K_1 + (K_1 \oplus (2K_1 + C_4))$, $K_1 + (K_1 \oplus 2P_3)$, or $K_1 + (K_1 \oplus (K_1 + \overline{K_2 + P_3}))$.*

In consequence, a graph G is a cograph minimal $(\infty, 2)$-polar obstruction if and only if it is isomorphic to some of the 23 cographs listed before.

Proof. Item 1 follows from Theorems 2.3 and 3.3, item 2 follows from Theorems 3.8 and 3.13, items 3 and 4 follow from Theorem 3.16 and Corollaries 3.20 and 3.23, and item 5 follows from Theorem 3.15.

By Lemma 3.2 we have that every cograph minimal $(\infty, 2)$-polar obstruction has at most 4 connected components, so the listed graphs are all the cograph minimal $(\infty, 2)$-polar obstructions. □

Theorem 3.25. *Let G be a cograph minimal $(\infty, 3)$-polar obstruction. Then*

1. *G is connected if and only if $\overline{G} \cong P_3 + H$, where H is isomorphic to $K_1 \oplus C_4$, $\overline{2P_3}$, $K_2 \oplus 2K_2$, or $K_1 \oplus (K_2 + P_3)$,*

2. *G is disconnected and has neither isolated vertices nor components isomorphic to P_3 if and only if G is isomorphic to $2C_4$, $2(K_1 \oplus 2K_2)$, or $C_4 + (K_1 \oplus 2K_2)$,*

3. *G is disconnected and has at least one component isomorphic to P_3 if and only if $G \cong P_3 + H$, where H is isomorphic to $P_3 + 2K_2$, $K_2 + (K_1 \oplus 2K_2)$, $K_2 + C_4$, $K_{3,3}$, $2K_1 \oplus (K_2 + 2K_1)$, $K_1 \oplus (K_1 + 2K_2)$, $K_1 \oplus C_4$, $\overline{2P_3}$, $K_2 \oplus 2K_2$, or $K_1 \oplus (K_2 + P_3)$,*

4. *G has exactly 5 connected components if and only if G is isomorphic to $4K_1 + (K_1 \oplus C_4)$, or $pK_1 + (4 - p)K_2 + K_{p,p}$ for some integer p with $p \in \{1, 2, 3, 4\}$,*

5. *G has exactly 4 connected components and at least one isolated vertex if and only if either $G \cong K_1 + H$, where H is isomorphic to $\overline{2P_3}$, $K_1 \oplus (P_3 + K_2)$, $K_2 \oplus 2K_2$, $K_1 \oplus (C_4 + K_1)$, $K_1 \oplus \overline{P_3 + K_2}$, or $\overline{K_2} \oplus (P_3 + K_1)$, or G is isomorphic to $pK_1 + (3 - p)K_2 + (2K_1 \oplus (K_2 + pK_1))$ for some integer p with $p \in \{1, 2, 3\}$,*

6. *G has exactly 3 connected components and at least one isolated vertex if and only if either $G \cong 2K_1 + H$, where H is isomorphic to $K_1 \oplus (C_4 + 2K_1)$, $K_1 \oplus 2P_3$, $K_1 \oplus (K_1 + \overline{P_3 + K_2})$, $K_1 \oplus (K_2 + \overline{P_3 + K_1})$, $K_2 \oplus (K_1 + 2K_2)$,*

$K_1 \oplus (K_1 + K_2 + P_3)$, or $K_1 \oplus (K_1 + (K_1 \oplus 2K_2))$, or G is isomorphic to $pK_1 + (2-p)K_2 + (K_1 \oplus (2K_1 + pK_2))$ for some integer p with $p \in \{1, 2\}$,

7. *G has exactly 2 connected components and one isolated vertex if and only if $G \cong K_1 + (K_1 \oplus H)$, where H is isomorphic to $P_3 + C_4$, $P_3 + (K_1 \oplus 2K_2)$, $2P_3 + K_2$, $K_1 + 3K_2$, $2K_1 + K_2 + C_4$, $3K_1 + K_{3,3}$, $K_1 + K_2 + (2K_1 \oplus (K_2 + K_1))$, $2K_1 + (2K_1 \oplus (K_2 + 2K_1))$, or $K_1 + (K_1 \oplus (K_1 + 2K_2))$.*

In consequence, a graph G is a cograph minimal $(\infty, 3)$-polar obstruction if and only if it is isomorphic to some of the 49 cographs listed before.

Proof. Item 1 follows from Theorems 2.3 and 3.3, item 2 follows from Theorem 3.8, item 3 follows from Theorem 3.13, items 4, 5 and 6 follow from Theorem 3.16 and Corollaries 3.20, 3.21 and 3.23, and item 7 follows from Theorem 3.15 and Corollary 3.21.

From Lemma 3.2 we have that every cograph minimal $(\infty, 3)$-polar obstruction has at most 5 connected components, so the listed graphs are all the cograph minimal $(\infty, 3)$-polar obstructions. □

3.4 Open problems and conjectures

Although the results given in this chapter are not enough to give exhaustive lists of cograph minimal (∞, k)-polar obstructions for an arbitrary integer k, we think it might be possible to have a general formula to describe them, so we pose it as an open problem.

Problem 3.26. *For a positive integer k, find a recursive characterization for the cograph minimal (∞, k)-polar obstructions.*

As we observed in Section 3.2.3, for some specific values of c and i it can be proved that there exists exactly one cograph minimal (∞, k)-polar obstruction of type (c, i). We conjecture that our result can be extended in the following way.

Conjecture 3.27. *Let k, c and i be integers such that $1 \leq i \leq c - 2 \leq k$. Then, there exists exactly one cograph minimal (∞, k)-polar obstruction of type (c, i).*

Additionally, results on exact lists of cograph minimal (∞, k)-polar obstructions, for $k \in \{1, 2, 3\}$, supports the following assertions.

Conjecture 3.28. *For every cograph minimal (∞, k)-polar obstruction G, the order of G is at most $3(k+1)$.*

Conjecture 3.29. *If G is a cograph minimal (∞, k)-polar obstruction, then G is a cograph minimal (s, k)-polar obstruction for any integer s greater than k.*

Finally, known characterizations of minimal (s, k)-polar obstructions [9, 19, 46] and Theorems 2.2, 3.24 and 3.25 support the following conjecture.

Conjecture 3.30. *A cograph minimal (s, k)-polar obstruction that is not a minimal (∞, k)-polar obstruction must admit an $(s+1, k)$-polar partition.*

Chapter 4

P_4-sparse and P_4-extendible graphs

Throughout this section, we study polarity on two cograph superclasses, namely P_4-sparse and P_4-extendible graphs. In Section 4.2 we prove that any hereditary property has finitely many minimal obstructions in the mentioned graph classes. Then, in Sections 4.3 to 4.6 we give complete lists of minimal obstructions for the properties of being $(s,1)$-polar, 2-polar, unipolar, monopolar, or polar, when restricted to the P_4-sparse and P_4-extendible graphs. Finally, in Section 4.7 we provide linear time algorithms to decide whether P_4-sparse and P_4-extendible graphs are unipolar, monopolar or polar graphs. It is worth mentioning that we recently submitted for publication the original results in this chapter in two separate documents; extended verisons of the manuscripts submitted can be found in [20, 21].

4.1 Structural characterizations

A graph was defined to be P_4-sparse if any vertex subset with at most five vertices induces at most one P_4. Clearly, P_4-sparse graphs are precisely the $\{C_5, P_5, \overline{P_5}, P, \overline{P}, F, \overline{F}\}$-free graphs (see Figure 4.1). Additionally, Jamison and Olariu [49] provided a connectedness characterization of P_4-sparse graphs based on some special graphs called spiders, which we now introduce.

A graph G is a **spider** if its vertex set admits a partition (S,K,R) such that $G[S \cup K]$ is a headless spider with partition (S,K), and R is both, completely adjacent to K and completely nonadjacent to S. For a spider $G = (S,K,R)$

we will say that S is its *legs set*, K is its *body*, and R is its *head*. A spider is called **thin** (respectively **thick**) if $d(s) = 1$ (respectively $d(s) = |K| - 1$) for any $s \in S$. Notice that the complement of a thin spider is a thick spider, and vice versa, and that a headless spider is precisely a spider with an empty head.

Theorem 4.1 ([49]). *A graph G is a P_4-sparse graph if and only if for every nontrivial induced subgraph H of G, exactly one of the following statements is satisfied*

1. *H is disconnected.*

2. *$\overline{H}$ is disconnected.*

3. *H is an spider.*

Given a graph G and a vertex subset W, we denote by $S(W)$ the set of vertices $x \in V_G - W$ such that x belongs to a P_4 sharing vertices with W. If a vertex subset W inducing P_4 is such that $S(W)$ has at most one vertex, we say that $W \cup S(W)$ is an **extension set**. In the above terms, P_4-extendible graphs were defined as the graphs such that, for every set W inducing a P_4, $W \cup S(W)$ is an extension set. As Jamison and Olariu noticed in [50], any extension set must induce one of the eight graphs depicted in Figure 4.1, namely P_4, C_5, P_5, P, F or their complements. We call these graphs **extension graphs**.

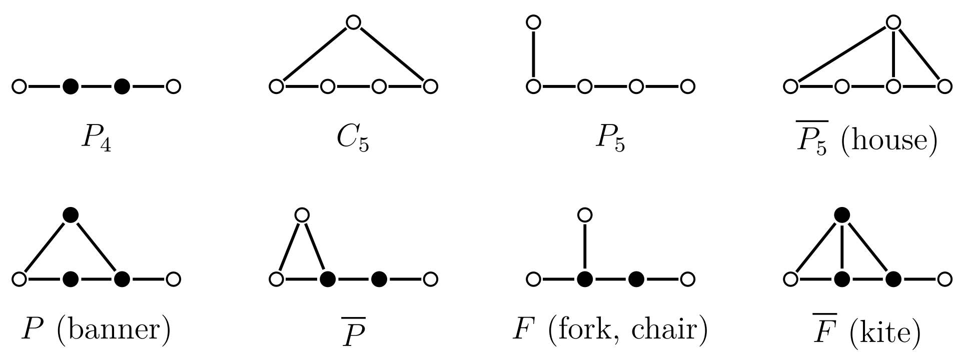


Figure 4.1: The eight extension graphs. Black vertices are the midpoints of separable extension graphs.

An extension set D is **separable** if no vertex of D is both an endpoint of some P_4 and a midpoint of some P_4 in $G[D]$. Observe that separable extension sets must induce one of P_4, P, F or their complements; these graphs are called **separable extension graphs**.

For a separable extension graph X with midpoints set K and endpoints set S, a graph H is said to be an X**-spider** if H is an induced supergraph of X such that $V_H \setminus V_X$, denoted R, is completely adjacent to K but completely nonadjacent to S. If H is an X-spider, we say that (S, K, R) is an X**-spider partition** of H, and we refer to S, K and R as the **legs set**, the **body**, and the **head** of H, respectively. From now on, every time we use the term X-spider, we are assuming that X is a separable extension graph.

Jamison and Olariu [50] gave the following connectedness characterization for the class of P_4-extendible graphs.

Theorem 4.2 ([50]). *If G is a graph, then G is a P_4-extendible graph if and only if, for every nontrivial induced subgraph H of G, precisely one of the following conditions is satisfied:*

1. *H is disconnected.*
2. *$\overline{H}$ is disconnected.*
3. *H is an extension graph.*
4. *There is a unique separable extension graph X such that H is an X-spider with nonempty head.*

Observe that every extension graph is trivially a P_4-extendible graph but the headless spiders on six vertices are examples of minimal P_4-extendible obstructions. Thus, since any headless spider is a P_4-sparse graph and all the forbidden P_4-sparse graphs are P_4-extendible, the classes of P_4-sparse graphs and P_4-extendible graphs are incomparable.

In the next section, we prove that, for any hereditary property $\mathcal{P}$ of graphs, there is only a finite number of minimal $\mathcal{P}$-obstructions that are P_4-sparse or P_4-extendible graphs.

4.2 Hereditary properties

We start this section with an easy observation that will often be used in the rest of the text without any explicit mention.

Remark 4.3. *Let $\mathcal{P}$ be a hereditary property of graphs, and let H be a $\mathcal{P}$-obstruction. If G is a minimal $\mathcal{P}$-obstruction such that $H \leq G$, then $G \cong H$.*

Now, we introduce a definition of Order Theory that allow us to characterize the classes of graphs for which any hereditary property has only a finite number of minimal obstructions. A partially ordered set $(M, \leq)$ is called a **well-quasi-ordering** (**WQO**) if any infinite sequence of elements $\{a_i\}_{i \in \mathbb{N}}$ from M contains an increasing pair, that is to say, a pair $a_i \leq a_j$ such that $i < j$. Equivalently, $(M, \leq)$ is a WQO if and only if M contains neither an infinite decreasing chain nor an infinite antichain.

Let $\mathcal{G}$ be a graph class ordered by the induced subgraph relation, and let $\mathcal{P}$ be a hereditary property on $\mathcal{G}$. By Remark 4.3, the family of minimal $\mathcal{P}$-obstructions is an antichain. Moreover, any antichain in $(\mathcal{G}, \leq)$ is the family of minimal $\mathcal{Q}$-obstructions for a hereditary property $\mathcal{Q}$. Then, since graphs ordered by the induced subgraph relation do not have infinite decreasing chains, $\mathcal{G}$ is WQO by the induced subgraph relation if and only if it contains no infinite antichain, or equivalently, if every hereditary property on $\mathcal{G}$ has only finitely many minimal obstructions. Peter Damaschke [28] used the following theorem to prove that cographs and P_4-reducible graphs are WQO under the induced subgraph relation.

Theorem 4.4 ([28])**.** *Let $\mathcal{G}$ be a family of graphs, and let Σ and Π be sets of unary and binary graph operations, respectively. Define partial orderings on Σ and Π as follows:*

$$\sigma \leq \sigma' \text{ if and only if } \sigma(G) \leq \sigma'(G) \text{ for all graphs } G, \text{ and}$$

$$\pi \leq \pi' \text{ if and only if } \pi(G, H) \leq \pi'(G, H) \text{ for all graphs } G, H.$$

Suppose that the following assertions are satisfied:

1. *$\mathcal{G}$ is WQO by the induced subgraph relation.*
2. *Any $\sigma \in \Sigma$ is monotonous (that is, $H \leq G$ implies $\sigma(H) \leq \sigma(G)$), and extensive (that is, for any graph G, $G \leq \sigma(G)$).*
3. *Any $\pi \in \Pi$ is commutative, associative, and satisfies:*
   (a) *if $G \leq G'$ and $H \leq H'$, then $\pi(G, H) \leq \pi(G', H')$, and*
   (b) *$G, H \leq \pi(G, H)$.*

4. $(\Sigma, \preceq)$ *and* $(\Pi, \preceq)$ *are WQO.*

Then, the class $\Gamma(\mathcal{G}, \Sigma, \Pi)$ *of all graphs obtained by start graphs from* $\mathcal{G}$ *using operations from* Σ *and* Π*, is WQO under the induced subgraph relation.*

Next, we provide new characterizations for both, P_4-sparse and P_4-extendible graphs, in order to show that Theorem 4.4 can be used to prove that such graph families (which are P_4-reducible superclasses), are WQO under the induced subgraph relation.

4.2.1 Hereditary properties on P_4-sparse graphs

Jamison and Olariu [49] gave a constructive characterization for P_4-sparse graphs starting with trivial graphs and using three binary operations. Nevertheless, the third operation used in such a characterization is not commutative, so it does not satisfy the hypotheses of Theorem 4.4, and we cannot use that characterization with Damaschke's theorem to conclude that P_4-sparse graphs are WQO. But, not everything is lost. Next, we establish a different constructive characterization for P_4-sparse graphs that is more appropriate for said purpose. Our characterization starts with trivial graphs and headless spiders, and it involves two binary operations as well as two infinite families of unary graph operations. We start with the following straightforward observation.

Remark 4.5. *Disjoint union and join operations preserve* P_4*-sparse graphs.*

Let H be a graph, and let j be an integer, $j \geq 2$. The graph $\sigma_j(H)$ is the thin spider $G = (S, K, R)$ such that $|S| = |K| = j$ and $G[R] = H$. Analogously, the graph $\tau_j(G)$ is the thick spider $G = (S, K, R)$ such that $|S| = |K| = j$ and $G[R] = H$. Notice that $\sigma_2(H) = \tau_2(H)$ for any graph H. The following observation follows directly from the definition of P_4-sparse graphs.

Remark 4.6. *Let* j *be an integer,* $j \geq 2$*. The graphs* $\sigma_j(H)$ *and* $\tau_j(H)$ *are* P_4*-sparse graphs if and only if* H *is a* P_4*-sparse graph. In addition, any headless spider is a* P_4*-sparse graph.*

Let Π be the set of binary operations whose only elements are the disjoint union and join graph operations, and let $\Sigma = \{\sigma_j\}_{j\geq 2} \cup \{\tau_j\}_{j\geq 3}$. Let us define the following partial order on Σ:

$$\sigma \preceq \sigma' \text{ if and only if } \sigma(H) \leq \sigma'(H) \text{ for all graphs } H.$$

It is straightforward to show that $\sigma_2 \preceq \sigma_3 \preceq \sigma_4 \preceq \cdots$ and $\sigma_2 \preceq \tau_3 \preceq \tau_4 \preceq \tau_5 \preceq \cdots$, so it trivially follows that Σ is WQO by $\preceq$. Analogously, it is easy to show that the family of graphs $\mathcal{G}$ whose only elements are the trivial graph and all headless spiders is WQO under the induced subgraph relation. Now we give our characterization of P_4-sparse graphs.

Theorem 4.7. *Let G be a graph. The following statements are equivalent.*

1. *G is a P_4-sparse graph.*

2. *G is obtained from trivial graphs by a finite sequence of Σ- and Π-operations.*

Proof. We have from Theorem 4.1 and Remarks 4.5 and 4.6 that 2 implies 1. The converse implication can be easily proved proceeding by induction on the order of G and using Theorem 4.1. □

The theorem above shows that $\Gamma(\mathcal{G}, \Sigma, \Pi)$ is precisely the class of P_4-sparse graphs. As we pointed before $(\mathcal{G}, \leq)$ and $(\Sigma, \leq)$ are WQO and, since Π is a finite set, $(\Pi, \leq)$ is too, so the following corollary is a simple application of Theorem 4.4.

Corollary 4.8. *The class of P_4-sparse graphs is WQO under the induced subgraph relation. Equivalently, any hereditary property on P_4-sparse graphs admits a finite forbidden induced subgraph characterization.*

4.2.2 Hereditary properties on P_4-extendible graphs

A constructive characterization for P_4-extendible graphs starting with trivial graphs and using four binary operations was given in [50]. As well as the constructive characterization for P_4-sparse graphs given in [49], this characterization for P_4-extendible graphs does not fit the hypotheses of Theorem 4.4, so we are unable to conclude that P_4-extendible graphs are WQO in this way. With this purpose in mind, we establish a new constructive characterization for P_4-extendible graphs, which starts from a set of nine basic graphs and involves two binary operations as well as five unary operations.

Let $\mathcal{G}$ be the set of graphs whose elements are the trivial graph K_1 and the eight extension graphs, that is, $\mathcal{G} = \{K_1, P_4, C_5, P_5, \overline{P_5}, P, \overline{P}, F, \overline{F}\}$. For each separable extension graph X (see Figure 4.1) and any graph G, we define the graph $\sigma_X(G)$ as the graph with vertex set $V_X \cup V_G$ and edge set

$$E_X \cup E_G \cup \{xy : x \text{ is a midpoint of } X \text{ and } y \in V_G\}.$$

For each separable extension graph X, the unary operation σ_X is its associated **separable!separable extension operation**. Let Σ be the set of the five separable extension operations σ_X, and let Π be the set of binary operations whose only elements are the disjoint union and join operations.

Remark 4.9 ([50])**.** *Let G be a graph whose vertex set admits a partition into two nonempty disjoint sets V' and V'' such that no P_4 in G contains vertices from both V' and V''. Then G is P_4-extendible if and only if the subgraphs of G induced by V' and V'' are.*

Observe that, from the remark above, P_4-extendible graphs are clearly closed under join and disjoint union operations. Now we use such remark for proving that separable extension operations also preserve P_4-extendible graphs.

Lemma 4.10. *The class of P_4-extendible graphs is closed under separable extension operations, that is to say, for any P_4-extendible graph G and any separable extension graph X, $\sigma_X(G)$ is a P_4-extendible graph.*

Proof. By definition of σ_X, the vertex set of $\sigma_X(G)$ is partitioned into V_X and V_G, and by hypothesis the graphs induced by these sets are P_4-extendible. Now, by Remark 4.9 we only need to prove that no P_4 has vertices in both V_X and V_G. Assume the contrary to obtain a contradiction.

Let M be the set of midpoints of X, and let W be a vertex set inducing a P_4 such that $W \cap V_X \neq \emptyset \neq W \cap V_G$. It is an easy observation that, since W induces a P_4, $|W \cap V_G| = 1$, $|W \cap V_X| = 3$, and $|W \cap M| \leq 2$. So we have only two possible cases, either $|W \cap M| = 1$ or $|W \cap M| = 2$. Let u be the only vertex in $W \cap V_G$.

First, assume that W has only one endpoint x of X and that y and z are both midpoints of X. Let E be the edge set of $\sigma_X(G)$. By definition of σ_X we have that $uy, uz \in E$ and $ux \notin E$. Moreover, since W induces a P_4, we have that $yz \notin E$ and x is adjacent to exactly one of y and z. But this is impossible, because the only separable extension graph with two nonadjacent midpoints is P, but no endpoint of P distinguishes between its nonadjacent midpoints.

Otherwise, W has two endpoints, y and z, and one endpoint, x, of X. By definition of σ_X we have that $ux \in E$ and $uy, uz \notin E$. Moreover, since W

induces a P_4, we have that $yz \in E$ and x is adjacent to exactly one of y and z. Here we have a contradiction, because the only separable extension graph with two adjacent endpoints is $\overline{P}$, but no midpoint of P distinguishes between its adjacent endpoints. □

Theorem 4.11. *Let G be a graph. The following statements are equivalent.*

1. *G is a P_4-extendible graph.*
2. *G is obtained from $\mathcal{G}$ by a finite sequence of Σ- and Π-operations.*

Proof. The fact that 2 implies 1 follows easily from Lemma 4.10, the observation after Remark 4.9, and since $\mathcal{G}$ is a subset of P_4-extendible graphs. For the converse implication we proceed by induction on the order of G. From Theorem 4.2 we have that, if G is not trivial, one of the following cases is satisfied:

1. G is disconnected.
2. $\overline{G}$ is disconnected.
3. G is an extension graph.
4. there is a unique separable extension graph X such that $G = \sigma_X(H)$ for some graph H.

In the first (second) case, G is the disjoint union (join) of two P_4-extendible graphs G_1 and G_2, which by induction hypothesis can be constructed from $\mathcal{G}$ by a finite sequence of Σ- and Π-operations, so the result follows in this case. The remaining cases are immediate. □

The theorem above shows that $\Gamma(\mathcal{G}, \Sigma, \Pi)$ is precisely the class of P_4-extendible graphs. In addition, considering that $\mathcal{G}, \Sigma$ and Π are finite sets, it is easy to justify the following consequence of Theorem 4.4.

Corollary 4.12. *The class of P_4-extendible graphs is WQO under the induced subgraph relation. Equivalently, any hereditary property on P_4-extendible graphs admits a finite forbidden subgraph characterization.*

The following sections are devoted to the characterizations by forbidden induced subgraphs of properties associated with polarity on P_4-sparse and P_4-extendible graphs. We start with the characterization of minimal unipolar obstructions on the mentioned graph classes.

4.3 Unipolarity

In this section we provide complete lists of minimal unipolar obstructions that are P_4-sparse or P_4-extendible graphs. With that purpose in mind we introduce some minimal unipolar obstructions that do not necessarily belong to the mentioned graph classes.

Remember that a hole is a cycle of length at least 5. An **antihole** is the complement of a hole, and it is said to be even or odd accordingly to its order.

Proposition 4.13. *The graphs depicted in Figure 4.2 are minimal unipolar obstructions.*

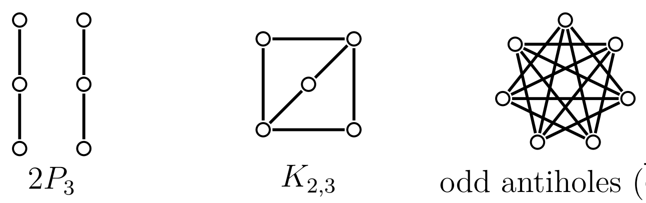


Figure 4.2: Some minimal unipolar obstructions.

Proof. To prove that these graphs are not unipolar, it is enough to observe that for any clique K, $G - K$ has an induced P_3, so $G - K$ is not a cluster. It is also easy to verify that any vertex-deleted subgraph of these graphs is a unipolar graph, so the result follows. □

The following two lemmas completely characterize minimal unipolar obstructions G (on general graphs) such that either G or $\overline{G}$ is disconnected. We use such characterizations as the base to provide complete lists of minimal unipolar obstructions for cographs, P_4-sparse graphs and P_4-extendible graphs.

Lemma 4.14. *If G is a graph, then G is a disconnected minimal unipolar obstruction if and only if $G \cong 2P_3$.*

Proof. Let G be a disconnected minimal unipolar obstruction. By the minimality of G, any of its components is a unipolar graph. In consequence, G has at least two components that are not complete graphs, otherwise G would be unipolar. Then, G has $2P_3$ as an induced subgraph, so $G \cong 2P_3$. The converse implication follows from Proposition 4.13. □

Lemma 4.15. *Let G be a graph. If $\overline{G}$ is disconnected, then G is a minimal unipolar obstruction if and only if $G \cong K_{2,3}$.*

Proof. First, suppose that G is a minimal unipolar obstruction. Notice that $\overline{G}$ is not a bipartite graph, or G would admit a partition into two cliques, so it would be a unipolar graph, which is impossible. Hence $\overline{G}$ contains an odd cycle as an induced subgraph. Moreover, since odd antiholes are minimal unipolar obstructions and $\overline{G}$ is disconnected, $\overline{G}$ does not contain odd cycles of length greater than three as induced subgraphs. Thus, $\overline{G}$ contains a triangle. In addition, since minimal unipolar obstructions do not have universal vertices, $\overline{G}$ does not have isolated vertices, and any component of $\overline{G}$ has order at least two. Therefore, since $\overline{G}$ has at least two connected components, it contains $K_2 + K_3$ as an induced subgraph, so $G \cong K_{2,3}$. The converse implication follows from Proposition 4.13. □

Since the complement of any nontrivial connected cograph is a disconnected cograph we have the following direct consequence of Lemmas 4.14 and 4.15.

Corollary 4.16. *If G is a cograph, then G is a minimal unipolar obstruction if and only if $G \cong 2P_3$ or $G \cong K_{2,3}$.*

Now, we use the characterization of P_4-sparse graphs given in Theorem 4.1 to give the explicit list of P_4-sparse minimal unipolar obstructions.

Lemma 4.17. *If $G = (S, K, R)$ is a spider, then G is a unipolar graph if and only if $R = \varnothing$ or $G[R]$ is unipolar.*

Proof. Since unipolarity is a hereditary property, we have that $G[R]$ is unipolar whenever G is. Conversely, for any unipolar partition (A, B) of $G[R]$, $(K \cup A, S \cup B)$ is a unipolar partition of G. □

Corollary 4.18. *If G is a P_4-sparse graph, then G is a minimal unipolar obstruction if and only if $G \cong 2P_3$ or $G \cong K_{2,3}$. In consequence, any P_4-sparse minimal unipolar obstruction is a cograph.*

Proof. The first statement follows from Lemmas 4.14, 4.15 and 4.17, since we have by Theorem 4.1 that any connected P_4-sparse graph with connected complement is a spider. The second statement follows directly from Corollary 4.16. □

We end this section by proving a result analogous to Corollary 4.18 for P_4-extendible graphs. Notice that, for any extension graph but P, its midpoints set is a clique while its endpoints induce a cluster (see Figure 4.1). Then, the proof of the following proposition is exactly the same as the proof of Lemma 4.17. The case of P-spiders is covered in Lemma 4.20.

Lemma 4.19. *Let $H \in \{P_4, \overline{P}, F, \overline{F}\}$. If $G = (S, K, R)$ is an H-spider, then G is a unipolar graph if and only if $R = \varnothing$ or $G[R]$ is unipolar.*

Lemma 4.20. *If $G = (S, K, R)$ is a P-spider, then G is a unipolar graph if and only if either R is an empty set or a clique. In consequence, if G is a P-spider, then it is not a minimal unipolar obstruction.*

Proof. Let w be the only vertex of $G[S \cup K]$ of degree 2 that is not adjacent to a vertex of degree three, and let u and v be its neighbors. If R has two nonadjacent vertices x and y, then $G[\{u, v, w, x, y\}]$ is isomorphic to $K_{2,3}$. Therefore, if G is a unipolar graph, then $R = \varnothing$ or R is a clique. Conversely, if R is a clique and z is the only vertex of $G[S \cup K]$ of degree three, then $(R \cup \{z, u\}, (S \cup K) \setminus \{z, u\})$ is a unipolar partition of G. Hence, if G is not a unipolar graph, R contains two nonadjacent vertices and G properly contains $K_{2,3}$, so G is not a minimal unipolar obstruction. □

Corollary 4.21. *Let G be a P_4-extendible graph. Then, G is a minimal unipolar obstruction if and only if $G \in \{2P_3, K_{2,3}, C_5\}$.*

Proof. We have from Theorem 4.2 that any connected P_4-extendible graph with connected complement is either an extension graph, or an X-spider for some separable extension graph X. It is easy to verify that the only extension graph that is a minimal unipolar obstruction is C_5, so the result follows from Lemmas 4.14, 4.15, 4.19 and 4.20. □

In the next sections, we give complete lists of minimal (s, k)-polar obstructions for some specific values of s and k when restricted to either P_4-sparse or P_4-extendible graphs, generalizing several results previously known for cographs.

4.4 $(s, 1)$-polarity

The following five lemmas completely characterize disconnected minimal $(s, 1)$-polar obstructions **for general graphs**. They are simple generalizations of Lemmas 2 to 5 from [19], so we will only sketch the proofs.

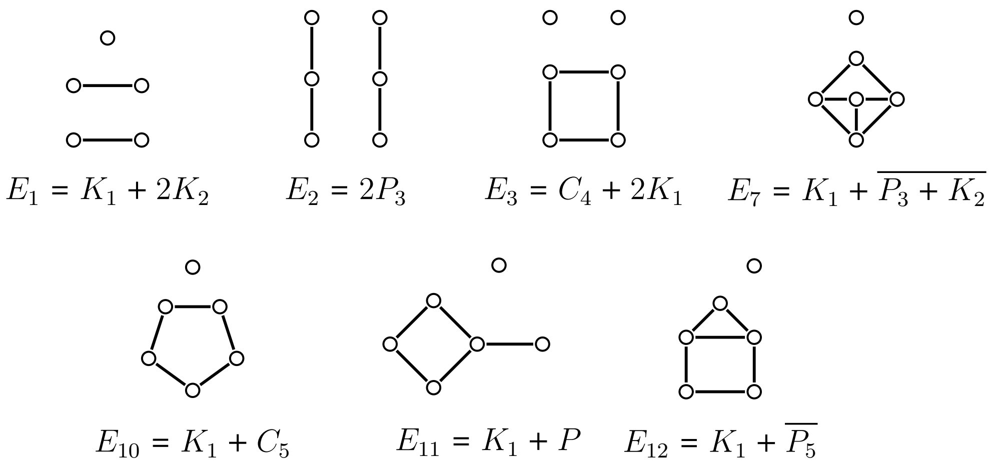


Figure 4.3: Some minimal $(\infty, 1)$-polar obstructions.

Lemma 4.22. *The seven graphs depicted in Figure 4.3 are minimal $(s, 1)$-polar obstructions for every integer s, $s \geq 2$. Hence, these graphs are minimal $(\infty, 1)$-polar obstructions.*

Proof. It is routine to verify that, for each graph G in Figure 4.3, the following assertions are satisfied: For any maximal clique K, $G - K$ contains an induced $\overline{P_3}$, and for any vertex v, $G - v$ is a $(2, 1)$-polar graph. The result follows easily from here. □

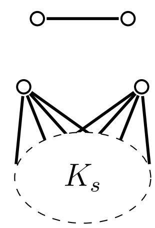
In [19], a proof of the following two lemmas restricted to the family of cographs was given. A minor change in such a proof brings us the more general results that we state here.

Lemma 4.23. *Let s be an integer, $s \geq 2$. Every minimal $(s, 1)$-polar obstruction different from $K_1 + 2K_2$ and $2K_1 + C_4$ has at most two connected components.*

Proof. Let G be a minimal $(s, 1)$-polar obstruction different from the graphs depicted in Figure 4.3. Aiming for a contradiction, assume that G has at least three connected components. Since $s \geq 2$, we have that G is not a split graph, so it contains $2K_2, C_4$ or C_5 as an induced subgraph. Having at least three connected components, G contains some of $K_1 + 2K_2, 2K_1 + C_4$ or $K_1 + C_5$ as an induced subgraph. This results in a contradiction, because these graphs are minimal $(s, 1)$-polar obstructions. Thus, G has at most two components. □

Lemma 4.24 ([19])**.** *Let s be an integer, $s \geq 2$. If a minimal $(s, 1)$-polar obstruction G distinct to the graphs depicted in Figure 4.3 has two connected components and it is not $2K_{s+1}$, then $G \cong K_r + H$, where $r \in \{1, 2\}$ and H is a connected graph that is not a complete s-partite graph.*

The proof of the following lemma has the same spirit than the proof of Lemma 4 in [19], but it has been rewritten for the sake of clarity.



$K_2 + (2K_1 \oplus K_s)$

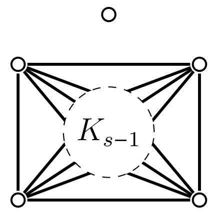
Figure 4.4: The only minimal $(s, 1)$-polar obstruction different from $2K_{s+1}$ with exactly two connected components one of them being isomorphic to K_2.

Lemma 4.25. *Let s be an integer, $s \geq 2$. If H is a connected graph such that $G = K_2 + H$ is a minimal $(s, 1)$-polar obstruction other than $2K_{s+1}$, then H is isomorphic to $2K_1 \oplus K_s$.*

Proof. It is routine to verify that $K_2 + (2K_1 \oplus K_s)$ is a minimal $(s, 1)$-polar obstruction. From Lemma 4.24 we know that H is not a complete s-partite graph, so H contains a copy of either $\overline{P_3}$ or K_{s+1} as an induced subgraph. Nevertheless, H is a $\overline{P_3}$-free graph, for otherwise G would contain $K_1 + 2K_2$ as a proper induced subgraph. Thus, H contains a copy of K_{s+1} as a proper induced subgraph. Let K be a maximum clique in H, and let $v \in V_H - K$. As we argued above, H is a $\overline{P_3}$-free graph, so v is adjacent to all but one vertex w in K. Hence, for any s-subset V' of $K \cap N(v)$, the graph $H[V' \cup \{v, w\}]$ is isomorphic to $2K_1 \oplus K_s$, so $G \cong K_2 + (2K_1 \oplus K_s)$. □

The following lemma is a slight generalization of Lemma 5 in [19]. Since the main ideas of the proof are very similar, we will only explain the significant differences.

Lemma 4.26. *Let s be an integer, $s \geq 2$. If H is a connected graph such that $G = K_1 + H$ is a minimal $(s, 1)$-polar obstruction isomorphic to none of the graphs depicted in Figure 4.3, then G is isomorphic to $K_1 + (C_4 \oplus K_{s-1})$.*



$K_1 + (C_4 \oplus K_{s-1})$

Figure 4.5: The only minimal $(s, 1)$-polar obstruction different from those on Figure 4.3 with exactly two connected components one of them being isomorphic to K_1.

Proof. The graph H cannot be a split graph, so it contains an induced copy of either $2K_2, C_4$ or C_5. Nevertheless, by the minimality of G and since $G \not\cong K_1 + C_5$ we know that H is $\{2K_2, C_5\}$-free, so it contains an induced cycle on four vertices, $C = (c_1, c_2, c_3, c_4)$. Let v be a vertex in $H - V_C$, which must exist since H is not a complete bipartite graph. Observe that, since G contains no graph depicted in Figure 4.3 as an induced subgraph, v only could be adjacent to either two nonadjacent vertices of C or to every vertex of C.

Let V_1, V_2 and V_3 be the subsets of vertices of H that are not in C and that are adjacent to c_1 and c_3, to c_2 and c_4, and to c_i for every $i \in \{1, 2, 3, 4\}$, respectively. Notice that since H contains no induced $\overline{K_2 + P_3}$, V_1 an V_2 are both independent sets, and V_3 is completely adjacent to $V_1 \cup V_2$. In addition, since H is P-free, we have that V_1 and V_2 are completely adjacent. From here is straightforward to notice that $H - V_3$ is a complete bipartite graph.

Hence, H is the join of the complete bipartite graph $H - V_3$ with $H[V_3]$, which implies that $H[V_3]$ is not a complete $(s-2)$-partite graph. One more time, since $\overline{K_2 + P_3}$ is not an induced subgraph of H we have that H is a $\overline{P_3}$-free graph, so $H[V_3]$ is too. Therefore, $H[V_3]$ contains a copy of K_{s-1} as an induced subgraph, so the result follows. □

So far, we have characterized all disconnected minimal $(s, 1)$-polar obstructions, which are a constant number for any choice of s. We summarize this result as follows.

Theorem 4.27. *Let s be an integer, $s \geq 2$, and let G be a disconnected minimal $(s, 1)$-polar obstruction. Then G satisfies one of the following assertions:*

1. *G is isomorphic to one of the graphs depicted in Figure 4.3.*

2. $G \cong 2K_{s+1}$.

3. $G \cong K_2 + (2K_1 \oplus K_s)$.

4. $G \cong K_1 + (C_4 \oplus K_{s-1})$.

For any nontrivial cograph G, either G or its complement is disconnected [23], so the complement of any nontrivial connected cograph is disconnected. This fact was used in [19] to give a recursive characterization of all cograph minimal $(s,1)$-polar obstructions. After giving a complete characterization of the disconnected cograph minimal $(s,1)$-polar obstructions, the authors provided a recursive construction for the disconnected cograph minimal $(1,s)$-polar obstructions (which are precisely the complements of connected cograph minimal $(s,1)$-polar obstructions).

Theorem 4.27 characterizes disconnected minimal $(s,1)$-polar obstructions for general graphs. Thus, to completely characterize minimal $(s,1)$-polar obstructions for a given class of graphs it suffices to characterize connected minimal $(s,1)$-polar obstructions. To this end, in order to follow the strategy described in the previous paragraph for P_4-sparse and P_4-extendible graphs, we notice that the following lemma, which was stated in [19] for the special case of cographs, is also valid for general graphs.

Lemma 4.28 ([19]). *Let t be an integer, $t \geq 2$, and for each $i \in \{1,\dots,t\}$, let G_i be a minimal $(1,k_i)$-polar obstruction that is a $(1,k_i+1)$-polar graph. Then, for $k = t-1+\sum_{i=1}^{t} k_i$, the graph $G = G_1 + \cdots + G_t$ is a minimal $(1,k)$-polar obstruction that is a $(1,k+1)$-polar graph.*

In the following sections we show that the converse of Lemma 4.28 holds for P_4-sparse and P_4-extendible graphs, that is to say, that any disconnected minimal $(1,k)$-polar obstruction on such classes is the disjoint union of minimal $(1,k_i)$-polar obstructions for some integers $k_i < k$.

4.4.1 P_4-sparse minimal $(s,1)$-polar obstructions

The induced path on three vertices is a minimal $(0,k)$-polar obstruction for any integer $k \geq 2$ so, if a graph G contains P_3 as a proper induced subgraph, then G is not a minimal $(0,k)$-polar obstruction. Similarly, if G contains $\overline{P_3}$ as a proper induced subgraph, then G is not a minimal $(s,0)$-polar obstruction. From here, the following observation follows easily.

Remark 4.29. *Let G be a spider. If G is a headless spider or the head of G induces a split graph, then G is a split graph that has both, P_3 and its complement, as proper induced subgraphs. Hence, G is not a minimal (s,k)-polar obstruction for any choice of s and k.*

The following two propositions provide the basis for showing that any connected P_4-sparse minimal $(s,1)$-polar obstruction has a disconnected complement.

Proposition 4.30. *Let k be a positive integer, and let $G = (S,K,R)$ be a spider with nonempty head. Then, G is not a minimal $(1,k)$-polar obstruction.*

Proof. Aiming for a contradiction, suppose that G is a minimal $(1,k)$-polar obstruction, and let $\sigma \in S$ be a leg of G. Let (A,B) be a $(1,k)$-polar partition of $G-\sigma$. Notice that $|K \cap A| \leq 1$ because K is a clique and A is an independent set. Therefore, since K has at least two vertices, $K \cap B \neq \varnothing$. Moreover, since B induces a cluster, R is completely adjacent to K, and $K \cap B \neq \varnothing$, we have that $R \cap B$ is a clique. Also notice that either $K \cap A = \varnothing$ or $R \cap A = \varnothing$.

Now, if $K \cap A \neq \varnothing$, then R is a clique, G is a split graph, and therefore G is a $(1,k)$-polar graph, which is impossible. Otherwise, if $K \subseteq B$, then $(A \cup \{\sigma\}, B)$ is a $(1,k)$-polar partition of G, a contradiction. □

Since the complement of a spider is also a spider, and any minimal $(\infty,1)$-polar obstruction is a minimal $(s,1)$-polar obstruction for some positive integer s, we have the following simple consequences of the previous proposition.

Corollary 4.31. *Let s be a positive integer. If G is a spider, then G is neither a minimal $(s,1)$-polar obstruction nor a minimal $(\infty,1)$-polar obstruction.*

Corollary 4.32. *Let s be a positive integer. If G is a P_4-sparse minimal $(s,1)$-polar obstruction, then G or its complement is disconnected.*

Proof. Since G is a P_4-sparse graph, if G and $\overline{G}$ are connected, we have from Theorem 4.1 that G is a spider, but that is impossible by Corollary 4.31. Therefore, either G or its complement is disconnected. □

The next two results, together with Lemma 4.28, provide us with a complete structural characterization for disconnected P_4-sparse minimal $(1,k)$-polar obstructions.

Lemma 4.33. *Let t be an integer, $t \geq 2$, and for each $i \in \{1, \ldots, t\}$, let G_i be a connected P_4-sparse minimal $(1, k_i)$-polar obstruction that is a $(1, k_i + 1)$-polar graph. If $G = G_1 + \cdots + G_t$, then G is a minimal $(1, k)$-polar obstruction if and only if $k = t - 1 + \sum_{i=1}^{t} k_i$.*

Proof. Let $k = t - 1 + \sum_{i=1}^{t} k_i$. We have from Lemma 4.28 that G is a minimal $(1, k)$-polar obstruction that is $(1, k + 1)$-polar, so we just need to show that G is not a minimal $(1, \kappa)$-polar obstruction for any $\kappa < k$.

Let G be a connected P_4-sparse minimal $(1, k_i)$-polar obstruction that is $(1, k_i + 1)$-polar. By Corollary 4.32, $\overline{G}$ is a disconnected minimal $(k_i, 1)$-polar obstruction that is a $(k_i + 1, 1)$-polar graph. Then, it follows from Theorem 4.27 that, for any nonnegative integer κ_i such that $\kappa_i < k_i$, G contains a proper induced subgraph G' that is both, a P_4-sparse minimal $(\kappa_i, 1)$-polar obstruction and a $(\kappa_i + 1, 1)$-polar graph. From here on, the proof follows as the proof of Lemma 8 in [19]. □

Lemma 4.34. *Let k be a nonnegative integer. If G is a disconnected P_4-sparse minimal $(1, k)$-polar obstruction with components $G_1, \ldots, G_t$, then there exist nonnegative integers $k_1, \ldots, k_t$ such that for each $i \in \{1, \ldots, t\}$, G_i is a connected minimal $(1, k_i)$-polar obstruction that is a $(1, k_i + 1)$-polar graph, and $\sum_{i=1}^{t} k_i = k - t + 1$. (Notice that $k_i < k$ for any $i \in \{1, \ldots, t\}$, and G is a $(1, k + 1)$-polar graph.)*

Proof. This is a generalization of Lemma 9 in [19], which states the same result for cographs. As in the original proof, it is easy to argue that each component G_i is a minimal $(1, k_i)$-polar obstruction that is $(1, k)$-polar, where k_i is the maximum integer such that any proper induced subgraph of G_i is $(1, k_i)$-polar.

Then, by Corollary 4.32, $\overline{G_i}$ is a disconnected minimal $(k_i, 1)$-polar obstruction that is $(k, 1)$-polar, and we have from Theorem 4.27 that $\overline{G_i}$ is a $(k_i + 1, 1)$-polar graph, so G_i is $(1, k_i + 1)$-polar. Finally, the result follows from Lemma 4.33. □

The following result provides a complete recursive construction of P_4-sparse minimal $(s, 1)$-polar obstructions.

Theorem 4.35. *Let s be an integer, $s \geq 2$. If G is a P_4-sparse graph, then G is a minimal $(s, 1)$-polar obstruction if and only if G satisfies exactly one of the following assertions:*

1. *G is isomorphic to one of the four cographs depicted in Figure 4.3.*

2. *G is isomorphic to some of $2K_{s+1}, K_2 + (K_s \oplus 2K_1)$ or $K_1 + (K_{s-1} \oplus C_4)$.*

3. *The complement of G is disconnected with components $G_1, \dots, G_t$, each G_i is a minimal $(1, s_i)$-polar obstruction whose complement is different from the graphs in Figure 4.3, and $s = t - 1 + \sum_{i=1}^{t} s_i$.*

Proof. If G is disconnected, it follows from Theorem 4.27 that G is a minimal $(s, 1)$-polar obstruction if and only if G is either a P_4-sparse graph depicted in Figure 4.3 (which can easily be checked to be a cograph), or it is isomorphic to some of $2K_{s+1}, K_2 + (K_s \oplus 2K_1)$ or $K_1 + (K_{s-1} \oplus C_4)$. Otherwise, if G is connected, Corollary 4.32 implies that $\overline{G}$ is a disconnected P_4-sparse minimal $(1, s)$-polar obstruction, and the result follows from Lemma 4.34. □

A result analogous to Theorem 4.35 will be given for P_4-extendible graphs in the next section. As the reader will notice, the technique used to obtain the results for both classes is the same, and the differences come only from the connectedness characterization for each graph class.

Before developing the aforementioned result, we prove that any P_4-sparse minimal $(s, 1)$-polar obstruction is a cograph. In Corollary 4.18 was proved a similar property for unipolarity, and analogous characteristics will be proven later for $(\infty, 1)$-, (∞, ∞)-, and $(2, 2)$-polarity. In contrast, it will be evident that none of these properties is satisfied when restricted to P_4-extendible graphs instead of P_4-sparse graphs.

For any hereditary property $\mathcal{P}$ and any graph classes $\mathcal{G}$ and $\mathcal{H}$ such that $\mathcal{G} \subseteq \mathcal{H}$, the set of minimal $\mathcal{P}$-obstructions in $\mathcal{G}$ clearly is a (possibly proper) subset of the set of minimal $\mathcal{P}$-obstructions in $\mathcal{H}$. The class of P_4-sparse graphs has been observed to have a behavior similar to cographs when computing their minimal obstructions with respect to some hereditary properties. For example, Hannnebauer [45] proved that every P_4-sparse minimal obstruction for (k, ℓ)-coloring is a cograph. Now, we prove that the same phenomenon occurs for $(s, 1)$-polarity.

Theorem 4.36. *Let s be a nonnegative integer. Any P_4-sparse minimal $(s, 1)$-polar obstruction is a cograph.*

Proof. Let G be a P_4-sparse minimal $(s, 1)$-polar obstruction. We proceed by induction on s. The statement is clearly true for $s \leq 1$. Let $s \geq 2$. It follows from Corollary 4.32 that G is not a spider, hence G or its complement is disconnected.

If G is disconnected, it follows from Theorem 4.27 that G is a cograph. Otherwise, if $\overline{G}$ is disconnected, Lemma 4.34 implies that any component H of $\overline{G}$ is a P_4-sparse minimal $(1,k_i)$-polar obstruction for a nonnegative integer k_i with $k_i < k$. Thus, $\overline{H}$ is a P_4-sparse minimal $(k_i,1)$-polar obstruction, and by induction hypothesis $\overline{H}$ (hence H) is a cograph. Since the disjoint union of cographs is also a cograph, $\overline{G}$ (hence G) is a cograph. □

4.4.2 P_4-extendible minimal $(s,1)$-polar obstructions

We begin with some easily verifiable facts, stated without proof and bundled to facilitate future references.

Remark 4.37. *Let s,k be either in $\mathbb{N}$ or equal to ∞.*

1. *P_4 and F are split graphs but they are neither $(0,\infty)$- nor $(\infty,0)$-polar graphs.*

2. *C_5, P_5, and P are $(1,2)$- and $(2,1)$-polar, but they are neither $(1,1)$-, $(\infty,0)$- nor $(0,\infty)$-polar graphs.*

3. *An extension graph G is a minimal (s,k)-polar obstruction if and only if $G \cong C_5$ and $s = k = 1$.*

The following proposition allow us to show that any connected P_4-extendible minimal $(1,k)$-polar obstruction, other than C_5, has a disconnected complement.

Lemma 4.38. *Let k be a nonnegative integer, and let X be a separable extension graph. If $H = (S,K,R)$ is an X-spider with nonempty head, then H is not a minimal $(1,k)$-polar obstruction.*

Proof. The proof is divided in three cases, depending on X. If X is isomorphic to P_4, F or $\overline{F}$ then the midpoints set of X conform a clique with at least two vertices, while its endpoints set is an independent set, in which case the proof is the same as Proposition 4.30.

Now, assume that $X \cong P$. Let v be the only vertex of X of degree one, and let w be the support vertex of v; notice that w is a midpoint of X. Aiming for a contradiction, suppose that H is a minimal $(1,k)$-polar obstruction, and let (A,B) be a $(1,k)$-polar partition of $H - v$. If $w \in A$, then there are two midpoints of X in B, but in such a case $R \cap A$ and $R \cap B$ are both empty sets, which is impossible. Then, $w \in B$ and $(A \cup \{v\}, B)$ is a $(1,k)$-polar partition of H, a contradiction. Hence, H is not a minimal $(1,k)$-polar obstruction.

Finally, assume that $X \cong \overline{P}$, and let v and w as in the previous paragraph. Aiming for a contradiction, suppose that H is a minimal $(1,k)$-polar obstruction, and let (A,B) be a $(1,k)$-polar partition of $H - v$. If $w \in A$, the other midpoint of X, w', is in B and at least one of the endpoints of X that is adjacent to w' is also in B. Therefore, $R \cap B = \varnothing$. But $w \in A$, so also $R \cap A = \varnothing$, which is impossible. Hence, $w \in B$ and $(A \cup \{v\}, B)$ is a $(1,k)$-polar partition of H, a contradiction. Then, H is not a minimal $(1,k)$-polar obstruction. □

Corollary 4.39. *Let k be a nonnegative integer, and let H be a P_4-extendible minimal $(1,k)$-polar obstruction. If $H \ncong C_5$, then H or its complement is disconnected.*

Proof. Since H is a P_4-extendible graph, we have from Theorem 4.2 that, if H and $\overline{H}$ are connected, then H is either an extension graph or an X-spider (with nonempty head) for some separable extension graph X. Nevertheless we have from Item 3 of Remark 4.37 and Lemma 4.38 that this is not the case, so either H or its complement is disconnected. □

In the last two results of this section we provide a complete structural characterization for disconnected P_4-extendible minimal $(1,k)$-polar obstructions. It is worth noticing that statements in Lemmas 4.40 and 4.41 are the same as those in Lemmas 4.33 and 4.34, respectively, except by the obvious difference of the graph class.

Lemma 4.40. *Let t be an integer, $t \geq 2$, and for each $i \in \{1, \ldots, t\}$, let G_i be a connected P_4-extendible minimal $(1,k_i)$-polar obstruction that is a $(1,k_i+1)$-polar graph. If $G = G_1 + \cdots + G_t$, then G is a minimal $(1,k)$-polar obstruction if and only if $k = t - 1 + \sum_{i=1}^{t} k_i$.*

Proof. Let $k = t - 1 + \sum_{i=1}^{t} k_i$. We have from Lemma 4.28 that G is a minimal $(1,k)$-polar obstruction that is $(1,k+1)$-polar, so we just need to show that G is not a minimal $(1,\kappa)$-polar obstruction for any $\kappa < k$.

Let G_i be a connected P_4-extendible minimal $(1,k_i)$-polar obstruction that is $(1,k_i+1)$-polar. By Corollary 4.39, we have that either $G_i \cong C_5$ or $\overline{G_i}$ is a disconnected minimal $(k_i,1)$-polar obstruction that is a $(k_i+1,1)$-polar graph. However, it follows from Theorem 4.27 that, for any nonnegative integer κ_i such that $\kappa_i < k_i$, G_i contains a proper induced subgraph G_i' that is both, a

P_4-extendible minimal $(1, \kappa_i)$-polar obstruction and a $(1, \kappa_i + 1)$-polar graph. From here on, the proof follows as the proof of Lemma 8 in [19]. □

Lemma 4.41. *Let k be a nonnegative integer. If G is a disconnected P_4-extendible minimal $(1, k)$-polar obstruction with components $G_1, \dots, G_t$, then there exist nonnegative integers $k_1, \dots, k_t$ such that for each $i \in \{1, \dots, t\}$, G_i is a connected minimal $(1, k_i)$-polar obstruction that is a $(1, k_i + 1)$-polar graph, and $\sum_{i=1}^{t} k_i = k - t + 1$. (Notice that $k_i < k$ for any $i \in \{1, \dots, t\}$, and G is a $(1, k + 1)$-polar graph.)*

Proof. This is a generalization of Lemma 9 in [19], that states the same result for cographs. As in the original proof, it is easy to argue that each component G_i is a minimal $(1, k_i)$-polar obstruction that is $(1, k)$-polar, where k_i is the maximum integer such that any proper induced subgraph of G_i is $(1, k_i)$-polar.

Then, by Corollary 4.39, $\overline{G_i}$ is either C_5 or a disconnected P_4-extendible minimal $(k_i, 1)$-polar obstruction that is $(k, 1)$-polar. However, it follows from Theorem 4.27 and Item 3 of Remark 4.37 that $\overline{G_i}$ is a $(k_i + 1, 1)$-polar graph, so G_i is a connected P_4-extendible minimal $(1, k_i)$-polar obstruction that is a $(1, k_i)$-polar graph. Finally, the result follows from Lemma 4.40. □

The following result is analogous to Theorem 4.35; it provides a complete recursive construction of P_4-extendible minimal $(s, 1)$-polar obstructions. Notice that, since C_5 is a P_4-extendible minimal $(1, 1)$-polar obstruction, there are P_4-extendible minimal $(s, 1)$-polar obstructions that are not cographs for each positive integer s.

Theorem 4.42. *Let s be an integer, $s \geq 2$. If G is a P_4-extendible graph, then G is a minimal $(s, 1)$-polar obstruction if and only if G satisfies exactly one of the following assertions:*

1. *G is isomorphic to one of the seven graphs depicted in Figure 4.3.*
2. *G is isomorphic to some of $2K_{s+1}, K_2 + (K_s \oplus 2K_1)$ or $K_1 + (K_{s-1} \oplus C_4)$.*
3. *The complement of G is disconnected with components $G_1, \dots, G_t$, each G_i is a minimal $(1, s_i)$-polar obstruction whose complement is different from the graphs in Figure 4.3, and $s = t - 1 + \sum_{i=1}^{t} s_i$.*

Proof. If G is disconnected, it follows from Theorem 4.27 that G is a minimal $(s, 1)$-polar obstruction if and only if G is either a graph depicted in Figure 4.3, or it is isomorphic to some of $2K_{s+1}, K_2 + (K_s \oplus 2K_1)$ or $K_1 + (K_{s-1} \oplus C_4)$.

Otherwise, if G is connected, Corollary 4.39 and Item 3 of Remark 4.37 imply that $\overline{G}$ is a disconnected P_4-extendible minimal $(1, s)$-polar obstruction, and the result follows from Lemma 4.41. □

4.5 Polarity and monopolarity

In order to analyze the minimal obstructions for polarity and monopolarity on the classes of P_4-sparse and P_4-extendible graphs we need a final lemma. Notice that it holds for general graphs.

Lemma 4.43. *If G is a graph, then G is a disconnected minimal polar obstruction if and only if $G \cong P_3 + H$ where H is a minimal monopolar obstruction that is not a minimal polar obstruction.*

Proof. First, assume that H is a minimal $(1, \infty)$-polar obstruction that is not a minimal polar obstruction, and let $G = P_3 + H$. Aiming for a contradiction, assume that G has a polar partition (A, B). Notice that $G[A]$ is not an empty graph because H is not a $(1, \infty)$-polar graph. Then $G[A]$ is completely contained in a component of G. Moreover, since any component of G is either P_3 or a component of H, and $G[B]$ is a P_3-free graph, we have that $A \cap V_H = \varnothing$ so H is a cluster, a contradiction. Hence, G is not a polar graph.

Let $v \in V_G$. If $v \in V_H$, let (A, B) be a $(1, \infty)$-polar partition of $H - v$, and let $w \in V_G - V_H$ be a vertex of degree 1. Then $(A', V_G - (A' \cup \{v\}))$, where $A' = A \cup \{w\}$, is a $(1, \infty)$-polar partition of $G - v$. Now, let $v \in V_G - V_H$. Then, since H is a polar graph and $P_3 - v$ is a cluster, $G - v$ is a polar graph. Therefore G is a disconnected minimal polar obstruction.

For the converse, assume that G is a disconnected minimal polar obstruction. Notice that, if all the components of G are $(1, \infty)$-polar graphs, then G is also a $(1, \infty)$-polar graph, so G has a component H' that contains a minimal $(1, \infty)$-polar obstruction H as an induced subgraph. Notice that by the minimality of G, H is a polar graph. In addition, G has no complete components, so any component of G contains an induced P_3, and therefore G contains the disjoint union of P_3 with a minimal $(1, \infty)$-obstruction that is a polar graph (H). Together with the minimality of G, this implies that $G \cong P_3 + H$. □

4.5.1 P_4-sparse minimal polar obstructions

A graph G is a P_4-sparse minimal monopolar obstruction if and only if $\overline{G}$ is a P_4-sparse minimal $(\infty, 1)$-polar obstruction. Hence, the following consequence of Theorems 2.2 and 4.36 completely characterizes the minimal monopolar obstructions in P_4-sparse graphs.

Corollary 4.44. *If G is a P_4-sparse graph, then G is a minimal $(\infty, 1)$-polar obstruction if and only if G is one of the four cographs depicted in Figure 4.3.*

Proof. Let G be P_4-sparse minimal $(\infty, 1)$-polar obstruction. Then G is a minimal $(s, 1)$-polar obstruction for some nonnegative integer s. Moreover, by Theorem 4.36 we have that G is a cograph minimal $(\infty, 1)$-polar obstruction. Then, from Theorem 2.2, we have that G is isomorphic to one of the cographs depicted in Figure 4.3. The converse is proved in Lemma 4.22. □

Next, we prove that, as well as P_4-sparse minimal $(s, 1)$- and $(\infty, 1)$-polar obstructions, every P_4-sparse minimal polar obstruction is a cograph.

Theorem 4.45. *If G is a P_4-sparse minimal polar obstruction, then G is a cograph.*

Proof. First, aiming for a contradiction, suppose that G is a spider, say $G = (S, K, R)$. Since headless spiders are split graphs, and thus polar graphs, R is not an empty set. Moreover, by the minimality of G, $G[R]$ admits a polar partition (A, B), and then $(A \cup K, B \cup S)$ would be a polar partition of G, contradicting the choice of G. Therefore G is not a spider. Thus, by Theorem 4.1, G or its complement is disconnected. However, in both cases Lemma 4.43 and Corollary 4.44 imply that G is a cograph. □

We obtain the complete list of P_4-sparse minimal polar obstructions as an immediate consequence of Theorem 2.3 and the previous proposition.

Corollary 4.46. *A graph G is a P_4-sparse minimal polar obstruction if and only if G or its complement is isomorphic to $P_3 + H$, where H is any P_4-sparse minimal $(1, \infty)$-polar obstruction.*

4.5.2 P_4-extendible minimal polar obstructions

Unlike P_4-sparse graphs, there are P_4-extendible minimal monopolar and polar obstructions that are not cographs. We give complete lists of such minimal obstructions in the next results.

Corollary 4.47. *If G is a P_4-extendible graph, then G is a minimal $(\infty, 1)$-polar obstruction if and only if G is one of the graphs depicted in Figure 4.3.*

Proof. Let G be a P_4-extendible minimal $(\infty, 1)$-polar obstruction. Then G is a minimal $(s, 1)$-polar obstruction for some integer s, $s \geq 2$. By Lemma 4.41 and Theorem 4.42 we conclude that G is isomorphic to one of the seven graphs depicted in Figure 4.3. The converse is proved in Lemma 4.22. □

Theorem 4.48. *If H is a P_4-extendible minimal polar obstruction, then H or its complement is the disjoint union of P_3 with the complement of one of the graphs depicted in Figure 4.3.*

Proof. First, aiming for a contradiction, let us assume that H is a G-spider for some separable extension G, say $H = (S, K, R)$. By Items 1 and 2 of Remark 4.37, we have that $R \neq \varnothing$, and by the minimality of H, $H[R]$ admits a polar partition (A, B). But, no matter what separable extension G is, its midpoints induce a complete multipartite graph while its endpoints induce a cluster, so $(A \cup K, B \cup S)$ is a polar partition of H, contradicting the assumption that H was a G-spider. Thus, by Theorem 4.2, either H or its complement is disconnected, and the result follows from Lemma 4.43 and Corollary 4.47. □

The next section is devoted to provide complete characterizations of minimal $(2, 2)$-polar obstructions on both, P_4-sparse and P_4-extendible graphs. As we have pointed before, we will notice that P_4-sparse minimal 2-polar obstructions are cographs, but there are P_4-extendible minimal 2-polar obstructions that are not.

4.6 2-polarity

Throughout this section we generalize the characterization of cograph minimal 2-polar obstructions given in [46]. In fact, we base our results in the following propositions, most of them taken from the aforementioned paper.

We start with two lemmas that provide us of some useful general structural properties about minimal k-polar obstructions in general graphs.

Lemma 4.49 ([46])**.** *Let H be a minimal k-polar obstruction. The following statements are true*

1. *H has at most $k + 2$ components.*

2. *H has at least one nontrivial component.*

3. *H has at most $k + 1$ trivial components.*

4. *If H has at least one trivial component, H has at most one noncomplete component.*

5. *If $H \ncong (k+1)K_{k+1}$, every complete component of H is isomorphic to K_1 or K_2.*

Lemma 4.50 ([46])**.** *Let H be a minimal 2-polar obstruction.*

1. *H has at least seven vertices.*

2. *If H has seven vertices and three connected components, then at least one of them is an isolated vertex.*

Next, we give a slight correction to Lemma 2 in [46], which characterize the minimal k-polar obstructions with the maximum possible number of components; it is worth noticing that it does not affect the main results in such paper.

Lemma 4.51. *Let k be an integer, $k \geq 2$, and let G be graph. Then, G is a minimal k-polar obstruction with exactly $k+2$ connected components if and only if $G \cong \ell K_1 + (k-\ell+1)K_2 + G'$, where ℓ is an integer in the set $\{1,\dots,k+1\}$ and G' is a connected complete k-partite graph that is a minimal $(1,\ell-1)$-polar obstruction and such that, if $\ell \leq k$, G' is a $(1,\ell)$-polar graph.*

Proof. Suppose $G \cong \ell K_1 + (k-\ell+1)K_2 + G'$, where ℓ is an integer in the set $\{1,\dots,k+1\}$ and G' is a connected complete k-partite graph that is a minimal $(1,\ell-1)$-polar obstruction such that, if $\ell \leq k$, it is a $(1,\ell)$-polar graph. If G is a $(1,k)$-polar graph, then G' is a $(1,\ell-1)$-polar graph, but it is not. Thus, since G is not $(1,k)$-polar, if its admits a k-polar partition (A,B), the subgraph $G[A]$ is a connected graph and hence it is completely contained in some component of G. But then, G would have at most $k+1$ connected components, which is not the case. Hence, G is not a k-polar graph.

Let v be an isolated vertex of G. Then $G-v$ is the disjoint union of a k-cluster with G', and since G' is a complete k-partite graph, then $G-v$ is a k-polar graph. Now, since G' is a minimal $(1,\ell-1)$-polar obstruction, for any vertex w of G', $G'-w$ can be partitioned into an stable set and an $(\ell-1)$-cluster, so $G-w$ is a $(1,k)$-polar graph, and then a k-polar graph. Finally, if at least one component of G is a copy of K_2, then $\ell \leq k$ and we have that G' is a $(1,\ell)$-polar graph. Thus, for any vertex u in a K_2-component of G, $G-u$ is a $(1,k)$-polar graph. Hence, G is a minimal k-polar obstruction that clearly has exactly $k+2$ connected components.

For the converse implication, assume that G is a minimal k-polar obstruction with precisely $k+2$ components and ℓ isolated vertices. If $\ell = 0$ then G properly contains $K_1 + (k+1)K_2$ as an induced subgraph, but that is impossible from the first part of this proof. Then, G has at least one isolated vertex, and G clearly is not an empty graph, so $\ell \leq k+1$. We know by Lemma 4.49 that G has at most one noncomplete connected component and that any complete component of G has at most two vertices, so $G \cong \ell K_1 + (k-\ell+1)K_2 + G'$ where $\ell \in \{1, \ldots, k+1\}$ and G' is a connected graph.

Notice that G' is not a $(1, \ell-1)$-polar graph, otherwise G would be a $(1,k)$-polar graph, and hence a k-polar graph. Let u be a vertex of G'. By the minimality of G, we have that $G-u$ is a k-polar graph. Moreover, since $G-u$ has at least $k+2$ connected components, any k-polar partition of $G-u$ is necessarily a $(1,k)$-polar partition, which implies that $G'-u$ is a $(1,\ell-1)$-polar graph. Then G' is a minimal $(1,\ell-1)$-polar obstruction. Now, let v be an isolated vertex of G. By the minimality of G, $G-u$ has a k-polar partition (A,B), but it cannot be a $(1,k)$-polar partition or G would be a $(1,k)$-polar graph. Thus, either $G' \cong K_2$ and $\ell = 1$, or $A = V(G')$ and hence G' is a complete k-partite graph. Finally, if $l \leq k$, G has at least one K_2-component. Let w be a vertex in one of such components. Then $G-w$ is a k-polar graph with $k+2$ connected components, which implies that in fact $G-w$ is a $(1,k)$-polar graph, and hence G' is a $(1,\ell)$-polar graph. □

A **partial complement** of a graph H is either the usual complement of H, or a graph $\overline{H_1} + \overline{H_2}$, where H_1 and H_2 are subgraphs of H obtained by splitting the components of H into two parts, H_1 and H_2. The next result shows how partial complements preserves 2-polarity, which will be useful to give compact lists of minimal 2-polar obstructions on P_4-sparse and P_4-extendible graphs. Remarkably, this lemma was originally proven for the special class of cographs, but the same proof works for any hereditary class of graphs closed under complement and disjoint union operations, particularly, it works for the classes of P_4-sparse and P_4-extendible graphs.

Lemma 4.52 ([46]). *Let $\mathcal{G}$ be a hereditary class of graphs closed under complement and disjoint union operations, and let $G \in \mathcal{G}$ be a 2-polar graph. Then, any partial complement of G is a 2-polar graph belonging to $\mathcal{G}$.*

Based on the previous propositions, in the following sections we provide complete lists of minimal 2-polar obstructions that are P_4-sparse or P_4-extendible graphs.

4.6.1 P_4-sparse minimal 2-polar obstructions

Throughout this section we characterize P_4-sparse graphs admitting a 2-polar partition by means of its family of minimal obstructions. At the end of the section we conclude that any P_4-sparse minimal 2-polar obstruction is in fact a cograph, which is interesting since any known P_4-sparse minimal (s,k)-polar obstruction is a cograph. We start by proving that the complement of any connected P_4-sparse minimal 2-polar obstruction is a disconnected graph.

Proposition 4.53. *If G is a spider, then G is 2-polar if and only if G is a split graph.*

Proof. Let (S,K,R) be the spider partition of G. We only need to prove that any 2-polar spider is, in fact, a split graph. Since k-polar graphs are closed under complements, and headless spiders trivially are split graphs, we can assume that G is a thin spider with nonempty head. Let (V_1,V_2,V_3,V_4) be a 2-polar partition of G, and for any $i \in \{1,2,3,4\}$, let $R_i = V_i \cap R$. Notice that, since K is completely adjacent to R, $R_i = \varnothing$ for some $i \in \{1,\dots,4\}$.

First, suppose that (R_1,R_3,R_4) is a $(1,2)$-polar partition of $G[R]$. Again, some of R_1, R_3 and R_4 must be empty because K and R are completely adjacent and K has at least two vertices. Thus, either (R_1,R_3) is a split partition of $G[R]$, or (R_3,R_4) is a $(0,2)$-polar partition of $G[R]$. But the second case is not possible since then, $S \cup K \subseteq V_1 \cup V_2$, which is impossible since $G[S \cup K]$ is not a complete multipartite graph. Hence, $G[R]$ is a split graph and, by Remark 4.29, also is G. The case in which (R_1,R_2,R_3) is a $(2,1)$-polar partition of $G[R]$ can be treated in a similar way. □

Corollary 4.54. *If G is a spider, then G is not a minimal 2-polar obstruction. In consequence, for any P_4-sparse minimal 2-polar obstruction H, either H or its complement is disconnected.*

Proof. Let (S,K,R) be the spider partition of G. As in the lemma above, we can suppose that G is a thin spider. Aiming for a contradiction, assume that G is a minimal 2-polar obstruction so, by the previous lemma and Remark 4.29, we have that $G[R]$ is not a split graph. Then, for any $r \in R$, $G - r$ is a spider that is 2-polar, so $G[R] - r$ is a split graph. Thus $G[R]$ is a P_4-sparse minimal split obstruction, that is to say, G is isomorphic to either $2K_2$ or C_4. From here is easy to prove that deleting either one leg or one vertex of the body of G the resulting graph is not a 2-polar graph, contradicting the minimality of G. Hence, a P_4-sparse minimal 2-polar obstruction is not a spider, and the result directly follows from Theorem 4.1. □

By Lemma 4.49, any P_4-sparse minimal 2-polar obstruction has at most four connected components. The following propositions show that there are exactly three P_4-sparse graphs attaining such bound.

Proposition 4.55. *Let ℓ be a positive integer. If G is a connected P_4-sparse minimal $(1, \ell - 1)$-polar obstruction that is a complete multipartite graph, then G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$.*

Proof. Clearly, if $\ell = 1$, $G \cong K_2$, while if $\ell = 2$, $G \cong C_4$. For $\ell \geq 3$, we have from Theorem 4.36 that $\overline{G}$ (hence G) is a cograph. Then, since G is connected, $\overline{G}$ is a disconnected graph, and it follows from Theorem 4.27 that G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$. □

Corollary 4.56. *If G is a P_4-sparse graph, then G is a minimal 2-polar obstruction with exactly 4 connected components if and only if $G \cong \ell K_1 + (3 - \ell)K_2 + K_{\ell,\ell}$ for some integer $\ell \in \{1, 2, 3\}$.*

Proof. Let G be a P_4-sparse graph. By Lemma 4.51 we have that G is a minimal 2-polar obstruction with precisely four connected components if and only if $G \cong \ell K_1 + (3 - \ell)K_2 + G'$, where $\ell \in \{1, 2, 3\}$, and G' is a connected complete bipartite graph that is a minimal $(1, \ell - 1)$-polar obstruction such that, if $\ell \neq 3$, G' is a $(1, \ell)$-polar graph. In addition, we have from Proposition 4.55 that the only connected P_4-sparse minimal $(1, \ell - 1)$-polar obstruction that is a complete bipartite graph is $K_{\ell,\ell}$. The result follows since $K_{\ell,\ell}$ trivially is a $(1, \ell)$-polar graph. □

The following proposition is a direct consequence of Theorem 4.35 that will be useful to give the complete list of P_4-sparse minimal 2-polar obstructions.

Proposition 4.57. *There are exactly nine P_4-sparse minimal $(2, 1)$-polar obstructions; they are the graphs $E_1, \dots, E_9$ depicted in Figures 4.3 and 4.6.*

Hannnebauer [45] proved that, for any nonnegative integers s and k, any P_4-sparse minimal (s, k)-polar obstruction has at most $(s + 1)(k + 1)$ vertices. Thus, we have by Lemma 4.50 that any P_4-sparse minimal 2-polar obstruction has at least seven and at most nine vertices. The following three lemmas completely characterize such minimal obstructions depending on their order; the proofs are simple generalizations of the analogous proofs given in [46] for cographs.

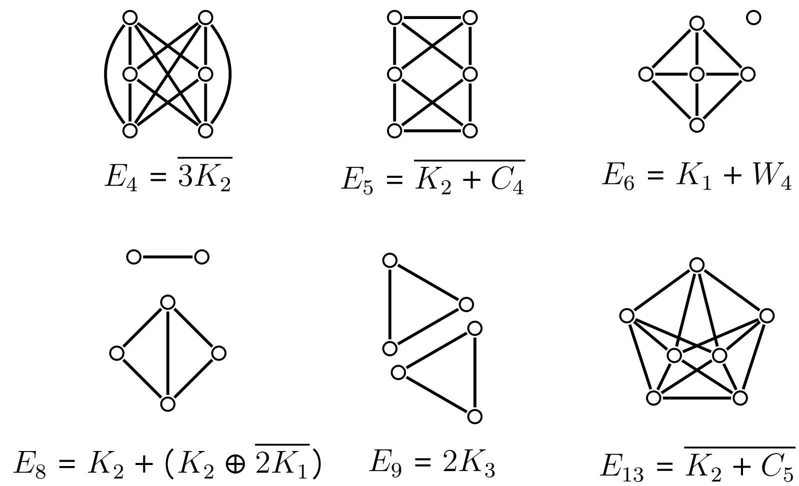


Figure 4.6: Some minimal $(2, 1)$-polar obstructions.

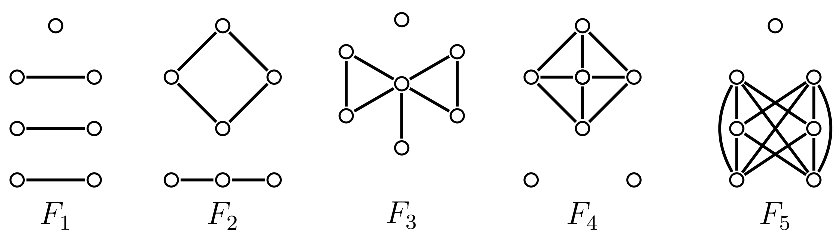


Figure 4.7: P_4-sparse minimal 2-polar obstructions on 7 vertices.

Lemma 4.58. *The disconnected P_4-sparse minimal 2-polar obstructions on 7 vertices are exactly the graphs $F_1, \dots, F_5$ depicted in Figure 4.7.*

Proof. Let H be a disconnected P_4-sparse minimal 2-polar obstruction on seven vertices. If H has four connected components or it can be transformed by a sequence of partial complementations into a graph with four components, it follows from Corollary 4.56 and Lemma 4.52 that H is isomorphic to F_i for some $i \in \{1, \dots, 5\}$. Thus, we can assume that any graph obtained from H by partial complementations has at most three components; from here we can replicate the argument in Lemma 7 of [46] to assume that H is a graph with precisely two connected components, one of them being a trivial graph.

Since H is not a 2-polar graph, its nontrivial component must contain a minimal $(2, 1)$-polar obstruction H' as an induced subgraph. Moreover, H'

cannot be a disconnected graph on six vertices, so we have from Proposition 4.57 that $H \in \{K_1 + 2K_2, \overline{3K_2}, 2K_2 \oplus 2K_1\}$. If $H' \cong \overline{3K_2}$, H is the graph F_5 in Figure 4.7. If $H' \cong 2K_2 \oplus 2K_1$, is straightforward to verify that H is a $(1,2)$-polar graph, which cannot occur. Otherwise, if $H' \cong K_1 + 2K_2$, we have that $H \cong F_3$, because P_4-sparse graphs are $\{\overline{P}, P_5\}$-free and H' is contained in a connected component of H on six vertices. □

Lemma 4.59. *The disconnected P_4-sparse minimal 2-polar obstructions on 9 vertices are exactly the graphs $F_{21}, \ldots, F_{24}$ depicted in Figure 4.8.*

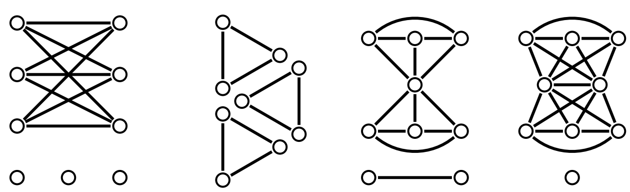


Figure 4.8: P_4-sparse minimal 2-polar obstructions on 9 vertices.

Proof. Almost all the arguments used in the proof of Lemma 8 in [46] are still valid for P_4-sparse graphs. We only have to care about the case when H is a P_4-sparse minimal 2-polar obstruction on 9 vertices with three connected components and precisely two isolated vertices. In such a case the nontrivial connected component of H, B_3, is either a spider or the join of two smaller P_4-sparse graphs T_1 and T_2. In the former case, since the head of B_3 has at most three vertices, B_3 is a split graph, so H is too. The latter case follows as in the original proof. □

Lemma 4.60. *The disconnected P_4-sparse minimal 2-polar obstructions on 8 vertices are exactly the graphs $F_6, \ldots, F_{20}$ and F_{25}, depicted in Figures 4.9 and 4.10.*

Proof. The proof of Lemma 9 in [46] is still valid for P_4-sparse graphs with the only addition of the graph F_{25} as a partial complement of the graph F_{19}, which was omitted by mistake in [46]. The main arguments are similar to those used in the proof of Lemma 4.59. □

We summarize the results of this section as follows.

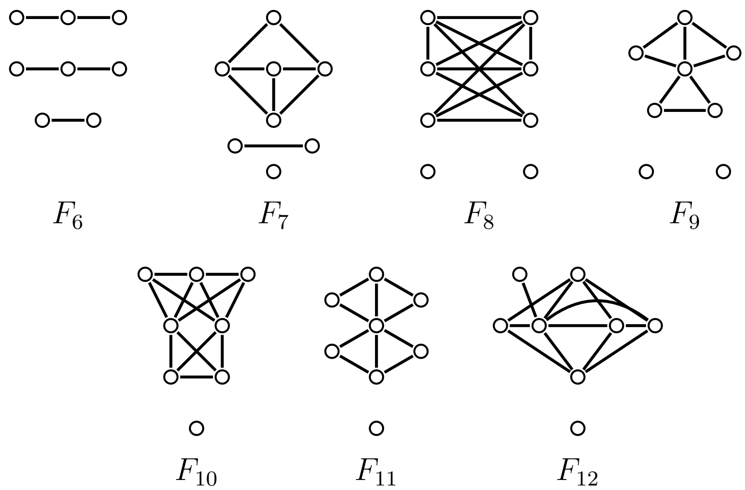


Figure 4.9: Family A of P_4-sparse minimal 2-polar obstructions on 8 vertices.

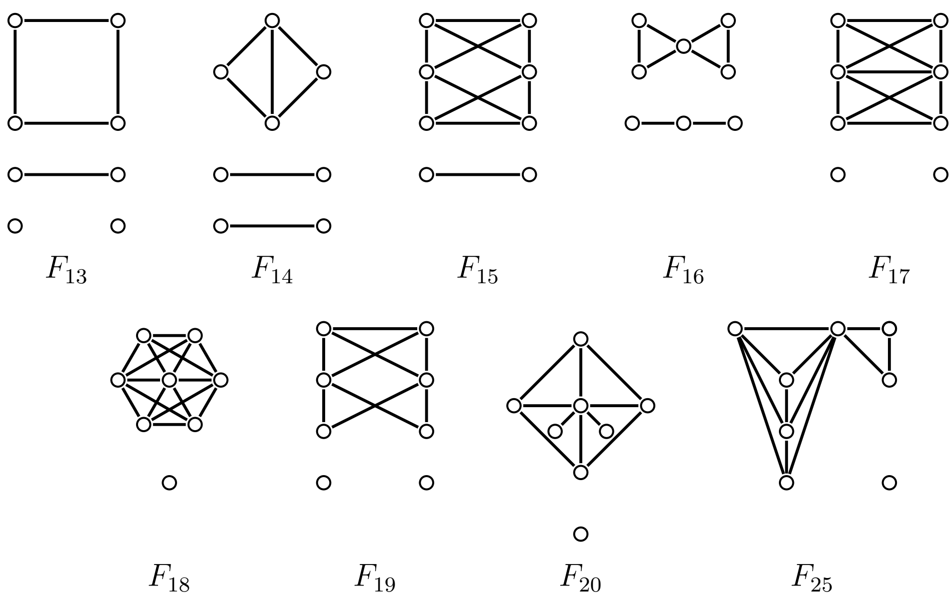


Figure 4.10: Family B of P_4-sparse minimal 2-polar obstructions on 8 vertices.

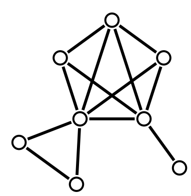
Theorem 4.61. *There are exactly 50 P_4-sparse minimal 2-polar obstructions, and each of them is a cograph. The disconnected P_4-sparse minimal 2-polar obstructions are the graphs $F_1, \ldots, F_{25}$ depicted in Figures 4.7 to 4.10.*

4.6.2 P_4-extendible minimal 2-polar obstructions

In Sections 4.4 and 4.5 it was observed that the set of cograph minimal (s, k)-polar obstructions is a proper subset of the set of P_4-extendible minimal (s, k)-polar obstructions for the cases $\min\{s, k\} = 1$ and $s = k = \infty$. In the present section we give the complete family of P_4-extendible minimal 2-polar obstructions, and show that also in the case $s = k = 2$ there are P_4-extendible minimal (s, k)-polar obstructions that are not cographs. Indeed, each graph depicted in Figures 4.11 to 4.14 is a P_4-extendible minimal 2-polar obstruction that is not a cograph.

We start by proving that there exists only one P_4-extendible connected minimal 2-polar obstruction whose complement is also a connected graph.

Lemma 4.62. *Let $G = (S, K, R)$ be a $\overline{P}$-spider. If $H = G[R]$, then G is a minimal 2-polar obstruction if and only if $H \cong P_3$, that is, if G is isomorphic to the graph F_{26} in Figure 4.11.*



F_{26}

Figure 4.11: A connected P_4-extendible minimal 2-polar obstruction with connected complement.

Proof. If $H \cong P_3$, then $G \cong F_{26}$, so G is a minimal 2-polar obstruction. Aiming for a contradiction, suppose that G is another $\overline{P}$-spider minimal 2-polar obstruction. Being P_3-free, H is a cluster. Moreover, if H is not a complete multipartite graph, then G properly contains F_3 as an induced subgraph, which is impossible. Then, H is a cluster that is a complete multipartite graph, so it is either a complete or an empty graph. However, it is easy to check that in both cases G is a 2-polar graph, contradicting our original assumption. □

The proofs of the next proposition and its corollaries are very similar to the proofs of Proposition 4.53 and its corollaries, so we only sketch them without going into details.

Proposition 4.63. *Let $X \in \{P_4, F\}$. If G is an X-spider, then G is a 2-polar graph if and only if R induces a split graph.*

Proof. Let (S, K, R) be the spider partition of G. First, assume that (A, B) is a split partition of $G[R]$. Then, $(A \cup S, B \cup K)$ is a split partition of G, so G is a split graph, and hence a 2-polar graph. Now, suppose that G has a 2-polar partition (V_1, V_2, V_3, V_4), and let $R_i = V_i \cap R$ for each $i \in \{1, \dots, 4\}$. Notice that, if R_1 and R_2 are both nonempty, then $S \cup K \subseteq V_3 \cup V_4$, which is impossible since X is not a cluster. Analogously, since X is not a complete multipartite graph, R_3 and R_4 cannot be both nonempty. Therefore $G[R]$ is a split graph. □

Corollary 4.64. *Let $X \in \{P_4, F\}$. If G is an X-spider, then it is not a minimal 2-polar obstruction.*

Proof. Let (S, K, R) be the spider partition of G. In order to reach a contradiction, suppose that G is a minimal 2-polar obstruction. By Proposition 4.63, $G[R]$ is not a split graph, but for any vertex $v \in R$, $G[R] - v$ is. Hence, $G[R]$ is a minimal split obstruction, i.e., $G[R]$ is isomorphic to some of $2K_2, C_4$ or C_5. But then, G contains $F_3, \overline{F_3}$ or F_{27}, respectively, as a proper induced subgraph, contradicting the minimality of G. □

Corollary 4.65. *If G is a P_4-extendible minimal 2-polar obstruction different from F_{26} and its complement, then G or its complement is disconnected.*

Proof. It is a simple exercise to verify that any extension graph is a 2-polar graph. In addition, by Lemma 4.62 and Proposition 4.63, the only X-spiders that are minimal 2-polar obstructions are F_{26} and its complement. Therefore, by Theorem 4.2, any other P_4-extendible minimal 2-polar obstruction is disconnected or has a disconnected complement. □

Now, we characterize the P_4-extendible minimal 2-polar obstructions with the maximum possible number of connected components. As the reader will notice, the proofs of the next proposition and its corollary are analogous to those of Proposition 4.55 and Corollary 4.56.

Proposition 4.66. *Let ℓ be a positive integer. If G is a connected P_4-extendible minimal $(1, \ell - 1)$-polar obstruction that is a complete multipartite graph, then G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$.*

Proof. Clearly, if $\ell = 1$, then $G \cong K_2$, while if $\ell = 2$, we have $G \cong C_4$. By Theorem 4.42, if $\ell \geq 3$, G is isomorphic to either $K_{\ell,\ell}$ or $K_1 \oplus C_4$. □

Corollary 4.67. *If G is a P_4-extendible graph, then G is a minimal 2-polar obstruction with exactly 4 connected components if and only if $G \cong \ell K_1 + (3 - \ell)K_2 + K_{\ell,\ell}$ for some integer $\ell \in \{1, 2, 3\}$.*

Proof. This result is to P_4-extendible graphs as Corollary 4.56 is to P_4-sparse graphs. In fact, the proof of this result is basically the same as that of Corollary 4.56, but using instead Proposition 4.66, which is to P_4-extendible graphs as Proposition 4.55 is to P_4-sparse graphs. □

The following proposition will be useful to give the complete list of P_4-extendible minimal 2-polar obstructions. It is a direct consequence of Theorem 4.42.

Corollary 4.68. *There are exactly 13 P_4-extendible minimal $(2, 1)$-polar obstructions; they are the graphs $E_1, \dots, E_{13}$ depicted in Figures 4.3 and 4.6.*

By Lemma 4.50, we have that no P_4-extendible minimal 2-polar obstruction has less than seven vertices. In the rest of this section we give the complete list of such obstructions, obtaining as a consequence that they have at most 9 vertices, as in the case of P_4-sparse graphs. We remark that these proofs are very similar in flavor to the analogous proofs for P_4-sparse graphs.

Lemma 4.69. *The disconnected P_4-extendible minimal 2-polar obstructions on 7 vertices are exactly the graphs $F_1, \dots, F_5$ depicted in Figure 4.7.*

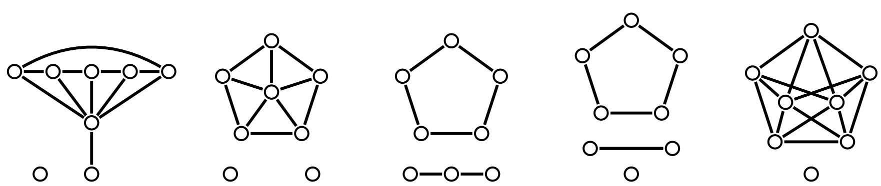
Proof. Let H be a disconnected P_4-extendible minimal 2-polar obstruction on 7 vertices. It follows from Corollary 4.67 that, if H has four components, or it can be transformed into a graph with four components through a sequence of partial complementations, then it is one of $F_1, \dots, F_5$.

So, assume that none of the graphs that can be obtained from H by means of partial complements has more than three connected components. Notice that any P_4-extendible graph H on seven vertices with exactly two components, can be transformed by partial complementation into a graph with at least three components, one of which is an isolated vertex, except in the case that H is the disjoint union of K_1 with an X-spider on 6 vertices, in which case it can be checked that H is a $(1, 2)$-polar graph. Taking a partial complementation separating one isolated vertex of H from the rest of the graph, we obtain a graph with two components, one of them being an isolated vertex. Let us suppose without loss of generality that H has this form.

Since H is not 2-polar, its non trivial component must contain a P_4-extendible minimal $(2,1)$-polar obstruction H' as an induced subgraph. Moreover, H' cannot be a disconnected graph on six vertices so, by Corollary 4.68, $H \in \{K_1 + 2K_2, \overline{3K_2}, 2K_2 \oplus 2K_1\}$. If $H' \cong \overline{3K_2}$, then H is the graph F_5 in Figure 4.7. If $H' \cong 2K_2 \oplus 2K_1$, it is straightforward to verify that H is a $(1,2)$-polar graph. Otherwise, $H' \cong K_1 + 2K_2$. But H' is contained in a connected component of H on six vertices, which must be isomorphic to $K_1 \oplus (K_1 + 2K_2)$ because H is a P_4-extendible graph. Then, H is isomorphic to F_3. □

The next technical lemma will be needed to give the complete list of P_4-extendible minimal 2-polar obstructions with at least eight vertices.

Lemma 4.70. *Let H be a disconnected minimal 2-polar obstruction. If H has a component H' that is not a cograph, then $H - H'$ is a split graph. In consequence, at most one component of H is not a cograph.*



F_{27} F_{28} F_{29} F_{30} F_{31}

Figure 4.12: Family C of P_4-extendible minimal 2-polar obstructions on 8 vertices.

Proof. If $H - H'$ is not a split graph it contains $2K_2, C_4$ or C_5 as an induced subgraph, and H would contain F_1, F_2 or F_{29} as a proper induced subgraph, respectively (see Figures 4.7 and 4.12). Now, aiming for a contradiction, suppose that H has at least two components, H_1 and H_2, which are not cographs. By the first part of this lemma, $H - H_1$ and $H - H_2$ (and hence H_1) are split graphs, so H is the disjoint union of two split graphs, which implies that it is a $(1,2)$-polar graph, contradicting that H is a 2-polar obstruction. □

Lemma 4.71. *The only disconnected P_4-extendible minimal 2-polar obstructions with at least 8 vertices are the graphs $F_6, \dots, F_{41}$ depicted in Figures 4.8 to 4.14*

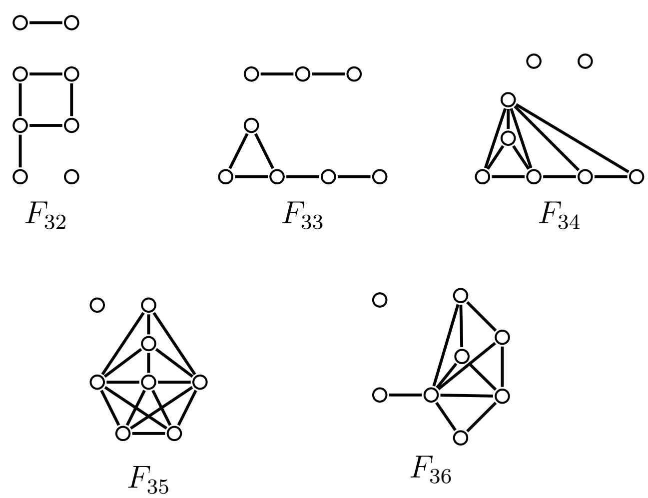


Figure 4.13: Family D of P_4-extendible minimal 2-polar obstructions on 8 vertices.

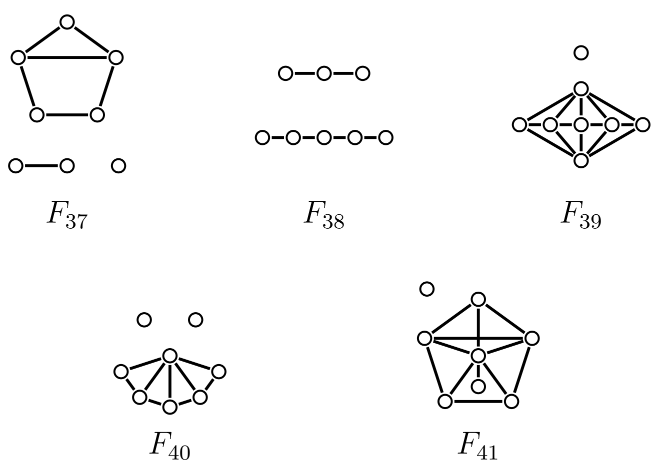


Figure 4.14: Family E of P_4-extendible minimal 2-polar obstructions on 8 vertices.

Proof. Let H be a P_4-extendible disconnected minimal 2-polar obstruction with at least eight vertices. If H can be transformed by means of partial complementations into a graph with four connected components, we have by Corollary 4.67 that H is one of $F_{13}, \ldots, F_{25}$.

Now, assume that H can be transformed by partial complementations into a graph H' with three components, but it cannot be transformed into a graph with four connected components. Notice that at least one component of H' is a cograph, otherwise $3P_4$ is an induced subgraph of H', but F_1 is a proper induced subgraph of $3P_4$, contradicting that H is a minimal 2-polar obstruction. Having a cograph component, H' can be transformed by a finite sequence of partial complementations into a graph H'' with three connected components where at least one of them, B_3, is a trivial component. Moreover, since H'' is also a minimal 2-polar obstruction, $H'' - B_3$ is 2-polar but it is neither a $(2,1)$- nor a $(1,2)$-polar graph. Therefore, a component B_2 of $H'' - B_3$, is a complete graph while its other component, B_1, is a $(2,1)$-polar graph that is neither a split nor a complete bipartite graph. Without loss of generality we can assume that B_1, B_2 and B_3 are the components of H itself. Denote by m the order of B_2.

Suppose first that $m \geq 2$. Since B_1 is not a split graph, then it contains some of $2K_2, C_5$ or C_4 as an induced subgraph. If $2K_2 \leq B_1$, then H properly contains a copy of F_1, while if $C_5 \leq B_1$, then H must be isomorphic to F_{30}. Otherwise, B_1 contains a copy C of C_4. Observe that if B_1 contains $K_1 + C_4$ as an induced subgraph, then H properly contains a copy of F_{13}, which is impossible. Hence, any vertex in B_1 not in C is adjacent to some vertex of C. Let u be a vertex in B_1 not in C. If u is adjacent to exactly one vertex of C, then $H \cong F_{32}$; if u is adjacent to two adjacent vertices of C, then $H \cong F_{37}$; if u is adjacent to exactly three vertices of C, $H \cong F_7$; and if u is adjacent to all vertices of C, then H properly contains a copy of F_4. Thus, if H is none of the graphs mentioned before, any vertex u in B_1 not in C is adjacent to two antipodal vertices in C. In addition, two vertices adjacent to the same pair of antipodal vertices cannot be adjacent to each other, otherwise H contains F_7 as a proper induced subgraph. Furthermore, any two vertices adjacent to distinct pairs of antipodal vertices in C must be adjacent to each other, or H would contain F_{32} as a proper induced subgraph. It is easy to observe that under such restrictions B_1 is a complete bipartite graph, which is impossible.

Now let us consider the case $m = 1$. We have that B_1 is a connected P_4-extendible graph with at least six vertices, so B_1 is either an X-spider or the join of two smaller P_4-extendible graphs. Suppose first that B_1 is an X-spider and let R be its head. If R contains $2K_2, C_4$ or C_5 as an induced subgraph, then H properly contains F_3, F_4 or F_{28}, respectively, but this is impossible. Then, R is a split graph, which implies that $X \notin \{P_4, F, \overline{F}\}$, or H would be a split graph. We can assume that $S = \overline{P}$. If R contains an induced P_3, then

H properly contains an induced copy of F_{26}, so R must be a cluster. Hence, R is a split graph that is a cluster, so $R = K_a + bK_1$ for some nonnegative integers a and b. Observe that $a \geq 2$ and $b \geq 1$, otherwise H is a 2-polar graph or it contains F_9 as a proper induced subgraph. Then, R contains an induced copy of $\overline{P_3}$, but this implies that H has a proper induced copy of F_3. Hence, B_1 is not an X-spider, so B_1 is the join of two smaller P_4-extendible graphs, T_1 and T_2, and hence $H = T_1 \oplus T_2 + B_2 + B_3$. If the complement of T_i is disconnected for some $i \in \{1, 2\}$, then $\overline{B_1} + \overline{B_2 + B_3}$ has four connected components, a contradiction. Then each T_i has a connected complement, so it is isomorphic to K_1 or it contains P_4 as an induced subgraph. Clearly, at least one of T_1 and T_2 is a nontrivial graph. First assume, without loss of generality, that T_1 is an isolated vertex, then $\overline{B_1} + \overline{B_2 + B_3}$ has three connected components, one of them isomorphic to K_2, and other isomorphic to K_1, so we are in the case $m = 2$. Otherwise, each of T_1 and T_2 contain an induced copy of P_4, so $\overline{B_1} + \overline{B_2 + B_3}$ contains F_1 as a proper induced subgraph, which is impossible.

Finally, assume that H cannot be transformed by partial complementations into a graph with at least three connected components. Notice that H has two connected components and the complement of any of them is connected. Then, by Lemma 4.70, H is the disjoint union of K_1 and an X-spider, but exactly as in the case $m = 1$, it can be proved that this is impossible for a P_4-extendible minimal 2-polar obstruction. □

We summarize the results of this section in the following theorem.

Theorem 4.72. *There are exactly 82 P_4-extendible minimal 2-polar obstructions, corresponding to the graphs $F_1, \ldots, F_{41}$ and their complements.*

4.7 Largest polar subgraphs

In this section, we give algorithms to find maximum order induced subgraphs with some given properties (related to polarity) in P_4-sparse and P_4-extendible graphs using their tree representations. Ekim, Mahadev and de Werra [32] previously obtained similar results for cographs using the cotree. Given a graph G, we denote by $\mathsf{MC}(G), \mathsf{MI}(G)$, and $\mathsf{MS}(G)$ a maximum subset of V_G inducing a complete graph, an empty graph, and a split graph, respectively. We use $\mathsf{MB}(G)$ and $\mathsf{McB}(G)$ to denote a maximum subset of V_G inducing a bipartite and a co-bipartite graph, respectively. We also use $\mathsf{MUC}(G)$ and $\mathsf{MJI}(G)$ to denote maximum subsets of V_G inducing a cluster and a complete multipartite graph, respectively; $\mathsf{MM}(G), \mathsf{McM}(G)$, and $\mathsf{MP}(G)$ stand for maximum subsets of V_G inducing a monopolar, a co-monopolar and a polar subgraph of G, while $\mathsf{MU}(G)$ and $\mathsf{McU}(G)$ are used to denote maximum subsets of V_G inducing a unipolar or a co-unipolar graph, respectively. To simplify the notation, when we are working with preset subgraphs G_i of G, we write $\mathsf{MC_i}$ instead of $\mathsf{MC}(G_i)$ and, if there is no possibility of confusion, we write MC instead of $\mathsf{MC}(G)$; we use an analogous notation for all other maximal subgraphs. Given a family $\mathcal{F}$ of subsets of V_G, a **witness** of $M = \max_{F\in\mathcal{F}}\{|F|\}$ *in* $\mathcal{F}$ is an element F' of $\mathcal{F}$ such that $|F'| = M$.

The following proposition provides recursive characterizations for the aforementioned maximum subgraphs in a disconnected graph.

Proposition 4.73. *Let $G = G_0 + G_1$ be a graph, and let W be a subset of V_G. The following statements hold true.*

1. *W is a maximum clique of G if and only if W is a witness of*
$$\max\{|\mathsf{MC_0}|, |\mathsf{MC_1}|\}.$$
2. *W is a maximum independent set of G if and only if W is a witness of*
$$\max\{|\mathsf{MI_0} \cup \mathsf{MI_1}|\}.$$
3. *W induces a maximum bipartite subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{MB_0} \cup \mathsf{MB_1}|\}.$$
4. *W induces a maximum co-bipartite subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{McB_0}|, |\mathsf{McB_1}|, |\mathsf{MC_0} \cup \mathsf{MC_1}|\}.$$
5. *W induces a maximum split subgraph of G if and only if W is a witness of*
$$\max_{i\in\{0,1\}}\{|\mathsf{MI_i} \cup \mathsf{MS_{1-i}}|\}.$$
6. *W induces a maximum cluster in G if and only if W is a witness of*
$$\max\{|\mathsf{MUC_0} \cup \mathsf{MUC_1}|\}.$$

7. *W induces a maximum complete multipartite subgraph of G if and only if W is a witness of*

$$\max\{|\mathsf{MI}|, |\mathsf{MJI}_0|, |\mathsf{MJI}_1|\}.$$

8. *W induces a maximum monopolar subgraph of G if and only if W is a witness of*

$$\max\{|\mathsf{MM}_0 \cup \mathsf{MM}_1|\}.$$

9. *W induces a maximum co-monopolar subgraph of G if and only if W is a witness of*

$$\max_{i\in\{0,1\}}\{|\mathsf{MS}_i \cup \mathsf{MI}_{1-i}|, |\mathsf{McM}_i|, |\mathsf{MC}_i \cup \mathsf{MJI}_{1-i}|\}.$$

10. *W induces a maximum polar subgraph of G if and only if W is a witness of*

$$\max\{|\mathsf{MM}|, |\mathsf{MP}_0 \cup \mathsf{MUC}_1|, |\mathsf{MP}_1 \cup \mathsf{MUC}_0|\}.$$

11. *W induces a maximum unipolar subgraph of G if and only if W is a witness of*

$$\max_{i\in\{0,1\}}\{|\mathsf{MU}_i \cup \mathsf{MUC}_{1-i}|, |\mathsf{MU}_{1-i} \cup \mathsf{MUC}_i|\}.$$

12. *W induces a maximum co-unipolar subgraph of G if and only if W is a witness of*

$$\max\{|\mathsf{MB}|, |\mathsf{MI}_0 \cup \mathsf{McU}_1|, |\mathsf{MI}_1 \cup \mathsf{McU}_0|\}.$$

Proof. 1. Let W be a maximum clique of G. Clearly, for some $i \in \{0,1\}$, $W \cap V_{G_i} = \varnothing$ and $W \cap V_{G_{1-i}}$ is a clique of G_{1-i}. It follows that W is a maximum clique for either G_0 or G_1 such that $|W| = \max\{|\mathsf{MC}_0|, |\mathsf{MC}_1|\}$.

2. Let W be a maximum independent set of G. Clearly, $W \cap V_{G_i}$ is an independent set of G_i for each $i \in \{0,1\}$. It follows that W is the union of a maximum independent set of G_0 with a maximum independent set of G_1.

3. Let W be a set inducing a maximum bipartite subgraph of G. For each $i \in \{0,1\}$, $G[W \cap V_{G_i}]$ is a bipartite graph, and the disjoint union of two bipartite graphs clearly is a bipartite graph, so the result follows.

4. Let W be a set inducing a maximum co-bipartite subgraph of G, and let (A, B) be a partition of W into two cliques. Clearly, each of A and B is completely contained in one of V_{G_1} or V_{G_2}. If both A and B are contained in V_{G_i} for some $i \in \{0, 1\}$, then W induces a maximum co-bipartite subgraph of G_i. Otherwise, $A \subseteq V_{G_i}$ and $B \subseteq V_{G_{1-i}}$ for some $i \in \{0, 1\}$, so $G[A]$ is a maximum clique in G_i and $G[B]$ is a maximum clique in G_{1-i}. The result easily follows from here.

5. Let W be a set inducing a maximum split subgraph of G, and let (A, B) be a split partition of $G[W]$. Since B is a clique, B is contained in either V_{G_0} or V_{G_1}. Hence, for some $i \in \{0, 1\}$, $W \cap V_{G_i}$ induces a split graph while $W \cap V_{G_{1-i}}$ is an independent set. It follows that $W = V_i \cup V_{1-i}$, where V_i is a subset of V_{G_i} inducing a maximum split graph, V_{1-i} is a maximum independent subset of $V_{G_{1-i}}$, and $|W| = \max_{i \in \{0,1\}}\{|\mathsf{MI}_i \cup \mathsf{MS}_{1-i}|\}$.

6. Let W be a set inducing a maximum cluster of G. Clearly, for each $i \in \{0, 1\}$, $W \cap V_{G_i}$ induces a cluster. It follows that W is the union of a set inducing a maximum cluster of G_0 with a set inducing a maximum cluster of G_1.

7. Let W be a set inducing a maximum complete multipartite subgraph of G. If W is an independent set, it clearly is a maximum independent set of G. Otherwise, $G[W]$ is a connected graph, so W is completely contained in V_{G_i} for some $i \in \{0, 1\}$, and therefore, W induces a maximum complete multipartite subgraph of G_i. In any case we have that $|W| = \max\{|\mathsf{MI}|, |\mathsf{MJI}_0|, |\mathsf{MJI}_1|\}$.

8. Let W be a set inducing a maximum monopolar subgraph of G. Clearly, for any $i \in \{0, 1\}$, $W \cap V_{G_i}$ induces a monopolar graph, so we have that W is the union of a set inducing a maximum monopolar subgraph of G_0 with a set inducing a maximum monopolar subgraph of G_1.

9. Let W be a set inducing a maximum co-monopolar subgraph of G, and let (A, B) be a partition of W such that A induces a complete multipartite graph and B is a clique. Since B is a clique, it is completely contained in either V_{G_0} or V_{G_1}. Now, if A is an independent set, then $W = V_i \cup V_{1-i}$ for some $i \in \{0, 1\}$, where V_i induces a maximum split subgraph of G_i and V_{1-i} induces a maximum independent set of G_{1-i}. Otherwise, if A is not an independent set, it induces a connected graph and is contained

in either V_{G_0} or V_{G_1}; hence, either W induces a maximum co-monopolar subgraph of G_i for some $i \in \{0,1\}$, or there exists $i \in \{0,1\}$ such that W is the union of a maximum clique in G_i and a set inducing a maximum complete multipartite subgraph of G_{1-i}.

10. Let W be a set inducing a maximum polar subgraph of G, and let (A,B) be a polar partition of $G[W]$. If A is an independent set, then $W \cap V_{G_i}$ induces a monopolar subgraph of G_i for each $i \in \{0,1\}$, so W induces a maximum monopolar subgraph of G. Otherwise, if A is not an independent set, $G[A]$ is connected and A is completely contained in V_{G_i} for some $i \in \{0,1\}$; hence, W is the union of a set inducing a maximum polar subgraph of G_i with a set inducing a maximum cluster of G_{1-i}.

11. Let W be a set inducing a maximum unipolar subgraph of G, and let (A,B) be a unipolar partition of $G[W]$. Since A is a clique, it is completely contained in V_{G_i} for some $i \in \{0,1\}$. Thus, $W \cap V_{G_{1-i}}$ induces a cluster and $W \cap V_{G_i}$ induces a unipolar graph, so W is the union of a set inducing a maximum unipolar subgraph of G_i with a set inducing a maximum cluster in G_{1-i}.

12. Let W be a set inducing a maximum co-unipolar subgraph of G, and let (A,B) be a unipolar partition of $\overline{G[W]}$. Since $G[B]$ is a complete multipartite graph, if $B \cap V_{G_1} \neq \varnothing$ and $B \cap V_{G_2} \neq \varnothing$, B is an independent set, so W induces a bipartite graph. Otherwise, $B \cap V_{G_i} = \varnothing$ for some $i \in \{0,1\}$, and we have that $W \cap V_{G_i}$ is an independent set and $W \cap V_{G_{1-i}}$ induces a co-unipolar graph. The result follows easily from here.

□

Since $G \oplus H = \overline{\overline{G} + \overline{H}}$ for any pair of graphs G and H, the following statement is an immediate consequence of the previous proposition, so we omit the details of the proof. Notice that, by Theorem 2.1, Propositions 4.73 and 4.74 can be used together in a mutual recursive algorithm to determine the maximum subgraphs listed in them for any cograph.

Proposition 4.74. *Let $G = G_0 \oplus G_1$ be a graph, and let W be a subset of V_G. The following statements hold true.*

1. *W is a maximum clique of G if and only if W is a witness of*

$$\max\{|\mathsf{MC}_0 \cup \mathsf{MC}_1|\}.$$

2. *W is a maximum independent set of G if and only if W is a witness of*
$$\max\{|\mathsf{MI}_0|, |\mathsf{MI}_1|\}.$$
3. *W induces a maximum bipartite subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{MB}_0|, |\mathsf{MB}_1|, |\mathsf{MI}_0 \cup \mathsf{MI}_1|\}.$$
4. *W induces a maximum co-bipartite subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{McB}_0 \cup \mathsf{McB}_1|\}.$$
5. *W induces a maximum split subgraph of G if and only if, W is a witness of*
$$\max_{i\in\{0,1\}}\{|\mathsf{MC}_\mathsf{i} \cup \mathsf{MS}_{1-\mathsf{i}}|\}.$$
6. *W induces a maximum cluster in G if and only if W is a witness of*
$$\max\{|\mathsf{MC}|, |\mathsf{MUC}_0|, |\mathsf{MUC}_1|\}.$$
7. *W induces a maximum complete multipartite graph of G if and only if W is a witness of*
$$\max\{|\mathsf{MUI}_0 \cup \mathsf{MUI}_1|\}.$$
8. *W induces a maximum monopolar subgraph of G if and only if W is a witness of*
$$\max_{i\in\{0,1\}}\{|\mathsf{MS}_\mathsf{i} \cup \mathsf{MC}_{1-\mathsf{i}}|, |\mathsf{MM}_\mathsf{i}|, |\mathsf{MI}_\mathsf{i} \cup \mathsf{MUC}_{1-\mathsf{i}}|\}.$$
9. *W induces a maximum co-monopolar subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{McM}_0 \cup \mathsf{McM}_1|\}.$$
10. *W induces a maximum polar subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{McM}|, |\mathsf{MP}_0 \cup \mathsf{MJI}_1|, |\mathsf{MP}_1 \cup \mathsf{MJI}_0|\}.$$
11. *W induces a maximum unipolar subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{McB}|, |\mathsf{MU}_1 \cup \mathsf{MC}_0|, |\mathsf{MU}_0 \cup \mathsf{MC}_1|\}.$$

12. *W induces a maximum co-unipolar subgraph of G if and only if W is a witness of*
$$\max\{|\mathsf{McU}_0 \cup \mathsf{MJI}_1|, |\mathsf{McU}_1 \cup \mathsf{MJI}_0|\}.$$

In the next sections, we characterize maximum subgraphs related to polarity properties in both P_4-sparse and P_4-extendible graphs, and we use such characterizations to give linear time algorithms to find the largest subgraphs with such properties in a given graph of the mentioned graph families.

4.7.1 Largest polar subgraph in P_4-sparse graphs

We start by introducing a tree representation for P_4-sparse graphs that is the base for our algorithms. Let $G_1 = (V_1, \varnothing)$ and $G_2 = (V_2, E_2)$ be disjoint graphs such that $V_2 = K \cup R \cup \{s_0\}$, where K is a clique completely adjacent to R, $|K| = |V_1| + 1 \geq 2$ and either $N_{G_2}(s_0) = \{k_0\}$ or $N_{G_2}(s_0) = K \setminus \{k_0\}$ for some vertex k_0 in K. Let f be a bijection from V_1 to $K \setminus \{k_0\}$. We define $G_1 \circledast G_2$ as the graph G with vertex set $V_1 \cup V_2$ such that $G[V_1] \cong G_1$, $G[V_2] \cong G_2$ and, for each $s \in V_1$, either $N_G(s) = \{f(s)\}$, provided $N_{G_2}(s_0) = \{k_0\}$, or $N_G(s) = K \setminus \{f(s)\}$ otherwise.

Proposition 4.75 ([49])**.** *If G is a graph, then G is a spider if and only if there exist graphs G_1 and G_2 such that $G = G_1 \circledast G_2$.*

By Theorem 4.1, for any nontrivial P_4-sparse graph G, either G is disconnected, or $\overline{G}$ is disconnected, or G is an spider. Hence, for each P_4-sparse graph G, a labeled tree T with G as its root and some subgraphs of G as each node can be constructed in the following way. Let H be a node of T. If H is a trivial graph, it is an unlabeled node in T with no children. If H is a disconnected graph, it is labeled as a 0-*node* and its children are its connected components. If $\overline{H}$ is disconnected, H is labeled as a 1-*node* and its children are the complements of the connected components of $\overline{H}$. Finally, if H is a spider, let us say $H = H_1 \circledast H_2$, H is labeled as a 2-*node* and its children are H_1 and H_2. The labeled tree constructed in this way is called the **ps-tree** of G. The ps-tree of a P_4-sparse graph was introduced by Jamison and Olariu in [51], where they proved that such representation can be computed in linear time. In what follow, we assume that if T is the ps-tree of G, and x is a node of T, then $c_1x, c_2x, \ldots$ denote the children of x. We will use G_x to represent the subgraph of G induced by the leaf descendants of x in T.

The next proposition shows that the ps-tree of any P_4-sparse graph of order n has $O(n)$ nodes. Particularly, it implies that we can compute the lists of children for each node of a ps-tree T in linear time and provide each node with such list preserving the linear space representation for T. Additionally, having the lists of children for each node of a ps-tree, we can compute in $O(n)$ time the number of unlabeled children that each node has. This will be helpful later.

Proposition 4.76. *Let G be a P_4-sparse graph, and let T be its ps-tree. If G has order n, then T has order at most $2n-1$ and height at most n.*

Proof. The bound for the order of T follows by an easy induction argument on n by noticing that for any vertex v of G, the ps-tree of $G-v$ has order $n-1$ if the parent of v in T has at least three children, and otherwise it has order $n-2$. The bound for the height of T follows by contradiction using the bound for the order of T and the fact that any internal node of T has at least two children. □

The following proposition implies that, given a ps-tree, we can decide in linear time whether the graphs associated to its nodes labeled 2 are thin spiders or thick spiders.

Proposition 4.77. *Let $G = G_1 \circledast G_2$ be a spider, and let T be its ps-tree. Let w be the only child of G with label 1 in T. If w has two or more unlabeled children, then G is a thick spider. Otherwise, G is a thin spider.*

Proof. Let v be the only leg of G in G_2. Observe that a vertex of G_2 is a universal vertex if and only if it is adjacent to v. Additionally, a vertex of G_2 is universal if and only if it is an unlabeled child of w. Hence, if w has two or more unlabeled children, the degree of v in G is at least two, so G is a thick spider. Otherwise, if w has precisely one unlabeled child, $d_G(v) = 1$ so that G is a thin spider. □

Some of the algorithms we give in this section require us to be able to recognize the spider partition of any spider from its associated ps-tree. Nevertheless, this is not always possible, for instance, if we consider any thin spider whose head complement is disconnected, there will be vertices for which it is impossible to decide from the associated ps-tree if they belong to the body or the head of the spider (see Figure 4.15).

However, it is clear that, given a ps-tree T, there is a unique P_4-sparse graph (up to isomorphism) associated with T, and it results that if we fix a

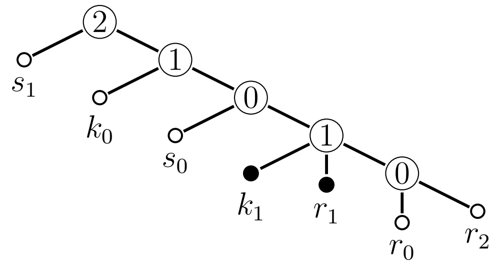


Figure 4.15: The ps-tree associated to the thin spider with 2 legs whose head is isomorphic to P_3. The solid vertices are indistinguishable, but one of them belong to the body of the spider, and the other one belongs to its head.

spider partition for any node labeled 2 in T, the graph is completely determined. Next, we explain how to fix the spider partition for such nodes, and how to save this data maintaining the linear space needed to store T.

Let $G = G_1 \circledast G_2$ be a thin spider, and let T be its associated ps-tree. Let V_1, V_2, K, R, and s_0 be like in the definition of $G_1 \circledast G_2$, and assume that $N_{G_2}(s_0) = \{k_0\}$. Clearly, the root r of T is labeled 2, and it has precisely two children in T, namely a child v labeled 1 such that $G_v \cong G_2$, and a child u, which is unlabeled if $|V_1| = 1$, or it is labeled 0 otherwise; we call v the 1-child of r. In addition, since G is a thin spider, G_2 can be obtained from $G[R \cup K \setminus \{k_0\}]$ by adding first an isolated vertex s_0 and then a universal vertex k_0. Thus, v has precisely two children, namely an unlabeled child (k_0) and a 0-labeled child w, which we will call the 0-child of v. Finally, if $|K| > 2$ or $R \neq \varnothing$, w has exactly two children, one unlabeled (s_0) and one child x labeled 1 ($G[R \cup K \setminus \{k_0\}]$) called the 1-child of w. Otherwise, if $|K| = 2$ and $R = \varnothing$ (in which case $G \cong P_4$), w has exactly two unlabeled children, namely s_0 and the only vertex x in the singleton $K \setminus \{k_0\}$ (see Figure 4.16).

As we mentioned before, if $|K| = 2$ and $R = \varnothing$, then w has precisely two children, s_0 and x, both of them unlabeled. Notice that in G, precisely one child of w is adjacent to the 0-child of r, but we are not able to distinguish from the ps-tree which child of w is such vertex, so we must choose arbitrarily some of them to fix a spider partition (which will completely determine a graph G' isomorphic to G, but possibly different from it, whose ps-tree is T and has the fixed spider partition). Now, if R induces either a disconnected graph or a spider, then x has precisely $|K| - 1$ unlabeled children, all of them elements of K. Nevertheless, if the complement of R is disconnected, then

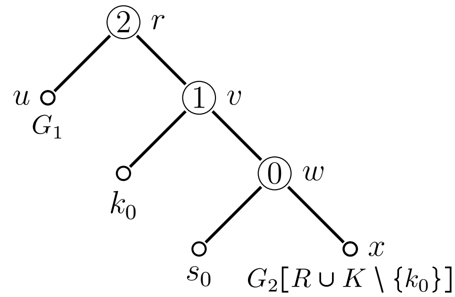


Figure 4.16: General structure of the ps-tree of a thin spider.

there are potentially more than $|K| - 1$ unlabeled children of x, and they will be indistinguishable, so we must choose arbitrarily $|K| - 1$ of them to fix a spider partition.

Now, let $G = G_1 \circledast G_2$ be a thick spider that is not a thin spider, and let T be its associated ps-tree. Let V_1, V_2, K, R, and s_0 be like in the definition of $G_1 \circledast G_2$, and assume that $N_{G_2}(s_0) = K \setminus \{k_0\}$. As before, the root r of T is labeled 2, and it has a child v labeled 1, and a child u labeled 0. Since G is a thick spider, G_v is the join of $G[K] - k_0$ with the disjoint union of the graph obtained from $G[R \cup \{k_0\}]$ by adding an isolated vertex s_0. Thus, v has precisely $|K|$ children, $|K| - 1$ unlabeled children and a 0-labeled child w. Finally, since $|K| \geq 3$ (because G is not a thin spider), w has exactly two children, one unlabeled (s_0) and one child x labeled 1 ($G[R \cup \{k_0\}]$). Similarly to the case of thin spiders, if R induces either a disconnected graph or a spider, then x has precisely one unlabeled child, k_0. Nevertheless, if the complement of R is disconnected, then there are potentially more than one unlabeled children of x, and they will be indistinguishable, so we must chose arbitrarily one unlabeled child to fix a spider partition.

As we have seen, to fix the spider partition of a node labeled 2 it is enough to select some unlabeled descendants of such node that will completely determine the body of the associated spider, as well as the entire spider partition. Moreover, we can simply mark the selected vertices for the body of any node labeled 2 and, since these marked vertices are considered only for the spider partition of their great great grandfather (or great grandfather) in the ps-tree, we can save and process the vertices of the bodies of each node labeled 2 in $O(n)$ space and time, in such a way that any time we need a spider partition of such nodes we use the same fixed partition. It is worth noticing that we could simultaneously mark the vertices of the spider bodies while constructing the ps-tree of a P_4-sparse graph, avoiding the extra processing time and ensuring that we can recover with precision the original graph from the ps-tree.

The following proposition is to thin spiders as Proposition 4.73 is to disconnected graphs. In it, we characterize maximum subgraphs of thin spiders with some properties related to polarity.

Proposition 4.78. *Let $G = (S, K, R)$ be a thin spider and let $f\colon S \to K$ be the bijection such that $N(s) = \{f(s)\}$ for each $s \in S$. The following statements hold for any subset W of V_G. Let H be the subgraph of G induced by R.*

1. *W is a maximum clique of G if and only if W is a witness of*

$$\max_{s \in S}\{|\{s, f(s)\}|, |K \cup \mathsf{MC}(H)|\}.$$

2. *W is a maximum independent set in G if and only if W is a witness of*

$$\max_{s \in S}\{|\{f(s)\} \cup S \setminus \{s\}|, |S \cup \mathsf{MI}(H)|\}.$$

3. *W induces a maximum bipartite subgraph of G if and only if W is a witness of*

$$\max_{k_1, k_2 \in K}\{|S \cup \{k_1, k_2\}|, |\mathsf{MI}(H) \cup S \cup \{k_1\}|, |\mathsf{MB}(H) \cup S|\}.$$

4. *W induces a maximum co-bipartite subgraph of G if and only if W is a witness of*

$$\max_{s_1, s_2 \in S}\{|\{s_1, s_2, f(s_1), f(s_2)\}|, |\mathsf{MC}(H) \cup K \cup \{s_1\}|, |\mathsf{McB}(H) \cup K|\}.$$

5. *W induces a maximum split subgraph of G if and only if W is a witness of*

$$\max\{|S \cup K \cup \mathsf{MS}(H)|\}.$$

6. *W induces a maximum cluster in G if and only if W is a witness of*

$$\max_{\substack{k \in K\\ W' \in \mathcal{X}}}\{|S \cup \{k\}|, |S \cup \mathsf{MUC}(H)|, |\mathsf{MC}(H) \cup W'|\},$$

   where $\mathcal{X}$ is the family of all $|S|$-subsets W' of $S \cup K$ such that $\{s, f(s)\} \nsubseteq W'$ for any $s \in S$.

7. *W induces a maximum complete multipartite subgraph of G if and only W is a witness of*

$$\max_{s_1,s_2 \in S}\{|\{s_1, f(s_1), f(s_2)\}|, |\{f(s_1)\} \cup S \setminus \{s_1\}|,$$
$$|\{s_1, f(s_1)\} \cup \mathsf{MI}(H)|, |S \cup \mathsf{MI}(H)|, |K \cup \mathsf{MJI}(H)|\}.$$

8. *W induces a maximum monopolar subgraph of G if and only W is a witness of*

$$\max_{k \in K}\{|S \cup K \cup \mathsf{MS}(H)|, |S \cup \{k\} \cup \mathsf{MUC}(H)|, |S \cup \mathsf{MM}(H)|\}.$$

9. *W induces a maximum co-monopolar subgraph of G if and only W is a witness of*

$$\max_{s \in S}\{|S \cup K \cup \mathsf{MS}(H)|, |K \cup \{s\} \cup \mathsf{MJI}(H)|, |K \cup \mathsf{McM}(H)|\}.$$

10. *W induces a maximum polar subgraph of G if and only if W is a witness of*

$$\max\{|S \cup K \cup \mathsf{MP}(H)|\}.$$

11. *W induces a maximum unipolar subgraph of G if and only if W is a witness of*

$$\max\{|S \cup K \cup \mathsf{MU}(H)|\}.$$

12. *W induces a maximum co-unipolar subgraph of G if and only if W is a witness of*

$$\max\{|S \cup K \cup \mathsf{McU}(H)|\}.$$

Proof. 1. Let W be a maximum clique of G. If $s \in W \cap S$, then $W \cap S = \{s\}$, $W \cap K \subseteq \{f(s)\}$, and $W \cap R = \varnothing$, so in this case $W = \{s, f(s)\}$. Otherwise, $W \cap S = \varnothing$, and since the union of any clique of H with K is a clique, we have that W is the union of K with a maximum clique of H.

2. Let W be a maximum stable set in G. If $f(s) \in W \cap K$ for some $s \in S$, then $W \cap K = \{f(s)\}$, $s \notin W \cap S$, and $W \cap R = \varnothing$, so in this case $W = \{f(s)\} \cup S \setminus \{s\}$. Otherwise, $W \cap K = \varnothing$, and since the union of any independent set in H with S is an independent set, we have that W is the union of S with a maximum independent set of H.

3. Let W be a set inducing a maximum bipartite subgraph of G. If $|W \cap K| \geq 2$, then $W \cap K = \{k_1, k_2\}$, and $W \cap R = \varnothing$. In addition, since the union of S with any 2-subset of K induces a bipartite graph, we have that W is the union of S with a 2-subset of K. Else, if $W \cap R$ is a non empty independent set, $|W \cap K| \leq 1$. Moreover, since the union of an independent subset of R with $S \cup \{k\}$ induces a bipartite graph for any $k \in K$, in this case we have that W is the union of $S \cup \{k\}$ with a maximum independent set of H. Otherwise, if $W \cap R$ induces a nonempty bipartite graph, then $W \cap K = \varnothing$ and W clearly is the union of S with a maximum subset of R inducing a bipartite graph.

4. Let W be a set inducing a maximum co-bipartite subgraph of G. If $|W \cap S| \geq 2$, then $W \cap S = \{s_1, s_2\}$, $W \cap R = \varnothing$, and $W \cap K \subseteq \{f(s_1), f(s_2)\}$. From here, it is clear that in this case $W = \{s_1, s_2, f(s_1), f(s_2)\}$ for some $s_1, s_2 \in S$. Else, if $W \cap R$ is a non empty clique, $|W \cap S| \leq 1$. Moreover, since the union of a clique in H with $K \cup \{s\}$ induces co-bipartite graph for any $s \in S$, in this case we have that W is the union of $K \cup \{s\}$ with a maximum clique of H. Otherwise, if $W \cap R$ induces a co-bipartite graph that is not a clique, then $W \cap S = \varnothing$ and W clearly is the union of K with the vertex set of a maximum co-bipartite subgraph of H.

5. For any subset W' of R inducing a graph with split partition (A, B), the graph $G[S \cup K \cup W']$ has $(A \cup S, K \cup B)$ as a split partition. Thus, if W is a set inducing a maximum split subgraph of G, $W \cap R$ is a maximum split subgraph of H, $W \setminus R = S \cup K$, and the result follows.

6. Let W be a set inducing a maximum cluster of G. First, assume that $W \cap R = \varnothing$. Since $S \cup \{k\}$ induces a cluster of G for any $k \in K$, we have that $|W| \geq |S| + 1$, so $\{s, f(s)\} \subseteq W$ for some $s \in S$. Moreover, since clusters are P_3-free graphs, if $\{s_1, f(s_1)\} \subseteq W$, $W \cap K = \{f(s_1)\}$. Thus, in this case $W = S \cup \{k\}$ for some $k \in K$. Otherwise, if $W \cap R \neq \varnothing$, $W \cap R$ induces a cluster and $\{s, f(s)\} \nsubseteq W$ for every $s \in S$, so $|W \setminus R| \leq |S|$. It follows that, if $W \cap R$ is a clique, then $W \setminus R$ is an $|S|$-subset of $K \cup S$ such that $\{s, f(s)\} \nsubseteq W \setminus R$ for any $s \in S$, and $W \cap R$ is a maximum clique of H. Otherwise, if $W \cap R$ has at least two connected components, then $W \cap K = \varnothing$, $W \setminus R = S$, and $W \cap R$ induces a maximum cluster in H.

7. Let W be a set inducing a maximum complete multipartite subgraph of

G. Notice that, for any subset R' of R inducing a complete multipartite graph, $G[K \cup R']$ is a complete multipartite graph. In consequence, if $W \cap S = \varnothing$, then W is the union of K with a maximum subset of R inducing a complete multipartite graph. Also observe that, since complete multipartite graphs are $\overline{P_3}$-free graphs, either $W \cap S = \varnothing$ or $W \cap R$ is an independent set.

If $|W \cap K| \geq 3$, then $W \cap S = \varnothing$, so we are done. Now, suppose that $W \cap K = \{f(s_1), f(s_2)\}$ for some $s_1, s_2 \in S$. Observe that in this case $W \cap S$ must be contained in either $\{s_1\}$ or $\{s_2\}$. In addition, some of $W \cap S$ or $W \cap R$ must be an empty set. As in the former case, if $W \cap S = \varnothing$, W is the union of K with a maximum subset of R inducing a complete multipartite graph. Otherwise, if $W \cap R = \varnothing$, thus $W = \{s_1, f(s_1), f(s_2)\}$ for some $s_1, s_2 \in S$.

Now, suppose that $W \cap K = \{f(s_1)\}$ for some $s_1 \in S$. Notice that either $s_1 \notin W$ or $W \cap S \subseteq \{s_1\}$. Also, $W \cap S \neq \varnothing$, otherwise K would be a subset of W, but $|K| \geq 2$ and we are assuming $|W \cap K| = 1$. Thus, if $W \cap S \subseteq \{s_1\}$, then $W \cap S = \{s_1\}$ and $W \cap R$ is a maximum independent subset of R. Else, if $W \cap S \nsubseteq \{s_1\}$, then $s_1 \notin W$ and there is a vertex $s_2 \in W \cap (S \setminus \{s_1\})$. Hence, $W \cap R = \varnothing$ and $W \cap S = S \setminus \{s_1\}$.

Finally, if $W \cap K = \varnothing$, then $W \cap S \neq \varnothing$, and W is the union of S with a maximum independent subset of R.

8. Let W be a set inducing a maximum monopolar subgraph of G, and let $W' = W \cap R$. If W' induces a graph with split partition (A, B), then $G[S \cup K \cup W']$ is a graph with monopolar partition $(A \cup S, B \cup K)$. Thus, if W' induces a split graph, W is the union of $S \cup K$ with a maximum subset of R inducing a split graph.

   Otherwise, if W' induces a cluster that is not a split graph, then W' has a subset inducing a $2K_2$; from here, since $K_2 \oplus 2K_2$ is not a monopolar graph, we have that $|W \cap K| \leq 1$, and it follows that $W = W' \cup S \cup \{k\}$ for some $k \in K$.

   Finally, if W' induces a monopolar graph that is neither a cluster or a split graph, then any monopolar partition (A, B) of $G[W']$ is such that $A \neq \varnothing$ and B has at least one pair of nonadjacent vertices; it follows that $W \cap K = \varnothing$, so W is the union of S with a maximum monopolar subgraph of H.

9. Let W be a set inducing a maximum co-monopolar subgraph of G, and let $W' = W \cap R$. If W' induces a graph with split partition (A, B), then $G[S \cup K \cup W']$ is a graph with co-monopolar partition $(B \cup K, A \cup S)$. Thus, if W' induces a split graph, W is the union of $S \cup K$ with a maximum subset of R inducing a split graph.

   Otherwise, if W' induces a complete multipartite graph that is not a split graph, then W' has a subset inducing a C_4; from here, since $2K_1 + C_4$ is not a co-monopolar graph, we have that $|W \cap S| \leq 1$, and it follows that $W = W' \cup K \cup \{s\}$ for some $s \in S$.

   Finally, if W' induces a co-monopolar graph that is neither a complete multipartite graph or a split graph, then any monopolar partition (A, B) of $\overline{G[W']}$ is such that $A \neq \varnothing$ and B has at least one pair adjacent vertices; it follows that $W \cap S = \varnothing$, so W is the union of K with a maximum co-monopolar subgraph of H.

10. Let W be a set inducing a maximum polar subgraph of G. Notice that the union of $S \cup K$ with any subset of R inducing a graph with polar partition (A, B), is a graph with polar partition $(A \cup S, B \cup K)$. Hence, W is the union of $S \cup K$ with a maximum polar subgraph of H.

11. For any subset R' of R inducing a graph with unipolar partition (A, B), the graph $G[S \cup K \cup R']$ has unipolar partition $(A \cup K, B \cup S)$. Thus, if W is a set inducing a maximum unipolar subgraph of G, $W = S \cup K \cup R'$, for some subset R' of R inducing a maximum unipolar graph.

12. For any subset R' of R inducing a graph with co-unipolar partition (A, B), the graph $G[S \cup K \cup R']$ has co-unipolar partition $(A \cup S, B \cup K)$. Thus, if W is a set inducing a maximum co-unipolar subgraph of G, $W = S \cup K \cup R'$, for some subset R' of R inducing a maximum co-unipolar graph.

□

In the following propositions we strongly use the fact that a thin spider is the complement of a thick spider and vice versa. Notice that by a simple complementary argument, analogous results can be given for computing $\mathsf{MI}(G_x)$, $\mathsf{McB}(G_x)$, $\mathsf{MJI}(G_x)$, $\mathsf{McM}(G_x)$, and $\mathsf{McU}(G_x)$.

Proposition 4.79. *Let G be a P_4-sparse graph, and let T be its ps-tree. For any node x of T the following assertions hold true.*

1. $\mathsf{MC}(G_x)$ *can be found in linear time.*

2. $\mathsf{MB}(G_x)$ *can be found in linear time.*

3. $\mathsf{MS}(G_x)$ *can be found in linear time.*

4. $\mathsf{MUC}(G_x)$ *can be found in linear time.*

5. $\mathsf{MM}(G_x)$ *can be found in linear time.*

6. $\mathsf{MP}(G_x)$ *can be found in linear time.*

7. $\mathsf{MU}(G_x)$ *can be found in linear time.*

Proof. 1. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 1 from Proposition 4.73 that $\mathsf{MC}(G_x)$ is a set realizing $\max_i\{\mathsf{MC}(G_{c_ix})\}$. If x has type 1, we have by part 1 from Proposition 4.74 that $\mathsf{MC}(G_x) = \bigcup_i \mathsf{MC}(G_{c_ix})$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. If G_x is a thin spider, we have from item 1 of Proposition 4.78 that $\mathsf{MC}(G_x)$ is a witness of $\max_{s\in S}\{|\{s, f(s)\}|, |K \cup \mathsf{MC}(G[R])|\}$, where $f(s)$ is the only neighbor of s in K for each $s \in S$. Otherwise, if G_x is a thick spider, we have from item 2 of Proposition 4.78 that $\mathsf{MC}(G_x)$ is a witness of $\max_{s\in S}\{|\{s\} \cup K \setminus \{f(s)\}|, |K \cup \mathsf{MC}(G[R])|\}$, where, for each $s \in S$, $f(s)$ is the only vertex in K that is not a neighbor of s. The result follows since G_x has $O(n)$ descendants.

2. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 3 from Proposition 4.73 that $\mathsf{MB}(G_x) = \bigcup_i \mathsf{MB}(G_{c_ix})$. If x has type 1, we have by part 3 from Proposition 4.74 that $\mathsf{MB}(G_x)$ is a set realizing $\max_{i,j}\{\mathsf{MB}(G_{c_ix}), \mathsf{MI}(G_{c_ix}) \cup \mathsf{MI}(G_{c_jx})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. If G_x is a thin spider, we have from item 3 of Proposition 4.78 that $\mathsf{MB}(G_x)$ is a witness of $\max_{k_1,k_2\in K}\{|S \cup \{k_1, k_2\}|, |\mathsf{MI}(H) \cup S \cup \{k_1\}|, |\mathsf{MB}(G[R]) \cup S|\}$. Otherwise, if G_x is a thick spider, we have from item 4 of Proposition 4.78 that $\mathsf{MB}(G_x)$ is a witness of $\max_{s_1,s_2\in S}\{|\{f(s_1), f(s_2), s_1, s_2\}|, |\mathsf{MI}(G[R]) \cup S \cup \{f(s_1)\}|, |\mathsf{MB}(G[R]) \cup S|\}$, where f is the bijection from S to K such that $N(s) = K \setminus \{f(s)\}$ for each $s \in S$. The result follows since G_x has $O(n)$ descendants.

3. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 5 from Proposition 4.73 that $\mathsf{MS}(G_x)$ is a set realizing $\max_i\{MS(G_{c_ix}) \cup \bigcup_{j\neq i} MI(G_{c_jx})\}$. If x has type 1, we have by part 5 from Proposition 4.74 that $\mathsf{MS}(G_x)$ is a set realizing $\max_i\{MS(G_{c_ix}) \cup \bigcup_{j\neq i} MC(G_{c_jx})\}$. If x has type 2, we have from item 5 of Proposition 4.78 that $\mathsf{MS}(G_x)$ is the union of a maximum subset of R inducing a split graph with $S \cup K$. The result follows since G_x has $O(n)$ descendants.

4. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 6 of Proposition 4.73 that $\mathsf{MUC}(G_x)$ is a set realizing $\bigcup_i \mathsf{MUC}(G_{c_ix})$. If x has type 1, we have by part 6 of Proposition 4.74 that $\mathsf{MUC}(G_x)$ is a set realizing $\max_i\{\mathsf{MC}(G_x), \mathsf{MUC}(G_{c_ix})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. If G_x is a thin spider, we have from item 6 of Proposition 4.78 that $\mathsf{MUC}(G_x)$ is a witness of

$$\max_{\substack{k\in K\\ X\in\mathcal{X}}}\{|S \cup \{k\}|, |S \cup \mathsf{MUC}(G[R])|, |\mathsf{MC}(G[R]) \cup X|\},$$

where $\mathcal{X}$ is the family of all $|S|$-subsets X of $S \cup K$ such that $\{s, f(s)\} \notin X$ for any $s \in S$, being f as usual. Otherwise, if G_x is a thick spider, we have from item 7 of Proposition 4.78 that $\mathsf{MUC}(G_x)$ is a witness of

$$\max_{s_1,s_2\in S}\{|\{f(s_1), s_1, s_2\}|, |\{s_1\} \cup K \setminus \{f(s_1)\}|, |\{s_1, f(s_1)\} \cup \mathsf{MC}(G[R])|,$$
$$|K \cup \mathsf{MC}(G[R])|, |S \cup \mathsf{MUC}(G[R])|\},$$

where f is the bijection from S to K such that $N(s) = K \setminus \{f(s)\}$ for each $s \in S$. The result follows since G_x has $O(n)$ descendants.

5. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 8 of Proposition 4.73 that $\mathsf{MM}(G_x)$ is a set realizing $\bigcup_i \mathsf{MM}(G_{c_ix})$. If x has type 1, we have by part 8 from Proposition 4.74 that $\mathsf{MM}(G_x)$ is a set realizing $\max_{i,j}\{\mathsf{MM}(G_{c_ix}), \mathsf{MS}(G_{c_ix}) \cup \bigcup_{j\neq i} \mathsf{MC}(G_{c_jx}), \mathsf{MI}(G_{c_ix}) \cup \bigcup_{j\neq i} \mathsf{MUC}(G_{c_jx})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. No matter if G_x is a thin or a thick spider, we have from items 8 and 9 of Proposition 4.78 that $\mathsf{MM}(G_x)$ is a witness of

$$\max_{k\in K}\{|S \cup K \cup \mathsf{MS}(G[R])|, |S \cup \{k\} \cup \mathsf{MUC}(G[R])|, |S \cup \mathsf{MM}(G[R])|\}.$$

The result follows since G_x has $O(n)$ descendants.

6. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 10 from Proposition 4.73 that $\mathsf{MP}(G_x)$ is a set realizing $\max_i\{\mathsf{MM}(G_x), \mathsf{MP}(G_{c_ix}) \cup \bigcup_{j\neq i} \mathsf{MUC}(G_{c_jx})\}$. If x has type 1, we have by part 10 from Proposition 4.74 that $\mathsf{MP}(G_x)$ is a set realizing $\max_i\{\mathsf{McM}(G_x), \mathsf{MP}(G_{c_ix}) \cup \bigcup_{j\neq i} \mathsf{MJI}(G_{c_jx})\}$. Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. No matter if G_x is a thin or a thick spider, we have from item 10 of Proposition 4.78 that $\mathsf{MP}(G_x)$ is the union of $S \cup K$ with a maximum subset of R inducing a polar graph. The result follows since G_x has $O(n)$ descendants.

7. The assertion is trivially satisfied if x is a leaf of T. If x has type 0, we have by part 11 from Proposition 4.73 that $\mathsf{MU}(G_x)$ is a set realizing $\max_i\{\mathsf{MU}(G_{c_ix}) \cup \bigcup_{j\neq i} \mathsf{MUC}(G_{c_jx})\}$. If x has type 1, we have by part 11 from Proposition 4.74 that $\mathsf{MU}(G_x)$ is a set realizing

$$\max_i\{\mathsf{McB}(G_x), \mathsf{MU}(G_{c_ix}) \cup \bigcup_{j\neq i} \mathsf{MC}(G_{c_jx})\}.$$

   Finally, let us assume that x has type 2, and let (S, K, R) be the spider partition of G_x. No matter if G_x is a thin or a thick spider, we have from items 11 and 12 of Proposition 4.78 that $\mathsf{MU}(G_x)$ is the union of $S \cup K$ with a maximum subset of R inducing a unipolar graph. The result follows since G_x has $O(n)$ descendants.The result follows since G_x has $O(n)$ descendants.

□

We obtain the main result of this section as a direct consequence of the proposition above.

Theorem 4.80. *The problems of deciding whether a P_4-sparse graph is either a complete multipartite graph, a monopolar graph, a unipolar graph, or a polar graph are linear-time solvable.*

Proof. From Proposition 4.79, $\mathsf{MJI}(G_x), \mathsf{MM}(G_x), \mathsf{MU}(G_x)$ and $\mathsf{MP}(G_x)$ can be found in linear time for any node x of the ps-tree associated to a P_4-sparse graph. Particularly, it can be done for the root of the ps-tree, so the result follows. □

4.7.2 Largest polar subgraph in P_4-extendible graphs

Based on Theorem 4.2, it is possible to represent each P_4-extendible graph G by means of a labeled tree T with root G, which can be constructed in the following way. Let H be a node of T. If H is a trivial graph, it is an unlabeled node of T with no children. If H is a disconnected graph, it is labeled 0 and its children are its connected components. If $\overline{H}$ is disconnected, then H is labeled 1 and its children are the components of $\overline{H}$. If H is an extension graph, it is a node labeled 2 with as many children as the order of H that has additional information encoding the graph induced by its children. Finally, if H is an X-spider with non empty head whose spider partition is (S, K, R), H is a node labeled 3 and has exactly two children: its left child, $H[S \cup K]$, and its right child, $H[R]$. We will call the tree constructed in this way the **parse tree** of G. Hochstättler and Schindler [47] showed that P_4-extendible graphs can be recognized and the parse tree representation can be simultaneously computed in linear time.[1]

The next proposition shows that the parse tree of a P_4-extendible graph of order n has $O(n)$ nodes. It implies that it takes linear time to compute the lists of children for all nodes of the parse tree. Since such lists can be considered additional information for each node, this preserves the condition that the tree uses only linear space. The proof is analogous to that of Proposition 4.76, so we omit it for the sake of length.

Proposition 4.81. *Let G be a P_4-extendible graph, and let T be its associated parse tree. If G has order n, then T has order at most $2n-1$.*

Next, we provide characterizations for maximal substructures associated to polarity on extension graphs and X-spiders. Propositions 4.82 and 4.83 are really easy to check, so their proof is omitted.

Proposition 4.82. *Let G be a 5-cycle, say $G = (v_0, v_1, \ldots, v_4, v_0)$. The following statements hold true for any subset W of V_G, where i is any integer in $\{0, 1, 2, 3, 4\}$ and the sums are considered modulo 5.*

1. *W is a maximum independent set of G if and only if $W = \{v_i, v_{i+2}\}$.*

[1] Actually, the parse tree defined in [47] is slightly different than the one we introduce, due to the fact that they assume by convention that the father of a node labeled 2 is always a node labeled 3, that the root is always a node labeled 1, and that nodes labeled 1 and 3 may have only one child. Nevertheless, with some minor changes, the algorithm in [47] can be adapted to construct our version of the parse tree.

2. *W is a maximum clique of G if and only if $W = \{v_i, v_{i+1}\}$.*

3. *W induces a maximum bipartite graph if and only if $W = V \setminus \{v_i\}$.*

4. *W induces a maximum co-bipartite graph if and only if $W = V \setminus \{v_i\}$.*

5. *W induces a maximum split graph in G if and only if $W = V \setminus \{v_i\}$.*

6. *W induces a maximum cluster in G if and only if $W = \{v_i, v_{i+2}, v_{i+3}\}$.*

7. *W induces a maximum complete multipartite graph in G if and only if $W = \{v_i, v_{i+1}, v_{i+2}\}$.*

8. *W induces a maximum unipolar graph in G if and only if $W = V \setminus \{v_i\}$.*

9. *W induces a maximum co-unipolar graph in G if and only if $W = V \setminus \{v_i\}$.*

Additionally, G is a monopolar graph, hence a co-monopolar and a polar graph.

Proposition 4.83. *Let G be a path of order five, let us say $G = (v_0, v_1, v_2, v_3, v_4)$. The following statements hold true for any subset W of V_G.*

1. *W is a maximum independent set of G if and only if $W = \{v_0, v_2, v_4\}$.*

2. *W is a maximum clique of G if and only if $W = \{v_i, v_{i+1}\}$ for some $i \in \{0, 1, 2, 3\}$.*

3. *W induces a maximum co-bipartite graph if and only if $W = V \setminus \{v_i\}$ for some $i \in \{0, 2, 4\}$.*

4. *W induces a maximum split graph in G if and only if $W = V \setminus \{v_i\}$ for some $i \in \{0, 1, 3, 4\}$.*

5. *W induces a maximum cluster in G if and only if $W = V \setminus \{v_2\}$.*

6. *W induces a maximum complete multipartite graph in G if and only if either $W = \{v_0, v_2, v_4\}$ or $W = \{v_i, v_{i+1}, v_{i+2}\}$ for some $i \in \{0, 1, 2\}$.*

In addition, G is a monopolar, a co-monopolar, a unipolar, a co-unipolar and a polar graph, as well as a bipartite graph.

Since P_4-spiders are special cases of thin (and thick) spiders, the following lemma is a trivial consequence of Proposition 4.78.

Lemma 4.84. *Let $G = (S, K, R)$ be a P_4-spider, and let (a, b, c, d) be the P_4 induced by $S \cup K$. Let W be a subset of V_G, and let $H = G[R]$. The following statements hold true.*

1. *W is a maximum clique of G if and only if W is a witness of*

$$\max\{|\mathsf{MC}(H) \cup K|, |\{a, b\}|, |\{c, d\}|\}.$$

2. *W is a maximum independent set of G if and only if W is a witness of*

$$\max\{|\mathsf{MI}(H) \cup S|, |\{a, c\}|, |\{b, d\}|\}.$$

3. *W induces a maximum bipartite graph if and only if W is a witness of*

$$\max_{k \in K}\{|S \cup K|, |\mathsf{MI}(H) \cup S \cup \{k\}|, |\mathsf{MB}(H) \cup S|\}.$$

4. *W induces a maximum co-bipartite graph if and only if W is a witness of*

$$\max_{s \in S}\{|S \cup K|, |\mathsf{MC}(H) \cup K \cup \{s\}|, |\mathsf{McB}(H) \cup K|\}.$$

5. *W induces a maximum split graph in G if and only if W is the union of $S \cup K$ with a set inducing a maximum split subgraph of H.*

6. *W induces a maximum cluster in G if and only if W is a witness of*

$$\max\{|\mathsf{MUC}(H) \cup S|, |\mathsf{MC}(H) \cup K|, |\mathsf{MC}(H) \cup \{a, c\}|,$$
$$|\mathsf{MC}(H) \cup \{b, d\}|, |\{a, c, d\}|, |\{a, b, d\}|\}.$$

7. *W induces a maximum complete multipartite graph in G if and only if W is a witness of*

$$\max\{|\mathsf{MJI}(H) \cup K|, |\mathsf{MI}(H) \cup S|, |\mathsf{MI}(H) \cup \{a, b\}|,$$
$$|\mathsf{MI}(H) \cup \{c, d\}|, |\{a, b, c\}|, |\{b, c, d\}|\}.$$

8. *W induces a maximum monopolar graph in G if and only if W is a witness of*

$$\max\{|\mathsf{MM}(H) \cup S|, |\mathsf{MS}(H) \cup S \cup K|,$$
$$|\mathsf{MUC}(H) \cup \{a, b, d\}|, |\mathsf{MUC}(H) \cup \{a, c, d\}|\}.$$

9. *W induces a maximum co-monopolar graph in G if and only if W is a witness of*

$$\max\{|\mathsf{McM}(H) \cup K|, |\mathsf{MS}(H) \cup S \cup K|, |\mathsf{MJI}(H) \cup \{a,b,c\}|, |\mathsf{MJI}(H) \cup \{b,c,d\}|\}.$$

10. *W induces a maximum polar graph in G if and only if W is the union of $S \cup K$ with a set inducing a maximum polar subgraph of H.*

11. *W induces a maximum unipolar graph in G if and only if W is the disjoint union of $S \cup K$ with a set inducing a maximum unipolar subgraph of H.*

12. *W induces a maximum co-unipolar graph in G if and only if W is the union of $S \cup K$ with a set inducing a maximum unipolar subgraph of H.*

We continue with propositions characterizing maximal substructures associated to polarity on both P-spiders and F-spiders. As the reader can notice, the proofs are very similar in nature to those of Proposition 4.78.

Lemma 4.85. *Let $G = (S, K, R)$ be a $\overline{P}$-spider, where $S = \{a, a', d\}$, $K = \{b, c\}$ and $\{a, a', b\}$ induces C_3. Let W be a subset of V_G, and let $H = G[R]$. The following statements hold.*

1. *W is a maximum clique of G if and only it is a witness of*

$$\max\{|\{a, a', b\}|, |\mathsf{MC}(H) \cup K|\}.$$

2. *W is a maximum independent set of G if and only if it is a witness of*

$$\max\{|\{a,c\}|, |\{a,d\}|, |\{a',c\}|, |\{a',d\}|, |\{b,d\}|, |\mathsf{MI}(H) \cup \{a,d\}|, |\mathsf{MI}(H) \cup \{a',d\}|\}.$$

3. *W induces a maximum bipartite graph if and only if W is a witness of*

$$\max\{|S \cup K \setminus \{a\}|, |S \cup K \setminus \{a'\}|, |\mathsf{MI}(H) \cup S \cup K \setminus \{b\}|, |\mathsf{MB}(H) \cup S|\}.$$

4. *W induces a maximum co-bipartite graph if and only if W is a witness of*

$$\max\{|S \cup K|, |\mathsf{MC}(H) \cup S \cup (K \setminus \{d\})|, |\mathsf{McB}(H) \cup \{c, b\}|\}.$$

5. *W induces a maximum split graph in G if and only it is a witness of*

$$\begin{aligned}\max\{&|\{a,a',b,d\}|,|\{a,a',b,c\}|,|\{a',b,c,d\}|,\\ &|\{a,b,c,d\}|,|\mathsf{MI}(H)\cup\{a,a',b,d\}|,\\ &|\mathsf{MS}(H)\cup\{a',b,c,d\}|,|\mathsf{MS}(H)\cup\{a,b,c,d\}|\}.\end{aligned}$$

6. *W induces a maximum cluster in G if and only if it is a witness of*

$$\begin{aligned}\max\{&|\{a,a',b,d\}|,|\{a,a',c,d\}|,|\mathsf{MC}(H)\cup\{a,a',c\}|,\\ &|\mathsf{MC}(H)\cup S|,|\mathsf{MUC}(H)\cup S|\}.\end{aligned}$$

7. *W induces a maximum complete multipartite graph in G if and only if it is a witness of*

$$\begin{aligned}\max\{&|\{a,a',b\}|,|\{a,b,c\}|,|\{a',b,c\}|,|\{b,c,d\}|,\\ &|\mathsf{MI}(H)\cup\{a,b\}|,|\mathsf{MI}(H)\cup\{a',b\}|,|\mathsf{MI}(H)\cup\{c,d\}|,\\ &|\mathsf{MI}(H)\cup\{a,d\}|,|\mathsf{MI}(H)\cup\{a',d\}|,|\mathsf{MJI}(H)\cup K|\}.\end{aligned}$$

8. *W induces a maximum monopolar graph in G if and only if it is a witness of*

$$\begin{aligned}\max\{&|\mathsf{MC}(H)\cup S\cup K|,|\mathsf{MUC}(H)\cup\{a,a',c,d\}|,\\ &|\mathsf{MUC}(H)\cup\{a,a',b,d\}|,|\mathsf{MS}(H)\cup\{a,b,c,d\}|,\\ &|\mathsf{MS}(H)\cup\{a',b,c,d\}|,|\mathsf{MS}(H)\cup\{a,a',c,d\}|,\\ &|\mathsf{MM}(H)\cup S|\}.\end{aligned}$$

9. *W induces a maximum co-monopolar graph in G if and only if it is a witness of*

$$\begin{aligned}\max\{&|\mathsf{MI}(H)\cup S\cup K|,|\mathsf{MS}(H)\cup\{a,b,c,d\}|,\\ &|\mathsf{MS}(H)\cup\{a',b,c,d\}|,|\mathsf{MJI}(H)\cup\{a,a',b,c\}|,\\ &|\mathsf{McM}(H)\cup K|\}.\end{aligned}$$

10. *W induces a maximum polar graph in G if and only if W is the union of a maximum subset of R inducing a polar graph with $S\cup K$.*

11. *W induces a maximum unipolar graph in G if and only if W is the union of a maximum subset of R inducing a unipolar graph with $S \cup K$.*

12. *W induces a maximum co-unipolar graph in G if and only if and only if W is a witness of*

$$\max\{\mathsf{MI}(H) \cup S \cup K, \mathsf{MB}(H) \cup S \cup \{b\},\\ \mathsf{McU}(H) \cup K \cup \{a, d\}, \mathsf{McU}(H) \cup K \cup \{a', d\}\}.$$

Proof. 1. Let W be a maximum clique of G. If $W \cap R = \varnothing$, then $W = \{a, a', b\}$. Otherwise, if $W \cap R \neq \varnothing$, then $W \cap S = \varnothing$ and W is the union of K with a maximum clique in H.

2. Let W be a maximum independent set of G. If $W \cap R = \varnothing$, then W is a maximum independent subset of $S \cup K$, i.e.,

$$W \in \{\{a, c\}, \{a, d\}, \{a', c\}, \{a', d\}, \{b, d\}\}.$$

Otherwise, if $W \cap R \neq \varnothing$, then $W \cap K = \varnothing$, and W is the union of a maximum independent set in H with a maximum independent subset of S.

3. Let W be a set inducing a maximum bipartite subgraph of G. Notice that, since $\{a, a', b\}$ induces a triangle, $|W \cap \{a, a', b\}| \leq 2$. It follows from the previous observation that, if $W \cap R = \varnothing$, W is some of $S \cup K \setminus \{a\}$, $S \cup K \setminus \{a'\}$, or $S \cup K \setminus \{b\}$. Else, if $W \cap R$ is a nonempty independent set, $|W \cap K| \leq 1$. Moreover, it is a simple observation that the union of any independent subset of R with $S \cup K \setminus \{b\}$ induces a bipartite graph, but the union of an independent subset of R with any other 4-subset of $S \cup K$ does not induce a bipartite graph. Thus, when $W \cap R$ is a nonempty independent set, W is the union of a maximum independent subset of R with $S \cup K \setminus \{b\}$. Finally, if $W \cap R$ induces a nonempty bipartite graph, then $W \cap K = \varnothing$ and we trivially have that $W \setminus R = S$.

4. Let W be a set inducing a maximum co-bipartite subgraph of G. Since $\overline{P}$ admits a partition in two cliques, if $W \cap R = \varnothing$, $W = S \cup K$. Else, if $W \cap R$ induces a nonempty clique, neither $\{a, d\}$ or $\{a', d\}$ is a subset of W. Moreover, the union of a clique contained in R with $S \cup K \setminus \{d\}$ induces a co-bipartite graph, and the union of a nonempty subset of R

with any other 4-subset of $S \cup K$ does not induce a co-bipartite graph. Thus, if $W \cap R$ induces a nonempty clique, W is the union of a maximum clique in H with $S \cup K \setminus \{d\}$. Otherwise, $W \cap R$ is a co-bipartite graph that is not a clique, and then $W \cap S = \emptyset$, so clearly $W \setminus R = K$; the result follows.

5. Let W be a set inducing a split subgraph of G. If $W \cap R = \emptyset$, W is a maximum subset of $S \cup K$ inducing a split graph, so W is one of $\{a, a', b, d\}$, $\{a, a', b, c\}$, $\{a', b, c, d\}$, or $\{a, b, c, d\}$. Now, assume that $W \cap R \neq \emptyset$. Notice that in this case $\{a, a', c\} \nsubseteq W$, otherwise $\{a, a', c, r\}$ would induce $2K_2$ for any $r \in W \cap R$. Thus, if $W \cap R$ is an independent set, $W \setminus R$ is any of $\{a, b, c, d\}, \{a', b, c, d\}$, or $\{a, a', b, d\}$. Else, if $W \cap R$ induces a split graph that is not empty, $\{a, a'\}$ could not be a subset of W, because $\{a, a', r, r'\}$ would induce $2K_2$ for any adjacent vertices $r, r' \in W \cap R$. Thus, if $W \cap R$ is not an independent set, $W \setminus R$ must be one of $\{a, b, c, d\}$, or $\{a', b, c, d\}$, and the result follows.

6. Let W be a set inducing a maximum cluster of G. If $W \cap R = \emptyset$, W is a maximum subset of $S \cup K$ inducing a cluster, i.e.,

$$W \in \{\{a, a', b, d\}, \{a, a', c, d\}\}.$$

Now, assume that $W \cap R \neq \emptyset$. If $W \cap R$ is a clique, then W cannot have simultaneously c and d, or b and any of a or a'. Thus, in this case W is the union of a maximum subset of R inducing a clique with one of $\{a, a', c\}$ or $\{a, a', d\}$. Otherwise, if $W \cap R$ induces a cluster that is not a complete graph, then $W \cap K = \emptyset$, and W is the union of S with a maximum subset of R inducing a cluster.

7. Let W be a set inducing a maximum complete multipartite subgraph of G. If $W \cap R = \emptyset$, W is a maximum subset of $S \cup K$ inducing a complete multipartite graph, i.e., W is one of $\{a, a', b\}, \{a, b, c\}, \{a', b, c\}$, or $\{b, c, d\}$. Now, assume that $W \cap R \neq \emptyset$. Notice that in this case, $W \cap S$ is completely adjacent to $W \cap K$. In addition, W cannot have both, a and a'. It follows that, if $W \cap R$ is an independent set, then $W \setminus R$ is one of $\{a, b\}, \{a', b\}, \{c, d\}, \{a, d\}, \{a', d\}$, or K. Otherwise, if $W \cap R$ induces a maximum complete multipartite graph of R that is not empty, $W \cap S = \emptyset$ and $W \setminus R = K$, so the result follows.

8. Let W be a set inducing a maximum monopolar subgraph of G. If $W \cap R$ is a clique, then $\{a, a', c\} \cup (W \cap R)$ induces a cluster and, since $\{b, d\}$ is an independent set, we have that $W \setminus R = S \cup K$, so W is the union of a maximum clique of H with $S \cup K$.

   When $W \cap R$ induces a non-complete graph that is simultaneously a split graph and a cluster, since $W \cap R$ is not a clique, $\{a, a', b, c\}$ could not be a subset of W or, for any nonadjacent vertices $r, r' \in W \cap R$, $\{a, a', b, c, r, r'\}$ would induce $K_1 \oplus (K_2 + P_3)$, which is not a monopolar graph. Moreover, some simple verifications show that $W \setminus R$ is any of $\{a, b, c, d\}, \{a', b, c, d\}, \{a, a', c, d\}$, or $\{a, a', b, d\}$.

   Else, if $W \cap R$ induces a cluster that is not a split graph, then it has a subset U inducing $2K_2$, so $\{b, c\} \nsubseteq W$, or $G[\{b, c\} \cup U] \cong K_2 \oplus 2K_2$, which is not a monopolar graph. From here, it is easy to verify that $W \setminus R$ is any of $\{a, a', b, d\}$, or $\{a, a', c, d\}$.

   Now, assume that $W \cap R$ induces a split graph that is not a cluster. Since $K_1 \oplus (K_2 + P_3)$ is not a monopolar graph and $W \cap R$ has a subset W' inducing P_3, we have that $\{a, a', b\}$ is not a subset of W. From here, we can easily check that $W \setminus R$ is any of $\{a, b, c, d\}$, $\{a', b, c, d\}$, or $\{a, a', c, d\}$.

   Finally, suppose that $W \cap R$ induces a monopolar graph that is neither a cluster or a split graph. Suppose that there exists $k \in K \cap W$, and let (A, B) be a monopolar partition of $G[W]$. If $k \in A$, then $W \cap R \subseteq B$, implying that $W \cap R$ induce a cluster, which is not the case. Then, it must be that $k \in B$, but then $W \cap R \cap B$ would be a clique and, since $(W \cap R) \setminus B \subseteq A$, we have that $W \cap R$ would induce a split graph, but we are assuming it does not. Therefore, $K \cap W = \varnothing$, and it follows that W is the union of S with a maximum subset of R inducing a monopolar graph.

9. Let W be a set inducing a maximum co-monopolar subgraph of G. If $W \cap R$ is an independent set, then $(\{a, a', b\}, \{c, d\} \cup (W \cap R))$ is a co-monopolar partition of $G[(W \cap R) \cup S \cup K]$. Hence, if $W \cap R$ is an independent set, $W \setminus R = S \cup K$.

   Notice that, if $W \cap R$ is not an independent set, then $S \nsubseteq W$, otherwise W would have a subset inducing $K_1 + 2K_2$, which is not a co-monopolar graph. In addition, if $W \cap R$ induces a graph with split partition (A, B) and $W \setminus R$ is any of $\{a, b, c, d\}$, or $\{a', b, c, d\}$, then W induces a graph with co-monopolar partition $(A \cup \{a, d\}, B \cup \{b, c\})$ or $(A \cup \{a', d\}, B \cup \{b, c\})$. Also, if $W \cap R$ induces a complete multipartite graph and $W \setminus R = \{a, a', b, c\}$, then $G[W]$ has the co-monopolar partition $(\{a, a'\}, (W \cap R) \cup \{b, c\})$.

If $W \cap R$ induces a split graph that is not a complete multipartite graph, then $\{a, a'\} \nsubseteq W$ or, for any subset $\{r_1, r_2, r_3\}$ of W inducing $\overline{P_3}$, $\{a, a', r_1, r_2, r_3\}$ would induce $K_1 + 2K_2$, which is not a co-monopolar graph. In addition, since $W \cap R$ is a split graph, $\{a, b, c, d\} \cup (W \cap R)$ and $\{a', b, c, d\} \cup (W \cap R)$ induce split graphs, and hence co-monopolar graphs, so in this case W is the union of a maximum subset of R inducing a split graph with one of $\{a, b, c, d\}$ or $\{a', b, c, d\}$.

Else, if $W \cap R$ induces a complete multipartite graph that is not a split graph, then W has a subset W' inducing C_4. Therefore, neither $\{a, d\} \subseteq W$ or $\{a', d\} \subseteq W$, otherwise W would have a subset inducing $C_4 + 2K_1$, which is not a co-monopolar graph. Moreover, the union of any subset of R inducing a complete multipartite graph with $\{a, a', b, c\}$ induces a co-monopolar graph, so in this case W is precisely the union of a maximal subset of R inducing a complete multipartite graph with $\{a, a', b, c\}$.

Finally, assume that $W \cap R$ induces a co-monopolar graph that is neither a split graph or a complete multipartite graph. Aiming for a contradiction, suppose that there exists a vertex $s \in S \cap W$, and let (A, B) be a co-monopolar partition of $G[W]$. If $s \in A$, then $W \cap R \subseteq B$, which is impossible since $G[W \cap R]$ is not a complete multipartite graph. Then, $s \in B$, but in such a case $B \cap W \cap R$ is an independent set, and $(W \cap R) \setminus B \subseteq A$, implying that $W \cap R$ induces a split graph, contradicting our initial assumption. Hence $S \cap W = \varnothing$. Additionally, for any subset W' of R inducing a co-monopolar graph, $W' \cup K$ is also a co-monopolar graph, so in this case W is the union of K with a maximum subset of R inducing a co-monopolar graph.

10. Let W be a set inducing a maximum polar subgraph of G. If (A, B) is a polar partition of $G[W \cap R]$, then $(A \cup K, B \cup S)$ is a polar partition of $G[W]$.

11. Let W be a set inducing a maximum unipolar subgraph of G. If (A, B) is a unipolar partition of $G[W \cap R]$, then $(A \cup K, B \cup S)$ is a polar partition of $G[W]$.

12. Let W be a set inducing a maximum co-unipolar subgraph of G. Notice

that, for any independent subset R' of R, $(\{a,d\} \cup R', \{a'\} \cup K)$ is a co-unipolar partition of $G[S \cup K \cup R']$. Therefore, if $W \cap R$ is an independent subset of R, we have that W is the union of a maximum independent subset of R with $S \cup K$.

Observe that $K_2 + K_3$ is not a co-unipolar graph. Hence, if $W \cap R$ is not an independent set, either $\{a,a'\} \nsubseteq W$ or $c \notin W$. It easily follows from the previous observation that, if $W \cap R$ induces a nonempty graph with bipartition (A,B), then W is the union of a maximum subset of R inducing a bipartite graph with some of $K \cup \{a,d\}$, $K \cup \{a',d\}$, or $S \cup \{b\}$.

Now, assume that $W \cap R$ induces co-unipolar graph that is a nonempty bipartite graph. We claim that, in such case, $\{a,a'\} \nsubseteq W$, and we prove it by means of contradiction. Suppose that $a, a' \in W$, and let (A,B) be a co-unipolar partition of $G[W]$. Since $G[W \cap R]$ is not an empty graph, $W \cap B \neq \varnothing$, and thus, either $a \in A$ and $a' \in B$, or vice versa. However, due to $B \cap \{a,a'\} \neq \varnothing$ we have that $W \cap R \cap B$ is an independent set, but then $W \cap R$ induces a bipartite graph, reaching a contradiction. From here, it is easy to conclude that, in this case, W is the union of a maximum subset of R inducing a co-unipolar graph with some of $K \cup \{a,d\}$ or $K \cup \{a',d\}$.

□

Lemma 4.86. *Let $G = (S,K,R)$ be an F-spider, where $S = \{a,a',d\}$, $K = \{b,c\}$ and $\{a,a',b\}$ induces P_3. Let W be a subset of V_G, and let $H = G[R]$. The following statements hold true.*

1. *W is a maximum clique of G if and only if W is a witness of*

$$\max\{|\{a,b\}|, |\{a',b\}|, |\{b,c\}|, |\{c,d\}|, |\mathsf{MC}(H) \cup K|\}.$$

2. *W is a maximum independent set of G if and only if W is a witness of*

$$\max\{|\{a,a',c\}|, |\mathsf{MI}(H) \cup S|\}.$$

3. *W is a set inducing a maximum bipartite subgraph of G if and only if W is a witness of*

$$\max\{|S \cup K|, |\mathsf{MI}(H) \cup S \cup K \setminus \{b\}|, |\mathsf{MI}(H) \cup S \cup K \setminus \{c\}|, |\mathsf{MB}(H) \cup S|\}.$$

4. *W is a set inducing a maximum co-bipartite subgraph of G if and only if W is a witness of*

$$\max\{|S \cup K \setminus \{a\}|, |S \cup K \setminus \{a'\}|, |\mathsf{MC}(H) \cup K \cup \{a\}|, |\mathsf{MC}(H) \cup K \cup \{a'\}|, |\mathsf{MC}(H) \cup K \cup \{d\}|, |\mathsf{McB}(H) \cup K|\}.$$

5. *W induces a maximum split graph in G if and only if W is the union of a maximum subset of R inducing a split graph with $S \cup K$.*

6. *W induces a maximum cluster of G if and only if W is a witness of*

$$\max\{|\{a, a', c, d\}|, |\mathsf{MC}(H) \cup \{a, a', c\}|, |\mathsf{MUC}(H) \cup S|\}.$$

7. *W induces a maximum complete multipartite graph in G if and only if W is a witness of*

$$\max\{|\{a, a', b, c\}|, |\mathsf{MI}(H) \cup S|, |\mathsf{MI}(H) \cup \{a, a', b\}|, |\mathsf{MIJ}(H) \cup K|\}.$$

8. *W induces a maximum monopolar graph in G if and only if W is a witness of*

$$\max\{|\mathsf{MS}(H) \cup S \cup K|, |\mathsf{MUC}(H) \cup \{a, a', b, d\}|, |\mathsf{MUC}(H) \cup \{a, a', c, d\}|, |\mathsf{MM}(H) \cup S|\}.$$

9. *W induces a maximum co-monopolar graph in G if and only if W is a witness of*

$$\max\{|\mathsf{MS}(H) \cup S \cup K|, |\mathsf{MJI}(H) \cup \{a, b, c\}|, |\mathsf{MJI}(H) \cup \{a', b, c\}|, |\mathsf{MJI}(H) \cup \{b, c, d\}|, |\mathsf{McM}(H) \cup K|\}.$$

10. *W induces a maximum polar graph in G if and only if W is the union of a maximum subset of R inducing a polar graph with $S \cup K$.*

11. *W induces a maximum unipolar graph in G if and only if W is the union of a maximum subset of R inducing a unipolar graph with $S \cup K$.*

12. *W induces a maximum co-unipolar graph in G if and only if W is the union of a maximum subset of R inducing a co-unipolar graph with $S \cup K$.*

Proof. 1. Let W be a maximum clique of G. If $R = \varnothing$, W clearly is one of $\{a, b\}$, $\{a', b\}$, $\{b, c\}$, or $\{c, d\}$. Otherwise, if $R \neq \varnothing$, $R' \cup K$ is a clique, for any clique R' contained in R, so in this case $W \cap R$ is a nonempty clique. It follows that $W \cap S = \varnothing$ and W is the union of K a maximum clique contained in R.

2. Let W be a maximum independent set of G. If $R = \varnothing$, W clearly is one of $\{a, a', c\}$ or S. Otherwise, if $R \neq \varnothing$, $R' \cup S$ is an independent set, for any independent subset R' of R. Thus, if $R \neq \varnothing$, $W \cap R$ is a nonempty independent subset of R, so $W \cap K = \varnothing$. Hence, in this case W is the union of S with a maximum independent subset of R.

3. Let W be a set inducing a maximum bipartite subgraph of G. If $W \cap R = \varnothing$, then clearly $W = S \cup K$. Else, if $W \cap R$ is a non empty independent set, then $|W \cap K| \leq 1$. In addition, for any independent subset R' of R, both $R' \cup S \cup \{b\}$ and $R' \cup S \cup \{c\}$ induce bipartite graphs, so in this case W is the union of a maximum independent set of R with either $S \cup \{b\}$ or $S \cup \{c\}$. Otherwise, $W \cap R$ induces a nonempty bipartite graph and $W \cap K = \varnothing$, where it easily follows that W is the union of S with a maximum bipartite subgraph of H.

4. Let W be a set inducing a maximum co-bipartite subgraph of G. It is an easy observation that the only subsets of $S \cup K$ inducing a maximum co-bipartite graph are $S \cup K \setminus \{a\}$ and $S \cup K \setminus \{a'\}$; hence, if $W \cap R = \varnothing$, W must be one of these sets. Notice that if $W \cap R \neq \varnothing$ then $|W \cap S| \leq 1$. From here, it is easy to observe that if $W \cap R$ is a nonempty clique, then $W \setminus R$ is one of $K \cup \{a\}$, $K \cup \{a'\}$, or $K \cup \{d\}$, so in this case W is the union of one of these sets with a maximum clique of H. Finally, if $W \cap R$ induces a co-bipartite graph that is not a clique, then $W \cap S = \varnothing$ and W clearly is the union of K with a maximum set inducing a co-bipartite subgraph of H.

5. Let W be a set inducing a maximum split subgraph of G. Just notice that, for any subset R' of R inducing a graph with split partition (A, B), $(A \cup S, B \cup K)$ is a split partition of $G[S \cup K \cup R']$.

6. Let W be a set inducing a maximum cluster of G. If $R = \varnothing$, then $W = \{a, a', c, d\}$. Otherwise, the union of S with any subset of R inducing a cluster is also a cluster. Thus, we may assume that $|W \setminus R| \geq 3$.

Moreover, if $W \neq \{a, a', c, d\}$, then $W \cap R \neq \varnothing$ and then, none of $\{a, b\}$, $\{a', b\}$, or $\{c, d\}$, is a subset of W, or W would have a subset inducing P_3. From here, it is a easy to conclude that, if $W \cap R$ is a clique, then $W \setminus R \in \{S, \{a, a', c\}\}$, while, if $W \cap R$ induces a cluster that is not complete graph, then $W \setminus R = S$.

7. Let W be a set inducing a maximum complete multipartite subgraph of G. If $W \cap R = \varnothing$, then W is a maximum subset of $S \cup K$ inducing a complete multipartite graph, so $W = \{a, a', b, c\}$. Otherwise, $W \cap R \neq \varnothing$, and since $G[W]$ is $\overline{P_3}$-free, none of $\{c, a\}$, $\{c, a'\}$, or $\{b, d\}$, could be a subset of W. It follows that, in this case, $|W \setminus R| \leq 3$. Notice that the union S with any independent subset of R is an independent set, so it induces a complete multipartite graph. Hence, if $W \cap R$ is an independent set, $|W \setminus R| = 3$ and a simple verification yields that $W \setminus R$ can be any of S or $\{a, a', b\}$. Finally, if $W \cap R$ induces a complete multipartite graph that is not an empty graph, then $W \cap S = \varnothing$, and $W \setminus R = K$.

8. Let W be a set inducing a maximum monopolar subgraph of G. If $W \cap R$ induces a graph with split partition (A, B), then $(A \cup S, B \cup K)$ is a split partition of $G[S \cup K \cup (W \cap R)]$. Else, if $W \cap R$ induces a cluster that is not a split graph, then $W \cap R$ has a subset inducing $2K_2$, so $K \notin W$, because $K_2 \oplus 2K_2$ is not a monopolar graph. In addition, it is easy to corroborate that for any subset R' of R inducing a cluster, $R \cup \{a, a', b, d\}$ and $R' \cup \{a, a', c, d\}$ induce monopolar graphs. Finally, assume that $W \cap R$ induces a monopolar graph that is neither a split graph or a cluster. Aiming for a contradiction, suppose that there exists a vertex $k \in K \cap W$, and let (A, B) be a monopolar partition of $G[W]$. If $k \in A$, then $W \cap R \subseteq B$, so $W \cap R$ induces a cluster, but we are assuming this is not the case. Thus, $k \in B$, but then, $B \cap W \cap R$ is a clique, and $(W \cap R) \setminus B \subseteq A$, so $W \cap R$ induces a split graph, which is impossible. Therefore, $K \cap W = \varnothing$. Moreover, if R' is a subset of R inducing a graph with monopolar partition (A, B), then $(A \cup S, B)$ is a monopolar partition of $G[R' \cup S]$, where the result follows.

9. Let W be a set inducing a maximum co-monopolar subgraph of G. If a subset R' of R induces a graph with split partition (A, B), then $(B \cup K, A \cup S)$ is a co-monopolar partition of $G[R' \cup S \cup K]$. Thus, if $W \cap R$ induces a split graph, then $W \setminus R = S \cup K$.

Now, if $W \cap R$ induces a complete multipartite graph that is not a split graph, there exists a subset W' of $W \cap R$ inducing a 4-cycle. Hence, since $C_4 + 2K_1$ is not a co-monopolar graph, $|W \cap S| \leq 1$. Moreover, for any subset R' of R inducing a complete multipartite graph and any $s \in S$, $(\{s\}, R' \cup K)$ is a co-monopolar partition of $G[R' \cup K \cup \{s\}]$. Thus, if $W \cap R$ induces a complete multipartite graph that is not a split graph, then $W \setminus R$ is one of $\{a, b, c\}$, $\{a', b, c\}$, $\{b, c, d\}$.

Finally, assume that $W \cap R$ induces a co-monopolar graph that is neither a complete multipartite graph or a split graph. Aiming for a contradiction, suppose that there exists a vertex $s \in S \cap W$, and let (A, B) be a co-monopolar partition of $G[W]$. If $s \in A$, then $W \cap R \cap A = \varnothing$, so $W \cap R$ must induce a complete multipartite graph, which is not the case. Thus, $s \in B$, so $B \cap W \cap R$ is an independent set, because complete multipartite graphs are $\overline{P_3}$-free graphs. But then, $W \cap R$ induces a split graph, which is impossible. Therefore $W \cap S = \varnothing$. In addition, if R' is any subset of R inducing a graph with co-monopolar partition (A, B), then $(A \cup K, B)$ is a co-monopolar partition of $G[R' \cup K]$. Hence, if $W \cap R$ induces a co-monopolar graph that is neither a split graph or a complete multipartite graph, then $W \setminus R = K$.

10. Let W be a set inducing a maximum polar subgraph of G. The result follows since, for any subset R' of R inducing a graph with polar partition (A, B), $(A \cup K, B \cup S)$ is a polar partition of $G[S \cup K \cup R']$.

11. Let W be a set inducing a maximum unipolar subgraph of G. It is enough to notice that, for any subset R' of R inducing a graph with unipolar partition (A, B), $(A \cup K, B \cup S)$ is a unipolar partition of $G[S \cup K \cup R']$.

12. Let W be a set inducing a maximum co-unipolar subgraph of G. The result follows since, for any subset R' of R inducing a graph with co-unipolar partition (A, B), we have that $(A \cup S, B \cup K)$ is a co-unipolar partition of $G[S \cup K \cup R']$.

□

For the proof of the next proposition we strongly use, without explicit mention, that the complements of P-spiders and the complements of F-spiders are, respectively, $\overline{P}$-spiders and $\overline{F}$-spiders. Notice that by a simple complementary argument, analogous results can be given for computing $\mathsf{MI}(G_x)$, $\mathsf{McB}(G_x)$, $\mathsf{MJI}(G_x)$, $\mathsf{McM}(G_x)$, and $\mathsf{McU}(G_x)$.

Proposition 4.87. *Let G be a P_4-extendible graph, and let T be its associated parse tree. For any node x of T the followings assertions are satisfied.*

1. $\mathsf{MC}(G_x)$ *can be computed in linear time.*
2. $\mathsf{MB}(G_x)$ *can be computed in linear time.*
3. $\mathsf{MS}(G_x)$ *can be computed in linear time.*
4. $\mathsf{MUC}(G_x)$ *can be computed in linear time.*
5. $\mathsf{MM}(G_x)$ *can be computed in linear time.*
6. $\mathsf{MP}(G_x)$ *can be computed in linear time.*
7. $\mathsf{MU}(G_x)$ *can be computed in linear time.*

Proof. The assertions trivially hold whenever x is a leaf of T. Also, if x is a node labeled 0 or 1, the proof follows exactly as in Proposition 4.79. Thus, we will assume for the rest of the proof that x has label either 2 or 3. Even in these cases the proof is similar in flavor to Proposition 4.79, but we use Propositions 4.82 and 4.83 and Lemmas 4.84 to 4.86 instead of Proposition 4.78. Hence, we only write the proof for item 6.

6. If x is a node labeled 2, Propositions 4.82 and 4.83 implies that $\mathsf{MP}(G_x) = G_x$. Otherwise, x is a node labeled 3, so G_x is an X-spider. By Lemmas 4.84 to 4.85, if G_x is a graph with X-spider partition (S, K, R), then $\mathsf{MP}(G_x)$ is the union of $S \cup K$ with a maximum subset of R inducing a polar graph. The result follows since G_x has $O(n)$ descendants.

□

The main results of this section are summarized in the next theorem, which is a direct consequence of the proposition above.

Theorem 4.88. *The problems of deciding whether a P_4-extendible graph is either a complete multipartite graph, a monopolar graph, a unipolar graph, or a polar graph are linear-time solvable.*

Proof. From Proposition 4.87, $\mathsf{MJI}(G_x), \mathsf{MM}(G_x), \mathsf{MU}(G_x)$ and $\mathsf{MP}(G_x)$ can be found in linear time for any node x of the parse tree associated to a P_4-extendible graph. Particularly, it can be done for the root of the parse tree, so the result follows. □

4.8 Open problems and conjectures

In this chapter we generalized some results related to hereditary properties in cographs, providing similar results for two superclasses of P_4-free graphs, namely P_4-sparse and P_4-extendible graphs.

Particularly, we showed that any P_4-sparse minimal obstruction for unipolarity, monopolarity, polarity, $(s, 1)$-polarity and 2-polarity, is a cograph. Hannnebauer [45] showed the following interesting result that generalize its analogue for cographs, which was previously proved in [37].

Theorem 4.89 ([45]). *Let H be a P_4-sparse minimal (s, k)-polar obstruction. Then H has at most $(s + 1)(k + 1)$ vertices.*

The observations above make us pose the following question.

Problem 4.90. *Can we establish an $O(sk)$ upper bound for the order of the P_4-extendible minimal (s, k)-polar obstructions?*

It was independently shown in [7] and [45] that any P_4-sparse minimal obstruction for (k, ℓ)-coloring is a cograph too, so we propose the following problems.

Problem 4.91. *Is every P_4-sparse minimal (s, k)-polar obstruction a cograph, for any positive integers s and k?*

Problem 4.92. *Which hereditary properties $\mathcal{P}$ satisfy that every P_4-sparse minimal $\mathcal{P}$-obstruction is a cograph?*

In Section 4.7, we presented linear time algorithms to find largest subgraphs with properties related to polarity on any graph being either P_4-sparse or P_4-extendible. Such algorithms can be easily adapted to give back yes-certificates, so we wonder whether they can be adapted, preserving its time-complexity, to return also no-certificates.

Problem 4.93. *Can we adapt our algorithms to make them linear-time certifying algorithms?*

We also think it is possible to extend our algorithms to wider classes of graphs having a simple enough tree representation. Specifically, we pose the next problem.

Problem 4.94. *Give a linear time algorithm to find maximum monopolar, maximum unipolar, and maximum polar subgraphs on P_4-tidy or extended P_4-laden graphs.*

In the context of matrix partitions was shown that, for any pair of fixed nonnegative integers, s and k, there is only a finite number of minimal (s,k)-polar obstructions [39], so that theoretically there is a polynomial-time brute force algorithm to decide whether a given graph is an (s,k)-polar graph. Moreover, in [38] an explicit polynomial-time algorithm to solve the problem of deciding whether an input graph admits a fixed sparse-dense partition was given. In particular, since both, complete s-partite graphs and k-clusters can be recognized in quadratic time, we have that (s,k)-polar graphs can be recognized in $O(|V|^{4+2\max\{s,k\}})$-time. The aforementioned results make us wonder if it is possible to improve the time complexity of such algorithms by restricting the input graph to some of the graph classes with relatively few induced paths on four vertices.

Problem 4.95. *Given arbitrary fixed nonnegative integers s and k, can we a give linear-time algorithm to find a maximum order (s,k)-polar subgraph of a cograph G?*

We also propose to solve the next natural problem, which is closely related to the previous question.

Problem 4.96. *Give an efficient algorithm to compute the minimum value of $z = s + k$ such that an input cograph G is an (s,k)-polar graph.*

Finally, we think that an approach similar to the one used in Section 4.6 can be helpful to find the complete family of minimal 2-polar obstructions for general graphs, so we pose such problem as a future line of work.

Part III

Polarity on $\mathcal{H}$-split graphs

Chapter 5

Pseudo-split graphs

Split graphs were defined in Section 1.1 as those graphs whose vertex set admits a partition (S, K) where S is an independent set, and K is a clique. The following marvelous characterizations of split graphs were provided by Foldes, Hammer, and Simeone.

Theorem 5.1 ([41, 44]). *Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, where d_i is the degree of vertex v_i. Set $p = \max\{i : d_i \geq i - 1\}$. The following conditions are equivalent:*

1. *G is a split graph;*
2. *G is a $\{2K_2, C_4, C_5\}$-free graph;*
3. *$\sum_{i=1}^{p} d_i = p(p-1) + \sum_{i=p+1}^{n} d_i$.*

Additionally, if G is a split graph, then $(\{v_1, \ldots, v_p\}, \{v_{p+1}, \ldots, v_n\})$ is a split partition of G, $\omega(G) = \chi(G) = p$, and $\alpha(G) = \theta(G) = n - \min\{p, d_p\}$.

Note that, once the degree sequence of a graph G is known, computing the value of p, as well as verifying the condition in item 3, can be done in $O(|V|)$-time, so split graphs are recognizable and a split partition can be found in linear time from their degree sequences.

Maffray and Preissmann introduced in [57] the following generalization of split graphs. Given a fixed graph H, a graph G is said to be H-**split** if V_G admits a partition (C, S, I) such that C is a clique, I is an independent set, either $S = \varnothing$ or $G[S] \cong H$, C is completely adjacent to S, and I is completely nonadjacent to S. A partition (C, S, I) as described above is called an H-**split**

partition of G. Given a family of graphs $\mathcal{H}$, we say that G is **$\mathcal{H}$-split** if it is H-split for some $H \in \mathcal{H}$. The next theorem implies that, for a graph H whose degree sequence is uniquely realizable, the class of H-split graphs is recognizable and an H-split partition can be found in $O(|V|)$-time from their degree sequences.

Theorem 5.2 ([57]). *Let $d_1^* \geq \cdots \geq d_h^*$ be a realizable degree sequence and let $\mathcal{H}$ be the class of all realizations of this sequence. Let G be any graph with n vertices and degree sequence $d_1 \geq \cdots \geq d_n$. Set $q = \max\{i : d_i \geq i - 1 + h\} \cup \{0\}$. Then, G is an $\mathcal{H}$-split graph if and only if G is split or*

$$\sum_{i=1}^{q} d_i = q(q-1) + qh + \sum_{j=q+h+1}^{n} d_i$$

and $d_{q+i} = q + d_i^*$ for each $i \in \{1, \dots, h\}$. Additionally, if the condition on the degrees holds, then the sets $C = \{v_1, \dots, v_q\}$, $S = \{v_{q+1}, \dots, v_{q+h}\}$ and $I = \{v_{q+h+1}, \dots, v_n\}$ conform an $\mathcal{H}$-partition of G, the subgraph induced by S being isomorphic to some graph $H \in \mathcal{H}$. Moreover, if d^* is a uniquely realizable degree sequence, then $\omega(G) = q + \omega(H)$, $\chi(G) = q + \chi(H)$, $\alpha(G) = \alpha(H) + n - q - h$ and $\theta(G) = \theta(H) + n - q - h$.

Notice that H-split graphs conform a hereditary class of graphs if and only if either H is a split graph, in which case H-split graphs coincide with split graphs, or H is one of the three minimal split obstructions mentioned in Theorem 5.1, i.e., if $H \in \{2K_2, C_4, C_5\}$. Additionally, the class of H-split graphs is self-complementary if and only if H is. From the above observations, it is not strange that the most studied H-split graphs are the C_5-split graphs, which, as we will notice in Theorem 5.4, are the **pseudo-split graphs** introduced in Section 2.1. Naturally, a C_5-split partition of a graph is called a **pseudo-split partition**.

The following remark will be frequently used without any explicit mention.

Remark 5.3. *Let H be some of $2K_2, C_4$, or C_5, and let $G = (C, S, I)$ be an H-split graph. Then, the only induced copy of H in G is $G[S]$ and, if $S \neq \varnothing$, the H-split partition of G is unique.*

If $H \in \{2K_2, C_4, C_5\}$ and $G = (C, S, I)$ is an H-split graph with $S \neq \varnothing$, we say that G is an **strict H-split** graph. Observe that a pseudo-split graph $G = (C, S, I)$ is perfect if and only if $S = \varnothing$, reason why strict pseudo-split graphs are usually called **imperfect**.

Additionally to the characterization of pseudo-split graphs by their degree sequences provided by Theorem 5.2, Maffray and Preissmann also gave in [57] the complete list of minimal pseudo-split obstructions, namely $\{2K_2, C_4\}$. The following proposition summarize such characterizations to facilitate future references.

Theorem 5.4 ([57]). *Let G be a graph of order at least five with vertex set $\{v_1, v_2, \dots, v_n\}$ and degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, where d_i is the degree of vertex v_i. Set $q = \max\{i : d_i \geq i + 4\} \cup \{0\}$. The following conditions are equivalent:*

1. *G is an imperfect pseudo-split graph;*
2. *G is a $\{2K_2, C_4\}$-free graph that has an induced C_5;*
3. *$\sum_{i=1}^{q} d_i = q(q+4) + \sum_{i=q+6}^{n} d_i$, and $d_j = q + 2$ whenever $q + 1 \leq j \leq q + 5$.*

Additionally, if G is an imperfect pseudo-split graph, then

$$(\{v_1, v_2, \dots, v_q\}, \{v_{q+1}, v_{q+2}, v_{q+3}, v_{q+4}, v_{q+5}\}, \{v_{q+6}, v_{q+7}, \dots, v_n\})$$

is the pseudo-split partition of G, $\omega(G) = q+2$, $\chi(G) = q+3$, $\alpha(G) = n-q-3$ and $\theta(G) = n - q - 2$.

5.1 Polarity on pseudo-split graphs

As we have observed before, split graphs are precisely the 1-polar graphs, so split graphs trivially are polar, monopolar, unipolar, and (s,k)-polar for any positive integers s and k. In the next sections we study polarity on pseudo-split graphs. As our main results we give complete lists of pseudo-split minimal (s,k)-polar obstructions for the cases $\min\{s,k\} \leq 2$, $s = \infty$, and $k = \infty$, we prove tight upper bounds for the order of pseudo-split minimal (s,k)-polar obstruction, and provide a $O(|V|)$-time algorithm to decide whether a pseudo-split graph is (s,k)-polar from its degree sequence.

5.1.1 Algorithms for polarity on pseudo-split graphs

The next observation is basic to obtain $O(|V|)$-time recognition algorithms for (s,k)-polarity on pseudo-split graphs; it follows directly from Theorem 5.4.

Remark 5.5. *Let G be an imperfect pseudo-split graph with pseudo-split partition (C, S, I). A vertex u of G has degree $|C| + 4$ if and only if u is a vertex in C that is completely nonadjacent to I. A vertex v of G has degree $|C|$ if and only if v is a vertex in I that is completely adjacent to C.*

In the following theorem we give a necessary and sufficient condition for a pseudo-split graph to be (s, ∞)-polar. Notice that such a condition can be verified in $O(|V|)$-time from the degree sequence of a graph. Additionally, since the class of pseudo-split graphs is self-complementary, and a graph is (s, ∞)-polar if and only if its complement is (∞, s)-polar, we have that by a simple argument of complements an analogous characterization can be given for (∞, k)-polarity on pseudo-split graphs.

Theorem 5.6. *Let s be a nonnegative integer, and let G be an imperfect pseudo-split graph with pseudo-split partition (C, S, I). Then, G is an (s, ∞)-polar graph if and only either $s > |C|$, or $|C| \geq s \geq 2$ and there are at least $|C| - s + 2$ vertices of G with degree exactly $|C| + 4$.*

Proof. Let us denote $|C|$ by c. Suppose that G is an (s, ∞)-polar graph, with polar partition (A, B), such that $s \leq c$. Observe that, if the restriction of (A, B) to S is a $(1, 2)$-polar partition, then $C \cap B = \varnothing$, or $G[B]$ has P_3 as an induced subgraph, but then $C \subseteq A$, which is impossible since $G[A]$ would have K_{c+1} as an induced subgraph and $s \leq c$. Thus, $G[S]$ is covered by a $(2, 1)$-polar partition, so $s \geq 2$. Notice that $I \subseteq B$, otherwise $\overline{P_3}$ would be an induced subgraph of $G[A]$, which cannot occur.

Then, since A induces a complete s-partite graph, at most $s - 2$ vertices of C belong to A. It implies that there is a subset C' of $C \cap B$ with at least $c - s + 2$ vertices. Moreover, if there exist adjacent vertices $c \in C'$ and $i \in I$, then $G[B]$ would have P_3 as an induced subgraph, which is impossible, so C' is completely nonadjacent to I. Hence, by Remark 5.5, G has at least $c - s + 2$ vertices of degree $c + 4$.

For the converse implication, let A and B be a maximum independent set and a maximum clique in $G[S]$, respectively. If $s > c$, then $(C \cup A, I \cup S \setminus A)$ is an (s, ∞)-polar partition of G. Otherwise, we have that $c \geq s$ and there are at least $c - s + 2$ vertices of G with degree exactly $c + 4$. But then, if C' is the subset of C consisting of the vertices of degree $c + 4$, we have by Remark 5.5 that $((S \setminus B) \cup (C \setminus C'), I \cup C' \cup S \setminus B)$ is an (s, ∞)-polar partition of G. □

Now, we present a necessary and sufficient condition for a pseudo-split graph to be (s, k)-polar. Once again, this condition can be verified in $O(|V|)$-time from the degree sequence of a graph, so it implies that (s,k)-polarity can be efficiently decided on pseudo-split graphs.

Theorem 5.7. *Let G be an imperfect pseudo-split graph with pseudo-split partition (C,S,I), and let c and i be the cardinalities of C and I, respectively. Let M_C be the number of vertices of G whose degree is exactly $c+4$, and M_I be the number of vertices of G whose degree is exactly c. Let s and k be nonnegative integers such that $s+k \geq 1$. Then, G is an (s,k)-polar graph if and only if either*

1. *$k \geq i+1$ and $s \geq c - M_C + 2$, or*

2. *$s \geq c+1$ and $k \geq i - M_I + 2$.*

Proof. First, let us suppose that G admits an (s,k)-polar partition (A,B). There are two possible cases, either $G[S]$ inherits a $(2,1)$-polar partition from (A,B), or it inherits a $(1,2)$-polar partition (see Figure 5.1). We will show that in the first case, $k \geq i+1$ and $s \geq c - M_C - 2$, while in the latter case $s \geq c+1$ and $k \geq i - M_I + 2$.

Thus, suppose that $G[S]$ inherits a $(2,1)$-polar partition from (A,B). Note that in such a case $I \subseteq B$, otherwise $G[A]$ would have $\overline{P_3}$ as an induced subgraph, which is impossible. Moreover, since $G[B]$ is a P_3-free graph, we have that every vertex $v \in C \cap B$ is completely nonadjacent to I, and then, by Remark 5.5, $|C \cap B| \leq M_C$. Thus, it occurs that

$$|C \cap A| = |C| - |C \cap B| \geq c - M_C,$$

where we conclude that $s \geq c - M_C + 2$. Furthermore, in this case $G[B]$ is a cluster with exactly $I + 1$ components, so $k \geq i+1$. Hence, we have proved that, if $G[S]$ inherits a $(2,1)$-polar partition from (A,B), then $k \geq i+1$ and $s \geq c - M_C + 2$. It can be proved in a similar way that $s \geq c+1$ and $k \geq i - M_I + 2$ whenever $G[S]$ inherits a $(1,2)$-polar partition from (A,B).

Conversely, let us assume that $k \geq i+1$ and $s \geq c - M_C + 2$. By definition of M_C and Remark 5.5, there exists a subset C' of C of cardinality M_C that is completely nonadjacent to I. Let B_1 be a set of two adjacent vertices of $G[S]$, and let $A_1 = S \setminus B_1$. Then, we have that $(A_1 \cup C \setminus C', B_1 \cup I \cup C')$ is a $(c - M_C + 2, i+1)$-polar partition of G, so G is an (s,k)-polar graph, as we had to prove. The result follows analogously if we assume that $s \geq c+1$ and $k \geq i - M_I + 2$, only taking a $(1,2)$-polar partition (A_1, B_1) of $G[S]$ instead of a $(2,1)$-polar partition. □

Now, we present some results about pseudo-split minimal (s,k)-polar obstructions, which include complete lists of minimal obstructions for some values of s and k, upper and lower bounds for any values of s and k, as well as a program that we designed to compute pseudo-split minimal (s,k)-polar obstructions for small values of s and k.

5.1.2 Pseudo-split minimal (s,k)-polar obstructions

If G is a graph with pseudo-split partition (C,S,I), and u and v are two adjacent vertices in $G[S]$, then $(C \cup S \setminus \{u,v\}, I \cup \{u,v\})$ is a polar partition of G. Hence, pseudo-split graphs are polar. In addition, since split graphs are precisely the 1-polar graphs, for every pair of positive integers s and k, any pseudo-split minimal (s,k)-polar obstruction is necessarily imperfect. Moreover, since split graphs are monopolar and unipolar, every pseudo-split minimal monopolar (unipolar) obstruction is also imperfect. Considering above observations, it seems natural to ask about the polar partitions of C_5.

Figure 5.1: The only two polar partitions of a 5-cycle. Shaded vertices induce complete multipartite graphs, while white vertices induce clusters.

Notice that a 5-cycle admits only two essentially different polar partitions, which are depicted in Figure 5.1. Observe that, if an imperfect pseudo-split graph $G = (C,S,I)$ has a polar partition (A,B), then (A,B) must inherit either a $(1,2)$- or a $(2,1)$-polar partition to $G[S]$. In the first case, since $G[B]$ is a P_3-free graph and C is completely adjacent to S, we have that $C \cap B$ must be an empty set, so $C \subseteq A$. Analogously, when $G[S]$ inherits a $(2,1)$-polar partition from (A,B), we have that $I \subseteq B$, because I is completely nonadjacent to S and A induces a $\overline{P_3}$-free graph. These observations are going to be used without any explicit mention in most of the proofs of this section.

Monopolar an unipolar pseudo-split graphs admit a very simple characterization by forbidden induced subgraphs, which we summarize in the following proposition.

Theorem 5.8. *Let G be a pseudo-split graph. Then,* *1. G is a minimal monopolar obstruction if and only if $G \cong K_1 \oplus C_5$, and*

2. G is a minimal unipolar obstruction if and only if $G \cong C_5$.

In consequence, the problems of deciding whether a pseudo-split graph is monopolar or unipolar are solvable in $O(|V|)$-time from its degree sequence.

Proof. It is a routine job to show that $K_1 \oplus C_5$ is a minimal monopolar obstruction. Moreover, if G has pseudo-split partition (C, S, I), and G does not have $K_1 \oplus C_5$ as an induced subgraph, then either G is a split graph, or G is an imperfect pseudo-split graph with $C = \varnothing$. In the first case, G trivially is a monopolar graph, while in the second case G is isomorphic to $nK_1 + C_5$ for some nonnegative integer n, and therefore, it is a monopolar graph. The proof of item 2 is similar and even simpler.

By item 1, a pseudo-split graph $G = (C, S, I)$ is monopolar if and only if either $S = \varnothing$ or $C = \varnothing$. Thus, it follows from Theorems 5.1 and 5.4 that deciding whether a pseudo-split graph is monopolar can be done in $O(|V|)$-time from its degree sequence. Analogously, by item 2, a pseudo-split graph is unipolar if and only if it is split, so in this case the result follows from Theorem 5.1. □

Minimal (s, k)-polar obstructions on general graphs are known only for the cases $\min\{s, k\} = 0$, and $s = k = 1$, which correspond to clusters, complete multipartite graphs, and split graphs. In the following proposition we give complete lists of pseudo-split minimal (s, k)-polar obstructions for the case $s \in \{1, 2\}$, which can be extrapolated to case $k \in \{1, 2\}$ by simple arguments of complements. Before presenting such results, we introduce notation for some particular graphs.

For each positive integer s, let us denote by G_s^0 the imperfect pseudo-split graph whose (C, S, I)-partition satisfies that $|C| = s$, $I = 1$, and C is completely adjacent to I. We will also use G_s^1 to denote the graph obtained from G_s^0 by deleting one edge incident with the only vertex of I. Notice that, by Theorem 5.7, for any integers $s, k \geq 2$, the graphs G_s^0 and G_s^1 are minimal (s, k)-polar obstructions.

For positive integers s and k, with $k \geq s$, let $H_s^k = (C, S, I)$ be the imperfect pseudo-split graph such that $|C| = s - 1$, $|I| = k - 1$ and, for an injection $f\colon C \to I$, a vertex $v \in C$ is adjacent to a vertex $u \in I$ if and only if $u = f(v)$. It follows from Theorem 5.7 that H_s^k is a minimal (s, k)-polar obstruction provided $k \geq s \geq 2$.

Theorem 5.9. *Let k be an integer, $k \geq 2$, and let G be a pseudo-split graph. Then,*

1. *G is a minimal $(1,k)$-polar obstruction if and only if $G \cong K_1 \oplus C_5$.*
2. *G is a minimal $(2,k)$-polar obstruction if and only if G is isomorphic to some of $G_2^0, G_2^1, \overline{G_k^0}$ or H_1^k.*

Proof. It is a routine to verify that $K_1 \oplus C_5$ is a minimal $(1,k)$-polar obstruction. In addition, G is $K_1 \oplus C_5$-free if and only if $S = \varnothing$ or $C = \varnothing$, but in both cases G is a $(1,2)$-polar graph, hence a $(1,k)$-polar graph.

Previously, we observed that the graphs G_2^0, G_2^1 and H_2^k are all of them $(2,k)$-polar obstructions. We also observed that G_k^0 is a minimal $(k,2)$-polar obstruction, so $\overline{G_k^0}$ is a minimal $(2,k)$-polar obstruction.

Now, to reach a contradiction, let us assume that G is a minimal (s,k)-polar obstruction different to $G_2^0, G_2^1, \overline{G_k^0}$ and H_1^k. Let (C,S,I) be the pseudo-split partition of G, and let us denote by c and i the cardinalities of C and I, respectively. Notice that G is imperfect, otherwise it would be a 1-polar graph, and hence a $(2,k)$-polar graph. Also observe that, if either $i = 0$, or both $c \leq 1$ and $i \leq k-1$, then G would admit a $(2,k)$-polar partition, which is impossible. From the previous observation we have that either $c \geq 2$ and $i \geq 1$, or $c \leq 1$ and $i \geq k$.

Suppose that $c \geq 2$ and $i \geq 1$. Since G is a $\{G_2^0, G_2^1\}$-free graph, we have that C is completely nonadjacent to I. Notice that $i \geq k$, otherwise G would be a $(2,k)$-polar graph. But then, G has $\overline{G_k^0}$ as an induced subgraph, which is impossible. Thus, it must be the case that $c \leq 1$ and $i \geq k$. Since G is not a $(2,k)$-polar graph, $c \geq 1$, so $c = 1$. Let v be the only vertex in C. If $|N(v) \cap I| \leq k-2$, then G is a $(2,k)$-polar graph, which is not possible, so that $|N(v) \cap I| \geq k-1$, but then G contains an induced subgraph isomorphic to either $\overline{G_k^0}$ or H_1^k, contradicting that G is not isomorphic to these graphs. The contradiction arose from supposing the existence of a pseudo-split minimal $(2,k)$-polar obstruction different to $G_2^0, G_2^1, \overline{G_k^0}$ and H_1^k, so it does not exist. □

As we will notice, the following remark on general graphs is a key ingredient to prove things about (s,∞)-polar graphs.

Remark 5.10. *For any nonnegative integer s, a graph G is a minimal (s,∞)-polar obstruction if and only if there is a nonnegative integer k_0 such that, for any integer $k \geq k_0$, G is a minimal (s,k)-polar obstruction.*

Since monopolar graphs are by definition the $(1, \infty)$-polar graphs, item 1 of Theorem 5.8 can be deduced as a consequence of the previous observation and item 1 of Theorem 5.9. The following corollary also follows from Remark 5.10 and Theorem 5.9.

Corollary 5.11. *There are only two pseudo-split minimal $(2, \infty)$-polar obstructions, namely G_2^0 and G_2^1.*

It seems that there is not an easy way to describe the complete lists of pseudo-split minimal (s, k)-polar obstructions when s and k are arbitrary nonnegative integers, but, as we observed at the start of Section 2.3, we known there is just a finite number of them, so it becomes natural to ask about upper bounds for their order. In the following propositions we use Theorem 5.7 to give tight upper bounds for the order of pseudo-split minimal (s, k)- and (s, ∞)-polar obstructions. We start with the following technical observation.

Lemma 5.12. *Let $G = (C, S, I)$ be an imperfect pseudo-split graph, and let c and i be the cardinalities of C and I, respectively. Let s and k be nonnegative integers such that $s + k \geq 1$. The following assertions hold true.*

1. *If $c > s$, then G is not a minimal (s, k)-polar obstruction.*
2. *If $i > k$, then G is not a minimal (s, k)-polar obstruction.*

Proof. We only prove item 1 because the proof of item 2 is analogous. Notice that, if $c > s$ and $i \geq k$, it follows from Theorem 5.7 that, for every vertex $v \in C$, $G - v$ is not an (s, k)-polar graph, which clearly implies that G is not a minimal (s, k)-polar obstruction. Thus, we can assume that $i < k$.

Aiming for a contradiction, let us assume that G is a minimal (s, k)-polar obstruction. Observe that for every vertex v of C, $G - v$ has pseudo-split partition $(C \setminus \{v\}, S, I)$, and $|C \setminus \{v\}| = c - 1 \geq s$. Since G is a minimal (s, k)-polar obstruction, we have that, for every vertex v of G, $G - v$ admits an (s, k)-polar partition. That is true in particular if $v \in C$. Then, we have from Remark 5.5, Theorem 5.7, and our previous observations that, for any vertex v of C, there are at least $c - s + 1$ vertices of $C \setminus \{v\}$ that are completely nonadjacent to I; let C'_v be the set of these vertices.

Notice that no vertex v of C is completely nonadjacent to I, otherwise $C'_v \cup \{v\}$ would be a subset of C of cardinality at least $c - s + 2$ that is completely nonadjacent to I, but then, by Remark 5.5 and Theorem 5.7, G would be an

(s,k)-polar graph, and we are assuming it is not. Thus, we conclude that each vertex of C is adjacent to at least one vertex of I.

Here is the desired contradiction. Let H be a graph obtained from G by removing any $c-s$ vertices of C. Thus, H is a proper induced subgraph of G with a pseudo-split partition (C^*, S, I) such that any vertex of C^* is adjacent to at least one vertex of I. But then, we have from Theorem 5.7 that H is not an (s,k)-polar graph, contradicting the minimality of G. □

By itself, Lemma 5.12 implies that, for any nonnegative integers s and k, a pseudo-split minimal (s,k)-polar obstruction has order at most $s+k+5$. Nevertheless, as we can corroborate in Theorem 5.9 and the observations that precede it, if $\min\{s,k\} \leq 2$, each minimal (s,k)-polar obstruction has order strictly lower than $s+k+5$. In Lemma 5.13 and Theorem 5.14 we will prove that this is true for general values of s and k, and not only when $\min\{s,k\} \leq 2$.

Lemma 5.13. *Let s and k be integers, $s,k \geq 2$, and let $G = (C,S,I)$ be a pseudo-split minimal (s,k)-polar obstruction. Then, G is imperfect, $|C| \leq s$, $|I| \leq k$ and $|C| + |I| \leq s+k-1$.*

Consequently, any pseudo-split minimal (s,k)-polar obstruction has order at most $s+k+4$, and this bound is tight when $\min\{s,k\} = 2$.

Proof. As we noticed at the beginning of this chapter, split graphs are 1-polar, hence (s,k)-polar, so G is an imperfect pseudo-split graph. Let $c = |C|$ and $i = |I|$. Observe that Lemma 5.12 implies that $s \geq c$ and $k \geq i$, so $|V_G| = |C| + |I| + |S| \leq s+k+5$ and this bound is attained if and only if $c = s$ and $i = k$.

Aiming for a contradiction, assume that $c = s$ and $i = k$, so G has order $s+k+5$. Let $v \in C$, and let us use C' to denote $C \setminus \{v\}$. Let (A,B) be an (s,k)-polar partition of $G-v$. Observe that $G[S]$ inherit a $(1,2)$-polar partition from (A,B), otherwise $G[S]$ would inherit a $(2,1)$-polar partition, but then $I \subseteq B$ implying that B has an independent subset of size $k+1$, which is impossible. Moreover, since $G[B]$ is $\{(k+1)K_1, P_3\}$-free, we have that $C' \subseteq A$ and there is at least one vertex u of I in the part A. Notice that u is completely adjacent to C', because $G[A]$ does not have induced copies of $\overline{P_3}$.

Here we have the desired contradiction, because $G[C \cup S \cup \{u\}]$ is isomorphic to either G_s^0 or G_s^1, depending on whether u is adjacent or not to v, but then G has an (s,k)-polar obstruction as a proper induced subgraph, contradicting the minimality of G. The contradiction arose from assuming that $|V_G| > s+k+4$, so it is not the case. Notice that G_s^0 and G_s^1 attain the bound when $s = 2$, so the bound is tight. □

Theorem 5.14. *Let s and k be integers, $s, k \geq 3$. Then, any pseudo-split minimal (s,k)-polar obstruction has order at most $s + k + 3$, and the bound is tight.*

Proof. Let G be an (s,k)-polar obstruction with pseudo-split partition (C, S, I), and let c and i be the cardinalities of C and I, respectively. By Lemma 5.13, G is an imperfect pseudo-split graph with $c \leq s$, $i \leq k$ and, either $c \leq s-1$ or $i \leq k-1$. Notice that, if $c < s-1$ or $i < k-1$, then $|V_G| \leq s+k-3$, so we are done. Thus we can assume that, either $c = s-1$ or $i = k-1$. Let us assume that $i = k-1$, the case $c = s-1$ is analogous.

To obtain a contradiction, let us suppose that G has at least $s + k + 4$ vertices, which implies by the previous observations that $c = s$. Let v be a vertex in C, and let (A, B) be an (s,k)-polar partition of $G - v$. We have two cases: either $G[S]$ inherits a $(1,2)$- or a $(2,1)$-polar partition from (A, B).

In the first case, since $G[B]$ is $\{(k+1)K_1, P_3\}$-free, we have that $C \setminus \{v\} \subseteq A$ and there exists a vertex $u \in I \cap A$. Moreover, $G[A]$ is a $\overline{P_3}$-free graph, so u is completely adjacent to $C \setminus \{v\}$. But then, $G[C \cup S \cup \{u\}]$ is isomorphic to either G_s^0 or G_s^1, so G properly contains an (s,k)-polar obstruction, which is impossible.

Hence, it must be the case that $G[S]$ inherits a $(2,1)$-polar partition (A', B') from (A, B), in which case $I \subseteq B$ and there exists a vertex $u \in B \cap (C \setminus \{v\})$, so that u is completely non adjacent to I. Additionally, repeating the argument, but using u instead of v, we have that there exists a vertex $w \in B \cap (C \setminus \{u\})$, so that w is completely non adjacent to I. But then,

$$(A' \cup C \setminus \{u, w\}, B' \cup I \cup \{u, w\})$$

is an (s,k)-polar partition of G, a contradiction. The contradiction arose from supposing that $|V_G| \geq s + k + 4$, so it must be the case that G has at most $s + k + 3$ vertices.

To bound is tight since H_s^k is a pseudo-split minimal (s,k)-polar obstruction whenever $k \geq s \geq 3$, and $\overline{H_k^s}$ is a pseudo-split minimal (s,k)-polar obstruction provided $s \geq k \geq 3$. □

In contrast with minimal (s,k)-polar obstructions when s and k are integers, it is unknown whether the number of minimal (s,∞)-polar obstructions is finite. In the following propositions we prove that, restricted to the class of pseudo-split graphs, minimal (s,∞)-polar obstructions are, all of them, minimal $(s,s+1)$-minimal obstructions, implying that there is only a finite number of them. We start with some technical propositions.

Lemma 5.15. *Let s and k be positive integers, and let $G=(C,S,I)$ be an imperfect pseudo-split graph such that $|C|=s$ and $0<|I|<k-1$. For each vertex $v\in I$, let $C_v=\{w\in C : w\notin N(v)\}$ and let C'_v be the set of all vertices in C that are completely nonadjacent to $I\setminus\{v\}$. Then, G is a minimal (s,k)-polar obstruction if and only if for each $v\in I$, both $|C'_v|\geq 2$ and $|C_v\cap C'_v|\leq 1$.*

Proof. Suppose that G is a minimal (s,k)-polar obstruction. From the minimality of G we have that, for each vertex $v\in I$, $G-v$ is an (s,k)-polar graph. Then, by Theorem 5.7, $|C'_v|\geq 2$. Moreover, from the same proposition we have that, if $|C_v\cap C'_v|\geq 2$ for some $v\in I$, then G is an (s,k)-polar graph, which is impossible. Then it must be the case that, for each vertex $v\in I$, $|C_v\cap C'_v|\leq 1$.

For the converse, assume that $|C'_v|\geq 2$ and $|C_v\cap C'_v|\leq 1$, for each $v\in I$. For any vertex $v\in C$, $G-v$ is a pseudo-split graph whose complete part has $s-1$ vertices and whose independent part has at most $k-2$ vertices, so it follows from Theorem 5.7 that $G-v$ admits an (s,k)-polar partition. For any vertex $v\in I$, the set C'_v has at least two vertices, so it also follows from Theorem 5.7 that $G-v$ is an (s,k)-polar graph. For any vertex $v\in S$, $G-v$ is a split graph, so $G-v$ is an (s,k)-polar graph. In summary, for every vertex v of G, $G-v$ is an (s,k)-polar graph. Furthermore, Theorem 5.7 implies that G is an (s,k)-polar graph if and only if there are at least two vertices of C that are completely nonadjacent to I. Nevertheless, if C' is any subset of C that is completely nonadjacent to I, then $C'\subseteq C_v\cap C'_v$ for any vertex $v\in I$, and therefore $|C'|\leq 1$. Hence, G is not an (s,k)-polar graph, and we conclude that G is a minimal (s,k)-polar obstruction. □

For each integer $s\geq 2$, let $F_s=(C,S,I)$ be the imperfect pseudo-split graph such that $|C|=s$, $|I|=s-1$ and, for an injection $f\colon I\to C$, a vertex $v\in I$ is adjacent to a vertex $u\in C$ if and only if $u=f(v)$. Notice that, from Theorem 5.7, we have that for any nonnegative integer k, F_s is not an (s,k)-polar graph. Moreover, F_s is a minimal (s,k)-polar obstruction if and only if $k>s$.

Lemma 5.16. *Let s and k be integers, $s, k \geq 3$, and let $G = (C, S, I)$ be a pseudo-split minimal (s, k)-polar obstruction such that $|C| = s$. Then, $|I| \leq s - 1$. In addition, if $|I| = s - 1$, then $s < k$ and $G \cong F_s$.*

Proof. Let $c = |C|$ and $i = |I|$. For each $v \in I$, $G - v$ has an (s, k)-polar partition (A, B). Moreover, since $c = s$, $G[S]$ inherits a $(2, 1)$-polar partition from (A, B), so we have that $I \subseteq B$. Additionally, at least two vertices of C belong to B, and any vertex in $C \cap B$ is completely nonadjacent to $I \setminus \{v\}$.

Observe that, if two vertices in $C \cap B$ are nonadjacent to v, then G would have an (s, k)-polar partition, but this is not the case. Hence, for each vertex $v \in I$, there is a vertex $u \in C$ whose only neighbor in I is v. Therefore, if $i \geq s$, G properly contains the (s, k)-polar obstruction F_s, contradicting the minimality of G. Thus, we conclude that $i \leq s - 1$.

Finally, if $i = s - 1$, G contains the (s, k)-polar obstruction F_s. But G is a minimal (s, k)-polar obstruction, so $G \cong F_s$, and then F_s is a minimal (s, k)-polar obstruction, which implies that $k > s$. □

Lemma 5.17. *Let s and k be integers, $k > s \geq 3$, and let G be a graph with pseudo-split partition (C, S, I).*

1. *If G is a minimal (s, k)-polar obstruction such that $|C| = s$, then G is a minimal (s, k')-polar obstruction for each integer $k' \geq k$.*

   Particularly, if G is a minimal $(s, s + 1)$-polar obstruction with $|C| = s$, then it is a minimal (s, ∞)-polar obstruction.

2. *If G is a minimal (s, k)-polar obstruction such that $|C| = s$, then G is a minimal $(s, s + 1)$-polar obstruction.*

   Consequently, if G is a minimal (s, ∞)-polar obstruction, then it a minimal $(s, s + 1)$-polar obstruction with $|C| = s$.

In consequence, G is a minimal (s, ∞)-polar obstruction if and only if G is an $(s, s + 1)$-polar obstruction with $|C| = s$.

Proof. Let k' be an integer, $k' \geq k$, and suppose that G is a minimal (s, k)-polar obstruction such that $|C| = s$. Thus, we have from Lemma 5.16 that, either $|I| < s - 1$, or $k > s$ and $G \cong F_s$. In the latter case the result follows because F_s is a minimal (s, k')-polar obstruction. Otherwise, we have that $|I| < s - 1 < k - 1$ and, by Lemma 5.15 and the observation that precede it, G is a minimal (s, k')-polar obstruction. From here, the rest of item 1 follows from Remark 5.10.

To prove item 2, assume again that G is a minimal (s,k)-polar obstruction such that $|C| = s$. Thus, since $s+1 \leq k$, it is clear that G is not an $(s,s+1)$-polar graph. Let v be a vertex of G. Clearly, if $v \in S$, then $G - v$ is a split graph, hence a $(s,s+1)$-polar graph. Additionally, we have from Lemma 5.16 that $|I| \leq s-1$, so it follows from Theorem 5.7 that $G - v$ is $(s,s+1)$-polar whenever $v \in C$. Notice that, from Theorem 5.14, we have that $|I| < k-1$. Thus, it follows from Lemma 5.15 and Theorem 5.7 that, if $v \in I$, then $G - v$ also is an $(s,s+1)$-polar graph. Therefore, G is not an $(s,s+1)$-polar graph but any vertex deleted subgraph of G is, so we have that G is a minimal $(s,s+1)$-polar obstruction. The rest of the proof for item 2 follows from Remark 5.10.

The last statement is an immediate consequence of items 1 and 2. □

Corollary 5.18. *Let s be an integer, $s \geq 3$. Any pseudo-split minimal (s,∞)-polar obstruction has order at most $2s+4$, and the bound is tight. In consequence, there are finitely many minimal (s,∞)-polar obstructions.*

Proof. Let G be a pseudo-split minimal (s,∞)-polar obstruction. We have from Lemma 5.17 that G is a minimal $(s,s+1)$-polar obstruction so, by Theorem 5.14, the order of G is at most $2s+4$. The bound is tight because F_s is a pseudo-split minimal (s,∞)-polar obstruction with $2s+4$ vertices. □

When we started our study of (s,k)-polarity on pseudo-split graphs, one goal was give characterizations of pseudo-split minimal (s,k)-polar obstructions, so we looked up by hand for such obstructions for very small values of s and k. Since we did not find a general pattern for arbitrary values of s and k, we decided to implement a computer program to help us find the desired obstructions for larger values of s and k. From the outputs obtained by running such a program, we made the observations that raised some of the theoretical results in this chapter. In the following small section, we present a brief explanation of how the mentioned program works. Readers interested in having access to the repository containing the program can contact the author of this text.

A program to compute minimal (s,k)-polar obstructions

The program was designed to compute pseudo-split minimal (s,k)-polar obstructions for integers s and k with $s,k \geq 2$. The reason for the restriction "$s,k \geq 2$" is that in such a case said obstructions are imperfect, so its (C,S,I)-partition is unique.

Moreover, there is a natural bijection from the class of imperfect pseudo-split graphs to the class of bipartite graphs with a prescribed bipartition that distinguishes the order of the parts in the bipartition, which we will call **specified-bipartite graphs**. An imperfect pseudo-split graph $G = (C, S, I)$ is completely determined by the specified-bipartite graph $B = (C, I)$ obtained from $G[C \cup I]$ by deleting the edges of $G[C]$. Conversely, a specified-bipartite graph $B = (X, Y)$ can be associated to the imperfect pseudo-split graph $G = (X, S, Y)$ such that $G[X \cup Y]$ is the graph obtained from B by adding edges between any pair of distinct vertices in X.

The advantage we took from the described relationship between imperfect pseudo-split graphs and specified-bipartite graphs is that, in contrast with pseudo-split graphs, bipartite graphs conform a widely studied class of graphs, so there are open access algorithms to generate them. Particularly, we used the software *Nauty* [58] to generate the complete lists of bipartite graphs of order n, for each integer $n \leq 16$[1].

The program starts with a three-phase preprocessing of the lists of bipartite graphs, which we explain below. The first phase consists of taking the list of bipartite graphs of order k, and use it to generate a list with all the specified-bipartite graphs of order k. The list produced in the first phase possibly contains repeated specified-bipartite graphs so, in the second phase, the program debugs such list and returns the list of all specified-bipartite graphs of order k, without repetitions. Phases 1 and 2 are carried out separately for each integer k. In the third phase of preprocessing, the program considers the lists of specified-bipartite graphs of order k and $k + 1$ generated in phase 2, and uses them to identify the vertex-deleted subgraphs of each specified-bipartite graph of order $k + 1$. Thus, at the end of the preprocessing, we have for each integer k, a list L_k of all specified-bipartite graphs of order k, and a list L_k^s where the vertex-deleted subgraphs of each graph in L_k have been identified.

Once the preprocessing has finished, we create lists L and L^s by concatenating the lists L_1, L_2, L_3, etc., and the lists L_1^s, L_2^s, L_3^s, etc., respectively. At this point, the program uses L and L^s to recursively identify pseudo-split minimal (s, k)-polar obstructions in the following way. For each graph G in L, the program check in L^s whether G has a vertex deleted subgraph marked as an (s, k)-obstruction. If any, G is marked as an (s, k)-obstruction. Otherwise, it is

[1]We really used only bipartite graphs of order at most 14, because the files with the lists of bipartite graphs of order 15 and 16 were really big (12 and 327 Gb, respectively).

verified by a brute force algorithm[2] whether G admits an (s, k)-polar partition. If G does, it is an (s, k)-polar graph, but if G does not, it is identified as a minimal (s, k)-polar obstruction. Notice that, in this step, it is a key point that the graphs in L and L^s are ordered from the smallest to the largest order.

Once the program has identified all the minimal (s, k)-polar obstructions in L, it is time to show them to the user by representing them in an understandable format[3]. Then, the last step in the program execution is to draw the specified-bipartite graphs associated to the minimal (s, k)-polar obstructions, returning an output as the image depicted in Figure 5.2.

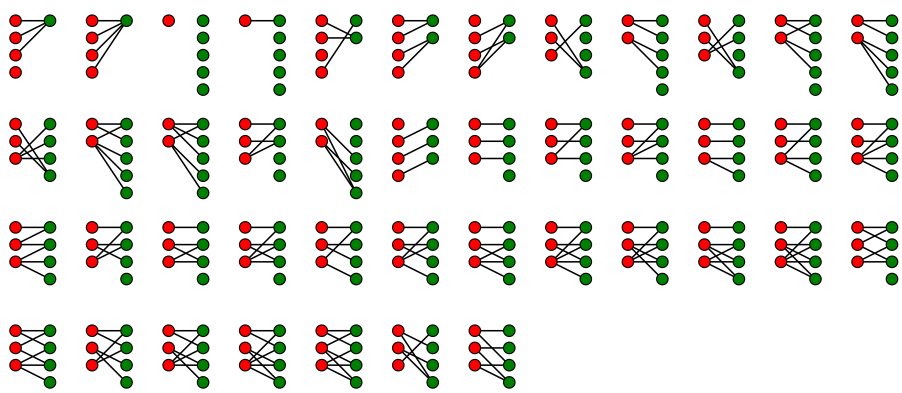


Figure 5.2: The output obtained from our program when computing minimal $(4, 5)$-polar obstructions. Vertices on the left sides represent the clique parts of the (C, S, I)-partitions, while vertices on the right sides represent the independent parts.

[2]As we mentioned at the beginning of this section, this program was implemented before many properties of (s, k)-polarity on pseudo-split graphs were known, included Theorem 5.7, which provide us of an efficient way of verifying whether a pseudo-split graph is (s, k)-polar.

[3]The graphs in the lists provided by Nauty, as well as those in L and L^s, are represented in the compact format *g6* (see `http://users.cecs.anu.edu.au/~bdm/data/formats.html` for more on g6 coding). The program converts such representation into adjacency matrices when some computation is necessary, but, for practical purposes, none of these graph codings is considered as a truly understandable representation of graphs.

5.2 Colorings of pseudo-split graphs

The (k, ℓ)-colorings were introduced in the 1990s by Brandstädt [3, 4, 5], who proved that the problem of deciding whether a graph admits a (k, ℓ)-coloring is polynomial time solvable if and only if $k, \ell \leq 2$.

The (k, ℓ)-coloring problem has been studied in graph classes with few induced P_4s. An efficient algorithm to solve (k, ℓ)-coloring problem in cographs, as well as an algorithm to find a maximal (k, ℓ)-colorable induced subgraph were given in [29]. In the same paper, the $(2, 1)$- and $(2, 2)$-colorable cographs were characterized by means of its family of minimal obstructions[4]. At the same time, finite forbidden subgraph characterizations of (k, ℓ)-colorable cographs for arbitrary values of k and ℓ was given in [6]. Moreover, the results on [6] was generalized some years later in [7], proving that the family of minimal (k, ℓ)-obstructions for cographs equals the family of minimal (k, ℓ)-obstructions for P_4-sparse graphs. Apparently, it was unknown for the authors of [7] that their result has been proven a little while before in [45]. The authors of [7] also gave a linear time recognition algorithm for P_4-sparse graphs that are (k, ℓ)-colorable. The class of extended P_4-laden graphs that admit a (k, ℓ)-coloring was studied for Bravo et. al. in [8]; there, it was given a linear time algorithm to decide (k, ℓ)-colorability, and as a consequence, it was also exhibited polynomial time algorithms to determine the chromatic and split-chromatic numbers; additionally, a polynomial time algorithm to find a maximal induced (k, ℓ)-colorable subgraph on an extended P_4-laden graph was given. Since the class of extended P_4-laden graphs extends both cographs and P_4-sparse graphs, the results in [8] generalize all known results of recognition algorithms on graphs with few induced P_4s. It is remarkable that the linear (k, ℓ)-recognition algorithms for all, cographs, P_4-sparse graphs, and extended P_4-laden graphs, are based on its respective tree representation and the fact that such representations can be computed in linear time.

In this brief section we study some coloring parameters of pseudo-split graphs, including (k, ℓ)-colorings, co-chromatic number, and bi-chromatic number.

Lemma 5.19. *Let G be a pseudo-split graph, and let k be an integer, $k \geq 2$. Then, G is k-colorable if and only if G is a $(K_{k+1}, C_5 \oplus K_{k-2})$-free graph.*

[4] In [27], structural characterizations of $(2, 1)$- and $(1, 2)$-colorable cographs were proven using its modular decomposition. A linear time recognition algorithm based on such characterizations was given.

Proof. It is easy to verify that both, K_{k+1} and $C_5 \oplus K_{k-2}$, are not k-colorable graphs. Thus, since being k-colorable is a hereditary property, it follows that any k-colorable graph is $(K_{k+1}, C_5 \oplus K_{k-2})$-free.

Conversely, suppose that G is a $(K_{k+1}, C_5 \oplus K_{k-2})$-free pseudo-split graph, and let (C, S, I) be a pseudo-split partition of G. Since split graphs are perfect, if $S = \varnothing$, then G is a K_{k+1}-free perfect graph, hence a k-colorable graph. Otherwise, G is an imperfect pseudo-split graph such that $|C| \leq k - 3$, or G would have $C_5 \oplus K_{k-2}$ as an induced subgraph; in this case a proper k-coloring of G could be obtained assign colors $1, \dots, k-3$ to the vertices of C, coloring S in a proper way with colors $k-2, k-1$ and k, and assigning color k to every vertex of I. □

Theorem 5.20. *Let G be an imperfect pseudo-split graph with pseudo-split partition (C, S, I), and let k and ℓ be nonnegative integers. The following statements hold true.*

1. *G is a $(k, 0)$-graph if and only if $|C| \leq k - 3$;*
2. *G is a $(0, \ell)$-graph if and only if $|I| \leq \ell - 3$;*
3. *G is not a $(1, 1)$-graph;*
4. *If k and ℓ are positive integers, and $k + \ell \geq 3$, then G is a (k, ℓ)-graph.*

Particularly, $\chi(G) = |C| + 3$ and $\theta(G) = |I| + 3$.

Proof. It follows from Lemma 5.19 that, for any integer $k \geq 2$, an imperfect pseudo-split graph is k-colorable if and only if it is a $C_5 \oplus K_{k-2}$-free graph. Thus, G is a $(k, 0)$-graph if and only if $|C| < k - 2$.

The second item follows from the first one since a graph G is $(0, \ell)$-colorable if and only if $\overline{G}$ is $(\ell, 0)$-colorable, and the complement of an imperfect pseudo-split graph is also an imperfect pseudo-split graph. Item 3 is due to G has an induced C_5, and C_5 is not a $(1, 1)$-graph.

Notice that for the last item it is enough to prove that G is a $(1, 2)$-graph, but this is trivially true since, for any $(1, 2)$-coloring (A, B) of $G[S]$, $(A \cup I, B \cup C)$ is a $(1, 2)$-coloring of G. The last statement is a direct consequence of the first two items, although it is also a direct consequence of Theorem 5.4. □

Corollary 5.21. *If G is an imperfect pseudo-split graph, then $\chi^c(G) = 3$.*

Proof. As we proved in Theorem 5.20, G is a $(1,2)$-colorable graph, hence a 3-cocolorable graph. Thus $\chi^c(G) \geq 3$. In addition, C_5 is an induced subgraph of G, but it is not a 2-cocolorable graph, so $\chi^c(G) > 2$, and the result follows. □

Lemma 5.22. *Let z be a positive integer, and let $\mathcal{F}^b(z)$ be the set of minimal z-bicolorable obstructions. Then,*

$$\mathcal{F}^b(z) \subseteq \bigcup_{i=0}^{z} \mathcal{F}(i, z-i),$$

where $\mathcal{F}(k,\ell)$ stands for the set of minimal (k,ℓ)-obstructions.

Proof. Notice that a graph H is a minimal z-bicolorable obstruction if and only if there exists an integer i with $0 \leq i \leq z$, such that H contains a graph $F_i \in \mathcal{F}(i, z-i)$ as an induced subgraph and, for any vertex v of H and every integer $j \in \{0, \dots, z\}$, $H - v$ does not contain any graph of $\mathcal{F}(j, z-j)$ as an induced subgraph. Particularly, for some integer $i \in \{0, \dots, z\}$, H is not an $(i, z-i)$-graph, but every vertex-deleted subgraph of H is, so H is a minimal $(i, z-i)$-obstruction. □

Theorem 5.23. *Let z be a positive integer, and let k and ℓ be nonnegative integers. Let $\mathcal{F}^b_{ps}(z)$ be the set of pseudo-split minimal z-bicolorable obstructions, and let $\mathcal{F}_{ps}(k,\ell)$ be the set of pseudo-split minimal (k,ℓ)-obstructions. Then,*

$$\mathcal{F}^b_{ps}(1) = \{K_2, \overline{K_2}\}, \quad \mathcal{F}^b_{ps}(2) = \{K_3, C_5, \overline{K_3}\},$$

and for any integer z with $z \geq 3$,

$$\mathcal{F}^b_{ps}(z) = \{K_{z+1}, C_5 \oplus K_{z-2}, \overline{K_{z+1}}, C_5 + \overline{K_{z-2}}\}.$$

Proof. It follows from Theorem 5.20 that $\mathcal{F}_{ps}(1,0) = \{K_2\}, \mathcal{F}_{ps}(1,1) = \{C_5\}$, and $\mathcal{F}_{ps}(k,0) = \{K_{k+1}, C_5 \oplus K_{k-2}\}$ for any integer $k \geq 2$. In addition, it also follows from Theorem 5.20 that, for any positive integers k and ℓ with $k + \ell \geq 3$, $\mathcal{F}_{ps}(k,\ell) = \varnothing$.

Then, since $H \in \mathcal{F}_{ps}(k,\ell)$ if and only if $\overline{H} \in \mathcal{F}_{ps}(\ell,k)$, we have from Lemma 5.22 and the observations in the above paragraph that, for any positive integer z,

$$\mathcal{F}^b_{ps}(z) = \bigcup_{i=0}^{z} \mathcal{F}_{ps}(i, z-i),$$

where the result follows. □

Corollary 5.24. *Let G be an imperfect pseudo-split graph with pseudo-split partition (C, S, I). Then $\chi^b(G) = \max\{|C| + 3, |I| + 3\} = \max\{\chi(G), \theta(G)\}$. Particularly, G is not a 2-bicolorable graph.*

Corollary 5.25. *Chromatic and bichromatic numbers can be determined in $O(|V|)$-time on imperfect pseudo-split graphs from their degree sequences. Additionally, the cochromatic number of imperfect pseudo-split graphs can be determined in constant time.*

Proof. From Theorem 5.4, we have that $\chi(G)$ and $\theta(G)$ can be computed in $O(|V|)$-time from the degree sequence of an imperfect pseudo-split graph. Then, we have from Corollary 5.24 that also the cochromatic number of imperfect pseudo-split graphs can be computed in $O(|V|)$-time. The last part of the statement follows from Corollary 5.21. □

Chapter 6

$2K_2$- and C_4-split graphs

Theorem 5.2 provide us of a characterization of $2K_2$-split graphs based on their degree sequences. Next, we characterize $2K_2$-split graphs by their forbidden induced subgraphs.

Theorem 6.1. *If G is a graph, then G is a $2K_2$-split graph if and only if G has not induced subgraphs isomorphic to the graphs depicted in Figure 6.1.*

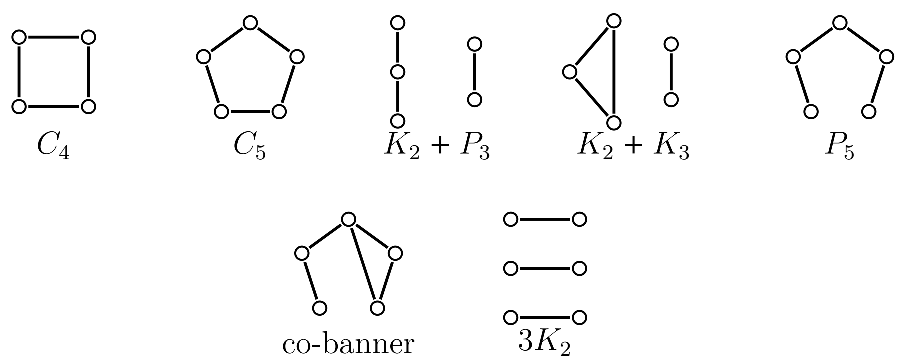


Figure 6.1: Minimal $2K_2$-split obstructions.

Proof. It is a routine job to check that any graph in Figure 6.1 is a minimal $2K_2$-split obstruction. To prove the necessary condition, let G be a graph without induced subgraphs isomorphic to the graphs in Figure 6.1. If G is $2K_2$-free, then G is a split graph, and hence a $2K_2$-split graph, so let us assume that G has an induced copy of $2K_2$ with vertex set S. Notice that any vertex in $V_G \setminus S$ is either completely adjacent to S or completely nonadjacent to it, or G would have some of the forbidden induced subgraphs.

Since G is a $3K_2$-free graph, any two vertices $u, v \in V_G \setminus S$ that are completely nonadjacent to S must be nonadjacent to each other. In addition, if there exist two nonadjacent vertices $u, v \in V_G \setminus S$ that are completely adjacent to S, then G has an induced C_4, which is impossible.

Hence, if we set C the set of all vertices in $V_G \setminus S$ that are completely adjacent to S, and I the set of vertices in $V_G \setminus S$ that are completely nonadjacent to S, then (C, S, I) is a $2K_2$-split partition of G. □

In this chapter we study polarity on $2K_2$-split graphs. Observe that a graph is $2K_2$-split if and only if its complement is C_4-split. Thus, since the complement of an (s, k)-polar graph is a (k, s)-polar graph, we have that by simple arguments of complements any result about (s, k)-polarity on $2K_2$-split graphs is equivalent to a dual result on C_4-split graphs. As the reader will notice, although some results are similar to those proved in Chapter 5 for pseudo-split graphs, there are also remarkable differences.

6.1 Polarity on $2K_2$-split graphs

As in the case of pseudo-split graphs, any $2K_2$-split graph is polar. Moreover, if $G = (C, S, I)$ is a $2K_2$-split graph, then $(C, S \cup I)$ is a unipolar partition of G, so G is unipolar, and hence polar. Additionally, it is clear that a $2K_2$-split graph is 1-polar if and only if it is $2K_2$-free. In the next proposition we give the complete sets of $2K_2$-split minimal $(1, k)$-polar obstructions for any positive integer k and, as a consequence, we derive the complete list of $2K_2$-split minimal monopolar obstructions.

Proposition 6.2. *Let k be an integer $k \geq 2$. A $2K_2$-split graph G is a minimal $(1, k)$-polar obstruction if and only if G is isomorphic to either $K_2 \oplus 2K_2$ or $K_1 \oplus (2K_2 + (k-1)K_1)$.*

In consequence, the only minimal monopolar obstruction is $K_2 \oplus 2K_2$, and monopolar $2K_2$-split graphs can be recognized in linear time from its degree sequence.

Proof. It is a routine to prove that both, $K_2 \oplus 2K_2$ and $K_1 \oplus (2K_2 + (k-1)K_1)$ are $2K_2$-split minimal $(1, k)$-polar obstructions. For the converse, let us assume that $G = (C, S, I)$ is a $2K_2$-split minimal $(1, k)$-polar obstruction; notice that

$S \neq \varnothing$, or G would be a 1-polar graph, and hence a $(1,k)$-polar graph. Also, notice that G does not have isolated vertices, because if v was a vertex of degree zero in G, and (A,B) is a $(1,k)$-polar partition of $G-v$, then $(A\cup\{v\},B)$ would be a $(1,k)$-polar partition of G, contradicting the election of G. Particularly, each vertex in I has a neighbor in C.

Now, if $|C| \geq 2$, then $K_2 \oplus 2K_2$ is an induced subgraph of G, so $G \cong K_2 \oplus 2K_2$. In addition, if $C = \varnothing$, then G clearly is a $(1,2)$-polar graph, and hence a $(1,k)$-polar graph, which is impossible. Thus, if $G \ncong K_2 \oplus 2K_2$, $|C| = 1$. Additionally, if $|I| < k-1$, then $(C, S \cup I)$ would be a $(1,k)$-polar partition of G, but that is absurd, so it must be the case that $|I| \geq k-1$, and it follows from the previous observations that G has $K_1 \oplus (2K_2 + (k-1)K_1)$ as an induced subgraph, so $G \cong K_1 \oplus (2K_2 + (k-1)K_1)$ by the minimality of G.

The second part of the statement follows from Remark 5.10 and Theorem 5.2. □

Results about unipolarity and monopolarity on C_4-split graphs cannot be deduced from those on $2K_2$-split graphs, so we develop them separately. The first thing we must observe is that any C_4-split graph $G = (C,S,I)$ has a unipolar partition namely $(C \cup \{u,v\}, I \cup (S \setminus \{u,v\}))$, where u and v are two adjacent vertices of S. Thus, any C_4-split graph is unipolar. The next proposition characterize C_4-split graphs that are monopolar.

Proposition 6.3. *Let k be an integer, $k \geq 2$. A C_4-split graph G is a minimal $(1,k)$-polar obstruction if and only if $G \cong C_4 \oplus K_1$.*

Consequently, the only C_4-split minimal monopolar obstruction is $C_4 \oplus K_1$, and monopolar C_4-split graphs can be recognized in linear time from its degree sequence.

Proof. It is a routine to prove that $C_4 \oplus K_1$ is a $(1,\infty)$-polar obstruction such that any vertex-deleted subgraph is $(1,2)$-polar, so we have that $C_4 \oplus K_1$ is a minimal $(1,k)$-polar obstruction for any integer $k \geq 2$.

Now, let $G = (C,S,I)$ be a C_4-split graph. If $C = \varnothing$, for any two nonadjacent vertices $u,v \in S$, $(I \cup (S \setminus \{u,v\}), \{u,v\})$ is a $(1,2)$-polar partition of G, so G is $(1,k)$-polar. Otherwise, we have that $|C| \geq 1$, so $C_4 \oplus K_1$ is an induced subgraph of G and G is not a $(1,k)$-polar graph. The rest of the proposition follows easily from Remark 5.10 and Theorem 5.2. □

Now, we give a complete characterization of $2K_2$-split graphs that admit an (s,k)-polar partition.

Theorem 6.4. *Let s and k be integers, $s, k \geq 2$, and let $G = (C, S, I)$ be an strict $2K_2$-split graph. Let c and i be the cardinalities of C and I, respectively. The following statements hold true.*

1. *If $s \geq c$ and $k \geq i + 2$, then G is an (s, k)-polar graph.*
2. *If $s \geq c + 2$ and $k \geq i + 1$, then G is an (s, k)-polar graph.*
3. *If $s \leq c - 1$ and $k \leq i$, then G is not an (s, k)-polar graph.*
4. *If $s \leq c - 1$ and $k \geq i + 1$, then G is an (s, k)-polar graph if and only if there is a subset $C^{'}$ of C with at least $c - s + 2$ vertices that is completely nonadjacent to I.*
5. *If $s \geq c + 1$ and $k \leq i$, then G is an (s, k)-polar graph if and only if there exists a subset $I^{'}$ of I with at least $i - k + 2$ vertices that satisfies some of the following conditions:*
   (a) *$I^{'}$ is completely adjacent to C.*
   (b) *There exists a vertex $v \in C$ such that $I^{'}$ is completely adjacent to $C \setminus \{v\}$ and v is completely nonadjacent to $I^{'}$.*
6. *If $s = c$ and $k \leq i$, then G is an (s, k)-polar graph if and only if there exists a subset $I^{'}$ of I with at least $i - k + 2$ vertices and a vertex $v \in C$ such that $I^{'}$ is completely adjacent to $C \setminus \{v\}$ and v is completely nonadjacent to $I^{'}$.*
7. *If $s = c$ and $k = i + 1$, then G is an (s, k)-polar graph if and only if some of the following statements is satisfied:*
   (a) *there exists a subset $C^{'}$ of C with at least $c - s + 2$ vertices that is completely nonadjacent to I.*
   (b) *there exists a nonempty subset $I^{'}$ of I and a vertex $v \in C$ such that $I^{'}$ is completely adjacent to $C \setminus \{v\}$ and v is completely nonadjacent to $I^{'}$.*
8. *If $s = c + 1$ and $k = i + 1$, then G is an (s, k)-polar graph if and only if some of the following statements is satisfied:*
   (a) *there exists a nonempty subset $C^{'}$ of C that is completely nonadjacent to I.*

(b) *there exists a nonempty subset I' of I such that satisfies some of the following conditions:*

  i. *I' is completely adjacent to C.*

  ii. *There is a vertex $v \in C$ that is completely nonadjacent to I' and such that I' is completely adjacent to $C \setminus \{v\}$.*

Proof. Let $S = \{u, v, x, y\}$ and assume that $uv, xy \in E_G$.

1. It is enough to notice that $(C, S \cup I)$ is a $(c, i + 2)$-polar partition of G.

2. It is enough to notice that $(C \cup \{u, v\}, I \cup \{x, y\})$ is a $(c + 2, i + 1)$-polar partition of G.

3. Aiming for a contradiction, assume that G admits an (s, k)-polar partition (A, B). Notice that $C \nsubseteq A$, so $C \cap B \neq \varnothing$, in which case the vertices of one component of $G[S]$ are in A. Suppose without loss of generality that $u, v \in A$. Then $I \cup \{x, y\} \subseteq B$, but in such a case $G[B]$ has at least $i + 1$ components, a contradiction.

4. For the necessary condition it is enough to notice that $(\{u, v\} \cup C \setminus C', \{x, y\} \cup C' \cup I)$ is an $(s, i + 1)$-polar partition of G. For the sufficient condition, let us assume that (A, B) is a polar partition of G. Since $s \leq c - 1$, $C \cap B \neq \varnothing$. Thus, we can assume without loss of generality that $\{u, v\} \in A$ and $\{x, y\} \in B$, and we have that $I \subseteq B$. Let $C' = C \cap B$. Hence, C' is completely nonadjacent to I and $|C'| \geq c - s + 2$, so the result follows.

5. For the necessary condition, notice that $(C \cup I', S \cup I \setminus I')$ is an $(c + 1, k)$-polar partition of G. For the sufficient condition let (A, B) be an (s, k)-polar partition of G. Since $2K_2$ is not a complete multipartite graph, we have that $B \cap S \neq \varnothing$, which implies that $I \cap A \neq \varnothing$, and then both, $B \cap \{u, v\} \neq \varnothing$ and $B \cap \{x, y\} \neq \varnothing$. In consequence $C \subseteq A$. Set $I' = I \cap A$, and notice that, since $k \leq i$, then $|I'| \geq i - k + 2$. Moreover, since $G[A]$ is a complete s-partite graph and $I' \cup C \subseteq A$ we have that every vertex of I' is either completely adjacent to C or it is adjacent to any vertex of C except for a vertex v. In addition, if a vertex $w \in C$ is adjacent to a vertex $z \in I'$, then w is completely adjacent to I'. The statement easily follows form the above observations.

6. For the necessary condition, notice that $(C \cup I', S \cup I \setminus I')$ is a (c,k)-polar partition of G. For the sufficient condition let (A,B) be an (s,k)-polar partition of G. Since $2K_2$ is not a complete multipartite graph, $B \cap S \neq \varnothing$, which implies that $I \cap A \neq \varnothing$. But then, $B \cap \{u,v\} \neq \varnothing$ and $B \cap \{x,y\} \neq \varnothing$, implying that $C \subseteq A$. Set $I' = I \cap A$. Since $k \leq i$, $|I'| \geq i - k + 2$. Moreover, since $G[A]$ is a complete c-partite graph and $I' \cup C \subseteq A$ we have that every vertex of I' is adjacent to any vertex of C except for a vertex v. In addition, if a vertex $w \in C$ is adjacent to a vertex $z \in I'$, then w is completely adjacent to I'. The conclusion follows easily from here.

7. For the necessary condition we only have to notice that, in case of (a) occurs, $(\{u,v\} \cup C \setminus C', \{x,y\} \cup C' \cup I)$ is an (s,k)-polar partition of G while, if (b) occurs, then $(C \cup I', S \cup I \setminus I')$ is an (s,k)-polar partition of G. For the sufficient condition let (A,B) be an (s,k)-polar partition of G. If $A \cap S \neq \varnothing$, then $B \cap C \neq \varnothing$, which implies, without loss of generality, that $\{u,v\} \subseteq A$ and $\{x,y\} \subseteq B$, and therefore $I \subseteq B$. Then, if $C' = C \cap B$, we have that $|C'| \geq c - s + 2$ and C' is completely nonadjacent to I. Otherwise, if $A \cap S = \varnothing$, $S \subseteq B$, and therefore $C \subseteq A$. Notice that, since $k = i + 1$, $A \cap I \neq \varnothing$. Let $I' = A \cap I$. Since $C \cup I' \subseteq A$ and A induces a complete c-partite graph, there exist a vertex $v \in C$ such that I' is completely adjacent to $C \setminus \{v\}$ but v is completely nonadjacent to I'.

8. For the necessary condition, notice that in case that (a) occurs, $(\{u,v\} \cup C \setminus C', \{x,y\} \cup C' \cup I)$ is an (s,k)-polar partition of G while, if (b) occurs, then $(C \cup I', S \cup I \setminus I')$ is an (s,k)-polar partition of G. For the sufficient condition, let us consider an (s,k)-polar partition (A,B) of G. If $G[B \cap S]$ is connected, then the vertex set of one of the connected components of $G[S]$ is completely contained in A, let us say, without loss of generality, that $\{u,v\} \subseteq A$. Observe that, in this case, $C \nsubseteq A$, so $C' = C \cap B$ is not empty. In addition, $I \cap A = \varnothing$, so $I \subseteq B$. Thus, since $G[B]$ is a P_3-free graph, C' is completely nonadjacent to I, and we have the case (a) of the statement. Otherwise, $G[B]$ is disconnected, which implies that $C \cap B = \varnothing$ and $I \nsubseteq B$. Then, we have that $C \subseteq A$ and the set $I' = I \cap A$ is not empty. Hence, $C \cup I' \subseteq A$ and, since $G[A]$ is $\overline{P_3}$-free, we have that either I' is completely adjacent to C, or there is a vertex $v \in C$ such that I' is completely adjacent to $C \setminus \{v\}$ and v is completely nonadjacent to I', so item (b) of the statement follows.

$\square$

6.1.1 $2K_2$-split minimal (s,k)-polar obstructions

The following propositions are consequences of Theorem 6.4. They are directed to prove an upper bound for the order of $2K_2$-split minimal (s,k)-polar obstructions for arbitrary integers s and k.

Observe that, for a $2K_2$-split graph $G = (C, S, I)$, if some of C, S or I is an empty set, then G is a 2-polar graph. Hence, for any integers s and k with $s, k \geq 2$, any $2K_2$-split minimal (s,k)-polar obstruction $G = (C, S, I)$ is such that C, S and I are all of them nonempty sets. We will use this observation in the following proofs without any explicit mention.

We start with a direct consequence of item 3 of Theorem 6.4.

Lemma 6.5. *Let s and k be integers, $s, k \geq 2$, and let $G = (C, S, I)$ be a strict $2K_2$-split graph. Let $c = |C|$ and $i = |I|$. The following assertions hold.*

1. *If $c \geq s + 2$ and $i \geq k$, for each vertex $v \in C$, $G - v$ is not an (s,k)-polar graph, so G is not a minimal (s,k)-polar obstruction.*

2. *If $c \geq s+1$ and $i \geq k+1$, for each vertex $v \in I$, $G-v$ is not an (s,k)-polar graph, so G is not a minimal (s,k)-polar obstruction.*

Lemma 6.6. *Let s and k be integers, $s, k \geq 2$, and let $G = (C, S, I)$ be a $2K_2$-split graph. Let $c = |C|$ and $i = |I|$. If $c \geq s + 2$ and $i \leq k - 1$, then G is not a minimal (s,k)-polar obstruction.*

Proof. Aiming for a contradiction, suppose that G is a minimal (s,k)-polar obstruction. Notice that, for each vertex $v \in C$, $(C \setminus \{v\}, S, I)$ is the $2K_2$-split partition of $G - v$ and $|C \setminus \{v\}| = c - 1 \geq s + 1$.

Since G is a minimal (s,k)-polar obstruction, for each $v \in C$, the graph $G - v$ is an (s,k)-polar graph, which implies, by item 4 of Theorem 6.4, that there is a subset C'_v of $C \setminus \{v\}$ with at least $c - s + 1$ vertices that is completely nonadjacent to I. In addition, also by item 4 of Theorem 6.4, since G is not an (s,k)-polar graph, each vertex $v \in C$ is adjacent to at least one vertex in I.

Let H be a graph obtained from G by deleting $c - s - 1$ vertices of C. Then H has a $2K_2$-split partition (C^*, S, I), with $|C^*| = s + 1$. Notice that each $v \in C^* \subseteq C$ has at least one neighbor in I, which implies that the only subset C' of C^* that is completely nonadjacent to I is the empty set. Thus, we have from item 4 of Theorem 6.4 that H is nos an (s,k)-polar graph, but that is impossible since H is a proper induced subgraph of G, which is by assumption a minimal (s, k)-polar obstruction. □

In the next remark we identify some distinguished $2K_2$-split minimal (s, k)-polar obstructions.

Remark 6.7. *Let s and k be integers, $s \geq 2$.*

1. *The strict $2K_2$-split graph $G = (C, S, I)$ such that $|C| = s$, $|I| = 1$, and C is completely adjacent to I, is a minimal $(s, 2)$-polar obstruction.*

2. *Let $k \geq 2$, and let $G = (C, S, I)$ be the strict $2K_2$-split graph such that $|C| = s + 1$, $|I| = 1$, and for two vertices u and v in C, $C' = C \setminus \{u, v\}$ is completely adjacent to I, and $\{u, v\}$ is completely nonadjacent to I. Then, G is a minimal (s, k)-polar obstruction.*

3. *Let $k \geq 3$, and let $G = (C, S, I)$ be the strict $2K_2$-split graph such that $|C| = s + 1$, $|I| = 1$, and for a vertex $u \in C$, $C' = C \setminus \{u\}$ is completely adjacent to I, and u is completely nonadjacent to I. Then, G is a minimal (s, k)-polar obstruction.*

Lemma 6.8. *Let s and k be integers, $s, k \geq 2$, and let $G = (C, S, I)$ be a $2K_2$-split graph. Let $c = |C|$ and $i = |I|$. If $c = s + 1$ and $i = k$, then G is not a minimal (s, k)-polar obstruction.*

Proof. In order to reach a contradiction, let us assume that G is a minimal (s, k)-polar obstruction, and for a vertex $v \in C$, let $C' = C \setminus \{v\}$. Let (A, B) be an (s, k)-polar partition of $G - v$. Since $i = k$ and $2K_2$ is not a complete multipartite graph, we have that $A \cap I \neq \varnothing$, which implies that B is present in both components of $2K_2$, and therefore $C' \subseteq A$.

Let $u \in A \cap I$. Since $C' \subseteq A$ and $|C'| = s$, we have that there is a vertex $w \in C'$ such that $C' \setminus \{w\}$ is completely adjacent to u and $wu \notin E$. Now, since $i = k \geq 2$, we have that $G[C \cup S \cup \{v\}]$ is a proper induced subgraph of G that contains one of the minimal (s, k)-polar obstruction mentioned in Remark 6.7, a contradiction. □

Lemma 6.9. *Let s and k be integers, $s, k \geq 2$, and let $G = (C, S, I)$ be a $2K_2$-split graph. Let $c = |C|$ and $i = |I|$. If $c = s$ and $i \geq 2k - 1$, then, G is not a minimal (s, k)-polar obstruction.*

Proof. Aiming for a contradiction, assume that G is a minimal (s,k)-polar obstruction, and let $u \in I$. Then, $G - u$ is an (s,k)-polar graph and, by item 6 of Theorem 6.4, there is a subset I'_u of $I \setminus \{u\}$ with at least $i - k + 1$ vertices, and a vertex $v_u \in C$ such that, I'_u is completely adjacent to $C \setminus \{v_u\}$ and v_u is completely nonadjacent to I'_u. Now, let $x \in I'_u$. By the same argument of the paragraph above, there is a subset I'_x of $I \setminus \{x\}$ with at least $i - k + 1$ vertices, and a vertex $v_x \in C$ such that, I'_x is completely adjacent to $C \setminus \{v_x\}$ and v_x is completely nonadjacent to I'_x.

Observe that $2i - 2k + 2 \geq i + 1$, because $i \geq 2k - 1$. Thus, we have that $I'_x \cap I'_u \neq \varnothing$, otherwise

$$i = |I| \geq |I'_x \cup I'_u| \geq 2(i - k + 1) \geq i + 1,$$

which is absurd. Since $x \in I'_u \setminus I'_x$ and $I'_x \cap I'_u \neq \varnothing$, we have that $|I'_x \cup I'_u| \geq |I'_x| + 1 \geq i - k + 2$, $v_u = v_x$, $I'_x \cup I'_u$ is completely adjacent to $C \setminus \{v_u\}$, and v_u is completely nonadjacent to $I'_x \cup I'_u$. This is impossible, since item 6 of Theorem 6.4 implies that in such a case G is an (s,k)-polar graph, contradicting our initial assumption. □

Lemma 6.10. *Let s and k be integers, $s,k \geq 2$, and let $G = (C,S,I)$ be a $2K_2$-split graph. Let $c = |C|$ and $i = |I|$. If $c \leq s - 1$ and $i \geq 2k - 1$, then G is not a minimal (s,k)-polar obstruction.*

Proof. Aiming for a contradiction, assume that G is a minimal (s,k)-polar obstruction, and let $u \in I$. Then $G - u$ is an (s,k)-polar graph and, by item 5 of Theorem 6.4, there exists a subset I'_u of $I \setminus \{u\}$ with at least $i - k + 1$ vertices and a vertex $v_u \in C$ such that I'_u is completely adjacent to $C \setminus \{v_u\}$ and, v_u is either completely adjacent or completely nonadjacent to I'_u.

We claim that v_u is completely adjacent to I'_u, and we prove it by means of contradiction. Let $x \in I'_u$. Then $G - x$ is an (s,k)-polar graph and again, we have from item 5 of Theorem 6.4 that there exists a subset I'_x of $I \setminus \{x\}$ with at least $i - k + 1$ vertices and a vertex $v_x \in C$ such that I'_x is completely adjacent to $C \setminus \{v_x\}$ and, v_x is either completely adjacent or completely nonadjacent to I'_x.

Observe that, as it occurred in Lemma 6.9, since $i \geq 2k - 1$, there is a vertex $w \in I'_x \cap I'_u$. Since we are assuming v_u is completely nonadjacent to I'_u, we have that v_u is not adjacent to w, and due to $w \in I'_x$, we have that $v_u = v_x$. But then, $I'_u \cup I'_x$ is completely adjacent to $C \setminus v_u$ and v_u is completely nonadjacent to $I'_u \cup I'_x$. Moreover, since $x \notin I'_x$, the set $I'_u \cup I'_x$ has at least $i - k + 2$ vertices.

But then, item 5 of Theorem 6.4 implies that G is an (s,k)-polar graph, a contradiction. The contradiction arose from assuming that v_u is completely nonadjacent to I'_u, so it must be the case that, for every vertex $u \in I$, there exists a subset I'_u of I with at least $i-k+1$ vertices such that I'_u is completely adjacent to C.

But then, for any $x \in I'_u$ and any subset I'_x of $I \setminus \{x\}$ with $i-k+1$ vertices such that I'_x is completely adjacent to C, we have that $I'_x \cup I'_u$ is a subset of I with at least $i-k+2$ vertices that is completely adjacent to C, which implies by Theorem 6.4 that G is an (s,k)-polar graph, contradicting our initial assumption. □

Now, we are ready to give an upper bound for the order of the $2K_2$-split minimal (s,k)-polar obstructions.

Theorem 6.11. *Let s and k be integers, $s,k \geq 2$. Any $2K_2$-split minimal (s,k)-polar obstruction has order at most $s+2k+2$.*

Proof. Let $G=(C,S,I)$ be a $2K_2$-split minimal (s,k)-polar obstruction. It follows from Lemmas 6.5, 6.6 and 6.8 that, if $|C| \geq s+1$, then $|V_G| \leq s+k+4$. Additionally, we conclude from Lemmas 6.9 and 6.10 that $|V_G| \leq s+2k+2$ whenever $|C| \leq s$. Hence, we have that $|V_G| \leq \max\{s+k+4, s+2k+2\}$. However, since $k \geq 2$, we have that $s+2k+2 \geq s+k+4$, so the result follows. □

We continue with a characterization for $2K_2$-split (s,∞)-polar graphs, and then with an upper bound for the order of $2K_2$-split minimal (s,∞)-minimal obstructions.

Lemma 6.12. *Let s be an integer, $s \geq 2$, and let $G=(C,S,I)$ be an strict $2K_2$-split graph. Let $c=|C|$ and $i=|I|$. Then, G is an (s,∞)-polar graph if and only if either $s \geq c$ or there is a subset C' of C with at least $c-s+2$ vertices that is completely nonadjacent to I.*

Proof. Suppose that G has an (s,∞)-polar partition (A,B). If $c>s$, since $G[A]$ is K_{s+1}-free, then $C \nsubseteq A$, so $C' = C \cap B \neq \varnothing$. Moreover, $G[B]$ is P_3-free and A induces a $\overline{P_3}$-free graph, which implies, without loss of generality, that $\{u,v\} \in A$ and $\{x,y\} \in B$. Thus, $I \cap A = \varnothing$ because $G[A]$ is $\overline{P_3}$-free, so $I \subseteq B$, and $|C'| \geq c-s+2$ because A induces a K_{s+1}-free graph. Additionally, since $C' \cup I \subseteq B$, we have that C' is completely nonadjacent to I, and we are done.

For the converse implication, if $s \le c$, then $(C, S \cup I)$ is an $(s, i+2)$-polar partition of G. Otherwise, there is a set C' of C with at least $c - s + 2$ vertices that is completely nonadjacent to I. In this case, $(\{u,v\} \cup C \setminus C', \{x,y\} \cup C' \cup I)$ is an $(s, i+1)$-polar partition of G, and the result follows. □

For each integer $s \ge 2$, let $H_s = (C, S, I)$ be the strict $2K_2$-split graph such that $|C| = s+1$, $|I| = s-1$, and for an injection $f\colon I \to C$, a vertex $v \in I$ is adjacent to a vertex $u \in C$ if and only if $f(v) = u$. Notice that, by Lemma 6.12, H_s is a minimal (s, ∞)-polar obstruction.

Theorem 6.13. *Let s be an integer, $s \ge 2$. Any $2K_2$-split minimal (s, ∞)-polar obstruction has order at most $2s+4$, and the bound is tight.*

Consequently, there is only a finite number of $2K_2$-split minimal (s, ∞)-polar obstructions.

Proof. Let $G = (C, S, I)$ be a $2K_2$-split minimal (s, ∞)-polar obstruction, and let c and i be the number of vertices in C and I, respectively. From Lemma 6.12, we have that $c > s$. In addition, since G is a minimal (s, k)-polar obstruction for some positive integer k, we have from Lemmas 6.5 and 6.6, that $c \le s+1$, so we conclude that $c = s+1$.

By the minimality of G, we have from Lemma 6.12 that, for each $u \in I$, there is a subset C'_u of C, with at least three vertices, that is completely nonadjacent to $I \setminus \{u\}$. Additionally, since G does not admit an (s, k)-polar partition, Lemma 6.12 implies that at most two vertices of C are completely nonadjacent to I, so each vertex $u \in I$ is adjacent to at least one vertex of C'_u. Moreover, it follows from the previous observations that, for each $u \in I$, there is at least one vertex in C'_u that is not in C'_v for any $v \in I \setminus \{u\}$. Therefore, $|\bigcup_{u \in I} C'_u| \ge i+2$, so $c \ge i+2$, and it follows that $|V_G| = |C| + |S| + |I| \le 2s+4$.

The bound is tight since H_s is a K_2-split minimal (s, ∞)-polar obstruction of order $2s+4$. □

Unlike pseudo-split graphs, $2K_2$-split graphs does not constitute a self-complementary class of graphs, so we cannot use simple arguments of complements to conclude results for (∞, k)-polarity from those of (s, ∞)-polarity on this class. Next, we provide an upper bound for the order of $2K_2$-split minimal (∞, k)-minimal obstructions by proving similar results to Lemma 6.12 and Theorem 6.13 for (∞, k)-polarity on $2K_2$-split graphs.

Lemma 6.14. *Let k be an integer, $k \ge 2$, and let $G = (C, S, I)$ be an strict $2K_2$-split graph. Let $c = |C|$ and $i = |I|$. Then, G is an (∞, k)-polar graph if* *and only if either $i \leq k-1$ or there exists a subset I' of I with at least $i-k+2$ vertices such that $G[C \cup I']$ is a complete multipartite graph.*

Proof. Suppose that G has an (∞, k)-polar partition (A, B). Since $G[S]$ is not a complete multipartite graph, we have that $S \nsubseteq A$, so $S \cap B \neq \varnothing$. From here, if $i \geq k$, then $I \nsubseteq A$ because $G[B]$ is $(k+1)K_1$-free, so $I' = I \cap A \neq \varnothing$. Hence, since $G[A]$ is a $\overline{P_3}$-free graph, we have that $A \cap S$ is an independent set, so B intersects the vertex sets of both of the connected components of $G[S]$. But then, $C \cap B = \varnothing$, because B induces a P_3-free graph. Therefore $C \cup I' \subseteq A$, and $C \cup I'$ induces a complete multipartite graph. Notice that, due to $G[B]$ is $(k+1)K_1$-free and B intersects the vertex sets of both components of $G[S]$, $|I'| \geq i-k+2$.

For the converse implication, let S' be a maximum clique of $G[S]$. If $i \leq k-1$, then $(C \cup S', I \cup S \setminus S')$ is an (∞, k)-polar partition of G. Otherwise, there is a subset I' of I with at least $i-k+2$ vertices such that $G[C \cup I']$ is a complete multipartite graph, so in this case $(C \cup I', S \cup I \setminus I')$ is an (∞, k)-polar partition of G. □

Theorem 6.15. *Let k be an integer, $k \geq 2$. Any $2K_2$-split minimal (∞, k)-polar obstruction has order at most $2 + 2k + 2^{2k-1}$. In consequence, there is only a finite number of $2K_2$-split minimal (∞, k)-polar obstructions.*

Proof. Let $G = (C, S, I)$ be a $2K_2$-split minimal (∞, k)-polar obstruction, and let c and i be the number of vertices in C and I, respectively. From Lemma 6.14 we have that $i \geq k$. Moreover, since G is a minimal (s, k)-polar obstruction for some positive integer s, we have from Lemmas 6.5, 6.9 and 6.10, that $i \leq 2k-2$.

By the minimality of G, for each vertex $x \in C$, $G - x = (C \setminus \{x\}, S, I)$ is an (∞, k)-polar graph with at least k vertices in its stable part, so it follows from Lemma 6.14 that there is a subset I'_x of I with at least $i-k+2$ vertices such that $G[I'_x \cup C \setminus \{x\}]$ is a complete multipartite graph.

We claim that, for any two different vertices $u, v \in C$, if I'_v is a subset of I'_u, then the neighborhood of each vertex in I'_v is precisely $C \setminus \{u, v\}$. Notice that this would imply that there are not three vertices $u, v, w \in I$ such that $I'_u = I'_v = I'_w$.

To prove our claim, suppose that u and v are vertices in C such that $I'_v \subseteq I'_u$. Since $G[I'_v \cup C \setminus \{v\}]$ is a complete multipartite graph we have two possibilities, either I'_v is completely adjacent to $C \setminus \{v\}$ or there is a vertex $w \in C \setminus \{v\}$ such that I'_v is completely adjacent to $C \setminus \{v, w\}$ and w is completely nonadjacent to I'_v. Notice that, regardless of the case, since $v \in C \setminus \{u\}$ and $G[I'_u \cup C \setminus \{u\}]$

is $\overline{P_3}$, v is either completely adjacent or completely nonadjacent to I'_u, and therefore, v is either completely adjacent or completely nonadjacent to I'_v. But we have from the previous observation that, if I'_v is completely adjacent to $C \setminus \{v\}$, then $G[I'_v \cup C]$ is a complete multipartite graph, which implies by Lemma 6.14 that G is an (∞, k)-polar graph, contradicting the election of G.

Thus, I'_v is not completely adjacent to $C \setminus \{v\}$, so there is a vertex $w \in C \setminus \{v\}$ such that I'_v is completely adjacent to $C \setminus \{v, w\}$ and w is completely nonadjacent to I'_v. Observe that we have two cases depending on whether $w = u$. Since $G[I'_u \cup C \setminus \{u\}]$ is a complete multipartite graph and $I'_v \subseteq I'_u$, we have that $G[I'_v \cup C \setminus \{u\}]$ is also a complete multipartite graph. Then, if $w \neq u$, we have that v is adjacent to every vertex of I'_v, but this would imply that $G[I'_v \cup C]$ is a complete multipartite graph, and we previously noticed that this is impossible. Hence $w = u$ and, since $G[I'_v \cup C \setminus \{u\}]$ is a complete multipartite graph but $G[I'_v \cup C]$ is not, we have that v is completely nonadjacent to I'_v, and it follows that the neighborhood of each vertex in I'_v equals $C \setminus \{u, v\}$.

By our previous arguments, there are at least $\lceil c/2 \rceil$ vertices of $u \in C$ whose associated sets I'_u are pairwise different. Therefore, since $I'_u \subseteq I$, we have that $\lceil C/2 \rceil \leq |\mathcal{P}(I)| = 2^{|I|} \leq 2^{2k-2}$, from which we conclude that

$$|V_G| = |C| + |S| + |I| \leq 2 + 2k + 2^{2^{k-1}}.$$

□

It is worth noticing that, unlike the upper bound for the order of $2K_2$-split minimal (s, ∞)-polar obstructions provided in Theorem 6.13, which is linear on s, the bound given in Theorem 6.15 for the order of $2K_2$-split minimal (∞, k)-polar obstructions is exponential on k. Moreover, we know that the first of these bounds is tight, but we cannot guarantee the same for the second one. We think that the next question can be answered in an affirmative way by imitating the proof of Theorem 6.13, which is very different than the one we used in Theorem 6.15.

Problem 6.16. *Is the order of the $2K_2$-split minimal (∞, k)-polar obstructions upper bounded by a function linear on k?*

Some initial explorations allow us to pose the following conjecture.

Conjecture 6.17. *Let k be an integer, $k \geq 3$, and let $G = (C, S, I)$ be a $2K_2$-split minimal (∞, k)-polar obstruction. Then $k \leq i \leq 2k - 2$ and $c \leq 2k - i - 1$, where c and i stands for the number of vertices in C and I, respectively.*

We also pose the following question.

Problem 6.18. *Can Lemmas 6.9 and 6.10 be improved by replacing the condition $i \geq 2k - 1$ for a stronger one like $i \geq k + c$, for a constant c?*

Notice that, from the proof used for Theorem 6.15, an affirmative answer to Problem 6.18 would imply an improvement of the bound provided in the mentioned theorem. Nevertheless, the next observation make us think the answer to Problem 6.18 is in a negative way.

Remark 6.19. *Let s and k be integers, $s, k \geq 2$.*

1. *If $G = (C, S, I)$ is the strict $2K_2$-split graph such that $C = \{w\}$, $I = \{i_1, \dots, i_{2k-2}\}$, and $wi_j \in E$ if and only if $1 \leq j \leq k - 1$, then G is a minimal (s, k)-polar obstruction, and hence it is a minimal (∞, k)-polar obstruction.*

2. *If $G = (C, S, I)$ is the strict $2K_2$-split graph such that $C = \{c_1, \dots, c_{k-1}\}$, $I = \{i_1, \dots, i_k\}$ and, for each $j \in \{1, \dots, k - 1\}$, $N(c_j) \cap I = I \setminus \{i_j\}$, then G is a minimal (∞, k)-polar obstruction.*

6.1.2 Algorithms for polarity on $2K_2$-split graphs

We have observed before that $2K_2$-split graphs are unipolar and co-unipolar, and hence polar graphs. Additionally, we proved that deciding monopolarity and co-monopolarity on $2K_2$-split graphs can be done in linear time from its degree sequence. In this section we prove that deciding whether a $2K_2$-split graph is (s, ∞)-, (∞, k)- or (s, k)-polar, also can be done efficiently.

The next observation follows directly from Theorem 5.2. It will be used along this section without any explicit mention.

Remark 6.20. *Let $G = (C, S, I)$ be a strict $2K_2$-split graph, and let c and i be the number of vertices in C and I, respectively. The following statements hold true.*

1. *Every vertex $v \in C$ has degree at least $c + 3$.*

2. *Every vertex $v \in S$ has degree exactly $c + 1$.*

3. *Every vertex $v \in I$ has degree at most c.*

4. *A vertex v of G has degree $c + 3$ if and only if $v \in C$ and it is completely nonadjacent to I,*

5. *A vertex v of G has degree c if and only if $v \in I$ and it is completely adjacent to C.*

We start proving that, for any positive integer s, the $2K_2$-split graphs that admit an (s, ∞)-polar partition can be recognized in linear time from their degree sequence.

Proposition 6.21. *Let s be an integer, $s \geq 2$. The problem of deciding whether a $2K_2$-split graph is (s, ∞)-polar is linear-time solvable from its degree sequence.*

Proof. Let $G = (C, S, I)$ be a $2K_2$-split graph, and let c and i be the number of vertices in C and I, respectively. If $c \leq s$, then $(C, S \cup I)$ is an (s, ∞)-polar partition of G. Otherwise, we have from Lemma 6.12 that G is an (s, ∞)-polar graph if and only if there exist at least $c - s + 2$ vertices of G whose degree is exactly $c + 3$. By Theorem 5.2, these verifications can be done in linear time from the degree sequence of G. □

We do not known whether (∞, k)-polarity can be decided in linear time for $2K_2$-split graphs, but in the next proposition we prove that this problem can be solved in polynomial time.

Proposition 6.22. *Let k be an integer, $k \geq 2$. The problem of deciding whether a $2K_2$-split graph is (∞, k)-polar is solvable in quadratic time.*

Proof. Let $G = (C, S, I)$ be a $2K_2$-split graph, and let c and i be the number of vertices in C and I, respectively. If $i \leq k - 1$ and $\{u, v\}$ is a maximum clique of $G[S]$, then $(C \cup \{u, v\}, I \cup S \setminus \{u, v\})$ is an (∞, k)-polar partition of G. Else, if the subset I' of all vertices of degree c in G has at least $i - k + 2$ elements, then $(C \cup I', S \cup I \setminus I')$ is an (∞, k)-polar partition of G. Hence, if $i \leq k - 1$ or there are at most $i - k + 2$ vertices of degree c in G, then G is an (∞, k)-polar graph. Now, let us assume that $i \geq k$ and there are at most $i - k + 1$ vertices of G whose degree is c.

For each vertex $v \in C$, let I_v^* be the set of all vertices whose neighborhood is $C \setminus \{v\}$. It follows from Lemma 6.14 that G is an (∞, k)-polar graph if and only if I_v^* has at least $i - k + 2$ vertices for some $v \in C$. The result follows since all the verifications can be performed in quadratic time. □

As a consequence of Remark 5.5 and Theorem 5.7, deciding whether a pseudo-split graph admits an (s,k)-polar partition can be done in linear time from its degree sequence. In contrast, it cannot be decided in general whether a $2K_2$-split graph is (s,k)-polar only from its degree sequence. For instance, in Figure 6.2 are depicted two strict $2K_2$-split graphs with the same degree sequence such that the left one is $(5,4)$-polar but the right one is not. Despite of that, through the following propositions we will prove that the problem of recognizing $2K_2$-polar graphs that admit an (s,k)-polar partition is solvable in polynomial time.

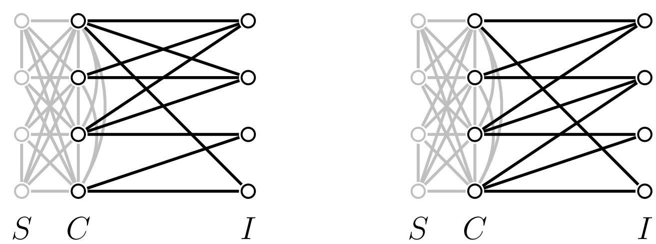


Figure 6.2: Two $2K_2$-split graphs with the same degree sequence such that the one on the left side is $(5,4)$-polar but the one on the right side is not.

Lemma 6.23. *Let $G = (S, K)$ be a split graph, and let k be a positive integer. Let S' be the set of all vertices in S that are completely nonadjacent to K. Then, G is a k-cluster if and only if the following statements hold true.*

1. *For each vertex $w \in S$, either $N(w) = \varnothing$ or $N(w) = K$.*
2. *$|S \setminus S'| \le 1$.*
3. *If $K \ne \varnothing$, then $|S'| \le k-1$. Otherwise $|S'| \le k$.*

Consequently, it can be decided whether a split graph is a k-cluster in linear time from its degree sequence.

Proof. The proposition can be easily verified if $K = \varnothing$, so we will assume for the proof that $K \ne \varnothing$. Notice that $K \cup S \setminus S'$ induces a component of G and the other components of G are trivial graphs induced by the singletons $\{w\}$ such that $w \in S'$. In consequence, G has exactly $1 + |S'|$ components.

Assume that G is a k-cluster. Since G is P_3-free, any vertex $w \in S$ is either completely adjacent or completely nonadjacent to K, and there is at most one vertex of S that is not an isolated vertex. Moreover, since $K \neq \varnothing$, it follows from our initial observation about the components that $|S'| \leq k-1$. Therefore, if G is a k-cluster, the three listed conditions hold. The consverse implication follows follows from the observations in the first paragraph of this proof and the third statment.

For the last part, suppose that G has degree sequence $d_1 \geq \cdots \geq d_n$, and let $p = \max\{i \colon d_i \geq i-1\}$. We have from Theorem 5.1 that $(S, K) = (\{v_{p+1}, \dots, v_n\}, \{v_1, \dots, v_p\})$ is a split partition of G such that K is a maximum clique. Then, we have from the characterization above that G is a k-cluster if and only if G has at most k components and S is completely nonadjacent to K. For the first condition, notice that G has at most k connected components if and only if either $p = 1$ and $n \leq k$ or $p > 1$ and $n - p \leq k - 1$, and this can be checked in linear time. The second condition is satisfied if and only if $d_1 = \cdots = d_p = p - 1$, which also can be verified in linear time. □

Lemma 6.24. *Let s and k be nonnegative integers such that $s + k \geq 1$. It can be decided whether a split graph is (s, k)-polar in linear time from its degree sequence.*

Proof. Let G be a split graph. Since split graphs are precisely the 1-polar graphs, if s and k are both positive integers, then G is (s, k)-polar. Otherwise, $s = 0$ or $k = 0$, and this case follows from Lemma 6.23. □

Theorem 6.25. *Let s and k be nonnegative integers such that $s + k \geq 1$. Deciding whether a $2K_2$-split graph is (s, k)-polar can be done in polynomial-time.*

Proof. Let $G = (C, S, I)$ be a $2K_2$-split graph, and let c and i be the number of vertices in C and I, respectively. Let us denote by I^* the set of all vertices of G of degree $c - 1$ and, for each vertex v in C, let I_v^* be the set of all vertices $w \in I$ such that $N(w) = C \setminus \{v\}$.

From Lemma 6.24 we have the result for the case in which $S = \varnothing$, so we can assume that G is a strict $2K_2$-split graph. In addition we have the following particular cases.

1. $2K_2$ is $(0, 2)$- and $(2, 1)$-polar but it is neither $(1, 1)$- nor $(\infty, 0)$-polar.

2. $K_1 \oplus 2K_2$ is $(1, 2)$- and $(2, 1)$-polar but it is not $(\infty, 0)$-, $(0, \infty)$-, or $(1, 1)$-polar.

3. For $c \geq 2$, $K_c \oplus 2K_2$ is $(2,1)$-polar but it is neither $(1,\infty)$- nor $(\infty,0)$-polar.

4. For $c \geq 1$, $iK_1 + 2K_2$ is $(0, i+2)$- and $(1,2)$-polar but it is neither $(0, i+1)$- nor $(\infty, 1)$-polar.

These cases correspond to the conditions $C = \varnothing$ or $I = \varnothing$, that can be checked in linear time from the degree sequence of G from Theorem 5.2.

From the above observations, we can assume for the rest of the proof that the sets C, S and I are all of them nonempty. We consider the following particular cases.

- If $s, k \leq 1$, then G is not (s,k)-polar, because $2K_2 \leq G$.

- If $k = 0$, then G is not a (s,k)-polar, because $2K_2 \leq G$.

- If $s = 0$ and $k \geq 2$, then G is not (s,k)-polar, because $K_1 \oplus 2K_2 \leq G$.

- If $k = 1$ and $s \geq 2$, then G is not (s,k)-polar, because $2K_2 + K_1 \leq G$.

- If $s = 1$ and $k \geq 2$, we have from Proposition 6.2 that G is an (s,k)-polar graph if and only if $c = 1$ and $|\{w \in I : \mathrm{d}(w) > 0\}| \leq k - 2$. This condition can be verified from the degree sequence of G in linear time.

Notice that, if none of the cases listed before occurs, then $s, k \geq 2$, so we can use the characterizations provided by Theorem 6.4. The following cases are based on the that characterizations.

1. If $c \leq s$ and $i \leq k - 2$, then G is an (s,k)-polar graph.

2. If $c \leq s - 2$ and $i \leq k - 1$, then G is an (s,k)-polar graph.

3. If $c \geq s + 1$ and $i \geq k$, then G is not an (s,k)-polar graph.

4. If $c > s$ and $i < k$, then G is an (s,k)-polar graph if and only if there exist at least $c - s + 2$ vertices whose degree is exactly $c + 3$. This condition can be verified from the degree sequence of G in linear time.

5. If $c < s$ and $i \geq k$. We can verify from the degree sequence of G if there exist at least $i - k + 2$ vertices of degree exactly c; if such vertices exist G is an (s,k)-polar graph. Otherwise, if $|I^*| < i - k + 2$, G is not an

(s, k)-polar graph, and if $|I^*| \geq i - k + 2$ we can check, for each vertex $v \in C$, whether the set I^*_v has at least $i - k + 2$ vertices; in this point G is an (s, k)-polar graph if and only if $|I^*_v| \geq i - k + 2$ for some $v \in C$. These verifications can be done in polynomial time.

6. If $c = s$ and $i \geq k$. If $|I^*| < i - k + 2$, G is not an (s, k)-polar graph. Otherwise, if $|I^*| \geq i - k + 2$, we can check for each vertex $v \in C$ whether the set I^*_v has at least $i - k + 2$ vertices; G is an (s, k)-polar graph if and only if $|I^*_v| \geq i - k + 2$ for some $v \in C$.

7. If $c = s$ and $i = k - 1$. If there exist at least two vertices of degree exactly $c + 3$, G is an (s, k)-polar graph. Otherwise, if $I^* = \varnothing$, G is not an (s, k)-polar graph, and if $I^* \neq \varnothing$, we can check for each vertex $v \in C$ whether the set I^*_v is empty; G is an (s, k)-polar graph if and only if $I^*_v \neq \varnothing$ for some $v \in C$.

8. If $c = s - 1$ and $i = k - 1$. First, if there exists a vertex of degree $c + 3$ or a vertex of degree c, G is an (s, k)-polar graph. Otherwise, if $I^* = \varnothing$, G is not an (s, k)-polar graph, and if $I^* \neq \varnothing$, we can check for each vertex $v \in C$ whether the set I^*_v is empty; G is an (s, k)-polar graph if and only if $I^*_v \neq \varnothing$ for some $v \in C$.

The result follows since all verifications can be performed in polynomial-time. □

Part IV

General obstructions for small patterns

Chapter 7

Some patterns of size at most 3

There are only two patterns of size one, namely (0) and (1), whose sets of minimal obstructions clearly are $\{K_2\}$ and $\{\overline{K_2}\}$, respectively. It can be easily checked that there are eight patterns of size two that cannot be reduced to a pattern of size 1, namely

$$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and their complements. These patterns describe the class of bipartite graphs, whose family of minimal obstructions is conformed by the odd length cycles, the class of complete bipartite graphs with minimal obstruction set $\{K_3, \overline{P_3}\}$, the class of split graphs, whose set of minimal obstructions is $\{2K_2, C_4, C_5\}$ [41], the class of complete split graphs, which can be characterized as the $\{C_4, \overline{P_3}\}$-free graphs, and the complement of some of the previous classes.

Complete lists of minimal obstructions are not known for all patterns of size three. Nevertheless, in [40] was given the complete list of minimal obstructions for each of the 12 patterns of size three without $*$ entries that cannot be reduced to a pattern of smaller size. We summarize these results in the following list by writing 6 of the mentioned patterns together with their minimal obstructions; the other six patterns are the complements of the listed ones, so their minimal obstructions can be easily obtained by a simple argument of graph complements.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad K_3, P_4, 2K_2 \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad K_4, \overline{P_3}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad P_4, K_1 + 2K_2, P_3 + 2K_1, 2K_3, \overline{K_1 + P_3}, K_2 \oplus \overline{K_2}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad C_4, P_4, 2K_2, \overline{K_1 + P_3} \qquad\qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \overline{3K_2}, \overline{P_3}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad 2K_2, K_1 + P_3, \overline{2K_1 + P_3}, K_1 \oplus C_4, P_4$$

In this part of the document, we make our contribution to the characterization of minimal obstructions for small patterns by giving complete lists of minimal obstructions for the patterns of size three with a single $*$ off the main diagonal.

Notice that, each pattern M of size k can be graphically represented with a bicolored graph G on the vertex set $\{v_1, \dots, v_k\}$ such that, for any distinct integers i and j with $i, j \in \{1, \dots, k\}$, v_i is a shaded vertex if and only if $M_{i,i} = 1$, v_i is adjacent to v_j if $M_{i,j} = 1$, v_i is not adjacent to v_j if $M_{i,j} = 0$, and v_i is possibly adjacent to v_j (represented with a dashed line) if $M_{i,j} = *$.

Thus, if we want to obtain a list of all patterns of size three with exactly one $*$ off the main diagonal, we can look up for all bicolored graphs of order three with exactly one dashed edge. It is easy to convince ourselves that there are exactly 20 of these graphs, namely the 10 graphs depicted in Figures 7.1 and 7.2, and their complements[1].

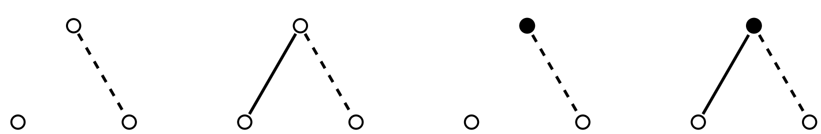


Figure 7.1: Some patterns of size three that can be reduced to patterns of size two.

[1] The complement of a bicolored graph with dashed edges is computed as the usual graph complement, but dashed edges remains unchanged, and vertices swap their colors.

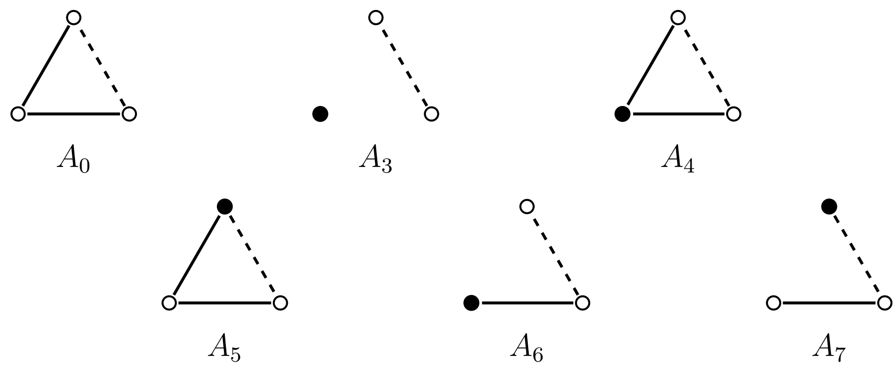


Figure 7.2: Patterns of size 3 with a single $*$ off the main diagonal that cannot be reduced to smaller patterns.

As we observed before, the minimal obstructions of a pattern can be deduced from the minimal obstructions of its complement so, for each pattern M in Figures 7.1 and 7.2 we will give complete lists of minimal obstructions only for one of M or $\overline{M}$. Moreover, patterns in Figure 7.1 can be reduced to smaller patterns by identifying both vertices in the bottom: the first two patterns reduce to the pattern associated to bipartite graphs, while the last two reduce to the pattern associated to split graphs. Thus, we only need to give the complete lists of minimal obstructions for the patterns in Figure 7.2.

In the next section we provide complete lists of minimal obstructions for patterns A_0, A_3, A_4 and $\overline{A_5}$. Then, in Section 7.2, we prove the analogous result for A_6, which requires a more involved - but still short - proof. Finally, in Chapter 8 we give the complete list of minimal $\overline{A_7}$-obstructions, for which a lot of technical lemmas are needed.

Some small graphs receive special names because of their drawings. Graphs in Figure 7.3 will appear frequently in this and the next chapter.

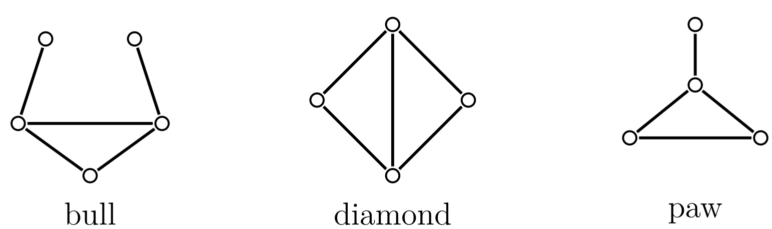
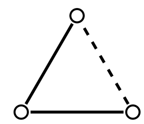


Figure 7.3: Some graphs with special names.

7.1 Four easy patterns

We start considering the pattern A_0:

$$A_0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & * \\ 1 & * & 0 \end{pmatrix}$$



Theorem 7.1. *If G is a graph, then G admits an A_0-partition if and only if G is an $\mathcal{F}$-free graph, where*

$$\mathcal{F} = \{\text{odd-hole}, K_4, K_1 + K_3, \text{house}, \text{bull}\}.$$

Proof. Clearly, a graph G admits an A_0-partition if and only if G has a (possible empty) independent set S such that $G - S$ is a bipartite graph and S is completely adjacent to $V_G \setminus S$. It easily follows from the previous observation that all graphs in $\mathcal{F}$ are minimal A_0-obstructions. Now, we prove that each $\mathcal{F}$-free graph G admits an A_0-partition.

If G is bipartite, it clearly has an A_0-partition. Otherwise, since G is odd-hole-free, G has a clique $C = \{v_1, v_2, v_3\}$. Notice that any vertex of G either is in C or is adjacent to some vertex in C, because G is $K_1 + K_3$-free. Additionally, since G does not have cliques on four vertices, any vertex of $G - C$ is adjacent to at most two vertices of C. For each $i \in \{1, 2, 3\}$, let us denote by V_i the subset of vertices of G whose only neighbor in C is v_i, and by $\overline{V_i}$ the subset of vertices $x \in V_G$ such that $C \setminus N(x) = \{v_i\}$. Observe that, if V_i and V_{i+i} are both nonempty sets, then G has either the house or the bull graph as an induced subgraph, contrary to our assumptions. Thus, we can assume without loss of generality that $V_2 = V_3 = \varnothing$, so $V_G = C \cup V_1 \cup \overline{V_1} \cup \overline{V_2} \cup \overline{V_3}$. Additionally, notice that $\overline{V_i}$ is an independent set for any $i \in \{1, 2, 3\}$ because G is a K_4-free graph.

First, let us assume $V_1 = \varnothing$, so $V_G = C \cup \overline{V_1} \cup \overline{V_2} \cup \overline{V_3}$. Notice that, $\mathcal{C} = (\{v_1\} \cup \overline{V_1}, \{v_2\} \cup \overline{V_2}, \{v_3\} \cup \overline{V_3})$ is a 3-proper coloring of G. Moreover, if $\overline{V_i}$ is completely adjacent to $\overline{V_{i+1}}$ for each $i \in \{1, 2, 3\}$, where additions are considered modulo 3, then G is a complete 3-partite graph with the same vertex partition, so $\mathcal{C}$ is also an A_0-partition of G. Thus, we can assume without loss of generality that there are nonadjacent vertices $u \in \overline{V_2}$ and $v \in \overline{V_3}$. We claim that in such a case $\mathcal{C}$ is still an A_0-partition of G. To prove our claim we only need to verify that $\overline{V_1}$ is completely adjacent to $\overline{V_2} \cup \overline{V_3}$. We start proving that any vertex in $\overline{V_1}$ is completely adjacent to $\{u, v\}$.

Let $w \in \overline{V_1}$ and, aiming for a contradiction, assume that w is not completely adjacent to $\{u, v\}$. Suppose without loss of generality that $wu \notin E$. As can be seen in the left part of Figure 7.4, if $wv \notin E$, then G has an induced copy of the bull graph, and otherwise G has an induced house graph,. Since both cases are impossible, we have a contradiction. Thus, it must be the case that $\overline{V_1}$ is completely adjacent to $\{u, v\}$.

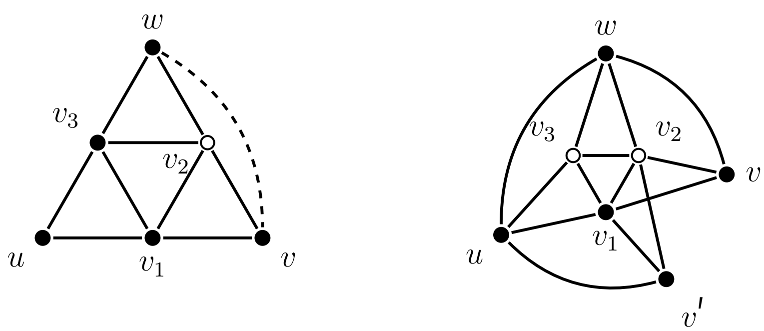


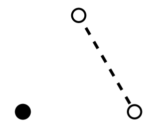
Figure 7.4: Cases of Theorem 7.1.

Now, suppose for a moment that $\overline{V_1}$ is not completely adjacent to $\overline{V_2} \cup \overline{V_3}$. Then, we can assume without loss of generality that there exist nonadjacent vertices $w \in \overline{V_1}$ and $v' \in \overline{V_3} \setminus \{v\}$. Notice that we can assume $v'u \in E$, otherwise we would be in the case of the previous paragraph. But then, as can be verified in the right side of Figure 7.4, G would have the house graph as an induced subgraph, but we are assuming this is not the case. Hence, $\overline{V_1}$ is completely adjacent to $\overline{V_2} \cup \overline{V_3}$, so $\mathcal{C}$ is an A_0-partition of G.

Since the case $V_1 = \varnothing$ is impossible, it must occur that $V_1 \neq \varnothing$. We claim that in this case there is a bipartition (X, Y) of $G - (\{v_1\} \cup \overline{V_1})$ such that $(\{v_1\} \cup \overline{V_1}, X, Y)$ is an A_0-partition of G. Clearly, $\{v_1\} \cup \overline{V_1}$ is an independent set, and v_1 is completely adjacent to $V_G \setminus (\{v_1\} \cup \overline{V_1}) = \{v_2, v_3\} \cup V_1 \cup \overline{V_2} \cup \overline{V_3}$. We have that $\overline{V_1}$ is completely adjacent to V_1 because G does not have any induced $K_1 + K_3$. Additionally, since G is a $\{\text{bull}, \text{house}\}$-free graph, it follows that $\overline{V_1}$ is completely adjacent to $\overline{V_2} \cup \overline{V_3}$. Thus, $\{v_1\} \cup \overline{V_1}$ is an independent set of G that is completely adjacent to $V_G \setminus (\{v_1\} \cup \overline{V_1})$, so we only need to prove that $G - (\{v_1\} \cup \overline{V_1})$ is a bipartite graph. But, G is a K_4-free graph and v_1 is completely adjacent to $\{v_2, v_3\} \cup V_1 \cup \overline{V_2} \cup \overline{V_3}$, so it follows that $G - (\{v_1\} \cup \overline{V_1})$ is triangle-free. Since G is an odd-hole-free graph, we conclude that $G - (\{v_1\} \cup \overline{V_1})$ is a bipartite graph. $\square$

Now, let us consider the pattern A_3:

$$A_3 = \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Theorem 7.2. *If G is a graph, then G admits an A_3-partition if and only if G is an $\mathcal{F}$-free graph, where*

$$\mathcal{F} = \{\text{odd-hole}, 2K_3, \text{diamond}, \text{paw}\}.$$

Proof. Clearly, a graph G is a minimal A_3-obstruction if and only if

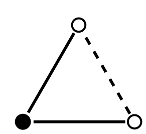
- G is not a complete graph,
- G is not a bipartite graph,
- G does not have a complete component K such that $G - K$ is a bipartite graph,
- Any vertex-deleted subgraph of G has an A_3-partition.

From here, it is a routine exercise to verify that all the graphs in $\mathcal{F}$ are minimal A_3-obstructions. Now let us prove that these are all of them. Let G be a minimal obstruction for A_3. Since G is not a bipartite graph, it contains an induced odd-cycle C. If C is a hole, then $G = C$, otherwise C is a triangle. Notice that, if G has two components, each one having a triangle, then $G \cong 2K_3$.

Assume that G is neither an odd-hole nor a copy of $2K_3$. By our previous observations, G has a non-complete component G_1 with a triangle C. Let K be the largest clique containing C. Since G_1 is non-complete, there must be a vertex v in G_1 that is not in K, and is adjacent to some but not all the vertices in K. If v is not adjacent to more than one vertex of K, then G is a paw. If v is not adjacent to exactly one vertex of K, then G is a diamond graph. □

We now turn our attention to the pattern A_4:

$$A_4 = \begin{pmatrix} 0 & * & 1 \\ * & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



Theorem 7.3. *If G is a graph, then G admits an A_4-partition if and only if G is an $\mathcal{F}$-free graph, where*

$$\mathcal{F} = \{\text{odd-hole}, K_1 + K_3, \text{bull}, \text{house}, \overline{K_2 + P_3}, \overline{3K_2}\}.$$

Proof. The property of admitting an A_4-partition is closed under addition and deletion of universal vertices. Hence, a graph admits an A_4-partition if and only if the deletion of all its universal vertices results in a (possibly null) bipartite graph. Also, it is clear that no minimal A_4-obstruction has universal vertices.

From here, it is a simple exercise to show that every graph in $\mathcal{F}$ is a minimal A_4-obstruction, so we only prove the converse implication. Let G be a minimal A_4-obstruction. Since G is not a bipartite graph, it has an induced odd-cycle C. If C is an odd-hole, then $G = C$. Otherwise, G contains a triangle with vertex set $\{u, v, w\}$. Since there are no universal vertices in G, each of u, v and w has a nonneighbor in G. Let x, y and z be nonneighbors of u, v and w, respectively.

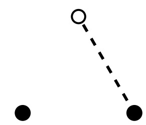
If $x = y = z$, then $\{u, v, w, x\}$ induces a $K_1 + K_3$. In the case that $\{x, y, z\}$ has cardinality 2, we can assume without loss of generality that $y = z$, and to avoid falling in the previous case, we have $uy \in E_G$. We again have two cases, namely, both v and w are adjacent to x, or precisely one of them is adjacent to x, since otherwise we may choose $x = y = z$. Assume first, without loss of generality that $vx \in E_G$ and $wx \notin E_G$. Hence, $\{u, v, w, x, y\}$ induces either a bull or a house, depending on whether $xy \in E_G$. If $vx, wx \in E_G$, then either $xy \in E_G$ and $\{u, v, w, x, y\}$ induces a copy of $\overline{K_2 + P_3}$, or $xy \notin E_G$ and $\{v, w, x, y\}$ induces a copy of $K_1 + K_3$.

Finally, if $\{x, y, z\}$ has cardinality 3, then we can assume that uy, uz, vx, vz, wx, and wy, are edges of G. Depending on which of the edges from the set $\{xy, yz, zx\}$ are present, we have the following possibilities. If all af them are present, then $\{u, v, w, x, y, z\}$ induces a copy of $\overline{3K_2}$. If none is present, then $\{v, w, x, y, z\}$ induces a copy of the bull graph. If only yz is present, then $\{v, w, x, y, z\}$ induces a copy of the house graph. If both xy and xz are present, then $\{u, v, w, x, y\}$ induces a copy of $\overline{K_2 + P_3}$.

Since the causes are exhaustive, the family $\mathcal{F}$ is the complete list of A_4-minimal obstructions. $\square$

To finish this section, we consider the pattern A_5, but we characterize the minimal $\overline{A_5}$-obstructions instead of the minimal A_5-obstructions.

$$\overline{A_5} = \begin{pmatrix} 0 & * & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Theorem 7.4. *If G is a graph, then G admits an $\overline{A_5}$-partition if and only if G is an $\mathcal{F}$-free graph, where*

$$\mathcal{F} = \{C_4, C_5, 3K_2, 2P_3, P_5, \overline{P}, K_1 \oplus 2K_2\}.$$

Proof. A graph G admits an $\overline{A_5}$-partition if and only if G is the disjoint union of a complete graph with a split graph. From here, it is a routine exercise to verify that any graph in $\mathcal{F}$ is a minimal $\overline{A_5}$-obstruction. Now we prove that any $\mathcal{F}$-free graph admits an $\overline{A_5}$-partition.

Let G be an $\mathcal{F}$-free graph. Since $3K_2 \in \mathcal{F}$, then G has at most two nontrivial components. Assume first that G has two nontrivial components, say G_1 and G_2. Because $2P_3 \in \mathcal{F}$, at most one of G_1 and G_2 is not a complete graph; assume without loss of generality that G_2 is a complete graph. Now, G_2 has at least one edge, so G_1 must be $2K_2$-free, as otherwise G would contain a copy of $3K_2$. Therefore, G_1, and hence $G - V_{G_2}$, is a split graph.

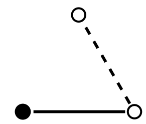
Suppose now that G has a unique nontrivial component, so we may actually assume that G is connected. We will show that, if G is not complete, then it is $2K_2$-free, and thus, it is a split graph. Consider two edges xy and zw in E_G, and, aiming for a contradiction, suppose that they are neither adjacent nor share an end. Let P be a shortest path from xy to zw, and assume without loss of generality that y and z are its first and last vertices, respectively. Thus, $P = (y, v_1, \dots, v_k, z)$. If $k \geq 3$, then the fist five vertices of P induce a copy of P_5, which is impossible. If $k = 2$, then $\{x, y, v_1, v_2, y\}$ induces either a copy of $\overline{P}$ or a copy of P_5, depending on whether $xv_1 \in E_G$ or not. Finally, if $k = 1$, then, depending on the presence of the edges xv_1 and wv_1, we have that $\{x, y, v_1, z, w\}$ induces a copy of P_5 (if none is present), $\overline{P}$ (if exactly one is present), or $K_1 \oplus 2K_2$ (if both are present). Since in any case a contradiction is reached, we conclude that xy and zw must be adjacent or share an end. Therefore, G is $2K_2$-free, and hence it is a split graph. □

7.2 Pattern A_6

This section is devoted to the pattern A_6. We will say that a partition (A, B, C) of the vertex set V of a graph G is an A_6-partition if and only if A and B

are independent sets, C is a clique, A is completely adjacent to C and B is completely nonadjacent to C.

$$A_6 = \begin{pmatrix} 0 & * & 1 \\ * & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$



Lemma 7.5. *Any odd-hole and any graph in Figure 7.5 is a minimal A_6-obstruction.*

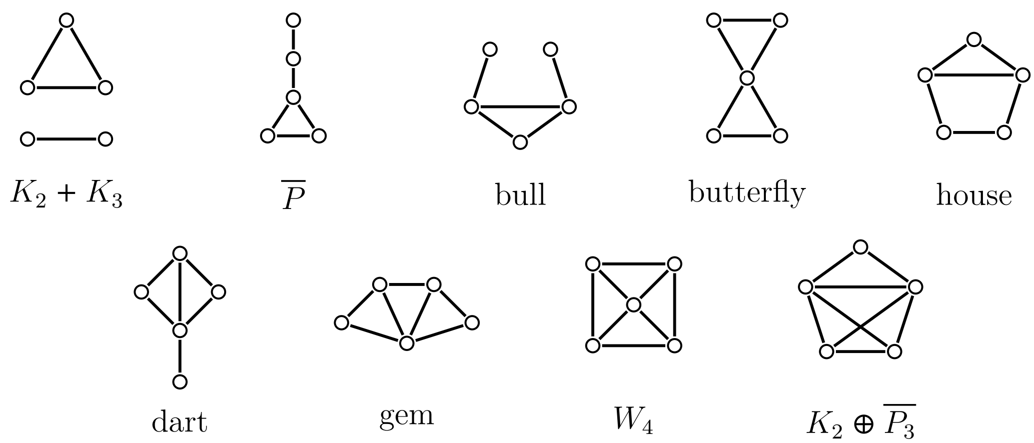


Figure 7.5: Some minimal A_6-obstructions.

Proof. Let G be either an odd-hole or a graph depicted in Figure 7.5. Since G has an induced odd cycle, G is not a bipartite graph so, for any A_6-partition (A, B, C) of G, C is a nonempty set. In addition, it is clear that for any $c \in C$, $N[c] = C \cup A$, so $N[c]$ induces a complete split graph, i.e., a $\{C_4, \overline{P_3}\}$-free graph, while $V_G \setminus N[c]$ is an independent set. It is a routine work to use the previous observations to corroborate that G has no A_6-partition. It is also an straightforward exercise to check that any vertex-deleted subgraph $G - v$ has such a partition. □

Along this section and the next chapter we will use the following notation. Let G be a graph and let H be an induced subgraph of G on the vertex set $\{v_1, \ldots, v_k\}$. For each subset S of $\{1, \ldots, k\}$ we define V_S to be the set of all vertices in $V_G \setminus V_H$ such that $N(v) \cap V_H = \{v_j : j \in S\}$, and $\overline{V_S}$ to be the set of all vertices in $V_G \setminus V_H$ such that $N(v) \cap V_H = V_H \setminus \{v_j : j \in S\}$. If S is a nonempty set, let us say $S = \{s_1, \ldots, s_t\}$, we will write $V_{s_1,\ldots,s_t}$ instead of V_S. Anytime, H will be an induced cycle or path, and it will be clear from the context.

Lemma 7.6. *Let G be a minimal A_6-obstruction with a triangle $C = (v_1, v_2, v_3)$. If G is not a graph mentioned in Lemma 7.5, then the following assertions hold for any integer $i \in \{1, \ldots, 4\}$:*

1. *$V_\varnothing$ is an independent set.*
2. *V_i is an independent set.*
3. *$\overline{V_i}$ is an independent set.*
4. *$\overline{V_\varnothing}$ induces a complete split graph.*
5. *V_i is completely nonadjacent to $V_\varnothing$.*
6. *$\overline{V_\varnothing}$ is completely nonadjacent to V_i.*
7. *$\overline{V_i}$ is completely adjacent to $\overline{V_\varnothing}$.*

Proof. We proceed by contradiction. The following assertions can be corroborated in Figure 7.6: if $V_\varnothing$ is not an independent set, then G has a copy of $K_2 + K_3$; if V_i is not an independent set, then G has the butterfly graph as an induced subgraph; if $\overline{V_i}$ is not an independent set, then G has a copy of $K_2 \oplus \overline{P_3}$; if $\overline{V_\varnothing}$ does not induce a complete split graph, G has some of $K_2 \oplus \overline{P_3}$ or W_4 as an induced subgraph.

Assertions below can be checked in Figure 7.7: if V_i is not completely nonadjacent to $V_\varnothing$, then G has $\overline{P}$ as an induced subgraph; if $\overline{V_\varnothing}$ is not completely nonadjacent to V_i, then G has an induced subgraph isomorphic to $K_2 \oplus \overline{P_3}$; if $\overline{V_i}$ is not completely adjacent to $\overline{V_\varnothing}$, then G has a copy of $K_2 \oplus \overline{P_3}$. □

Lemma 7.7. *Let G be a minimal A_6-obstruction with a triangle $C = (v_1, v_2, v_3)$. If G is not a graph mentioned in Lemma 7.5 and $V_1 \neq \varnothing$, then:*

1. *$V_2 = V_3 = \varnothing$.*
2. *$\overline{V_2} = \overline{V_3} = \varnothing$.*
3. *$\overline{V_\varnothing}$ is a clique.*

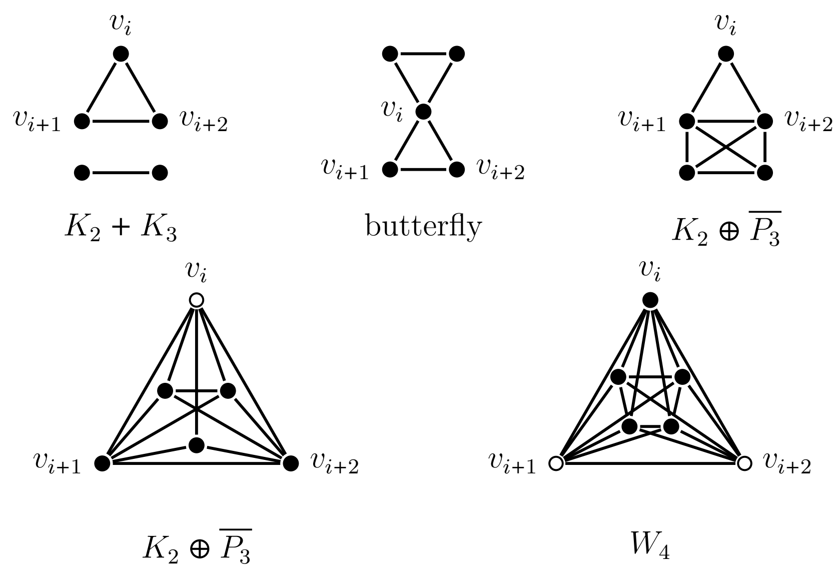


Figure 7.6: Cases 1 to 4 of Lemma 7.6.

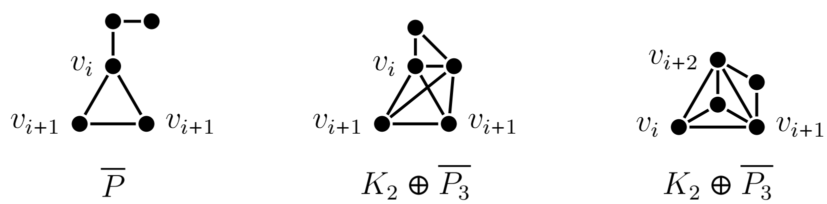


Figure 7.7: Cases 5 to 7 of Lemma 7.6.

4. *$\overline{V_\varnothing}$ is completely nonadjacent to $V_\varnothing$.*

5. *$(A, B, C) = (\{v_1\} \cup \overline{V_1}, V_1 \cup V_\varnothing, \{v_2, v_3\} \cup \overline{V_\varnothing})$ is an A_6-partition of G.*

Proof. We proceed by contradiction. The following assertions can be corroborated in Figure 7.8: if $V_2 \neq \varnothing$, then G has a copy of either the bull or the house graph; if $\overline{V_2} \neq \varnothing$, G has either the dart or the gem graph as an induced subgraph. By the symmetries of C_3 we have that also $V_3 = \overline{V_3} = \varnothing$. If $\overline{V_\varnothing}$ is not a clique, then G has a copy of the dart graph; if $\overline{V_\varnothing}$ is not completely nonadjacent to $V_\varnothing$, then G has a copy of either the bull or the house graph.

Item 5 follows from Lemma 7.6 an the first four items of this lemma. $\square$

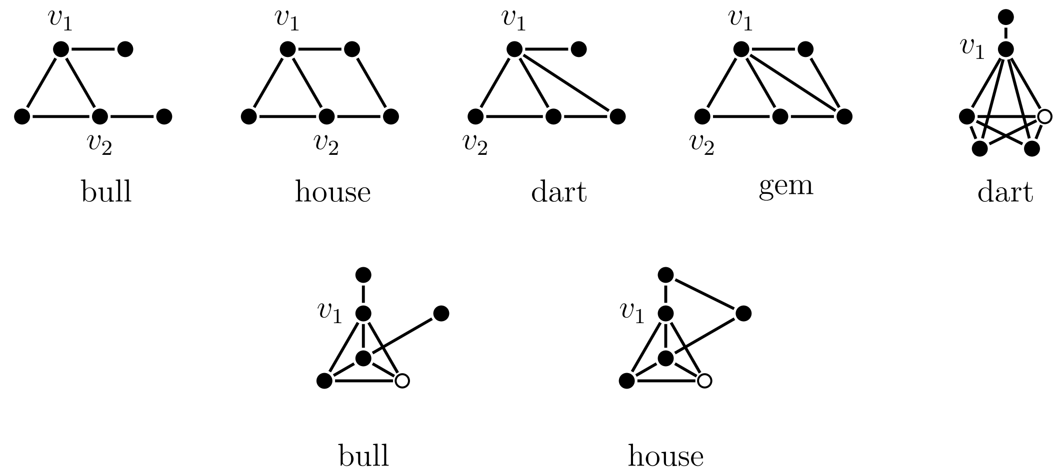


Figure 7.8: Cases of Lemma 7.7.

Lemma 7.8. *Let G be a minimal A_6-obstruction with a triangle $C = (v_1, v_2, v_3)$. If G is not a graph mentioned in Lemma 7.5, $\overline{V_1} \neq \varnothing$ and $V_i = \varnothing$ for any $i \in \{1, 2, 3\}$, then:*

1. $\overline{V_2} = \overline{V_3} = \varnothing$.
2. $\overline{V_\varnothing}$ *is a clique.*
3. $\overline{V_\varnothing}$ *is completely nonadjacent to* $V_\varnothing$.
4. $(A, B, C) = (\{v_1\} \cup \overline{V_1}, V_\varnothing, \{v_2, v_3\} \cup \overline{V_\varnothing})$ *is an A_6-partition of G.*

Proof. We proceed by contradiction. The following assertions can be corroborated in Figure 7.9. If $\overline{V_2} \neq \varnothing$, then G has a copy of either the gem graph or W_4, thus $\overline{V_2} = \varnothing$ and by the symmetries of C_3, $\overline{V_3} = \varnothing$ too. If $\overline{V_\varnothing}$ is not a clique, then G has W_4 as an induced subgraph; if $\overline{V_\varnothing}$ is not completely nonadjacent to $V_\varnothing$, then G has a copy of either the dart or the gem graph. Item 4 follows from Lemma 7.6 and the first three items of this lemma. □

Lemma 7.9. *Let G be a complete split graph that is not complete graph, and let M be the set of all vertices v of G such that there exists an induced path (u, v, w) in G. Then, $(K, S) = (M, V_G \setminus M)$ is the only complete split partition of G.*

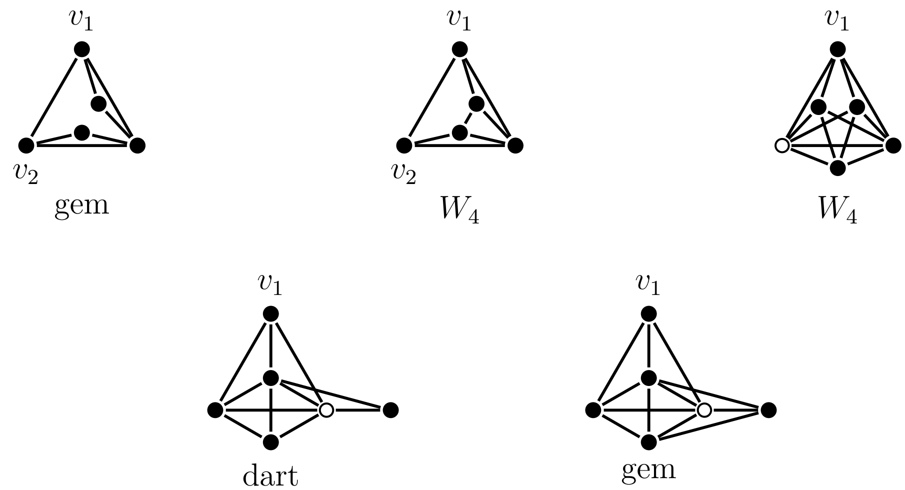


Figure 7.9: Cases of Lemma 7.8.

Proof. Let (K, S) be a complete split partition of G. Since G is not a complete graph, G has at least two non-universal vertices u and w, which clearly belongs to S. Therefore, for any $v \in K$, (u, v, w) is an induced path in G, so $K \subseteq M$.

Now, let $v \in M$, and let (u, v, w) be an induced path in G. Notice that v is adjacent to any vertex x of $G - v$, otherwise $G[\{u, v, w, x\}]$ would have some of $\overline{P_3}$ or C_4 as an induced subgraph, but that is impossible since G is a complete split graph. Hence, v is a universal vertex of G, and therefore it belongs to K because, due to G is not a complete graph, S has not universal vertices of G. We conclude that $M \subseteq K$, and therefore $K = M$, so the result follows. □

Theorem 7.10. *If G is a graph, then G admits an A_6-partition if and only if G is an $\mathcal{F}$-free graph, where $\mathcal{F}$ is the family of all odd-holes and all graphs depicted in Figure 7.5.*

Proof. Let H be a minimal A_6-obstruction. Notice that, by Lemma 7.5, it is enough to prove that H belongs to $\mathcal{F}$. Since H does not admit an A_6-partition, it is not a bipartite graph, so it has an odd-cycle as an induced subgraph. If H has an induced odd-hole, we have finished, so let us assume that H has no induced odd-hole, in which case H has a triangle $C = (v_1, v_2, v_3)$. Now, aiming for a contradiction, suppose that H is not a graph in Figure 7.5. If $V_1 \neq \varnothing$, we have from Lemma 7.7 that H has an A_6-partition, which is impossible. By the symmetries of C, the cases $V_2 \neq \varnothing$ and $V_3 \neq \varnothing$ are analogous, so we can assume that $V_i = \varnothing$ for any integer i, with $i \in \{1,2,3\}$. Now, if $\overline{V_1} \neq \varnothing$, if follows form Lemma 7.8 that H has an A_6-partition, what cannot occur. Again, by the symmetries of C we can assume that $\overline{V_i} = \varnothing$ for any $i \in \{1,2,3\}$.

At this point, we have that $V_H \setminus V_C = \overline{V_\varnothing} \cup V_\varnothing$. Observe that, if $\overline{V_\varnothing}$ is a clique, each vertex of $V_\varnothing$ is adjacent to at most one vertex of $\overline{V_\varnothing}$, otherwise H has a copy of $K_2 \oplus \overline{P_3}$ (see Figure 7.10). Moreover, there exists a vertex $v \in \overline{V_\varnothing}$ such that $V_\varnothing$ is completely nonadjacent to $\overline{V_\varnothing} \setminus \{v\}$, or H would have the bull graph as an induced subgraph as can be checked in Figure 7.10. But then, it is clear that $(A, B, C) = (\{v\}, V_\varnothing, V_C \cup \overline{V_\varnothing} \setminus \{v\})$ is an A_6-partition of H, which is not possible by the election of H.

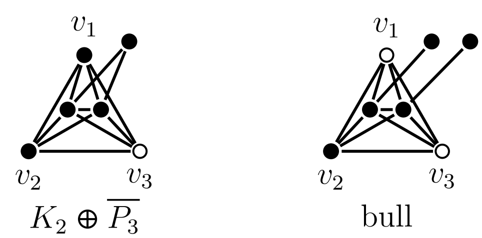


Figure 7.10: Some cases of Theorem 7.10.

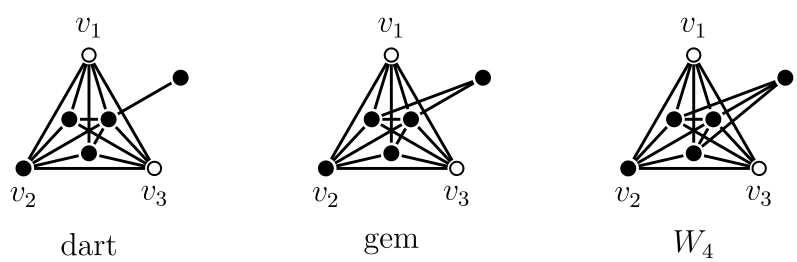


Figure 7.11: Some cases of Theorem 7.10.

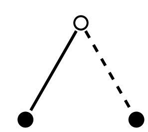
Thus, it must be the case that $\overline{V_\varnothing}$ is not a clique. As can be verified in Figure 7.11, $V_\varnothing$ is completely nonadjacent to the set M of all vertices v of $H[\overline{V_\varnothing}]$ such that there exists an induced path (u, v, w) in $H[\overline{V_\varnothing}]$, otherwise H would have a copy of the dart graph, the gem graph, or W_4. But then, we have from Lemma 7.9 that $(A, B, C) = (\overline{V_\varnothing} \setminus M, V_\varnothing, V_C \cup M)$ is an A_6-partition of H, which is absurd since H is an A_6-obstruction by hypothesis. The contradiction arose from assuming that H is not a graph in Figure 7.5, so it is. $\square$

Chapter 8

Pattern A_7

This chapter is devoted to pattern A_7 but, instead of characterizing the minimal A_7-obstructions, we will give the complete list of minimal $\overline{A_7}$-obstructions. We will say that a partition (A, B, C) of the vertex set V of a graph G is an $\overline{A_7}$-partition of G if and only if A is a stable set, B and C are cliques, A is completely adjacent to B, and B is completely nonadjacent to C.

$$\overline{A_7} = \begin{pmatrix} 0 & 1 & * \\ 1 & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$$



The characterization of minimal $\overline{A_7}$-obstructions is much more difficult to prove than any characterization given in Chapter 7, so we divided it in parts. We prove that the next statements hold for a family $\mathcal{F}$ of minimal $\overline{A_7}$-obstructions, concluding that $\mathcal{F}$ is the complete list of minimal $\overline{A_7}$-obstructions.

1. A minimal $\overline{A_7}$-obstruction with an induced C_5 is in $\mathcal{F}$.

2. A C_5-free minimal $\overline{A_7}$-obstruction with an induced C_4 is in $\mathcal{F}$.

3. A $\{C_4, C_5\}$-free minimal $\overline{A_7}$-obstruction with an induced P_5 is in $\mathcal{F}$.

4. A $\{P_5, C_4, C_5\}$-free minimal $\overline{A_7}$-obstruction with an induced P_4 is in $\mathcal{F}$.

5. A P_4-free minimal $\overline{A_7}$-obstruction is in $\mathcal{F}$.

Next lemma introduce the family $\mathcal{F}$ mentioned above.

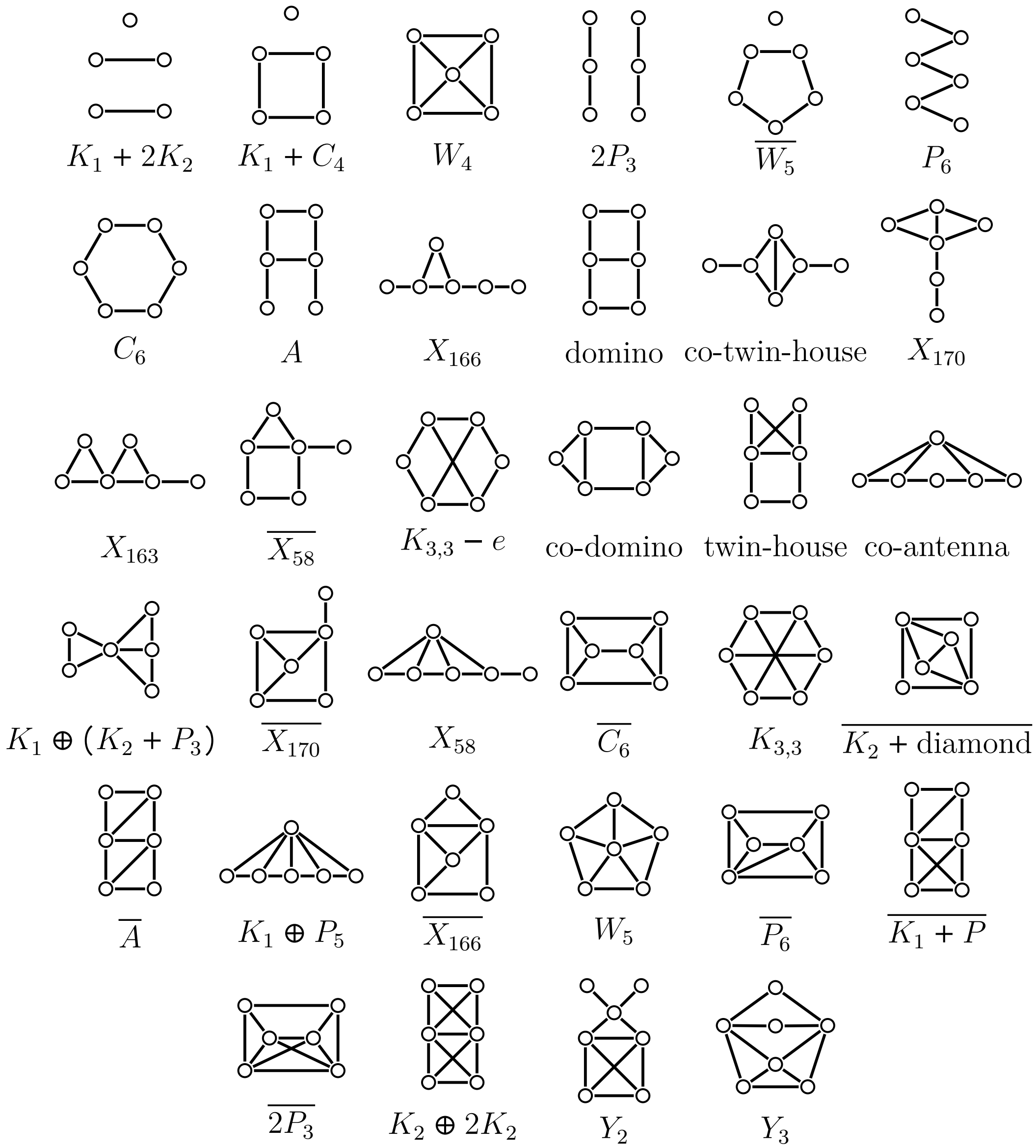


Figure 8.1: Minimal $\overline{A_7}$-obstructions.

Lemma 8.1. *The graphs depicted in Figure 8.1 are minimal $\overline{A_7}$-obstructions.*

Proof. It is a routine to verify that no graph G in Figure 8.1 admits an $\overline{A_7}$-partition, but any vertex-deleted subgraph does. Now, we describe a way to verify that G is an $\overline{A_7}$-obstruction. Clearly, G has a copy of $2K_2, C_4$ or C_5, so it is not a split graph. Therefore, if G has an $\overline{A_7}$-partition (A, B, C), B cannot be an empty set. In addition, since B is a clique and it is completely adjacent to A, for any $b \in B$, $A \cup B \subseteq N[b]$, so $V \setminus N[b] \subseteq C$ and then $V \setminus N[b]$ must be a clique. Also, since B is completely nonadjacent to C, for any $b \in B$, $N[b] \cap C = \varnothing$, so $N[b] \subseteq A \cup B$, in particular, since $A \cup B$ induces a complete split graph, $N[b]$ must induce a $\{\overline{P_3}, C_4\}$-free graph. From the previous observations, one can check that no vertex of G belongs to B, and therefore G does not admit an $\overline{A_7}$-partition. □

As we did in Section 7.2, for the rest of this chapter we use the following notation. Given a graph G, a fixed induced subgraph H of G with vertex set $\{v_1, \ldots, v_k\}$, and a subset S of $\{1, \ldots, k\}$, we denote by V_S and $\overline{V_S}$ the sets

$$\{u \in V_G \setminus V_H : N(v) \cap V_H = \{v_j : j \in S\}\},$$

and

$$\{u \in V_G \setminus V_H : N(v) \cap V_H = \{v_j : j \in \{1, \ldots, k\} \setminus S\}\},$$

respectively. To lighten up the notation, if $S = \{s_1, \ldots, s_t\}$ is not an empty set, we write $V_{s_1,\ldots,s_t}$ instead of V_S.

8.1 Obstructions with a C_5

In this section we prove that any minimal $\overline{A_7}$-obstruction that has an induced cycle of length five is a graph in Figure 8.1.

Lemma 8.2. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 5-cycle, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. If G is not a graph in Figure 8.1 the following statements are satisfied for any $i \in \{1, \ldots, 5\}$ (where the subscripts are considered modulo 5):*

1. *V_i is an independent set.*
2. *$V_{i,i+1}$ is a clique.*
3. *$V_{i,i+2}$ is an independent set.*
4. *$\overline{V_{i,i+1}}$ is a clique.*
5. *$\overline{V_{i,i+3}}$ is an independent set.*
6. *$\overline{V_i}$ is a clique.*

Proof. The following facts can be easily checked in Figure 8.2:

1. If V_i is not an independent set, then G contains a copy of $K_1 + 2K_2$.

2. If $V_{i,i+1}$ is not a clique, then G contains a copy of X_{170}.
3. If $V_{i,i+2}$ is not an independent set, then G contains a copy of $K_1 + 2K_2$.
4. If $\overline{V_{i,i+1}}$ is not a clique, then G contains a copy of W_4.
5. If $\overline{V_{i,i+3}}$ is not an independent set, then G contains a copy of $\overline{X_{170}}$.
6. If $\overline{V_i}$ is not a clique, then G contains a copy of W_4.

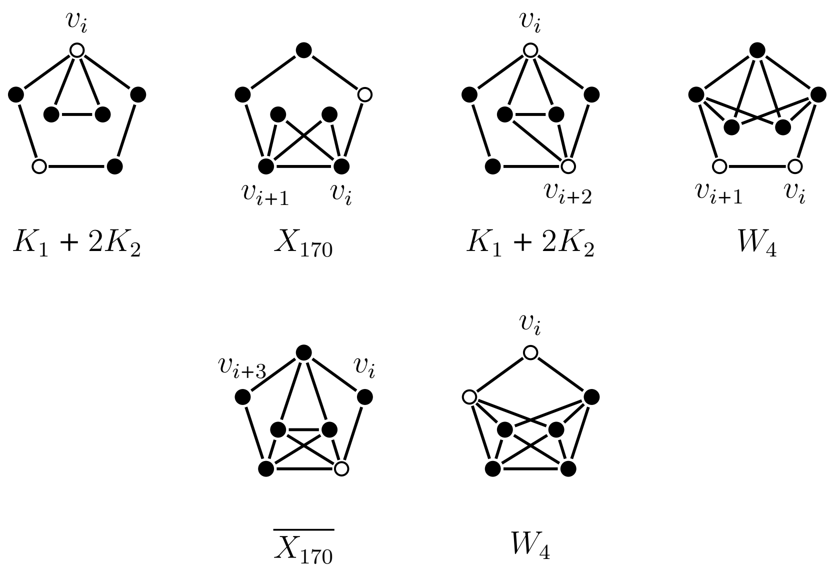


Figure 8.2: Cases for Lemma 8.2.

□

Lemma 8.3. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 5-cycle, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. If G is not a graph in Figure 8.1, and $i \in \{1, \ldots, 5\}$, the following pairs of sets are completely adjacent (where the subscripts are considered modulo 5):*

1. *V_i with $\overline{V_{i+2,i+3}}$.*
2. *$V_{i,i+1}$ with $\overline{V_{i+3,i+4}}$.*
3. *$V_{i,i+1}$ with $\overline{V_{i+3}}$.*
4. *$\overline{V_{i+1,i+2}}$ with $\overline{V_{i,i+3}}$.*
5. *$\overline{V_{i,i+1}}$ with $\overline{V_{i+1,i+2}}$.*
6. *$\overline{V_{i,i+1}}$ with $\overline{V_i}$.*

By symmetry, also the pairs $(V_{i,i+1}, \overline{V_{i+2,i+3}})$ *and* $(\overline{V_{i,i+1}}, \overline{V_{i+1}})$ *are completely adjacent sets of vertices.*

Proof. The following facts can be checked in Figure 8.3.

1. If V_i is not completely adjacent to $\overline{V_{i+2,i+3}}$, then G contains a copy of $\overline{W_5}$.
2. If $V_{i,i+1}$ is not completely adjacent to $\overline{V_{i+3,i+4}}$, then G contains a copy of X_{58}.
3. If $V_{i,i+1}$ is not completely adjacent to $\overline{V_{i+3}}$, then G contains a copy of X_{58}.
4. If $\overline{V_{i+1,i+2}}$ is not completely adjacent to $\overline{V_{i,i+3}}$, then G contains a copy of the co-domino graph.
5. If $\overline{V_{i,i+1}}$ is not completely adjacent to $\overline{V_{i+1,i+2}}$, then G contains a copy of X_{58}.
6. If $\overline{V_{i,i+1}}$ is not completely adjacent to $\overline{V_i}$, then G contains a copy of W_4.

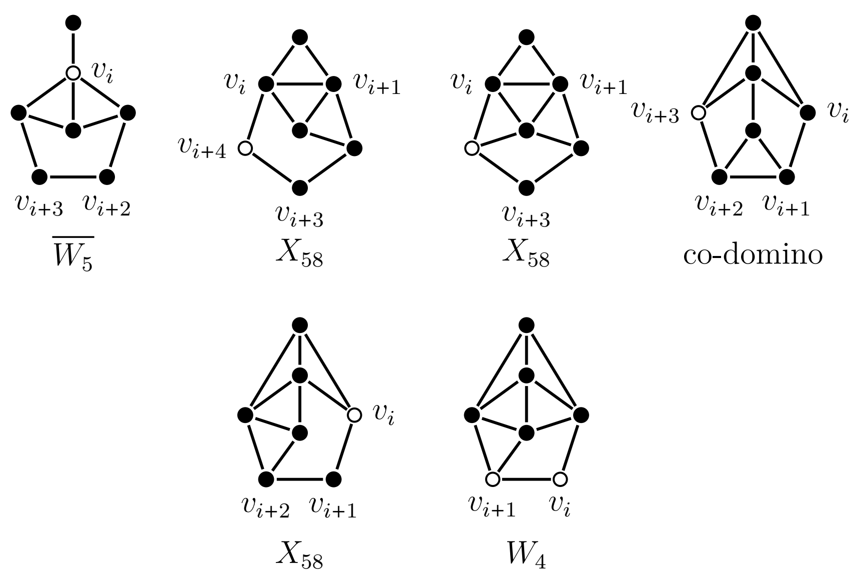


Figure 8.3: Cases for Lemma 8.3.

□

Lemma 8.4. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 5-cycle, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. If G is not a graph in Figure 8.1, and $i \in \{1, \ldots, 5\}$, the following pairs of sets are completely nonadjacent (where the subscripts are considered modulo 5):*

1. *V_i with $V_{i,i+2}$.*
2. *V_i with $V_{i+1,i+2}$.*
3. *V_i with $\overline{V_{i+1,i+4}}$.*
4. *$V_{i,i+1}$ with $\overline{V_{i,i+1}}$.*
5. *$V_{i,i+2}$ with $V_{i,i+3}$.*
6. *$V_{i,i+2}$ with $\overline{V_{i+1,i+4}}$.*
7. *$V_{i+1,i+2}$ with $\overline{V_{i+1,i+4}}$.*
8. *$\overline{V_{i,i+1}}$ with $\overline{V_{i+3,i+4}}$.*
9. *$\overline{V_{i,i+1}}$ with $\overline{V_{i+3}}$.*

By symmetry, also the pairs $(V_i, V_{i,i+3})$, $(V_i, V_{i+3,i+4})$, $(V_{i,i+2}, \overline{V_{i+1,i+3}})$, and $(V_{i+1,i+2}, \overline{V_{i+2,i+4}})$ are pairs of completely nonadjacent sets of vertices.

Proof. The following facts can be checked in Figure 8.4.

1. V_i is completely nonadjacent with $V_{i,i+2}$, otherwise G contains a copy of $K_1 + 2K_2$.
2. V_i is completely nonadjacent with $V_{i+1,i+2}$, otherwise G contains a copy of C_6.
3. V_i is completely nonadjacent with $\overline{V_{i+1,i+4}}$, otherwise G contains a copy of the co-antenna graph.
4. $V_{i,i+1}$is completely nonadjacent with $\overline{V_{i,i+1}}$, otherwise G contains a copy of the co-domino graph.
5. $V_{i,i+2}$ is completely nonadjacent with $V_{i,i+3}$, otherwise G contains a copy of A.
6. $V_{i,i+2}$ is completely nonadjacent with $\overline{V_{i+1,i+4}}$, otherwise G contains a copy of $\overline{X_{58}}$.
7. $V_{i+1,i+2}$ is completely nonadjacent with $\overline{V_{i+1,i+4}}$, otherwise G contains a copy of X_{58}.
8. $\overline{V_{i,i+1}}$ is completely nonadjacent with $\overline{V_{i+3,i+4}}$, otherwise G contains a copy of the co-domino graph.

9. $\overline{V_{i,i+1}}$ is completely nonadjacent with $\overline{V_{i+3}}$, otherwise G contains a copy of W_4.

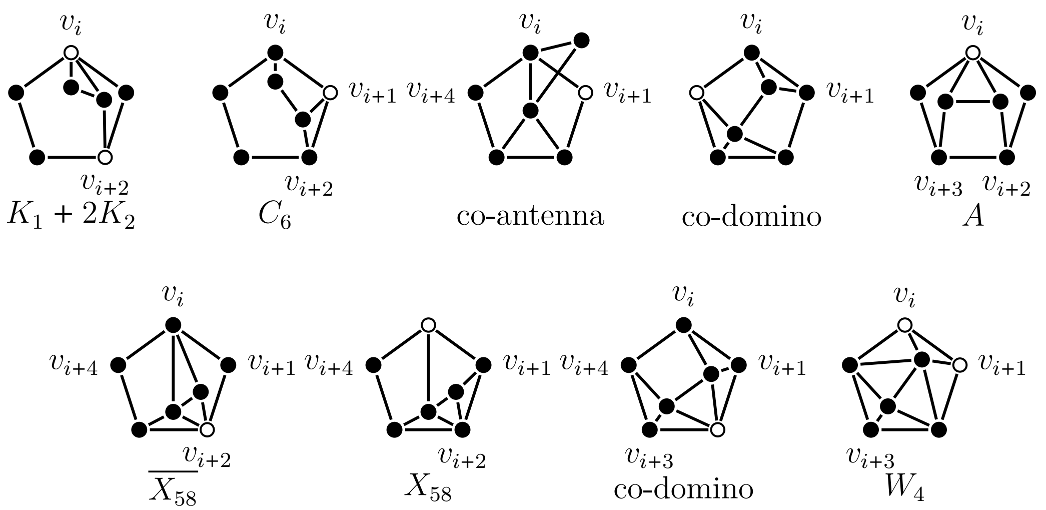


Figure 8.4: Cases for Lemma 8.4.

□

Lemma 8.5. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 5-cycle, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. If G is not a graph in Figure 8.1, and $i \in \{1, \ldots, 5\}$, the following statements are satisfied (where the subscripts are considered modulo 5):*

1. *If $V_i \neq \varnothing$, then V_{i+1} and V_{i+2} are empty sets. By symmetry, V_{i+3} and V_{i+4} are empty sets too.*

2. *If $V_i \neq \varnothing$, then $V_{i,i+1}$ and $V_{i+1,i+2}$ are empty sets. By symmetry, $V_{i,i+4}$ and $V_{i+3,i+4}$ are empty sets too.*

3. *If $V_i \neq \varnothing$, then $V_{i+1,i+3}$ and $V_{i+1,i+4}$ are empty sets. By symmetry, $V_{i+2,i+4}$ is also an empty set.*

4. *If $V_i \neq \varnothing$, then $\overline{V_{i+1,i+2}}$ is an empty set. By symmetry, $\overline{V_{i+3,i+4}}$ is also an empty set.*

5. *If $V_i \neq \varnothing$, then $\overline{V_{i,i+2}}$ and $\overline{V_{i+1,i+3}}$ are empty sets. By symmetry, $\overline{V_{i,i+3}}$ and $\overline{V_{i+2,i+4}}$ are empty sets too.*

6. *If* $V_i \neq \varnothing$*, then* $\overline{V_{i+1}}$ *and* $\overline{V_{i+2}}$ *are empty sets. By symmetry,* $\overline{V_{i+4}}$ *and* $\overline{V_{i+3}}$ *are empty sets too.*

7. *If* $V_{i,i+1} \neq \varnothing$*, then* $V_{i+1,i+2}$ *and* $V_{i+2,i+3}$ *are empty sets.*

8. *If* $V_{i,i+1} \neq \varnothing$*, then* $V_{i+1,i+4}$ *and* $V_{i+2,i+4}$ *are empty sets. By symmetry,* $V_{i,i+2}$ *is also an empty set.*

9. *If* $V_{i,i+1} \neq \varnothing$*, then* $\overline{V_{i,i+4}}$ *is an empty set. By symmetry,* $\overline{V_{i+1,i+2}}$ *is also an empty set.*

10. *If* $V_{i,i+1} \neq \varnothing$*, then* $\overline{V_{i,i+3}}$ *and* $\overline{V_{i+1,i+4}}$ *are empty sets. By symmetry,* $\overline{V_{i+1,i+3}}$ *and* $\overline{V_{i,i+2}}$ *are empty sets.*

11. *If* $V_{i,i+1} \neq \varnothing$*, then* $\overline{V_i}$ *and* $\overline{V_{i+4}}$ *are empty sets. By symmetry,* $\overline{V_{i+1}}$ *and* $\overline{V_{i+2}}$ *are empty sets.*

12. *If* $\overline{V_{i,i+3}} \neq \varnothing$*, then* $V_{i,i+2}$ *and* $V_{i,i+3}$ *are empty sets. By symmetry,* $V_{i+1,i+3}$ *is also an empty set.*

13. *If* $\overline{V_{i,i+3}} \neq \varnothing$*, then* $\overline{V_{i,i+1}}$ *is an empty set. By symmetry,* $\overline{V_{i+2,i+3}}$ *is an empty set.*

14. *If* $\overline{V_{i,i+3}} \neq \varnothing$*, then* $\overline{V_{i+1,i+4}}$ *and* $\overline{V_{i,i+2}}$ *are empty sets.*

15. *If* $\overline{V_{i,i+3}} \neq \varnothing$*, then* $\overline{V_i}$ *and* $\overline{V_{i+1}}$ *are empty sets. By symmetry,* $\overline{V_{i+3}}$ *and* $\overline{V_{i+2}}$ *are empty sets too.*

16. *If* $\overline{V_i} \neq \varnothing$*, then* $V_{i+1,i+3}$ *and* $V_{i+1,i+4}$ *are empty sets. By symmetry,* $V_{i+2,i+4}$ *is also an empty set.*

17. *If* $\overline{V_i} \neq \varnothing$*, then* $\overline{V_{i+1,i+2}}$ *is an empty set. By symmetry,* $\overline{V_{i+3,i+4}}$ *is also an empty set.*

18. *If* $\overline{V_i} \neq \varnothing$ *, then* $\overline{V_{i+1}}$ *and* $\overline{V_{i+2}}$ *are empty sets. By symmetry,* $\overline{V_{i+4}}$ *and* $\overline{V_{i+3}}$ *are empty sets too.*

19. *If* $V_{i,i+2} \neq \varnothing$*, then* $V_{i+1,i+3}$ *is an empty set.*

20. *If* $\overline{V_{i+1,i+2}} \neq \varnothing$*, then* $V_{i,i+3}$ *is an empty set.*

21. *If* $\overline{V_{i,i+1}}$ *and* $\overline{V_{i+1,i+2}}$ *are both nonempty sets, then* $\overline{V_{i+2,i+3}} = \varnothing$.

Proof. 1. If V_{i+1} is not empty, then G contains a copy of either $K_1 + 2K_2$ or P_6; if V_{i+2} is not empty, then G contains a copy of either P_6 or C_6 (see Figure 8.5).

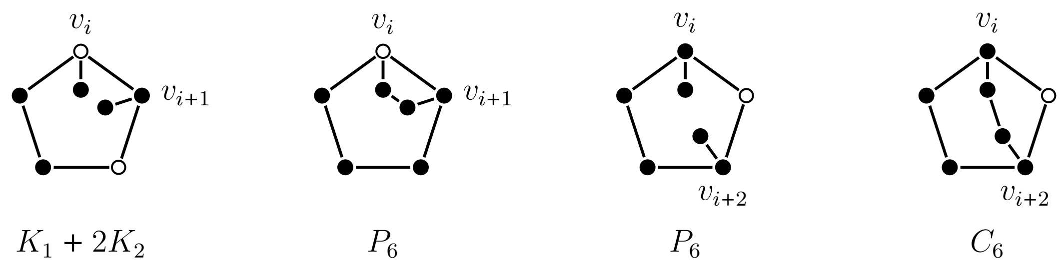


Figure 8.5: Cases for part 1 of Lemma 8.5.

2. If $V_{i,i+1}$ is not empty, then G contains a copy of either $K_1 + 2K_2$ or P_6; if $V_{i+1,i+2}$ is not empty, then G contains a copy of either P_6 or C_6 (see Figure 8.6).

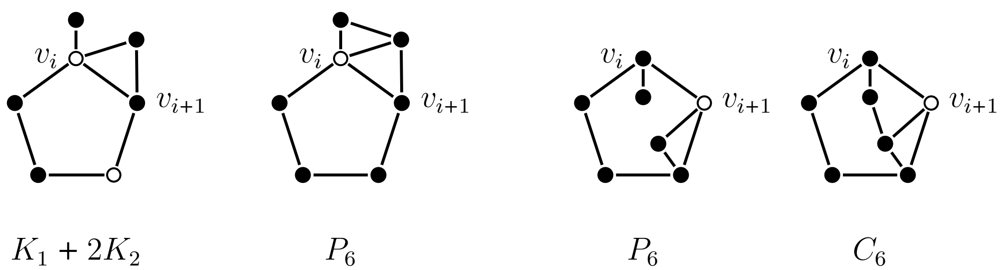


Figure 8.6: Cases for part 2 of Lemma 8.5.

3. If $V_{i+1,i+3}$ is not empty, then G contains a copy of either $K_1 + C_4$ or the domino graph; if $V_{i+1,i+4}$ is not empty, then G contains a copy of either $\overline{W_5}$ or $K_1 + C_4$ (see Figure 8.7).

4. If $\overline{V_{i+1,i+2}}$ is not an empty set, then G contains a copy of either $K_1 + 2K_2$ or X_{166} (see Figure 8.8).

5. If $\overline{V_{i,i+2}}$ is not an empty set, then G contains a copy of $K_1 + C_4$; if $\overline{V_{i+1,i+3}}$ is not an empty set, then G contains a copy of either $K_1 + C_4$ or A (see Figure 8.9).

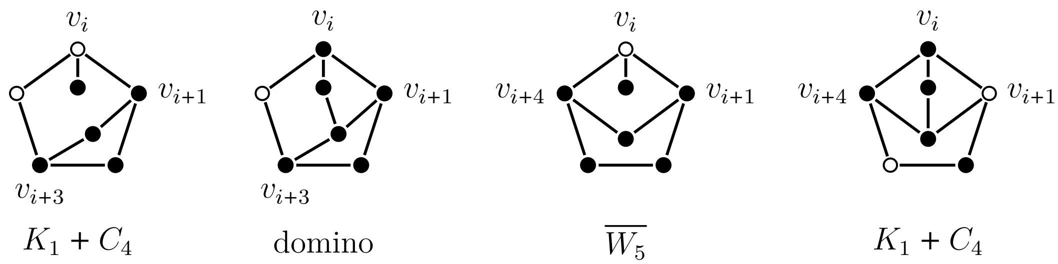


Figure 8.7: Cases for part 3 of Lemma 8.5.

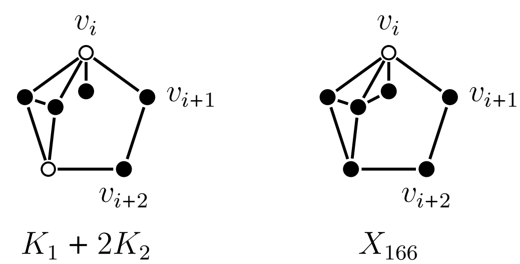


Figure 8.8: Cases for part 4 of Lemma 8.5.

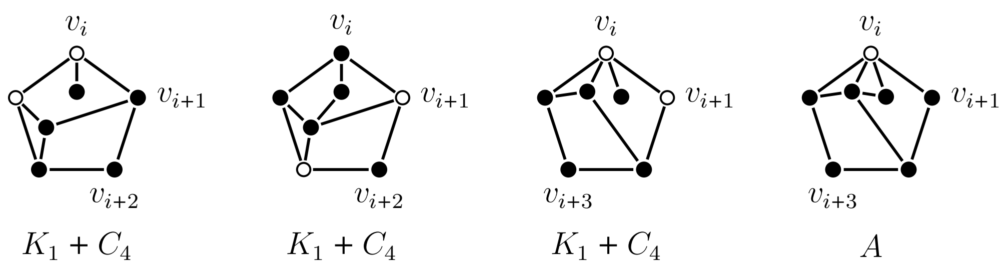


Figure 8.9: Cases for Part 5 of Lemma 8.5.

6. If $\overline{V_{i+1}}$ is not an empty set, then G contains a copy of either X_{58} or $K_1 \oplus P_5$; if $\overline{V_{i+2}}$ is not an empty set, then G contains a copy of either $K_1 + C_4$ or $\overline{X_{58}}$ (see Figure 8.10).

7. If $V_{i+1,i+2}$ is not empty, then G contains a copy of either P_6 or C_6; if $V_{i+2,i+3}$ is not empty, then G contains a copy of either X_{166} or $K_1 + C_4$ (see Figure 8.11).

8. If $V_{i+1,i+4}$ is not an empty set, then G contains a copy of either $\overline{X_{58}}$ or

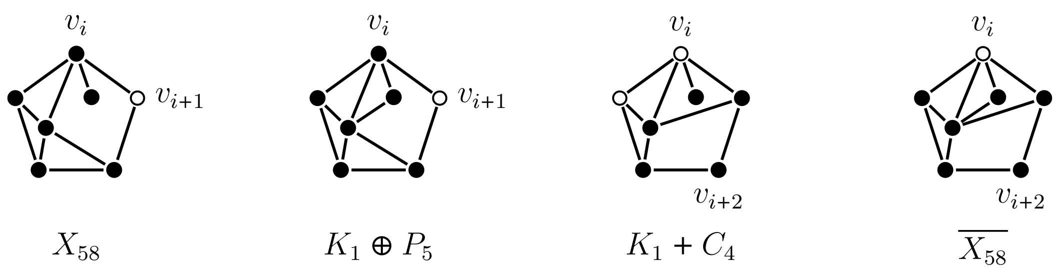


Figure 8.10: Cases for part 6 of Lemma 8.5.

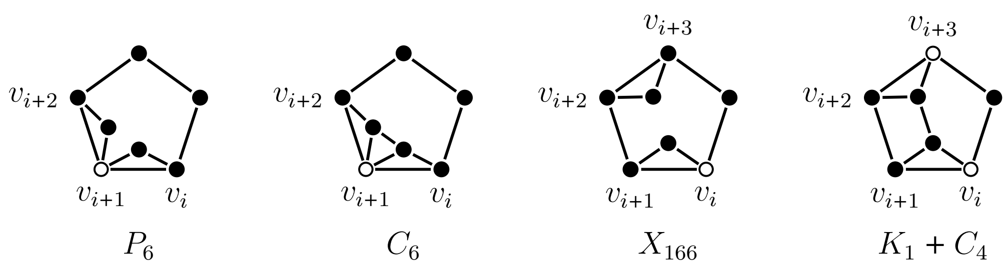


Figure 8.11: Cases for part 7 of Lemma 8.5.

$K_1 + C_4$; if $V_{i+2,i+4}$ is not an empty set, then G contains a copy of either $K_1 + C_4$ or the domino graph (see Figure 8.12).

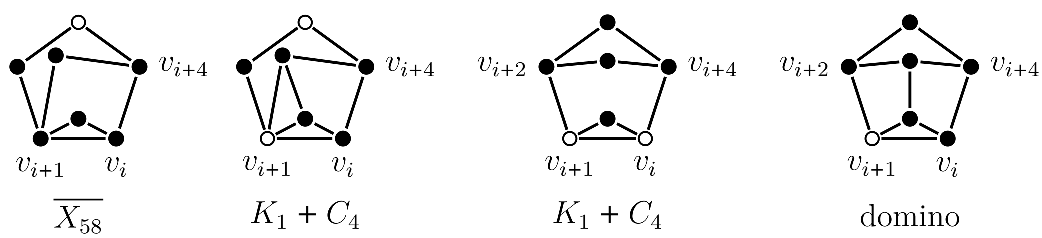


Figure 8.12: Cases for part 8 of Lemma 8.5.

9. If $\overline{V_{i,i+4}}$ is not an empty set, then G contains a copy of either the co-twin-house or X_{58} (see Figure 8.13).

10. If $\overline{V_{i,i+3}}$ is not an empty set, then G contains a copy of either $K_1 + C_4$ or X_{58}; if $\overline{V_{i+1,i+4}}$ is not an empty set, then G contains a copy of either the co-domino graph or $K_1 + C_4$ (see Figure 8.14).

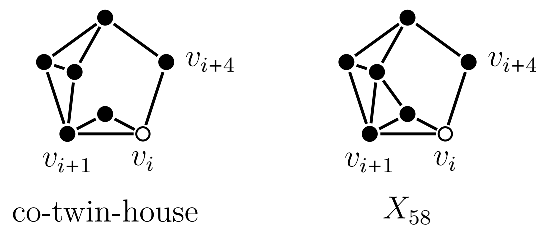


Figure 8.13: Cases for part 9 of Lemma 8.5.

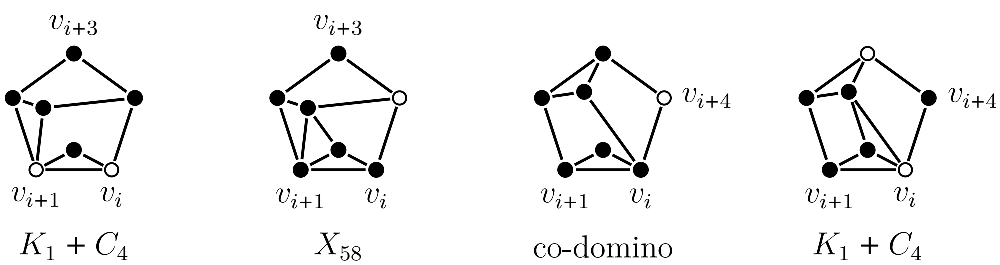


Figure 8.14: Cases for part 10 of Lemma 8.5.

11. If $\overline{V_i}$ is not an empty set, then G contains a copy of either X_{58} or $K_1 \oplus P_5$; if $\overline{V_{i+4}}$ is not an empty set, then G contains a copy of either the co-twin-house or the co-antenna graph (see Figure 8.15).

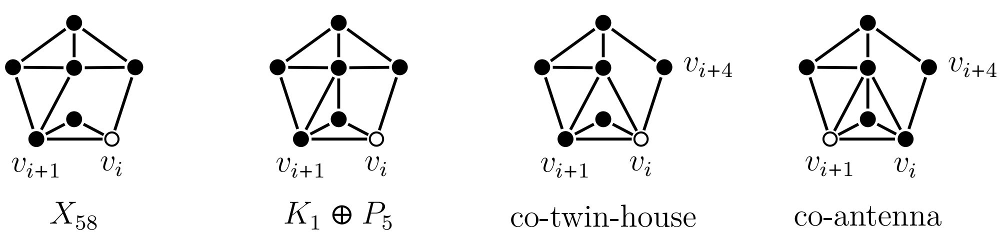
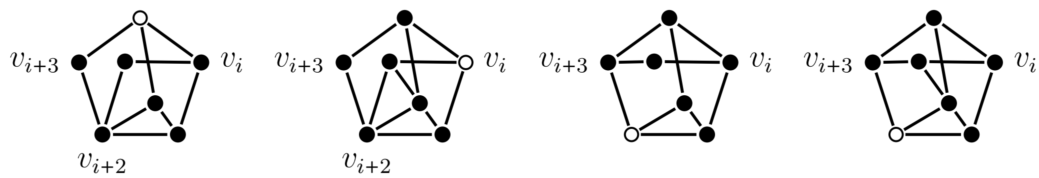


Figure 8.15: Cases for part 11 of Lemma 8.5.

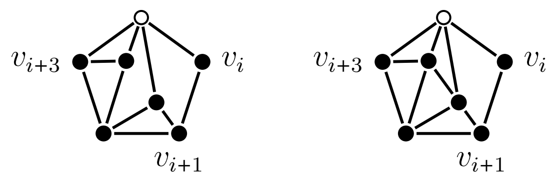
12. If $V_{i,i+2}$ is not an empty set, then G contains a copy of either $\overline{X_{58}}$ or the twin-house graph; if $V_{i,i+3}$ is not an empty set, then G contains a copy of either the domino graph or $K_{3,3} - e$ (see Figure 8.16).

13. If $\overline{V_{i,i+1}}$ is not an empty set, then G contains a copy of either X_{163} or X_{58} (see Figure 8.17).



$\overline{X_{58}}$ twin-house domino $K_{3,3} - e$

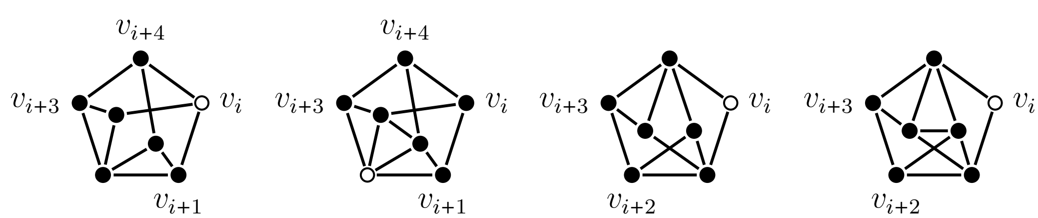
Figure 8.16: Cases for part 12 of Lemma 8.5.



X_{163} X_{58}

Figure 8.17: Cases for part 13 of Lemma 8.5.

14. If $\overline{V_{i+1,i+4}}$ is not an empty set, then G contains a copy of either the co-antenna graph or $K_{3,3}-e$; if $\overline{V_{i,i+2}}$ is not an empty set, then G contains a copy of either $\overline{C_6}$ or $\overline{P_6}$ (see Figure 8.18).



co-antenna $K_{3,3} - e$ $\overline{C_6}$ $\overline{P_6}$

Figure 8.18: Cases for part 14 of Lemma 8.5.

15. If $\overline{V_i}$ is not an empty set, then G contains a copy of either $\overline{P_6}$ or $\overline{X_{166}}$; if $\overline{V_{i+1}}$ is not an empty set, then G contains a copy of either $\overline{C_6}$ or $\overline{P_6}$ (see Figure 8.19).

16. If $V_{i+1,i+3}$ is not an empty set, then G contains a copy of either H or H; if $V_{i+1,i+4}$ is not an empty set, then G contains a copy of either H or H

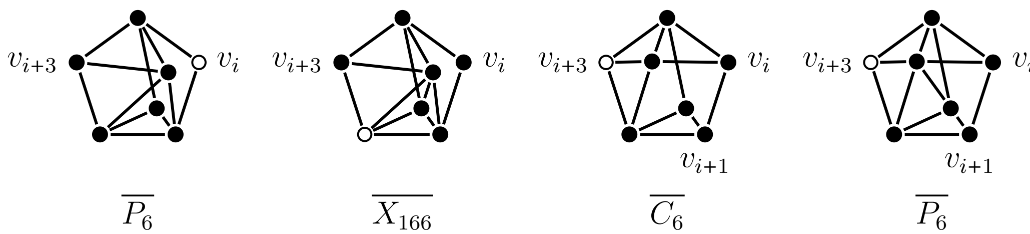


Figure 8.19: Cases for part 15 of Lemma 8.5.

(see Figure 8.20).

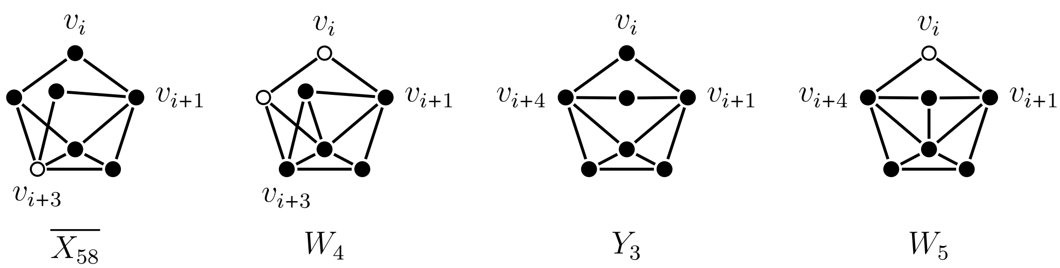


Figure 8.20: Cases for part 16 of Lemma 8.5.

17. If $\overline{V_{i+1,i+2}}$ is not an empty set, then G contains a copy of either $\overline{A}$ or $\overline{K_1 + P}$ (see Figure 8.21).

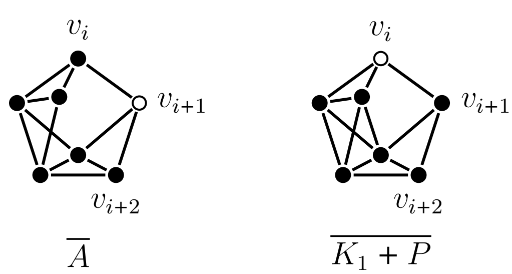


Figure 8.21: Cases for part 17 of Lemma 8.5.

18. If $\overline{V_{i+1}}$ is not empty, then G contains a copy of either W_4 or $\overline{P_6}$; if $\overline{V_{i+2}}$ is not empty, then G contains a copy of either $\overline{P_6}$ or W_4 (see Figure 8.22).

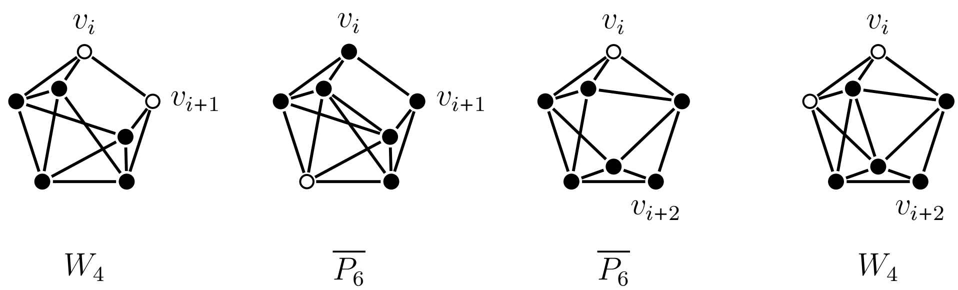


Figure 8.22: Cases for part 18 of Lemma 8.5.

19. If $V_{i+1,i+3}$ is not empty, then G contains a copy of either A or $K_{3,3} - e$ (see Figure 8.23).

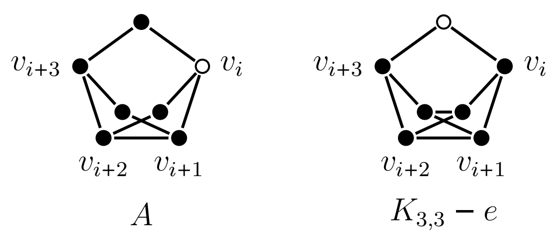


Figure 8.23: Cases for part 19 of Lemma 8.5.

20. If $V_{i,i+3}$ is not an empty set, then G contains a copy of either $\overline{X_{170}}$ or W_4 (see Figure 8.24).

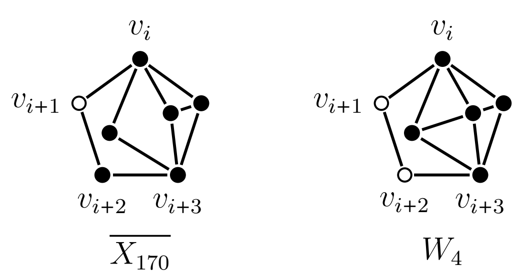
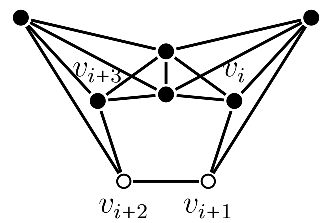


Figure 8.24: Cases for part 20 of Lemma 8.5.

21. Considering Lemma 8.3 and Lemma 8.4, we have that if $\overline{V_{i+2,i+3}}$ is not empty, then G contains a copy of $K_2 \oplus 2K_2$ (see Figure 8.25).



$K_2 \oplus 2K_2$

Figure 8.25: Cases for part 21 of Lemma 8.5.

□

Lemma 8.6. *Let G be a minimal $\overline{A_7}$-obstruction different from the graphs in Figure 8.1 and such that it contains an induced 5-cycle, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. Suppose that some of the sets $V_1, V_{3,4}, \overline{V_{2,5}}$, or $\overline{V_1}$ is not empty. Then, the following statements hold.*

1. *Any vertex of $G - V_C$ lies in some of $V_1, V_{3,4}, V_{1,3}, V_{1,4}, \overline{V_{3,4}}, \overline{V_{1,2}}, \overline{V_{1,5}}, \overline{V_{2,5}}$, or $\overline{V_1}$.*

2. *$\overline{V_{3,4}}$ is completely adjacent to $V_{1,3} \cup V_{1,4}$.*

Proof. Since G is a minimal $\overline{A_7}$-obstruction different from the graphs depicted in Figure 8.1, G is a $\{W_5, \overline{W_5}\}$-free graph, so $V_\varnothing = \overline{V_\varnothing} = \varnothing$. First, suppose that $V_1 \neq \varnothing$.

1. By Item 1 of Lemma 8.5, $V_2 = V_3 = V_4 = V_5 = \varnothing$.

2. By Item 2 of Lemma 8.5, $V_{1,2} = V_{1,5} = V_{2,3} = V_{4,5} = \varnothing$.

3. By Item 3 of Lemma 8.5, $V_{2,4} = V_{2,5} = V_{3,5} = \varnothing$.

4. By Item 4 of Lemma 8.5, $\overline{V_{2,3}} = \overline{V_{4,5}} = \varnothing$.

5. By Item 5 of Lemma 8.5, $\overline{V_{1,3}} = \overline{V_{2,4}} = \overline{V_{3,5}} = \overline{V_{1,4}} = \varnothing$.

6. By Item 6 of Lemma 8.5, $\overline{V_2} = \overline{V_5} = \overline{V_3} = \overline{V_4} = \varnothing$.

Thus, any vertex of $G - V_C$ lies in some of $V_1, V_{3,4}, V_{1,3}, V_{1,4}, \overline{V_{3,4}}, \overline{V_{1,2}}, \overline{V_{1,5}}, \overline{V_{2,5}}$, or $\overline{V_1}$. In a similar way, Items 2 and 7 to 11 of Lemma 8.5 imply the same result when $V_{3,4} \neq \varnothing$, Items 5, 10 and 12 to 15 do the same if $\overline{V_{2,5}} \neq \varnothing$, and Items 6, 11 and 15 to 18 allow us to conclude the same in the case $\overline{V_1} \neq \varnothing$.

For the second statement we only need to check that $\overline{V_{3,4}}$ is completely adjacent to $V_{1,3}$, because from here it follows by symmetry that $\overline{V_{3,4}}$ is completely adjacent to $V_{1,4}$. This can be done using Lemmas 8.3 and 8.4 as follows (see Figure 8.26). Aiming for a contradiction, suppose that there exist nonadjacent vertices $u \in \overline{V_{3,4}}$ and $v \in V_{1,4}$: if $V_1 \neq \varnothing$, G contains a copy of X_{170}; if $V_{3,4} \neq \varnothing$, G contains a copy of either X_{166} or the co-domino graph; if $\overline{V_{2,5}} \neq \varnothing$, G contains a copy of $K_1 + C_4$; if $\overline{V_1} \neq \varnothing$, G contains a copy of either $K_1 + C_4$ or $\overline{X_{58}}$. □

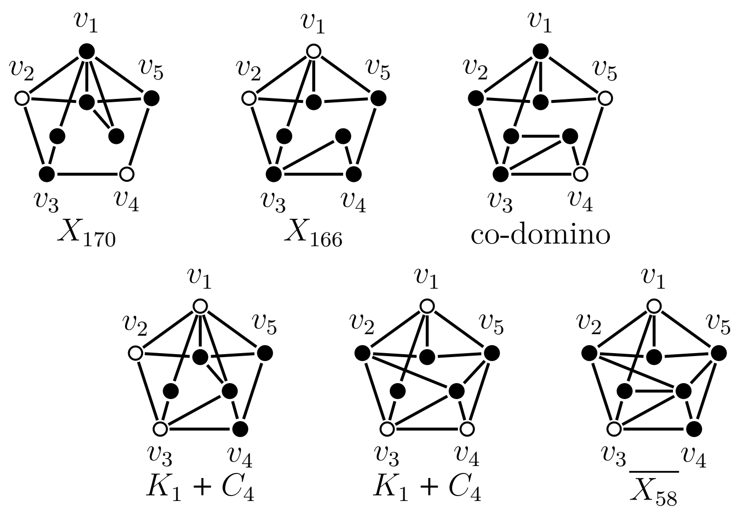


Figure 8.26: Cases for Lemma 8.6.

The following useful observation follows directly from Lemmas 8.2 to 8.4

Remark 8.7. *Let G be a minimal $\overline{A_7}$-obstruction having an induced 5-cycle, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$. The following statements hold.*

1. *The set $A = \{v_2, v_5\} \cup V_1 \cup V_{1,3} \cup V_{1,4} \cup \overline{V_{2,5}}$ is an independent set.*

2. *The set $B = \{v_1\} \cup \overline{V_{3,4}}$ is a clique.*

3. *The set $C = \{v_3, v_4\} \cup V_{3,4} \cup \overline{V_{1,2}} \cup \overline{V_{1,5}} \cup \overline{V_1}$ is a clique.*

4. *B is completely nonadjacent to C.*

5. *A is completely adjacent to $B \setminus \overline{V_{3,4}}$.*

6. *$A \setminus (V_{1,3} \cup V_{1,4})$ is completely adjacent to B.*

Therefore, in order for (A, B, C) to be an $\overline{A_7}$-partition of $G[A \cup B \cup C]$, we only need to check that $\overline{V_{3,4}}$ is completely adjacent to $V_{1,3} \cup V_{1,4}$.

Lemma 8.8. *Let G be a minimal $\overline{A_7}$-obstruction. If G has an induced 5-cycle, then G is isomorphic to some graph in Figure 8.1.*

Proof. Aiming for a contradiction, assume that there exists a minimal $\overline{A_7}$-obstruction G with an induced C_5, $C = (v_1, v_2, v_3, v_4, v_5, v_1)$, and different from the graphs in Figure 8.1. Since G does not admit an $\overline{A_7}$-partition, we have that $V_G \setminus V_C \neq \varnothing$. Notice that $\overline{V_\varnothing} = V_\varnothing = \varnothing$, otherwise G would contain a copy of either W_5 or its complement.

Case 1. Let us suppose that some of $V_1, V_{3,4}, \overline{V_{2,5}}$, or $\overline{V_1}$ is not an empty set. By Lemma 8.6 we have that any vertex of $G - V_C$ lies in some of $V_1, V_{3,4}, V_{1,3}, V_{1,4}, \overline{V_{3,4}}, \overline{V_{1,2}}, \overline{V_{1,5}}, \overline{V_{2,5}}$, or $\overline{V_1}$. But in such a case it follows from Remark 8.7 and Lemma 8.6 that

$$(A, B, C) = (\{v_2, v_5\} \cup V_1 \cup V_{1,3} \cup V_{1,4} \cup \overline{V_{2,5}}, \{v_1\} \cup \overline{V_{3,4}}, \{v_3, v_4\} \cup V_{3,4} \cup \overline{V_{1,2}} \cup \overline{V_{1,5}} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, contradicting that G is an $\overline{A_7}$-obstruction.

Case 2. If Case 1 does not occur, then $V_1 = V_{3,4} = \overline{V_{2,5}} = \overline{V_1} = \varnothing$. Moreover, by the symmetries of C_5, we can assume that any vertex of $G - V_C$ lies in a set of the the form $V_{i,i+2}$ or $\overline{V_{i,i+1}}$ for some $i \in \{1, \ldots, 5\}$, where the subscripts are considered modulo 5. Notice that, from Item 19 of Lemma 8.5, we have that for some $i \in \{1, \ldots, 5\}$, $V_{i+1,i+3} = V_{i+2,i+4} = V_{i+1,i+4} = \varnothing$.

Now, if $\overline{V_{i,i+1}} = \varnothing$ for every integer i, $1 \leq i \leq 5$, we can assume without loss of generality that $V_G \setminus V_C \subseteq V_{1,3} \cup V_{1,4}$, but then it follows from Remark 8.7 that

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,3} \cup V_{1,4}, \{v_1\}, \{v_3, v_4\})$$

is an $\overline{A_7}$-partition of G, in contradiction with our hypotheses. Thus $\overline{V_{i,i+1}} \neq \varnothing$ for some subscript i.

Case 2.1. Suppose that $\overline{V_{i,i+1}}$ and $\overline{V_{i+2,i+3}}$ are both nonempty sets for some integer i, so by our previous observations $\overline{V_{i+1,i+2}} = \varnothing$; assume $i = 1$ without loss of generality. By Item 20 of Lemma 8.5, $V_{3,5}$ and $V_{2,5}$ are both empty sets. In addition, from Item 19 of Lemma 8.5, some of $V_{2,4}$ or $V_{1,3}$ is empty, and by symmetry we can assume without loss of generality that $V_{2,4} = \varnothing$. Notice that $V_{1,3}$ is completely adjacent to $\overline{V_{3,4}}$, otherwise G would contain either $K_1 + 2K_2$ or X_{170} as an induced subgraph as can be verified in Figure 8.27.

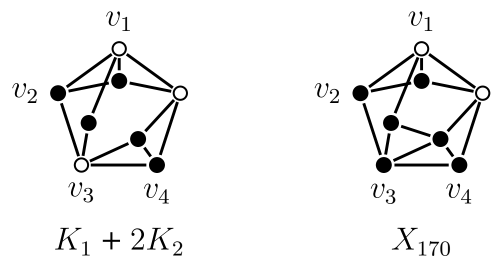


Figure 8.27: Cases for part 2.1 of Lemma 8.8.

Also notice that, by Item 20 of Lemma 8.5, $V_{1,3}$ and $\overline{V_{4,5}}$ cannot be simultaneously nonempty sets, so we have the following cases.

Case 2.1.1. Assume that $V_{1,3} \neq \varnothing$, so $\overline{V_{4,5}} = \varnothing$. We claim that in this case $V_{1,4}$ is completely adjacent to $\overline{V_{3,4}}$, otherwise G would contain a copy of X_{170}, as can be verified in Figure 8.28.

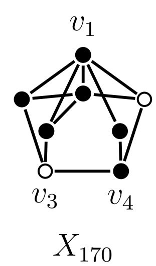


Figure 8.28: Cases for part 2.1.1 of Lemma 8.8.

But then, we have from Remark 8.7 that

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,3} \cup V_{1,4}, \{v_1\} \cup \overline{V_{3,4}}, \{v_3, v_4\} \cup \overline{V_{1,2}} \cup \overline{V_{1,5}})$$

is an $\overline{A_7}$-partition of G, which cannot be.

Case 2.1.2. We assume that $V_{1,3} = \varnothing$. Aiming for a contradiction, suppose that $V_{1,4}$ is completely adjacent to neither $\overline{V_{3,4}}$ or $\overline{V_{1,2}}$. If there are vertices $u \in V_{1,4}$, $v \in \overline{V_{3,4}}$ and $w \in \overline{V_{1,2}}$ such that $uv, uw \notin E$, then G would have X_{166}

as an induced subgraph. Otherwise, there exist vertices $u_1, u_2 \in V_{1,4}$, $v \in \overline{V_{3,4}}$ and $w \in \overline{V_{1,2}}$ such that $u_1w, u_2v \notin E$ and $u_1v, u_2w \in E$, but in such a case G would contain a copy of the co-domino graph (see Figure 8.29). Therefore, we have that $V_{1,4}$ is completely adjacent to some of $\overline{V_{3,4}}$ or $\overline{V_{1,2}}$; by symmetry we assume without loss of generality that $V_{1,4}$ is completely adjacent to $\overline{V_{3,4}}$.

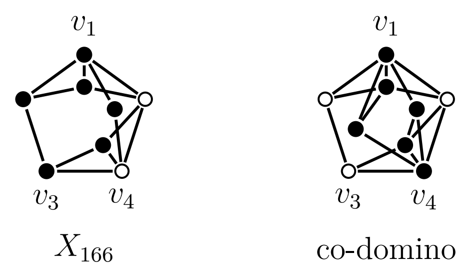


Figure 8.29: Cases for part 2.1.2 of Lemma 8.8.

Observe that it is impossible for $V_{1,4}$ to be completely adjacent to $\overline{V_{3,4}} \cup \overline{V_{1,2}}$: in such a case, due to $\overline{V_{1,2}} \neq \varnothing$ and $\overline{V_{3,4}} \neq \varnothing$, we have from Item 21 of Lemma 8.5 that some of $\overline{V_{1,5}}$ or $\overline{V_{4,5}}$ is an empty set, and by symmetry we can suppose that $\overline{V_{4,5}} = \varnothing$, but then, by Remark 8.7,

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,4}, \{v_1\} \cup \overline{V_{3,4}}, \{v_3, v_4\} \cup \overline{V_{1,2}} \cup \overline{V_{1,5}})$$

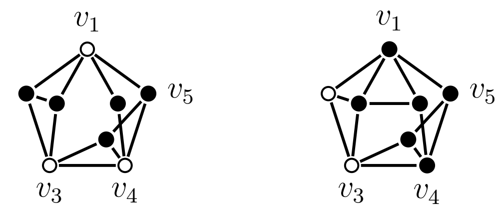
is an $\overline{A_7}$-partition of G, an absurd.

Thus, we know that $V_{1,4}$ is not completely adjacent to $\overline{V_{3,4}} \cup \overline{V_{1,2}}$, and $V_{1,4}$ is completely adjacent to $\overline{V_{3,4}}$, so $V_{1,4}$ is not completely adjacent to $\overline{V_{1,2}}$. Notice that in such a case, $\overline{V_{4,5}} = \varnothing$, otherwise G would contain a copy of either $K_1 + 2K_2$ or the co-domino graph (see Figure 8.30). But then, again

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,4}, \{v_1\} \cup \overline{V_{3,4}}, \{v_3, v_4\} \cup \overline{V_{1,2}} \cup \overline{V_{1,5}})$$

is an $\overline{A_7}$-partition of G, which is impossible.

Case 2.2. In this case we assume that there is no integer i such that $\overline{V_{i,i+1}}$ and $\overline{V_{i+2,i+3}}$ are both nonempty sets. By Item 21 of Lemma 8.5 we have that there is an integer i such that $\overline{V_{i+1,i+2}} = \overline{V_{i+2,i+3}} = \overline{V_{i+3,i+4}} = \varnothing$; assume $i = 1$. In addition, we previously proved that, if $\overline{V_{i,i+1}} = \varnothing$ for every integer i, $1 \leq i \leq 5$, then G admits an $\overline{A_7}$-partition; then we assume without loss of generality that



$K_1 + 2K_2$ co-domino

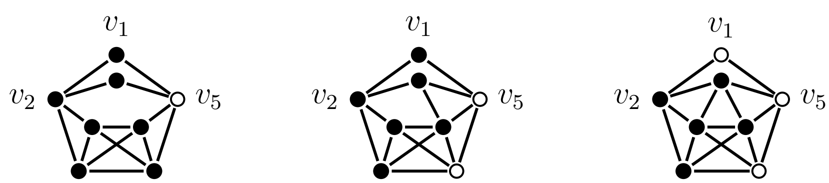
Figure 8.30: More cases for part 2.1.2 of Lemma 8.8.

$\overline{V_{1,2}} \neq \varnothing$. Observe that, by Item 20 of Lemma 8.5, $V_{3,5} = \varnothing$ and, either $\overline{V_{1,5}} = \varnothing$ or $V_{2,4} = \varnothing$.

Let us assume first that $\overline{V_{1,5}} \neq \varnothing$, in which case $V_{2,4} = \varnothing$. Observe that $V_{2,5} = \varnothing$, or G contains some of Y_2, $\overline{X_{170}}$ or W_4 as can be checked in Figure 8.31. But then, by Remark 8.7,

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,3} \cup V_{1,4}, \{v_1\}, \{v_3, v_4\} \cup \overline{V_{1,2}} \cup \overline{V_{1,5}})$$

is an $\overline{A_7}$-partition of G.



Y_2 $\overline{X_{170}}$ W_4

Figure 8.31: Cases for part 2.2 of Lemma 8.8.

Thus, we must suppose that $\overline{V_{1,5}} = \varnothing$. Notice that by Item 19 of Lemma 8.5, either $V_{1,3} = \varnothing$ or $V_{2,5} \cup V_{2,4} = \varnothing$. If $V_{1,3} \neq \varnothing$, it follows from Remark 8.7 that

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,3} \cup V_{1,4}, \{v_1\}, \{v_3, v_4\} \cup \overline{V_{1,2}})$$

is an $\overline{A_7}$-partition of G. Thus, $V_{1,3} = \varnothing$ and again, by Item 19 of Lemma 8.5, some of $V_{2,5}$ or $V_{1,4}$ is an empty set. If $V_{1,4} = \varnothing$, by increasing in 1 the subscripts of each set, Remark 8.7 yields

$$(A, B, C) = (\{v_1, v_3\} \cup V_{2,5} \cup V_{2,4}, \{v_2\}, \{v_{4,5}\} \cup \overline{V_{1,2}})$$

is an $\overline{A_7}$-partition of G, so we can suppose that $V_{1,4} \neq \varnothing$ and then, $V_{2,5} = \varnothing$.

If $V_{2,4} = \varnothing$, we have from Remark 8.7 that

$$(A, B, C) = (\{v_2, v_5\} \cup V_{1,4}, \{v_1\}, \{v_3, v_4\} \cup \overline{V_{1,2}})$$

is an $\overline{A_7}$-partition of G, so we can suppose that $V_{2,4} \neq \varnothing$. Nevertheless, in such a case, we have that $\overline{V_{1,2}}$ is completely adjacent to $V_{1,4} \cup V_{2,4}$, or G would contain either $\overline{X_{58}}$ or X_{170} as an induced subgraph (see Figure 8.32).

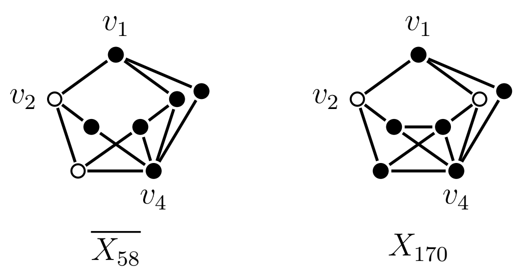


Figure 8.32: More cases for part 2.2 of Lemma 8.8.

But then, by increasing the subscripts by 3, Remark 8.7 implies that

$$(A, B, C) = (\{v_3, v_5\} \cup V_{1,4} \cup V_{2,4}, \{v_4\} \cup \overline{V_{1,2}}, \{v_1, v_2\})$$

is an $\overline{A_7}$-partition of G.

Since we have made an exhaustive verification of cases, and we obtain in each of them that G either contains an copy of a graph in Figure 8.1 or it has an $\overline{A_7}$-partition, we conclude that the only minimal $\overline{A_7}$-obstructions with an induced 5-cycle are those depicted in Figure 8.1. □

8.2 C_5-free obstructions with a C_4

Now, we prove that any C_5-free minimal $\overline{A_7}$-obstruction with an induced cycle on four vertices is a graph in Figure 8.1.

Lemma 8.9. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1 the following statements are satisfied for any $i \in \{1, \ldots, 4\}$ (where the subscripts are considered modulo 4):*

1. *V_i induces both, a cluster and a complete multipartite graph, and hence, V_i is either an independent set or a clique.*

2. *$V_{i,i+1}$ is a clique.*

3. *$V_{i,i+2}$ is an independent set.*

4. *$\overline{V_i}$ is a clique.*

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.33: if V_i is not a cluster or a complete multipartite graph, G contains a copy of either $2P_3$ or $K_1 + 2K_2$, respectively; if $V_{i,i+1}$ is not a clique, G has the twin-house graph as an induced subgraph; if $V_{i,i+2}$ is not an independent set, G has an induced subgraph isomorphic to $\overline{K_2 + \text{diamond}}$; if $\overline{V_i}$ is not a clique, G has W_4 as an induced subgraph. □

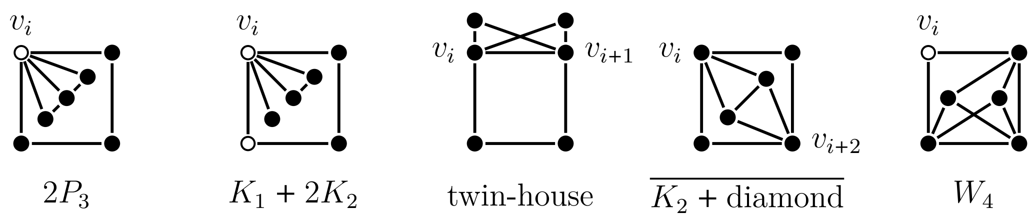


Figure 8.33: Cases of Lemma 8.9.

Lemma 8.10. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1, and $i \in \{1, \dots, 4\}$, the following pairs of sets are completely adjacent (where the subscripts are considered modulo 4):*

1. *V_{i+1} with $V_{i,i+1}$.*

2. *V_i with $\overline{V_{i+2}}$.*

3. *$V_{i,i+1}$ with $\overline{V_{i+3}}$.*

By symmetry, also the following pairs of sets are completely adjacent: $(V_i, V_{i,i+1})$, and $(V_{i,i+1}, \overline{V_{i+2}})$.

Proof. We proceed by contradiction.The following facts can be verified in Figure 8.34: if V_{i+1} is not completely adjacent to $\overline{V_{i,i+1}}$, then G has a copy of $\overline{X_{58}}$; if V_i is not completely adjacent to $\overline{V_{i+2}}$, G has $K_1 + C_4$ as an induced subgraph; if $V_{i,i+1}$ is not completely adjacent to $\overline{V_{i+3}}$, G has an induced graph isomorphic to $\overline{X_{166}}$. □

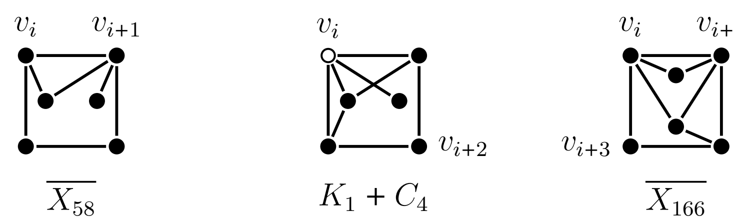


Figure 8.34: Cases of Lemma 8.10.

Lemma 8.11. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1, and $i \in \{1, \dots, 4\}$, the following pairs of sets are completely nonadjacent (where the subscripts are considered modulo 4):*

1. *$V_{i,i+1}$ with $\overline{V_{i+1}}$.*
2. *$\overline{V_i}$ with $\overline{V_{i+2}}$.*
3. *$V_{i,i+1}$ with V_{i+3}.*
4. *V_i with V_{i+2}.*

By symmetry, also the following pairs of sets are completely nonadjacent: $(V_{i,i+1}, \overline{V_i})$, and $(V_i, V_{i+2,i+3})$.

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.35: if there is an edge from $V_{i,i+1}$ to $\overline{V_{i+1}}$, G has $\overline{P_6}$ as an induced subgraph; if there is an edge from $\overline{V_i}$ to $\overline{V_{i+2}}$, G has W_4 as an induced subgraph; if there is an edge from $V_{i,i+1}$ to V_{i+3}, or from V_i to V_{i+2}, then G has an induced C_5. □

Lemma 8.12. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not any graph in Figure 8.1 the following statements are satisfied:*

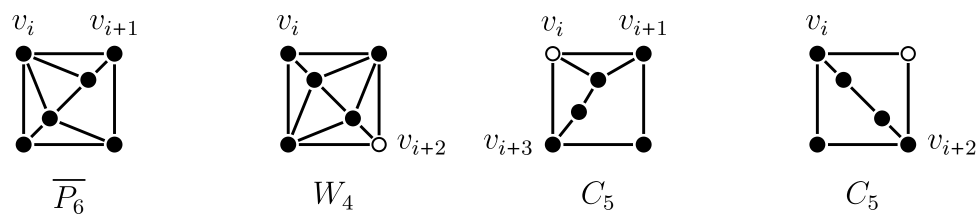


Figure 8.35: Cases of Lemma 8.11.

1. *$V_{1,3}$ is completely adjacent to either $\overline{V_3}$ or $\overline{V_1}$.*

2. *$V_{1,3}$ is completely nonadjacent to either V_1 or V_3.*

3. *If $V_{1,3}$ is not completely adjacent to $\overline{V_3}$, then $V_{1,3}$ is completely nonadjacent to V_3.*

4. *If $V_{3,4}$ is not completely adjacent to $V_{1,3}$, then $\overline{V_3}$ is completely adjacent to $V_{1,3}$.*

Proof. The following facts from Lemmas 8.9 and 8.11 are implicitly used in this proof:

1. $\overline{V_1}$ is completely non adjacent to $\overline{V_3}$.

2. $V_{1,3}$ is an independent set.

3. V_1 is completely non adjacent to V_3.

We proceed by contradiction. Assertions listed below can be verified in Figure 8.36.

1. Assume that $V_{1,3}$ is completely adjacent to neither $\overline{V_3}$ or $\overline{V_1}$. If there exists a vertex $u \in V_{1,3}$ that is completely adjacent to neither $\overline{V_3}$ or $\overline{V_1}$, then G has a copy of $K_1 + C_4$, otherwise there exist vertices $u, v \in V_{1,3}$, $x \in \overline{V_1}$ and $y \in \overline{V_3}$ such that $ux, vy \notin E$, u is completely adjacent to $\overline{V_3}$, and v is completely adjacent to $\overline{V_1}$, but then G has the co-domino graph as an induced subgraph.

2. Assume that there are edges from $V_{1,3}$ to V_1 and from $V_{1,3}$ to V_3. If there exist vertices $u \in V_{1,3}$, $x \in V_1$, and $y \in V_3$, such that $ux, uy \in E$, then G has a copy of the co-antenna graph; otherwise, there exist vertices

$u, v \in V_{1,3}$, $x \in V_1$, and $y \in V_3$, such that $ux, vy \in E$, u is completely nonadjacent to V_3, and v is completely nonadjacent to V_1, but in such a case G has the co-domino graph as an induced subgraph.

3. Suppose that $V_{1,3}$ is not completely adjacent to $\overline{V_3}$, and let $u \in V_{1,3}$ and $v \in \overline{V_3}$ be nonadjacent vertices. Assume that there is an edge from $V_{1,3}$ to V_3. If there is an edge from u to a vertex $x \in V_3$, then G has a copy of either the co-domino graph or $\overline{C_6}$, depending on whether v is adjacent to x. Otherwise, u is completely nonadjacent to V_3 and there exist adjacent vertices $x \in V_3$ and $y \in V_{1,3}$; notice that we can assume that y is adjacent to v, or we are in analogous case to the previous one. Nevertheless, in this case G has a copy of the co-antenna graph, or $\overline{X_{170}}$, depending on whether v is adjacent to x.

4. If we assume that $V_{3,4}$ is not completely adjacent to $V_{1,3}$, then $\overline{V_3}$ is completely adjacent to $V_{1,3}$, otherwise G has either $\overline{X_{58}}$ or X_{170} as an induced subgraph.

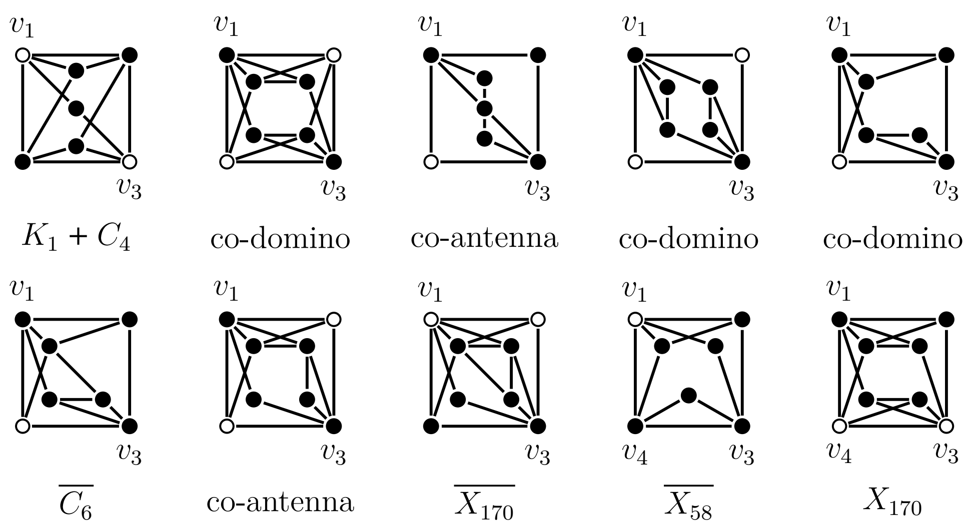


Figure 8.36: Counterexamples.

□

Lemma 8.13. *Let G be a minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1 and $V_1 \neq \varnothing$, then:*

1. $V_2 = \varnothing$.

2. $V_{2,4} = \varnothing$.

3. $\overline{V_2} = \varnothing$.

By symmetry also $V_4 = \overline{V_4} = \varnothing$.

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.37: if $V_2 \neq \varnothing$, then G has a copy of either A or the domino graph; if $V_{2,4} \neq \varnothing$, G has either $K_1 + C_4$ or $K_{3,3} - e$ as an induced subgraph; if $\overline{V_2} \neq \varnothing$, then G has a subgraph isomorphic to either $\overline{X_{170}}$ or $\overline{X_{166}}$. □

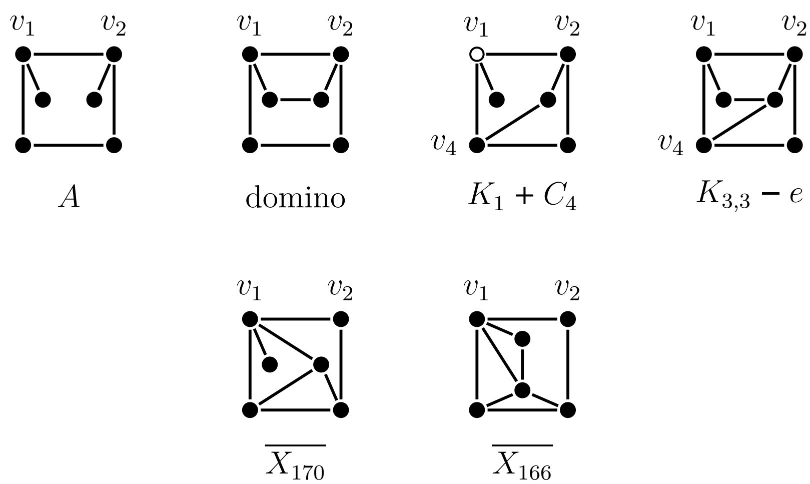


Figure 8.37: Cases of Lemma 8.13.

Lemma 8.14. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1 and $V_{i,i+1} \neq \varnothing$, the following statements are satisfied for any $i \in \{1, \dots, 4\}$ (where the subscripts are considered modulo 4):*

1. $V_{i+2,i+3} = \varnothing$.

2. $V_{i+1,i+2} = \varnothing$.

3. V_{i+3} *is completely nonadjacent to* $V_{i+1,i+3}$.

4. V_{i+3} *is an independent set.*

5. *V_{i+1} is a clique.*

6. *$V_{i+1,i+3}$ is completely adjacent to $\overline{V_{i+1}}$.*

7. *V_{i+1} is completely nonadjacent to $\overline{V_{i+1}}$.*

By symmetry, $V_{i+3,i} = \varnothing$.

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.38: if $V_{i+2,i+3} \neq \varnothing$, G contains either the co-domino graph or $\overline{C_6}$ as an induced subgraph; if $V_{i+1,i+2} \neq \varnothing$, G has a copy or either the co-antenna graph or C_5; if there exists an edge from V_{i+3} to $V_{i+1,i+3}$, G has an induced co-domino or co-antenna graph; if V_{i+3} is not an independent set, then G has an induced subgraph isomorphic to $K_1 + 2K_2$; if V_{i+1} is not a clique, G has a copy of X_{170}; if $V_{i+1,i+3}$ is not completely adjacent to $\overline{V_{i+1}}$, then G has an induced co-domino or co-antenna graph; if there exists an edge from V_{i+1} to $\overline{V_{i+1}}$, then G has a copy of the co-domino graph. □

Lemma 8.15. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1 and V_i is not an independent set, the following statements are satisfied for any $i \in \{1, \ldots, 4\}$ (where the subscripts are considered modulo 4):*

1. *$V_{i+2,i+3} = \varnothing$.*

2. *$\overline{V_i}$ is completely nonadjacent to V_i.*

3. *$V_{i,i+2}$ is completely adjacent to $\overline{V_i}$.*

4. *V_{i+2} is completely nonadjacent to $V_{i,i+2}$.*

5. *V_{i+2} is an independent set.*

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.39. By Lemma 8.11, $V_{i+2,i+3}$ is completely non adjacent to V_i so, if $V_{i+2,i+3} \neq \varnothing$, G has $K_1 + 2K_2$ as an induced subgraph; if there exists an edge from $\overline{V_i}$ to V_i, G has a copy of either the co-domino graph or $\overline{K_2 + \text{diamond}}$; if $V_{i,i+2}$ is not completely adjacent to $\overline{V_i}$, then G has some of $K_1 + 2K_2$ or X_{170} as an induced subgraph; if there exists an edge from V_{i+2} to $V_{i,i+2}$, G has a copy of either $K_1 + 2K_2$ or the co-antenna graph; if V_{i+2} is not an independent set, G has $K_1 + 2K_2$ as an induced subgraph. □

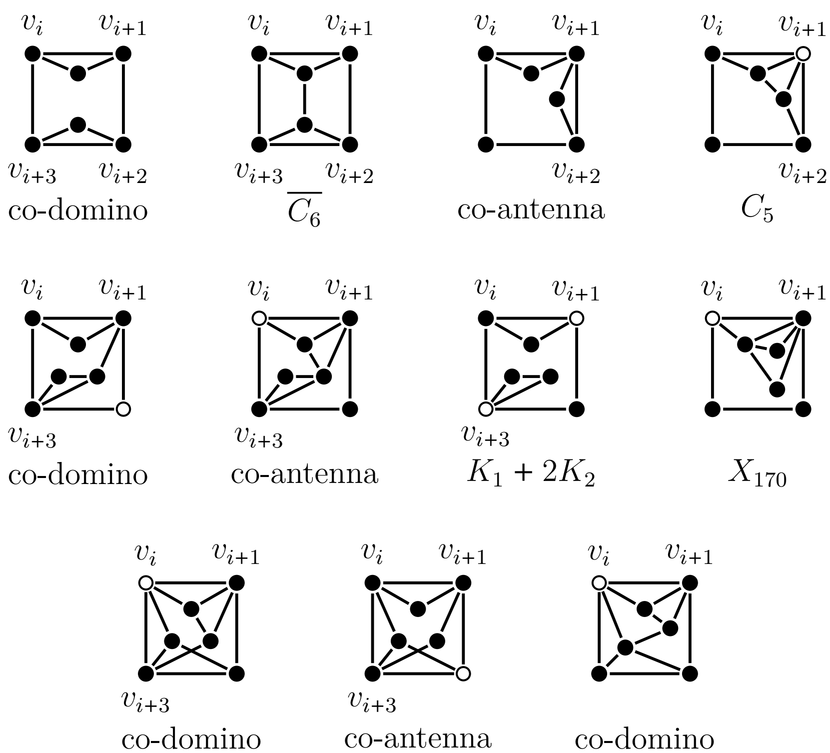


Figure 8.38: Cases of Lemma 8.14.

Lemma 8.16. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not a graph in Figure 8.1 and V_i is not an clique, the following statements are satisfied:*

1. *V_{i+2} is a clique.*

2. *$V_{i,i+2}$ is completely nonadjacent to V_i.*

3. *$V_{i,i+2}$ is completely adjacent to $\overline{V_{i+2}}$*

4. *V_{i+2} is completely nonadjacent to $\overline{V_{i+2}}$*

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.40: if V_{i+2} is not a clique, G contains a copy of $2P_3$; if there exists an edge from $V_{i,i+2}$ to V_i, G has a copy of either $\overline{X_{58}}$ or the twin-house graph; if $V_{i,i+2}$ is not completely adjacent to $\overline{V_{i+2}}$, G has some of X_{170}, X_{58} or W_4 as an

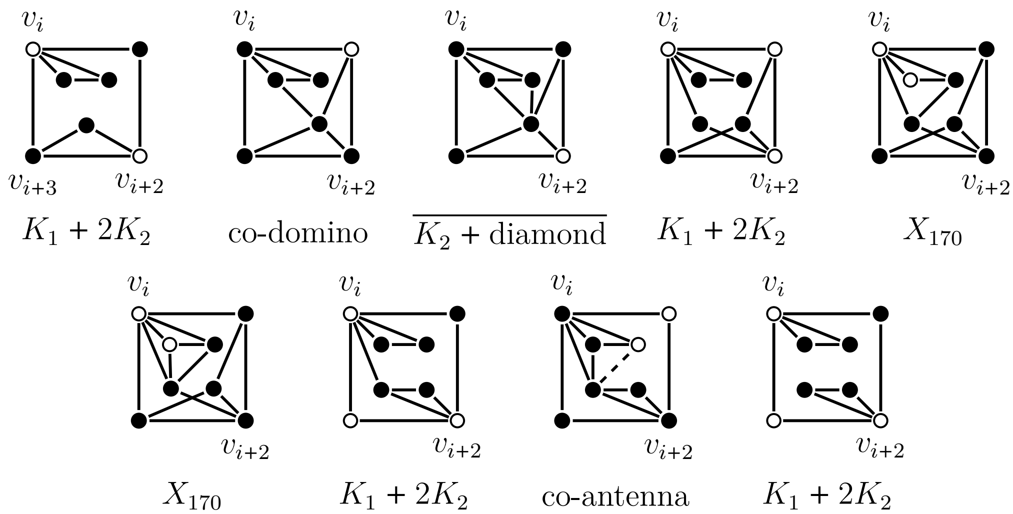


Figure 8.39: Counterexamples.

induced subgraph; if there exists an edge from V_{i+2} to $\overline{V_{i+2}}$, G has an induced subgraph isomorphic to some of X_{170}, X_{58} or W_4. □

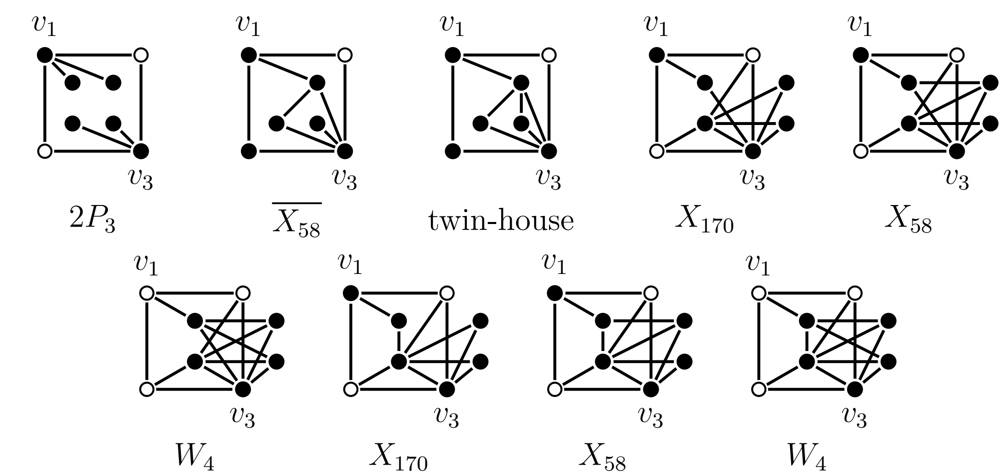


Figure 8.40: Counterexamples.

Lemma 8.17. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not any graph in Figure 8.1 and there exists an edge with ends in V_i and $\overline{V_i}$, the following statements are satisfied:*

1. *V_{i+2} is completely nonadjacent to $\overline{V_{i+2}}$.*

2. *$V_{i,i+2}$ is completely adjacent to $\overline{V_{i+2}}$.*

3. *If V_i is a singleton, $V_{i,i+2}$ is completely nonadjacent to V_i.*

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.41: if there exists an edge from V_{i+2} to $\overline{V_{i+2}}$, then G has the co-domino graph as an induced subgraph; if $V_{i,i+2}$ is not completely adjacent to $\overline{V_{i+2}}$, G has a copy of some of $C_5, K_1 + C_4$, the co-domino graph or $\overline{P_6}$; if V_i is a singleton, but there exists an edge from $V_{i,i+2}$ to V_i, then G has either the twin-house graph or $\overline{P_6}$ as an induced subgraph. □

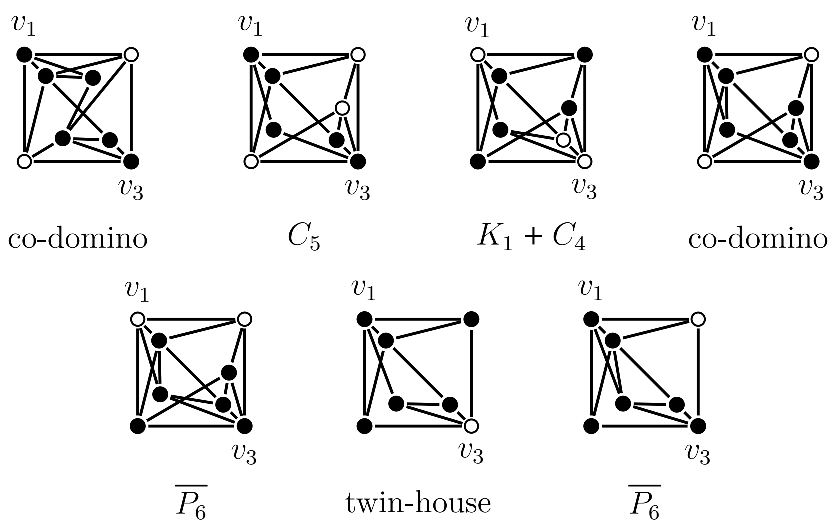


Figure 8.41: Cases of Lemma 8.17.

Lemma 8.18. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not any graph in Figure 8.1, and there exists an edge from V_1 to $V_{1,3}$, the following statements are satisfied:*

1. *$V_{1,3}$ is completely adjacent to $V_{3,4}$.*

2. *V_1 is completely nonadjacent to $\overline{V_1}$.*

3. *$V_{1,3}$ is completely adjacent to $\overline{V_1}$.*

Proof. The following facts from Lemmas 8.9 and 8.11 are implicitly used in this proof:

1. V_1 is completely nonadjacent to $V_{3,4}$.

2. $V_{1,3}$ is an independent set.

We proceed by contradiction. Assertions listed below can be verified in Figure 8.42. If $V_{1,3}$ is not completely adjacent to $V_{3,4}$, G has a copy of some of the co-domino graph, the co-antenna graph, or $K_1 + 2K_2$; if there is an edge from V_1 to $\overline{V_1}$, G has an induced subgraph isomorphic to either the twin-house or $\overline{K_2 + \text{diamond}}$; if $V_{1,3}$ is not completely adjacent to $\overline{V_1}$; G has either X_{170} or $\overline{X_{58}}$ as an induced subgraph. $\square$

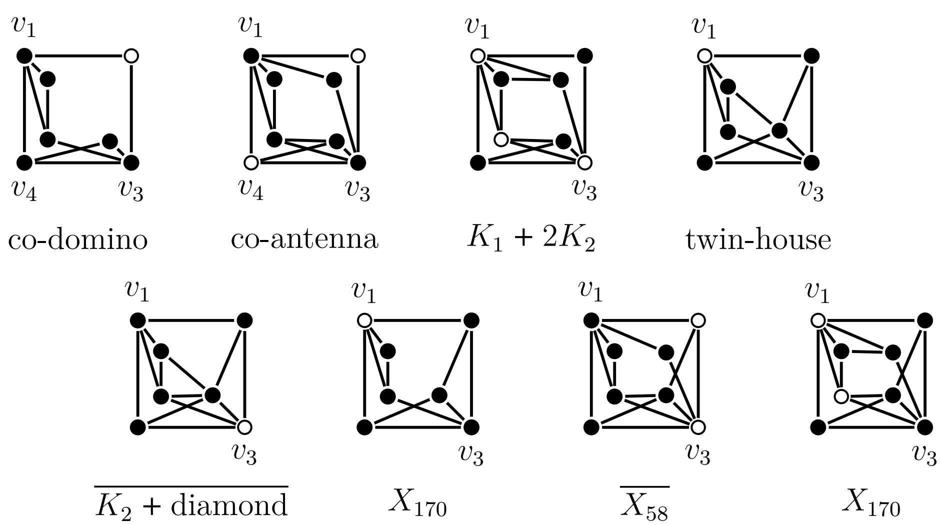


Figure 8.42: Counterexamples.

Lemma 8.19. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction containing an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. If G is not any graph in Figure 8.1, and $V_{1,3} \neq \varnothing$, the following statements are satisfied:*

1. *$V_{2,4} = \varnothing$.*

2. *$\overline{V_2} = \varnothing$.*

By symmetry, also $\overline{V_4} = \varnothing$.

Proof. We proceed by contradiction. Assertions listed below can be verified in Figure 8.43. If $V_{2,4} \neq \varnothing$, G has either $K_{3,3} - e$ or $K_{3,3}$ as an induced subgraph; if $\overline{V_2} \neq \varnothing$, G has a copy of either $\overline{K_2 + \text{diamond}}$ or W_4. □

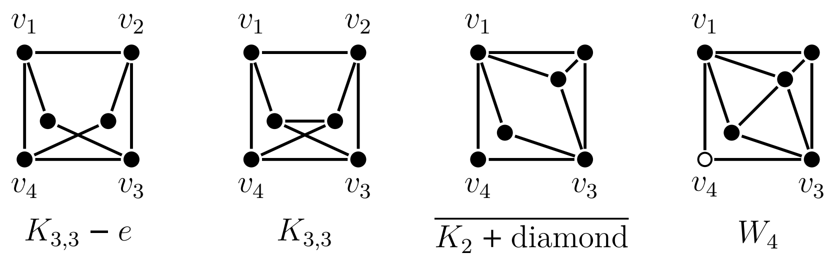


Figure 8.43: Counterexamples.

The following observations are simple consequences of Lemmas 8.9 to 8.11.

Remark 8.20. *Let G be a minimal $\overline{A_7}$-obstruction having an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. Then,*

$$(A, B, C) = (\{v_2, v_4\} \cup V_3 \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup V_{1,4} \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of $G[A \cup B \cup C]$ provided that

1. *V_1 is a clique,*
2. *V_3 is an independent set,*
3. *$V_{1,3}$ is completely adjacent to $\overline{V_1}$,*
4. *V_1 is completely nonadjacent to $\overline{V_1}$, and*
5. *V_3 is completely nonadjacent to $V_{1,3}$.*

Proof. We have from Lemma 8.9 that $V_{1,3}$ is an independent set, and $\overline{V_1}, V_{1,4}$ and $\overline{V_3}$ are all of them cliques. In addition, by Lemma 8.10 we know that the following pairs of vertex subsets are completely adjacent: $(V_1, V_{1,4} \cup \overline{V_3})$, $(V_{1,4}, \overline{V_3})$, and $(V_3, \overline{V_1})$. Similarly, $\overline{V_1}$ is completely nonadjacent to $V_{1,4} \cup \overline{V_3}$ by Lemma 8.11. □

Remark 8.21. *Let G be a minimal $\overline{A_7}$-obstruction having an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. Then,*

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of $G[A \cup B \cup C]$ provided that

1. *V_1 is a clique,*
2. *$\overline{V_1}$ is completely adjacent to $V_{1,3}$,*
3. *$\overline{V_1}$ is completely nonadjacent to V_1.*

Proof. We have from Lemma 8.9 that $V_{1,3}$ is an independent set, and $\overline{V_1}$ and $\overline{V_3}$ both cliques. In addition, by Lemma 8.10 we know that V_1 is completely adjacent to $\overline{V_3}$, and by Lemma 8.11 we have that $\overline{V_1}$ is completely nonadjacent to $\overline{V_3}$. □

Remark 8.22. *Let G be a minimal $\overline{A_7}$-obstruction having an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. Then,*

$$(A, B, C) = (\{v_2, v_4\} \cup V_1 \cup V_{1,3}, \{v_1\} \cup \overline{V_3}, \{v_3\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of $G[A \cup B \cup C]$ provided that

1. *V_1 is an independent set,*
2. *V_1 is completely nonadjacent to $V_{1,3}$,*
3. *$V_{1,3}$ is completely adjacent to $\overline{V_3}$,*

Proof. We have from Lemma 8.9 that $V_{1,3}$ is an independent set, and $\overline{V_1}, \overline{V_3}$ and $V_{3,4}$ are cliques. In addition, by Lemma 8.10 we know that the following pairs of vertex subsets are completely adjacent: $(V_{3,4}, \overline{V_1})$ and $(V_1, \overline{V_3})$. Similarly, $\overline{V_3}$ is completely nonadjacent to $\overline{V_1} \cup V_{3,4}$ by Lemma 8.11. □

Remark 8.23. *Let G be a minimal $\overline{A_7}$-obstruction having an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. Then,*

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of $G[A \cup B \cup C]$ provided that $V_{1,3}$ is completely adjacent to $\overline{V_1}$.

Proof. We have from Lemma 8.9 that $V_{1,3}$ is an independent set and both $\overline{V_1}$ and $\overline{V_3}$ are cliques. In addition, by Lemma 8.10 we know that $\overline{V_1}$ is completely nonadjacent to $\overline{V_3}$. □

Lemma 8.24. *Let G be a C_5-free minimal $\overline{A_7}$-obstruction having an induced 4-cycle, $C = (v_1, v_2, v_3, v_4, v_1)$. Then, G is isomorphic to some graph in Figure 8.1.*

Proof. Assume, to reach a contradiction, that G is different from any graph in Figure 8.1. Notice that $V_\varnothing, \overline{V_\varnothing} = \varnothing$, otherwise G would have either $K_1 + C_4$ or W_4 as an induced subgraph. Now, we have the following cases.

Case 1. Assume that there exists an integer $i \in \{1, 2, 3, 4\}$ such that $V_i \neq \varnothing$; we can suppose without loss of generality that $i = 1$. From Lemma 8.13 we have that $V_2 = V_4 = V_{2,4} = \overline{V_2} = \overline{V_4} = \varnothing$. In addition, we have from Lemma 8.14 that there exists at most one $i \in \{1, 2, 3, 4\}$ such that $V_{i,i+1} \neq \varnothing$. We identify two subcases: either $V_{2,3} \cup V_{3,4} = \varnothing$ (Case 1.a) or $V_{1,4} \cup V_{1,2} = \varnothing$ (Case 1.b).

Case 1.a. Suppose that $V_{2,3} \cup V_{3,4} = \varnothing$. Notice that, by symmetry, we can assume without loss of generality that $V_{1,2}$ is an empty set too, so $V_G = V_C \cup V_1 \cup V_3 \cup V_{1,4} \cup V_{1,3} \cup \overline{V_1} \cup \overline{V_3}$. Observe that, if $V_{1,4} \neq \varnothing$, we have from Lemma 8.14 and Remark 8.20 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_3 \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup V_{1,4} \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, contradicting our the election of G. Thus, we can assume that $V_{1,4} = \varnothing$, and then $V_G = V_C \cup V_1 \cup V_3 \cup V_{1,3} \cup \overline{V_1} \cup \overline{V_3}$.

Now, if V_1 is not an independent set, it follows from Lemmas 8.9 and 8.15 and Remark 8.20 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_3 \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup V_{1,4} \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, which is impossible. Analogously, if V_3 is not an independent set, then G is not an $\overline{A_7}$-obstruction, so it must be the case that V_1 and V_3 are both independent sets.

Similarly, if V_3 is not a clique, it is easy to corroborate from Lemmas 8.9 and 8.16 and Remark 8.20 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_3 \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$$

conform an $\overline{A_7}$-partition of G, which cannot occur. Thus, V_3 is a clique and, by an analogous argument, V_1 is too. Notice that, from the previous arguments, each of V_1 and V_3 has at most one vertex.

From here, we consider two distinct subcases, depending on whether V_3 and $\overline{V_3}$ are completely nonadjacent. If there is an edge with ends in V_3 and $\overline{V_3}$, we have from Remark 8.20 and Lemma 8.17 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_3 \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, an absurd. The case in which there exists an edge with ends in V_1 and $\overline{V_1}$ is analogous, so we can assume that V_i is completely nonadjacent to $\overline{V_i}$ for $i \in \{1, 3\}$.

If $V_{1,3}$ is not completely adjacent to $\overline{V_3}$, we have by Lemma 8.12 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_3 \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, which is absurd. The case in which $V_{1,3}$ is not completely adjacent to $\overline{V_1}$ is analogous, so we can assume that $V_{1,3}$ is completely adjacent to $\overline{V_1} \cup \overline{V_3}$. However, by Lemma 8.12 we can assume without loss of generality that $V_{1,3}$ is completely nonadjacent to V_3, in which case (A, B, C) is still an $\overline{A_7}$-partition of G.

Case 1.b. Suppose that $V_{1,4} \cup V_{1,2} = \varnothing$. By Lemma 8.14, we can assume without loss of generality that $V_{2,3} = \varnothing$. Notice that, if $V_3 \neq \varnothing$, we are in a case analogous to the Case 1.a. Thus, we assume that $V_3 = \varnothing$, so $V_G = V_C \cup V_1 \cup V_{3,4} \cup V_{1,3} \cup \overline{V_1} \cup \overline{V_3}$. We distinguish the following subcases.

If V_1 is not an independent set, it follows from Lemmas 8.9 and 8.15 and Remark 8.21 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, which is impossible. Similarly, if V_1 is not an clique, we have from Lemmas 8.9 and 8.16 and Remark 8.22 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_1 \cup V_{1,3}, \{v_1\} \cup \overline{V_3}, \{v_3\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, which cannot occur. Thus, V_1 is both a clique and an independent set, so V_1 is a singleton.

Now, if there is an edge from V_1 to $V_{1,3}$, we have from Lemma 8.18 and Remark 8.21 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, which is absurd. Hence, V_1 is completely nonadjacent to $V_{1,3}$. Notice that, if $V_{1,3}$ is completely adjacent to $\overline{V_3}$, then it follows from Remark 8.22 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_1 \cup V_{1,3}, \{v_1\} \cup \overline{V_3}, \{v_3\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, which is impossible. Thus, we can assume that $V_{1,3}$ is not completely adjacent to $\overline{V_3}$. This implies, from Lemma 8.12, that $V_{1,3}$ is completely adjacent to $\overline{V_1} \cup V_{3,4}$. Additionally, we have from Lemma 8.11 that $V_{3,4}$ is completely nonadjacent to $\overline{V_3}$. The previous observations imply that $V_{3,4} = \varnothing$, or G would have an induced copy of the co-domino graph, as is shown in Figure 8.44.

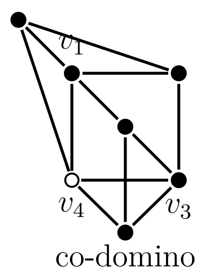


Figure 8.44: Case 1b of Lemma 8.24.

Observe that when $V_{1,3}$ is not completely adjacent to $\overline{V_3}$, we have that V_1 is completely nonadjacent to $\overline{V_1}$, otherwise G would have the co-domino as an induced subgraph (see Figure 8.45).

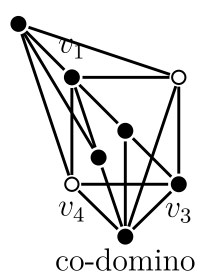


Figure 8.45: Case 1b of Lemma 8.24.

From here, it follows from Remark 8.20 that $(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_1 \cup \overline{V_3})$ is an $\overline{A_7}$-partition of G, which cannot occur.

Case 2. We assume that Case 1 do not occur, so $V_i = \varnothing$ for every $i \in \{1, 2, 3, 4\}$. Notice that, by Lemma 8.14 and the symmetries of C_4, we can assume without loss of generality that $V_{1,2} = V_{2,3} = V_{1,4} = \varnothing$. We distinguish two different cases, depending on whether $V_{1,3}$ is an empty set.

Case 2.a. If $V_{1,3} \neq \varnothing$, we have from Lemma 8.19 that $V_{2,4} = \overline{V_2} = \overline{V_4} = \varnothing$, so $V_G = V_C \cup V_{1,3} \cup V_{3,4} \cup \overline{V_1} \cup \overline{V_3}$.

First, suppose that $V_{3,4}$ is completely adjacent to $V_{1,3}$. If $V_{1,3}$ is completely adjacent to $\overline{V_3}$, we have from Remark 8.22 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_1\} \cup \overline{V_3}, \{v_3\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, an absurd. Thus, it must be the case that $V_{1,3}$ is not completely adjacent to $\overline{V_3}$, but then, we have by Lemma 8.12 that $V_{1,3}$ is completely adjacent to $\overline{V_1}$. Moreover, as we explained in Case 1.b, when $V_{1,3}$ is not completely adjacent to $\overline{V_3}$ we have that $V_{3,4} = \varnothing$, and then it follows from Remark 8.23 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup \overline{V_3})$$

is an $\overline{A_7}$-partition of G, which is impossible. Therefore, $V_{3,4}$ is not completely adjacent to $V_{1,3}$. Nevertheless, in such a case we have from Lemma 8.12 and Remark 8.22 that

$$(A, B, C) = (\{v_2, v_4\} \cup V_{1,3}, \{v_1\} \cup \overline{V_3}, \{v_3\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, contradicting the election of G. Hence, the case in which $V_{1,3}$ is not an empty set is impossible and, by the symmetries of the 4-cycle we can assume that $V_{2,4}$ is empty too.

Case 2.b. Assume that $V_{1,3} = V_{2,4} = \varnothing$. Observe that $\overline{V_i}$ and $\overline{V_{i+1}}$ cannot be simultaneously nonempty sets, or G would have either $\overline{P_6}$ or $\overline{2P_3}$ as an induced subgraph as can be seen in Figure 8.46.

Thus, by the symmetries of C_4, we can assume without loss of generality that $\overline{V_2} = \overline{V_4} = \varnothing$. But in this case, it follows from Lemmas 8.9 and 8.11 that

$$(A, B, C) = (\{v_2, v_4\}, \{v_1\} \cup \overline{V_3}, \{v_3\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, which is a contradiction.

Since we made an exhaustive consideration of cases and none of them is possible, the result follows. □

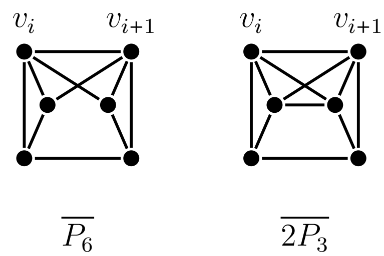


Figure 8.46: Counterexamples.

8.3 (C_4, C_5)-free obstructions with a P_5

In this section, we prove that any $\{C_4, C_5\}$-free minimal $\overline{A_7}$-obstruction with an induced path of length 4 is a graph in Figure 8.1.

Lemma 8.25. *Let G be a (C_4, C_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4, v_5)$. If G is not a graph in Figure 8.1, any vertex of $G - V_P$ lies in some of the following sets: $V_2, V_4, V_{1,2}, V_{4,5}, \overline{V_{1,2}}$, or $\overline{V_{4,5}}$.*

Proof. We will prove that if a vertex v of $G - V_P$ is such that $N(v) \cap V_P$ is different from the listed sets, then G is not a (C_4, C_5)-free graph, or it has a copy of some graph in Figure 8.1. The following statements can be easily corroborated:

1. If some of the sets $V_{1,4}, V_{2,5}, \overline{V_{2,3}}$, or $\overline{V_{3,4}}$ is not empty, then G has an induced 5-cycle.

2. If some of the sets $V_{1,3}, V_{2,4}, \overline{V_{1,3}}, \overline{V_{1,4}}, \overline{V_{2,4}}, \overline{V_2}$, and $\overline{V_3}$ is not an empty set, then G has a copy of C_4; by the symmetries of P_5 we also have that if some of the sets $V_{3,5}, \overline{V_{3,5}}, \overline{V_{2,5}}$, and $\overline{V_4}$ is not empty, then G has an induced 4-cycle.

3. If $V_{1,5} \neq \varnothing$, G has a copy of C_6; if there exist vertices completely adjacent to P, G has $K_1 \oplus P_5$ as an induced subgraph; if V_1 (or V_5) is not empty, G has a copy of P_6; if either $V_3 \neq \varnothing$ or there are vertices completely nonadjacent to P, then G has $K_1 + 2K_2$ an an induced subgraph; if $V_{2,3}$ (or $V_{3,4}$) is not an empty-set, G has a copy of X_{166}; if $\overline{V_{1,5}} \neq \varnothing$, then G has a copy of the co-twin-house graph; if $\overline{V_1}$ (or $\overline{V_5}$) is not empty, G has a copy of X_{58}.

□

Lemma 8.26. *Let G be a (C_4, C_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4, v_5)$. If G is not a graph in Figure 8.1, then V_2 is an independent set, while the sets $V_{1,2}$ and $\overline{V_{4,5}}$ are cliques. Analogously, V_4 is an independent set, and the sets $V_{4,5}$ and $\overline{V_{1,2}}$ are cliques.*

Proof. The following facts can be verified in Figure 8.47: if V_2 is not an independent set, G has $K_1 + 2K_2$ as an induced subgraph; if any of $V_{1,2}$ or $\overline{V_{4,5}}$ is not a clique, G contains a copy of X_{170}. □

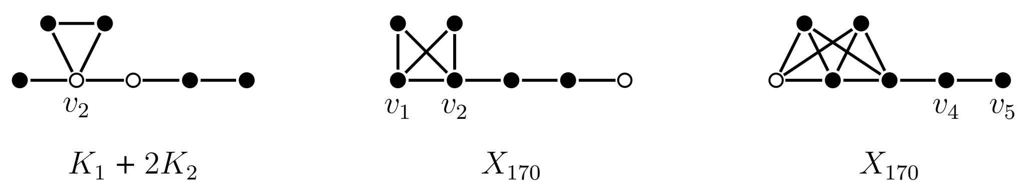


Figure 8.47: Counterexamples.

Lemma 8.27. *Let G be a (C_4, C_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4, v_5)$. If G is not a graph in Figure 8.1, the following statements hold.*

1. *V_2 is completely adjacent to $\overline{V_{4,5}}$, and $V_{1,2}$ is completely adjacent to $\overline{V_{4,5}}$. By the symmetries of P_5, also V_4 is completely adjacent to $\overline{V_{1,2}}$, and $V_{4,5}$ to $\overline{V_{1,2}}$.*

2. *$V_{1,2}$ is completely nonadjacent to $\overline{V_{1,2}}$, and $\overline{V_{1,2}}$ is completely nonadjacent to $\overline{V_{4,5}}$. By the symmetries of P_5, also $V_{4,5}$ is completely nonadjacent to $\overline{V_{4,5}}$.*

Proof. We proceed by contradiction. The following claims can be verified in Figure 8.48: if V_2 is not completely adjacent to $\overline{V_{4,5}}$, G has a copy of $K_1 + 2K_2$; if $V_{1,2}$ is not completely adjacent to $\overline{V_{4,5}}$, G has a copy of X_{58}; if $V_{1,2}$ is not completely nonadjacent to $\overline{V_{1,2}}$, G has a copy of C_4; if $\overline{V_{1,2}}$ is not completely nonadjacent to $\overline{V_{4,5}}$, G has a copy of X_{58}. □

Lemma 8.28. *Let G be a (C_4, C_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4, v_5)$. If G is not a graph in Figure 8.1, the following statements are satisfied.*

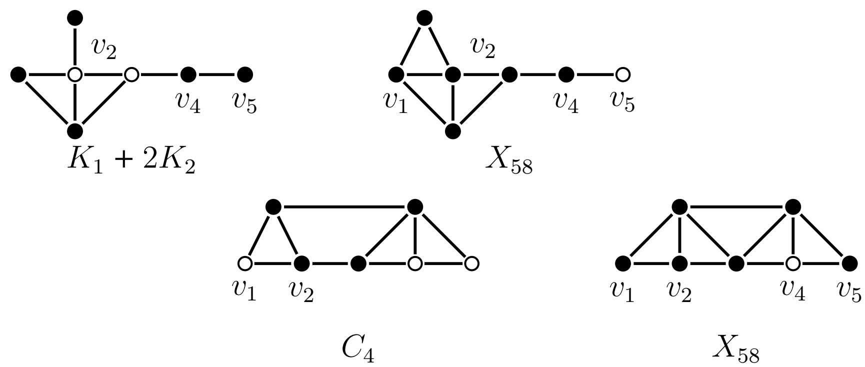


Figure 8.48: Counterexamples.

1. *$V_2 = \varnothing$ or $V_4 = \varnothing$.*

2. *$V_{1,2} = \varnothing$ or $V_{4,5} = \varnothing$.*

3. *$V_2 = \varnothing$ or $V_{1,2} = \varnothing$ (Symmetrically, $V_4 = \varnothing$ or $V_{4,5} = \varnothing$).*

Proof. We proceed by contradiction. The following statements can be corroborated in Figure 8.49: if both, V_2 and V_4, are not empty sets, then G contains a copy of either copy of $2P_3$ or C_5; if both, $V_{1,2}$ and $V_{4,5}$, are not empty sets, G contains either $K_1 + 2K_2$ or C_5 as an induced subgraph if both, V_2 and $V_{1,2}$, are not empty sets, G contains a copy of either $K_1 + 2K_2$ or $2P_3$. □

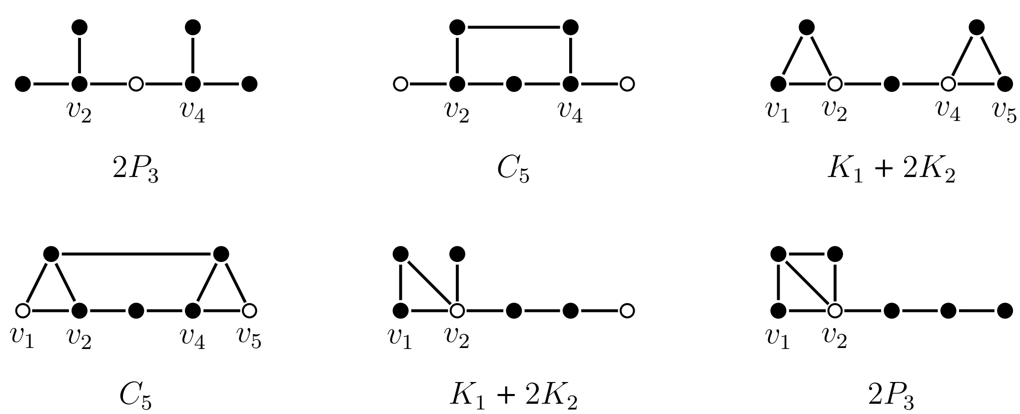


Figure 8.49: Counterexamples.

Lemma 8.29. *Let G be a (C_4, C_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4, v_5)$. Then, G is isomorphic to some graph in Figure 8.1.*

Proof. By Lemma 8.25 we have that any vertex of $G - V_P$ lies in $V_2 \cup V_4 \cup V_{1,2} \cup V_{4,5} \cup \overline{V_{1,2}} \cup \overline{V_{4,5}}$. First, let us assume that $V_2 \cup V_4 \neq \varnothing$; we can suppose without loss of generality that $V_4 \neq \varnothing$, so we have by Lemma 8.28 that $V_2 = V_{4,5} = \varnothing$. Therefore, any vertex of $G - V_P$ lies in $V_4 \cup V_{1,2} \cup \overline{V_{1,2}} \cup \overline{V_{4,5}}$. Nevertheless, in such a case it follows from Lemmas 8.26 and 8.27 that

$$(A, B, C) = (\{v_3, v_5\} \cup V_4, \{v_4\} \cup \overline{V_{1,2}}, \{v_1, v_2\} \cup V_{1,2} \cup \overline{V_{4,5}})$$

is an $\overline{A_7}$-partition of G, which is impossible since G is an $\overline{A_7}$-obstruction.

Now assume that $V_2 \cup V_4 = \varnothing$. By Lemma 8.27 we know that at most one of $V_{1,2}$ and $V_{4,5}$ is not an empty set. Supposing without loss of generality that $V_{4,5}$ is empty, we have by the arguments in the paragraph before that $(A, B, C) = (\{v_3, v_5\}, \{v_4\} \cup \overline{V_{1,2}}, \{v_1, v_2\} \cup V_{1,2} \cup \overline{V_{4,5}})$ is an $\overline{A_7}$-partition of G, which is absurd. □

8.4 (C_4, C_5, P_5)-free obstructions with a P_4

Now, we prove that any (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction with an induced path on four vertices is a graph in Figure 8.1.

Lemma 8.30. *Let G be a (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4)$. If G is not a graph in Figure 8.1, the sets $V_1, V_4, V_{1,3}, V_{2,4}, V_{1,4}, \overline{V_2}$, and $\overline{V_3}$ are empty. Additionally, either $V_3 = \varnothing$ or $V_{3,4} = \varnothing$.*

Proof. The following facts can be easily corroborated: if V_1 (or V_4) is not an empty set, then G has P_5 as an induced subgraph; if some of $V_{1,3}$ or $\overline{V_2}$ is not an empty set, then G has C_4 as an induced subgraph (symmetrically, $V_{2,4}$ and $\overline{V_3}$ are both empty sets); if $V_{1,4}$ is not empty, then G a copy of C_5; if V_3 and $V_{3,4}$ are both nonempty sets, then G has either $K_1 + 2K_2$ or X_{170} as an induced subgraph. □

Lemma 8.31. *Let G be a (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4)$. If G is not a graph in Figure 8.1 the following affirmations hold.*

1. *$V_\varnothing$ and V_2 are independent sets. Analogously, V_3 is also an independent set.*

2. *$V_{1,2}, \overline{V_1}$, and $\overline{V_\varnothing}$ are cliques. By the symmetries of P_4, also the sets $V_{3,4}$ and $\overline{V_4}$ are cliques.*

3. *$V_{2,3}$ induces a split graph.*

Proof. We proceed by contradiction. The following claims can be verified in Figure 8.50: if some of $V_\varnothing$ or V_2 is not an independent set, G contains $K_1 + 2K_2$ as an induced subgraph; if $V_{1,2}$ is not a clique, G has a copy of X_{170}; if some of $\overline{V_1}$ or $\overline{V_\varnothing}$ is not clique, G contains W_4 as an induced subgraph.

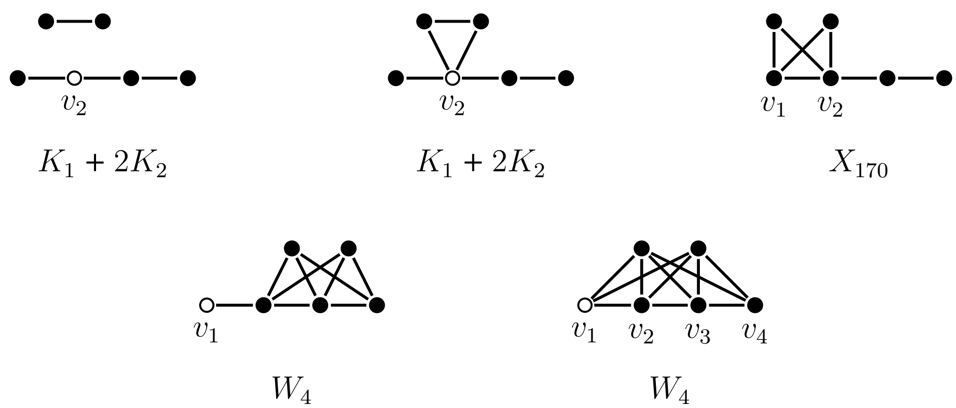


Figure 8.50: Counterexamples.

Finally, if a subset V' of $V_{2,3}$ induces $2K_2, C_4$ or C_5, then $G[V' \cup \{v_1\}]$ is isomorphic to $K_1 + 2K_2, K_1 + C_4$ or $\overline{W_5}$, respectively. $\square$

Lemma 8.32. *Let G be a (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4)$. If G is not a graph in Figure 8.1 then:*

1. *Then following pairs of sets are completely adjacent:*

   (a) *$V_{1,2}$ with $\overline{V_4}$ (analogously, $V_{3,4}$ with $\overline{V_1}$).*

   (b) *$\overline{V_1}$ with $\overline{V_\varnothing}$ (analogously, $\overline{V_4}$ with $\overline{V_\varnothing}$).*

   (c) *$\overline{V_1}$ with $\overline{V_4}$.*

2. *The following pairs of sets are completely nonadjacent:*

(d) $V_{1,2}$ *with* $V_{3,4}$.

(e) $V_{1,2}$ *with* $\overline{V_1}$ *(analogously,* $V_{3,4}$ *with* $\overline{V_4}$*).*

(f) $V_{1,2}$ *with* V_3 *(symmetrically,* $V_{3,4}$ *with* V_2*).*

(g) V_2 *with* V_3.

(h) $V_\varnothing$ *with* V_2 *(analogously,* $V_\varnothing$ *with* V_3*).*

Proof. We proceed by contradiction. The following facts can be corroborated in Figure 8.51: if $V_{1,2}$ is not completely adjacent to $\overline{V_4}$, G has X_{58} as an induced subgraph; if $\overline{V_1}$ is not completely adjacent to $\overline{V_\varnothing}$, G has a copy of W_4; if $\overline{V_1}$ is not completely adjacent to $\overline{V_4}$, $\overline{A}$ is an induced subgraph of G.

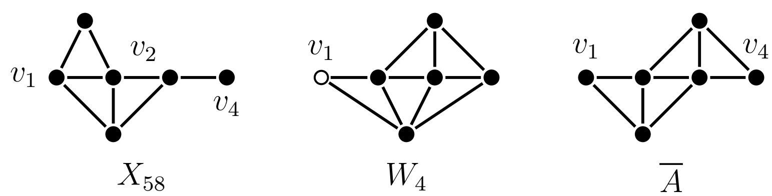


Figure 8.51: Counterexamples.

The following facts can be corroborated in Figure 8.52: since G is C_4-free, $V_{1,2}$ is completely nonadjacent to $V_{3,4}$; if $V_{1,2}$ is not completely nonadjacent to $\overline{V_1}$, G has a copy of $\overline{A}$; if there is an edge with and end in $V_{1,2}$ and the other end in V_3, then G has an induced C_4; since G is C_4-free, V_2 is completely nonadjacent to V_3; if $V_\varnothing$ is not completely nonadjacent to V_2, $K_1 + 2K_2$ is an induced subgraph of G. □

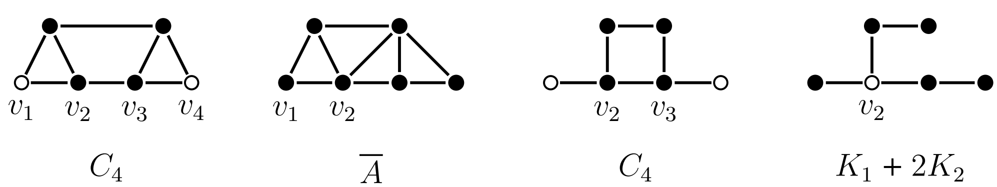


Figure 8.52: Counterexamples.

Lemma 8.33. *Let* G *be a* (C_4, C_5, P_5)*-free minimal* $\overline{A_7}$*-obstruction having an induced path* $P = (v_1, v_2, v_3, v_4)$*. If* G *is not a graph in Figure 8.1 such that* $V_{1,2} \neq \varnothing$*:*

1. *$V_2 = \varnothing$.*
2. *$V_{2,3} = \varnothing$.*
3. *$V_{\varnothing} = \varnothing$.*
4. *$\overline{V_{\varnothing}} = \varnothing$.*
5. *$\overline{V_1} = \varnothing$ or $\overline{V_4} = \varnothing$.*
6. *V_3 is completely adjacent to $\overline{V_1}$.*
7. *Either $V_3 = \varnothing$ or $\overline{V_4} = \varnothing$.*

Proof. We proceed by contradiction. The following facts can be verified in Figure 8.53: if $V_2 \neq \varnothing$, G has a copy of either $K_1 + 2K_2$ or X_{170}; if $V_{2,3} \neq \varnothing$, then G has some of X_{163} or X_{58} as an induced subgraph; if $V_{\varnothing} \neq \varnothing$, G has an induced subgraph isomorphic to either $K_1 + 2K_2$ or X_{166}; if $\overline{V_{\varnothing}} \neq \varnothing$, then G has a copy of $\overline{A}$ or $\overline{K_1 + P}$; if $\overline{V_1}$ and $\overline{V_4}$ are simultaneously nonempty sets, it follows from Lemma 8.32 that G has $K_2 \oplus 2K_2$ as an induced subgraph.

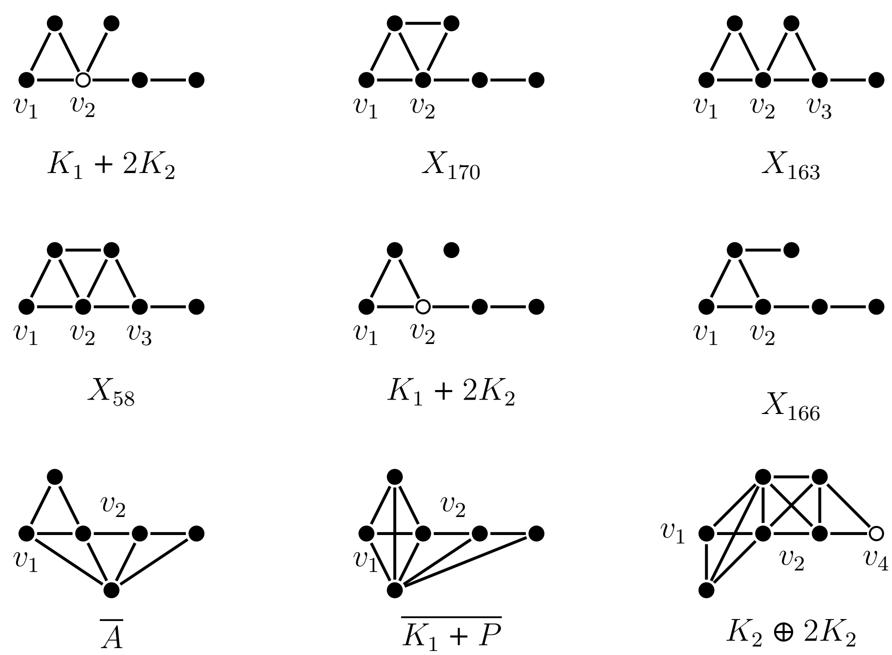


Figure 8.53: Counterexamples.

If V_3 is not completely adjacent to $\overline{V_1}$, then G has $K_1 + 2K_2$ as an induced subgraph; if V_3 and $\overline{V_4}$ are simultaneously nonempty sets, then G has either Y_2 or $\overline{K_1 + P}$ as an induced subgraph. These assertions can ve verified in Figure 8.54. □

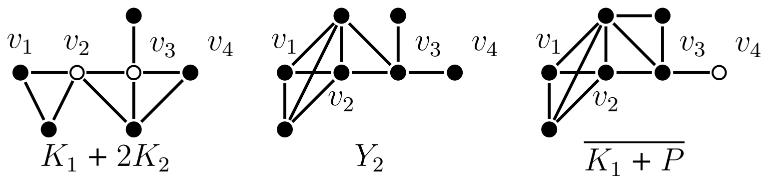


Figure 8.54: Cases of Lemma 8.33.

The following observation can be easily deduced from Lemmas 8.31 and 8.32.

Remark 8.34. *Let G be a (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4)$. Then*

$$(S, K) = (\{v_1, v_4\} \cup V_\varnothing \cup V_2 \cup V_3, \{v_2, v_3\} \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing})$$

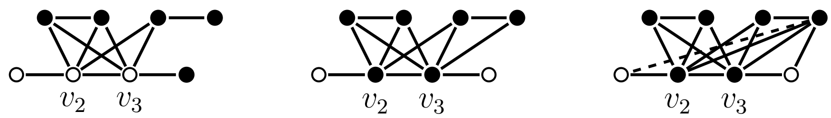
is a split partition of $G[V_P \cup V_\varnothing \cup V_2 \cup V_3 \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}]$.

Lemma 8.35. *Let G be a (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4)$. If G is not a graph in Figure 8.1, either $V_{1,2} \cup V_{3,4} \neq \varnothing$ or $V_{2,3} = \varnothing$.*

Proof. We proceed by contradiction: assuming that $V_{1,2} = V_{3,4} = \varnothing$ and $V_{2,3} \neq \varnothing$ we will prove that G has a split partition, and hence that G is not an $\overline{A_7}$-obstruction. Notice that, in order to prove that G is a split graph, it is enough to show that G is a $2K_2$-free graph because we have by hypothesis that G is a (C_4, C_5)-free graph.

So, aiming for a contradiction, suppose that $V' = \{u, v, x, y\}$ is a subset of V_G inducing $2K_2$; let us say without loss of generality that $uv, xy \in E_G$. Observe that, from Lemma 8.30, $V_G \setminus V_P = V_\varnothing \cup V_2 \cup V_3 \cup V_{2,3} \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$. Moreover, it follows from Remark 8.34 that $(S, K) = (\{v_1, v_4\} \cup V_\varnothing \cup V_2 \cup V_3, \{v_2, v_3\} \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing})$ is a split partition of $G - V_{2,3}$, so $V' \cap V_{2,3} \neq \varnothing$. Additionally, we have from Lemma 8.31 that $G[V_{2,3}]$ (and hence $G[V_P \cup V_{2,3}]$) is a split graph, so $V' \notin V_{2,3}$. Hence, $1 \leq |V' \cap V_{2,3}| \leq 3$. We distinguish the following cases.

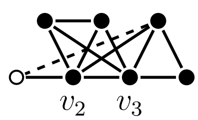
Case 1. $V' \cap V_{2,3}$ induces $\overline{P_3}$. Assume without loss of generality that u is the only vertex of V' not in $V_{2,3}$. Clearly, $u \in V_\varnothing \cup V_2 \cup V_3 \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$, and by the symmetries of P_4 we can suppose that $u \in V_\varnothing \cup V_3 \cup \overline{V_1} \cup \overline{V_\varnothing}$. Nevertheless, as can be checked in Figure 8.55, in any of such cases G contains a graph in Figure 8.1, which cannot occur.



$K_1 + 2K_2$ $\qquad$ $\overline{K_1 + P}$ $\qquad$ $K_2 \oplus 2K_2$

Figure 8.55: Counterexamples.

Case 2. $V' \cap V_{2,3}$ induces K_2. Assume without loss of generality that $V' \setminus V_{2,3} = \{u, v\}$. As we have observed, $\{v_1, v_4\} \cup V_\varnothing \cup V_2 \cup V_3$ and $\{v_2, v_3\} \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$ are an independent set and a clique, respectively. From the previous observation, and due to the fact that $\{u, v\} \cap \{v_2, v_3\} = \varnothing$, it follows that at least one of u or v belongs to $\overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$. Let us assume without loss of generality that $u \in \overline{V_1} \cup \overline{V_\varnothing}$. As can be seen in Figure 8.56, since $\{u, v\}$ is completely nonadjacent to $\{x, y\}$, in this case G has a copy of $\overline{K_1 + P}$, which is impossible.



$\overline{K_1 + P}$

Figure 8.56: Counterexamples.

Case 3. $V' \cap V_{2,3}$ induces $\overline{K_2}$. Assume without loss of generality that $V' \setminus V_{2,3} = \{u, x\}$. We claim that some of u or x belongs to $V_\varnothing \cup V_2 \cup V_3$, otherwise either both vertices belong to $\overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$ (which cannot occur since such a set is a clique), or one of them belongs to V_P (which is absurd since $\{u, v, x, y\}$ induces a $2K_2$, and both u and x are neither completely adjacent nor completely nonadjacent to $V_{2,3}$). By the symmetries of P_4 we can assume without loss of generality that either $u \in V_\varnothing$ and $x \in V_\varnothing \cup V_2 \cup \overline{V_4} \cup \overline{V_\varnothing}$, or $u \in V_2$ and $x \in V_2 \cup V_3 \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$. But, if $u \in V_\varnothing$ and $x \in V_\varnothing \cup V_2 \cup \overline{V_4}$, G has a copy of $K_1 + 2K_2$; if $u \in V_\varnothing$ and $x \in \overline{V_\varnothing}$, G has X_{170} as an induced subgraph; if $u \in V_2$ and $x \notin V_2 \cup V_3$, G has a subgraph isomorphic to $\overline{K_1 + P}$; if $u \in V_2$ and $x \in V_3$, G has a copy of $\overline{A}$; and finally, if $u, x \in V_2$, then G has $K_1 \oplus P_5$ as an induced subgraph. The previous claims can be verified in

Figure 8.57.

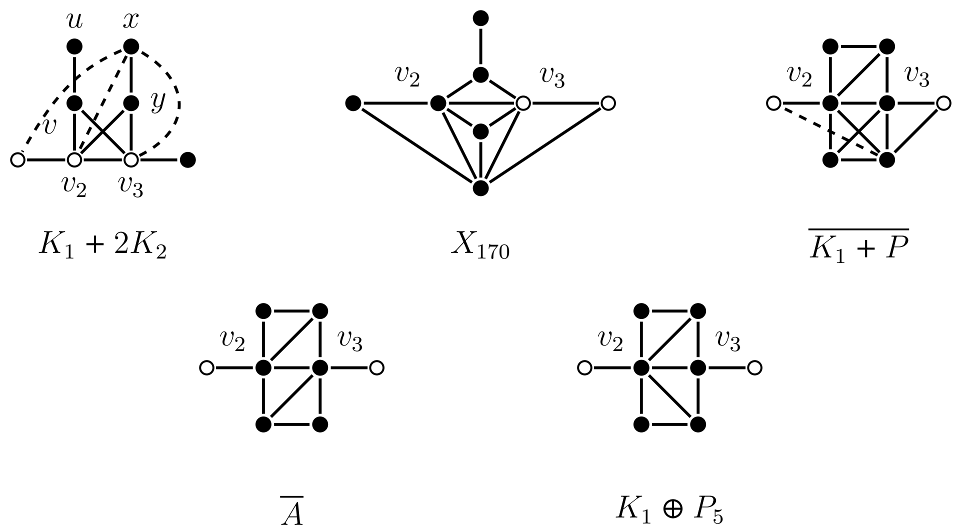


Figure 8.57: Counterexamples.

Case 4. $V' \cap V_{2,3}$ induces K_1. Assume without loss of generality that $V' \setminus V_{2,3} = \{u, x, y\}$. First, since x is adjacent to y and $\{x, y\}$ is completely non adjacent to v, it is easy to notice that we can assume one of the following eleven subcases (any other case violates some of the properties mentioned above or is analogous by symmetry to one of the cases in this list):

1. $x = v_1$ and $y \in \overline{V_4}$.
2. $x = v_1$ and $y \in \overline{V_\varnothing}$.
3. $x \in V_\varnothing$ and $y \in \overline{V_1}$.
4. $x \in V_\varnothing$ and $y \in \overline{V_\varnothing}$.
5. $x \in V_2$ and $y \in \overline{V_1}$.
6. $x \in V_2$ and $y \in \overline{V_4}$.
7. $x \in V_2$ and $y \in \overline{V_\varnothing}$.
8. $x, y \in \overline{V_1}$.
9. $x \in \overline{V_1}$ and $y \in \overline{V_4}$.
10. $x \in \overline{V_1}$ and $y \in \overline{V_\varnothing}$.
11. $x, y \in \overline{V_\varnothing}$.

In addition, u must be adjacent to v and completely nonadjacent to $\{x, y\}$. Using such an observation it is easy to notice that u necessarily belongs to $V_\varnothing \cup V_2 \cup V_3$, no matter the election of x and y. Nevertheless, we claim that none of these 33 cases is possible:

a. Suppose that $x = v_1$ and $y \in \overline{V_4}$. The following facts can be verified in Figure 8.58: if $u \in V_\varnothing$, G contains a copy of X_{58}; if $u \in V_2$, G has $K_1 \oplus P_5$

as an induced subgraph; if $u \in V_3$, G contains a copy of $\overline{A}$. The case in which $x = v_1$ and $y \in \overline{V_\varnothing}$ follows in the same way.

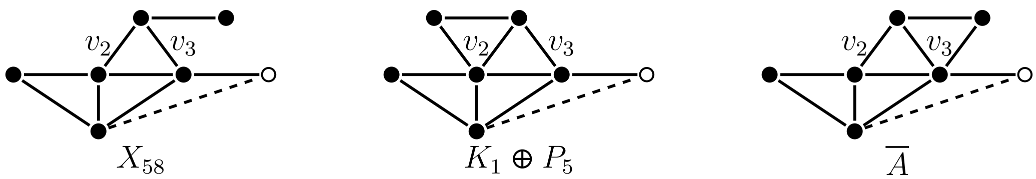


Figure 8.58: Counterexamples.

b. Suppose that $x \in V_\varnothing$ and $y \in \overline{V_1}$. The following facts can be verified in Figure 8.59: if $u \in V_\varnothing$, G contains a copy of X_{58}; if $u \in V_2$, G contains a copy of $\overline{A}$; if $u \in V_3$, G has $K_1 \oplus P_5$ as an induced subgraph. The cases in which either $x \in V_\varnothing$ and $y \in \overline{V_\varnothing}$, or $x \in V_2$ and $y \in \overline{V_1}$ follow in the same way.

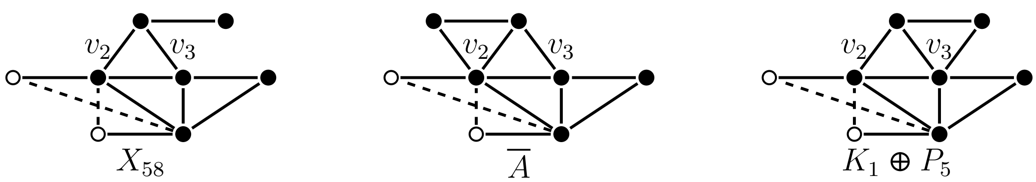


Figure 8.59: Counterexamples.

c. Suppose that $x \in V_2$ and $y \in \overline{V_4}$. The following facts can be verified in Figure 8.60: if $u \in V_\varnothing$, G contains a copy of X_{58}; if $u \in V_2$, G has $K_1 \oplus P_5$ as an induced subgraph; if $u \in V_3$, G contains a copy of $\overline{A}$. The case in which $x \in V_2$ and $y \in \overline{V_\varnothing}$ follows in the same way.

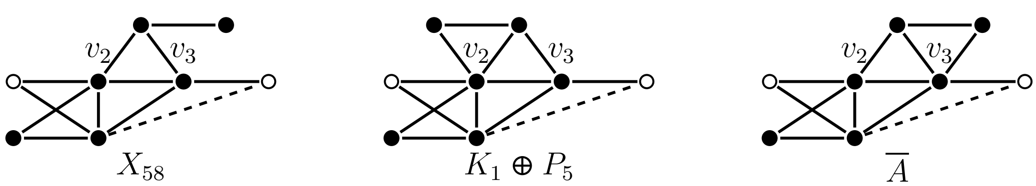


Figure 8.60: Counterexamples.

d. Suppose that $x, y \in \overline{V_1}$. The following facts can be verified in Figure 8.61: if $u \in V_\varnothing$, G contains a copy of X_{58}; if $u \in V_2 \cup V_3$, then G contains a copy of $\overline{K_1 + P}$. The cases in which either $x \in \overline{V_1}$ and $y \in \overline{V_4}$, or $x \in \overline{V_1}$ and $y \in \overline{V_\varnothing}$, or $x, y \in \overline{V_\varnothing}$ follow in the same way.

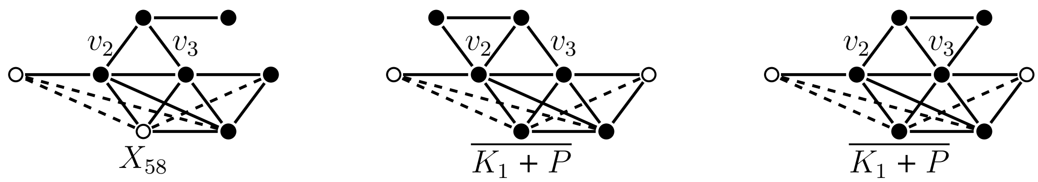


Figure 8.61: Counterexamples.

□

Lemma 8.36. *Let G be a (C_4, C_5, P_5)-free minimal $\overline{A_7}$-obstruction having an induced path $P = (v_1, v_2, v_3, v_4)$. Then, G is some of the graphs in Figure 8.1.*

Proof. By Lemma 8.30 we have that $V_G = V_P \cup V_\varnothing \cup V_2 \cup V_3 \cup V_{1,2} \cup V_{3,4} \cup V_{2,3} \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing}$. Let us start supposing that $V_{1,2} \cup V_{3,4} \neq \varnothing$; we assume without lose of generality that $V_{1,2} \neq \varnothing$. In this case, it follows from Lemma 8.33 that $V_G = V_P \cup V_3 \cup V_{1,2} \cup V_{3,4} \cup \overline{V_1} \cup \overline{V_4}$, and V_3 is completely adjacent to $\overline{V_1}$.

First, assume that $\overline{V_4} = \varnothing$. If $V_3 = \varnothing$, it follows from Lemmas 8.31 and 8.32 that

$$(A, B, C) = (\{v_2\}, \{v_1\} \cup V_{1,2}, \{v_3, v_4\} \cup V_{3,4} \cup \overline{V_1})$$

is an $\overline{A_7}$-partition of G, which is impossible. Hence, $V_3 \neq \varnothing$, and we have from Lemma 8.30 that $V_{3,4} = \varnothing$ but then, Lemmas 8.31 and 8.32 imply that

$$(A, B, C) = (\{v_2, v_4\} \cup V_3, \{v_3\} \cup \overline{V_1}, \{v_1\} \cup V_{1,2})$$

is an $\overline{A_7}$-partition of G, an absurd. Thus, it must be the case that $\overline{V_4} \neq \varnothing$, and it follows from Lemma 8.33 that $\overline{V_1} = V_3 = \varnothing$, so

$$(A, B, C) = (\{v_3\}, \{v_4\} \cup V_{3,4}, \{v_1, v_2\} \cup V_{1,2} \cup \overline{V_4})$$

is an $\overline{A_7}$-partition of G, what cannot occur.

We conclude from the previous cases that it is impossible that $V_{1,2} \cup V_{3,4} \neq \varnothing$, and we have from Lemma 8.35 that $V_{2,3} = \varnothing$. Nevertheless, in such a case, it follows from Remark 8.34 that

$$(A, B, C) = (\{v_1, v_4\} \cup V_\varnothing \cup V_2 \cup V_3, \varnothing, \{v_2, v_3\} \cup \overline{V_1} \cup \overline{V_4} \cup \overline{V_\varnothing})$$

is an $\overline{A_7}$-partition of G, contradicting that G is an $\overline{A_7}$-obstruction. □

8.5 P_4-free obstructions and main result

We start this section by proving that any cograph minimal $\overline{A_7}$-obstruction is a graph in Figure 8.1. We finish the chapter proving that the set of minimal $\overline{A_7}$-obstructions is precisely the set of graphs depicted in Figure 8.1.

Lemma 8.37. *The only disconnected minimal $\overline{A_7}$-obstructions are $K_1 + 2C_4$, $\overline{W_4}$, $\overline{W_5}$, and $2P_3$.*

Proof. Let G be a disconnected minimal $\overline{A_7}$-obstruction. First, suppose that G has at least three connected components. If at least two components of G are nontrivial, then $G \cong K_1 + 2K_2$. Otherwise, $G = H + \ell K_1$ for some nontrivial graph H and some integer $l \geq 2$. Nevertheless, if H is a split graph, then G is too, but if it is not, G properly contains $K_1 + 2K_2, K_1 + C_4$ or $K_1 + C_5$. Thus, the only minimal $\overline{A_7}$-obstruction with at least three components is $K_1 + 2K_2$.

Now assume that G has exactly two components, G_1 and G_2. If neither component is a complete graph, $G \cong 2P_3$, so let us suppose without loss of generality that G_1 is a complete component. In addition, if we assume that G is different from $K_1 + 2K_2, K_1 + C_4$ and $K_1 + C_5$, we have that G_2 is a split graph, contradicting the choice of G. Thus, G_1 has at least two vertices. Since G does not have $K_1 + 2K_2$ as an induced subgraph, G_2 is a $\overline{P_3}$-free graph. Moreover, G_2 is a complete split graph because it is a C_4-free graph. This is a contradiction, because in such a case G admits an $\overline{A_7}$-partition. □

Lemma 8.38. *The only minimal $\overline{A_7}$-obstructions whose complement is disconnected are W_4, $K_2 \oplus 2K_2$, $\overline{2P_3}$, W_5, $\overline{K_1 + P}$, $K_{3,3}$, $\overline{K_2 + \text{diamond}}$, $K_1 \oplus P_5$, and $K_1 \oplus (K_2 + P_3)$.*

Proof. We will prove the following equivalent statement: The only disconnected minimal A_7-obstructions are $K_1 + 2K_2$, $\overline{C_4 + 2K_1}$, $2P_3$, $K_1 + C_5$, $K_1 + P$, $2K_3$, $K_2 + \text{diamond}$, $K_1 + \text{house}$, and $K_1 + \overline{K_2 + P_3}$.

Let G be a disconnected minimal A_7-obstruction. Let us start assuming that G has at least three connected components. Since G is not a split graph, it contains a copy of $2K_2, C_4$ or C_5, and then G has $K_1 + 2K_2, C_4 + 2K_1$ or $K_1 + C_5$ as an induced subgraph.

Now, let us assume that G has exactly two connected components, G_1 and G_2. On one hand, if both components of G are complete graphs, each of them has at least three vertices, otherwise G would be the disjoint union of a complete graph with a complete bipartite graph, so G would have an A_7-partition, that is absurd; therefore, if both components of G are complete graphs, $G \cong 2K_3$. On the other hand, if both components of G are non-complete graphs, then G contains $2P_3$. So we can assume for the rest of the proof that G_1 is a complete graph, and G_2 is neither a complete graph or a complete bipartite graph.

Aiming for a contradiction, assume that G_1 has at least three vertices. Let $v \in V_{G_1}$ and let (A, B, C) be an A_7-partition of $G - v$. If $A \cap V_{G_1 - v} \neq \varnothing$, then $(A \cup \{v\}, B, C)$ is an A_7-partition of G. Otherwise $G_1 - v$ has precisely two vertices, one of them in B and the other in C, but then $(A, B \cup \{v\}, C)$ is an A_7-partition of G, and we have a contradiction. Thus, G_1 has one or two vertices.

First suppose that $G_1 \cong K_2$. Observe that in this case G_2 is a $\overline{P_3}$-free graph, that is to say, a complete multipartite graph. Now, since G_2 is a connected non-complete graph, it has an induced $P_3 = (u, v, w)$ such that any other vertex of G_2 is at distance one from { u,v,w }. If there is a vertex that is completely adjacent to P_3, then $G \cong K_2 +$ diamond, otherwise, since G_2 is a complete multipartite graph, any vertex in $G_2 - P_3$ is either adjacent to both ends of P_3 but not to its center, or adjacent to only the center of P_3. Once again, since G_2 is an $\overline{P_3}$-free graph, the set of vertices adjacent to the ends of P_3, V_1, and the set of vertices adjacent to the center of P_3, V_2, are both independent sets. Moreover, if there are nonadjacent vertices $v_1 \in V_1$ and $v_2 \in V_2$, then $P \lneq G_2$ so G properly contains $K_1 + P$, an absurd. Hence, V_1 and V_2 are completely adjacent. Here we have a contradiction, because then G_2 is a complete bipartite graph with bipartition $(V_1 \cup \{v\}, V_2 \cup \{u, w\})$.

The only remaining case is that $G \cong K_1$. If G_2 is a split graph, G is too, so this is not the case and G_2 contains some of $2K_2, C_5$ or C_4. In the first cases G contains $K_1 + 2K_2$ or $K_1 + C_5$, so we can assume that G contains an induced $C_4 = (u, v, w, z, u)$ such that any other vertex is at distance 1 from { u,v,w,z }. Unless G has some of $K_1 + P, K_1 +$ house, $K_1 + \overline{K_2 + P_3}$ or W_4 as an induced subgraph (all of them minimal A_7-obstructions), the vertices of $G_2 - C_4$ are all of them adjacent to precisely two antipodal vertices of C_4. In addition, by assuming that G_2 does not have $\overline{K_2 + P_3}$, we have that the set V_1 of vertices of $G_2 - C_4$ that are adjacent to u and w, is an independent set, as well as the set V_2 of the vertices of $G_2 - C_4$ that are adjacent to the vertices v and z. Moreover, if V_1 and V_2 are not completely adjacent, G properly contains $K_1 + P$, so they are. In this case it is easy to notice that G_2 is a complete bipartite graph with bipartition $(V_1 \cup \{v, z\}, V_2 \cup \{u, w\})$, which is impossible. □

Lemma 8.39. *The only P_4-free minimal $\overline{A_7}$-obstructions are the cographs mentioned in Lemmas 8.37 and 8.38.*

Proof. The result follows from the well-known fact that, for any nontrivial cograph G, either G or $\overline{G}$ is disconnected. □

After all work developed in this chapter, we can finally state a complete characterization of graphs admitting an $\overline{A_7}$-partition by means of their minimal forbidden induced subgraphs.

Theorem 8.40. *A graph G admits an $\overline{A_7}$-partition if and only if its is an $\mathcal{F}$-free graph, where $\mathcal{F}$ is the family of graphs depicted in Figure 8.1.*

Proof. It is enough to prove that $\mathcal{F}$ is the set of minimal $\overline{A_7}$-obstructions. It follows from Lemma 8.1 that any graph in $\mathcal{F}$ is a minimal $\overline{A_7}$-obstruction, so we only need to prove that there are no more of such obstructions, but that follows directly from Lemmas 8.8, 8.24, 8.29, 8.36 and 8.39. □

Conclusions

The study of polarity on cographs started in [34], were the complete list of cograph minimal polar obstructions was given. In [9] and [46] were exposed the complete lists of cograph minimal $(2,1)$- and 2-polar obstructions, respectively. Then, in [19] was developed a recursive characterization of minimal $(s,1)$-polar obstructions and, as a byproduct, the list of cograph minimal $(\infty,1)$-obstructions was given. Additionally, in [28] was proved that any hereditary property has only finitely many minimal obstructions when restricted to cographs. Part of this work is a natural continuation of these lines of research.

In Chapter 3, we studied the minimal obstructions for (∞,k)-polarity on cographs, providing a partial recursive characterization for them (Theorems 3.3, 3.8, 3.13 and 3.22 and Lemma 3.14) and giving complete lists for the cases $k = 2$ and $k = 3$ (Theorems 3.24 and 3.25). We started Chapter 4 proving that any hereditary property has finitely many minimal obstructions when restricted to two cograph superclasses, namely P_4-sparse and P_4-extendible graphs (Corollaries 4.8 and 4.12). Then, we generalized almost all known results on characterizations by minimal obstructions of cographs with properties related to polarity on both, P_4-sparse graphs and P_4-extendible graphs. Specifically, we gave complete lists of minimal obstructions for P_4-sparse and P_4-extendible graphs having some of the following partitions: unipolar (Corollaries 4.18 and 4.21), monopolar (Corollaries 4.44 and 4.47), polar (Corollary 4.46 and Theorem 4.48), $(s,1)$-polar (Theorems 4.35 and 4.42), and 2-polar (Theorems 4.61 and 4.72). On these results, it is worth emphasizing that all P_4-sparse minimal obstructions for said properties are cographs, in contrast with the case of P_4-extendible graphs. In Chapter 4 we also gave linear time algorithms to find maximum subgraphs with some given properties related to polarity on P_4-sparse and P_4-extendible graphs, particularly, we proved that deciding polarity, monopolarity and unipolarity can be done in linear time on this graph classes (Theorems 4.80 and 4.88).

As we mentioned in Section 2.3, the problems of deciding whether an arbitrary graph is either polar or monopolar are NP-complete and, although unipolarity and (s,k)-polarity can be solved in polynomial time, the sets of minimal obstructions for most of these properties remain unknown in general graphs. In Chapters 5 and 6 we considered the aforementioned problems on $\mathcal{H}$-split graphs, specifically, in pseudo-split graphs (C_5-split graphs) and $2K_2$-split graphs (from which we can easily deduce analogous results for C_4-split graphs).

In Chapter 5, we developed linear time recognizing algorithms for pseudo-split (s,k)-polar graphs, where s and k are arbitrary nonnegative integers (Theorem 5.7) or some of them is ∞ (Theorem 5.6). Additionally, in the same chapter we observed that any pseudo-split graph is polar, and gave complete lists of pseudo-split minimal monopolar (unipolar) obstructions, from which we derived linear time recognition algorithms for such graph families (Theorem 5.8). In Theorem 5.9 we exhibited the pseudo-split minimal (s,k)-polar obstructions for $s \in \{1,2\}$, and we obtained as a byproduct the complete list of pseudo-split minimal $(2,\infty)$-polar obstructions (Corollary 5.11). In Theorem 5.14 we proved that any pseudo-split minimal (s,k)-polar obstruction has at most $s+k+3$ vertices, and this bound is tight. Moreover, we demonstrated that there is only a finite number of pseudo-split minimal (s,∞)-polar obstructions by proving that any of such obstructions has order at most $2s+4$, and this bound is also tight (Corollary 5.18); it is worth noticing that by a simple argument of complements an analogous result can be deduced for pseudo-split minimal (∞,k)-polar obstructions. We finished our study of polarity on pseudo-split graphs by describing a computer program we implemented for obtaining complete lists of pseudo-split minimal (s,k)-polar obstructions from bipartite graphs. In the small Section 5.2 we gave some results about (k,ℓ)-colorings on pseudo-split graphs; we proved that for any pair of nonnegative integers k and ℓ, it can be decided in linear time from its degree sequence whether a pseudo-split graph is a (k,ℓ)-graph (Theorem 5.20), and we concluded that the cochromatic and bichromatic numbers of a pseudo-split graph can de found in linear time too (Corollary 5.25).

Chapter 6 was devoted to the study of polarity on $2K_2$-split graphs. In that chapter we obtained analogous results to those we mentioned above for pseudo-split graphs. We proved that $2K_2$-split graphs are unipolar and co-unipolar, and hence polar graphs. We gave complete lists of $2K_2$-split minimal $(1,k)$- and $(s,1)$-polar obstructions, and we conclude that there is just one $2K_2$-split minimal monopolar (co-monopolar) obstruction, and $2K_2$-split monopolar (co-monopolar) graphs can be recognized in linear time from its degree sequence (Propositions 6.2 and 6.3). We provided a complete characterization of $2K_2$-split graphs with an (s, k)-polar partition in Theorem 6.4, and we used this result to demonstrate that any $2K_2$-split minimal (s, k)-polar obstruction has order at most $s + 2k + 2$ (Theorem 6.11), $2K_2$-split minimal (s, ∞)-polar obstructions have order at most $2s + 4$ (Theorem 6.13), and the order of the $2K_2$-split minimal (∞, k)-polar obstructions is upper-bounded by an $O(2^k)$-function (Theorem 6.15). In Section 6.1.2 we proved that the properties of being (s, ∞)-, (∞, k), and (s, k)-polar can be efficiently recognized in $2K_2$-split graphs (Propositions 6.21 and 6.22 and Theorem 6.25), but this cannot always be done from their degree sequences as in the case of pseudo-split graphs.

The last two chapters were devoted to give the lists of minimal obstructions for some small patterns, namely, those patterns of size 3×3 with exactly one entry $*$ off the main diagonal. In Chapter 7 we gave the mentioned lists for patterns A_0 (Theorem 7.1), A_3 (Theorem 7.2), A_4 (Theorem 7.3), $\overline{A_5}$ (Theorem 7.4), and A_6 (Theorem 7.10). Due to the length of the proof needed to demonstrate the characterization by minimal obstructions of the graphs admitting an A_7-partition, Chapter 8 is entirely devoted to that proof, which culminates in Theorem 8.40. It is worth mentioning that, with these new results, the only patterns of size at most three that remain pending to be characterized by their sets of minimal obstructions are those patterns of size 3×3 with two or three entries $*$ off the main diagonal, which include the patterns associated to $(1, 2)$-polar graphs, $(1, 2)$-colorable graphs, and 3-colorings.

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Glossary of symbols

$\cap$	Intersection	
$\cup$	Union	
$\in$	Element of a set	
$\lvert X \rvert$	Cardinality of the set X	
$\setminus$	Set-theoretic difference	
$\subseteq$	Subset	
$\subsetneq$	Proper subset	
$\alpha(G)$	Independence number of G	6
$\chi(G)$	Chromatic number of G	6
$\chi^b(G)$	Bichromatic number of G	6
$\chi^c(G)$	Cochromatic number of G	6
$\mathrm{d}(u)$	Degree of u	3
$\omega(G)$	Clique number of G	6
$\overline{G}$	Graph complement	5
$\theta(G)$	Clique covering number of G	6
C_k	The cycle of order k	5

E_G	Edge set of the graph G	3
F_s	A pseudo-split minimal (s,k)-polar obstruction	134
$G+H$	Disjoint union of graphs	5
$G=(V,E)$	A graph with vertex set V and edge set E	3
$G \cong H$	Graph isomorphism	4
$G \oplus H$	Join of graphs	5
$G-v$	Vertex deleted subgraph	4
$G-V'$	Subgraph of G induced by the vertex subset $V_G \setminus V'$	4
$G[V']$	Subgraph of G induced by the vertex subset V'	4
$G_1 * G_2$	Spider operation	92
G_s^0	A pseudo-split minimal (s,k)-polar obstruction	129
G_s^1	A pseudo-split minimal (s,k)-polar obstruction	129
$H \leq G$	Induced subgraph relationship	4
H_s	A $2K_2$-split minimal (s,∞)-polar obstruction	153
H_s^k	A pseudo-split minimal (s,k)-polar obstruction	129
K_n	The complet graph of order n	5
$N_G(u)$	Open neighborhood of u	3
$N_G[u]$	Closed neighborhood of u	3
nG	Disjoint union of n copies of G	5
$O(f(n))$	Big O notation	7
P_k	The path of order k	4
V_G	Vertex set of the graph G	3
W_n	The wheel graph of order $n+1$	5

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