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COLORFUL THEOREMS IN DISCRETE AND CONVEX GEOMETRY

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Contents

Introduction

Convex sets are geometrical objects with very interesting properties and have been studied in several branches of mathematics. In particular, discrete and convex geometry is a branch of discrete mathematics with the goal of studying the combinatorial properties of convex sets. The classical theorems in discrete and convex geometry are the theorems of Carathéodory [8] (in 1907), Radon [34] (in 1921) and Helly [18] (in 1923). In order to start seeing the geometrical and combinatorial ideas we state Carathéodory's theorem [8] and Helly's theorem [18].

Carathédory's theorem states that if a point $a \in \mathbb{R}^d$ is in the convex hull of some set $A \subset \mathbb{R}^d$, then there exist at most $d+1$ points in A such that their convex hull contains the point a. In other words, in order to know if a point is in the convex hull of some set A we only need to know if the point is in the convex hull of a finite subset (of size at most $d+1$) of A.

Helly's theorem states that if a finite family of convex sets in $\mathbb{R}^d$ satisfies that every $d+1$ or fewer of them have non-empty intersection, then the whole family has non-empty intersection. In fact, the result is also true for infinite families of compact convex sets. In other words, Helly's theorem states that we only need information concerning the intersection of finite subfamilies (of size $d+1$) in order to know if the whole family has non-empty intersection.

On the other hand, there are several theorems in discrete mathematics which have *colorful* versions. For example, Lovász proved the Colorful Helly theorem in 1973 (see [4]). In addition, in 1982 Bárány [4] proved the Colorful Carathéodory theorem.

The Colorful Helly theorem states that if we have $d+1$ finite families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty, then there exists $i \in \{1, \ldots, d+1\}$ such that the family $\mathcal{F}_i$ has non-empty intersection.

The Colorful Carathéodory theorem states that if we have $d+1$ finite sets $A_1, \ldots, A_{d+1}$ of points in $\mathbb{R}^d$ such that the origin is contained in the convex hull of every set A_i, for $i = 1, \ldots, d+1$, then there exist $d+1$ points $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ such that the origin is contained in the convex hull of $\{a_1, \ldots, a_{d+1}\}$.

Note that we recover Helly's theorem from the Colorful Helly theorem when all the families are equal. Therefore, the Colorful Helly theorem is a generalization of Helly's theorem. By a similar argument, the Colorful Carathéodory theorem is a generalization of Carathéodory's theorem. In general, *Colorful theorems* are usually generalizations of their uncolored versions.

The name *colorful* comes from thinking that every family is colored (and every family has a different color). Then *colorful theorems* follow some of the following two ideas.

- In the hypothesis of colorful theorems we have information concerning rainbow subfamilies and the conclusion is concerning subfamilies of the same color.

- In the hypothesis of colorful theorems we have information concerning subfamilies of the same color and the conclusion is concerning rainbow subfamilies.

For example, in the Colorful Helly theorem we have information concerning the intersection of rainbow subfamilies and the conclusion is concerning the intersection of a subfamily of the same color. On the other hand, in the Colorful Carathéodory theorem we have information concerning sets of the same color and the conclusion is concerning a rainbow set.

This work has two purposes. On the one hand, we present a collection of several colorful theorems in discrete and convex geometry by introducing the ideas of proofs intuitively in low dimensions. On the other hand, we also present new results concerning colorful theorems and improve bounds of colorful theorems.

In Chapter 1 we see an introduction to convex geometry and present the definitions and notation that we use in this work. We present the classical theorems of convex geometry: Carathéodory [8], Helly [18] and Radon [34]. We also see Eckhoff's theorems concerning transversals.

In Chapter 2 we prove the classical colorful theorems in convex geometry: Colorful Helly (done by Lovász in 1973, see [4]), Colorful Carathéodory (done by Bárány [4] in 1982) and Colorful Radon (done by Lovász in 1989, see [2]).

In Chapter 3 we see the following two generalizations of the classical colorful theorems.

- Pach, Holmsen and Tverberg [21] (in 2008) and independently Arocha, Bárány, Bracho, Fabila and Montejano [1] (in 2009) proved that we can weaken the hypothesis of the Colorful Carathéodory theorem and obtain the same conclusion. They proved that if we have $d+1$ finite sets $A_1, \ldots, A_{d+1}$ of points in $\mathbb{R}^d$ such that the origin is contained in the convex hull of $A_i \cup A_j$, for every $1 \leq i < j \leq d+1$, then there exist $d+1$ points $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ such that the origin is contained in the convex hull of $\{a_1, \ldots, a_{d+1}\}$. In addition, we prove that this result cannot be generalized in two different senses.

- In 2020, Martínez-Sandoval, Roldán-Pensado and Rubin [28] wondered if there are further consequences with the hypothesis of the Colorful Helly theorem. They proved that for each dimension $d \geq 2$ there exist numbers $f(d)$ and $g(d)$ with the following property. If $\mathcal{F}_1, \ldots, \mathcal{F}_d$ are finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_d \in \mathcal{F}_d$ the intersection $\bigcap_{i=1}^d C_i$ is non-empty, then either there is a family $\mathcal{F}_j$ that can be pierced by $f(d)$ points, or the family $\bigcup_{i=1}^d \mathcal{F}_i$ can be crossed by $g(d)$ lines. In particular, they proved that their result in the plane ($d=2$) holds with $f(2)=1$ and $g(2)=4$.

In Chapter 4 we present our results. First, we see the topological preliminaries that we use to prove our results. Then we do the following:

- We improve the 2-dimensional case of the theorem by Martínez-Sandoval, Roldán-Pensado and Rubin [28]; we prove that the 2-dimensional case of their theorem also holds with

$f(2) = 1$ and $g(2) = 2$. We prove that if $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are finite families of convex sets in $\mathbb{R}^2$ (with $n \geq 2$), such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ (with $i \neq j$), then either there exists $j \in \{1, 2, \ldots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 1 point, or the family $\bigcup_{i=1}^{n} \mathcal{F}_i$ can be crossed by 2 lines. Furthermore, we also prove that if K is a compact convex set in the plane and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are finite families of translates of K (with $n \geq 2$), such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ (with $i \neq j$), then either there exists $j \in \{1, 2, \ldots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 3 points, or the family $\bigcup_{i=1}^{n} \mathcal{F}_i$ can be crossed by 1 line. We also prove similar results for families of homothets, circles and rectangles.

- We state an open problem proposed by Martínez-Sandoval, Roldán-Pensado and Rubin [28]. The problem is if there exists $n \in \mathbb{Z}^+$ such that for any two families $\mathcal{A}$, $\mathcal{B}$ of convex sets in $\mathbb{R}^3$ so that $A \cap B \neq \emptyset$ holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one of the families $\mathcal{A}$ or $\mathcal{B}$ can be crossed by n lines. We show a particular case of this problem (for small families) solved by Montejano and Karasev ([32], [33]) and give an elementary proof (for small families) by Strausz [41]. In addition, we propose a geometrical idea to reduce the open problem to a topological problem.

- We prove colorful versions of Eckhoff's theorems.

  We prove that if $\mathcal{F}_1, \ldots, \mathcal{F}_4$ are finite families of connected sets in $\mathbb{R}^2$ such that every four sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, \ldots, A_4 \in \mathcal{F}_4$ have a line transversal, then there is a family $\mathcal{F}_i$ that can be crossed by 2 lines.

  We also prove that if $\mathcal{F}_1, \ldots, \mathcal{F}_6$ are finite families of connected sets in $\mathbb{R}^2$ such that every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, for $1 \leq i_1 < i_2 < i_3 \leq 6$, have a line transversal, then there is a family $\mathcal{F}_i$ that can be crossed by 3 lines.

  In addition, we prove the following theorem. Let K be a compact convex set in $\mathbb{R}^2$. If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are finite families of translates of K such that every three sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ have a line transversal, then there is a family $\mathcal{F}_i$ that can be crossed by 4 lines.

  Finally, we present new problems and conjectures related to these colorful versions of Eckhoff's theorems.

Introducción

Los conjuntos convexos son objetos geométricos con propiedades muy interesantes y han sido estudiados en varias áreas de las matemáticas. En particular, geometría discreta y convexa es una rama de las matemáticas discretas que tiene el propósito de estudiar las propiedades combinatorias de conjuntos convexos. Los teoremas clásicos en geometría discreta y convexa son los teoremas de Carathéodory [8] (en 1907), Radon [34] (en 1921) y Helly [18] (en 1923). Para empezar a ver las ideas geométricas y combinatorias enunciamos el teorema de Carathéodory [8] y el teorema de Helly [18].

El teorema de Carathédory nos dice que si un punto $a \in \mathbb{R}^d$ está en la envolvente convexa de un conjunto $A \subset \mathbb{R}^d$, entonces existen a lo más $d+1$ puntos en A tal que su envolvente convexa contiene el punto a. En otras palabras, para saber si un punto está en la envolvente convexa de un conjunto A solo necesitamos saber si el punto está en la envolvente convexa de un subconjunto finito (de tamaño a lo más $d+1$) de A.

El teorema de Helly nos dice que si una familia finita de conjuntos convexos en $\mathbb{R}^d$ cumple que cada $d{+}1$ o menos de ellos tienen intersección no vacía, entonces toda la familia tiene intersección no vacía. De hecho, el resultado también es cierto para familias infinitas de conjuntos convexos y compactos. En otras palabras, el teorema de Helly nos dice que solo necesitamos información sobre la intersección de subfamilias finitas (de tamaño $d+1$) para saber si toda la familia tiene intersección no vacía.

Por otro lado, hay muchos teoremas en matemáticas discretas que tiene versiones *coloreadas*. Por ejemplo, Lovász probó el teorema de Helly coloreado en 1973 (ver [4]). Además, en 1982 Bárány [4] probó el teorema de Carathéodory coloreado.

El teorema de Helly coloreado nos dice que si tenemos $d+1$ familias finitas $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ de conjuntos convexos en $\mathbb{R}^d$ tal que para cada elección de conjuntos $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, la intersección $\bigcap_{i=1}^{d+1} C_i$ es no vacía, entonces existe $i \in \{1, \ldots d+1\}$ tal que la familia $\mathcal{F}_i$ tiene intersección no vacía.

El teorema de Carathéodory coloreado nos dice que si tenemos $d+1$ conjuntos finitos $A_1, \ldots, A_{d+1}$ de puntos en $\mathbb{R}^d$ tal que el origen está contenido en la envolvente convexa de cada conjunto A_i, para $i = 1, \ldots, d+1$, entonces existen $d+1$ puntos $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ tal que el origen está contenido en la envolvente convexa de $\{a_1, \ldots, a_{d+1}\}$.

Note que el caso particular del teorema de Helly coloreado donde todas las familias son la misma familia, es el teorema de Helly. Por lo tanto, el teorema de Helly coloreado es una generalización del teorema de Helly. Por un argumento similar, el teorema de Carathéodory coloreado es una generalización del teorema de Carathéodory. En general, los *teoremas coloreados* la mayoría de las veces son generalizaciones de sus versiones no coloreadas.

El nombre *coloreado* es porque podemos pensar que cada familia está coloreada (y cada familia tiene un color diferente). Entonces los *teoremas coloreados* siguen alguna de las siguientes dos ideas.

- En las hipótesis de los teoremas coloreados tenemos información sobre subfamilias arcoíris y la conclusión es sobre subfamilias del mismo color.

- En las hipótesis de los teoremas coloreados tenemos información sobre subfamilias del mismo color y la conclusión es sobre subfamilias arcoíris.

Por ejemplo, en el teorema de Helly coloreado tenemos información sobre la intersección de subfamilias arcoíris y la conclusión es sobre la intersección de una subfamilia del mismo color. Por otro lado, en el teorema de Carathéodory coloreado tenemos información sobre conjuntos del mismo color y la conclusión es sobre un conjunto arcoíris.

Este trabajo tiene dos propósitos. Por un lado, presentamos una colección de varios teoremas coloreados en geometría discreta y convexa, introduciendo las ideas de las pruebas intuitivamente en dimensiones bajas. Por otro lado, también presentamos nuevos resultados sobre teoremas coloreados y mejoramos cotas de teoremas coloreados.

En el Capítulo 1 vemos una introducción a geometría convexa y presentamos las definiciones y la notación que usamos en este trabajo. Presentamos los teoremas clásicos de geometría convexa: Carathéodory [8], Helly [18] y Radon [34]. También vemos los teoremas de Eckhoff sobre transversales.

En el Capítulo 2 probamos los teoremas clásicos coloreados en geometría convexa: Helly coloreado (por Lovász en 1973, ver [4]), Carathéodory coloreado (por Bárány [4] en 1982) y Radon coloreado (por Lovász en 1989, ver [2]).

En el Capítulo 3 vemos las siguientes dos generalizaciones de los teoremas coloreados clásicos.

- Pach, Holmsen y Tverberg [21] (en 2008) e independientemente Arocha, Bárány, Bracho, Fabila y Montejano [1] (en 2009) probaron que podemos debilitar las hipótesis del teorema de Carathéodory coloreado y obtener la misma conclusión. Ellos probaron que si tenemos $d+1$ conjuntos finitos $A_1, \ldots, A_{d+1}$ de puntos en $\mathbb{R}^d$ tal que el origen está contenido en la envolvente convexa de $A_i \cup A_j$ para cada $1 \leq i < j \leq d+1$, entonces existen $d+1$ puntos $a_1 \in A_1, \ldots, a_{d+1} \in A_{d+1}$ tal que el origen está contenido en la envolvente convexa de $\{a_1, \ldots, a_{d+1}\}$. Además, nosotros probamos que este resultado no puede ser generalizado en dos sentidos diferentes.

- En el 2020, Martínez-Sandoval, Roldán-Pensado y Rubin [28] se preguntaron si hay más consecuencias usando las mismas hipótesis del teorema de Helly coloreado. Ellos probaron que para cada dimensión $d \geq 2$ existen números $f(d)$ y $g(d)$ con la siguiente propiedad. Si $\mathcal{F}_1, \ldots, \mathcal{F}_d$ son familias finitas de conjuntos convexos en $\mathbb{R}^d$ tal que para cada elección de conjuntos $C_1 \in \mathcal{F}_1, \ldots, C_d \in \mathcal{F}_d$ la intersección $\bigcap_{i=1}^d C_i$ es no vacía, entonces hay una familia $\mathcal{F}_j$ que puede ser pinchada por $f(d)$ puntos, o la familia $\bigcup_{i=1}^d \mathcal{F}_i$ puede ser atravezada por $g(d)$ líneas. En particular, ellos probaron que su resultado en el plano $(d = 2)$ se cumple con $f(2) = 1$ y $g(2) = 4$.

En el Capítulo 4 presentamos nuestros resultados. Primero vemos los preliminares topológicos que usamos para probar nuestros resultados. Después, vemos los siguientes resultados y problemas:

- Nosotros mejoramos el caso 2-dimensional del teorema de Martínez-Sandoval, Roldán-Pensado y Rubin [28]; nosotros probamos que el caso 2-dimensional de su teorema también se cumple con $f(2) = 1$ y $g(2) = 2$. Probamos que si $\mathcal{F}_1, \dots, \mathcal{F}_n$ son familias finitas de conjuntos convexos en $\mathbb{R}^2$ (con $n \geq 2$), tal que $A \cap B \neq \emptyset$ para cada $A \in \mathcal{F}_i$ y $B \in \mathcal{F}_j$ (con $i \neq j$), entonces existe $j \in \{1, 2, \dots, n\}$ tal que $\bigcup_{i \neq j} \mathcal{F}_i$ puede ser pinchado por 1 punto, o la familia $\bigcup_{i=1}^{n} \mathcal{F}_i$ puede ser atravezada por 2 líneas. Además, también probamos que si K es un conjunto convexo y compacto en el plano y $\mathcal{F}_1, \dots, \mathcal{F}_n$ son familias finitas de trasladados de K (con $n \geq 2$), tal que $A \cap B \neq \emptyset$ para cada $A \in \mathcal{F}_i$ y $B \in \mathcal{F}_j$ (con $i \neq j$), entonces existe $j \in \{1, 2, \dots, n\}$ tal que $\bigcup_{i \neq j} \mathcal{F}_i$ puede ser pinchado por 3 puntos, o la familia $\bigcup_{i=1}^{n} \mathcal{F}_i$ puede ser atravezada por 1 línea. También probamos resultados similares para familias de homotéticos, círculos y rectángulos.

- Vemos un problema abierto propuesto por Martínez-Sandoval, Roldán-Pensado y Rubin [28]. El problema es si existe $n \in \mathbb{Z}^+$ tal que para todas dos familias $\mathcal{A}$, $\mathcal{B}$ de conjuntos convexos en $\mathbb{R}^3$ tal que $A \cap B \neq \emptyset$ se cumple para cada $A \in \mathcal{A}$ y $B \in \mathcal{B}$, una de las familias $\mathcal{A}$ o $\mathcal{B}$ puede ser cruzado por n líneas. Vemos un caso particular de este problema (para familias pequeñas) resuelto por Montejano y Karasev ([32], [33]) y vemos una prueba elemental (para familias pequeñas) por Strausz [41]. Además, nosotros proponemos una idea geométrica para reducir el problema abierto a un problema topológico.

- Nosotros probamos versiones coloreadas de los teoremas de Eckhoff.

  Probamos que si $\mathcal{F}_1, \dots, \mathcal{F}_4$ son familias finitas de conjuntos conexos en $\mathbb{R}^2$ tal que cada cuatro conjuntos $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, \dots, A_4 \in \mathcal{F}_4$ tienen una línea transversal, entonces hay una familia $\mathcal{F}_i$ que puede ser cruzada por 2 líneas.

  Probamos que si $\mathcal{F}_1, \dots, \mathcal{F}_6$ son familias finitas de conjuntos conexos en $\mathbb{R}^2$ tal que cada tres conjuntos $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, para $1 \leq i_1 < i_2 < i_3 \leq 6$, tienen una línea transversal, entonces hay una familia $\mathcal{F}_i$ que puede ser cruzada por 3 líneas.

  Además, nosotros también probamos el siguiente teorema. Sea K un conjunto compacto y convexo en $\mathbb{R}^2$. Si $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ son familias finitas de trasladados de K tal que cada tres conjuntos $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ tienen una línea transversal, entonces hay una familia $\mathcal{F}_i$ que puede ser cruzada por 4 líneas.

  Finalmente, presentamos nuevos problemas y conjeturas relacionados con estas versiones coloreadas de los teoremas de Eckhoff.

Chapter 1

Convex geometry

In this chapter we give an introduction to convex geometry. In Section 1.1 we see the definitions and the notation that we use in this thesis. In Sections 1.2 and 1.3 we see some of the classical theorems in convex geometry.

1.1. Convexity

Most of the results in this work are concerning *convex sets*. Intuitively, a convex set is a set without holes. To formally define a convex set, we have to recall the definition of a *segment*. For any two points $a, b \in \mathbb{R}^d$, we define the segment $[a, b]$ as the set

$$[a, b] = \{\alpha a + \beta b : \alpha, \beta \geq 0, \alpha + \beta = 1\}.$$

Definition 1.1. *A set $C \subset \mathbb{R}^d$ is* convex *if for every two points $a, b \in C$, the segment $[a, b]$ is contained in C.*

To give the first example of a convex set, let us recall that a hyperplane H in $\mathbb{R}^d$ is a set

$$H = \{x \in \mathbb{R}^d \ : \ u \cdot x = a\},$$

for some $u \in \mathbb{R}^d \setminus \{0\}$ and $a \in \mathbb{R}$. The hyperplane $H = \{x \in \mathbb{R}^d \ : \ u \cdot x = a\}$ define two half spaces denoted as $H^+ = \{x \in \mathbb{R}^d \ : \ u \cdot x \geq a\}$ and $H^- = \{x \in \mathbb{R}^d \ : \ u \cdot x \leq a\}$. An immediate observation is that the hyperplane H and the half spaces H^+ and H^- are convex sets.

For the second example of a convex set, we need the following definition.

Definition 1.2. *Let $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ be two different points. Let $j = \min\{i \ : \ x_i \neq y_i\}$. If $x_j < y_j$ we say that x is less than y with the lexicographical order and denote it as $x <_{lex} y$ (if $x_j > y_j$, then $x >_{lex} y$). For every two points $x, y \in \mathbb{R}^d$, we denote $x \leq_{lex} y$ if x is less than y with the lexicographic order or $x = y$.*

Note that for any point $q \in \mathbb{R}^d$, the set $C = \{x \in \mathbb{R}^d \ : \ x <_{lex} q\}$ is a convex set.

Let $A \subset \mathbb{R}^d$. We say that

$$\alpha_1 a_1 + \cdots + \alpha_n a_n$$

is a *convex combination* of elements of A, if $\{a_1, \ldots, a_n\} \subset A$, $\alpha_1 + \cdots + \alpha_n = 1$ and $\alpha_i \geq 0$ for all $i = 1, \ldots, n$.

The *convex hull* of A, denoted by $\operatorname{conv}(A)$, is the set of all the convex combinations of elements of A. Note that $\operatorname{conv}(A)$ is the smallest convex set that contains A.

In this work we focus on the combinatorial properties of convex sets. For instance, an important observation is that the intersection of convex sets is also a convex set. We are interested in the necessary conditions for a family of convex sets to have non-empty intersection. Another goal is to impose hypotheses to a family to get weaker conclusions; for example, sometimes the goal is to intersect the family by few points or few lines instead of having it intersect. Now, we present the definitions and the notation we use in these geometric results.

An *affine subspace* is a set $x + L$, where $x \in \mathbb{R}^d$ is some vector and L is a linear subspace of $\mathbb{R}^d$. A *k-flat* is an affine subspace of dimension k. For instance, a line is a 1-flat and an hyperplane is a $(d-1)$-flat in $\mathbb{R}^d$.

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $T \subset \mathbb{R}^d$ is a *transversal* to the family $\mathcal{F}$ if every set $C \in \mathcal{F}$ intersects the set T. Additionally, if T is a k-flat and is transversal to the family $\mathcal{F}$, we say that T is a k-flat transversal. For example, a *line transversal* is a line that intersects every member of $\mathcal{F}$, and a *hyperplane transversal* is a hyperplane that intersects every member of $\mathcal{F}$.

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ and let $n \in \mathbb{Z}^+$. We say that the family $\mathcal{F}$ can be *pierced* by n points if there exist n points $p_1, \ldots, p_n \in \mathbb{R}^d$ such that every set $C \in \mathcal{F}$ intersects some of the points $p_1, \ldots, p_n$ (in other words, $P = \{p_1, \ldots, p_n\}$ is a transversal to the family $\mathcal{F}$). We say that the family $\mathcal{F}$ can be *crossed* by n lines if there exist n lines $l_1, \ldots, l_n \in \mathbb{R}^d$ such that every set $C \in \mathcal{F}$ intersects some of the lines $l_1, \ldots, l_n$ (in other words, $l_1 \cup \cdots \cup l_n$ is a transversal to the family $\mathcal{F}$).

Although the results in this thesis are concerning convex sets, sometimes convexity is not sufficient to prove the results. For that reason there are results where we need additional hypotheses. For example, sometimes the results hold only for families of *translates* of some compact convex set or families of *constant width* sets.

Given $K \subset \mathbb{R}^d$, we say that $K + x$ (for some $x \in \mathbb{R}^d$) is a *translate* of K. Given $K \subset \mathbb{R}^d$ compact convex set, for every direction $u \in \mathbb{S}^{d-1}$, there are two tangent hyperplanes l_1, l_2 to K such that l_1 and l_2 are orthogonal to u and the strip bounded by the two hyperplanes l_1, l_2 contains K. The width of the strip is the *width of K in the direction u* (see Figure 1.1).

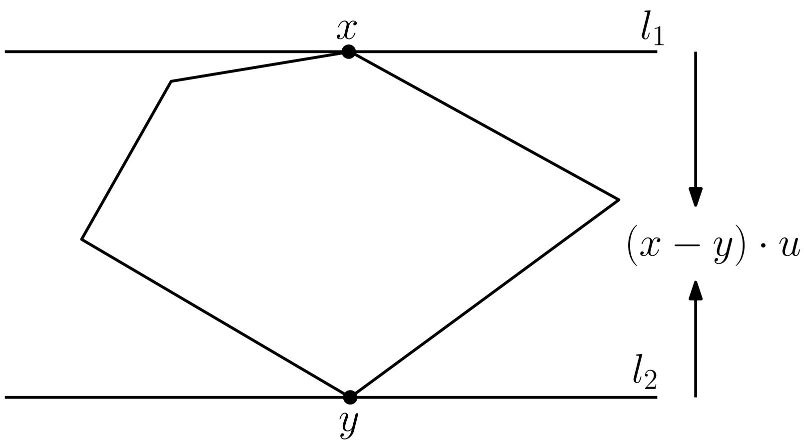

Figure 1.1: Definition of the width of K in the direction u.

Formally, we define the *width of K in the direction u* as

$$\max\{(x-y)\cdot u \ : \ x, y \in K\}.$$

We say that $K \subset \mathbb{R}^d$ is of *constant width* if K has the same width in every direction.

Let K, L be two disjoint compact convex sets in the plane. It is known that there are exactly four tangent lines to K and L (see [13] or [29, page 260]). These lines are called the *inner or outer common tangents* of K and L (see Figure 1.2).

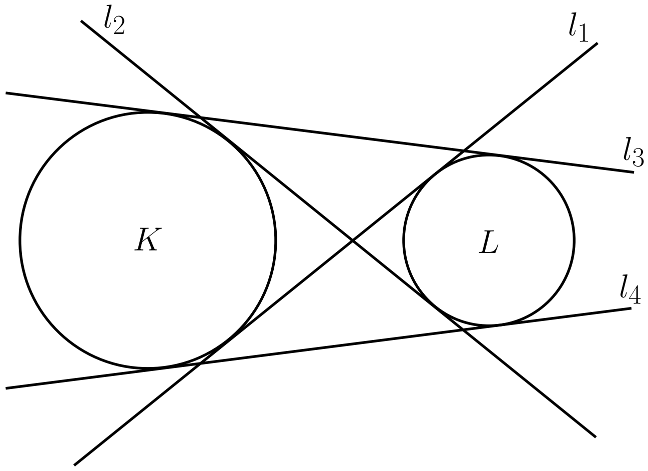

Figure 1.2: The lines l_1 and l_2 are the inner common tangents of K and L, and the lines l_3 and l_4 are the outer common tangents of K and L.

1.2. Carathéodory, Radon and Helly

In this section we state the three classical theorems in convex geometry: Carathéodory's theorem [8], Helly's theorem [18] and Radon's lemma [34]. We do not prove these theorems in this section. Despite this, the reader interested in the proofs of the theorems in this section can consult [29] and [5].

We begin with Carathédory's theorem, proved by Carathédory [8] in 1907. According to Bárány (see [5, page 9]), this theorem is probably the first result in combinatorial convexity.

Carathédory's theorem states that if a point $a \in \mathbb{R}^d$ is in the convex hull of some set $A \subset \mathbb{R}^d$, then there exist at most $d+1$ points in A such that their convex hull contains the point a (see Figure 1.3). In other words, in order to know if a point is in the convex hull of some set A we only need to know if the point is in the convex hull of a finite subset (of size at most $d+1$) of A.

Theorem 1.3. *(Carathéodory's theorem) Let $A \subset \mathbb{R}^d$ and $a \in \operatorname{conv}(A)$. Then there exists $B \subset A$ with $|B| \leq d+1$ such that $a \in \operatorname{conv}(B)$.*

In Section 2.1 we show the Colorful Carathéodory theorem (proved by Bárány [4] in 1982), that implies the Carathéodory theorem (Theorem 1.3). Additionally, in Section 3.1 we give a generalization of the Colorful Carathéodory theorem (proved by Holmsen, Pach and Tverberg [21] in 2008 and independently by Arocha, Bárány, Bracho, Fabila and Montejano [1] in 2009).

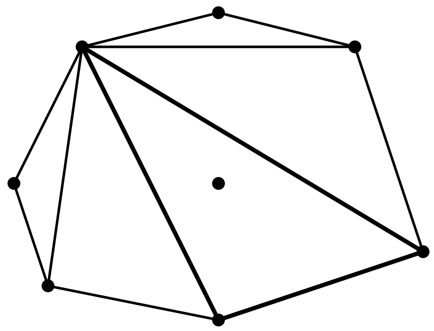

Figure 1.3: In the plane, Caratheodory's theorem states that if a point is in the convex hull of some set of points, then the point is in the convex hull of at most 3 points.

Now, we shall see Helly's theorem, which is probably the most famous theorem in convex and discrete geometry. According to Bárány (see [5, page 29]), Helly discovered his theorem in 1913, but he did not publish it until 1923 [18] due to the First World War.

Helly's theorem states that if we have a finite family of convex sets in $\mathbb{R}^d$ such that every $d+1$ or fewer of them have a non-empty intersection, then the whole family has a non-empty intersection (see Figure 1.4). In other words, Helly's theorem states that we only need information concerning the intersection of finite subfamilies of size $d+1$ in order to know if the whole family has a non-empty intersection.

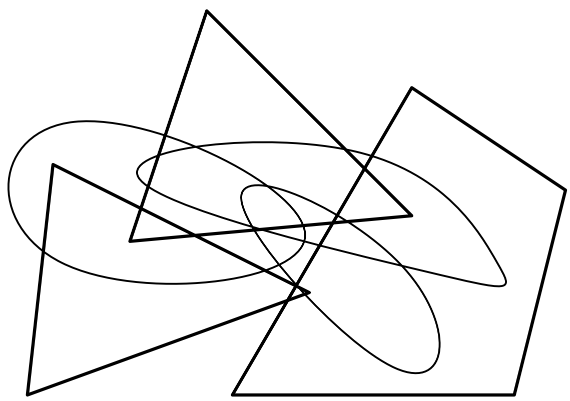

Figure 1.4: In the plane, Helly's theorem states that if a family has empty intersection, then there are three sets in the family with empty intersection.

Theorem 1.4. *(Helly's theorem). Let $\mathcal{F} = \{C_1, C_2, \dots, C_n\}$ be a finite family of convex sets in $\mathbb{R}^d$ such that for every $I \subset \{1, 2, \dots, n\}$ with $|I| \leq d+1$, the intersection $\bigcap_{i \in I} C_i$ is non-empty. Then $\bigcap \mathcal{F} \neq \emptyset$.*

In Section 2.2 we show the Colorful Helly theorem (proved by Lovász in 1973), that implies Helly's theorem. Additionally, in Section 3.2 we give a generalization of the Colorful Helly theorem (proved by Martínez-Sandoval, Roldán-Pensado and Rubin [28] in 2020).

Note that Helly's theorem holds for finite families, however Helly's theorem does not hold for infinite families as we see in Example 1.5.

Example 1.5. *For any $i \in \mathbb{Z}^+$, let H_i^+ be the half space defined by the equation $x_1 \geq i$. Let $\mathcal{F} = \{H_i^+\}_{i \in \mathbb{Z}^+}$. For $i_1 < i_2 < \cdots < i_{d+1}$ positive integers, we have that*

$$H_{i_1}^+ \cap H_{i_2}^+ \cap \cdots \cap H_{i_{d+1}}^+ = H_{i_{d+1}}^+ \neq \emptyset,$$

then the family $\mathcal{F}$ satisfies the hypothesis of Helly's theorem (Theorem 1.4). However, $\bigcap \mathcal{F} = \emptyset$.

Despite this, Helly's theorem is also true for infinite families if we assume that the sets are compact (see [5, Theorem 7.3] or [29, Theorem 1.3.3]).

Theorem 1.6. *(Infinite Helly theorem) Let $\mathcal{F} = \{C_i : i \in J\}$ be a infinite family of compact convex sets in $\mathbb{R}^d$ such that for every $I \subset J$ with $|I| = d+1$, the intersection $\bigcap_{i \in I} C_i$ is non-empty. Then $\bigcap \mathcal{F} \neq \emptyset$.*

In fact, although in Helly's theorem (Theorem 1.4) we do not need the sets to be compact, we can assume that the sets are compact without losing generality. Indeed, if $\mathcal{F} = \{C_1, C_2, \ldots, C_n\}$ are convex sets satisfying the hypothesis of Theorem 1.4 and we denote $C_I = \bigcap_{i \in I} C_i$, then for each I so that $C_I \neq \emptyset$, we can choose $p_I \in C_I$. Then, let

$$K_i = \operatorname{conv}(\{p_I \ : \ C_I \neq \emptyset, \ i \in I\}).$$

We notice that the sets K_i are compact convex sets satisfying the hypothesis of Theorem 1.4 and $K_i \subset C_i$. Hence, if $\bigcap_{i=1}^n K_i$ is non-empty, $\bigcap_{i=1}^n C_i$ is non-empty. Therefore, in Helly's theorem (Theorem 1.4) we can assume without loss of generality that the sets are also compact.

On the other hand, we have a connection between intersections of convex sets and intersections of half-spaces.

Theorem 1.7. *Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$, with $n \geq 2$. Then $\bigcap_{i=1}^n C_i = \emptyset$ if and only if there are hyperplanes $H_1, \ldots, H_n$ such that the closed half-spaces $H_1^+, \ldots, H_n^+$ satisfy $C_i \subset H_i^+$ for all i and $\bigcap_{i=1}^n H_i^+ = \emptyset$.*

The proof of Theorem 1.7 can be consulted in [5, page 30]. The particular case when $n = 2$ is a very useful lemma in discrete geometry and is known as the separation theorem.

We finish this section with Radon's lemma. In 1921, Radon [34] gave an elementary and beautiful proof of Helly's theorem (Theorem 1.4). The idea was to take a point in the intersection of every subfamily of size $d+1$, then Radon studied the combinatorial properties of the points in order to prove Helly's theorem. In particular, Radon [34] proved the following theorem (see Figure 1.5).

Theorem 1.8. *(Radon's theorem). Let A be a set of $d+2$ points in $\mathbb{R}^d$. Then there exists two disjoint subsets $A_1, A_2 \subset A$ such that $\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset$.*

In Section 2.3 we see a Colorful Radon theorem (by Lovász, see [2]) and a proof by Soberón [37], which is very similar to the proof of Radon's lemma.

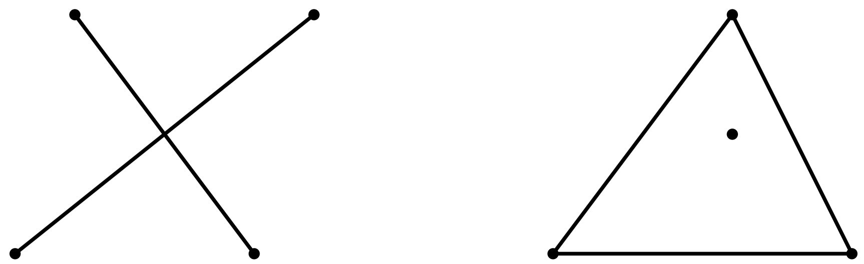

Figure 1.5: The two examples of Radon's theorem in the plane.

1.3. Transversals

In Section 1.2 we saw Helly's theorem (Theorem 1.4) which states that a finite family of convex sets in $\mathbb{R}^d$ has non-empty intersection if each $d+1$ sets of the family have non-empty intersection. A natural question is whether we have a similar conclusion if in the hypothesis we only have that each d sets of the family have non-empty intersection. A first answer is the following Proposition.

Proposition 1.9. *Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ such that the intersection of every d sets in $\mathcal{F}$ is non-empty. Then $\mathcal{F}$ has a line transversal.*

Sometimes, mathematicians restrict the results to some families and obtain stronger results for these families. For instance, Grunbaum [16] (in 1959) proved the following theorem for families of homothets of a convex set.

Theorem 1.10. *For any integer $d \geq 1$ there exists an integer $c = c(d)$ such that the following holds. If $\mathcal{F}$ is a finite family of homothets of a convex set in $\mathbb{R}^d$ and any two members of $\mathcal{F}$ have non-empty intersection, then $\mathcal{F}$ can be pierced by c points.*

The special case of circles in the plane was solved by Danzer [10] (in 1956, but not published until 1986).

Theorem 1.11. *Let $\mathcal{F}$ be a finite family of circles in $\mathbb{R}^2$ such that the intersection of every 2 sets in $\mathcal{F}$ is non-empty. Then $\mathcal{F}$ can be pierced by 4 points.*

In addition, Karasev [23] (in 2000) proved the following theorem for families of translates of a compact convex set in the plane (in fact, Karasev [24] also proved similar results in higher dimensions).

Theorem 1.12. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}$ be a finite family of translates of K such that the intersection of every 2 sets in $\mathcal{F}$ is non-empty. Then $\mathcal{F}$ can be pierced by 3 points.*

It is also known (see [17, page 7]) the following result concerning rectangles with sides parallel to the coordinate axes in the plane.

Theorem 1.13. *Let $\mathcal{F}$ be a finite family of rectangles with sides parallel to the coordinate axes in $\mathbb{R}^2$ such that the intersection of every 2 sets in $\mathcal{F}$ is non-empty. Then $\mathcal{F} \neq \emptyset$ (in other words, $\mathcal{F}$ can be pierced by 1 point).*

In Section 3.2 we see colorful versions of Proposition 1.9. In Section 4.2 we see colorful versions of Theorems 1.10, 1.11, 1.12 and 1.13.

Proposition 1.9 shows that we can weaken the hypothesis of Helly's theorem (Theorem 1.4) and obtain interesting results regarding transversals. Now, we want to answer what conclusions we can obtain if now the hypothesis concerns line transversals (instead of information concerning intersections).

Let $\mathcal{F}$ be a family of sets in the plane. We say that the family $\mathcal{F}$ has the property $T(r)$ if every r sets in $\mathcal{F}$ have a line transversal. Motivated by Helly's theorem (Theorem 1.4), we wonder whether there exists $r \in \mathbb{Z}^+$ such that for any finite family $\mathcal{F}$ (of convex sets) in the plane, if the family $\mathcal{F}$ has the property $T(r)$, then $\mathcal{F}$ has a line transversal. However, Santaló [35] (in 1940) and Danzer [9] (in 1957) observed that for any integer $n \geq 3$ there is a family of n convex sets in the plane satisfying the property $T(n-1)$ while the family does not have a line transversal.

Even though a family satisfying the property $T(r)$ does not necessary have a line transversal, we wonder if such families can be crossed by few lines. In 1969, Eckhoff [11] proved that any finite family of convex sets in the plane satisfying the property $T(4)$ can be crossed by 2 lines.

Theorem 1.14. *Let $\mathcal{F}$ be a finite family of connected sets in $\mathbb{R}^2$. If every four sets in $\mathcal{F}$ have a line transversal (in other words, $\mathcal{F}$ has the property $T(4)$), then the family $\mathcal{F}$ can be crossed by 2 lines (the lines can actually be chosen to be orthogonal).*

The following question was if there exists $n \in \mathbb{Z}^+$ such that any finite family (of convex sets) in the plane satisfying the property $T(3)$ can be crossed by n lines. In 1973, Eckhoff [12] gave an example of a finite family of compact convex sets in the plane satisfying the property $T(3)$ and the family cannot be crossed by 2 lines. Fortunately, in 1974, Kramer [26] proved that any finite family of convex sets in the plane satisfying the property $T(3)$ can be crossed by 5 lines. In 1993, Eckhoff [13] proved that actually the finite families of convex sets in the plane with the property $T(3)$ can be crossed by 4 lines. So during several years an open question was if such families can be crossed by 3 lines. In 2021, McGinnis and Zerbib [31] proved that any finite family of connected sets in the plane that satisfies the property $T(3)$ can be crossed by 3 lines, and the problem is over.

Theorem 1.15. *Let $\mathcal{F}$ be a finite family of connected sets in $\mathbb{R}^2$. If every three sets in $\mathcal{F}$ have a line transversal (in other words, $\mathcal{F}$ has the property $T(3)$), then the family $\mathcal{F}$ can be crossed by 3 lines.*

It is known that Theorems 1.14 and 1.15 are also true for infinite families if we assume that the sets are compact (see [11]).

In addition, in Theorems 1.14 and 1.15 we can assume that the sets are compact and we do not lose generality. This is because for a finite family $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ satisfying the hypothesis of Theorem 1.15 (or Theorem 1.14) we define another family of compact sets as follows. For every three sets in the family $\mathcal{F}$ we choose a line transversal for these sets. Then, for every set in the family $\mathcal{F}$ and every one of these lines that intersects the set, take a point in the intersection of the set and the line. Let X be the set of all these points. Note that the family

$$\mathcal{G} = \{\operatorname{conv}(A_i \cap X) \ : \ A_i \in \mathcal{F}\}$$

of compact sets satisfies the hypothesis of Theorem 1.15 (or Theorem 1.14). Besides, for each $A_i \in \mathcal{F}$, we have that $\operatorname{conv}(A_i \cap X) \subset \operatorname{conv}(A_i)$. Hence, if the family $\mathcal{G}$ can be crossed by few lines, $\mathcal{F}$ can also be crossed by few lines. Therefore, in Theorems 1.14 and 1.15 we can assume without loss of generality that the sets are also compact.

In Section 4.4 we prove colorful versions of Theorems 1.14 and 1.15, which are generalizations of these theorems. The main tools to prove our results (the colorful versions of Theorems 1.14 and 1.15) are the Colorful KKM theorem (Theorem 4.4 in Section 4.1) and the following lemma. In fact, the following lemma has been very useful to prove theorems concerning line transversals in the plane, in particular was used in [26] and [13].

Lemma 1.16. *Let $C_1, C_2, C_3 \subset \mathbb{R}^2$ be convex sets. The following two conditions are equivalent:*

(i) There is no line transversal for C_1, C_2, C_3.

(ii) For each $i \in \{1, 2, 3\}$, the set C_i can be strictly separated by a line of $\bigcup_{j \neq i} C_j$.

In fact, Lemma 1.16 is also true in arbitrary dimension (proved by Goodman, Pollack and Wenger in 1996 [15]), although we do not use it in this work.

Lemma 1.17. *Let $C_1, C_2, \ldots, C_{d+1} \subset \mathbb{R}^d$ be convex sets. The following two conditions are equivalent:*

(i) There is no hyperplane transversal for $C_1, C_2, \ldots, C_{d+1}$.

(ii) For each non-empty $I \subset \{1, 2, \ldots, d+1\}$, the sets $\bigcup_{i \in I} C_i$ can be strictly separated by a hyperplane of $\bigcup_{j \notin I} C_j$.

Furthermore, Goodman, Pollack and Wenger proved the following stronger result concerning k-flats.

Lemma 1.18. *Let $C_1, C_2, \ldots, C_{k+2} \subset \mathbb{R}^d$ be convex sets. The following two conditions are equivalent:*

(i) There is no k-flat transversal for $C_1, C_2, \ldots, C_{k+2}$.

(ii) For each non-empty $I \subset \{1, 2, \ldots, k+2\}$, the sets $\bigcup_{i \in I} C_i$ can be strictly separated by a hyperplane of $\bigcup_{j \notin I} C_j$.

Chapter 2

Colorful theorems

In this chapter we see generalizations of the classical theorems in convex geometry, which are known as *colorful theorems*. According to Bárány (see [5, page 49]), Lovász proved the Colorful Helly theorem in 1973, although he did not publish it. In 1982, Bárány [4] proved the Colorful Carathéodory theorem and gave another proof of the Colorful Helly theorem. In addition, Lovász proved a Colorful Radon theorem and his proof was published in [2]. In the next sections we prove these colorful theorems by introducing the ideas of the proofs in low dimensions.

2.1. Colorful Carathéodory

Imagine we have red points and blue points in the real line. Suppose that the convex hull of the red points contains the origin and the convex hull of the blue points also contains the origin (see Figure 2.1). Since the convex hull of the red points contains the origin, there must be a negative red point and a positive red point. Analogously, there must be a negative blue point and a positive blue point. Therefore, we can take a negative red point and a positive blue point and the convex hull of them also contains the origin. If we have a red point and a blue point we say that the segment joining these two points is a *rainbow segment*. We can notice that we proved that there is a rainbow segment containing the origin. In fact, if now we take a negative blue point and a positive red point, then the convex hull of them also contains the origin. Therefore, there are at least two rainbow segments containing the origin.

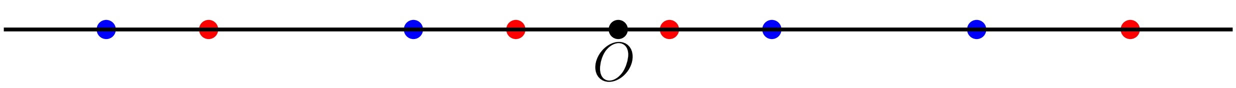

Figure 2.1: An example of blue points containing the origin and red points containing the origin.

A natural question is if we have similar results in higher dimensions. Let $A_1, A_2, \ldots, A_n$ be finite point sets in $\mathbb{R}^d$. We say that $T \subset \bigcup_{i=1}^{n} A_i$ is a *rainbow set* and $\operatorname{conv}(T)$ is a *rainbow simplex* if $|T \cap A_i| \leq 1$ for each $1 \leq i \leq n$. If we suppose that $a \in \bigcap_{i=1}^{n} \operatorname{conv}(A_i)$ for some point $a \in \mathbb{R}^d$, the question is if there are rainbow simplexes containing the point a. For example, we have the following result in the plane.

Proposition 2.1. *Let A_1, A_2, A_3 be finite point sets in $\mathbb{R}^2$ and $a \in \mathbb{R}^2$ so that $a \in \bigcap_{i=1}^3 \operatorname{conv}(A_i)$. Then, for each $a_1 \in A_1$, there exist $a_2 \in A_2, a_3 \in A_3$ so that $a \in \operatorname{conv}(\{a_1, a_2, a_3\})$. In particular, there are at least as many rainbow simplexes containing a as the cardinality of A_1.*

Proof. Let $a_1 \in A_1$. Since $a \in \operatorname{conv}(A_2)$, by the 2-dimensional case of Carathéodory's theorem (Theorem 1.3), there are three points $b_1, b_2, b_3 \in A_2$ such that $a \in \operatorname{conv}(\{b_1, b_2, b_3\})$. Set $B = \{b_1, b_2, b_3\}$.

Let l be the line through a_1 and a. Denote by l^+ and l^- the closed half-planes bounded by l. By the pigeon-hole principle one of the half-planes bounded by l contains at least 2 points of B, without loss of generality l^+ contains at least 2 points of B. Since $a \in \operatorname{conv}(B)$, then l^- contains at least 1 point of B. Without loss of generality, b_1, b_2 are in l^+ and b_3 is in l^-.

For any $i = 1, 2, 3$, let l_i be the line trough b_i and a, and let l_i^+ be the closed half-plane bounded by l_i that does not contain a_1 (see Figure 2.2). Let $R_1 = l_1^+ \cap l^-$, $R_2 = l_2^+ \cap l^-$, $R_3 = l_3^+ \cap l^+$, and $R_4 = \mathbb{R}^2 \setminus (\bigcup_{i=1}^3 R_i)$.

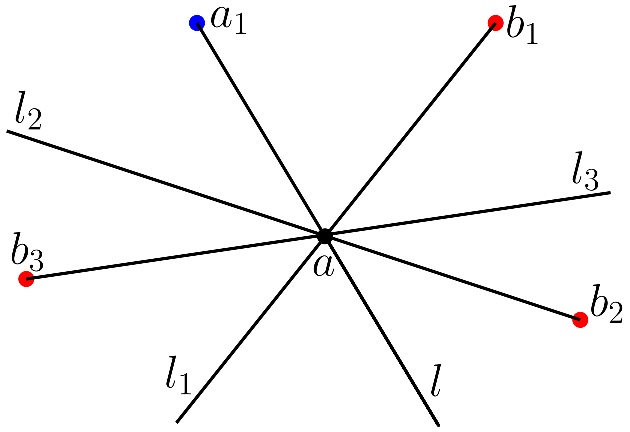

Figure 2.2: Illustration for the proof of Theorem 2.1.

If $A_3 \cap (\bigcup_{i=1}^3 R_i) = \emptyset$, then $A_3 \subset R_4$ and hence A_3 does not contain the point a, a contradiction. Otherwise, we can assume that $A_3 \cap (\bigcup_{i=1}^3 R_i) \neq \emptyset$. Then there is $j \in \{1, 2, 3\}$ such that $A_3 \cap R_j \neq \emptyset$.

Let $c_j \in A_3 \cap R_j$. Therefore, $\operatorname{conv}(\{a_1, b_j, c_j\})$ is a rainbow simplex that contains a. □

Note that we have the following reformulation of Proposition 2.1 using only 2 colors.

Proposition 2.2. *Let A_1, A_2 be finite point sets in $\mathbb{R}^2$ and $a \in \mathbb{R}^2$ so that $a \in \bigcap_{i=1}^2 \operatorname{conv}(A_i)$. Then, for each $x \notin \bigcup_{i=1}^2 A_i$, there exist $a_1 \in A_1, a_2 \in A_2$ so that $a \in \operatorname{conv}\{x, a_1, a_2\}$.*

We have a similar result in arbitrary dimensions and it is known as the Colorful Carathéodory theorem made by Bárány [4] in 1982. The idea of the proof is to use the *extremal principle*.

Theorem 2.3. *(Colorful Carathéodory). Let $A_1, A_2, \ldots, A_{d+1}$ be finite point sets in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ so that $a \in \bigcap_{i=1}^{d+1} \operatorname{conv}(A_i)$. Then there exist $a_1 \in A_1, a_2 \in A_2, \ldots, a_{d+1} \in A_{d+1}$ so that $a \in \operatorname{conv}\{a_1, a_2, \ldots, a_{d+1}\}$.*

Proof. Let $T = \{a_1, a_2, \ldots, a_{d+1}\}$ be a rainbow set (with $a_i \in A_i$ for every $i = 1, \ldots, d+1$) such that the distance of $\operatorname{conv}(T)$ to a is the smallest possible. If the distance of $\operatorname{conv}(T)$ to a

is 0, then $a \in \mathrm{conv}(T)$ and we are done. Otherwise, we assume that the distance of $\mathrm{conv}(T)$ to a is greater than 0 in order to get a contradiction.

Let $p \in \mathrm{conv}(T)$ be the point of $\mathrm{conv}(T)$ closest to a. Let H be the hyperplane containing p and orthogonal to the segment $[a, p]$, and denote by H^+ the closed halfspace bounded by H and containing a. Let

$$B(a, p) = \{x \in \mathbb{R}^d \ : \ \|a - x\| < \|a - p\|\}$$

be the open ball centered at a through p (see Figure 2.3).

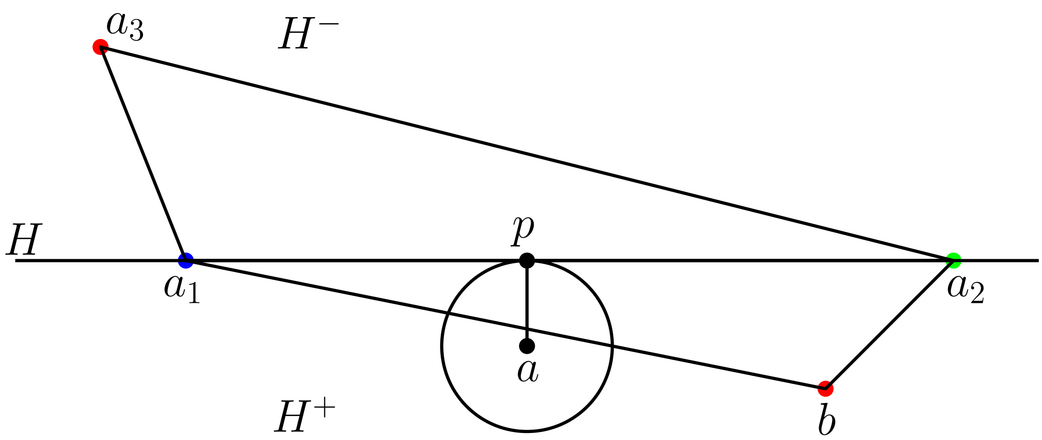

Figure 2.3: Illustration for the proof of Theorem 2.3 (in the plane).

We claim that $\mathrm{conv}(T) \subset H^-$, where H^- is the closed halfspace bounded by H which does not contain a. Indeed, if there exists $y \in \mathrm{conv}(T) \cap (H^+ \setminus H)$, then the segment $[p, y]$ meets $B(a, p)$ at a point q. Note that $q \in B(a, p)$ is a point in $\mathrm{conv}(T)$ closer to a than p, contradicting that $p \in \mathrm{conv}(T)$ is the point of $\mathrm{conv}(T)$ closest to a. Then $\mathrm{conv}(T) \subset H^-$.

Since $\mathrm{conv}(T) \subset H^-$, we have that $p \in H \cap \mathrm{conv}(T) = \mathrm{conv}(H \cap T)$. By the Carathéodory theorem (Theorem 1.3) in dimension $d - 1$, there exist at most d points in $T \cap H$ such that the convex hull of them contains p. Without loss of generality,

$$p \in \mathrm{conv}(\{a_1, a_2, \ldots, a_d\}).$$

Since $a \in \mathrm{conv}(A_{d+1})$, then there exists $b \in A_{d+1} \cap (H^+ \setminus H)$. Let

$$S = \{a_1, a_2, \ldots, a_d, b\}.$$

Since $p \in \mathrm{conv}(\{a_1, a_2, \ldots, a_d\})$, then the segment $[b, p]$ is contained in $\mathrm{conv}(S)$. Note that $[b, p]$ meets $B(a, p)$ at a point r, then $r \in B(a, p)$ is a point in $\mathrm{conv}(S)$ closer to a than p. Hence $\mathrm{conv}(S)$ is a rainbow simplex closer to a than $\mathrm{conv}(T)$, a contradiction. $\square$

Note that Theorem 2.3 implies Theorem 1.3. Indeed, if $A \subset \mathbb{R}^d$ and $a \in \mathrm{conv}(A)$, then the sets $A_i = A$ for $i = 1, \ldots, d + 1$ satisfy the hypothesis of Theorem 2.3, thus there exist at most $d + 1$ points in A such that their convex hull contains the point a.

2.2. Colorful Helly

Imagine we have red intervals and blue intervals in the real line. Suppose that every red interval and every blue interval have non-empty intersection. Then we want to prove that either the red intervals have non-empty intersection or the blue intervals have non-empty intersection (see Figure 2.4). We begin this section with two different proofs of this fact.

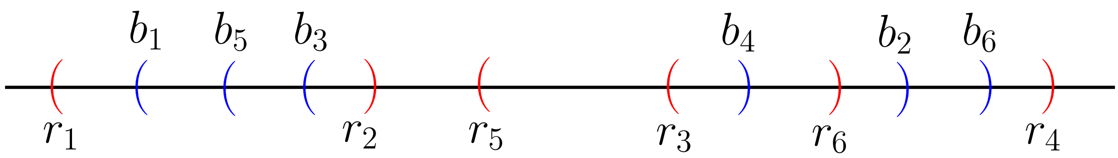

Figure 2.4: An example of three red intervals, $(r_1, r_2), (r_3, r_4), (r_5, r_6)$, and three blue intervals, $(b_1, b_2), (b_3, b_4), (b_5, b_6)$, such that every red interval intersect every blue interval. In this example the blue intervals have non-empty intersection.

In the first proof we use the 1-dimensional case of Helly's theorem. If every two red intervals have non-empty intersection, then by Helly's theorem (Theorem 1.4) the red intervals have non-empty intersection. Otherwise, there are two disjoint red intervals $(r_1, r_2), (r_3, r_4)$, with $r_1 < r_2 < r_3 < r_4$. Since every blue interval intersects the intervals $(r_1, r_2), (r_3, r_4)$, then $\dfrac{r_2 + r_3}{2}$ is in every blue interval. Therefore, the blue intervals have non-empty intersection.

In the second proof we do not use the 1-dimensional case of Helly's theorem, however it is less intuitive. The idea is to use the *extremal principle*. For every interval (x_1, x_2) (either red interval or blue interval), we consider its minimal point with the lexicographic order, that is the point x_1. Let X be the set of all the points which are the minimal point of some interval (either red interval or blue interval). In other words,

$$X = \{x_1 \in \mathbb{R} : \textit{there is } x_2 \in \mathbb{R} \textit{ such that } (x_1, x_2) \textit{ is either a red interval or a blue interval}\}.$$

Let $x = \max X$. Without loss of generality, we assume x is the minimal point of some red interval (x, y). We claim that x is in every blue interval. Let (b_1, b_2) be any blue interval. Since $x = \max X$, then $b_1 < x$. In addition, the intervals (b_1, b_2) and (x, y) have non-empty intersection, then $b_2 > x$. Therefore, $x \in (b_1, b_2)$ and the blue intervals have non-empty intersection.

As in the last section, we have a similar result in arbitrary dimensions and it is known as the Colorful Helly theorem made by Lovász (see [4] and [5, page 49]). Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be families of convex sets in $\mathbb{R}^d$. We say that $\mathcal{F} \subset \bigcup_{i=1}^n \mathcal{F}_i$ is a *rainbow subfamily* if $|\mathcal{F} \cap \mathcal{F}_i| \leq 1$ for each $1 \leq i \leq n$. The Colorful Helly theorem states that if every rainbow subfamily of size $d+1$ has non-empty intersection, then there is some family $\mathcal{F}_i$ with non-empty intersection. The proof is very similar to the second proof that we saw of the 1-dimensional case, in particular, we use the *extremal principle*.

Theorem 2.4. *(Colorful Helly). Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty. Then there exists $i \in \{1, \ldots, d+1\}$ such that $\bigcap \mathcal{F}_i \neq \emptyset$.*

Proof. We can assume, without loss of generality, that the sets in the families $\mathcal{F}_i$ are compact (see Section 1.2). Then for every rainbow subfamily of size d, we consider the lexicographically minimum of its intersection. We take a rainbow subfamily of size d such that the lexicographically minimum of its intersection is maximum among all the lexicographically minimum of intersections of rainbow subfamilies. Without loss of generality, we assume that

$$\{C_1, C_2, \ldots, C_d\}$$

is a rainbow subfamily, with $C_i \in \mathcal{F}_i$ for each $1 \leq i \leq d$, and such that the lexicographically minimum p of $\bigcap_{i=1}^{d} C_i$ is maximum. We claim that $p \in \bigcap \mathcal{F}_{d+1}$.

Let $C_{d+1} \in \mathcal{F}_{d+1}$ be any convex set in the family $\mathcal{F}_{d+1}$. By hypothesis, $\bigcap_{i=1}^{d+1} C_i \neq \emptyset$, then let q be the lexicographically minimum of $\bigcap_{i=1}^{d+1} C_i$ (note that in particular $q \in C_{d+1}$). We will prove that $p = q$.

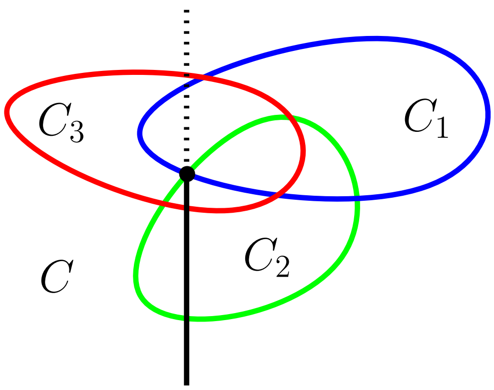

Figure 2.5: Illustration for the proof of Theorem 2.4 (in the plane).

Since $\bigcap_{i=1}^{d+1} C_i \subset \bigcap_{i=1}^{d} C_i$, then $p \leq_{lex} q$. In order to prove that $q \leq_{lex} p$, we define the convex set $C = \{x \in \mathbb{R}^d : x <_{lex} q\}$ (see Figure 2.5). Then

$$\mathcal{F} = \{C_1, \ldots, C_d, C_{d+1}, C\}$$

is a family of convex sets such that $\bigcap \mathcal{F} = \bigcap_{i=1}^{d+1} C_i \cap C = \emptyset$. By Helly's theorem (Theorem 1.4) and since $\bigcap_{i=1}^{d+1} C_i \neq \emptyset$, we have that there is a subfamily $\mathcal{G}$ of $\{C_1, \ldots, C_d, C_{d+1}\}$ of size d such that $\bigcap \mathcal{G} \cap C = \emptyset$. Let r be the minimum lexicographically of $\bigcap \mathcal{G}$, then $q \leq_{lex} r$. By the maximality of p, we have that $r \leq_{lex} p$. Thus, $q \leq_{lex} r \leq_{lex} p$ and $p = q$ (note that then $\mathcal{G} = \{C_1, C_2, \ldots, C_d\}$). Therefore, $p \in \bigcap \mathcal{F}_{d+1}$ and $\bigcap \mathcal{F}_{d+1} \neq \emptyset$. $\square$

Note that Theorem 2.4 implies Theorem 1.4. Indeed, if $\mathcal{F}$ is a family of convex sets satisfying the hypothesis of Theorem 1.4, then the families $\mathcal{F}_i = \mathcal{F}$ for $i = 1, \ldots, d+1$ satisfy the hypothesis of Theorem 2.4, thus $\bigcap \mathcal{F} \neq \emptyset$.

2.3. Colorful Radon

The result of this section is known as the colorful Radon theorem and was proved by Lovász (see [2]) using the Borsuk-Ulam theorem [3]. The ingenious proof presented here is by Soberón [37] and is very elemental.

Theorem 2.5. *(Colorful Radon) Let $F_1, \ldots, F_{d+1}$ be sets of 2 points each of $\mathbb{R}^d$. Then the union $\bigcup_{i=1}^{d+1} F_i$ can be partitioned into 2 sets A_1, A_2 such that $|A_i \cap F_j| = 1$ for each i, j and* $\operatorname{conv}(A_1) \cap \operatorname{conv}(A_2) \neq \emptyset$.

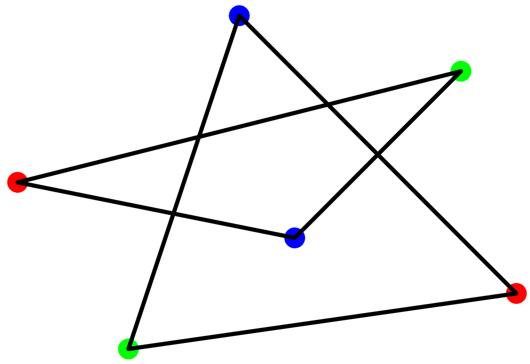

Figure 2.6: An example of the colorful Radon theorem in the plane.

Proof. Let $F_i = \{x_i, y_i\}$ be a set of two points in $\mathbb{R}^d$, for each $1 \leq i \leq d+1$. Since $\{x_1 - y_1, x_2 - y_2, \ldots, x_{d+1} - y_{d+1}\}$ is a set of $d+1$ points in $\mathbb{R}^d$, then the $d+1$ vectors $x_i - y_i$ are linearly dependent. Then there exists $d+1$ real numbers $\alpha_1, \ldots, \alpha_{d+1} \in \mathbb{R}$ such that not all of them are 0 and

$$\sum_{i=1}^{d+1} \alpha_i (x_i - y_i) = 0.$$

If there is $\alpha_i < 0$, we relabel the names of the points x_i, y_i and change the sign of α_i, then we can assume that all the α_i are non negative. In addition, since each $\alpha_i \geq 0$ and not all of them are 0, we can use scalar multiplication and so we can also assume that $\sum_{i=1}^{d+1} \alpha_i = 1$. Then we have the following convex combinations:

$$p = \sum_{i=1}^{d+1} \alpha_i x_i = \sum_{i=1}^{d+1} \alpha_i y_i.$$

Therefore, $A_1 = \{x_1, \ldots, x_{d+1}\}$ and $A_2 = \{y_1, \ldots, y_{d+1}\}$ satisfy that we want with

$$p \in \text{conv}(A_1) \cap \text{conv}(A_2).$$

$\square$

The proof of Soberón is very similar to the proof of Radon's theorem (Theorem 1.8), despite that no one had thought of this clever proof.

Chapter 3

Very colorful theorems

Pach, Holmsen and Tverberg [21] (in 2008) and independently Arocha, Bárány, Bracho, Fabila and Montejano [1] (in 2009) proved a generalization of the Colorful Carathéodory theorem (Theorem 2.3). In 2020, Martínez-Sandoval, Roldán-Pensado and Rubin [28] proved a generalization of the Colorful Helly theorem (Theorem 2.4).

In this chapter we prove these generalizations of the Colorful Carathéodory theorem and the Colorful Helly theorem.

3.1. Colorful Carathéodory-type theorems

In Section 2.1 we proved Proposition 2.1, which is stronger than the 2-dimensional Colorful Carathéodory theorem (Theorem 2.3). The generalization of Proposition 2.1 to arbitrary dimensions is also true. In other words, if $A_1, A_2, \dots, A_{d+1}$ are finite point sets in $\mathbb{R}^d$ such that $a \in \bigcap_{i=1}^{d+1} \operatorname{conv}(A_i)$, then for each $a_1 \in A_1$, there is a rainbow set $T = \{a_1, a_2, \dots, a_{d+1}\}$ such that $a \in \operatorname{conv}(T)$. In particular, there are at least as many rainbow simplexes containing a as the cardinality of A_1. This result is stronger than the Colorful Carathéodory theorem (Theorem 2.3).

Proposition 3.1. *Let $A_1, A_2, \dots, A_{d+1}$ be finite point sets in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ such that $a \in \bigcap_{i=1}^{d+1} \operatorname{conv}(A_i)$. Then, for each $a_1 \in A_1$, there exist $a_2 \in A_2, a_3 \in A_3, \dots, a_{d+1} \in A_{d+1}$ so that*

$$a \in \operatorname{conv}(\{a_1, a_2, \dots, a_{d+1}\}).$$

Notice that when $|A_i| = d+1$ for all $i = 1, \dots, d+1$, then there are at least $d+1$ rainbow simplexes containing a. Moreover, Sarrabezolles [36] (in 2015) proved that, in this case, there are at least d^2+1 rainbow simplexes containing a.

In this section, we prove the following stronger result (Theorem 3.2) found by Arocha, Bárány, Bracho, Fabila and Montejano [1] and independently by Holmsen, Pach and Tverberg [21]. First, we see the proof of Theorem 3.2 by Arocha, Bárány, Bracho, Fabila and Montejano [1]. Then, we prove that Theorem 3.2 implies Proposition 3.1.

Theorem 3.2. *Let $A_1, A_2, \dots, A_{d+1}$ be finite point sets in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ such that $a \in \operatorname{conv}(A_i \cup A_j)$ for all $1 \le i < j \le d+1$. Then there exist $a_1 \in A_1, a_2 \in A_2, \dots, a_{d+1} \in A_{d+1}$ so that*

$$a \in \operatorname{conv}(\{a_1, a_2, \dots, a_{d+1}\}).$$

Proof. Let $T = \{a_1, a_2, \dots, a_{d+1}\}$ be a rainbow set (with $a_i \in A_i$ for every $i = 1, \dots, d+1$) such that $\operatorname{conv}(T)$ is closest to a among all the rainbow simplexes. If the distance of $\operatorname{conv}(T)$ to a is 0, then $a \in \operatorname{conv}(T)$ and we are done. Otherwise, we assume that the distance of $\operatorname{conv}(T)$ to a is greater than 0 in order to get a contradiction.

Let $p \in \operatorname{conv}(T)$ be the point of $\operatorname{conv}(T)$ closest to a. Let H be the hyperplane containing p and orthogonal to the segment $[a, p]$, and denote by H^+ the closed halfspace bounded by H and containing a.

By the same arguments of the proof of Theorem 2.3 we have that $\operatorname{conv}(T) \subset H^-$ (where H^- is the closed halfspace bounded by H which does not contain a) and p is in the convex hull of at most d points in $T \cap H$. Without loss of generality,

$$p \in \operatorname{conv}(\{a_1, \dots, a_d\}).$$

If there exists $b \in A_{d+1} \cap (H^+ \setminus H)$, then $\operatorname{conv}(\{a_1, \dots, a_d, b\})$ is a rainbow simplex closer to a than $\operatorname{conv}(T)$ (by the same argument of the last paragraph of the proof of Theorem 2.3), contradicting the minimality of $\operatorname{conv}(T)$. Hence $A_{d+1} \subset H^-$.

Since $A_{d+1} \subset H^-$ and $a \in \operatorname{conv}(A_i \cup A_j)$ for all $1 \leq i < j \leq d+1$, then for each $i = 1, \dots, d$, there is a point $b_i \in A_i$ such that $b_i \in H^+ \setminus H$.

We claim that p is in the relative interior of $\operatorname{conv}(\{a_1, \dots, a_d\})$. We assume on the contrary that p is in the convex hull of at most $d-1$ points of $\{a_1, \dots, a_d\}$, without loss of generality $p \in \operatorname{conv}(\{a_1, \dots, a_{d-1}\})$. Then $\operatorname{conv}(\{a_1, \dots, a_{d-1}, b_d, a_{d+1}\})$ is a rainbow simplex closer to a than $\operatorname{conv}(T)$ (by the same argument of the last paragraph of the proof of Theorem 2.3), contradicting the minimality of $\operatorname{conv}(T)$. Thus, p is in the relative interior of $\operatorname{conv}(\{a_1, \dots, a_d\})$.

Let L_1 be the half-line starting at a and containing p. Let b_{d+1} be the point such that a_{d+1}, a, b_{d+1} are in the same line and $\|a - a_{d+1}\| = \|a - b_{d+1}\|$. Since $A_{d+1} \subset H^-$, then $a_{d+1} \in H^-$ and $b_{d+1} \in H^+$. Let L_2 be the half-line starting at a and containing b_{d+1}. Let $L = L_1 \cup L_2$ (see Figure 3.1). The homotopy group $\Pi_{d-2}(\mathbb{R}^d \setminus L)$ is non-zero and in fact, the $(d-1)$-dimensional simplex $\operatorname{conv}(\{a_1, \dots, a_d\})$ is an essential $(d-2)$-cycle (because p is in the relative interior of $\operatorname{conv}(\{a_1, \dots, a_d\})$).

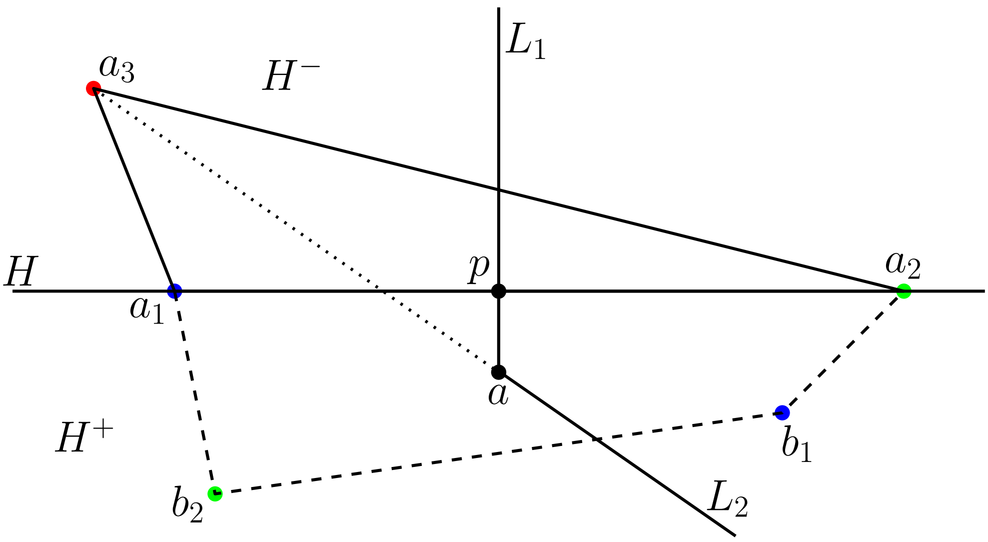

Figure 3.1: Illustration for the proof of Theorem 3.2 (in the plane). In this example $\operatorname{conv}(\{b_1, b_2, a_3\})$ is a rainbow simplex that contains a.

Denote by $E = \{e_1, \dots, e_d\}$ the standard orthonormal basis of $\mathbb{R}^d$. The d-dimensional cross-polytope is $\operatorname{conv}(E \cup -E)$. Let Ω^{d-1} be the boundary of the d-dimensional cross-polytope. Observe that a subset of d vertices of Ω^{d-1} spans a facet if and only if it does not contain antipodal points. For any $i \in \{1, \dots, d\}$, we color e_i and $-e_i$ of the same color of a_i. Then every facet of Ω^{d-1} is a $(d-1)$-dimensional rainbow simplex. Let U be the interior of the facet $\operatorname{conv}(E)$.

Let $f : \Omega^{d-1} \setminus U \to \mathbb{R}^d$ be the piecewise linear map defined as follows. For $i = 1, \dots, d$, we set $f(e_i) = a_i$ and $f(-e_i) = b_i$. Then we extend f linearly. Note that f preserves colors, hence f sends rainbow simplexes in rainbow simplexes.

By definition, $f(E) = \{a_1, \dots, a_d\}$ and $\operatorname{conv}(\{a_1, \dots, a_d\})$ is an essential $(d-2)$-cycle of $\Pi_{d-2}(\mathbb{R}^d \setminus L)$, then $f(\Omega^{d-1} \setminus U)$ must intersect L. Let $\sigma = \{z_1, \dots, z_d\} \neq E$ be a set of vertices of Ω^{d-1} such that $\operatorname{conv}(\sigma)$ is a facet of $\Omega^{d-1} \setminus U$ and $L \cap \operatorname{conv}(f(\sigma)) \neq \emptyset$. Since every facet of Ω^{d-1} is a $(d-1)$-dimensional rainbow simplex and f preserves colors, then $\operatorname{conv}(f(\sigma))$ is a $(d-1)$-dimensional rainbow simplex. Since $L \cap \operatorname{conv}(f(\sigma)) \neq \emptyset$, we have that either

$$L_1 \cap \operatorname{conv}(f(\sigma)) \neq \emptyset \quad \textit{or} \quad L_2 \cap \operatorname{conv}(f(\sigma)) \neq \emptyset.$$

If $L_1 \cap \operatorname{conv}(f(\sigma)) \neq \emptyset$, then there exists a point $q \in L_1 \cap \operatorname{conv}(f(\sigma))$ which is closer to a than p. Hence the rainbow simplex $\operatorname{conv}(f(\sigma) \cup a_{d+1})$ is closer to a than T, contradicting the minimality of T.

If $L_2 \cap \operatorname{conv}(f(\sigma)) \neq \emptyset$, then $\operatorname{conv}(f(\sigma) \cup a_{d+1})$ is a rainbow simplex that contains a. □

The following proposition is a reformulation of Proposition 3.1 and is a corollary of Theorem 3.2.

Proposition 3.3. *Let $A_1, A_2, \dots, A_d$ be finite point sets in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ such that $a \in \bigcap_{i=1}^{d} \operatorname{conv}(A_i)$. Then, for each $x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{d} A_i$, there exist $a_1 \in A_1, a_2 \in A_2, \dots, a_d \in A_d$ so that*

$$a \in \operatorname{conv}(\{a_1, a_2, \dots, a_d, x\}).$$

Proof. Let $x \in \mathbb{R}^d \setminus \bigcup_{i=1}^{d} A_i$ and $A_{d+1} = \{x\}$. The sets $A_1, A_2, \dots, A_d, A_{d+1}$ satisfy the hypothesis of Theorem 3.2, then there exist $a_1 \in A_1, a_2 \in A_2, \dots, a_d \in A_d, x \in A_{d+1}$ such that

$$a \in \operatorname{conv}(\{a_1, a_2, \dots, a_d, x\}).$$

□

Let $A_1, A_2, \dots, A_{d+1}$ be finite point sets in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$. The Colorful Carathéodory theorem (Theorem 2.3) states that if $a \in \bigcap_{i=1}^{d+1} \operatorname{conv}(A_i)$, then there is a rainbow simplex containing a. Theorem 3.2 states that if $a \in \operatorname{conv}(A_i \cup A_j)$ for all $1 \le i < j \le d+1$, then there is a rainbow simplex containing a. A natural question is whether we only need that $a \in \operatorname{conv}(A_i \cup A_j \cup A_k)$ for all $1 \le i < j < k \le d+1$, to conclude that there is a rainbow simplex containing a. However, the answer to this question is no. The counterexample is given in Example 3.4 (see Figure 3.2).

Example 3.4. *For each $1 \le i \le d-1$, let $A_i = \{e_i, -e_i\}$, where $\{e_1, e_2, \dots, e_d\}$ is the standard basis of $\mathbb{R}^d$. Let $A_d = \{e_d\}$ and $A_{d+1} = \{2e_d\}$. The sets $A_1, A_2, \dots, A_{d+1}$ satisfy that the origin is in the convex hull of $\operatorname{conv}(A_i \cup A_j \cup A_k)$ for all $1 \le i < j < k \le d+1$. However, there is* *no rainbow simplex containing the origin. Indeed, if the origin 0 was a convex combination of a rainbow simplex, then*

$$0 = \alpha_1 a_1 + \cdots + \alpha_{d-1} a_{d-1} + \alpha_d e_d + \alpha_{d+1} 2e_d,$$

where $a_i \in \{e_i, -e_i\}$, for $1 \le i \le d-1$. Then $\alpha_1 = \alpha_2 = \cdots = \alpha_{d-1} = 0$ and $\alpha_d + 2\alpha_{d+1} = 0$. However, $\alpha_d + \alpha_{d+1} = 1$ and $\alpha_{d+1} \ge 0$ (because it is a convex combination), this contradicts that $\alpha_d + 2\alpha_{d+1} = 0$. Therefore, there is no rainbow simplex containing the origin.

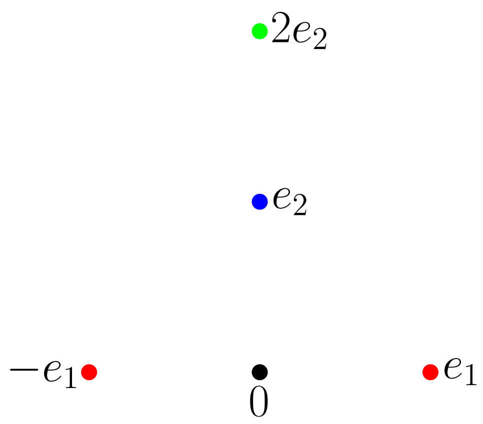

Figure 3.2: The 2-dimensional case of Example 3.4.

Despite this, it is true that if we have $d+2$ finite point sets $A_1, A_2, \ldots, A_{d+2}$ in $\mathbb{R}^d$ (instead of $d+1$ finite point sets) such that $a \in \operatorname{conv}(A_i \cup A_j \cup A_k)$ for all $1 \le i < j < k \le d+2$, then there is a rainbow simplex containing a. In general, Soberón [38] (in 2018) and Holmsen [19] (in 2016) found the following result, which can be proved following the same arguments of the proof of Theorem 3.2 (as was noted in [38]).

Theorem 3.5. *Let $A_1, A_2, \ldots, A_n$ be finite point sets in $\mathbb{R}^d$, with $n \ge d+1$, and $a \in \mathbb{R}^d$ such that $a \in \operatorname{conv}\left(\bigcup_{i \in I} A_i\right)$ for all $I \subset \{1, \ldots, n\}$ with $|I| = n-d+1$. Then there exist $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$ so that*

$$a \in \operatorname{conv}(\{a_1, a_2, \ldots, a_n\}).$$

Another natural question is if in Theorem 3.2 there are several rainbow simplexes containing a. However, Example 3.6 shows that it is possible to have only 1 rainbow simplex containing a (see Figure 3.3).

Example 3.6. *For each $1 \le i \le d$, let $A_i = \{e_i, -e_i\}$, and let $A_{d+1} = \{(e_1 + e_2 + \cdots + e_d)\}$. The sets $A_1, A_2, \ldots, A_{d+1}$ satisfy the hypothesis of Theorem 3.2, where a is the origin. However, there is only 1 rainbow simplex containing the origin. Indeed, if the origin 0 is a convex combination of a rainbow simplex, then*

$$0 = \alpha_1 a_1 + \cdots + \alpha_d a_d + \alpha_{d+1}(e_1 + e_2 + \cdots + e_d),$$

where $a_i \in \{e_i, -e_i\}$, for $1 \le i \le d$. Then, for each $1 \le i \le d$, $\alpha_{d+1} \in \{\alpha_i, -\alpha_i\}$. Since $\alpha_1 + \cdots + \alpha_{d+1} = 1$ and $\alpha_i \ge 0$ for every $i \in \{1, \ldots, d+1\}$, then $\alpha_i = \alpha_{d+1} = \frac{1}{d+1}$ and $a_i = -e_i$ for every $i \in \{1, \ldots, d\}$. Therefore,

$$\operatorname{conv}(\{-e_1, -e_2, \ldots, -e_d, e_1 + e_2 + \cdots + e_d\})$$

is the only rainbow simplex containing the origin.

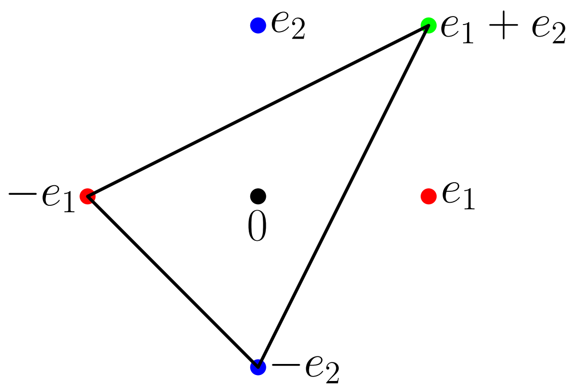

Figure 3.3: The 2-dimensional case of Example 3.6.

3.2. Colorful Helly-type theorems

Motivated by Theorem 3.2, we could think that, in the Colorful Helly theorem (Theorem 2.4), there is a second color that can be pierced by few points. Unfortunately, that is false as we see in Example 3.7.

Example 3.7. *For each $1 \leq i \leq d$, let $\mathcal{F}_i$ be a set of hyperplanes orthogonal to the x_i-axis, and let $\mathcal{F}_{d+1} = \{\mathbb{R}^d\}$. The families satisfy the hypothesis of the Colorful Helly theorem (Theorem 2.4) and the family $\mathcal{F}_{d+1}$ is the family with non-empty intersection. Moreover, every family $\mathcal{F}_i$, for $1 \leq i \leq d$, needs an arbitrary large number of points in order to be pierced (see Figure 3.4).*

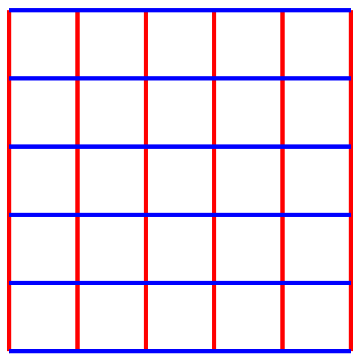

Figure 3.4: The 2-dimensional case of Example 3.7.

Despite that it is not true that a second color can be crossed by few points in the Colorful Helly theorem, the purpose of this section is prove that we can say something concerning the remaining colors.

Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty. By the Colorful Helly theorem (Theorem 2.4), there is a family with non-empty intersection, without loss of generality the family $\mathcal{F}_1$ has non-empty intersection.

We project the families $\mathcal{F}_2, \mathcal{F}_3, \ldots, \mathcal{F}_{d+1}$ to a $(d-1)$-dimensional hyperplane H and, for each $i \in \{2, 3, \ldots, d+1\}$, let $\mathcal{G}_i$ be the family of all the projections of sets in the family $\mathcal{F}_i$. Note that the families $\mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_{d+1}$ satisfy the hypothesis of the $(d-1)$-dimensional Colorful Helly theorem in the hyperplane H, then all the sets in one of the families $\mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_{d+1}$ have a common point, without loss of generality the sets in the family $\mathcal{G}_2$ have a common point g_2. The line whose the projection is the point g_2 is a line transversal of the family $\mathcal{F}_2$.

If now we consider the remaining $d-1$ families, projecting to a $(d-2)$-dimensional hyperplane and applying the Colorful Helly theorem in dimension $d-2$, we obtain that there is another family with a 2-flat transversal. We can follow the same argument d times to say something concerning all the families. We have proved the following proposition.

Proposition 3.8. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty. Then there exists a permutation $\pi \in S_{d+1}$ such that for each $i \in \{1, \ldots, d+1\}$ the family $\mathcal{F}\pi(i)$ has a $(i-1)$-flat transversal.*

We have the following reformulation of Proposition 3.8 using only d colors.

Proposition 3.9. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_d$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_d \in \mathcal{F}_d$, the intersection $\bigcap_{i=1}^{d} C_i$ is non-empty. Then there exists a permutation $\pi \in S_d$ such that for each $i \in \{1, \ldots, d\}$ the family $\mathcal{F}\pi(i)$ has a i-flat transversal.*

Note that Proposition 3.9 is a colorful version of Proposition 1.9.

Motivated by Proposition 3.8, a natural question is what additional consequences we can obtain with the same hypothesis of the Colorful Helly theorem (Theorem 2.4). Let $\mathcal{F}_1, \ldots, \mathcal{F}_d$ be finite families of convex sets in $\mathbb{R}^d$ satisfying the hypothesis of the Colorful Helly theorem. By Theorem 2.4 we know that there is a family with non-empty intersection. Martínez-Sandoval, Roldán-Pensado and Rubin [28] proved that either there is an additional family whose sets can be pierced by few points or all the sets in the union of the $d+1$ families can be crossed by few lines.

Theorem 3.10. *For each dimension $d \geq 2$ there exist numbers $f(d)$ and $g(d)$ (depending only on the dimension) with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_{d+1} \in \mathcal{F}_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is non-empty. Let $i \in \{1, \ldots, d+1\}$ such that $\bigcap \mathcal{F}_i \neq \emptyset$ (by Theorem 2.4). Then one of the following statements must also hold:*

1. *an additional family $\mathcal{F}_j$, for $j \in \{1, \ldots, d+1\} \setminus \{i\}$, can be pierced by $f(d)$ points, or*
2. *the family $\bigcup_{i=1}^{d+1} \mathcal{F}_i$ can be crossed by $g(d)$ lines.*

We have the following reformulation of Theorem 3.10 using only d colors.

Theorem 3.11. *For each dimension $d \geq 2$ there exist numbers $f(d)$ and $g(d)$ (depending only on the dimension) with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_d$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice of sets $C_1 \in \mathcal{F}_1, \ldots, C_d \in \mathcal{F}_d$, the intersection $\bigcap_{i=1}^{d} C_i$ is non-empty. Then one of the following statements holds:*

1. *there is a family $\mathcal{F}_j$, for $j \in \{1, \ldots, d\}$, that can be pierced by $f(d)$ points, or*
2. *the family $\bigcup_{i=1}^{d} \mathcal{F}_i$ can be crossed by $g(d)$ lines.*

Note that Theorem 3.11 is a strong colorful version of Proposition 1.9. Since the proof of Theorem 3.11 is very involved, we only see the main ideas of its proof. In particular, we prove the 2-dimensional case of Theorem 3.11. We begin with the following lemma that was proved in [28] in order to prove Theorem 3.11.

Lemma 3.12. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite families of convex sets in $\mathbb{R}^d$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then either*

(1) $\bigcap \mathcal{A} \neq \emptyset$, or

(2) $\mathcal{B}$ can be crossed by d hyperplanes.

Proof. If $\bigcap \mathcal{A} \neq \emptyset$, we are done. Otherwise, we assume that $\bigcap \mathcal{A} = \emptyset$. Then by Helly's theorem (Theorem 1.4) there are n convex sets $A_1, \ldots, A_n \in \mathcal{A}$, with $2 \leq n \leq d+1$, such that $\bigcap_{i=1}^n A_i = \emptyset$. By Theorem 1.7, there exist n hyperplanes $H_1, \ldots, H_n$ such that the closed half-spaces $H_1^+, \ldots, H_n^+$ satisfy $A_i \subset H_i^+$ for all i and $\bigcap_{i=1}^n H_i^+ = \emptyset$. We claim that the family $\mathcal{B}$ can be crossed by the $n-1$ hyperplanes $H_1, H_2, \ldots, H_{n-1}$, where $n-1 \leq d$ (see Figure 3.5).

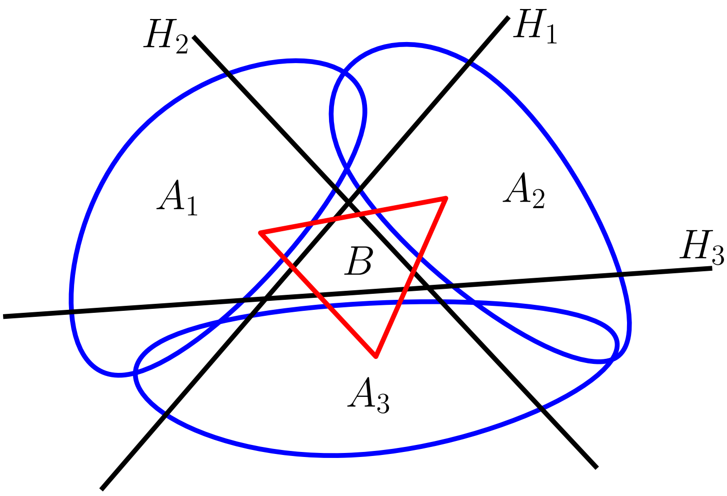

Figure 3.5: Illustration for the proof of Lemma 3.12.

By contradiction, suppose that a set $B \in \mathcal{B}$ does not intersect any of the hyperplanes H_i, for $i = 1, 2, \ldots, n-1$. Then B must be contained in an open region bounded by the $n-1$ hyperplanes $H_1, H_2, \ldots, H_{n-1}$, in other words, $B \subset \bigcap_{i=1}^{n-1} H_i^*$ where $* \in \{+, -\}$. By hypothesis, $A_i \cap B \neq \emptyset$ for each $1 \leq i \leq n-1$, then $B \subset \bigcap_{i=1}^{n-1} H_i^+$. Since $\bigcap_{i=1}^n H_i^+ = \emptyset$ and $B \subset \bigcap_{i=1}^{n-1} H_i^+$, then $B \cap H_n^+ = \emptyset$. Hence $B \cap A_n = \emptyset$, contradicting the hypothesis. □

Applying Lemma 3.12 twice in dimension $d = 2$, we have that either one of the two families has non-empty intersection or every one of the families can be crossed by 2 lines. Then as a corollary of Lemma 3.12, Martínez-Sandoval, Roldán-Pensado and Rubin also proved that the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 4$. In Section 4.2 we improve the last result; we prove that the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 2$.

On the other hand, the following lemma [28] is a generalization of Lemma 3.12 and is the main result needed to prove Theorem 3.11.

Lemma 3.13. *For any $1 \leq k \leq d$ and $m \geq 1$ there exist numbers $F(m, k, d)$ and $G(m, k, d)$ with the following property. Let $\mathcal{A}$ and $\mathcal{B}$ be finite families of convex sets in $\mathbb{R}^d$ so that the family*

$$\mathcal{I}(\mathcal{A}, \mathcal{B}) = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$$

can be crossed by m k-flats. Then one of the following conditions is satisfied:

1. $\mathcal{A}$ can be pierced by $F(m,k,d)$ points, or

2. $\mathcal{B}$ can be crossed by $G(m,k,d)$ $(k-1)$-flats.

The proof of Lemma 3.13 is very sophisticated and can be consulted in [28]. Note that in Lemma 3.13 the hypothesis that $\mathcal{I}(\mathcal{A},\mathcal{B})$ can be crossed by m k-flats implies that every two sets $A \in \mathcal{A}, B \in \mathcal{B}$ intersect, then the family $\mathcal{I}(\mathcal{A},\mathcal{B})$ can be crossed by $\mathbb{R}^d$. Thus, by Lemma 3.12, the particular case of Lemma 3.13 in which $k = d$ and $m = 1$ holds with $F(1,d,d) = 1$ and $G(1,d,d) \leq d$.

We are now ready to prove Theorem 3.11. Before we see the rigorous proof, we will see the idea of the proof. The idea is to apply Lemma 3.13 $d-1$ times. For each $1 \leq i \leq d$ we define

$$\mathcal{I}(\mathcal{F}_i,\dots,\mathcal{F}_d) = \left\{ \bigcap_{j=i}^{d} A_j : A_j \in \mathcal{F}_j \right\}.$$

First, we apply Lemma 3.13 (or Lemma 3.12) to the families $\mathcal{A} = \mathcal{F}_1$ and $\mathcal{B} = \mathcal{I}(\mathcal{F}_2,\dots,\mathcal{F}_d)$. If $\mathcal{A} = \mathcal{F}_1$ can be pierced by 1 point, we are done. Otherwise, the family $\mathcal{B} = \mathcal{I}(\mathcal{F}_2,\dots,\mathcal{F}_d)$ can be crossed by d hyperplanes. Then, in the case where the family $\mathcal{B} = \mathcal{I}(\mathcal{F}_2,\dots,\mathcal{F}_d)$ can be crossed by d hyperplanes, we apply Lemma 3.13 to the families $\mathcal{A} = \mathcal{F}_2$ and $\mathcal{B} = \mathcal{I}(\mathcal{F}_3,\dots,\mathcal{F}_d)$. Thus, $\mathcal{A} = \mathcal{F}_2$ can be pierced by $F(d,d-1,d)$ points or $\mathcal{B} = \mathcal{I}(\mathcal{F}_3,\dots,\mathcal{F}_d)$ can be crossed by $G(d,d-1,d)$ $(d-2)$-flats. In the first case we are done, in the second case we continue applying Lemma 3.13 using the families $\mathcal{I}(\mathcal{F}_i,\dots,\mathcal{F}_d)$. Following the same argument several times we have that either there is a family that can be pierced by few points or the last family $(\mathcal{F}_d)$ can be crossed by few lines. Since the labeling of the families was arbitrary, we can choose d different labelings such that every family is the last family in one of the labelings. Therefore, there is a family that can be pierced by few points or all the families can be crossed by few lines.

Proof of Theorem 3.11. Let $\mathcal{F}_1,\dots,\mathcal{F}_d$ be finite families of convex sets in $\mathbb{R}^d$ satisfying the hypothesis of Theorem 3.11. We set

$$M(i,d) = \begin{cases} 1 & for\ i = 1, \\ d & for\ i = 2, \\ G(M(i-1,d), d-i+2, d) & for\ 3 \leq i \leq d-1. \end{cases}$$

Since for every choice of sets $C_1 \in \mathcal{F}_1,\dots,C_d \in \mathcal{F}_d$ the intersection $\bigcap_{i=1}^{d} C_i$ is non-empty, the families $\mathcal{A} = \mathcal{F}_1$ and $\mathcal{B} = \mathcal{I}(\mathcal{F}_2,\dots,\mathcal{F}_d)$ satisfy the hypothesis of Lemma 3.13. Then, $\mathcal{A} = \mathcal{F}_1$ can be pierced by 1 point or $\mathcal{B} = \mathcal{I}(\mathcal{F}_2,\dots,\mathcal{F}_d)$ can be crossed by $M(2,d) = d$ hyperplanes. In the first case we are done. Otherwise, we assume that none of the families $\mathcal{F}_i$, for $1 \leq i \leq d-1$, can be pierced by $F(M(i,d), d-i+1, d)$ points, then applying $d-1$ times Lemma 3.13, the last family $\mathcal{F}_d$ can be crossed by $G(M(d-1,d),2,d)$ lines.

Applying the last argument to d different labelings such that every family is the last family in one of the labelings, we prove Theorem 3.11 with

$$f(d) = \max\{F(M(i,d), d-i+1, d) : 1 \leq i \leq d-1\},\ and$$
$$g(d) = d \cdot G(M(d-1,d),2,d).$$

□

Finally, Martínez-Sandoval, Roldán-Pensado and Rubin [28] also proved that if $f(d), g(d)$ are numbers satisfying Theorem 3.11, then $g(d) \geq \left\lceil \frac{d+1}{2} \right\rceil$. In Section 4.2, we see the construction of Martínez-Sandoval, Roldán-Pensado and Rubin in the plane that shows that if $f(2), g(2)$ are numbers satisfying the 2-dimensional case of Theorem 3.11, then $g(2) \geq \left\lceil \frac{2+1}{2} \right\rceil = 2$.

Chapter 4

Colorful theorems for line transversals

In this chapter we prove colorful theorems in low dimensions. Most of the results in this chapter are our own, although there are also other people's results in this chapter. Some of our main tools are the KKM theorem and the Colorful KKM theorem. For that reason, in Section 4.1 we see the KKM theorem and the colorful KKM theorem. In Section 4.2 we improve the 2-dimensional case of Theorem 3.11; in fact, we give the best numbers satisfying the 2-dimensional case of Theorem 3.11. In Section 4.3 we see an open problem proposed in [28], some ideas that have been used to prove particular cases and we give an idea in order to prove the general case of the open problem. In Section 4.4 we prove colorful versions of Theorems 1.14 and 1.15, we also present new problems and conjectures.

4.1. KKM theorem

There are a lot of problems in discrete mathematics that have been solved using tools from topology. For instance, the Borsuk-Ulam theorem [3] and the KKM theorem [25] are some of the results from topology most used to solve problems from discrete mathematics. The reader interested in the history of applications of topology in discrete mathematics can consult [30]. In this section we state the KKM theorem and some of its equivalent theorems.

In 1929, Knaster, Kuratowski and Mazurkiewicz [25] proved the called KKM theorem. Let

$$\Delta^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ x_i \geq 0, \ \sum_{i=1}^{n+1} x_i = 1 \right\}$$

denote the n-dimensional simplex in $\mathbb{R}^{n+1}$ that is the convex hull of $\{e_1, \ldots, e_{n+1}\}$, the standard orthonormal basis of $\mathbb{R}^{n+1}$.

Theorem 4.1. *(KKM theorem). Let $\{O_1, O_2, \ldots, O_{n+1}\}$ be an open cover (or closed cover) of Δ^n such that:*

a) $e_i \in O_i$ for each $i \in \{1, 2, \ldots, n+1\}$, and

b) $\operatorname{conv}\{e_i : i \in I\} \subset \bigcup_{i \in I} O_i$ for each $I \subset \{1, 2, \ldots, n+1\}$.

Then $\bigcap_{i=1}^{n+1} O_i \neq \emptyset$.

The KKM theorem is equivalent to Sperner's lemma [40] and Brouwer's fixed point theorem [7]. In fact, Knaster, Kuratowski and Mazurkiewicz [25] proved the KKM theorem using Sperner's lemma (a combinatorial lemma). In 1994, Krasa and Yannelis [27] gave another elementary proof of the KKM theorem using Brouwer's fixed point theorem.

Theorem 4.2. *(Sperner's lemma). Let Δ be a n-dimensional simplex with vertices $v_1, v_1, \ldots, v_{n+1}$. Let T be a triangulation of Δ and let*

$$f : V(T) \longrightarrow \{1, 2, \ldots, n+1\}$$

be a coloration such that:

a) $f(v_i) = i$ for each $1 \leq i \leq n+1$, and

b) if $x \in \operatorname{conv}(\{v_i : i \in I\})$, then $f(x) \in I$, for each $I \subset \{1, 2, \ldots, n+1\}$.

Then there are an odd number of rainbow simplexes of the triangulation T of dimension n. In particular, there is at least one rainbow simplex in the triangulation.

Theorem 4.3. *(Brouwer's fixed point theorem). Every continuous function $f : B^d \to B^d$, where $B^d = \{x \in \mathbb{R}^d \ : \ \|x\| \leq 1\}$ is a closed ball, has a fixed point; that is, there exists $x \in B^d$ such that $f(x) = x$.*

In 1984, Gale [14] proved a colorful version of the KKM theorem (Theorem 4.1).

Theorem 4.4. *(Colorful KKM theorem). For $i, j = 1, \ldots, n+1$ let O_i^j be open sets (or closed sets) such that for every j, $\{O_1^j, O_2^j, \ldots, O_{n+1}^j\}$ is an open cover (or closed cover) of Δ^n such that:*

a) $e_i \in O_i^j$ for each $i \in \{1, 2, \ldots, n+1\}$, and

b) $\operatorname{conv}\{e_i : i \in I\} \subset \bigcup_{i \in I} O_i^j$ for each $I \subset \{1, 2, \ldots, n+1\}$.

Then there exists a permutation $\pi \in S_{n+1}$ such that $\bigcap_{i=1}^{n+1} O_i^{\pi(i)} \neq \emptyset$.

Note that Theorem 4.4 implies Theorem 4.1. Indeed, if $\mathcal{O} = \{O_1, O_2, \ldots, O_{n+1}\}$ is an open cover satisfying the hypothesis of Theorem 4.1, then the open covers $\mathcal{O}^j = \mathcal{O}$ for $j = 1, \ldots, n+1$ satisfy the hypothesis of Theorem 4.4 and thus $\bigcap_{i=1}^{n+1} O_i \neq \emptyset$.

Recently, Soberón [39] proved the following generalization of Theorem 4.4.

Theorem 4.5. *Let $k \leq n$ be positive integers. For $i = 1, \ldots, k$ and $j = 1, \ldots, n$, let O_i^j be open sets (or closed sets) such that for every $I \in \binom{[n]}{n-k+1}$, the family*

$$\left\{ \bigcup_{j \in I} O_1^j, \ldots, \bigcup_{j \in I} O_k^j \right\}$$

is an open cover (or closed cover) of Δ^{k-1} that satisfies the hypothesis of Theorem 4.1. Then there exists an injective function $\pi : [k] \longrightarrow [n]$ such that $\bigcap_{i=1}^{k} O_i^{\pi(i)} \neq \emptyset$.

Notice that we recover Theorem 4.4 from Theorem 4.5 when $k = n$.

4.2. A colorful Helly-type theorem in $\mathbb{R}^2$

In Section 3.2 we saw that Martínez-Sandoval, Roldán-Pensado and Rubin [28] proved that the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 4$. In this section, using the KKM theorem (Theorem 4.1), we improve this result. We prove that the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 2$.

We begin with the following particular case of our result which has a very elementary proof.

Theorem 4.6. *Let $\mathcal{A}, \mathcal{B}$ be finite families of rectangles with sides parallel to the coordinate axes in $\mathbb{R}^2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then one of the following statements holds:*

1. *one of the families $\mathcal{A}$ or $\mathcal{B}$ can be pierced by 1 point, or*
2. *the family $\mathcal{A} \cup \mathcal{B}$ can be crossed by 2 lines (the lines can actually be chosen to be orthogonal).*

Proof. We project all the rectangles in the family $\mathcal{A} \cup \mathcal{B}$ to the x axis, then we obtain two finite families of intervals satisfying the hypothesis of the 1-dimensional Colorful Helly theorem (Theorem 2.4). Thus one of the families, $\mathcal{A}$ or $\mathcal{B}$, has a line transversal orthogonal to the x axis. Without loss of generality, the family $\mathcal{A}$ has a line transversal l_1 orthogonal to the x axis.

Now we project all the rectangles in the family $\mathcal{A} \cup \mathcal{B}$ to the y axis and, by the same argument, one of the families, $\mathcal{A}$ or $\mathcal{B}$, has a line transversal l_2 orthogonal to the y axis. Note that l_1 is orthogonal to l_2. Let p be the point intersection of the lines l_1 and l_2 (see Figure 4.1).

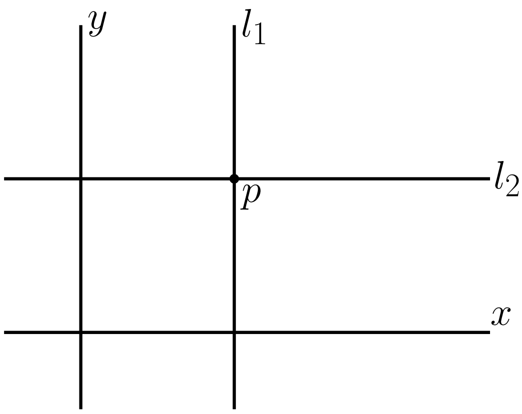

Figure 4.1: Illustration for the proof of Theorem 4.6.

First we suppose that l_2 is transversal to $\mathcal{A}$. Then l_1 and l_2 are line transversals to the family $\mathcal{A}$. Since $\mathcal{A}$ is a family of rectangles with parallel sides to the axes and l_1, l_2 are parallel to the axes, then every rectangle in the family $\mathcal{A}$ contains the point p.

Now we suppose that l_2 is transversal to $\mathcal{B}$. Then l_1 is transversal to the family $\mathcal{A}$ and l_2 is transversal to the family $\mathcal{B}$. Therefore, the family $\mathcal{A} \cup \mathcal{B}$ can be crossed by two orthogonal lines (l_1 and l_2). □

Note that we proved Theorem 4.6 using only tools from combinatorial geometry. To prove the general case we also need tools from topology. We prove that the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 2$. In other words, we prove that if $\mathcal{A}, \mathcal{B}$ are finite families of convex sets in $\mathbb{R}^2$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then either one of the families $\mathcal{A}$ or $\mathcal{B}$ can be pierced by 1 point or the family $\mathcal{A} \cup \mathcal{B}$ can be crossed by 2 lines.

The geometrical idea to prove our result is the following. If there are two lines crossing the family $\mathcal{A} \cup \mathcal{B}$, we are done. Otherwise, for any two lines in the plane with non-empty intersection there is a set in the family $\mathcal{A} \cup \mathcal{B}$ contained in the interior of one of the 4 regions bounded by the two lines. Then, by using the KKM theorem (Theorem 4.1) and following ideas similar to the ones used in [31], we prove that, in this case, there are two lines l_1, l_2 with point intersection p and four sets C_1, C_2, C_3, C_4 in the same family (either $\mathcal{A}$ or $\mathcal{B}$) so that every set C_i is contained in the interior of one of the 4 regions bounded by the lines l_1, l_2, each set C_i in a different region (see Figure 4.2). Using the convexity, we prove that the point p is contained in every set in the other family.

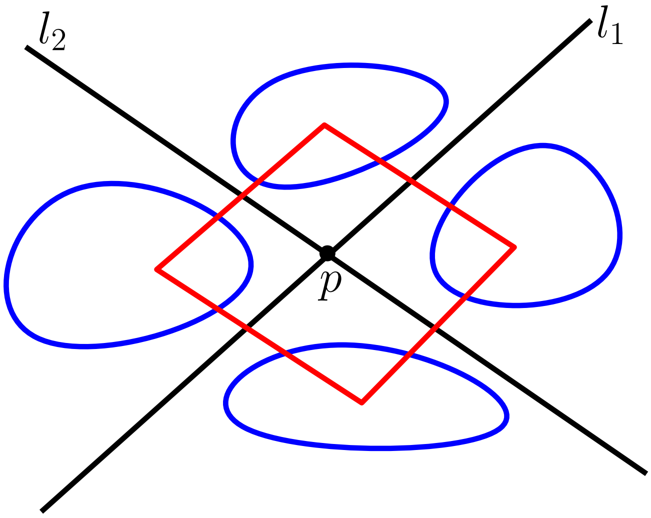

Figure 4.2: If there are no two lines crossing the family $\mathcal{A} \cup \mathcal{B}$, then there are two lines l_1, l_2 with point intersection p such that $p \in \bigcap \mathcal{A}$ or $p \in \bigcap \mathcal{B}$.

Theorem 4.7. *Let $\mathcal{A}, \mathcal{B}$ be finite families of convex sets in $\mathbb{R}^2$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then one of the following statements holds:*

1. *one of the families $\mathcal{A}$ or $\mathcal{B}$ can be pierced by 1 point, or*
2. *the family $\mathcal{A} \cup \mathcal{B}$ can be crossed by 2 lines.*

Proof. We can assume, without loss of generality, that the sets in both finite families are compact (see Section 1.2). Hence, we may scale the plane such that every set in $\mathcal{A} \cup \mathcal{B}$ is contained in the unit disk. Let $f(t)$ be a parametrization of the unit circle defined by

$$f(t) = (\cos(2\pi t),\ \sin(2\pi t)).$$

To each point $x = (x_1, \ldots, x_4) \in \Delta^3$ we associate 4 points on the unit circle given by

$$f_i(x) = f\left(\sum_{j=1}^{i} x_j\right),$$

for $1 \leq i \leq 4$. Let $l_1(x) = l_3(x) = [f_1(x), f_3(x)]$ and $l_2(x) = l_4(x) = [f_2(x), f_4(x)]$. For $i = 1, \ldots, 4$ let R_x^i be the interior of the region bounded by $l_{i-1}(x), l_i(x)$ and the arc on the unit circle connecting $f_{i-1}(x)$ and $f_i(x)$, where $i-1$ is taken modulo 4 (see Figure 4.3).

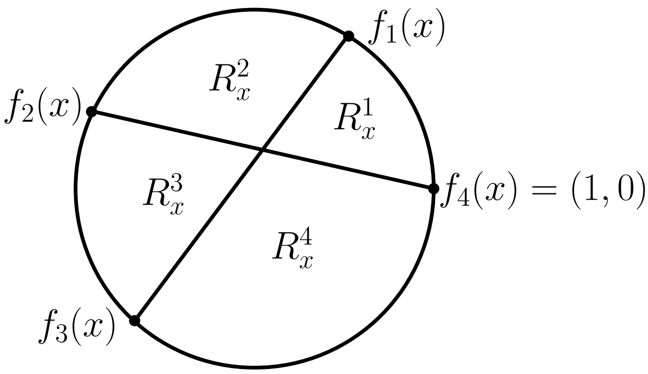

Figure 4.3: Illustration for the proof of Theorem 4.7.

Notice that $f_4(x) = (1,0)$ for each $x \in \Delta^3$. Also, the points x_1, x_2, x_3, x_4 are always in counter-clockwise order.

For $i = 1, \ldots, 4$, let O_i be the set of points $x \in \Delta^3$ such that R_x^i contains a set $C \in \mathcal{A} \cup \mathcal{B}$. Since the sets $C \in \mathcal{A} \cup \mathcal{B}$ are compact, O_i is open. If there is some $x \in \Delta^3$ for which $x \notin \bigcup_{i=1}^4 O_i$, then since the sets in $\mathcal{A} \cup \mathcal{B}$ are convex, every set in $\mathcal{A} \cup \mathcal{B}$ must intersect $\bigcup_{i=1}^2 l_i(x)$, and we are done. Otherwise, we assume that $\Delta^3 = \bigcup_{i=1}^4 O_i$. Observe that if $x \in \operatorname{conv}\{e_i : i \in I\}$ for some $I \subset \{1, \ldots, 4\}$, then $R_x^j = \emptyset$ for $j \notin I$, and therefore $x \in \bigcup_{i \in I} O_i$.

The last paragraph shows that the second statement of the theorem holds or $\{O_1, \ldots, O_4\}$ is an open cover that satisfies the hypothesis of the KKM theorem (Theorem 4.1). Then there is a point $y = (y_1, \ldots, y_4) \in \Delta^3$ such that $y \in \bigcap_{i=1}^4 O_i$. In other words, each one of the open regions R_y^i contains a set $C_i \in \mathcal{A} \cup \mathcal{B}$ (in particular $R_y^i \neq \emptyset$ and $y_i \neq 0$ for all $i \in \{1, \ldots, 4\}$).

Since $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (by hypothesis), then the sets $C_1, \ldots, C_4$ are in the same family (either $\mathcal{A}$ or $\mathcal{B}$). Without loss of generality, we assume $C_1, \ldots, C_4$ are in the family $\mathcal{A}$. Let p be the point intersection of the lines $l_1(y) = l_3(y)$ and $l_2(y) = l_4(y)$. Take any set $B \in \mathcal{B}$. Since $B \cap C_i \neq \emptyset$ for every $i \in \{1, \ldots, 4\}$ and B is convex, then B intersects the line segments $[p, f_4(y)]$ and $[p, f_2(y)]$. Therefore, $p \in B$ and the first statement of the theorem holds. $\square$

In fact, using the same proof we have proved the following stronger result.

Theorem 4.8. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite families of convex sets in $\mathbb{R}^2$ with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

1. *there exists $j \in \{1, 2, \ldots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 1 point, or*
2. *the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 2 lines.*

Proof. If the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 2 lines, we are done. Otherwise, following the same ideas of the proof of Theorem 4.7 and using the same notation, we have that there is a point $y = (y_1, \ldots, y_4) \in \Delta^3$ such that each one of the open regions R_y^i contains a set $C_i \in \bigcup_{i=1}^n \mathcal{F}_i$.

Since $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$, then the sets $C_1, \ldots, C_4$ are in the same family. Without loss of generality, we assume $C_1, \ldots, C_4$ are in the family $\mathcal{F}_1$. Let p be the point intersection of the lines $l_1(y) = l_3(y)$ and $l_2(y) = l_4(y)$. Take any set $B \in \bigcup_{i \neq 1} \mathcal{F}_i$. Since $B \cap C_i \neq \emptyset$ for every $i \in \{1, \ldots, 4\}$ and B is convex, then B intersects the line segments $[p, f_4(y)]$ and $[p, f_2(y)]$. Therefore, $p \in B$ and the family $\bigcup_{i \neq 1} \mathcal{F}_i$ can be pierced by 1 point. $\square$

We already proved that the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 2$. Now a natural question is if there exists $n \in \mathbb{Z}^+$ such that the 2-dimensional case of Theorem 3.11 holds with $f(2) = n$ and $g(2) = 1$, however this is false. In Section 3.2 we saw that Martínez-Sandoval, Roldán-Pensado and Rubin [28] proved that if $f(d), g(d)$ are numbers satisfying Theorem 3.11, then $g(d) \geq \lceil \frac{d+1}{2} \rceil$. In Example 4.9 we see the construction of Martínez-Sandoval, Roldán-Pensado and Rubin in the plane that prove that if $f(2), g(2)$ are numbers satisfying the 2-dimensional case of Theorem 3.11, then $g(2) \geq \lceil \frac{2+1}{2} \rceil = 2$.

Example 4.9. *Let $n \in \mathbb{Z}^+$. We want to construct two families, $\mathcal{A}, \mathcal{B}$ satisfying the hypothesis of the 2-dimensional case of Theorem 3.11 while neither $\mathcal{A}$ or $\mathcal{B}$ can be pierced by n points nor $\mathcal{A} \cup \mathcal{B}$ can be crossed by 1 line.*

Let A be a triangle in the plane with one edge parallel to the x-axis. Let $\mathcal{A} = \{A_1, \ldots, A_{2n+1}\}$ be a family of triangles in the plane such that any triangle A_i has an edge parallel to the x-axis, the three vertices of A_i are in the relative interior of the edges of A (two vertices of A_i are in different edges of A), and any three different triangles A_i, A_j, A_k have empty intersection. We can construct the family $\mathcal{A}$ recursively, as follows. We start with two arbitrary triangles A_1 and A_2 satisfying the conditions of the family $\mathcal{A}$, and at each step $i > 2$ we place the horizontal edge of A_i sufficiently close to the horizontal edge of A so that A_i does not intersect all previous pairwise intersections (see Figure 4.4). By construction, we need at least $n + 1$ points to pierce the family $\mathcal{A}$.

Let E_1, E_2, E_3 be the edges of A. Since every triangle A_i intersects the edges of A, we can take three segments F_1, F_2, F_3, every segment F_i contained in the relative interior of E_i, such that every segment F_i intersects every triangle in the family $\mathcal{A}$. For each segment F_i, we take n translates of F_i pairwise disjoint so that they still intersect every triangle in the family $\mathcal{A}$. Let $\mathcal{B}$ be the set of the $3n$ defined segments (see Figure 4.4). By construction, we need at least $3n > n + 1$ points to pierce the family $\mathcal{B}$.

In order to cross $\mathcal{A} \cup \mathcal{B}$, in particular we have to cross the interiors of the three edges E_1, E_2, E_3, hence we need at least 2 lines to cross the family $\mathcal{A} \cup \mathcal{B}$.

Theorem 4.7 and Example 4.9 show that the best numbers $f(2), g(2)$ satisfying the 2-dimensional case of Theorem 3.11 are $f(2) = 1$ and $g(2) = 2$.

Despite that the 2-dimensional case of Theorem 3.11 does not hold with $g(2) = 1$, we prove that in the case where the families are translates of a compact convex set in the plane, Theorem 3.11 holds with $f(2) = 3$ and $g(2) = 1$. This result is joint work with Edgardo Roldán-Pensado.

Theorem 4.10. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite families of translates of K, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

1. *there exists $j \in \{1, 2, \ldots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 3 points, or*

2. *the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 1 line.*

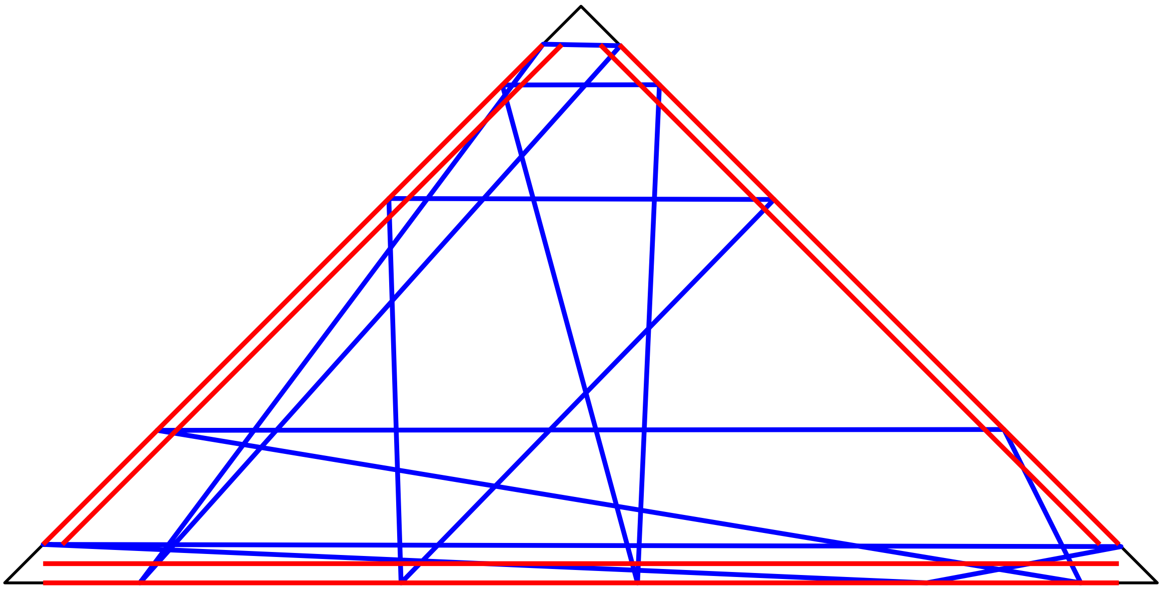

Figure 4.4: Illustration for Example 4.9 in the case where $n = 2$. The family $\mathcal{A}$ is the set of blue triangles and the family $\mathcal{B}$ is the set of red segments.

Proof. For every direction $u \in \mathbb{S}^1$, let l_u be the line through u and the origin. For every $u \in \mathbb{S}^1$, we project all the sets in the family $\bigcup_{i=1}^n \mathcal{F}_i$ to the line l_u, then we obtain a finite family of intervals in the line l_u.

If there exists $u \in \mathbb{S}^1$ such that every two intervals in the line l_u have non-empty intersection, then by the 1-dimensional Helly theorem (Theorem 1.4), the intervals in the line l_u have a common point p. Thus, if k_u is the line whose the projection (to the line l_u) is the point p, then k_u is a line transversal to the family $\bigcup_{i=1}^n \mathcal{F}_i$, and we are done (see Figure 4.5).

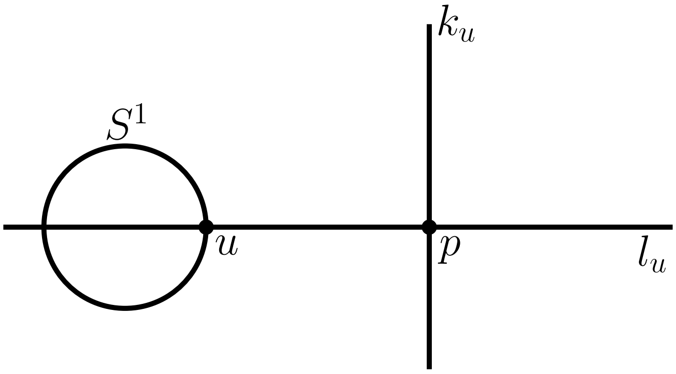

Figure 4.5: If the intervals in the line l_u have a common point p, then k_u is a line transversal to the family $\bigcup_{i=1}^n \mathcal{F}_i$.

Otherwise, for every $u \in \mathbb{S}^1$, there are two disjoint intervals in the line l_u. Hence, for every $u \in \mathbb{S}^1$, there are two sets A_u, B_u in the family $\bigcup_{i=1}^n \mathcal{F}_i$ that are separated by a line m_u orthogonal to l_u (see Figure 4.6). Since $A_u \cap B_u = \emptyset$, then A_u and B_u must be in the same family $\mathcal{F}_i$, for some $i \in \{1, \dots, n\}$.

We color the sphere $\mathbb{S}^1$ as follows. We color $u \in \mathbb{S}^1$ of color i if there are two sets $A_u, B_u \in \mathcal{F}_i$ that are separated by a line m_u orthogonal to l_u. Let O_i be the set of $u \in \mathbb{S}^1$ such that u has

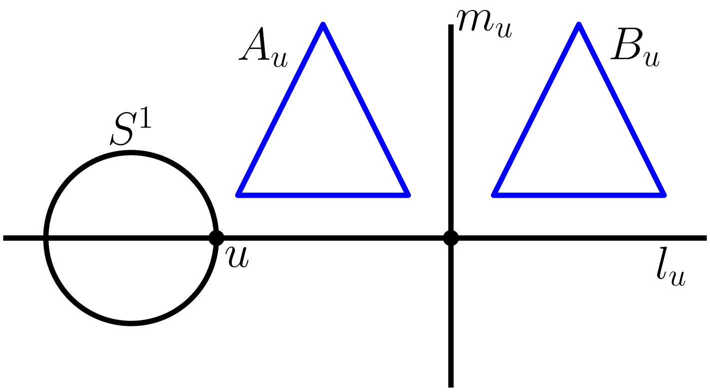

Figure 4.6: If there are two disjoint intervals in the line l_u, then there are two sets A_u, B_u in the family $\bigcup_{i=1}^{n} \mathcal{F}_i$ that are separated by a line m_u orthogonal to l_u.

color i. Since K is a compact set, then the sets O_i are open. Notice that we already proved that $\mathbb{S}^1 = \bigcup_{i=1}^{n} O_i$. We have two cases.

First we suppose that at least two sets from $O_1, \ldots, O_n$ are non-empty. Since the sets $O_1, \ldots, O_n$ are open and $\mathbb{S}^1 = \bigcup_{i=1}^{n} O_i$, then there exist $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $u \in O_i \cap O_j \neq \emptyset$. Hence there exist two lines m_u, n_u orthogonal to l_u and sets $A_u, B_u \in \mathcal{F}_i$ and $C_u, D_u \in \mathcal{F}_j$ such that $A_u \subset m_u^+ \setminus m_u$, $B_u \subset m_u^- \setminus m_u$, $C_u \subset n_u^+ \setminus n_u$ and $D_u \subset n_u^- \setminus n_u$ (see Figure 4.7). Notice that we can assume without loss of generality that $(m_u^+ \setminus m_u) \cap (n_u^- \setminus n_u) = \emptyset$. Then $A_u \in \mathcal{F}_i$ and $D_u \in \mathcal{F}_j$ satisfy that $A_u \cap D_u = \emptyset$, contradicting the hypothesis.

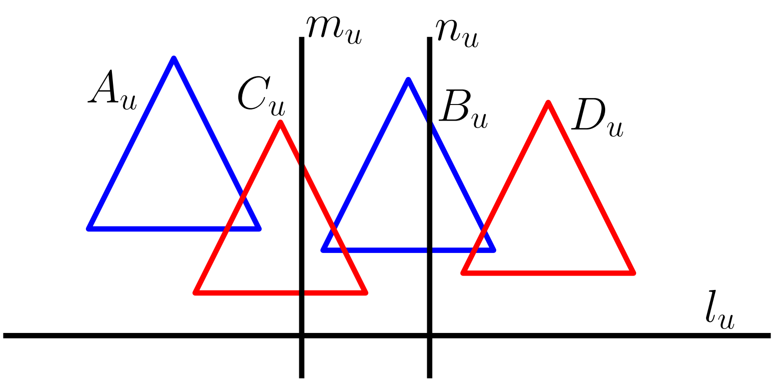

Figure 4.7: In this example, $A_u \cap D_u = \emptyset$.

Then there exists $j \in \{1, \ldots, n\}$ such that $\mathbb{S}^1 = O_j$. Hence for every $u \in \mathbb{S}^1$ there exists a line m_u orthogonal to l_u and sets $A_u, B_u \in \mathcal{F}_j$ such that $A_u \subset m_u^+ \setminus m_u$ and $B_u \subset m_u^- \setminus m_u$. We claim that, for every $u \in \mathbb{S}^1$, the line m_u is transversal to $\bigcup_{i \neq j} \mathcal{F}_i$. Indeed, if $u \in \mathbb{S}^1$ and $C \in \bigcup_{i \neq j} \mathcal{F}_i$, by hypothesis we have that $A_u \cap C \neq \emptyset$ and $B_u \cap C \neq \emptyset$. Since $A_u \subset m_u^+ \setminus m_u$ and $B_u \subset m_u^- \setminus m_u$, then C must intersect the line m_u (see Figure 4.8). Thus, the family $\bigcup_{i \neq j} \mathcal{F}_i$ has a line transversal in every direction.

Now we claim that every two sets in $\bigcup_{i \neq j} \mathcal{F}_i$ intersect. If there are two sets $C, D \in \bigcup_{i \neq j} \mathcal{F}_i$ such that $C \cap D = \emptyset$, then C and D can be separated by a line l. Hence C and D do not have line transversal in the same direction of l, a contradiction (see Figure 4.9).

Therefore, every two sets in $\bigcup_{i \neq j} \mathcal{F}_i$ intersect and by Theorem 1.12, the family $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 3 points. □

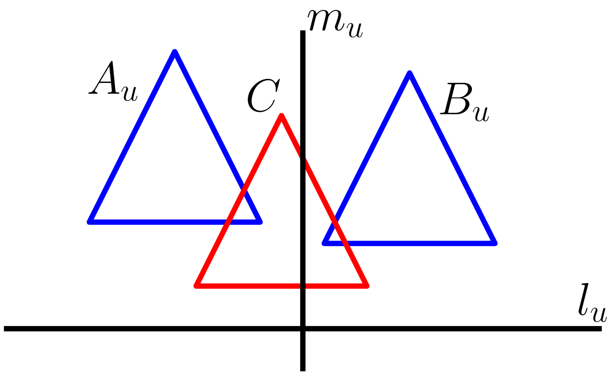

Figure 4.8: For every $u \in \mathbb{S}^1$ and $C \in \bigcup_{i \neq j} \mathcal{F}_i$, the line m_u intersects C.

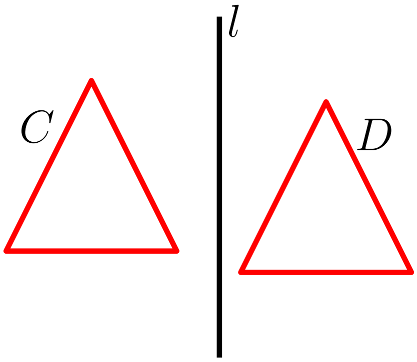

Figure 4.9: If the convex sets $C, D \in \bigcup_{i \neq j} \mathcal{F}_i$ can be separated by a line l, then $\bigcup_{i \neq j} \mathcal{F}_i$ does not have line transversal in the same direction of l.

Notice that we used that the families are translates of K in the last part of the proof. Actually, we used that every two sets in $\bigcup_{i \neq j} \mathcal{F}_i$ intersect. Grunbaum [16], Danzer [10] and Karasev [23] [24] have studied families (in the plane) such that every two sets in the family have non-empty intersection, and for some families of this type they proved that the whole family can be pierced by few points. For example, Theorems 1.10, 1.11, 1.12 and 1.13 show some families of this type. Then we observe that with the same proof of Theorem 4.10 we have similar results for families of this type.

Theorem 4.11. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite families of compact convex sets in $\mathbb{R}^2$, with $n \geq 2$ and the following property: if there exists $j \in \{1, 2, \ldots, n\}$ such that every two sets in $\bigcup_{i \neq j} \mathcal{F}_i$ intersect, then $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by c points. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

1. *there exists $j \in \{1, 2, \ldots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by c points, or*
2. *the family $\bigcup_{i=1}^{n} \mathcal{F}_i$ can be crossed by 1 line.*

Proof. If the family $\bigcup_{i=1}^{n} \mathcal{F}_i$ can be crossed by 1 line, we are done. Otherwise, as in the proof of Theorem 4.10, we have that there exists $j \in \{1, \ldots, n\}$ such that every two sets in $\bigcup_{i \neq j} \mathcal{F}_i$ intersect. Therefore, by hypothesis, the family $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by c points. □

In particular, by Theorems 1.10, 1.11 and 1.13, we have the following corollaries.

Theorem 4.12. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite families of homothets of K, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

1. *there exists $j \in \{1, 2, \dots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by c points (where c is the number that there exists in Theorem 1.10), or*
2. *the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 1 line.*

Theorem 4.13. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite families of circles in $\mathbb{R}^2$, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

1. *there exists $j \in \{1, 2, \dots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 4 points, or*
2. *the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 1 line.*

Theorem 4.14. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite families of rectangles with sides parallel to the coordinate axes in $\mathbb{R}^2$, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then one of the following statements holds:*

1. *there exists $j \in \{1, 2, \dots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 1 point, or*
2. *the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 1 line.*

In summary, Theorems 4.10, 4.11, 4.12, 4.13 and 4.14 show some special families where the 2-dimensional case of Theorem 3.11 holds with $g(2) = 1$. In particular, Theorem 4.14 shows that for rectangles with sides parallel to the coordinate axes in $\mathbb{R}^2$, the 2-dimensional case of Theorem 3.11 holds with $f(2) = 1$ and $g(2) = 1$, hence Theorem 4.14 improves the result of Theorem 4.6.

On the other hand, we wonder if we can improve the numbers in Theorem 4.10. We propose the following problem.

Problem 4.15. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite families of translates of K, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Is it true that one of the following statements holds?*

1. *there exists $j \in \{1, 2, \dots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 1 point, or*
2. *the family $\bigcup_{i=1}^n \mathcal{F}_i$ can be crossed by 1 line.*

Jerónimo-Castro, Magazinov and Soberón [22] proved that in the case where the families are circles of the same radius there is a stronger conclusion.

Theorem 4.16. *Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be finite families of circles of the same radius in $\mathbb{R}^2$, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then there exists $j \in \{1, 2, \dots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 3 points.*

Theorem 4.16 is a colorful version of Theorem 1.12 in the case of circles with the same radius. Jerónimo-Castro, Magazinov and Soberón [22] also conjectured that Theorem 4.16 holds for families of translates of K, for any compact convex set K in $\mathbb{R}^2$. In other words, they conjectured a colorful version of Theorem 1.12.

Conjecture 4.17. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite families of translates of K, with $n \geq 2$. Suppose that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$ with $i \neq j$. Then there exists $j \in \{1, 2, \ldots, n\}$ such that $\bigcup_{i \neq j} \mathcal{F}_i$ can be pierced by 3 points.*

We believe that Theorem 4.10 can be useful to prove Conjecture 4.17.

4.3. A colorful Helly-type problem in $\mathbb{R}^3$

We already proved the 2-dimensional case of Theorem 3.11. Another natural question is if with 2 color classes we have a similar conclusion in $\mathbb{R}^3$. Martínez Sandoval, Roldán-Pensado and Rubin [28] presented the following problem.

Problem 4.18. *Is it true that there exists $n \in \mathbb{Z}^+$ such that for any two families $\mathcal{A}$, $\mathcal{B}$ of convex sets in $\mathbb{R}^3$ so that $A \cap B \neq \emptyset$ holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, one of the families $\mathcal{A}$ or $\mathcal{B}$ can be crossed by n lines?*

The last problem is still open. Actually, the next weaker problem is also open.

Problem 4.19. *Is it true that there exists $n \in \mathbb{Z}^+$ such that for any family $\mathcal{F}$ of convex sets in $\mathbb{R}^3$ so that $A \cap B \neq \emptyset$ holds for all $A, B \in \mathcal{F}$, the family $\mathcal{F}$ can be crossed by n lines?*

In this section we prove a particular case of Problem 4.18 by Montejano [32] and we give an idea to solve Problem 4.19.

Montejano [32] proved that Problem 4.18 holds for small families of convex sets. Imagine three red convex sets and three blue convex sets in $\mathbb{R}^3$ such that every red set and every blue set have non-empty intersection. Montejano proved that either there is a line transversal to the red sets or there is a line transversal to the blue sets.

If we project the two families to a line, we obtain two finite families of intervals satisfying the hypothesis of the 1-dimensional Colorful Helly theorem (Theorem 2.4), then the intervals of one of the families have a common point, without loss of generality the blue intervals have a common point b. The plane whose the projection is the point b is a plane transversal to the blue sets (see Figure 4.10). However, we only have a plane transversal to one of the families and we want a line transversal to one of the families. Montejano and Karasev ([32], [33]), following the last argument in every direction and using topology proved that there is a line transversal to one of the families. Although Montejano and Karasev proved the last particular case of Problem 4.18, Montejano wanted to see an elementary proof. Strausz [41] gave an elementary proof based on the non-planarity of the complete bipartite graph $K_{3,3}$. We will see the elementary proof by Strausz [41].

Theorem 4.20. *Let A, B, C be three red convex sets in $\mathbb{R}^3$ and let U, V, W be three blue convex sets in $\mathbb{R}^3$. Suppose that every red convex set intersect every blue convex set. Then either there is a line transversal to the red convex sets or there is a line transversal to the blue convex sets.*

Proof. By contradiction, suppose that there are no line transversals to the red convex sets nor to the blue convex sets. Then, by Lemma 1.18, each red convex set can be separated by a plane from the other two red convex sets. Analogously, each blue convex set can be separated by a plane from the other two blue convex sets. Let H_A be a plane such that $A \subset H_A^+$ and $B \cup C \subset H_A^-$, and let $a \in \mathbb{S}^2$ be the unitary normal vector of the plane H_A. Analogously, we

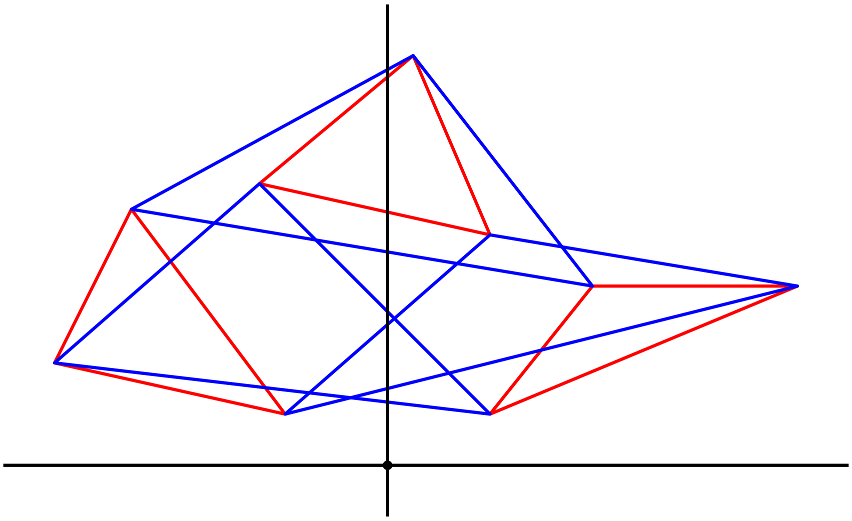

Figure 4.10: Montejano's problem in $\mathbb{R}^2$ is easy projecting the sets to a line and applying the 1-dimensional case of Colourful Helly. However, with the same argument in $\mathbb{R}^3$ we only have a plane transversal to one of the families instead of a line transversal to one of the families.

define the planes H_B, H_C, H_U, H_V, H_W and the unitary normal vectors $b, c, u, v, w \in \mathbb{S}^2$. We color the vectors a, b, c red and the vectors u, v, w blue.

Now, we join each red vector to each blue vector with a spherical segment. Then we have $K_{3,3}$ drawn in $\mathbb{S}^2$, thus there must be a crossing (by Kuratowski's theorem). Without loss of generality, the spherical segment through $a \in \mathbb{S}^2$ and $u \in \mathbb{S}^2$ intersects the spherical segment through $c \in \mathbb{S}^2$ and $w \in \mathbb{S}^2$.

Since $A \cap U \neq \emptyset$, then there exists $q \in A \cap U \subset (H_A^+ \cap H_U^+)$. Since $A \subset H_C^-$ and $U \subset H_W^-$, then $A \cap U \subset H_C^- \cap H_W^-$. Therefore, $q \in (H_A^+ \cap H_U^+) \setminus (H_C^+ \cup H_W^+)$. Analogously, there exists $r \in (H_C^+ \cap H_W^+) \setminus (H_A^+ \cup H_U^+)$. In order to prove that this is a contradiction, we prove the following lemma in arbitrary dimension.

Lemma. Let $H_A^+, H_U^+, H_C^+, H_W^+$ be half spaces in $\mathbb{R}^d$ with unitary normal vectors $a, u, c, w \in \mathbb{S}^{d-1}$, respectively. If

$$(H_A^+ \cap H_U^+) \setminus (H_C^+ \cup H_W^+) \neq \emptyset,$$

and

$$(H_C^+ \cap H_W^+) \setminus (H_A^+ \cup H_U^+) \neq \emptyset,$$

then the spherical segment au and the spherical segment cw are disjoint.

Proof. Let $q \in (H_A^+ \cap H_U^+) \setminus (H_C^+ \cup H_W^+)$ and $r \in (H_C^+ \cap H_W^+) \setminus (H_A^+ \cup H_U^+)$. Without loss of generality, we suppose that $0 \in H_A \cap H_U$ and let $p \in H_C \cap H_W$. In other words,

$$\begin{aligned}
H_A &= \{x \in \mathbb{R}^d : a \cdot x = 0\}, \\
H_U &= \{x \in \mathbb{R}^d : u \cdot x = 0\}, \\
H_C &= \{x \in \mathbb{R}^d : c \cdot x = c \cdot p\} = \{x \in \mathbb{R}^d : c \cdot (x - p) = 0\}, \\
H_W &= \{x \in \mathbb{R}^d : w \cdot x = w \cdot p\} = \{x \in \mathbb{R}^d : w \cdot (x - p) = 0\}.
\end{aligned}$$

Then, since $q \in (H_A^+ \cap H_U^+) \setminus (H_C^+ \cup H_W^+)$ and $r \in (H_C^+ \cap H_W^+) \setminus (H_A^+ \cup H_U^+)$, we have the following eight inequalities:

$$\begin{array}{rr} a \cdot q > 0 & u \cdot q > 0 \\ c \cdot (q-p) < 0 & w \cdot (q-p) < 0 \\ c \cdot (r-p) > 0 & w \cdot (r-p) > 0 \\ a \cdot r < 0 & u \cdot r < 0 \end{array}$$

To prove the lemma, we suppose, by contradiction, that z is in the intersection of the spherical segment au and the spherical segment cw. Then there exists $i, j, m, n > 0$ such that $z = ia + ju = mc + nw$.

On the one hand, $z \cdot q = (ia + ju) \cdot q > 0$ and $z \cdot (q - p) = (mc + nw) \cdot (q - p) < 0$. Thus,

$$0 < z \cdot q < z \cdot p.$$

On the other hand, $z \cdot r = (ia + ju) \cdot r < 0$ and $z \cdot (r - p) = (mc + nw) \cdot (r - p) > 0$. Thus,

$$0 > z \cdot r > z \cdot p,$$

a contradiction. □

Since the spherical segment through $a \in \mathbb{S}^2$ and $u \in \mathbb{S}^2$ intersects the spherical segment through $c \in \mathbb{S}^2$ and $w \in \mathbb{S}^2$, by the lemma, we have a contradiction. □

Recently, Holmsen [20] gave a new proof of Theorem 4.20 (which is a particular case of Problem 4.18) using the Borsuk-Ulam theorem [3]. On the other hand, Bárány [6] proved a particular case of Problem 4.19. However, both Problem 4.18 and Problem 4.19 are still open. We finish this section with an idea to prove Problem 4.19.

An idea to Problem 4.19.

We suspect Problem 4.19 holds with $n \geq 3$. The idea goes as follows. For every orthonormal base $\{u, v, w\}$ in $\mathbb{R}^3$ let l_u, l_v, l_w be lines through the origin such that $u \in l_u, v \in l_v, w \in l_w$. We project the convex sets in the family $\mathcal{F}$ to the line l_u, then we obtain a family of intervals in the line l_u satisfying the hypothesis of the 1-dimensional Helly theorem (Theorem 1.4). Since every two sets in $\mathcal{F}$ have a non-empty intersection, then the intervals have a common point $f \in l_u$ (by Theorem 1.4). Let H_u be the hyperplane where the projection is the point $f \in l_u$. Note that H_u is orthogonal to $u \in \mathbb{S}^2$ and is transversal to the family $\mathcal{F}$. Analogously, there exist hyperplanes H_v, H_w such that H_v, H_w are orthogonal to $v, w \in \mathbb{S}^2$ (respectively) and are transversal to the family $\mathcal{F}$. Let k_u, k_v, k_w be the pairwise intersections of the hyperplanes H_u, H_v, H_w.

We believe that there exists an orthogonal base $\{u, v, w\}$ such that the three lines k_u, k_v, k_w and perhaps together with other lines cross the family $\mathcal{F}$. A new topological result might be needed to prove our claim.

4.4. Colorful Eckhoff

In 2021, McGinnis and Zerbib [31] proved Theorem 1.15 using the KKM theorem (Theorem 4.1). We observed that following the same ideas of McGinnis and Zerbib and using the Colorful KKM theorem (Theorem 4.4) we can obtain colorful versions of Theorems 1.14 and 1.15. Afterwards, McGinnis and Zerbib also noticed the colorful versions and uploaded a second version of their paper. In this section we prove colorful versions of Theorems 1.14 and 1.15.

We begin with the colorful version of Theorem 1.14. Let $\mathcal{F}_1, \dots, \mathcal{F}_4$ be finite families of connected sets in $\mathbb{R}^2$. Suppose that every four sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3, A_4 \in \mathcal{F}_4$ have a line transversal. We prove that there exists $i \in \{1, \dots, 4\}$ such that the family $\mathcal{F}_i$ can be crossed by 2 lines. This result is joint work with Edgardo Roldán-Pensado.

The geometrical idea of our proof goes as follows. If there are two lines crossing one of the families, we are done. Otherwise, we suppose, by contradiction, that for each family $\mathcal{F}_i$ and for each two lines in the plane with non-empty intersection there is a set in the family $\mathcal{F}_i$ contained in the interior of one of the 4 regions bounded by the two lines. Then using the colorful KKM theorem (Theorem 4.4) we prove that there are two lines l_1, l_2 (with non-empty intersection) and four sets $C_i \in \mathcal{F}_i$, for $i = 1, \dots, 4$, so that every set C_i is contained in the interior of one of the 4 regions bounded by the lines l_1, l_2, each set C_i in a different region (see Figure 4.11). Then the sets C_1, C_2, C_3, C_4 do not have a line transversal, a contradiction.

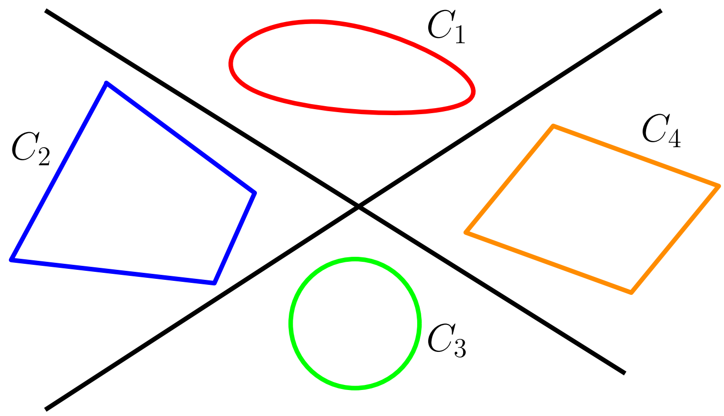

Figure 4.11: If there are no two lines crossing one of the families $\mathcal{F}_i$, then there are four sets $C_i \in \mathcal{F}_i$, for $i = 1, \dots, 4$, such that the sets C_1, C_2, C_3, C_4 do not have a line transversal.

Theorem 4.21. *Let $\mathcal{F}_1, \dots, \mathcal{F}_4$ be finite families of connected sets in $\mathbb{R}^2$. If every four sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, \dots, A_4 \in \mathcal{F}_4$ have a line transversal, then there exists $i \in \{1, \dots, 4\}$ such that the family $\mathcal{F}_i$ can be crossed by 2 lines.*

Proof. We can assume, without loss of generality, that the sets in the four finite families are compact (see section 1.3). Hence, we may scale the plane such that every set in $\mathcal{F}_j$ is contained in the unit disk, for each $j \in \{1, \dots, 4\}$. For each point $x = (x_1, \dots, x_4) \in \Delta^3$ and every $i = 1, \dots, 4$ we define $f_i(x)$, $l_i(x)$ and R_x^i as in the Proof of Theorem 4.7 (see Figure 4.12).

For $i, j = 1, \dots, 4$, let O_i^j be the set of points $x \in \Delta^3$ such that R_x^i contains a set $F \in \mathcal{F}_j$. Since the sets $F \in \mathcal{F}_j$ are compact, O_i^j is open for all i, j. If there is some $x \in \Delta^3$ and $j \in \{1, \dots, 4\}$ for which $x \notin \bigcup_{i=1}^4 O_i^j$, then since the sets in $\mathcal{F}_j$ are connected, every set in $\mathcal{F}_j$ must intersect $\bigcup_{i=1}^2 l_i(x)$, and we are done. Otherwise, we assume for contradiction that

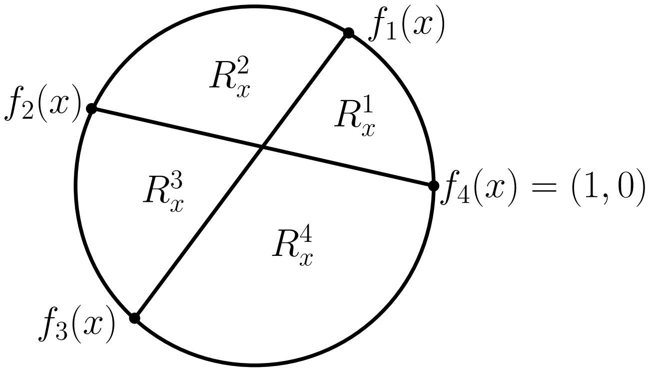

Figure 4.12: Illustration for the proof of Theorem 4.21.

$\Delta^3 = \bigcup_{i=1}^4 O_i^j$ for all j. Observe that if $x \in \operatorname{conv}\{e_i : i \in I\}$ for some $I \subset \{1, \ldots, 4\}$, then $R_x^k = \emptyset$ for $k \notin I$, and therefore $x \in \bigcup_{i \in I} O_i^j$ for all j.

The last paragraph shows that, for all j, $\{O_1^j, \ldots, O_4^j\}$ is an open cover that satisfies the hypothesis of the Colorful KKM theorem (Theorem 4.4), then there exists some permutation $\pi \in S_4$ and a point $y = (y_1, \ldots, y_4) \in \bigcap_{i=1}^4 O_i^{\pi(i)}$. In other words, each of the open regions R_y^i contains a set $C_i \in \mathcal{F}_{\pi(i)}$ (in particular $R_y^i \neq \emptyset$ and $y_i \neq 0$ for all $i \in \{1, \ldots, 4\}$).

Then the sets C_1, C_2, C_3, C_4 do not have a line transversal, a contradiction. □

Note that Theorem 4.21 implies Theorem 1.14. Indeed, if $\mathcal{F}$ is a family satisfying the hypothesis of Theorem 1.14, then $\mathcal{F}_i = \mathcal{F}$ for $i = 1, \ldots, 4$ satisfy the hypothesis of Theorem 4.21 and thus the family $\mathcal{F}$ can be crossed by 2 lines.

Further, if we use Theorem 4.5 (instead of Theorem 4.4) we have the following stronger result.

Theorem 4.22. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be finite families of connected sets in $\mathbb{R}^2$, with $n \geq 4$. Suppose that every four sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}, A_4 \in \mathcal{F}_{i_4}$, for $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$, have a line transversal. Then there exists $I \subset \{1, \ldots, n\}$, with $|I| = n - 3$, such that the family $\bigcup_{i \in I} \mathcal{F}_i$ can be crossed by 2 lines.*

Proof. We assume for contradiction that for every $I \subset \{1, \ldots, n\}$, with $|I| = n - 3$, the family $\bigcup_{i \in I} \mathcal{F}_i$ cannot be crossed by 2 lines. Then, following the same arguments of the proof of Theorem 4.21 and using the same notation, we have that for every $I \in \binom{[n]}{n-4+1}$, the family

$$\left\{ \bigcup_{j \in I} O_1^j, \ldots, \bigcup_{j \in I} O_4^j \right\}$$

is an open cover of Δ^3 that satisfies the hypothesis of Theorem 4.1. Then, by Theorem 4.5, there is an injective function $\pi : [4] \longrightarrow [n]$ and a point $y \in \bigcap_{i=1}^4 O_i^{\pi(i)} \neq \emptyset$. In other words, each of the open regions R_y^i contains a set $C_i \in \mathcal{F}_{\pi(i)}$. Then the sets $C_1, \ldots, C_4$ do not have a line transversal, a contradiction. □

Now we prove a colorful version of Theorem 1.15. Let $\mathcal{F}_1, \ldots, \mathcal{F}_6$ be finite families of connected sets in $\mathbb{R}^2$. Suppose that every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, for $1 \leq i_1 <$

$i_2 < i_3 \le 6$, have a line transversal. We prove that there exists $i \in \{1, \ldots, 6\}$ such that the family $\mathcal{F}_i$ can be crossed by 3 lines. This result is joint work with Edgardo Roldán-Pensado.

The proof is very similar to the proof of Theorem 4.21. The idea goes as follows. If there are three lines crossing one of the families, we are done. Otherwise, we suppose, by contradiction, that for each family $\mathcal{F}_i$ and for every three lines in the plane there is a set in the family $\mathcal{F}_i$ not crossing the three lines. Then using the Colorful KKM theorem (Theorem 4.4) we prove that there are three lines l_1, l_2, l_3 separating three sets C_1, C_2, C_3 from three different families (see Figure 4.13). By Lemma 1.16, the sets C_1, C_2, C_3 do not have a line transversal, a contradiction.

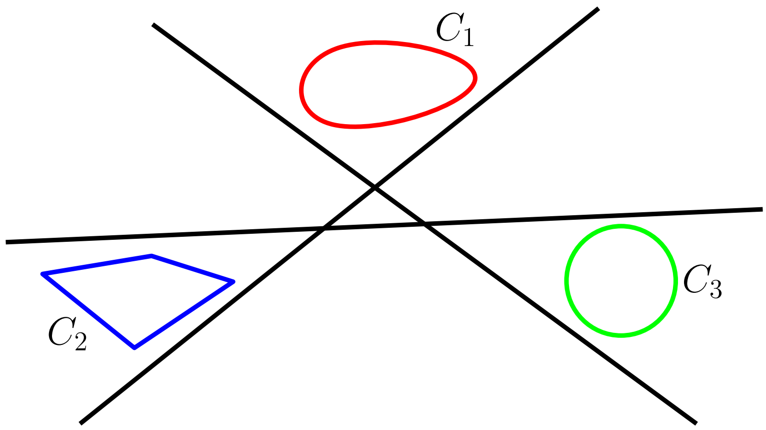

Figure 4.13: If there are no three lines crossing one of the families $\mathcal{F}_i$, then there are three sets C_1, C_2, C_3 from three different families such that the sets C_1, C_2, C_3 do not have a line transversal.

Theorem 4.23. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_6$ be finite families of connected sets in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, for $1 \le i_1 < i_2 < i_3 \le 6$, have a line transversal, then there exists $i \in \{1, \ldots, 6\}$ such that the family $\mathcal{F}_i$ can be crossed by 3 lines.*

Proof. We can assume, without loss of generality, that the sets in the six finite families are compact (see section 1.3). Hence, we may scale the plane such that every set in $\mathcal{F}_j$ is contained in the unit disk, for each $j \in \{1, \ldots, 6\}$. Let $f(t)$ be a parametrization of the unit circle defined by $f(t) = (\cos(2\pi t), \sin(2\pi t))$.

To each point $x = (x_1, \ldots, x_6) \in \Delta^5$ we associate 6 points on the unit circle given by

$$f_i(x) = f\left(\sum_{j=1}^{i} x_j\right),$$

for $1 \le i \le 6$. Let $l_1(x) = l_4(x) = [f_1(x), f_4(x)]$, $l_2(x) = l_5(x) = [f_2(x), f_5(x)]$ and $l_3(x) = l_6(x) = [f_3(x), f_6(x)]$. For $i = 1, \ldots, 6$ let R_x^i be the interior of the region bounded by $l_{i-1}(x), l_i(x)$ and the arc on the unit circle connecting $f_{i-1}(x)$ and $f_i(x)$, where $i - 1$ is taken modulo 6 (see Figure 4.14).

Notice that $f_6(x) = (1, 0)$ for each $x \in \Delta^5$. Also, the points $x_1, x_2, \ldots, x_6$ are always in counter-clockwise order.

For $i, j = 1, \ldots, 6$, let O_i^j be the set of points $x \in \Delta^5$ such that R_x^i contains a set $F \in \mathcal{F}_j$. Since the sets $F \in \mathcal{F}_j$ are compact, O_i^j is open for all i, j. If there is some $x \in \Delta^5$ and $j \in \{1, \ldots, 6\}$ for which $x \notin \bigcup_{i=1}^{6} O_i^j$, then since the sets in $\mathcal{F}_j$ are connected, every set in

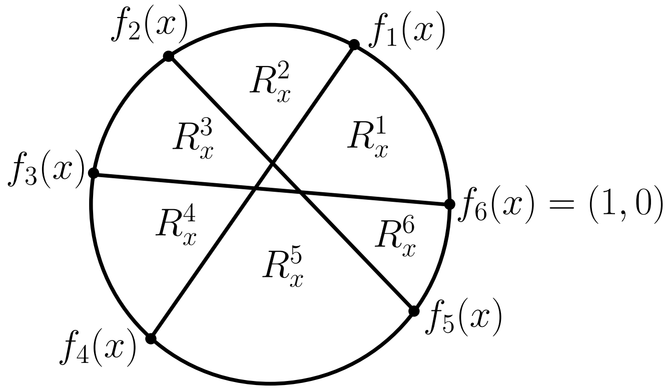

Figure 4.14: Illustration for the proof of Theorem 4.23.

$\mathcal{F}_j$ must intersect $\bigcup_{i=1}^{3} l_i(x)$, and we are done. Otherwise, we assume for contradiction that $\Delta^5 = \bigcup_{i=1}^{6} O_i^j$ for all j. Observe that if $x \in \text{conv}\{e_i : i \in I\}$ for some $I \subset \{1,\ldots,6\}$, then $R_x^k = \emptyset$ for $k \notin I$, and therefore $x \in \bigcup_{i\in I} O_i^j$ for all j.

The last paragraph shows that, for all j, $\{O_1^j,\ldots,O_6^j\}$ is an open cover that satisfies the hypothesis of the colorful KKM theorem (Theorem 4.4), then there exists some permutation $\pi \in S_6$ and a point $y = (y_1,\ldots,y_6) \in \bigcap_{i=1}^{6} O_i^{\pi(i)}$. In other words, each of the open regions R_y^i contains a set $C_i \in \mathcal{F}_{\pi(i)}$ (in particular $R_y^i \neq \emptyset$ and $y_i \neq 0$ for all $i \in \{1,\ldots,6\}$).

Observe that the regions R_y^1, R_y^3, R_y^5 are pairwise disjoint or the regions R_y^2, R_y^4, R_y^6 are pairwise disjoint. Without loss of generality, we assume R_y^1, R_y^3, R_y^5 are pairwise disjoint. Then by Lemma 1.16, the sets C_1, C_3, C_5 do not have a line transversal, a contradiction. □

Note that Theorem 4.23 implies Theorem 1.15. Indeed, if $\mathcal{F}$ is a family satisfying the hypothesis of Theorem 1.15, then $\mathcal{F}_i = \mathcal{F}$ for $i = 1,\ldots,6$ satisfy the hypothesis of Theorem 4.23 and thus the family $\mathcal{F}$ can be crossed by 3 lines.

Further, if we use Theorem 4.5 (instead of Theorem 4.4) we have the following stronger result.

Theorem 4.24. *Let $\mathcal{F}_1,\ldots,\mathcal{F}_n$ be finite families of connected sets in $\mathbb{R}^2$, with $n \geq 6$. Suppose that every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, for $1 \leq i_1 < i_2 < i_3 \leq n$, have a line transversal. Then there exists $I \subset \{1,\ldots,n\}$, with $|I| = n-5$, such that the family $\bigcup_{i\in I} \mathcal{F}_i$ can be crossed by 3 lines.*

Proof. We assume for contradiction that for every $I \subset \{1,\ldots,n\}$, with $|I| = n-5$, the family $\bigcup_{i\in I} \mathcal{F}_i$ cannot be crossed by 3 lines. Then, following the same arguments of the proof of Theorem 4.23 and using the same notation, we have that for every $I \in \binom{[n]}{n-6+1}$, the family

$$\left\{\bigcup_{j\in I} O_1^j,\ldots,\bigcup_{j\in I} O_6^j\right\}$$

is an open cover of Δ^5 that satisfies the hypothesis of Theorem 4.1. Then, by Theorem 4.5, there is an injective function $\pi : [6] \longrightarrow [n]$ and a point $y \in \bigcap_{i=1}^{6} O_i^{\pi(i)} \neq \emptyset$. In other words, each of the open regions R_y^i contains a set $C_i \in \mathcal{F}_{\pi(i)}$.

Observe that the regions R_y^1, R_y^3, R_y^5 are pairwise disjoint or the regions R_y^2, R_y^4, R_y^6 are pairwise disjoint. Without loss of generality, we assume R_y^1, R_y^3, R_y^5 are pairwise disjoint. Then by Lemma 1.16, the sets C_1, C_3, C_5 do not have a line transversal, a contradiction. □

The colorful version of Theorem 1.15 that we wanted to prove uses 3 colors instead of 6 colors. Although we have not been able to prove a colorful version of Theorem 1.15 using only 3 colors, we prove the following theorem concerning finite families of translates of a compact convex set using only 3 colors.

Theorem 4.25. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be finite families of translates of K in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ have a line transversal, then there exists $i \in \{1, 2, 3\}$ such that the family $\mathcal{F}_i$ can be crossed by 4 lines.*

Proof. Let $\mathcal{C}$ be the set of all the rainbow pairs (C_i, C_j) such that C_i and C_j are disjoint. In other words,

$$\mathcal{C} = \{(C_i, C_j) \mid C_i \in \mathcal{F}_i,\ C_j \in \mathcal{F}_j,\ i \neq j,\ C_i \cap C_j = \emptyset\}.$$

If $\mathcal{C}$ is empty, then we project the sets of the three families in some fixed line and so we obtain three families of segments satisfying the hypothesis of the 1-dimensional case of the Colorful Helly theorem (Theorem 2.4). Thus, by the Colorful Helly theorem (Theorem 2.4) on the line, there is $i \in \{1, 2, 3\}$ such that $\mathcal{F}_i$ has a line transversal, and we are done.

Otherwise, choose a pair $(C_i, C_j) \in \mathcal{C}$ for which the angle between the inner common tangents of (C_i, C_j) is minimal among all pairs in $\mathcal{C}$. Without loss of generality, we assume that $(C_1, C_2) \in \mathcal{C}$, with $C_1 \in \mathcal{F}_1, C_2 \in \mathcal{F}_2$, is one of the pairs for which the angle between the inner common tangents is minimal.

We claim that the 4 common tangents of (C_1, C_2) cross the family $\mathcal{F}_3$. Let l_1, l_2 be the inner common tangents of C_1 and C_2. Let l_3, l_4 be the outer common tangents of C_1 and C_2. We denote by $l_1^+, l_2^+, l_3^+, l_4^+$ the half-planes bounded by l_1, l_2, l_3, l_4, respectively, such that $C_1 \subset \bigcap_{i=1}^4 l_i^+$.

We define the regions $R_1, \ldots, R_6$ as follows. Let R_1 be the interior of $l_1^+ \cap l_2^+$, let R_2 the interior of $l_1^- \cap l_2^-$, let R_3 be the interior of $l_1^+ \cap l_2^- \cap l_3^-$, let R_4 be the interior of $l_1^- \cap l_2^+ \cap l_4^-$, let R_5 be the interior of $l_1^+ \cap l_2^- \cap l_3^+$ and let R_6 be the interior of $l_1^- \cap l_2^+ \cap l_4^+$. See Figure 4.15.

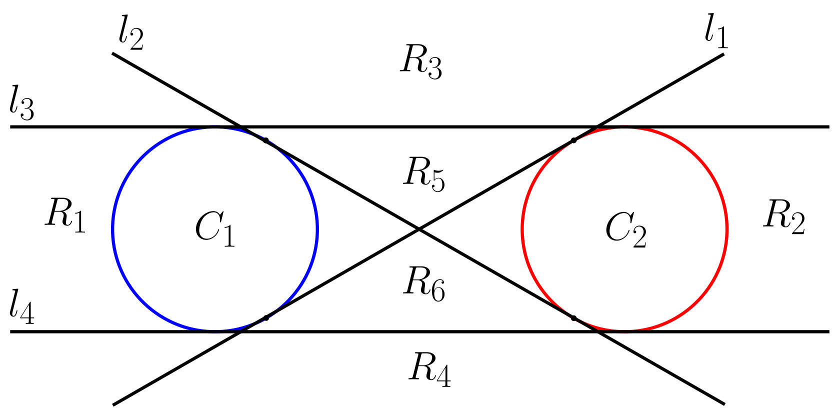

Figure 4.15: Illustration for the proof of Theorem 4.25.

If there is a set $C_3 \in \mathcal{F}_3$ contained in the region R_1, then the angle between the inner common tangents of $(C_2, C_3) \in \mathcal{C}$ is less than the angle between the inner common tangents of (C_1, C_2), contradicting the choice of the pair (C_1, C_2), see Figure 4.16. Analogously, if there is a set $C_3 \in \mathcal{F}_3$ contained in the region R_2, then the angle between the inner common tangents of $(C_1, C_3) \in \mathcal{C}$ is less than the angle between the inner common tangents of (C_1, C_2), contradicting the choice of the pair (C_1, C_2). Thus, there are no sets in the family $\mathcal{F}_3$ contained in the regions R_1 or R_2.

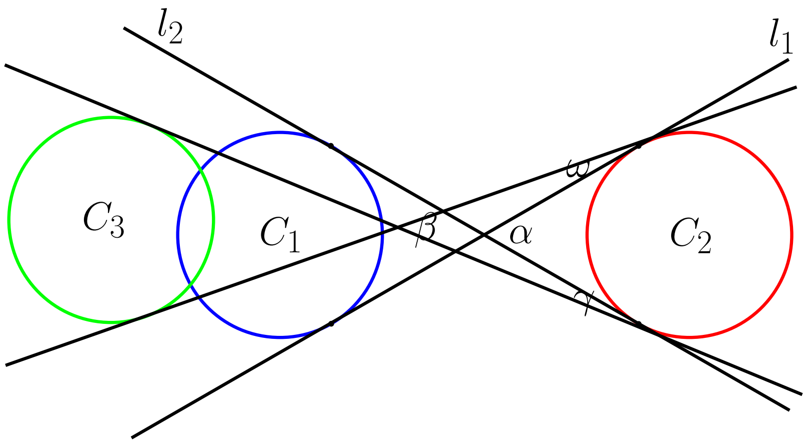

Figure 4.16: Notice that $\alpha = \beta + \omega + \gamma$, then $\alpha > \beta$.

If there is a set $C_3 \in \mathcal{F}_3$ contained in the region R_3, then by Lemma 1.16, the sets $C_1 \in \mathcal{F}_1, C_2 \in \mathcal{F}_2, C_3 \in \mathcal{F}_3$ do not have a line transversal, a contradiction. Analogously, if there is a set $C_3 \in \mathcal{F}_3$ contained in the region R_4, then by Lemma 1.16, the sets $C_1 \in \mathcal{F}_1, C_2 \in \mathcal{F}_2, C_3 \in \mathcal{F}_3$ do not have a line transversal, a contradiction. Thus, there are no sets in the family $\mathcal{F}_3$ contained in the regions R_3 or R_4.

Since the sets in the three families are translates of K, then every set in the family $\mathcal{F}_3$ has the same width of C_1 and C_2 in the direction orthogonal to the lines l_3, l_4. Thus, there are no sets in the family $\mathcal{F}_3$ contained in the regions R_5 or R_6.

Therefore, every set in the family $\mathcal{F}_3$ must intersects some of the lines l_1, l_2, l_3, l_4. □

Note that using the same proof, we have that Theorem 4.25 is also true for finite families of congruent copies of a compact convex set of constant width.

Theorem 4.26. *Let K be a compact convex set of constant width in $\mathbb{R}^2$. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be finite families of congruent copies of K in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ have a line transversal, then there exists $i \in \{1, 2, 3\}$ such that the family $\mathcal{F}_i$ can be crossed by 4 lines.*

In particular, Theorems 4.25 and 4.26 hold for translates of circles of the same radius.

Theorem 4.27. *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be finite families of circles of the same radius in $\mathbb{R}^2$. If every three circles $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ have a line transversal, then there exists $i \in \{1, 2, 3\}$ such that the family $\mathcal{F}_i$ can be crossed by 4 lines.*

An interesting question is if we can improve the number of lines in the conclusion of Theorems 4.25, 4.26 and 4.27.

Problem 4.28. *Let K be a compact convex set in $\mathbb{R}^2$. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be finite families of translates of K in $\mathbb{R}^2$ such that every three sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ have a line transversal. Is there exists $i \in \{1, 2, 3\}$ such that the family $\mathcal{F}_i$ can be crossed by 3 (or 2) lines?*

On the other hand, we still wonder if there is a colorful version of Theorem 1.15 using 3 colors.

Problem 4.29. *Is there exists $n \in \mathbb{Z}^+$ with the following property? Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be finite families of connected sets in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2, A_3 \in \mathcal{F}_3$ have a line transversal, then there exists $i \in \{1, 2, 3\}$ such that the family $\mathcal{F}_i$ can be crossed by n lines.*

In addition, we conjecture the following colorful version of Theorem 1.15 that uses 4 colors.

Conjecture 4.30. *There exists $n \in \mathbb{Z}^+$ with the following property. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be finite families of connected sets in $\mathbb{R}^2$. If every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, for $1 \leq i_1 < i_2 < i_3 \leq 4$, have a line transversal, then there exists $i \in \{1, \ldots, 4\}$ such that the family $\mathcal{F}_i$ can be crossed by n lines.*

We tried to prove that Conjecture 4.30 holds with $n = 3$, however we did not succeed. The idea was the following: for each $i \in \{1, ..., 4\}$ we applied the KKM theorem (Theorem 4.1) to the family $\mathcal{F}_i$, then following the same ideas of the proof of Theorem 4.23, we have that there are three sets A_1^i, A_2^i, A_3^i in the family $\mathcal{F}_i$ such that the sets A_1^i, A_2^i, A_3^i do not have a line transversal. Using the 12 convex sets A_j^i, for $i = 1, 2, 3, 4$ and $j = 1, 2, 3$, we wanted to prove that there are 3 sets in different families without line transversal, which would be a contradiction. It gave rise to the following conjecture.

Conjecture 4.31. *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be families of connected sets in $\mathbb{R}^2$, each family with 3 sets. If every three sets $A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}$, for $1 \leq i_1 < i_2 < i_3 \leq 4$, have a line transversal, then there exists $i \in \{1, 2, 3, 4\}$ such that $\mathcal{F}_i$ has a line transversal.*

However, Conjecture 4.31 is false. The counterexample is given in Example 4.32 and Figure 4.17.

Example 4.32. *Let*

$$\begin{aligned}
\mathcal{F}_1 &= \{[(-20, 6), (-2, 6)], [(2, 6), (20, 6)], [(-8, 12), (8, 12)]\},\\
\mathcal{F}_2 &= \{[(-8, 2), (8, 2)], [(-8, 6), (-8, 24)], [(8, 6), (8, 24)]\},\\
\mathcal{F}_3 &= \{[(-8, 8), (8, 8)], [(-17, 10), (-1, 10)], [(1, 10), (17, 10)]\},\\
\mathcal{F}_4 &= \{[(-8, 4), (8, 4)], [(-10, 6), (-10, 24)], [(10, 6), (10, 24)]\}
\end{aligned}$$

be families of segments in the plane. The families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ satisfy the hypothesis of Conjecture 4.31. However, none of the four families has a line transversal (see Figure 4.17).

Even though Conjecture 4.31 is false, it does not imply that Conjecture 4.30 is false. Motivated by Theorem 4.25, we still believe Conjecture 4.30 is true, although maybe n will be very large.

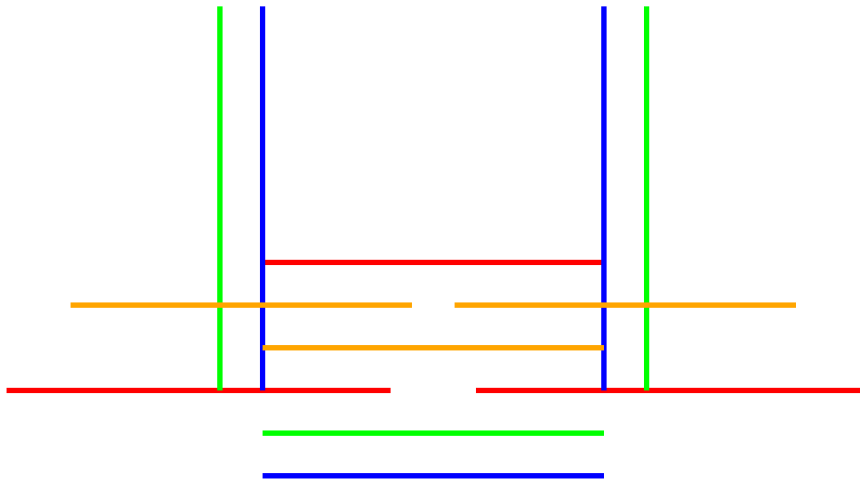

Figure 4.17: Illustration for Example 4.32.

Chapter 5

Conclusions

As we mentioned in the Introduction, this work is a survey of colorful theorems in discrete and convex geometry. Chapters 1 and 2 were introductory. In Chapter 3 we saw Theorems 3.2 ([1], [21]) and 3.11 ([28]) which are generalizations of the Colorful Carathéodory theorem and the Colorful Helly theorem, respectively.

The original contributions of this Thesis are the following theorems and observations.

- In Example 3.4 we showed that we can not continue to weaken the hypothesis of Theorem 3.2. In Example 3.6 we showed that in Theorem 3.2 we can only ensure the existence of 1 rainbow simplex.

- In Theorem 4.7 we gave and proved the best numbers satisfying the 2-dimensional case of Theorem 3.11 ($f(2) = 1$ and $g(2) = 2$). Furthermore, we proved Theorem 4.8 which is stronger than Theorem 4.7. In addition, in Theorem 4.6 we proved a particular case of Theorem 4.7 with an elementary proof. We also proved Theorems 4.10, 4.11, 4.12, 4.13 and 4.14 concerning some special families where the 2-dimensional case of Theorem 3.11 holds with $g(2) = 1$. Finally, we proposed Problem 4.15 and we believe that Theorem 4.10 can be useful to prove Conjecture 4.17.

- We proved Theorems 4.21 and 4.23 (colorful versions of Eckhoff's theorems), although McGinnis and Zerbib [31] after wrote a paper with these results. Additionally, we proved Theorems 4.25 and 4.26 (and Theorem 4.27 which is a particular case of Theorems 4.25 and 4.26) concerning families of translates of a compact convex set or families of congruent copies of a compact convex set of constant width. An interesting question is if in Theorem 4.25 we can improve the 4 lines (in the conclusion) by 3 or 2 lines (Problem 4.28). Finally, in Problem 4.29 and Conjecture 4.30 we wonder if we can improve the number of families (or colors) in Theorem 4.23.

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