UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN INGENIERÍA
INGENIERÍA ELÉCTRICA – CONTROL

DISEÑO DE ALGORITMOS CONTINUOS POR MODOS DESLIZANTES MULTIDIMENSIONALES

TESIS
QUE PARA OPTAR POR EL GRADO DE:
MAESTRO EN INGENIERÍA

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CIUDAD DE MÉXICO, TBA 2022

CIUDAD DE MÉXICO, SEPTIEMBRE, 2022

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La tesis se realizó en el Posgrado de Ingeniería, UNAM.

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FIRMA

Acknowledgements

Al CONACyT y al Posgrado de Ingeniería Eléctrica de la UNAM,
que me apoyaron a través del PAPIIT-UNAM, proyecto IN102121.

Contents

Glossary

- Boundary matrix: A term invented in this work. For a set of bounded matrices, the boundary matrix is defined with diagonals as the lower bounds of the diagonals of the set and the off-diagonals as the negative upper bounds of the off-diagonals of the set. See subsection 3.1.1.

- Chattering: High-frequency, sometimes dangerous, vibrations in the system. In sliding mode control it's caused by the high frequency of control switching. There are three distinct kinds: infinitesimal chattering, bounded chattering and unbounded chattering [1].

- Compensator: Synonym of controller. The use of this word is in decline.

- Definite matrix: A matrix is said to be positive (or negative) definite if it's symmetric and its quadratic form is positive (or negative) for any nonzero vector. A positive definite matrix's eigenvalues are all positive real. A negative definite matrix's eigenvalues are all negative real.

- Generalized diagonal dominance: A matrix property. A matrix is said to have generalized diagonal dominance if there exists a positive diagonal matrix that makes the product of both matrices strictly diagonally dominant. See section 2.4. In this work this property is only used on matrices with positive diagonal elements.

- Matched perturbations/uncertainties: Perturbations/uncertainties that satisfy the matching condition, that is, they can be represented as a vector times the input matrix [2].

- Order of sliding mode: A sliding mode is said to be of order r if the rth total derivative of the sliding variable is the first total derivative which contains a discontinuity [2].

- Positive stable matrix: A matrix is said to be positive stable if the real part of all of its eigenvalues is positive. If one multiplies a positive stable matrix times a negative scalar, a Hurwitz matrix is obtained.

- Quasi-definite matrix: A matrix is said to be quasi-definite if this matrix plus its transpose results in a definite matrix.

- Sliding mode: System trajectory movement along the discontinuity surface, the *sliding surface*.

- Sliding surface: The attractive and invariant subset of the state space that we want the *sliding variable* to reach to solve the control problem. Here the system must enter into *sliding mode*.

- Sliding variable: A measurable state variable. It is such that when this variable reaches zero, the control problem (whatever it may be) is solved. The sliding variable may be equal to the system's output, the tracking error or a function of the state variables.

- Strict diagonal dominance: A matrix property. A matrix has strict row (or column) diagonal dominance if the absolute value of every diagonal is greater than the sum of the absolute value of the elements of its row (or column). See section 2.4. In this work this property is only used on matrices with positive diagonal elements.

- Symmetric part of a matrix: The symmetric part of a matrix is this matrix plus its transpose divided by two.

Chapter 1

Introduction

1.1 State of the Art

Multi-input multi-output (MIMO) systems appear naturally in control problems and the main difference between these systems and the scalar single-input single-output (SISO) case is the presence of *directions*, which affect both vectors and matrices in MIMO cases [3].

The idea to decouple such a system into a multitude of SISO systems and control each one separately is appealing, but there are several difficulties, mainly that: decoupling is very sensitive to modelling errors and uncertainties, it may not be desirable for disturbance rejection and quite simply it may not be desired in practice [3].

In the particular area of sliding mode control, the super-twisting algorithm is a classic [4, 2]. It's been used for SISO control [5] and robust differentiation [6]. In fact, as a control algorithm it is a popular choice because of its unique features and advantages for systems of relative degree 1, mainly that the algorithm can compensate matched Lipschitz perturbations and uncertainties; it forces the output and its derivative to zero in finite-time, while only requiring knowledge of the output; and it generates a continuous control signal [4, 2].

Because of all of this, there's been interest in generalizing such control algorithm to the MIMO case, especially for systems with uncertain input matrices.

Nevertheless, only a few generalizations have been made: the first one proposed in [7] can stabilize the origin of a general nonlinear system, meaning the input matrix could be dependent on time or state variables. However they assume no model uncertainties which would allow for perfect decoupling. Stability is achieved by using scalar constant gains in both the integral and static feedback. In addition, they use extra linear PI terms in their algorithm.

This approach was combined with so called Integral sliding mode control in [8], that results in a control law consisting of a nominal control (which could be linear or nonlinear) and a sliding mode control to compensate for perturbations and effectively stabilizes the nominal control, in this case they use the MIMO super-twisting algorithm previously discussed. This clever combination extended its possible use to any MIMO system.

While the proposed selection of gains could compensate for perturbations linearly bounded by the sliding output variable and Lipschitz perturbations, a similar version presented in [9] allows for a broader class of nonlinear perturbations to be compensated thanks to not being restricted to constant gains. This was also the first multivariable super-twisting algorithm for uncertain systems, however they did restrict themselves to linear time invariant systems, meaning the input matrix is constant. They also assumed this matrix is symmetric, and in [10] they found this condition to be fragile since a nonsymmetric input matrix easily leads to instability.

A different type of controller this time for general nonlinear systems was presented in [11] where they modelled an uncertain description of the input matrix, which is a uncertain term, that can be time-varying and state dependent, and is factorized from the left of the nominal input matrix. A key restriction in their design (although still allowing for very general systems) is that the input matrix should be positive quasi-definite in spite of the uncertainties.

Further generalization using a gain in the integral variable itself, not in the integral injection term, was studied in [12], where they found the proper design of this gain could help compensate a broader class of nonlinear perturbations.

All of the previous controllers used nonsmooth Lyapunov functions for their stability analysis. They also share the use of unit vector style or quasi-continuous style feedback injection that doesn't require decoupling and uses scalar gains, in fact it induces coupling into all of the subsystems whether this is desired or not.

This type of feedback injection results in a different behaviour from a collection of SISO super-twisting controllers for a decoupled system, as it was seen in the comparisons made in [13]. Among the differences they reported were: the unit vector style feedback makes all the states converge together with less gains to design, whereas the SISO ones converge independently and have more gains to design, but they work better with observers and could be endowed with different dynamical and robustness properties.

In the field of SISO super-twisting controllers we have for example [14], where they show a decoupled design for a system that is allowed to have some model uncertainties as long as they're not in the input matrix, allowing them to perfectly decouple the system. Here the gains are found by solving an implicit Lyapunov function. This last algorithm also showcases the passivity of the super-twisting algorithm.

Conversely, in [12] they allow uncertainties in the input matrix and use SISO super-twisting controllers in systems that are already known to be diagonal or at least block diagonal with diagonal blocks. They found that a diagonal constant gain matrix in the static feedback and a scalar matrix gain in the integral one can stabilize the origin of the system, meaning in this case the subsystems are completely decoupled.

1.2 Contribution

Here in Chapter 3, the problem of a MIMO dynamical system represented by differential equations and an uncertain input matrix is tackled. The input matrix is assumed to be time and state dependent, with an *uncertain but constant* matrix factor.

With the objective of designing a continuous control law to robustly stabilize the origin of such a system in spite of the uncertainties in the input matrix and in the presence of matched Lipschitz perturbations, a new family of MIMO homogeneous controllers was developed. This family contains not only a multivariable super-twisting algorithm for nonlinear systems, but also a full family of continuous and homogeneous approximations, including a linear PI-control law.

Although the new controllers use SISO style discontinuous feedback terms, they're not restricted to diagonal gain matrices, in contrast to the usual unit-vector-like ones of the previous MIMO algorithms or the completely decoupled SISO ones.

The added degrees of freedom in the form of nondiagonal matrix elements in the gains enables the designer to adjust the performance of the controller, reducing overshoot for example. This is seen in Chapter 5, which contains some illustrative examples with numerical simulations of two different real word systems where the new control algorithms are used and their main features are highlighted.

On another note, these new controllers were developed from a smooth homogeneous Lyapunov function, motivated by a passivity interpretation that's already been shown in the SISO case [15], which is an approach quite different than what's been done in other generalizations.

On top of that, in Chapter 4 an arbitrary relative degree generalization for these controllers is presented, although under stricter conditions than the relative degree one case.

To complement the results, Chapter 2 contains most of the theoretical background needed for a controls engineer to understand the results and their mathematical proofs.

Finally in Chapter 6 the effectiveness of the controllers is discussed along with how they fare against other similar generalizations: what are the advantages and disadvantages.

Chapter 2

Theoretical Background

2.1 MIMO Sliding Mode Control Development

Sliding mode control is a well established collection of nonlinear control techniques that were developed in five distinct generations [16]:

The first generation corresponds to the classical theory of first order sliding mode control which offers theoretical exact compensation of bounded matched uncertainties and finite time convergence to the sliding surface at the cost of chattering and having to design a surface of relative degree one with respect to the control input.

The second generation was developed with the goal of reducing chattering, which is mainly caused by unmodeled dynamics that increase the relative degree of the sliding surface. It was understood that if the derivative of the sliding variable could be forced to converge to zero along with the sliding variable itself, then the chattering could be adjusted. In order to do this, second order controllers were developed. In the end, they still produced a discontinuous control signal so chattering wasn't reduced substantially and they required additional measurements of the sliding variable's derivative.

The third generation was the Super-Twisting Algorithm which is a second order sliding mode controller that uses only the measurements of the sliding variable and is capable of making both the sliding variable and its derivative converge to zero in finite time, in turn allowing the development of a robust differentiator. This controller also managed to significantly attenuate the chattering but not to completely eliminate it. Nevertheless, a sliding surface design of relative degree one was still required.

The fourth and fifth generations corresponded to the arbitrary relative degree generalizations. It is safe to say that the Quasi-continuous controller is the flagship of the fourth generation, it serves as generalized version of the first generation controllers. On the other hand, the fifth generation has been dealing with controllers that provide the main advantages of the Super-Twisting Algorithm for any relative degree [16].

All of this has been the development of sliding mode controllers for SISO systems, in contrast MIMO sliding mode controllers can be considered to still be in their infancy.

MIMO controllers of first generation are already well established even for uncertain systems, mainly the algorithm named *Unit Vector Control* attributed to Utkin [17]. Here, the idea of a scalar sliding surface is extended to the multivariable case as the intersection of multiple discontinuous surfaces and the sign function is extended as the sliding output vector divided by its norm [17].

There's even been extensions of this algorithm to uncertain systems for decades now, for example the ones reported in [18, 19], both from 1979. These controllers use variable gains to compensate the uncertainties more effectively without overkilling with gains designed for the worst case scenario.

When reaching the third generation it becomes clear that MIMO sliding mode control has been lagging behind the SISO developments: of all of the MIMO super-twisting controllers mentioned in the introduction, the first reported one [7] is just from 2014. And in the case of fourth generation controllers the first MIMO quasi-continuous controller developed for general uncertain systems was reported in 2018 [20]. This is just to say that new generation MIMO sliding mode control is still a fresh and active research field.

One thing that all MIMO generalizations of sliding mode controllers share is that they must impose some restrictions on the uncertain input matrix to be able to prove the stability of the controller, meaning that even though the exact value of the matrix's elements isn't known, some matrix properties are guaranteed. The work of [21] includes a brief summary on the matrix properties that have been assumed to be preserved under parametric uncertainties, these are: positive definiteness, positive quasi-definiteness and positive stability, the latter being the most general and the former the most restrictive. They also add that these conditions can be complemented with time or state dependent perturbations with a known norm bound that multiplies the constant input matrix. For example, in their own work they are able to prove the stability of their controller under the restriction that the variable input matrix is positive stable for all time and any state.

2.2 The Super-Twisting Algorithm

As mentioned before, the Super-Twisting Algorithm played such a seminal role in sliding mode control theory that it is worthy of its own generation. Its main characteristics were briefly mentioned but the theory that explains them was not. In this section the theory behind this controller is explored.

The SISO super-twisting control law, first introduced by Levant in [5], can be expressed as:

$$\begin{aligned} u &= -k_1 \lceil x \rfloor^{\frac{1}{2}} + v \\ \dot{v} &= -k_2 \lceil x \rfloor^{0} \end{aligned} \tag{2.1}$$

with x as the sliding variable.

A main feature of this controller is the use of an integrator, and in fact, this is what explains how it can reject Lipschitz perturbations, provide a continuous control signal and make the derivative of the sliding variable go to zero.

Firstly, it is known that the use of integrators in higher order sliding mode control produces a continuous control signal. With conventional sliding mode control, this comes at the cost of artificially raising the relative degree of the system by one. Such strategy requires taking another derivative and using a new control variable $v = \dot{u}$ a higher order sliding mode controller is proposed for v [15, 22]. As a consequence, it is necessary to also get the rth derivative of sliding variable which involves taking the derivative of the perturbations and uncertainties [15].

In contrast, in the super-twisting case, the additional dynamic equation does not alter the relative degree with respect to u since instead of using the derivative of the control law, it goes in the other direction, using an integral control law. Still, the discontinuous part of the control law is behind an integrator, making the result a continuous signal. This makes it a second order sliding mode controller even though it is made for relative degree 1 control problems.

Secondly, the derivative of the sliding variable can go to zero thanks to the control signal being continuous, in turn allowing the derivative of the sliding variable to be continuous, therefore the sliding variable is continuously differentiable finally implying that when the sliding variable is zero, its derivative must be zero too. To see these properties more clearly consider a relative degree one system with a constant coefficient controlled by a super-twisting controller:

$$\begin{aligned} \dot{x} &= b \left[-k_1 \lceil x \rfloor^{\frac{1}{2}} + v \right] \\ \dot{v} &= -k_2 \lceil x \rfloor^{0} \end{aligned} \tag{2.2}$$

where x is the scalar output, $b > 0$ is the control coefficient and the input has been substituted by the super-twisting in (2.1).

There is a discontinuous part in $\dot{v}$ that can be seen as a conventional sliding mode controller, but as we've seen, due to the integrator, v is a continuous signal. This added to the other term in $\dot{x}$ results in the derivative of x being continuous, which is what we wanted to show.

Finally, the complete rejection of Lipschitz perturbations is explained by the internal model interpretation of this control law. The internal model principle was introduced into linear control theory as a viable tool to design servocompensators [23, 24, 25]. In general, the servocompensator is incorporated as an *exosystem* and its dynamics are designed such that they describe the class of signals corresponding to the references to be tracked or the perturbations to be rejected [26]. Under this lens, $\dot{v}$ in equation (2.1) is the internal model of Lipschitz signals, this is, signals with bounded derivatives [26].

To see this better, the multivalued interpretation of the signum function is used, this means the servocompensator is given by the differential inclusion $\dot{v} \in k_2[-1,1]$, therefore the signal v is allowed to be any absolutely continuous signal with a derivative, where it exists, bounded by k_2 [26].

Given that the construction of a proper internal model is the main problem when designing a servocompensator for a nonlinear system, the fact that this Lipschitz internal model is so simple yet very general is a great property [26]. So the invariance of the super-twisting algorithm against Lipschitz signals has been explained, in turn explaining why this algorithm has found so many different applications.

On that note, a servocompensator can be unstable on its own, so an additional controller is needed to guarantee stability [24, 26]. And this is what the other term of the super-twisting controller does. The stability of the super-twisting algorithm has been demonstrated in a variety of different ways, first by geometric means, then via homogeneity properties, and at the end using Lyapunov methods [27, 7, 28]. It's been shown that in the unperturbed case, it is sufficient that $k_1, k_2 > 0$ to have stability [28, 27], nevertheless the performance and robustness of the controller will be affected by the choice of these gains, for example in [29] frequency domain analysis was used to find gains that adjust chattering, minimizing the amplitude of the oscillations or the average power lost to chattering.

The stability proof for the multivariable extension of the super-twisting that will be presented in Chapter 3 shares a similarity with the ones for the SISO case in [28] and [27], there they've proposed a Lyapunov function for the unperturbed case, and based on this the analysis for the perturbed case gives the conditions for the stability of the controller in spite of the perturbations. In fact one of the authors of these papers has confirmed that the idea of passivity was in their mind when designing the Lyapunov functions. Nevertheless, for many reasons this wasn't shown explicitly until [15].

2.2.1 Passivity interpretation

A passivity based stability analysis of the generalized super-twisting algorithm has been reported in [15] and will be presented here. But firstly a reminder of what passivity means in this context.

Definition 1. *[30, Definition 6.3] A dynamical system represented by a state model*

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

with output $y \in \mathbb{R}^n$ and input $u \in \mathbb{R}^n$ is passive if there exists a continuously differentiable positive semi-definite scalar function $V(x)$ such that

$$\langle u, y \rangle \geq \dot{V} \tag{2.3}$$

With that out of the way, consider again the unperturbed SISO dynamical system of relative degree one and constant coefficient controlled by the super-twisting algorithm in (2.2) and consider the scalar function:

$$V = |x| \tag{2.4}$$

This is not continuously differentiable at $x = 0$, so it isn't a true storage function, however it is Lipschitz continuous, so in some sense it preserves the spirit of the definition. This is enough to show where the main idea of the passivity based design comes from. A more rigorous analysis could be made with passivity generalizations that allow for nonsmooth storage functions, but this is not done here.

Taking the derivative (wherever the derivative exists) the following is obtained

$$\begin{aligned}\dot{V} &= \dot{x}\,sign(x) = b\left[-k_1\lceil x\rfloor^{\frac{1}{2}} + v\right]\lceil x\rfloor^0 \\ &= -bk_1|x|^{\frac{1}{2}} + bv\lceil x\rfloor^0 \\ &\leq vb\lceil x\rfloor^0\end{aligned}$$

with respect to the output $y = b\lceil x\rfloor^0$ and the input v, this system is passive and V is their storage function.

The passivity of the complete system can then be deduced form the passive interconnection of the previous subsystem with the integrator, which is known to be passive.

In addition, the passivity interpretation of a whole family of generalized SISO super-twisting homogeneous controllers is shown in [15], they include more homogeneity degrees and arbitrary relative degrees. Here only the discontinuous case of relative degree one was shown to illustrate the main idea. The passivity of the generalized MIMO controllers will be shown in later chapters.

2.3 Homogeneous Dynamical Systems

First of all, the mathematical property known as *homogeneity* is a dilation symmetry first described by none other than Leonhard Euler in the 18th century. However, it was Vladimir Zubov who used a generalized version of homogeneity to study nonlinear systems [31]. In particular, the generalization known as *weighted homogeneity* allows us to define the concept of weighted-homogeneous functions.

Definition 2. *[32, 15] For a fix set of coordinates $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$, and the* coordinate weights $\boldsymbol{r} \triangleq [r_1, ..., r_n]^\top \in \mathbb{R}^n_{>0}$. *Define the family of dilations $\Delta^{\mathbf{r}}_\varepsilon$ such that*

$$\Delta^{\mathbf{r}}_\varepsilon x \triangleq [\varepsilon^{r_1}x_1, ..., \varepsilon^{r_n}x_n]^\top \tag{2.5}$$

Now, a scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be $\mathbf{r}$-homogeneous of degree $l \in \mathbb{R}$, commonly abbreviated to $(\mathbf{r}, l)$-homogeneous, if for all $\varepsilon > 0$ and all $x \in \mathbb{R}^n$ the equality $V(\Delta^{\mathbf{r}}_\varepsilon x) = \varepsilon^l V(x)$ holds.

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\mathbf{r}$-homogeneous of degree $l \in \mathbb{R}$ if for all $\varepsilon > 0$ and all $x \in \mathbb{R}^n$ the equality $f(\Delta^{\mathbf{r}}_\varepsilon x) = \varepsilon^l \Delta^{\mathbf{r}}_\varepsilon f(x)$ holds.

With this, a dynamical system

$$\dot{x} = f(x) \tag{2.6}$$

is a *homogeneous dynamical system* if $f(x)$ is a homogeneous vector field [2].

It's been established that this concept can simplify the stability and robustness analysis of control systems as well as the design of nonlinear controllers and observers. In fact, the homogeneity degree together with the weights specifies the type of convergence rate of any asymptotically stable homogeneous system, and under some restrictions homogeneous systems can even be finite time stable [31].

An unofficial convention in sliding mode control, that'll be used here too, is to set the weight of the last coordinate (state variable) and of the time variable to be equal to 1, for the second to last coordinate the weight is $1-l$, then $1-2l$ and so on. With this selection the homogeneous degree of linear systems is pegged to $l=0$ and discontinuous systems to $l=-1$. Such convention is useful given that the degree of this type of functions would be different if the weights were selected in another manner. This is similar to how a musical melody can be transposed into a different key, changing all of its notes but still having the same melodic meaning. By this analogy sliding mode control is always played starting on the same note.

In summary, although homogeneity isn't required to understand, design or analyse nonlinear systems (like sliding mode controllers) it provides many useful tools to this. In particular a key homogeneity property and a mathematical lemma for this work were the following.

Definition 3. *[15] Given a dialation $\Delta_\varepsilon^{\mathbf{r}}$ and for $p \geq 1$ the homogeneous norm is defined as*

$$\| x \|_{\boldsymbol{r}p} := \left(\sum_{i=1}^{n} |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}, \forall x \in \mathbb{R}^n \tag{2.7}$$

The homogeneous norm is an r-homogeneous function of degree 1. In the following work, no ambiguity of weight vectors may arise, so when the homogeneous norm is presented and p could be any number, it'll simply be presented as $\| x \|$. In any other case the full form would be used. And when talking about a conventional p-norm, $\| x \|_p$ will be used.

Finally, sometimes called unofficially the *Homogeneous Domination Lemma*, we have:

Lemma 1. *[33, 15]Let $\eta : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ be two continuous homogeneous functions, with weights $\boldsymbol{r} = (r_1, ..., r_n)$ and degrees m, with $\gamma(x) \geq 0$, such that the following holds*

$$\{x \in \mathbb{R}^n \setminus \{0\} : \gamma(x) = 0\} \subseteq \{x \in \mathbb{R}^n \setminus \{0\} : \eta(x) < 0\}.$$

then, there exists a real number λ^* such that, for all $\lambda > \lambda^*$, for all $x \in \mathbb{R}^n \setminus \{0\}$ and some $c > 0$, the following inequality is true

$$\eta(x) - \lambda\gamma(x) < -c\,\|x\|^m$$

2.4 On Diagonal Dominance

Surprisingly enough, a key matrix property that was found to be necessary (for the method used) to prove the stability of the MIMO super-twisting algorithm, presented in the following chapters, was the so called *strict diagonal dominance*. For this reason some mathematical properties associated with matrix diagonal dominance are presented in this section. But first of all, the main definition of diagonal dominance is presented.

Definition 4. *[34, 35] A matrix $A = [\alpha_{i,j}] \in \mathbb{C}^{n\times n}$ is said to be strictly* row *diagonally dominant if the inequality*

$$|\alpha_{i,i}| > \sum_{j\neq i} |\alpha_{i,j}| \tag{2.8}$$

is true $\forall i = 1, 2, ..., n$.

In addition to that, a matrix $A = [\alpha_{i,j}] \in \mathbb{C}^{n\times n}$ is said to be strictly *column* diagonally dominant if the inequality

$$|\alpha_{i,i}| > \sum_{i\neq j} |\alpha_{i,j}| \tag{2.9}$$

is true $\forall i = 1, 2, ..., n$.

Notice that the two last inequalities are exactly the same except for the subscript of the sum operator: in column diagonal dominance i (which indicates the rows) changes and can't be equal to j (which indicates the column). This is reversed in the row diagonal dominance case.

The set of Z-matrices is denoted as:

$$\mathfrak{Z}^{n\times n} = \{A = [\alpha_{i,j}] \in \mathbb{R}^{n\times n} | \alpha_{i,j} \leq 0, i \neq j\} \tag{2.10}$$

A Z-matrix can be represented as

$$A \in \mathfrak{Z}^{n\times n} = s\mathbb{I}_n - B \tag{2.11}$$

where s is a positive scalar, $\mathbb{I}_n$ is the identity matrix of dimension $n \times n$ and B is a matrix whose elements are all nonnegative [35].

M-matrices are the subset of Z-matrices for which s is greater or equal to the spectral radius of B, and nonsingular M-matrices occur when s is strictly greater than the spectral radius of B [35]. Nonsingular M-matrices have many special characteristics, those of greater importance for the research at hand are summarized in the next lemma:

Lemma 2. *[35, Theorem 2.3 (M35, H24, G20, A1, A5, N38)] For a nonsingular M-matrix $A \in \mathfrak{Z}^{n\times n}$ the following are equivalent. Moreover for a general $A \in \mathbb{R}^{n\times n}$ the first one implies the rest:*

1. *A has all positive diagonal elements and there exist a positive diagonal matrix D such that AD is strictly row diagonally dominant.*
2. *There is a positive diagonal matrix D such that $DA + A^T D$ is positive definite.*
3. *A is positive stable; that is, the real part of each eigenvalue of A is positive.*
4. *All leading principal minors of A are positive.*
5. *A is inverse positive; that is, A^{-1} exists and all of its elements are nonnegative*

Throughout the following, the first property will be simply referred to as *generalized diagonal dominance*, because only matrices with positive real diagonals were needed to prove the main result. This is compatible but less general than other better established forms of generalized diagonal dominance, such as the one that defines H-matrices (see [34, 35]) where the diagonals are not required to be positive and could be complex numbers.

There are many consequences of the previous properties that are important to understand the contents of the next chapters. For example, any matrix with positive diagonals and strict diagonal dominance also has generalized diagonal dominance.

More importantly, an M-matrix multiplied by a positive diagonal matrix results in another M-matrix. The proof for this is very simple but not obvious: suppose AD has strict diagonal dominance for a general A and a diagonal D, if A were to be multiplied by a diagonal B, the generalized diagonal dominance condition for AB asks us to find a diagonal C such that ABC has strict diagonal dominance, choose $C = B^{-1}D$ to solve the generalized diagonal dominance condition, and by equivalence AB is an M-matrix.

Another thing is that the transpose of an M-matrix is also an M-matrix, this can be deduced from Lemma 2 and the fact that square matrices and its transposes share the same eigenvalues and determinants. A direct consequence of this is that any matrix with generalized row diagonal dominance must also have generalized column diagonal dominance.

We can further deduce that for an M-matrix A there exists a positive diagonal D such that DA is strictly column diagonally dominant, since the transpose of a strictly row diagonally square matrix must be strictly column diagonally dominant.

Some of these properties are also expressed in the following lemma:

Lemma 3. *[36, Theorem 2.5.3 item 2.5.3.15] If Γ is an M-matrix with diagonal positive entries, there exist positive diagonal matrices B and $\mathscr{D}$ such that $B\Gamma\mathscr{D}$, and also $\mathscr{D}\Gamma^T B$, are both strictly row diagonally dominant and strictly column diagonally dominant.*

Apart from that, a symmetric matrix with generalized diagonal dominance is also positive definite. This comes from the fact that the eigenvalues of a symmetric matrix have no imaginary part, and from Lemma 2 we see that generalized diagonal dominance implies that the eigenvalues of the matrix have positive real part, ergo a symmetric matrix with generalized diagonal dominance has only positive real eigenvalues so it is positive definite.

2.4.1 Calculating matrix D

Given the definitions and theory presented so far, a natural question is how can one determine matrix D for a given M-matrix A such that AD is strictly diagonally dominant.

For small systems this isn't really an issue, since the inverse of an M-matrix is nonnegative (and couldn't possibly have a row of all zeros), then for some positive vector $y \in \mathbb{R}^n$ the vector $x = A^{-1}y$ is positive, this means $D = diag(x)$ is a matrix D that solves the condition. In conclusion, solving the linear system

$$Ax = y, \qquad y > 0 \tag{2.12}$$

for some M-matrix A, returns the values x of the diagonal matrix $D = diag(x)$ that makes AD strictly diagonally dominant. This is also reported in [37].

However, this requires that we previously know that A is an M-matrix, which again for small systems is easy since checking its eigenvalues, for example, gives us an answer immediately.

For larger systems determining if A were an M-matrix is a task in and of itself. If we know $A \in \mathfrak{Z}^{n\times n}$ then we can use some of the algorithms developed for identifying H-matrices, because an H-matrix in $\mathfrak{Z}^{n\times n}$ is an M-matrix.

Many such methods have been developed [37, 38, 39, 40, 41, 42]. The so-called *algorithm* $\mathbb{H}$ [37, 38] is of particular interest for it returns the matrix D we're after, in contrast, other algorithms use other H-matrix criteria to identify them, leaving the calculation of D to the side.

This algorithm can be more efficient (in terms of calculations required) than solving the linear equation (2.12) if the number of iterations required is smaller than one third of the dimension of the matrix [37]. As we can see, this will only be advantageous in really large matrices, so it will probably not be required in the majority of control applications, the important thing is to be aware that this algorithm exists and is very efficient if one is presented with a really large matrix.

Chapter 3

Homogeneous MIMO PI Controllers

"I warn the reader that this chapter requires careful reading, for I do not know the art of making myself clear to those that will not be attentive." - J. J. Rousseau ([43], book 3.1)

3.1 Relative Degree 1 Systems

Suppose an uncertain MIMO system is given by the differential equation

$$\dot{x} = G(t,x)\,u + \delta(t)\ , \tag{3.1}$$

where $x \in \mathbb{R}^m$ is the state, $u \in \mathbb{R}^m$ is the control input, $\delta(t) \in \mathbb{R}^m$ is a matched perturbation and $G(t,x) \in \mathbb{R}^{m\times m}$ is the uncertain input matrix.

The control objective is to robustly stabilize the origin of the system despite uncertainties and perturbations.

The perturbation vector signal $\delta(t)$ is supposed to be a Lipschitz function of time, so there is a constant $C \geq 0$ such that its derivative (where it exists) is bounded

$$\| \dot{\delta}(t) \|_p \leq C, \forall t \geq 0 \tag{3.2}$$

3.1.1 Description of the Uncertain Input Matrix

The following uncertain representation for the input matrix was proposed:

$$G(x,t) = \Upsilon G_0(x,t) \tag{3.3}$$

where $G_0(x,t) \in \mathbb{R}^{m\times m}$ is regular, and is the nominal and known part of the matrix, while $\Upsilon \in \mathbb{R}^{m\times m}$ is constant and regular but uncertain. This implies that $\det G(t,x) \neq 0$ for all (t,x). We also assume that $0 < c_1 \leq \|G_0(t,x)\|_p \leq c_2$ for all (t,x), for some positive constants.

It is assumed that the sign of the elements of Υ is only known for its diagonals, and that the magnitude of its elements falls under the following bounds:

$$\upsilon_{ii} \geq \underline{\upsilon}_{ii} > 0,\ \forall i = 1,\ldots,m \tag{3.4}$$

$$|\upsilon_{ij}| \leq \bar{\upsilon}_{ij},\ \forall i = 1,\ldots,m, \forall j = 1,\ldots,m \tag{3.5}$$

Additionally, the set $\mathscr{Y}$ is the set that contains all the matrices Υ that fall under these conditions. Finally, the *boundary matrix* is defined as $\mathscr{B}(\Upsilon) = [\upsilon_{b,ij}] \in \mathfrak{Z}^{m\times m}$

$$\upsilon_{b,ii} = \underline{\upsilon}_{i,i} \qquad \upsilon_{b,ij} = -\bar{\upsilon}_{ij},\ i \neq j \tag{3.6}$$

3.2 Passive Homogeneous PI Controllers

The control law can be described as

$$u = -K(t,x)\lceil x \rfloor^{\frac{1}{1-l}} + B(t,x)v \tag{3.7}$$
$$\dot{v} = -K_I(t,x)\lceil x \rfloor^{\frac{1+l}{1-l}} \tag{3.8}$$

with $l \in [-1,0]$ being the selectable homogeneous degree of the controller. The gains $K(t,x) \in \mathbb{R}^{m\times m}$, $K_I(t,x) \in \mathbb{R}^{m\times m}$, and $B(t,x) \in \mathbb{R}^{m\times m}$ are continuous and regular gain matrices to be designed.

Remark 1. *The feedback injection functions use a multivariable generalization of the scalar $\lceil\cdot\rfloor$ function. This function and the absolute value function are applied element wise to a vector $z \in \mathbb{R}^n$*

$$\lceil z \rfloor^a = \begin{bmatrix} \lceil z_1 \rfloor^a & \cdots & \lceil z_n \rfloor^a \end{bmatrix}^T, \ |z|^a = \begin{bmatrix} |z_1|^a & \cdots & |z_n|^a \end{bmatrix}^T$$

Where $\lceil z_1 \rfloor^a = |z_1|^a sign(z_1)$.

An additional requirement is that the gain matrices are bounded: $0 < c_3 \leq \|A(t,x)\|_p \leq c_4$ for all (t,x), for some positive constants, with $A \in \{K, K_I, B\}$.

The controller described previously can stabilize the origin of the extended system for $x_E \in \mathbb{R}^{2m} = \begin{bmatrix} x \\ x_I \end{bmatrix}$ with

$$x_I \in \mathbb{R}^m = v + (GB)^{-1}\delta(t) \tag{3.9}$$

are the added integral variables. The closed loop system is

$$\dot{x}_E = \begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} G(t,x)[-K(t,x)\lceil x \rfloor^{\frac{1}{1-l}} + B(t,x)x_I] \\ -K_I(t,x)\lceil x \rfloor^{\frac{1+l}{1-l}} + \Delta \end{bmatrix} \tag{3.10}$$

where

$$\Delta(t,x,\dot{x}) = \frac{d}{dt}\left(B^{-1}(t,x)\,G^{-1}(t,x)\,\delta(t)\right) \tag{3.11}$$

Remark 2. *The definition of Δ in (3.11) raises immediately the issue, that in general Δ depends on (t,x) and $\dot{x}$, even when $\delta(t)$ is assumed only to be a function of time. Since $\dot{x}$ depends on the control law, an algebraic loop appears. In order to avoid this, an additional requirement is that $G(t,x)B(t,x)$ has to be constant, that is,*

$$G(t,x)B(t,x) = \mathcal{N}, \tag{3.12}$$

where $\mathcal{N} \in \mathbb{R}^{m\times m}$ is a constant and regular matrix. If (3.12) is satisfied, then

$$\Delta = \frac{d}{dt}\left(\mathcal{N}^{-1}\delta(t)\right) = \mathcal{N}^{-1}\dot{\delta}(t)\,. \tag{3.13}$$

In this case, the boundedness of $\dot{\delta}(t)$ (3.2) implies the boundedness of Δ.

Remark 3. *Note that for $l = -1$ the closed-loop system (3.10) has a discontinuous right-hand side. In this case the solutions are understood in the sense of Filippov [44]. In the scalar discontinuous case, the classic Super-Twisting Algorithm [5] is obtained, proving (3.8) is a MIMO generalization.*

Besides that, (3.10) has a switching surface at $x = 0$, in contrast to the quasi-continuous MIMO versions in [7, 9, 12, 11], having a discontinuity only at $x_E = 0$. For $l \in (-1,0]$ (3.8) can be seen as a family of continuous approximations *of the discontinuous Super-Twisting Algorithm. For $l = 0$ it becomes a linear PI-controller. Hence why it's been called a family of homogeneous PI controllers.*

3.2.1 Gain design

Theorem 1. *Consider the system in (3.10) and suppose that all of the restrictions* (3.3)*,* (3.5) *and* (3.2) *presented previously are satisfied. Now, if the uncertain matrix* Υ *is such that its* boundary matrix *is an M-matrix, then there exists gain matrices* $K(t,x)$*,* $B(t,x)$ *and* $K_I(t,x)$ *such that the control problem can be solved. In particular:*

1. *If* $C = 0$ *and* $l = 0$ *then the linear controller makes the origin* $x_E = 0$ *a exponentially stable equilibrium point.*
2. *If* $C = 0$ *then the origin* $x_E = 0$ *is a finite-time stable equilibrium point for all homogeneity degrees* $l \in (-1,0)$.
3. *If* $C > 0$ *then for the continuous homogeneous controllers of degree* $l \in (-1,0]$ *provide input to state stability from* Δ *to* x_E.
4. *If* $C \geq 0$ *and* $l = -1$ *the discontinuous controller makes* $x_E = 0$ *finite-time stable.*

A constructive proof of Theorem 1 is given in section 3.3 of this same chapter.
When the conditions of the theorem are met, a particular but still very general selection of gains is:

- $B(t,x) = G_0^{-1}(x,t)D$ with a diagonal $D > 0$
- $K(t,x) = G_0^{-1}(x,t)K_O K_D$
- The columns of K_O satisfy the following inequality for all i:

$$2k_{ii}q_i > \left\langle |\mathbf{k}_i|, \sum_{j=1}^{m} \frac{|\hat{\mathbf{y}}_j|}{\underline{\upsilon}_{jj}} q_j \right\rangle \tag{3.14}$$

  where $\hat{\mathbf{y}}_j$ are the rows of $\mathscr{B}(\Upsilon)$ and $\mathbf{k}_i$ are the columns of K_O

  - if $l \in (-1,0)$ then K_D must be such that $\mathscr{B}(\Upsilon K_O)K_D$ is strictly row diagonally dominant
  - if $l = -1 \vee l = 0$ then K_D is an arbitrary positive diagonal matrix

- $K_I(t,x) = K_I$ is constant and positive quasi-definite

In other words, both the gains $B(t,x)$ and $K(t,x)$ contain a time varying term to decouple the nominal part of the uncertain control matrix after which additional constant gain matrices appear. The constant gain K_O contains the off-diagonal terms of the static gain, which are restricted under 3.14 to not disturb the generalized diagonal dominance of Υ, while K_D is positive diagonal matrix whose job is to appropriately scale the gain K_O, because of this, matrix K_O can always have its diagonal elements equal to one.

Similarly, the positive diagonal gain D can be used for gain scaling but it could also be seen as a part of the integral gain K_I and moved to the integral part of the controller.

Finally, the integral gain K_I is free to be any constant positive quasi-definite matrix so, together with K_O, it allows the design of full gain matrices that can be used to adjust the interaction between subsystems.

Although these conditions can look intimidating, there are several cases where the gain design is simplified. As an example, when diagonal gain matrices are desired, K_O can be equal to the identity matrix, and choosing the linear ($l = 0$) or discontinuous ($l = -1$) controller the design of K_D is also simplified. Additionally, the gain D can always be the identity matrix. All of this reduces the gain design to selecting two positive diagonal gains K_D and K_I, just like a PI controller.

Some other special cases are presented in section 3.4.

3.2.2 Gain scaling

For a given set of appropriate gains and perturbation (K, B, K_I, Δ) stabilizing the closed-loop system, it is possible to obtain a new stabilizing set by introducing a gain scaling. First select *time* and *perturbation* scaling factors $T > 0$ and $\kappa > 0$, respectively. Then, two possible gain and perturbation scalings are considered:

$$(K, B, K_I, \Delta) \to \left(LK, B, L^2 K_I, \kappa\Delta\right) \tag{3.15}$$

where $L = \left(\kappa^{-l} T^{(1+l)}\right)^{\frac{1}{1-l}}$, and

$$(K, B, K_I, \Delta) \to (LK, LB, LK_I, \kappa\Delta) \tag{3.16}$$

where $L = \kappa^{-l} T^{(1+l)}$.

The change of coordinates in state and time, given by

$$x \to \frac{T^2}{\kappa} x, \qquad x_I \to \frac{T}{\kappa} x_I, \qquad t \to Tt,$$

for (3.15), and

$$x \to \left(\frac{T}{\kappa}\right)^{1-l} x, \qquad x_I \to \frac{T}{\kappa} x_I, \qquad t \to Tt,$$

for (3.16), transform the gain scaled system into (3.10), so that they are equivalent.

And therefore, the gain scaled system can compensate a perturbation κ times larger, and it accelerates the convergence by T. For (3.15) the selected T and κ have to be constant, while for (3.16) they can be functions of (t, x). In particular, if $\kappa(t,x) \geq \bar{\kappa} > 0$ is a continuous function, uniformly bounded by a constant $\bar{\kappa} > 0$, and if $T(t,x) = \bar{T}\kappa(t,x) > 0$ with a constant $\bar{T} > 0$, then the transformation above is valid. The gain scaling becomes $L(t,x) = \kappa(t,x)\bar{T}^{(1+l)}$. This allows, for example, to deal with a perturbation Δ with a time and/or state dependent but known upper bound.

3.3 Stability Analysis

To prove Theorem 1, a Lyapunov function candidate is presented in four steps:

1. A weak Lyapunov function candidate is proposed for the unperturbed system.
2. The weak Lyapunov function is shown to also be the storage function that proves the passivity of the controller.
3. Adding a cross-term, a strong Lyapunov function candidate that accounts for perturbation is derived.
4. Conditions to make the candidate functions into true Lyapunov functions are described.

These functions were constructed by adding pondered SISO Lyaponov functions designed in [15]. For simplicity's sake, the arguments of the functions may be omitted.

3.3.1 A weak Lyapunov function

Consider the positive definite scalar function

$$\mathscr{W}(x_E) = \frac{1-l}{2} \left\lceil x^T \right\rfloor^{\frac{1}{1-l}} P \left\lceil x \right\rfloor^{\frac{1}{1-l}} + \frac{1}{2} x_I^T \Gamma x_I, \tag{3.17}$$

where $P \in \mathbb{R}^{m\times m}$ is a positive diagonal matrix and $\Gamma \in \mathbb{R}^{m\times m}$ is a constant and positive quasi-definite matrix. Notice that this scalar function is a sum of two quadratic forms, therefore it is not necessary for the matrix P to be diagonal to make it a positive definite function, it would be sufficient that P was also a positive quasi-definite matrix. Nonetheless, in the following sections a diagonal P is used, an alternative form will only be considered in section 3.4.4.

This is a homogeneous function of $d_{\mathscr{W}} = 2$ with weight vectors following the sliding mode control convention: $\mathbf{r}_1 = (1-l)\mathbf{1}_m$ and $\mathbf{r}_2 = \mathbf{1}_m$ for vectors x and x_I, respectively. It is also continuously differentiable for $l \in (-1,0]$ and Lipschitz continuous for $l = -1$.

In the case that $l = -1$ and $m = 1$ (SISO super-twisting algorithm case) then the storage function from 2.4 is obtained, and in fact this is a generalized storage function too.

The derivative of $\mathscr{W}$ along the trajectories of the closed loop in the absence of perturbations is

$$\dot{\mathscr{W}} = \lceil x^T \rfloor^{\frac{1+l}{1-l}} P\dot{x} + \frac{1}{2}\dot{x}_I^T \Gamma x_I + \frac{1}{2} x_I^T \Gamma \dot{x}_I$$

expanding the terms gets

$$\dot{\mathscr{W}} = \lceil x^T \rfloor^{\frac{1+l}{1-l}} PG\left[-K\lceil x \rfloor^{\frac{1}{1-l}} + Bx_I\right] + \frac{1}{2}\left[-K_I \lceil x^T \rfloor^{\frac{1+l}{1-l}}\right]^T \Gamma x_I + \frac{1}{2} x_I^T \Gamma [-K_I] \lceil x \rfloor^{\frac{1+l}{1-l}}$$

splitting the terms with the gain B and rearranging them results in

$$\dot{\mathscr{W}} = -\lceil x^T \rfloor^{\frac{1+l}{1-l}} PGK\lceil x \rfloor^{\frac{1}{1-l}} + \frac{1}{2}\lceil x^T \rfloor^{\frac{1+l}{1-l}}\left[PGB - K_I^T \Gamma\right] x_I + \frac{1}{2} x_I^T \left[B^T G^T P - \Gamma K_I\right] \lceil x \rfloor^{\frac{1+l}{1-l}}$$

This is because we can transpose our terms without problem since they're all scalar functions.

Finally, choosing $K_I^T \Gamma = PGB$ cancels all terms that depend on integral variables. Doing so, and again, in the absence of perturbations, will only leave

$$\dot{\mathscr{W}} = -\lceil x^T \rfloor^{\frac{1+l}{1-l}} PGK\lceil x \rfloor^{\frac{1}{1-l}}$$

Although function $\dot{\mathscr{W}}$ itself is not assured to be homogeneous (due to the dependence on time and x of G, and the matrix gains), it can be bounded by a homogeneous function of degree $d_{\dot{\mathscr{W}}} = 2 + l$. For now, it is only assumed that the first term of the function expressed above can always be made negative $\forall x \neq 0$, leading to the following bound:

$$\dot{\mathscr{W}} \leq -\varepsilon \parallel x \parallel^{2+l}$$

This means that $\mathscr{W}$ is a weak Lyapunov function. If the closed-loop system is time-invariant, then Lasalle's invariance principle will imply asymptotic stability, but not in the general case.

3.3.2 Passivity interpretation

When the assumption that the form $-\lceil x^T \rfloor^{\frac{1+l}{1-l}} PGK\lceil x \rfloor^{\frac{1}{1-l}}$ is negative definite holds, then the same thing as in section 2.2.1 can be done:

$$\begin{aligned} \nu(x) &:= \frac{1-l}{2}\lceil x^T \rfloor^{\frac{1}{1-l}} P\lceil x \rfloor^{\frac{1}{1-l}} \\ \dot{\nu}(x) &= -\lceil x^T \rfloor^{\frac{1+l}{1-l}} PGK\lceil x \rfloor^{\frac{1}{1-l}} + \lceil x^T \rfloor^{\frac{1+l}{1-l}} PGBx_I \\ &\leq \lceil x^T \rfloor^{\frac{1+l}{1-l}} PGBx_I \end{aligned}$$

so for the output $y^T = \lceil x^T \rfloor^{\frac{1+l}{1-l}} PGB$ and input $x_I = v$ (for the unperturbed case) this system is passive with the storage function $\nu(x)$. And by the same argument as for the SISO case, the interconnection with the passive integrator with storage function $\frac{1}{2}x_I^T \Gamma x_I$ results in the total system being a true passive system for $l \in (-1,0]$ and passive-like in $l = -1$.

3.3.3 A strong Lyapunov function

A cross-term is added to the weak Lyapunov function $\mathscr{W}$ to render it strong:

$$\mathscr{U}(x_E) = \mu \frac{2}{2-l} \mathscr{W}^{\frac{2-l}{2}} - x_I^T M x \tag{3.18}$$

where $M \in \mathbb{R}^{m\times m}$ is a constant regular matrix and $\mu \in \mathbb{R}_+$ is a constant positive scalar. Since $\frac{2-l}{2} \geq 1$, $\mathscr{U}$ is continuously differentiable for $l \in (-1,0]$ and Lipschitz continuous for $l = -1$. $\mathscr{U}$ is also homogeneous of degree $d_{\mathscr{U}} = 2-l$. To render $\mathscr{U}$ positive definite it is necessary to select $\mu > 0$ sufficiently large, what can be shown using the homogeneous domination lemma (Lemma 1).

The derivative is

$$\dot{\mathscr{U}} = \mu \mathscr{W}^{\frac{-l}{2}}(x_E)\dot{\mathscr{W}}(x_E)n - x_I^T M\dot{x}_n + \lceil x^T \rfloor^{\frac{1+l}{1-l}} K_I^T Mx + \rho(x_E)\Delta$$

where

$$\rho(x_E) \triangleq \frac{\mu}{2}\mathscr{W}^{\frac{-l}{2}}(x_E)x_I^T\left[\Gamma + \Gamma^T\right] - x^T M^T$$

Using the knowledge gained previously, we see it's possible to bound the first term of $\dot{\mathscr{U}}$ with a homogeneous function, this time with homogeneous degree $d_{\dot{\mathscr{U}}} = 2$, while each component of the row vector $\rho(x_E)$ is homogeneous of degree $d_\rho = 1-l$. This leads to

$$\dot{\mathscr{U}} \leq -\mu\varepsilon \parallel x \parallel^2 - \dot{x}_I^T Mx - x_I^T MG[-K\lceil x \rfloor^{\frac{1}{1-l}} + Bx_I] + \rho(x_E)\Delta$$

That being said, first the analysis for the unperturbed case (when $\Delta = 0$) is presented.

Due to the assumption that the weak Lyaponov function is already of a defined sign for $x \neq 0$, the first term of $\mathscr{U}$ can be used to dominate the sign of the whole function if the rest of the function is positive on the set $\mathfrak{S} = \left\{x_E \in \mathbb{R}^{2m} | x = 0\right\}$, then using Lemma 1[1] the whole function could be rendered positive. Incidentally, in this set and without perturbations the function is

$$\dot{\mathscr{U}}\big|_{\mathfrak{S}} = -x_I^T MG(t,0)B(t,0)x_I$$

which is positive for $x_I \neq 0$ if $MG(t,x)B(t,x)$ is positive quasi-definite.

For the perturbed case, when $\delta(t) \neq 0$, condition (3.2) together with (3.13) implies that Δ is bounded

$$\|\Delta\|_p \leq \tilde{C}$$

for some constant $\tilde{C} \geq 0$. Thus, there is a constant $\rho \geq 0$ such that

$$\rho(x_E)\Delta \leq \rho\tilde{C}\mathscr{U}^{\frac{1-l}{2-l}}(x_E)\,.$$

When $l \in (-1,0]$, $d_\rho < d_{\dot{\mathscr{U}}}$, and therefore near $x_E = 0$ the term $\rho(t,x)$ dominates and $\dot{\mathscr{U}}$ cannot be rendered negative near zero. However, for large values of (x_E) it becomes negative. And using standard Lyapunov

[1] Note that this Lemma has to be applied not directly to function $\mathscr{W}$ but to its homogeneous bounding function. This lack of precision is justified for simplicity of the presentation.

arguments, we conclude that the system (3.10) is input to state stable with respect to the input Δ. Or, equivalently, that the trajectories of the closed-loop system are ultimately and uniformly bounded [30].

When $l = -1$, then $d_\rho = d_{\dot{\mathscr{U}}}$ and $\dot{\mathscr{U}}$ is negative definite for $\tilde{C}$ sufficiently small. This can be shown again using Lemma 1 (see again footnote 1). This proves item 4) of Theorem 1. Note that the previous argument assures stability only for a perturbation bound $\tilde{C}$ sufficiently small. Using the scaling of the gains, either (3.15) or (3.16), an arbitrary size of the perturbation can be accommodated. Even a perturbation which grows with time and the state, when (3.16) is used.

3.3.4 Stability requirements and a feasible solution

In sum, the conditions to have a true Lyaponov function and prove the stability of the controllers are the following:

$$\Gamma + \Gamma^T > 0, \tag{3.19}$$

$$PG(t,x)B(t,x) = K_I^T(t,x)\Gamma \tag{3.20}$$

$$\lceil x^T \rfloor^{1+l} PG(t,x)K(t,x)x > 0, \forall x \neq 0 \tag{3.21}$$

$$z^T MG(t,x)B(t,x)z > 0, \forall x,t, \forall z \neq 0 \tag{3.22}$$

An important reminder: only B, K and K_I need to be known explicitly for the design problem. In contrast, showing that the other matrices exist even if they're unknown is enough, in other words, we may select them dependent on the uncertain matrix Υ. With this insight, a *particular* admissible solution of these relations is given by

$$B(t,x) = G_0^{-1}(t,x)D \tag{3.23}$$

$$K(t,x) = G_0^{-1}(t,x)K_O K_D \tag{3.24}$$

$$K_I(t,x) = K_I,\; K_I + K_I^T > 0 \tag{3.25}$$

with positive diagonal D, $M = D\Upsilon^T$ to solve (3.22), $\mathscr{N} = \Upsilon D$, and Γ as the solution to the algebraic Lyapunov equation below

$$[P\Upsilon D]\Gamma^{-1} + \left[\Gamma^{-1}\right]^T [P\Upsilon D]^T = K_I^T + K_I \tag{3.26}$$

Said equation is obtained by solving for the integral gain in (3.20) getting $P\Upsilon D\Gamma^{-1} = K_I^T$, thus writing the symmetric part of K_I results in a Lyapunov equation.

For conditions (3.19) and (3.20) to be solved simultaneously, assume a symmetric Γ. Now, from classic results we know a positive definite solution to (3.26) exists if $P\Upsilon D$ is always positive stable and $K_I + K_I^T$ is positive definite.

On that topic, Lemma 2 shows generalized diagonal dominance implies stability, we also know multiplying by a positive diagonal matrix won't affect this condition. So demonstrating that all $\Upsilon \in \mathscr{Y}$ have generalized diagonal dominance will prove the stability of the whole set.

Lemma 4. *All $\Upsilon \in \mathscr{Y}$ have generalized diagonal dominance if and only if $\mathscr{B}(\Upsilon)$ has generalized diagonal dominance.*

Proof. Necessity. $\mathscr{B}(\Upsilon)$ belongs to the set $\mathscr{Y}$.

Sufficiency. Since $\mathscr{B}(\Upsilon)$ has generalized diagonal dominance then $\exists D \in \mathbb{R}^{m\times m} > 0 = diag\{d_1, \ldots, d_m\}$ such that

$$\underline{\upsilon}_{i,i} d_i > \sum_{j\neq i} |-\bar{\upsilon}_{i,j}| d_j,\; i = 1, \ldots, m$$

And given the restrictions of the set $\mathscr{Y}$ we know:

$$\upsilon_{i,i}d_i \geq \underline{\upsilon}_{i,i}d_i \ \wedge \ |\upsilon_{i,j}|d_j \leq \bar{\upsilon}_{i,j}d_j = |-\bar{\upsilon}_{i,j}|d_j$$

From the inequalities presented it is possible to get that

$$\upsilon_{i,i}d_i \geq \underline{\upsilon}_{i,i}d_i > \sum_{j\neq i}\bar{\upsilon}_{i,j}d_j \geq \sum_{j\neq i}|\upsilon_{i,j}|d_j$$

Therefore, all $\Upsilon \in \mathscr{Y}$ have generalized diagonal dominance with the same D. □

In conclusion, if the boundary matrix $\mathscr{B}(\Upsilon)$ is an M-matrix, selecting K_I positive quasi-definite ensures that conditions (3.19) and (3.20) are simultaneously satisfied.

Finally, condition (3.21) becomes

$$\lceil x^T \rfloor^{1+l} P\Upsilon K_O K_D x > 0, \forall x \neq 0 \tag{3.27}$$

where $K_D \in \mathbb{R}^{m\times m}$ is the diagonal part of the static gain matrix, it is a positive diagonal matrix. Meanwhile $K_O \in \mathbb{R}^{m\times m}$ is a general matrix with diagonals equal to one and represents the off-diagonal part of the static gain. This representation can be interpreted as K_O containing the columns of the total gain matrix in terms of its diagonals, and K_D being the one who sets the magnitude of these columns.

In the next Lemma, sufficient conditions to satisfy the inequality are presented.

Lemma 5. *Suppose that $\mathscr{B}(\Upsilon K_O)$ is an M-matrix and that $l \in [-1,0]$. Then there is a positive diagonal matrices $P, K_D \in \mathbb{R}^{m\times m}$, such that (3.21) is satisfied.*

Proof. Define:

$$\Omega := P\Upsilon K_O K_D$$

This is a matrix with arbitrary elements, given that all of the matrices being multiplied are of full rank.

1. Discontinuous case

   When $l = -1$ the whole function can be expressed as:

   $$\lceil x^T \rfloor^{1+l}\Omega x = \sum_{i=1}^{m} f(x_i) = \sum_{i=1}^{m}\left(\omega_{1,i}\lceil x_1 \rfloor^0 x_i + \cdots + \omega_{i,i}|x_i|\right)$$

   Given that $\lceil x_i \rfloor^0 \in [-1,1], \forall x_i, \forall i$ one can take into account all of the possible values of these feedback injection terms at once by substituting with this multivalued set. So now we have for each term in the sum operator

   $$f(x_i) = \omega_{1,i}[-1,1]x_i + \cdots + \omega_{i,i}|x_i| \tag{3.28}$$

   which is always positive if and only if $\omega_{i,i} > \sum_{j\neq i}|\omega_{j,i}|$

   If this column diagonal dominance is met, it turns the function in question into a sum of positive numbers, ergo it is always positive.

2. nonlinear continuous case

   First we see that the cross product terms can be bounded by its absolute value:

$$-|x_i|^{1+l}|x_j| \leq \lceil x_i \rfloor^{1+l} x_j \leq |x_i|^{1+l}|x_j|$$

   Then, using Young's inequality with $q = 2+l$ and $p = \frac{2+l}{1+l}$ we can write

$$\lceil x_i \rfloor^{1+l} x_j \geq -\frac{1+l}{2+l}|x_i|^{2+l} - \frac{1}{2+l}|x_j|^{2+l}$$

   and multiplying by their associated coeficient

$$|\omega_{i,j}|\lceil x_i \rfloor^{1+l} x_j \geq -|\omega_{i,j}|\frac{1+l}{2+l}|x_i|^{2+l} - |\omega_{i,j}|\frac{1}{2+l}|x_j|^{2+l}$$

   Notice how the terms have been separated into *row terms* and *column terms* because the variables are no longer mixed. We can now rearrange the terms as

$$\lceil x^T \rfloor^{1+l}\Omega x \geq \sum_{i=1}^{m} f(x_i)$$

   this time defining $f(x_i)$ as

$$f(x_i) := \left(\omega_{ii} - \frac{1+l}{2+l}\sum_{j \neq i}|\omega_{i,j}| - \frac{1}{2+l}\sum_{j \neq i}|\omega_{j,i}|\right)|x_i|^{2+l}$$

   Furthermore, since $\frac{1+l}{2+l} + \frac{1}{2+l} = 1$ we can give the *diagonal terms* the same treatment and separate the function according to these coeficients too. This is

$$f(x_i) = \frac{1+l}{2+l}\left(\omega_{ii} - \sum_{j \neq i}|\omega_{i,j}|\right)|x_i|^{2+l} + \frac{1}{2+l}\left(\omega_{ii} - \sum_{i \neq j}|\omega_{i,j}|\right)|x_i|^{2+l} \tag{3.29}$$

   Finally we make a similar argument as in the previous case and declare that if Ω were to have not only column but also row diagonally dominant then these functions would all be positive and the total function would be a sum of positive real numbers, thus fulfilling the condition.

   This is consistent with previous results given that, although Young's inequality doesn't hold for $l = -1$ since that would imply $q \leq 1$, the limit of condition (3.29) when $l \to -1$ is column diagonal dominance.

3. Linear case

   When $l = 0$ the function turns into an all too familiar quadratic form

$$\lceil x^T \rfloor^{1+l}\Omega x = x^T \Omega x$$

   It is known that to make this function positive $\forall x \neq 0$ it's both necessary and sufficient for Ω to be positive quasi-definite, this is:

$$\Omega + \Omega^T > 0 \tag{3.30}$$

   This is also consistent with the previous result, for it is known that a symmetric strictly diagonal dominant matrix with positive diagonals is always positive definite. So if Ω were strictly diagonally dominant in rows and columns, then the sum of $\Omega + \Omega^T$ would be symmetric and strictly diagonally dominant.

For all of the cases presented it is sufficient that $\mathscr{B}(\Upsilon K_O)$ is an M-matrix since by Lemma 2 there exists positive diagonal matrices that can make the M-matrix strictly diagonally dominant, and it is also possible to choose them such that the M-matrix is positive quasi-definite.

In summary, under the hypothesis of the Lemma:

- For $l=-1$, there always exists P can make $P\mathscr{B}(\Upsilon K_O)K_D$ strictly column diagonally dominant.
- For $l\in(-1,0)$ there always exists P and K_D that make $P\mathscr{B}(\Upsilon K_O)K_D$ strictly diagonally dominant in columns and rows respectively.
- And for $l=0$ there always exists P that makes $P\mathscr{B}(\Upsilon K_O)K_D$ positive quasi-definite.

□

Notice that there is a gap in the conditions between homogeneity degrees. In the linear and discontinuous case, necessary conditions have been found for the form in question to always be positive, nevertheless, these conditions are completely different (though they may be satisfied at the same time).

It would be expected that the conditions exists in a spectrum and that if one were to go from $l=-1$ to $l=0$ column diagonal dominance slowly loses to positive definiteness, however this is not confirmed.

Apart from that, from the previous proof a natural question arises: How can we choose K_O to preserve the M-matrix condition for $\mathscr{B}(\Upsilon K_O)$?

To solve this query, Lemma 4 is key, since it shows that it's possible to find a diagonal matrix that makes a whole set of bounded matrices strictly diagonally dominant by only studying their boundary matrix. To find the bounds of the elements of ΥK_O, the matrices in question are written in the form of vectors:

$$K_O := \begin{bmatrix}\mathbf{k}_1 & \cdots & \mathbf{k}_m\end{bmatrix}, \qquad \Upsilon := \begin{bmatrix}\mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_m^T\end{bmatrix} \tag{3.31}$$

And $\Lambda = [\lambda_{ij}] \in \mathbb{R}^{m\times m}$ is defined as

$$[\lambda_{ij}] := \langle \mathbf{y}_i, \mathbf{k}_j\rangle \tag{3.32}$$

where $\langle\cdot,\rangle$ denotes the inner product.

For the diagonals, we want them to be positive, and the worse case scenario would be that all of the off-diagonal terms collude with each other to become negative, this is:

$$\lambda_{ii} \geq k_{ii}\upsilon_{ii} - \sum_{j\neq i}|k_{ij}||\upsilon_{ji}| = 2k_{ii}\upsilon_{ii} - \langle|\mathbf{y}_i|,|\mathbf{k}_i|\rangle \tag{3.33}$$

and if we allow $k_{ii}=1$ to simplify the analysis and let K_D scale the diagonals later we get:

$$\lambda_{ii} \geq 2\upsilon_{ii} - \langle|\mathbf{y}_i|,|\mathbf{k}_i|\rangle \tag{3.34}$$

For the off-diagonal terms, the worse case scenario is that all of the terms of the inner product have the same sign:

$$|\lambda_{ij}| \leq \langle|\mathbf{y}_i|,|\mathbf{k}_j|\rangle \tag{3.35}$$

We could now write the condition for generalized diagonal dominance and see what's needed to satisfy the M-matrix condition. Here, the inequality for column diagonal dominance is used:

$$(2\upsilon_{ii} - \langle|\mathbf{y}_i|,|\mathbf{k}_i|\rangle)p_i > \sum_{i\neq j}\langle|\mathbf{y}_j|,|\mathbf{k}_i|\rangle p_j \tag{3.36}$$

Moving the second term to the righthand side, and defining $p_i = \dfrac{q_i}{\upsilon_{ii}}$ gives us:

$$2q_i > \left\langle |\mathbf{k}_i|, \sum_{j=1}^{m} \frac{|\mathbf{y}_j|}{\upsilon_{jj}} q_j \right\rangle \tag{3.37}$$

where $|\mathbf{k}_i|$ has been factorized.

Having the inequality in this form allows us to see that the worst case scenario for Λ is indeed when the diagonals of Υ are equal to their lower bounds, since the smallest diagonal elements will lead to the biggest value of each term in the sum above; and without a question the biggest off diagonal elements are the most problematic ones.

All of this implies that the vectors of $\mathscr{B}(\Upsilon)$ are sufficient to calculate the bounds of Λ and find its boundary matrix, no additional bounds or other information needs to be known. This last fact is how we get the inequality:

$$2q_i > \left\langle |\mathbf{k}_i|, \sum_{j=1}^{m} \frac{|\hat{\mathbf{y}}_j|}{\underline{\upsilon}_{jj}} q_j \right\rangle \tag{3.38}$$

where $q_i > 0$ are the elements of an arbitrary positive diagonal matrix $Q \in \mathbb{R}^{m \times m}$ and $\hat{\mathbf{y}}_j^T \in \mathbb{R}^m$ are the rows of $\mathscr{B}(\Upsilon)$.

Under the hypothesis that $\mathscr{B}(\Upsilon)$ is an M-matrix then it's always possible to select K_O equal to the identity matrix to solve the inequalities and an appropriate Q will exist and is no longer required to be known explicitly to design the gains, most importantly this selection allows the selection of arbitrary K_D for the linear and discontinuous cases, where strict row diagonal dominances is not necessary.

One may think it is possible to choose K_O to make $\mathscr{B}(\Upsilon K_O)$ into an M-matrix even when $\mathscr{B}(\Upsilon)$ didn't already fulfil the condition. Unfortunately, it seems this is only possible if Υ were a known matrix.

Proof. Contradiction: Suppose an K_O that turns $\Lambda = \Upsilon K_O$ into M-matrices $\forall \Upsilon \in \mathscr{Y}$. Now from Lemma 2 we know that all of the principal minors of Λ are positive, this implies that their determinant is $det(\Lambda) > 0$ and thus $sign(det(K_O)) = sign(det(\Upsilon))$.

Now suppose $\mathscr{B}(\Upsilon)$ is not an M-matrix, then there is a subset $\mathscr{Y}^- \subset \mathscr{Y} = \{\Upsilon \in \mathscr{Y} | det(\Upsilon) < 0\}$ and its complementary subset $\mathscr{Y}^+ \subset \mathscr{Y} = \{\Upsilon \in \mathscr{Y} | det(\Upsilon) > 0\}$.

If sign of $det(K_O)$ is constant, then there exists some Λ with negative determinant $\Rightarrow\Leftarrow$

If the sign of $det(K_O)$ is always equal to the sign of $det(\Upsilon)$, then the restriction that the gains can only be dependent on the bounds of Υ is violated, K_O would be designed as a function of the uncertain matrix $\Rightarrow\Leftarrow$ □

Notice that none of the restrictions presented lock K_O or K_D into being constant, but if they were time varying, they'd need to preserve the bounds for K_O and the *direction* of K_D for all time.

It is inferred that, when the off-diagonal terms of $\mathscr{B}(\Upsilon)$ are smaller, the off-diagonals of K_O can be bigger. Indeed, in the extreme case that $\mathscr{B}(\Upsilon)$ is a diagonal matrix (3.38) turns into the generalized diagonal dominance condition for K_O, this implicitly tells us that it must always be generalized diagonal dominance.

3.4 Special Cases

3.4.1 Systems with decoupled uncertainties or no uncertainties

When one has a good understanding of the systems and the model has no uncertainties, then the matrix $\Upsilon = \mathbb{I}_m$. On top of that, if one could be sure that the uncertainties of the model are decoupled, meaning Υ is a diagonal matrix, then the matrix P can completely compensate for the model uncertainties since

$$P = Q\Upsilon^{-1} \tag{3.39}$$

with an arbitrary positive diagonal Q still makes P a positive diagonal matrix, so no conditions are broken.

With that out of the way, to satisfy the design restrictions one can select the gains K_I, $K(t,x)$ and $B(t,x)$ the same way as in the general case (see equations (3.23) to (3.25)) and then design D with generalized diagonal dominance, and K_O as an arbitrary strictly diagonally dominant matrix in both rows and columns and positive diagonals with $K_D = \mathbb{I}_m$. In other words, it is not necessary to solve the inequalities in (3.14) even if a coupled gain matrices are required.

Please notice that this set of solutions includes any arbitrary positive diagonal gains K_I, K_O and K_D for any homogeneity degree.

Besides that, in the linear case $K(t,x) = G_0^T(t,x)$ always achieves the positive definite condition thus satisfying (3.21).

3.4.2 Time invariant known input matrix

When there are no uncertainties (or they're decoupled, see 3.4.1) and $G(t,x) = G$ is a constant known matrix, the control algorithm proposed really shines. The gains can be designed as

- $B(t,x) = \mathbb{I}_m$
- $K_I(t,x) = K_I$ is constant and positive quasi-definite
- $K(t,x) = K_O K_D$
- K_O is such that GK_O has generalized diagonal dominance
  - if $l \in (-1,0)$ then K_D must be such that $GK_O K_D$ is strictly row diagonally dominant
  - if $l = -1 \vee l = 0$ then K_D is an arbitrary positive diagonal matrix

With this selection the matrix P exists such that condition (3.21) is satisfied, since by Lemma 2 it could make the whole form strictly column diagonally dominant for $l \in [-1,0)$ or positive quasi-definite for $l = 0$.

Notice that in this nominal case no decoupling of any kind is required.

3.4.3 Time invariant unknown input matrix

When the matrix $G(t,x) = G$ is a time invariant matrix, the system doesn't really need to be decoupled. Not much knowledge of the system is required and the uncertain description could be

$$G_0 \in \mathbb{R}^{m\times m} = diag\{sign(g_{ii})\}, \qquad \Upsilon = GG_0^{-1} \tag{3.40}$$

in other words, only the sign of the diagonals of G are assumed to be known.

With that selection an Υ with positive diagonal elements is obtained and if the boundary matrix, defined in (3.6), is an M-matrix then the rest of the general solution with (3.23) to (3.25) can be designed with no more knowledge than the bounds of G.

If this selection of G_0 doesn't lead to a stable family of $\Upsilon \in \mathscr{Y}$ then it may still be possible to find a different G_0 that cancels the instability of the input matrix, then the general solution can be used. However, this is not guaranteed.

3.4.4 Symmetric uncertainties

If in addition to knowing that Υ is an M-matrix it is known that this matrix is symmetric, then it is possible to show that uncertainties can be completely compensated. Recalling the weak Lyaponov function in (3.17), it was mentioned that the matrix P wasn't locked into being a positive diagonal matrix. Here a symmetric positive definite P is considered.

If for some reason it is possible to guarantee that all $\Upsilon \in \mathscr{Y}$ are symmetric and the main hypothesis that the boundary matrix defined in (3.6) is an M-matrix holds, then by Lemma 4 the whole set $\mathscr{Y}$ is composed of generalized diagonally dominant matrices, and by hypothesis they are also symmetric.

What is more, a symmetric generalized diagonally dominant matrix is also positive definite so in fact, under the hypothesis of symmetry, all $\Upsilon \in \mathscr{Y}$ are positive definite matrices. Finally since the inverse of a positive definite matrix is also positive definite, it is possible to then define

$$P = \Upsilon^{-1} \tag{3.41}$$

In conclusion, the matrix P is capable of completely compensating positive definite uncertainties. However, contrary to all of the other cases presented previously, here P cannot be used to achieve the column diagonal dominance condition, since in general a positive diagonal matrix times a positive definite matrix is not positive definite, so adding an extra positive diagonal matrix to P (like in (3.39)) is not possible.

In this case to satisfy the design restrictions (3.19), (3.20), (3.21) and (3.22) with the general solution in equations (3.23), (3.24) and (3.25) the matrix D could be any matrix with generalized diagonal dominance, K_O must be strictly column diagonally dominant, K_D arbitrary for $l = -1 \vee l = 0$ or such that $K_O K_D$ is strictly row diagonally dominant for $l \in (-1, 0)$, and finally K_I can be any positive quasi-definite matrix.

Although this case seems more general than the case for diagonal uncertainties presented previously in section 3.4.1, a similar extension into linear time invariant systems like the one done in section 3.4.2 is more restrictive. In fact, even if G_0 is constant and known but it isn't already strictly column diagonally dominant it is not possible to satisfy the restriction (3.21) for $l \neq 0$ without decoupling, for here the matrix P cannot be used to make G strictly column diagonally dominant.

Even so, decoupling shouldn't be a problem in this case since the uncertainties of the input matrix are cancelled and even if interaction between subsystems is a design requirement, the proposed control scheme enables the designer to choose the strength of these interactions via the off-diagonal terms of gain matrices.

Chapter 4

An Arbitrary Relative Degree Generalization

4.1 General MIMO Normal From

Consider a coupled $m \times m$ MIMO system with the outputs vector $y \in \mathbb{R}^m$ being

$$y_1 = x_{11}, \quad \dots \quad , y_m = x_{m,1} \tag{4.1}$$

each output has a well defined relative degree n_i and the dynamics of each subsystem $x_i \in \mathbb{R}^{n_i} = [x_{i1}, \dots, x_{i,n_i}]^T$ are given by

$$\dot{x}_i = \begin{bmatrix} \dot{x}_{i,1} = x_{i,2} \\ \vdots \\ \dot{x}_{i,n_i} = g_i^T(t,x)u \end{bmatrix} \tag{4.2}$$

where $g_i^T(t,x) \in \mathbb{R}^{1 \times m}$ is the coupled input matrix for each subsystem.

For the vector $x = [x_1, \dots, x_m]^T$ the dynamics can be written as

$$\dot{x} = Ax + \gamma(t,x)u \tag{4.3}$$

where the matrix $A \in \mathbb{R}^{n \times n}$ is a block diagonal with m blocks $A_i \in \mathbb{R}^{n_i \times n_i}$

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.4}$$

While matrix $\gamma(t,x) \in \mathbb{R}^{n \times m}$ is expressed as

$$\gamma(t,x) \in \mathbb{R}^{n \times m} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{bmatrix} \qquad \gamma_i \in \mathbb{R}^{n_i \times m} = \begin{bmatrix} \mathbf{0}_m^T \\ \vdots \\ g_i^T(t,x) \end{bmatrix} \tag{4.5}$$

Defining the vector

$$x_n = \begin{bmatrix} x_{1,n} \\ x_{2,n} \\ \vdots \\ x_{m,n} \end{bmatrix} := \begin{bmatrix} x_{1,n_1} \\ x_{2,n_2} \\ \vdots \\ x_{m,n_m} \end{bmatrix} \tag{4.6}$$

and focusing on its dynamics, it is possible to write

$$\dot{x}_n = G(x,t)u + \delta, \qquad G(x,t) = \begin{bmatrix} g_1^T(t,x) \\ \vdots \\ g_m^T(t,x) \end{bmatrix} \tag{4.7}$$

where $G(x,t) \in \mathbb{R}^{m\times m}$ is the input matrix and is assumed to have the property of $det\, G(x,t) \neq 0, \forall t, x$, and $\delta \in \mathbb{R}^m$ is the perturbations vector.

The perturbation vector signal $\delta(t)$ is supposed to be a Lipschitz function of time, so there is a constant $C \geq 0$ such that its derivative (where it exists) is bounded

$$\left\|\dot{\delta}(t)\right\|_p \leq C, \forall t \geq 0. \tag{4.8}$$

Note that the rest of the state variables' dynamics are integrator chains.

The *control objective* is to design a control law for u, such that $u(t)$ is a continuous function of time and renders $x = 0$ robustly asymptotically stable, in spite of the perturbations $\delta(t)$ and the uncertainties in the input matrix.

4.2 Uncertain Description of the Input Matrix

Consider the following uncertain representation for our input matrix:

$$G(x,t) = \Upsilon G_0(x,t) \tag{4.9}$$

where $G_0(x,t) \in \mathbb{R}^{m\times m}$ is regular, and is the nominal and known part of the matrix, while $\Upsilon \in \mathbb{R}^{m\times m}$ is constant and regular but uncertain under the bounds:

$$\upsilon_{ii} \geq \underline{\upsilon}_{ii} > 0,\ \forall i = 1,...,m \tag{4.10}$$

$$|\upsilon_{ij}| \leq 0,\ \forall i = 1,...,m,\ \forall j = 1,...,m \tag{4.11}$$

And the set $\mathscr{Y}$ is the set that contains all the matrices Υ that fall under these conditions. Finally, the *boundary matrix* is defined as $\mathscr{B}(\Upsilon) = [\upsilon_{b,ij}] \in \mathfrak{Z}^{m\times m}$

$$\upsilon_{b,ii} = \underline{\upsilon}_{i,i} \qquad \upsilon_{b,ij} = -\bar{\upsilon}_{ij},\ i \neq j \tag{4.12}$$

4.3 Passive Homogeneous Controllers

The control law can be described as

$$u = -K(t,x)\sigma_n(x) + B(t,x)v \tag{4.13}$$

$$\dot{v} = -K_I(t,x)\iota_n(x) \tag{4.14}$$

These are MIMO generalizations for the SISO passivity based controller in [15].This means that

$$\sigma_n(x) \in \mathbb{R}^m = \begin{bmatrix} \lceil \sigma_{n_1}(x_1) \rfloor^{\frac{1}{r_{11}}} \\ \vdots \\ \lceil \sigma_{n_m}(x_m) \rfloor^{\frac{1}{r_{m1}}} \end{bmatrix}, \qquad \iota_n(x) \in \mathbb{R}^m = \begin{bmatrix} \iota_{n_1}(x_1) \\ \vdots \\ \iota_{n_m}(x_m) \end{bmatrix} \tag{4.15}$$

The iota functions are composed of functions from the original work:

$$\iota_{n_i}(x_i) = V_{n_i}^{\frac{2}{1+r_1}-1} W_{n_i}^{\frac{1+r_1}{1+r_{i,n_i}}-1} \sigma_{n_i}(x_i) \tag{4.16}$$

These functions are described in detain in the next section 4.3.1.

The controller is homogeneous in nature (for time varying gains that preserve this property) and the weights are defined as

$$\mathbf{r}_S \in \mathbb{R}^n = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} \qquad r_i \in \mathbb{R}^{n_i} = \begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{n_i} \end{bmatrix} := \begin{bmatrix} 1 - n_i \times l \\ \vdots \\ 1-2l \\ 1-l \end{bmatrix} \tag{4.17}$$

with $l \in [-1,0]$ being the selectable homogeneous degree of the controller.

The controller is also dynamic, since an integral variable is added for each subsystem, the weights assigned to these variables are

$$\mathbf{r}_I \in \mathbb{R}^m = \mathbf{1}_m \tag{4.18}$$

This way the weights for the first variables are the largest and decrease until 1 for the added integral variable. Moreover, the weight of every $x_{i,n}$ is $r_{n_i} = 1-l$.

Additionally, the iota functions now always have an homogeneous degree of $1+l$ while the sigma functions always have a homogeneous degree of 1.

The coupled controller described previously can stabilize the origin of the extended system for $x_E \in \mathbb{R}^{n+m} = \begin{bmatrix} x \\ x_I \end{bmatrix}$ where $x_I \in \mathbb{R}^m = v + (GB)^{-1}\delta(t)$ are the added integral variables. The closed loop system is

$$\dot{x}_E = \begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} Ax + \gamma(t,x)[-K\sigma_n(x) + Bx_I] \\ -K_I \iota_n(x) + \Delta \end{bmatrix} \tag{4.19}$$

where

$$\Delta(t,x,\dot{x}) = \frac{d}{dt}\left(B^{-1}(t,x)\, G^{-1}(t,x)\, \delta(t)\right). \tag{4.20}$$

In particular, the closed loop of x_n is

$$\dot{x}_n = G(x,t)[-K\sigma_n(x) + Bx_I] \tag{4.21}$$

and the rest of the dynamics of the original states are integrator chains.

Recalling remark 2, B is restricted to being such that $GB = \mathscr{N}$ is a constant matrix, so $\Delta := \mathscr{N}^{-1}\dot{\delta}(t)$.

Again, for $l = -1$ the closed-loop system (4.19) has a discontinuous right-hand side. In this case the solutions are understood in the sense of Filippov [44].

In the relative degree one case, the classic homogeneous PI controllers of the previous chapter are obtained, and thus (4.14) is an arbitrary relative degree generalization.

4.3.1 Auxiliary functions

First of all, the auxiliary functions will be defined in terms of the arbitrary vector $z \in \mathbb{R}^{n_i}$ that is used to represent any vector $x_i \in \mathbb{R}^{n_i}$ with $i = 1, 2, ..., m$ with the same homogeneous weights r_i. This is only to simplify the presentation, avoiding the use of more subscripts.

If one were to read the original work for the SISO case, one would notice that the functions were presented in a way that allowed for polynomial generalizations, here we've presented only the case when $\alpha_i = r_1, \forall i$ and $m = 2r_1$.

The sigma function is defined as

$$\sigma_1(z_1) = z_1, j = 1 \tag{4.22}$$

$$\sigma_j(z) = \lceil z_j \rfloor^{\frac{r_1}{r_j}} + \beta_{j-1}^{\frac{r_1}{r_j}} \sigma_{j-1}(z), j = 2, 3, ..., n_i \tag{4.23}$$

and the gains $\beta_{j-1} > 0$ will be referred as *internal gains* of the sigma function.

To define the iota functions we must define each of their components. For all $j \geq 2$ we define

$$W_j(z) = \frac{r_j}{r_1 + r_j} |z_j|^{\frac{r_1+r_j}{r_j}} + \beta_{j-1}^{\frac{r_1}{r_j}} \sigma_{j-1}(z) z_j + \frac{r_j}{r_1 + r_j} \beta^{\frac{r_1+r_j}{r_j}} |\sigma_{j-1}(z)|^{\frac{r_1+r_j}{r_j}} \tag{4.24}$$

and finally with this we define

$$V_1(z_1) = \frac{1}{2}|z_1|^2, j = 1 \tag{4.25}$$

$$V_j(z) = \frac{r_1 + r_j}{2r_1} W_j^{\frac{2r_1}{r_1+r_j}}(z) + \nu_{j-1} V_{j-1}(z), j = 2, 3, ..., n_i \tag{4.26}$$

$$\tag{4.27}$$

with an arbitrary $\nu_j > 0$.

The derivative along the trajectories of these last functions will be required for the stability analysis, it's written in terms of more auxiliary functions that will be also defined here.

$$\dot{V}_{n_i}(z) = W_{n_i}^{\frac{2r_1}{r_1+r_{n_i}}-1}(z) \dot{W}_{n_i}(z) + \nu_{n_i-1} H_{n_i-1}(z) + \nu_{n_i-1} \partial_{z_{n_i-1}} V_{n_i-1}(z) s_{n_i}(z) \tag{4.28}$$

where functions H and s are defined as follows:

$$s_j(z) = z_j + \beta_{j-1} \lceil \sigma_{j-1}(z) \rfloor^{\frac{r_j}{r_1}} \tag{4.29}$$

$$H_1(z_1) = -\beta_1 \partial_{z_1} V_1(z_1) \lceil \sigma_{j-1}(z) \rfloor^{\frac{r_2}{r_1}} \tag{4.30}$$

$$H_j(z_j) = \sum_{q=1}^{j-1} \left(\partial_{z_q} V_j(z_j) z_{q+1} \right) - \beta_j \partial_{z_j} V_j(z_j) \lceil \sigma_j(z) \rfloor^{\frac{r_{j+1}}{r_1}} \tag{4.31}$$

and also

$$\dot{W}_{n_i}(z) = \sigma_{n_i}(z) \dot{z}_{n_i} + \beta_{n_i-1}^{\frac{r_1}{r_{n_i}}} |\sigma_{n_i-1}(z)|^0 s_n(z) \dot{\sigma}_{n_i-1}(z) \tag{4.32}$$

4.3.2 Gain design

Theorem 2. *Consider the system in (4.19) and suppose that all of the restrictions presented previously are satisfied. Now, if the uncertain matrix Υ is such that its boundary matrix is a diagonal matrix, then there exists gains such that the control problem can be solved.*

1. *If $C = 0$ and $l = 0$ then the linear controller makes the origin $x_E = 0$ a exponentially stable equilibrium point.*

2. *If $C = 0$ then the origin $x_E = 0$ is a finite-time stable equilibrium point for all homogeneity degrees $l \in (-1, 0)$.*

3. *If $C > 0$ then for the continuous homogeneous controllers of degree $l \in (-1, 0]$ provide input to state stability from Δ to x_E.*

4. *If $C \geq 0$ and $l = -1$ the discontinuous controller makes $x_E = 0$ finite-time stable.*

A constructive proof of Theorem 2 is given in section 4.4 of this same chapter.
When the conditions of the theorem are met, a particular but still very general selection of gains is:

- $B(t,x) = G_0^{-1}(x,t)D$ with a generalized diagonally dominant D
- $K(t,x) = G_0^{-1}(x,t)K_O$
- $K_I(t,x) = K_I$ is constant and positive quasi-definite

One may notice that this is a generalization of the special case presented in 3.4.1, where only diagonal uncertainty matrices are considered. Woefully, sufficient conditions for a more general case haven't been found, so the arbitrary degree generalization was limited to this special case.

4.3.3 Gain scaling

The same gain scaling proposed before can be done here.

For a given set of appropriate gains and perturbation (K, B, K_I, Δ) stabilizing the closed-loop system, it is possible to obtain a new stabilizing set by introducing a gain scaling. First select *time* and *perturbation* scaling factors $T > 0$ and $\kappa > 0$, respectively. Then, two possible gain and perturbation scalings are considered:

$$(K, B, K_I, \Delta) \rightarrow \left(LK, B, L^2 K_I, \kappa\Delta\right) \tag{4.33}$$

where $L = \left(\kappa^{-l} T^{(1+l)}\right)^{\frac{1}{1-l}}$, and

$$(K, B, K_I, \Delta) \rightarrow (LK, LB, LK_I, \kappa\Delta) \tag{4.34}$$

where $L = \kappa^{-l} T^{(1+l)}$.

The change of coordinates in state and time, given by

$$x \rightarrow \frac{T^2}{\kappa} x,\ x_I \rightarrow \frac{T}{\kappa} x_I,\ t \rightarrow Tt,$$

for (4.33), and

$$x \rightarrow \left(\frac{T}{\kappa}\right)^{1-l} x,\ x_I \rightarrow \frac{T}{\kappa} x_I,\ t \rightarrow Tt,$$

for (4.34), transform the gain scaled system into (4.19), so that they are equivalent.

And therefore, the gain scaled system can compensate a perturbation κ times larger, and it accelerates the convergence by T. For (4.33) the selected T and κ have to be constant, while for (4.34) they can be functions of (t,x). In particular, if $\kappa(t,x) \geq \bar{\kappa} > 0$ is a continuous function, uniformly bounded by a constant $\bar{\kappa} > 0$, and if $T(t,x) = \bar{T}\kappa(t,x) > 0$ with a constant $\bar{T} > 0$, then the transformation above is valid. The gain scaling becomes $L(t,x) = \kappa(t,x)\bar{T}^{(1+l)}$. This allows, for example, to deal with a perturbation Δ with a time and/or state dependent but known upper bound.

4.4 Stability Analysis

To prove Theorem 2, a Lyapunov function candidate is presented in three steps:

1. A weak Lyapunov function candidate is proposed.
2. Adding a cross-term, a strong Lyapunov function candidate is derived.
3. Conditions to make the candidate functions into true Lyapunov functions are derived

These functions were constructed by adding pondered SISO Lyaponov functions designed in [15].

4.4.1 A weak Lyapunov function

Consider the positive definite scalar function

$$\mathscr{W} = \sum_{i=1}^{m}\left(p_i r_{i1} V_{n_i}^{\frac{1}{r_{i1}}}(x_i)\right) + \frac{1}{2}x_I^T \Gamma x_I \tag{4.35}$$

Where $p_i > 0,\ \forall i$ and $\Gamma \in \mathbb{R}^{m\times m}$ is symmetric and positive definite. This function can be bounded by a homogeneous function of degree $d_{\mathscr{W}} = 2$.

The derivative of the fist terms is

$$\frac{d}{dt}\left(r_{i1} V_{n_i}^{\frac{1}{r_{i1}}}(x_i)\right) = V_{n_i}^{\frac{1}{r_{i1}}-1}(x_i)\dot{V}_{n_i}(x_i)$$

From the analysis in [15] we know that completely expanding the terms then gets

$$\begin{aligned}
\dot{V}_{n_i}(x_i) = \quad & W_{n_i}^{\frac{2r_{i1}}{r_{i1}+r_{n_i}}-1}(x_i)\sigma_{n_i}(x_i)\dot{x}_{i,n} \\
+ \quad & W_{n_i}^{\frac{2r_{i1}}{r_{i1}+r_{n_i}}-1}(x_i)\beta_{n_i-1}^{\frac{r_{i1}}{r_{n_i}}}|\sigma_{n_i-1}(x_i)|^0 s_n(x_i)\dot{\sigma}_{n_i-1}(x_i) \\
+ \quad & \nu_{n_i-1}H_{n_i-1}(x_i) + \nu_{n_i-1}\partial_{x_{n_i-1}}V_{n_i-1}(x_i)s_{n_i}(x_i)
\end{aligned}$$

Except for when $\dot{x}_{i,n}$ appears, no other term depends on variables from the other subsystems, only on variables of their own system. Additionally, those terms are of homogeneous degree $2+l$.

Contrary to the SISO analysis where the whole function is bounded by a negative scalar and the homogeneous norm to the $2+l$ power, here instead only the decoupled terms are bounded resulting in

$$V_{n_i}^{\frac{1}{r_{i1}}-1}(x_i)\dot{V}_{n_i}(x_i) \leq V_{n_i}^{\frac{1}{r_{i1}}-1}(x_i)W_{n_i}^{\frac{2r_{i1}}{r_{i1}+r_{n_i}}-1}(x_i)\sigma_{n_i}(x_i)\dot{x}_{i,n} + \eta_i \parallel x_i \parallel^{2+l}$$

And using the definition of $\iota_{n_i}(x_i)$ in (4.16), simple substitution results in

$$V_{n_i}^{\frac{1}{r_{i1}}-1}(x_i)\dot{V}_{n_i}(x_i) \leq \iota_{n_i}(x_i)\dot{x}_{i,n} + \eta_i \parallel x_i \parallel^{2+l} \tag{4.36}$$

Returning to the full Lyapunov function, it will be clearer now that its derivative is

$$\dot{\mathscr{W}} \leq \iota_n^T(x)P\dot{x}_n + \langle \mathbf{p}, \eta(x)\rangle + x_I^T\Gamma\dot{x}_I$$

where $P = \text{diag}\{p_1, \cdots, p_m\}$ is a diagonal matrix with elements p_i and

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix}, \qquad \eta(x) = \begin{bmatrix} \eta_1 \parallel x_1 \parallel^{2+l} \\ \vdots \\ \eta_m \parallel x_m \parallel^{2+l} \end{bmatrix}$$

This then leads to

$$\dot{\mathscr{W}} \leq \quad -\iota_n^T(x) PGK\sigma_n(x) + \iota_n^T(x) PGBx_I + \langle \mathbf{p}, \eta(x) \rangle - x_I^T \Gamma K_I \iota_n(x)$$

Just as in the relative degree one case, choosing $K_I^T \Gamma^T = PGB$ cancels all terms that depend on integral variables, only leaving

$$\dot{\mathscr{W}} = -\iota_n^T(x) PGK\sigma_n(x) + \langle \mathbf{p}, \eta(x) \rangle$$

Through the proper selection of K the sign of the first term can dominate the sign of the second term, since both are homogeneous functions of degree $d_{\mathscr{W}} = 2 + l$, what's not clear yet is if the first term has a definite sign. For now, it is only assumed that the first term of the function expressed above can always be made negative $\forall x \neq 0$ and can then be used to homogeneously dominate the second term, leading to the following bound:

$$\dot{\mathscr{W}} \leq -\varepsilon \parallel x \parallel^{2+l}$$

This means that $\mathscr{W}$ is a weak Lyapunov function. If the closed-loop system is time-invariant, then Lasalle's invariance principle will imply asymptotic stability, but not in the general case.

Note that in the passivity interpretation, $\mathscr{W}$ is a storage function for the interconnected system. This can be deduced by performing a similar analysis as in 3.3.2, here the passive output would be $y_p^T = \iota_n^T(x) PGB$.

4.4.2 A strong Lyapunov function

A cross-term is added to the weak Lyapunov function $\mathscr{W}$ to render it strong:

$$\mathscr{U} = \mu \frac{2}{2-l} \mathscr{W}^{\frac{2-l}{2}} - x_I^T M x_n \tag{4.37}$$

where $M \in \mathbb{R}^{m \times m}$ is a constant regular matrix and $\mu \in \mathbb{R}_+$ is a constant positive scalar. Since $\frac{2-l}{2} \geq 1$, $\mathscr{U}$ is continuously differentiable for $l \in (-1, 0]$ and Lipschitz continuous for $l = -1$. $\mathscr{U}$ is also bounded by a homogeneous function, in this case of degree $d_{\mathscr{U}} = 2 - l$. To render $\mathscr{U}$ positive definite it is necessary to select $\mu > 0$ sufficiently large, what can be shown using the homogeneous domination lemma.

The derivative is

$$\dot{\mathscr{U}} = \mu \mathscr{W}^{\frac{-l}{2}}(x_E) \dot{\mathscr{W}}(x_E) - x_I^T M \dot{x}_n + \iota_n^T(x) K_I^T M x_n + \rho(x_E) \Delta$$

where

$$\rho(x_E) \triangleq \mu \mathscr{W}^{\frac{-l}{2}}(x_E) x_I^T \Gamma - x_n^T M^T$$

Using the knowledge gained previously, we see it's possible to bound the first term with a homogeneous function, this time with homogeneous degree $d_{\dot{\mathscr{U}}} = 2$, while each component of the row vector $\rho(x_E)$ is homogeneous of degree $d_\rho = 1 - l$. This leads to

$$\dot{\mathscr{U}} \leq -\mu\varepsilon \parallel x \parallel^2 - \dot{x}_I^T M x_n - x_I^T MG[-K\sigma_n(x) + Bx_I] + \rho(x_E) \Delta$$

That being said, first the analysis for the unperturbed case (when $\Delta = 0$) is presented.

Due to the assumption that the weak Lyaponov function is already of a defined sign for $x \neq 0$, the first term of $\mathscr{U}$ can be used to dominate the sign of the whole function if the rest of the function is negative on the set $\mathscr{S} = \{x_E \in \mathbb{R}^{n+m} | x = 0\}$, then using Lemma 1 the whole function could be rendered negative. Incidentally, in this set and without perturbations the function is

$$\mathscr{U}|_{\mathscr{S}} = -x_I^T MG(t,0)B(t,0)x_I$$

which is positive for $x_I \neq 0$ if $MG(t,x)B(t,x)$ is positive quasi-definite.

For the perturbed case, when $\delta(t) \neq 0$, condition (3.2) together with (3.13) implies that Δ is bounded

$$\|\Delta\|_p \leq \tilde{C}$$

for some constant $\tilde{C} \geq 0$. Thus, there is a constant $\rho \geq 0$ such that

$$\rho(x_E)\Delta \leq \rho \tilde{C} \mathscr{U}^{\frac{1-l}{2-l}}(x_E) .$$

When $l \in (-1,0]$, $d_\rho < d_{\mathscr{W}}$, and therefore near $x_E = 0$ the term $\rho(t,x)$ dominates and $\dot{\mathscr{U}}$ cannot be rendered negative near zero. However, for large values of x_E it becomes negative. And using standard Lyapunov arguments, we conclude that the system (4.19) is input to state stable with respect to the input Δ. Or, equivalently, that the trajectories of the closed-loop system are ultimately and uniformly bounded [30].

When $l = -1$, then $d_\rho = d_{\dot{\mathscr{U}}}$ and $\dot{\mathscr{U}}$ is negative definite for $\tilde{C}$ sufficiently small. This can be shown again using Lemma 1 (see again footnote 1). This proves item 4) of Theorem 2.

Using the scaling of the gains, either (4.33) or (4.34), an arbitrary size of the perturbation can be accommodated. Even a perturbation which grows with time and the state, when (4.34) is used.

4.4.3 Stability requirements and a feasible solution

In sum, the conditions to have a true Lyaponov function and prove the stability of the controllers are the following:

$$\Gamma + \Gamma^T > 0 , \tag{4.38}$$
$$PG(t,x)B(t,x) = K_I^T(t,x)\Gamma , \tag{4.39}$$
$$-\iota_n^T(x)PG(t,x)K(t,x)\sigma_n(x) + \langle \mathbf{p}, \eta(x) \rangle < 0, \forall x \neq 0 , \tag{4.40}$$
$$z^T MG(t,x)B(t,x)z > 0, \forall x,t , \forall z \neq 0 . \tag{4.41}$$

Also, since the uncertain matrix is assumed to be diagonal, the selection of P from (3.39) can be used to cancel the uncertainties. Together with the general solution

$$\begin{aligned} B(t,x) &= G_0^{-1}(t,x)D, \\ K(t,x) &= G_0^{-1}(t,x)K_O, \\ K_I(t,x) &= K_I \end{aligned} \tag{4.42}$$

the conditions become

$$\Gamma + \Gamma^T > 0 , \tag{4.43}$$
$$QD = K_I^T \Gamma , \tag{4.44}$$
$$-\iota_n^T(x)QK_O\sigma_n(x) + \langle \mathbf{p}, \eta(x) \rangle < 0, \forall x \neq 0 , \tag{4.45}$$
$$z^T M\Upsilon Dz > 0, \forall x,t , \forall z \neq 0 . \tag{4.46}$$

with the new arbitrary positive diagonal Q matrix.

The last one is easy, for $M = D^T \Upsilon^T$ always solves the condition.

To solve (4.43) and (4.44) simultaneously, the same trick as in the relative degree one case can be used. The symmetric part of K_I is

$$QD\Gamma^{-1} + \Gamma^{-1}DQ = K_I + K_I^T \tag{4.47}$$

a positive definite solution for Γ^{-1} (and therefore for Γ) exists if QD is always positive stable and $K_I + K_I^T$ is positive definite.

If D is selected with generalized diagonal dominance, then the positive diagonal Q won't affect this property, and from Lemma 2 it is concluded that QD is positive stable.

Finally, for condition (4.45) a sufficient condition to satisfy the inequality is presented in the next lemma.

Lemma 6. *Suppose that $\mathscr{B}(\Upsilon)$ is a diagonal matrix and that $l \in [-1,0]$. Then there is a positive diagonal matrix $P = \operatorname{diag}\{p_1, \cdots, p_m\}$, $p_i > 0$, and a matrix $K_O = [k_{ij}] \in \mathbb{R}^{m\times m}$, with positive diagonal elements $k_{ii} > 0$, such that* (4.45) *is satisfied.*

Proof. Before we start the proof proper, some properties of the functions in question are recalled: From the SISO analysis we know that when a ι_{n_i} and a $\lceil \sigma_{n_i} \rfloor^{\frac{1}{r_{i1}}}$ function depend on the same nonzero variable, their product is always positive; we also know that the sigma functions are always zero when their argument is zero, but in the discontinuous cases this is not true for the iota functions; finally, the whole function is of the same homogeneous degree $d_{\mathscr{W}} = 2 + 1$.

With that out of the way, define:

$$\Omega := P\Upsilon K_O \tag{4.48}$$

This is a matrix with arbitrary elements, given that all of the matrices being multiplied are of full rank.

With this, all of the conditions to apply the homogeneous domination lemma are met. In order to use this lemma define without loss of generality

$$d(x) := \sum_{i=1}^{m-1} \omega_{i,i} \iota_{n_i}(x_i) \lceil \sigma_{n_i}(x_i) \rfloor^{\frac{1}{r_{i1}}}$$

$$h(x) := \iota_n^T(x)\Omega\sigma_n(x) - \langle \mathbf{p}, \eta(x)\rangle - d(x)$$

When $d(x) = 0 \Rightarrow x \in \mathscr{X}_m = \{x = (0,\dots,0,x_m)^T\}$ we get

$$h(x)|_{\mathscr{X}_m} = \lceil \sigma_{n_m}(x_m) \rfloor^{\frac{1}{r_{m1}}} \left[\omega_{m,m}\iota_{n_m}(x_m) + \sum_{j\neq m} \omega_{j,m}\iota_{n_j}(0) \right] - p_m \eta_m \parallel x_m \parallel^{2+l}$$

for $l = -1$ it is positive if $\omega_{m,m} > \eta_m + \sum_{j\neq m} |\omega_{j,m}|$. For the rest of homogeneous degrees $l \in (-1,0]$ it is positive if $\omega_{m,m} > \eta_m$.

All of this is sufficient to claim that a big enough selection of the diagonal elements ω_{ii} gives the whole function a definite sign, fulfilling the inequality in (4.46).

What's not clear from this proof so far is under what uncertainty conditions for the matrices $\Upsilon \in \mathscr{Y}$ is it possible to find feasible gains that make the diagonals of Ω reach this condition.

If all $\Upsilon \in \mathscr{Y}$ then after choosing $P = Q\Upsilon^{-1}$ the gain K_O can be freely chosen with diagonal elements of sufficient magnitude to reach stability.

In this case Q could be the identity matrix or some scalar matrix, for example.

Therefore, under the hypothesis of Theorem 2, it is possible to find gains that satisfy (4.45). □

To add to what's been in the proof above, if the off-diagonal elements of Υ were really big and we don't know their direction, then how could we find a matrix K_O that makes the diagonals of $P\Upsilon K_O$ of sufficient magnitude to achieve stability for all $\Upsilon \in \mathscr{Y}$? In the relative degree one case, it was found that strict diagonal dominance was sufficient to achieve stability, but here there is no proof that this condition is sufficient.

The difference is mainly due to the fact that the magnitude of $\iota_n(x)$ and $\sigma_n(x)$ is altered not only by the matrices of the function $P\Upsilon K_O$, but also by other terms and internal gains of these feedback injection functions. In contrast, in the relative degree one case the feedback injection functions are $\lceil x \rfloor^{\frac{1}{1-l}}$ and $\lceil x \rfloor^{\frac{1+l}{1-l}}$, here the magnitude of one function with respect to the other is altered only by the exponents and the coefficients of the function's matrices, therefore it was easier to find clear conditions for the stability of the controllers.

Nevertheless it is expected that, if $\mathscr{B}(\Upsilon)$ weren't diagonal but its off-diagonals were small enough, stability would be preserved, but after a certain magnitude, it would become impossible to find stabilizing gains.

Chapter 5

Case Studies

In this chapter, two real systems are simulated and the controllers described in the previous sections are implemented to illustrate their properties.

In the first example, a distillation process of two outputs and relative degree one is presented with the control objective of making the closed loop more robust to perturbations and uncertainties. For this, integral sliding modes are used, with a nominal linear controller and a homogeneous PI controller as the sliding mode control.

In the second example, a decoupled omnidirectional robot of relative degree 2 and three outputs is presented. Nondiagonal gains are designed for the multivariable super-twisting controller to show how coupling two subsystems can modify the performance of the closed loop.

5.1 LV Distillation Process

Consider the idealized model of an LV distillation process with an output gain uncertainty from [3] converted to a state variable model:

$$\dot{x} = Ax + \Upsilon G_0 u \tag{5.1}$$

where $x \in \mathbb{R}^2 = [y_D \, x_B]^T$ are the top composition and bottom composition respectively, and $u \in \mathbb{R}^2 = [L \, V]^T$ with L as the reflux and V the boilup, hence this is an LV configuration. The system's matrices are

$$A = -\frac{1}{75}\mathbb{I}_2 \tag{5.2}$$

$$\Upsilon = \mathrm{diag}\{1 + \hat{\varepsilon}_i\} \tag{5.3}$$

$$G_0 = \frac{1}{75}\begin{bmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{bmatrix} \tag{5.4}$$

where $\hat{\varepsilon}_i > -1$ are the output gain errors.

This is a subsystem of a larger model of 5 outputs and 5 inputs, but the first 3 states are normally controlled in independent SISO loops.

There are two main problems when controlling this system: First of all, the system is nominally stable but its poles are very near to the imaginary axis, therefore some stabilization is required. Secondly, the effect of disturbances and uncertainties on the output must be reduced.

These objectives must be achieved with a plant that is strongly interactive even at steady state, so true MIMO tools are required to control it. On top of that, in [3] it's been shown that decoupling can easily lead to instability under some model uncertainties, so that isn't an option either.

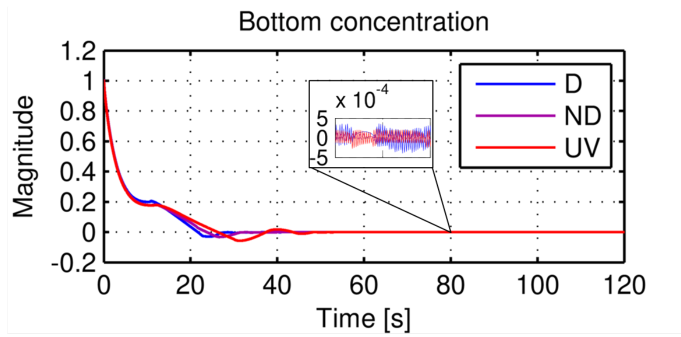

Figure 5.1: Bottom concentration output chattering. D for the super-twisting algorithm with diagonal gains, ND for the one with nondiagonal gains and UV for the unit-vector like super-twisting algorithm

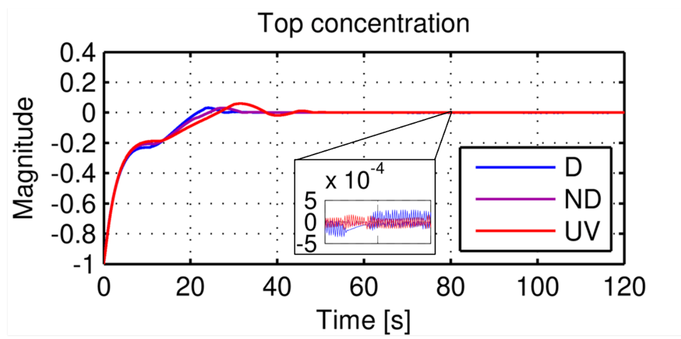

Figure 5.2: Top concentration output chattering. Same key as before.

Linear controllers can be used to move the poles of the system, and a linear servomechanism can be designed to improve disturbance rejection. However, here the use of integral sliding mode control is proposed to handle uncertainties and perturbations, with a nominal linear control to place the poles in a more favourable position.

The nominal control was obtained solving an algebraic Riccati equation with an added degree of stability of $\alpha = 0.1$. The matrices used in the equation were: $A_{ricc} = A + \alpha \mathbb{I}_2$, G_0 unchanged, $R = \mathbb{I}_2$ and the matrix Q was

$$Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{5.5}$$

this results in a constant gain matrix K_n that will make the real part of the poles of the closed loop system no more than -0.1. The robustness properties of this nominal control law won't be discussed here.

For the integral sliding mode the projection matrix was the identity:

$$s = x(t) - x_0(t) - \int_0^t (A - G_0 K_n) x \, d\tau \tag{5.6}$$

where the nominal control law has been substituted in. The dynamics of this sliding surface can be reduced to:

$$\dot{s} = \Upsilon G_0 u_c + \delta \tag{5.7}$$

where $\delta \in \mathbb{R}^m$ is a perturbations vector that accounts for other model uncertainties and unknown inputs. Now the task is to design a controller u_c that can deal with these perturbations.

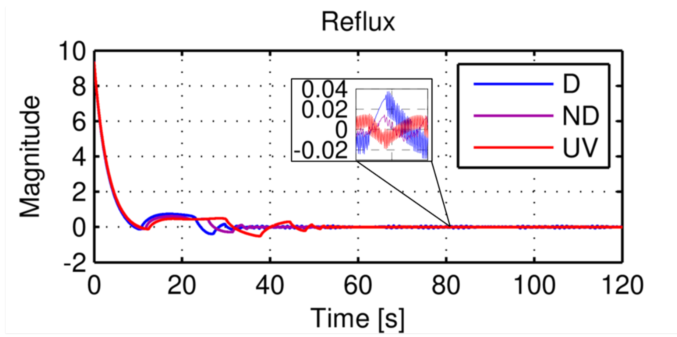

Figure 5.3: Reflux input chattering. Same key as before.

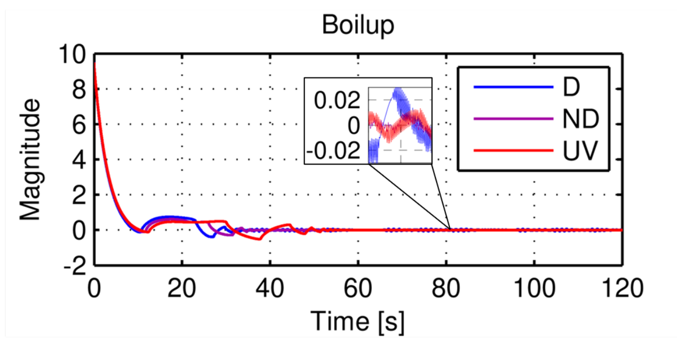

Figure 5.4: Boilup input chattering. Same key as before.

A possible choice would be the super-twisting algorithm. Since the output uncertainty proposed in [3] is diagonal, the matrix P can completely compensate it and it's possible to design as if there were no uncertainties in G.

For the passive super-twisting algorithm the gains found by trial and error were:

$$K_O = \begin{bmatrix} 0.85 & -0.425 \\ 0.425 & -0.85 \end{bmatrix} \tag{5.8}$$

$$K_I = \begin{bmatrix} 0.25 & 0.05 \\ 0.05 & 0.25 \end{bmatrix} \tag{5.9}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5.10}$$

B is only used to make the diagonals of GB positive, a similar thing happens with the sign of the diagonals of K_O which also preserve the row diagonal dominance of G_0. Notice that no decoupling of any kind was performed and even then the design conditions are met.

Additionally, a controller that uses only the diagonal part of the previous matrices was tested too. The controllers are also compared with the nominal control working alone, and a unit-vector-like super-twisting algorithm with no linear gains. This is [7] [8]:

$$\begin{aligned} u_c &= B\left[-k_1 \frac{s}{\| s \|^{\frac{1}{2}}} + v - k_2 s\right] \\ \dot{v} &= -k_3 \frac{s}{\| s \|} - k_4 s \end{aligned} \tag{5.11}$$

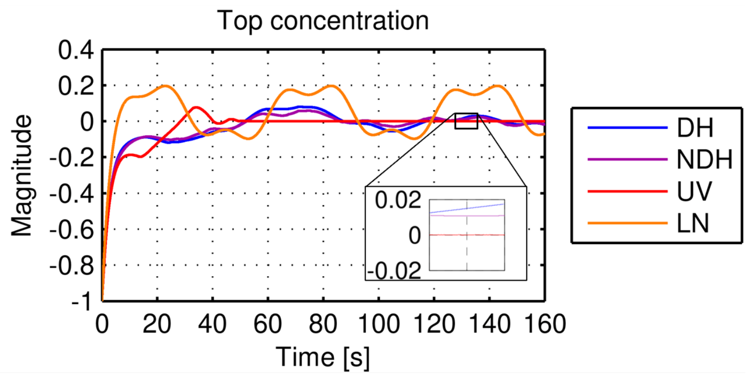

Figure 5.5: Top concentration output. DH for the homogeneous controller with diagonal gains, NDH for the homogeneous controller with nondiagonal gains, UV for the unit-vector like super-twisting algorithm, and LN for the nominal control acting alone.

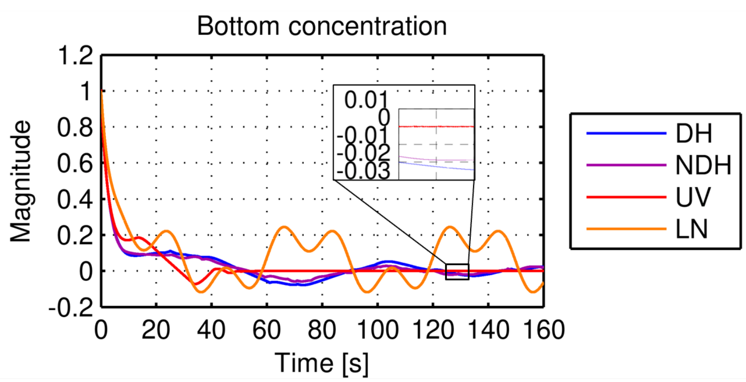

Figure 5.6: Bottom concentration output, same key as before.

with $k_2 = k_4 = 0$ and B as in (5.10). The other gains $k_1 = 0.85$ and $k_3 = 0.25$ were chosen similarly to the other controller.

For the simulation, the value of the output gain errors were $\hat{\varepsilon}_1 = 0.5$ and $\hat{\varepsilon}_2 = 1.4$ and the matched perturbations were:

$$\delta = \begin{bmatrix} 0.5\sin\frac{\tau_c}{60} + 0.1 \\ -0.25\sin\frac{\tau_c}{20} - 0.1 \end{bmatrix} \tag{5.12}$$

with the circle constant $\tau_c = 2\pi$. Finally, the initial conditions chosen were $x_0 = [-1\ 1]^T$.

As expected the discontinuous controller produces chattering in the input as seen in figures 5.1 to 5.4. Lets say the additional stress on the actuators caused by the chattering effect outweigh the benefits of complete Lipschitz perturbation rejection, in this case, a homogeneous approximation is proposed, that is, a controller from (3.8) with $l = -0.8$, for example. The results of using this degree of homogeneity are shown in figures 5.5, 5.6, 5.7 and 5.8.

All simulations were preformed with the ode1 Euler solver in Simulink with a fixed step of 1×10^{-2}.

The integrated square error was calculated over 160 s and the results are summarized in Table 5.1. The ideal control was the nominal control without perturbations or uncertainties.

In terms of the error comparisons, although all of the controllers preformed similarly, the one with nondiagonal gains was the best one, what's surprising is that with similar gain magnitudes both homogeneous controllers of degree $l = -0.8$ outperformed the unit-vector like STA. To be fair, the unit-vector-like super-

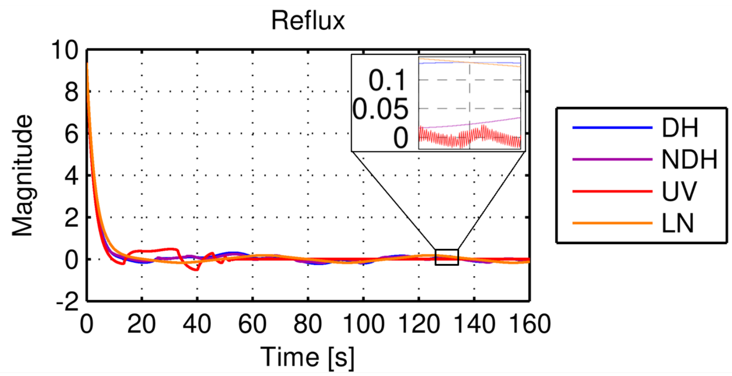

Figure 5.7: Reflux input, same key as before.

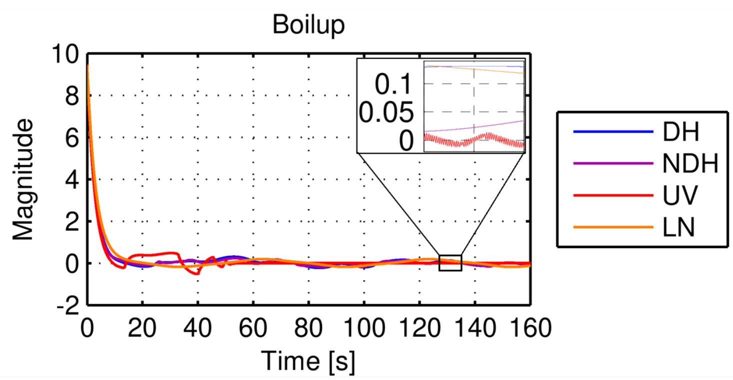

Figure 5.8: Boilup input, same key as before.

Table 5.1: Distillation Process Integrated Square Error

Output	DH	NDH	UV	LN	Ideal
y_D	2.1877	2.0342	2.2486	3.4731	1.6217
x_B	1.9941	1.8718	1.9975	5.4362	1.3671
Total	4.1818	3.9060	4.2461	8.9093	2.9888

twisting controller proposed in the other papers allows for the selection of some linear feedback injection gains that were ignored here to make an even comparison, nevertheless it is possible that the use of these additional terms would improve the performance of this controller.

5.2 Omnidirectional Robot

Consider the dynamics of a omnidirectional mobile robot in the inertial frame, taking into account the actuator dynamics with identical motors [45]:

$$M\ddot{\xi} + C(\dot{\xi})\dot{\xi} + D\dot{\xi} = \tau + \rho \tag{5.13}$$

where $\xi = [x\ y\ \theta]^T$ is the state vector and the states are the coordinates of the robot on the plane (x,y) and the angle θ where it's facing. The input $\tau \in \mathbb{R}^3$ is a vector of generalized forces, $\rho \in \mathbb{R}^3$ contains

Lipschitz continuous disturbance forces. M, $C(\dot{\xi})$, $D \in \mathbb{R}^{3\times 3}$ are given by

$$\begin{aligned} M &= M_R + (J_2 + J_m r_e^2) E E^T \\ C(\dot{\xi}) &= \tfrac{4\dot{\theta}}{r^2}(J_2 + J_m r_e^2) \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ D &= r_e^2 \left(\tfrac{k_a k_b}{R_a} + k_v \right) E E^T \end{aligned} \tag{5.14}$$

with $M_R = diag\{m, m, 4m_2(l_1^2 + l_2^2) + J_1 + 4J_3\}$, and $m = m_1 + 4m_2$ where m_1 is the mass of the body, m_2 is the mass of each wheel, J_1 is the inertia of the body, J_2 is the inertia of the wheels over the motor's shaft, J_3 is the inertia of the wheels perpendicular to the motor's shaft, l_1 is the robot's width of the robot and l_2 is the robot's length, r is the radius of the wheels. The other parameters come from the actuator motors: J_m is the inertia of the shaft of the motors, k_b is the back EMF constant, k_a is the torque constant, R_a is the armature resistance, k_v is the viscous friction of the motor and r_e is the gear ratio.

The matrix $E \in \mathbb{R}^{3\times 4}$ is defined as

$$E = \frac{1}{r} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ l_1 & -l_1 & -l_1 & l_1 \end{bmatrix}$$

A more detailed explanation of the model is described in [45] and the same numerical values for them were used for the following simulation, except for the gear ratio that was changed from 64 to 6.4 to improve numerical conditioning. One key property of the mass matrix M is that it's diagonal and constant.

In order not to distract from this dissertation's subject, the following assumptions are made: position tracking is the control problem; velocity is available as an output; the unknown parameters are in M and the rest have no uncertainties; the control law is required in generalized forces.

If velocity wasn't available, an exact differentiator could be used to obtain it, and the control law in generalized forces can be easily converted into armature voltages using a pseudo-inverse transformation matrix as it's shown in [45], so these two assumptions aren't very restrictive.

That being said, to solve a tracking problem for the reference $\xi_d \in \mathbb{R}^3$, the error dynamics $e_p = \xi - \xi_d$ are written as

$$\ddot{e}_p = M^{-1}[\tau + \rho - C(\dot{\xi})\dot{\xi} - D\dot{\xi}] - \ddot{\xi}_d \tag{5.15}$$

Since we assumed no uncertainties in $C(\dot{\xi})$ and D they can be cancelled using

$$\tau = C(\dot{\xi})\dot{\xi} + D\dot{\xi} + u \tag{5.16}$$

where u is a new free control law. Moreover, defining $\delta := M^{-1}\rho - \ddot{\xi}_d$ leaves us with

$$\ddot{e}_p = M^{-1}u + \delta \tag{5.17}$$

The uncertainty description means that $M^{-1} = \Upsilon G_0$, assuming that only the fact that M is diagonal, positive and constant is known thus setting $G_0 = \mathbb{I}_3$ and $M^{-1} = \Upsilon$, now the control algorithm can be applied.

The reference to track and the perturbations for the simulation were

$$x_d = \begin{bmatrix} \sin\omega t \\ \cos\omega t \\ -\omega t \end{bmatrix}, \qquad \rho = \begin{bmatrix} 2.5\sin\frac{\tau_c}{15} + 0.1 \\ -2.5\sin\frac{\tau_c}{20} + 0.1 \\ 2.5\sin\frac{\tau_c}{25} + 0.1 \end{bmatrix} \tag{5.18}$$

with $\omega = \frac{\tau_c}{20}$ and the circle constant $\tau_c = 2\pi$. What is more, the total parameters of the system used in the simulation were:

$$M = \begin{bmatrix} 4.8809 & 0 & 0 \\ 0 & 4.8809 & 0 \\ 0 & 0 & 0.1445 \end{bmatrix} \tag{5.19}$$

$$C(\dot{\xi}) = \begin{bmatrix} 0 & 0.5609 & 0 \\ -0.5609 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} \tag{5.20}$$

$$D = \begin{bmatrix} 101.3319 & 0 & 0 \\ 0 & 101.3319 & 0 \\ 0 & 0 & 2.3535 \end{bmatrix} \tag{5.21}$$

Since Υ is a diagonal matrix, the controllers proposed are insensitive to the uncertainties therefore it's possible to find a suitable gain matrix K_O with sufficiently big diagonal elements. For K_I a positive quasi-definite matrix was tuned to provide good convergence and reduce oscillations, similarly to how the integral part of classic PI-controllers has to be tuned.

Given that a method to finding these gains is not known, they were find via simulation trial and error without perturbations and then they were scaled to provide stability in spite of these perturbations.

Something that was kept in mind while doing this was the desire to couple the x and y subsystems using the off-diagonal gains, and to couple the third subsystem only in one direction, meaning it would be affected by the other subsystems but at the same time this subsystem couldn't affect the others. This was mainly to showcase the unique ways the proposed control structure can alter the interaction between subsystems, for other MIMO super-twisting controllers have no freedom in this category.

By trial and error, a set of good gains was found:

$$K_O = \begin{bmatrix} 4.2750 & 2.1375 & 0 \\ -2.1375 & 4.2750 & 0 \\ -1.7100 & -1.7100 & 3.2250 \end{bmatrix}, \qquad K_I = \begin{bmatrix} 1.25 & -0.5 & 0 \\ 0.5 & 1.25 & 0 \\ 0.4 & 0.4 & 0.7 \end{bmatrix}$$

This selection of gains was compared with gains corresponding to the diagonals of these matrices to see the effect of the coupled dynamics that were introduced.

The internal gains of the sigma function where all the same $\beta_1 = 1.25$ for all subsystems and both simulations.

All simulations were preformed with the ode1 Euler solver in Simulink with a fixed step of 1×10^{-4}.

Both controllers solved the tracking problem in a satisfactory manner, but the nondiagonal gains changed the performance of the controller and coupled two outputs of a decoupled system, this is better seen in figures 5.10, 5.11 and 5.12 where the differences between the two controllers is due to the added nondiagonal gains. From the behaviour in figure 5.12, it is undeniable that coupled dynamics were induced.

In the (x, y) plane of figure 5.9 the trajectory followed looks very different for each controller, and here the benefits of coupling are more clear: the decoupled controller suffers from more overshoot and undershoot after the initial spike, it even crosses the reference several times before converging, in contrast, the coupled controller goes straight to the reference.

Disclaimer: the initial conditions were chosen deliberately to make the effect of nondiagonal gains more noticeable, I'm not claiming this selection of gains *always* outperforms a diagonal selection.

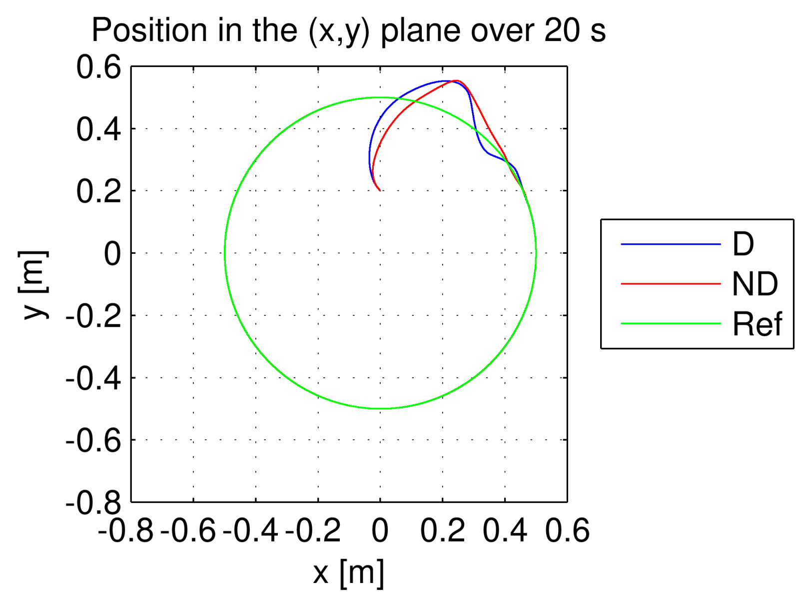

Figure 5.9: Position tracking in the (x, y) plane. Green for the reference to track, blue for the controller with diagonal gains and red for the one with nondiagonal gains.

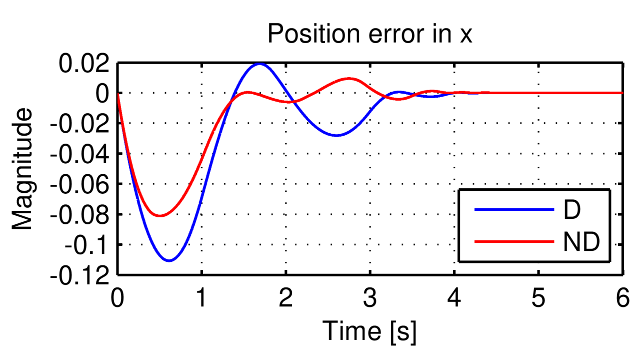

Figure 5.10: Position error in x. Blue for the controller with diagonal gains and red for the one with nondiagonal gains.

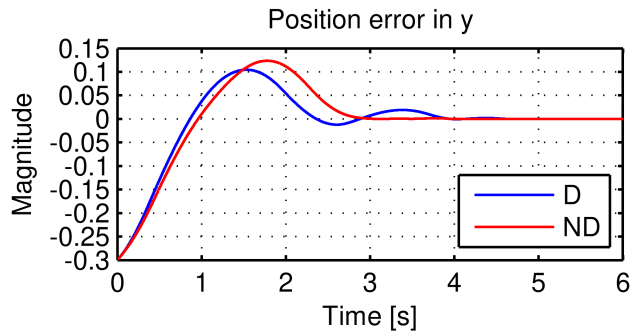

Figure 5.11: Position error in y. Same key as before.

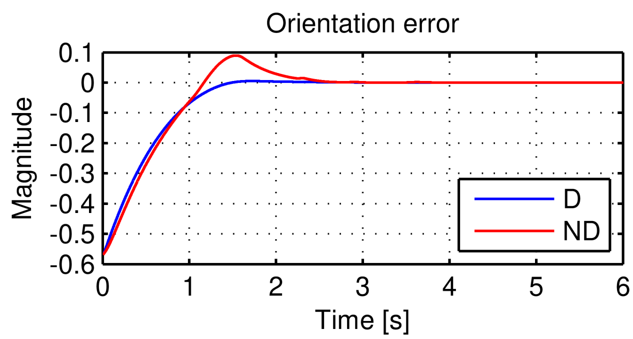

Figure 5.12: Orientation error. Same key as before.

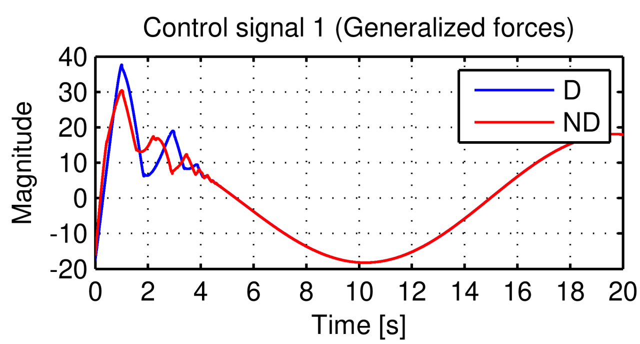

Figure 5.13: Control signal 1. Same key as before.

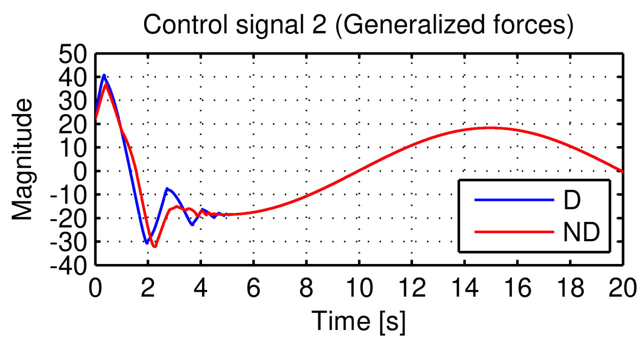

Figure 5.14: Control signal 2. Same key as before.

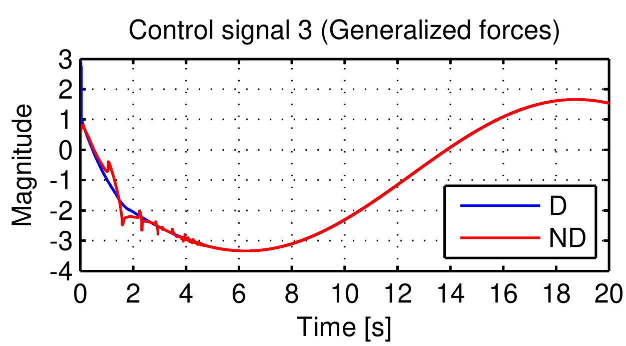

Figure 5.15: Control signal 3. Same key as before.

Chapter 6

Discussion

In Chapter 3 a new kind of multivariable super-twisting algorithm has been presented as part of a family of homogeneous controllers also including a linear case. The stability theory behind this new controller lead to the discovery of its main features that set this controller apart from other similar works.

There are two main restrictions for the new controllers, a very strong one and a very general one:

- The strong one is that the uncertain matrix factor of the input matrix *must* be constant so, in general, the time or state varying nominal input matrix can be decoupled.
- The general one is that the whole set of possible uncertain matrix factors must have generalized diagonal dominance.

Even so, if decoupling is not desired, another feature of this family of controllers is the ability to select nondiagonal gain matrices, so the strength of the interactions between subsystems can be adjusted to some degree. This is something that has never been done before for the super-twisting algorithm, for there have been generalizations that either force all of the outputs to converge at the same time or completely decouples all of the subsystems. Still, some of the other generalizations allow for other types of uncertainties in the input matrix that this controller has no ability to compensate.

Be that as it may, in the case of a time invariant input matrix the stronger restriction is always satisfied and the controller gains can be designed without any kind of decoupling. This special case has been presented in 3.4.2 for known input matrices and in 3.4.3 for unknown ones. Since in this case the main restriction of the controller is effectively negated, it should be here where more applications could be found, some of which were presented in Chapter 5 where two real systems were simulated to illustrate the use of the new controllers. Those simulations also showed that the new controllers are comparable in performance to other MIMO generalizations that had been proposed previously.

Another interesting characteristic is that, after decoupling the nominal part, positive diagonal gain matrices can be designed in spite of the uncertainties, and in the linear and discontinuous case these gains can even be arbitrary.

Additionally, it was shown that the whole family of controllers is insensitive not only to constant perturbations (which was widely known before) but they're all also insensitive to diagonal uncertain matrix factors, this fact enabled an arbitrary relative degree generalization, even if its only for this more restricted set of unknown nonlinear systems. It was designed for a system in normal form and stability was proven just for the case when the whole set of possible uncertain matrix factors are diagonal. Another disadvantage of this generalization is that the integral controller is more complex than what similar controllers have proposed. Nevertheless, this generalization inherited the main property of allowing nondiagonal gain matrices.

In summary, the new homogeneous super-twisting controller is comparable to other multivariable generalizations, and under some restrictions of the uncertain input matrix, it allows for tuning the interaction between subsystems with the use of nondiagonal gain matrices. This is also true for the rest of the homogeneity degrees presented, moreover the homogeneity degree is designable and can be used to change the convergence type and speed or to approximate either a super-twisting controller or a linear one hopefully retaining some of their advantages. The controllers can also be interpreted as a homogeneous generalization of MIMO linear controllers, since many linear design techniques make use of the off-diagonal gains to modify the performance of the controller, a behaviour that was also observed in the nonlinear controllers proposed, unfortunately a theory on how to design these gains in the nonlinear case to get a desired performance has not been developed yet.

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