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BL-HOMOGENEOUS SLIDING-MODE OBSERVERS DESIGN FOR MIMO LINEAR SYSTEMS

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Resumen

En el presente trabajo se expone una novedosa idea en el desarrollo de estimadores de estado para sistemas lineales invariantes en el tiempo en presencia de disturbios modelados como entradas desconocidas, tales estimadores son bien conocidos en la literatura como Observadores con Entradas Desconocidas (UIO, por sus siglas en ingles). La suposición esencial sobre la familia de sistemas analizados es la observabilidad fuerte, sin embargo, será claro que el resultado puede ser extendido fácilmente a sistemas fuertemente detectables, una propiedad menos restrictiva la asumida aquí. El tipo de entradas desconocidas aceptables son aquellas señales medibles uniformemente acotadas.

La idea central en la estructura del observador es la aplicación de propiedades de homogeneidad en el bi-límite, un tipo de generalización de homogeneidad. Lo que permite una asignación independiente del comportamiento en los términos de corrección cerca y lejos del origen, dando al observador la capacidad de dominar términos con crecimiento lineal y al mismo tiempo compensar los efectos producidos por las entradas desconocidas del sistema.

Los observadores propuestos se introducen primero para el caso SISO donde las propiedades de bl-homogeneidad permiten que el observador estime global y al menos asintóticamente los estados del sistema. Se muestra que la estimación se lleva a cabo de forma efectiva aun cuando el sistema está puesto en la forma de observabilidad.

En el caso MIMO el sistema es transformado en una forma adecuada para el diseño que reexpresa el sistema en un conjunto de subsistemas interconectados por términos lineales en función de los estados de todos los subsistemas. La estructura del observador, por lo tanto, puede verse como un conjunto de subobservadores interconectados entre sí. En general, el tipo de interconexiones no beneficia a la estabilidad de la dinámica del error. Se muestra que los términos no lineales de corrección con propiedades de bl-homogeneidad son capaces de compensar tales interconexiones y asegurar que los estados estimados converjan global y al menos asintóticamente a los estados verdaderos.

Se mostrará que el tipo de convergencia del UIO está en función de la elección de un conjunto de parámetros. En general el observador converge de manera asintótica, en tiempo finito o más aun, en tiempo fijo en ausencia de entradas desconocidas. Por otro lado, se mostrará que, con una selección adecuada, en el observador se induce un Modo Deslizante de Alto Orden (HOSM) capaz de compensar el efecto de disturbios o entradas desconocidas convergiendo en tiempo finito o en tiempo fijo.

Una característica importante es que el orden del observador es el mismo que del sistema, es decir, no se requiere de estructuras adicionales que estabilicen la dinámica del error y por lo tanto el orden del sistema completo no se incrementa innecesariamente.

Las pruebas necesarias de convergencia para los observadores se realizan a través de Funciones bl-homogéneas de Lyapunov, lo cual a su vez permite definir una metodología intuitiva para el ajuste de ganancias dentro del UIO.

Finalmente, se dan algunos ejemplos que denotan la efectividad de los observadores presentados.

Abstract

In the present work, a novel idea is exposed for the development of state estimators for linear time-invariant systems in presence of external effects modeled as unknown inputs, such estimators are well known in the literature as Unknown Input Observers (UIO). The essential assumption about the family of analyzed systems is strong observability, however, it will be clear that the presented result can be easily extended to strongly detectable systems, a less restrictive property than the assumed here. Additionally, the family of admissible unknown input signals are those uniformly bounded measurable signals.

The central idea in the observer structure is the application of homogeneity in the bi-limit properties, a kind of homogeneity generalization. This allows an independent assignment of the behavior in the correction terms near and far from the origin, giving the observer the ability to dominate terms with linear growth and at the same time compensate for the effects produced by the unknown inputs of the system. The proposed observers are first introduced for the SISO case where the bl-homogeneity properties allow the observer to estimate globally and at least asymptotically the states of the system. It is shown that the estimation is carried out effectively even when the system is put into observability form.

In the MIMO case, the system is transformed into a design-appropriate form that re-expresses the system into a set of interconnected subsystems by linear terms as a function of the states of all subsystems. Therefore, the observer structure can be viewed as a set of interconnected sub-observers. In general, the type of interconnections does not benefit the stability of the error dynamics. It is shown that nonlinear correction terms with bl-homogeneity properties can compensate for such interconnections and ensure that the estimated states converge globally and at least asymptotically to the true states.

It will be shown that the type of convergence of the UIO is a function of the choice of a set of parameters. In general, the observer converges asymptotically, in finite time or even more, in fixed time in the absence of unknown inputs. On the other hand, it will be shown that, with an adequate selection, a High Order Sliding Mode (HOSM) is induced in the observer, allowing compensation for the effect of disturbances or unknown inputs and converging in finite or fixed time.

An important feature is that the order of the observer is the same as that of the system, that is, no additional structures are required to stabilize the dynamics of the error and therefore the order of the entire system is not increased unnecessarily.

The necessary convergence proofs for the observers are performed through bl-homogeneous Lyapunov Functions, which in turn allows for defining an intuitive methodology for gain adjustment within the UIO.

Finally, some examples are given that denote the effectiveness of the presented observers.

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List of acronyms

HOSM	Higher-Order Sliding Modes
UIO	Unknown Input Observer
SMO	Sliding Mode Observer
Bl	Bi-limit
LTI	Linear Time Invariant
SCB	Special Coordinate Basis
MIMO	Multiple-input Multiple-output
LF	Lyapunov Function
Bl-LF	Bl-homogeneous Lyapunov Function
SM	Sliding mode
FT	Finite-Time
FxT	Fixed-Time
RED	Robust Exact Differentiator

Chapter 1

Introduction

The estimation of state variables from known inputs and outputs of dynamic systems plays a very important role, since in general, not all state variables are available in a control system. This problem is of great interest in various applications, for example, in detection and reconstruction faults, or in the design of robust feedback control laws.

In the case of Linear Time Invariant (LTI) systems, there are well-established approaches such as the Luenberger observer or the Kalman Filter, which provide an asymptotic estimation of the real states of the system. However, in presence of parametric uncertainties or unknown external disturbances, robust approaches have been developed, such as the extended Kalman filter or high-gain observers, which suppress the effect of unknown inputs through a linear injection with a high gain.

In the last decades, in parallel with the development of controllers based upon Sliding Modes (SM) techniques focused on dealing with perturbations in the model, observers based on these tools have been widely considered in reconstruction states and unknown inputs problem [39][42][3][8] due to the (more than robust) insensitivity to uncertainties[19]. Observers based on SM techniques have been shown to achieve very interesting results, because in contrast to those based on linear techniques, they are capable of estimating state variables exactly and in finite time despite the presence of unknown disturbances belonging to a certain family of functions, which in practical terms is very extensive.

The necessary and sufficient conditions for the existence of Unknown Inputs Observers (UIO) have been well defined, tacking into account arbitrary signal inputs, and are limited to Strongly Observable systems that also satisfy the so-called Observer Matching Condition (OMC), which translates to conditions of minimum phase or invariant zeros and relative degree one condition with respect to the unknown input[21][19]. Unfortunately, most physical systems do not fulfill this condition. On the other hand, with this motivation, schemes based upon High-Order Sliding Modes (HOSM) approach have been proposed [19][16][15][13][17], which, although they cannot relax the minimum phase condition, do not require that OMC fulfilled throughout some extra conditions, this increases greatly the family of systems for which UIO with the robust properties inherited from HOSM can be developed.

In general, HOSM observers are based on Levant's Robust Exact Differentiator (RED)[26],[27][28] which estimates robustly, exactly and in finite time the $n-1$ derivatives of a signal, provided the n derivative is uniformly bounded. That is, in terms of UIO, Levant's RED opened the possibility of relaxing the condition of relative degree one, to arbitrary relative degree, through the extra requirement of having a uniformly bounded unknown input. Therefore, RED can be applied directly as an observer as long as the state variables can be expressed as a function of the outputs, their derivatives and the control inputs only, a condition that turns out to be very restrictive. Moreover, in general this scheme only allows local stability in the dynamics of the observation error due to the necessary bounding conditions in the variables, in addition, the unknown inputs must be differentiable.

To address some of these problems, observation schemes have been proposed consisting of a cascade of a Luenberger observer which ensures global convergence to a bounded region, plus a RED providing exact convergence in finite time in the presence of unknown inputs[19][16][17][38]. However, this construction notably increases the order of the observer, thereby increasing the number of parameters that need to be adjusted. Moreover, due to the presence of the linear Luenberger observer, the estimation is delayed.

In homogeneous systems, this property of homogeneity has important implications, local stability implies global stability, and in the case of systems with a homogeneity negative degree, asymptotic stability translates into stability in finite time[5][6].

Hence, continuous homogeneous differentiators with homogeneity degree $d \in (-1, 0]$ (linear with $d = 0$) can estimate in finite time (exponentially) the derivatives of a signal, provided that $f^{(n)} = 0$[35]. While in the discontinuous case ($d = -1$), the RED can estimate robustly, exactly and in finite time the derivatives of a signal $f(t)$ subject to the aforementioned conditions, that is, $f^{(n)}$ uniformly bounded[27].

An extension to the homogeneous functions has been recently introduced, the concept of bl-homogeneity refers not to a homogeneous function, but to a homogeneous function in the limit[1], that is, near the origin it can be approximated by a function with homogeneity degree d_0 and away from the origin it can be approximated by a function with homogeneity degree d_∞, such that $d_0 \leq d_\infty$. The application of this idea to the control [10] differentiation[31] and observation problems, results in a dominance effect being produced in the non-linear injection terms of the error, such that for values close to the origin the approximation in 0 of the injection term dominates, which may contain discontinuous terms capable of dealing with bounded unknown inputs, while for values far from the origin the infinity approximation of the injection terms is capable of ensuring global stability. That is, the immediate advantage is that the design of bl-homogeneous systems is more flexible, since the properties near and far from the origin can be assigned independently, in addition, with the appropriate selection of the parameters d_0 and d_∞, the results results of continuous and discontinuous homogeneous differentiators can be recovered.

We recall, in bl-homogeneous differentiators, a case of particular interest occurs when the homogeneity degree for the approximation at 0, that is, d_0, is equal to -1. Since, in this case, a HOSM is induced at the origin, allowing the estimation of the the derivatives (in the absence of noise) to be exact, robust and in finite time[31].

A disadvantage of homogeneous differentiators and observers (including Levant's RED) is that the convergence time, despite being finite, grows unboundedly (and faster than linearly) with the size of the initial estimation error[31]. This situation has been counteracted with another property of the bl-homogeneous systems design, which in particular for homogeneous differentiators, assigning a positive homogeneity degree to the approximation in the infinite limit and a negative homogeneity degree to the approximation in the zero limit convergence of the estimation will be achieved in Fixed Time (FxT), that is, the estimation error converges globally in finite time, and also the settling-time function is globally bounded by a constant $\bar{T}$, regardless of the size of the initial condition in estimation error. This is important, since the differentiator parameters can be designed such that, after an arbitrarily assigned time $\bar{T}$, we can be sure that the estimation of the derivatives of a signal is correct and exact, regardless of the initial conditions. The idea is to extend these results obtained for bl-homogeneous differentiators towards observers with the same properties, in fact, the observation schemes that will be proposed are closely related.

Furthermore, the extension of the HOSM observer design problem to the Multiple-Input, Multiple-Output (MIMO) case with unknown inputs is not fully available and still has several weak points.

Similar to the SISO case, in the MIMO case the transformation of the system to a suitable representation for the design of observers with all known inputs has been widely described by Luenberger in his works, whose construction is based on the observability indices of the system, however, in such a representation the impact of unknown inputs is not taken into account[33]. On the other hand, the Sannuti and Saberi's so-called Special Coordinate Basis (SCB) transformation [37][9] decomposes the system into a set of inter-coupled integrator chains. Such a structure is impossible for the direct application of the RED as an observer, since it requires the bounding of the state variables for a global convergence. Recent works [33][40] have developed observation schemes based on a inferior blocks triangular structure, obtained through a set of modifications to the algorithm that builds the SCB representation, such representation is suggested as a new MIMO observer form which allows the direct application of RED as an observer with unknown inputs, achieving convergence in finite time. However, in addition to inheriting the disadvantages of a homogeneous observer, already mentioned previously, the design is restricted to systems expressed in a very particular representation.

The objective of this work is to extend the results in the UIO design obtained so far for LTI MIMO systems with unknown inputs based upon the original SCB representation. Inspired by bl-homogeneous function tools, which in addition to achieving fixed time (FxT) convergence, have the possibility of designing such UIOs in a more general framework, that is, without the need for lower triangular structures. And even more, whose size does not exceed that of the original system.

The convergence and stability proofs will be built from a set of recently proposed smooth Lyapunov functions, whose structure also has terms with bl-homogeneous properties.

1.1 Literature review

In [21][41] the necessary and sufficient conditions for the existence of observers for systems in which the input is not completely available for measurement were introduced, i.e. observers in the presence of unknown inputs signals. Such conditions are described in terms of three concepts directly related to the structure of the system; strong detectability, strong detectability* and strong observability. The system Σ_s

$$\Sigma_s : \begin{cases} \dot{x} &= Ax + D\omega, \quad x(0) = x_0 \\ y &= Cx \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are the state, unknown input, and output respectively, is strongly detectable if the output $y = 0 (t > 0)$ implies $x \to 0 (t \to \infty)$ irrespective of the input and the initial condition. Strong detectability is a weaker property than strong observability, which can be defined either by the condition that $y(t) = 0 (t > 0)$ implies $x(t) = 0 (t > 0)$ for any input and initial state. Additionally, a system is said to be strong* detectable if $y \to 0 (t \to \infty)$ implies $x \to 0 (t \to \infty)$ irrespective of the input and initial condition. Therefore strong* detectable implies strong detectable.

This concepts have also been given in terms of the zeros of the system. To explain this we recall that the zeros of the system (1.1) correspond to the values $s \in \mathbb{C}$ for which the Rosenbrock's matrix

$$R(s) = \begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix}, \quad \forall s \in \mathbb{C} \tag{1.2}$$

loses rank, i.e. $rank[R(s)] < n + m$.

It has been shown that the system (1.1) is strongly detectable if and only if all its zeros satisfy $\text{Re}[s] < 0$ (equivalently $rank[R(s)] = n + m$), which correspond to minimum-phase condition. In similar way, the system is strong* detectable if and only if it is strongly detectable and in addition

$$rank(CD) = rank(D) \tag{1.3}$$

The main result in [21] is that in presence of arbitrary unknown input signal the system (1.1) has a strong observer (estimate only based on the output) if and only if it is strong* detectable.

The condition (1.3) is referred as the Observer Matching Condition (OMC)[43], which is equivalent to have relative degree with respect to ω equal to 1. Unfortunately, in most of practical systems OMC does not old.

These conditions are very restrictive and because they are necessary, it is impossible to overcome them without imposing some further restrictions on the system or relaxing the desired properties of the observer. The condition of strong* detectablitity is impossible to overcome, but the condition of relative degree 1 w.r.t. ω can be relaxed imposing some bounding conditions as it will be shown.

Differentiation of signals in real time is an old and well-known problem. Let an input signal $f(t)$, which is assumed to be decomposed as $f(t) = f_0(t) + \nu(t)$. The first term is the unknown base signal $f_0(t)$ to be differentiated and belonging to the class $\mathscr{F}^n_\Delta$ of signals which are $n-1$ times differentiable and with a $(n-1)th$ derivative having a known Lipschitz constant $\Delta > 0$, i.e. the $n-th$ derivative is bounded, $|f_0^{(n)}(t)| < \Delta$.

The continuous differentiators are the most common in practice, as shown in [23][25][24] the linear and homogeneous ones can estimate asymptoticaly the $n-1$ derivatives of a signal when the $n-th$ derivative is bounded. The most popular is the High Gain diferentiator which has the form

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \frac{\alpha_i}{\epsilon^i}(y - x_1), \quad i = 1, ..., n-1 \\ \dot{\hat{x}}_n &= \frac{\alpha_n}{\epsilon^n}(y - x_1) \end{aligned} \tag{1.4}$$

where the positive constants α_i are chosen such that the polynomial

$$s^n + \alpha_1 s^{n-1} + ... + \alpha_{n-1}s + \alpha_n \tag{1.5}$$

is Hurwitz and ϵ is a small positive constant and when $\epsilon \to 0$ system (1.4) acts as a differentiator with asymptotically convergence.

The continuous and homogeneous differentiation algorithms presented in [34] and [1] converge in finite-time, in contrast to the exponential convergence of the linear ones. More recently in [29] extend [34] and develop continuous differentiators converging in fixed-time.

However, Levant [26] has shown that differentiators with continuous dynamics are only exact for the rather thin class of signals having vanishing $n-th$ derivative. He has the proposed a High-Order Sliding Mode (HOSM) differentiator which is a discontinuous system that can estimate exactly, robustly and in finite-time the $n-1$ derivatives of a signal when the $n-th$ one is uniformly bounded, which is given by

$$\begin{aligned} \dot{\hat{x}}_i &= -k_i L^{\frac{i}{n}} \lceil \hat{x}_1 - f \rfloor^{\frac{n-i}{n}} + \hat{x}_{i+1}, \quad i = 1, ..., n-1 \\ \dot{\hat{x}}_n &= -k_i L \lceil \hat{x}_1 - y \rfloor^0 \end{aligned} \tag{1.6}$$

As we say in the introduction, Levant's RED opened the possibility of relaxing the condition of relative degree one in the observers design, to arbitrary relative degree, through the extra requirement of having a uniformly bounded unknown input. Therefore, RED can be applied directly as an observer as long as the state variables can be expressed as a function of the outputs, their derivatives and the control inputs only, a condition that in general is not fulfilled.

In [18][19][16] in addition to present a characterization of strong observability and strong detectability in terms of the relative degree w.r.t the unknown input, a possible solution to he previous problem, it was proposed a novel scheme of observation composed by a cascade of a Luenberger Observer and a HOSM differentatiator which provided global finite-time exact observation of the state vector of strongly observable systems. The observer is built in the form

$$\begin{aligned} \dot{z} &= Az + Bu + L(y - Cz) \\ \hat{x} &= z + K\nu \\ \dot{\nu} &= W(y - Cz, \nu) \end{aligned} \tag{1.7}$$

where $\hat{x}$ is the estimation of x, and the column matrix $L = [l_1, l_2, ..., l_n]^T \in \mathbb{R}^n$ is a correction factor chosen so that the eigenvalues of the matrix $A - LC$ have negetive real part. The nonlinear part of (1.7) is chosen in the form of the $(n-1)th$-order RED of Levant [26] described in (1.6). The disadvantage of this observer scheme is the strong increment in the order of the system and the delay introduced in the estimation.

Recently, a new idea in the observers construction have been presented in [33] for MIMO LTI systems. For this purpose, a new observer normal form is proposed where the system is represented by means of p coupled single- output systems which allow for a straightforward design of a robust observer. The corresponding transformation is derived from a modification to the classical Special Coordinate Basis (SCB) [37][9]. It is summarized as follows: Let the LTI system (1.1) be strongly observable, then, there exist non-singular transformation matrices $T \in \mathbb{R}^{n\times n}$ and $\Gamma \in \mathbb{R}^{n\times n}$ such that teh state transformation $\bar{x} = T^{-1}x$ and the output transformation $\bar{y} = \Gamma y$ yield the system in observer normal form

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{D}\omega \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{1.8}$$

with the dynamic matrix $\bar{A}$ in (1.13), the unknown-input matrix $\bar{D}$ in (1.14) and the output matrix $\bar{C}$ (1.15).

Where the order of the subsystems are given by the integers $\mu_j, j = 1, ..., p$, with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 0, \quad \sum_{j=1}^{p} \mu_j = n$.

The proposed observer relies on the RED [26] and it is given by

$$\begin{aligned} \dot{\hat{\bar{x}}} &= \bar{A}\hat{\bar{x}} + \bar{\Phi}\sigma_{\bar{y}} + \bar{l}(\sigma_{\bar{y}}) \\ \hat{\bar{y}} &= \bar{C}\hat{\bar{x}} \end{aligned} \tag{1.9}$$

where

$$\sigma_{\bar{y}} = \bar{y} - \hat{\bar{y}} \tag{1.10}$$

is the output error. Additionally, the term

$$\bar{\Pi} = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{p,1} \\ \vdots & & \vdots \\ \alpha_{1,n} & \cdots & \alpha_{p,n} \end{bmatrix} \tag{1.11}$$

output systems provides for a linear output injection in order to compensate for the couplings between the single-output systems. And $\bar{l}(\sigma_{\bar{y}})$ is the nonlinear output injection based upon RED, which is described by

$$\bar{l}(\sigma_{\bar{y}}) = \left[\begin{array}{ccc|c|ccc} \kappa_{1,\mu_1-1}\lceil\sigma_1\rfloor^{\frac{\mu_1-1}{\mu_1}} & \cdots \;\; \kappa_{1,1}\lceil\sigma_1\rfloor^{\frac{1}{\mu_1}} & \kappa_{1,0}\lceil\sigma_1\rfloor^{0} & \cdots & \kappa_{p,\mu_p-1}\lceil\sigma_{\mu_1+\mu_{p-1}+1}\rfloor^{\frac{\mu_p-1}{\mu_p}} & \cdots & \kappa_{p,0}\lceil\sigma_{\mu_1+\mu_{p-1}+1}\rfloor^{0} \end{array} \right]^T \tag{1.12}$$

$$\bar{A} = \left[\begin{array}{ccccc|ccccc||ccccc}
\alpha_{1,1} & 1 & 0 & \cdots & 0 & \alpha_{2,1} & 0 & \cdots & \cdots & 0 & \alpha_{p,1} & 0 & \cdots & \cdots & 0 \\
\alpha_{1,2} & 0 & 1 & \ddots & \vdots & \alpha_{2,2} & \vdots & & & \vdots & \alpha_{p,2} & \vdots & & & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & & \ddots & 1 & \vdots & \vdots & & & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & 0 & \cdots & \cdots & 0 & \vdots & 0 & \cdots & \cdots & 0 & \vdots & 0 & \cdots & \cdots & 0 \\
\hline
\vdots & 0 & 0 & \cdots & 0 & \vdots & 1 & 0 & \cdots & 0 & \vdots & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & 0 & 1 & \ddots & \vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & & & 0 & \vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & & & \vdots \\
\vdots & 0 & \cdots & \cdots & 1 & \vdots & \vdots & & \ddots & 1 & \vdots & \vdots & & & \vdots \\
\vdots & \beta_{1,2,1} & \cdots & \cdots & \beta_{1,2,\mu_1-1} & \vdots & 0 & \cdots & \cdots & 0 & \vdots & 0 & \cdots & \cdots & 0 \\
\hline
\vdots & \beta_{1,2,1} & \cdots & \cdots & \beta_{1,2,1} & \vdots & 1 & 0 & \cdots & 0 & \vdots & 1 & 0 & \cdots & 0 \\
\hline
\vdots & 1 & 0 & \cdots & 0 & \vdots & 0 & \cdots & \cdots & 0 & \vdots & 1 & 0 & \cdots & 0 \\
\vdots & 0 & 1 & \ddots & \vdots & \vdots & \vdots & & & \vdots & \vdots & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\
\vdots & \vdots & & \ddots & 1 & \vdots & 0 & \cdots & \cdots & 0 & \vdots & \vdots & & \ddots & 1 \\
\alpha_{1,n} & \beta_{1,p,1} & \cdots & \cdots & \beta_{1,p,\mu_1-1} & \alpha_{2,n} & \beta_{2,p,1} & \cdots & \cdots & \beta_{2,p,\mu_2-1} & \alpha_{p,n} & 0 & \cdots & \cdots & 0
\end{array}\right] \tag{1.13}$$

$$\bar{D} = \left[\begin{array}{ccc}
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
\bar{d}_{\mu_1,1} & \cdots & \bar{d}_{\mu_1,m} \\
\hline
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
\bar{d}_{\mu_1+\mu_2,1} & \cdots & \bar{d}_{\mu_1+\mu_2,m} \\
\hline
\vdots & & \vdots \\
\hline
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0 \\
\bar{d}_{n,1} & \cdots & \bar{d}_{n,m}
\end{array}\right] \tag{1.14}$$

$$\bar{C} = \left[\begin{array}{ccccc|ccccc|c|ccccc}
1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & & & & & & & & & & \ddots & & & & \vdots & \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & \cdots & 1 & 0 & \cdots & \cdots & 0
\end{array}\right] \tag{1.15}$$

The error dynamics for each subsystem in terms of structure coincides with the estimation error dynamics of the RED with additional couplings in the last differential equation. Since the unknown inputs are bounded, the error dynamics present a sequential convergence from the subsystem 1 to p due to the lower triangular structure of the transformed system (1.8). Therefore, the convergence is achieved exactly in finite time.

Additionally to being attached to a particular lower triangular structure, the disadvantage of this observer and in general the homogeneous ones is that the convergence time, although finite, grows unboundedly (and faster than linearly) with the size of the initial estimation error.

A kind of generalization to homogeneous systems have recently been presented in [31]. One of the nice properties of the bl-homogeneous design in general [1], and of the proposed differentiator in [31], is that assigning a positive homogeneity degree to the ∞-limit approximation $d_\infty > 0$ and a negative homogeneity degree to the 0-limit approximation $d_0 < 0$, it is possible to counteract the unbounded increasing effect of the convergence time, i.e. convergence of the estimation will be achieved in Fixed-Time (FxT), that is, the estimation error converges globally, in finite-time and the settling-time function is globally bounded by a positive constant T, independent of the initial estimation error.

The diferentiator introduced is a dynamic system with bl-homogeneous properties, which in absence of noise is able to estimate asymptotically the $n-1$ derivatives of a based signal $f_0(t)$ coming from a function $f(t) = f_0(t) + \nu(t)$ with $f_0(t)$ n-times differentiable and $|f_0^{(n)}(t)| < \Delta$, and $\nu(t)$ is a uniformly bounded measurable signal.

The diferentiatiator is given by

$$\begin{aligned} \dot{\hat{x}}_i &= -k_i \phi_i(\hat{x}_1 - f) + \hat{x}_{i+1}, \quad i = 1, ..., n-1 \\ \dot{\hat{x}}_n &= -k_n \phi_n(\hat{x}_1 - f) \end{aligned} \tag{1.16}$$

where the nonlinear output injection terms, given by

$$\phi_i(z) = \varphi_i \circ ... \circ \varphi_2 \circ \varphi_1(z) \tag{1.17}$$

are the composition of the monotonic growing functions

$$\varphi_i(s) = \kappa_i \lceil s \rfloor^{\frac{r_{0,i+1}}{r_{0,i}}} + \theta_i \lceil s \rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,i}}} \tag{1.18}$$

with powers selected as $r_{0,n} = r_{\infty,n} = 1$, and for $i = 1, ..., n+1$

$$\begin{aligned} r_{0,i} &= r_{0,i+1} - d_0 = 1 - (n-i)d_0 \\ r_{\infty,i} &= r_{\infty,i+1} - d_\infty = 1 - (n-i)d_\infty \end{aligned} \tag{1.19}$$

which are completely defined by two parameters $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$.

Selecting $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ and choosing arbitrary positive (internal) gains $\kappa_i > 0$ and $\theta_i > 0$, for $i = 1, ..., n$. It is supposed that either $\Delta = 0$ or $d_0 = -1$. Under these conditions and in the absence of noise $\nu(t) \equiv 0$, then, there exist appropriate gains $k_i > 0$, for $i = 1, ..., n$, such that the bl-homogeneous differentiator (1.16) converge globally and asymptotically to the derivatives of the signal. Moreover, it converges in Fixed-Time if either

$$\begin{aligned} &(a) -1 < d_0 < 0 < d_\infty < \frac{1}{n-1} \quad \text{and} \quad f(t) \in \mathscr{F}_0^n, \text{or} \\ &(b) -1 = d_0 < 0 < d_\infty < \frac{1}{n-1} \quad \text{and} \quad f(t) \in \mathscr{F}_\Delta^n \end{aligned} \tag{1.20}$$

where $\mathscr{F}_0^n \triangleq \left\{ f^{(n)}(t) \equiv 0 \right\}$ represent the class of polinomial signals and $\mathscr{F}_\Delta^n \triangleq \left\{ \left| f^{(n)}(t) \right| \leq \Delta \right\}$ corresponds to the class of n-Lipschitz signals [31]. This differentiator can be seen has an observer for a particular kind of systems as a result of the present work.

1.2 Motivational examples

1.2.1 Example 1. RED Observer

Consider a *strongly observable* LTI-SISO system

$$\Sigma : \begin{cases} \dot{x} &= Ax + D\omega \\ y &= Cx \end{cases} \tag{1.21}$$

where $x \in \mathbb{R}^4, \omega \in \mathbb{R}, y \in \mathbb{R}$ are the states, unknown input and output respectively, we do not consider known input since it does not modify the observability properties. Note that the system is stable since the matrix A has eigenvalues $\Lambda = \{-4.056, -0.246, -0.346 \pm 0.937i\}$ and it is put in observability canonical form.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -5 & -5 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ \omega(t) &= \cos(0.5t) + 0.5\sin(3t) + 0.5, \quad |\omega(t)| \leq 2 \end{aligned} \tag{1.22}$$

it can be proposed a Robust Exact Differentiator (RED) as observer, given by

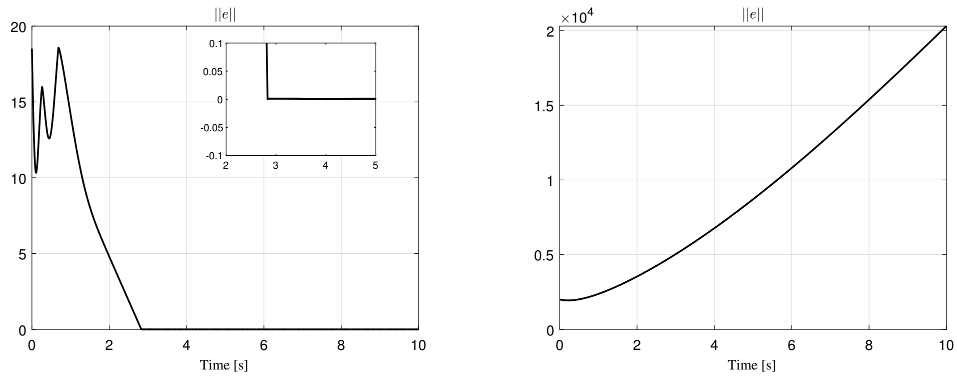
$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1 L^{\frac{1}{4}} \lceil \hat{x}_1 - y \rfloor^{\frac{3}{4}} + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -k_2 L^{\frac{1}{2}} \lceil \hat{x}_1 - y \rfloor^{\frac{1}{2}} + \hat{x}_3 \\ \dot{\hat{x}}_3 &= -k_3 L^{\frac{3}{4}} \lceil \hat{x}_1 - y \rfloor^{\frac{1}{4}} + \hat{x}_4 \\ \dot{\hat{x}}_4 &= -k_4 L \lceil \hat{x}_1 - y \rfloor^{0} \end{aligned} \tag{1.23}$$

The initial conditions of the plant states are $x_0 = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$. The gains k are fixed as

$$\left\{ k_1 = 8.6 k_4^{\frac{1}{4}} \quad k_2 = 21 k_4^{\frac{1}{2}} \quad k_3 = 16.25 k_4^{\frac{1}{3}} \quad k_4 = 1 \right\} \tag{1.24}$$

and parameter $L = 1$. We perform simulations along 5 seconds. We have used a fixed-step explicit Euler method, with integration step $\tau = 1 \times 10^{-5}$.

In Figure1.1(a) it is illustrated the error norm of the states $\|e\| = \sqrt{\sum_{j=1}^{4} e_j^2}$ for the case when initial condition of the observer is $\hat{x}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times 10^1$ which means that the initial error estimation starts near to zero. In this case the observer is able to estimate exactly, in finite time the states of the plant. Nevertheless, in 1.1(b) the initial condition of the observer is $x_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times 10^3$ and the observer can not converge. This shows that the observer does not converge globally despite having stable plants.



(a) Initial error near to 0. (b) Initial error far from 0.

Figure 1.1: Estimation error with different initial values in stable plant.

In the case of unstable plants the estimation is even less satisfactory. That is, the presence of state trajectories that grow unboundedly cause divergence in the estimation error even though they had previously converged. In order to illustrate this fact consider now the following system taken from [18], note that it is unstable since the matrix A has eigenvalues $\Lambda = \{-3, -2, -1, 1\}$.

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\
\omega(t) &= \cos(0.5t) + 0.5\sin(3t) + 0.5, \quad |\omega(t)| \leq 2
\end{aligned}
\tag{1.25}
$$

The situation is shown in Figure 1.2, where the initial state of the observer is $\hat{x}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \times 10^1$ for all cases. It is clear that even with a large increase in the value of the gains through the parameter L in (1.23) it is impossible to maintain the convergence of the observer.

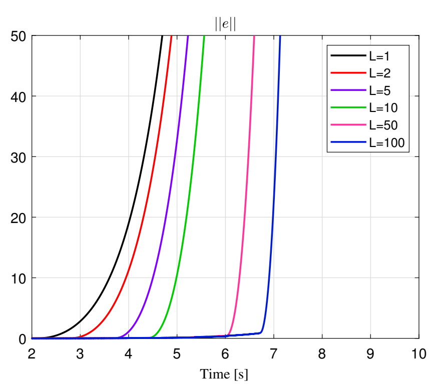


Figure 1.2: Estimation error in unstable plant and different gains.

1.2.2 Example 2. Luenberger + RED Observer

Consider the non-stable plant of the example 1 in (1.25) taken from [18] whose eigenvalues are $\Lambda = \{-3, -2, -1, 1\}$.

The observer has the form

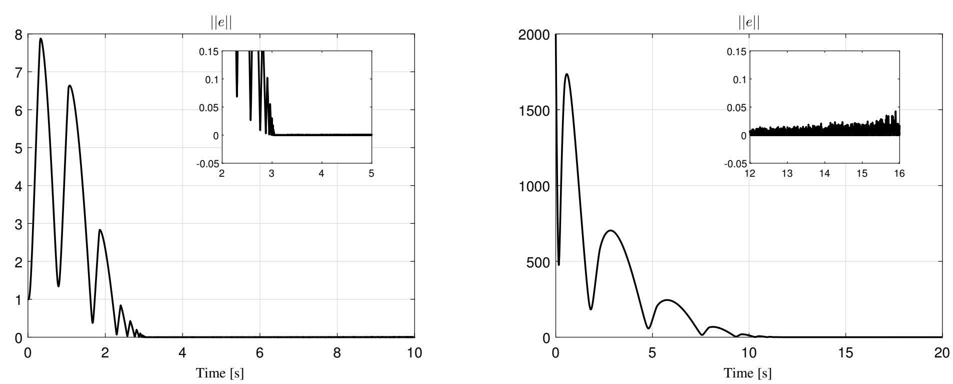
$$\begin{aligned} \dot{z} &= Az + L(y - Cz) \\ \hat{x} &= z + Kv \\ \dot{v} &= W(y - Cz, v) \end{aligned} \tag{1.26}$$

where the correction factor $L = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$ provides for the eigenvalues $\Lambda_o = \{-1, -2, -3, -4\}$ of the matrix $A - LC$. And the gain matrix K is chosen as

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 5 & 5 & 1 & 0 \\ 5 & 5 & 5 & 1 \end{bmatrix} \tag{1.27}$$

the nonlinear part W in the observer is given by the Levant's RED of order 4, where the parameters $\alpha_1 = 1.1$, $\alpha_2 = 1.5$, $\alpha_3 = 2$, $\alpha_4 = 3$, $M = 2$.

$$\begin{aligned} &\dot{v}_1 = w_1 = -\alpha_4 M^{\frac{1}{4}} \lceil v_1 - y + Cz \rfloor^{\frac{3}{4}} + v_2, \\ &\dot{v}_2 = w_2 = -\alpha_3 M^{\frac{1}{3}} \lceil v_2 - w_1 \rfloor^{\frac{2}{3}} + v_3, \\ &\dot{v}_3 = w_3 = -\alpha_2 M^{\frac{1}{2}} \lceil v_3 - w_2 \rfloor^{\frac{1}{2}} + v_4, \\ &\dot{v}_4 = -\alpha_1 M \lceil v_4 - w_3 \rfloor^{0} \end{aligned} \tag{1.28}$$



(a) Initial error near to 0. (b) Initial error far from 0.

Figure 1.3: Estimation error of Luenberger + RED observer with different initial values.

Figure 1.3 shows that the observation error globally converges to 0, that is, regardless of whether the estimated states are close or far from the true values, the observer brings the estimation error to 0 in finite time even in presence of the unknown input. Unfortunately, this convergence time is a function of the initial condition of the error, which can grow more than linearly with e_0. Additionally, the number of parameters of the observer is increased due to the structure of the observer.

1.3 Problem Statement

Consider a general strictly proper MIMO Linear Time Invariant system Σ with unknown inputs

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu + D\omega, \quad D \neq 0, \quad x(0) = x_0 \\ y &= Cx \\ \omega &= [\omega_1 ... \omega_m]^T, \quad |\omega_i| \leq \Delta_i \end{cases} \tag{1.29}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^q$ the known input vector, $\omega \in \mathbb{R}^m$ the unknown input vector and $y \in \mathbb{R}^p$ is the output vector. Accordingly $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}, D \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}$. Without loss of generality we assume that all the inputs and outputs are linearly independent,i.e. $rank(D) = m$, $rank(C) = p$.

For simplicity, the system is considered without feedthrough (the arguments are equally applicable with some extra steps in transformation). For the analysis, it can also be assumed that the known inputs u are equal to 0, i.e. $u = 0$, since it does not modify the observability properties and the effect of these completely known signals can be easily added in the observer formulation.

Assuming Σ to be Strongly Observable, the problem is defined as the construction of a state observer Ω with properties of homogeneity in the bi-limit, capable of estimating exactly in finite time, or preferably in fixed time (FxT) the states of the system even in presence of unknown inputs that satisfy a uniform bounded condition, in general given by

$$\Omega : \left\{ \dot{\hat{x}} = -K\Phi(y, \hat{x}, u) + Ax, \quad \hat{x}(0) = x_0 \right. \tag{1.30}$$

where $\hat{x}$ are the estimated states, K is a design gain matrix and $\Phi(\cdot)$ represents the correction terms.

The design task is based on the transformation in Special Coordinate Basis (SCB) [37][9] which express the original system into a set of interconnected subsystems, each one of them put in observability form.

The order of the observer is desired to be at most the order of the system.

The equations in observer are understood in the Filippov sense [14], in order to provide the possibility of using discontinuous signals. Note that the Filippov solutions coincide with the usual solutions when the right-hand side of the expressions are continuous.

1.4 Objectives

1.4.1 Overall Objective

Design observers for MIMO-LTI systems with unknown inputs assuming to have strongly observability, based on classical and current results on homogeneous and bl-homogeneous systems, such that they offer global exact convergence in finite-time or preferably in fixed-time.

1.4.2 Specific Objectives

- Propose an observer with bl-homogeneous properties for strongly observable SISO-LTI systems. The design is built on the well known SISO Special Coordinate Basis (SCB) [37][9] putting the system in observability canonical form.

- Propose an observer scheme with bl-homogeneous properties for MIMO-LTI systems. The design is based on the well known MIMO-SCB transformation, which decomposes the original system into a set of interconnected subsystems, they are expressed in observability canonical form. The general structure of the designed observer is made up of a set of observers with interconnection terms between them.

- Define a design methodology for the observer parameters, so that the tuning process of gains and adjustable parameters is simple and intuitive for the designer.
- Give a rigorous proof of the error estimation convergence based upon a Lyapunov approach.
- Show the effectiveness of the proposed observers through some examples of physical and academic systems.

1.5 Contributions

- This work presents a family of observers applicable to strongly observable Linear Time Invariant (SISO and MIMO) systems with arbitrary relative degree with respect to the unknown input.
- The structure and design of the Unknown Input Observers (UIO) uses properties of homogeneity in the bi-limit applied to functions, vector fields and dynamic systems.
- We use flexible injection nonlinear terms in the observer to accelerate (when possible) the convergence of the estimation dynamics. Consequently, finite-time or fixed-time convergence can be achieved by selecting gains and parameters appropriately.
- The design starts with a transformation of the original system into Special Coordinate Basis (SCB), resulting in a set of interconnected subsystems associated with the observable and strongly observable dynamics of the system.
- The subsystems are put in observability form, since it requires the bl-homogeneity of the observer, it is shown that bl-homogeneous observer's structure is able assure convergence without the need to bring the system to observer form as reported in previous works [33].
- The structure of the proposed observers does not unnecessarily raise the order of the whole system, that is, the order of the observer is at most the order of the plant.
- A convergence proof of the estimation error with a Lyapunov approach is presented, which, despite being very detailed, is very intuitive. This results in a simple methodology for observer gain adjustment.

1.6 Thesis structure

This thesis is organized as follows. Chapter 2 provides some notations, definitions and preliminaries which are necessary to present the observers design and proofs. In Chapter 3 is presented the first part of main result in this work, the design of the observers for strongly observable SISO-LTI systems, additionally we give some examples to show the effectiveness in solving the observation problem in presence of unknown input. Chapter 4 presents the second part of main result, it is an extension to the MIMO case. In Chapter 5 are given some conclusions and possible future work opportunities. Finally, in Appendix are provided all the proofs in detail.

Chapter 2

Preliminaries and Theoretical framework

In this chapter some definitions and a brief review of necessary mathematical tools will be given in order to have a clear exposition of the problem and results. First we start with the description of systems we are working on and the Special Coordinate Basis (SCB) transformation of linear systems, required in this work to the observer design. Then, we recall the well known concepts of strong observability and strong detectability giving a characterization of them in terms of the zeros and relative degree of the system. Later we state the concepts of classical and weighted homogeneity and some relevant results in functional analysis with this property, immediately we give an extension to homogeneity in the bi-limit, which is part of the central axis of this work. Additionally we have to remember a few important ideas on the Lyapunov stability issue, some recent concepts such as finite-time (FT) stability and fixed-fime (FxT) stability are given formally. Finally we explain how the recently introduced Bl-homogeneous differentiators are built.

Although they have already been used in the previous chapter, some important notations are as follows. For a real variable $z \in \mathbb{R}$ and a real number $p \in \mathbb{R}$ the symbol $\lceil z \rfloor^p = |z|^p \mathrm{sign(z)}$ is the signed power p of z. According to this $\lceil z \rfloor^0 = \mathrm{sign(z)}$, additionally $\frac{d}{dz}\lceil z \rfloor^m = m|z|^{m-1}$ and $\frac{d}{dz}|z|^m = m\lceil z \rfloor^{m-1}$. Note that $\lceil z \rfloor^2 = |z|^2 \mathrm{sign(z)} \neq z^2$, and if p is an odd number then $\lceil z \rfloor^p = z^p$ and $|z|^p = z^p$ for any even integer p. Moreover, $\lceil z \rfloor^p \lceil z \rfloor^q = |z|^{p+q}$, $\lceil z \rfloor^p \lceil z \rfloor^0 = |z|^p$ and $\lceil z \rfloor^0 |z|^p = \lceil z \rfloor^p$.

2.1 Description of systems and properties about observability

In contrast to the Single Input - Single Output (SISO) case, in the Multi Input - Multi Output (MIMO) case existing normal forms are not clearly defined in order to design Unknown Input Observers (UIO) for example the well-known classical observability canonical form by Luenberger [30][20] which is based on the observability indices does not take the impact of the unknown inputs into account.

2.1.1 Special Coordinate Basis

Essentially, the so-called Special Coordinate Basis (SCB) [9][37] decomposes the multivariable linear system into coupled chains of integrators. Such that several fundamental properties of linear systems regarding controllability (stabilisability), observability (detectability), invariant zeros, decoupling zeros, infinite zero structure, effect of feedback on zero structure, squaring down, diagonal and triangular decoupling, etc. can be directly displayed in terms of the Special Coordinate Basis.

Consider a general strictly proper linear system Σ characterized by

$$\Sigma : \begin{cases} \dot{x} &= Ax + D\omega \\ y &= Cx \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n, \omega \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output respectively. Without loss of generality, we assume that all inputs and outputs are linearly independent, i.e. both D and C are full rank. Then we have the following structural or Special Coordinate Basis decomposition of Σ.

Theorem 2.1. *Consider the strictly proper system Σ characterized by (2.3). There exist a nonsingular state transformation, $\Gamma_s \in \mathbb{R}^{n\times n}$, a nonsingular output transformation, $\Gamma_o \in \mathbb{R}^{p\times p}$, and a nonsingular input transformation, $\Gamma_i \in \mathbb{R}^{m\times m}$, that will reveal all the structural properties of Σ. More specifically, we have*

$$x = \Gamma_s \bar{x}, \quad y = \Gamma_o \bar{y}, \quad \omega = \Gamma_i \bar{\omega}, \tag{2.2}$$

which transform the system into

$$\Sigma_{SCB} : \begin{cases} \dot{\bar{x}} & = \bar{A}\bar{x} + \bar{D}\bar{\omega} \\ \bar{y} & = \bar{C}\bar{x} \end{cases} \tag{2.3}$$

with the new state variables

$$\bar{x} = \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}, x_a \in \mathbb{R}^{n_a}, \quad x_b \in \mathbb{R}^{n_b}, \quad x_c \in \mathbb{R}^{n_c}, \quad x_d \in \mathbb{R}^{n_d} \tag{2.4}$$

the new output variables

$$\bar{y} = \begin{bmatrix} y_d \\ y_b \end{bmatrix}, y_d \in \mathbb{R}^{p_d}, \quad y_b \in \mathbb{R}^{p_b} \tag{2.5}$$

and the new input variables

$$\bar{\omega} = \begin{bmatrix} \omega_d \\ \omega_c \end{bmatrix}, \omega_d \in \mathbb{R}^{p_d}, \quad \omega_c \in \mathbb{R}^{m_c} \tag{2.6}$$

Further, the stare variables x_b can be decomposed as

$$x_b = \begin{bmatrix} x_{b,1} \\ x_{b,2} \\ \vdots \\ x_{b,p_b} \end{bmatrix}, \quad y_b = \begin{bmatrix} y_{b,1} \\ y_{b,2} \\ \vdots \\ y_{b,p_b} \end{bmatrix}, \tag{2.7}$$

$$x_{b,\iota} \in \mathbb{R}^{n_{b,i}}, \quad x_{b,\iota} = \begin{bmatrix} x_{b,\iota,1} \\ x_{b,\iota,2} \\ \vdots \\ x_{b,\iota,n_{b,\iota}} \end{bmatrix}, \quad \iota = 1, 2, ..., p_b, \tag{2.8}$$

with $n_{b,1} \leq n_{b,2} \leq \ldots \leq n_{b,p_b}$ and $\sum_{\iota=1}^{p_b} n_{b,\iota} = n_b$.

The state variables x_c *can be decomposed as*

$$x_c = \begin{bmatrix} x_{c,1} \\ x_{c,2} \\ \vdots \\ x_{c,m_c} \end{bmatrix}, \quad \omega_c = \begin{bmatrix} \omega_{c,1} \\ \omega_{c,2} \\ \vdots \\ \omega_{c,m_c} \end{bmatrix}, \tag{2.9}$$

$$x_{c,k} \in \mathbb{R}^{n_{c,k}}, \quad x_{c,k} = \begin{bmatrix} x_{c,k,1} \\ x_{c,k,2} \\ \vdots \\ x_{c,k,n_{c,k}} \end{bmatrix}, \quad k = 1, 2, ..., m_c, \tag{2.10}$$

with $n_{c,1} \leq n_{c,2} \leq \ldots \leq n_{c,m_c}$ *and* $\sum_{k=1}^{m_c} n_{c,k} = n_c$.

And finally, the state variable x_d *can be decomposed as:*

$$x_d = \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ \vdots \\ x_{d,p_d} \end{bmatrix}, \quad y_d = \begin{bmatrix} y_{d,1} \\ y_{d,2} \\ \vdots \\ y_{d,p_d} \end{bmatrix}, \quad \omega_d = \begin{bmatrix} \omega_{d,1} \\ \omega_{d,2} \\ \vdots \\ \omega_{d,p_d} \end{bmatrix}, \tag{2.11}$$

$$x_{d,i} \in \mathbb{R}^{n_{d_i}}, \quad x_{d,i} = \begin{bmatrix} x_{d,i,1} \\ x_{d,i,2} \\ \vdots \\ x_{d,i,n_{d,i}} \end{bmatrix}, \quad i = 1, 2, ..., p_d, \tag{2.12}$$

with $n_{d_1} \leq n_{d_2} \leq \ldots \leq n_{d_{p_d}}$ *and* $\sum_{i=1}^{p_d} n_{d,i} = n_d$.

The decomposed system can be expressed in the following dynamical subsystems. First Σ_a

$$\dot{x}_a = A_{aa} x_a + H_{ab} y_b + H_{ad} y_d \tag{2.13}$$

Σ_b *composed by each subsystem* $\Sigma_{b,\iota}$ *associated with* $x_{b,\iota}, \iota = 1, 2, ..., p_b$,

$$\Sigma_{b,\iota} : \begin{cases} \dot{x}_{b,\iota,1} &= x_{b,\iota,2} + H_{bd,\iota,1} y_d, \quad y_{b,\iota} = x_{b,\iota,1}, \\ \dot{x}_{b,\iota,j} &= x_{b,\iota,j+1} + H_{bd,\iota,j} y_d, \\ &\vdots \quad j = 2, ..., n_{b,\iota} - 1 \\ \dot{x}_{b,\iota,n_{b,\iota}} &= A_{bb,\iota} x_b + H_{bd,\iota,n_{b,\iota}} y_d, \end{cases} \tag{2.14}$$

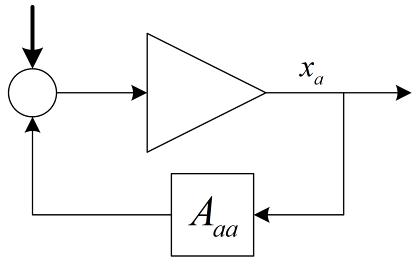
Σ_c *composed by each subsystem* $\Sigma_{c,k}$ *associated with* $x_{c,k}, k = 1, 2, ..., m_c$

$$\Sigma_{c,k} : \begin{cases} \dot{x}_{c,k,1} &= x_{c,k,2} + H_{cb,k,1} y_b + H_{cd,k,1} y_d, \\ \dot{x}_{c,k,j} &= x_{c,k,j+1} + H_{cb,k,j} y_b + H_{cd,k,j} y_d, \\ &\vdots \quad j = 2, ..., n_{c,k} - 1 \\ \dot{x}_{c,k,n_{c,k}} &= A_{ca,k} x_a + A_{cc,k} x_c + H_{cb,k,n_{c,k}} y_b + H_{cd,k,n_{c,k}} y_d + \omega_{c,k}, \end{cases} \tag{2.15}$$

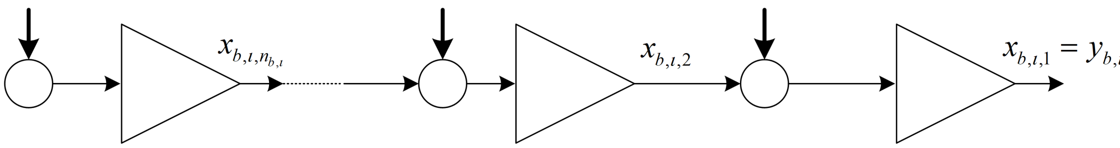
and finally, Σ_d *composed by each subsystem* $\Sigma_{d,i}$ *associated with* $x_{d,i}, i = 1, 2, ..., p_d$

$$\Sigma_{d,i} : \begin{cases} \dot{x}_{d,i,1} &= x_{d,i,2} + H_{dd,i,1} y_d, \quad y_{d,i} = x_{d,i,1} \\ \dot{x}_{d,i,j} &= x_{c,i,j+1} + H_{dd,i,j} y_d, \\ &\vdots \quad j = 2, ..., n_{c,i} - 1 \\ \dot{x}_{d,i,n_{d,i}} &= A_{da,i} x_a + A_{dc,i} x_c + A_{db,i} x_b + A_{dd,i} x_d + w_{d,i}, \end{cases} \tag{2.16}$$

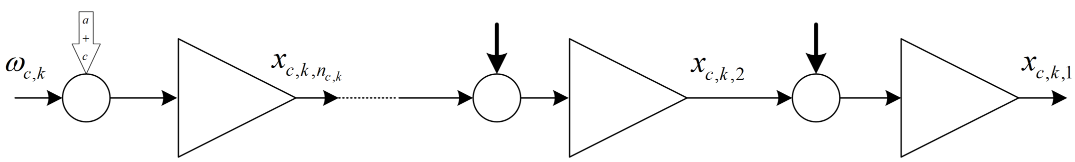
where $A_{aa}, H_{ab}, H_{ad}, A_{bb}, H_{bd,i,j}, A_{cc}, A_{ca}, H_{cb,k}, H_{cd,k}, A_{dd}, A_{da}, A_{dc}, A_{db}, H_{dd,\iota}$ *are constant row vectors of appropriate dimensions. And we consider* $|w_{d,i}(t)| < \Delta_i \in \mathbb{R}_{\geq 0}$.



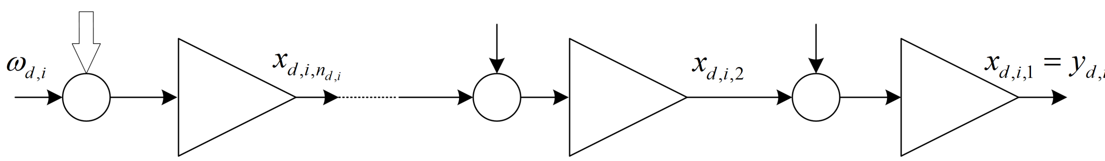
(a) x_a the subsystem without direct input and output.



(b) $x_{b,\iota}$ the chain of integrators without a direct input.



(c) $x_{c,k}$ the chain of integrators without a direct output.



(d) $x_{d,i}$ the chain of integrators with direct input and output.

Figure 2.1: Graphic interpretation of structural SCB decomposition of a MIMO system.

The proof of this Theorem 2.1 is given in [9]. For simplicity, the system (2.1) is considered without feedthrough (the arguments in following chapters are equally applicable with some simple extra steps in SCB transformation). Although the procedure for the decomposition of MIMO systems is complicated, the main idea is the identification of chains of integrators between the system inputs and outputs variables. Three different types of chains of integrators can be identified:

1. Chains that start from an input channel and end with an output. This type of chain gives the infinite zero structures of the given system and covers the subspace corresponding to x_d.

2. Chains that start from an input channel but do not end with an output. This type of chain covers the subspace corresponding to x_c.

3. Chains that do not start from an input but end with an output variable. This type of chain covers the subspace corresponding to x_b.

These subspaces do not cover the whole state space of the given system. The remaining part forms a subspace corresponding to x_a, which is related to the invariant zeros of the system, i.e. the zero dynamics. These four subsystems x_a, x_b, x_c, x_d are depicted in graphical form in Figure 2.1.

where the signal indicated by the double-edged arrow in x_d is a linear combination of all the state variables; the signal indicated by the double-edged arrow marked with $a + c$ in x_c is a linear combination of the state variables x_a and x_c; the signals indicated by the thick vertical arrows are some linear combinations of the output variables y_d and y_b; and the signals indicated by the thin vertical arrows are some linear combinations of the output variable y_d.

The subsystems have the following properties

1. Σ_a corresponds to the zero dynamics. If A_{aa} is Hurwitz then it is detectable. If not it is undetectable.

2. Subsystem Σ_b is not affected by the unknown input vector, and it is observable. Therefore it can be expressed in observer or observability canonical form. Subsystems (2.16) are expressed as the latter.

3. Subsystem Σ_c is affected by the unknown input, it is not strongly observable.

4. Subsystem Σ_d is affected by the unknown input vector, it is strongly observable, and because the number of inputs and outputs is the same p_d then the subsystem is square.

For obtaining the SCB transformation of a system, the original analytic procedure can be followed. However, for simplicity, Professor Chen has published a Matlab toolkit at http://linearsystemskit.net.

2.1.2 Properties and definitions

Several important properties of linear systems related to this work can be displayed in the SCB, nonetheless, before that, some definitions about observability and detectability for systems with unknown inputs have to be remembered.

Definition 2.1. *The zeros of the system (2.1) correspond to the values $s \in \mathbb{C}$ for which the Rosenbrock's matrix*

$$R(s) = \begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix}, \quad \forall s \in \mathbb{C} \tag{2.17}$$

loses rank, i.e. $rank[R(s)] < n + rank[D]$.

Definition 2.2. *The system (2.1) is strongly observable if*

$$y(t) = 0 \text{ for } t > 0 \quad \text{implies} \quad x(t) = 0(t > 0) \tag{2.18}$$

for any input and initial state.
Equivalently, the system is strongly observable if and only if it has no zeros.

Definition 2.3. *The system (2.1) is strongly detectable if*

$$y(t) = 0 \text{ for } t > 0 \quad \text{implies} \quad x(t) \to 0(t \to 0) \tag{2.19}$$

for all inputs and initial states.
Equivalently, the system is strongly detectable if and only if all its zeros s satisfy $Re[s] < 0$.

Definition 2.4. *The system (2.1) is strongly* detectable if*

$$y(t) \to 0(t \to 0) \quad \text{implies} \quad x(t) \to 0(t \to 0) \tag{2.20}$$

for all inputs and initial states.
Equivalently, the system is strongly detectable if and only if it is strongly detectable, i.e.*

$$rank \begin{bmatrix} sI - A & -D \\ C & 0 \end{bmatrix} = n + m, \quad \forall s \in \mathbb{C} \tag{2.21}$$

and additionally

$$rank(CD) = rank(D) \tag{2.22}$$

Consequently of these definitions strong observability implies strong detectability but it does not imply strong* detectability.

As mentioned before, all the invariant properties of the given system can be easily obtained from the structural decomposition. It can be now stated the next property of the system and subsystems in SCB.

Property 2.1. *The system Σ_{SCB} in (2.3) is strongly observable if and only if, x_a and x_c are non-existent.*

2.2 Conditions for the existence of Unknown Input Observers

In [21][41] the necessary and sufficient conditions for the existence of observers were introduced for systems where the input is not completely available for measurement, i.e. Observers with Unknown Inputs (UIO) signals. Such conditions are described in terms of the aforementioned properties related to the structure of the system.

Theorem 2.2. *Under the assumption that the unknown input $\omega(t)$ is a completely arbitrary signal, e.g. it may be unbounded. The system Σ in (2.1) has a UIO if and only if it is strongly detectable*.*

In Definition 2.4, equation (2.21) is equivalent to have minimum phase condition, since the rank of the Rosenbrock matrix has to be equal to $n + m, \forall s \in \mathbb{C}$ or the absence of invariant zeros, and from (2.22) a relative degree one condition is required. Based on the definition we can emphasize:

Observation 2.1. *Strong detectability or even strong observability is not sufficient for the existence of an Unknown Input Observer.*

Since the conditions of minimum phase and relative degree one are necessary and sufficient, it is impossible to overcome them without imposing another restrictions to the problem formulation. Therefore, hereafter the following is assumed

Assumption 2.1. *The unknown input $\omega(t)$ is uniformly bounded, i.e. there exist some $\Delta \in \mathbb{R}_{\geq 0}$ such that $\|\omega(t)\| \leq \Delta$.*

2.3 Homogeneity and Bl-homogeneity

Homogeneity is the property whereby objects such as functions or vector fields scale in a consistent fashion with respect to a scaling operation called a dilation [4], which is essentially an action of the multiplicative group of positive real numbers on the state space [7]. Homogeneity with respect to the standard dilation is one of the two axioms for linearity, the other being additivity. Many familiar properties of linear systems follow, in fact, from homogeneity alone. The first step of homogeneity consists in homogeneous polynomials. The Euler's homogeneous function theorem was the first result linking homogeneity with analysis. And in control theory, homogeneity appeared with Massera and Hahn in the 50's.

Definition 2.5. *Let n and m be two positive integers. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be homogeneous (in the classical sense) with degree $l \in \mathbb{R}$ if and only if $\forall \epsilon > 0 : \quad f(\epsilon x) = \epsilon^l f(x)$.*

The main issue with the classical homogeneity was its very restrictive field of use. Hence, a generalization of the classical homogeneity was proposed by V.I. Zubov in 50s and developed by H. Hermes in the 90's using different weights, leading to weighted homogeneity. Nowadays, this is the most popular definition of homogeneity [4].

Definition 2.6. *Fix a set of Coordinate $(x_1, ..., x_n) \in \mathbb{R}^n$. Let $\epsilon > 0$ all real numbers and $r = (r_1, ..., r_n)$ be a n-upled of positive real numbers. The dilation operator is defined as $\Delta_\epsilon^r x = [\epsilon^{r_1} x_1, ..., \epsilon^{r_n} x_n]^T$, where the numbers r_i are the weights of the Coordinate. The map also can be written as $\Delta_\epsilon^r x = diag(\epsilon^{r_1}, ..., \epsilon^{r_n})x$, where Δ_ϵ^r is the dilation matrix and x the vector of Coordinate.*

Definition 2.7. *It is said that*

- *A function $V : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous of degree l or (r, l)-homogeneous for short, if the equality $V(\Delta_\epsilon^r x) = \epsilon^l V(x), \forall x \in \mathbb{R}^n \backslash \{0\}, \forall \epsilon > 0$ holds.*
- *A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous of degree l, if the equality $f(\Delta_\epsilon^r x) = \epsilon^l \Delta_\epsilon^r f(x), \forall x \in \mathbb{R}^n \backslash \{0\}, \forall \epsilon > 0$ holds.*
- *A vector-set field $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $F(x), \subset \mathbb{R}^n$ is r-homogeneous of degree l, if the equality $F(\Delta_\epsilon^r x) = \epsilon^l \Delta_\epsilon^r F(x), \forall x \in \mathbb{R}^n \backslash \{0\}, \forall \epsilon > 0$ holds.*
- *A system $\dot{x} = f(x)$ is homogeneous if and only if f is so.*

An extension to this concept has is the homogeneity in the bi-limit or bl-homogeneity for short.

Definition 2.8. *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be homogeneous in the 0-limit with associated triple (r_0, l_0, φ_0), if it is approximated near $x = 0$ by the (r_0, l_0)-homogeneous function φ_0. It is said to be homogeneous in the ∞-limit with associated triple $(r_\infty, l_\infty, \varphi_\infty)$, if it is approximated near $x = \infty$ by the (r_∞, l_∞)-homogeneous function φ_∞. Similar definitions apply for vector fields and set-valued vector fields.*

Consequently, a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ (or a vector field or set-valued vector field) is said to be homogeneous in the bi-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit.

There are several results related to the homogeneity of functions, which are going to be useful for the stability analysis in the following chapters. Here we recall some of them. Fisrstly, let us mention that the regularity of a homogeneous mapping f is related to its degree:

Theorem 2.3. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n \backslash \{0\}$ and homogeneous of degree l. Then*

- *If $l < 0$, the f is continuous on $\mathbb{R}^n$ if and only if $V \equiv 0$.*
- *If $l = 0$, the f is continuous on $\mathbb{R}^n$ if and only if $V \equiv V(0)$.*
- *If $l > 0$, the f is continuous on $\mathbb{R}^n$.*

The proof of this Theorem 2.3 is given in [7].

The following lemma asserts that sign-definite, homogeneous functions are radially unbounded.

Lemma 2.1. *Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is continuous and homogeneous, then*

- *If V is sign definite, then V is radially unbounded.*
- *If $n > 1$ and V is proper, then V is sign definite.*

This property is useful in the task of Lyapunov functions construction. Another useful result, but in bl-homogeneous functions, which going to be used in the proof of main result is as follows.

Lemma 2.2. *Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ and $\eta : \mathbb{R}^n \to \mathbb{R}_{\leq 0}$ be two upper semicontinuous (u.s.c.) single-valued bl-homogeneous functions, with the same weights r_0 and r_1, degrees m_0 and m_∞, and approximating functions η_0, η_∞ and γ_0, γ_∞ which are u.s.c. Suppose that $\forall x \in \mathbb{R}^n$, $\gamma(x) \leq 0$, $\gamma_0(x) \leq 0$, $\gamma_\infty(x) \leq 0$. If $\gamma(x) = 0 \wedge x \neq 0 \Rightarrow \eta(x) < 0$, $\gamma_\iota(x) = 0 \wedge x \neq 0 \Rightarrow \eta_\iota(x) < 0$ for $\iota \in \{0, \infty\}$, then there are constants $\lambda^* \in \mathbb{R}$, $c_0 > 0, c_\infty > 0$ such that for all $\lambda \geq max\{\lambda_0, \lambda_\infty\}, \lambda_0 \geq \lambda^*$, $\lambda_\infty > \lambda^*$ and for all $x \in \mathbb{R}^n \backslash \{0\}$,*

$$
\begin{aligned}
\eta(x) + \lambda\gamma(x) &\leq -c_0\|x\|^{m_0}_{r_0,p} - c_\infty\|x\|^{m_\infty}_{r_\infty,p}, \\
\eta_\iota(x) + \lambda\gamma_\iota(x) &\leq -c_\iota\|x\|^{m_\iota}_{r_\iota,p}, \quad \iota \in \{0, \infty\}
\end{aligned}
\tag{2.23}
$$

2.3.1 Stability of homogeneous systems

There are some crucial stability results that appear in the literature for the special case of systems that are homogeneous with respect to dilations of the form $\Delta^r_\epsilon x$. But before presenting them, we formalize the concepts of stability, and the classical results in Lyapunov stability will be remembered.

Definition 2.9. *Consider the autonomous system*

$$\dot{x} = f(x) \tag{2.24}$$

whit $f : \mathbb{R}^n \to \mathbb{R}^n$. Then
The equilibrium point $x = 0$ of (2.24) is

- *Stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) >$0 such that*

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0 \tag{2.25}$$

- *Unstable if it is not stable.*
- *Asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0 \tag{2.26}$$

The well known Lyapunov's stability theorem is as follows, taken from [23]

Theorem 2.4. *Let $x = 0$ be an equilibrium point for (2.24) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} V(0) = 0 \quad \textit{and} \quad V(x) > 0 \quad \textit{in} \quad D\backslash\{0\} \\ \dot{V}(x) \leq 0 \quad \textit{in} \quad D \end{aligned} \tag{2.27}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \quad \textit{in} \quad D\backslash\{0\} \tag{2.28}$$

then, $x = 0$ is asymptotically stable.

This is local result, which can be extended to globally stability as shown in the next theorem

Theorem 2.5. *Let $x = 0$ be an equilibrium point for (2.24). Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function such that*

$$V(0) = 0 \quad \textit{and} \quad V(x) > 0 \quad \forall x \neq 0 \tag{2.29}$$

V is radially unbounded, i.e.

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{2.30}$$

and

$$\dot{V}(x) < 0 \quad \forall x \neq 0 \tag{2.31}$$

then, $x = 0$ is globally asymptotically stable.

Additionally, we will recall a few definitions about stability in some stronger sense.

Definition 2.10. *[7] The system (2.24) is said to be finite-time stable (FTS) at the origin (on an open neighborhood $\mathcal{V} \subset \mathbb{R}^n$ of the origin) if:*

- *There exists a funtion $\delta \in \mathcal{K}$ such that for all $x_0 \in \mathcal{V}$ we have $\|(x_0)\| \leq \delta(\|x_0\|)$ for all $t \geq 0$.*
- *There exists a function $T : \mathcal{V}\backslash\{0\} \to \mathbb{R}_+$ such that for all $x_0 \in \mathcal{V}\backslash\{0\}$, $x(x_0)$ is defined, unique, nonzero on $[0, T(x_0))$ and $\lim_{t \to T(x0)} x(x_0) = 0$. $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the settling-time function.*

If $\mathcal{V} = \mathbb{R}^n$, the the system is called globally FTS.

For differential inclusions (DI) the notion has been defined to deals with all solutions originated from a given initial condition. Details can be seen in the reference.

Finally, the fixed-time (FxT) stability is a particular case of the FTS property.

Definition 2.11. *The system (2.24) is said to be FxT stable at the origin if it is globally FTS and the settling-time function T is bounded, i.e. $\exists \bar{T} > 0$ such that $T(x) < \bar{T}$ for all $x \in \mathbb{R}^n$.*

2.3.2 Homogeneous Lyapunov functions

Now we are completely ready to set down the some principal implications about stability for homogeneous and bl-homogeneous systems, these are important because the observer construction in the next chapter keeps this properties, which will be useful in the mathematical analysis of stability and convergence.

Theorem 2.6. *Let (2.24) be a homogeneous system, if the origin a locally stable equilibrium point, then the origin is globally asymptotically stable.*

It is well known that an asymptotically stable linear system possesses a strict Lyapunov function which is a quadratic form. It turns out that any homogeneous asymptotically stable system admits a homogeneous strict Lyapunov function, not necessarily quadratic. The following theorem formalizes the existence of a Lyapunov function for a homogeneous system

Theorem 2.7. *[2] Let f a continuous vector field on $\mathbb{R}^n$ such that the origin is a locally asymptotically stable equilibrium point. Assume that f is r-homogeneous of degree l with $r \in (0, +\infty)^n$. Then, for any $k \in \mathbb{N}$ and any $p > k \cdot max_i\{r_i\}$, there exists a strict Lyapunov function V for the system (2.24), which is r-homogeneous of degree p and of class $\mathcal{C}^k$. As a direct consequence, the time derivative $V = \langle \nabla V, f \rangle$ is r-homogeneous of degree $l + p$.*

The following corollary shows that the rate convergence of trajectories for homogeneous asymptotically stable system is completely characterized by the degree of the vector field.

Corollary 2.1. *Let f, l defined as in Theorem 2.7*

- *If $l > 0$, then the origin is asymptotically stable.*
- *If $l = 0$, then the origin es exponentially stable.*
- *If $l < 0$, then the origin is finite-time stable.*

There are some important points

Observation 2.2. *In homogeneous systems we have that*

- *Finite Time Stability (FTS) is equivalent to an infinite eigenvalue assignation for the closed-loop system at the origin, therefore the right-hand side of the ordinary differential equation cannot be locally Lipschitz at the origin.*
- *There exists the settling time function $T(x_0)$ that determines the time for a solution to reach the equilibrium, this function depends on the initial condition of a solution. In general this function T can grow unboundedly (possibly more than linearly).*

The main issue with T is its continuity at the origin. For continuous systems, the continuity of T at 0 is equivalent to the continuity of T everywhere. The bi-limit homogeneity application allows us to have a globally bounded T, which means that in practice one gets a FxT convergence to the origin for all initial conditions.

A recently application of bl-homogeneity to the observation problem is in the differentiators developed with correction terms having property of being homogeneous in the bi-limit, therefore, under some assumptions of uniform bounding the differentiator is able to estimate exactly in FxT the true derivatives of a signal f. In fact, these results are closely linked to this work.

2.4 Arbitrary Order Fixed-Time Differentiators

[31] Given a signal $f(t)$ defined on $[0,\infty)$, the objective is to estimate some of its time derivatives. $f(t)$ is composed of a base signal f_0 n-times differentiable, and a uniformly bounded noise $v(t)$, i.e.$f(t) = f_0(t) + v(t)$ and $|f_0^{(n)}(t)| \leq \Delta$ with $\Delta \geq 0$.

Defining the variables $\varsigma_1 = f_0(t), \varsigma_2 = \dot{f}_0(t), ..., \varsigma_n = f_0^{(n-1)}(t)$, where $f_0^{(i)}(t) = \frac{d^i}{dt^i} f_0(t)$. A state representation of f_0 is

$$\begin{aligned} \dot{\varsigma}_i &= \varsigma_{i+1} \qquad i = 1, ..., n-1 \\ \dot{\varsigma}_n &= f_0^{(n)}(t) \end{aligned} \tag{2.32}$$

In order to estimate the derivatives $f_0^{(i)}(t)$ for $i = 1, ..., n-1$ we have the following nonlinear family of differentiators

$$\begin{aligned} \dot{x}_i &= -k_i \phi_i(x_1 - y) + x_{i+1}, \qquad i = 1, ..., n-1 \\ \dot{x}_n &= -k_n \phi_n(x_1 - y) \end{aligned} \tag{2.33}$$

where the nonlinear output injection terms, given by

$$\phi_i(z) = \varphi_i \circ ...\varphi_2 \circ \varphi_1(z) \tag{2.34}$$

are the composition of the monotonic growing functions

$$\varphi_i(s) = \kappa_i \lceil s \rfloor^{\frac{r_{0,i+1}}{r_{0,i}}} + \theta_i \lceil s \rfloor^{\frac{r_{\infty,i+1}}{r_{\infty,i}}} \tag{2.35}$$

with powers selected as $r_{0,n} = r_{\infty,n} = 1$, and for $i = 1, ..., n+1$

$$\begin{aligned} r_{0,i} &= r_{0,i+1} - d_0 = 1 - (n-i)d_0 \\ r_{\infty,i} &= r_{\infty,i+1} - d_\infty = 1 - (n-i)d_\infty \end{aligned} \tag{2.36}$$

which are completely defined by two parameters $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$. With this selection the first term in (2.35) is dominating for small values of s, while the second one is dominating for large values of s.

Differentiator (2.33) is not homogeneous, but it is homogeneous in the bi-limit, that is, near to the origin it is approximated by a homogeneous system of degree d_0 and far from the origin it is approximated by a homogeneous system of degree d_∞. Although the scaling properties of the homogeneous systems are lost, the design of bl-homogeneous differentiators is more flexible, since the properties near the origin and far from it can be assigned independently.

If we select $d_0 = d_\infty = d$ the differentiator (2.33) becomes homogeneous. And making $d = 0$ it is obtained the High-Gain differentiator, for $d = -1$ Levant's Robust and Exact Differentiator (RED) is recovered and for other values of d the family of continuous differentiators in [11][36][12][22] is attained. For polynomial signals note that if $d < 0$ (resp. $d = 0$) the estimation converges in finite-time (resp. exponentially). For $d > 0$ the convergence is asymptotic, but it attains any neighborhood of zero in a time which is uniform in the initial conditions.

A particular case of interest for the differentiator is a property that is only achieved when $d_0 = -1$. In that case ϕ_n is discontinuous and it induces a Higher-Order Sliding-Mode at the origin, allowing the estimation to converge (in the absence of noise) exactly, robustly and in finite-time to the real values of the signal derivatives when the n-th derivative of the signal is bounded by a non zero constant $\Delta \in \mathbb{R}_{\geq 0}$, i.e. $|f_0^{(n)}(t)| \leq \Delta$. For all other values of $d_0 > -1$, the convergence is only achieved if $\Delta = 0$.

As we mentioned in Observation 2.2, one of the disadvantages in homogeneous (including Levant's RED) differentiators with $d_0 < 0$, is that the convergence time, although finite, grows unboundedly (and faster than linearly) with the size of the initial estimation error. One of the nice features of the bl-homogeneous design in general and of the proposed differentiator (2.33) in particular, is that assigning a positive homogeneity degree to the ∞-limit approximation $d_\infty > 0$ and a negative homogeneity degree to the 0-limit approximation $d_0 < 0$, it is possible to counteract this effect: Convergence of the estimation will be achieved in Fixed-Time [31].

The main result of this differentiators (2.33) can be expressed formally as follows. In the absence of noise, it is able to estimate asymptotically the first $n-1$ derivatives of the signal $f_0(t)$. Let $\mathscr{F}_0^n \triangleq \left\{ f^{(n)}(t) \equiv 0 \right\}$ represent the class of polynomial signals and $\mathscr{F}_\Delta^n \triangleq \left\{ \left| f^{(n)}(t) \right| \leq \Delta \right\}$ corresponds to the class of n-Lipschitz signals.

Assumption 2.2. *$f(t) = f_0(t) + \nu(t)$, with $f_0(t)$ $n-$times differentiable, $|f^{(n)}(t)| \leq \Delta$, and $\nu(t)$ a uniformly bounded measurable signal.*

Then it is possible to have the following statement

Theorem 2.8. *[31] Let the function $f(t) = f_0(t)$ be such that Assumption 2.2 is fulfilled. Select $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ and choose arbitrary positive (internal) gains $\kappa_i > 0$ and $\theta_i > 0$, for $i = 1, ..., n$. Suppose that either $\Delta = 0$ or $d_0 = -1$. Under this conditions, and in the absence of noise ($\nu(t) \equiv 0$) there exist appropriate gains $k_i > 0$, for $i = 1, ..., n$, such that the solutions bl-homogeneous differentiator (2.33) converge globally and asymptotically to the derivatives of signal, i.e. $x_i(t) \to f_0^{(i-1)}(t)$ as $t \to \infty$. In particular, they converge in Fixed-Time, i.e. $\exists \bar{T} > 0$ such that for any $x_i(0) \in \mathbb{R}^n$, $x_i(t) \equiv f_0^{(i-1)}(t)$ for $t \geq \bar{T}$ for $i = 1, ..., n$ if either*

$$(a) \qquad -1 < d_0 < 0 < d_\infty < \frac{1}{n-1} \quad \text{and} \quad f(t) \in \mathscr{F}_0^n, \quad \text{or}$$

$$(b) \qquad -1 = d_0 < 0 < d_\infty < \frac{1}{n-1} \quad \text{and} \quad f(t) \in \mathscr{F}_\Delta^n.$$

The proof of this Theorem 2.8, which is of essential importance in this work is given in the the Apendix A and detailed in [31].

This differentiator can be seen as an observer for a special type of SISO systems, composed by a chain of n integrators and with a unknown input. The idea of this work is generalizing this observer to a family of MIMO-LTI systems by decomposing the original system in a set of subsystems and designing a bl-homogeneous observer composed by a set of sub-observers with unknown inputs. And proof that the convergence is achieved exactly and in fixed time by appropriately selecting the set of gains in the observer.

With the necessary theory of homogeneous and bl-homogeneous systems given in this chapter, we are ready to present the main contribution of this work in Chapters 3 and 4.

Chapter 3

Bl-Homogeneous observers for SISO Linear Time Invariant systems

In this chapter we present the first part of main result of this work. We introduce the design of Bl-homogeneous observers for SISO-LTI systems with bounded unknown inputs assuming strong observability. The idea is to transform the system in to a Special Coordinate Basis, (detailed in Chapter 2 for the MIMO general case) obtaining a representation of the system in which it is possible to design an UIO.

Here we use directly a discontinuous nonlinear observer instead of differentiators. This fact suppress the necessity of using a cascade scheme composed by a linear observer and a discontinuous differentiator.

The nonlinear injection terms can be designed to accelerate the convergence as much as we want by selecting appropriate and sufficiently large gains. Even more, due to the assignability of bl-homogeneous degrees in the observer we can reach and assure exact and finite-time (or moreover fixed-time) stability of the error estimation dynamics in presence of unknown inputs.

3.1 System transformation

Before attacking the MIMO case we will introduce the Single-Input Single-Output (SISO) case, which going to be useful in order to give the basic idea in solving the estimation problem.

Consider the SISO-LTI system without feedthrough (for simplicity) given by

$$\Sigma : \begin{cases} \dot{x} & = Ax + D\omega \\ y & = Cx \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\omega \in \mathbb{R}$ the unknown input and $y \in \mathbb{R}$ is the output of the system. Accordingly, the matrices A, D, C have appropriate dimensions. For simplicity in the development we do not consider a known input u, since it does not modify the observability properties and it is simple to include it in the observer design.

The task is to build an observer providing for finite-time (preferably fixed-time convergent and exact) estimation of the states in presence of the unknown input. In the previous chapters we have stated the general conditions for the existence and characterization of unknown input observers (UIO). Here we will recall this conditions in the particular case we are working on.

The equations in the observer will be understood in the Filippov sense [14] in order to provide for the possibility to use discontinuous signals. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous.

Accordingly to the Definitions 2.1 and 2.2 the system (3.1) is strongly observable if the triple (A, D, C) has no invariant zeros. Unfortunately, this definition does not give specific nor convenient form to the system matrices. Special Coordinate Basis for SISO case (a particular case of MIMO-SCB presented in Chapter 2) clarifies this problem.

Theorem 3.1. *Consider the system (3.1). There exist nonsingular state, input and output transformations $\Gamma_s \in \mathbb{R}^{n\times n}, \Gamma_i \in \mathbb{R}, \Gamma_o \in \mathbb{R}$, which decompose the state space of Σ into two subspaces, x_a and x_d. These two subspaces correspond to the finite zero and infinite zero structures of Σ, respectively. The new satate spaces, input and output spaces of the decomposed system are described by the following set of equations:*

$$x = \Gamma_s \bar{x}, \quad y = \Gamma_o \bar{y}, \quad u = \Gamma_i \bar{u}, \tag{3.2}$$

$$\bar{x} = \begin{bmatrix} x_a \\ x_d \end{bmatrix}, x_a \in \mathbb{R}^{n_a}, \quad x_d \in \mathbb{R}^{n_d}, \quad x_d = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_d} \end{bmatrix}, \tag{3.3}$$

and

$$\Sigma_{SCB} : \begin{cases} \dot{x}_a &= A_{aa}x_a + H_{ad}y \\ \dot{x}_{d,1} &= x_{d,2}, \quad y = x_1, \\ \dot{x}_{d,j} &= x_{d,j+1} \\ & \vdots \quad j = 2, ..., n_d - 1 \\ \dot{x}_{d,n_b} &= a_{d,1}x_{d,1} + a_{d,2}x_{d,2} + ... + a_{d,n_d}x_{d,n_d} + \omega \end{cases} \tag{3.4}$$

Similar to Property 2.1 we have:

Property 3.1. *The system Σ_{SCB} in (3.4) is strongly observable if and only if x_a is non-existent.*

This is equivalent to have relative degree n with respect to the unknown input $\omega(t)$. This latter is a sufficient condition of strong observability presented in [18].

If we assume strong observability, then we can apply an extra transformation $\Gamma_{\mathcal{O}} = \mathcal{O}^{-1}$, where $\mathcal{O}$ is the observability matrix and puts the system in observability canonical form.

3.2 Unknown Input Observer design

Given a strongly observable system Σ in (3.1) under the SCB transformation (3.2), and suppressing the subscript d, the system is then given by

$$\Sigma_s : \begin{cases} \dot{x}_1 &= x_2, \quad y = x_1, \\ \dot{x}_j &= x_{j+1} \\ & \vdots \quad j = 2, ..., n_d - 1 \\ \dot{x}_n &= A_{dd}x + \omega, \end{cases} \tag{3.5}$$

where $A_{dd} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ and we have by Property 3.1 that $n_d = n$ and it is supposed the following.

Assumption 3.1. *Unknown input $\omega(t)$ a is uniformly bounded function, $|\omega(t)| \leq \Delta, \quad \Delta \in \mathbb{R}_{\geq 0}$*

This allow us to relax the existence conditions of UIO's in o order to have an observer under strong observability only, see Section 2.2. It has to be noted that the system (3.5) is in the observability canonical form, which requires the bl-homogeneity of the observer, since the observer canonical form can be implemented with a homogeneous differentiator (as already done in [33]). It is clear that in observer form the bl-homogeneous observer can also be implemented.

The observer is given by

$$\Omega : \begin{cases} \dot{\hat{x}}_1 & = -k_1 L \tilde{\phi}_1(\hat{x}_1 - y) + \hat{x}_2 \\ \dot{\hat{x}}_j & = -k_j L^j \tilde{\phi}_j(\hat{x}_1 - y) + \hat{x}_{j+1} \\ \vdots & j = 2, ..., n-1 \\ \dot{\hat{x}}_n & = -k_n L^n \tilde{\phi}_n(\hat{x}_1 - y) + A_{dd}\hat{x} \end{cases} \tag{3.6}$$

with positive external gains $k_j > 0$ and positive tuning gains $\alpha, L > 0$, appropriately selected as it will be show latter. The output injection terms $\tilde{\phi}_j(\cdot)$ are obtained from the functions

$$\phi_j(s) = \kappa_j \lceil s \rfloor^{\frac{r_{0,j+1}}{r_{0,1}}} + \theta_j \lceil s \rfloor^{\frac{r_{\infty,j+1}}{r_{\infty,1}}} \tag{3.7}$$

by scaling the positive internal gains $\kappa_j > 0, \theta_j > 0$

$$\kappa_j \to \left(\frac{L^n}{\alpha}\right)^{\frac{jd_0}{r_{0,1}}} \kappa_j, \qquad \theta_j \to \left(\frac{L^n}{\alpha}\right)^{\frac{jd_\infty}{r_{\infty,1}}} \theta_j \tag{3.8}$$

with powers selected as $r_{0,n} = r_{\infty,n} = 1$, and

$$\begin{aligned} r_{0,j} &= r_{0,j+1} - d_0 = 1 - (n-j)d_0 \\ r_{\infty,j} &= r_{\infty,j+1} - d_\infty = 1 - (n-j)d_\infty \end{aligned} \tag{3.9}$$

which are completely defined by two parameters d_0, d_∞. They have to satisfy $-1 \le d_0 \le d_\infty < \frac{1}{n-1}$.

We have to highlight the fact that injection terms in (3.6) and (3.7) are very similar to them in the bl-homogeneous differentiator (2.33) but the ones here are simpler. This simplifies the task of implementation.

3.2.1 Gain Selection

Each type of gain in the observer has a different role, and the idea in the gain tuning is very intuitive.

1. The internal gains $\kappa_j > 0, \theta_j > 0$ can be selected arbitrary. They can be selected as arbitrary positive real values, and correspond to the desired weighting of each term of low degree and high degree respectively in ϕ_j.

2. The external gains $k_j > 0$ have the objective of stabilizing the observer in absence of interconnections and external perturbations, i.e. when $A_{dd} = 0$ and $\omega(t) = 0$.

3. Parameter L is selected large enough to assure the convergence in presence of interconnections, but not of the bounded perturbations $\omega(t)$. Setting its value grater than minimal value to assure stability the convergence velocity will be increased.

4. The tuning parameter α is selected large enough to assure the convergence in presence of the bounded unknown input $\omega(t)$.

3.2.2 Estimation in original coordinates

The estimated state obtained from the observer Ω in (3.6) corresponds to the transformed system Σ_{SCB} in (3.5) represented in SCB coordinates through the state Γ_s, input Γ_i and output Γ_o transformations (3.2), moreover it was applied an extra transformation $\Gamma_{\mathcal{O}} = \mathcal{O}^{-1}$ which puts the system in observability canonical form.

The states in original coordinates can be computed as

$$x = \Gamma_s \Gamma_{\mathcal{O}} \hat{x} \tag{3.10}$$

Therefore, the observer in original coordinates for the system (3.1) takes the form

$$\begin{aligned} \dot{\hat{x}} &= -\Gamma_s \Gamma_{\mathcal{O}} K \Phi(e_y) + A\hat{x} + Bu \\ \hat{y} &= \Gamma_o C \hat{x} \end{aligned} \tag{3.11}$$

with $e_y = \hat{y} - y$ and

$$\begin{aligned} K &= \mathrm{diag}(k_1 L, k_2 L^2, ..., k_n L^n) \\ \Phi(\hat{y} - y) &= \begin{bmatrix} \tilde{\phi}_1(e_y) & \tilde{\phi}_2(e_y) & ,\dots, & \tilde{\phi}_n(e_y) \end{bmatrix}^T \end{aligned} \tag{3.12}$$

3.2.3 Main result. UIO - SISO case

The main result of this work in the SISO case establishes that the Unknown Input Observer (3.6) is able to estimate at least asymptotically the states of a strongly observable linear system.

Theorem 3.2. *Let the strongly observable SISO-LTI system* Σ (3.1) *has an UIO given by* (3.11),(3.12). *Select* $-1 \leq d_0 \leq d_\infty < \frac{1}{n-1}$ *and chose arbitrary (internal gains)* $\kappa_j > 0$ *and* $\theta_j > 0$, *for* $j = 1, ..., n$. *Suppose that either* $\Delta = 0$ *or* $d_0 = -1$. *Under this conditions, there exist appropriate gains* $k_j > 0$, *for* $j = 1, ..., n$, *and parameters* $L > 0, \alpha > 0$ *sufficiently large such that the solutions of bl-homogeneous UIO* (3.6) *converge globally and asymptotically to the true states of* Σ_s, *i.e.* $\hat{x}_j(t) \to x_j(t)$ *as* $t \to \infty$.

In particular, it can converge globally and

- *exponentially if*

$$d_0 = 0 \quad \textit{with} \quad \Delta \equiv 0, \tag{3.13}$$

- *finite-time if*

$$\begin{aligned} &(a) \quad -1 < d_0 < 0 \quad \textit{with} \quad \Delta \equiv 0, \quad \textit{or} \\ &(b) \quad d_0 = -1 \quad \textit{with} \quad \Delta \neq 0 \end{aligned} \tag{3.14}$$

- *fixed-time if*

$$\begin{aligned} &(a) \quad -1 < d_0 < 0 < d_\infty < \tfrac{1}{n-1} \quad \textit{with} \quad \Delta \equiv 0, \quad \textit{or} \\ &(b) \quad -1 = d_0 < 0 < d_\infty < \tfrac{1}{n-1} \quad \textit{with} \quad \Delta \neq 0. \end{aligned} \tag{3.15}$$

3.2.4 Proof of Theorem 3.2

The proof will be carried out in a Lyapunov framework through a bl-homogeneous Lyapunov function, this one can be used to realize an estimation fo the convergence time and calculation of gains k_j moreover in an optimal sense. Part of this work had been presented in [31]. This work does not address the problem.

To study the error system in a more suitable form, we are going to take the system and observer in the transformed SCB coordinates, i.e. the system (3.5) and observer (3.6). It is clear that the analysis is completely equivalent in original coordinates.

Let the estimation error $e_j = \hat{x}_j - x_j$. The dynamics error are described by

$$\Xi : \begin{cases} \dot{e}_1 &= -k_1 L \tilde{\phi}_1(e_1) + \hat{e}_2 \\ \dot{e}_j &= -k_j L^j \tilde{\phi}_j(e_1) + \hat{e}_{j+1} \\ \vdots & j = 2, ..., n-1 \\ \dot{e}_n &= -k_n L^n \tilde{\phi}_n(e_1) + A_{dd}\hat{e} - \omega \end{cases} \tag{3.16}$$

where, by Assumption 3.1 $\omega(t) \leq \Delta$. Applying the time scaling via the next transformation

$$\epsilon_j = \frac{L^{n-j+1}}{\alpha} e_j, \quad j = 1, ..., n \tag{3.17}$$

we obtain

$$\Xi_{d,i} : \begin{cases} \dot{\epsilon}_1 &= L\left[-k_1\phi_1(\epsilon_1) + \epsilon_2\right] \\ \dot{\epsilon}_j &= L\left[-k_j\phi_j(\epsilon_1) + \epsilon_{j+1}\right] \\ \vdots & j = 2, ..., n-1 \\ \dot{\epsilon}_n &= L\left[-k_n\phi_n(\epsilon_1) + \frac{1}{\alpha}\Psi_i(\epsilon, \omega)\right] \end{cases} \tag{3.18}$$

where

$$\Psi(\epsilon, \omega) = A_{dd} e - \omega = \sum_{j=1}^{n} a_j e_j - \omega = \alpha \sum_{j=1}^{n} \frac{a_j}{L^{n-j+1}} \epsilon_j - \omega \tag{3.19}$$

the fact that $\tilde{\phi}_j(\frac{\alpha}{L^n} s) = \frac{\alpha}{L^n} \tilde{\phi}_j(s)$ has been used.

For the convergence proof, it is convenient to perform another state transformation

$$z_j = \frac{\epsilon_j}{k_{j-1}}, \quad k_0 = 1, \quad j = 1, ..., n \tag{3.20}$$

Then (3.18) become

$$\Xi^* : \begin{cases} z_1' &= -\tilde{k}_1\left(\phi_1(z_1) + z_2\right) \\ z_j' &= -\tilde{k}_j\left(\phi_j(z_1) + z_{j+1}\right) \\ \vdots & j = 2, ..., n-1 \\ z_n' &= -\tilde{k}_n \phi_n(z_1) + \tilde{\Psi}(z, \omega) \end{cases} \tag{3.21}$$

with $\tilde{k}_j = \frac{k_j}{k_{j-1}}, \quad k_0 = 1, \quad j = 1, ..., n$ and

$$\tilde{\Psi}(z, \omega) = \frac{1}{k_{n-1}} \sum_{j=1}^{n} \frac{a_j k_{j-1}}{L^{n-j+1}} z_j - \frac{1}{\alpha k_{n-1}} \omega \tag{3.22}$$

Lyapunov analysis

Before presenting the Lyapunov function we have to recall that the output injection terms in (3.7) are much simpler than those described in [31]. However, the stability proof in [31] for the differentiator is applicable to the case with the simpler injection terms (3.7), since the same requirements and properties are fulfilled. The functions (3.7) can be written as a composition of functions $\varphi_j(s)$. Such that

$$\phi_j(s) = \varphi_j \circ ... \circ \varphi_2 \circ \varphi_1(s) \tag{3.23}$$

where

$$\begin{aligned}
\varphi_1(s) &= \phi_1(s) \\
\varphi_2(s) &= \phi_2 \circ \phi^{-1}(s) \\
&\vdots \quad j = 2, ..., n \\
\varphi_j(s) &= \phi_j \circ \phi_{j-1}^{-1}(s), \quad j = 2, ..., n
\end{aligned} \tag{3.24}$$

We will use a (smooth) bl-homogeneous Lyapunov Function (bl-LF) V, which was introduced in [31]. Selecting for $n \geq 2$ two positive real numbers $p_0, p_\infty \in \mathbb{R}_+$ that correspond to the homogeneity degrees of the 0-limit and the ∞-limit approximations of V, such that

$$\begin{aligned}
p_0 &\geq \max_{j\in\{1,...,n\}} \{r_{0,j}\} + d_0 \\
p_\infty &\geq \max_{j\in\{1,...,n\}} \left\{ 2r_{\infty,j} + \frac{r_{\infty,j}}{r_{0,j}} d_0 \right\} \\
\frac{p_0}{r_{0,j}} &\leq \frac{p_\infty}{r_{\infty,j}}
\end{aligned} \tag{3.25}$$

For $i = 1, ..., n$ choosing arbitrary positive real numbers $\beta_{0,i}, \beta_{\infty,i} > 0$ such that the following functions are defined

$$\begin{aligned}
Z_j(z_j, z_{j+1}) &= \sum_{k\in\{0,\infty\}} \beta_{k,j} \left[\frac{r_{k,j}}{p_k} |z_j|^{\frac{p_k}{r_{k,j}}} - z_j \lceil \xi_j \rfloor^{\frac{p_k - r_{k,j}}{r_{k,j}}} + \frac{p_k - r_{k,j}}{p_k} |\xi_j|^{\frac{p_k}{r_{k,j}}} \right] \\
\xi_j &= \varphi_j^{-1}(z_{j+1}) \quad j = 1, ..., n-1 \\
\xi_j &= z_{n+1} = 0, \quad j = n \\
Z_n(z_n) &= \beta_{0,n} \frac{1}{p_0} |z_n|^{p_0} + \beta_{\infty,n} \frac{1}{p_\infty} |z_n|^{p_\infty}
\end{aligned} \tag{3.26}$$

where we have

Lemma 3.1. *[31] $Z_j(z_j, z_{j+1}) \geq 0$ for every $j = 1, ..., n$ and $Z_j(z_j, z_{j+1}) = 0$ if and only if $\varphi_j(z_j) = z_{j+1}$.*

The Bl-homogeneous Lyapunov Function (Bl-LF) is defined as

$$V(z) = \sum_{j=1}^{n-1} Z_j(z_j, z_{j+1}) + Z_n(z_n) \tag{3.27}$$

For the partial derivatives we introduce the following variables

$$\begin{aligned} \sigma_j(z_j,z_{j+1}) &\triangleq \frac{\partial Z_j(z_j,z_{j+1})}{\partial z_j} = \sum_{k\in\{0,\infty\}} \beta_{k,j}\left(\lceil z_j\rfloor^{\frac{p_k-r_{k,j}}{r_{k,j}}} - \lceil \xi_j\rfloor^{\frac{p_k-r_{k,j}}{r_{k,j}}}\right) \\ s_j(z_j,z_{j+1}) &\triangleq \frac{\partial Z_j(z_j,z_{j+1})}{\partial z_{j+1}} = \sum_{k\in\{0,\infty\}} -\beta_{k,j}\frac{p_k-r_{k,j}}{r_{k,j}}(z_j-\xi_j)|\xi_j|^{\frac{p_k-2r_{k,j}}{r_{k,j}}}\frac{\partial \xi_j}{z_{j+1}} \end{aligned} \tag{3.28}$$

where $\xi_i = \varphi_i^{-1}(z_{i+1})$. Note that $Z_{b,\iota,j}, \sigma_{b,\iota,j}, s_{b,\iota,j}$ vanish when $\varphi_{b,\iota,j}(z_{b,\iota,j}) = z_{b,\iota,j+1}$. Performing time derivative with respect the new time variable τ

$$V'(z) = -W(z) + \frac{\partial V(z)}{\partial z_n}\tilde{\Psi}(z,\omega) \tag{3.29}$$

where $\frac{\partial V(z)}{\partial z_n} = [s_{n-1}+\sigma_n]$ and

$$\begin{aligned} W(z) &= \tilde{k}_1\sigma_1(\phi_1(z_1)-z_2) \\ &+ \sum_{j=2}^{n-1}\tilde{k}_j\left[s_{j-1}+\sigma_j\right](\phi_j(z_1)-z_{j+1}) \\ &+ \tilde{k}_n\left[s_{n-1}+\sigma_n\right]\phi_n(z_1) \end{aligned} \tag{3.30}$$

Due to the definition of s_j in (3.28), $s_n \equiv 0$ and functions $s_j, \sigma_j \in \mathcal{C}$ in $\mathbb{R}$, are r-bl-homogeneous of degrees $p_0 - r_{0,j}, p_0 - r_{0,j+1}$ for the 0-approximation and $p_\infty - r_{\infty,j}, p_\infty - r_{\infty,j+1}$ for the ∞-approximation, respectively. Additionally, for $j = 1, ..., n$ we have $\sigma_j = 0$ on the same set as $s_j = 0$, i.e. they become both zero at the points where Z_j achieves its minimum, $Z_j = 0$.

V is bl-homogeneous of degrees p_0 and p_∞ and $\mathcal{C}$ on $\mathbb{R}$. It is also non negative, since it is a positive combination of non negative terms. Moreover, V is positive definite since $V(z) = 0$ only if all $Z_j = 0$, what only happens at $z = 0$. Due to bl-homogeneity it is also radially unbounded.

If we analyze (3.30), $W(z)$ is bl-homogeneous of degree $p_0 + d_0$ for the 0-approximation and $p_\infty + d_\infty$ for the ∞-approximation.

It has been shown in [31] that there exists appropriate gains $\tilde{k}_j$ such that $W(z)$ in (3.30) is rendered positive definite. The idea in the following is to prove that there exist gains L, α sufficiently large such that the negative definiteness of $-W(z)$ and therefore $V'(z)$ is hold.

From (3.29), we are now interested in finding an upper bound of $\tilde{\Psi}$. Assuming $L \geq 1$, and $\alpha \geq 1$. Due to the power of L we can write

$$\begin{aligned} \tilde{\Psi}(z,\omega) &= \sum_{j=1}^{n}\frac{a_jk_{j-1}}{k_{n-1}L^{n-j+1}}z_j - \frac{1}{\alpha k_{n-1}}\omega = \frac{1}{L}\tilde{\Psi}_s + \frac{1}{\alpha}\tilde{\Psi}_\omega \\ \tilde{\Psi}_s &= \sum_{j=1}^{n}\frac{a_jk_{j-1}}{k_{n-1}L^{n-j}}z_j, \quad \tilde{\Psi}_\omega = -\frac{1}{k_{n-1}}\omega \end{aligned} \tag{3.31}$$

The term $\frac{\partial V(z)}{\partial z_n}$ is bl-homogeneous of degree $p_0 - r_{0,n} = p_0 - 1$ for the 0-approximation and $p_\infty - r_{\infty,n} = p_\infty - 1$ for the ∞-approximation. Using the properties of bl-homogeneous functions, it is clear that each term $\frac{\partial V(z)}{\partial z_n}z_j$ is bl-homogeneous of degree $p_0 - r_{0,n} + r_{0,j} = p_0 - (n-j)d_0$ for the 0-approximation and $p_\infty - r_{\infty,n} + r_{\infty,j} = p_\infty - (n-j)d_\infty$ for the ∞-approximation. Finally, since $d_0 \leq 0$ and $d_\infty \geq 0$ we can conclude that

$$\begin{aligned} p_0 + d_0 &\leq p_0 - (n-j)d_0 \\ p_\infty + d_\infty &\geq p_\infty - (n-j)d_\infty \end{aligned} \tag{3.32}$$

and by the properties of bl-homogeneous functions, there exists a positive real number $\lambda_1 > 0$ which satisfy

$$\frac{\partial V(z)}{\partial z_n}\frac{1}{L}\tilde{\Psi}_s \leq \frac{\lambda_1}{L}W(z) \tag{3.33}$$

furthermore, there exists $\lambda_2 > 0$ such that

$$\frac{\partial V(z)}{\partial z_n} \leq -\lambda_2\left[W(z)^{\frac{p_0-1}{p_0+d_0}} + W(z)^{\frac{p_\infty-1}{p_\infty+d_\infty}}\right] \tag{3.34}$$

If we put everything together, $V'(z)$ can be bounded as

$$\begin{aligned} V'(z) &\leq -W(z) + \frac{\lambda_1}{L}W(z) + \frac{\lambda_2}{\alpha}\left[W(z)^{\frac{p_0-1}{p_0+d_0}} + W(z)^{\frac{p_\infty-1}{p_\infty+d_\infty}}\right]\|\tilde{\Psi}_\omega\|_\infty \\ &= -\left(1-\frac{\lambda_1}{L}\right)W(z) + \frac{\lambda_2}{\alpha}\left[W(z)^{\frac{p_0-1}{p_0+d_0}} + W(z)^{\frac{p_\infty-1}{p_\infty+d_\infty}}\right]\|\tilde{\Psi}_\omega\|_\infty \end{aligned} \tag{3.35}$$

If we apply Lyapunov arguments we can conclude that we can chose L sufficiently large such that the first term become negative definite. In absence of $\tilde{\Psi}_\omega$, $z = 0$ is asymptotic stable. With $d_0 < 0$ it converges in finite time, moreover, with $d_\infty > 0$ it converges in fixed-time.

In the case $\tilde{\Psi}_\omega \neq 0$ but ultimately bounded and selecting $d_0 = -1$ we can chose α sufficiently large, such that $V'(z)$ is negative definite. And then, finite-time or fixed-time stability is achieved.

■

3.2.5 Some comments

- Although the observation problem for linear systems has been solved through the direct application of a differentiator putting the system in observer form, the same differentiator cannot globally converge with the system in observability form, since bounded state variables would be necessary. We have proved that the UIO presented here globally converges with the system expressed in observability form.

- The bl-homogeneous correction terms seen as a composition of two homogeneous systems of degrees d_0 and d_∞ respectively allow the observer to deal with the linear terms of the model far from the origin and in turn guarantee convergence near the origin even in presence of a bounded unknown input.

- Each type of gain in the observer has a specific role. The proof showed that the tuning process is very intuitive for the designer. Once the gains k_j (which ensure observer stability in absence of A_{dd}) have been selected, the designer can decrease the effect of this linear terms by increasing the values of α and L, then velocity of convergence can be accelerated when possible by further increasing their value.

- The observer has been designed for strongly observable systems, i.e. systems without zeros at all, which translates into systems without internal dynamics. This is clarified under the SCB transformation where the strong observability condition results in the non-existence of the subsystem x_a in (3.4). However, the results presented can be easily extended to strongly detectable systems (having only stable invariant zeros), that is, systems whose internal dynamics are stable (A_{aa} Hurwitz in (3.4)). The convergence of the observer would then be subject to the asymptotic convergence of the part associated with x_a. Since it is an inaccessible dynamic, there is no freedom to accelerate the speed of convergence.

This idea will be applied in the MIMO case, but before that, it will be present some examples to illustrate the effectiveness of the observer.

3.3 Example

Let a *strongly observable* system taken from [18] and given by

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu + D\omega \\ y &= Cx \end{cases} \tag{3.36}$$

where $x \in \mathbb{R}^4, u \in \mathbb{R}, \omega \in \mathbb{R}, y \in \mathbb{R}$ are the states, known input, unknown input and output respectively. Note that the system is unstable since the matrix A has eigenvalues $\Lambda = \{-3, -2, -1, 1\}$ and one of them is positive, it means that one or more state trajectories can grow unboundedly.

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 5 & -5 & -5 \end{bmatrix}, \quad B = D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ \omega(t) &= \cos(0.5t) + 0.5\sin(3t) + 0.5, \quad |\omega(t)| \leq 2 \end{aligned} \tag{3.37}$$

which can be written as

$$\Sigma : \begin{cases} \dot{x}_{b,1} &= x_{b,2}, \qquad y_1 = x_{b,1} \\ \dot{x}_{b,2} &= x_{b,3} \\ \dot{x}_{b,3} &= x_{b,4} \\ \dot{x}_{b,4} &= [6 \quad 5 \quad -5 \quad -5]x + \omega + u \end{cases} \tag{3.38}$$

The system is already in observability canonical form. Furthermore, the UIO can be designed as

$$\Omega : \begin{cases} \dot{\hat{x}}_1 &= -k_1 L\tilde{\phi}_1(\hat{x}_1 - y_1) + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -k_2 L^2\tilde{\phi}_2(\hat{x}_1 - y_1) + \hat{x}_3 \\ \dot{\hat{x}}_3 &= -k_3 L^3\tilde{\phi}_3(\hat{x}_1 - y_1) + \hat{x}_4 \\ \dot{\hat{x}}_4 &= -k_4 L^4\tilde{\phi}_4(\hat{x}_1 - y_1) + [6 \quad 5 \quad -5 \quad -5]\hat{x} + u \end{cases} \tag{3.39}$$

where the nonlinear output injection terms $\tilde{\phi}.(\cdot)$ are as follows

$$\tilde{\phi}_j(s) = \left(\frac{L^4}{\alpha}\right)^{\frac{jd_0}{1-3d_0}} \kappa_j \lceil s \rfloor^{\frac{1-(3-j)d_0}{1-3d_0}} + \left(\frac{L^4}{\alpha}\right)^{\frac{jd_\infty}{1-3d_\infty}} \theta_j \lceil s \rfloor^{\frac{1-(3-j)d_\infty}{1-3d_\infty}}, \quad j = 1, ..., 4 \tag{3.40}$$

The assigned homogeneity degrees d_0, d_∞ in 0 and ∞ respectively in (4.63) have to satisfy

$$-1 \leq d_0 \leq d_\infty < \frac{1}{3} \tag{3.41}$$

We present three cases:

1. Linear UIO. Homogeneity degrees $d_0 = d_\infty = 0$.
2. Continuous UIO. Homogeneity degrees $d_0 = -\frac{1}{8}, d_\infty = 0$
3. HOSM-UIO. Homogeneity degrees $d_0 = -1$, $d_\infty = \frac{1}{8}$. With this selection we get a discontinuous observer. Note that $d_0 < 0 < d_\infty$.

The initial conditions of the plant states are $x_0 = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}$ and $\hat{x}_j = 5, j = 1, ..., 4$ for the observer. For all cases the values of gains are fixed as

$$\left\{ k_1 = 8.6 k_4^{\frac{1}{4}} \quad k_2 = 21 k_4^{\frac{1}{2}} \quad k_3 = 16.25 k_4^{\frac{1}{3}} \quad k_4 = 1 \right\} \tag{3.42}$$

internal gains $\kappa = \theta = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and parameters $L = 1, \alpha = 50$. We perform simulations along 5 seconds. We have used a fixed-step explicit Euler method, with integration step $\tau = 1 \times 10^{-5}$.

The Figures 3.2, 3.3 related to the linear and continuous cases respectively show that the observer can not exactly estimate the states of the plant, i.e. although the estimation error converges to a neighborhood of zero, it is not able to converge exactly to zero, this is due to non-compensation of unknown input.

In the third case, with the selection of homogeneity degree $d_0 = -1$ for the zero approximation (discontinuous observer) a HOSM is induced which allows the observer to compensate exactly the effect of unknown input. It is shown in Figure 3.4 that the observer achieve exact estimation of the estates, even in presence of the unknown input. This is illustrated in the last subfigures where the error norm converges to zero exactly in finite time.

In fact, in this case we get more than finite time stability, fixed time stability, i.e. there exist a $\bar{T}$ independent of e_0 such that for any initial error we have exact convergence in a time less than $\bar{T}$. In order to show this, With $L = 5$ now, the Figure 3.1 shows the norm of error vector $\|e\| = \sqrt{\sum_{j=1}^{4} e_j^2}$ for a wide range of magnitude orders in initial error $e_0 \times 10^p, p = 0, 2, ..., 14$. Despite of this, the convergence time does not increase beyond an upper bound in $\bar{T} = 5s$, more over, and as we said before this can be reduced arbitrary by increasing appropriately the value of parameter L.

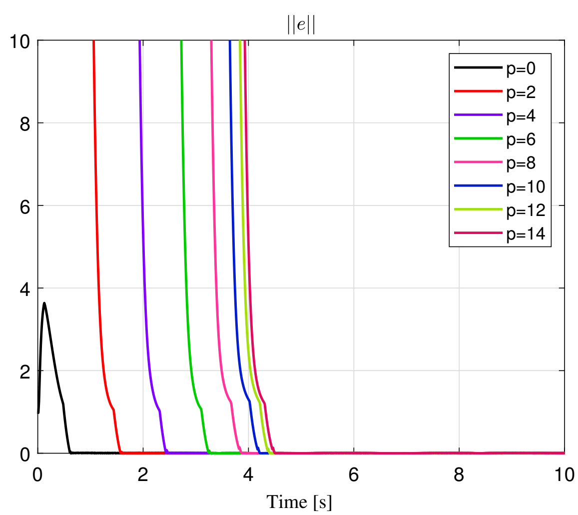


Figure 3.1: HOSM-UIO. Estimation error with different orders at initial error, showing fixed-time estimation.

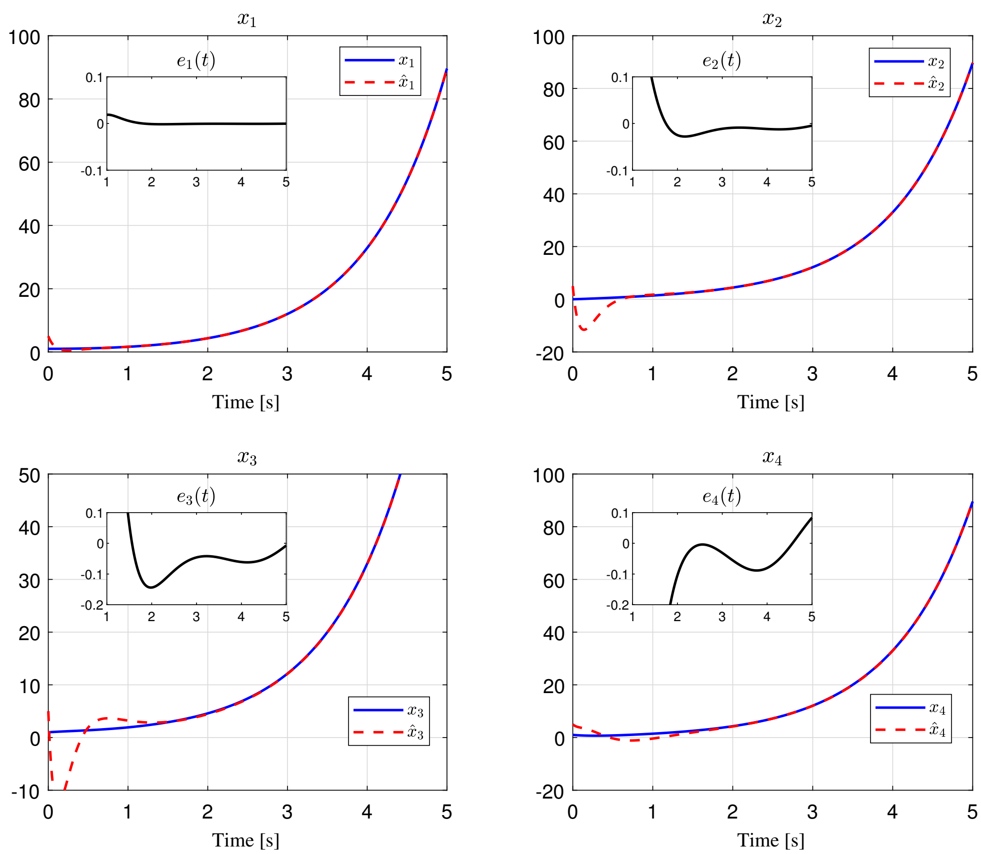


Figure 3.2: Linear UIO. Plant state x, state estimation $\hat{x}$ and estimation errors $e = \hat{x} - x$.

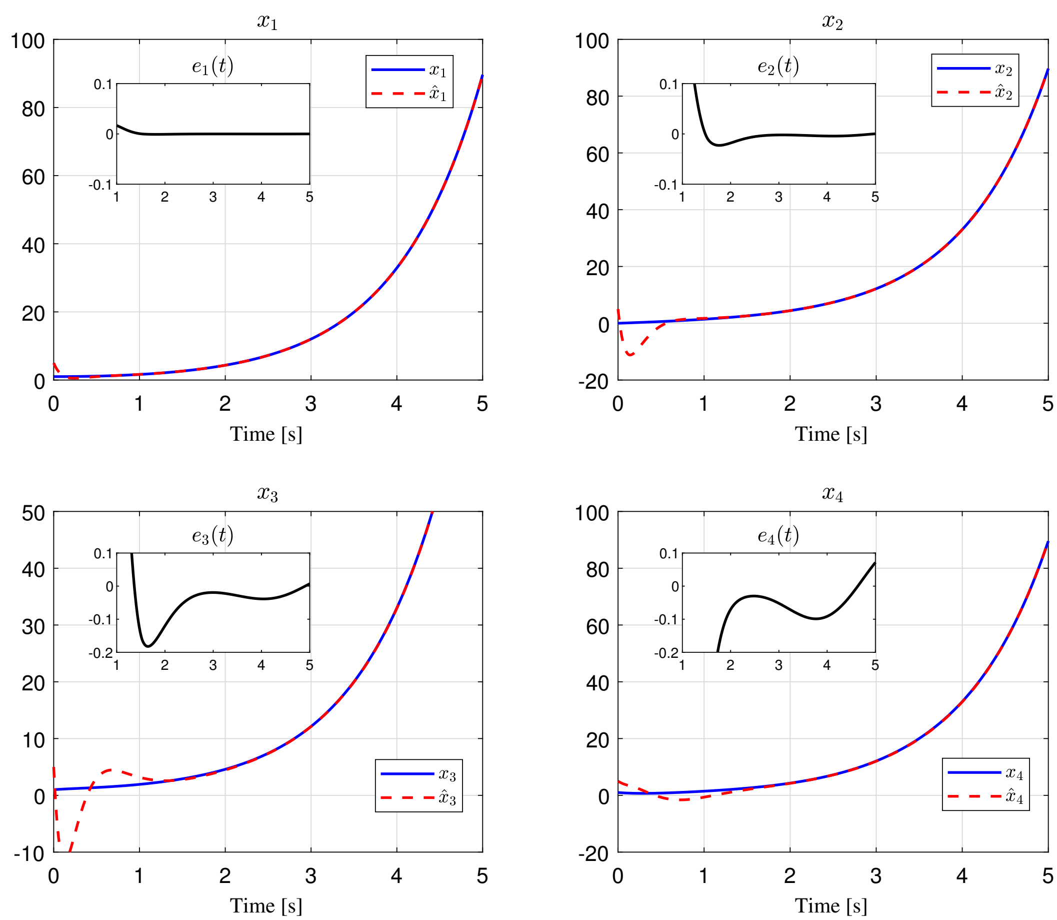


Figure 3.3: Continuous UIO. Plant state x, state estimation $\hat{x}$ and estimation errors $e = \hat{x} - x$.

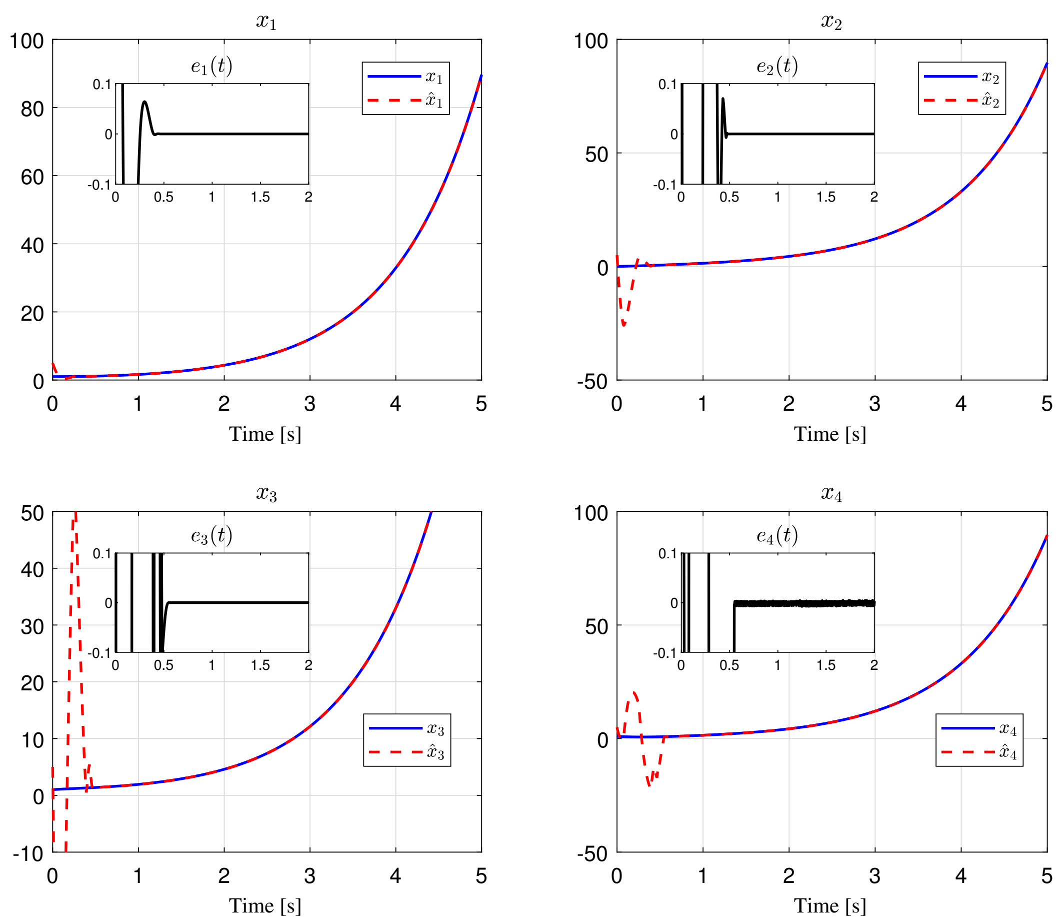


Figure 3.4: HOSM-UIO. Plant state x, state estimation $\hat{x}$ and estimation errors $e = \hat{x} - x$.

Chapter 4

Bl-Homogeneous observers for MIMO linear time invariant systems

In this chapter we present the second part of main result in this work. We introduce the design of Bl-homogeneous Unknown Input Observers (UIO) for MIMO-LTI systems assuming strong observability. The idea is to transform the system in to a Special Coordinate Basis, (detailed in Chapter 2) obtaining a convenient representation of the system for the observer design, but more general than the used in previous works, for example the MIMO observer form used in [33] which decompose the system in a set of subsystems conveniently interconnected in a 'triangular' form (see Section 1.1, Equation (1.8)). Even though such an observer form of the system is a great feature obtained for linear systems and greatly simplifies convergence analysis, it is shown that the observers designed here can deal with a more general type of interconnections between subsystems.

Here we use directly a discontinuous nonlinear observer instead of differentiators. This fact suppress the necessity of using a cascade scheme composed by a linear observer and a discontinuous differentiator.

As in the SISO case, the nonlinear injection terms will be designed to accelerate the convergence as much as we want by selecting appropriate and sufficiently large gains. Even more, due to the assignability of bl-homogeneous degrees in the observer, we can reach and assure exactly and finite-time (or moreover fixed-time) estimation of the states in presence of unknown inputs.

4.1 Unknown input observers for LTI-MIMO systems

Consider the MIMO-LTI system without feedthrough (for simplicity) given by

$$\Sigma : \begin{cases} \dot{x} & = Ax + D\omega \\ y & = Cx \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $\omega \in \mathbb{R}^m$ the unknown input vector and $y \in \mathbb{R}^p$ is the output vector. Accordingly, the matrices A, D, C have appropriate dimensions. For simplicity in the development we do not consider known inputs u, since it does not modify the (observability) properties and it is simple to include it in the observer design.

The task is to build an unknown input observer (UIO) providing for finite-time (preferably fixed-time) estimation of the states in presence of the unknown inputs. In chapter 2 we have stated the necessary and sufficient conditions for the existence and of UIO with arbitrary unknown inputs. We assume to have strong observability only. In Chapter 3 we have introduced this observers in the SISO case. This Chapter generalize the results to arbitrary number of inputs and outputs.

The equations in the observer are understood in the Filippov sense [14] in order to provide for the possibility to use discontinuous signals. Note that Filippov solutions coincide with the usual solutions, when the right-hand sides are continuous.

Special Coordinate Basis (SCB) is a useful tool to represent a system in an appropriate form for the observer design.

4.1.1 Unknown Input Observer design

Given a strongly observable system Σ in (4.1) under SCB transformation detailed in Section 2.1.1 a transformed system is obtained, if we take into account the Property 2.1 this transformed system $\Sigma_{SCB}(\Sigma_b, \Sigma_d)$ is described by the following set.

Subsystems $\Sigma_{b,\iota}$ with associated states $x_{b,\iota} \in \mathbb{R}^{n_{b,\iota}}, \iota = 1, ..., p_b$

$$\Sigma_{b,\iota} : \begin{cases} \dot{x}_{b,\iota,1} &= x_{b,\iota,2} + H_{bd,\iota,1} y_d, \quad y_{b,\iota} = x_{b,\iota,1}, \\ \dot{x}_{b,\iota,j} &= x_{b,\iota,j+1} + H_{bd,\iota,j} y_d, \\ & \vdots \quad j = 2, ..., n_{b,\iota} - 1 \\ \dot{x}_{b,\iota,n_{b,\iota}} &= A_{bb,\iota} x_b + H_{bd,\iota,n_{b,\iota}} y_d, \end{cases} \tag{4.2}$$

with $\sum_{\iota=1}^{p_b} n_{b,\iota} = n_b$.

and subsystems $\Sigma_{d,i}$ with associated states $x_{d,i} \in \mathbb{R}^{n_{d,i}}, i = 1, ..., p_d$

$$\Sigma_{d,i} : \begin{cases} \dot{x}_{d,i,1} &= x_{d,i,2} + H_{dd,i,1} y_d, \quad y_{d,i} = x_{d,i,1} \\ \dot{x}_{d,i,j} &= x_{c,i,j+1} + H_{dd,i,j} y_d, \\ & \vdots \quad j = 2, ..., n_{d,i} - 1 \\ \dot{x}_{d,i,n_{d,i}} &= A_{db,i} x_b + A_{dd,i} x_d + w_{d,i}, \end{cases} \tag{4.3}$$

similarly $\sum_{i=1}^{p_d} n_{d,i} = n_d$. And therefore $n_b + n_d = n$.

Where $A_{bb,\iota}, H_{bd,\iota,j}, A_{dd,i}, A_{db,i}, H_{dd,i,j}$ are constant row vectors of appropriate dimensions. Recall that Σ_b corresponds to the observable dynamics of the system, and Σ_d corresponds to the strongly observable dynamics. It is clear that the subsystems can be expressed in observer or observability form, as we say before, the first option allows us to apply directly an homogeneous differentiator as an state observer [33], but it can not be applied in the observability form. The observability form requires the bl-homogeneity of the observer. This fact have been shown in SISO case at previous Chapter.

Assumption 4.1. *Unknown input $\omega(t)$ is a uniformly bounded function $\|\omega(t)\| \leq \Delta$, equivalently, each element of the vector $|w_{d,i}(t)| \leq \Delta_i \in \mathbb{R}_{\geq 0}, i = 1, ..., p_d$.*

This allow us to relax the existence conditions of UIO in other to have an observer under strong observability only, see Section 2.2. The relative degree of the outputs $y_{d,i}$ with respect to the unknown input is $n_{d,i}$.

The observer $\Omega(\Omega_b, \Omega_d)$ is given by

$$\Omega_{b,\iota} : \begin{cases} \dot{\hat{x}}_{b,\iota,1} &= -k_{b,\iota,1} L \tilde{\phi}_{b,\iota,1}(\hat{x}_{b,\iota,1} - y_{b,\iota}) + \hat{x}_{b,\iota,2} + H_{bd,\iota,1} y_d \\ \dot{\hat{x}}_{b,\iota,j} &= -k_{b,\iota,j} L^j \tilde{\phi}_{b,\iota,j}(\hat{x}_{b,\iota,1} - y_{b,\iota}) + \hat{x}_{b,\iota,j+1} + H_{bd,\iota,j} y_d \\ & \vdots \quad j = 2, ..., n_{b,\iota} - 1 \\ \dot{\hat{x}}_{b,\iota,n_{b,\iota}} &= -k_{b,\iota,n_{b,\iota}} L^{n_{b,\iota}} \tilde{\phi}_{b,\iota,n_{b,\iota}}(\hat{x}_{b,\iota,1} - y_{b,\iota}) + A_{bb,\iota} \hat{x}_b + H_{bd,\iota,n_{b,\iota}} y_d \end{cases} \tag{4.4}$$

$$\Omega_{d,i}:\begin{cases}\dot{\hat{x}}_{d,i,1} &= -k_{d,i,1}L\tilde{\phi}_{d,i,1}(\hat{x}_{d,i,1}-y_{d,i})+\hat{x}_{d,i,2}+H_{dd,i,1}y_d,\\ \dot{\hat{x}}_{d,i,j} &= -k_{d,i,j}L^j\tilde{\phi}_{d,i,j}(\hat{x}_{d,i,1}-y_{d,i})+\hat{x}_{d,i,j+1}+H_{dd,i,j}y_d,\\ \vdots & j=2,...,n_{d,i}-1\\ \dot{\hat{x}}_{d,i,q_i} &= -k_{d,i,q_i}L^{n_{d,i}}\tilde{\phi}_{d,i,q_i}(\hat{x}_{d,i,1}-y_{d,i})+A_{db,i}\hat{x}_b+A_{dd,i}\hat{x}_d\end{cases} \tag{4.5}$$

with positive external gains $k_{b,\iota,j}>0, k_{d,i,j}>0$ and positive tuning gains $\alpha>0, L>0$, appropriately selected as it will be show latter.

In Ω_b the nonlinear output injection terms $\tilde{\phi}_{b,\iota,j}(\cdot)$ are obtained from the functions

$$\phi_{b,\iota,j}(s)=\kappa_{b,\iota j}\lceil s\rfloor^{\frac{r_{(b,\iota),0,j+1}}{r_{(b,\iota),0,1}}}+\theta_{b,\iota j}\lceil s\rfloor^{\frac{r_{(b,\iota),\infty,j+1}}{r_{(b,\iota),\infty,1}}} \tag{4.6}$$

by scaling the positive internal gains $\kappa_{b,\iota j}>0, \theta_{b,\iota j}>0$

$$\kappa_{b,\iota j}\to\left(\frac{L^{n_{b,\iota}}}{\alpha}\right)^{\frac{jd_0}{r_{(b,\iota),0,1}}}\kappa_{b,\iota j},\qquad \theta_{b,\iota j}\to\left(\frac{L^{n_{b,\iota}}}{\alpha}\right)^{\frac{jd_\infty}{r_{(b,\iota),\infty,1}}}\theta_{b,\iota j} \tag{4.7}$$

with powers selected as $r_{(b,\iota),0,n_{b,\iota}}=r_{(b,\iota),\infty,n_{b,\iota}}=1$, and

$$\begin{aligned} r_{(b,\iota),j,n_{b,\iota}}&=r_{(b,\iota),0,j+1}-d_0=1-(n_{b,\iota}-j)d_0\\ r_{(b,\iota),j,n_{b,\iota}}&=r_{(b,\iota),\infty,j+1}-d_\infty=1-(n_{b,\iota}-j)d_\infty\end{aligned} \tag{4.8}$$

which are completely defined by two parameters d_0, d_∞.

Similarly, for Ω_d, the nonlinear output injection terms $\tilde{\phi}_{d,i,j}$ are obtained from the functions

$$\phi_{d,i,j}(s)=\kappa_{d,ij}\lceil s\rfloor^{\frac{r_{(d,i),0,j+1}}{r_{(d,i),0,1}}}+\theta_{d,ij}\lceil s\rfloor^{\frac{r_{(d,i),\infty,j+1}}{r_{(d,i),\infty,1}}} \tag{4.9}$$

by scaling the positive internal gains $\kappa_{d,ij}>0, \theta_{d,ij}>0$

$$\kappa_{d,ij}\to\left(\frac{L^{n_{d,i}}}{\alpha}\right)^{\frac{jd_0}{r_{(d,i),0,1}}}\kappa_{d,ij},\qquad \theta_{d,ij}\to\left(\frac{L^{n_{d,i}}}{\alpha}\right)^{\frac{jd_\infty}{r_{(d,i),\infty,1}}}\theta_{d,ij} \tag{4.10}$$

with powers selected as $r_{(d,i),0,n_{d,i}}=r_{(d,i),\infty,n_{d,i}}=1$, and

$$\begin{aligned} r_{(d,i),j,n_{d,i}}&=r_{(d,i),0,j+1}-d_0=1-(n_{d,i}-j)d_0\\ r_{(d,i),j,n_{d,i}}&=r_{(d,i),\infty,j+1}-d_\infty=1-(n_{d,i}-j)d_\infty\end{aligned} \tag{4.11}$$

which are completely defined by the same parameters d_0, d_∞. They have to satisfy

$$-1\le d_0\le d_\infty<\min_{\substack{\iota=1,...,p_b,\\ i=1,...,p_d}}\left\{\frac{1}{n_{b,\iota}-1},\frac{1}{n_{d,i}-1}\right\} \tag{4.12}$$

We have to highlight the fact that nonlinear injection terms in (4.6) and (4.9) are very similar to them in the bl-homogeneous differentiator [31] but the terms given here are simpler. This simplifies the task of implementation.

4.1.2 Gain Selection

Each type of gains in the observer has a different role, and the idea in the gain tuning is very intuitive. For simplicity in explanation we take hereafter $\psi = \{(b,\iota),(d,i)\}$ to refer to both types of subsystems.

1. The internal gains $\kappa_{\psi,j} > 0, \theta_{\psi,j} > 0$ can be selected arbitrary. These positive real values correspond to the desired weighting of each term of low degree and high degree respectively of $\phi_{\psi,j}$.

2. The external gains $k_{\psi,j} > 0$ have the objective of stabilizing the observer in absence of interconnections and external perturbations, i.e. when $A_{dd} = A_{db} = A_{dd} = 0$ and $\omega(t) = 0$.

3. Parameter L is selected large enough to assure the convergence in presence of interconnections, but not of the bounded perturbations $\omega(t)$. Setting its value grater than minimal value to assure stability the convergence velocity will be increased.

4. The tuning parameter α is selected large enough to assure the convergence in presence of the unknown bounded inputs $\omega(t)$.

4.1.3 Estimation in original coordinates

The estimated state obtained from the observer $\Omega(\Omega_b, \Omega_d)$ in (4.4),(4.5) corresponds to the transformed system Σ_{SCB} in (4.2),(4.3) represented in SCB coordinates through the state Γ_s, input Γ_i and output Γ_o transformations (2.2), moreover it was applied an extra transformation $\Gamma_{\mathcal{O}}$ given by $\Gamma_{\mathcal{O}} = \text{diag}\left\{\mathcal{O}_{b,1}^{-1},...,\mathcal{O}_{b,p_b}^{-1},\mathcal{O}_{d,1}^{-1},...,\mathcal{O}_{d,p_d}^{-1}\right\}$ which puts the subsystems in observability canonical form.

The states in original coordinates can be computed as

$$x = \Gamma_s \Gamma_{\mathcal{O}} \hat{x} \tag{4.13}$$

Therefore, the observer in original coordinates for the system (4.1) takes the form

$$\begin{aligned} \dot{\hat{x}} &= -\Gamma_s \Gamma_{\mathcal{O}} K \Phi(e_y) + A\hat{x} + Bu \\ \hat{y} &= \Gamma_o C \hat{x} \end{aligned} \tag{4.14}$$

with $e_y = \hat{y} - y$ and

$$\begin{aligned} K &= \text{diag}(K_{b,1},...,K_{b,p_b},K_{d,1},...,K_{d,p_d}) \\ K_{b,\iota} &= \begin{bmatrix} k_\iota L & & & \\ & k_{b,\iota}L^2 & & \\ & & \ddots & \\ & & & k_{b,\iota}L^{n_\iota} \end{bmatrix}, \quad \iota = 1,...,p_b \\ K_{d,i} &= \begin{bmatrix} k_i L & & & \\ & k_{d,i}L^2 & & \\ & & \ddots & \\ & & & k_{d,i}L^{n_i} \end{bmatrix}, \quad i = 1,...,p_d \end{aligned} \tag{4.15}$$

$$\Phi(e_y) = \Big[\Phi_{b,1}(e_y) \quad \Phi_{b,2}(e_y) \quad ,\ldots, \quad \Phi_{b,n_b}(e_y), \\ \Phi_{d,1}(e_y) \quad \Phi_{d,2}(e_y) \quad ,\ldots, \quad \Phi_{d,n_d}(e_y)\Big]^T \tag{4.16}$$

$$\Phi_{b,\iota}(e_y) = \left[\tilde{\phi}_{b,\iota,1}(e_y) \quad \tilde{\phi}_{b,\iota,2}(e_y) \quad ,\ldots, \quad \tilde{\phi}_{b,\iota,n_b}(e_y)\right]^T, \quad \iota = 1, ..., p_b \\ \Phi_{d,i}(e_y) = \left[\tilde{\phi}_{d,i,1}(e_y) \quad \tilde{\phi}_{d,i,2}(e_y) \quad ,\ldots, \quad \tilde{\phi}_{d,i,n_d}(e_y)\right]^T, \quad i = 1, ..., p_d \tag{4.17}$$

4.1.4 Main result. UIO - MIMO case

The main result of this work in the MIMO case establishes that the bl-homogeneous Unknown Input Observer (4.14), is able to estimate at least asymptotically the states of a strongly observable linear system (4.1).

Theorem 4.1. *Let the strongly observable MIMO-LTI system Σ (4.1) in original coordinates has an UIO given by (4.14)-(4.17). Selecting d_0, d_∞ as in (4.12) and choosing arbitrary (internal gains) $\kappa_{\psi,j} > 0$ and $\theta_{\psi,j} > 0$, with $\psi = \{(b,\iota),(d,i)\}$. Suppose that either $\Delta_i = 0, i = 1, ..., p_d$ or $d_0 = -1$. Under this conditions, there exist appropriate gains $k_{\psi,j} > 0$ with $\psi = \{(b,\iota),(d,i)\}$, and parameters $L > 0, \alpha > 0$ sufficiently large such that the solutions of bl-homogeneous UIO (4.4)(4.5) converge globally and asymptotically to the true states of Σ_{SCB}, i.e. $\hat{x}_j(t) \to x_j(t)$ as $t \to \infty$.*

In particular, we have we have the following convergence properties and assignment.

- *The observer $\Omega_{b,\iota}$ in (4.4) has assignable dynamics and they can converge globally and*
  1. *Exponentially if $d_0 = 0$*
  2. *Finite-time if $d_0 < 0$*
  3. *Fixed-time if $d_0 < 0 < d_\infty$*
- *The observer $\Omega_{d,i}$ in (4.5) has assignable dynamics and they can converge globally and*
  1. *Exponentially if*
  $$d_0 = 0 \quad \text{with} \quad \Delta_i \equiv 0, \tag{4.18}$$
  2. *Finite-time if*
  $$(a) \quad -1 < d_0 < 0, \quad \Delta_i \equiv 0, \quad \text{or} \\ (b) \quad d_0 = -1, \quad \Delta_i \neq 0 \tag{4.19}$$
  3. *Fixed-time if*
  $$(a) \quad -1 < d_0 < 0 < d_\infty, \quad \Delta_i \equiv 0, \quad \text{or} \\ (b) \quad -1 = d_0 < 0 < d_\infty, \quad \Delta_i \neq 0. \tag{4.20}$$

  with d_∞ always subject to (4.12) be fullfiled.

4.1.5 Proof of Theorem 4.1

The proof, similar to the SISO case will be carried out in a Lyapunov framework through a bl-homogeneous Lyapunov function (Bl-LF), composed of a sum of Bl-LFs related to each subsystem.

To study the error system in a more suitable form, we are going to take the system and observer in the transformed SCB coordinates, i.e. the system (4.2),(4.3) and observer (4.4),(4.5). It is clear that the analysis is completely equivalent in original coordinates.

Let the estimation error variables

$$\begin{aligned} e_{b,\iota,j} &= \hat{x}_{b,\iota,j} - x_{b,\iota,j}, \quad \iota = 1, ..., p_b, j = 1, ..., n_{b,\iota}, \\ e_{d,i,j} &= \hat{x}_{d,i,j} - x_{d,i,j}, \quad i = 1, ..., p_b, j = 1, ..., n_{d,i}, \end{aligned} \tag{4.21}$$

The dynamics error are described by

$$\Xi_{b,i} : \begin{cases} \dot{e}_{b,\iota,1} &= -k_{b,\iota,1} L\tilde{\phi}_{b,\iota,1}(e_{b,\iota,1}) + e_{b,\iota,2} \\ \dot{e}_{b,\iota,j} &= -k_{b,\iota,j} L^j \tilde{\phi}_{b,\iota,j}(e_{b,\iota,1}) + e_{b,\iota,j+1} \\ \quad\vdots & j = 2, ..., n_{b,\iota} - 1 \\ \dot{e}_{b,\iota,n_{b,\iota}} &= -k_{b,\iota,n_{b,\iota}} L^{n_{b,\iota}} \tilde{\phi}_{b,\iota,n_{b,\iota}}(e_{b,\iota,1}) + A_{bb,\iota} e_b \end{cases} \tag{4.22}$$

$$\Xi_{d,i} : \begin{cases} \dot{e}_{d,i,1} &= -k_{d,i,1} L\tilde{\phi}_{d,i,1}(e_{d,i,1}) + e_{d,i,2} \\ \dot{e}_{d,i,j} &= -k_{d,i,j} L^j \tilde{\phi}_{d,i,j}(e_{d,i,1}) + e_{d,i,j+1} \\ \quad\vdots & j = 2, ..., n_{d,i} - 1 \\ \dot{e}_{d,i,q_i} &= -k_{d,i,q_i} L^{n_{d,i}} \tilde{\phi}_{d,i,q_i}(e_{d,i,1}) + A_{db,i} e_b + A_{dd,i} e_d - \omega_{d,i} \end{cases} \tag{4.23}$$

for $\iota = 1, 2, ..., p_b, i = 1, 2, ..., p_d$.

Applying the time scaling via the next transformation

$$\epsilon_{b,\iota,j} = \frac{L^{n_{b,\iota}-j+1}}{\alpha} e_{b,\iota,j}, \qquad \epsilon_{d,i,j} = \frac{L^{n_{d,i}-j+1}}{\alpha} e_{d,i,j} \tag{4.24}$$

we obtain

$$\Xi_{b,\iota} : \begin{cases} \dot{\epsilon}_{b,\iota,1} &= L \left[-k_{b,\iota,1} \phi_{b,\iota,1}(\epsilon_{b,\iota,1}) + \epsilon_{b,\iota,2} \right] \\ \dot{\epsilon}_{b,\iota,j} &= L \left[-k_{b,\iota,j} \phi_{b,\iota,j}(\epsilon_{b,\iota,1}) + \epsilon_{b,\iota,j+1} \right] \\ \quad\vdots & j = 2, ..., n_{b,\iota} - 1 \\ \dot{\epsilon}_{b,\iota,n_{b,\iota}} &= L \left[-k_{b,\iota,n_{b,\iota}} \phi_{b,\iota,n_{b,\iota}}(\epsilon_{b,\iota,1}) + \frac{1}{\alpha} \mu_\iota(\epsilon_b) \right] \end{cases} \tag{4.25}$$

$$\Xi_{d,i} : \begin{cases} \dot{\epsilon}_{d,i,1} &= L \left[-k_{d,i,1} \phi_{d,i,1}(\epsilon_{d,i,1}) + \epsilon_{d,i,2} \right] \\ \dot{\epsilon}_{d,i,j} &= L \left[-k_{d,i,j} \phi_{d,i,j}(\epsilon_{d,i,1}) + \epsilon_{d,i,j+1} \right] \\ \quad\vdots & j = 2, ..., n_{d,i} - 1 \\ \dot{\epsilon}_{d,i,q_i} &= L \left[-k_{d,i,q_i} \phi_{d,i,q_i}(\epsilon_{d,i,1}) + \frac{1}{\alpha} \Psi_i(\epsilon_b, \epsilon_d, \omega_{d,i}) \right] \end{cases} \tag{4.26}$$

where

$$\begin{aligned}
\mu_\iota(\epsilon_b) &= A_{bb,\iota}e_b = \sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} a_{(bb,\iota),j,k}e_{b,j,k} = \alpha\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} \frac{a_{(bb,\iota),j,k}}{L^{n_{b,\iota}-k+1}}\epsilon_{b,j,k} \\
\Psi_i(\epsilon_b,\epsilon_d,\omega_{d,i}) &= A_{db,i}e_b + A_{dd,i}e_d - \omega_{d,i} \\
&= \sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} a_{(db,i),j,k}e_{b,j,k} + \sum_{j=1}^{p_d}\sum_{k=1}^{n_{d,j}} a_{(dd,i),j,k}e_{d,j,k} - \omega_{d,i} \\
&= \alpha\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k}}{L^{n_{b,j}-k+1}}\epsilon_{b,j,k} + \alpha\sum_{j=1}^{p_d}\sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k}}{L^{n_{d,j}-k+1}}\epsilon_{d,j,k}
\end{aligned} \tag{4.27}$$

the fact that $\tilde{\phi}_{\psi,j}(\frac{\alpha}{L^n}s) = \frac{\alpha}{L^n}\tilde{\phi}_{\psi,j}(s)$ has been used.

For the convergence proof, it is convenient to perform another state transformation

$$\begin{aligned}
z_{b,\iota,j} &= \frac{\epsilon_{b,i,j}}{k_{b,\iota,j-1}}, \quad \iota = 1,...,p_b, j = 1,...,n_{b,\iota} \\
z_{d,i,j} &= \frac{\epsilon_{d,i,j}}{k_{d,i,j-1}}, \quad i = 1,...,p_d, j = 1,...,n_{d,i}
\end{aligned} \tag{4.28}$$

Then (4.25) and (4.26) become

$$\Xi^*_{b,\iota} : \begin{cases}
z'_{b,\iota,1} &= -\tilde{k}_{b,\iota,1}\left(\phi_{b,\iota,1}(z_{b,\iota,1}) + z_{b,\iota,2}\right) \\
z'_{b,\iota,j} &= -\tilde{k}_{b,\iota,j}\left(\phi_{b,\iota,j}(z_{b,\iota,1}) + z_{b,\iota,j+1}\right) \\
\vdots & j = 2,...,n_{b,\iota}-1 \\
z'_{b,\iota,n_{b,\iota}} &= -\tilde{k}_{b,\iota,n_{b,\iota}}\phi_{b,\iota,n_{b,\iota}}(z_{b,\iota,1}) + \tilde{\mu}_\iota(z_b)
\end{cases} \tag{4.29}$$

$$\Xi^*_{d,i} : \begin{cases}
z'_{d,i,1} &= -\tilde{k}_{d,i,1}\left(\phi_{d,i,1}(z_{d,i,1}) + z_{d,i,2}\right) \\
z'_{d,i,j} &= -\tilde{k}_{d,i,j}\left(\phi_{d,i,j}(z_{d,i,1}) + z_{d,i,j+1}\right) \\
\vdots & j = 2,...,n_{d,i}-1 \\
z'_{d,i,q_i} &= -\tilde{k}_{d,i,q_i}\phi_{d,i,q_i}(z_{d,i,1}) + \tilde{\Psi}_i(z_b,z_d,\omega_{d,i})
\end{cases} \tag{4.30}$$

with $\tilde{k}_{b,\iota,j} = \frac{k_{b,\iota,j}}{k_{b,\iota,j-1}}$, $\tilde{k}_{d,i,j} = \frac{k_{d,i,j}}{k_{d,i,j-1}}$, $\quad k_{b,\iota,0} = k_{d,i,0} = 1$

where

$$\tilde{\mu}_\iota(z_b) = \frac{1}{k_{b,\iota,n_{b,\iota-1}}}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} \frac{a_{(bb,\iota),j,k}k_{b,j,k-1}}{L^{n_{b,j}-k+1}}z_{b,j,k} \tag{4.31}$$

$$\begin{aligned}
\tilde{\Psi}_i(z_b,z_d,\omega_{d,i}) &= \frac{1}{k_{d,i,n_{d,i-1}}}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k}k_{b,j,k-1}}{L^{n_{b,j}-k+1}}z_{b,j,k} \\
&+ \frac{1}{k_{d,i,n_{d,i}-1}}\sum_{j=1}^{p_d}\sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k}k_{d,j,k-1}}{L^{n_{d,j}-k+1}}z_{d,j,k} - \frac{1}{\alpha k_{d,i,n_{d,i}-1}}\omega_{d,i}
\end{aligned} \tag{4.32}$$

Lyapunov analysis

Before presenting the Lyapunov function we have to recall that the output injection terms in (4.6),(4.9) are much simpler than those described in [31]. However, the stability proof for the bl-homogeneous differentiator in [31] is applicable to the case with the simpler injection terms (4.6),(4.9), since the same requirements and properties are fulfilled. These functions $\phi_{\psi,j}(s)$ can be written as a composition of functions $\varphi_{\psi,j}(s)$ with $\psi = \{(b,\iota),(d,i)\}$ where ψ again refers to the case of both subsystems. Such that

$$\phi_{\psi,j}(s) = \varphi_{\psi,j} \circ ... \circ \varphi_{\psi,2} \circ \varphi_{\psi,1}(s) \tag{4.33}$$

where

$$\begin{aligned} \varphi_{\psi,1}(s) &= \phi_{\psi,1}(s) \\ \varphi_{\psi,2}(s) &= \phi_{\psi,2} \circ \phi^{-1}_{\psi,1}(s) \\ &\vdots \quad j = 2, ..., n_\psi \\ \varphi_{\psi,j}(s) &= \phi_{\psi,j} \circ \phi^{-1}_{\psi,j-1}(s) \end{aligned} \tag{4.34}$$

It will be used a (smooth) bl-homogeneous Lyapunov Function (bl-LF) V composed by a sum of bl-LFs, which were introduced in [31]. Selecting, for $n \geq 2$ two positive real numbers $p_0, p_\infty \in \mathbb{R}_+$ that correspond to the homogeneity degrees of the 0-limit and the ∞-limit approximations of V, such that

$$\begin{aligned} p_0 &\geq \max_{\substack{i=1,...p_b \\ j=1,...,n_{b,\iota}}} \left\{ \frac{r_{(b,\iota),0,j}}{r_{(b,\iota),\infty,j}} \left(2r_{(b,\iota),\infty,j} + d_\infty\right) \right\} \\ p_0 &\geq \max_{\substack{i=1,...p_d \\ j=1,...,n_{d,i}}} \left\{ \frac{r_{(d,i),0,j}}{r_{(d,i),\infty,j}} \left(2r_{(d,i),\infty,j} + d_\infty\right) \right\} \end{aligned} \tag{4.35}$$

$$\begin{aligned} p_\infty &\geq 2 \max_{\substack{i=1,...p_d \\ j=1,...,n_{d,i}}} \left\{ r_{(b,\iota),\infty,j} \right\} + d_\infty \\ p_\infty &\geq 2 \max_{\substack{i=1,...p_d \\ j=1,...,n_{d,i}}} \left\{ r_{(d,i),\infty,j} \right\} + d_\infty \\ \frac{p_0}{r_{(b,\iota),0,j}} &\leq \frac{p_\infty}{r_{(b,\iota),\infty,j}}, \quad \frac{p_0}{r_{(d,i),0,j}} \leq \frac{p_\infty}{r_{(d,i),\infty,j}} \end{aligned} \tag{4.36}$$

For $\iota = 1, ..., p_b$ and $j = 1, ..., n_{b,\iota}$ choose arbitrary positive real numbers $\beta_{(b,\iota),0,j}, \beta_{(b,\iota),\infty,j} > 0$

such that the following functions are defined

$$\begin{aligned} Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) &= \sum_{k\in\{0,\infty\}} \beta_{(b,\iota),k,j} \left[\frac{r_{(b,\iota),k,j}}{p_k} |z_{b,\iota,j}|^{\frac{p_k}{r_{(b,\iota),k,j}}} - z_{b,\iota,j} \lceil \xi_{b,\iota,j} \rfloor^{\frac{p_k - r_{(b,\iota),k,j}}{r_{(b,\iota),k,j}}} \right. \\ &\left. + \frac{p_k - r_{(b,\iota),k,j}}{p_k} |\xi_{b,\iota,j}|^{\frac{p_k}{r_{(b,\iota),k,j}}} \right], \\ \xi_{b,\iota,j} &= \varphi^{-1}_{b,\iota,j}(z_{b,\iota,j+1}) \quad j = 1, ..., n_{b,\iota} - 1 \\ \xi_{b,\iota,n_{b,\iota}} &= z_{b,\iota,n_{b,\iota}+1} \equiv 0, \\ Z_{b,\iota,n_{b,\iota}}(z_{b,\iota,n_{b,\iota}}) &= \beta_{0,\iota,n_{b,\iota}} \frac{1}{p_0} |z_{b,\iota,n_{b,\iota}}|^{p_0} + \beta_{\infty,\iota,n_{b,\iota}} \frac{1}{p_\infty} |z_{b,\iota,n_{b,\iota}}|^{p_\infty} \end{aligned} \tag{4.37}$$

and similarly, associated to Σ_d we construct for $i = 1, ..., p_d$ and $j = 1, ..., n_{d,i}$ choose arditrary positive real numders $\beta_{(d,i),0,j}, \beta_{(d,i),\infty,j} > 0$ such that the following functions are defined

$$\begin{aligned} Z_{d,i,j}(z_{d,i,j}, z_{d,i,j+1}) &= \sum_{k\in\{0,\infty\}} \beta_{(d,i),k,j} \left[\frac{r_{(d,i),k,j}}{p_k} |z_{d,i,j}|^{\frac{p_k}{r_{(d,i),k,j}}} - z_{d,i,j} \lceil \xi_{d,i,j} \rfloor^{\frac{p_k - r_{(d,i),k,j}}{r_{(d,i),k,j}}} \right. \\ &\left. + \frac{p_k - r_{(d,i),k,j}}{p_k} |\xi_{d,i,j}|^{\frac{p_k}{r_{(d,i),k,j}}} \right], \\ \xi_{d,i,j} &= \varphi^{-1}_{d,i,j}(z_{d,i,j+1}) \quad j = 1, ..., n_{d,i} - 1 \\ \xi_{d,i,n_{d,i}} &= z_{d,i,n_{d,i}+1} \equiv 0, \\ Z_{d,i,n_{d,i}}(z_{d,i,n_{d,i}}) &= \beta_{0,i,n_{d,i}} \frac{1}{p_0} |z_{d,i,n_{d,i}}|^{p_0} + \beta_{\infty,i,n_{d,i}} \frac{1}{p_\infty} |z_{d,i,n_{d,i}}|^{p_\infty} \end{aligned} \tag{4.38}$$

it is easy to check the following

Lemma 4.1. *Similar to the SISO case we have*

[31] $Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) \geq 0$ for every $\iota = 1, ..., p_b$, $j = 1, ..., n_{b,\iota}$ and $Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) = 0$ if and only if $\varphi_{b,\iota,j}(z_{b,\iota,j}) = z_{b,\iota,j+1}$.

Similarly for $Z_{d,i,j}(z_{d,i,j}, z_{b,i,j+1})$, for every $i = 1, ..., p_d$, $j = 1, ..., n_{d,i}$

The Bl-LF for each subsystem of the error system $\Xi^*_{b,\iota}$ associated to observable one is defined as

$$V_{b,\iota}(z_{b,\iota}) = \sum_{j=1}^{n_{b,\iota}-1} Z_{b,\iota,j}(z_{b,\iota,j}, z_{b,\iota,j+1}) + Z_{b,\iota,n_{b,\iota}}(z_{b,\iota,n_{b,\iota}}) \tag{4.39}$$

So that, the Bl-LF for the error observable system Ξ^*_b is

$$V_b(z_b) = \sum_{\iota=1}^{p_b} V_{b,\iota}(z_{b,\iota}), \tag{4.40}$$

In a similar way, the Bl-LF for each subsystem of the error system associated to strongly observable one $\Xi^*_{d,i}$ is given by

$$V_{d,i}(z_{d,i}) = \sum_{j=1}^{n_{d,i}-1} Z_{d,i,j}(z_{d,i,j}, z_{d,i,j+1}) + Z_{d,i,n_{d,i}}(z_{d,i,n_{d,i}}) \tag{4.41}$$

and the Bl-LF for the error observable system Ξ_b^* is

$$V_d(z_d) = \sum_{i=1}^{p_d} V_{d,i}(z_{d,i}), \tag{4.42}$$

finally, the Bl-LF candidate for the whole system composed by the interconnection of the observable and strongly observable error systems Ξ_b^* in (4.29) and Ξ_d^* in (4.30) is given by the sum of them, i.e.

$$V(z) = V_b(z_b) + V_d(z_d) \tag{4.43}$$

For the partial derivatives we introduce the following variables associated with both type of systems, i.e. taking $\psi = \{(b,\iota),(d,i)\}$

$$\begin{aligned}
\sigma_{\psi,j} &= \frac{\partial Z_{\psi,j}(z_{\psi,j}, z_{\psi,j+1})}{\partial z_{\psi,j}} = \sum_{k\in\{0,\infty\}} \beta_{k,i,j}\left(\lceil z_{\psi,j} \rfloor^{\frac{p_k - r_{\psi,k,j}}{r_{\psi,k,j}}} - \lceil \xi_{\psi,j} \rfloor^{\frac{p_k - r_{\psi,k,j}}{r_{\psi,k,j}}} \right) \\
s_{\psi,j} &= \frac{\partial Z_{\psi,j}(z_{\psi,j}, z_{\psi,j+1})}{\partial z_{\psi,j+1}} = \sum_{k\in\{0,\infty\}} -\beta_{k,i,j} \frac{p_k - r_{\psi,k,j}}{r_{\psi,k,j}} (z_{\psi,j} - \xi_{\psi,j}) |\xi_{\psi,j}|^{\frac{p_k - 2r_{\psi,k,j}}{r_{\psi,k,j}}} \frac{\partial \xi_{\psi,j}}{z_{\psi,j+1}}
\end{aligned} \tag{4.44}$$

Note that $Z_{\psi,j}, \sigma_{\psi,j}, s_{\psi,j}$ vanish when $\varphi_{\psi,j}(z_{\psi,j}) = z_{\psi,j+1}$.
Performing time derivative of $V(z)$ with respect to the new time variable τ

$$\begin{aligned}
V'(z) = V_b'(z_b) + V_d'(z_d) &= \sum_{\iota=1}^{p_b} V_{b,\iota}'(z_{b,\iota}) + \sum_{i=1}^{p_d} V_{d,i}'(z_{d,i}) \\
&= - W_b(z_b) + \sum_{\iota=1}^{p_b} \frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}} \tilde{\mu}_\iota(z_b) \\
&\quad - W_d(z_d) + \sum_{i=1}^{p_d} \frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} \tilde{\Psi}_i(z_b, z_d, \omega_{d,i})
\end{aligned} \tag{4.45}$$

where

$$W_b(z_b) = \sum_{\iota=1}^{p_b} W_{b,\iota}(z_{b,\iota}), \quad W_d(z_d) = \sum_{i=1}^{p_d} W_{d,i}(z_{d,i}) \tag{4.46}$$

$$\begin{aligned}
W_{b,\iota}(z_{b,\iota}) = &\, \tilde{k}_{b,\iota,1}\sigma_{b,\iota,1}(\phi_{b,\iota,1}(z_{b,\iota,1}) - z_{b,\iota,2}) \\
&+ \sum_{j=2}^{n_{b,\iota}-1} \tilde{k}_{b,\iota,j}\left[s_{b,\iota,j-1} + \sigma_{b,\iota,j}\right](\phi_{b,\iota,j}(z_1) - z_{b,\iota,j+1}) \\
&+ \tilde{k}_{b,\iota,n_{b,\iota}}\left[s_{n_{b,\iota}-1} + \sigma_{n_{b,\iota}}\right]\phi_{n_{b,\iota}}(z_{b,\iota,n_{b,\iota}})
\end{aligned} \tag{4.47}$$

and

$$\begin{aligned}
W_{d,i}(z_{d,i}) = &\, \tilde{k}_{d,i,1}\sigma_{d,i,1}(\phi_{d,i,1}(z_{d,i,1}) - z_{d,i,2}) \\
&+ \sum_{j=2}^{n_{d,i}-1} \tilde{k}_{d,i,j}\left[s_{d,i,j-1} + \sigma_{d,i,j}\right](\phi_{d,i,j}(z_1) - z_{d,i,j+1}) \\
&+ \tilde{k}_{d,i,n_{d,i}}\left[s_{n_{d,i}-1} + \sigma_{n_{d,i}}\right]\phi_{n_{d,i}}(z_{d,i,n_{d,i}})
\end{aligned} \tag{4.48}$$

Due to the definition of $s_{\psi,j}$ in (4.44), $s_{\psi,n} \equiv 0$ and functions $s_{\psi,j}, \sigma_{\psi,j} \in \mathcal{C}$ in $\mathbb{R}$ are r-bl-homogeneous of degrees $p_0 - r_{\psi,0,j}, p_0 - r_{\psi,0,j+1}$ for the 0-approximation and $p_\infty - r_{\psi,\infty,j}, p_\infty - r_{\psi,\infty,j+1}$ for the ∞-approximation, respectively. Additionally, we have $\sigma_{\psi,j} = 0$ on the same set as $s_{\psi,j} = 0$, i.e. they become both zero at the points where $Z_{\psi,j}$ achieves its minimum, $Z_{\psi,j} = 0$.

Each function $V_{b,\iota}(z_{b,\iota})$ and $V_{d,i}(z_{d,i})$ in (4.39),(4.41) are bl-homogeneous of degrees p_0 and p_∞ and $\mathcal{C}$ on $\mathbb{R}$. They are also non negative, since they are positive combinations of non negative terms $Z_{\psi,j}$ with $\psi = \{(b,\iota),(d,i)\}$ respectively. Moreover, $V_\psi(z_\psi)$ are positive definite since $V_\psi(z_\psi) = 0$ only if all $Z_{\psi,j} = 0$, what only happens at $z_b = 0$, $z_d = 0$ respectively. Then, $V_b(z_b)$, $V_d(z_d)$ and therefore $V(z)$ in (4.43) as a sum of them are positive definite. Due to bl-homogeneity they are also radially unbounded.

If we analyze the terms $W_{b,\iota}(z_{b,\iota}), W_{d,i}(z_{d,i})$, in (4.47),(4.48) they are both bl-homogeneous of degrees $p_0 + d_0$ for their 0-approximations and $p_\infty + d_\infty$ for their ∞-approximations. Therefore $W_b(z_b)$ and $W_d(z_d)$ in (4.46) as well as its sum $W_{bd}(z_b, z_d) = W_b(z_b) + W_d(z_d)$ are also bl-homogeneous of degrees $p_0 + d_0$ for their 0-approximations and $p_\infty + d_\infty$ for their ∞-approximations.

As a previous related result, it has been shown in [31] that there exists appropriate gains $\tilde{k}_{\psi,j}$ such that $W_\psi(z_\psi)$ in (4.47),(4.48) are rendered positive definite. Now, the idea in the following is to prove that there exist gains L, α sufficiently large such that the negative definiteness of the terms $-W_\psi(z_\psi)$ in (4.45) and therefore $V'(z)$ is hold.

We are now interested in finding an upper bound of $\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\tilde{\mu}_\iota(z_b)$. Assuming $L \geq 1$, and $\alpha \geq 1$. Taking into account (4.31), due to the power of L we can write

$$\begin{aligned}\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\tilde{\mu}_\iota(z_b) &= \frac{1}{k_{b,\iota,n_{b,\iota-1}}}\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}}\frac{a_{(bb,\iota),j,k}k_{b,j,k-1}}{L^{n_{b,j}-k+1}}z_{b,j,k} \\ &\leq \frac{1}{L}\sum_{j=1}^{p_b}\sum_{k=1}^{n_{b,j}}\frac{a_{(bb,\iota),j,k}k_{b,j,k-1}}{k_{b,\iota,n_{b,\iota-1}}}\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}z_{b,j,k}\end{aligned} \tag{4.49}$$

where $\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}$ is bl-homogeneous of degree $p_0 - r_{(b,\iota),0,n_{b,\iota}} = p_0 - 1$ for the 0-approximation and $p_\infty - r_{(b,\iota),\infty,n_{b,\iota}} = p_\infty - 1$ for the ∞-approximation, additionally, each term $z_{b,j,k}$ is bl-homogeneous of degree $r_{(b,j),0,k}$ and $r_{(b,j),\infty,k}$ for the 0 and ∞ approximation respectively.

Then, $\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}z_{b,j,k}$ is bl-homogeneous of degree $p_0 - r_{(b,\iota),0,n_{b,\iota}} + r_{(b,j),0,k} = p_0 - (n_{b,j} - k)d_0$ for the 0-approximation and $p_\infty - r_{(b,\iota),\infty,n_{b,\iota}} + r_{(b,j),\infty,k} = p_\infty - (n_{b,j} - k)d_\infty$ for the $\infty-$approximation. We can conclude that

$$p_0 + d_0 \leq p_0 - (n_{b,j} - k)d_0, \qquad p_\infty - (n_{b,j} - k)d_\infty \leq p_\infty + d_\infty \tag{4.50}$$

And therefore, by the property of bl-homogeneous functions, there exist positive real numbers $\lambda_{b,\iota}$ such that

$$\frac{\partial V_{b,\iota}(z_{b,\iota})}{\partial z_{b,\iota,n_{b,\iota}}}\tilde{\mu}_\iota(z_b) \leq \frac{\lambda_{b,\iota}}{L}W_b(z_b) \tag{4.51}$$

For the upper bound analysis of $\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}}\tilde{\Psi}_i(z_b, z_d, \omega_{d,i})$, it is convenient to write $\tilde{\Psi}_i(z_b, z_d, \omega_{d,i})$

separately, i.e

$$\begin{aligned}
\tilde{\Psi}_i(z_b, z_d, \omega_{d,i}) &= \frac{1}{k_{d,i,n_{d,i-1}}} \sum_{j=1}^{p_b} \sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k} k_{b,j,k-1}}{L^{n_{b,j}-k+1}} z_{b,j,k} \\
&+ \frac{1}{k_{d,i,n_{d,i-1}}} \sum_{j=1}^{p_d} \sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k} k_{d,j,k-1}}{L^{n_{d,j}-k+1}} z_{d,j,k} - \frac{1}{\alpha k_{d,i,n_{d,i-1}}} \omega_{d,i} \\
&= \frac{1}{L} \tilde{\Psi}_{b,i} + \frac{1}{L} \tilde{\Psi}_{d,i} + \frac{1}{\alpha} \tilde{\Psi}_{\omega,i} \\
\tilde{\Psi}_{b,i}(z_b) &= \sum_{j=1}^{p_b} \sum_{k=1}^{n_{b,j}} \frac{a_{(db,i),j,k} k_{b,j,k-1}}{k_{d,i,n_{d,i-1}} L^{n_{b,j}-k}} z_{b,j,k} \\
\tilde{\Psi}_{d,i}(z_d) &= \sum_{j=1}^{p_d} \sum_{k=1}^{n_{d,j}} \frac{a_{(dd,i),j,k} k_{d,j,k-1}}{k_{d,i,n_{d,i-1}} L^{n_{d,j}-k}} z_{d,j,k} \\
\tilde{\Psi}_{\omega,i}(\omega_d, i) &= -\frac{1}{k_{d,i,n_{d,i}-1}} \omega_{d,i}
\end{aligned} \tag{4.52}$$

where $\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}}$ is bl-homogeneous of degree $p_0 - r_{(d,i),0,n_{d,i}} = p_0 - 1$ for the 0-approximation and $p_\infty - r_{(d,i),\infty,n_{d,i}} = p_\infty - 1$ for the ∞-approximation, additionally, each term $z_{b,j,k}$ is bl-homogeneous of degree $r_{(b,j),0,k}$ and $r_{(b,j),\infty,k}$ for the 0 and ∞ approximation respectively, similarly each term $z_{d,j,k}$ is bl-homogeneous of degree $r_{(d,j),0,k}$ and $r_{(d,j),\infty,k}$ for the 0 and ∞ approximation respectively.

$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} z_{b,j,k}$ is bl-homogeneous of degree $p_0 - r_{(d,i),0,n_{d,i}} - r_{(b,j),0,k} = p_0 - (n_{b,j} - k)d_0$ for the 0-approximation and $p_\infty - r_{(d,i),\infty,n_{d,i}} - r_{(b,j),\infty,k} = p_\infty - (n_{b,j} - k)d_\infty$ for the ∞-approximation. In a similar way, $\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} z_{d,j,k}$ is bl-homogeneous of degree $p_0 - r_{(d,i),0,n_{d,i}} - r_{(d,j),0,k} = p_0 - (n_{d,j} - k)d_0$ for the 0-approximation and $p_\infty - r_{(d,i),\infty,n_{d,i}} - r_{(d,j),\infty,k} = p_\infty - (n_{d,j} - k)d_\infty$ for the ∞-approximation.

It is clear

$$\begin{aligned}
p_0 + d_0 \leq p_0 - (n_{b,j} - k)d_0 \qquad p_\infty - (n_{b,j} - k)d_\infty \leq p_\infty + d_\infty \\
p_0 + d_0 \leq p_0 - (n_{d,j} - k)d_0 \qquad p_\infty - (n_{d,j} - k)d_\infty \leq p_\infty + d_\infty
\end{aligned} \tag{4.53}$$

Therefore, by the property of bl-homogeneous functions, there exist $\lambda_{d,i}, \lambda_{bd,i}, \bar{\lambda}_{d,i} > 0$ such that

$$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} \frac{1}{L} \tilde{\Psi}_{b,i}(z_b) \leq \frac{\lambda_{bd,i}}{L} W_{bd}(z_b, z_d) \tag{4.54}$$

$$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} \frac{1}{L} \tilde{\Psi}_{d,i}(z_d) \leq \frac{\lambda_{d,i}}{L} W_d(z_d) \tag{4.55}$$

$$\frac{\partial V_{d,i}(z_{d,i})}{\partial z_{d,i,n_{d,i}}} \leq \bar{\lambda}_{d,i} \left(W_d^{\frac{p_0-1}{p_0+d_0}}(z_d) + W_d^{\frac{p_\infty-1}{p_\infty+d_\infty}}(z_d) \right) \tag{4.56}$$

If we put everything together, $V'(z)$ in (4.45) can be bounded as

$$\begin{aligned} V'(z) \leq & - W_b(z_b) - W_d(z_d) + \frac{\lambda_b}{L} W_b(z_b) + \frac{\lambda_{bd}}{L} W_{bd}(z_b, z_d) + \frac{\lambda_d}{L} W_d(z_d) \\ & + \frac{\bar{\lambda}_d}{\alpha} \left(W_d^{\frac{p_0 - 1}{p_0 + d_0}}(z_d) + W_d^{\frac{p_\infty - 1}{p_\infty + d_\infty}}(z_d) \right) \|\tilde{\Psi}_\omega(\omega_d)\|_\infty \\ = & - \left(1 - \frac{\lambda_b + \lambda_{bd}}{L}\right) W_b(z_b) - \left(1 - \frac{\lambda_d + \lambda_{bd}}{L}\right) W_d(z_d) \\ & + \frac{\bar{\lambda}_d}{\alpha} \left(W_d^{\frac{p_0 - 1}{p_0 + d_0}}(z_d) + W_d^{\frac{p_\infty - 1}{p_\infty + d_\infty}}(z_d) \right) \|\tilde{\Psi}_\omega(\omega_d)\|_\infty \end{aligned} \tag{4.57}$$

where $\|\tilde{\Psi}_\omega(\omega_d)\|_\infty = \max\left\{\sum_{i=1}^{p_d} |\tilde{\Psi}_{\omega,i}(\omega_{d,i})|\right\}$.

If we apply Lyapunov stability arguments we conclude that it can be chosen L sufficiently large such that the first two terms become negative definite. In absence of $\tilde{\Psi}_\omega$, the origin $z = 0$ is asymptotic stable. With $d_0 < 0$ it converges in finite time, moreover, with $d_\infty > 0$ it converges in fixed-time.

In the case $\tilde{\Psi}_\omega \neq 0$ but ultimately bounded, by selecting $d_0 = -1$ we can chose α sufficiently large, such that $V'(z)$ is negative definite. And then, finite-time or fixed-time stability is achieved.

■

4.1.6 Some comments

- It has been shown that the proposed observer is capable of estimating the state variables of linear MIMO systems with uniformly bounded unknown inputs. And that the resulting interconnections between subsystems can be globally dominated through bl-homogeneous terms.

- In the design we have assumed for simplicity that the parameters α and L are the same for all subsystems, however, this is not strictly necessary, the designer could design a pair of parameters α and L for each subsystem. Therefore, the speed of convergence of each of them can be chosen independently in a certain sense.

- As we said before, the subsystem Σ_b corresponds to the observable part of the system, that is, the observation dynamics is completely assignable and since there is no effect of unknown inputs on it, the presence of discontinuous terms is never necessary, that is, a continuous observer for Σ_b is sufficient to ensure convergence in fixed time. However in the case of the subsystems $\Sigma_{d,i}$ when $\Delta_i \neq 0$ there is exact convergence only when $d_0 = -1$ which produces a discontinuous HOSM.

- It has just been shown that the order of the complete observer is at most of the same order as the system, that is, the direct application of the bl-homogeneous UIO does not increase the order unnecessarily. This also eliminates the delay effect introduced by the Luenberger observer.

- The observer was proposed for strongly observable systems, i.e. systems without zeros at all, which means non-existence of internal dynamics. This is clarified under the SCB transformation where the strong observability Property 2.1 states the non-existence of x_a and x_c.

  However, the results presented can be easily extended to strongly detectable systems (having only stable invariant zeros or in SCB A_{aa} Hurwitz in (2.13)). The convergence of the observer would be subject to the asymptotic convergence of the part associated with x_a. Since it is an inaccessible dynamic, there is no freedom to accelerate the convergence velocity

4.2 Example

Let a *strongly observable* system, given by a linearized model of the lateral motion of a light aircraft taken from [32][33] in original Coordinate with unknown inputs

$$\Sigma : \begin{cases} \dot{x} &= Ax + Bu + D\omega \\ y &= Cx \end{cases} \tag{4.58}$$

where

$$A = \begin{bmatrix} -0.3 & 0 & -33 & 9.81 & 0 & -5.4 & 0 \\ -0.1 & -8.3 & 3.75 & 0 & 0 & 0 & -28.6 \\ 0.37 & 0 & -0.64 & 0 & 0 & -9.5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 10 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 20 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \omega(t) = 0.08 + 0.1sin(2t) + 0.02cos(13t), \quad |\omega(t)| \leq 0.2 \tag{4.59}$$

The state vector $x = \begin{bmatrix} x_1 & \dots & x_7 \end{bmatrix}^T$ consists of sideslip velocity x_1, the roll rate x_2, the yaw rate x_3, the roll angle x_4, the yaw angle x_5, the rudder angle x_6 and the aileron angle x_6. The control input $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ is given by the rudder angle demand u_1 and the aileron angle demand u_2. The unknown input ω is a bounded actuator fault in the rudder is considered. The output $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ provide measurements of the roll rate x_2 and the yaw angle x_5.

It is important to note that the system (4.58) is unstable since the matrix A has to eigenvalues with non-negative real part in $s_1 = 0.1219$ and $s_2 = 0$, them all the states of the system can be unbounded. Therefore, the direct application of a homogeneous differentiator like the Levant´s one is impossible.

Applying the SCB change of Coordinate, and expressing each subsystem in observability canonical form through $T = \text{diag}\left\{\mathcal{O}_1^{-1}, \mathcal{O}_2^{-1}\right\}$, the transformed system is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 2.006 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1.665 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2.6 & 0 & 0 \\ 0 & 0 & -7.009 & -6.401 & -3.645 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 488.789 & -3.81 & -14.933 & -183.158 & -87.597 & -17.838 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 \\ 0 & 77.44 \\ 0 & -456.244 \\ 0 & 2378 \\ 0 & 0 \\ 0 & -286 \\ -701.7 & 3803.8 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{4.60}$$

which can be written as

$$\Sigma_b : \begin{cases} \dot{x}_{b,1} &= x_{b,2} + [2.006]y_2, \qquad y_1 = x_{b,1} \\ \dot{x}_{b,2} &= x_{b,3} + [-1.665]y_2 + [0 \quad 77.44]u \\ \dot{x}_{b,3} &= x_{b,4} + [2.6]y_2 + [0 \quad -456.244]u \\ \dot{x}_{b,4} &= [0 \quad 0 \quad -7.009 \quad -6.401]x_1 + [-3.645]y_2 + [0 \quad 2378]u \end{cases}$$
$$\Sigma_d : \begin{cases} \dot{x}_{d,1} &= x_{d,2}, \qquad y_2 = x_{d,1} \\ \dot{x}_{d,2} &= x_{d,3} + [0 \quad -286]u \\ \dot{x}_{d,3} &= [0 \quad 488.789 \quad -3.81 \quad -14.933]x_b + \\ & \dots + [-183.158 \quad -87.597 \quad -17.838]x_d + [-701.7 \quad 3803.8]u + [1]\omega \end{cases} \tag{4.61}$$

where we have one observable subsystem $x_b \in \mathbb{R}^4$ and one strong observable subsystem $x_d \in \mathbb{R}^3$. In this example we omit the sub indices of the subsystem number, due to we have only $p_b = p_d = 1$. Note that the fact that the subsystems are in observability canonical form make impossible to apply the methodology and observer proposed in [33]. The terms in the last channel can be seen as interconnection ones, i.e. in some sense, the following methodology can be thought of as the observer design of linear interconnected systems.

The observer is given by

$$\Omega_b : \begin{cases} \dot{\hat{x}}_{b,1} &= -k_{b,1}L\tilde{\phi}_{b,1}(\hat{x}_{b,1} - y_1) + \hat{x}_{1,2} + [2.006]y_2 \\ \dot{\hat{x}}_{b,2} &= -k_{b,2}L^2\tilde{\phi}_{b,2}(\hat{x}_{b,1} - y_1) + \hat{x}_{1,3} + [-1.665]y_2 + [0 \quad 77.44]u \\ \dot{\hat{x}}_{b,3} &= -k_{b,3}L^3\tilde{\phi}_{b,3}(\hat{x}_{b,1} - y_1) + \hat{x}_{1,4} + [2.6]y_2 + [0 \quad -456.244]u \\ \dot{\hat{x}}_{b,4} &= -k_{b,4}L^4\tilde{\phi}_{b,4}(\hat{x}_{b,1} - y_1) + [0 \quad 488.789 \quad -3.81 \quad -14.933]\hat{x}_b + \\ & \dots + [-3.645]y_2 + [0 \quad 2378]u \end{cases}$$
$$\Omega_d : \begin{cases} \dot{\hat{x}}_{d,1} &= -k_{d,1}L\tilde{\phi}_{d,1}(\hat{x}_{d,1} - y_2) + \hat{x}_{1,2} \\ \dot{\hat{x}}_{d,2} &= -k_{d,2}L^2\tilde{\phi}_{d,2}(\hat{x}_{d,1} - y_2) + \hat{x}_{1,3} + [0 \quad -286]u \\ \dot{\hat{x}}_{d,3} &= -k_{d,3}L^3\tilde{\phi}_{d,3}(\hat{x}_{d,1} - y_2) + [0 \quad 488.789 \quad -3.810 \quad -14.933]\hat{x}_b + \\ & \dots + [-183.158 \quad -87.597 \quad -17.838]\hat{x}_d + [-701.7 \quad 3803.8]u \end{cases} \tag{4.62}$$

where the nonlinear output injection terms $\tilde{\phi}.(\cdot)$ are as follows

$$\tilde{\phi}_{b,j}(s) = \left(\frac{L^4}{\alpha}\right)^{\frac{jd_0}{1-3d_0}} \kappa_{b,\iota j} \lceil s \rfloor^{\frac{1-(3-j)d_0}{1-3d_0}} + \left(\frac{L^4}{\alpha}\right)^{\frac{jd_\infty}{1-3d_\infty}} \theta_{b,\iota j} \lceil s \rfloor^{\frac{1-(3-j)d_\infty}{1-3d_\infty}}, \quad j = 1, ..., 4 \tag{4.63}$$

and

$$\tilde{\phi}_{d,j}(s) = \left(\frac{L^3}{\alpha}\right)^{\frac{jd_0}{1-2d_0}} \kappa_{d,j} \lceil s \rfloor^{\frac{1-(2-j)d_0}{1-2d_0}} + \left(\frac{L^3}{\alpha}\right)^{\frac{jd_\infty}{1-2d_\infty}} \theta_{d,j} \lceil s \rfloor^{\frac{1-(2-j)d_\infty}{1-2d_\infty}}, \quad j = 1, ..., 3 \tag{4.64}$$

The assigned homogeneity degrees d_0, d_∞ in 0 and ∞ in these terms (4.63),(4.64) have to satisfy

$$-1 \leq d_0 \leq d_\infty < \min\left\{\frac{1}{3}, \frac{1}{2}\right\} = \frac{1}{3} \tag{4.65}$$

We present three cases:

1. Linear UIO. Homogeneity degrees $d_0 = d_\infty = 0$.
2. Continuous UIO. Homogeneity degrees $d_0 = -\frac{1}{9}, d_\infty = 0$
3. HOSM-UIO. Homogeneity degrees $d_0 = -1$, $d_\infty = \frac{1}{9}$. With this selection we get a discontinuous observer. Note that $d_0 < 0 < d_\infty$.

The initial conditions of the plant states are $x_0 = \begin{bmatrix} -0.5 & 0.1 & 0.02 & 0.2 & -0.1 & -0.3 & 0.2 \end{bmatrix}$ and $\hat{x}_j = 0, j = 1, ..., 7$ for the observer. For all cases the values of gains $k_{b,j}, k_{d,j}$ are fixed as

$$\left\{k_{b,1} = 8.6k_{b,4}^{\frac{1}{4}} \quad k_{b,2} = 21k_{b,4}^{\frac{1}{2}} \quad k_{b,3} = 16.25k_{d,4}^{\frac{1}{3}} \quad k_{b,4} = 2\right\} \tag{4.66}$$

$$\left\{k_{d,1} = 3.34k_{d,3}^{\frac{1}{3}} \quad k_{d,2} = 5.3k_{d,3}^{\frac{2}{3}} \quad k_{d,3} = 5\right\} \tag{4.67}$$

In te case of Figures 4.1-4.3 a set of 5 seconds simulations are shown, where $L = 1, \alpha = 10$ and $\kappa_{b,j} = \kappa_{d,j} = 1$ have been set. We can see that in the cases 1 and 2 (linear and continuous) the observer does not accurately estimate the states of the plant due to the non-compensation of unknown input, i.e. no matter how big the gains can be, with this homogeneity degrees selection, the observer will never be able to compensate the effect of the unknown input. This is cleaner in the last subfigure, where $\|e\|$ is illustrated.

On the other hand, in the third case with $d_0 = -1$ (discontinuous observer), the induced HOSM allows the observer to compensate exactly the unknown input effect. It is shown in Figure 4.3, that after 0.5s exact convergence of the error norm to zero even in presence of unknown input is achieved, therefore, the states are estimated exactly in finite time.

In all cases, with appropriately gains selection the error converges to a neighborhood of zero in about one second but it is not clear what happens when the initial estimate is not close to the true state. For the third case, i.e. with $d_0 = -1$, $d_\infty = \frac{1}{9}$ and setting now $L = 5, \alpha = 10$ we run a set of simulations, Figure 4.4 shows the norm of the error vector over a wide range orders of initial estimation error. Here, the error converge to zero in less than 4s for all cases.

In other words, we achieve more than finite-time, fixed-time stability of the estimation error, i.e. regardless of the initial magnitude of the error, the observer converges before certain time value $\bar{T}$, in this case we can see that the value of this upper cote of estimation time is $\bar{T} = 4s$, but this can be reduced arbitrary by increasing appropriately the value of parameter L as we said in the parameters tuning section.

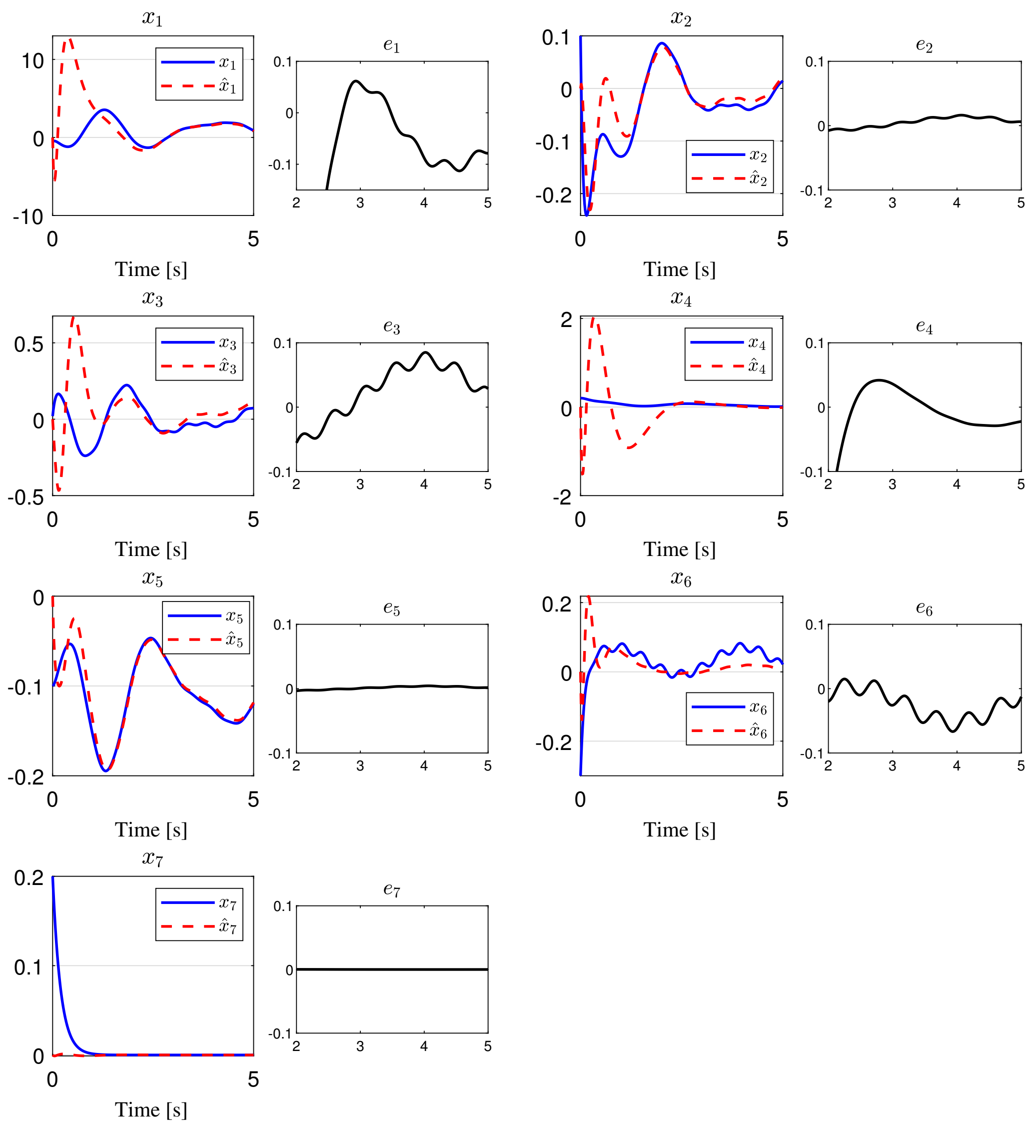


Figure 4.1: Linear UIO. Plant state x, state estimation $\hat{x}$ and estimation errors $e = \hat{x} - x$.

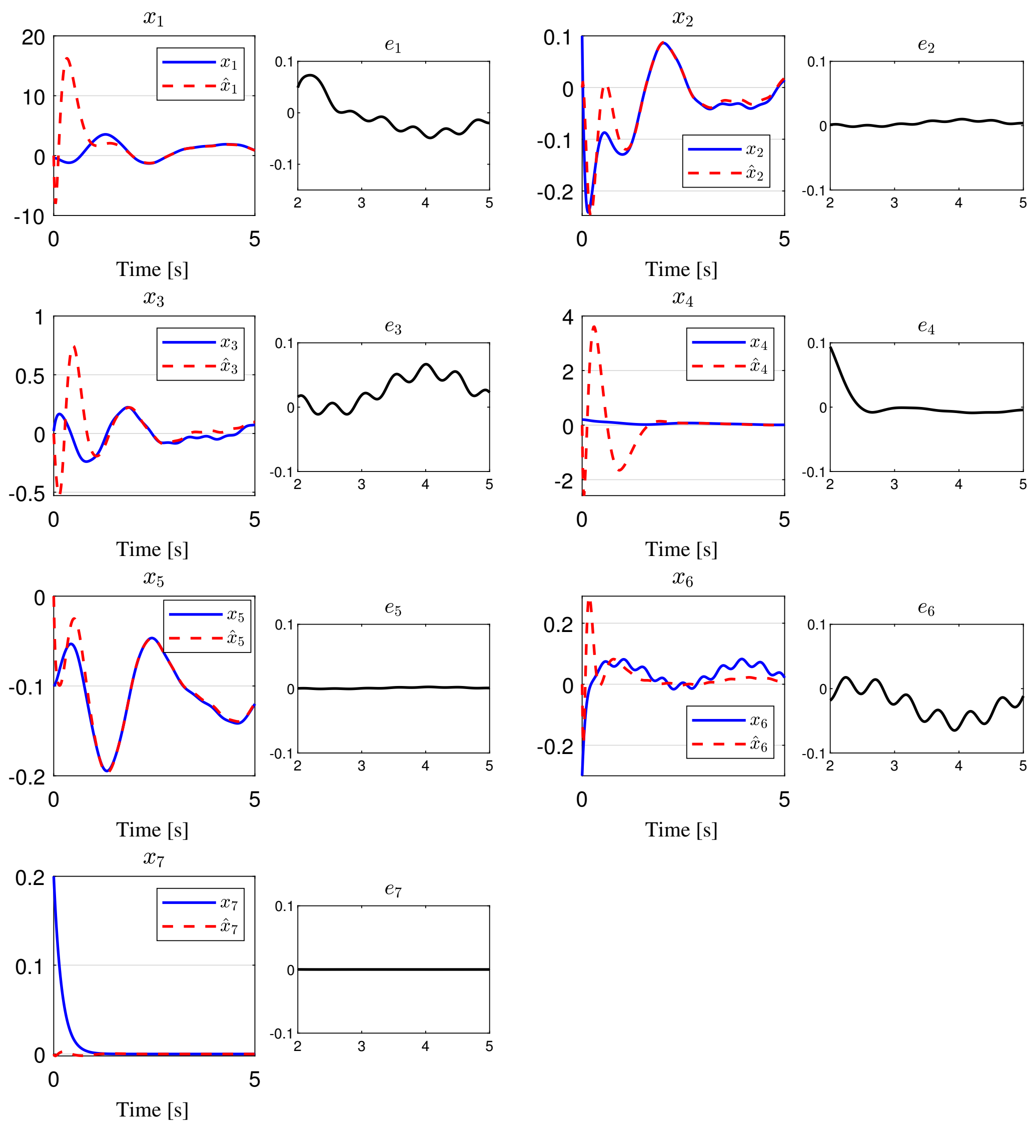


Figure 4.2: Continuous UIO. Plant state x, state estimation $\hat{x}$ and estimation errors $e = \hat{x} - x$.

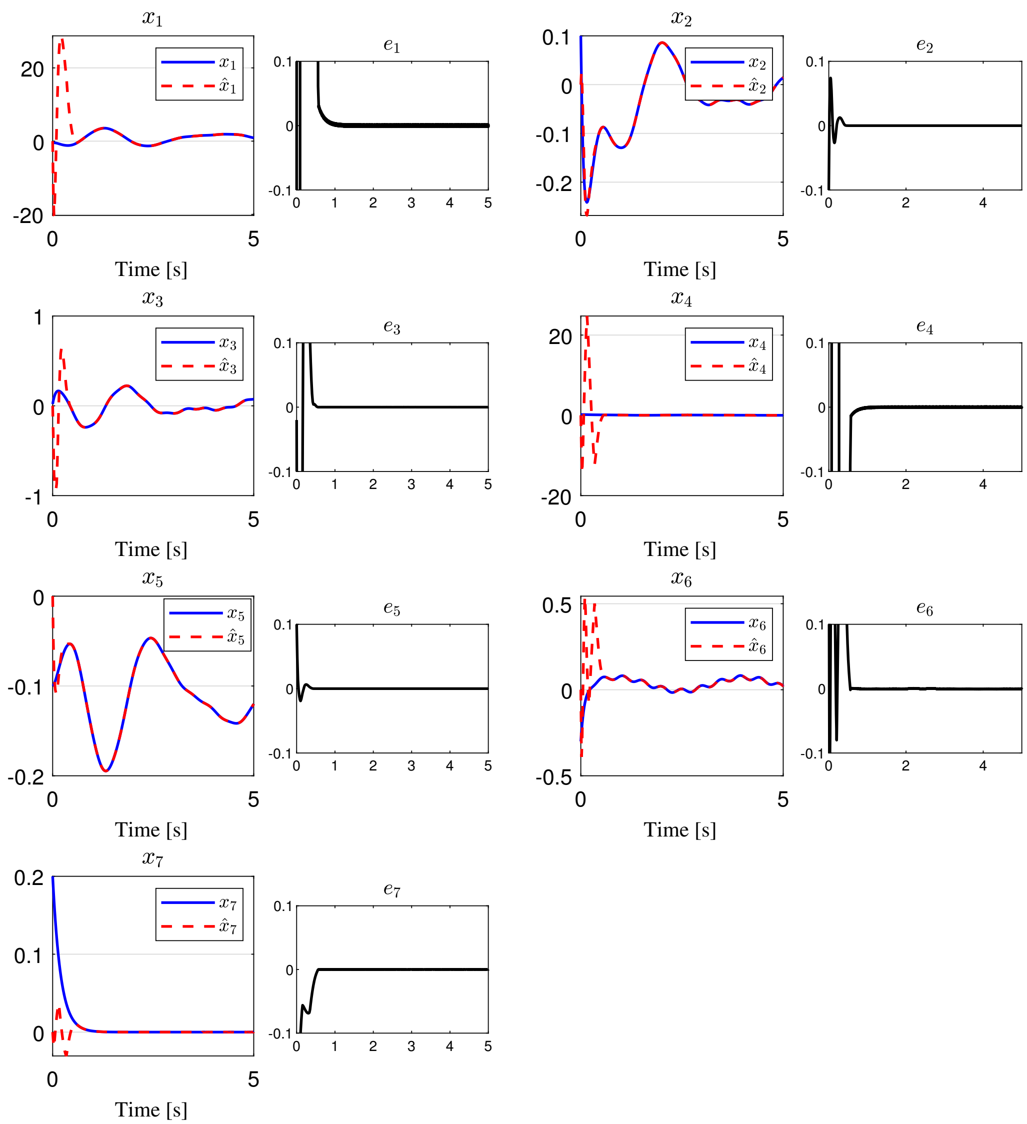


Figure 4.3: HOSM-UIO. Plant state x, state estimation $\hat{x}$ and estimation errors $e = \hat{x} - x$.

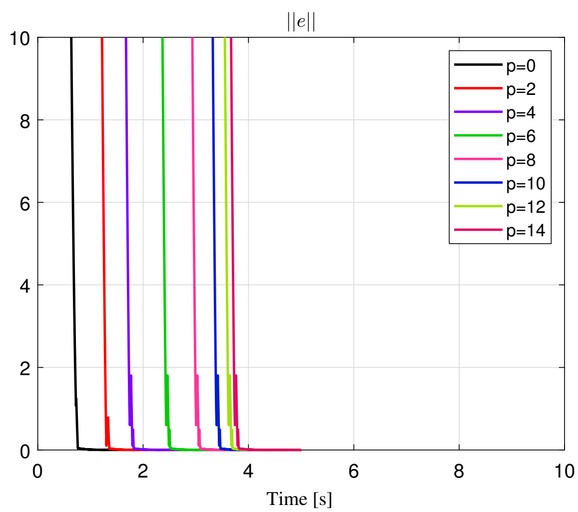


Figure 4.4: HOSM-UIO. Estimation error with different orders at initial error, showing fixed-time estimation.

Chapter 5

Conclusions

In this work we have proposed a family of observers for state estimation of strongly observable SISO and MIMO Linear Time Invariant systems, the proposed scheme has bl-homogeneous properties, which allow us to assign behavior near and far from the origin independently, this feature allows us the possibility of accelerating when possible the convergence velocity of the observer achieving finite time or more fixed-time convergence. Additionally, the induced HOSM at the origin allow us to deal with bounded unknown inputs having theoretically exact convergence.

The tuning parameters of the observer are composed by a set of gains, which have an intuitive roll each one, the internal gains $\kappa_{\psi,j}, \theta_{psi,j}$ with $\psi = \{(b,\iota),(d,i)\}$ give a relative weight of the low degree terms and the high degree ones in the nonlinear injection terms. The external gains $k_{\psi,j}$ are responsible for ensuring stability of the observer considering absence of interconnections and perturbations, and finally the gains L, α are in charge of dealing with the interconnection and unknown inputs terms respectively by setting them sufficiently large. Thus, the tuning procedure of the observer is simple and structured.

The boundedness of the state variables is not required for global finite-time or fixed-time stability of the estimation error dynamics which allows for its application to unstable plants.

In the SISO case the state estimation for LTI systems with unknown inputs problem can be solved by directly applying a HOSM differentiator (like Levant's RED) if we put the system in observer form. Nevertheless, this idea can not be applied if the system is in observability form, since to have global convergence we should have bounded state variables, and this reduces its applicability. In the MIMO case this problem is the same, extended to a set of subsystems. Base on this problem, the construction of bl-homogeneous observers allow us to deal with unbounded functions far of the origin, i.e. with $d_\infty \geq 0$ we can deal with Lipschitz and then linear functions of the states in last channel of each subsystem and in the case of selecting $d_0 = -1$ we have proof that the observer can deal with the effect of unknown inputs due to the HOSM, obviously, in the absence of unknown inputs a linear $d_0 = d_\infty = 0$ or continuous $-1 < d_0 < 0 < d_\infty$ observer is sufficient to achieve asymptotic or exact and finite time convergence, respectively.

This work results in a more general observation scheme, with respect to the already existing in literature offering global convergence. In contrast to the cascade scheme [18] composed by a Luenberger observer and a HOSM differentiator the structure is much simpler because the order is not unnecessarily increased, therefore the number of parameters is significantly less. Moreover, the Luenberger observer introduce a delay in the estimation, that is avoided in the present scheme.

On the other hand, for the MIMO case, the observer presented in [33] is based on a MIMO observer normal form which allows to apply directly an homogeneous differentiator (Levant's RED is applied), this is because in this observer normal form the resulting subsystems ares interconnected in a convenient form, i.e. the convergence is achieved in a sequential way, when the first subsystem has converged the second one can do, and then sequentially. This observer can not deal with another kind of interconnection terms, for example, those given in observability form. The observer proposed here can deal with both interconnections and in fact, even more general.

The effectiveness of the observer has been illustrated in the presented examples, both of them have unstable dynamics and bounded unknown inputs. The results showed that, although the unbounded state variables it has global convergence. Also, it was shown that linear and continuous observers cannot completely compensate the effect of unknown inputs, keeping the error in a neighborhood of zero only. But the High Order Sliding Mode produced by having $d_0 = -1$ homogeneity degree in the 0-approximation terms allows the observers to compensate these effects.

5.1 Future works

The present work opens the possibility to several extensions.

- Design of bl-homogeneous observers for strongly observable SISO and MIMO Linear Time Variant (LTV) systems, i.e. linear systems whose parameters vary with time. In the SISO case the problem can be stated as the observer design for a system given by

$$\Sigma : \begin{cases} \dot{x}_1 &= x_2, \quad y = x_1, \\ \dot{x}_j &= x_{j+1} \\ & \vdots \quad j = 2, ..., n-1 \\ \dot{x}_n &= a_1(t)x_1 + a_2(t)x_2 + ... + a_n(t)x_n + \omega \end{cases} \tag{5.1}$$

- Design of bl-homogeneous UIO for strongly detectable SISO and MIMO Linear Time Invariant systems, that is, systems with inaccessible but stable internal dynamics. This is an immediate extension that does not require a lot of extra work. For example, in SISO case, the problem boils down to designing observers for systems given by

$$\Sigma : \begin{cases} \dot{x}_a &= A_{aa}x_a + H_{ad}y \\ \dot{x}_{d,1} &= x_{d,2}, \quad y = x_1, \\ \dot{x}_{d,j} &= x_{d,j+1} \\ & \vdots \quad j = 2, ..., n_d - 1 \\ \dot{x}_{d,n_b} &= a_{d,1}x_{d,1} + a_{d,2}x_{d,2} + ... + a_{d,n_d}x_{d,n_d} + \omega \end{cases} \tag{5.2}$$

- The design of bl-homogeneous UIO for nonlinear systems with unknown inputs. Although this issue is currently being addressed by colleagues in the same working group, the MIMO non-linear case is still an undeveloped problem. The problem can be seen as the design of observers for a system given by

$$\begin{aligned} \dot{x} &= f(t, x, u) + g(x)\omega(t), \quad x(0) = x_0 \\ y &= h(x) \end{aligned} \tag{5.3}$$

- Design of UIO for networks of non-linear systems in general. The methodology presented in this work in the MIMO case under the SCB transformation is a first approach in the development of this topic, since it can be seen as the design of observers for linear subsystems with also linear interconnections in the states. That is, for future works the task can be extended to the development of observers for more general systems with non-linear interconnection functions between them. The problem can be stated a the design for a general system Σ given by N non-linear systems with interconnection terms $\Psi_i(\cdot)$ and unknown inputs $\omega(t)$.

$$\Sigma_i : \begin{cases} \dot{x}_i &= F_i(x_i, u_i) + \Psi_i(x_1, ..., x_N) + G(x_i)\omega_i(t), \quad x_i(0) = x_{i,0} \\ y_i &= H_i(x_i), \quad i = 1, ..., N \end{cases} \tag{5.4}$$

$\Psi_i(\cdot)$ is an interconnection function that contains all the states $x_i \in \mathbb{R}^{n_i}, i = 1, ..., N$.

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