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MORSE INEQUALITIES: AN ANALYTIC PROOF

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Contents

List of Symbols

(U,φ)	chart of a manifold
$(\Omega^\bullet(M),d)$	De Rham complex of M
$(\Omega^\bullet(M),d_{Tf})$	deformed of De Rham complex
(E,π,M)	differentiable vector bundle $\pi\colon E\longrightarrow M$
$[X,Y]$	Lie bracket of two vector fields X,Y
Alg	category of real algebras of type $C_p^\infty(M)$
$\beta_k(M)$	k–th Betti number of M
$\chi(M)$	Euler characteristic
$\mathrm{Cl}(V,q)$	the Clifford algebra of vector space V
deg	degree of a form
det( )	the determinant
Diff	category of differentiable manifolds
$\mathbf{Diff}_*$	category of pointed differentiable manifolds
$\Gamma(E)$	space of sections of vector bundle (E,π,B)
$\mathrm{grad} f$	gradient of f
$\mathrm{H}^k_{\mathrm{DR}}(M)$	k–th group of De Rham cohomology of M
Hess_f	Hessian matrix of f
id	identity map
ι	inclusion map
$\Lambda^k V$	the k–th exterior power of V

$\mathbf{T}^k(V)$	k–th tensor power of V
$\mathrm{n}_f(p)$	Morse index of f at p
∇^{Cl}	Clifford connection
∇^{LC}	Levi-Civita connection
∇^E	connection of a vector bundle (E, π, M)
$\Omega^k(E)$	set of differentiable sections of the vector bundle $\Lambda^k(T^*M) \otimes E$
$\Omega^k(M)$	space of differentiable k–forms of M
$\Omega^k_c(M)$	set of differentiable k–forms of M with compact support
$\overline{f}, \overline{\varphi}, \overline{\phi}$	germs
∂M	boundary of M
$\mathrm{pr}, \mathrm{pr}^\perp, \mathrm{Pr}, \mathrm{Pr}_k$	projections
$\operatorname{supp} f$	support of a differentiable function f
$\Box$	Laplace–Beltrami operator
$\Box_{Tf}$	deformed the Laplace operator
$\star$	Hodge $\star$–operator
$\mathbf{VB}(B)$	category of vector bundles over a fixed base space B
$\mathbf{Vect}_{\mathbb{R}}$	category of finite dimensional vector spaces over $\mathbb{R}$
Vol_M	volume form of M
$B^k(M)$	set of exact k–forms
$C^\infty(M)$	set of differentiable functions $M \longrightarrow \mathbb{R}$
$C^\infty(M, N)$	set of differentiable maps $M \longrightarrow N$
$C^\infty_p(M)$	set of function germs around $p \in M$
D	De Rham-Hodge operator
d	exterior derivative
$d^\star$	adjoint operator of d
$d^\star_{Tf}$	adjoint operator of d_{Tf}

D_{Tf}	deformed the De Rham–Hodge operator
d_{Tf}	deformed exterior derivative
Df	differential of f
g	Riemannian metric
$H^l_k(M)$	l–Sobolev space of differentiable k–forms
m_k	number of critical points p of f such that $\mathrm{n}_f(p) = k$
$R_\lambda(T)$	resolvent operator of T
$S(k, l)$	set of shuffles of the set $\{1, \ldots, k + l\}$
S^kV	k–th symmetric power of V
S_k	set of permutations of the set $\{1, \ldots, k\}$
T^2	2-torus
TM	tangent bundle
$v \lrcorner$	contraction by v
$Z^k(M)$	set of closed k–forms

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Introduction

Differentiable manifolds are the object of study of differential geometry. We can think of these objects as subsets of $\mathbb{R}^n$ smoothly glued together by homeomorphisms. Differentiable manifolds are generalizations to higher dimensions of curves and surfaces.

To classify differentiable manifolds (and mathematical objects in general) we use equivalence relations. In the case of differentiable manifolds, the equivalence relations can be given by homeomorphism or diffeomorphisms. Note that diffeomorphic differentiable manifolds imply that they are homeomorphic. However, having homeomorphic differentiable manifolds does not imply that they are diffeomorphic. For example, there exist differentiable manifolds that are homeomorphic to the 7–sphere but not diffeomorphic, these manifolds are called *exotic spheres* and they where discovered by Milnor in 1956.

To distinguish two differentiable manifolds we can use topological invariants. A *topological invariant* associates to a differentiable manifold M an algebraic object $I(M)$, say a number, group, vector space, etc. If we have a continuous map $f\colon M \longrightarrow N$ between two differentiable manifolds M and N, to such f we associate a morphism $I(f)\colon I(M) \longrightarrow I(N)$ that preserves the algebraic structure, say bijection, homomorphism, linear isomorphism, etc; so if two differentiable manifolds are homeomorphic, then $I(M)$ and $I(N)$ are isomorphic, that is, $I(M)$ and $I(N)$ are equivalent as algebraic objects. If the reciprocal implication also holds, that is, if $I(M) \cong I(N)$ then M and N are homeomorphic, the topological invariant is called *complete*. Topological invariants are used in the following way: if $I(M) \neq I(N)$, then M and N cannot be homeomorphic, and of course they are not diffeomorphic.

Some examples of topological invariants are:

1. The number of connected components. This is a basic topological invariant.

2. Cohomology associates to a topological space X a family of modules $H^k(X)$ for $k \geq 0$. There are different cohomology theories which all are isomorphic on "reasonable spaces". "Furthermore, in the realm of differentiable manifolds, all these theories coincide with the De Rham theory which makes its appearance there and constitutes in some sense the most perfect example of a cohomology theory. The De Rham theory is also unique in that it stands at the crossroads of topology, analysis, and physics, enriching all three disciplines." [1]

   De Rham cohomology is defined as follows:

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[1] R. Bott and L. W. Tu. Differential Forms in Algebraic Topology, p. 3 [8].

Let M be a differentiable manifold, k be an integer $0 \leq k \leq \dim M$ and $\Omega^k(M)$ be the real vector space of differentiable k–forms on M. We consider the *exterior derivative* $d\colon \Omega^k(M) \longrightarrow \Omega^k(M)$ and since $d \circ d = 0$ we obtain a complex called the *De Rham complex* of M given by

$$(\,\Omega^\bullet(M),\, d\,): \qquad 0 \longrightarrow C^\infty(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} \Omega^n(M) \longrightarrow 0.$$

We denote by $\mathrm{H}^k_{\mathrm{DR}}(M) := \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}}$ the *k–th group of De Rham cohomology* with real coefficients of M.

Note that to define the De Rham cohomology we need the differentiable structure of the differentiable manifold, however, De Rham showed that if the manifold is differentiable this cohomology is isomorphic to the singular cohomology, see [8, Cor. 8.9.2] and [39, Thm. 5.36]. Then De Rham cohomology is only a topological invariant, and not an invariant of the differentiable structure.

3. The k–th Betti number of M is the dimension of the k–th group of De Rham cohomology and it is denoted by $\beta_k(M)$.

4. The *Euler characteristic*: Let M be a differentiable n–manifold, the Euler characteristic of M is the alternating sum of its Betti numbers, that is,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k(M).$$

In the case of connected, closed surfaces, the Euler characteristic is a complete topological invariant.

There is a relationship between the k–th group of the De Rham cohomology and the kernel of the Laplace-Beltrami differentiable operator on $\Omega^k(M)$. Let $d^\star$ be the adjoint operator of d. The *Laplace-Beltrami operator* is the operator $\Box_k\colon \Omega^k(M) \longrightarrow \Omega^k(M)$, defined by $\Box_k \omega = (d + d^\star)^2 \omega$. This operator is an extension of the classical Laplace on differentiable functions on $\mathbb{R}^n$.

Hodge theorem says that for all oriented compact Riemannian manifold of dimension n, a De Rham cohomology class of M can be represented by a unique element of $\operatorname{Ker} \Box_k$, moreover,

$$\mathrm{H}^k_{\mathrm{DR}}(M) \cong \operatorname{Ker} \Box_k.$$

Hodge theorem is important in geometric analysis and harmonic analysis, we will use it constantly.

By Hodge theorem we have $\beta_k(M) = \dim(\operatorname{Ker} \Box_k)$.

To calculate the k–th Betti number of an orientable, closed differentiable n–manifold we can use results as the Poincaré duality for De Rham cohomology or De Rham theorem, but in general it can be difficult. It is possible to give upper bounds to the Betti numbers of M in terms of critical points of some special differentiable functions: Morse functions.

A differentiable function $f\colon M \longrightarrow \mathbb{R}$ is called a *Morse function* on M if all its critical points are non-degenerate, that is, the symmetric matrix of second order partial derivatives called the *Hessian matrix* is invertible. Morse functions can be expressed locally as quadratic polynomials near a critical point (Morse Lemma).

Let f be a Morse function and p be a critical point of f. The *index* of p respect to f, denoted by $\mathrm{n}_f(p)$, is the number of negative eigenvalues of the Hessian matrix at p. We will denote by m_k the number of critical points of f with index k.

Morse inequalities give upper bounds for the Betti numbers:

Theorem (Morse inequalities)**.** *Let M be an oriented, closed Riemannian n–manifold. For any Morse function on M one has*

1. *(Weak Morse inequalities) For any $0 \leq k \leq n$, we have*

$$\beta_k(M) \leq m_k. \tag{1}$$

2. *(Strong Morse inequalities) For any $0 \leq k \leq n$, we have*

$$\beta_k(M) - \beta_{k-1}(M) + \ldots + (-1)^k \beta_0(M) \leq m_k - m_{k-1} + \ldots + (-1)^k m_0. \tag{2}$$

Moreover, for $k = n$:

$$\beta_n(M) - \beta_{n-1}(M) + \ldots + (-1)^n \beta_0(M) = m_n - m_{n-1} + \ldots + (-1)^n m_0. \tag{3}$$

Note that by (7) the Euler characteristic $\chi(M)$ of M can be calculated (up to sign) using the number of critical points of Morse functions.

There are several proofs of this theorem, there is a proof by Marston Morse making use of the notion of subadditive function, for more details see [25] and [30]. Another proof is using Morse Homology, see [5]. These proofs of the Morse inequalities are more general than the one presented here because we need the extra hypothesis of M to be orientable.

The aim of this thesis is to develop the analytic proof of Morse inequalities using the Witten Deformation given on the paper "Supersymmetry and Morse theory" [40] in 1982.

Witten's ideas created relationships between analysis, geometry, topology and mathematical physics. The reader may consult developments or consequences of his ideas in [41], [5], and [6].

Witten's proof consists of studying the deformed De Rham complex of M with the deformed exterior derivative, we deform d by taking a positive real parameter T and a Morse function f on M,

$$d_{Tf}\omega := \exp(-Tf)d\exp(Tf)\omega, \quad \omega \in \Omega^k(M).$$

The important fact is that the cohomology spaces of De Rham complex and the deformed De Rham complex are isomorphic seen as vector spaces, therefore the k–th Betti numbers are the same.

There is an analogue of Hodge Theorem for $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ the k–th cohomology space of the deformed the De Rham complex and $\Box_{Tf,\,k}$ the deformed Laplace–Beltrami operator, that is, $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) \cong \mathrm{Ker}\,\Box_{Tf,\,k}$. Hence

$$\beta_k(M) = \dim \mathrm{Ker}\,\Box_{Tf,\,k}.$$

Thus, it is enough to give bounds for $\dim \operatorname{Ker} \square_{Tf,\,k}$.

To do this, let $c \in \mathbb{R}$, $c > 0$ and A_ν be the eigenspace of $\square_{Tf,\,k}$ associated to the eigenvalue ν. We define $\mathrm{F}^{[0,c]}_{Tf,\,k} \subset \Omega^\bullet(M)$ the direct sum of the the eigenspaces of $\square_{Tf,\,k}$ associated with eigenvalues in $[0,c]$ with $0 \le k \le n$,

$$\mathrm{F}^{[0,c]}_{Tf,\,k} = \bigoplus_{\nu \in [0,c]} A_\nu .$$

The following theorem is the key to the proof of the Morse inequalities:

Theorem. *Let M be an oriented, closed Riemannian n–manifold, $0 < T \in \mathbb{R}$ and $f\colon M \longrightarrow \mathbb{R}$ be a Morse function. For any $0 < c \in \mathbb{R}$ there exist a $0 < T_0 \in \mathbb{R}$ such that for every $T \ge T_0$*

$$\dim\,(\mathrm{F}^{[0,\,c]}_{Tf,\,k}) = m_k. \tag{4}$$

Proof of the Morse inequalities:

1. To prove weak Morse inequality (5), we note that $A_0 \subset \mathrm{F}^{[0,\,c]}_{Tf,\,k}$ the eigenspace of $\square_{Tf,\,k}$ associated to the eigenvalue 0 and $\operatorname{Ker}(\square_{Tf,\,k}) = A_0$. By Hodge Theorem and T large enough such that (8) holds we conclude that $\beta_k(M) \le \dim\,(\mathrm{F}^{[0,\,c]}_{Tf,\,k}) = m_k$.

2. By Rank–Nullity Theorem we have

$$m_k = \dim \mathrm{F}^{[0,c]}_{Tf,\,k} = \dim \operatorname{Ker}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) + \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}).$$

   By the dimension of the quotient vector space we have

$$\begin{aligned} m_k &= \dim \left(\frac{\operatorname{Ker} d_{Tf}\Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}}{\operatorname{Im} d_{Tf}\Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}} \right) + \dim \operatorname{Im}\,\left(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}\right) + \dim \operatorname{Im}\,\left(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}\right) \\ &= \beta_k(M) + \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}) + \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}). \end{aligned}$$

   For $0 \le l \le n$, we take alternating the sum of the m_k to get

$$\begin{aligned} \sum_{k=0}^{l} (-1)^k m_{l-k} &= \sum_{k=0}^{l} (-1)^k \Big(\beta_{l-k}(M) + \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k-1}}) + \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k}}) \Big) \\ &= \sum_{k=0}^{l} (-1)^k \beta_{l-k}(M) + \sum_{k=0}^{l} (-1)^k \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k-1}}) \\ &\quad + \sum_{k=0}^{l} (-1)^k \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k}}) \\ &= \sum_{k=0}^{l} (-1)^k \beta_{l-k}(M) + \dim \operatorname{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l}}). \end{aligned}$$

We have the last equality by cancelling the dimensions of the images of the respective operators and by noticing that $\dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,-1}}) = \dim 0 = 0$.

In particular, for all $0 \leq l \leq n$, we have

$$\sum_{k=0}^{l}(-1)^k \beta_{l-k}(M) \leq \sum_{k=0}^{l}(-1)^k m_{l-k}.$$

This proves the inequality (6).

3. For $l = n$, since $\operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,n}}) = 0$ we get

$$\sum_{k=0}^{n}(-1)^k m_{n-k} = \sum_{k=0}^{n}(-1)^k \beta_{n-k}(M).$$

Then the equality (7) is proved.

Outline:

In Chapter 1 we define the De Rham complex and the k–th group of De Rham cohomology. We will present some relevant results of this cohomology, for example: De Rham Theorem, Poincaré duality for De Rham cohomology and Mayer-Vietoris Theorem. We do the explicit calculations to obtain the k–th groups of De Rham cohomology of some surfaces.

Chapter 2 presents preliminaries of Morse theory. It shows that Morse functions are characterized by locally being quadratic polynomials and we will describe their critical points. This include figures to illustrate examples of Morse functions. We enunciate the Morse inequalities.

In Chapter 3 we proceed with the study of differentiable operators $d^\star, \square_k$. Also, it contains the proof of Hodge theorem which tells us that $\mathrm{H}^k_{\mathrm{DR}}(M) \cong \operatorname{Ker} \square_k$.

In Chapter 4 we study connections on vector bundles and Clifford algebras which in Chapter 6 we will use to do an explicit description of the deformed operator $\square_{Tf,k}$.

In Chapter 5 we present the Witten deformation of the exterior derivative and we define the corresponding deformed operator $\square_{Tf,k}$. The main result in this Chapter is that the cohomology spaces $\mathrm{H}^k_{\mathrm{DR}}(M)$ of De Rham complex and $\mathrm{H}^k_{Tf,\mathrm{DR}}(M)$ the deformed De Rham complex are isomorphic, that is,

$$\mathrm{H}^k_{\mathrm{DR}}(M) \cong \mathrm{H}^k_{Tf,\mathrm{DR}}(M).$$

In Chapter 6 we compute a local description of the deformed Laplace-Beltrami operator on differentiable k–forms.

In chapter 7 we describe the eigenspaces of the deformed Laplace-Beltrami operator on differentiable k–forms.

In Chapter 8 we prove the Morse inequalities.

Also four appendices are included, we recall notions and results for the following topics: Multilinear Algebra, Differential Geometry, Vector bundles and Functional Analysis.

Introducción

Las variedades diferenciables son el objeto de estudio de la geometría diferencial. Podemos pensar estos objetos como subconjuntos de $\mathbb{R}^n$ suavemente pegados por homeomorfismos. Las variedades diferenciables son las generalizaciones a altas dimensiones de curvas y superficies.

Para clasificar variedades diferenciables (y objetos matemáticos en general) usamos relaciones de equivalencia. En el caso de variedades diferenciables, las relaciones de equivalencia pueden estar dadas por homeomorfismos o difeomorfismos. Notar que variedades diferenciables difeomorfas implica que son homeomorfas. Sin embargo, tener variedades diferenciables homeomorfas no implica que sean difeomorfas. Por ejemplo, existen variedades diferenciables que son homeomorfas a la 7–esfera pero no son difeomorfas, estas variedades se llaman *esferas exóticas* y fueron descubiertas por Milnor en 1956.

Para distinguir dos variedades diferenciables podemos usar invariantes topológicos. Un *invariante topológico* asocia a una variedad diferenciable M un objeto algebraico $I(M)$, digamos un número, grupo, espacio vectorial, etc. Si tenemos una aplicación continua $f\colon M \longrightarrow N$ entre dos variedades diferenciables M y N, a tal f le asociamos un morfismo $I(f)\colon I(M) \longrightarrow I(N)$ que preserva la estructura algebraica, digamos biyección, homomorfismo, isomorfismo lineal, etc; así si dos variedades diferenciables son homeomorfas entonces $I(M)$ y $I(N)$ son isomorfas, es decir, $I(M)$ y $I(N)$ son equivalentes como objetos algebraicos. Si la implicación recíproca también se cumple, es decir, si $I(M) \cong I(N)$ entonces M y N son homeomorfas, el invariante topológico es llamado *completo*.

Los invariantes topológicos son usados de la siguiente manera: si $I(M) \neq I(N)$, entonces M y N no pueden ser homeomorfos, y por supuesto las variedades no son difeomorfas.

Algunos ejemplos de invariantes topológicos son:

1. El número de componentes conexas. Este es un invariante topológico básico.

2. La cohomología asocia a un espacio topológico X una familia de módulos $H^k(X)$ para $k \geq 0$. Existen diferentes teorías de cohomología las cuales son todas isomorfas sobre "espacios razonables". "Más aún, en el reino de las variedades diferenciables, todas estas teorías coinciden con la teoría de De Rham que hace su aparición allí y constituye en algun sentido el ejemplo más perfecto de una teoría cohomológica. La teoría de De Rham es también única porque se encuentra en la intersección de la topología, el análisis y la física, enriqueciendo a las tres disciplinas.." [2]

   La cohomología de De Rham es definida como sigue:

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[2] R. Bott and L. W. Tu. Differential Forms in Algebraic Topology, pág. 3 [8].

Sea M una variedad diferenciable, k un entero $0 \leq k \leq \dim M$ y $\Omega^k(M)$ es el espacio vectorial real de k–formas diferenciables sobre M. Consideramos la *derivada exterior* $d\colon \Omega^k(M) \longrightarrow \Omega^k(M)$ y dado que $d \circ d = 0$ obtenemos un complejo llamado el *complejo De Rham* de M dado por

$$(\,\Omega^{\bullet}(M),\, d\,): \qquad 0 \longrightarrow C^{\infty}(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Omega^n(M) \longrightarrow 0.$$

Denotamos por $\mathrm{H}^k_{\mathrm{DR}}(M) := \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}}$ el *k–ésimo grupo de cohomología de De Rham* con coeficientes reales sobre M.

Notar que para definir la cohomología De Rham cohomology necesitamos la estructura diferenciable de la variedad diferenciable, sin embargo, De Rham mostró que si la variedad es diferenciable esta cohomología es isomorfa a la cohomología singular, ver [8, Cor. 8.9.2] y [39, Teo. 5.36]. Entonces la cohomología De Rham es sólo un invariante topológico, y no un invariante de la estructura diferenciable.

3. El k–ésimo número de Betti de M es la dimensión de el k–ésimo grupo de la cohomología De Rham y es denotado por $\beta_k(M)$.

4. La *característica de Euler*: Sea M una n–variedad diferenciable, la característica de Euler de M es la suma alternada de sus números de Betti, es decir,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k(M).$$

En el caso de superficies conexas cerradas, la característica de Euler es un invariante topológico completo.

Existe una relación entre el k–ésimo grupo de cohomología de De Rham y el kernel del operador diferenciable Laplace-Beltrami sobre $\Omega^k(M)$. Sea $d^{\star}$ el operador adjunto de d. El *operador Laplace-Beltrami* es el operador $\Box_k\colon \Omega^k(M) \longrightarrow \Omega^k(M)$, definido por $\Box_k \omega = (d + d^{\star})^2 \omega$. Este operador es una extension del Laplaciano clásico sobre funciones diferenciables sobre $\mathbb{R}^n$.

El teorema de Hodge dice que para toda variedad Riemanniana compacta, orientada de dimensión n, una clase de cohomología De Rham de M puede ser representada por un único elemento de $\operatorname{Ker} \Box_k$, más aún,

$$\mathrm{H}^k_{\mathrm{DR}}(M) \cong \operatorname{Ker} \Box_k.$$

El teorema de Hodge es importante en el análisis geométrico y el análisis armónico, lo usaremos constantemente.

Por el teorema de Hodge tenemos $\beta_k(M) = \dim(\operatorname{Ker} \Box_k)$.

Para calcular el k–ésimo número de Betti de una n–variedad diferenciable cerrada y orientada podemos usar resultados como la dualidad de Poincaré para la cohomología de De Rham o el teorema de De Rham, pero en general esto puede ser difícil. Es posible dar cotas superiores de los números de Betti de M en términos de los puntos críticos de algunas funciones diferenciables especiales: funciones de Morse.

Una función diferenciable $f\colon M \longrightarrow \mathbb{R}$ es llamada *función de Morse* sobre M si todos sus puntos criticos son no degenerados, es decir, la matriz simétrica de derivadas parciales de segundo orden llamada la *matriz Hessiana* es invertible. Las funciones de Morse pueden ser expresadas localmente como polinomios cuadráticos cerca de un punto crítico (Lema de Morse).

Sea f una función de Morse y p un punto crítico de f. El *índice* de p respecto a f, denotado por $\mathrm{n}_f(p)$, es el número de eigenvalores negativos de la matriz Hessiana en p. Denotaremos por m_k el número de puntos críticos de f con índice k.

Las desigualdades de Morse dan cotas superiores para los números de Betti:

Theorem (Desigualdades de Morse)**.** *Sea M una n–variedad Riemanniana cerrada y orientada. Para cualquier función de Morse sobre M uno tiene:*

1. *(Desigualdad de Morse débil) Para cualquier $0 \leq k \leq n$, tenemos*

$$\beta_k(M) \leq m_k. \tag{5}$$

2. *(Desigualdades de Morse fuertes) Para cualquier $0 \leq k \leq n$, tenemos*

$$\beta_k(M) - \beta_{k-1}(M) + \ldots + (-1)^k \beta_0(M) \leq m_k - m_{k-1} + \ldots + (-1)^k m_0. \tag{6}$$

Más aún, para $k = n$:

$$\beta_n(M) - \beta_{n-1}(M) + \ldots + (-1)^n \beta_0(M) = m_n - m_{n-1} + \ldots + (-1)^n m_0. \tag{7}$$

Notar que por la igualdad (7) la característica de Euler $\chi(M)$ de M puede ser calculada (salvo un signo) usando el número de puntos críticos de funciones de Morse.

Hay varias pruebas de este teorema, hay una prueba por Marston Morse que hace uso de la noción de función subaditiva, para más detalles ver [25] y [30]. Otra prueba es usando homología de Morse, ver [5]. Estas pruebas de las desigualdades son más generales a la que presentaremos aquí ya que necesitamos la hipótesis extra sobre M de ser orientable.

El objetivo de esta tesis es desarrollar la prueba analítica de las desigualdades de Morse usando la Deformación de Witten dada en el artículo “Supersymmetry and Morse theory” [40] en 1982.

Las ideas de Witten crearon relaciones entre análisis, geometría, topología y física matemática. El lector puede consultar desarrollos o consecuencias de sus ideas en [41], [5], y [6].

La demostración de Witten consiste en estudiar el complejo de De Rham deformado de M con la derivada exterior deformada, deformamos d tomando un parámetro real positivo T y una función de Morse f sobre M,

$$d_{Tf}\omega := \exp(-Tf)d\exp(Tf)\omega, \quad \omega \in \Omega^k(M).$$

El hecho importante es que los espacios de cohomología del complejo de De Rham y el complejo de De Rham deformado son isomorfos vistos como espacios vectoriales, por lo tanto, los k–ésimos números Betti son los mismos.

Hay un análogo del Teorema de Hodge para $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ el k–ésimo espacio de cohomología del complejo de De Rham deformado y $\square_{Tf,\,k}$ el operador Laplace–Beltrami deformado, es decir, $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) \cong \operatorname{Ker} \square_{Tf,\,k}$. Entonces

$$\beta_k(M) = \dim \operatorname{Ker} \square_{Tf,\,k}.$$

Por lo tanto, es suficiente dar cotas para $\dim \operatorname{Ker} \square_{Tf,\,k}$.

Para ello, sea $c \in \mathbb{R}$, $c > 0$ y A_ν el eigenespacio de $\square_{Tf,\,k}$ asociado al eigenvalor ν. Definimos $\mathrm{F}^{[0,c]}_{Tf,\,k} \subset \Omega^\bullet(M)$ la suma directa de los eigenspacios de $\square_{Tf,\,k}$ asociado con eigenvalues en $[0, c]$ con $0 \leq k \leq n$,

$$\mathrm{F}^{[0,c]}_{Tf,\,k} = \bigoplus_{\nu \in [0,c]} A_\nu.$$

El siguiente teorema es la clave para la demostración de las desigualdades de Morse:

Theorem. *Sea M una n–variedad Riemanniana cerrada y orientada, $0 < T \in \mathbb{R}$ y $f \colon M \longrightarrow \mathbb{R}$ una función de Morse. Para cualquier $0 < c \in \mathbb{R}$ existe un $0 < T_0 \in \mathbb{R}$ tal que para cualquier $T \geq T_0$*

$$\dim(\mathrm{F}^{[0,\,c]}_{Tf,\,k}) = m_k. \tag{8}$$

Prueba de las desigualdades de Morse:

1. Para probar la desigualdad de Morse débil (5), notemos que $A_0 \subset \mathrm{F}^{[0,\,c]}_{Tf,\,k}$ el eigenspacio de $\square_{Tf,\,k}$ asociado al eigenvalor 0 y $\operatorname{Ker}(\square_{Tf,\,k}) = A_0$. Por el teorema de Hodge, para T lo suficientemente grande tal que (8) se cumpla, concluimos que $\beta_k(M) \leq \dim(\mathrm{F}^{[0,\,c]}_{Tf,\,k}) = m_k$.

2. Por el teorema de Rango–Nulidad tenemos

$$m_k = \dim \mathrm{F}^{[0,c]}_{Tf,\,k} = \dim \operatorname{Ker}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}).$$

   Por la dimensión del espacio vectorial cociente tenemos

$$\begin{aligned} m_k &= \dim \left(\frac{\operatorname{Ker} d_{Tf} \Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}}{\operatorname{Im} d_{Tf} \Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}} \right) + \dim \operatorname{Im} \left(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}} \right) + \dim \operatorname{Im} \left(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}} \right) \\ &= \beta_k(M) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}). \end{aligned}$$

   Para $0 \leq l \leq n$, tomamos la suma alternada de los m_k para obtener

$$\begin{aligned} \sum_{k=0}^{l} (-1)^k m_{l-k} &= \sum_{k=0}^{l} (-1)^k \Big(\beta_{l-k}(M) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k-1}}) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k}}) \Big) \\ &= \sum_{k=0}^{l} (-1)^k \beta_{l-k}(M) + \sum_{k=0}^{l} (-1)^k \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k-1}}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{l}(-1)^k \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k}}) \\
=\;&\sum_{k=0}^{l}(-1)^k \beta_{l-k}(M) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l}}).
\end{aligned}
$$

Tenemos la última igualdad cancelando las dimensiones de las imágenes de los respectivos operadores y notando que $\dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,-1}}) = \dim 0 = 0$.

En particular, para toda $0 \leq l \leq n$, tenemos

$$\sum_{k=0}^{l}(-1)^k \beta_{l-k}(M) \leq \sum_{k=0}^{l}(-1)^k m_{l-k}.$$

Esto demuestra la desigualdad (6).

3. Para $l = n$, dado que $\operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,n}}) = 0$ obtenemos

$$\sum_{k=0}^{n}(-1)^k m_{n-k} = \sum_{k=0}^{n}(-1)^k \beta_{n-k}(M).$$

Entonces la igualdad (7) es probada.

Esquema:

En el Capítulo 1 definimos el complejo de De Rham y el k–ésimo grupo de cohomología de De Rham. Presentaremos algunos resultados relevantes de esta cohomología, por ejemplo: el Teorema de De Rham, la dualidad de Poincaré para la cohomología de De Rham y el Teorema de Mayer-Vietoris. Hacemos los cálculos explícitos para obtener los k–ésimos grupos de cohomología de De Rham de algunas superficies.

El capítulo 2 presenta los preliminares de la teoría de Morse. Muestra que las funciones de Morse estan caracterizadas por ser localmente polinomios cuadráticos y describiremos sus puntos críticos. Este incluye figuras para ilustrar ejemplos de funciones Morse. Enunciamos las desigualdades de Morse.

En el Capítulo 3 procedemos con el estudio de los operadores diferenciables $d^\star, \Box_k$. Además, contiene la prueba del teorema de Hodge que nos dice que $\mathrm{H}^k_{\mathrm{DR}}(M) \cong \operatorname{Ker} \Box_k$.

En el Capítulo 4 estudiamos conexiones sobre haces vectoriales y álgebras de Clifford que en el Capítulo 6 usaremos para hacer una descripción explícita del operador deformado $\Box_{Tf,\,k}$. En el Capítulo 5 presentamos la deformación de Witten de la derivada exterior y definimos el operador deformado correspondiente $\Box_{Tf,\,k}$. El principal resultado de este capítulo es que los espacios de cohomología $\mathrm{H}^k_{\mathrm{DR}}(M)$ del complejo De Rham y $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ el complejo de De Rham deformado son isomorfos, es decir,

$$\mathrm{H}^k_{\mathrm{DR}}(M) \cong \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M).$$

En el Capítulo 6 calculamos una descripción local del operador deformado de Laplace-Beltrami en k–formas diferenciables.

En el capítulo 7 describimos los eigenespacios del operador deformado de Laplace-Beltrami en k–formas diferenciables.

En el Capítulo 8 demostramos las desigualdades de Morse.

También se incluyen cuatro apéndices, recordamos nociones y resultados para los siguientes temas: Álgebra multilineal, Geometría diferencial, Haces vectoriales y Análisis funcional.

Chapter 1

De Rham cohomology

The objective of this chapter is to describe the De Rham cohomology. One can consult books [18], [37], [31] and [24].

1.1 Tangent space and tangent bundle

Let M be a differentiable manifold and $p \in M$ be a point, we will denote by $C^\infty_p(M)$ the set of all differentiable germs of $\phi\colon M \longrightarrow \mathbb{R}$, (see Definitions B.2.1 and B.2.4).

The *tangent space* at the point p is the real vector space of derivations of the algebra $C^\infty_p(M)$, that is, $X\colon C^\infty(M) \longrightarrow \mathbb{R}$ which satisfies the *Leibniz rule*, (see B.2.5),

$$X(\overline{\phi} \circ \overline{\psi}) = X(\overline{\phi}) \circ \overline{\psi}(p) + \overline{\phi}(p) \circ X(\overline{\psi}). \tag{1.1}$$

This space is denoted by T_pM.

The readers interested in these notions, see section B.2 in particular definition B.2.6.

Let TM be the *tangent bundle* of M whose fibers are the tangent spaces at each point, (see example C.1.3). A differentiable section of the tangent bundle TM of M is called a *vector field*, see Definition C.0.9.

Analogously, we have T^*M the *cotangent bundle* of M whose fibers are the cotangent spaces of M, (see example C.1.5), the vector spaces dual to the tangent space T_pM. The sections of T^*M are called 1-*forms*.

For more details on vector bundles see the Appendix C.

1.2 Differentiable forms on M

In this section we will define differentiable k–forms, that is, differentiable sections of the k–th exterior power of the cotangent bundle, (see example C.1.6).

First, we give a characterization of differentiable 1-forms.

Let M be a differentiable manifold and $(U, \varphi) = (U, x_1, \ldots, x_n)$ be a chart on M, the value of the 1-form ω at $p \in U$ is a linear combination

$$\omega(p) = \sum_{i=1}^{n} a_i(p)(dx_i)_p.$$

As p varies in U, the coefficients a_i become functions on U.

We will extend the definition of the support of a function to k–forms as follows.

Definition 1.2.1. Let M be a differentiable manifold, we define the *support of a k–form ω* to be

$$\operatorname{supp} \omega = \overline{\{p \in M | \omega(p) \neq 0\}}.$$

Definition 1.2.2. If f is a differentiable function on a differentiable manifold M, its differential is defined to be the 1–form df on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(df)_p(X_p) = X_p f.$$

Where X_p is a derivation (see Definition B.2.5), making abuse of notation $X_p f$ denote $X_p \overline{f}_p$ with $\overline{f} \in C_p^\infty(M)$.

Proposition 1.2.3 (Linearity of a 1–form over functions)**.** *Let ω be a 1–form on a differentiable manifold M. If f is a differentiable function and X is a vector field on M, then $\omega(fX) = f\omega(X)$.*

Proof. At each point $p \in M$, since $\omega(X)$ is defined pointwise, and at each $\omega(p)$ is $\mathbb{R}$–linear in its argument:

$$\omega(fX)(p) = \omega(p)(f(p)X_p) = f(p)\omega(p)(X_p) = (f\omega(X))(p).$$

$\square$

The objective now is to generalize the construction of 1–forms on a differentiable manifold to k–forms.

We apply the construction of exterior algebra to the tangent space T_pM of a differentiable manifold M at a point p.

The *k-th exterior power bundle* of T^*M is the vector bundle over M with fibers $\Lambda^k T_p^* M$ over $p \in M$, denoted by $\Lambda^k T^* M$. The fiber $\Lambda^k T_p^* M$ at a point $p \in M$ is isomorphic to the vector space of all alternating k–forms on the tangent space T_pM, see Remark A.3.33.

A *k–form* on a differentiable manifold M is a section ω of the vector bundle $\Lambda^k T^* M$. The space of sections of $\Lambda^k T^* M$ is denoted by $\Omega^k(M) = \Gamma(\Lambda^k T^* M)$.

Suppose $(U, x_1, \ldots, x_n)$ is a chart on a differentiable manifold M, we know that $dx_1, \ldots, dx_n$ are 1–forms on U. Since at each point $p \in U$, $(dx_1)_p, \ldots, (dx_n)_p$ is a basis for $T_p^* M$, then a basis for $\Lambda^k T_p^* M$ is the set

$$(dx_{i_1})_p \wedge \ldots \wedge (dx_{i_k})_p, \quad 1 \leq i_1 < \ldots < i_k \leq n.$$

Let

$$\mathcal{J}_{k,n} = \{I = (i_1, \ldots, i_k) | 1 \leq i_1 < \ldots < i_k \leq n\} \tag{1.2}$$

be the set of all strictly ascending multiindices between 1 and n of length k and let dx_I denote $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. Thus, locally a k–form on U can be written as

$$\omega = \sum_{I \in \mathcal{J}_{k,n}} a_I dx_I$$

where the a_I are functions on U.

Proposition 1.2.4 ([37, Prop. 18.7])**.** *Let M be a differentiable manifold, let ω be a k–form on M. The following affirmations are equivalent:*

1. *The k–form ω is differentiable on M.*
2. *For every chart $(U, x_1, \ldots, x_n)$ on M, the coefficients a_I of $\omega = \sum a_I dx_I$ relative to the local frame $\{dx_I\}_{I \in \mathcal{J}_{k,n}}$ are all differentiable.*
3. *For any k vector fields $X_1, \ldots, X_k$ on M, the function $\omega(X_1, \ldots, X_k)$ is differentiable on M.*

Remark 1.2.5. Let M be a differentiable manifold, the set of differentiable forms is endowed with a wedge product induced by the wedge product of alternating multilinear maps, (see A.3.41). The pointwise wedge product of differentiable forms on M is given as follows: let ω be a k–form and η be an l–form on M, their wedge product $\omega \wedge \eta \in \Lambda^{\bullet} T^* M$ is the $(k + l)$–form on M such that

$$(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p)$$

at all $p \in M$.

Proposition 1.2.6. *If ω and η are differentiable forms on a differentiable manifold M, then $\omega \wedge \eta$ is also differentiable.*

Proof. Let $(U, x_1, \ldots, x_n)$ be a chart on a differentiable M. On U we have:

$$\omega = \sum_{I \in \mathcal{J}_{k,n}} a_I dx_I \quad \text{and} \quad \eta = \sum_{J \in \mathcal{J}_{l,n}} b_J dx_J,$$

where $a_I, b_J \in C^\infty(U)$, then

$$\begin{aligned} \omega \wedge \eta &= \left(\sum_{I \in \mathcal{J}_{k,n}} a_I dx_I \right) \wedge \left(\sum_{J \in \mathcal{J}_{l,n}} b_J dx_J \right) \\ &= \sum_{J \in \mathcal{J}_{l,n}} \sum_{I \in \mathcal{J}_{k,n}} a_I b_J dx_I \wedge dx_J. \end{aligned}$$

In the sum, $dx_I \wedge dx_J = 0$ if I and J have an index in common. If I and J are disjoint then $dx_I \wedge dx_J = \pm dx_K$ where $K = I \cup J$ but reordered as an increasing multiindex. Thus,

$$\omega \wedge \eta = \sum_{K \in \mathcal{J}_{k+l,n}} \pm a_I b_J dx_K.$$

Since the coefficients of dx_K are differentiable functions on U, by the Proposition 1.2.4, $\omega \wedge \eta$ is differentiable.

□

Let M be a differentiable manifold, the set of all the differentiable forms on M is denoted by $\Omega^\bullet(M)$, this is an anticommutative algebra over $C^\infty(M)$ with the wedge product.

1.2.1 Pullback of k–forms

Let us see how to pull differentiable forms from one manifold to another. Let $f\colon M \longrightarrow N$ be a differentiable function and $\phi \in C^\infty(N)$, the *pullback* $f^*\phi$ is the composition $f^*\phi = \phi \circ f \in C^\infty(M)$.

First, let any $k \geq 1$, we consider $T\colon V \longrightarrow W$ a linear map of vector spaces. It induces a pullback map

$$\begin{array}{rcl} T^*\colon \Lambda^k W^* & \longrightarrow & \Lambda^k V^*, \\ (T^*\eta)(v_1, \ldots, v_k) & = & \eta(T(v_1), \ldots, T(v_n)), \end{array}$$

for $\eta \in \Lambda^k W^*$ and $v_1, \ldots, v_k \in V$. Let $f\colon M \longrightarrow N$ be a differentiable function, at each point $p \in M$, the differential $D_p f\colon T_pM \longrightarrow T_{f(p)}N$ is a linear map of tangent spaces, see definition B.2.7. There is a pullback map

$$f_p^*\colon \Lambda^k T^*_{f(p)} N \longrightarrow \Lambda^k T^*_p M.$$

Thus, let $\omega(f(p)) \in \Lambda^k T^*_{f(p)} N$, then its *pullback* $f^*(\omega(f(p))) \in \Lambda^k T^*_p M$ given by

$$f^*(\omega(f(p)))(X_1, \ldots, X_k) = \omega(f(p))(D_p f X_1, \ldots, D_p f X_k)$$

for all $X_i \in T_pM$. Now, if $\omega \in \Omega^k(N)$, then its *pullback* $f^*\omega \in \Omega^k(M)$ defined pointwise by $(f^*\omega)_p = f^*(\omega(f(p)))$ for all $p \in M$. Equivalently,

$$(f^*\omega)(p)(X_1, \ldots, X_k) = \omega(f(p))(D_p f(X_1), \ldots, D_p f(X_k)). \tag{1.3}$$

1.3 Exterior derivative

Now, we want to extend the differential of differentiable functions to differentiable k–forms.

Definition 1.3.1. Let M be a differentiable manifold and $k = 0, 1, \ldots, \dim M$. We define a linear map $d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ called the *exterior derivative* by

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \quad f \in C^\infty(M), \tag{1.4}$$

and extended by linearity to all of $\Omega^k(M)$.

We have the following fundamental result.

Theorem 1.3.2. *Let M be a differentiable manifold and the exterior derivative $d\colon \Omega^\bullet(M) \longrightarrow \Omega^{\bullet+1}(M)$ of (1.4), then d satisfies:*

1. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta.$$
Then d is of degree 1, that is $\deg d\omega = \deg \omega + 1$.
2. *$(d \circ d)\omega = 0$, for all $\omega \in \Omega^\bullet(M)$.*
3. *If f is a differentiable function and X a vector field on M, then $(df)(X) = Xf$.*

Proof. Let $(U, x_1, \ldots, x_n)$ be a chart on a differentiable manifold M.

1. By linearity of d, we consider $\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ and $\eta = g dx_{j_1} \wedge \ldots \wedge dx_{j_r}$, with $f, g \in C^\infty(U)$. We set for simplicity
$$dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k} \quad \text{and} \quad dx_J = dx_{j_1} \wedge \ldots \wedge dx_{j_r}.$$
Since $\omega \wedge \eta = fg dx_I \wedge dx_J$, by product rule of differentiable functions and anticommutativity of the wedge product of forms we have
$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{r=1}^n \frac{\partial fg}{\partial x_r} dx_r \wedge dx_I \wedge dx_J \\ &= \sum_{r=1}^n \left(g \frac{\partial f}{\partial x_r} + f \frac{\partial g}{\partial x_r} \right) dx_r \wedge dx_I \wedge dx_J \\ &= \sum_{r=1}^n g \frac{\partial f}{\partial x_r} dx_r \wedge dx_I \wedge dx_J + \sum_{r=1}^n f \frac{\partial g}{\partial x_r} dx_r \wedge dx_I \wedge dx_J \\ &= \left(\sum_{r=1}^n \frac{\partial f}{\partial x_r} dx_r \wedge dx_I \right) \wedge g dx_J + (-1)^k f dx_I \left(\sum_{r=1}^n \frac{\partial g}{\partial x_r} dx_r \wedge dx_J \right) \\ &= d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta. \end{aligned}$$

2. By linearity of d, it suffices to check the asserted identity on forms of the type:
$$\omega = f dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \quad f \in C^\infty(U).$$
By definition of d, 1.3.1, then
$$\begin{aligned} d\omega &= \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \\ d(d\omega) &= \sum_{l=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_l \partial x_j} dx_l \wedge dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= \sum_{l<j} \frac{\partial^2 f}{\partial x_l \partial x_j} dx_l \wedge dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} - \sum_{l>j} \frac{\partial^2 f}{\partial x_l \partial x_j} dx_j \wedge dx_l \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &\quad + \sum_{l=j}^n \frac{\partial^2 f}{\partial^2 x_l} dx_l \wedge dx_l \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \\ &= 0. \end{aligned}$$

3. Let $X \in \Gamma(TM)$ be a vector field, then $X = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$ with $g_i \in C^\infty(U)$. By definition 1.2.2, we obtain
$$df(X) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} g_i = \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}.$$

□

Theorem 1.3.3 ([24, Thm. 3.7]). *Let M be a differentiable manifold, there is precisely one linear map $d\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, such that it satisfies the properties of the Theorem 1.3.2.*

Corollary 1.3.4 ([18, Cor. 2.1.2]). *d is independent of the choice of charts.*

In section 4.2 we will see that we can express d in other ways.

1.4 De Rham cohomology

Now, the objective is to define the De Rham cohomology and describe some results and examples.

Definition 1.4.1. Let us consider the vector space $\{\Omega^k(M)\}_{k=0}^n$ of differentiable forms on a differentiable manifold M of dimension n together with the exterior derivative d, the *De Rham complex* of M is the complex defined by

$$(\Omega^\bullet(M), d): \qquad 0 \longrightarrow C^\infty(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Omega^n(M) \longrightarrow 0. \tag{1.5}$$

Where by Theorem 1.3.2-2. $d_{n+1} \circ d_n = 0$.

Definition 1.4.2. Let M be a differentiable manifold, a differentiable k–form $\omega \in \Omega^k(M)$ is said to be a *closed k–form* if $d\omega = 0$. A differentiable k–form $\beta \in \Omega^k(M)$ is an *exact k–form* if $\beta = d\tau$ for some form $\tau \in \Omega^{k-1}(M)$.

The vector space of all closed k–forms is denoted by $Z^k(M)$; and the vector space of all exact k–forms on M is denoted by $B^k(M)$. Since $d \circ d = 0$, (see Theorem 1.3.2-2.), every exact form is closed but not every closed form is exact, then $B^k(M)$ is a subspace of $Z^k(M)$,

Definition 1.4.3. Let M be a differentiable manifold, for all $k \in \mathbb{Z}$, with $0 \leq k \leq \dim M$ the *k–th group of De Rham cohomology* with real coefficients of M is defined by the quotient vector space
$$\mathrm{H}^k_{\mathrm{DR}}(M, \mathbb{R}) := \frac{Z^k(M)}{B^k(M)} = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}}. \tag{1.6}$$

We shall simply write $\mathrm{H}^k_{\mathrm{DR}}(M)$.

We have an equivalence relation given by the quotient $\frac{Z^k(M)}{B^k(M)}$ on $Z^k(M)$, as follows:

$$\omega' \sim \omega \text{ in } Z^k(M) \text{ if and only if } \omega' - \omega \in B^k(M).$$

The equivalence class of a closed form ω is called its *cohomology class* and denoted by $[\omega]$. Also, two closed forms ω and ω' determine the same cohomology class if and only if they differ by an exact form $\omega' = \omega + d\beta$. In this case, we say that two closed forms ω and ω' are *cohomologous*.

Proposition 1.4.4. *If the differentiable manifold M has r connected components, then the 0–th group of De Rham cohomology is $\mathrm{H}^0_{\mathrm{DR}}(M) = \mathbb{R}^r$.*

Proof. Let M be a differentiable manifold.

By complex (1.5) and definition 1.4.2, there are no nonzero exact 0–forms, then by definition (1.6) $\mathrm{H}^0_{\mathrm{DR}}(M) = Z^0(M)$.

Let $(U, x_1, \ldots, x_n)$ be any chart on M and $f \in Z^0(M)$, that is, $f \in C^\infty(M)$ such that $df = 0$. We have

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Since $f \in Z^0(M)$, $df = 0$ on U if and only if all the partial derivatives $\frac{\partial f}{\partial x_i}$ vanish identically on U. This is equivalent to f be locally constant on U.

Let C_i be a connected component of M, then $M = \bigcup\limits_{i=1}^{r} C_i$.

Let $U, V \subset C_i$ such that $U \cap V \neq \emptyset$, since f is locally constant $f|_{U\cap V} = f|_U = f|_V$, we set $f\colon C_i \longrightarrow \mathbb{R}$ defined by $f(p) = c_i$ for some $c_i \in \mathbb{R}$, the same for each connected component C_i, with $i = 1, \ldots, r$.

If M has r connected components, then a closed 0–form is a constant differentiable function on the connected components, which can be specified by $(c_1, \ldots, c_r) \in \mathbb{R}^r$. Therefore $\mathrm{H}^0_{\mathrm{DR}}(M) = Z^0(M) = \mathbb{R}^r$.

□

Proposition 1.4.5. *Let M be a differentiable manifold of dimension n, then the k–th group of De Rham cohomology $\mathrm{H}^k_{\mathrm{DR}}(M) = 0$ for $k > n$.*

Proof. At any point $p \in M$, the tangent space T_pM is a vector space of dimension n. Let $\omega \in \Omega^k(M)$, then $\omega(p)\colon T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$ is a k–multilinear map on T_pM by Proposition A.3.31 if $k > n$, then $\Lambda^k T_pM = 0$. Hence, $M \cong M \times \{0\}$, that is, M is the trivial bundle of rank 0. Then for $k > n$, the only k–form on M is the zero form. □

Let M be a differentiable manifold of dimension n, the *total De Rham cohomology* of M is given by

$$\mathrm{H}^\bullet_{\mathrm{DR}}(M) = \bigoplus_{k=0}^{n} \mathrm{H}^k_{\mathrm{DR}}(M).$$

We consider the wedge product of differentiable forms, (see Remark 1.2.5), it induces a product on $\mathrm{H}^\bullet_{\mathrm{DR}}(M)$ in the following way: let $x \in \mathrm{H}^k_{\mathrm{DR}}(M)$ and $y \in \mathrm{H}^l_{\mathrm{DR}}(M)$ be represented by closed forms $\omega \in Z^k(M), \beta \in Z^l(M)$ respectively, then we set

$$xy = [\omega] \wedge [\beta] = [\omega \wedge \beta] \in \mathrm{H}^{k+l}_{\mathrm{DR}}(M).$$

Lemma 1.4.6. *The product*

$$xy = [\omega \wedge \beta] \in \mathrm{H}_{\mathrm{DR}}^{k+l}(M)$$

is well defined.

Proof. Let $x \in \mathrm{H}_{\mathrm{DR}}^{k}(M)$ and $y \in \mathrm{H}_{\mathrm{DR}}^{l}(M)$ be represented by closed forms $\omega \in Z^k(M), \beta \in Z^l(M)$ respectively. Since ω and β are closed forms, then

$$d(\omega \wedge \beta) = (d\omega) \wedge \beta + (-1)^k \omega \wedge d\beta = 0$$

The $\omega \wedge \beta$ is a closed $k+l$–form.

Let us see that xy does not depend of ω and β. Assume that $\omega' = \omega + d\alpha, \beta' = \beta + d\eta$, by Theorem 1.3.2-1. and by linearity of d then

$$\begin{aligned}
\omega' \wedge \beta' &= (\omega + d\alpha) \wedge (\beta + d\eta) \\
&= \omega \wedge \beta + d\alpha \wedge \beta + \omega \wedge d\eta + d\eta \wedge d\eta \\
&= \omega \wedge \beta + d(\alpha \wedge \beta) + (-1)^k d(\omega \wedge \eta) + d(\alpha \wedge d\eta) \\
&= \omega \wedge \beta + d((-1)^k \omega \wedge \eta + \alpha \wedge \beta + \alpha \wedge d\eta).
\end{aligned}$$

Then $\omega' \wedge \beta'$ and $\omega \wedge \beta$ are cohomologous.

Hence the product xy is determined independently of the choice of closed forms representing x, y. ☐

Also, by Proposition A.3.44 we have: $yx = (-1)^{kl} xy$.

An element $\omega \in \mathrm{H}_{\mathrm{DR}}^{\bullet}(M)$ is a finite sum of cohomology classes in $\mathrm{H}_{\mathrm{DR}}^{k}(M)$ for several $k \in \{0, \ldots, n\}$:

$$\omega = \omega_0 + \ \ldots \ + \omega_k \in \mathrm{H}_{\mathrm{DR}}^{\bullet}(M).$$

This is similar to operating with polynomials, except that the multiplication operation is the wedge product. Then under addition and multiplication, $\mathrm{H}_{\mathrm{DR}}^{\bullet}(M)$ is a ring, called the *cohomology ring* of M. Also, since the wedge product of differentiable forms is anticommutative then the ring is anticommutative.

Note that the ring $\mathrm{H}_{\mathrm{DR}}^{\bullet}(M)$ has a natural grading by the degree of a closed form.

Then, $\mathrm{H}_{\mathrm{DR}}^{\bullet}(M)$ is an anticommutative graded algebra.

A priori, the spaces $\mathrm{H}_{\mathrm{DR}}^{k}(M)$ may be infinite dimensional. The number $\dim(\mathrm{H}_{\mathrm{DR}}^{k}(M))$, denoted by $\beta_k(M)$, is called the *k–th Betti number* of M.

For each integer $k \geq 0$, we denote by $\mathrm{H}_{\mathrm{Sing}}^{k}(M)$ the k–th group of singular cohomology with real coefficients, see the section The Classical Cohomology Theories in [39], [12] and [16].

Theorem 1.4.7 (De Rham, [8, Cor. 8.9.2] and [39, Thm. 5.36])**.** *Let M be a compact differentiable manifold, then for any integer k with $0 \leq k \leq \dim M$,*

1. *$\beta_k(M) < +\infty$.*
2. *$\mathrm{H}_{\mathrm{DR}}^{k}(M)$ is canonically isomorphic to $\mathrm{H}_{\mathrm{Sing}}^{k}(M)$.*

Remark 1.4.8. Since the De Rham cohomology of a differentiable manifold is defined using differentiable forms, it would seem to depend significantly on the differentiable structure of M. However, in reality, it is determined only by properties of M as a topological space. The De Rham theorem expresses this fact concretely.

Theorem 1.4.9 (Homotopic invariance, [33, Thm. 1.44])**.** *If M and N are smoothly homotopy equivalent manifolds, then $\mathrm{H}^k_{\mathrm{DR}}(M) = \mathrm{H}^k_{\mathrm{DR}}(N)$.*

Theorem 1.4.10 (Poincaré duality for De Rham cohomology, [33, Thm. 1.48])**.** *Let M be a compact, connected, oriented differentiable n–manifold, then $\mathrm{H}^k_{\mathrm{DR}}(M) \cong \mathrm{H}^{n-k}_{\mathrm{DR}}(M)$.*

If $k = 0$, by Proposition 1.4.4 and Theorem 1.4.10 we have:

Corollary 1.4.11. *Let M be a compact, connected, oriented differentiable n–manifold, then* $\dim\left(\mathrm{H}^n_{\mathrm{DR}}(M)\right) = 1$.

1.4.1 Computation of the De Rham cohomology

To understand the De Rham complex and the information obtained in De Rham cohomology groups we will compute some examples.

Example 1.4.12 (De Rham cohomology of the real line.)**.** Since the real line $\mathbb{R}^1$ is connected, by Proposition 1.4.4

$$\mathrm{H}^0_{\mathrm{DR}}(\mathbb{R}^1) = \mathbb{R}.$$

For dimensional reasons, on $\mathbb{R}^1$ there are no nonzero differentiable 2–forms. This implies that every differentiable 1–form on $\mathbb{R}^1$ is closed. A differentiable 1-form $f(x)dx$ on $\mathbb{R}^1$ is exact if and only if there is a differentiable function $g(x)$ on $\mathbb{R}^1$ such that

$$f(x)dx = dg = g'(x)dx,$$

where $g'(x)$ is the derivative of g with respect to x. Such a function $g(x)$ is simply an antiderivative of $f(x)$, for example

$$g(x) = \int_0^x f(t)dt.$$

This proves that every differentiable 1–form on $\mathbb{R}^1$ is exact. Therefore, $\mathrm{H}^1_{\mathrm{DR}}(\mathbb{R}^1) = 0$ and by Proposition 1.4.5 we have

$$\mathrm{H}^k_{\mathrm{DR}}(\mathbb{R}^1) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k \geq 1. \end{cases} \tag{1.7}$$

Example 1.4.13 (The cohomology of the Circle)**.** Cover the circle S^1 with two open arcs U and V, the intersection $U \cap V$ is the disjoint union of two open arcs, which we call A and B.

Since S^1 is connected, by Proposition 1.4.4 $\mathrm{H}^0_{\mathrm{DR}}(S^1) = \mathbb{R}$.

We know that S^1 is a compact, connected, oriented differentiable manifold of dimension 1, by Corollary 1.4.11, $\mathrm{H}^1_{\mathrm{DR}}(S^1) = \mathbb{R}$, and by Proposition 1.4.5, $\mathrm{H}^k_{\mathrm{DR}}(S^1) = 0$ for $k > 1$. Therefore, we have

$$\mathrm{H}^k_{\mathrm{DR}}(S^1) = \begin{cases} \mathbb{R} & \text{if } k = 1, 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Then $\beta_0(S^1) = 1$ and $\beta_1(S^1) = 1$.

The Mayer-Vietoris Sequence

In the example of the cohomology of the real line $\mathbb{R}^1$ we can see that calculating the cohomology of a differentiable manifold by solving a given system of differential equations on the manifold and, in case it is not solvable, perhaps we find obstructions to its solvability. This is usually quite difficult to do directly. We introduce one of the most useful tools in the calculation of de Rham cohomology, the Mayer–Vietoris sequence.

Let M be a differentiable manifold and $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\Lambda}$ be an open cover of M, let $\iota_U\colon U \to M$ be the inclusion map give by $\iota_U(p) = p$ where $p \in U$. Then the pullback

$$\iota_U^*\colon \Omega^k(M) \longrightarrow \Omega^k(U)$$

is the restriction map that restricts the domain of a differentiable k–form on M to U: $\iota_U^*\omega = \omega|_U$. In fact, there are four inclusion maps that form a commutative diagram:

$$\begin{tikzcd}
& U \arrow[dr, "\iota_U"] & \\
U \cap V \arrow[ur, "j_U"] \arrow[dr, "j_V"'] & & M \\
& V \arrow[ur, "\iota_V"'] &
\end{tikzcd}$$

By restricting a k–form from M to U and to V, we get a homomorphism of vector spaces

$$\begin{aligned} \iota\colon\ & \Omega^k(M) \longrightarrow \Omega^k(U) \oplus \Omega^k(V), \\ & \sigma \mapsto (\iota_U^*\sigma, \iota_V^*\sigma) = (\sigma|_U, \sigma|_V). \end{aligned}$$

Define the map

$$\begin{aligned} j\colon\ & \Omega^k(U) \oplus \Omega^k(V) \longrightarrow \Omega^k(U\cap V) \\ & (\omega, \eta) \mapsto j_V^*\eta - j_U^*\omega = \eta|_{U\cap V} - \omega|_{U\cap V}. \end{aligned}$$

If $U \cap V$ is empty, we define $\Omega^k(U\cap V) = 0$. In this case, j is simply the zero map. We call ι the *restriction map* and j is the *difference map.*

Theorem 1.4.14 (Mayer-Vietoris, [31, Thm. 7.1.29])**.** *Let M be a differentiable manifold and $M = U \cup V$ be an open cover of M. Then there exists a long exact sequence*

$$\ldots \longrightarrow \mathrm{H}^k_{\mathrm{DR}}(M) \xrightarrow{\iota^*} \mathrm{H}^k_{\mathrm{DR}}(U) \oplus \mathrm{H}^k_{\mathrm{DR}}(V) \xrightarrow{j^*} \mathrm{H}^k_{\mathrm{DR}}(U\cap V) \xrightarrow{d_k^*} \mathrm{H}^{k+1}_{\mathrm{DR}}(M) \longrightarrow \ldots,$$

called the Mayer-Vietoris sequence.

Lemma 1.4.15. *Let* $0 \longrightarrow A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \xrightarrow{d_2} \ldots \xrightarrow{d_{m-1}} A_m \longrightarrow 0$ *be an exact sequence of finite dimensional vector spaces. Then* $\sum\limits_{k=0}^{m}(-1)^k \dim A^k = 0$.

The proof is by Rank-Nullity Theorem and the fact that $\dim \operatorname{Ker} d_k = \dim \operatorname{Im} d_k$.

Proposition 1.4.16 (Mayer-Vietoris, [37, Prop. 26.4]). *In the Mayer–Vietoris sequence, if U, V and $U \cap V$ are connected and nonempty, then*

1. *M is connected and*

$$0 \longrightarrow \mathrm{H}^0_{\mathrm{DR}}(M) \longrightarrow \mathrm{H}^0_{\mathrm{DR}}(U) \oplus \mathrm{H}^0_{\mathrm{DR}}(V) \longrightarrow \mathrm{H}^0_{\mathrm{DR}}(U \cap V) \longrightarrow 0$$

   is exact.

2. *We may start the Mayer–Vietoris sequence with*

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{DR}}(M) \xrightarrow{\iota^*} \mathrm{H}^1_{\mathrm{DR}}(U) \oplus \mathrm{H}^1_{\mathrm{DR}}(V) \xrightarrow{j^*} \mathrm{H}^1_{\mathrm{DR}}(U \cap V) \longrightarrow \dots .$$

Example 1.4.17 (The cohomology of the 2-sphere). Consider the 2–sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We will use the Mayer–Vietoris sequence to deduce the cohomology groups of the 2–sphere.

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be points on S^2, we note that $S^2 = U \cup V$ where $U = S^2 - \{N\}$ and $V = S^2 - \{S\}$.

Since S^2 is connected, by Proposition 1.4.4 $\mathrm{H}^0_{\mathrm{DR}}(S^2) = \mathbb{R}$.

On the other hand, U is homeomorphic to $\mathbb{R}^2$, which is connected, by Theorem 1.4.9 and Proposition 1.4.4, we have $\mathrm{H}^0_{\mathrm{DR}}(U) = \mathbb{R}$, analogously for V, we get $\mathrm{H}^0_{\mathrm{DR}}(V) = \mathbb{R}$. Also, $\mathbb{R}^2$ is contractible to a point and by Theorem 1.4.9

$$\mathrm{H}^k_{\mathrm{DR}}(U) = \mathrm{H}^k_{\mathrm{DR}}(V) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k > 1. \end{cases}$$

Now, $U \cap V$ is homotopically equivalent to S^1, by Theorem 1.4.9 and by equality (1.8) we have

$$\mathrm{H}^k_{\mathrm{DR}}(U \cap V) = \begin{cases} \mathbb{R} & \text{if } k = 0, 1 \\ 0 & \text{if } k > 1. \end{cases}$$

Since U, V and $U \cap V$ are connected we can apply Proposition 1.4.16, we have the Mayer–Vietoris sequence:

$$0 \longrightarrow \mathrm{H}^0_{\mathrm{DR}}(S^2) = \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0,$$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1_{\mathrm{DR}}(S^2) \arrow[r] & 0 \oplus 0 \arrow[r] & \mathbb{R} \arrow[r] & {} \\ {} \arrow[r] & \mathrm{H}^2_{\mathrm{DR}}(S^2) \arrow[r] & 0 \oplus 0 \arrow[r] & 0 & \end{tikzcd}$$

Then $\mathrm{H}^1_{\mathrm{DR}}(S^2) = 0$ and for the exactness of the sequence $\mathrm{H}^2_{\mathrm{DR}}(S^2) = \mathbb{R}$. Therefore

$$\mathrm{H}^k_{\mathrm{DR}}(S^2) = \begin{cases} \mathbb{R} & \text{if } k = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\beta_0(S^2) = 1$, $\beta_1(S^2) = 0$ and $\beta_2(S^2) = 1$.

Example 1.4.18 (The cohomology of the 2-Torus)**.** We consider the 2-Torus.

Cover the torus T^2 with two open subsets U and V, both U and V are homeomorphic to $S^1 \times I$, where $I = [0,1]$.

Note that T^2 is connected, by Proposition 1.4.4 $\mathrm{H}^0_{\mathrm{DR}}(T^2) = \mathbb{R}$.

Also, U and V are homotopically equivalent to S^1, by Theorem 1.4.9

$$\mathrm{H}^k_{\mathrm{DR}}(U) = \mathrm{H}^k_{\mathrm{DR}}(V) = \begin{cases} \mathbb{R} & \text{if } k = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

Now, $U \cap V$ is the disjoint union of two S^1, then

$$\mathrm{H}^k_{\mathrm{DR}}(U \cap V) = \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{if } k = 0,1 \\ 0 & \text{if } k > 1. \end{cases}$$

By Theorem 1.4.14 we have the long Mayer–Vietoris sequence of T^2:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "t"] & \mathbb{R} \oplus \mathbb{R} \\
\arrow[r, "d_0^*"] & \mathrm{H}^1_{\mathrm{DR}}(T^2) \arrow[r, "\iota^*"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "s"] & \mathbb{R} \oplus \mathbb{R} \\
\arrow[r, "d_1^*"] & \mathrm{H}^2_{\mathrm{DR}}(T^2) \arrow[r] & 0 \oplus 0 \arrow[r] & 0
\end{tikzcd}$$

By Proposition 1.4.4 $\mathrm{H}^0_{\mathrm{DR}}(M)$ is the vector space of constant functions on the manifold, if $a \in \mathrm{H}^0_{\mathrm{DR}}(U)$ is the constant function with value a on U, $j_U^*\colon \mathrm{H}^0_{\mathrm{DR}}(U) \longrightarrow \mathrm{H}^0_{\mathrm{DR}}(U \cap V)$ then $j_U^* a = a|_{U \cap V}$ is the constant function with the value a on each component of $U \cap V$, that is, $j_U^* a = (a,a)$.

Then, for $(a,b) \in \mathrm{H}^0_{\mathrm{DR}}(U) \oplus \mathrm{H}^0_{\mathrm{DR}}(V)$, $t(a,b) = b|_{U\cap V} - a|_{U \cap V} = (b,b) - (a,a) = (b-a, b-a)$.

Analogously, we describe the map $s\colon \mathrm{H}^1_{\mathrm{DR}}(U) \oplus \mathrm{H}^1_{\mathrm{DR}}(V) \longrightarrow \mathrm{H}^1_{\mathrm{DR}}(U \cap V)$.

Let $U \cap V = A \sqcup B$, A and B the connected components. We have the inclusions $j_{U,A}\colon A \longrightarrow U$, $j_{U,B}\colon B \longrightarrow U$, if ω_U generates $\mathrm{H}^1_{\mathrm{DR}}(U)$, we define

$$\begin{aligned} j_{U,A}^*\colon \quad \mathrm{H}^1_{\mathrm{DR}}(U) &\longrightarrow \mathrm{H}^1_{\mathrm{DR}}(A) \\ \omega_U &\mapsto \omega_A. \end{aligned}$$

Then $j_{U,A}^* \omega_U = \omega_A$ is a generator of $\mathrm{H}^1_{\mathrm{DR}}(A)$, the same for $\mathrm{H}^1_{\mathrm{DR}}(B)$.

$$\begin{aligned} j_U^*\colon \quad \mathrm{H}^1_{\mathrm{DR}}(U) &\longrightarrow \mathrm{H}^1_{\mathrm{DR}}(U \cap V) \\ c\omega_U &\mapsto (c\omega_A, c\omega_B). \end{aligned}$$

The pair of real numbers $(a,b) \in \mathrm{H}^1_{\mathrm{DR}}(U) \oplus \mathrm{H}^1_{\mathrm{DR}}(V)$ stands for $(a\omega_U, b\omega_V)$.

Then

$$s(a,b) = j_V^*(b\omega_V) - j_U^*(a\omega_U) = (b,b) - (a,a) = (b-a, b-a).$$

By Rank-Nullity Theorem, s and the exactness of the sequence we have $\mathrm{H}^2_{\mathrm{DR}}(T^2) = \mathbb{R}$. And by Lemma 1.4.15 we obtain $\dim \mathrm{H}^1_{\mathrm{DR}}(T^2) = 2$, therefore $\mathrm{H}^1_{\mathrm{DR}}(T^2) = \mathbb{R} \oplus \mathbb{R}$.

1.5 Other expression of d

A generalization of the definition of contraction, A.3.34, is as follows:

Definition 1.5.1. Let M be a differentiable manifold and $X \in \Gamma(TM)$ be a vector field on M. The *contraction* or *interior product* by X is a linear map

$$X\lrcorner\colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$

defined by

$$X\lrcorner\omega(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1})$$

for $\omega \in \Omega^k(M)$, $X_1, \ldots, X_{k-1} \in \Gamma(TM)$.

Note that if $k = 0$, we define $X\lrcorner = 0$.

Let $f \in C^\infty(M)$, by definition $X\lrcorner(f\omega) = f \cdot [X\lrcorner\omega]$, then $X\lrcorner$ is linear with respect to differentiable functions.

Lemma 1.5.2 ([37, Proposition 20.8])**.** *Let M be a differentiable manifold, for all $X \in \Gamma(TM)$ a vector field on M, then $X\lrcorner\colon \Omega^k(M) \to \Omega^{k-1}(M)$ the contraction by X satisfies:*

1. $X\lrcorner \circ X\lrcorner = 0.$
2. *$X\lrcorner$ is of degree -1, such that, for each $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$,*

$$X\lrcorner(\omega \wedge \eta) = (X\lrcorner\omega) \wedge \eta + (-1)^k \omega \wedge (X\lrcorner\eta).$$

In the tangent space we have an anticonmutative bilinear map $[\ ,\ ]$.

Definition 1.5.3. Let M be a differentiable manifold and $p \in M$. Let $(U, x_1, \ldots, x_n)$ be a chart around $p \in U$, the *Lie bracket* between two vector fields $[\ ,\ ]\colon \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$ is a bilinear map defined by

$$[X,Y] = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(a_i \frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\frac{\partial}{\partial x_i}\right), \quad \text{where } X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},\ Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j},\ a_i, b_i \in C^\infty(M).$$

We say that the vector fields X and Y *commute if* $[X,Y] = 0$.

Lemma 1.5.4. *The Lie bracket $[\ ,\ ]$ is $\mathbb{R}$–bilinear. Also, for any differentiable function $f\colon M \longrightarrow \mathbb{R}$, we have $[X,Y]f = X(Y(f)) - Y(X(f))$. Furthermore, we have that the Jacobi identity holds*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

for any three vector fields X, Y, Z.

Proof. Let $p \in M$, $(U, x_1, \ldots, x_n)$ be a chart at p and $X = \sum\limits_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$, $Y = \sum\limits_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$ vector fields. We have

$$[X, Y]f = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} \right) = X(Y(f)) - Y(X(f)).$$

And this is $\mathbb{R}$–bilinear in X, Y. This implies the first two claims. By computation follows the Jacobi identity. ☐

Theorem 1.5.5 ([28, Thm. 42.9])**.** *Let M be a differentiable manifold, $\omega \in \Omega^k(M)$ an arbitrary differentiable k–form on M. Then for any vector fields $X_j \in \Gamma(TM)$, with $0 \leq j \leq k$, we have*

$$\begin{aligned} d\omega(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k). \end{aligned}$$

Chapter 2

Morse theory

The idea of Morse Theory is that global invariants of a compact differentiable manifold can be recovered from the local analysis at the critical points of a differentiable function on that manifold, for example the Morse inequalities.

In this chapter we will give the terminology, results and examples of Morse theory.

For more references consult [25], [30] and [26].

Definition 2.0.1. Let M be a differentiable manifold of dimension n, let $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function on M. For each point $p \in M$, we choose a chart around p, $\varphi\colon U \longrightarrow V \subset \mathbb{R}^n$. Consider $F = f \circ \varphi^{-1}\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and its differential

$$D_{\varphi(p)}F\colon T_{\varphi(p)}\mathbb{R}^n \longrightarrow T_{f(p)}\mathbb{R}.$$

1. p is a *critical point* of f if $D_{\varphi(p)}F$ is not surjective, that is, the partial derivatives vanishes
$$\frac{\partial F}{\partial x_1}(\varphi(p)) = 0, \dots, \frac{\partial F}{\partial x_n}(\varphi(p)) = 0.$$
2. The *Hessian matrix* of f with respect to φ is defined as the symmetric matrix of second order partial derivatives:
$$\mathrm{Hess}_F = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq n}.$$
3. p is a *non-degenerate critical point* of f if the Hessian is invertible, that is,
$$\det(\mathrm{Hess}_F(\varphi(p))) \neq 0.$$

Definition 2.0.2. A differentiable function is called a *Morse function* if all its critical points are non-degenerate.

Lemma 2.0.3. *The critical points and non-degenerate critical points do not depend on the choice of chart.*

Proof. Let M be a differentiable manifold of dimension n and $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function.

Let (U,φ) with $\varphi = (x_1,\ldots,x_n)$ and (V,ψ) with $\psi = (y_1,\ldots,y_n)$ be charts around a critical point p of f. We note that

$$\begin{aligned} f\circ\varphi^{-1}|_{\varphi(U\cap V)} &= (f\circ\psi^{-1})\circ(\psi\circ\varphi^{-1})|_{\varphi(U\cap V)}. && (2.1)\\ f\circ\psi^{-1}|_{\psi(U\cap V)} &= (f\circ\varphi^{-1})\circ(\varphi\circ\psi^{-1})|_{\psi(U\cap V)}. && (2.2)\end{aligned}$$

Let us see that

$$\frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) = 0 \quad \text{if and only if} \quad \frac{\partial(f\circ\psi^{-1})}{\partial y_i}(\psi(p)) = 0 \quad \text{for all } i = 1,\ldots,n.$$

Suppose that for all $i = 1,\ldots,n$,

$$\frac{\partial(f\circ\psi^{-1})}{\partial y_i}(\psi(p)) = 0.$$

We get $\frac{\partial(f\circ\psi^{-1})}{\partial y_i}(\psi(p))$ and let $(\psi\circ\varphi^{-1})_j$ be the j–th coordinate function of $\psi\circ\varphi^{-1}$.

By equality (2.1) and the chain rule we have

$$\begin{aligned}\frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) &= \frac{\partial((f\circ\psi^{-1})\circ(\psi\circ\varphi^{-1}))}{\partial x_i}(\varphi(p))\\ &= \sum_{j=1}^{n}\left(\frac{\partial(f\circ\psi^{-1})}{\partial y_j}(\psi\circ\varphi^{-1})(\varphi(p))\right)\frac{\partial(\psi\circ\varphi^{-1})_j}{\partial x_i}(\varphi(p)).\end{aligned}$$

We note that $\psi\circ\varphi^{-1}(\varphi(p)) = \psi(p)$ and we evaluate

$$\frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) = \sum_{j=1}^{n}\left(\frac{\partial(f\circ\psi^{-1})}{\partial y_j}(\psi(p))\right)\left(\frac{\partial(\psi\circ\varphi^{-1})_j}{\partial x_i}(\varphi(p))\right). \tag{2.3}$$

By hipothesis, we obtain

$$\frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) = 0.$$

The same in the other direction.

Therefore, a critical point of f does not depend on the choice of chart.

Now, suppose that p is a non-degenerate critical point of f.

Let (U,φ) with $\varphi = (x_1,\ldots,x_n)$ and (V,ψ) with $\psi = (y_1,\ldots,y_n)$ be charts around of p.

By equation (2.3) and Leibniz rule we obtain that for all $1 \le j \le n$

$$\begin{aligned}\frac{\partial^2(f\circ\varphi^{-1})}{\partial x_i\partial x_j}(\varphi(p)) &= \sum_{r=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial(f\circ\psi^{-1})}{\partial y_r}(\psi(p))\right)\left(\frac{\partial(\psi\circ\varphi^{-1})_r}{\partial x_i}(\varphi(p))\right)\\ &= \sum_{r=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial^2(f\circ\psi^{-1})}{\partial y_l\partial y_r}(\psi(p))\frac{\partial(\psi\circ\varphi^{-1})_l}{\partial x_j}(\varphi(p))\right)\left(\frac{\partial(\psi\circ\varphi^{-1})_r}{\partial x_i}(\varphi(p))\right)\end{aligned}$$

$$
\begin{aligned}
& +\sum_{r=1}^{n}\left(\frac{\partial(f\circ\psi^{-1})}{\partial y_r}(\psi(p))\right)\left(\frac{\partial^2(\psi\circ\varphi^{-1})_r}{\partial x_i\partial x_j}(\varphi(p))\right)\\
= & \sum_{r=1}^{n}\sum_{l=1}^{n}\left(\frac{\partial^2(f\circ\psi^{-1})}{\partial y_l\partial y_r}\psi(p)\right)\left(\frac{\partial(\psi\circ\varphi^{-1})_l}{\partial x_j}\varphi(p)\right)\left(\frac{\partial(\psi\circ\varphi^{-1})_r}{\partial x_i}(\varphi(p))\right)\\
& +\sum_{r=1}^{n}\frac{\partial(f\circ\psi^{-1})}{\partial y_r}\psi(p)\frac{\partial^2(\psi\circ\varphi^{-1})}{\partial x_i\partial x_j}(\varphi(p)).
\end{aligned}
$$

Since p is a critical point, the second term vanishes. Then

$$\frac{\partial^2(f\circ\varphi^{-1})}{\partial x_i\partial x_j}(\varphi(p))=\sum_{r=1}^{n}\sum_{l=1}^{n}\frac{\partial^2(f\circ\psi^{-1})}{\partial y_l\partial y_r}\psi(p)\frac{\partial(\psi\circ\varphi^{-1})_l}{\partial x_j}\varphi(p)\frac{\partial(\psi\circ\varphi^{-1})_r}{\partial x_i}(\varphi(p)).$$

We consider the Jacobian of $\psi\circ\varphi^{-1}$ at $\varphi(p)=0$

$$
\begin{aligned}
J &= J_{\psi\circ\varphi^{-1}}(0)\\
&= \begin{pmatrix}
\frac{\partial(\psi\circ\varphi^{-1})_1}{\partial x_1} & \frac{\partial(\psi\circ\varphi^{-1})_1}{\partial x_2} & \dots & \frac{\partial(\psi\circ\varphi^{-1})_1}{\partial x_n}\\
\vdots & \vdots & & \vdots\\
\frac{\partial(\psi\circ\varphi^{-1})_n}{\partial x_1} & \frac{\partial(\psi\circ\varphi^{-1})_n}{\partial x_2} & \dots & \frac{\partial(\psi\circ\varphi^{-1})_n}{\partial x_n}
\end{pmatrix}.
\end{aligned}
$$

We denoted by J^t its transpose, we have

$$\mathrm{Hess}_{f\circ\varphi}(\varphi(p))=J^t\mathrm{Hess}_{f\circ\psi}(\psi(p))J.$$

Since $\psi\circ\varphi^{-1}$ is a differentiable function with differentiable inverse function, the matrix J and J^t have non-zero determinant.

Therefore, $\det(\mathrm{Hess}_{f\circ\varphi^{-1}}(\varphi(p)))\neq 0$ if and only if $\det(\mathrm{Hess}_{f\circ\psi^{-1}}(\psi(p)))\neq 0$.

□

The existence of Morse functions is guaranteed by [30, Thm. 1.21]. In fact, by Sard Theorem the majority of differentiable functions are actually Morse functions, see [15, Section 1.7].

Morse functions have a very simple local structure: up to a change of coordinates all Morse functions are quadratic polynomials. This is the content of the *Morse Lemma.*

Theorem 2.0.4 (Morse Lemma)**.** *Let M be an n–dimensional differentiable manifold. Suppose $f\colon M\longrightarrow\mathbb{R}$ is a differentiable function and p is a non-degenerate critical point of f. Then there exists an open neighbourhood U of p and a chart $\varphi\colon U\longrightarrow V\subset\mathbb{R}^n$ such that $\varphi(p)=0$ and in this chart we have the equality*

$$(f\circ\varphi^{-1})(\mathbf{y})=f(p)-y_1^2-\ldots-y_k^2+y_{k+1}^2+\ldots+y_n^2,\quad \mathbf{y}=(y_1,\ldots,y_n)\in V. \tag{2.4}$$

Proof. Without loss of generality, assume that $f(p)=0$, otherwise we can take the function $g:=f-f(p)$.

Since the problem is local and invariant under local diffeomorphisms we can also assume that $f\colon W\longrightarrow\mathbb{R}$ where W is an open connected neighborhood around 0 in $\mathbb{R}^n$ and 0 is a non-degenerate critical point of f.

By Lemma B.2.8, there exist differentiable functions $g_i\colon W \longrightarrow \mathbb{R}$, $1 \leq i \leq n$ such that

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i g_i(\mathbf{x}), \quad \mathbf{x} \in W.$$

Since 0 is a critical point, $g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$.

Again, for g_i by Lemma B.2.8 there exist differentiable functions $g_{ij}\colon W \longrightarrow \mathbb{R}$ with $1 \leq j \leq n$ such that

$$g_i(\mathbf{x}) = \sum_{j=1}^{n} x_j g_{ij}(\mathbf{x}), \quad \mathbf{x} \in W.$$

Then

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j g_{ij}(\mathbf{x}) \\ &= \sum_{i=1}^{n} x_i^2 g_{ii}(\mathbf{x}) + \sum_{i<j} x_i x_j (g_{ij} + g_{ji})(\mathbf{x}). \end{aligned}$$

We define $h_{ij}(\mathbf{x}) = \frac{1}{2}(g_{ij} + g_{ji})(\mathbf{x})$, then we rewrite

$$f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 h_{ii}(\mathbf{x}) + \sum_{i<j} x_i x_j h_{ij}(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j h_{ij}(\mathbf{x}). \tag{2.5}$$

Hence $(h_{ij}(\mathbf{x}))$ is a symmetric $n \times n$ matrix of differentiable functions.

Let us calculate the second derivatives of f:

$$\begin{aligned} \frac{\partial f}{\partial x_i}(\mathbf{x}) &= 2x_i h_{ii}(\mathbf{x}) + x_i^2 \frac{\partial h_{ii}}{\partial x_i}(\mathbf{x}) + \sum_{j=1}^{n}\left(x_j h_{ij}(\mathbf{x}) + x_i x_j \frac{\partial h_{ij}}{\partial x_i}(\mathbf{x})\right). \\ \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}) &= x_i^2 \frac{\partial^2 h_{ii}}{\partial x_j \partial x_i}(\mathbf{x}) + h_{ij}(\mathbf{x}) + x_j \frac{\partial h_{ij}}{\partial x_j}(\mathbf{x}) + x_i \frac{\partial h_{ij}}{\partial x_i}(\mathbf{x}) + x_i x_j \frac{\partial^2 h_{ij}}{\partial x_j \partial x_i}(\mathbf{x}). \\ \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x}) &= 2h_{ii}(\mathbf{x}) + 4x_i \frac{h_{ii}}{\partial x_i}(\mathbf{x}) + x_i^2 \frac{\partial^2 h_{ii}}{\partial x_i}(\mathbf{x}) + 2x_j \frac{\partial h_{ij}}{\partial x_i}(\mathbf{x}) + x_i x_j \frac{\partial^2 h_{ij}}{\partial x_i^2}(\mathbf{x}). \end{aligned}$$

We get

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0) = \begin{cases} h_{ij}(0) & \text{if } i \neq j \\ 2h_{ii}(0) & \text{if } i = j. \end{cases}$$

Since 0 is a non-degenerate critical point of f, then $\mathrm{Hess}_f(0) = (h_{ij}(0))$ is an invertible matrix.

We will do the proof by induction, let us see that the chart φ of the Theorem can be chosen in such a way that it is given by equality (2.5) with

$$(h_{ij}(\mathbf{x})) = \begin{pmatrix} D & 0 \\ 0 & S \end{pmatrix}$$

with D an $(l-1)\times(l-1)$ matrix with diagonal $(\pm 1,\dots,\pm 1)$ and S some symmetric $(n-l-1)\times(n-l-1)$ matrix of differentiable functions. Then we assume the induction hypothesis

$$f(\mathbf{x}) = \sum_{i=1}^{l-1}\delta_i x_i^2 + \sum_{i=l}^{n}\sum_{j=l}^{n} x_i x_j h_{ij}(\mathbf{x}), \quad \delta = \pm 1. \tag{2.6}$$

Remark 2.0.5. We can always find $h_{ss}(0) \neq 0$ with $l \leq s \leq n$, the arguments are the following:

1. If some $h_{rr}(0) \neq 0$ for some $l \leq r \leq n$, we only make a change of rows and columns.

2. If $h_{ll}(0) = h_{l+1l+1}(0) = 0 = \dots = h_{nn}(0) = 0$, as among the coefficients of the double summation of the equality (2.5) it must be coefficients different from 0, otherwise the $\mathrm{Hess}_f(0) = 0$.

   For example, suppose that $h_{rs}(0) \neq 0$, with $l \leq r, s \leq n$. Then it is sufficient to consider the differentiable function $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined as

   $$T(z_1,\dots,z_n) = \begin{cases} x_r = z_r - z_s, & \\ x_s = z_r + z_s, & \\ x_i = z_i, & i \neq r, i \neq s. \end{cases}$$

   Then the Jacobian matrix of T at 0 is

   $$J_T(0) = \begin{pmatrix} 1 & \dots & \overbrace{0}^{l-1} & \overbrace{0}^{l} & \dots & \overbrace{0}^{r} & \dots & \overbrace{0}^{s} & \dots & \overbrace{0}^{n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & \dots & -1 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 1 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

   $J_T(0)$ is a non-degenerate matrix with determinant 2.

   So the term $x_r x_s h_{rs}(\mathbf{x})$, we rewrite it as

   $$x_r x_s h_{rs}(\mathbf{z}) = (z_r - z_s)(z_r + z_s)h_{rs}(\mathbf{z}) = z_r^2 h_{rs}(\mathbf{z}) - z_s^2 h_{rs}(\mathbf{z}).$$

   Replacing it in the equality (2.6)

   $$f(T(\mathbf{x})) = \sum_{i=1}^{l-1}\delta_i z_i^2 + \sum_{\substack{i=l \\ i\neq s,r}}^{n} \sum_{\substack{j=l \\ j\neq r,s}}^{n} x_i x_j h_{ij}(\mathbf{z}) + (z_r^2 - z_s^2)(h_{rs} + h_{sr})(\mathbf{z}), \quad \delta = \pm 1.$$

If $l = 1$, we get

$$f(\mathbf{x}) = \delta_1 x_1^2 + \sum_{i=2}^{n}\sum_{j=2}^{n} x_i x_j h_{ij}(\mathbf{x}), \quad \delta = \pm 1.$$

By Remark 2.0.5, we can assume $h_{11}(0) \neq 0$, by continuity of h_{11} we can also assume that $h_{11}(\mathbf{x})$ has a constant sign $\delta_1 = \pm 1$ on some smaller neighborhood $W_1 \subset W$. Then $\sqrt{|h_{11}(\mathbf{x})|}$ is a non zero differentiable function at $\mathbf{x}$ over W_1.

We have the new variables through the differentiable function $R\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\begin{aligned} y_1 &= \sqrt{|h_{11}(\mathbf{x})|}\left(x_1 + \sum_{i=2}^{n} x_j \frac{h_{i1}(\mathbf{x})}{h_{11}(\mathbf{x})}\right) \\ y_j &= x_j, \quad \text{for all} \quad j = 2, \dots, n. \end{aligned}$$

Note that $\det J_R(0) = \sqrt{|h_{11}(0)|} \neq 0$, then R is an invertible function. Also, $\dim(T_0\mathbb{R}^n) = \dim(T_{R(0)}\mathbb{R}^n)$, we have D_0R is a linear isomorphism. By Theorem B.2.11, R is a local diffeomorphism. Then

$$\begin{aligned} f \circ R^{-1}(\mathbf{y}) &= f(\mathbf{x}) \\ &= \delta_1 x_1^2 + \sum_{i=2}^{n}\sum_{j=2}^{n} x_i x_j h_{ij}(\mathbf{x}) \end{aligned}$$

Now, we assume the induction hypothesis (2.6), let us see that it is true for l.

By Remark 2.0.5, we suppose that $h_{ll}(0) \neq 0$ and by continuity of $h_{ll}(\mathbf{x})$ we can assume that $h_{11}(\mathbf{x})$ has a constant sign $\delta_l = \pm 1$ on some smaller neighborhood $W_1 \subset W$.

We define

$$q(\mathbf{x}) := \sqrt{|h_{ll}(\mathbf{x})|}.$$

Since $h_{ll}(0) \neq 0$, $q(\mathbf{x})$ is a differentiable function no zero at $\mathbf{x}$ over W_1.

Introducing the new variables through the differentiable function $S\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\begin{aligned} y_l &= q(\mathbf{x})\left(x_l + \sum_{i=l+1}^{n} x_j \frac{h_{il}(\mathbf{x})}{h_{ll}(\mathbf{x})}\right) && (2.7) \\ y_j &= x_j, \quad \text{for all} \quad j = 1, \dots, n,\ j \neq l. && (2.8) \end{aligned}$$

We calculate

$$J_S(0) = \begin{pmatrix} 1 & \dots & \overbrace{0}^{l-1} & \overbrace{0}^{l} & \overbrace{0}^{l+1} & \dots & \overbrace{0}^{n} \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & q(0) & \frac{h_{l+1l}(0)}{h_{ll}(0)} & \dots & \frac{h_{nl}(0)}{h_{ll}(0)} \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

We note that $\det J_S(0) = q(0) \neq 0$, then S is an invertible function. Also, $\dim(T_0\mathbb{R}^n) = \dim(T_{S(0)}\mathbb{R}^n)$ and by Theorem B.2.11, S is a local diffeomorphism.

By equality (2.6), then

$$\begin{aligned}
f \circ S^{-1}(\mathbf{y}) &= f(\mathbf{x}) \\
&= \sum_{i=1}^{l-1} \delta_i x_i^2 + x_l^2 h_{ll}(\mathbf{x}) + 2x_l \sum_{j=l+1}^{n} x_j h_{jl}(\mathbf{x}) + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} x_i x_j h_{ij}(\mathbf{x}) \\
&= \sum_{i=1}^{l-1} \delta_i x_i^2 + h_{ll}(\mathbf{x}) \left(x_l^2 + 2x_l \sum_{j=l+1}^{n} x_j \frac{h_{jl}(\mathbf{x})}{h_{ll}(\mathbf{x})} \right) + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} x_i x_j h_{ij}(\mathbf{x}) \\
&= \sum_{i=1}^{l-1} \delta_i x_i^2 + h_{ll}(\mathbf{x}) \left[x_l^2 + 2x_l \sum_{j=l+1}^{n} x_j \frac{h_{jl}(\mathbf{x})}{h_{ll}(\mathbf{x})} + \left(\sum_{j=l+1}^{n} x_j \frac{h_{jl}(\mathbf{x})}{h_{ll}(\mathbf{x})} \right)^2 \right] \\
&\quad - h_{ll}(\mathbf{x}) \left(\sum_{j=l+1}^{n} x_j \frac{h_{jl}(\mathbf{x})}{h_{ll}(\mathbf{x})} \right)^2 + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} x_i x_j h_{ij}(\mathbf{x}).
\end{aligned}$$

By squaring equality (2.7), we obtain:

$$\begin{aligned}
f \circ S^{-1}(\mathbf{y}) &= \sum_{i=1}^{l-1} \delta_i x_i^2 + h_{ll}(\mathbf{x}) \left(x_l^2 + \sum_{j=l+1}^{n} x_j \frac{h_{jl}(\mathbf{x})}{h_{ll}(\mathbf{x})} \right)^2 - h_{ll}(\mathbf{x}) \left(\sum_{j=l+1}^{n} x_j \frac{h_{jl}(\mathbf{x})}{h_{ll}(\mathbf{x})} \right)^2 \\
&\quad + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} x_i x_j h_{ij}(\mathbf{x}) \\
&= \sum_{i=1}^{l-1} \delta_i y_i^2 + \frac{h_{ll}(\mathbf{y})}{|h_{ll}(\mathbf{y})|} y_l^2 - h_{ll}(\mathbf{y}) \left(\sum_{j=l+1}^{n} y_j \frac{h_{jl}(\mathbf{y})}{h_{ll}(\mathbf{y})} \right)^2 + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} y_i y_j h_{ij}(\mathbf{y}) \\
&= \sum_{i=1}^{l} \delta_i y_i^2 - \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} y_i y_j \frac{h_{jl}(y) h_{il}(\mathbf{y})}{h_{ll}(\mathbf{y})} + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} y_i y_j h_{ij}(\mathbf{y}) \\
&= \sum_{i=1}^{l} \delta_i y_i^2 + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} y_i y_j \left(h_{ij}(\mathbf{y}) - \frac{h_{jl}(\mathbf{y}) h_{il}(\mathbf{y})}{h_{ll}(\mathbf{y})} \right).
\end{aligned}$$

We define

$$\tilde{h}_{ij}(\mathbf{x}) = h_{ij}(\mathbf{x}) - \frac{h_{jl}(\mathbf{y}) h_{il}(\mathbf{y})}{h_{ll}(\mathbf{y})}.$$

Therefore

$$f \circ S^{-1}(\mathbf{y}) = \sum_{i=1}^{l} \delta_i y_i^2 + \sum_{i=l+1}^{n}\sum_{j=l+1}^{n} y_i y_j \tilde{h}_{ij} \circ S^{-1}(\mathbf{y}).$$

□

Definition 2.0.6. Let M be an n–dimensional differentiable manifold, let $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function and p be a non-degenerate critical point to f. The *index* of p respect to f is the number of negative eigenvalues of the Hessian $\mathrm{Hess}_F(\varphi(p))$, where $F = f \circ \varphi^{-1}$ for any chart $\varphi\colon U \subset M \longrightarrow V \subset \mathbb{R}^n$ around p. We denoted the index of f at p by $\mathrm{n}_f(p)$.

Note that in the Morse Lemma (equality (2.4)), the index k coincides with $\mathrm{n}_f(p)$. This index, by Sylvester's law of inertia is invariant under diagonalization, see [11, Thm. 6.38].

Considering the linear change of variables $T(y_1, \dots, y_n) = (\frac{y_1}{\sqrt{2}}, \dots, \frac{y_n}{\sqrt{2}})$ and adding the notion of index, we reformulate the Morse Lemma as follows:

Corollary 2.0.7. *Let M be an n–dimensional differentiable manifold. Suppose $f\colon M \longrightarrow \mathbb{R}$ is a differentiable function and p is a non-degenerate critical point of f. Then there exists an open neighbourhood U of p and a chart $\varphi\colon U \longrightarrow V \subset \mathbb{R}^n$ such that $\varphi(p) = 0$ and in this chart we have the equality*

$$(f\circ\varphi^{-1})(\mathbf{x}) = f(p) - \frac{1}{2}x_1^2 - \dots - \frac{1}{2}x_{\mathrm{n}_f(p)}^2 + \frac{1}{2}x_{\mathrm{n}_f(p)+1}^2 + \dots + \frac{1}{2}x_n^2, \quad \mathbf{x} = (x_1, \dots, x_n) \in V. \tag{2.9}$$

Now, we will describe the non-degenerate critical points of differentiable functions.

Corollary 2.0.8. *Let M be a differentiable manifold and $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function. Every non-degenerate critical point of f is isolated. In particular, if f is a Morse function and M is compact, then f has a finite number of critical points.*

Proof. By Corollary 2.0.7, there exist a chart (U, φ) around p and by equality (2.9)

$$D(f \circ \varphi^{-1})(\mathbf{x}) = (-x_1, \dots, -x_{\mathrm{n}_f(p)}, x_{\mathrm{n}_f(p)}, \dots x_n).$$

Note that $D(f \circ \varphi^{-1})(\mathbf{x}) = 0$ if and only if $\mathbf{x} = 0$.

Then the chart does not contain another critical point, that is, $\varphi^{-1}(0) = p$ is the only critical point of f in U, therefore, p is isolated.

Now suppose that M is compact and f is a Morse function.

By contradiction.

We assume that the set of critical points is infinite, since M is a compact space, by Theorem (see [29, Thm. 28.1]) the set has an accumulation point, we say q.

Let $(U, \varphi = (x_1, \dots, x_n))$ be a chart about p, since f is a differentiable function then $\frac{\partial}{\partial x_i}\Big|_p \overline{f} := D_0\varphi^{-1}(\frac{\partial}{\partial r_i})$ depends smoothly on $p \in M$, where $\overline{f}$ is the germ of f and $r_1, \dots, r_n$ the standard coordinates on $\mathbb{R}^n$. For each critical point p of f we get $\frac{\partial}{\partial x_i}\Big|_p \overline{f} = 0$. Then at the accumulation point $\frac{\partial}{\partial x_i}\Big|_q \overline{f} = 0$, therefore q is also a critical point of f and by definition of Morse function 2.0.2, q is a non-degenerate critical point.

Without loss of generality, let $V \subset M$ be an open neighborhood around q. By definition of accumulation point V contains at least one other critical point close to q. Then q is a non-degenerate critical point not isolated, which is a contradiction to the first statement. ☐

2.1 Height function

By Corollary 2.0.7 we have that Morse functions have a simple local structure. Also, the existence of Morse functions is guaranteed by Whitney embedding Theorem, for more details see [30, Sec. 1.2].

The objective of this section is to describe examples of Morse functions and to see the information obtained. We will consider compact manifolds, so by Corollary 2.0.8, their Morse functions have a finite number of critical points.

The following Morse functions can be thought as "height functions".

Definition 2.1.1. Let M be a differentiable manifold and $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function. Assume $M \subset \mathbb{R}^k$, for some integer $k > 0$, f is a *height function* if f is a projection on to the last coordinate axis of $\mathbb{R}^k$.

Example 2.1.2. Now, consider the 2–sphere in $\mathbb{R}^3$

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Let $f\colon S^2 \longrightarrow \mathbb{R}$ the height function, define by $f(x_1, x_2, x_3) = x_3$.

Let us see that f is a Morse function and the indices of f at its critical points.

Let $N = (0, 0, 1)$ and $S = (0, 0, -1)$ be the north pole and the south pole of S^2, respectively.

Through stereographic projection we have two charts of S^2, $\varphi_1\colon S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ and $\varphi_2\colon S^2 \setminus \{S\} \longrightarrow \mathbb{R}^2$ give by

$$\varphi_1(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right) \quad \text{and} \quad \varphi_2(x_1, x_2, x_3) = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}\right).$$

The inverses of φ_1 and φ_2 are

$$\varphi_1^{-1}(x_1, x_2) = \left(\frac{2x_1}{x_1^2 + x_2^2 + 1}, \frac{2x_2}{x_1^2 + x_2^2 + 1}, \frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2 + 1}\right)$$

and

$$\varphi_2^{-1}(x_1, x_2) = \left(\frac{2x_1}{x_1^2 + x_2^2 + 1}, \frac{2x_2}{x_1^2 + x_2^2 + 1}, \frac{1 - x_1^2 - x_2^2}{x_1^2 + x_2^2 + 1}\right)$$

respectively.

To determine the critical points of f, considerer the map $F_i = f \circ \varphi_i^{-1}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ for each $i = 1, 2$.

We consider the map $F_1 = f \circ \varphi_1^{-1}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$F(x_1, x_2) = \frac{x_1^2 + x_2^2 - 1}{x_1^2 + x_2^2 + 1}.$$

Since

$$D_{\mathbf{x}} F_1 = \left(\frac{4x_1}{(x_1^2 + x_2^2 + 1)^2}, \frac{4x_2}{(x_1^2 + x_2^2 + 1)^2}\right).$$

We have that $D_{\mathbf{x}} F_1 = 0$ if and only if $x_1 = 0 = x_2$. Then $\varphi_1^{-1}(0, 0) = (0, 0, -1) = S$ is the only critical point of f in $S^2 \setminus \{N\}$. Now, let us see the Hessian of f at S,

$$\mathrm{Hess}_F(\varphi_2(S)) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

So, S is a non-degenerate critical point of f in $S^2 \setminus \{N\}$ with index 0.

Similar calculation shows that N is the only non-degenerate critical point of f in $S^2 \setminus \{S\}$ with index 2.

Therefore, f is a Morse function.

The critical points of a height function are characterized by the tangent spaces at the points, that is, let f be a height function and p a critical point of f, then T_pM is orthogonal to the axis onto which f is projected, that is, $D_pf = 0$.

Remark 2.1.3. Let S be a surface, $f \colon S \longrightarrow \mathbb{R}$ be a Morse function and $p \in S$ be a critical point of f. Let $(U, \varphi = (x_1, x_2, x_3))$ be a chart around p and $F = f \circ \varphi$. We have the following cases:

1. We will say that p is a *minimum point* of f if $\frac{\partial^2 F}{\partial x_i \partial x_j}(\varphi(q)) > 0$ for all $q \in U$ and for all $i, j = 1, 2, 3$.
2. We will say that p is a *maximum point* of f if $\frac{\partial^2 F}{\partial x_i \partial x_j}(\varphi(q)) < 0$ for all $q \in U$ and for all $i, j = 1, 2, 3$.
3. Otherwise, we will say that p is a *saddle point* of f.

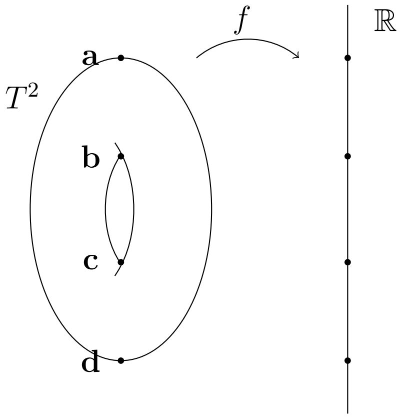

Figure 2.1: Height function on T^2

Example 2.1.4. Analogously to the 2–sphere, one can see that if r and R are real numbers satisfying $0 < r < R$, consider the 2–torus $T^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + (\sqrt{x_2^2 + x_3^2} - R)^2 = r^2\}$.

The function $f \colon T^2 \longrightarrow \mathbb{R}$ defined by $f(x_1, x_2, x_3) = x_3$ is a Morse function which has 4 non-degenerate critical points, (see Figure 2.1),

$$\mathbf{a} = (0, 0, R + r), \ \mathbf{b} = (0, 0, R - r), \ \mathbf{c} = (0, 0, -(R - r)), \ \mathbf{d} = (0, 0, -(R + r)).$$

Since a is a maximum of f, $\mathrm{n}_f(\mathbf{a}) = 2$, d is a minimum of f then $\mathrm{n}_f(\mathbf{d}) = 0$, while b and c are saddle points of f, then $\mathrm{n}_f(\mathbf{b}) = 1, \mathrm{n}_f(\mathbf{c}) = 1$.

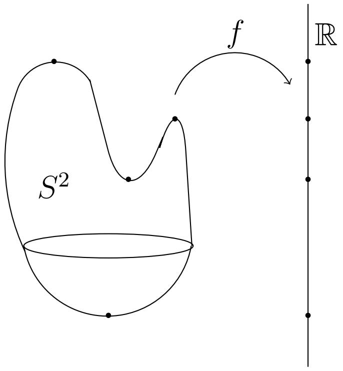

Figure 2.2: Height function on S^2 with a saddle at the top

Example 2.1.5. Let M be S^2 with a saddle at the top, this has four critical points: two maxima points, one saddle point and one minimum point, see Figure 2.2.

Example 2.1.6. On the other hand, if we take the 2–torus with a saddle at the top, then the height function is a Morse function. The function has two maxima points, three saddle points and one minimum point.

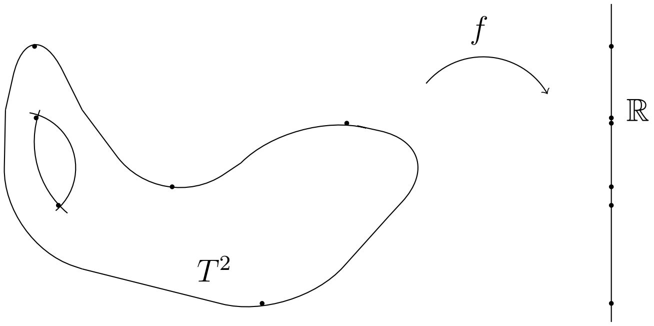

Figure 2.3: Height function on T^2 with a saddle at the top

2.2 Morse inequalities

Let M be an n–dimensional differentiable manifold, remember that for any integer k such that $0 \leq k \leq n$, $\beta_k(M) = \dim \mathrm{H}^k_{\mathrm{DR}}(M)$ is the k–th Betti number.

Let m_k denote the number of critical points $p \in M$ of f such that $\mathrm{n}_f(p) = k$.

The Morse inequalities establish a relationship between the number of critical points of index k of a real valued Morse function on M and the k–th Betti number on M.

Theorem 2.2.1 (Morse inequalities, Thm. 5.2,[41])**.** *Let M be an oriented, closed Riemannian n–manifold. For any Morse function on M one has*

1. *(Weak Morse inequalities) For any $0 \leq k \leq n$, we have*

$$\beta_k(M) \leq m_k. \tag{2.10}$$

2. *(Strong Morse inequalities) For any* $0 \leq k \leq n$, *we have*

$$\beta_k(M) - \beta_{k-1}(M) + \ldots + (-1)^k \beta_0(M) \leq m_k - m_{k-1} + \ldots + (-1)^k m_0. \tag{2.11}$$

Moreover, for $k = n$:

$$\beta_n(M) - \beta_{n-1}(M) + \ldots + (-1)^n \beta_0(M) = m_n - m_{n-1} + \ldots + (-1)^n m_0. \tag{2.12}$$

Let us see that Morse inequalities hold for the examples S^2 and T^2.

Example 2.2.2. Consider the 2–sphere S^2.
By example 1.4.17 we obtain $\beta_0(S^2) = 1$, $\beta_1(S^2) = 0$ and $\beta_2(S^2) = 1$.
By example 2.1.2 we have $m_0 = 1, m_1 = 0$ and $m_2 = 1$.
One can see that the inequalities and equality of Theorem 2.2.1 are satisfied.

Example 2.2.3. We consider the 2–torus, T^2.
By example 1.4.18 we obtain $\beta_0(T^2) = 1$, $\beta_1(T^2) = 2$ and $\beta_2(T^2) = 1$.
By example 2.1.4 we have $m_0 = 1, m_1 = 2$ and $m_2 = 1$.
We obtain the equality and inequalities of Theorem 2.2.1.

A proof of Theorem 2.2.1 using topological tools and further development of Morse theory can be found in [25].

In the present text we will follow the ideas of Witten, to obtain an analytic proof for the Morse inequalities (2.10) and (2.11).

Chapter 3

Hodge theory

In this chapter we will describe the adjoint operator of the exterior derivative and extend the Laplace operator to differentiable forms.

For more details see [28], [18] and [1].

3.1 $\star$–Operator

In this section we define an isomorphism of vector spaces that we will extend to the space of forms.

Let V be a real vector space of dimension n with an inner product $\langle\, , \rangle$. Also, for $\Lambda^k V$ with $1 < k \leq n$, we can define an inner product

$$\langle\, , \rangle_{\Lambda^k V} \colon \Lambda^k V \times \Lambda^k V \longrightarrow \mathbb{R}.$$

Let $v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \in \Lambda^k V$, with $v_i, w_j \in V$, we define their inner product as

$$\langle v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k \rangle_{\Lambda^k V} = \det(\langle v_i, w_j \rangle). \tag{3.1}$$

The value is independent of the way the two elements are represented, this follows from the properties of wedge product and determinant.

If $e_1 \ldots, e_n$ is an orthonormal basis of V, then all the elements of the form

$$e_{i_1} \wedge \ldots \wedge e_{i_k}, \quad 1 \leq i_1 < \ldots i_k \leq n,$$

form an orthonormal basis of $\Lambda^k V$.

Given an orientation in V which is the choice of an equivalence class of an ordered basis, see section A.4, we have an orientation in $\Lambda^n V$, taking the equivalence class of the ordered basis of $\Lambda^n V$ induced by the ordered basis of V.

Let $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n \in V$ be an arbitrary positively oriented orthonormal basis. We define $\mathrm{Vol}_V := e_1 \wedge \ldots \wedge e_n$ the *volume form* of V.

We define a linear map

$$\star \colon \Lambda^k V \longrightarrow \Lambda^{n-k} V$$

such that for each $w, u \in \Lambda^k V$

$$w \wedge \star u = \langle w, u\rangle_{\Lambda^k V} \mathrm{Vol}_V. \tag{3.2}$$

This operator is called the $\star$–*operator.*

In elements of the oriented orthonormal basis of $\Lambda^k V$ the map $\star$ is given by:

$$\star(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}) = \mathrm{sgn}\sigma e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}, \tag{3.3}$$

where $\sigma \in S(k, n-k)$, the set of $(k, n-k)$–shuffles, see definition A.2.4. Where (3.3) follows from (3.2),

$$\begin{aligned} e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)} \wedge \star(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}) &= e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)} \wedge \mathrm{sgn}\sigma(e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}) \\ &= \mathrm{sgn}\sigma e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(n)} \\ &= (\mathrm{sgn}\sigma)^2 \mathrm{Vol}_V \\ &= \mathrm{Vol}_V. \end{aligned}$$

And

$$\langle e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}, e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}\rangle_{\Lambda^k V} = 1.$$

By condition (3.3), we consider $1 \in \mathbb{R} = \Lambda^0 V$, we have $\star 1 = e_1 \wedge \ldots \wedge e_n$ and $\star(e_1 \wedge \ldots \wedge e_n) = 1$.

Also, by equality (3.3) in basic elements, we get that $\star$ is surjective and since the vector spaces $\Lambda^k V$ and $\Lambda^{n-k} V$ are of the same dimension hence $\star$ is a linear isomorphism.

Proposition 3.1.1. *Let V be a real vector space of dimension n. The $\star$–operator has the following properties. For any $r, t \in \mathbb{R}$ and for any w and u in $\Lambda^k V$ we have*

1. $\star(rw + tu) = r \star w + t \star u$.
2. $\star \star w = (-1)^{k(n-k)} w$.
3. $w \wedge \star u = u \wedge \star w$.
4. $\star(w \wedge \star u) = \star(u \wedge \star w) = \langle w, u\rangle_{\Lambda^k V}$.
5. $\langle \star w, \star u\rangle_{\Lambda^k V} = \langle w, u\rangle_{\Lambda^k V}$.

Proof. 1. By linearity of $\star$, it satisfies $\star(rw + tu) = r \star w + t \star u$, for all $r, t \in \mathbb{R}$.

2. Let $e_1, \ldots e_n$ be an oriented orthonormal basis of V. Assume that $w = e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(k)}$, then $\star w = \mathrm{sgn}\sigma e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}$. By condition (3.2), we have

$$\begin{aligned} \star w \wedge \star \star w &= (\mathrm{sgn}\sigma e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}) \wedge \star(\mathrm{sgn}\sigma e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}) \\ &= (\mathrm{sgn}\sigma)^2 e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)} \wedge \star(e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)}) \\ &= e_{\sigma(k+1)} \wedge \ldots \wedge e_{\sigma(n)} \wedge \mathrm{sgn}\sigma e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(n)} \\ &= (\mathrm{sgn}\sigma)^2 e_1 \wedge \ldots \wedge e_n \\ &= \mathrm{Vol}_V. \end{aligned}$$

On the other hand, since the basis of V is orthonormal

$$\begin{aligned}\langle\star w,\star w\rangle\mathrm{Vol}_V &= \langle \mathrm{sgn}\sigma e_{\sigma(k+1)}\wedge\ldots\wedge e_{\sigma(n)},\mathrm{sgn}\sigma e_{\sigma(k+1)}\wedge\ldots\wedge e_{\sigma(n)}\rangle_{\Lambda^k V}\\ &= (\mathrm{sgn}\sigma)^2\langle e_{\sigma(k+1)}\wedge\ldots\wedge e_{\sigma(n)},e_{\sigma(k+1)}\wedge\ldots\wedge e_{\sigma(n)}\rangle_{\Lambda^k V}\mathrm{Vol}_V\\ &= \mathrm{Vol}_V.\end{aligned}$$

If we consider $(-1)^{k(n-k)}w$, we obtain:

$$\begin{aligned}\star w\wedge\star\star w &= \star w\wedge(-1)^{k(n-k)}w\\ &= (-1)^{k(n-k)}\star w\wedge w\\ &= (-1)^{k(n-k)}\mathrm{sgn}\sigma e_{k+1}\wedge\ldots\wedge e^n\wedge e_{\sigma(1)}\wedge\ldots\wedge e_{\sigma(k)}\\ &= ((-1)^{k(n-k)})^2\mathrm{sgn}\sigma e_{\sigma(1)}\wedge\ldots\wedge e_{\sigma(n)}\\ &= (\mathrm{sgn}\sigma)^2 e_1\wedge\ldots\wedge e_n\\ &= \mathrm{Vol}_V.\end{aligned}$$

Finally:

$$\begin{aligned}\star w\wedge(-1)^{k(n-k)}w &= ((-1)^{k(n-k)})^2 w\wedge\star w\\ &= \langle w,w\rangle_{\Lambda^k V}\mathrm{Vol}_V\\ &= \mathrm{Vol}_V.\end{aligned}$$

Therefore

$$\star\star w=(-1)^{k(n-k)}w.$$

3. By condition (3.2) and the symmetry of inner product of $\Lambda^k V$, we have

$$\langle w,u\rangle_{\Lambda^k V}\mathrm{Vol}_V = w\wedge\star u = u\wedge\star w.$$

4. Applying $\star$ to 3.1.1-3. in $w\wedge\star u=u\wedge\star w$, we have $\star(w\wedge\star u)=\star(u\wedge\star w)$. Also, by (3.2) we have $u\wedge\star w=\langle w,u\rangle_{\Lambda^k V}\mathrm{Vol}_V$, since $\mathrm{Vol}_V=\star 1$ and 3.1.1-3., we get

$$\star(u\wedge\star w)=\star(\langle w,u\rangle\mathrm{Vol}_V)=\star\mathrm{Vol}_V\langle w,u\rangle_{\Lambda^k V}=\langle w,u\rangle_{\Lambda^k V}.$$

5. Item 5 holds by Proposition 3.1.1-2. and -4.

$$\begin{aligned}\langle\star w,\star u\rangle_{\Lambda^k V} &= \star(\star w\wedge\star\star u)\\ &= \star(\star w\wedge(-1)^{k(n-k)}u)\\ &= (-1)^{k(n-k)}\star(\star w\wedge u)\\ &= (-1)^{k(n-k)}\star(-1)^{k(n-k)}(u\wedge\star w)\\ &= \star(u\wedge\star w)\\ &= \langle w,u\rangle_{\Lambda^k V}.\end{aligned}$$

□

3.2 Hodge $\star$–operator

Using the properties of the $\star$–operator we will define the Hodge $\star$–operator on differentiable forms. For this reason, we will first study the inner product we need, the Riemannian metric.

3.2.1 Riemannian metric

A Riemannian metric on a differentiable manifold M is a section g of S^2T^*M which is pointwise positive definite, (see definition C.2.5). Now we will describe the Riemannian metric locally.

Let $(U, x_1, \ldots, x_n)$ be a chart on M. If we set $g_{ij}\colon U \longrightarrow \mathbb{R}$

$$g_{ij}(p) = g_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \quad p \in U. \tag{3.4}$$

Then g_{ij} is a function of $x_1, \ldots, x_n$. We say that g is *differentiable* if the functions g_{ij} are differentiable in all charts.

Example 3.2.1. One example of a Riemannian manifold is $\mathbb{R}^n$ with its Euclidean metric g, which is just the usual inner product on each tangent space $T_p\mathbb{R}^n$ under the natural identification $T_p\mathbb{R}^n = \mathbb{R}^n$. In standard coordinates, let be a chart $(\mathbb{R}^n, x_1, \ldots, x_n)$, g can be written in several ways:

$$g = \sum_{i=1}^{n} dx_i \otimes dx_i = \sum_{i=1}^{n} (dx_i)^2.$$

g viewed as a 2 degree polynomial in the variables $\{dx_1, \ldots, dx_n\}$.

By Proposition C.2.4 we have that for every differentiable manifold M there always exists a Riemannian metric.

Proposition 3.2.2. *Let (M, g) be a Riemannian manifold. For each point $p \in M$, consider the inner product $g_p\colon T_pM \times T_pM \longrightarrow \mathbb{R}$. The linear map $\hat{g}_p\colon T_pM \longrightarrow T_p^*M$, given by $\hat{g}_p(X)(Y) = g_p(X, Y)$, $X, Y \in T_pM$, is an isomorphism.*

Proof. Assume that $\hat{g}_p(X) = 0$, then $\hat{g}_p(X)(X) = 0$, that is, $g_p(X, X) = 0$, since g_p is positive definite then $X = 0$. So $\hat{g}_p$ is injective. Also, we have $\dim(T_pM) = \dim(T_p^*M)$, hence $\hat{g}_p$ is an isomorphism. ☐

Then the metric g_p identifies the tangent space T_pM and the cotangent space T_p^*M. Moreover, we may extend this identification to the space $\Gamma(TM)$ of all vector fields on M and the space $\Omega^1(M)$ of all differentiable forms of degree 1 on M. For example, for each differentiable function f on M, $df\colon TM \longrightarrow T\mathbb{R} \cong \mathbb{R}$ is a differentiable form of degree 1 on M and by the isomorphism $\Gamma(TM) \cong \Omega^1(M)$, there is a unique vector field called the *gradient* of f, denoted by $\mathrm{grad} f$, such that

$$g(\mathrm{grad} f, X) = df(X) = Xf,$$

for every vector field X on M. For a differentiable function $f = f(x_1, \dots, x_n)$ on the Euclidean space $\mathbb{R}^n$, we have

$$\mathrm{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

3.2.2 Hodge $\star$–operator

Let (M, g) be an oriented Riemannian n–manifold.

For any integer k, with $0 \leq k \leq n$, we have an inner product on $\Lambda^k T_p^* M$ for each $p \in M$. There is a natural linear isomorphism

$$\star \colon \Lambda^k T_p^* M \longrightarrow \Lambda^{n-k} T_p^* M$$

for each point $p \in M$. That induces the vector bundle isomorphism $\star \colon \Lambda^k T^* M \longrightarrow \Lambda^{n-k} T^* M$. By varying $p \in M$, we have the linear isomorphism

$$\star \colon \Omega^k(M) \longrightarrow \Omega^{n-k}(M), \quad (\star\omega)(p) = \star(\omega(p)),$$

called the *Hodge $\star$–operator*.

Moreover, if $(U, x_1, \dots, x_n)$ is an oriented chart assume that $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ form a positive local frame. Take the Gram-Schmidt orthogonalization process and get an oriented orthonormal local frame $e_1, \dots, e_n$ of TM. That is, we let $e_1 = \frac{\frac{\partial}{\partial x_1}}{||\frac{\partial}{\partial x_1}||}$ and inductively define with g the Riemannian metric

$$Y_i = \frac{\partial}{\partial x_i} - \sum_{j=1}^{i-1} g\left(\frac{\partial}{\partial x_i}, e_j\right) e_j, \quad e_i = \frac{Y_i}{||Y_i||}, \quad i = 2, 3, \dots, n.$$

Let $\{e^1, \dots, e^n\}$ be the dual oriented orthonormal basis of T^*M. Now, if

$$\omega = \sum_{\sigma \in S(k, n-k)} f_{\sigma(1) \dots \sigma(k)} e^{\sigma(1)} \wedge \dots \wedge e^{\sigma(k)},$$

then we have

$$\star\omega = \sum_{\sigma \in S(k, n-k)} \mathrm{sgn}\sigma f_{\sigma(1) \dots \sigma(k)} e^{\sigma(k+1)} \wedge \dots \wedge e^{\sigma(n)}.$$

Let $1 \in C^\infty(M)$ be the constant function with value 1, we have $\star 1 \in \Omega^n(M)$, which is called the *volume form* and will be denoted by Vol_M, a concrete expression is given by $\mathrm{Vol}_M = e^1 \wedge \dots \wedge e^n$. In terms of the metric (3.4) we have

$$\mathrm{Vol}_M = \sqrt{\det(g_{ij})} dx_1 \wedge \dots \wedge dx_n.$$

3.3 The Laplace–Beltrami operator and harmonic forms

Using the Hodge $\star$–operator, we can define the adjoint operator of the exterior derivative and with these two operators we will extend the Laplace operator to forms.

Let (M, g) be an oriented Riemannian n–manifold without boundary, in addition, we also need it to be compact. Using the inner product on $\Lambda^k T_p^* M$ for each $p \in M$ we can define an inner product in $\Omega^k(M)$. Let $\omega, \eta \in \Omega^k(M)$ by integrating the function $\langle \omega(p), \eta(p) \rangle$ over M, we define

$$\langle \omega, \eta \rangle_{\Omega^k(M)} = \int_M \langle \omega(p), \eta(p) \rangle_{\Lambda^k T_p^* M} \mathrm{Vol}_M. \tag{3.5}$$

where Vol_M is the volume form of M.

The inner product on $\Omega^k(M)$ will be denoted simply by $\langle\ ,\ \rangle$.

According to Proposition 3.1.1-3., the inner product (3.5) can also be written in the form

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star\eta = \int_M \eta \wedge \star\omega. \tag{3.6}$$

Furthermore, by Proposition 3.1.1-5., $\langle \star\omega, \star\eta \rangle = \langle \omega, \eta \rangle$, which means that the Hodge $\star$–operator $\star\colon \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ is an isometry relative to the inner product (3.5).

By convention, we define the inner product between differentiable forms of two different degrees to be zero, so that the entire vector space $\Omega^\bullet(M)$ is provided with an inner product.

Now we study how the exterior derivative $d\colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$ is transformed by the Hodge $\star$–operator.

Definition 3.3.1. Let $d^\star\colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ be the differentiable linear operator defined as follows: let $\omega \in \Omega^k(M)$,

$$d^\star \omega = (-1)^{n(k+1)+1} \star d \star \omega \in \Omega^{k-1}(M). \tag{3.7}$$

Lemma 3.3.2. *d and d^* satisfy the following equalities: let $\omega \in \Omega^k(M)$*

$$\begin{aligned} \star d^\star \omega &= (-1)^k d \star \omega, && (3.8)\\ d^\star \star \omega &= (-1)^{k+1} \star d\omega, && (3.9)\\ d^\star \circ d^\star \omega &= 0. && (3.10) \end{aligned}$$

Proof. Let $\omega \in \Omega^k(M)$.

By equality (3.7) and Proposition 3.1.1, we see that

1.

$$\begin{aligned} \star d^\star \omega &= \star(-1)^{n(k+1)+1} \star d \star \omega \\ &= (-1)^{n(k+1)+1} \star \star d \star \omega \\ &= (-1)^{n(k+1)+1} (-1)^{(n-k+1)(k-1)} d \star \omega \\ &= (-1)^k d \star \omega. \end{aligned}$$

2.

$$\begin{aligned} d^\star \star \omega &= (-1)^{n(n-k+1)+1} \star d \star (\star\omega) \\ &= (-1)^{nk+1} \star d(-1)^{k(n-k)}\omega \\ &= (-1)^{k+1} \star d\omega. \end{aligned}$$

3.

$$\begin{aligned} (d^\star \circ d^\star)\omega &= d^\star[(-1)^{n(k+1)+1} \star d \star \omega] \\ &= (-1)^{n(k+1)+1}(-1)^{nk+1} \star d \star (\star d \star \omega) \\ &= (-1)^{n} \star d(-1)^{(n-k+1)(k-1)} d \star \omega \\ &= 0. \end{aligned}$$

□

Stokes theorem is a fundamental formula concerning the integral of differentiable forms and we will use it to describe the adjoint operator of d. First we describe the case of manifolds with boundary.

Theorem 3.3.3 (Stokes Theorem, [28, Thm. 3.6]). *Let M be an oriented differentiable n–manifold with boundary and ω a differentiable $(n-1)$–form on M with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Here the right-hand side is the integral of ω on the boundary ∂M of M, and we assume that ∂M is equipped with an orientation induced from that of M.

The next corollary follows immediately from Theorem 3.3.3.

Corollary 3.3.4 ([28, Cor. 3.7]). *Let M be an oriented differentiable n–manifold without boundary. Then for an arbitrary differentiable $(n-1)$–form ω on M with compact support, we have*

$$\int_M d\omega = 0.$$

Proposition 3.3.5. *Let M be an oriented Riemannian n–manifold without boundary. Relative to the inner product $\langle\, , \rangle$ in $\Omega^\bullet(M)$, $d^\star$ is an adjoint operator of the exterior derivative d, that is, we have*

$$\langle d\omega, \eta\rangle = \langle \omega, d^\star\eta\rangle.$$

Proof. It suffices to take $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k+1}(M)$. By Theorem 1.3.2-1. and equality (3.8) we have

$$\begin{aligned} d\omega \wedge \star\eta &= d(\omega \wedge \star\eta) - (-1)^k \omega \wedge d \star \eta \\ &= d(\omega \wedge \star\eta) + (-1)^{k+1} \omega \wedge d \star \eta \\ &= d(\omega \wedge \star\eta) + \omega \wedge \star d^\star \eta. \end{aligned}$$

Integrating each side over M, we have

$$\int_M d\omega \wedge \star\eta = \int_M d(\omega \wedge \star\eta) + \int_M \omega \wedge \star d^\star \eta.$$

Since $\omega \wedge \star\eta$ is an $(n-1)$–form by Corollary 3.3.4, we have

$$\int_M d(\omega \wedge \star\eta) = 0.$$

Now, by definition of the inner product in $\Omega^k(M)$ and Proposition 3.1.1-4.,

$$\langle d\omega, \eta\rangle = \langle \omega, d^\star\eta\rangle.$$

□

Definition 3.3.6. Let (M, g) be an oriented Riemannnian n–manifold, the *De Rham-Hodge operator*

$$\mathrm{D}\colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

associated to g is defined by

$$\mathrm{D}\omega := d\omega + d^\star\omega. \tag{3.11}$$

Lemma 3.3.7. D *is a self-adjoint operator over* $\Omega^\bullet(M)$.

Proof. Let $\omega, \eta \in \Omega^\bullet(M)$, by Proposition 3.3.5 we have

$$\begin{aligned} \langle \omega, \mathrm{D}\eta\rangle &= \langle \omega, (d + d^\star)\eta\rangle \\ &= \langle \omega, d\eta\rangle + \langle \omega, d^\star\eta\rangle \\ &= \langle d^\star\omega, \eta\rangle + \langle d\omega, \eta\rangle \\ &= \langle (d^\star + d)\omega, \eta\rangle \\ &= \langle \mathrm{D}\omega, \eta\rangle. \end{aligned}$$

□

Definition 3.3.8. Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function, we define the *Laplacian* of f by

$$\Box f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

With the Hodge $\star$–operator and the exterior derivative and its adjoint we can extend the Laplacian operator to differentiable forms.

Definition 3.3.9. Let M be an oriented Riemannian n–manifold, the *Laplace–Beltrami operator* or *Laplacian* $\Box_k\colon \Omega^k(M) \longrightarrow \Omega^k(M)$ is defined by

$$\Box_k\omega = \mathrm{D}^2\omega = dd^\star\omega + d^\star d\omega, \tag{3.12}$$

for all $\omega \in \Omega^k(M)$ and is a linear operator for each k with $0 \le k \le n$.

It is also called *the Hodge-De Rham Laplacian.*

Note that $\Box_k$ preserves each $\Omega^k(M)$ with $0 \leq k \leq n$.

Definition 3.3.10. A form $\omega \in \Omega^k(M)$ such that $\Box_k\omega = 0$ is called a *harmonic* k*–form.*

In particular, a differentiable function such that $\Box_0 f = 0$ is called a *harmonic function.*

Proposition 3.3.11. *Let* $V \subset \mathbb{R}^n$ *be an open subset of* $\mathbb{R}^n$, $\omega = f_I dx_1 \wedge \ldots \wedge dx_k \in \Omega^k(V)$, *then the Laplace–Beltrami operator on* $\mathbb{R}^n$ *is as follows:*

$$\Box_k\omega = -\sum_{i=1}^{n} \frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_k.$$

Proof. We consider the Euclidean metric, $\star$ and $d^\star$ with respect to the Euclidean metric.

Let $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ be a positive orthonormal basis of $\mathbb{R}^n$.

It is sufficient to compute $\Box_k\omega$ for a differentiable k–form written as

$$\omega = f_I dx_1 \wedge \ldots \wedge dx_k.$$

By definition (3.3) $\star\omega = f_I dx_{k+1} \wedge \ldots \wedge dx_n$. We apply the exterior derivative, see 1.3.1,

$$d \star \omega = \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n.$$

By equality (3.2), we have:

$$\begin{aligned}
\mathrm{Vol}_{\mathbb{R}^n} &= dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n \wedge \star(dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n) \\
&= dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k \\
&= (-1)^{(n-k+1)(i-1)} dx_1 \wedge \ldots \wedge dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n \\
&= (-1)^{(n-k+1)(i-1)}(-1)^{(n-k)(k-i)} \mathrm{Vol}_{\mathbb{R}^n}.
\end{aligned}$$

Where $(n-k+1)(i-1)+(n-k)(k-i) = ni-ik+i-n+k-1+nk-k^2-ni+ki = nk+i-n-1$. We obtain:

$$\star d \star \omega = \sum_{i=1}^{k} \frac{\partial f_I}{\partial x_i} (-1)^{nk-n+i-1} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k.$$

We have $(-1)^{nk-n+i-1}(-1)^{n(k+1)+1} = (-1)^i$. By definition 3.3.1

$$\begin{aligned}
d^\star\omega &= \sum_{i=1}^{k} (-1)^i \frac{\partial f_I}{\partial x_i} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots dx_k. \\
dd^\star\omega &= \sum_{i=1}^{k} \frac{\partial^2 f_I}{\partial x_i^2} (-1)^i dx_i \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k \\
&\quad + \sum_{i=1}^{k} \sum_{j=k+1}^{n} \frac{\partial^2 f_I}{\partial x_j \partial x_i} (-1)^i dx_j \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k \\
&= -\sum_{i=1}^{k} \frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_k \\
&\quad + \sum_{i=1}^{k} \sum_{j=k+1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} (-1)^i dx_j \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k.
\end{aligned}$$

On the other hand, by definition 1.3.1 we get

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_1 \ldots \wedge dx_k \\ &= \sum_{i=k+1}^{n} \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_1 \wedge \ldots \wedge dx_k. \end{aligned}$$

Later, by equality (3.2)

$$\begin{aligned} \mathrm{Vol}_{\mathbb{R}^n} &= dx_i \wedge dx_1 \wedge \ldots \wedge dx_k \wedge \star(dx_i \wedge dx_1 \wedge \ldots \wedge dx_k) \\ &= (-1)^{i-1} dx_1 \wedge \ldots \wedge dx_n. \end{aligned}$$

We apply $\star$ and d, hence

$$\begin{aligned} \star d\omega &= \sum_{i=k+1}^{n} (-1)^{i-1} \frac{\partial f_I}{\partial x_i} dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n. \\ d \star d\omega &= \sum_{j=1}^{n} \sum_{i=k+1}^{n} (-1)^{i-1} \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_j \wedge dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\ &= \sum_{i=k+1}^{n} (-1)^{i-1} \frac{\partial^2 f_I}{\partial x_i^2} dx_i \wedge dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\ &\quad + \sum_{j=1}^{k} \sum_{i=k+1}^{n} (-1)^{i-1} \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_j \wedge dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\ &= \sum_{j=1}^{k} \sum_{i=k+1}^{n} (-1)^{i-1} \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_j \wedge dx_{k+1} \wedge dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \\ &\quad + \sum_{i=k+1}^{n} (-1)^{k} \frac{\partial^2 f_I}{\partial x_i^2} dx_{k+1} \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_n. \end{aligned}$$

Again, by equality (3.2) we have

$$\begin{aligned} \mathrm{Vol}_{\mathbb{R}^n} &= dx_j \wedge dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \wedge \star(dx_j \wedge dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n) \\ &= (-1)^{n-k-1+j} dx_{k+1} \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \wedge dx_1 \wedge \ldots \wedge dx_j \wedge \ldots \wedge dx_k \wedge dx_i \\ &= (-1)^{n-k+j-1} (-1)^{k+n-k-i} dx_{k+1} \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_n \wedge dx_1 \wedge \ldots \wedge dx_k \\ &= (-1)^{n-k+j-1} (-1)^{k+n-k-i} (-1)^{(n-k)k} \mathrm{Vol}_{\mathbb{R}^n}. \end{aligned}$$

Where $(-1)^{n-k+j-1}(-1)^{k+n-k-i}(-1)^{(n-k)k} = (-1)^{nk+j}$. Also, we apply the equality (3.2) to $dx_{k+1} \wedge \ldots \wedge dx_n$, then

$$\begin{aligned} \star d \star d\omega &= \sum_{j=1}^{k} \sum_{i=k+1}^{n} (-1)^{nk+j} \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k \wedge dx_i \\ & \sum_{i=k+1}^{n} (-1)^{k+k(n-k)} \frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_k. \end{aligned}$$

Now, we note that $d\omega \in \Omega^{k+1}(V)$, then we take $d^\star\colon \Omega^{k+1}(V) \longrightarrow \Omega^{k+2}(V)$, that is,

$$d^\star(d\omega) = (-1)^{n(k+2)+1} \star d \star d\omega.$$

Then $(-1)^{n(k+2)+1}(-1)^{nk+j} = (-1)^{j+1}$ and $(-1)^{n(k+2)+1}(-1)^{k+k(n-k)} = -1$.

$$\begin{aligned} d^\star d\omega &= \sum_{j=1}^{k}\sum_{i=k+1}^{n} (-1)^{j+1}\frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k \wedge dx_i \\ &\quad - \sum_{i=k+1}^{n} \frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_k. \end{aligned}$$

Therefore, adding the two calculations, we obtain:

$$\begin{aligned} \Box_k\omega &= dd^\star\omega + d^\star d\omega \\ &= -\sum_{i=1}^{k}\frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_k \\ &\quad + \sum_{i=1}^{k}\sum_{j=k+1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(-1)^i dx_j \wedge dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k \\ &\quad + \sum_{j=1}^{k}\sum_{i=k+1}^{n} (-1)^{j+1}\frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k \wedge dx_i \\ &\quad - \sum_{i=k+1}^{n}\frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_k \\ &= -\sum_{i=1}^{n}\frac{\partial^2 f_I}{\partial x_i^2} dx_1 \wedge \ldots \wedge dx_k. \end{aligned}$$

From the penultimate equality, the double summations are canceled.

□

Proposition 3.3.12 ([28, Prop. 4.13])**.** *The Laplace–Beltrami operator has the following properties: let $\omega \in \Omega^k(M)$*

1. $\star\Box_k\omega = \Box_k \star \omega$. *If ω is a harmonic k–form, so is $\star\omega$.*
2. $\Box_k$ *is self-adjoint, that is,* $\langle \Box_k\omega, \eta\rangle = \langle \omega, \Box_k\eta\rangle$ *for all* $\omega, \eta \in \Omega^k(M)$.
3. $\Box_k\omega = 0$ *if and only if* $d\omega = 0$ *and* $d^\star\omega = 0$.

Proof. 1. Let $\omega \in \Omega^k(M)$, we have

$$\begin{aligned} \star d^\star\omega &= \star\overbrace{(-1)^{n(k+1)+1} \star d \star \omega}^{\in\Omega^{k-1}(M)} \\ &= (-1)^{n(k+1)+1}(-1)^{(k-1)(n-(k-1))} d \star \omega \end{aligned}$$

But $n(k+1)+1+(k-1)(n-(k-1)) = nk+n+1+kn-k^2+2k-n-1 = 2nk-k^2+2k$, then by the axioms of the exponents and since $(2i)^2$ is even and $(2i+1)^2$ is odd:

$$\begin{aligned}\star d^\star\omega &= (-1)^{2nk}(-1)^{-k^2}(-1)^{2k}d\star\omega\\ &= (-1)^{-k^2}d\star\omega\\ &= (-1)^k d\star\omega.\end{aligned}$$

On the other hand,

$$\begin{aligned}d^\star \underbrace{\star\omega}_{\in\Omega^{n-k}(M)} &= (-1)^{n(n-k+1)}\star d\star\star\omega\\ &= (-1)^{n^2-kn+n+1}(-1)^{k(n-k)}\star d\omega.\end{aligned}$$

But $n^2-kn+n+1+k(n-k) = n^2-kn+n+1+kn-k^2 = n(n+1)-k^2+1$, note that if n is even then $n+1$ is odd and reciprocally. Then,

$$\begin{aligned}d^\star\star\omega &= (-1)^{-k^2}(-1)\star d\omega\\ &= (-1)^{k+1}\star d\omega.\end{aligned}$$

Then, by the second calculation,

$$\begin{aligned}\star d\ \overbrace{d^\star\omega}^{\in\Omega^{k-1}(M)} &= (-1)^k d^\star\star d^\star\omega\\ &= d^\star d\star\omega.\end{aligned}$$

Analogously, by the first calculation,

$$\begin{aligned}\star d^\star d\omega &= (-1)^{k+1}d\star d\omega\\ &= dd^\star\star\omega.\end{aligned}$$

Therefore,

$$\begin{aligned}\star\Box_k\omega &= \star(dd^\star\omega + d^\star d\omega)\\ &= \star dd^\star\omega + \star d^\star d\omega\\ &= d^\star d\star\omega + dd^\star\star\omega\\ &= (dd^\star + d^\star d)\star\omega\\ &= \Box_k\star\omega.\end{aligned}$$

2. It is a consequence of Definition A.3.12 and Proposition 3.3.5. Let $\omega, \eta \in \Omega^k(M)$, then

$$\begin{aligned}\langle\Box_k\omega,\eta\rangle &= \langle(dd^\star\omega + d^\star d\omega),\eta\rangle\\ &= \langle dd^\star\omega,\eta\rangle + \langle d^\star d\omega,\eta\rangle\\ &= \langle d^\star\omega, d^\star\eta\rangle + \langle d\omega, d\eta\rangle\\ &= \langle\omega, dd^\star\eta\rangle + \langle\omega, d^\star d\eta\rangle\\ &= \langle\Box_k\omega,\eta\rangle.\end{aligned}$$

3. Let $\omega \in \Omega^k(M)$.

   Note that if $d\omega = 0$ and $d^\star\omega = 0$ then $\Box_k\omega = 0$.

   Now, assume that $\Box_k\omega = 0$. By Definition A.3.12, we have

$$\begin{aligned}\langle \Box\omega, \omega\rangle &= \langle (dd^\star + d^\star d)\omega, \omega\rangle \\ &= \langle d\omega, d\omega\rangle + \langle d^\star\omega, d^\star\omega\rangle\end{aligned}$$

   The last equality follows by Lemma 3.3.5, since

$$\begin{aligned}\langle d\omega, d\omega\rangle &= \langle \omega, d^\star d\omega\rangle \\ &= \langle d^\star d\omega, \omega\rangle\end{aligned}$$

   and

$$\langle d^\star\omega, d^\star\omega\rangle = \langle dd^\star\omega, \omega\rangle.$$

   Since $\langle \cdot, \cdot\rangle$ is definite positive, so $d\omega = 0$ and $d^\star\omega = 0$.

$\square$

3.4 Sobolev spaces on k–forms

In section D.3 we define k–Sobolev spaces of the L^2–space of functions with compact support on $\mathbb{R}^n$, we will now extend the definition of k–Sobolev space to differentiable forms with compact support.

Definition 3.4.1. Let M be a differentiable manifold, an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ is a *locally finite cover* if every point $p \in M$ has a neighborhood that meets only finitely many of the sets U_α.

Definition 3.4.2. Let M be a differentiable manifold and $\{U_\alpha\}_{\alpha\in\Lambda}$ a locally finite open cover of M. A *partition of unity subordinate* to $\{U_\alpha\}_{\alpha\in\Lambda}$ is a collection of non negative differentiable functions $\{\rho_\alpha\}_{\alpha\in\Lambda}$ satisfying

1. $\sum \rho_\alpha = 1$.
2. $\operatorname{supp} \rho_\alpha \subset U_\alpha$.

Given an open cover of M, one can construct a locally finite subcover of M, see [9, Thm. 7.1].

Let $p \in M$, consider the inner product on $\Lambda^k T_p^* M$

$$\langle\ ,\ \rangle_p \colon \Lambda^k T_p^* M \times \Lambda^k T_p^* M \longrightarrow \mathbb{R},$$

defined by (3.1).

In a natural way, this inner product induces the norm

$$||\ \ || \colon \Lambda^k T_p^* M \longrightarrow \mathbb{R}, \quad ||\omega(p)|| = \langle \omega(p), \omega(p)\rangle_p^{\frac{1}{2}}.$$

Note that this inner product and norm depends smoothly on $p \in M$. We denote this function by $f_\omega \colon M \longrightarrow \mathbb{R}$, given by $f_\omega(p) = ||\omega(p)||$.

Definition 3.4.3. Let (M, g) be a Riemannian n–manifold with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ where $\{U_\alpha\}_{\alpha\in\Lambda}$ is a locally finite open cover of M and $\varphi_\alpha\colon U_\alpha \longrightarrow V_\alpha$ with $\overline{V}_\alpha$ compact in $\mathbb{R}^n$. Take a partition of unity $\{\rho_\alpha\colon U_\alpha \longrightarrow [0,1]\}_{\alpha\in\Lambda}$. We define the l–norm of a compactly supported k–form ω to be the l–norm,

$$||\omega||_l = \left(\sum_{\alpha\in\Lambda} ||(\rho_\alpha f_\omega) \circ \varphi_\alpha^{-1}||^2_{l,\, H^l(\mathbb{R}^n)}\right)^{\frac{1}{2}} \tag{3.13}$$

And $||\;||_{l,\,H^l(\mathbb{R}^n)}$ is the l–norm of functions defined by the equality (D.7).

We denote the set of all differentiable k–forms with compact support contained in M by $\Omega^k_c(M)$. Note that $\Omega^k_c(M) \subset \Omega^k(M)$.

Definition 3.4.4. The completation of $\Omega^k_c(M)$ with respect to the l–norm (3.13) is the *l–Sobolev space of differentiable k–forms*, denoted by $H^l_k(M)$.

The inner product on $\Omega^k(M)$ defined in equality (3.6) induce the L^2–norm

$$||\omega||_0 := \langle \omega, \omega\rangle^{1/2}. \tag{3.14}$$

With respect to the $||\;||_0$–norm we have the 0–Sobolev space $H^0(M)$, by Remark D.3.4 $H^0_k(M) = L^2(\Omega^k_c(M))$, see section D.2.

On $\Omega^k(M)$, we define inner product

$$\langle \omega, \omega\rangle_1 := \langle d\omega, d\omega\rangle + \langle d^\star\omega, d^\star\omega\rangle + \langle \omega, \omega\rangle. \tag{3.15}$$

And

$$||\omega||_1 := \langle \omega, \omega\rangle_1^{1/2}. \tag{3.16}$$

By straightforward calculations we can see that if $l = 1$ the 1–norm (3.13) coincides with the norm (3.16).

We complete the space $\Omega^k_c(M)$ of differentiable k–forms with respect to the norm $||\;||_1$, the resulting vector space is the 1–Sobolev space of $\Omega^k(M)$, denoted by $H^1_k(M)$.

Also, one can extend the inner products (3.6) and (3.15) to $\Omega^\bullet(M)$, we will denote by $H^i_\bullet(M)$ the i–Sobolev space of $\Omega^\bullet(M)$, where $i = 0, 1$.

3.5 Hodge theorem

The objective of the section is to see that each De Rham cohomology class contains a harmonic representative, this result relates differential geometry and geometric analysis.

Lemma 3.5.1 ([18, Lemma. 3.4.2])**.** *Let $\{\omega_n\}_{n\in\mathbb{N}} \subset H^1_k(M)$ be bounded. Then a subsequence of $\{\omega_n\}$ converges with respect to (3.14) to some $\omega \in H^1_k(M)$.*

Lemma 3.5.2. *There exists a constant $c > 0$, depending only on the Riemannian metric of M, with the property that for all closed k–form ω that is orthogonal to the kernel of $d^\star$,*

$$\langle \omega, \omega\rangle \leq c\langle d^\star\omega, d^\star\omega\rangle. \tag{3.17}$$

Proof. If (3.17) is not true, suppose there exists a sequence of closed k–forms $\{\beta_n\}$ orthogonal to $\operatorname{Ker} d^\star$ with

$$\langle \beta_n, \beta_n\rangle \geq n\langle d^\star\beta_n, d^\star\beta_n\rangle. \tag{3.18}$$

We define $\lambda_n := \langle \beta_n, \beta_n\rangle^{-1/2} \in \mathbb{R}$. Then

$$\begin{aligned} \langle \lambda_n\beta_n, \lambda_n\beta_n\rangle &= \lambda_n\langle \beta_n, \lambda_n\beta_n\rangle \\ &= \lambda_n^2\langle \beta_n, \beta_n\rangle \\ &= (\langle \beta_n, \beta_n\rangle)^{-1}\langle \beta_n, \beta_n\rangle \\ &= 1. \end{aligned}$$

By equality (3.18)

$$1 = \langle \lambda_n\beta_n, \lambda_n\beta_n\rangle \geq n\langle d^\star(\lambda_n\beta_n), d^\star(\lambda_n\beta_n)\rangle. \tag{3.19}$$

By hypothesis β_n is a closed k–form, since d is $\mathbb{R}$–linear, then $d(\lambda_n\beta_n) = 0$. One has that

$$\frac{1}{n} \geq \langle d^\star(\lambda_n\beta_n), d^\star(\lambda_n\beta_n)\rangle + \langle d(\lambda_n\beta_n), d(\lambda_n\beta_n)\rangle$$

We add the term $\langle \lambda_n\beta_n, \lambda_n\beta_n\rangle$,

$$\frac{1}{n} + 1 \geq \langle d^\star(\lambda_n\beta_n), d^\star(\lambda_n\beta_n)\rangle + \langle d(\lambda_n\beta_n), d(\lambda_n\beta_n)\rangle + \langle \lambda_n\beta_n, \lambda_n\beta_n\rangle = \langle \lambda_n\beta_n\rangle_1 = ||\lambda_n\beta_n||_1^2.$$

Since $\{\lambda_n\beta_n\}$ is a bounded sequence, by Lemma 3.5.1, there exist a subsequence of $\{\lambda_n\beta_n\}$ that converges with respect to the 0–norm $||\ \ ||_0$ to some $\psi \in H_k^1(M)$.

By inequality (3.19), $\frac{1}{n} \geq \langle d^\star(\lambda_n\beta_n), d^\star(\lambda_n\beta_n)\rangle$, then $d^\star(\lambda_n\beta_n)$ converges to 0 with respect to 0–norm.

Since a subsequence $\lambda_n\beta_n$ converges to ψ, we get that for all $\omega \in \Omega^{k-1}(M)$

$$\begin{aligned} 0 &= \lim_{n\to\infty}\langle d^\star(\lambda_n\beta_n), \omega\rangle \\ &= \lim_{n\to\infty}\langle \lambda_n\beta_n, d\omega\rangle \\ &= \langle \psi, d\omega\rangle \\ &= \langle d^\star\psi, \omega\rangle. \end{aligned}$$

Then $d^\star\psi = 0$.

Since $d^\star\psi = 0$ and β_n is orthogonal to $\operatorname{Ker} d^\star$, then

$$\langle \psi, \lambda_n\beta_n\rangle = 0. \tag{3.20}$$

On the other hand, since $\langle \lambda_n\beta_n, \lambda_n\beta_n\rangle = 1$ and $\lambda_n\beta_n$ converges to ψ with respect to $||\ \ ||_0$, then

$$\lim_{n\to\infty}\langle \psi, \lambda_n\beta_n\rangle = \lim_{n\to\infty}\langle \lambda_n\beta_n, \lambda_n\beta_n\rangle = \lim_{n\to\infty} 1 = 1.$$

Which is a contradiction to (3.20). □

Theorem 3.5.3 (Hodge Theorem)**.** *Let M be an oriented, compact Riemannian n–manifold. An arbitrary De Rham cohomology class of M can be represented by a unique harmonic form, that is*

$$\operatorname{Ker}\square_k \cong \mathrm{H}^k_{\mathrm{DR}}(M).$$

Proof. Uniqueness:

Let $[\omega_1], [\omega_2] \in \mathrm{H}^k_{\mathrm{DR}}(M)$ such that ω_1, ω_2 are cohomologous and harmonic k–forms. Since $\omega_1, \omega_2 \in \Omega^k(M)$ are cohomologous, then

$$\omega_1 = \omega_2 + d\eta \tag{3.21}$$

for some $\eta \in \Omega^{k-1}(M)$.

We have the following cases:

1. If $k = 0$, by hipothesis $\omega_1, \omega_2 \in \Omega^0(M) = C^\infty(M)$ and the equality (3.21) is satisfied for some $\eta \in \Omega^{-1}(M) = 0$, then $\eta = 0$. Therefore, $\omega_1 = \omega_2$.

2. If $k \neq 0$. By equality (3.21) for some $\eta \in \Omega^{k-1}(M)$ and Proposition 3.3.5, we have

$$\langle \omega_1 - \omega_2, \omega_1 - \omega_2 \rangle = \langle \omega_1 - \omega_2, d\eta \rangle = \langle d^\star(\omega_1 - \omega_2), \eta \rangle.$$

   Since $d^\star$ is a linear map, ω_1 and ω_2 are harmonic k–forms and by Proposition 3.3.12-3 we obtain

$$||\omega_1 - \omega_2||_0 = \langle \omega_1 - \omega_2, \omega_1 - \omega_2 \rangle = \langle d^\star\omega_1, \eta \rangle - \langle d^\star\omega_2, \eta \rangle = 0$$

   Therefore $\omega_1 = \omega_2$.

Existence:

Let ω_0 be a closed differentiable form representing of $[\omega_0] \in \mathrm{H}^k_{\mathrm{DR}}(M)$.

Note that all forms cohomologous to ω_0 are of the form

$$\omega = \omega_0 + d\alpha \tag{3.22}$$

for some $\alpha \in \Omega^{k-1}$, where ω is also a closed form.

We denote by $Z^k_{\omega_0}(M)$ the vector space of all closed k–forms cohomologous to ω_0. We consider the functional

$$\begin{aligned} N\colon\ & Z^k_{\omega_0}(M) \longrightarrow \mathbb{R} \\ & \omega \mapsto \langle \omega, \omega \rangle. \end{aligned}$$

We want to minimize N, that is we want to see the infimum is achieved by a differentiable form $\eta \in Z^k_{\omega_0}(M)$, such that η must satisfy the following equation: for all $\beta \in \Omega^{k-1}(M)$,

$$\left.\frac{d}{dt} N(\eta + td\beta)\right|_{t=0} = 0 \tag{3.23}$$

$$
\begin{aligned}
0 &= \frac{d}{dt}\langle \eta + td\beta, \eta + td\beta\rangle|_{t=0} \\
&= \frac{d}{dt}(\langle \eta, \eta + td\beta\rangle + \langle td\beta, \eta + td\beta\rangle)|_{t=0} \\
&= \frac{d}{dt}(\langle \eta, \eta\rangle + \langle \eta, td\beta\rangle + \langle td\beta, \eta\rangle + \langle td\beta, td\beta\rangle)|_{t=0} \\
&= \frac{d}{dt}(\langle \eta, \eta\rangle + 2\langle \eta, td\beta\rangle + t^2\langle d\beta, d\beta\rangle)|_{t=0} \\
&= (2\langle \eta, d\beta\rangle + 2t\langle d\beta, d\beta\rangle)|_{t=0} \\
&= 2\langle \eta, d\beta\rangle \\
&= 2\langle d^\star\eta, \beta\rangle \\
0 &= \langle d^\star\eta, \beta\rangle.
\end{aligned}
$$

$$0 = \langle \eta, d\beta\rangle. \tag{3.24}$$

Since this holds for all $\beta \in \Omega^{k-1}(M)$, $d^\star\eta = 0$. Since η is a closed form, then $d\eta = 0$. By Proposition 3.3.12-3, η is a harmonic k–form.

If we prove that there exists the infimum of N by the equation (3.23), it will already be a harmonic k–form.

Let $\{\omega_n\}_{n\in\mathbb{N}} \subset Z^k_{\omega_0}(M)$ be a sequence such that

$$\omega_n = \omega_0 + d\alpha_n, \tag{3.25}$$

for some $\alpha_n \in \Omega^{k-1}(M)$, $N(\omega_n)$ converges to $\inf\limits_{\omega=\omega_0+d\alpha} N(\omega) = \kappa$.

So $\langle \omega_n, \omega_n\rangle = N(\omega_n) \leq \kappa + 1$.

Since $\{\omega_n\}_{n\in N}$ is bounded, by Theorem D.1.13 then there exist converges weakly subsequence $\{\omega_n\}_{n\in\mathbb{N}}$ to some $\omega \in H^0_k(M)$, see Definition D.1.12.

Since $\langle \omega_n - \omega_0, \varphi\rangle = \langle d\alpha_n, \varphi\rangle = \langle \alpha_n, d^\star\varphi\rangle$ for all $\varphi \in \Omega^k(M)$. Then

$$\langle \omega - \omega_0, \varphi\rangle = 0 \tag{3.26}$$

if and only if $d^\star\varphi = 0$, $\varphi \in \Omega^k(M)$.

Set $\tau := \omega - \omega_0$.

We define the functional

$$
\begin{aligned}
A\colon\ & \operatorname{Im} d^\star \longrightarrow \mathbb{R} \\
& d^\star\varphi \longrightarrow \langle \tau, \varphi\rangle.
\end{aligned} \tag{3.27}
$$

Let us see that A is well defined. If $d^\star\varphi_1 = d^\star\varphi_2$, since $d^\star$ is a linear map, then $0 = d^\star\varphi_1 - d^\star\varphi_2 = d^\star(\varphi_1 - \varphi_2)$.

By equality (3.26) and by definition (3.27) then

$$A(d^\star(\varphi_1 - \varphi_2)) = \langle \tau, (\varphi_1 - \varphi_2)\rangle = 0.$$

Since $\langle\ ,\ \rangle$ is bilinear

$$\begin{aligned} 0 &= \langle \tau, (\varphi_1 - \varphi_2)\rangle = \langle \tau, \varphi_1\rangle - \langle \tau, \varphi_2\rangle, \quad \text{then} \quad \langle \tau, \varphi_1\rangle = \langle \tau, \varphi_2\rangle, \\ A(d^\star\varphi_1) &= A(d^\star\varphi_2). \end{aligned}$$

Therefore, A is well defined.

Consider $\mathrm{pr}\colon \Omega^k(M) \longrightarrow \operatorname{Ker} d^\star$ be the orthogonal projection onto $\operatorname{Ker} d^\star$

Let $\varphi \in \Omega^k(M)$, we define $\psi := \varphi - \mathrm{pr}(\varphi) \in (\operatorname{Ker} d^\star)^\perp$. Note that

$$d^\star\psi = d^\star(\varphi - \mathrm{pr}(\varphi)) = d^\star\varphi - d^\star(\mathrm{pr}(\varphi)) = d^\star\varphi. \tag{3.28}$$

Then

$$A(d^\star\varphi) = A(d^\star\psi) = \langle \tau, \psi\rangle. \tag{3.29}$$

In equality (3.29) apply Cauchy-Schwarz inequality

$$|A(d^\star\varphi)| = |\langle \tau, \psi\rangle| \leq ||\tau||_0||\psi||_0. \tag{3.30}$$

Since ψ is a closed k–form and ψ is orthogonal to the kernel of $d^\star$, by Lemma 3.5.2 there is a constant $c > 0$ such that $\langle \psi, \psi\rangle \leq c\langle d^\star\psi, d^\star\psi\rangle$. By definition (3.14) and equality (3.28), then

$$||\psi||_0 \leq \sqrt{c}||d^\star\psi||_0 = \sqrt{c}||d^\star\varphi||_0. \tag{3.31}$$

By equalities (3.30) and (3.31) then $|A(d^\star\varphi)| \leq \sqrt{c}||\tau||_0||d^\star\varphi||_0$, therefore A is a bounded functional, see definition D.1.17.

Since A is a bounded functional, it is a continuous functional, then A can be extended to the L^2-closure of $\operatorname{Im} d^\star$. By Riesz Theorem D.1.19, there exist $\alpha \in L^2(\operatorname{Im} d^\star)$ such that

$$\langle \alpha, d^\star\varphi\rangle = \langle \tau, \varphi\rangle$$

for all $\varphi \in \Omega^k(M)$. Since d is the adjoint operator of $d^\star$, see Proposition 3.3.5, rewrite

$$\langle d\alpha, \varphi\rangle = \langle \tau, \varphi\rangle,$$

then $d\alpha = \tau$. Therefore $\omega = \omega_0 + \tau \in Z^k_{\omega_0}(M)$.

By Theorem D.3.9 we have the regularity of the solutions of equality (3.24). ☐

Some consequences of Hodge Theorem are Theorems 1.4.10 and 1.4.7-1.

Chapter 4

More expressions for d, $d^\star$ and D

In this chapter, we shall omit the word "differentiable" for a vector bundle, form and section, since we will only deal with differentiable objects.

Through the notions and properties of connections and Clifford algebras we will give expressions for d, $d^\star$ and $\Box_k$ that we need, the equalities (4.20), (4.21) and (4.46).

4.1 Connections

To review topics related to this section see [18], [28], [14] and [24].

First, let us mention a result of isomorphisms of $C^\infty(M)$–modules.

Theorem 4.1.1 ([24, Prop. 16.13]). *Let (E, π, M) and (F, π', M) be two vector bundles, there are the following isomorphisms:*

1. $\Gamma(\mathrm{Hom}_{\mathbb{R}}(E, F)) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), \Gamma(F))$.
2. $\Gamma(E \otimes F) \cong \Gamma(E) \otimes_{C^\infty(M)} \Gamma(F)$.
3. $\Gamma(E^*) \cong \mathrm{Hom}_{C^\infty(M)}(\Gamma(E), C^\infty(M))$.
4. $\Gamma(\Lambda^i E) \cong \Lambda^i_{C^\infty(M)}(\Gamma(E))$.

Definition 4.1.2. Let E and M be differentiable manifolds and $\pi\colon E \longrightarrow M$ be a real vector bundle over M. The set of all *k–forms with values in E* is

$$\Omega^k(E) := \Gamma(\Lambda^k(T^*M) \otimes E).$$

That is, by Proposition 1.2.4-2 an arbitrary element of $\Omega^k(E)$ can be written as a linear combination of elements of the form $\omega \otimes s$, where $\omega \in \Omega^k(M), s \in \Gamma(E)$. Let $\omega \in \Omega^k(M)$, $\omega(p)\colon T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$, generalizing this, for a vector bundle $\pi\colon E \longrightarrow M$ we obtain a k–form with values in E, taking $\omega(p) \otimes \pi^{-1}(p)$.

The space of sections of $\Lambda^\bullet(T^*M) \otimes E$ the tensor product vector bundle is denoted by

$$\Omega^\bullet(E) = \Gamma(\Lambda^\bullet(T^*M) \otimes E). \tag{4.1}$$

A connection on E may be thought of, in some sense, as an extension of the exterior derivative d to include coefficients in E.

Definition 4.1.3. A *connection* in a vector bundle (E, π, M) over a differentiable manifold M, is a linear map

$$\nabla^E \colon \Gamma(E) \longrightarrow \Omega^1(E)$$

satisfying the following condition: (Leibniz rule) for each $f \in C^\infty(M)$, $s \in \Gamma(E)$,

$$\nabla^E(fs) = df \cdot s + f\nabla^E s. \tag{4.2}$$

If $X \in \Gamma(TM)$, then a connection ∇^E induces a canonical map

$$\nabla^E_X \colon \Gamma(E) \longrightarrow \Gamma(E)$$

via the contraction between TM and T^*M, that is, let $s \in \Gamma(E)$ $\nabla^E_X s = X \lrcorner \nabla^E s$, (see definition of contraction 1.5.1).

∇^E_X is called the *covariant derivative* of ∇^E along X.

Proposition 4.1.4 ([20, Prop. 1.2, Prop. 2.7])**.** *Let (E, π, M) be a vector bundle, let X and Y be vector fields on a differentiable manifold M, then the covariant derivative has the following properties: for all $s \in \Gamma(E)$,*

1. $\nabla^E_{X+Y} s = \nabla^E_X s + \nabla^E_Y s.$
2. $\nabla^E_{fX} s = f\nabla^E_X s$ *and* $\nabla^E_{\lambda X} s = \lambda \nabla^E_X s$ *for each* $f \in C^\infty(M)$ *and* $\lambda \in \mathbb{R}$.
3. $\nabla^E_X f = Xf$ *for every function* $f \in C^\infty(M)$.

Elements of $\Omega^1(E|_U)$ are written uniquely as $\sum\limits_{i=1}^{k} \eta_i \otimes s_i$ for some $\eta_i \in \Omega^1(U)$.

Definition 4.1.5. Let (E, π, M) be a vector bundle of rank k, $U \subset M$ an open subset and $s_1, \ldots, s_k \in \Gamma(E\mid_U)$ be a local frame. For a connection ∇^E on E we have

$$\nabla^E s_i = \sum_{j=1}^{k} A_{ij} \otimes s_j \tag{4.3}$$

where $A_{ij} \in \Omega^1(U)$ is a $k \times k$ matrix of 1–forms, which is called the *connection matrix* with respect to the local frame $\{s_1, \ldots, s_k\}$ and it is denoted by A.

Conversely, given an arbitrary matrix A of 1–forms on U and a local frame $\{s_1, \ldots, s_k\}$ for $E\mid_U$, then equality (4.3) defines a connection on $\Gamma(E\mid_U)$. Let $s \in \Gamma(E\mid_U)$ we can write it as $s = \sum\limits_{i=1}^{k} a_i s_i$, with $a_i \in C^\infty(U)$. By equalities (4.2) and (4.3), we get

$$\nabla^E s = \nabla^E\left(\sum_{i=1}^{k} a_i s_i\right) = \sum_{i=1}^{k}\left(da_i \cdot s_i + a_i \nabla^E s_i\right).$$

Then

$$\nabla^E s = \sum_{i=1}^{k} da_i \cdot s_i + \sum_{i=1}^{k}\sum_{j=1}^{k} a_i A_{ij} \otimes s_j. \tag{4.4}$$

With respect to $s = (a_1, \ldots, a_k)$, equation (4.4) can be written in matrix for as

$$\nabla^E (a_1, \ldots, a_k) = (da_1, \ldots, da_k) + (a_1, \ldots, a_k)A.$$

We consider a trivial bundle $\pi\colon E \longrightarrow M$, that is, it has a trivialization $E \cong M \times \mathbb{R}^n$. One can see that $\Gamma(E) = C^\infty(M, \mathbb{R}^n)$, we have

$$\nabla^E = \nabla^{M\times\mathbb{R}^n}\colon C^\infty(M, \mathbb{R}^n) \longrightarrow \Gamma(T^*M) \times C^\infty(M, \mathbb{R}^n).$$

Let $f_1, \ldots, f_n \in C^\infty(M, \mathbb{R}^n)$ be a local frame, for any $f \in C^\infty(M, \mathbb{R}^n)$ $f = \sum\limits_{i=1}^{n} a_i f_i$ with $a_i \in C^\infty(M)$, by equality (4.4), we have

$$\nabla^E f = \sum_{i=1}^{n} (da_i) f_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i A_{ij} \otimes f_j. \tag{4.5}$$

Suppose A_{ij} is the zero matrix, then for every vector field X, $\nabla^E_X f$ is just the directional derivative of f in the direction of X. In this case, ∇^E is called the *trivial connection* in the product bundle.

Lemma 4.1.6. *Any vector bundle over a differentiable manifold admits a connection.*

Proof. Let (E, π, M) be a vector bundle over a differentiable manifold M, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a locally finite open cover. By the local trivializations we have $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^n$, we denoted by ∇^α a trivial connection for each $\pi^{-1}(U_\alpha) \longrightarrow U_\alpha$.

Let $\{g_\alpha\}_{\alpha\in\Lambda}$ be a partition of unity for the cover $\{U_\alpha\}_{\alpha\in\Lambda}$, (see definition 3.4.2). By equality (4.5) we define

$$\nabla^E s := \sum_{\alpha\in\Lambda} g_\alpha \nabla^\alpha s = \sum_{\alpha\in\Lambda}\sum_{i=1}^{n} g_\alpha da_i f_i + \sum_{\alpha\in\Lambda}\sum_{i=1}^{n}\sum_{j=1}^{n} g_\alpha a_i A_{ij} \otimes f_j.$$

□

Remark 4.1.7. In this way, we construct a connection on E. Since we have an infinite number of connection matrices, there are infinitely many connections on E.

4.1.1 Connections on the tangent bundle

Connections on the tangent bundle TM are particularly important. Also on the tangent bundle there is the Levi-Civita connection.

The Levi-Civita Connection

Definition 4.1.8. The *torsion of a connection* ∇^{TM} on TM is defined as

$$T(X,Y) := \nabla^{TM}_X Y - \nabla^{TM}_Y X - [X,Y], \quad X,Y \in \Gamma(TM).$$

∇^{TM} is called *torsion free* if $T(X,Y) = 0$ for all $X,Y \in \Gamma(TM)$.

Definition 4.1.9. Let TM be the tangent bundle on a Riemannian manifold (M,g). A connection ∇^{TM} on TM is called *metric* if

$$Xg(Y,Z) = g(\nabla^{TM}_X Y, Z) + g(Y, \nabla^{TM}_X Z), \quad X,Y,Z \in \Gamma(TM). \tag{4.6}$$

Theorem 4.1.10 ([18, Thm. 4.3.1])**.** *On each Riemannian manifold (M,g), there is precisely one metric and torsion free connection ∇^{LC} on TM. It is determined by the formula:*

$$g(\nabla^{\mathrm{LC}}_X Y, Z) = \frac{1}{2}(Xg(Y,Z) - Zg(X,Y) + Yg(Z,X) - g(X,[Y,Z]) + g(Z,[X,Y]) + g(Y,[Z,X]), \tag{4.7}$$

for all $X,Y,Z \in \Gamma(TM)$. The formula (4.7) is called the Koszul formula.

Definition 4.1.11. The connection ∇^{LC} determined by (4.7) is called the *Levi-Civita connection* of M.

Definition 4.1.12. Let ∇^{TM} be a connection on TM, the *Christoffel symbols* Γ^k_{ij} are given by

$$\nabla^{TM}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma^k_{ij} \frac{\partial}{\partial x_k}. \tag{4.8}$$

It is possible to characterize a torsion free connection ∇^{TM} in terms of its Christoffel symbols. In local coordinates, by equality (4.8) the components of the torsion T are given by

$$T_{ij} = T\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \nabla^{TM}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla^{TM}_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \sum_{k=1}^{n} (\Gamma^k_{ij} - \Gamma^k_{ji}) \frac{\partial}{\partial x_k}. \tag{4.9}$$

Theorem 4.1.13 ([18, Cor. 4.3.1])**.** *The connection ∇^{TM} on TM is torsion free if and only if*

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad \textit{for all } i,j,k. \tag{4.10}$$

Let M be a differentiable manifold, $c\colon I \longrightarrow M$ be a differentiable curve and $(U, x_1, \dots, x_n)$ be a chart on M. We consider $x(t) := x(c(t))$ where $x = (x_1, \dots, x_n)$. Then,

$$\dot{c} = \sum_{i=1}^{n} \dot{c}_i \frac{\partial}{\partial x_i}.$$

By equalities (4.2), (4.8) and Proposition 4.1.4-2. we get

$$\begin{aligned}
\nabla^{TM}_{\dot{c}(t)}\dot{c}(t) &= \nabla^{TM}_{\sum_{i=1}^{n}\dot{c}_i(t)\frac{\partial}{\partial x_i}} \sum_{k=1}^{n} \dot{c}_k(t)\frac{\partial}{\partial x_k} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \nabla^{TM}_{\dot{c}_i(t)\frac{\partial}{\partial x_i}} \dot{c}_k(t)\frac{\partial}{\partial x_k} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\dot{c}(t)\frac{\partial}{\partial x_k}(c(t)) + \dot{c}_k(t)\nabla^{TM}_{\dot{c}_i(t)\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_k}\right) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n} \ddot{c}(t)\frac{\partial}{\partial x_j} + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} \dot{c}_i(t)\dot{c}_k(t)\Gamma^{j}_{ik}(c(t))\frac{\partial}{\partial x_j}.
\end{aligned}$$

Definition 4.1.14. Let M be a differentiable manifold and ∇^{TM} be a connection on the tangent bundle TM. A *geodesic* is a differentiable curve $c\colon I \longrightarrow M$ with respect to ∇^{TM} if

$$\nabla^{TM}_{\dot{c}}\dot{c} \equiv 0.$$

That is,

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \ddot{c}(t) + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} \dot{c}_i(t)\dot{c}_k(t)\Gamma^{j}_{ik}(c(t))\right)\frac{\partial}{\partial x_j} = 0. \tag{4.11}$$

Theorem 4.1.15 ([18, Thm. 1.4.2])**.** *Let M be a Riemannian manifold, for each $p \in M$, $v \in T_pM$ there exist a maximal interval $\epsilon > 0$ and precisely one geodesic $c\colon [0,\epsilon] \longrightarrow M$ with $c(0) = p$, $\dot{c}(0) = v$. In addition, c depends smoothly on p and v.*

The geodesic of Theorem 4.1.15 will be denoted by c_v, also, we have that for $\lambda > 0, t \in [0,\epsilon]$

$$c_v(t) = c_{\lambda v}\left(\frac{t}{\lambda}\right).$$

By Heine-Borel Theorem, see [4, Thm. 3.3.1], the set $\{v \in T_pM \mid ||v|| = 1\}$ is compact and since c_v depends smoothly on v, there exists $\epsilon_0 > 0$ with the property that for $||v|| = 1$, c_v is defined at least on $[0,1]$.

Definition 4.1.16. Let M be a Riemannian manifold, $p \in M$.

Let $V_p = \{v \in T_pM | c_v \text{ is defined on } [0,1]\}$, we define

$$\begin{aligned}
\exp_p\colon V_p &\longrightarrow M \\
v &\mapsto c_v(1).
\end{aligned}$$

Called the *exponential map* of M at p. If $v \in V_p$, $0 \le t \le 1$, then $\exp_p(tv) = c_v(t)$.

Theorem 4.1.17 ([18, Thm. 1.4.3])**.** *The exponential map $\exp_p$ maps a neighborhood of $0 \in T_pM$ diffeomorphically onto a neighborhood of $p \in M$.*

Let $X_1, \dots, X_n$ be an orthonormal basis of T_pM with respect to the Riemannian metric. For each $v \in T_pM$, we can write $v = \sum\limits_{i=1}^{n} a_i X_i$ with $a_i \in \mathbb{R}$. We have a linear map:

$$\begin{aligned} \psi\colon\ & T_pM \longrightarrow \mathbb{R}^n \\ & v \mapsto (a_1, \dots, a_n). \end{aligned}$$

By the linear map ψ we identify T_pM with $\mathbb{R}^n$.

By Theorem 4.1.17 there exists a neighborhood V of p such that is mapped by $\exp_p^{-1}$ diffeomorphically onto a neighborhood $W \subset C$ of $0 \in T_pM$ and by $\psi \circ \exp_p^{-1}$ we have a neighborhood V of p diffeomorphic onto a neighborhood U of $0 \in \mathbb{R}^n$. In particular, p is mapped to 0.

Definition 4.1.18. Let M be a Riemannian n–manifold, the local coordinates defined by the charts $(U, \psi \circ \exp_p^{-1})$ are called *normal coordinates* with center p.

Theorem 4.1.19. *Let M be a Riemannian n–manifold, in normal coordinates we have:*

$$\Gamma_{ij}^k(0_{\mathbb{R}^n}) = 0_{\mathbb{R}}, \quad \textit{for all} \quad i, j, k. \tag{4.12}$$

Proof. Let M be a Riemannian n–manifold, $p \in M$ and (U, x), where $x = (x_1, \dots, x_n)$, normal coordinates with center p. In this coordinates, the straight lines throught the origin of $\mathbb{R}^n$, (or, more precisely, their portions contained in the chart image) are geodesic. Namely, the line $t\mathbf{x}$, $t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n$ is mapped (for sufficiently small t) onto $c_{t\mathbf{x}}(1) = c_{\mathbf{x}}(t)$, where $c_{\mathbf{x}}(t)$ is the geodesic, parametrized by arc length, with $\dot{c}_{\mathbf{x}}(0) = \mathbf{x}$.

We consider $x(c(t)) = tv$, with $v \in T_pM$, since is a normal coordinate $x(p) = 0$.

We have

$$\dot{c}(t) = \frac{d}{dt} x(c(t)) = (\dot{c}_1(t), \dots, \dot{c}_n(t)), \text{ where } \dot{c}_i(t) = \frac{d}{dt} x_i(c(t)).$$

Then $\dot{c}_j(t) = \frac{d}{dt}(tv_j) = v_j$ and $\ddot{c}_j(t) = 0$, we substitute this in equality (4.11) and have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \Gamma_{ij}^k(x(c(t))) v_i v_j \frac{\partial}{\partial x_k} = 0.$$

Then $\Gamma_{ij}^k(tv) v_i v_j = 0$ for all $k = 1, \dots, n$. In particular at $t = 0$, $\Gamma_{ij}^k(0_{\mathbb{R}^n}) v_i v_j = 0$ for all $\mathbf{v} \in \mathbb{R}^n$ and $k = 1, \dots, n$.

Let $X_1, \dots, X_n$ be an orthonormal basis of T_pM with respect to the Riemannian metric. We put $v = \frac{1}{2}(X_l + X_m)$ with $m, l = 1, \dots, n$ and since x is a normal coordinate we obtain

$$\Gamma_{ij}^k(0_{\mathbb{R}^n}) = 0_{\mathbb{R}}$$

for all $k = 1, \dots, n$.

Also, since M is a Riemannian manifold, for ∇^{LC}, by Theorem 4.1.13 $\Gamma_{ij}^k = \Gamma_{ji}^k$. Therefore $\Gamma_{ij}^k(0_{\mathbb{R}^n}) = 0$ for all i, j, k. □

4.1.2 Induced connections

In this section ∇^E will be a connection on a vector bundle (E, π, M) over a differentiable manifold M and ∇^F will be a connection on the vector bundle (F, π', M), these connections induce connections on the vector bundles that we build in the section C.1, for example the connection in the cotangent bundle and in the k–th exterior bundle of T^*M.

Let (E, π, M) and (F, π', M) be two vector bundles over M. The usual wedge product induces a natural map

$$\wedge\colon \Omega^i(M) \times \Omega^j(E) \longrightarrow \Omega^{i+j}(E),$$

defined by

$$\wedge(\eta, \omega \otimes s) = (\eta \wedge \omega) \otimes s.$$

That induces a $C^\infty(M)$–pairing

$$\wedge\colon \Omega^i(E) \otimes \Omega^j(F) \longrightarrow \Omega^{i+j}(E \otimes F), \quad (\omega \otimes s) \wedge (\eta \otimes t) = \omega \wedge \eta \otimes (s \otimes t). \tag{4.13}$$

Where $\omega \in \Omega^i(M), \eta \in \Omega^j(M), s \in \Gamma(E)$ and $t \in \Gamma(F)$ and $\omega \wedge \eta$ is the wedge product.

Let (E^*, π', M) be the dual vector bundle of (E, π, M), we define the evaluation map as

$$\begin{gathered} \mathrm{Ev}\colon \Omega^{i+j}(E \otimes E^*) \longrightarrow \Omega^{i+j}(M), \\ \omega \otimes (s \otimes s^*) \mapsto \omega(s^*(s)). \end{gathered}$$

With respect to (E^*, π', M) and (E, π, M) we have the pairing $(\ ,\ ) = \mathrm{Ev} \circ \wedge$ of $\Omega^i(E)$ and $\Omega^j(E^*)$ defined by

$$\begin{gathered} (\ ,\ )\colon \Omega^i(E) \otimes \Omega^j(E^*) \longrightarrow \Omega^{i+j}(M) \\ (\omega \otimes s, \eta \otimes s^*) \mapsto \omega \wedge \eta(s^*(s)). \end{gathered}$$

Since $\wedge$ is a non-singular pairing, then $(\ ,\ )$ is also a non-singular pairing.

Let ∇^E be a connection on (E, π, M), using the pairing $(\ ,\ )$ we define the connection, ∇^{E^*} on E^* such that

$$d(s, s^*) = (\nabla^E s, s^*) + (s, \nabla^{E^*} s^*), \quad s^* \in \Gamma(E^*), s \in \Gamma(E). \tag{4.14}$$

On the right side of the equality (4.14) the first pairing is $(\ ,\ )\colon \Omega^1(E) \otimes \Gamma(E^*) \longrightarrow \Omega^1(M)$, and the second is $(\ ,\ )\colon \Gamma(E) \otimes \Omega^1(E^*) \longrightarrow \Omega^1(M)$. Since the pairing $(\ ,\ )$ is non-singular, the connection ∇^{E^*} is unique and will be called the *dual* connection of ∇^E.

We can rewrite the equality (4.14) as

$$d(s^*(s)) = s^*(\nabla^E s) + (\nabla^{E^*} s^*)(s). \tag{4.15}$$

If we return to the matrix of the connection we have the following.

Lemma 4.1.20. *Let (E, π, M) be a vector bundle, $U \subset M$ be an open subset, $s_1, \ldots, s_k$ be a local frame over U. Let $A = (A_{ij})$ be the connection matrix of ∇^E with respect to the local frame, then for E^* the dual vector bundle has the connection matrix $\tilde{A} = -A^t = (-A_{ji})$.*

Proof. Let $s_1, \dots, s_k$ be a local frame over $U \subset M$ and let $s_1^*, \dots, s_k^*$ be the dual local frame on E^*.

By equality (4.14) for the connection ∇^{E^*} we get

$$d(s_i, s_j^*) = (\nabla^E s_i, s_j^*) + (s_i, \nabla^{E^*} s_j^*).$$

Let $1, 0 \in C^\infty(M)$ be the constant functions with values 1 and 0 respectively, from first term, if $i = j$, then $(s_i, s_j^*) = 1$, while if $i \neq j$ then $(s_i, s_j^*) = 0$, in both cases $d(s_i, s_j^*) = 0$. By equality (4.3) and since the pairing is bilinear we have

$$\begin{aligned} (s_r, \nabla^{E^*} s_i^*) &= -(\nabla^E s_r, s_i^*) \\ (s_r, \sum_{j=1}^k \tilde{A}_{ij} \otimes s_j^*) &= -(\sum_{l=1}^k A_{rl} \otimes s_l, s_i^*) \\ \sum_{j=1}^k \tilde{A}_{ij}(s_r, s_j^*) &= -\sum_{l=1}^k A_{rl}(s_l, s_i^*). \end{aligned}$$

On both sides of the expressions are nonzero if the indices coincide, that is, $r = j$ and $i = l$. Therefore $\tilde{A}_{ir} = -A_{ri}$. ☐

Remark 4.1.21. We consider $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$ the local frame over $TM|_U$, with $U \subset M$ an open subset.

In relation with equality (4.3) and the Christoffel symbols we obtain

$$\nabla^{TM}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{l=1}^k A_{jl}\left(\frac{\partial}{\partial x_i}\right) \frac{\partial}{\partial x_l} \tag{4.16}$$

that is, $A_{jl}(\frac{\partial}{\partial x_i}) = \Gamma_{ij}^l \in C^\infty(U)$, we obtain a $k \times k$ matrix $A = \left(\Gamma_{ij}^l\right)_{1 \leq j, l \leq k}$.

Remark 4.1.22. By lemma 4.1.20 and Remark 4.1.21, we can obtain a matrix of $\nabla^{T^*M}_{\frac{\partial}{\partial x_i}}$. Let $A = (\Gamma_{ij}^l)_{1 \leq j, l \leq k}$ be the matrix of $\nabla^{TM}_{\frac{\partial}{\partial x_i}}$, then $-A^t$ is the matrix of $\nabla^{T^*M}_{\frac{\partial}{\partial x_i}}$. We get

$$\nabla^{T^*M}_{\frac{\partial}{\partial x_i}} dx_j = -\sum_{l=1}^k \Gamma_{il}^j dx_l. \tag{4.17}$$

Lemma 4.1.23. *Let (M, g) be a Riemannian manifold, if ∇^{TM} is metric then: for all $X, Y \in \Gamma(TM)$, where α is the dual of Y with respect to g,*

$$\left(\nabla^{TM}_X Y\right)^* = \nabla^{T^*M}_X \alpha.$$

Proof. Let $X, Y, s \in \Gamma(TM)$, where $\alpha \in \Gamma(T^*M)$ is the dual section of Y with respect to g.

Using the pairings and equality (4.15) we have:

$$\begin{aligned} (\nabla^{T^*M}_X \alpha)(s) &= X(s, \alpha) - (\nabla^{TM}_X s, \alpha) \\ &= Xg(s, Y) - g(\nabla^{TM}_X s, Y). \end{aligned}$$

On the other hand, since ∇^{TM} is metric, see equality (4.6), we have

$$\begin{aligned}(\nabla^{TM}_X Y)^*(s) &= g(s, \nabla^{TM}_X Y)\\ &= Xg(s,Y) - g(\nabla^{TM}_X s, Y).\end{aligned}$$

□

Definition 4.1.24. The *Hessian* of a differentiable function $f\colon M \longrightarrow \mathbb{R}$ on a Riemannian manifold M is $\nabla^{T^*M} df$.

In local coordinates $(U, x_1, \dots, x_n)$ we have:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

By Leibniz rule and equality (4.17), we have:

$$\begin{aligned}\nabla^{T^*M}_{\frac{\partial}{\partial x_j}} df &= \nabla^{T^*M}_{\frac{\partial}{\partial x_j}} \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \right)\\ &= \sum_{i=1}^{n} \nabla^{T^*M}_{\frac{\partial}{\partial x_j}} \left(\frac{\partial}{\partial x_i} dx_i \right)\\ &= \sum_{i=1}^{n} \left(d\left(\frac{\partial f}{\partial x_i} \right) \right) \left(\frac{\partial}{\partial x_j} \right) dx_i - \sum_{i=1}^{n}\sum_{l=1}^{n} \frac{\partial f}{\partial x_i} \Gamma^i_{jl} dx_l.\end{aligned}$$

In particular, by Lemma 4.1.23, we have:

Corollary 4.1.25. *Let (M,g) be a Riemannian manifold and $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function. If ∇^{TM} is a metric connection and any $X \in \Gamma(TM)$ then $(\nabla^{TM}_X \operatorname{grad} f)^* = \nabla^{T^*M}_X df$.*

Definition 4.1.26. Let ∇^E be a connection in (E, π, M). Then there a unique connection $\nabla^{\Lambda^k E}$ such that

$$\nabla^{\Lambda^k E}(s_1 \wedge \ldots \wedge s_k) = \sum_{i=1}^{k} s_1 \wedge \ldots \wedge \nabla^E s_i \wedge \ldots \wedge s_k. \tag{4.18}$$

Where $s_1, \dots, s_k \in \Gamma(E)$.

By Theorem 4.1.1-4. $s_1 \wedge \ldots \wedge s_k \in \Gamma(\Lambda^k E)$.
And by equality (4.18) we have: let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$,

$$\nabla^{\Lambda^{k+l}E}(\omega \wedge \eta) = \nabla^{\Lambda^k E}\omega \wedge \eta + \omega \wedge \nabla^{\Lambda^l E}\eta.$$

Lemma 4.1.27. *Let ∇^{TM} be a connection on the tangent bundle TM. Then it induces canonically a unique connection $\nabla^{\Lambda^k T^*M}$ on $\Lambda^k T^*M$.*

Proof. Let $s_1, \ldots, s_k \in \Gamma(TM)$ be a local frame and the dual local frame $s^1, \ldots, s^k \in \Gamma(T^*M)$, by equalities (4.18), (4.15) and Lemma A.3.42 we obtain:

$$\begin{aligned}
\nabla^{\Lambda^k T^*M}(s^1 \wedge \ldots \wedge s^k)(s_1, \ldots, s_k) &= \sum_{i=1}^{k}(s^1 \wedge \ldots \wedge \nabla^{T^*M} s^i \wedge \ldots \wedge s^k)(s_1, \ldots, s_k) \\
&= \sum_{i=1}^{k} s^1(s_1) \wedge \ldots \wedge (\nabla^{T^*M} s^i)(s_i) \wedge \ldots \wedge s^k(s_k) \\
&= \sum_{i=1}^{k} (\nabla^{T^*M} s^i)(s_i) \\
&= \sum_{i=1}^{k} (d(s^i(s_i)) - s^i(\nabla^{TM} s_i)) \\
&= -\sum_{i=1}^{k} s^i(\nabla^{TM} s_i).
\end{aligned}$$

In fact, $\nabla^{\Lambda^k T^*M}(s^1 \wedge \ldots \wedge s^k)(s_1, \ldots, s_k) = -\sum_{i=1}^{k} s^i(\nabla^{TM} s_i)$. ☐

Remark 4.1.28. $\nabla^{\Lambda^k T^*M}$ coincides with the induced connection of T^*M from definition 4.1.26.

By linearity of the connections, Leibniz rule, Lemmas 4.1.27 and A.3.42, we have:

Corollary 4.1.29. *If $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \Gamma(TM)$, then*

$$\nabla_{X_0}^{\Lambda^k T^*M}\omega(X_1, \ldots, X_k) = X_0(\omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \ldots, X_{i-1}, \nabla_{X_0}^{TM} X_i, X_{i+1}, \ldots, X_k) \tag{4.19}$$

4.2 Other expressions for d and $d^\star$

In sections 1.3 and 3.3 we described the exterior derivative and its adjoint operator, in this section we obtain other expressions for d and $d^\star$ using connections, properties of the wedge product and the contraction.

Proposition 4.2.1. *Let (M, g) be a Riemannian manifold of dimension n. Let $\omega \in \Omega^k(M)$, $X_0, \ldots, X_k \in \Gamma(TM)$. Then*

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k}(-1)^i \nabla_{X_i}^{\Lambda^k T^*M}\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k).$$

Proof. By Theorem 4.1.10 there is ∇^{LC} the Levi-Civita connection of TM, that is metric and torsion free.

We use the exterior derivative of Theorem 1.5.5 and since ∇^{LC} is torsion free, we obtain

$$\begin{aligned}
d\omega(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,\widehat{X_i},\dots,X_k)) \\
&\quad + \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k) \\
&= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,\widehat{X_i},\dots,X_k)) \\
&\quad + \sum_{0\le i<j\le k}(-1)^{i+j}\omega(\nabla^{\mathrm{LC}}_{X_i}X_j,X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k) \\
&\quad - \sum_{0\le i<j\le k}(-1)^{i+j}\omega(\nabla^{\mathrm{LC}}_{X_j}X_i,X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k).
\end{aligned}$$

We permute $\nabla^{\mathrm{LC}}_{X_i}X_j$ and $\nabla^{\mathrm{LC}}_{X_j}X_i$ to the entries $j-2$ and i respectively. After we develop and reorder the sums.

$$\begin{aligned}
d\omega(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,\widehat{X_i},\dots,X_k)) \\
&\quad + \sum_{0\le i<j\le k}(-1)^{i-1}\omega(X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\nabla^{\mathrm{LC}}_{X_i}X_j,X_{j+1}\dots,X_k) \\
&\quad - \sum_{0\le i<j\le k}(-1)^{j}\omega(X_0,\dots,\widehat{X_i},\nabla^{\mathrm{LC}}_{X_j}X_i,X_{i+1},\dots,\widehat{X_j},\dots,X_k) \\
&= \sum_{i=0}^{k}(-1)^i\Bigg[X_i\omega(X_0,\dots,\widehat{X_i},\dots,X_k)) \\
&\quad - \sum_{j=i+1}^{k}\omega(X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\nabla^{\mathrm{LC}}_{X_i}X_j,X_{j+1}\dots,X_k)\Bigg] \\
&\quad - \sum_{i=0}^{k-1}\sum_{j=i+1}^{k}(-1)^{j}\omega(X_0,\dots,\widehat{X_i},\nabla^{\mathrm{LC}}_{X_j}X_i,X_{i+1},\dots,\widehat{X_j},\dots,X_k) \\
&\quad + (-1)^k X_k\omega(X_0,\dots,X_{k-1},\widehat{X_k}) \\
&= (-1)^k X_k\omega(X_0,\dots,X_{k-1}) \\
&\quad + \sum_{i=0}^{k-1}(-1)^i\Bigg[X_i\omega(X_0,\dots,X_{k-1})) \\
&\quad - \sum_{j=i+1}^{k}\omega(X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\nabla^{\mathrm{LC}}_{X_i}X_j,X_{j+1}\dots,X_k)\Bigg] \\
&\quad - \sum_{i=0}^{k}\sum_{j=0}^{i-1}(-1)^{i}\omega(X_0,\dots,\widehat{X_i},\nabla^{\mathrm{LC}}_{X_i}X_j,X_{i+1},\dots,\widehat{X_j},\dots,X_k).
\end{aligned}$$

By equality (4.19), then

$$d\omega(X_0,\dots,X_k) = X_0\omega(X_1,\dots,X_k) - \sum_{j=1}^{k}\omega(X_1,\dots,X_{i-1},\nabla^{\mathrm{LC}}_{X_0}X_j,\dots,X_k)$$

$$\begin{aligned}
&+\sum_{i=1}^{k-1}(-1)^i\Bigg[X_i\omega(X_0,\dots,\widehat{X_i},\dots,X_k))\\
&-\sum_{j=i+1}^{k}\omega(X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\nabla^{\text{LC}}_{X_i}X_j,X_{j+1}\dots,X_k)\\
&-\sum_{j=0}^{i-1}\omega(X_0,\dots,\widehat{X_i},\nabla^{\text{LC}}_{X_i}X_j,X_{i+1},\dots,\widehat{X_j},\dots,X_k)\Bigg]\\
&+(-1)^k\left[X_k\omega(X_0,\dots,X_{k-1})-\sum_{j=1}^{k-1}\omega(X_0,\dots,\nabla^{\text{LC}}_{X_k}X_j,\dots,X_{k-1})\right]\\
=\;&\nabla^{\Lambda^kT^*M}_{Y_0}\omega(X_1,\dots,X_k)+\sum_{i=1}^{k-1}(-1)^i\nabla^{\Lambda^kT^*M}_{X_i}\omega(X_0,\dots,\widehat{X_i},\dots,X_k)\\
&+(-1)^k\nabla^{\Lambda^kT^*M}_{X_k}\omega(X_0,\dots,X_{k-1})\\
=\;&\sum_{i=0}^{k}(-1)^i\nabla^{\Lambda^kT^*M}_{X_i}\omega(X_0,\dots,\widehat{X_i},\dots,X_k).
\end{aligned}$$

□

Theorem 4.2.2. *Let (M,g) be a Riemannian manifold of dimension n, $e_1,\dots,e_n$ be a local frame of TM and $e^1,\dots,e^n$ be the dual local frame of T^*M. The exterior derivative satisfies*

$$d\omega=\sum_{i=1}^{n}e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\omega,\quad \omega\in\Omega^k(M) \tag{4.20}$$

Proof. Let $X_0,X_1,\dots,X_k\in\Gamma(TM)$.

Each $X_j=\sum_{i=1}^{n}a_{j_i}e_i$, in particular, $e^i(X_j)=a_{j_i}$, where $a_{j_i}\in C^\infty(M)$. By definition of wedge product, see the definition A.3.41 we obtain

$$\begin{aligned}
\sum_{i=1}^{n}e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\omega(X_0,\dots,X_k) &= \sum_{i=1}^{n}\sum_{\sigma\in S(1,k)}\operatorname{sgn}\sigma\, e^i(X_{\sigma(1)})\cdot\nabla^{\Lambda^kT^*M}_{e_i}\omega(X_{\sigma(0)},\dots,X_{\sigma(k)})\\
&= \sum_{i=1}^{n}\sum_{j=0}^{k}(-1)^j a_{j_i}\cdot\nabla^{\Lambda^kT^*M}_{e_i}\omega(X_0,\dots,\widehat{X_j},\dots,X_k).
\end{aligned}$$

By Proposition 4.1.4-2., the linearity of $\nabla^{\Lambda^kT^*M}$ and Proposition 4.2.1 we get

$$\begin{aligned}
\sum_{i=1}^{n}e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\omega(X_0,\dots,X_k) &= \sum_{i=1}^{n}\sum_{j=0}^{k}(-1)^j\nabla^{\Lambda^kT^*M}_{a_{j_i}e_i}\omega(X_0,\dots,\widehat{X_j},\dots,X_k)\\
&= \sum_{j=0}^{n}(-1)^j\nabla^{\Lambda^kT^*M}_{\sum_{i=1}^{n}a_{j_i}e_i}\omega(X_0,\dots,\widehat{X_j}\dots,X_k)\\
&= \sum_{j=0}^{n}(-1)^j\nabla^{\Lambda^kT^*M}_{X_j}\omega(X_0,\dots,\widehat{X_j},\dots,X_k)\\
&= d\omega(X_0,\dots,X_k).
\end{aligned}$$

□

Theorem 4.2.3. *Let M be an oriented Riemannian n–manifold without boundary, $e_1, \dots, e_n$ be an oriented orthonormal local frame of TM and $e^1, \dots, e^n$ be the dual oriented orthonormal local frame of T^*M. We have*

$$d^\star\omega = -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i}^{\Lambda^k T^*M}\omega, \quad \omega \in \Omega^k(M). \tag{4.21}$$

Proof. Let $\omega \in \Omega^k(M)$ we put

$$\tilde{d}^\star\omega := -\sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i}^{\Lambda^k T^*M}\omega.$$

First let us see that $\tilde{d}^\star$ does not depend on the choice of the local frame $e_1, \dots, e_n$.

Let $f_1, \dots, f_n$ be another oriented orthonormal local frame with dual orthonormal local frame $f^1, \dots, f^n$. Then

$$f_j = \sum_{k=1}^{n} a_j^k e_k, \quad f^j = \sum_{k=1}^{n} b_k^j e^k$$

for some coefficients $a_j^k, b_k^j \in C^\infty(M)$. Since the bases are orthonormal, the transition matrix is orthogonal, then $b_k^j = a_j^k$ and

$$\sum_{j=1}^{n} a_j^k a_j^l = \delta^{kl}. \tag{4.22}$$

Now, let $\omega \in \Omega^k(M)$, by Proposition 4.1.4-2., equality (4.22) and since $\lrcorner$ is a linear map we have

$$\begin{aligned} -\sum_{i=1}^{n} f_i \lrcorner \nabla_{f_i}^{\Lambda^k T^*M}\omega &= -\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} a_i^l e_l \lrcorner \nabla_{a_i^j e_j}^{\Lambda^k T^*M}\omega \\ &= -\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} a_i^l a_i^j e_l \lrcorner \nabla_{e_j}^{\Lambda^k T^*M}\omega \\ &= -\sum_{j=1}^{n}\sum_{l=1}^{n} \delta^{lj} e_l \lrcorner \nabla_{e_j}^{\Lambda^k T^*M}\omega \\ &= -\sum_{j=1}^{n} e_j \lrcorner \nabla_{e_j}^{\Lambda^k T^*M}\omega. \end{aligned}$$

Therefore, $\tilde{d}^\star$ does not depend on the choice of the local frame.

Since d is independent of the choice of charts, see the Corollary 1.3.4, then also $d^\star$.

We choose normal coordinates $(x_1, \dots, x_n)$ with center at $p \in M$ and we will consider the local frames $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ of TM and $\{dx_1, \dots, dx_n\}$ of T^*M.

We will show (4.21) at the point p for those bases, since $p \in M$ is arbitrary, it is sufficient.

At p, by Theorem 4.1.19 we have for all i, j

$$\nabla^{\mathrm{LC}}_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right)\Big|_p = 0$$

By equality (4.17) we get for all i, j

$$\nabla^{T^*M}_{\frac{\partial}{\partial x_i}}(dx_j)\Big|_p = 0 \tag{4.23}$$

Since $\tilde{d^\star}$ is a linear map, it suffices to verify the equality (4.21) on forms of type $f dx_{i_1} \wedge \ldots \wedge dx_{i_k} \in \Omega^k(M)$ with $f \in C^\infty(M)$, renumbering indices, it suffices to consider the form $f dx_1 \wedge \ldots \wedge dx_k$.

By equalities (4.2), (4.18), (4.23) and Lemma A.3.42 we obtain

$$\begin{aligned}
\tilde{d^\star}(f dx_1 \wedge \ldots \wedge dx_k)\Big|_p &= -\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i}\lrcorner \nabla^{\Lambda^k T^*M}_{\frac{\partial}{\partial x_i}}(f dx_1 \wedge \ldots \wedge dx_k)\right]\Bigg|_p \\
&= -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\lrcorner\left[\frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge dx_k + f\nabla^{\Lambda^k T^*M}_{\frac{\partial}{\partial x_i}}(dx_1 \wedge \ldots \wedge dx_k)\right]\Bigg|_p \\
&= -\left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}\lrcorner dx_1 \wedge \ldots \wedge dx_k\right)\Bigg|_p \\
&\quad + \left(\sum_{i=1}^{n}\sum_{j=1}^{n} f\frac{\partial}{\partial x_i}\lrcorner dx_1 \wedge \ldots \nabla^{T^*M}_{\frac{\partial}{\partial x_i}} dx_j \wedge \ldots \wedge dx_k\right)\Bigg|_p \\
&= -\left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i}\lrcorner dx_1 \wedge \ldots \wedge dx_k\right)\Bigg|_p \\
&= \left(\sum_{i=1}^{n}(-1)^i\frac{\partial f}{\partial x_i} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k\right)\Bigg|_p .
\end{aligned}$$

On the other hand, by Definition 3.3.1, equalities (4.20) and (4.2) we have

$$\begin{aligned}
d^\star(f dx_1 \wedge \ldots \wedge dx_k)\big|_p &= (-1)^{n(k+1)+1} \star d \star (f dx_1 \wedge \ldots \wedge dx_k)\Big|_p \\
&= (-1)^{n(k+1)+1} \star d(f dx_{k+1} \wedge \ldots \wedge dx_n)\Big|_p \\
&= (-1)^{n(k+1)+1} \star \left(\sum_{i=1}^{n} dx_i \wedge \nabla^{\Lambda^k T^*M}_{\frac{\partial}{\partial x_i}}(f dx_{k+1} \wedge \ldots \wedge dx_n)\right)\Bigg|_p \\
&= (-1)^{n(k+1)+1} \star \left(\sum_{i=1}^{n}\frac{\partial f}{\partial x_i} dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n\right. \\
&\quad \left. + \sum_{i=1}^{n} f dx_i \wedge \nabla^{\Lambda^k T^*M}_{\frac{\partial}{\partial x_i}}(dx_{k+1} \wedge \ldots \wedge dx_n)\right)\Bigg|_p
\end{aligned}$$

By equality (4.23) the second term is zero. Now, we consider

$$\omega = dx_i \wedge dx_{k+1} \wedge \ldots \wedge dx_n$$

with $i = 1, \ldots, k$. By equality (3.2) we have

$$\star\omega = (-1)^{(j-1)(n-k+1)+(n-k)(k-j)} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_k. \tag{4.24}$$

$$\begin{aligned} d^\star(f dx_1\wedge\ldots\wedge dx_k)\big|_p &= (-1)^{n(k+1)+1}\star\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\wedge dx_{k+1}\wedge\ldots\wedge dx_n\right)\Bigg|_p \\ &= (-1)^{n(k+1)+1+(j-1)(n-k+1)+(n-k)(k-j)}\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_1\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_k\Bigg|_p \\ &= \sum_{i=1}^n (-1)^i\frac{\partial f}{\partial x_i} dx_1\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_k\Bigg|_p . \end{aligned}$$

Therefore

$$d^\star\omega = -\sum_{i=1}^n \frac{\partial}{\partial x_i}\lrcorner\nabla^{\Lambda^k T^*M}_{\frac{\partial}{\partial x_i}}\omega,\quad \omega\in\Omega^k(M).$$

□

4.3 Clifford algebra and Clifford operators

We want to obtain an explicit expression of the Laplace-Beltrami operator (3.12), so we need to introduce the Clifford algebra and the Clifford operators.

The book where you can consult related topics is [22].

Definition 4.3.1. Let V be a finite dimensional real vector space with a non-degenerate symmetric bilinear form $q\colon V\times V\longrightarrow\mathbb{R}$. The *Clifford algebra* $\mathrm{Cl}(V,q)$ is the algebra over $\mathbb{R}$, with unit, generated by the elements of V, subject to the relation

$$ef+fe=-2q(e,f)\quad\text{with } e,f\in V. \tag{4.25}$$

Example 4.3.2. If $V=\mathbb{R}^n$ and $q=\langle\,,\rangle$ the standard inner product on $\mathbb{R}^n$, we denote the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n,\langle\,,\rangle)$ by Cl_n. Also, if we consider $\{e_1,\ldots,e_n\}$ the canonical basis for $\mathbb{R}^n$ then Cl_n is subject to the relations

$$(e_i)^2 = -1 \tag{4.26}$$
$$e_ie_j = -e_je_i\quad\text{with}\quad i\neq j \tag{4.27}$$

But if we consider the inner product as $q=-\langle\,,\rangle$, we have the Clifford algebra

$$\mathrm{Cl}_{n,-}=\mathrm{Cl}(\mathbb{R}^n,-\langle\,,\rangle)$$

that is subject to the relations

$$(e_i)^2 = 1 \tag{4.28}$$
$$e_ie_j = -e_je_i\quad\text{with}\quad i\neq j, \tag{4.29}$$

Examples 4.3.3.

$$\begin{aligned}
\mathrm{Cl}_0 &= \mathbb{R} \text{ with basis } 1\\
\mathrm{Cl}_1 &= \mathbb{C} \text{ with basis } 1, e_1\\
\mathrm{Cl}_2 &= \mathbb{H} \text{ with basis } 1, e_1, e_2, e_1e_2.
\end{aligned}$$

Let V be a real vector space of dimension n endowed with an inner product, note that since the vector spaces $\Lambda^\bullet V$ and $\mathrm{Cl}(V)$ have dimension 2^n, then there is a natural isomorphism of vector spaces

$$\gamma\colon \mathrm{Cl}(V) \longrightarrow \Lambda^\bullet V.$$

Note that $\mathrm{Cl}(V)$ contains V, let us denote by $c\colon V \hookrightarrow \mathrm{Cl}(V)$ the inclusion defined by $c(v) = v$. We consider the bilinear map $\cdot$ of the Clifford algebra as $\cdot\colon c(V) \times \mathrm{Cl}(V) \longrightarrow \mathrm{Cl}(V)$.

Analogously for $\mathbb{R}^{n*}$, we consider $\mathrm{Cl}^*_n = \mathrm{Cl}(\mathbb{R}^{n*}, \langle\, , \rangle_*)$. For the following proposition see the definition A.3.34.

Proposition 4.3.4. *With respect to the canonical isomorphism* $\mathrm{Cl}^*_n \cong \Lambda^\bullet\mathbb{R}^{n*}$*, Clifford multiplication between* $\mathbf{x} \in \mathbb{R}^n$ *and any* $v \in \mathrm{Cl}^*_n$ *can be written as*

$$\mathbf{x}\cdot v := \mathbf{x}^* \wedge v - \mathbf{x}\lrcorner v.$$

$$\begin{tikzcd}
V \times \mathrm{Cl}(V) \arrow[r, "\cdot"] \arrow[d, "\mathrm{id}_V \times \gamma"'] & \mathrm{Cl}(V) \arrow[d, "\gamma"] \\
V \times \Lambda^\bullet V^* \arrow[r] & \Lambda^\bullet V^*
\end{tikzcd}$$

Proof. Let $e_1, \ldots, e_n$ the canonical basis for $\mathbb{R}^n$ and $e^1, \ldots, e^n$ the dual basis for $\mathbb{R}^{n*}$, let $v = e^{i_1} \cdot \ldots \cdot e^{i_k}$ for $i_1 < \ldots < i_k$.

Set i, let $\mathbf{x} = te_i$ for some $t \in \mathbb{R}$. By equalities (4.26), (4.27) and the contraction (A.8), then we have the following cases:

1. If $i = 1$, we obtain

$$\mathbf{x}\cdot v = \begin{cases} -te^{i_2}\cdot\ldots\cdot e^{i_k} = \mathbf{x}^*\wedge v - \mathbf{x}\lrcorner v & \text{if } i_1 = 1\\ te_1\cdot e^{i_1}\cdot\ldots\cdot e^{i_k} = \mathbf{x}^*\wedge v - \mathbf{x}\lrcorner v & \text{if } i_1 > 1. \end{cases} \tag{4.30}$$

2. If $i = i_j$ for some $1 < j \leq k$, then

$$\mathbf{x}\cdot v = (-1)^j te^{i_1}\cdot\ldots\cdot \widehat{e^{i_j}}\cdot\ldots\cdot e^{i_k} = \mathbf{x}^*\wedge v - \mathbf{x}\lrcorner v.$$

3. If $i \neq i_r$ with $r = 1, \ldots, k$, then $i_j < i < i_{j+1}$. Hence

$$\mathbf{x}\cdot v = (-1)^j te^{i_1}\cdot\ldots\cdot e^{i_j}\cdot e^i\cdot e^{i_{j+1}}\cdot\ldots\cdot e^{i_k} = \mathbf{x}^*\wedge v - \mathbf{x}\lrcorner v.$$

$\square$

Analogously, we have the following result

Proposition 4.3.5. *With respect to the canonical isomorphism* $\mathrm{Cl}^*_{n,-} \cong \Lambda^\bullet\mathbb{R}^{n*}$*, Clifford multiplication between* $\mathbf{x} \in \mathbb{R}^n$ *and any* $v \in \mathrm{Cl}^*_{n,-}$ *can be written as*

$$\mathbf{x} \cdot v := \mathbf{x}^* \wedge v + \mathbf{x} \lrcorner v.$$

The proof is the same as the Proposition 4.3.4, but now we use the equalities (4.28) and (4.29).

4.3.1 The Clifford algebra of TM

Now, let (M, g) be a Riemannian n–manifold, we take the Clifford algebra of TM.

We consider $\pi' \colon \mathrm{Cl}(TM) \longrightarrow M$ the vector bundle whose fibers are the Clifford algebras $\mathrm{Cl}(T_pM)$ with respect to g, also we take $\pi \colon TM \longrightarrow M$, making abuse of notation, we have the inclusion vector bundle map $c \colon TM \hookrightarrow \mathrm{Cl}(TM)$. Fiber to fiber we have an isomorphism of vector bundles $h \colon \Lambda^\bullet T^*M \longrightarrow \mathrm{Cl}(TM)$.

Let $X \in \Gamma(TM)$, we have that $c(X) \in \Gamma(\mathrm{Cl}(TM))$.

On the other hand, for any $X \in \Gamma(TM)$, let $X^* \in \Gamma(T^*M)$ corresponds to X via g, that is, for any $Y \in \Gamma(TM)$,

$$X^*(Y) = g(X, Y).$$

Given the linear isomorphism $h' \colon \Omega^\bullet(M) \longrightarrow \mathrm{Cl}(TM)$, by Propositions 4.3.4 and 4.3.5 we have the diagram of vector spaces

$$\begin{tikzcd}
\Gamma(TM) \times \Omega^\bullet(M) \arrow[r] \arrow[d, "\mathrm{id}_{\Gamma(TM)} \times h'"'] & \Omega^\bullet(M) \arrow[d, "h'"] \\
\Gamma(TM) \times \mathrm{Cl}(TM) \arrow[r, "\cdot"'] & \mathrm{Cl}(TM)
\end{tikzcd}$$

Then $c(X)$ acting on $\Omega^\bullet(M)$, we define the Clifford multiplication between $X \in TM$ and any $\omega \in \mathrm{Cl}(V)$ as follows:

Definition 4.3.6. Let $X \in \Gamma(TM)$, we define the *Clifford operators* (*multiplications*)

$$\begin{aligned} c(X), \hat{c}(X) \colon \quad & \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M) \\ & \omega \mapsto c(X)\omega = X^* \wedge \omega - X \lrcorner \omega, \qquad (4.31) \\ & \omega \mapsto \hat{c}(X)\omega = X^* \wedge \omega + X \lrcorner \omega. \qquad (4.32) \end{aligned}$$

Where $\wedge$ and $\lrcorner$ are the wedge product and the contraction, respectively.

Since $\cdot$, $\mathrm{id}_{TM} \times h'$ are bilinear maps and h' linear isomorphism, then $c(X)$ is a linear map. Furthermore, since $\wedge$ is $C^\infty(M)$–bilinear and $\mathbb{R}$–bilinear and $\lrcorner$ is $C^\infty(M)$–linear and $\mathbb{R}$–linear, then $c(X)$ and $\hat{c}(X)$ are $C^\infty(M)$–linear and $\mathbb{R}$–linear for all $X \in \Gamma(TM)$.

Lemma 4.3.7. *The Clifford operators satisfy the following equalities: let* $X \in \Gamma(TM)$ *and* $\omega \in \Omega^k(M)$*,*

1. $(\hat{c}(X))^2\omega = |X|^2\omega$.
2. $(c(X))^2\omega = -|X|^2\omega$.

Proof. Let $\omega \in \Omega^k(M)$, $X \in \Gamma(TM)$. By equality (4.32) and Lemma 1.5.2-1 and -2 we have

$$\begin{aligned}
\hat{c}(X)\hat{c}(X)\omega &= \hat{c}(X)(X^* \wedge \omega + X \lrcorner \omega) \\
&= X^* \wedge (X^* \wedge \omega + X \lrcorner \omega) + X \lrcorner (X^* \wedge \omega + X \lrcorner \omega) \\
&= X^* \wedge X \lrcorner \omega + X \lrcorner (X^* \wedge \omega) \\
&= X \lrcorner X^* \wedge \omega \\
&= X^*(X)\omega \\
&= g(X,X)\omega \\
&= |X|^2\omega.
\end{aligned}$$

Analogously for the second equality, by equality (4.31) and Lemma 1.5.2-1 and -2 we have

$$\begin{aligned}
c(X)c(X)\omega &= c(X)(X^* \wedge w - X \lrcorner \omega) \\
&= X^* \wedge (X^* \wedge \omega - X \lrcorner \omega) - X \lrcorner (X^* \wedge \omega - X \lrcorner \omega) \\
&= -X^* \wedge e \lrcorner \omega - X \lrcorner (X^* \wedge \omega) \\
&= -X \lrcorner X^* \wedge \omega \\
&= -X^*(X)\omega \\
&= -g(X,X)\omega \\
&= -|X|^2\omega.
\end{aligned}$$

□

Lemma 4.3.8. *The Clifford operators $c(X)$ and $\hat{c}(X)$ satisfy the following relations: let $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^k(M)$ we have*

$$c(X)c(Y)\omega + c(X)c(Y)\omega = -2g(X,Y)\omega \tag{4.33}$$

$$\hat{c}(X)\hat{c}(Y)\omega + \hat{c}(Y)\hat{c}(X)\omega = 2g(X,Y)\omega \tag{4.34}$$

$$c(X)\hat{c}(Y)\omega + \hat{c}(Y)c(X)\omega = 0. \tag{4.35}$$

Proof. Note that, $c(X), \hat{c}(X) \in \mathrm{Cl}(TM)$, then the first two relations follows by equality (4.25) with g and $-g$.

While the third relation follows from the following: let $\omega \in \Omega^k(M)$

$$\begin{aligned}
c(X)\hat{c}(X)\omega &= c(X)(X^* \wedge \omega + X \lrcorner \omega) \\
&= X^* \wedge (X^* \wedge \omega + X \lrcorner \omega) - X \lrcorner (X^* \wedge \omega + X \lrcorner \omega) \\
&= X^* \wedge (X \lrcorner \omega) - X \lrcorner (X^* \wedge \omega).
\end{aligned}$$

$$\begin{aligned}
\hat{c}(X)c(X)\omega &= \hat{c}(X)(X^* \wedge \omega - X \lrcorner \omega) \\
&= X^* \wedge (X^* \wedge \omega - X \lrcorner \omega) + X \lrcorner (X^* \wedge \omega - X \lrcorner \omega) \\
&= -X^* \wedge (X \lrcorner \omega) + X \lrcorner (X^* \wedge \omega).
\end{aligned}$$

Then

$$c(X)\hat{c}(X)\omega + \hat{c}(X)c(X)\omega = 0.$$

□

For the next proof we will use the following remark.

Remark 4.3.9. Let (M, g) be a Riemannian n–manifold.

We consider the local frames $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ of TM and $\{dx_1, \dots, dx_n\}$ of T^*M .

We have $\{dx_{i_1} \wedge \dots \wedge dx_{i_k}\}_{(i_1,\dots,i_k)\in\mathcal{I}_{k,n}}$ the local frame of $\Lambda^k T^*M$, see notation 1.2. We choose an arbitrary element of the basis $dx_1 \wedge \dots \wedge dx_k$, (if is necessary, we reindex the multiindex).

1. If $j = 1, \dots, k$, by Lemma A.3.42 we have

$$\frac{\partial}{\partial x_j} \lrcorner (dx_1 \wedge \dots \wedge dx_k) = dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_k.$$

2. If $j = k+1, \dots, n$, by Lemma A.3.42 and since $dx_i(\frac{\partial}{\partial x_j}) = 0$ for $i = 1, \dots, k$, we have

$$\frac{\partial}{\partial x_j} \lrcorner (dx_1 \wedge \dots \wedge dx_k) = 0$$

Proposition 4.3.10. *Let (M, g) be a Riemannian manifold. Every connection $\nabla^{\Lambda^\bullet T^*M}$ on $\Lambda^\bullet T^*M$ and ∇^{TM} on TM satisfy the following formulas: for all $X, Y \in \Gamma(TM)$ and $\omega \in \Omega^k(M)$.*

$$\nabla_X^{\Lambda^\bullet T^*M}(\hat{c}(Y)\omega) = \hat{c}(\nabla_X^{TM} Y)\omega + \hat{c}(Y)\nabla_X^{\Lambda^k T^*M}\omega \tag{4.36}$$

$$\nabla_X^{\Lambda^\bullet T^*M}(c(Y)\omega) = c(\nabla_X^{TM} Y)\omega + c(Y)\nabla_X^{\Lambda^k T^*M}\omega, \tag{4.37}$$

Proof. Since $c(Y), \nabla_X^{\Lambda^\bullet T^*M}, \nabla_X^{TM}$ are linear maps it is sufficient do the proof in basic elements.

Let $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \in \Gamma(TM)$ and $dx_1 \wedge \dots \wedge dx_k \in \Omega^k(M)$.

For the proof of the expression (4.36), we have the cases $j = 1, \dots, k$ and $j = k+1, \dots, n$.

1. If $j = 1, \dots, k$, by equality (4.32) and Remark 4.3.9 we have:

$$\hat{c}\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_k = \frac{\partial}{\partial x_j} \lrcorner dx_1 \wedge \dots \wedge dx_k = dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_k.$$

By equalities (4.18) and (4.17) we obtain:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_i}}^{\Lambda^{k-1}T^*M}\left(\hat{c}\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_k\right) &= \nabla_{\frac{\partial}{\partial x_i}}^{\Lambda^{k-1}T^*M}(dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_k) \\ &= \sum_{\substack{r=1 \\ r\neq j}}^{k} dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge \nabla_{\frac{\partial}{\partial x_i}}^{T^*M} dx_r \wedge \dots \wedge dx_k \\ &= -\sum_{s=1}^{n} \sum_{\substack{r=1 \\ r\neq j}}^{k} \Gamma_{is}^{r} dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge \underbrace{dx_s}_{r} \wedge \dots \wedge dx_k \end{aligned}$$

Since $j = 1, \ldots, k$, hence $s = j, r, k+1, \ldots, n$. We have

$$\nabla^{\Lambda^{k-1}T^*M}_{\frac{\partial}{\partial x_i}} \left(\hat{c}\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \ldots \wedge dx_k \right) =$$

$$-\sum_{\substack{r=1\\ r\neq j}}^{k} \Gamma^r_{ij} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_j}_{r} \wedge \ldots \wedge dx_k \tag{4.38}$$

$$-\sum_{\substack{r=1\\ r\neq j}}^{k} \Gamma^r_{ir} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_r}_{r} \wedge \ldots \wedge dx_k \tag{4.39}$$

$$-\sum_{s=k+1}^{n} \sum_{\substack{r=1\\ r\neq j}}^{k} \Gamma^r_{is} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_s}_{r} \wedge \ldots \wedge dx_k \tag{4.40}$$

On the other hand, since that $\hat{c}(Y)$ is a linear map, by equality (4.8) and definition (4.32) we get

$$\begin{aligned}
\hat{c}\left(\nabla^{TM}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right) dx_1 \wedge \ldots \wedge dx_k &= \hat{c}\left(\sum_{s=1}^{n} \Gamma^s_{ij} \frac{\partial}{\partial x_s}\right) dx_1 \wedge \ldots \wedge dx_k \\
&= \sum_{s=1}^{n} \Gamma^s_{ij} \left(dx_s \wedge dx_1 \wedge \ldots \wedge dx_k + \frac{\partial}{\partial x_s} \lrcorner dx_1 \wedge \ldots \wedge dx_k \right) \\
&= \sum_{s=k+1}^{n} \Gamma^s_{ij} dx_s \wedge dx_1 \wedge \ldots \wedge dx_k \\
&\quad + \sum_{s=1}^{k} \Gamma^s_{ij} dx_1 \wedge \ldots \wedge \widehat{dx_s} \wedge \ldots \wedge dx_k
\end{aligned}$$

By equalities (4.18) and (4.17) we have:

$$\begin{aligned}
\hat{c}\left(\frac{\partial}{\partial x_j}\right) \nabla^{\Lambda^k T^*M}_{\frac{\partial}{\partial x_i}} dx_1 \wedge \ldots \wedge dx_k &= \hat{c}\left(\frac{\partial}{\partial x_j}\right) \left[\sum_{l=1}^{k} dx_1 \wedge \ldots \wedge \nabla^{T^*M}_{\frac{\partial}{\partial x_i}} dx_l \wedge \ldots \wedge dx_k \right] \\
&= \hat{c}\left(\frac{\partial}{\partial x_j}\right) \left[-\sum_{s=1}^{n} \sum_{l=1}^{k} \Gamma^l_{is} dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge dx_k \right] \\
&= -\sum_{s=1}^{n} \sum_{l=1}^{k} \Gamma^l_{is} \hat{c}\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge dx_k \\
&= -\sum_{l=1}^{k} \Gamma^l_{il} \hat{c}\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \ldots \wedge \underbrace{dx_l}_{l} \wedge \ldots \wedge dx_k \\
&\quad -\sum_{s=k+1}^{n} \sum_{l=1}^{k} \Gamma^l_{is} \hat{c}\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge dx_k
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\sum_{l=1}^{k} \Gamma_{il}^{l} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_l}_{l} \wedge \ldots \wedge dx_k \\
& -\sum_{s=k+1}^{n} \sum_{l=1}^{k} \Gamma_{is}^{l} \left(dx_j \wedge dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge dx_k \right. \\
& \left. + \frac{\partial}{\partial x_j} \lrcorner dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge dx_k \right) \\
= \quad & -\sum_{l=1}^{k} \Gamma_{il}^{l} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_l}_{l} \wedge \ldots \wedge dx_k \\
& -\sum_{s=k+1}^{n} \sum_{l=1}^{k} \Gamma_{is}^{l} dx_j \wedge dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge dx_k \\
& -\sum_{s=k+1}^{n} \sum_{l=1}^{k} \Gamma_{is}^{l} dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k \\
= \quad & -\sum_{l=1}^{k} \Gamma_{il}^{l} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_l}_{l} \wedge \ldots \wedge dx_k \\
& -\sum_{s=k+1}^{n} \Gamma_{is}^{j} dx_j \wedge dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{j} \wedge \ldots \wedge dx_k \quad \text{when } j = l \\
& -\sum_{s=k+1}^{n} \sum_{\substack{l=1 \\ l \neq j}}^{k} \Gamma_{is}^{l} dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{l} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_k
\end{aligned}
$$

Adding the two terms we have

$$
\hat{c}\left(\nabla^{TM}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right) dx_1 \wedge \ldots \wedge dx_k + \hat{c}\left(\frac{\partial}{\partial x_j} \right) \nabla^{\Lambda^k T^* M}_{\frac{\partial}{\partial x_i}} dx_1 \wedge \ldots \wedge dx_k =
$$

$$
\sum_{s=k+1}^{n} \Gamma_{ij}^{s} dx_s \wedge dx_1 \wedge \ldots \wedge dx_k \tag{4.41}
$$

$$
+\sum_{s=1}^{k} \Gamma_{ij}^{s} dx_1 \wedge \ldots \wedge \widehat{dx_s} \wedge \ldots \wedge dx_k \tag{4.42}
$$

$$
-\sum_{l=1}^{k} \Gamma_{il}^{l} dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge \underbrace{dx_l}_{l} \wedge \ldots \wedge dx_k \tag{4.43}
$$

$$
-\sum_{s=k+1}^{n} \Gamma_{is}^{j} dx_j \wedge dx_1 \wedge \ldots \wedge \underbrace{dx_s}_{j} \wedge \ldots \wedge dx_k \quad \text{when } j = l \tag{4.44}
$$

$$-\sum_{s=k+1}^{n}\sum_{\substack{l=1\\ l\neq j}}^{k}\Gamma_{is}^{l}dx_1\wedge\ldots\wedge\underbrace{dx_s}_{l}\wedge\ldots\wedge\widehat{dx_j}\wedge\ldots\wedge dx_k \tag{4.45}$$

We have that expressions (4.39) and (4.43) are equal, the same with (4.40) and (4.45). We permute dx_s and dx_j, later by Remark (4.1.22) the Christoffel symbols $\Gamma_{is}^{j}=-\Gamma_{ij}^{s}$, hence the expressions (4.38) and (4.42) are also equal. From term (4.44) we get:

$$\begin{aligned}
-\sum_{s=k+1}^{n}\Gamma_{is}^{j}dx_j\wedge dx_1\wedge\ldots\wedge\underbrace{dx_s}_{j}\wedge\ldots\wedge dx_k &=\\
-\sum_{s=k+1}^{n}\Gamma_{is}^{j}(-1)^{j-1+j}dx_s\wedge dx_1\wedge\ldots\wedge\underbrace{dx_j}_{j}\wedge\ldots\wedge dx_k&\\
=-(-1)\sum_{s=k+1}^{n}\Gamma_{is}^{j}dx_s\wedge dx_1\wedge\ldots\wedge\underbrace{dx_j}_{j}\wedge\ldots\wedge dx_k&\\
=\sum_{s=k+1}^{n}\Gamma_{is}^{j}dx_s\wedge dx_1\wedge\ldots\wedge\underbrace{dx_j}_{j}\wedge\ldots\wedge dx_k&\\
=-\sum_{s=k+1}^{n}\Gamma_{ij}^{s}dx_s\wedge dx_1\wedge\ldots\wedge\underbrace{dx_j}_{j}\wedge\ldots\wedge dx_k&
\end{aligned}$$

The last expression is canceled with the expression (4.41).

Therefore, if $j=1,\ldots,k$

$$\begin{aligned}
\nabla^{\Lambda^{k-1}T^*M}_{\frac{\partial}{\partial x_i}}\left(\hat{c}\left(\frac{\partial}{\partial x_j}\right)dx_1\wedge\ldots\wedge dx_k\right) &= \hat{c}\left(\nabla^{TM}_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right)dx_1\wedge\ldots\wedge dx_k\\
&+\hat{c}\left(\frac{\partial}{\partial x_j}\right)\nabla^{\Lambda^kT^*M}_{\frac{\partial}{\partial x_i}}dx_1\wedge\ldots\wedge dx_k.
\end{aligned}$$

2. The case $j=k+1,\ldots,n$ is analogous.

□

A connection on $\Lambda^\bullet T^*M$ that satisfies conditions (4.37) and (4.36) is called a *Clifford connection* on $\Lambda^\bullet T^*M$.

Let M be an oriented Riemannian n–manifold without boundary, $e_1,\ldots,e_n$ be an oriented orthonormal local frame of TM, let $e^1,\ldots,e^n$ be the corresponding dual local frame of T^*M with respect to g.

Let $\omega\in\Omega^k(M)$. Since $D\omega:=d\omega+d^\star\omega$, by equalities (4.20), (4.21), (4.31) we obtain

$$\begin{aligned}
D\omega &= \sum_{i=1}^{n}e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\omega-\sum_{i=1}^{n}e_i\lrcorner\nabla^{\Lambda^kT^*M}_{e_i}\omega\\
D\omega &= \sum_{i=1}^{n}c(e_i)\nabla^{\Lambda^kT^*M}_{e_i}\omega. \qquad (4.46)
\end{aligned}$$

Chapter 5

Witten Deformation

In this chapter we will deform the De Rham complex of a differentiable manifold (see the definitions 1.4.1 and 1.4.3) and we will define the deformed Laplace-Beltrami operator.

We will also see that the deformed De Rham complex has the same Betti numbers as the usual one.

Let M be an n–dimensional differentiable manifold and $f\colon M \longrightarrow \mathbb{R}$ be a differentiable function on M.

We define the deformed exterior derivative operator by conjugation, as follows: for any $T \in \mathbb{R}$, set

$$d_{Tf}\omega := \exp(-Tf)d\exp(Tf)\omega, \quad \omega \in \Omega^k(M). \tag{5.1}$$

Since the algebra of differentiable forms is a module over $C^\infty(M)$ and multiplication by a function does not affect the grading, the deformation defined above can still be seen as an operator $d_{Tf}\colon \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, for any $0 \leq k \leq n$.

Let $\omega \in \Omega^k(M)$, by Theorem 1.3.2-2. we see that

$$\begin{aligned} d^2_{Tf}\omega &= (\exp(-Tf)d\exp(Tf))(\exp(-Tf)d\exp(Tf))\omega \\ &= \exp(-Tf)d^2\exp(Tf)\omega \\ &= 0. \end{aligned}$$

Therefore, we get a deformation of the De Rham complex $(\Omega^\bullet(M), d)$, given by the cochain complex $(\Omega^\bullet(M), d_{Tf})$ defined by

$$(\,\Omega^\bullet(M),\, d_{Tf}\,) : 0 \longrightarrow C^\infty(\,M\,) \overset{d_{Tf,0}}{\longrightarrow} \Omega^1(M) \overset{d_{Tf,1}}{\longrightarrow} \ \ldots \ \overset{d_{Tf,\,n-1}}{\longrightarrow} \Omega^n(M) \longrightarrow 0\,.$$

Associated to this complex, for each $k = 0, \ldots, n$ we have the k–*the cohomology space*

$$\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) = \frac{\mathrm{Ker}\, d_{Tf,\,k}}{\mathrm{Im}\, d_{Tf,\,k-1}}.$$

The *total cohomology* is given by

$$\mathrm{H}^\bullet_{Tf,\,\mathrm{DR}}(M) = \bigoplus_{k=0}^{n} \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M).$$

The k–th cohomology spaces $\mathrm{H}^k_{\mathrm{DR}}(M)$ and $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ are the same viewed as vector spaces.

Proposition 5.0.1. *Let* M *be a differentiable manifold of dimension* n*,* $f\colon M \longrightarrow \mathbb{R}$ *differentiable function and* $T \in \mathbb{R}$*. For any integer* k *such that* $0 \leq k \leq n$*, the cohomologies* $\mathrm{H}^k_{\mathrm{DR}}(M)$ *and* $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ *are isomorphic. Therefore,*

$$\dim \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) = \beta_k(M).$$

Proof. We define the linear map

$$\begin{aligned} \varphi\colon \quad \Omega^k(M) &\longrightarrow \Omega^k(M), \\ \omega &\mapsto \exp(-Tf)\omega. \end{aligned}$$

We will see that φ induces a linear map $\mathrm{H}^k_{\mathrm{DR}}(M) \longrightarrow \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$.

Take $\omega \in \Omega^k(M)$ be a closed form, that is, $d\omega = 0$, we have

$$\begin{aligned} d_{Tf}(\exp(-Tf)\omega) &= \exp(-Tf)d\exp(Tf)(\exp(-Tf)\omega) \\ &= \exp(-Tf)d\omega \\ &= 0. \end{aligned}$$

Then, under φ $\operatorname{Ker} d\,|_{\Omega^k(M)}$ is mapped into $\operatorname{Ker} d_{Tf}\,|_{\Omega^k(M)}$.

On the other hand, let $\eta \in \Omega^{k-1}(M)$, we get

$$\begin{aligned} \varphi(d\eta) &= \exp(-Tf)d\eta \\ &= \exp(-Tf)d(\exp(Tf)\exp(-Tf))\eta \\ &= (\exp(-Tf)d\exp(Tf))\exp(-Tf)\eta \\ &= d_{Tf}\exp(-Tf)\eta. \end{aligned}$$

That is, φ maps $\operatorname{Im} d\,|_{\Omega^{k-1}(M)}$ into $\operatorname{Im} d_{Tf}\,|_{\Omega^{k-1}(M)}$. Therefore, φ induces a linear map in the quotient

$$\Phi\colon \mathrm{H}^k_{\mathrm{DR}}(M) \longrightarrow \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M).$$

Now, define the linear map $\psi\colon \Omega^k(M) \longrightarrow \Omega^k(M)$ by $\psi(\omega) = \exp(Tf)\omega$.

By doing a completely analogous reasoning we can see that the map ψ induces a linear map in the quotient

$$\Psi\colon \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) \longrightarrow \mathrm{H}^k_{\mathrm{DR}}(M).$$

Note that Φ and Ψ are the inverse of each other, then $\mathrm{H}^k_{\mathrm{DR}}(M)$ and $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ are isomorphic, in particular, have the same dimension. □

We can develop the Hodge Theory associated to the complex $(\Omega^\bullet(M), d_{Tf})$ in the same way as in the De Rham complex.

Let (M, g) be an oriented Riemannian n–manifold with boundary and $\langle\ ,\ \rangle$ the inner product on $\Omega^k(M)$, (see (3.6)).

Let $T \in \mathbb{R}$, for any $\omega \in \Omega^{k-1}(M), \eta \in \Omega^k(M)$. By Proposition 3.3.5 we get

$$\begin{aligned} \langle d_{Tf}\omega, \eta\rangle &= \langle \exp(-Tf)d\exp(Tf)\omega, \eta\rangle \\ &= \langle d\exp(Tf)\omega, \exp(-Tf)\eta\rangle \\ &= \langle \exp(Tf)\omega, d^\star\exp(-Tf)\eta\rangle \\ &= \langle \omega, \exp(Tf)d^\star\exp(-Tf)\eta\rangle. \end{aligned}$$

Thus, let $\omega \in \Omega^k(M)$, we define

$$d^\star_{Tf}\omega := \exp(Tf) d^\star \exp(-Tf)\omega. \tag{5.2}$$

In other words, the adjoint of d_{Tf} is $d^\star_{Tf}$.

For any $T \in \mathbb{R}$, let $\omega \in \Omega^k(M)$, set

$$\mathrm{D}_{Tf}\omega = d_{Tf}\omega + d^\star_{Tf}\omega. \tag{5.3}$$

Similarly to the Lemma 3.3.7, we have:

Lemma 5.0.2. *D_{Tf} is a self-adjoint differentiable operator over $\Omega^\bullet(M)$.*

The proof is analogous to that of the Lemma 3.3.7.

So the corresponding Laplace–Beltrami operator for $(\Omega^\bullet(M), d_{Tf})$ is

$$\square_{Tf}\omega := \mathrm{D}^2_{Tf}\omega = d_{Tf}d^\star_{Tf}\omega + d^\star_{Tf}d_{Tf}\omega, \quad \omega \in \Omega^\bullet(M). \tag{5.4}$$

By Definitions (5.1) and (5.2) one sees that $\square_{Tf}$ preserves each $\Omega^k(M)$, for $0 \le k \le n$, that is, $\square_{Tf,\,k}\colon \Omega^k(M) \longrightarrow \Omega^k(M)$, note that by restricting ourselves to the space of differentiable k–forms, we add a subscript in the notation.

Remark 5.0.3. Let $\omega \in \Omega^k(M)$ be an eigenform of D_{Tf} with eigenvalue λ, by Definition (5.4) then

$$\square_{Tf,\,k}\omega = \mathrm{D}^2_{Tf}\omega = \lambda(\mathrm{D}_{Tf}\omega) = \lambda^2\omega,$$

therefore, the deformed Laplace-Beltrami operator $\square_{Tf,\,k}$ on $\Omega^k(M)$ is a nonnegative operator for all $0 \le k \le n$.

Lemma 5.0.4. *Let (M, g) be an oriented Riemannian n–manifold without boundary, the operators d_{Tf} and $d^\star_{Tf}$ satisfy the following equalities: let $\omega \in \Omega^k(M)$,*

$$d_{Tf}\square_{Tf,\,k}\omega = \square_{Tf,\,k+1}d_{Tf}\omega, \tag{5.5}$$
$$d^\star_{Tf}\square_{Tf,\,k}\omega = \square_{Tf,\,k-1}d^\star_{Tf}\omega. \tag{5.6}$$

Proof. Let $\omega \in \Omega^k(M)$, by equalities (5.4), (5.1) (5.2) one can see that

$$\begin{aligned}
d_{Tf}\square_{Tf,\,k}\omega &= d_{Tf}\mathrm{D}^2_{Tf}\omega \\
&= d_{Tf}(d_{Tf}d^\star_{Tf}\omega + d^\star_{Tf}d_{Tf}\omega) \\
&= d_{Tf}d^\star_{Tf}d_{Tf}\omega. \\
\square_{Tf,\,k+1}d_{Tf}\omega &= \mathrm{D}^2_{Tf}d_{Tf}\omega \\
&= (d_{Tf}d^\star_{Tf} + d^\star_{Tf}d_{Tf})d_{Tf}\omega \\
&= d_{Tf}d^\star_{Tf}d_{Tf}\omega.
\end{aligned}$$

So

$$d_{Tf}\square_{Tf,\,k}\omega = \square_{Tf,\,k+1}d_{Tf}\omega, \quad \omega \in \Omega^k(M).$$

Similarly, we have

$$d^{\star}_{Tf}\Box_{Tf,\,k}\omega = \Box_{Tf,\,k-1}d^{\star}_{Tf}\omega, \quad \omega \in \Omega^k(M).$$

$\Box$

Moreover, we can also establish Hodge Theorem, see 3.5.3, for the complex $(\Omega^{\bullet}(M),\, d_{Tf})$, we have

$$\operatorname{Ker}\Box_{Tf,\,k} \cong \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M). \tag{5.7}$$

This implies that for any integer k such that $0 \leq k \leq n$,

$$\dim\left(\operatorname{Ker}\Box_{Tf,\,k}\right) = \dim\left(\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)\right).$$

By Proposition 5.0.1

$$\dim\left(\operatorname{Ker}\Box_{Tf,\,k}\right) = \beta_k(M). \tag{5.8}$$

Thus we reduced the problem of estimating the Betti numbers to analyzing the behavior of the kernel of $\Box_{Tf,\,k}$.

Chapter 6

Local behavior of $\square_{Tf,\,\mathrm{n}_f(p)}$

In this chapter we will focus on the local behavior of the deformed Laplace–Beltrami operator.

Proposition 6.0.1. *Let (M,g) be an oriented Riemannian n–manifold without boundary, $T\in\mathbb{R}$, $f\colon M\longrightarrow\mathbb{R}$ be a differentiable function. Let $\{e_1,\dots,e_n\}$ be a local frame of TM and $\{e^1,\dots,e^n\}$ be the dual local frame of T^*M. We have the following expressions on $\Omega^k(M)$: let $\omega\in\Omega^k(M)$*

$$\begin{aligned} d_{Tf}\omega &= d\omega + Tdf\wedge\omega. && (6.1)\\ d^\star_{Tf}\omega &= d^\star\omega + T\mathrm{grad}f\lrcorner\omega. && (6.2)\\ \mathrm{D}_{Tf}\omega &= \mathrm{D}\omega + T\hat{c}(\mathrm{grad}f)\omega. && (6.3)\\ \square_{Tf,\,k}\omega &= \square_k\omega + T\sum_{i=1}^n c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega + T^2|\mathrm{grad}f|^2\omega. && (6.4)\end{aligned}$$

Where $c(X),\hat{c}(X)$ are the operators (4.31) and (4.32).

Proof. Let $\omega\in\Omega^k(M)$, by definition (5.1), equality (4.20) and Leibniz rule (4.2) we see that

$$\begin{aligned} d_{Tf}\omega &= (\exp(-Tf)d\exp(Tf))\omega\\ &= \sum_{i=1}^n\Big(\exp(-Tf)\Big(e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\Big)\exp(Tf)\Big)\omega\\ &= \sum_{i=1}^n\exp(-Tf)e^i\wedge(\nabla^{\Lambda^kT^*M}_{e_i}\exp(Tf)\omega)\\ &= \sum_{i=1}^n\exp(-Tf)\ e^i\wedge\Big(T\exp(Tf)df(e_i)\omega+\exp(Tf)\nabla^{\Lambda^kT^*M}_{e_i}\omega\Big)\\ &= \sum_{i=1}^n T\exp(-Tf)\exp(Tf)e^i\wedge(df(e_i))\omega+\sum_{i=1}^n\exp(-Tf)\exp(Tf)e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\omega\\ &= T\sum_{i=1}^n(df(e_i))e^i\wedge\omega+\sum_{i=1}^n e^i\wedge\nabla^{\Lambda^kT^*M}_{e_i}\omega\\ &= Tdf\wedge\omega+d\omega.\end{aligned}$$

Then

$$d_{Tf}\omega = d\omega + Tdf \wedge \omega. \tag{6.5}$$

Similarly, let $\omega \in \Omega^k(M)$, we have that $\exp(-Tf) \in C^\infty(M)$, by Leibniz rule (4.2) notice that

$$\begin{aligned}\nabla_{e_i}^{\Lambda^k T^*M} \exp(-Tf)\omega &= d\exp(-Tf)(e_i)\omega + \exp(-Tf)\nabla_{e_i}^{\Lambda^k T^*M}\omega \\ &= -T\exp(-Tf)(df(e_i))\omega + \exp(-Tf)\nabla_{e_i}^{\Lambda^k T^*M}\omega.\end{aligned}$$

By equality (4.21), we have

$$\begin{aligned}d^\star_{Tf}\omega &= \left(\exp(Tf)\left(-\sum_{i=1}^n e_i\lrcorner\nabla_{e_i}^{\Lambda^k T^*M}\right)\exp(-Tf)\right)\omega \\ &= -\sum_{i=1}^n \exp(Tf)e_i\lrcorner\nabla_{e_i}^{\Lambda^k T^*M}(\exp(-Tf)\omega) \\ &= -\sum_{i=1}^n \exp(Tf)e_i\lrcorner\left(-T\exp(-Tf)df(e_i)\omega + \exp(-Tf)\nabla_{e_i}^{\Lambda^k T^*M}\omega\right) \\ &= T\sum_{i=1}^n \exp(Tf)\exp(-Tf)e_i\lrcorner(df(e_i))\omega - \sum_{i=1}^n \exp(Tf)\exp(-Tf)e_i\lrcorner\nabla_{e_i}^{\Lambda^k T^*M}\omega \\ &= T\sum_{i=1}^n e_i\lrcorner(df(e_i))\omega - \sum_{i=1}^n e_i\lrcorner\nabla_{e_i}^{\Lambda^k T^*M}\omega \\ &= T\mathrm{grad}f\lrcorner\omega + d^\star\omega.\end{aligned}$$

So,

$$d^\star_{Tf}\omega = d^\star\omega + T\mathrm{grad}f\lrcorner\omega. \tag{6.6}$$

Let $\omega \in \Omega^k(M)$, substituting in the equality (5.3) the expressions (6.5) and (6.6), by definitions (4.32) and (3.11), then

$$\begin{aligned}\mathrm{D}_{Tf}\omega &= d\omega + Tdf\wedge\omega + d^\star\omega + T\mathrm{grad}f\lrcorner\omega \\ &= d\omega + d^\star\omega + T(df\wedge\omega + \mathrm{grad}f\lrcorner\omega) \\ &= \mathrm{D}\omega + T\hat{c}(\mathrm{grad}f)\omega.\end{aligned}$$

We want to write $\Box_{Tf,\,k}$, see the definition (5.4). One gets

$$\Box_{Tf,\,k}\omega = \mathrm{D}^2\omega + T(D\hat{c}(\mathrm{grad}f)\omega + \hat{c}(\mathrm{grad}f)\mathrm{D}\omega) + T^2(\hat{c}(\mathrm{grad}f))^2\omega.$$

By expression (4.46) and since $\hat{c}(\mathrm{grad}f)\omega \in \Omega^{k+1}(M)\oplus\Omega^{k-1}(M)$, we take $\nabla^{\Lambda^\bullet T^*M}$ we obtain

$$\begin{aligned}T(\mathrm{D}\hat{c}(\mathrm{grad}f) + \hat{c}(\mathrm{grad}f)\mathrm{D})\omega &= T\left(\sum_{i=1}^n c(e_i)\nabla_{e_i}^{\Lambda^\bullet T^*M}\right)\hat{c}(\mathrm{grad}f)\omega + T\hat{c}(\mathrm{grad}f)\left(\sum_{i=1}^n c(e_i)\nabla_{e_i}^{\Lambda^k T^*M}\right)\omega \\ &= T\sum_{i=1}^n c(e_i)\nabla_{e_i}^{\Lambda^\bullet T^*M}(\hat{c}(\mathrm{grad}f)\omega) + T\sum_{i=1}^n \hat{c}(\mathrm{grad}f)c(e_i)\nabla_{e_i}^{\Lambda^k T^*M}\omega.\end{aligned}$$

By Clifford connection (4.36) and property (4.35) we get:

$$\begin{aligned}
T(\mathrm{D}\hat{c}(\mathrm{grad}f)+\hat{c}(\mathrm{grad}f)\mathrm{D})\omega &= T\sum_{i=1}^{n}c(e_i)(\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega+T\hat{c}(\mathrm{grad}f)\nabla^{\Lambda^kT^*M}_{e_i}\omega)\\
&\quad +T\hat{c}(\mathrm{grad}f)\sum_{i=1}^{n}c(e_i)\nabla^{\Lambda^kT^*M}_{e_i}\omega\\
&= T\sum_{i=1}^{n}c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega+T\sum_{i=1}^{n}c(e_i)\hat{c}(\mathrm{grad}f)\nabla^{\Lambda^kT^*M}_{e_i}\omega\\
&\quad +T\sum_{i=1}^{n}\hat{c}(\mathrm{grad}f)c(e_i)\nabla^{\Lambda^kT^*M}_{e_i}\omega\\
&= T\sum_{i=1}^{n}c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega\\
&\quad +T\sum_{i=1}^{n}(c(e_i)\hat{c}(\mathrm{grad}f)+\hat{c}(\mathrm{grad}f)c(e_i))\nabla^{\Lambda^kT^*M}_{e_i}\omega\\
&= T\sum_{i=1}^{n}c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega.
\end{aligned}$$

By Lemma 4.3.7-1 we get $(\hat{c}(\mathrm{grad}f))^2=|\mathrm{grad}f|^2$ and Definition (3.12), therefore we rewrite

$$\Box_{Tf,k}\omega=\Box_k\omega+T\sum_{i=1}^{n}c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega+T^2|\mathrm{grad}f|^2\omega.$$

□

Remark 6.0.2. Note that to prove the equality (6.1) we only need that (M,g) is a Riemannian n–manifold.

Theorem 6.0.3. *Let (M,g) be an oriented Riemannian n–manifold without boundary, $T\in\mathbb{R}$, $f\colon M\longrightarrow\mathbb{R}$ be a Morse function and p be a critical point of f. Then there is a chart $\varphi\colon U\longrightarrow V\subset\mathbb{R}^n$ around p such that the deformed Laplace-Beltrami operator $\Box_{Tf,\mathrm{n}_f(p)}$ on $\Omega^{\mathrm{n}_f(p)}(\mathbb{R}^n)$ is given by*

$$\Box_{Tf,\mathrm{n}_f(p)}\omega=-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\omega-nT\omega+T^2|\mathbf{x}|^2\omega+2T\left[\sum_{i=1}^{\mathrm{n}_f(p)}e_i\lrcorner(dx_i\wedge\omega)+\sum_{i=\mathrm{n}_f(p)+1}^{n}dx_i\wedge(e_i\lrcorner\omega)\right],\tag{6.7}$$

where $\mathbf{x}=(x_1,\dots,x_n)\in V$.

Proof. Let f be a Morse function and $p\in M$ a critical point of f, by Corollary 2.0.7, there is an open neighbourhood U of p and a chart $\varphi\colon U\longrightarrow V\subset\mathbb{R}^n$ around p such that $\varphi(p)=0$ and for all $\mathbf{x}\in V$ the equality (2.9) is satisfied, that is,

$$(f\circ\varphi^{-1})(\mathbf{x})=f(p)-\frac{1}{2}x_1^2-\ldots-\frac{1}{2}x^2_{\mathrm{n}_f(p)}+\frac{1}{2}x^2_{\mathrm{n}_f(p)+1}+\ldots+\frac{1}{2}x_n^2.$$

Let $\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\}$, be the oriented local frame of TU.
We will simply denote $e_i=\frac{\partial}{\partial x_i}$ for all $i=1,\dots,n$.
We will study the equality (6.4) in parts.
Let $V\subset\mathbb{R}^n$ and $\omega\in\Omega^{\mathrm{n}_f(p)}(V)$, by Proposition 3.3.11 the first part we already have it,

$$\Box_{\mathrm{n}_f(p)}\omega=-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\omega. \tag{6.8}$$

While the last term, by equality (2.9), we have

$$\mathrm{grad}f=(-x_1,\dots,-x_{\mathrm{n}_f(p)},x_{\mathrm{n}_f(p)+1},\dots,x_n). \tag{6.9}$$

Then

$$|\mathrm{grad}f|^2=(-x_1)^2+\ldots+(-x_{\mathrm{n}_f(p)})^2+x_{\mathrm{n}_f(p)}^2+\ldots+x_n^2=|\mathbf{x}|^2. \tag{6.10}$$

Now, we develop the middle term of the expression (6.4).
We know that the gradient of f is the dual of df under g, (see subsection 3.2.1).
Let ∇^{TM} be a metric connection on TM, by Corollary 4.1.25 $\nabla^{TM}_{e_i}\mathrm{grad}f=(\nabla^{T^*M}_{e_i}df)^*$. By equality (6.9) we have

$$\frac{\partial f}{\partial x_j}=\begin{cases}-x_j & \text{if } j\le \mathrm{n}_f(p),\\ \ \ x_j & \text{if } j>\mathrm{n}_f(p).\end{cases}$$

Fix i, then

$$d\left(\frac{\partial f}{\partial x_j}\right)(e_i)=\begin{cases}-1 & \text{if } i\le \mathrm{n}_f(p),\\ \ \ 1 & \text{if } i>\mathrm{n}_f(p).\end{cases}$$

We take the Hessian of f, see definition 4.1.24, since ∇^{T^*M} is a linear map and by Leibniz rule, then

$$\begin{aligned}\nabla^{T^*M}_{e_i}df &= \nabla^{T^*M}_{e_i}\left(\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}dx_j\right)\\ &= \sum_{j=1}^{n}\nabla^{T^*M}_{e_i}\left(\frac{\partial f}{\partial x_j}dx_j\right)\\ &= \sum_{j=1}^{n}\left(d\left(\frac{\partial f}{\partial x_j}\right)(e_i)\,dx_j+\frac{\partial f}{\partial x_j}\nabla^{T^*M}_{e_i}dx_j\right).\end{aligned}$$

Since p is a critical point of f, $\frac{\partial f}{\partial x_j}\Big|_p=0$ for all $j=1,\dots,n$, also since $(dx_i)^*=e_i$ and by equality (6.9), we obtain at the point p:

$$\nabla^{TM}_{e_i}\mathrm{grad}f=\begin{cases}-e_i & \text{if } i\le \mathrm{n}_f(p),\\ \ \ e_i & \text{if } i>\mathrm{n}_f(p).\end{cases}$$

Since $\hat{c}(e_i)$ is a $\mathbb{R}$–linear operator, then

$$T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega = -T\sum_{i=1}^{\mathrm{n}_f(p)} c(e_i)\hat{c}(e_i)\omega + T\sum_{i=\mathrm{n}_f(p)+1}^{n} c(e_i)\hat{c}(e_i)\omega.$$

We can write as $nT = T\sum\limits_{i=1}^{n} 1$, then

$$\begin{aligned} T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega &= T\sum_{i=1}^{\mathrm{n}_f(p)} \omega - T\sum_{i=1}^{\mathrm{n}_f(p)} c(e_i)\hat{c}(e_i)\omega + T\sum_{i=\mathrm{n}_f(p)+1}^{n} \omega \\ &\quad + T\sum_{i=\mathrm{n}_f(p)+1}^{n} c(e_i)\hat{c}(e_i)\omega - nT\omega \\ &= T\left[\sum_{i=1}^{\mathrm{n}_f(p)} (1 - c(e_i)\hat{c}(e_i))\omega + \sum_{i=\mathrm{n}_f(p)+1}^{n} (1 + c(e_i)\hat{c}(e_i))\omega\right] - nT\omega. \end{aligned}$$

Let $\omega \in \Omega^{\mathrm{n}_f(p)}(M)$, by Definitions (4.31), (4.32) and by Lemma 1.5.2-1. and -2., we have

$$\begin{aligned} c(e_i)\hat{c}(e_i)\omega &= c(e_i)(dx_i \wedge \omega + e_i \lrcorner \omega) \\ &= dx_i \wedge (dx_i \wedge \omega + e_i \lrcorner \omega) - e_i \lrcorner (dx_i \wedge \omega + e_i \lrcorner \omega) \\ &= dx_i \wedge (e_i \lrcorner \omega) - e_i \lrcorner (dx_i \wedge \omega) \\ &= dx_i \wedge (e_i \lrcorner \omega) + dx_i \wedge (e_i \lrcorner \omega) - (e_i \lrcorner dx_i) \wedge \omega \\ &= 2dx_i \wedge (e_i \lrcorner \omega) - \omega. \end{aligned}$$

Thus

$$\begin{aligned} \omega - c(e_i)\hat{c}(e_i)\omega &= \omega - 2dx_i \wedge (e_i \lrcorner \omega) + \omega = -2dx_i \wedge (e_i \lrcorner \omega) + 2\omega \\ \omega + c(e_i)\hat{c}(e_i)\omega &= \omega + 2dx_i \wedge (e_i \lrcorner \omega) - \omega = 2dx_i \wedge (e_i \lrcorner \omega). \end{aligned}$$

It follows that

$$T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega = 2T\left[\sum_{i=1}^{\mathrm{n}_f(p)} (\omega - dx_i \wedge (e_i \lrcorner \omega)) + \sum_{i=\mathrm{n}_f(p)+1}^{n} dx_i \wedge (e_i \lrcorner \omega)\right] - nT\omega.$$

Note that by Lemma 1.5.2-2., for $\omega \in \Omega^{\mathrm{n}_f(p)}(M)$ we obtain

$$\begin{aligned} e_i \lrcorner (dx_i \wedge \omega) &= (e_i \lrcorner dx_i) \wedge \omega + (-1)dx_i \wedge (e_i \lrcorner \omega) \\ &= dx_i(e_i)\omega - dx_i \wedge (e_i \lrcorner \omega) \\ &= \omega - dx_i \wedge (e_i \lrcorner \omega). \end{aligned}$$

Rewrite

$$T\sum_{i=1}^{n} c(e_i)\hat{c}(\nabla^{TM}_{e_i}\mathrm{grad}f)\omega = 2T\left[\sum_{i=1}^{\mathrm{n}_f(p)} e_i \lrcorner (dx_i \wedge \omega) + \sum_{i=\mathrm{n}_f(p)+1}^{n} dx_i \wedge (e_i \lrcorner \omega)\right] - nT\omega. \quad (6.11)$$

By equalities (6.4), (6.8), (6.11), (6.10) the deformed Laplace–Beltrami operator on $\Omega^{\mathrm{n}_f(p)}(\mathbb{R}^n)$ we have

$$\Box_{Tf,\mathrm{n}_f(p)}\omega = -\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\omega - nT\omega + T^2|\mathbf{x}|^2\omega + 2T\left[\sum_{i=1}^{\mathrm{n}_f(p)} e_i\lrcorner(dx_i\wedge\omega) + \sum_{i=\mathrm{n}_f(p)+1}^{n} dx_i\wedge(e_i\lrcorner\omega)\right]$$

□

The differentiable operator

$$-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - nT + T^2|\mathbf{x}|^2$$

acts only on differentiable functions and $T > 0$ is a harmonic oscillator operator, see [21, Example 11.3-1]. Let $\kappa(x)$ be a function such that

$$\left(-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - nT + T^2|\mathbf{x}|^2\right)\kappa(x) = 0.$$

Since it is a harmonic oscillator operator, the solution is given by

$$\kappa(x) = \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right).$$

Proposition 6.0.4 ([41, Prop. 5.4])**.** *Under the conditions of Theorem 6.0.3, $T > 0$,* Ker $(\Box_{Tf,\mathrm{n}_f(p)})$ *is generated by*

$$\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)} \in \Omega^{\mathrm{n}_f(p)}(\mathbb{R}^n).$$

That is, $\dim(\mathrm{Ker}\,\Box_{Tf,\mathrm{n}_f(p)}) = 1$.

Let us see that this differentiable $\mathrm{n}_f(p)$–form is in the kernel of $\Box_{Tf,\mathrm{n}_f(p)}$.

By Remark 4.3.9 we can see that

$$dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)} \in \mathrm{Ker}\left(\sum_{i=1}^{\mathrm{n}_f(p)} e_i\lrcorner(dx_i\wedge\ ) + \sum_{i=\mathrm{n}_f(p)+1}^{n} dx_i\wedge(e_i\lrcorner\ )\right).$$

1. If $i = 1,\ldots,\mathrm{n}_f(p)$, then $e_i\lrcorner(dx_i\wedge dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}) = e_i\lrcorner 0$.
2. If $i = \mathrm{n}_f(p)+1,\ldots,n$, then $dx_i\wedge(e_i\lrcorner dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}) = dx_i\wedge 0 = 0$.

Therefore $dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)} \in \mathrm{Ker}\left(\sum_{i=1}^{\mathrm{n}_f(p)} e_i\lrcorner(dx_i\wedge\ ) + \sum_{i=\mathrm{n}_f(p)+1}^{n} dx_i\wedge(e_i\lrcorner\ )\right)$.

Note that $dx_{\mathrm{n}_f(p)+1}\wedge\ldots\wedge dx_n \notin \mathrm{Ker}\left(\sum_{i=1}^{\mathrm{n}_f(p)} e_i\lrcorner(dx_i\wedge\ ) + \sum_{i=\mathrm{n}_f(p)+1}^{n} dx_i\wedge(e_i\lrcorner\ )\right)$, one can check it also using Remark 4.3.9.

Now, the case of $\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \in \operatorname{Ker}\left(-\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - nT + T^2|\mathbf{x}|^2\right)$, we have:

$$\begin{aligned}
\frac{\partial}{\partial x_i}\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) &= -Tx_i\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right),\\
\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) &= -T\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) + T^2x_i^2\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\left(-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - nT + T^2|\mathbf{x}|^2\right)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) &= -\sum_{i=1}^{n}\left[T^2x_i^2\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) - T\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right]\\
&\quad -nT\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) + T^2|\mathbf{x}|^2\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\\
&= -T^2|\mathbf{x}|^2\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) + nT\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\\
&\quad -nT\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) + T^2|\mathbf{x}|^2\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\\
&= 0.
\end{aligned}$$

Therefore, $\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)} \in \operatorname{Ker}\square_{Tf,\,\mathrm{n}_f(p)}$.

Remark 6.0.5. Theorem 6.0.3 and Proposition 6.0.4 tell us that:

1. For each critical point p of a Morse function f one can write a local description of the deformed Laplace-Beltrami operator $\square_{Tf,\,\mathrm{n}_f(p)}$.
2. There may be no critical point of some index $\mathrm{n}_f(p) = 0,\ldots,n = \dim(M)$.

To illustrate this observations, consider the following examples.

Example 6.0.6. Consider the 2–torus and $f\colon T^2 \longrightarrow \mathbb{R}$ the height function, we will use the information obtained from the examples 1.4.18 and 2.1.4.

The 2–torus is an oriented, closed Riemannian manifold of dimension 2, then we have $\Omega^k(T^2)$ with $k = 0, 1, 2$. We will denote the critical points by a, b, c, d, as in the Figure 2.1.

1. Let $k = 0$, we take $\square_{Tf,\,0}\colon C^\infty(T^2) \longrightarrow C^\infty(T^2)$.

   By Hodge Theorem (5.7) and Proposition 5.0.1 we have

   $$\dim(\operatorname{Ker}\square_{Tf,\,0}) = \dim(\mathrm{H}^0_{Tf,\,\mathrm{DR}}(T^2)) = 1.$$

   We have a critical point of index 0, the critical point d, by Theorem 6.0.3 there is a chart $\varphi\colon U_d \longrightarrow V_d \subset \mathbb{R}^2$ around d such that we get the equation of $\square_{Tf,\,0}$ for all $\mathbf{x} = (x_1, x_2) \in V_d$ and by Proposition 6.0.4 we know $\operatorname{Ker}\square_{Tk,\,0}$, that is, its generator is $g_d = \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)$. Which agrees with Hodge Theorem.

2. Let $k = 1$, consider $\Box_{Tf,1}\colon \Omega^1(T^2) \longrightarrow \Omega^1(T^2)$.

   By Hodge Theorem (5.7) and Proposition 5.0.1 we get

$$\dim(\mathrm{Ker}\,\Box_{Tf,1}) = \dim(\mathrm{H}^1_{Tf,\mathrm{DR}}(T^2)) = 2.$$

   There are two critical points of index 1, the critical points b and c, by Theorem 6.0.3 for each point there are charts $\phi\colon U_b \longrightarrow V_b \subset \mathbb{R}^2$ and $\phi'\colon U_c \longrightarrow V_c \subset \mathbb{R}^2$ around b and c respectively such that we have the equations of $\Box_{Tf,\mathrm{n}_f(b)}$ for all $\mathbf{y} = (y_1, y_2) \in V_b$ and $\Box_{Tf,\mathrm{n}_f(c)}$ for all $\mathbf{z} = (z_1, z_2) \in V_c$ and by Proposition 6.0.4 we know:

$$\begin{aligned}
\mathrm{Ker}\,\Box_{Tf,\mathrm{n}_f(b)} &= \langle g_b \rangle = \left\langle \exp\left(\frac{-T|\mathbf{y}|^2}{2}\right) dy_1 \right\rangle, \\
\mathrm{Ker}\,\Box_{Tf,\mathrm{n}_f(c)} &= \langle g_c \rangle = \left\langle \exp\left(\frac{-T|\mathbf{z}|^2}{2}\right) dz_1 \right\rangle.
\end{aligned}$$

   Then $2 = \dim(\mathrm{Ker}\,\Box_{Tf,1}) = \dim(\mathrm{Ker}\,\Box_{Tf,\mathrm{n}_f(b)}) + \dim(\mathrm{Ker}\,\Box_{Tf,\mathrm{n}_f(c)})$, which coincides with Hodge Theorem.

3. If $k = 2$, it is similar to the situation $k = 0$, for $\Box_{Tf,2}\colon \Omega^2(T^2) \longrightarrow \Omega^2(T^2)$ by Hodge Theorem (5.7) and Proposition 5.0.1 then

$$\dim(\mathrm{Ker}\,\Box_{Tf,2}) = \dim(\mathrm{H}^2_{Tf,\mathrm{DR}}(T^2)) = 1.$$

   The critical point a is the critical point of index 2, by Theorem 6.0.3 there is a chart $\psi\colon U_a \longrightarrow V_a \subset \mathbb{R}^2$ around a such that we get the equation of $\Box_{Tf,2}$ for all $\mathbf{v} = (v_1, v_2) \in V_a$ and by Proposition 6.0.4 the generator of $\mathrm{Ker}\,\Box_{Tk,2}$ is $g_a = \exp\left(\frac{-T|\mathbf{v}|^2}{2}\right) dv_1 \wedge dv_2$, that coincides with Hodge Theorem.

Example 6.0.7. Consider the 2–sphere and $f\colon S^2 \longrightarrow \mathbb{R}$ the height function, we will use the information obtained from examples 1.4.17 and 2.1.2.

Proceeding analogously to the example 6.0.6, we get:

1. Let $k = 0$, $\Box_{Tf,0}\colon C^\infty(S^2) \longrightarrow C^\infty(S^2)$, then by Hodge Theorem (5.7) and Proposition 5.0.1 one can obtain that $\dim(\mathrm{Ker}\,\Box_{Tf,0}) = 1$.

   There is a critical point of index 0, the south pole S, by Theorem 6.0.3 there is a chart $\varphi\colon U_S \longrightarrow V_S \subset \mathbb{R}^2$ around S such that we get the equation of $\Box_{Tf,0}$ for all $\mathbf{x} = (x_1, x_2) \in V_S$ and by Proposition 6.0.4 the generator of $\mathrm{Ker}\,\Box_{Tk,0}$ is $g_S = \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)$, then the dimensions of the vector spaces coincide with the given by Hodge Theorem.

2. Let $k = 2$, the critical point of index 2 is N the north pole, by Theorem 6.0.3 there is a chart $\phi\colon U_N \longrightarrow V_N \subset \mathbb{R}^n$ around N such that we obtain the equation of $\Box_{Tf,2}$ for all $\mathbf{y} = (y_1, y_2) \in V_N$ and by Proposition 6.0.4 then $\mathrm{Ker}\,\Box_{Tf,2} = \langle g_N \rangle = \left\langle \exp\left(\frac{-T|\mathbf{y}|^2}{2}\right) dy_1 \wedge dy_2 \right\rangle$. Hodge Theorem also holds.

3. This case is an example of Remark 6.0.5-2.

   Let $k = 1$, by Hodge Theorem (5.7) and Proposition 5.0.1 $\dim(\operatorname{Ker} \square_{Tf,1}) = 0$. Also, we have no critical points of index 1.

In the next chapter we will see that the set of generators of all kernels of the local operators $\square_{Tf,\mathrm{n}_f(p)}$ from each of the critical points p generate the kernel of $\square_{Tf,\bullet}$.

Chapter 7

Global description of $\square_{Tf,k}$

We will generate the eigenspaces of $\square_{Tf,\bullet}$ and $\square_{Tf,k}$ using functional analysis.

Fore more details see [41].

7.1 Bump functions

In differential geometry and analysis we use bump functions as tools. For example, they are used to define partitions of unity and for extending locally defined differentiable functions to globally defined differentiable functions.

Definition 7.1.1. Let f be a differentiable function on a differentiable manifold M. The *support* of f is defined to be the closure of the set on which $f(p) \neq 0$ for $p \in M$, that is: $\operatorname{supp} f = \overline{\{p \in M | f(p) \neq 0\}}$.

Definition 7.1.2. Let M be a differentiable manifold, $p \in M$ and U a neighbourhood of p. A *bump function* at p supported in U is any differentiable function $f \colon M \longrightarrow \mathbb{R}$ that is 1 in a neighbourhood of p with $\operatorname{supp} f \subset U$.

Example 7.1.3. Figure 7.1 is the graph of a bump function at 0 with support in $(-1, 1)$, the function is nonzero on the open interval $(-1, 1)$ and is zero otherwise. Its support is the closed interval $[-1, 1]$.

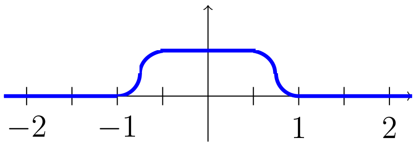

Figure 7.1: A bump function

7.2 Global description of $\square_{Tf,k}$

Now, let us consider differentiable k–forms with compact support, see definition 1.2.1.

Let M be an oriented, closed Riemannian n–manifold, $T \in \mathbb{R}, T > 0$.

We want to describe globally the operator $\square_{Tf,k}$ and relate it to the number of critical points $p \in M$ of $f \in C^\infty(M)$ such that $\mathrm{n}_f(p) = k$, that is, with m_k, this for every $0 \leq k \leq n$.

Let $f\colon M \longrightarrow \mathbb{R}$ be a Morse function and $p \in M$ a critical point of f. Let $W \subset \mathbb{R}$ be a neighborhood of 0.

We assume that $f(p) = 0$, if $f(p) \neq 0$, then we consider $g = f - f(p)$.

We consider $\gamma\colon \mathbb{R} \longrightarrow [0,1]$ a bump function such that

$$\gamma(t) = \begin{cases} 1 & \text{if } |t| \leq r \\ 0 & \text{if } |t| \geq 2r \end{cases}$$

for some radius r such that the ball of radius $2r$, $B_{2r}(0)$, is still contained in W.

By Corollary 2.0.7 there is a chart $\varphi\colon U \longrightarrow V$ around p such that (2.9) holds, that is,

$$(f \circ \varphi^{-1})(\mathbf{x}) = f(p) - \frac{1}{2}x_1^2 - \ldots - \frac{1}{2}x_{\mathrm{n}_f(p)}^2 + \frac{1}{2}x_{\mathrm{n}_f(p)+1}^2 + \ldots + \frac{1}{2}x_n^2, \quad \mathbf{x} = (x_1, \ldots, x_n) \in V.$$

By equalities (2.9) and (6.9) $|\mathrm{grad}(f \circ \varphi^{-1})| = |\mathbf{x}|^2 \in C^\infty(V), \mathrm{grad}(f \circ \varphi^{-1})(0_{\mathbb{R}}) = \mathrm{grad} f(p) = 0_{\mathbb{R}}$.

By Proposition 6.0.4 we define the real number $\lambda_{p,T}$ by

$$\lambda_{p,T} := \int_V \gamma(|\mathbf{x}|)^2 \exp(-T|\mathbf{x}|^2) dx_1 \wedge \ldots \wedge dx_n. \tag{7.1}$$

Let

$$\omega_{p,T} := \frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) dx_1 \wedge \ldots \wedge dx_{\mathrm{n}_f(p)} \in \Omega_c^{\mathrm{n}_f(p)}(V). \tag{7.2}$$

$\omega_{p,T}$ is a differentiable $\mathrm{n}_f(p)$–form with compact support contained in $\overline{B_{2r}(0)}$.

We want to extend this differentiable $\mathrm{n}_f(p)$–form to a differentiable $\mathrm{n}_f(p)$–form on M, so we take the pullback of the $\mathrm{n}_f(p)$–form, see 1.2.1 in particular (1.3), and we define

$$\widetilde{\omega}_{p,T}(q) = \begin{cases} \varphi^*(\omega_{p,T})(q) & \text{if } q \in \varphi^{-1}(\overline{B_{2r}(0)}), \\ 0 & \text{if } q \notin \varphi^{-1}(\overline{B_{2r}(0)}). \end{cases}$$

Note that $B_r(0) \subset B_{2r}(0) \subset V$.

Therefore, $\widetilde{\omega}_{p,T} \in \Omega^{\mathrm{n}_f(p)}(M)$ with compact support contained in $\varphi^{-1}(\overline{B_{2r}(0_{\mathbb{R}^n})})$.

Lemma 7.2.1. *For all* $p \in \mathrm{Crit}(f)$, *we have* $||\widetilde{\omega}_{p,T}||_0 = 1$.

Proof. Let $\widetilde{\omega}_{p,T} \in \Omega^{\mathrm{n}_f(p)}(M)$, by equalities (3.14) and (3.6)

$$\begin{aligned} ||\widetilde{\omega}_{p,T}||_0^2 &= \langle \widetilde{\omega}_{p,T}, \widetilde{\omega}_{p,T} \rangle \\ &= \int_M \widetilde{\omega}_{p,T} \wedge \star\widetilde{\omega}_{p,T}. \end{aligned}$$

Since φ is a local diffeomorphism

$$||\widetilde{\omega}_{p,T}||_0^2 = \int_{\overline{\varphi^{-1}(B_{2r}(0))}} \varphi^*(\omega_{p,T}) \wedge \star\varphi^*(\omega_{p,T}).$$

By equalities (7.1) and (7.2) and by product property of the exponential

$$\begin{aligned}
||\widetilde{\omega}_{p,T}||_0^2 &= \int_{\overline{B_{2r}(0)}} \omega_{p,T} \wedge \star\omega_{p,T} \\
&= \int_{\overline{B_{2r}(0)}} \left(\frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}}\right)^2 \left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right)\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) dx_1 \wedge \ldots \wedge dx_n \\
&= \frac{1}{\lambda_{p,T}} \int_{\overline{B_{2r}(0)}} (\gamma(|\mathbf{x}|))^2 \exp(-T|\mathbf{x}|^2) dx_1 \wedge \ldots \wedge dx_n \\
&= 1. \\
||\widetilde{\omega}_{p,T}||_0 &= 1.
\end{aligned}$$

□

Since $\lambda_{p,T}$ is the appropriate term such that $\widetilde{\omega}_{p,T}$ is a differentiable $\mathrm{n}_f(p)$–form of norm 1, then $\lambda_{p,T}$ is called the *normalization factor* of $\widetilde{\omega}_{p,T}$.

We will denote by $\mathrm{Crit}(f)$ the set of critical points of f.

Let E_T be the subspace of $\Omega_c^\bullet(M)$ generated by the $\widetilde{\omega}_{p,T}$ for all $p \in \mathrm{Crit}(f)$.

Since M is a compact manifold, by Corollary 2.0.8 the set $\mathrm{Crit}(f)$ is finite and the critical points are isolated, then if we take the domain of the charts that exists from Corollary 2.0.7, the domains of the charts are disjoint.

Lemma 7.2.2. *$\{\widetilde{\omega}_{p,T}\}_{p\in\mathrm{Crit}(f)}$ is an orthonormal set.*

Proof. By Lemma 7.2.1, it suffices to prove that $\langle\widetilde{\omega}_{p,T},\, \widetilde{\omega}_{q,T}\rangle = 0$ for all $p, q \in \mathrm{Crit}(f), p \neq q$, $\mathrm{n}_f(p) = \mathrm{n}_f(q)$.

Let $\widetilde{\omega}_{p,T},\, \widetilde{\omega}_{q,T} \in \Omega^{\mathrm{n}_f(p)}(M)$, by Corollary 2.0.7 there exists $\varphi_1\colon U_p \longrightarrow V_p$ and $\varphi_2\colon U_q \longrightarrow V_q$ charts around p and q, respectively, then

$$\begin{aligned}
\langle\widetilde{\omega}_{p,T},\, \widetilde{\omega}_{q,T}\rangle &= \int_M \varphi_1^*(\omega_{p,T}) \wedge \star\varphi_2^*(\omega_{q,T}) \\
&= \int_{\mathrm{supp}\,\varphi_1^*(\omega_{p,T}) \cup \mathrm{supp}\,\varphi_2^*(\omega_{q,T})} \varphi_1^*(\omega_{p,T}) \wedge \star\varphi_2^*(\omega_{q,T}) \\
&= 0.
\end{aligned}$$

□

We complete the space $\Omega_c^\bullet(M)$ of differentiable $\bullet$–forms with respect to the norm $||\ ||_i$, $i = 0, 1$, the resulting vector space is the i–Sobolev space of $\Omega_c^\bullet(M)$, denoted by $H_\bullet^i(M)$. Analogously, if we take $\Omega^k(M)$ and the norm $||\ ||_{i,k}$, we have the i–Sobolev space of differentiable k–forms $H_k^i(M)$.

By Remark D.3.4 $H_\bullet^0(M) = L^2(\Omega_c^\bullet(U))$ and by Corollary D.2.9 $H_\bullet^0(M)$ is a Hilbert space. In particular, $E_T \subset H_\bullet^0(M)$.

Remark 7.2.3. By Lemma 7.2.2, E_T is an orthonormal set and by Lemma A.5.4 E_T is linearly independent, therefore E_T is a finite dimensional vector space. Also, since E_T is a normed space with $|| \, ||_0$ by Theorem D.1.6 we have E_T is complete, then E_T is a Hilbert space.

Let $E_T^{\perp}$ be the orthogonal complement to E_T in $H^0_{\bullet}(M)$.

Since E_T is complete, by Theorem D.1.10 E_T is a closed space in $H^0_{\bullet}(M)$, by Theorem D.1.11 $H^0_{\bullet}(M)$ admits an orthogonal decomposition

$$H^0_{\bullet}(M) = E_T \oplus E_T^{\perp}. \tag{7.3}$$

We consider $\mathrm{pr}\colon H^0_{\bullet}(M) \longrightarrow E_T$, $\mathrm{pr}^{\perp}\colon H^0_{\bullet}(M) \longrightarrow E_T^{\perp}$ the projections, see the Definition D.1.23.

We decompose the deformed Witten operator

$$\begin{aligned} \mathrm{D}_{Tf}\colon \quad H^0_{\bullet}(M) &\longrightarrow \quad H^0_{\bullet}(M), \\ E_T \oplus E_T^{\perp} &\longrightarrow \quad E_T \oplus E_T^{\perp}. \end{aligned}$$

Let $\omega \in H^0_{\bullet}(M)$, set

$$\begin{aligned} \mathrm{D}_{T,1}\omega &= \mathrm{pr}\mathrm{D}_{Tf}\mathrm{pr}\omega, \quad &(7.4)\\ \mathrm{D}_{T,2}\omega &= \mathrm{pr}\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega, \quad &(7.5)\\ \mathrm{D}_{T,3}\omega &= \mathrm{pr}^{\perp}\mathrm{D}_{Tf}\mathrm{pr}\omega, \quad &(7.6)\\ \mathrm{D}_{T,4}\omega &= \mathrm{pr}^{\perp}\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega. \quad &(7.7) \end{aligned}$$

Lemma 7.2.4. $\mathrm{D}_{T,2}$ *is the adjoint operator of* $\mathrm{D}_{T,3}$.

Proof. Let $\omega, \eta \in H^0_{\bullet}(M)$.

By Theorem D.1.24 pr, $\mathrm{pr}^{\perp}$ are self-adjoint operators and by Lemma 5.0.2 D_{Tf} is a self-adjoint operator, all with respect to $\langle \, , \, \rangle$, (3.6), then

$$\begin{aligned} \langle \mathrm{D}_{T,3}\omega, \ \eta \rangle &= \langle \mathrm{pr}^{\perp}\mathrm{D}_{Tf}\mathrm{pr}\omega, \ \eta \rangle \\ &= \langle \mathrm{D}_{Tf}\mathrm{pr}\omega, \ \mathrm{pr}^{\perp}\eta \rangle \\ &= \langle \mathrm{pr}\omega, \ \mathrm{D}_{Tf}\mathrm{pr}^{\perp}\eta \rangle \\ &= \langle \omega, \ \mathrm{pr}\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\eta \rangle \\ &= \langle \omega, \ \mathrm{D}_{T,2}\eta \rangle. \end{aligned}$$

□

We describe some estimates for these operators

Proposition 7.2.5.

1. *For any* $T > 0$ *and for all* $\omega \in H^0_{\bullet}(M)$, *we have* $\mathrm{D}_{T,1}\omega = 0$.

2. *There exists a constant* $T_1 > 0$*, such that for any* $\omega \in E_T^{\perp} \cap H^1_{\bullet}(M)$*,* $\omega' \in E_T$ *and* $T \geq T_1$*, one has*

$$||\mathrm{D}_{T,2}\omega||_0 \leq \frac{||\omega||_0}{T}. \tag{7.8}$$

$$||\mathrm{D}_{T,3}\omega'||_0 \leq \frac{||\omega'||_0}{T}. \tag{7.9}$$

Proof. 1. By the definition of E_T and since pr is a projection, for all $\omega \in H^0_{\bullet}(M)$

$$\mathrm{pr}\omega = \sum_{p\in \mathrm{Crit}(f)} \langle \widetilde{\omega}_{p,T}, \omega\rangle \widetilde{\omega}_{p,T}.$$

Since $\langle \widetilde{\omega}_{p,T}, \omega\rangle \widetilde{\omega}_{p,T} \in \Omega^{\mathrm{n}_f(p)}(M)$ has compact support in U and its derivatives have compact support in U, then

$$\mathrm{D}_{Tf}(\langle \widetilde{\omega}_{p,T}, \omega\rangle \widetilde{\omega}_{p,T}) \in \Omega^{\mathrm{n}_f(p)-1}(M) \oplus \Omega^{\mathrm{n}_f(p)+1}(M)$$

has compact support in U. But inside U, pr maps into $\Omega^{\mathrm{n}_f(p)}(M)$, so

$$\mathrm{pr}\mathrm{D}_{Tf}(\langle \widetilde{\omega}_{p,T}, \omega\rangle \widetilde{\omega}_{p,T}) = 0$$

for each $p \in \mathrm{Crit}(f)$. Therefore, $\mathrm{D}_{T,1}\omega = 0$.

2. By Remark 7.2.3 E_T is a Hilbert space. By Definition D.1.21, Theorem D.1.22 and Lemma 7.2.4, it is enough to prove the estimate for $\mathrm{D}_{T,2}$ or $D_{T,3}$. We will prove the estimate for $\mathrm{D}_{T,2}$.

   Let $\omega \in E_T^{\perp} \cap H^1_{\bullet}(M)$, since $\omega \in E_T^{\perp}$ and by Lemma 5.0.2 we have

$$\begin{aligned} \mathrm{D}_{T,2}\omega &= \mathrm{pr}\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega \\ &= \mathrm{pr}\mathrm{D}_{Tf}\omega \\ &= \sum_{p\in \mathrm{Crit}(f)} \langle \widetilde{\omega}_{p,T}, \mathrm{D}_{Tf}\omega\rangle \widetilde{\omega}_{p,T} \\ &= \sum_{p\in \mathrm{Crit}(f)} \langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T}, \omega\rangle \widetilde{\omega}_{p,T}. \end{aligned} \tag{7.10}$$

   By Cauchy-Schwarz inequality one see that

$$|\langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T}, \omega\rangle| \leq ||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0 ||\omega||_0. \tag{7.11}$$

   By Lemma 5.0.2, then

$$\begin{aligned} ||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &= \langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T}, \mathrm{D}_{Tf}\widetilde{\omega}_{p,T}\rangle \\ &= \langle \mathrm{D}^2_{Tf}\widetilde{\omega}_{p,T}, \widetilde{\omega}_{p,T}\rangle \\ &= \int_M \mathrm{D}^2_{Tf}\widetilde{\omega}_{p,T} \wedge \star\widetilde{\omega}_{p,T} \\ &= \int_M \mathrm{D}^2_{Tf}\varphi^*(\omega_{p,T}) \wedge \star\varphi^*(\omega_{p,T}) \\ &= \int_V \mathrm{D}^2_{Tf}\omega_{p,T} \wedge \star\omega_{p,T}. \end{aligned}$$

Remember the equality (7.2):

$$\omega_{p,T} = \frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) dx_1 \wedge \ldots \wedge dx_{\mathrm{n}_f(p)},$$

where $\frac{1}{\sqrt{\lambda_{p,T}}} \in \mathbb{R}$, then

$$\star\omega_{p,T} = \frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n.$$

Also, since $\mathrm{D}^2_{Tf}(\omega_{p,T}) = \square_{Tf,\mathrm{n}_f(p)}(\omega_{p,T})$:

$$\begin{aligned}
||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||^2_0 &= \int_{B_{2r}(0)} \square_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge \star(\omega_{p,T}). \\
&= \int_{B_{2r}(0)} \square_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge \frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n \\
&= \int_{B_{2r}(0)} \frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \square_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n.
\end{aligned}$$

By equality (6.7) and Proposition 6.0.4 it is enough to see how

$$-\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - nT + T^2|\mathbf{x}|^2$$

acts on $\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \in C^\infty(V)$ on $B_r(0)$ and $B_{2r}(0) \setminus B_r(0)$. Since

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) &= \gamma(|\mathbf{x}|)\frac{\partial}{\partial x_i}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) \\
&\quad + \frac{\partial}{\partial x_i}(\gamma(|\mathbf{x}|)\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}\left(\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) &= \gamma(|\mathbf{x}|)\frac{\partial^2}{\partial x_i^2}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) \\
&\quad + 2\frac{\partial}{\partial x_i}(\gamma(|\mathbf{x}|)\frac{\partial}{\partial x_i}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) \\
&\quad + \frac{\partial^2}{\partial x_i^2}(\gamma(|\mathbf{x}|)\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right).
\end{aligned}$$

Where $\gamma(|\mathbf{x}|)\frac{\partial^2}{\partial x_i^2}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) \neq 0$ on $B_{2r}(0)$, $2\frac{\partial}{\partial x_i}(\gamma(|\mathbf{x}|)\frac{\partial}{\partial x_i}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) \neq B_{2r}(0) \setminus B_r(0)$ and $\frac{\partial^2}{\partial x_i^2}(\gamma(|\mathbf{x}|)\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right) \neq B_{2r}(0) \setminus B_r(0)$.

Note that on $B_r(0)$

$$\left(-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-nT+T^2|\mathbf{x}|^2\right)\left(\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\right)=0 \quad (7.12)$$

While that on $B_{2r}(0)\setminus B_r(0)$ we have that

$$\begin{aligned}
&\left(-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-nT+T^2|\mathbf{x}|^2\right)\left(\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\right)\\
=\;& -\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left[\gamma(|\mathbf{x}|)\frac{\partial^2}{\partial x_i^2}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right)+2\frac{\partial}{\partial x_i}(\gamma(|\mathbf{x}|))\frac{\partial}{\partial x_i}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right)\right.\\
&\left.+\frac{\partial^2}{\partial x_i^2}(\gamma(|\mathbf{x}|))\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right)\right]dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\\
&-nT\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\\
&+T^2|\mathbf{x}|^2\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\\
=\;& \gamma(|\mathbf{x}|)\left[-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right)-nT\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right.\\
&\left.+T^2\;|\mathbf{x}|^2\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right]dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\\
&-2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)\frac{\partial}{\partial x_i}\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\\
&-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}.
\end{aligned}$$

By Proposition 6.0.4, $\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\in\operatorname{Ker}\left(\square_{Tf,\mathrm{n}_f(p)}\right)$ then

$$\begin{aligned}
&\left(-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-nT+T^2|\mathbf{x}|^2\right)\left(\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\right)\\
=\;& -\sum_{i=1}^{n}\left[2\frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)\frac{\partial}{\partial x_i}\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)+\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right]dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}.
\end{aligned}$$

And

$$\frac{\partial}{\partial x_i}\left(\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)\right)=-Tx_i\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right). \quad (7.13)$$

Substituting

$$\begin{aligned}
&\left(-\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-nT+T^2|\mathbf{x}|^2\right)\left(\gamma(|\mathbf{x}|)\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}\right)\\
=\;& \sum_{i=1}^{n}\left[2Tx_i\frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)-\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\right]\exp\left(\frac{-T|\mathbf{x}|^2}{2}\right)dx_1\wedge\ldots\wedge dx_{\mathrm{n}_f(p)}. \quad (7.14)
\end{aligned}$$

Then

$$\begin{aligned}
||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &= \int_{B_{2r}(0)} \frac{\gamma(|\mathbf{x}|)}{\sqrt{\lambda_{p,T}}} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \Box_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n \\
&= \frac{1}{\sqrt{\lambda_{p,T}}} \int_{B_r(0)} \gamma(|\mathbf{x}|) \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \Box_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n \\
&\quad + \frac{1}{\sqrt{\lambda_{p,T}}} \int_{B_{2r}(0)\setminus B_r(0)} \gamma(|\mathbf{x}|) \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \Box_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n.
\end{aligned}$$

By equality (7.12) and since $\gamma(|\mathbf{x}|) \leq 1$

$$\begin{aligned}
||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &= \frac{1}{\sqrt{\lambda_{p,T}}} \int_{B_{2r}(0)\setminus B_r(0)} \gamma(|\mathbf{x}|) \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \Box_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n \\
&\leq \frac{1}{\sqrt{\lambda_{p,T}}} \int_{B_{2r}(0)\setminus B_r(0)} \exp\left(\frac{-T|\mathbf{x}|^2}{2}\right) \Box_{Tf,\mathrm{n}_f(p)}(\omega_{p,T}) \wedge dx_{\mathrm{n}_f(p)+1} \wedge \ldots \wedge dx_n
\end{aligned}$$

Also, using equalities (7.13) and (7.14) we have

$$\begin{aligned}
||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &\leq \frac{2T}{\lambda_{p,T}} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)\right) dx_1 \wedge \ldots \wedge dx_n \\
&\quad - \frac{1}{\lambda_{p,T}} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\right) dx_1 \wedge \ldots \wedge dx_n. \\
&\leq \left| \frac{2T}{\lambda_{p,T}} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)\right) dx_1 \wedge \ldots \wedge dx_n \right. \\
&\quad \left. - \frac{1}{\lambda_{p,T}} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\right) dx_1 \wedge \ldots \wedge dx_n \right| \\
&\leq \frac{2T}{|\lambda_{p,T}|} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n \left| x_i \frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)\right|\right) dx_1 \wedge \ldots \wedge dx_n \\
&\quad + \frac{1}{|\lambda_{p,T}|} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n \left|\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\right|\right) dx_1 \wedge \ldots \wedge dx_n
\end{aligned}$$

Since $|x_i| \leq 2r$. Then

$$\begin{aligned}
||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &\leq \frac{4rT}{|\lambda_{p,T}|} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n \left| \frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)\right|\right) dx_1 \wedge \ldots \wedge dx_n \\
&\quad + \frac{1}{|\lambda_{p,T}|} \int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2) \left(\sum_{i=1}^n \left|\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)\right|\right) dx_1 \wedge \ldots \wedge dx_n.
\end{aligned}$$

For all $r < |\mathbf{x}| < 2r$ we have $|\frac{\partial}{\partial x_i}\gamma(|\mathbf{x}|)| \leq s$ for some $s \in \mathbb{R}$. The same for $\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)$, for some $u \in \mathbb{R}$ $|\frac{\partial^2}{\partial x_i^2}\gamma(|\mathbf{x}|)| \leq u$ for all $r < |\mathbf{x}| < 2r$. The derivatives of $\gamma(|\mathbf{x}|)$ vanish everywhere except on $B_{2r}(0) \setminus B_r(0)$.

$$||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 \leq \frac{4Trsn+nu}{|\lambda_{p,T}|}\int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2)dx_1\wedge\ldots\wedge dx_n$$

By definition $\lambda > 0$, see (7.1). Set $C = \frac{4Trsn+nu}{|\lambda_{p,T}|}$.

$$||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 \ \leq \ C\int_{B_{2r}(0)\setminus B_r(0)} \exp(-T|\mathbf{x}|^2)dx_1\wedge\ldots\wedge dx_n$$

We can bound the function $\exp(-T|\mathbf{x}|^2)$ for $\exp(-T(2r)^2) = \exp(-4Tr^2)$ on $B_{2r}(0) \setminus B_r(0)$.

$$\begin{aligned} ||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &\leq C\exp(-4Tr^2)\int_{B_{2r}(0)\setminus B_r(0)} dx_1\wedge\ldots\wedge dx_n \\ &= C\exp(-4Tr^2)C'((2r)^n - r^n) \end{aligned}$$

Let $C_0 = CC'((2r)^n - r^n)$, since $\exp(-4Tr^2) = \frac{1}{\sum\limits_{n=0}^{\infty}\frac{(4Tr^2)^n}{n!}}$, we take the largest element of the sum, say $\frac{(4Tr^2)^N}{N!}$, there exist $T' > 0$ such that

$$\begin{aligned} ||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0^2 &\leq C_0\frac{1}{\frac{(4Tr^2)^N}{N!}} \leq \frac{1}{T'} \\ ||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0 &\leq \frac{1}{T'^{\frac{1}{2}}} \end{aligned} \tag{7.15}$$

Note that $\frac{1}{T'^{\frac{1}{2}}}$ is too small. This for all $p \in \mathrm{Crit}(f)$.

By equality (7.10) and Lemma 7.2.1 then

$$\begin{aligned} ||\mathrm{D}_{T,2}\omega||_0 &= ||\sum_{p\in\mathrm{Crit}(f)}\langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T},\omega\rangle\widetilde{\omega}_{p,T}||_0 \\ &\leq \sum_{p\in\mathrm{Crit}(f)}||\langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T},\omega\rangle\widetilde{\omega}_{p,T}||_0 \\ &\leq \sum_{p\in\mathrm{Crit}(f)}|\langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T},\omega\rangle|\,||\widetilde{\omega}_{p,T}||_0 \\ &= \sum_{p\in\mathrm{Crit}(f)}|\langle \mathrm{D}_{Tf}\widetilde{\omega}_{p,T},\omega\rangle|. \end{aligned}$$

And by inequalities (7.11), (7.15)

$$||\mathrm{D}_{T,2}\omega||_0 \leq \sum_{p\in\mathrm{Crit}(f)}||\mathrm{D}_{Tf}\widetilde{\omega}_{p,T}||_0||\omega||_0.$$

There is a constant $T'' > 0$ such that for all $T \geq T''$

$$||\mathrm{D}_{T,2}\omega||_0 \leq \frac{||\omega||_0}{T}.$$

□

Proposition 7.2.6 ([41, Prop. 4.12])**.** *There exist* $T_2 > 0$ *and* $C > 0$ *such that for any* $\omega \in E_T^{\perp} \cap H^1(M)$ *and* $T \geq T_2$

$$||\mathrm{D}_{Tf}\omega||_0 \geq C\sqrt{T}||\omega||_0.$$

For the following proof it will be necessary to change the field of the vector space $\Omega^{\bullet}(M)$ of real numbers to that of complex numbers and extend the inner product to $\mathbb{C}$.

We define

$$\Omega^{\bullet}_{\mathbb{C}}(M) := \mathbb{C} \otimes_{\mathbb{R}} \Omega^{\bullet}(M).$$

Where $\otimes_{\mathbb{R}}$ means that we see $\mathbb{C}$ as a real vector space (of real dimension 2) and we consider the tensor product with $\Omega^{\bullet}(M)$.

For all $\lambda \in \mathbb{C}$, $\lambda \cdot (z \otimes \omega) = (\lambda z) \otimes \omega$, with λz the multiplication of complex numbers. By doing this we have the *complexification* $\Omega^{\bullet}_{\mathbb{C}}(M)$.

Also, we define the inner product over $\Omega^{\bullet}_{\mathbb{C}}(M)$, using the inner product (3.6) over $\Omega^{\bullet}_{\mathbb{R}}(M)$,

$$\begin{aligned} \langle \omega, i \otimes \eta \rangle_{\mathbb{C}} &= -i\langle \omega, \eta \rangle_{\mathbb{R}}, && (7.16)\\ \langle i \otimes \omega, \eta \rangle_{\mathbb{C}} &= i\langle \omega, \eta \rangle_{\mathbb{R}}. && (7.17) \end{aligned}$$

Considering this inner product over $\Omega^{\bullet}_{\mathbb{C}}(M)$ and the associated norm $|| \ ||_{0,\mathbb{C}}$, we have the 0–Sobolev space of differentiable forms $\Omega^{\bullet}_{\mathbb{C}}(M)$, $H^0_{\bullet}(M)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} H^0_{\bullet}(M)_{\mathbb{R}}$.

We will specify if the norm and the Sobolev space are over the field $\mathbb{R}$ or $\mathbb{C}$ by writing it as a subscript.

With respect to $\mathbb{C}$, recall the *arc length* of a curve $z \colon [a,b] \longrightarrow \mathbb{C}$ given by equation

$$z(t) = x(t) + iy(t), \quad t \in [a,b]. \tag{7.18}$$

is define by

$$L = \int_a^b |z'(t)|dt, \tag{7.19}$$

where $|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ is the modulus of $z'(t)$.

Let C be the contour represented by the equation (7.18) and $f \colon \mathbb{C} \longrightarrow \mathbb{C}$ be a complex-valued function $f(z) = u(x,y) + iv(x,y)$ such that $u[x(t), y(t)]$ and $v[x(t), y(t)]$ of $f[z(t)]$ are piecewise continuous functions of t. Then

$$\left|\int_C f(z)dz\right| \leq \int_a^b |f[z(t)]z'(t)|dt. \tag{7.20}$$

If there exist a constant c such that $|f(z)| \leq c$ whenever z is on the contour C, by equalities (7.20) and (7.19) then

$$\left|\int_C f(z)dz\right| \leq c\int_a^b |z'(t)|dt = cL. \tag{7.21}$$

For more details see [10].

Let $c \in \mathbb{R}, c > 0$ be a constant, $E_T(c) \subset H^0_\bullet(M)_\mathbb{R}$ be the direct sum of eigenspaces of D_{Tf} corresponding to the eigenvalues in the interval $[-c, c]$. Note that by Lemma 5.0.2 D_{Tf} is self-adjoint and by Theorem D.1.26 the eigenvalues of D_{Tf} are real.

Let $c \in \mathbb{R}, c > 0$, we consider $\Pr(c)\colon H^0_\bullet(M)_\mathbb{R} \longrightarrow E_T(c)$ be the spectral projection onto $E_T(c)$, see Definition D.1.30.

Proposition 7.2.7. *There exist $C_1 > 0, T_3 > 0$ such that for any $T \geq T_3$ and any $\omega \in E_T$ holds that*

$$||\Pr(c)\omega - \omega||_{0,\mathbb{R}} \leq \frac{C_1}{T}||\omega||_{0,\mathbb{R}}.$$

Proof. Let $S = \{\lambda \in \mathbb{C} |\, |\lambda| = c\}$ be the counterclockwise oriented circle of radius c.

Let $\lambda \in S$, $T \geq T_1 + T_2$ as in Propositions 7.2.5-2 and 7.2.6.

Let $\omega \in H^1_\bullet(M)_\mathbb{R}$, by Remark D.3.8 $H^1_\bullet(M)_\mathbb{R} \subset H^0_\bullet(M)_\mathbb{R}$ and by decomposition (7.3) one can see that $\omega = \mathrm{pr}\omega + \mathrm{pr}^\perp\omega$.

Using the projections, definitions (7.4), (7.5) (7.6), (7.7) and Proposition 7.2.5-1, we have two cases:

1. Since $\mathrm{pr}\omega \in E_T$, we get

$$\mathrm{D}_{T,1}\mathrm{pr}\omega = 0, \quad \mathrm{D}_{T,2}\mathrm{pr}\omega = 0, \quad \mathrm{D}_{T,3}\mathrm{pr}\omega \neq 0 \quad \text{and} \quad \mathrm{D}_{T,4}\mathrm{pr}\omega = 0. \tag{7.22}$$

2. Since $\mathrm{pr}^\perp\omega \in E_T^\perp$, then

$$\mathrm{D}_{T,1}\mathrm{pr}^\perp\omega = 0, \quad \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega \neq 0, \quad \mathrm{D}_{T,3}\mathrm{pr}^\perp\omega = 0 \quad \text{and} \quad \mathrm{D}_{T,4}\mathrm{pr}^\perp\omega \neq 0. \tag{7.23}$$

We need to take the complexification $H^0_\bullet(M)_\mathbb{C}$.

Consider $\mathrm{D}_{Tf}\colon H^0_\bullet(M)_\mathbb{C} \longrightarrow H^0_\bullet(M)_\mathbb{C}$ given by $\mathrm{D}_{Tf}(z \otimes \omega) = z \otimes \mathrm{D}_{Tf}(\omega)$.

Since $\omega \in H^1_\bullet(M)_\mathbb{R}$, we can write $\omega = 1 \otimes \omega$ with $1 \in \mathbb{R}$, so that $\omega \in H^0_\bullet(M)_\mathbb{C}$.

$$\mathrm{pr}\omega, \mathrm{pr}^\perp\omega, \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega, \mathrm{D}_{T,3}\mathrm{pr}\omega, \mathrm{D}_{T,4}\mathrm{pr}^\perp\omega, \mathrm{D}_{Tf}\omega, \mathrm{D}_{Tf}\mathrm{pr}^\perp\omega \in H^0_\bullet(M)_\mathbb{R},$$

so if take the complex norm $||\ \ ||_{0,\mathbb{C}}$ of each of these elements it coincides with the real norm $||\ \ ||_{0,\mathbb{R}}$. Then we can use the estimation results 7.2.5 and 7.2.6 without problem.

By equalities (7.22) and (7.23), we get

$$\begin{aligned} ||(\lambda - \mathrm{D}_{Tf})\omega||_{0,\mathbb{C}} &= ||\lambda \otimes \mathrm{pr}\omega + \lambda \otimes \mathrm{pr}^\perp\omega - \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega - \mathrm{D}_{T,3}\mathrm{pr}\omega - \mathrm{D}_{T,4}\mathrm{pr}^\perp\omega||_{0,\mathbb{C}} \\ &= ||(\lambda \otimes \mathrm{pr}\omega - \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega) + (\lambda \otimes \mathrm{pr}^\perp\omega - \mathrm{D}_{T,3}\mathrm{pr}\omega - \mathrm{D}_{T,4}\mathrm{pr}^\perp\omega)||_{0,\mathbb{C}}. \end{aligned}$$

Since $\lambda \otimes \mathrm{pr}\omega - \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega \in E_T$, $\lambda \otimes \mathrm{pr}^\perp\omega - \mathrm{D}_{T,3}\mathrm{pr}\omega - \mathrm{D}_{T,4}\mathrm{pr}^\perp\omega \in E_T^\perp$ and $E_T, E_T^\perp$ are orthogonal, by Lemma A.5.3, then

$$||(\lambda - \mathrm{D}_{Tf})\omega||_{0,\mathbb{C}} \geq \frac{1}{2}||\lambda \otimes \mathrm{pr}\omega - \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega||_{0,\mathbb{C}} + \frac{1}{2}||\lambda \otimes \mathrm{pr}^\perp\omega - (\mathrm{D}_{T,3}\mathrm{pr}\omega + \mathrm{D}_{T,4}\mathrm{pr}^\perp\omega)||_{0,\mathbb{C}}.$$

First, of the term $\frac{1}{2}\Big(||\lambda \otimes \mathrm{pr}\omega - \mathrm{D}_{T,2}\mathrm{pr}^\perp\omega||_{0,\mathbb{C}}\Big)$.

By triangle inequality, see inequality (D.1),

$$\frac{1}{2}\Big(||\lambda \otimes \mathrm{pr}\omega - \mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}\Big) \geq \frac{1}{2}(||\lambda \otimes \mathrm{pr}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}).$$

Since $|\lambda| = c$,

$$\frac{1}{2}(||\lambda \otimes \mathrm{pr}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}) = \frac{1}{2}(c||\mathrm{pr}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}) \tag{7.24}$$

For the term $\frac{1}{2}(||\lambda \otimes \mathrm{pr}^{\perp}\omega - \mathrm{D}_{T,3}\mathrm{pr}\omega - \mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}})$. Note that by inequality (D.1):

$$\begin{aligned} ||\lambda \otimes \mathrm{pr}^{\perp}\omega - \mathrm{D}_{T,3}\mathrm{pr}\omega - \mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} &= ||-\lambda \otimes \mathrm{pr}^{\perp}\omega + \mathrm{D}_{T,3}\mathrm{pr}\omega + \mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \\ &= ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega - \lambda \otimes \mathrm{pr}^{\perp}\omega + \mathrm{D}_{T,3}\mathrm{pr}\omega||_{0,\mathbb{C}} \\ &\geq ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega - \lambda \otimes \mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,3}\mathrm{pr}\omega||_{0,\mathbb{C}} \\ &\geq ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - ||\lambda \otimes \mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,3}\mathrm{pr}\omega||_{0,\mathbb{C}} \end{aligned}$$

Since $|\lambda| = c$ and $\mathrm{pr}\omega \in E_T$, by Proposition 7.2.5-2 there exist a constant $T_1 > 0$ such that for any $T \geq T_1$ we obtain

$$||\lambda \otimes \mathrm{pr}^{\perp}\omega - \mathrm{D}_{T,3}\mathrm{pr}\omega - \mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \geq ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - c||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - \frac{||\mathrm{pr}\omega||_{0,\mathbb{C}}}{T} \tag{7.25}$$

By equalities (7.24) and (7.25) then:

$$||(\lambda - \mathrm{D}_{Tf})\omega||_{0,\mathbb{C}} \geq \frac{1}{2}\left(c||\mathrm{pr}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} + ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - c||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - \frac{||\mathrm{pr}\omega||_{0,\mathbb{C}}}{T}\right)$$

By definitions (7.4), (7.5), (7.6), (7.7) and triangle inequality we have

$$||\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} = ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega + \mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \leq ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} + ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}.$$

On the other hand, since $\mathrm{pr}^{\perp}\omega \in E_T^{\perp} \cap H^1_{\bullet}(M)_{\mathbb{R}}$ and by Proposition 7.2.6, there exist $T_2 > 0$ and $C > 0$ such that for all $T \geq T_2$, $C\sqrt{T}||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \leq ||\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}$. Then

$$C\sqrt{T}||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \leq ||\mathrm{D}_{T,4}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}.$$

Substituting this inequality and since $\mathrm{pr}^{\perp}\omega \in E_T^{\perp} \cap H^1_{\bullet}(M)_{\mathbb{R}}$, by Proposition 7.2.5-2 there exist $T_1' > 0$, for any $T \geq T_1'$

$$\begin{aligned} ||(\lambda - \mathrm{D}_{Tf})\omega||_{0,\mathbb{C}} &\geq \frac{1}{2}\left(c||\mathrm{pr}\omega||_{0,\mathbb{C}} - 2||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} + C\sqrt{T}||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - c||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - \frac{||\mathrm{pr}\omega||_{0,\mathbb{C}}}{T}\right) \\ &= \frac{1}{2}\left(c - \frac{1}{T}\right)||\mathrm{pr}\omega||_{0,\mathbb{C}} + \frac{1}{2}(C\sqrt{T} - c)||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} - ||\mathrm{D}_{T,2}\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \\ &\geq \frac{1}{2}\left(c - \frac{1}{T}\right)||\mathrm{pr}\omega||_{0,\mathbb{C}} + \frac{1}{2}\left(C\sqrt{T} - c - \frac{2}{T}\right)||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}} \\ &\geq \frac{1}{2}\left(c - \frac{1}{T}\right)||\mathrm{pr}\omega||_{0,\mathbb{C}} + \frac{1}{2}\left(C\sqrt{T} - c - \frac{1}{T}\right)||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}. \end{aligned}$$

There exist $A_2, B_2 > 0$ constants such that $(c-\frac{1}{T}) \geq A_2 > 0$ and $(C\sqrt{T}-c-\frac{1}{T}) \geq B_2 > 0$ for all $T > T_1 + T_2$, we take $C_2 = \min\{A_2, B_2\}$ such that

$$||(\lambda - \mathrm{D}_{Tf})\omega||_{0,\mathbb{C}} \geq \frac{C_2}{2}(||\mathrm{pr}\omega||_{0,\mathbb{C}} + ||\mathrm{pr}^{\perp}\omega||_{0,\mathbb{C}}).$$

There exist $C_3 > 0$ constant $C_3 \leq \frac{C_2}{2}$ such that

$$||(\lambda - \mathrm{D}_{Tf})\omega||_{0,\mathbb{C}} \geq C_3||\omega||_{0,\mathbb{C}}. \tag{7.26}$$

Therefore, $\lambda - \mathrm{D}_{Tf}$ is bounded below.

Therefore $\lambda - \mathrm{D}_{Tf}\colon H^1_\bullet(M) \longrightarrow H^0_\bullet(M)$ is a bounded operator.

By Lemma D.1.20 and inequality (7.26), the operator

$$(\lambda - \mathrm{D}_{Tf})^{-1}\colon H^0_\bullet(M) \longrightarrow H^1_\bullet(M),$$

exists and is bounded. Let $\lambda \in \rho(\lambda - D_{Tf})$, then we can define the resolvent operator, see definition D.1.27, by

$$R_\lambda(\mathrm{D}_{Tf})\omega := (\lambda - \mathrm{D}_{Tf})^{-1}\omega.$$

and by Theorem D.1.28 $R_\lambda(\mathrm{D}_{Tf})\colon H^0_\bullet(M)_\mathbb{C} \longrightarrow H^0_\bullet(M)_\mathbb{C}$.

We take $\lambda = r\exp(i\theta), r > 0, -\pi < \theta < \pi$, then

$$\int_S \lambda^{-1} d\lambda = \int_0^{2\pi} \exp(-i\theta) i \exp(i\theta) dt = 2\pi i.$$

Therefore $\frac{1}{2\pi i}\int_S \lambda^{-1}d\lambda = 1$. By Definition D.1.30 we get

$$\mathrm{Pr}(c)\omega - \omega = \frac{1}{2\pi i}\int_S \left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1}\right)\omega d\lambda, \quad \omega \in H^0_\bullet(M)_\mathbb{R} \tag{7.27}$$

Since $(\lambda - \mathrm{D}_{Tf})^{-1}(\lambda - \mathrm{D}_{Tf}) = \mathrm{id}$ and multiplying by λ^{-1}, we get

$$(\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1}\mathrm{id} = \lambda^{-1}(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{Tf}.$$

Applying to $\omega \in E_T$, and by Proposition 7.2.5-1, we have

$$\left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1}\right)\omega = \lambda^{-1}(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega.$$

Taking $\eta = (\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega$ and by inequality (7.26),

$$||(\lambda - \mathrm{D}_{Tf})(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}} \geq C_3||(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}},$$

and $||(\lambda - \mathrm{D}_{Tf})(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}} = ||\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}}$, then $||\mathrm{D}_{T,3}\omega||_0 \geq C_3||(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}}$. By Proposition 7.2.5-2 there exist $T_4 > 0$, such that for all $T \geq T_4 > 0$ and $\omega \in E_T$

$$||(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}} \leq \frac{||\omega||_{0,\mathbb{C}}}{C_3 T}$$

Then:

$$\begin{aligned} || \left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1} \right) \omega ||_{0,\mathbb{C}} &= ||\lambda^{-1}(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}} \\ &= |\lambda^{-1}| \, ||(\lambda - \mathrm{D}_{Tf})^{-1}\mathrm{D}_{T,3}\omega||_{0,\mathbb{C}}. \end{aligned}$$

$$|| \left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1} \right) \omega ||_{0,\mathbb{C}} \leq \frac{c}{C_3 T} ||\omega||_{0,\mathbb{C}}. \tag{7.28}$$

By equality (7.27)

$$\begin{aligned} ||\mathrm{Pr}(c)\omega - \omega||_{0,\mathbb{C}} &= \frac{1}{2\pi} || \frac{1}{i} \int_S \left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1} \right) \omega d\lambda ||_{0,\mathbb{C}} \\ &= \frac{1}{2\pi} \left| \frac{1}{i} \right| \, || \int_S \left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1} \right) \omega d\lambda ||_{0,\mathbb{C}} \end{aligned}$$

Therefore, by inequalities (7.20) and (7.28) we get

$$\begin{aligned} ||\mathrm{Pr}(c)\omega - \omega||_{0,\mathbb{C}} &\leq \frac{1}{2\pi} \int_S || \left((\lambda - \mathrm{D}_{Tf})^{-1} - \lambda^{-1} \right) z'(t)\omega ||_{0,\mathbb{C}} d\lambda \\ &\leq \frac{c}{2\pi C_3 T} ||\omega||_{0,\mathbb{C}} \int_0^{2\pi} |z'(t)| dt. \end{aligned}$$

Taking the length of the curve $z\colon [0, 2\pi] \longrightarrow \mathbb{R}^2$ given by $z(t) = (\cos t, \sin t)$, then $|z'(t)| = 1$ and

$$\int_0^{2\pi} |z'(t)| dt = \int_0^{2\pi} dt = 2\pi.$$

By inequality (7.21), therefore

$$||\mathrm{Pr}(c)\omega - \omega||_{0,\mathbb{C}} \leq \frac{c}{C_3 T} ||\omega||_{0,\mathbb{C}}. \tag{7.29}$$

Since $\omega, \mathrm{Pr}(c)\omega \in H^0_\bullet(M)_\mathbb{R}$ the norm $||\ ||_{0,\mathbb{C}}$ coincides with $||\ ||_{0,\mathbb{R}}$, Therefore Proposition 7.2.7 is satisfied. ☐

Let $\mathrm{F}^{[0,c']}_{Tf,k} \subset \Omega^k(M)$ be the vector space generated by the eigenspaces of $\square_{Tf,k}$ associated with eigenvalues in $[0, c']$ with $0 \leq k \leq n$. We will to describe this vector space of $\square_{Tf,k}$.

Theorem 7.2.8. *Let M be an oriented, closed Riemannian n–manifold, $T \in \mathbb{R}, T > 0$ and $f\colon M \longrightarrow \mathbb{R}$ be a Morse function. For any $0 < c' \in \mathbb{R}$ there exist a $0 < T_0 \in \mathbb{R}$ such that for every $T \geq T_0$*

$$\dim\left(\mathrm{F}^{[0,c']}_{Tf,k}\right) = m_k.$$

Proof. First let us see that there exists T sufficiently large such that $\{\mathrm{Pr}(c)\widetilde{\omega}_{p,T}\}_{p\in\mathrm{Crit}(f)}$ is a linearly independent set.

Since M is a compact manifold, by Corollary 2.0.8 the set of critical points of f is finite, we can assume that $|\mathrm{Crit}(f)| = r$.

Let us suppose that

$$\sum_{i=1}^{r} a_i \mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} = 0, \quad a_i \in \mathbb{R}.$$

Since $\operatorname{Pr}(c)$ is a linear map, then

$$\sum_{i=1}^{r} a_i \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T} = \operatorname{Pr}(c)\left(\sum_{i=1}^{r} a_i \widetilde{\omega}_{p_i,T}\right).$$

We denote by $\eta = \sum\limits_{i=1}^{r} a_i\widetilde{\omega}_{p_i,T}$, note that $\eta \in E_T$. Since $\operatorname{Pr}(c)\eta = 0$ then $\eta \in (E_T(c))^{\perp}$.

By contradiction, assume that $||\eta||_0 > 0$.

By Proposition 7.2.7 there exists $C_1 > 0, T_3 > 0$ such that for all $T \geq T_3$

$$||\operatorname{Pr}(c)\eta - \eta||_0 \leq \frac{C_1}{T}||\eta||_0.$$

But $||\operatorname{Pr}(c)\eta - \eta||_0 = ||\eta||_0$, then $||\eta||_0 \leq \frac{C_1}{T}||\eta||_0$ this is true if and only if $C_1 > T$, this contradicts the hypothesis that for all $T \geq T_3$. Then $||\eta||_0 = 0$, so $\eta = 0$.

Since $\{\widetilde{\omega}_{p_i,T}\}_{p_i \in \operatorname{Crit}(f)}$ is a linearly independent set, then $a_i = 0$ for all $i = 1, \ldots, r$. Therefore $\{\operatorname{Pr}(c)\widetilde{\omega}_{p,T}\}_{p \in \operatorname{Crit}(f)}$ is a linearly independent set if $T < C_1$.

Then, there must be a $T_5 > 0$ such that for $T \geq T_5$ implies

$$\dim E_T(c) \geq \dim \operatorname{Pr}(c)(E_T) = \dim E_T. \tag{7.30}$$

Let us see that the equality holds.

By contradiction, assume we have $\dim E_T(c) > \dim E_T$, there is a nonzero element $\omega \in E_T(c)$ such that $\omega \in (\operatorname{Pr}_T(c)(E_T))^{\perp}$, that is, for every $i = 1, \ldots, r$

$$\langle \omega, \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle = 0.$$

Also for all $i = 1, \ldots, r$, we get

$$\langle \omega, \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T} = 0.$$

In particular,

$$\sum_{i=1}^{r}\langle \omega, \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T} = 0. \tag{7.31}$$

By equality (7.31), adding and subtracting the term $\sum\limits_{i=1}^{r}\langle \omega, \widetilde{\omega}_{p_i,T}\rangle \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}$, implies

$$\begin{aligned}
\operatorname{pr}\omega &= \sum_{i=1}^{r}\langle \omega, \widetilde{\omega}_{p_i,T}\rangle \widetilde{\omega}_{p_i,T} - \sum_{i=1}^{r}\langle \omega, \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle \widetilde{\omega}_{p_i,T} \\
&= \sum_{i=1}^{r}\langle \omega, \widetilde{\omega}_{p_i,T}\rangle (\widetilde{\omega}_{p_i,T} - \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}) + \sum_{i=1}^{r}\langle \omega, \widetilde{\omega}_{p_i,T} - \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle \operatorname{Pr}(c)\widetilde{\omega}_{p_i,T}.
\end{aligned}$$

Since the inner product is bilinear and by the Cauchy-Schwarz inequality we have

$$||\operatorname{pr}\omega||_0^2 = \langle \operatorname{pr}\omega, \operatorname{pr}\omega\rangle$$

$$\begin{aligned}
&\leq \left|\sum_{i=1}^{r}\langle\omega,\widetilde{\omega}_{p_i,T}\rangle^2||\widetilde{\omega}_{p_i,T}-\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2\right.\\
&\qquad\left.+\sum_{i=1}^{r}\langle\omega,\widetilde{\omega}_{p_i,T}-\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle^2||\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2\right|\\
&\leq \left|\sum_{i=1}^{r}\langle\omega,\widetilde{\omega}_{p_i,T}\rangle^2\right|||\widetilde{\omega}_{p_i,T}-\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2\\
&\qquad+\left|\sum_{i=1}^{r}\langle\omega,\widetilde{\omega}_{p_i,T}-\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}\rangle^2\right|||\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2\\
&\leq \sum_{i=1}^{r}||\omega||_0^2||\widetilde{\omega}_{p_i,T}||_0^2||\widetilde{\omega}_{p_i,T}-\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2\\
&\qquad+\sum_{i=1}^{r}||\omega||_0^2||\widetilde{\omega}_{p_i,T}-\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2||\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2.
\end{aligned}$$

By Lemma 7.2.1 and Proposition 7.2.7, for each $p_i \in \mathrm{Crit}(f)$ there exists $C_i, T_i > 0$ such that for all $T \geq \sum_{i=1}^{r} T_i$,

$$\begin{aligned}
||\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}||_0^2 &= ||\widetilde{\omega}_{p_i,T}-(1-\mathrm{Pr}(c))\widetilde{\omega}_{p_i,T}||_0^2\\
&\leq (||\widetilde{\omega}_{p_i,T}||_0+||(1-\mathrm{Pr}(c))\widetilde{\omega}_{p_i,T}||_0)^2\\
&\leq \left(1+\frac{C_i}{T}||\widetilde{\omega}_{p_i,T}||_0\right)^2\\
&= \left(1+\frac{C_i}{T}\right)^2.
\end{aligned}$$

By Lemma 7.2.1, then

$$\begin{aligned}
||\mathrm{pr}\omega||_0^2 &\leq \sum_{i=1}^{r}||\omega||_0^2\left(\frac{C_i}{T}\right)^2||\widetilde{\omega}_{p_i,T}||_0^2+\sum_{i=1}^{r}||\omega||_0^2\left(\frac{C_i}{T}\right)^2||\widetilde{\omega}_{p_i,T}||_0^2\left(1+\frac{C_i}{T}\right)^2\\
&\leq \sum_{i=1}^{r}\left(\frac{C_i}{T}\right)^2\left(1+\left(1+\frac{C_i}{T}\right)^2\right)||\omega||_0^2
\end{aligned}$$

Since $i = 1,\ldots,r$, then $\left(\frac{C_i}{T}\right)^2\left(1+\left(1+\frac{C_i}{T}\right)^2\right) = \frac{C'}{T^2}$ for some $C' > 0$ such that for all $T > \sum_{i=1}^{r} T_i$ implies

$$||\mathrm{pr}\omega||_0 \leq \frac{\sqrt{C'}}{T}||\omega||_0. \tag{7.32}$$

Let $\omega \in H^1_\bullet(M)$, $\omega = \mathrm{pr}\omega + \mathrm{pr}^\perp\omega$, then

$$||\mathrm{pr}^\perp\omega||_0 = ||\omega - \mathrm{pr}\omega||_0 \geq ||\omega||_0 - ||\mathrm{pr}\omega||_0.$$

By inequality (7.32), let $C_4 = (1-\frac{\sqrt{C'}}{T}) > 0$ be a constant and when T is large enough such that

$$||\mathrm{pr}^\perp\omega||_0 \geq ||\omega||_0 - ||\mathrm{pr}\omega|| \geq ||\omega||_0 - \frac{\sqrt{C'}}{T}||\omega||_0 = C_4||\omega||_0$$

Now, by Proposition 7.2.6 there exists $C>0$ and $T'>0$ such that for any $T \geq T'$ we have

$$C\sqrt{T}C_4||\omega||_0 \leq C\sqrt{T}||\mathrm{pr}^{\perp}\omega||_0 \leq ||\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega||_0 \tag{7.33}$$

Since $\omega \in E_T(c)$, by definitions (7.4) - (7.7) and Proposition 7.2.5-1 we obtain $\mathrm{D}_{Tf}\mathrm{pr}\omega = \mathrm{D}_{T,3}\omega$. By Proposition 7.2.5-2, there exist $T_1' > 0$ such that for all $T \geq T_1'$ we have

$$||\mathrm{D}_{Tf}\mathrm{pr}^{\perp}\omega||_0 = ||\mathrm{D}_{Tf}\omega - \mathrm{D}_{Tf}\mathrm{pr}\omega||_0 \leq ||\mathrm{D}_{Tf}\omega||_0 + ||\mathrm{D}_{T,3}\mathrm{pr}\omega||_0 \leq ||\mathrm{D}_{Tf}\omega||_0 + \frac{||\omega||_0}{T}.$$

By equality (7.33) rewriting and taking $C_5 = CC_4\sqrt{T} - \frac{1}{T}$ and for all $T \geq T_1' + T'$ we get

$$C_5||\omega||_0 = CC_4\sqrt{T}||\omega||_0 - \frac{||\omega||_0}{T} \leq ||\mathrm{D}_{Tf}\omega||_0.$$

Since $\omega \in E_T(c), \omega = \sum\limits_{\lambda \in [-c,\,c]} a_\lambda \omega_\lambda$, where λ is the eigenvalue of the eigenform ω_λ, whose eigenspace we will denote by E_λ. Since D_{Tf} is a linear map, then

$$C_5||\omega||_0^2 = C_5||\sum_{\lambda \in [-c,\,c]} a_\lambda\omega_\lambda||_0^2 \leq ||\mathrm{D}_{Tf}\left(\sum_{\lambda \in [-c,\,c]} a_\lambda\omega_\lambda\right)||_0^2 = ||\sum_{\lambda \in [-c,\,c]} a_\lambda \mathrm{D}_{Tf}(\omega_\lambda)||_0^2 = ||\sum_{\lambda \in [-c,\,c]} a_\lambda\lambda\omega_\lambda||_0^2$$

For eigenforms corresponding to different eigenvalues of D_{Tf} by Theorem D.1.26-2, we have

$$C_5||\omega||_0^2 \leq \sum_{\lambda \in [-c,\,c]} |\lambda|^2||a_\lambda\omega_\lambda||_0^2 \leq c^2 \sum_{\lambda \in [-c,c]} ||a_\lambda\omega_\lambda||_0^2 = c^2||\omega||_0^2.$$

This is true if and only if $C_5 = CC_4\sqrt{T} - \frac{1}{T} \leq c$ this contradicts the hypothesis that for all $T \geq T_1' + T'$.

Therefore,

$$\dim E_T(c) \geq \dim \mathrm{Pr}(c)(E_T) = \dim E_T = \sum_{i=1}^{n} m_i. \tag{7.34}$$

Then $\{\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T}\}_{p\in\mathrm{Crit}(f)}$ generates $E_T(c)$, therefore, $\{\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T}\}_{p\in\mathrm{Crit}(f)}$ form a basis for $E_T(c)$.

We will give a decomposition of $E_T(c)$.

For each integer $0 \leq k \leq n$, we define $\mathrm{Pr}_k\colon H^0_{\bullet}(M) \longrightarrow H^0_k(M)$ the projection onto $H^0_k(M)$ the 0–Sobolev space of $\Omega^k(M)$ with respect to the $||\ ||_{0,\,k}$–norm.

First, since

$$\begin{aligned}
||\mathrm{Pr}_{\mathrm{n}_f(p)}\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T} - \widetilde{\omega}_{p,\,T}||_{0,\,\mathrm{n}_f(p)} &= ||\mathrm{Pr}_{\mathrm{n}_f(p)}\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T} - \mathrm{Pr}_{\mathrm{n}_f(p)}\widetilde{\omega}_{p,\,T}||_{0,\,\mathrm{n}_f(p)}\\
&= ||\mathrm{Pr}_{\mathrm{n}_f(p)}(\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T} - \widetilde{\omega}_{p,\,T})||_{0,\,\mathrm{n}_f(p)}\\
&\leq ||\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T} - \widetilde{\omega}_{p,\,T}||_{0,\,\bullet}
\end{aligned}$$

By Proposition 7.2.7 there exist $C_{\mathrm{n}_f(p)} > 0$, $T_{\mathrm{n}_f(p)} > 0$ such that for every $T \geq T_{\mathrm{n}_f(p)}$ and Lemma 7.2.1 we have that for every $p \in \mathrm{Crit}(f)$

$$||\mathrm{Pr}_{\mathrm{n}_f(p)}\mathrm{Pr}(c)\widetilde{\omega}_{p,\,T} - \widetilde{\omega}_{p,\,T}||_{0,\,\mathrm{n}_f(p)} \leq \frac{C_{\mathrm{n}_f(p)}}{T}||\widetilde{\omega}_{p,\,T}||_0 = \frac{C_{\mathrm{n}_f(p)}}{T}. \tag{7.35}$$

Also, we see that the set $\{\mathrm{Pr}_{\mathrm{n}_f(p)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T}\}_{i=1}^{r}$ is linearly independent.

Let us suppose that

$$\sum_{i=1}^{r} a_i \mathrm{Pr}_{\mathrm{n}_f(p)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} = 0, \quad a_i \in \mathbb{R}.$$

By Lemmas 7.2.2 and 7.2.1, then

$$||\sum_{i=1}^{r}\left(a_i\mathrm{Pr}_{\mathrm{n}_f(p_i)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} - a_i\widetilde{\omega}_{p_i,T}\right)||_{0,\bullet}^2 = ||\sum_{i=1}^{r} a_i\widetilde{\omega}_{p_i,T}||_{0,\bullet}^2 = \sum_{i=1}^{r}|a_i|^2\,||\widetilde{\omega}_{p_i,T}||_{0,\mathrm{n}_f(p_i)}^2 = \sum_{i=1}^{r}|a_i|^2.$$

On the other hand, by triangle inequality and inequality (7.35) for all $T \geq \sum_{i=1}^{r} T_i$

$$\begin{aligned} ||\sum_{i=1}^{r} a_i\left(\mathrm{Pr}_{\mathrm{n}_f(p_i)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} - \widetilde{\omega}_{p_i,T}\right)||_{0,\bullet} &\leq \sum_{i=1}^{r}||a_i\left(\mathrm{Pr}_{\mathrm{n}_f(p_i)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} - \widetilde{\omega}_{p_i,T}\right)||_{0,\bullet} \\ &= \sum_{i=1}^{r}||a_i\left(\mathrm{Pr}_{\mathrm{n}_f(p_i)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} - \widetilde{\omega}_{p_i,T}\right)||_{0,\mathrm{n}_f(p_i)} \\ &= \sum_{i=1}^{r}|a_i|\,||\left(\mathrm{Pr}_{\mathrm{n}_f(p_i)}\mathrm{Pr}(c)\widetilde{\omega}_{p_i,T} - \widetilde{\omega}_{p_i,T}\right)||_{0,\mathrm{n}_f(p_i)} \\ &\leq \sum_{i=1}^{r}|a_i|\frac{C_{\mathrm{n}_f(p_i)}}{T} \end{aligned}$$

Let $C_{\max} = \max\{C_{\mathrm{n}_f(p_i)}\}_{i=1}^{r}$, then $\sum_{i=1}^{r}|a_i|^2 \leq \frac{C_{\max}^2}{T^2}\left(\sum_{i=1}^{r}|a_i|\right)^2$ if and only if $T \leq rC_{\max}$.

We have that for T large enough, the $\{\mathrm{Pr}_{\mathrm{n}_f(p)}\mathrm{Pr}(c)\widetilde{\omega}_{p,T}\}_{p\in\mathrm{Crit}(f)}$ is a linearly independent set. We denote by p_{i_j} a critical point of $\mathrm{n}_f(p_{i_j}) = i$, where $j = 1,\dots,m_i$. Then

$$\mathrm{Crit}(f) = \{p_{1_1},\dots,p_{1_{m_1}},\dots,p_{r_1},\dots,p_{r_{m_r}}\}$$

Then

$$E_T(c) = \{\mathrm{Pr}(c)\widetilde{\omega}_{p_{1_1},T},\dots,\mathrm{Pr}(c)\widetilde{\omega}_{p_{1_{m_1}},T},\dots,\mathrm{Pr}(c)\widetilde{\omega}_{p_{r_1},T},\dots,\mathrm{Pr}(c)\widetilde{\omega}_{p_{r_{m_r}},T}\}$$

Now,

$$\mathrm{Pr}_k(E_T(c)) = \{\mathrm{Pr}_k(\mathrm{Pr}(c)\widetilde{\omega}_{p_{k_1},T}),\dots,\mathrm{Pr}_k(\mathrm{Pr}(c)\widetilde{\omega}_{p_{1_{m_k}},T})\}$$

Thus for each $0 \leq k \leq n$ we obtain

$$\dim \mathrm{Pr}_k(E_T(c)) \geq m_k.$$

Then

$$\sum_{k=0}^{n} m_k \leq \sum_{k=0}^{n}\dim \mathrm{Pr}_k(E_T(c)).$$

On the other hand, we define the operator

$$\mathrm{Pr} = \sum_{k=0}^{n} \mathrm{Pr}_k \colon H^0_\bullet(M) \longrightarrow H^0_\bullet(M).$$

Since for all $0 \leq k \leq n$, the $\mathrm{Pr}_k(E_T(c))$ are orthogonal to each other, by Theorem D.1.25-1. and -2. Pr is a projection onto $\bigoplus\limits_{k=0}^{n} \mathrm{Pr}_k(E_T(c))$.

And since Pr is a linear map, we get

$$\sum_{k=0}^{n} \dim \mathrm{Pr}_k(E_T(c)) = \dim \left(\bigoplus_{k=0}^{n} \mathrm{Pr}_k(E_T(c)) \right) = \dim \left(\mathrm{Pr}(E_T(c))\right) \leq \dim \left(E_T(c)\right) = \sum_{k=0}^{n} m_k.$$

Therefore, for any $0 \leq k \leq n$ we get

$$\dim \mathrm{Pr}_k(E_T(c)) = m_k. \tag{7.36}$$

Since $\Box_{Tf}$ preserves the grading of $\omega \in \Omega^\bullet(M)$, the following diagram commutes

$$\begin{tikzcd}
H^0_\bullet(M) \arrow[r, "\mathrm{D}^2_{Tf}"] \arrow[d, "\mathrm{Pr}_k"'] & H^0_\bullet(M) \arrow[d, "\mathrm{Pr}_k"] \\
H^0_k(M) \arrow[r, "\Box_{Tf,\,k}"'] & H^0_k(M)
\end{tikzcd}$$

Let $\omega \in E_T(c)$ an eigenform of D_{Tf} with eigenvalue $\lambda \in [-c,\, c]$, by the commutative diagram then

$$\Box_{Tf,\,k}\mathrm{Pr}_k\omega = \mathrm{D}^2_{Tf}\mathrm{Pr}_k\omega = \mathrm{Pr}_k\mathrm{D}^2_{Tf}\omega = \mathrm{Pr}_k\lambda(\mathrm{D}_{Tf}\omega) = \lambda^2\mathrm{Pr}_k\omega.$$

Then $\mathrm{Pr}_k E_T(c) = \mathrm{F}^{[0,\,c^2]}_{Tf,\,k}$.

Taking $c = \sqrt{c'}$, by equality (7.36), the Theorem follows.

$\Box$

Chapter 8

Proof of Morse Inequalities

Finally we will prove Morse inequalities mentioned in section 2.2.

As in the chapter 7, let $c \in \mathbb{R}$, $c > 0$ and A_ν be the eigenspace of $\square_{Tf,\,k}$ associated to the eigenvalue $\nu \in [0, c]$. We define $\mathrm{F}^{[0,c]}_{Tf,\,k} \subset \Omega^\bullet(M)$ the vector space generated by the eigenspaces of $\square_{Tf,\,k}$ associated with eigenvalues in $[0, c]$ with $0 \leq k \leq n$.

$$\mathrm{F}^{[0,c]}_{Tf,\,k} = \bigoplus_{\nu \in [0,c]} A_\nu. \tag{8.1}$$

Consider $\square_{Tf,k} \colon \mathrm{F}^{[0,c]}_{Tf,\,k} \longrightarrow \mathrm{F}^{[0,c]}_{Tf,\,k}$, equalities $d_{Tf}\square_{Tf,\,k}\omega = \square_{Tf,\,k+1}d_{Tf}\omega$ and $d^\star_{Tf}\square_{Tf,\,k}\omega = \square_{Tf,\,k-1}d^\star_{Tf}\omega$, see (5.5) and (5.6), imply that d_{Tf} and $d^\star_{Tf}$ restrict to

$$d_{Tf} \colon \mathrm{F}^{[0,c]}_{Tf,\,k} \longrightarrow \mathrm{F}^{[0,c]}_{Tf,\,k+1}$$

and

$$d^\star_{Tf} \colon \mathrm{F}^{[0,c]}_{Tf,\,k} \longrightarrow \mathrm{F}^{[0,c]}_{Tf,\,k-1}.$$

So we obtain a finite dimensional subcomplex of $(\Omega^\bullet(M), d_{Tf})$ defined by

$$(\mathrm{F}^{[0,c]}_{Tf,\,\bullet}, d_{Tf}) : \quad 0 \longrightarrow \mathrm{F}^{[0,c]}_{Tf,0} \xrightarrow{d_{Tf}} \mathrm{F}^{[0,c]}_{Tf,1} \xrightarrow{d_{Tf}} \mathrm{F}^{[0,c]}_{Tf,2} \xrightarrow{d_{Tf}} \ldots \xrightarrow{d_{Tf}} \mathrm{F}^{[0,c]}_{Tf,n} \longrightarrow 0 \,.$$

We define the k–th cohomology space by

$$H^k_F(M) = \frac{\operatorname{Ker} d_{Tf}\Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}}{\operatorname{Im} d_{Tf}\Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}}\,.$$

Remark 8.0.1. Remember that A_0 is the eigenspace of $\square_{Tf,\,k}$ associated to the eigenvalue 0, then $\operatorname{Ker}(\square_{Tf,\,k}) = A_0$ and by definition $A_0 \subset \mathrm{F}^{[0,c]}_{Tf,\,k}$ thus $A_0 = \operatorname{Ker}(\square_{Tf,\,k}\,|_{\mathrm{F}^{[0,c]}_{Tf,\,k}})$. Therefore $\operatorname{Ker}(\square_{Tf,\,k}) = \operatorname{Ker}(\square_{Tf,\,k}\,|_{\mathrm{F}^{[0,c]}_{Tf,\,k}})$.

Lemma 8.0.2. *Let M be a differentiable manifold of dimension n, $T \in \mathbb{R}$ and $f \colon M \longrightarrow \mathbb{R}$ be a Morse function. Then $H^k_F(M) \cong \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$. Therefore,*

$$\dim(H^k_F(M)) = \beta_k(M) \tag{8.2}$$

Proof. We denote the vector space of all closed k–forms under d_{Tf} by $Z^k_{Tf}(M)$, and we denote the vector space of all closed k–forms under $d_{Tf}\mid_{\mathrm{F}^{[0,c]}_{Tf,\,k}}$ by $Z^k_F(M)$.

On the other hand, we denote by $B^k_{Tf}(M)$ the vector space of all exact k–forms under d_{Tf}, and by $B^k_F(M)$ the vector space of all exact k–forms under $d_{Tf}\mid_{\mathrm{F}^{[0,c]}_{Tf,\,k}}$.

Since $\mathrm{F}^{[0,c]}_{Tf,k} \subset \Omega^k(M)$ we have $Z^k_{Tf}(M) \subset Z^k_F(M)$.

Consider the projections to the quotient spaces $\pi\colon Z^k_F(M) \longrightarrow H^k_F(M)$ and $\pi'\colon Z^k_{Tf}(M) \longrightarrow \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$.

Note that $B^k_F(M) = \mathrm{Ker}\,(\pi' \circ \iota)$, by First Isomorphism Theorem $\mathrm{Im}\,(\pi' \circ \iota) \cong H^k_F(M)$ and we have the following diagram

$$\begin{tikzcd}
Z^k_F(M) \arrow[r, "\iota"] \arrow[d, "\pi"'] & Z^k_{Tf}(M) \arrow[d, "\pi'"] \\
H^k_F(M) \arrow[r, "T"'] & \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)
\end{tikzcd}$$

Let us see that $\pi' \circ \iota$ is surjective.

Let $\alpha \in \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M)$ by Hodge Theorem (5.7) α has an harmonic representative, that is, $\alpha = [\omega]$ with $\omega \in A_0$. By Remark 8.0.1 $\omega \in Z^k_F(M)$. So $\pi' \circ \iota$ is surjective, therefore $\mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) \cong H^k_F(M)$.

By Proposition 5.0.1, $\dim\,(H^k_F(M)) = \beta_k(M)$. □

Corollary 8.0.3. *Let M be a differentiable manifold of dimension n, $T \in \mathbb{R}$ and $f\colon M \longrightarrow \mathbb{R}$ be a Morse function. Then*

$$\mathrm{Ker}\,(\square_{Tf,\,k}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) \cong H^k_F(M).$$

Theorem 8.0.4 (Morse inequalities)**.** *Let M be an oriented, closed Riemannian n–manifold. For any Morse function on M one has*

1. *(Weak Morse inequalities) For any $0 \le k \le n$, we have*

$$\beta_k(M) \le m_k. \tag{8.3}$$

2. *(Strong Morse inequalities) For any $0 \le k \le n$, we have*

$$\beta_k(M) - \beta_{k-1}(M) + \ldots + (-1)^k\beta_0(M) \le m_k - m_{k-1} + \ldots + (-1)^k m_0. \tag{8.4}$$

   Moreover, for $k = n$:

$$\beta_n(M) - \beta_{n-1}(M) + \ldots + (-1)^n\beta_0(M) = m_n - m_{n-1} + \ldots + (-1)^n m_0. \tag{8.5}$$

Now, we are ready to prove the Morse inequalities.

Proof. We will assume T large enough, so that Theorem 7.2.8 is true, that is,

$$\dim \mathrm{F}^{[0,c]}_{Tf,\,k} = m_k.$$

Since $\operatorname{Ker}(\square_{Tf,\,k}\,|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) \subset \mathrm{F}^{[0,c]}_{Tf,\,k}$, implies

$$\dim\left(\operatorname{Ker}(\square_{Tf,\,k}\,|_{\mathrm{F}^{[0,c]}_{Tf,\,k}})\right) \leq m_k. \tag{8.6}$$

Therefore, by inequality (8.6), Remark 8.0.1 and the analogue of Hodge Theorem (5.7), we get

$$\beta_k(M) = \dim \mathrm{H}^k_{Tf,\,\mathrm{DR}}(M) = \dim \operatorname{Ker}(\square_{Tf,\,k}) = \dim \operatorname{Ker}(\square_{Tf,\,k}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) \leq m_k.$$

This proves the weak Morse inequality.

Now, to show the inequalities (8.4) and (8.5), let us note that from the complex $(\mathrm{F}^{[0,c]}_{Tf,\,\bullet}, d_{Tf})$, by Rank-Nullity Theorem we have:

$$\begin{aligned} m_k &= \dim \mathrm{F}^{[0,c]}_{Tf,\,k} \\ &= \dim \operatorname{Ker}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}) \end{aligned}$$

By the dimension of the quotient vector space and Lemma 8.0.2

$$\begin{aligned} m_k &= \dim\left(\frac{\operatorname{Ker} d_{Tf}\Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}}{\operatorname{Im} d_{Tf}\Big|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}}\right) + \dim \operatorname{Im}\left(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}\right) + \dim \operatorname{Im}\left(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}\right) \\ &= \beta_k(M) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k-1}}) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,k}}). \end{aligned}$$

For $0 \leq l \leq n$, we take alternating the sum of the m_k to get

$$\begin{aligned} \sum_{k=0}^{l}(-1)^k m_{l-k} &= \sum_{k=0}^{l}(-1)^k\left(\beta_{l-k}(M) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k-1}}) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k}})\right) \\ &= \sum_{k=0}^{l}(-1)^k\beta_{l-k}(M) + \sum_{k=0}^{l}(-1)^k \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k-1}}) \\ &\quad + \sum_{k=0}^{l}(-1)^k \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l-k}}) \\ &= \sum_{k=0}^{l}(-1)^k\beta_{l-k}(M) + \dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,l}}). \end{aligned}$$

We have the last equality by cancelling the dimensions of the images of the respective operators and by noticing that $\dim \operatorname{Im}(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,-1}}) = \dim 0 = 0$.

In particular, for all $0 \leq l \leq n$, we have

$$\sum_{k=0}^{l}(-1)^k\beta_{l-k}(M) \leq \sum_{k=0}^{l}(-1)^k m_{l-k}.$$

For $l = n$, since $\mathrm{Im}\,(d_{Tf}|_{\mathrm{F}^{[0,c]}_{Tf,\,n}}) = 0$

$$\sum_{k=0}^{n}(-1)^k m_{n-k} = \sum_{k=0}^{n}(-1)^k \beta_{n-k}(M).$$

Therefore, the strong Morse inequalities hold. $\square$

Let M be a differentiable n–manifold, we define the *Euler characteristic* of M to be the alternating sum of its Betti numbers

$$\chi(M) = \sum_{k=0}^{n}(-1)^k \beta_k(M).$$

Remark 8.0.5. We can rewrite equality (8.5) by $(-1)^n\chi(M) = m_n - m_{n-1} + \ldots + (-1)^n m_0$.

Appendix A

Multilinear algebra

This appendix contains the definitions and results of multilinear algebra that we will use in the rest of the thesis.

For more details and proofs see [36], [24], [37], [11] and [13].

A.1 Categories

Sometimes it is helpful to use the language of category theory, in this section we give the basic definitions. For more references consult [23] and [2].

Definition A.1.1. A *category* $\mathscr{C}$ consists of the following:

1. A class $\mathscr{C}$, whose elements are called *objects*.
2. A set $Hom_{\mathscr{C}}(A, B)$ for any pair of objects A, B, whose elements are called *morphism* from A to B.
3. For any 3 objects A, B, C, a binary operation called *composition*

$$Hom_{\mathscr{C}}(A, B) \times Hom_{\mathscr{C}}(B, C) \longrightarrow Hom_{\mathscr{C}}(A, C)$$

   whose value in (f, g) is denoted by $g \circ f$. It satisties the following conditions:

   (a) For every object A, there exists a distinguished element $\mathrm{id}_A^{\mathscr{C}} \in Hom_{\mathscr{C}}(A, A)$, called the *identity* of A, such that: for any objects A, B and any $f \in Hom_{\mathscr{C}}(A, B)$, we have that

$$f \circ \mathrm{id}_A^{\mathscr{C}} = \mathrm{id}_B^{\mathscr{C}} \circ f = f.$$

   (b) For any objects A, B, C, D and $f \in Hom_{\mathscr{C}}(A, B)$, $g \in Hom_{\mathscr{C}}(B, C)$ and $h \in Hom_{\mathscr{C}}(C, D)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Example A.1.2. We denote by $\mathbf{Vect}_{\mathbb{R}}$ the category of finite dimensional vector spaces over $\mathbb{R}$ and linear maps.

Example A.1.3. The category **Set** whose objects are all sets and morphisms are functions between sets.

Example A.1.4. The category **Grp** of all groups and group homomorphisms.

In the following appendices we will describe other categories.

Definition A.1.5. Let $\mathscr{C}$ be a category, a morphism $f\colon A \longrightarrow B$ in $\mathscr{C}$ is called an *isomorphism* if there exists a morphism $g\colon B \longrightarrow A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Such a morphism g is called an *inverse* of f.

Proposition A.1.6. *Let $\mathscr{C}$ be a category, if $f\colon A \longrightarrow B, g\colon B \longrightarrow A$ and $h\colon B \longrightarrow A$ are morphism such that $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ then $g = h$.*

Proof. We have $h = \mathrm{id}_A \circ h = (g \circ f) \circ h = g \circ (f \circ h) = g \circ \mathrm{id}_B = g$. □

Definition A.1.7. Let $\mathscr{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F\colon \mathscr{C} \longrightarrow \mathcal{D}$ is a map which assigns

1. to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathcal{D}$,
2. to each morphism $f \in Hom_{\mathscr{C}}(A, B)$ a morphism $F(f) \in Hom_{\mathcal{D}}(F(A), F(B))$ so that
   (a) For any objects A, B, C in $\mathscr{C}$ and any $f \in Hom_{\mathscr{C}}(A, B)$ and $g \in Hom_{\mathscr{C}}(B, C)$
   $$F(g \circ f) = F(g) \circ F(f).$$
   (b) For every object A in $\mathscr{C}$ we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Definition A.1.8. Let $\mathscr{C}$ and $\mathcal{D}$ be two categories. A *contravariant functor* $F\colon \mathscr{C} \longrightarrow \mathcal{D}$ is a map which assigns

1. to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathcal{D}$,
2. to each morphism $f \in Hom_{\mathscr{C}}(A, B)$ a morphism $F(f) \in Hom_{\mathcal{D}}(F(B), F(A))$ such that
   (a) For any objects A, B, C in $\mathscr{C}$ and any $f \in Hom_{\mathscr{C}}(A, B)$ and $g \in Hom_{\mathscr{C}}(B, C)$
   $$F(g \circ f) = F(f) \circ F(g).$$
   (b) For every object A in $\mathscr{C}$ we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Also, in section A.3 and in the following appendices we will describe several functors.

Proposition A.1.9. *A functor preserves isomorphisms.*

Proof. Let $\mathscr{C}$ and $\mathcal{D}$ be two categories and $F\colon \mathscr{C} \longrightarrow \mathcal{D}$ be a covariant functor. Let $f\colon A \to A'$ be an isomorphism in $\mathscr{C}$ and f^{-1} the inverse of f then

$$F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(\mathrm{id}_{A'}) = \mathrm{id}_{F(A')}.$$

Similarly, $F(f^{-1}) \circ F(f) = \mathrm{id}_{F(A)}$.

Analogously if F is a contravariant functor. □

A.2 Symmetric group

For details of the symmetric group and shuffles see [34], [24].

Definition A.2.1. Fix a positive integer k. A *permutation* of the set $A = \{1, \ldots, k\}$ is a bijection $\sigma\colon A \longrightarrow A$.

Let S_k be the set of all permutations of the set $\{1, \ldots, k\}$, S_k is a group with the operation of composition.

Definition A.2.2. Let $i_1, \ldots, i_r$ be distinct integers between 1 and n. If $\sigma \in S_n$ fixes the remaining $n - r$ integers and if

$$\sigma(i_1) = i_2, \quad \sigma(i_2) = i_3, \ldots, \sigma(i_{r-1}) = i_r, \sigma(i_r) = i_1,$$

then σ is an r*–cycle* of *length* r.

Every 1–cycle fixes every element of A, and so all 1–cycles are equal is the identity. A 2–cycle, which merely interchanges a pair of elements, is called a *transposition*.

Every permutation $\sigma \in S_k$ is a product of transpositions, see [34, Thm. 1.3].

Definition A.2.3. A permutation $\sigma \in S_k$ is *even* if it is a product of an even number of transpositions; otherwise, σ is *odd*.

The *sign* of a permutation $\mathrm{sgn} : S_k \longrightarrow \{\pm 1\}$ is a homomorphism between S_k and the group $\{\pm 1\}$ defined by

$$\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Definition A.2.4. A (k, l)*–shuffle* σ is a permutation of $\{1, \ldots, k + l\}$ satisfying

$$\sigma(1) < \ldots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \ldots < \sigma(k+l).$$

The set of all such permutations is denoted by $S(k, l)$.

Since a (k, l)–shuffle is uniquely determined by the set $\{\sigma(1), \ldots, \sigma(k)\}$, the cardinality of $S(k, l)$ is $\begin{pmatrix} k+l \\ k \end{pmatrix}$.

A.3 Multilinear algebra

This section deals with various aspects of linear and multilinear maps.

Remark A.3.1. Let V be a real vector space of dimension n. Let $T\colon V \longrightarrow \mathbb{R}^n$ be a linear isomorphism, using T we can endow V with a topology. Let $U \subset \mathbb{R}^n$ be an open subset, we set that $T^{-1}(U) \subset V$ is an open subset. One can see that this topology of V does not depend on the linear isomorphism.

Let V and W be vector spaces over $\mathbb{R}$ of dimension n and m respectively, the set $\mathrm{Hom}_{\mathbb{R}}(V, W)$ of all linear maps $T\colon V \longrightarrow W$ is itself a vector space over $\mathbb{R}$ with the operations:

1. Sum of linear maps, that is, let $T, R\colon V \longrightarrow W$ be two linear maps, we define $T + R\colon V \longrightarrow W$ by $(T + R)(v) = T(v) + R(v)$ for all $v \in V$.
2. The scalar product of linear maps with a real number, that is, let $r \in \mathbb{R}$ and $T\colon V \longrightarrow W$, we define $rT\colon V \longrightarrow W$ by $(rT)(v) = rT(v)$, for all $v \in V$.

Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{w_1, \dots, w_m\}$ be a basis of W.

Define $T_{ij}\colon V \longrightarrow W$ as

$$T_{ij}(e_k) = \begin{cases} w_j & \text{for } i = k \\ 0 & i \neq k. \end{cases}$$

The set $\{T_{ij} \mid i = 1 \dots, m, j = 1, \dots n\}$ is a basis of $\mathrm{Hom}_{\mathbb{R}}(V, W)$, thus it has dimension mn over $\mathbb{R}$.

If $V = W$ we write $\mathrm{End}(V) := \mathrm{Hom}_{\mathbb{R}}(V, V)$.

Definition A.3.2. A functor $F\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ is called a *continuous functor* if for each pair $(V, W) \in \mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}}$, the natural map

$$\begin{aligned} F_{V,W}\colon \quad \mathrm{Hom}_{\mathbb{R}}(V, W) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(F(V), F(W)) \\ T &\mapsto F_{V,W}(T) \end{aligned}$$

is continuous with respect to the usual topology on finite dimensional vector spaces described in Remark A.3.1.

The concept of a functor and a continuous functor $F\colon \mathbf{Vect}_{\mathbb{R}} \times \ldots \times \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$ in k variables is defined similarly.

In the rest of this section we will define several continuous functors which will allow us to define different vector bundles, (see Appendix C).

A.3.1 Dual space V^*

Definition A.3.3. The *dual space* of a vector space V over $\mathbb{R}$ is the vector space of all real-valued linear functions on V,

$$V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}).$$

Let $\{e_1, \dots, e_n\}$ be a basis for V, then every $v \in V$ can be written uniquely as a linear combination $v = \sum\limits_{i=1}^{n} a_i e_i$ with $a_i \in \mathbb{R}$. Let $e^i\colon V \longrightarrow \mathbb{R}$ be the linear function that picks out the i–th coordinate, $e^i(v) = a_i$. Note that e^i is characterized by

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Proposition A.3.4 ([37, Prop. 3.1])**.** *The functions $e^1, \dots, e^n$ form a basis for V^*.*

This basis $\{e^1, \dots, e^n\}$ for V^* is called the *dual basis* of the basis $\{e_1, \dots, e_n\}$ for V.

Corollary A.3.5 ([37, Cor. 3.2])**.** *A vector space V and its dual V^* have the same dimension.*

Let V and W be vector spaces and $T\colon V \longrightarrow W$ be a linear map, T induces a linear map $T^*\colon W^* \longrightarrow V^*$ called the *adjoint* of T as follows: let $f \in W^*$, then $T^*(f) = f \circ T \in V^*$.

The linear map T is determined by a matrix A with respect to bases of V and W, where T^* is associated with the transposed matrix A^* with respect to the dual basis. Since the map

$$\begin{tikzcd} {}^*_{V,W}\colon \mathrm{Hom}(V,W) \arrow[r] & \mathrm{Hom}(W^*,V^*) \\ A \arrow[r, mapsto] & A^* \end{tikzcd}$$

is continuous we have a (contravariant) continuous functor.

A.3.2 $\mathrm{Hom}_{\mathbb{R}}(V,W)$

We have the continuous functor of two variables

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{R}}\colon \mathbf{Vect}_{\mathbb{R}} & \times\ \mathbf{Vect}_{\mathbb{R}} \arrow[rr] & & \mathbf{Vect}_{\mathbb{R}} \\
V & V' \arrow[dd, "R"'] & & \mathrm{Hom}_{\mathbb{R}}(V,V') \arrow[dd, "\mathrm{Hom}_{\mathbb{R}}(T,R)"] \\
 & & {} \arrow[r, mapsto] & {} \\
W \arrow[uu, "T"] & W' & & \mathrm{Hom}_{\mathbb{R}}(W,W')
\end{tikzcd}$$

Let us considerer

$$\mathrm{Hom}_{\mathbb{R}}(W,V) \times \mathrm{Hom}_{\mathbb{R}}(V',W) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathrm{Hom}_{\mathbb{R}}(V,V'), \mathrm{Hom}_{\mathbb{R}}(W,W')) \qquad \text{(A.1)}$$
$$(T,R) \mapsto \mathrm{Hom}_{\mathbb{R}}(T,R)$$

Where:

$$\mathrm{Hom}_{\mathbb{R}}(T,R)\colon \mathrm{Hom}_{\mathbb{R}}(V,V') \longrightarrow \mathrm{Hom}_{\mathbb{R}}(W,W'), \quad (\mathrm{Hom}_{\mathbb{R}}(T,R))(f) = R \circ f \circ T.$$

One can see that (A.1) is a continuous map, therefore the functor $\mathrm{Hom}_{\mathbb{R}}$ is a continuous functor.

Remark A.3.6. V^* is a particular case of the functor $\mathrm{Hom}_{\mathbb{R}}(V,W)$ taking $W = \mathbb{R}$.

A.3.3 Direct sum $V \oplus W$

Definition A.3.7. Let V and W be vector spaces over $\mathbb{R}$, the *direct sum* of V and W is defined by

$$V \oplus W = V \times W = \{(v,w) \mid v \in V, w \in W\}.$$

Let $(v_1,w_1),(v_2,w_2) \in V \oplus W$, $\lambda \in \mathbb{R}$, we define

$$(v_1,w_1) + (v_2,w_2) = (v_1 + v_2, w_1 + w_2), \quad \lambda(v_1,w_1) = (\lambda v_1, \lambda w_1).$$

By definition of this vector space we have $\dim(V \oplus W) = \dim(V) + \dim(W)$.

Let V, V', W and W' be vector spaces over $\mathbb{R}$, let $T\colon V \longrightarrow W$ and $R\colon V' \longrightarrow W'$ be linear maps, we define the linear map

$$T \oplus R\colon V \oplus V' \longrightarrow W \oplus W'$$

by $(T \oplus R)(v, v') = (T(v), R(v'))$. This defines the map.

$$\begin{tikzcd} V \bigoplus W \colon \mathrm{Hom}_{\mathbb{R}}(V, V') \times \mathrm{Hom}_{\mathbb{R}}(W, W') \arrow[r] & \mathrm{Hom}_{\mathbb{R}}(V \oplus W, V' \oplus W') \\ (T, R) \arrow[r, mapsto] & T \bigoplus R \end{tikzcd}$$

The linear maps T and R are determined by matrices A and B respectively, with respect to bases of V, V', W, W'. With respect to the bases of $V \oplus V'$ and $W \oplus W'$, $T \oplus R$ is associated with a matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ which continually depends on A and B. Therefore, $V \bigoplus W$ is a covariant continuous functor of two variables.

A.3.4 Multilinear maps

The *Cartesian product* of k copies of a vector space V is denoted by $V^k = V \times \ldots \times V$.

Definition A.3.8. Let $V_1, \ldots, V_k, W$ be vector spaces. A map $T \colon V_1 \times \ldots \times V_k \longrightarrow W$ is *k–multilinear* if it is linear on each of its k arguments: for each $i \in \{1, \ldots, k\}$, if all of the variables but v_i are held constant, then $T(v_1, \ldots, v_k)$ is a linear map of v_i.

Example A.3.9. The dot product $f \colon V^2 \longrightarrow \mathbb{R}^n$ denoted by $f(v, w) = v \cdot w$ is bilinear: let $v = \sum_{i=1}^{n} a_i e_i$ and $w = \sum_{i=1}^{n} b_i e_i$, $a_i, b_i \in \mathbb{R}$ then

$$v \cdot w = \sum_{i=1}^{n} a_i b_i.$$

Example A.3.10. The determinant $f(v_1, \ldots, v_n) = \det(v_1 \ldots v_n)$, viewed as a function of the n column vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ is n–linear.

Definition A.3.11. Let V and W be real finite dimensional vector spaces. A *pairing* of V and W is a bilinear map $(\ ,\ ) \colon V \times W \longrightarrow \mathbb{R}$. A pairing is called *non-singular* if whenever $w \neq 0$ in W, there exists an element $v \in V$ such that $(v, w) \neq 0$, and whenever $v \neq 0$ in V, there exists an element $w \in W$ such that $(v, w) \neq 0$.

Let $\lambda \in \mathbb{C}$, the bar $\overline{\lambda}$ denotes the conjugate of λ. If $\lambda \in \mathbb{R}$, $\overline{\lambda} = \lambda$.

Definition A.3.12. Let V be a vector space over the scalar field $\mathbb{K} = \mathbb{R}$, or $\mathbb{C}$. An *inner product* on V is a function $\langle\ ,\ \rangle \colon V \times V \longrightarrow \mathbb{K}$ that assigns to each pair of vectors $v, w \in V$ a scalar $\langle v, w \rangle$ in $\mathbb{K}$ with the following properties: For all $v, u, w \in V$ and $\alpha \in \mathbb{K}$

1.

$$\begin{aligned} \langle v + u, w \rangle &= \langle v, w \rangle + \langle u, w \rangle, \\ \langle \alpha v, w \rangle &= \alpha \langle v, w \rangle, \\ \langle v, u + w \rangle &= \langle v, u \rangle + \langle v, w \rangle, \\ \langle v, \alpha w \rangle &= \overline{\alpha} \langle v, w \rangle. \end{aligned}$$

2. $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

3. Positive Definiteness: $\langle v, v\rangle \geq 0$. If $\langle v, v\rangle = 0$ if and only if $v = 0$.

4. If $\langle v, u\rangle = \langle v, w\rangle$ for all $v \in V$, then $u = w$.

A vector space V endowed with a inner product is called an *inner product space*.

Example A.3.13. Let V be a real inner product space, by the positive definite condition we have that the real inner product of V is an example of a non-singular pairing.

Definition A.3.14. Let V be a vector space. We define

$$\mathbf{Mult}^k(V) = \{\eta \colon V^k \longrightarrow \mathbb{R} \mid \eta \text{ is a } k\text{–multilinear function}\}.$$

Also, let V, W be two finite dimensional vector spaces and $T \colon V \longrightarrow W$ be a lineal map, we have the linear map

$$\begin{aligned} \mathbf{Mult}^k(T)\colon \quad & \mathbf{Mult}^k(W) \longrightarrow \mathbf{Mult}^k(V) \\ & \eta \mapsto \eta \circ (T \times \ldots \times T). \end{aligned}$$

The functor $\mathbf{Mult}^k$ is a continuous functor, because the following map is continuous

$$\begin{tikzcd} \mathbf{Mult}^k\colon \mathrm{Hom}_{\mathbb{R}}(V, W) \arrow[r] & \mathrm{Hom}_{\mathbb{R}}(\mathbf{Mult}^k(W), \mathbf{Mult}^k(V)) \\ T \arrow[r, mapsto] & \mathbf{Mult}^k(T). \end{tikzcd}$$

A.3.5 Tensor product $V \otimes W$

Let V, W be two vector spaces over $\mathbb{R}$ of dimension n and m respectively. Let $\{e_1, \ldots, e_n\}$ be a basis of V and $\{f_1, \ldots, f_m\}$ be a basis of W. Let us consider the symbols of the form $e_i \otimes f_j$ with $1 \leq i \leq n$ and $1 \leq j \leq m$. Let $V \otimes W$ be the vector space generated by the symbols $e_i \otimes f_j$. The vector space $V \otimes W$ is called the *tensor product* of V and W.

Note that $\dim(V \otimes W) = \dim V \dim W = mn$, see [36, Thm. 8.3.1].

Let $v \in V$ and $w \in W$, then we have

$$v = \sum_{i=1}^{n} a_i e_i, \quad w = \sum_{j=1}^{m} b_j f_j, \quad a_i, b_j \in \mathbb{R}.$$

We define the bilinear map $\Upsilon \colon V \times W \longrightarrow V \otimes W$, by

$$\Upsilon(v, w) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j e_i \otimes f_j. \tag{A.2}$$

We will denote $\Upsilon(v, w)$ by $v \otimes w$.

Theorem A.3.15 (Universal property of tensor product, [31, Prop. 2.2.1])**.** *Let V, W, Z be finite dimensional vector spaces over $\mathbb{R}$ and let $\Upsilon\colon V \times W \longrightarrow V \otimes W$ be the bilinear map (A.2). It has the property that given any bilinear map $R\colon V \times W \longrightarrow Z$, there exists an unique linear map $S\colon V \otimes W \longrightarrow Z$ such that the diagram below is commutative.*

$$\begin{tikzcd} V \times W \arrow[rr, "\Upsilon"] \arrow[dr, "R"'] & {} \arrow[d, phantom, "\circlearrowleft"] & V \otimes W \arrow[dl, dotted, "S"] \\ & Z & \end{tikzcd}$$

Remark A.3.16. If $Z = \mathbb{R}$ and $W = V$, since the bilinear map R induces a linear map S, then $\mathbf{Mult}^2(V) = (V \otimes V)^*$.

Example A.3.17. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$ and let $\{e^1, \ldots, e^n\}$ be its dual basis. The dot product on $\mathbb{R}^n$ is the bilinear function $f\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ defined in example A.3.9 by

$$f(v, w) = v \cdot w.$$

We can express f in terms of the tensor product:

$$f(v, w) = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} e^i(v) e^i(w) = \sum_{i=1}^{n} (e^i \otimes e^i)(v, w).$$

Theorem A.3.18 ([36, Thm. 8.3.3])**.** *Let U, V and W be finite dimensional vector spaces. Then there are natural isomorphisms:*

$$\begin{aligned} U \otimes (V \oplus W) &\cong (U \otimes V) \oplus (U \otimes W). && \text{(A.3)} \\ U \otimes V &\cong V \otimes U. && \text{(A.4)} \\ (U \otimes V)^* &\cong U^* \otimes V^*. && \text{(A.5)} \end{aligned}$$

Let $T\colon V \longrightarrow V'$ and $R\colon U \longrightarrow U'$ be linear maps, they induce a linear map:

$$T \otimes R\colon V \otimes U \longrightarrow V' \otimes U', \quad (T \otimes R)(v \otimes u) = T(v) \otimes R(u)$$

which continually depends on T and R. Then, we have a continuous functor

$$\begin{gathered} \otimes\colon \mathrm{Hom}(V, V') \times \mathrm{Hom}_{\mathbb{R}}(W, W') \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V \otimes V', W \otimes W') \\ (T, S) \mapsto T \otimes S. \end{gathered}$$

By induction we can define the tensor product of n (possibly distinct) vector spaces which is a continuous functor of n variables.

A.3.6 k–th tensor power $\mathbf{T}^k(V)$

Let V be a vector space over $\mathbb{R}$, we define

$$\mathbf{T}^k(V) := \underbrace{V \otimes \ldots \otimes V}_{k}.$$

If $k = 0$, $\mathbf{T}^k(V) = \mathbb{R}$.

Let V be a vector space and $\{e_1, \ldots, e_n\}$ be a basis for V. It defines a basis for $\mathbf{T}^k(V)$ consisting of the n^k elements of the form $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_k}$ where $1 \leq i_1, \ldots, i_k \leq n$. In particular $\mathbf{T}^k(V)$ has dimension n^k.

Theorem A.3.15 can be extended to several vector spaces.

Theorem A.3.19 ([13, Sec. 1.20])**.** *Let $V_1, \ldots, V_k, Z$ be any $k+1$ vector spaces and let $\widehat{\Upsilon}\colon V_1 \times \ldots \times V_k \longrightarrow V_1 \otimes \ldots \otimes V_k$ be the k–multilinear map which generalizes (A.2). For any k–multilinear map $\widehat{R}\colon V_1 \times \ldots \times V_k \longrightarrow Z$, there exists an unique linear map $\widehat{S}\colon V_1 \otimes \ldots \otimes V_k \longrightarrow Z$ such that the diagram is commutative.*

$$\begin{tikzcd}
V_1 \times \ldots \times V_k \arrow[rr, "\widehat{\Upsilon}"] \arrow[dr, "\widehat{R}"'] & \circlearrowleft & V_1 \otimes \ldots \otimes V_k \arrow[dl, dotted, "\hat{S}"] \\
& Z &
\end{tikzcd}$$

Let V and W be vector spaces of dimension n and m respectively. Let $\{e_1, \ldots, e_n\}$ be a basis of V.

Let $T\colon V \longrightarrow W$ be a linear map, we have $\mathbf{T}^k(T)\colon \mathbf{T}^k(V) \longrightarrow \mathbf{T}^k(W)$ in basic elements is given by $\mathbf{T}^k(T)(e_{i_1} \otimes \ldots \otimes e_{i_k}) = T(e_{i_1}) \otimes \ldots \otimes T(e_{i_k})$.

$\mathbf{T}^k(T)$ is a continuous functor because the following map is continuous

$$\begin{aligned} \mathbf{T}^k_{V,W}\colon \mathrm{Hom}_{\mathbb{R}}(V, W) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(\mathbf{T}^k(V), \mathbf{T}^k(W)) \\ T &\longmapsto \mathbf{T}^k(T). \end{aligned}$$

Remark A.3.20. By Theorem A.3.19, if $V_i = V$ for all $i = 1, \ldots, k$ and $Z = \mathbb{R}$ we have the composition of continuous functors $\mathbf{Mult}^k(V) = (\mathbf{T}^k(V))^* = \mathbf{T}^k(V^*)$ and a correspondence between k–multilinear maps and linear maps.

A.3.7 k–th symmetric power S^kV

Definition A.3.21. Let V and W be vector spaces. A k–multilinear map of the form

$$\eta\colon V^k \longrightarrow W, \qquad (v_1, v_2, \ldots, v_k) \longmapsto \eta(v_1, v_2, \ldots, v_k)\ ,$$

is *symmetric* if

$$\eta(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}) = \eta(v_1, v_2, \ldots, v_k);$$

for every $\sigma \in S_k$ and any argument vectors $v_1, \ldots, v_k \in V$.

Example A.3.22. The dot product $f(v,w) = v \cdot w$ on $\mathbb{R}^n$ is symmetric.

Definition A.3.23. Let V be a vector space over $\mathbb{R}$ of dimension n. We define

$$\mathrm{Sym}^k V := \{\eta \colon V^k \to \mathbb{R} \mid \eta \text{ is a symmetric } k\text{–multilinear function}\}$$

for each $k \in \mathbb{N}$ with $\mathrm{Sym}^0 V := \mathbb{R}$.

The set $\mathrm{Sym}^k V$ is a vector space over $\mathbb{R}$ in the usual manner:

$$\begin{aligned} (\omega+\eta)(v_1,\ldots,v_k) &= \omega(v_1,\ldots,v_k) + \eta(v_1,\ldots,v_k), \\ (\lambda\omega)(v_1,\ldots,v_k) &= \lambda\omega(v_1,\ldots,v_k),\ \lambda \in \mathbb{R}. \end{aligned}$$

Let V be a vector space of dimension n with basis $\{e_1,\ldots,e_n\}$. The *k–th symmetric power* of V is S^kV the set of homogeneous polynomials of degree k in the variables $\{e_1,\ldots,e_n\}$. It is a vector space of dimension $\binom{n+k+1}{k}$.

Let $v_1,\ldots,v_n \in V$, then

$$\begin{aligned} v_1 &= a_{11}e_1 + \ldots + a_{1n}e_n \\ v_2 &= a_{21}e_1 + \ldots + a_{2n}e_n \\ &\vdots \\ v_n &= a_{n1}e_1 + \ldots + a_{nn}e_n \end{aligned}$$

One can see $v_i = a_{i1}e_1 + \ldots + a_{in}e_n$ as an homogeneous polynomial of degree 1 in the variables $\{e_1,\ldots,e_n\}$.

We define the linear map

$$\begin{gathered} S \colon V^k \longrightarrow S^kV \\ (v_1,\ldots,v_k) \mapsto v_1 \cdot \ldots \cdot v_k = (a_{11}e_1 + \ldots + a_{1n}e_n) \cdot \ldots \cdot (a_{n1}e_1 + \ldots + a_{nn}e_n). \end{gathered} \tag{A.6}$$

We denote $S(v^1,\ldots,v^k)$ by $v^1 \cdot \ldots \cdot v^k$.

We have the following universal property.

Theorem A.3.24 ([36, Thm. 10.1.3 and Thm. 10.5.1])**.** *Let V be a vector space of dimension n and an integer $k > 0$, let $S \colon V^k \longrightarrow S^kV$ be the map (A.6), then S is a symmetric k–multilinear map such that if W is a vector space of finite dimension and $R \colon V^k \longrightarrow W$ is a symmetric k–multilinear map, then there exists an unique linear map $T \colon S^kV \longrightarrow W$ such that the diagram below is commutative*

$$\begin{tikzcd} V^k \arrow[rr, "S"] \arrow[dr, "R"'] & \circlearrowright & S^kV \arrow[dl, dotted, "T"] \\ & W & \end{tikzcd}$$

If $W = \mathbb{R}$, one can see that $\mathrm{Sym}^k V = (S^kV)^*$.

Example A.3.25. Let V and W be vector spaces and $T\colon V \longrightarrow W$ be a linear map, it induces the linear map

$$\begin{aligned} S^kT\colon \quad & S^kV \longrightarrow S^kW \\ & v_1 \cdot \ldots \cdot v_k \mapsto T(v_1) \cdot \ldots \cdot T(v_k). \end{aligned}$$

The functor S^k is a continuous functor, because the following map is continuous

$$\begin{tikzcd} S^k_{V,W}\colon \mathrm{Hom}_{\mathbb{R}}(V,W) \arrow[r] & \mathrm{Hom}_{\mathbb{R}}(S^kV, S^kW) \\ T \arrow[r, mapsto] & S^kT. \end{tikzcd}$$

A.3.8 k–th exterior power Λ^kV

Definition A.3.26. Let V and W be vector spaces. A k–multilinear map $\eta\colon V^k \longrightarrow W$ is called *alternating* if

$$\eta(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}) = \mathrm{sgn}(\sigma)\eta(v_1, v_2, \ldots, v_k)$$

for every $\sigma \in S_k$ and any argument vectors $v_1, \ldots, v_k \in V$.

Proposition A.3.27 ([24, Lemma 2.7])**.** *Let $\omega\colon V^k \longrightarrow \mathbb{R}$ be a k–multilinear map, if $\omega(v_1, \ldots, v_k) = 0$ for all k–tuples with $v_i = v_{i+1}$ for all $1 \leq i \leq k-1$, then ω is alternating.*

Examples A.3.28.

1. The determinant $\omega\colon \mathbb{R}^{n^2} \longrightarrow \mathbb{R}$, $\omega(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \det(\mathbf{x}_1 \ldots \mathbf{x}_n)$, where $(\mathbf{x}_1 \ldots \mathbf{x}_n)$ denotes the $n \times n$ matrix whose columns are $\mathbf{x}_1, \ldots, \mathbf{x}_n$, then ω is alternating.
2. The cross product $v \times w$ on $\mathbb{R}^3$ is alternating.

The set

$$\mathbf{Alt}^k(V) = \{\eta\colon V^k \to \mathbb{R} \mid \eta \text{ is an alternating } k\text{–multilinear map}\},$$

is a vector space over $\mathbb{R}$ in the usual manner:

$$\begin{aligned} (\omega + \eta)(v_1, \ldots, v_k) &= \omega(v_1, \ldots, v_k) + \eta(v_1, \ldots, v_k), \\ (\lambda\omega)(v_1, \ldots, v_k) &= \lambda\omega(v_1, \ldots, v_k),\ \lambda \in \mathbb{R}. \end{aligned}$$

Definition A.3.29. Let V be a vector space over $\mathbb{R}$ of dimension n and $\{e_1, \ldots, e_n\}$ be a basis of V. We consider the symbols of the form $e_{i_1} \wedge \ldots \wedge e_{i_k}$ with $1 \leq i_1 < \ldots < i_k \leq n$. We have $\binom{n}{k}$ elements.

The *k–th exterior power* of V is the vector space generated by elements $e_{i_1} \wedge \ldots \wedge e_{i_k}$, this set is denoted by Λ^kV, for each $k \in \mathbb{N}$ with $\Lambda^0V := \mathbb{R}$.

Theorem A.3.30 ([24, Thm. 16.7])**.** *Let V be a vector space of dimension n, if $\{e_1, \ldots, e_n\}$ is a basis for V, then $\{e_{i_1} \wedge \ldots \wedge e_{i_k} | 1 \leq i_1 < \ldots < i_k \leq n\}$ is a basis for the vector space $\Lambda^k V$, this basis has dimension $\binom{n}{k}$. Also, there is a natural isomorphism $\Lambda^k V^* \cong (\Lambda^k V)^*$.*

Proposition A.3.31 ([36, Thm. 9.3.2])**.** *Let V be a vector space of dimension n, if $n < k$, then $\Lambda^k V = 0$. Also,* $\dim(\Lambda^n V) = 1$.

Let $v_1, \ldots, v_n \in V$, then

$$\begin{aligned} v_1 &= a_{11}e_1 + \ldots + a_{1n}e_n \\ v_2 &= a_{21}e_1 + \ldots + a_{2n}e_n \\ &\vdots \\ v_n &= a_{n1}e_1 + \ldots + a_{nn}e_n \end{aligned}$$

We define the alternating k–multilinear map

$$\begin{aligned} \Theta\colon V^k &\longrightarrow \Lambda^k V \\ (v_1, \ldots, v_k) &\mapsto \sum_{e_{i_1} \wedge \ldots \wedge e_{i_k}} \det(A_{i_1,\ldots,i_k})(e_{i_1} \wedge \ldots \wedge e_{i_k}). \end{aligned} \tag{A.7}$$

where $A_{i_1,\ldots,i_k}$ is a $k \times k$ submatrix of $A = (a_{ij})$ which is obtained by taking the columns $i_1, \ldots, i_k$.

Given the elements $v_1, \ldots, v_k \in V$ we denoted $\Theta(v_1, \ldots, v_k)$ by $v_1 \wedge \ldots \wedge v_k$. The map Θ and the vector space $\Lambda^k V$ satisfy the following property.

Theorem A.3.32 ([36, Thm. 9.1.3])**.** *Let V be a vector space of dimension n, let $k \in \mathbb{N}, 0 < k \leq n$ and $\Theta\colon V^k \longrightarrow \Lambda^k V$ the map (A.7). Then Θ is an alternating k–multilinear map, such that for every vector space Z of finite dimension and $R\colon V^k \longrightarrow Z$ an alternating k–multilinear map, then there exists an unique linear map $\widehat{R}\colon \Lambda^k V \longrightarrow Z$ such that the following diagram is commutative*

$$\begin{tikzcd} V^k \arrow[rr, "\Theta"] \arrow[dr, "R"'] & {} \arrow[d, phantom, "\circlearrowright"] & \Lambda^k V \arrow[dl, dotted, "\widehat{R}"] \\ & Z & \end{tikzcd}$$

Remark A.3.33. If $Z = \mathbb{R}$, we have $\mathbf{Alt}^k(V) \cong (\Lambda^k V)^*$.

Let V and W be vector spaces of dimension n and m respectively. Let $T\colon V \longrightarrow W$ be a linear map. It induces a linear map $\Lambda^k T\colon \Lambda^k V \longrightarrow \Lambda^k W$ which in basic elements is given by $\Lambda^k T(e_{i_1} \wedge \ldots \wedge e_{i_k}) = T(e_{i_1}) \wedge \ldots \wedge T(e_{i_k})$.

We get a continuous functor of one variable.

$$\begin{aligned} \Lambda^k_{V,W}\colon \mathrm{Hom}_{\mathbb{R}}(V, W) &\longrightarrow \mathrm{Hom}_{\mathbb{R}}(\Lambda^k V, \Lambda^k W) \\ T &\longmapsto \Lambda^k T. \end{aligned}$$

In relation to the k–th exterior power of a vector space we have the following bilinear map.

Definition A.3.34. Let V be a real vector space of dimension n. The *contraction* on $\Lambda^k V^*$ is a bilinear map

$$\lrcorner\colon V \times \Lambda^k V^* \longrightarrow \Lambda^{k-1} V^*,$$

it $v \in V$ and $v^1, \ldots, v^k \in V^*$ it is given by

$$v \lrcorner (v^1 \wedge \ldots \wedge v^k) = \sum_{i=1}^{k} (-1)^{i+1} v^i(v) v^1 \wedge \ldots \wedge \widehat{v^i} \wedge \ldots \wedge v^k. \tag{A.8}$$

A.3.9 Graded algebras

In the thesis we will consider some examples of algebras.

Definition A.3.35. An $\mathbb{R}$–*algebra* $\mathcal{A}$ consist of a vector space over $\mathbb{R}$ and a bilinear map $\mu\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ which is associative, that is, for every $a, b, c \in \mathcal{A}$

$$\mu(a, \mu(b, c)) = \mu(\mu(a, b), c).$$

Definition A.3.36. An $\mathbb{R}$–algebra $\mathcal{A}$ is *graded* if it can be written as a direct sum

$$\mathcal{A}^\bullet = \bigoplus_{k=0}^{\infty} \mathcal{A}^k$$

of vector spaces over $\mathbb{R}$ so that the multiplication map sends $\mathcal{A}^k \times \mathcal{A}^l$ to $\mathcal{A}^{k+l}$.

The notation $\mathcal{A}^\bullet = \bigoplus\limits_{k=0}^{\infty} \mathcal{A}^k$ means that each element of A is uniquely a finite sum

$$a = a_{i_1} + \ldots + a_{i_m},$$

where $a_{i_j} \in A^{i_j}$. The elements in $\mathcal{A}^k$ are said to have degree k.

Definition A.3.37. A graded $\mathbb{R}$–algebra $\mathcal{A}^\bullet$ is called graded *anticommutative* if

$$\mu(a, b) = (-1)^{kl} \mu(b, a)$$

for all $a \in \mathcal{A}^k$ and $b \in \mathcal{A}^l$.

Tensor algebra $\bigoplus_{k=0}^{\infty} \mathbf{T}^k(V)$

Definition A.3.38. We define

$$\mathbf{T}^\bullet(V) = \bigoplus_{k=0}^{\infty} \mathbf{T}^k(V).$$

On $\mathbf{T}^\bullet(V)$ we have a product map.

Definition A.3.39. Let V be a vector space, we define the bilinear map

$$\begin{aligned} \mu\colon \quad \mathbf{T}^k(V) \times \mathbf{T}^l(V) &\longrightarrow \mathbf{T}^{k+l}(V), \\ (v, w) &\mapsto v \otimes w. \end{aligned} \tag{A.9}$$

For every $v \in \mathbf{T}^k(V)$ and $w \in \mathbf{T}^l(V)$.

If we consider V^*, identifying $\mathbf{T}^k(V^*)$ with the k–multilinear functions (see Remark A.3.20), the product map (A.9) can be seen as

$$\mu\colon \mathbf{T}^k(V^*) \times \mathbf{T}^l(V^*) \longrightarrow \mathbf{T}^{k+l}(V^*), \quad (\omega, \eta) \mapsto \omega \otimes \eta.$$

where ω is a k–multilinear map and η an l–multilinear map on V and $\omega \otimes \eta$ is a $(k+l)$–multilinear function $\omega \otimes \eta$ defined by

$$(\omega \otimes \eta)(v_1, \dots, v_{k+l}) = \omega(v_1, \dots, v_k)\eta(v_{k+1}, \dots, v_{k+l}).$$

Where $(v_1, \dots, v_{k+l}) \in V^{k+l}$.

Theorem A.3.40 ([38, Cor. 18.19])**.** *The product (A.9) is associative: if $v, w, u \in V$, then*

$$(v \otimes w) \otimes u = v \otimes (w \otimes u).$$

By Theorem A.3.40 $\mathbf{T}^\bullet(V)$ is associative. The graded algebra $\mathbf{T}^\bullet(V)$ is called the *tensor algebra* over V.

Exterior algebra $\oplus_{k=0}^{\infty} \Lambda^k V$

We define

$$\Lambda^\bullet V := \bigoplus_{k=0}^{n} \Lambda^k V.$$

We want to give a structure of graded algebra to $\Lambda^\bullet V$, for that, we need to define a product.

Definition A.3.41. We define the *wedge product* as the bilinear map

$$\wedge\colon \Lambda^k V \times \Lambda^l V \longrightarrow \Lambda^{k+l} V, \quad (v, w) \mapsto v \wedge w. \tag{A.10}$$

If we take V^*, identifying $\Lambda^k(V^*)$ with the alternating k–multilinear functions the wedge product (A.10) can be seen as

$$\wedge\colon \Lambda^k V^* \times \Lambda^l V^* \longrightarrow \Lambda^{k+l} V^*,$$

for each $\omega \in \Lambda^k V^*$ and $\eta \in \Lambda^l V^*$ is defined by

$$(\omega \wedge \eta)(v_1, \dots, v_{k+l}) = \sum_{\sigma \in S(k,l)} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}).$$

Where $(v_1, \dots, v_{k+l}) \in V^{k+l}$.

When $k = l = 1$ it is given by

$$(\omega \wedge \eta)(v_1, v_2) = \omega(v_1)\eta(v_2) - \eta(v_1)\omega(v_2).$$

Where $v_1, v_2 \in V$.

Lemma A.3.42 ([24, Lemma 2.7]). *For any* $f_1, \ldots, f_k \in V^*$ *and any* $v_1, \ldots, v_k \in V$ *we have*

$$(f_1 \wedge \ldots \wedge f_k)(v_1, \ldots, v_k) = \det \begin{pmatrix} f_1(v_1) & f_1(v_2) & \cdots & f_1(v_k) \\ f_2(v_1) & f_2(v_2) & \cdots & f_2(v_k) \\ \vdots & \vdots & & \vdots \\ f_k(v_1) & f_k(v_2) & \cdots & f_k(v_k) \end{pmatrix}$$

In particular, we can express it as:

$$(f_1 \wedge \ldots \wedge f_k)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \operatorname{sgn}\sigma f_1(v_{\sigma(1)}) \cdot \ldots \cdot f_k(v_{\sigma(k)}). \tag{A.11}$$

We have that $\omega \wedge \eta$ is an alternating $(k+l)$–multilinear map, that is:

Proposition A.3.43 ([24, Lemma 2.6]). *If* $v \in \Lambda^k V$ *and* $w \in \Lambda^l V$ *then* $v \wedge w \in \Lambda^{k+l} V$.

Proposition A.3.44 ([24, Lemma 2.8]). *The wedge product is anticommutative, that is, if* $v \in \Lambda^k V$ *and* $w \in \Lambda^l V$, *then*

$$v \wedge w = (-1)^{kl} w \wedge v.$$

Proposition A.3.45 ([24, Lemma 2.9]). *Let* V *be a real vector space and* $v \in \Lambda^k V$, $w \in \Lambda^l V, u \in \Lambda^m V$. *Then*

$$(v \wedge w) \wedge u = v \wedge (w \wedge u).$$

The basic formal properties of $\Lambda^\bullet V$ can now be summarized in

Theorem A.3.46 ([24, Thm. 2.12]). $\Lambda^\bullet V$ *is an anticommutative graded algebra.*

$\Lambda^\bullet V$ is called the *exterior or alternating algebra* of V.

A.4 Orientation

Let V be a real vector space of finite dimension n, we considerer the set of all ordered bases of V.

Definition A.4.1. Let V be a vector space of dimension n with ordered basis α and β given by $\alpha = \{a_1, \ldots, a_n\}$ and $\beta = \{b_1, \ldots, b_n\}$. Let A be a matrix $n \times n$ such that $Ab_i = a_i$. The matrix A is called the *transition matrix* of β to α.

Note that every transition matrix is invertible, then any transition matrix has $\det(A) > 0$ or $\det(A) < 0$.

We define an equivalence relation of the set of all ordered bases of V as follows: two ordered bases of V being equivalent if and only if their transition matrix has positive determinant.

Definition A.4.2. An *orientation* of V is a choice of one of these equivalence classes. To indicate an orientation in a vector space we will generally give a basis representative of that equivalence class.

A.5 Inner product space

In this section we will focus on vector spaces endowed with an inner product and describe the linear applications over these spaces. In chapter D we will return to some of the following notions.

Definition A.5.1. Let V be an inner product space. For $v \in V$, we define the *norm* of v by $||v|| = \langle v, v\rangle^{\frac{1}{2}}$.

Definition A.5.2. Let V be an inner product space.

1. Two elements $v, u \in V$ are *orthogonal* if $\langle v, u\rangle = 0$.
2. Let S be a nonempty subset of V, we define the *orthogonal complement* of S:
$$S^{\perp} = \{v \in V | \langle v, u\rangle = 0 \text{ for all } u \in S\}.$$
3. A vector $v \in V$ is an *unit vector* if $||v|| = 1$.
4. A subset S of V is *orthonormal* if S is orthogonal and consists entirely of unit vectors.

Let V be an inner product space if v, w are orthogonal elements we have the *Pythagorean relation*

$$||v + w||^2 = ||v||^2 + ||w||^2. \tag{A.12}$$

More generally, if $\{v_1, \ldots, v_n\}$ is an set whose elements are orthogonal to each other, then

$$||v_1 + \ldots + v_n||^2 = ||v_1||^2 + \ldots + ||v_n||^2. \tag{A.13}$$

Lemma A.5.3. *Let V be an inner product space if v, w are orthogonal elements then*

$$||v + w|| \geq \frac{1}{2}(||v|| + ||w||). \tag{A.14}$$

Proof. By relation (A.12) we have $||v + w|| \geq ||v||$ and $||v + w|| \geq ||w||$ then $2||v + w|| \geq ||v|| + ||w||$, that is, $||v + w|| \geq \frac{1}{2}(||v|| + ||w||)$. ☐

Lemma A.5.4 (Linear independence)**.** *Let V be an inner product space and $S \subset V$. If S is an orthonormal set, then S is linearly independent.*

Proof. Let $\{v_1, \ldots, v_n\}$ be an orthonormal set and consider the equality

$$a_1 v_1 + \ldots + a_n v_n = 0.$$

Set v_j a fixed element, we take the inner product for this element, then

$$\left\langle \sum_{i=1}^{n} a_i v_i, v_j \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, v_j\rangle = a_j \langle v_j, v_j\rangle = a_j = 0.$$

Therefore any finite orthonormal set is linearly independent. ☐

Appendix B

Differential geometry

The objective of this appendix is to introduce the necessary definitions and results of differentiable manifolds, in particular, we are interested in describing the tangent space.

For topics related to this section consult [9] and [31].

B.1 Topological manifolds

We will first see the topological structure of a differentiable manifold.

Definition B.1.1. A topological space is *second countable* if it has a countable basis.

Definition B.1.2. A topological space M is *locally homeomorphic* to $\mathbb{R}^n$ if for each point $p \in M$ there exists an open neighbourhood U of p and a homeomorphism $h\colon U \longrightarrow U'$ onto an open set $U' \subset \mathbb{R}^n$.

Definition B.1.3. An *n–dimensional topological manifold M* is a Hausdorff and second countable topological space, which is locally homeomorphic to $\mathbb{R}^n$.

For the dimension to be well defined, it is important to know that for $n \neq m$ an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$, this result is called Invariance of dimension, see [37, Cor. 8.7]. However, if a topological manifold has several connected components, it is possible for each component to have a different dimension.

Examples B.1.4. Consider $S^n = \{x \in \mathbb{R}^{n+1} | ||x|| = 1\}$ and the 2-Torus as a closed surface defined as the product of two circles. Every open subset of Euclidean space, the n–sphere S^n and the 2-torus are examples of topological manifolds.

The requirement that the space must be Hausdorff does not follow from the local condition as the following example shows.

Example B.1.5. An example of a topological space locally homeomorphic to $\mathbb{R}^n$ that is not Hausdorff is to take the real line $\mathbb{R}$, together with an additional point p. Define the topology on $M = \mathbb{R} \cup \{p\}$ by saying that $\mathbb{R}$ is open and that the neighbourhoods of p are the sets $(U - \{0\}) \cup \{p\}$, where U is a neighbourhood of $0 \in \mathbb{R}$.

Recall that the Hausdorff condition and second contability are hereditary properties, that is, a subspace of a Hausdorff space is Hausdorff, analogously, a subspace of a second countable space is second countable. So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

Definition B.1.6. Let M be a topological manifold and $\varphi\colon U \longrightarrow U'$ a homeomorphism of an open subset $U \subset M$ onto an open subset $U' \subset \mathbb{R}^n$, then φ is called a *chart* of M and U is the associated *chart domain*, the chart is traditionally indicated by the pair (U, φ). A collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ with domains U_α is called an *atlas* for M if

$$\bigcup_{\alpha\in\Lambda} U_\alpha = M.$$

Example B.1.7. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$, where $\mathrm{id}_{\mathbb{R}^n}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the identity map. This space is a topological manifold. Also, every open subset of $\mathbb{R}^n$ is a topological manifold, with chart (U, id_U).

B.1.1 Differentiable manifolds

We want to introduce the notion of differentiable manifold.

Definition B.1.8. Let M be a topological manifold. Let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two charts of M such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$. We define the *chart transformation* $\varphi_{\alpha\beta} := \varphi_\beta \circ \varphi_\alpha^{-1}\colon \varphi_\alpha(U_{\alpha\beta}) \longrightarrow \varphi_\beta(U_{\alpha\beta})$ as a homeomorphism between open subsets on $\mathbb{R}^n$ by means of the commutative diagram:

$$\begin{tikzcd}
U'_\alpha \supset \varphi_\alpha(U_{\alpha\beta}) \arrow[rr, "\varphi_{\alpha\beta}"] & & \varphi_\beta(U_{\alpha\beta}) \subset U'_\beta \\
& U_{\alpha\beta} \arrow[ul, "\varphi_\alpha"] \arrow[ur, "\varphi_\beta"'] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

For the chart transformations $\varphi_{\alpha\beta}$, wherever the respective maps are defined, it is clear that $\varphi_{\alpha\alpha} = \mathrm{id}, \quad \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ where $U_\alpha \cap U_\beta \cap U_\gamma \neq 0$, it follows that $\varphi_{\alpha\beta}^{-1} = \varphi_{\beta\alpha}$.

Definition B.1.9. An atlas of a manifold is called *differentiable*, if all its chart transformations are differentiable.

Recall that a function between open subsets of $\mathbb{R}^n$ is differentiable if its partial derivatives exist and are continuous.

Definition B.1.10. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A differentiable function $f\colon U \longrightarrow V$ is called a *diffeomorphism* if it is bijective and has a differentiable inverse $f^{-1}\colon V \longrightarrow U$.

Since $\varphi_{\alpha\beta}^{-1} = \varphi_{\beta\alpha}$, the inverses of the chart transformations are also differentiable and the chart transformations are diffeomorphism.

Let $\mathcal{U}$ be a differentiable atlas on the manifold M. Let $\mathcal{D} = \mathcal{D}(\mathcal{U})$ be the atlas that contains precisely those charts for which every chart transformation with a chart from $\mathcal{U}$ is differentiable. The atlas $\mathcal{D}$ is then differentiable as well, since one can locally write a chart transformation $\varphi_{\beta\gamma}$ in $\mathcal{D}$ as a composition $\varphi_{\beta\gamma} = \varphi_{\alpha\gamma} \circ \varphi_{\beta\alpha}$ of chart transformations for a chart φ_α in $\mathcal{D}$, and differentiable atlases is a local property. As an element in the family of differentiable atlases, the atlas $\mathcal{D}$ can obviously not be enlarged by the addition of further charts, and it is the largest differentiable atlas which contains $\mathcal{U}$, thus each differentiable atlas unequivocally determines a *maximal differentiable atlas* $\mathcal{D}(\mathcal{U})$, so that $U \subset \mathcal{D}(\mathcal{U})$; and $\mathcal{D}(\mathcal{U}) = \mathcal{D}(\mathcal{B})$ if and only if the atlas $\mathcal{U} \cup \mathcal{B}$ is differentiable.

Definition B.1.11. A *differentiable structure* on a topological manifold is a maximal differentiable atlas. A *differentiable manifold* is a topological manifold, together with a differentiable structure.

Example B.1.12. As example B.1.7, the Euclidean space $\mathbb{R}^n$ is a differentiable manifold with a single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$, this atlas determines a differentiable structure.

Example B.1.13. Any open subset V of a differentiable manifold M is also a differentiable manifold. If $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is an atlas for M, then $\{(U_\alpha \cap V, \varphi_\alpha|_{U_\alpha\cap V})\}_{\alpha\in\Lambda}$ is an atlas for V, where

$$\varphi_\alpha|_{U_\alpha\cap V} \colon U_\alpha \cap V \longrightarrow \mathbb{R}^n$$

denotes the restriction of φ_α to the subset $U_\alpha \cap V$.

Definition B.1.14. Let M be an $n+k$–dimensional differentiable manifold. A subset $N \subset M$ is called an n–dimensional *differentiable submanifold* of M if for every point $p \in N$, there exists a chart around p $\varphi\colon U \longrightarrow U' \subset \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ with $\varphi(p) = 0 \in \mathbb{R}^{n+k}$ so that

$$\varphi(N \cap U) = U' \cap (\mathbb{R}^n \times \{0\}).$$

The number $k = \dim M - \dim N$ is called the *codimension* of the submanifold. That is, locally the submanifold N lies in M as $\mathbb{R}^n$ lies $\mathbb{R}^{n+k}$.

Definition B.1.15. Let M and N be differentiable manifolds of dimension m and n respectively. A continuous map $f\colon M \longrightarrow N$ between differentiable manifolds is said to be *differentiable at the point* $p \in M$ if for some (and therefore for every) chart $\varphi\colon U \longrightarrow U' \subset \mathbb{R}^m$, $p \in U$ and $\phi\colon V \longrightarrow V' \subset \mathbb{R}^n$, $f(p) \in V$ of M and N respectively, the composition $\phi \circ f \circ \varphi^{-1}\colon U' \subset \mathbb{R}^m \longrightarrow V' \subset \mathbb{R}^n$ is differentiable at the point $\varphi(p) \in U'$. The map f is called *differentiable* if it is differentiable at every point $p \in M$.

Note that this map is defined in the neighbourhood $\varphi(f^{-1}(V) \cap U)$ of $\varphi(p)$. This definition is independent of the choice the chart (U, φ), since the chart transformations are differentiable.

The identity map of a differentiable manifold is differentiable, the composition of differentiable maps is differentiable, see [31, Thm. 6.9].

In definition B.1.10 we define diffeomorphisms between open subsets of $\mathbb{R}^n$, but in general we have the notion of diffeomorphism between manifolds.

Definition B.1.16. The map $f\colon M \longrightarrow N$ is a *diffeomorphism* if there is a differentiable map $g\colon N \longrightarrow M$, so that $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$, in other words, f is bijective, and f^{-1} is also differentiable.

We can consider the category whose objects are manifolds and morphisms are differentiable maps, which we will denote by **Diff**.

B.1.2 Manifolds with boundary

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ be the closed Euclidean half-space.

Definition B.1.17. Let M be a second countable Hausdorff space, M is an n–dimensional *manifold with boundary* if is locally homeomorphic to $\mathbb{R}^n_+$. An n–dimensional *differentiable manifold with boundary* is a pair consisting of a n–dimensional manifold with boundary M and a maximal differentiable atlas $\mathcal{U}$ for M.

Definition B.1.18. Let M be an n–dimensional manifold with boundary. At each point $p \in M$, which is mapped by some (and hence by every) chart about p to a point with $x_n = 0$, is called a *boundary point* of M. The set of boundary points of M is canonically an $(n-1)$–dimensional manifold, denoted by ∂M and called the *boundary* of M.

Definition B.1.19. A *closed manifold* is a compact manifold without boundary.

B.2 Tangent space

Problems in differential topology often divide into local and a global parts, we will study the local part, then the key notion is the tangent space at a point.

For local descriptions in addition to considering maps $f\colon M \longrightarrow N$ defined on all M, also consider maps which are defined only in a neighbourhood of $p \in M$. Two such maps can be considered as equal if they agree in a neighbourhood. On the set of differentiable maps

$$\{f\colon U \longrightarrow N \,|\, U \text{ is a neighbourhood of } p \in M\}$$

we define the following relation: let $f\colon U \longrightarrow N$ and $g\colon U' \longrightarrow N$ be differentiable maps, then $f \sim g$ if and only if there is a neighbourhood V of p, $V \subset U \cap U'$, so that $f|_V = g|_V$.

The relation $\sim$ is an equivalence relation.

Definition B.2.1. An equivalence class for this relation is called the *germ* of a map $f\colon M \longrightarrow N$ at p. We denote such a germ by $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$.

Given germs $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ and $\overline{g}\colon (N, f(p)) \longrightarrow (L, g(f(p)))$, one obtains a composition $\overline{g} \circ \overline{f}\colon (M,p) \longrightarrow (L, g(f(p)))$ as the germ of the composition of suitable representatives.

We consider the category of all pointed differentiable manifolds and differentiable germs, which will be denoted by $\mathbf{Diff}_*$.

Definition B.2.2. Let $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ be a differentiable germ, we say that $\overline{f}$ is an *invertible germ* if there is a germ $\overline{g}\colon (N, f(p)) \longrightarrow (M,p)$ such that $\overline{f} \circ \overline{g} = \overline{\mathrm{id}}_N$ and $\overline{g} \circ \overline{f} = \overline{\mathrm{id}}_M$.

Remark B.2.3. If $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ is an invertible germ then there is a representative map $f\colon U \subset M \longrightarrow N$ which is a local diffeomorphism.

Definition B.2.4. A *function germ* is a differentiable germ $\overline{\phi}\colon (M,p) \longrightarrow (\mathbb{R}, \phi(p))$.

The set of all function germs at $p \in M$ will be denoted by $C_p^\infty(M)$, while the set of all differentiable germs $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ will be denoted by $C_p^\infty(M,N)$.

The addition and multiplication in the set $C_p^\infty(M)$ are defined by the corresponding operations on representatives, thus $C_p^\infty(M)$ has the structure of a real algebra.

A differentiable germ $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ induces by composition a homomorphism of algebras

$$\begin{aligned} f^*\colon\ & C_p^\infty(N) \longrightarrow C_p^\infty(M) \\ & \overline{\phi} \mapsto \overline{\phi} \circ \overline{f} = \overline{\phi \circ f}. \end{aligned} \tag{B.1}$$

Let us considerer $\overline{\mathrm{id}}_M\colon (M,p) \longrightarrow (M,p)$ this induces the homomorphism $\mathrm{id}^*\colon C_p^\infty(M) \longrightarrow C_p^\infty(M)$ then

$$\mathrm{id}^* = \mathrm{id}_{C_p^\infty(M)}. \tag{B.2}$$

Let $\overline{g}\colon (N, f(p)) \longrightarrow (L, g(f(p)))$ be a germ, we have

$$(g \circ f)^* = f^* \circ g^*. \tag{B.3}$$

Consider the category whose objects are real algebras of type $C_p^\infty(M)$ and morphisms are homomorphism of real algebras. We will denote this category by **Alg**.

Properties (B.2) and (B.3) imply that we have a functor from the category of pointed differentiable manifolds and differentiable germs to the category of algebras and homomorphisms defined by

$$\begin{tikzcd}
\mathbf{Diff}_* \arrow[r] & \mathbf{Alg} \\
(M,p) \arrow[r, mapsto] \arrow[d, "\overline{f}"'] & C_p^\infty(M) \\
(N,q) \arrow[r, mapsto] & C_q^\infty(N) \arrow[u, "f^*"']
\end{tikzcd}$$

By Proposition A.1.9 the functor applied to an invertible germ is an isomorphism of algebras: $\overline{f} \circ \overline{f}^{-1} = \overline{\mathrm{id}}_N$ then $(f^{-1})^* \circ f^* = \mathrm{id}_M$. For example, if we take a chart φ about p, which defines an invertible germ $\overline{\varphi}\colon (M,p) \longrightarrow (\mathbb{R}^n, 0)$, therefore we have an isomorphism $\varphi^*\colon C_0^\infty(\mathbb{R}^n) \longrightarrow C_p^\infty(M)$.

Let φ be a chart about p, taking the composition with translations, we have $\varphi(p) = 0$.

We simply denoted $C_n^\infty = C_0^\infty(\mathbb{R}^n)$, then to study $C_p^\infty(M)$ is equivalent to study C_n^∞.

Now, we will define the tangent space.

Definition B.2.5. A *derivation* of $C_p^\infty(M)$ is a linear map $X\colon C_p^\infty(M) \longrightarrow \mathbb{R}$ which satisfies the product rule (Leibniz rule)

$$X(\overline{\phi} \circ \overline{\psi}) = X(\overline{\phi}) \circ \overline{\psi}(p) + \overline{\phi}(p) \circ X(\overline{\psi}). \tag{B.4}$$

Definition B.2.6. The *tangent space* T_pM of the differentiable manifold M at a point p is the set of derivations of $C_p^\infty(M)$.

Definition B.2.7. Let $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ be a differentiable germ. Let $f^*\colon C^\infty_{f(p)}(N) \longrightarrow C_p^\infty(M)$ be the induced homomorphism given in (B.1). The *differential* of f at p (or the linear *tangent map*) is defined by

$$\begin{aligned} D_pf\colon \quad T_pM &\longrightarrow T_{f(p)}N, \\ X &\mapsto X \circ f^*. \end{aligned} \tag{B.5}$$

Note that a linear combination of derivations is again a derivation, then the set of derivations forms a vector space. We can see that the differential is linear.

The definition of the differential implies that for a function germ $\overline{\phi}\colon (N, f(p)) \longrightarrow (\mathbb{R}, \phi(f(p)))$

$$D_pf(X)(\overline{\phi}) = X \circ f^*(\overline{\phi}) = X(\overline{\phi \circ f}). \tag{B.6}$$

Consider the function germ of the constant function with value 1, $\overline{1}\colon (M,p) \longrightarrow (\mathbb{R}, 1)$, let $X \in T_pM$, by the Leibniz rule it follows that $X(\overline{1}) = X(\overline{1}) + X(\overline{1})$, therefore $X(\overline{1}) = 0$. Thus, for each function germ of a constant function with constant value $c \in \mathbb{R}$, we have by linearity that

$$X(\overline{c}) = 0. \tag{B.7}$$

From the functorial properties (B.2) and (B.3) of *, it follows that for the composition of $\overline{f}\colon (M,p) \longrightarrow (N, f(p))$ and $\overline{g}\colon (N, f(p)) \longrightarrow (L, g(f(p)))$, one has the property

$$D_p(\overline{g} \circ \overline{f}) = D_{f(p)}\overline{g} \circ D_p\overline{f}$$

for the differential of $g \circ f$. This property is called *the chain rule.*

Now, if $\overline{\varphi}\colon (N,p) \longrightarrow (\mathbb{R}^n, 0)$ is the germ of a chart, then the induced homomorphism $\varphi^*\colon C_n^\infty \longrightarrow C_p^\infty(N)$ is an isomorphism, as well as the differential of φ at p

$$D_p\varphi\colon T_pN \longrightarrow T_0\mathbb{R}^n.$$

Now, we will describe a basis of $T_0\mathbb{R}^n$.

Lemma B.2.8. *Let $\mathbf{x} \in U$ be an open ball around the origin of $\mathbb{R}^n$ or $\mathbb{R}^n$ itself, and $f\colon U \longrightarrow \mathbb{R}$ a differentiable function, then there exist differentiable functions $\phi_1, \ldots, \phi_n\colon U \longrightarrow \mathbb{R}$ so that*

$$f(\mathbf{x}) = f(0) + \sum_{i=1}^{n} x_i \phi_i(\mathbf{x})$$

Where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Proof. By Fundamental Theorem of Calculus:

$$f(\mathbf{x}) - f(0) = \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n) dt.$$

We can consider $g\colon \mathbb{R} \longrightarrow \mathbb{R}^n$ define by $g(t) = (tx_1, \ldots, tx_n) = t\mathbf{x}$. By the chain rule we have

$$Df(t\mathbf{x}) \cdot Dg(t) = \left(\frac{\partial f}{\partial x_1}(t\mathbf{x}), \ldots, \frac{\partial f}{\partial x_n}(t\mathbf{x}) \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t\mathbf{x}) x_i.$$

Then

$$\begin{aligned} f(\mathbf{x}) - f(0) &= \int_0^1 \frac{d}{dt} f(tx_1, \ldots, tx_n) dt \\ &= \int_0^1 Df(t\mathbf{x}) Dg(t) dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t\mathbf{x}) x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) dt. \end{aligned}$$

We define

$$\phi_i(\mathbf{x}) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) dt.$$

Therefore,

$$\begin{aligned} f(\mathbf{x}) - f(0) &= \sum_{i=1}^n x_i \phi_i(\mathbf{x}) \\ f(\mathbf{x}) &= f(0) + \sum_{i=1}^n x_i \phi_i(\mathbf{x}). \end{aligned}$$

□

Among the derivations of the algebra C_n^∞ are the partial derivatives, which we denoted by

$$\left. \frac{\partial}{\partial x_i} \right|_0 : C_n^\infty \longrightarrow \mathbb{R}, \quad \left. \frac{\partial}{\partial x_i} \right|_0 (\overline{\phi}) = \frac{\partial \phi}{\partial x_i}(0).$$

Theorem B.2.9. *The* $\left. \frac{\partial}{\partial x_i} \right|_0$, $i = 1, \ldots, n$, *form a basis of the vector space* $T_0\mathbb{R}^n$ *of derivations of* C_n^∞.

Proof. Let $a_i \in \mathbb{R}$.

If the derivation $\sum_{i=1}^n a_i \left(\left. \frac{\partial}{\partial x_i} \right|_0 \right) = 0$, then, in particular, one obtains for $\overline{x}_\mu\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, the μ–th coordinate function

$$a_\mu = \sum_{i=1}^n a_i \left(\left. \frac{\partial \overline{x}_\mu}{\partial x_i} \right|_0 \right) = 0$$

for each $\mu = 1, \ldots, n$.

Therefore the $\{\frac{\partial}{\partial x_i}\big|_0\}$ is a linearly independent set.

Now, let $X \in T_0\mathbb{R}^n$ and $X(\overline{x_i}) = a_i$. We see that $X = \sum\limits_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\big|_0$.

Set $Y := X - \sum\limits_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\big|_0$.

Since $\frac{\partial}{\partial x_i}\big|_0$ are derivations for each $i = 1, \ldots, n$, and X is a derivation of $T_0\mathbb{R}^n$, then Y is a derivation.

Now, by construction, $Y(\overline{x_i}) = 0$ for every coordinate function. If $\overline{\phi} \in C_n^\infty$ is an arbitrary function germ, then by Lemma B.2.8 we get

$$\overline{\phi} = \overline{\phi}(0) + \sum_{\nu=1}^{n} \overline{x_\nu}\overline{\phi_\nu}.$$

Since Y is a derivation, we apply the Leibniz rule, equation (B.7) and the definition of the ν–th coordinate function we obtain

$$\begin{aligned} Y(\overline{\phi}) &= Y(\overline{\phi}(0)) + \sum_{\nu=1}^{n} Y(\overline{x_\nu}\overline{\phi_\nu}) \\ &= Y(\overline{\phi}(0)) + \sum_{\nu=1}^{n} [Y(\overline{x_\nu})\overline{\phi_\nu}(0) + \overline{x_\nu}(0)Y(\overline{\phi_\nu})] \\ &= 0 \end{aligned}$$

Therefore,

$$X = \sum_{i=1}^{n} a_i \left.\frac{\partial}{\partial x_i}\right|_0 .$$

□

Let M be an n–dimensional differentiable manifold, note that the tangent space at a point has dimension n as vector space, so that the dimension is indeed unequivocally defined. If U is an open set containing p in M, then the algebra $C_p^\infty(U)$ of germs of differentiable functions in U at p is the same as $C_p^\infty(M)$, then $T_pU = T_pM$.

Let $(U, \varphi) = (U, x_1, \ldots, x_n)$ be a chart about a point p in a manifold M, where each x_i is a coordinate function of φ. Let $r_1, \ldots, r_n$ be the standard coordinates on $\mathbb{R}^n$. Then

$$x_i = r_i \circ \varphi \colon U \longrightarrow \mathbb{R}.$$

If f is a differentiable function in a neighbourhood of p, we define $\frac{\partial}{\partial x_i}\big|_p := D_0\varphi^{-1}\left(\frac{\partial}{\partial r_i}\right)$, by definition (see (B.2.4))

$$\left.\frac{\partial}{\partial x_i}\right|_p \overline{f} := \left.\frac{\partial}{\partial r_i}\right|_{\varphi(p)} (f \circ \varphi^{-1}).$$

Theorem B.2.10 ([9, Thm. 2.4])**.** *Let M and N be differentiable manifolds of dimension n and m respectively. Let $(U, x_1, \ldots, x_n)$ and $(V, y_1, \ldots, y_m)$ be two charts around $p \in M$ and $q \in N$ respectively, then the derivations $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_j}$ form bases of the vector spaces T_pM and*

T_qN respectively, and the differential of a germ $\overline{f}\colon (M,p) \longrightarrow (N,q)$ with respect to these bases is given by $J_f(0)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, where $J_f(0)$ is the matrix $J_f(0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i\leq n, 1\leq j\leq m}$.

The matrix $J_f(0)$ is called the *Jacobian matrix* of f.

Theorem B.2.11 (The inverse function Theorem, [9, Thm. 5.1])**.** *Let $f\colon M \longrightarrow N$ be a differentiable map between differentiable manifolds and suppose that $D_pf\colon T_pM \longrightarrow T_{f(p)}N$ is a linear isomorphism at a point $p \in M$. Then there exist a neighborhood U of p in M such that the restriction of f to U is a local diffeomorphism onto a neighborhood V of $f(p)$ in N.*

B.2.1 Orientation

Let us remember Definition A.4.2 of orientation of a vector space, now we will define orientation of a manifold.

Definition B.2.12. Let M be a differentiable manifold with boundary, an *orientation* of M is a differentiable choice of orientations for all the tangent spaces T_pM.

Also, we say M is *orientable* if it may be given an orientation. If so, then M admits at least two different orientations, for if one is specified we need only reverse the orientations of each tangent space to obtain the opposite orientation.

Theorem B.2.13 ([16, Prop. 3.25])**.** *A connected, orientable manifold admits exactly two orientations.*

Appendix C

Vector Bundles

In this appendix we will introduce vector bundles, we will also describe the ways of constructing these objects.

The books that the reader can consult are [17] and [9].

Definition C.0.1. Let E and B be topological spaces. A *real vector bundle* of rank n over B is a continuous surjective map $\pi\colon E \longrightarrow B$ such that it satisfies the following properties:

1. For each $b \in B$, the *fiber* over b, $E_b = \pi^{-1}(b)$, has a vector space structure of dimension n over $\mathbb{R}$.

2. *Local triviality*: There is an open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of B such that for each $\alpha \in \Lambda$ there exists a homeomorphism $h_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ which makes the following diagram commute

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "h_\alpha"] \arrow[ddr, "\pi"'] & & U_\alpha \times \mathbb{R}^n \arrow[ddl, "\pi_1"] \\
& \circlearrowright & \\
& U_\alpha &
\end{tikzcd}$$

   taking $\pi^{-1}(b)$ to $\{b\} \times \mathbb{R}^n$ by a vector space isomorphism for each $b \in U_\alpha$. Such an h_α is called a *local trivialization* of the vector bundle.

The space B is called the *base space*, E is the *total space*, π is the *projection*, the vector spaces E_b are the *fibers* and π_1 is the projection on the first factor. We denote the vector bundle by (E, π, B)

Example C.0.2. The *product bundle* $\pi_1\colon E = B \times \mathbb{R}^n \longrightarrow B$ with π_1 projection on the first factor.

Definition C.0.3. Let (E, π, B) be a vector bundle, a pair (U, h) where U is a open subset of B and h is a local trivialization such that satisfies the axiom of local triviality is called a *bundle chart*.

Definition C.0.4. A *vector bundle map* f between two vector bundles $\pi\colon E \longrightarrow B$ and $\pi'\colon E' \longrightarrow B$ with the same base space is a continuous map $f\colon E \longrightarrow E'$ such that $\pi = \pi' \circ f$, that is, the following diagram commute

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & \circlearrowright & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

Definition C.0.5. An *isomorphism* between vector bundles $\pi\colon E \longrightarrow B$ and $\pi'\colon E' \longrightarrow B$ is a homeomorphism $h\colon E \longrightarrow E'$ taking each fiber $\pi^{-1}(b)$ to the corresponding fiber $\pi'^{-1}(b)$ by a linear isomorphism.

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & \circlearrowright & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

If there is an isomorphism between two vector bundles, we say that they are *isomorphic.*

Definition C.0.6. Let $\pi\colon E \longrightarrow B$ be a vector bundle of rank n. (E, π, B) is called a *trivial bundle* if is isomorphic to the product bundle $\pi_1\colon B \times \mathbb{R}^n \longrightarrow B$.

If we consider vector bundles over a fixed base space B as objects and vector bundle maps as morphism, we have a category, denoted by $\mathbf{VB}(B)$.

Definition C.0.7. Let (E, π, B) be a vector bundle of rank n. A set $\{(U_\alpha, h_\alpha)\}_{\alpha\in\Lambda}$ of bundle charts is called a *bundle atlas* for E if $\cup_{\alpha\in\Lambda} U_\alpha = B$.

Now, let $\alpha, \beta \in \Lambda$ such that $U_\alpha \cap U_\beta \neq 0$. We have local trivializations

$$h_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n,$$
$$h_\beta\colon \pi^{-1}(U_\beta) \longrightarrow U_\beta \times \mathbb{R}^n.$$

Let $b \in U_\alpha \cap U_\beta$, consider the restrictions of h_α and h_β to $\pi^{-1}(b)$

$$h_{\alpha,b}\colon \pi^{-1}(b) \longrightarrow \{b\} \times \mathbb{R}^n,$$
$$h_{\beta,b}\colon \pi^{-1}(b) \longrightarrow \{b\} \times \mathbb{R}^n.$$

Then

$$h_{\alpha,b} \circ h_{\beta,b}^{-1}\colon \{b\} \times \mathbb{R}^n \longrightarrow \{b\} \times \mathbb{R}^n$$

is a linear isomorphism of $\mathbb{R}^n$, that is, $g_{\alpha\beta}(b) := h_{\alpha,b} \circ h_{\beta,b}^{-1} \in \mathbf{GL}(n, \mathbb{R})$. Now let us consider h_α and h_β restricted to $U_\alpha \cap U_\beta$, so

$$\begin{aligned} h_\alpha|_{U_\alpha\cap U_\beta} \circ (h_\beta|_{U_\alpha\cap U_\beta})^{-1} \quad &: (U_\alpha \cap U_\beta) \times \mathbb{R}^n \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n \\ &(b, v) \mapsto (b, g_{\alpha\beta}(b)(v)). \end{aligned}$$

Therefore we have maps

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbf{GL}(n, \mathbb{R}).$$

This continuous maps given by overlapping of the bundle charts are called the *transition functions* of the atlas and satisfy the following condition:

$$g_{\alpha\beta}(b)g_{\beta\gamma}(b) = g_{\alpha\gamma}(b), \quad b \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Definition C.0.8. A *bundle atlas* for a vector bundle (E, π, M) over a differentiable manifold M is differentiable if all its transition functions are differentiable. A *differentiable vector bundle* is a pair $(E, \mathcal{B})$ consisting of a vector bundle E over M and a maximal differentiable bundle atlas $\mathcal{B}$ for E.

Note that the total space of a differentiable vector bundle of rank k over an n–dimensional manifold M is an $(n+k)$–dimensional differentiable manifold.

Definition C.0.9. A (differentiable) *section* of a (differentiable) vector bundle $\pi\colon E \longrightarrow M$ is a (differentiable) continuous map $s\colon M \longrightarrow E$ assigning to each $p \in M$ a vector $s(p)$ in the fiber E_p, that is, $\pi \circ s = \mathrm{id}_M$.

In the thesis we will focus on differentiable sections.

The set of sections of E is denoted by $\Gamma(E)$.

The set of sections of a vector bundle $\pi\colon E \longrightarrow M$ is a real vector space, we can add sections by using the vector space structure of each fiber. The zero in $\Gamma(E)$ is the zero-section which to every $p \in M$ assigns the zero of the fiber $\pi^{-1}(p)$. Also, $\Gamma(E)$ has a structure of module not only over $\mathbb{R}$ but also over $C^\infty(M)$, with

$$(f_1 s_1 + f_2 s_2)(p) = f_1(p)s_1(p) + f_2(p)s_2(p), \quad f_1, f_2 \in C^\infty(M),\ p \in M.$$

Definition C.0.10. Let (E, π, M) be a differentiable vector bundle of rank k and U an open set in M. A *local frame* of E over U is an k–tuple $s_1, \ldots, s_k$ of differentiable sections of E over U so that for each $p \in U$, $s_1(p), \ldots, s_k(p)$ form a basis of E_p.

If $h\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ is a local trivialization and if we set $s_i(p) = h^{-1}(p, e_i)$, where e_i is a basis element of $\mathbb{R}^n$, then $s_1, \ldots, s_k$ form a local frame of E over U. Conversely, if $s_1, \ldots, s_k$ is a local frame of E over U, then for any $p \in U$ and any $v_p \in E_p$, there exists a unique k–tuple of scalars $c_1, \ldots, c_k$ so that $v_p = c_1 s_1(p) + \ldots + c_k s_k(p)$. From this, one can define a local trivialization of E over U by setting $h(v_p) = (p, c_1, \ldots, c_k)$. So the existence of a local frame of E over U is equivalent to the existence of a local trivialization over U.

Definition C.0.11. Let (E, π, M) be a differentiable vector bundle. A *global frame* is a frame defined on the entire manifold M.

Remark C.0.12. The collection of sections $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}$ of $T\mathbb{R}^3$ is a global frame on $\mathbb{R}^3$.

Corollary C.0.13. *If (E, π, M) has a global frame, then is a trivial bundle.*

C.1 Constructing bundles

In this section we will obtain vector bundles through pre-vector bundles and using the continuous functors of the section A.3.

C.1.1 Pre-vector bundles

Definition C.1.1. A *pre-vector bundle* of rank n is a quadruple $(E, \pi, B, \mathcal{B})$ consisting of a set E, a topological space B, a surjective map $\pi\colon E \longrightarrow B$ where $E_b = \pi^{-1}(b)$ has a real vector space structure of dimension n over $\mathbb{R}$ for every $b \in B$. A pre–bundle atlas $\mathcal{B}$, that is, a set $\{(U_\alpha, f_\alpha)\}_{\alpha\in\Lambda}$, where $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of B and $f_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ a bijective map which maps the fibre E_b linearly and isomorphically onto $\{b\} \times \mathbb{R}^n$ for every $b \in U_\alpha$ such that all the transition functions $U_\alpha \cap U_\beta \longrightarrow \mathbf{GL}(n, \mathbb{R})$ of $\mathcal{B}$ are continuous.

The important fact is to define a topology over E that makes it the total space of a vector bundle.

Proposition C.1.2 ([9, Note 3.17])**.** *If $(E, \pi, B, \mathcal{B})$ is a pre-vector bundle, then there is exactly one topology on E, relative to which (E, π, B) is a vector bundle and $\mathcal{B}$ is a bundle atlas.*

If M is a differentiable manifold and $(E, \pi, M, \mathcal{B})$ is differentiable pre–vector bundle, that is, if all the transition functions of $\mathcal{B}$ are differentiable, then by the maximal differentiable atlas $D(\mathcal{B})$ of $\mathcal{B}$ we clearly have a differentiable vector bundle $(E, D(\mathcal{B}))$ over M.

Example C.1.3. Let M be an n–dimensional differentiable manifold and $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ be a differentiable atlas of M. Then we can construct a pre-vector bundle $(TM, \pi, M, \mathcal{B})$ as follows:

$$TM := \bigsqcup_{p\in M} T_pM.$$

The surjective map $\pi\colon TM \longrightarrow M$, given by $v \in T_pM \longrightarrow p$. And $\mathcal{B} = \{(U_\alpha, f_\alpha)\}_{\alpha\in\Lambda}$ where

$$f_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n.$$

Let

$$v_p = \sum_{i=1}^{n} a_i \left.\frac{\partial}{\partial x_i}\right|_p \in \pi^{-1}(p) \in T_pM,$$

f_α is defined by $f_\alpha(v_p) = (p, a_1, \ldots, a_n)$ where each a_i is a real number on U, with respect to $(U_\alpha, \varphi_\alpha)$.

Note that the transition functions of TM correspond to the differential of the chart transformations of M, since M is a differentiable manifold, its chart transformations are differentiable, then the transition functions of TM are also differentiable. In addition, let $(U_\alpha, \varphi_\alpha) \in \mathcal{U}$, $\varphi_\alpha\colon U_\alpha \longrightarrow U'$ where $U' \subset \mathbb{R}^n$, by the composition $(\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^n}) \circ f_\alpha$ we have that TM is a differentiable manifold of dimension $2n$.

The differentiable vector bundle $\pi\colon TM \longrightarrow M$ of rank n obtained from this pre-vector bundle, is called the *tangent bundle* of M.

Definition C.1.4. If $f\colon M \longrightarrow N$ is a differentiable map, then the differentials

$$D_pf\colon T_pM \longrightarrow T_{f(p)}N$$

define a vector bundle map

$$Df\colon TM \longrightarrow TN,$$

which is called the *differential* of f.

C.1.2 Constructing new bundles using continuous functor

From one or more known vector bundles we can construct new ones applying continuous functors fiber by fiber. Thus, continuous functor on $\mathbf{Vect}_{\mathbb{R}}$ induce functors on $\mathbf{VB}(B)$.

Consider an arbitrary real vector bundle $\pi\colon E \longrightarrow B$ of rank n, suppose that we have a covariant continuous functor $F\colon \mathbf{Vect}_{\mathbb{R}} \longrightarrow \mathbf{Vect}_{\mathbb{R}}$, applying the functor to every fiber of the vector bundle we obtain a pre-vector bundle.

We define the set $F(E)$ as

$$F(E) := \bigsqcup_{b\in B} F(E_b).$$

Let $e \in F(E)$, then $e \in F(E_b)$ for some $b \in B$, we define $F(\pi)\colon F(E) \longrightarrow B$ by $F(\pi)(e) = b$, $F(\pi)$ is a surjective map. Note that given $b \in B$, $(F(\pi))^{-1}(b) = F(E)_b = F(E_b)$.

Since E_b is a real vector space of dimension n and F is a functor of $\mathbf{Vect}_{\mathbb{R}}$, $F(E_b)$ is a real vector space of dimension, let is say k. Let U_α be an open set in B, since

$$(F(\pi))^{-1}(U_\alpha) = \bigsqcup_{b\in U_\alpha} F(E_b),$$

then $f_\alpha\colon (F(\pi))^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$, where $f_\alpha(e) = (b, v)$, is a bijective map where $F(\mathbb{R}^n) = \mathbb{R}^k$ and isomorphically onto $F(\pi^{-1}(b)) \longrightarrow \{b\} \times \mathbb{R}^k$. Now, let U_α, U_β be open subsets of B such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, let $b \in U_{\alpha\beta}$, we take $f_\alpha\colon U_\alpha \longrightarrow U_\alpha \times \mathbb{R}^k$, $f_\beta\colon U_\beta \longrightarrow U_\beta \times \mathbb{R}^k$ be bijective maps, with $k > 0$. We consider (U_α, h_α) and (U_β, h_β) be local trivialization of vector bundle $\pi\colon E \longrightarrow B$, the transition function

$$\tilde{g}_{\alpha\beta}\colon U_{\alpha\beta} \longrightarrow \mathbf{GL}(k, \mathbb{R})$$

given by

$$f_{\beta,b} \circ f_{\alpha,b}^{-1} = \tilde{g}_{\alpha\beta}(b) = F(h_{\beta,b}) \circ F(h_{\alpha,b}^{-1}) = F(h_{\beta,b} \circ h_{\alpha,b}^{-1}) = F(g_{\alpha\beta}(b)).$$

Where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbf{GL}(n, \mathbb{R})$ is transition function of the atlas of $\pi\colon E \longrightarrow B$ and since F is a continuous functor

$$F_{\mathbb{R}^n,\mathbb{R}^n}\colon \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathrm{Hom}_{\mathbb{R}}(F(\mathbb{R}^n), F(\mathbb{R}^n))$$

is continuous, then $\tilde{g}_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbf{GL}(k, \mathbb{R})$ is a continuous map.

Therefore, $F(\pi)\colon F(E) \longrightarrow B$ is a pre–vector bundle of rank k and by the Proposition C.1.2, $(F(E), F(\pi), B)$ is a real vector bundle of rank k over B.

For instance, if $\pi\colon E \longrightarrow B$ and $\pi'\colon E' \longrightarrow B$ are (differentiable) vector bundles over B and we consider the functors defined in sections A.3.1, A.3.8, A.3.7, A.3.5, A.3.3 and A.3.2 then we get new (differentiable) vector bundles over B:

1. E^* is the real vector bundle with fiber $(E^*)_b = (E_b)^*$ the dual vector space.
2. $\Lambda^k E$ is the vector bundle with fiber at $b \in B$ is the k–th exterior power $(\Lambda^k E)_b$ of the fiber E_b.
3. $S^k E$ is the vector bundle with fiber at $b \in B$ is the k-th symmetric power $(S^k E)_b$ of the fiber E_b.
4. $E \otimes E'$ is the real vector bundle with fiber $(E \otimes E')_b = E_b \otimes E'_b$.
5. $E \oplus E' = \{(v, w) \in E \times E' | \pi(v) = \pi'(w)\}$ and the projection $\pi_{E\otimes E'}(v, w) = \pi(v) = \pi(w)$, where given $b \in B$ the fiber $(E \oplus E')_b$ is equal to $E_b \oplus E'_b$.
6. $\mathrm{Hom}(E, E') = \amalg_{b\in B}\mathrm{Hom}(E_b, E'_b)$, with $\pi\colon \mathrm{Hom}(E, E') \longrightarrow B$ the projection map onto B, which maps the entire vector space $\mathrm{Hom}(E_b, E'_b)$ to $b \in B$.

In particular, we have the following differentiable vector bundles:

Example C.1.5. Let M be an n–dimensional differentiable manifold and $\mathcal{U}$ be a differentiable atlas of M.

We take (TM, π, M) the tangent bundle of M, apply the dual continuous functor to it and by the Proposition C.1.2 we have the *cotangent bundle*, given by (T^*M, π^*, M). There is a natural surjective map $\pi^*\colon T^*M \longrightarrow M$ give by $\pi^*(\omega) = p$ if $\omega \in T^*_p M$.

Example C.1.6. We repeat the same construction, but now we take (T^*M, π^*, M) the cotangent bundle of M, we apply the continuous functor Λ^k to it and by the Proposition C.1.2 we obtain the *k–th exterior bundle* of T^*M, give by $(\Lambda^k T^*M, \overline{\pi}, M)$, where $\overline{\pi} = \Lambda^k(\pi^*)$. This for any $k = 1, \ldots, n$.

There are more examples of vector bundles that we can be build and that are important to this topic, the reader can find more constructions in [27] and [19].

C.2 Sections

Let M be a differentiable manifold, we will describe sections of differentiable vector bundles examples, remember the definition of section (Definition C.0.9).

Definition C.2.1. A differentiable section of the tangent bundle TM of M is called a *vector field* on M.

Note that a vector field assigns to a point $p \in M$ a vector in its tangent space T_pM.

Let X be a vector field on M. Let us see a local expression for X.

Let $(U, x_1, \ldots, x_n)$ be a chart of M, for each point $p \in U$, by Theorem B.2.10 $\{ \frac{\partial}{\partial x_1}\big|_p, \ldots, \frac{\partial}{\partial x_n}\big|_p\}$ is a basis for T_pM, therefore $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ is a local frame of TM over U. Since $X_p \in T_pM$ hence we have

$$X_p = \sum_{i=1}^{n} a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p$$

where $a_i \in C^\infty(U)$. This is called a *local expression* of X.

The condition of each coefficient a_i being a differentiable function does not depend on the choice of chart.

Let M be a differentiable manifold and $X \in \Gamma(TM)$, for each $f \in C^\infty(M)$, define Xf to be the function

$$(Xf)(p) = X_p f, \quad p \in M.$$

Where X_p is a derivation, see the definition B.2.5, and $X_p f$ is $X_p \overline{f}_p$ with $\overline{f}_p \in C_p^\infty(M)$.

In some books the set of all vector fields on M is denoted by $\mathfrak{X}(M)$, we will use $\Gamma(TM)$.

Definition C.2.2. A section of $\Lambda^k T^*M$ is called a *k–form*. The space of k–form is denoted by

$$\Omega^k(M) = \Gamma(\Lambda^k T^*M). \tag{C.1}$$

That is, $\omega \in \Omega^k(M)$ for all $p \in M$,

$$\omega(p) \colon \underbrace{T_pM \times \ldots \times T_pM}_{k\text{–times}} \longrightarrow \mathbb{R}.$$

is an alternating k–multilinear map.

Definition C.2.3. An *inner product* on a real differentiable vector bundle (E, π, M) is a section $g \in \Gamma(S^2E^*)$ such that, for any $p \in M$, $s(p)$ is positive definite on E_p.

Proposition C.2.4 ([28, Prop. 5.8])**.** *Every differentiable vector bundle admits an inner product.*

Definition C.2.5. An inner product in the tangent bundle TM of a differentiable manifold M is called a *Riemannian metric* on M.

Definition C.2.6. Let M be a differentiable manifold, if g is a *Riemannian metric* on M, we also say that (M, g) is a *Riemannian manifold*.

Appendix D

Functional analysis

This appendix contains a description of vector spaces endowed with an inner product.

The objective of this appendix is to describe Hilbert spaces, in particular, Sobolev spaces.

For topics developed in this appendix consult in [21] [3], and [33].

D.1 Operators and Hilbert spaces

Definition D.1.1. Let V be a real vector space, a real–valued function $|| \, ||\colon V \longrightarrow \mathbb{R}$ is called a *norm* if for all $v \in V$:

1. $||v|| \geq 0$.
2. $||v|| = 0$ if and only if $v = 0$.
3. $||tv|| = |t| ||v||$ for all $v \in V$ and $t \in \mathbb{R}$.
4. $||v + u|| \leq ||v|| + ||u||$ for all $u, v \in V$, the *triangle inequality.*

A *normed space* is a vector space V provided with a norm.

If take $||v + u - u||$, by triangle inequality we get

$$||v|| - ||u|| \leq ||v + u||. \tag{D.1}$$

Definition D.1.2. Let V be a normed space and $\{v_i\}_{i \in \mathbb{N}} \subset V$ be a sequence. We say $\{v_i\}_{i \in \mathbb{N}}$ *converges* to $v \in V$ if and only if

$$\lim_{i \to \infty} ||v_i - v|| = 0.$$

v is called a *limit point* of V.

Definition D.1.3. Let V be a normed space, V is *closed* if for all $\{v_i\}_{i \in \mathbb{N}} \subset V$ sequence such that $\{v_i\}_{i \in \mathbb{N}}$ converges to v implies $v \in V$.

Definition D.1.4. Let V be a normed space. A sequence $\{v_i\}_{i\in\mathbb{N}} \subset V$ is called a *Cauchy sequence* if and only if for every $\epsilon > 0$ there exist $0 < N \in \mathbb{N}$ such that

$$||v_k - v_l|| < \epsilon \quad \text{for all} \quad k, l \geq N.$$

Definition D.1.5. Let V be a normed space. V is called *complete* (or a *Cauchy space*) if every Cauchy sequence in V converges.

Theorem D.1.6 (Completeness, [21, Thm. 2.4-2])**.** *Let V be a normed space, $W \subset V$ be a subspace, if W is a finite dimensional subspace then is complete. In particular, every finite dimensional normed space is complete.*

Definition D.1.7. A *Banach space* V is a complete, normed space.

Definition D.1.8. Let V be a vector space endowed with an inner product $\langle\, , \rangle$, the *associated norm* is

$$||v|| := \langle v, v\rangle^{\frac{1}{2}}$$

The *Cauchy-Schwarz inequality* states for all $v, u \in V$

$$|\langle v, u\rangle| \leq ||v||||u||. \tag{D.2}$$

Definition D.1.9. A *Hilbert space* is a vector space with an inner product such that it is a Banach space with the associated norm.

Theorem D.1.10 (Subspace, [21, Thm. 3.2-4])**.** *Let V be a subspace of a Hilbert space H. Then V is complete if and only if V is closed in H.*

Theorem D.1.11 (Direct sum, [21, Thm. 3.3-4])**.** *Let V be any closed subspace of a Hilbert space H. Then $H = V \oplus V^{\perp}$, where $V^{\perp}$ is the orthogonal complement of V.*

Definition D.1.12. Let H be a Hilbert space, a sequence $\{v_n\}_{n\in\mathbb{N}}$ is said to be *weakly convergent* if and only if there is a $v \in H$ such that $\langle v_n, w\rangle \longrightarrow \langle v, w\rangle$ for all $w \in H$.

Theorem D.1.13. *Let H be a Hilbert space. Every bounded sequence $\{v_n\}_{n\in\mathbb{N}}$ in H contains a weakly convergent subsequence.*

With respect to real-valued functions and Hilbert spaces we have:

Definition D.1.14. A *linear functional* f is a real-valued function defined on a vector space V. The functional f is linear provided

$$f(tv + sw) = tf(v) + sf(w), \quad v, w \in V,\ s, t \in \mathbb{R}.$$

Definition D.1.15. Let V and W be two vector space and $T\colon V \longrightarrow W$ be a linear map. T is called a *linear operator* if:

1. The domain $\mathrm{Dom}(T)$ of T is a vector space and $\mathrm{Im}\,(T)$ lies in a vector space over the same field $\mathbb{K}$.

2. T is a linear map for all $v, w \in \mathrm{Dom}(T)$ and scalars $\alpha \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Example D.1.16. Let V be a normed space, the norm $|| \, ||\colon V \longrightarrow \mathbb{R}$ is a functional on V, by the triangle inequality we note $|| \, ||$ is not a linear operator.

Definition D.1.17. Let V be a normed space, a *bounded* linear functional $f\colon \mathrm{Dom}(f) \subset V \longrightarrow \mathbb{R}$ is a real-valued function such that: there exists a real number c such that for all $v \in \mathrm{Dom}(f)$,

$$|f(v)| \leq c||v||.$$

Definition D.1.18. Let V and W be normed spaces and $T\colon \mathrm{Dom}(T) \subset V \longrightarrow W$ be a linear operator. The linear operator T is *bounded* if there is $c \in \mathbb{R}$ such that for all $v \in \mathrm{Dom}(T)$

$$||T(v)||_W \leq c||v||_V.$$

Theorem D.1.19 (Riesz's Theorem (Functionals on Hilbert spaces), [21, Thm. 3.8.1])**.** *Every bounded linear functional f on a Hilbert space H can be represented in terms the inner product, namely,*

$$f(v) = \langle v, w \rangle$$

where w depends on f, is uniquely determined by f and has norm

$$||w|| = ||f||.$$

While operators between Hilbert spaces we have the following results and notion.

Lemma D.1.20 (Inverse operator, [21, Ex. 2.7.7])**.** *Let V and W be two normed spaces and $T\colon V \longrightarrow W$ be a bounded linear operator. If there is a positive $c \in \mathbb{R}$ such that for all $v \in V$*

$$||T(v)|| \geq c||v||.$$

Then $T^{-1}\colon W \longrightarrow V$ exist and is bounded.

Definition D.1.21. Let H_1, H_2 be two Hilbert spaces and $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator. The *adjoint operator* T^* of T is the operator $T^*\colon H_2 \longrightarrow H_1$ such that for all $v \in H_1$ and $w \in H_2$

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle.$$

Theorem D.1.22 (Existence, [21, Thm. 3.9-2])**.** *The adjoint operator T^* of T in Definition D.1.21 exists, is unique and is a bounded linear operator with norm $||T^*|| = ||T||$.*

If $T^* = T$, T is said to be *self-adjoint.*

By Theorem D.1.11, we have the direct sum $H = V \oplus V^{\perp}$, for any $x \in H$, there exist unique $v \in V$ and $w \in V^{\perp}$ such that $x = v + w$, then this direct sum defines a linear operator onto V:

$$\begin{aligned} P\colon \quad H &\longrightarrow H \\ x &\mapsto v. \end{aligned}$$

Definition D.1.23. Let H be a Hilbert space, a linear operator $P\colon H \longrightarrow H$ is called a *projection* of H if there is a closed subspace V of H such that V is the range of P and $V^{\perp}$ is the kernel of P and $P|_V$ is the identity operator on V.

Theorem D.1.24 ([21, Thm. 9.5.1]). *Let H be a Hilbert space, a bounded linear operator $P\colon H \longrightarrow H$ is a projection if and only if P is self-adjoint and $P^2 = P$.*

The sum of projections need not be a projection, we have the result:

Theorem D.1.25 ([21, Thm. 9.5.4]). *Let $P_0, \dots, P_n$ be projections on a Hilbert space H. Then*

1. *The sum $P = \sum\limits_{i=0}^{n} P_i$ is a projection on H if and only if $Y_i = P_i(H)$ and $Y_j = P_j(H)$ are orthogonal for all $i, j = 0, \dots, n$, $i \neq j$.*
2. *If $P = \sum\limits_{i=0}^{n} P_i$ is a projection, P projects H onto $Y = \bigoplus_{i=0}^{n} Y_i$.*

D.1.1 Spectral theory of bounded self-adjoint operator

Let $T\colon V \longrightarrow V$ on a complex vector space V. A nonzero vector $v \in V$ is called an *eigenvector* of T if there exists a scalar λ such that $T(v) = \lambda v$. The scalar λ is called the *eigenvalue* corresponding to the eigenvector v. The set $E_\lambda = \{v \in V : T(v) = \lambda v\} = \ker(\lambda \mathrm{id}_V - T)$ is called the *eigenspace* of T corresponding to the eigenvalue λ.

Theorem D.1.26. *Let H be a complex Hilbert space and $T\colon H \longrightarrow H$ be a bounded self-adjoint linear operator. Then*

1. *All the eigenvalues of T (if they exist) are real.*
2. *Eigenvectors corresponding to (numerically) different eigenvalues of T are orthogonal.*

Proof.

1. Let λ be any eigenvalue of T and v a corresponding eigenvector. Then $v \neq 0$ and $T(v) = \lambda v$. Since T is self-adjoint operator
$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle T(v), v\rangle = \langle v, T(v)\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle$$
Since $v \neq 0$, then $\langle v, v\rangle \neq 0$, we divide by $\langle v, v\rangle$ on both sides, then $\lambda = \overline{\lambda}$.
2. Let λ and ν be eigenvalues of T and let v and w be corresponding eigenvectors, that is, $T(v) = \lambda v$ and $T(w) = \nu w$, since T is self-adjoint and by item 1. ν is real, we get:
$$\lambda\langle v, w\rangle = \langle \lambda v, w\rangle = \langle T(v), w\rangle = \langle v, T(w)\rangle = \langle v, \nu w\rangle = \nu\langle v, w\rangle$$
Since $\lambda \neq \nu$, then $\langle v, w\rangle = 0$.

□

Definition D.1.27. Let V be a complex Banach space and $T\colon V \longrightarrow V$ be a bounded linear operator, the *resolvent set* of T is

$$\rho(T) = \{\lambda \in \mathbb{C} \mid \lambda \mathrm{id} - T\colon \mathrm{Dom}(T) \longrightarrow V \text{ is one-to-one and onto}\}.$$

If $\lambda \in \rho(T)$, the *resolvent operator* $R_\lambda(T)\colon V \longrightarrow V$ is defined by

$$R_\lambda(T)v := (\lambda \mathrm{id} - T)^{-1}v.$$

Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T.

Theorem D.1.28 (Domain of R_λ, [21, Lemma 7.2-3])**.** *Let V be a complex Banach space, $T\colon V \longrightarrow V$ be a linear operator and $\lambda \in \rho(T)$. Assume that T is closed or T is bounded, then $R_\lambda(T)$ is defined on the whole space V and is bounded.*

Definition D.1.29. Let H be a Hilbert space, a bounded self-adjoint operator $T\colon H \longrightarrow H$ is said to be *nonnegative* or *positive* if and only if its spectrum consists of nonnegative real values only.

Projections have simple properties, we can to obtain a representation of a self-adjoint operator on Hilbert spaces in terms of such operators.

For more details see [35, Sec. 6.4], [32, Sec. 148] and [21, Sec. 9.9].

Definition D.1.30. Let T be a self-adjoint operator of Hilbert spaces, $\sigma_1 \subset \sigma(T)$ part of the spectrum and there exist a domain D such that $\sigma_1 \subset D$, we define Pr_{σ_1} the projection onto the eigensubspace corresponding to σ_1 by

$$\mathrm{Pr}_{\sigma_1} = \frac{1}{2\pi i}\int_{\partial D} R_\lambda(T)d\lambda. \tag{D.3}$$

We call Pr_{σ_1} the *spectral projection* associated with σ_1.

D.2 The space $L^2(V)$

In this section we describe the L^2–space of real–valued functions on $\mathbb{R}^n$.

Definition D.2.1. A collection Σ of subsets of $\mathbb{R}^n$ is called a *σ–algebra* if the following conditions hold

1. $\mathbb{R}^n \in \Sigma$.
2. If $A \in \Sigma$, then its complement $A^c \in \Sigma$.
3. If $A_j \in \Sigma$, $j = 1, 2, \dots$ then $\bigcup\limits_{j=1}^{\infty} A_j \in \Sigma$.

It follows from 1. – 3. that

- The empty set $\emptyset \in \Sigma$.
- If $A_j \in \Sigma, j = 1, 2, \dots,$ then $\bigcap\limits_{j=1}^{\infty} A_j \in \Sigma$.
- If $A, B \in \Sigma$, then $A - B = A \cap B^c \in \Sigma$.

Definition D.2.2. A *measure* μ on a σ–algebra Σ is a function on Σ taking values in $\mathbb{R} \cup \{+\infty\}$ (a positive measure) which is *countably additive* in the sense that

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j}^{\infty} \mu(A_j)$$

whenever $A_j \in \Sigma, j = 1, 2, \ldots$ and the sets A_j are pairwise disjoint, that is, $A_j \cap A_k = \emptyset$ for $j \neq k$.

Definition D.2.3. If $B \subset A \subset \mathbb{R}^n$ and $\mu(B) = 0$, then any condition that holds on the set $A - B$ is said to hold *almost everywhere* in A.

Theorem D.2.4 (Existence of Lebesgue Measure, [3, Thm. 1.39])**.** *There exists a σ–algebra Σ of subsets of $\mathbb{R}^n$ and a positive measure μ on Σ having the following properties:*

1. *Every open set in $\mathbb{R}^n$ belongs to Σ.*
2. *If $A \subset B, B \in \Sigma$ and $\mu(B) = 0$, then $A \in \Sigma$ and $\mu(A) = 0$.*
3. *If $A = \{\mathbf{x} \in \mathbb{R}^n | a_j \leq x_j \leq b_j, j = 1, 2, \ldots, n\}$ then $A \in \Sigma$ and $\mu(A) = (b_1 - a_1) \ldots (b_n - a_n)$.*
4. *μ is translation invariant. That is, if $\mathbf{x} \in \mathbb{R}^n$ and $A \in \Sigma$, then $\mathbf{x} + A = \{\mathbf{x} + \mathbf{y} | \mathbf{y} \in A\} \in \Sigma$ and $\mu(\mathbf{x} + A) = \mu(A)$.*

The elements of Σ are called (Lebesgue) *measurable subsets* of $\mathbb{R}^n$ and μ is called the *(Lebesgue) measure* in $\mathbb{R}^n$.

Definition D.2.5. A function f defined on a measurable set and values in $\mathbb{R} \cup \{-\infty, +\infty\}$ is itself called *measurable* if the set $\{\mathbf{x} \mid f(\mathbf{x}) > t\}$ is measurable for every real t.

Definition D.2.6. Let $V \subset \mathbb{R}^n$, we denote by $L^2(V)$ the class of all measurable functions $f\colon V \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ defined on V for which

$$\int_V |f(\mathbf{x})|^2 d\mathbf{x} < \infty. \tag{D.4}$$

We identify in $L^2(V)$ functions that are equal almost everywhere in V, the elements of $L^2(V)$ are thus equivalence classes of measurable functions satisfying D.4. Two functions being equivalent if they are equal almost everywhere on V.

For convenience, we ignore this distinction and write $f \in L^2(V)$ if f satisfies D.4, and $f = 0$ in $L^2(V)$ if $f(x) = 0$ almost everywhere in V.

$L^2(V)$ is a real vector space.

Definition D.2.7. Let $V \subset \mathbb{R}^n$, the L^2*–norm* on $L^2(V)$ of $f\colon V \longrightarrow \mathbb{R}$ is defined by

$$||f||_{L^2} = \left(\int_V |f(\mathbf{x})|^2 d\mathbf{x}\right)^{1/2} \tag{D.5}$$

Theorem D.2.8 ([3, Thm. 2.16])**.** *$L^2(V)$ with the L^2–norm is a Banach space.*

Corollary D.2.9 ([3, Cor. 2.18]). *$L^2(V)$ is a real Hilbert space with respect to the inner product*

$$\langle f, g\rangle = \int_V f(\mathbf{x})g(\mathbf{x})d\mathbf{x}. \tag{D.6}$$

D.3 Sobolev space

We will continue studying real-valued function spaces endowed with a norm where we define a new norm so that it is a complete space. They are a important tool in the theory of partial differential equations and modern analysis.

In this section we introduce Sobolev spaces of integer order and establish some results.

Definition D.3.1. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multiindex and $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$. We consider

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

Let $V \subset \mathbb{R}^n$ be an open, for any nonnegative integer k let $C^k(V)$ the space consisting of all functions $f\colon V \longrightarrow \mathbb{R}$ which, together with all their partial derivatives $D^\alpha f$ of orders $|\alpha| \leq k$, are continuous on V. Let

$$C^\infty(V) := \bigcap_{k=0}^{\infty} C^k(U).$$

The subspaces $C_c^k(V)$ and $C_c^\infty(V)$ consist of all those functions in $C^k(V)$ and $C^\infty(V)$, respectively, that have compact support in V.

Let $V \subset \mathbb{R}^n$ be an open neighbourhood of $\mathbf{x} \in \mathbb{R}^n$, for each $f \in C^k(V)$, we define a function $||\ ||_k$ where k is a positive integer as follows:

$$||f||_k = \left(\sum_{|\alpha| \leq k} ||D^\alpha f||_{L^2}^2 \right)^{\frac{1}{2}} \tag{D.7}$$

This function defines a norm, the *k–Sobolev norm*, on any vector space of functions on which the right side takes finite values provided functions are identified in the space if they are equal almost everywhere in V.

Example D.3.2. Let $V \subset \mathbb{R}^n$ be an open subset and $f\colon V \longrightarrow \mathbb{R}$ be a differentiable function.

If $k = 1$, the 1–Sobolev norm of real-valued differentiable functions (see equalities (D.7) and (D.5)) is given by

$$||f||_{1,H^1(V)} = \left(\int_V |f(x)|^2 d\mathbf{x} + \sum_{i=1}^{n} \int_V \left| \frac{\partial f}{\partial x_i}(\mathbf{x}) \right|^2 d\mathbf{x} \right)^{\frac{1}{2}}. \tag{D.8}$$

Definition D.3.3. Let $V \subset \mathbb{R}^n$ be an open subset and $k \in \mathbb{N}$, we consider two vector spaces on which $||\ ||_k$ is a norm:

1. $H^k(V)$ the completion of $\{f \in C^k(V) : ||f||_k < \infty\}$ with respect to the norm $|| \,||_k$, see (D.7).

2. $W^k(V)$ the set of all $f \in L^2(V)$ such that $D^\alpha f \in L^2(V)$ for $0 \leq |\alpha| \leq k$.

This are *k–Sobolev spaces* over V.

Remark D.3.4. Note that the 0–Sobolev space, $H^0(V) = L^2(V)$.

Theorem D.3.5 ([3, Thm. 3.17])**.** *Let $V \subset \mathbb{R}^n$ be an open subset, if $1 \leq k < \infty$, then $H^k(V) = W^k(V)$.*

Characterizations of $H^k(\mathbb{R}^n)$

Let $V \subset \mathbb{R}^n$ be an open neighbourhood of $\mathbf{x} \in \mathbb{R}^n$ with compact closure $\overline{V}$, the set of infinitely differentiable functions $f\colon V \longrightarrow \mathbb{R}$ on V with compact support will be denoted by $C_c^\infty(V)$.

Let $u \in C_c^\infty(V)$, the *Fourier transform* of u is the function $\hat{u}$ defined on $\mathbb{R}^n$ by:

$$\hat{u}(\mathbf{y}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\mathbf{y}} u(\mathbf{x}) d\mathbf{x}.$$

Definition D.3.6. Let V be a vector space with two norms $|| \,||_{(1)}, || \,||_{(2)}$. The *norms are equivalent* if there are constants $C_1, C_2 > 0$ such that for all $f \in V$

$$C_1||f||_{(1)} \leq ||f||_{(2)} \leq C_2||f||_{(1)}.$$

If $|| \,||_{(1)}$ and $|| \,||_{(2)}$ are equivalent we denoted it by $|| \,||_{(1)} \approx || \,||_{(2)}$.

Proposition D.3.7 ([33, Lem. 1.18])**.** *Let $k \in \mathbb{N}$, for $f \in C_c^\infty(V)$, we have*

$$||f||_k \approx \left(\int_{\mathbb{R}^n} |\hat{f}(\mathbf{y})|^2 (1 + |\mathbf{y}|^2)^k d\mathbf{y} \right)^{1/2}$$

Remark D.3.8 ([7, Ex. 2])**.** Using the fact that $(1 + |\mathbf{y}|^2)^k > (1 + |\mathbf{y}|^2)^l$ for $k > l$, then if $k > l > 0 > r$, we have continuous inclusions of Sobolev spaces $H^k(V) \subset H^l(V) \subset H^0(V) = L^2(V) \subset H^r(V)$.

Theorem D.3.9 (Sobolev Embedding Theorem, [33, Thm. 1.20])**.** *If $f \in H^k(V)$ then $f \in C_c^t(\overline{V})$, for each $t < k - \frac{1}{2}$.*

Corollary D.3.10 ([33, Cor. 1.21])**.** *$f \in \bigcap\limits_{k \in \mathbb{R}} H^k(V)$ if and only if $f \in C^\infty(\overline{V})$.*

Theorem D.3.11 (Rellich-Kondarachov Compacteness Theorem, [33, Thm. 1.22])**.** *Let $k, t \in \mathbb{N}$, if $t > k$, then the inclusion $H^t(V) \longrightarrow H^k(V)$ is compact.*

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