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FLUCTUATIONS OF *P*-PARAMETER ADDITIVE LÉVY PROCESSES THROUGH CONVEX MINORANTS

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1 | Introduction

The focus of the present document lies on p-parameter **additive Lévy processes** on $\mathbb{R}$. That is, considering $p \geq 1$ independent Lévy processes $X^1, \ldots, X^p$ on $\mathbb{R}$, the multiparameter process $\{X_t : t \in [0,\infty)^p\}$ given by $X_{(t_1,\ldots,t_p)} = X^1_{t_1} + \cdots + X^p_{t_p}$ for $(t_1, \ldots, t_p) \in [0,\infty)^p$ is a p-parameter additive Lévy process and will be denoted as $X^1 \oplus \cdots \oplus X^p$. Such a definition is a particular case of additive Lévy processes on $\mathbb{R}^d$ studied mainly by Y. Xiao and D. Khoshnevisan in [10], [11] and [12], to mention a few. An example is given by the additive stable process $X = X^1 \oplus \cdots \oplus X^p$ for which X^κ is an isotropic stable process of index $\alpha \in (0,2]$ for each $\kappa \in [p] \stackrel{\text{def}}{=} \{1, \ldots, p\}$ in the paper [12]. I.e., the value of the characteristic function of X^κ_t at $\beta \in \mathbb{R}$ is $\exp\{-\sigma^\alpha |\beta|^\alpha t\}$, where $\sigma > 0$ is a constant. The case in which $\alpha = 2$ corresponds to the additive Brownian Motion.

As stated in [10], additive Lévy processes are closely related to the analysis of multiparameter processes such as Lévy sheets. Those studies are mainly focused on potential-theoretical level sets and capacity as the aforementioned papers [10] and [12] or, for example, the work of R. Dalang, J. Walsh and T. Mountford in Lévy and Brownian sheets (cf. [5] y [6]). Papers about additive Lévy processes outside potential theory can be found in [11] about local times or, e.g., in [7], in which L. Chaumont and M. Marolleau develop fluctuation theory for a particular notion of spectrally positive additive Lévy fields.

On a tangent line lies the research on the excursions of Lévy processes over the greatest **convex minorant** of its sample paths[1]. Several studies of the convex minorants for different kind of processes such as random walks, Brownian Motion and other Markov and Lévy processes have been provided by a broad list of authors at least since the decade of 1970 (see [2] for details). Recent papers that encompass a comprehensive description of such matter are: [1] for discrete random walks with exchangeable increments by J. Abramson and J. Pitman; later generalized for Lévy processes with continuous distribution in [17] by J. Pitman and G. Uribe Bravo; next extended to general Lévy processes on $\mathbb{R}$ by J. González Cázares and A. Mijatovic in [8].

Even though such line of work will be described in more detail in Section 2, it is worth mentioning that it provides a remarkable distributional representation for the lengths and heights of the excursions of Lévy processes via its sampling at a particular time partition. Among other virtues, such approach enables the study of well-known results of Lévy processes in a considerably direct way, as asserted in [8], such as fluctuation theory.

[1] The greatest convex function dominated by the path.

That being stated, the main purpose of this document is to develop[2] properties for the fluctuations of p-parameter additive Lévy processes on $\mathbb{R}$. The mean to this end will be the derivation of Poisson point processes associated to the heights and lengths of the excursions of the additive process over its convex minorant through the stick-breaking representation for Lévy processes mentioned above. This is possible due to the fact that it provides a representation for additive processes as well, since the convex minorant of the additive process $X^1 \oplus \cdots \oplus X^p$ coincides with the sum of the convex minorants associated to the components X^κ, as shown in Figure 1.1.

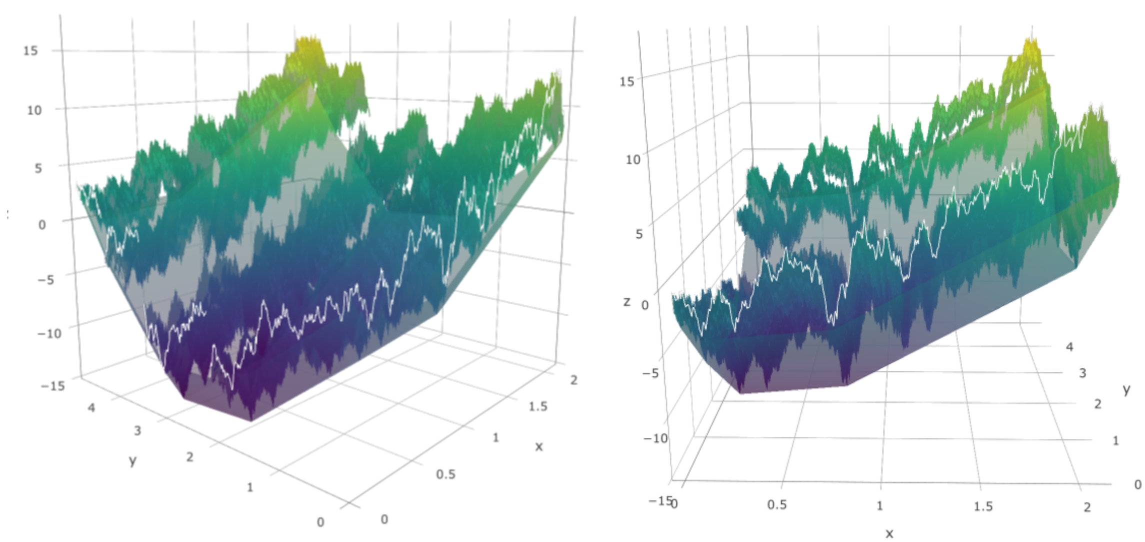

Figure 1.1: Sample path of the 2-parameter additive Lévy process $X^1 \oplus X^2$ where: X^1 is a Brownian Motion with drift 0.5 and standard deviation 4.5; X^2 is a compound Poisson process with intensity 2.5 and jump distribution $\mathcal{N}(6, 9^2)$, superposed with a Brownian Motion with standard deviation 6 and negative drift 5. The surface below corresponds to the greatest convex minorant associated to the path.

This document is organized as follows. First of all, Section 2 is devoted to the description of the aforementioned studies of the stick-breaking representation for the convex minorant of Lévy processes, as well as the extension of such representation to additive processes of p parameters. Then, the purpose of Section 3 is to give a natural generalization of a Poisson point process associated to the paths of Lévy processes [17, pp. 4, 18] for the case of additive Lévy processes on $\mathbb{R}$; that is, the point process with atoms at lengths and heights of excursion rectangles of the convex minorant of the process on a finite time horizon. Lastly, aspects of the fluctuation theory for additive Lévy processes are studied in Section 4. To be specific, well-known properties for Lévy processes on $\mathbb{R}$ are generalized for the additive kind:

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- Fluctuations up to an exponential time horizon (characteristic functions of the extremal vectors and Wiener-Hopf factorisation), and
- Asymptotic behavior (proposed notion of regularity of the origin for additive processes, Rogozin-like criterions, Laplace transforms for the limit of extremal vectors and a characterization for drifting and oscillating).

Beforehand, some notation and conventions will be introduced. On the elements of $\mathbb{R}^p$: the κ-th canonical vector for $\mathbb{R}^p$ will be denoted by e_κ, for every $z \in \mathbb{R}^p$ it will always be assumed that $z = (z_1, \ldots, z_p) \overset{\text{def}}{=} (z_\kappa)_1^p$ unless otherwise specified, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^p$ and $\cdot$ denotes the usual inner product. The Lebesgue measure on $(\mathbb{R}^n, \mathbb{B}_{\mathbb{R}^n})$ will be denoted as λ_n with the simplification $\lambda = \lambda_1$, as well as $\lambda^+ = \lambda_1\big|_{(0,\infty)}$ and $\lambda_+ = \lambda_1\big|_{[0,\infty)}$. In order to keep a neat presentation, Dirac's delta will sometimes be written as $\delta(x_1, \ldots, x_n; A)$ instead of $\delta_{(x_1,\ldots,x_n)}(A)$. Finally, $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}_+ = [0, \infty)$, even though they will be used indistinctly when convenient.

2 | Stick-breaking representation for the convex minorant of additive Lévy processes on $\mathbb{R}$

Geared towards fulfilling the objectives of this document, in this section a stick-breaking representation for the convex minorant, so as concave majorant, for the paths of additive Lévy processes on a finite time horizon will be established in the line of the work developed in [1], [2], [8] and [17], recent articles which have come to generalize previous studies in the matter, as described in [2].

First of all, consider $X = X^1 \oplus \cdots \oplus X^p$, a p-parameter additive Lévy Process on $\mathbb{R}$ and note that, by nature of the process, the **running infimum** of X at $t \in [0, T]$ can be expressed as

$$\underline{X}_t \overset{\text{def}}{=} \inf_{\substack{0 \le s_\kappa \le t_\kappa \\ 1 \le \kappa \le p}} \left\{ X^1_{s_1} + \cdots + X^p_{s_p} \right\} = \sum_{\kappa=1}^p \inf_{0 \le s_\kappa \le t_\kappa} X^\kappa_{s_\kappa} = \sum_{\kappa=1}^p \underline{X}^\kappa_{t_\kappa}$$

for fixed $T \in \mathbb{R}^p_+$. Then it makes sense to think about the **time of attainment**, in the sense of partial order in $\mathbb{R}^p_+$, that is

$$\underline{\tau}_t(X) \overset{\text{def}}{=} \left(\underline{\tau}_{t_1}\left(X^1\right), \ldots, \underline{\tau}_{t_p}\left(X^p\right) \right) = \sum_{\kappa=1}^p \inf\left\{ s \in [0, t_\kappa] : X^\kappa_{t_\kappa -} \wedge X^\kappa_{t_\kappa} = \underline{X}^\kappa_{t_\kappa} \right\} \mathrm{e}_\kappa.$$

So, $\left(X_t, \underline{X}_t, \underline{\tau}_t(X)\right)$ and $\left(X_t, \overline{X}_t, \overline{\tau}_t(X)\right)$, the **extremal vectors** of X at $t \in [0,T]$, can be considered as in the single parameter case of study. As a matter of fact, one can consider the extremal vectors associated to any càdlàg p-variate real function. In a similar way as for the infimum of the paths of X on $[0,T]$, let us denote

$$\overline{X}_t \stackrel{\text{def}}{=} \sup_{\substack{0\le s_\kappa \le t_\kappa \\ 1\le \kappa\le p}} \left\{X^1_{s_1} + \cdots + X^p_{s_p}\right\} = \sum_{\kappa=1}^{p} \overline{X}^\kappa_{s_\kappa}$$

and

$$\overline{\tau}_t(X) \stackrel{\text{def}}{=} \left(\overline{\tau}_{t_1}\left(X^1\right), \ldots, \overline{\tau}_{t_p}\left(X^p\right)\right) = \sum_{\kappa=1}^{p} \inf\left\{s \in [0, t_\kappa] : X^\kappa_{t_\kappa -} \vee X^\kappa_{t_\kappa} = \overline{X}^\kappa_{t_\kappa}\right\} \mathrm{e}_\kappa.$$

If the context is not ambiguous, it will be written $\underline{\tau}_t$ and $\overline{\tau}_t$ instead of $\underline{\tau}_t(X)$ and $\overline{\tau}_t(X)$, respectively.

Recall that by the (greatest) convex minorant associated to a function f we mean the greatest convex function dominated by f. A natural question is whether the convex minorant associated to the paths of X at $t \in \mathbb{R}^p_+$ can be written in terms of that of $X^1, \ldots, X^p$; and so is the case. As a matter of fact, it coincides with the sum of the convex minorants of $X^1, \ldots, X^p$ at the individual times $t_1, \ldots, t_p$, as stated next.

Lemma 2.1. *Consider a p-parameter, $\mathbb{R}$-valued, additive Lévy process $X = X^1 \oplus \cdots \oplus X^p$. Let $T = \left(T_1, \ldots, T_p\right) \in \mathbb{R}^p_+$ and $C_{T_\kappa,\kappa} : [0, T_\kappa] \to \mathbb{R}$ the convex minorant associated to the paths of X^κ on the finite horizon $[0, T_\kappa]$. The mapping $C_T : [0,T] \to \mathbb{R}$ defined as*

$$C_T(t) = C_{T_1,1}(t_1) + \cdots + C_{T_p,p}\left(t_p\right) \quad \textit{for} \quad t = \left(t_1, \ldots, t_p\right) \in \mathbb{R}^p_+$$

coincides with the convex minorant for the paths of X on given $[0,T]$.

***Proof*.** Clearly, C_T is dominated by the paths of X on $[0,T]$ due to the fact that $C_{T_\kappa,\kappa} \le X^\kappa$ on $[0, T_\kappa]$ a.s. In addition, C_T is convex, for it is the sum of the p convex functions defined on a convex set given as $[0,T] \ni t \mapsto C_{T_\kappa,\kappa}(t \cdot \mathrm{e}_\kappa)$. To verify that the latter is true, consider $t^1, t^2 \in [0,T]$ and $\lambda \in [0,1]$. Let us write $\overline{\lambda} = 1 - \lambda$ and note that

$$C_{T_\kappa,\kappa}\left(\left[\lambda t^1 + \overline{\lambda} t^2\right] \cdot \mathrm{e}_\kappa\right) = C_{T_\kappa,\kappa}\left(\lambda t^1_\kappa + \overline{\lambda} t^2_\kappa\right) \le \lambda C_{T_\kappa,\kappa}\left(t^1_\kappa\right) + \overline{\lambda} C_{T_\kappa,\kappa}\left(t^2_\kappa\right) = \lambda C_{T_\kappa,\kappa}\left(t^1 \cdot \mathrm{e}_\kappa\right) + \overline{\lambda} C_{T_\kappa,\kappa}\left(t^2 \cdot \mathrm{e}_\kappa\right).$$

The maximality property of the convex minorant will be verified by induction with respect to p, so assume it holds for $p \ge 2$, since there is nothing to prove when $p = 1$. Let $T \in \mathbb{R}^{p+1}_+$ and $C' : [0,T] \to \mathbb{R}$ be convex and dominated by the paths of X on $[0,T] \subset \mathbb{R}^{p+1}_+$. Denote $T' = \left(T_1, \ldots, T_p\right)$

and $X' = X^1 \oplus \cdots \oplus X^p$. For fixed $t_{p+1} \in [0, T_{p+1}]$, the mapping $(t_1, \ldots, t_p) \mapsto C'(t_1, \ldots, t_p, t_{p+1})$ is convex; then, so is

$$[0, T'] \ni (t_1, \ldots, t_p) \mapsto C'(t_1, \ldots, t_p, t_{p+1}) - X^{p+1}_{t_{p+1}} \leq X^1_{t_1} + \cdots + X^p_{t_p}.$$

Hence, $C'(t) - X^{p+1}_{t_{p+1}} \leq C_{T'}(t_1, \ldots, t_p)$ almost surely due to the hypothesis of induction, which implies

$$C'(t) - C_{T'}(t_1, \ldots, t_p) \leq X^{p+1}_{t_{p+1}} \quad \text{(a.s.)}. \tag{2.1}$$

for every $t_{p+1} \in [0, T_{p+1}]$. Furthermore, $[0, T_{p+1}] \ni t_{p+1} \mapsto C'(t) - C_{T'}(t_1, \ldots, t_p)$ defined for fixed $(t_1, \ldots, t_p) \in [0, T']$ is convex and dominated by X^{p+1} on $[0, T_{p+1}]$ by (2.1). Therefore,

$$C'(t) - C_{T'}(t_1, \ldots, t_p) \leq C_{T_{p+1}, p+1}(t_{p+1}),$$

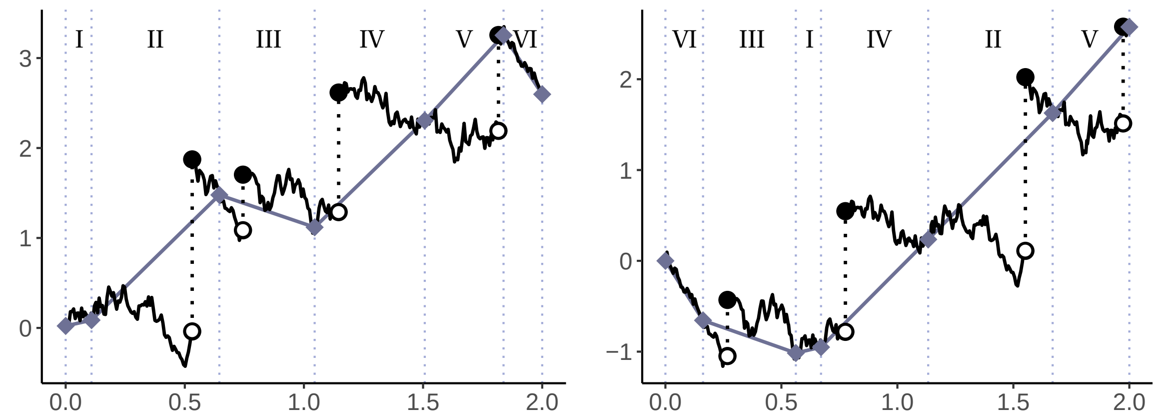
expression from which the statement follows, since it was assumed that $C_{T'} = C_{T_1,1} \oplus \cdots \oplus C_{T_p,p}(t_p)$. ◆

Remark 2.2. Noting that the concave majorant of X coincides with the convex minorant of $-X$, which is an additive Lévy Process as well, the concave majorant associated to the paths of X on given $[0, T]$ holds the same additive relationship with the majorants of $X^1, \ldots, X^p$ as the convex minorant. That is, $C^{\mathrm{maj}}_T = C^{\mathrm{maj}}_{T_1,1} \oplus \cdots \oplus C^{\mathrm{maj}}_{T_p,p}$ if $X = X^1 \oplus \cdots \oplus X^p$.

Next, the main idea after the stick-breaking representation for Lévy Processes on $\mathbb{R}$ will be described, for it will be the means for deriving a stick-breaking representation for additive Lévy Processes on $\mathbb{R}$. Beforehand, fix $[0, T] \subset \mathbb{R}$ and consider an uniform stick-breaking process $\{\ell_n\}_1^\infty$ on $[0, T]$ independent of X. That is, letting $(U_n)_1^\infty$ independent random variables with uniform distribution over $(0, T)$, the stick-breaking process, along with the associated remainder process L, are given recursively by

$$L_0 = T, \quad \ell_n = U_n L_{n-1} \quad \text{and} \quad L_n = L_{n-1} - \ell_n, \qquad n \in \mathbb{N}. \tag{2.2}$$

Note that L induces a partition of $[0, T]$ with lengths ℓ. So, given $\ell_n = L_{n-1} - L_n$, one can consider the increments $\Delta_n = X_{L_n} - X_{L_{n-1}}$ associated to a Lévy process X and construct a piecewise linear convex function (**pw.l.c.**) starting in 0, which faces are given by attaching the line segments with length ℓ_n, height Δ_n and slope Δ_n/ℓ_n in increasing order with respect to the magnitude of the slopes. This process is illustrated in Figure 1 where, as must be stated, the representation is purely illustrative since the partition of the time horizon is not derived from a stick-breaking process. The latter is due to the fact that, since $L_n \to 0$ as $n \to \infty$, such a partition (computed for some *large* n) does not result in a visually useful representation of the method.



(a) Before permutation of the segments.

(b) After permutation of the segments.

Figure 2.2: Example of the construction of a piecewise linear convex function described above for a Lévy Process with drift -1, standard diffusion and positive jumps with exponential distribution. On the left: A sample path of X and a random partition of $[0,2]$. On the right: The pw.l.c. function derived from the arragement of the segments in (a) increasingly with respect to the slopes Δ/ℓ. The roman enumeration of the subintervals of $[0,2]$ in (b) corresponds with those in (a).

Such a piecewise linear convex function on $[0,T]$ can be expressed as

$$t \mapsto \sum_{n\geq 1} \Delta_n \left[\frac{1}{\ell_n}\left(t - \sum_{k\geq 1} \ell_k \mathbb{1}_{\frac{\Delta_k}{\ell_k} < \frac{\Delta_n}{\ell_n}} - \sum_{k=1}^{n-1} \ell_k \mathbb{1}_{\frac{\Delta_k}{\ell_k} = \frac{\Delta_n}{\ell_n}} \right)^{+} \wedge 1 \right], \tag{2.3}$$

as in [8], and the associated distribution coincides in the space of continuous functions on $[0,T]$ with the distribution of the convex minorant of X on $[0,T]$. The analogous result for discrete random walks with exchangeable increments, which concerns the permutation of the ranked lengths of the process as described above, was firstly given in [1]. The equivalent result which involves the equality in (joint) distribution between the lengths and heights of the minorant and its lengths equals that of ℓ and the increments of X sampled at L can be found in [17] and [2] for continuously distributed Lévy processes. Whilst the stick-breaking representation of the convex minorant in the form of (2.3) for general Lévy processess was provided in [8] and is given as follows

Theorem 2.3. [8, Theorem 12, p. 6] *Let X be a Lévy process and fix $T>0$. Let $\{\ell_n\}_1^\infty$ be a uniform stick-breaking process on $[0,T]$ independent of X with remainder process $\{L_n\}_0^\infty$ and consider the increments $\Delta_n = X_{L_{n-1}} - X_{L_n}$ for $n \geq 1$. The convex minorant C_T of X has the same law in the space of continuous functions on $[0,T]$ as the piecewise linear convex function on $[0,T]$ defined in (2.3).*

Roughly speaking, the proof given in [8] for Theorem 2.3 consists on approximating the convex minorant associated to the process X through those of the random walk skeleton of X pointwise in the space of continuous functions on $[0,T]$. So as, simultaneously, approximating the distribution of the stick-breaking pw.l.c. function associated to X from that of the random walk skeleton of X in probability. The latter holds since the statement of the theorem is true for polygonal random walks thanks to Theorem 1 in [1, p. 4].

Next, relying on Theorem 2.3 and Lemma 2.1, a stick-breaking representation for additive Lévy Processes on $\mathbb{R}$ is given. It shows that the virtues of analysing the convex minorant of the paths of Lévy processes in $\mathbb{R}$ are inherited to the multiparameter case of study, for it is then possible to describe the characteristics of the extremal vectors of the process in terms of its increments sampled at L and the excursion lengths ℓ.

Corollary 2.4. *Let $X = X^1 \oplus \cdots \oplus X^p$ be a p-parameter Lévy process on $\mathbb{R}$. Fix a time horizon $T \in \mathbb{R}_+^p$ and let $\ell^\kappa = \left(\ell_n^\kappa\right)_{n=1}^\infty$ be p jointly independent stick-breaking processes on $[0,T_\kappa]$ independent of X. Then,*

$$C_T(\bullet) \stackrel{\mathrm{d}}{=} \sum_{n\geq 1}\left[\left(\Delta_n^1,\ldots,\Delta_n^p\right)\cdot\left(1\wedge\frac{\left(\bullet\cdot \mathrm{e}_1 - a_n^1\right)^+}{\ell_n^1},\cdots,1\wedge\frac{\left(\bullet\cdot \mathrm{e}_p - a_n^p\right)^+}{\ell_n^p}\right)\right], \tag{2.4}$$

where $L^\kappa = \left(L_n^\kappa\right)_{n=0}^\infty$ is the remainder process associated to ℓ^κ,

$$\Delta_n^\kappa = X^\kappa_{L^\kappa_{n-1}} - X^\kappa_{L^\kappa_n} \quad \textit{and} \quad a_n^\kappa = \sum_{k\geq 1}\ell_k^\kappa \mathbb{1}\left\{\tfrac{\Delta_k^\kappa}{\ell_k^\kappa} < \tfrac{\Delta_n^\kappa}{\ell_n^\kappa}\right\} + \sum_{k=1}^{n-1}\ell_k^\kappa \mathbb{1}\left\{\tfrac{\Delta_k^\kappa}{\ell_k^\kappa} = \tfrac{\Delta_n^\kappa}{\ell_n^\kappa}\right\} \quad \textit{for } \kappa\in[p],\ n\in\mathbb{N}.$$

Moreover,

$$\left(X_T, \underline{X}_T, \underline{\tau}_T(X)\right) \stackrel{\mathrm{d}}{=} \sum_{n\geq 1}\sum_{\kappa=1}^{p}\left(\Delta_n^\kappa,\ \Delta_n^\kappa \mathbb{1}_{\Delta_n^\kappa<0},\ \ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa<0}\, \mathrm{e}_\kappa\right). \tag{2.5}$$

***Proof*.** According to Lemma 2.1 and Theorem 2.3, due to the independence structure,

$$C_T(\bullet) = \sum_{\kappa=1}^{p} C_{T_\kappa,\kappa}(\bullet\cdot \mathrm{e}_\kappa) \stackrel{\mathrm{d}}{=} \sum_{\kappa=1}^{p}\sum_{n\geq 1}\Delta_n^\kappa\left[1\wedge\frac{\left(\bullet\cdot \mathrm{e}_\kappa - a_n^\kappa\right)^+}{\ell_n^\kappa}\right],$$

from which (2.4) follows iterating the sums and writing the quantities in vectorial form.

Furthermore, note that $C_{T_\kappa,\kappa}(T_\kappa) = X^\kappa_{T_\kappa-} \wedge X^\kappa_{T_\kappa}$. Nevertheless, for each $\kappa \in [p]$ the event in which $C_{T_\kappa,\kappa}(T_\kappa) = X^\kappa_{T_\kappa-}$ coincides with the event that X^κ jumps upwards exactly at T_κ which, according to the Lévy-Itô decomposition [14, Theorem 2.1, p. 37], has null probability. That is, $C_{T_\kappa,\kappa}(T_\kappa) = X^\kappa_{T_\kappa}$ almost surely for each $\kappa \in [p]$. Then,

$$\left(C_{T_\kappa,\kappa}(T_\kappa),\ \underline{C}_{T_\kappa,\kappa},\ \underline{\tau}_{T_\kappa}\left(C_{T_\kappa,\kappa}\right)\right) = \left(X^\kappa_{T_\kappa},\ \underline{X}^\kappa_{T_\kappa},\ \underline{\tau}_{T_\kappa}(X)\right),$$

due to the fact that, if the equation didn't hold, the maximality property of $C_{T_\kappa,\kappa}$ being convex minorant would be contradicted. Hence,

$$\left(C_T(T),\ \underline{C}_T(T),\ \sum_{\kappa=1}^{p} \underline{\tau}_{T_\kappa}\left(C_{T_\kappa,\kappa}\right) \mathrm{e}_\kappa\right) = \left(X_T, \underline{X}_T, \underline{\tau}_T(X)\right). \tag{2.6}$$

Consider $\mathcal{F}^{\Delta^\kappa,\ell^\kappa}$ for $\kappa \in [p]$, the pw.l.c. function associated to X^κ in $[0, T_\kappa]$ defined in (2.3). By its definition, the extremal vector associated to $\mathcal{F}^{\Delta^\kappa,\ell^\kappa}$ on $[0, T_\kappa]$ can be written as

$$\left(\mathcal{F}^{\Delta^\kappa,\ell^\kappa}(T_\kappa),\ \underline{\mathcal{F}}^{\Delta^\kappa,\ell^\kappa}_{T_\kappa},\ \underline{\tau}_{T_\kappa}\left(\mathcal{F}^{\Delta^\kappa,\ell^\kappa}\right)\right) = \sum_{n\geq 1}\left(\Delta^\kappa_n,\ \Delta^\kappa_n \mathbb{1}_{\Delta^\kappa_n<0},\ \ell^\kappa_n \mathbb{1}_{\Delta^\kappa_n<0}\right).$$

Therefore, (2.4) and (2.6) imply (2.5). ◆

Remark 2.5. Since $-X$ is a Lévy process as well, denoting the increments of $-X^\kappa$ sampled at L^κ as $\Delta^\kappa_n(-X^\kappa)$ and applying Corollary 2.4 to $-X$ gives

$$\left(-(-X_T),\ -\underline{(-X)}_T,\ \underline{\tau}_T(-X)\right) \stackrel{\mathrm{d}}{=} \sum_{n\geq 1}\sum_{\kappa=1}^{p}\left(-\Delta^\kappa_n(-X^\kappa),\ -\Delta^\kappa_n(-X^\kappa)\mathbb{1}_{\Delta^\kappa_n(-X^\kappa)<0},\ \ell^\kappa_n \mathbb{1}_{\Delta^\kappa_n(-X^\kappa)<0}\,\mathrm{e}_\kappa\right).$$

Hence,

$$\left(X_T, \overline{X}_T, \overline{\tau}_T\right) \stackrel{\mathrm{d}}{=} \sum_{n\geq 1}\sum_{\kappa=1}^{p}\left(\Delta^\kappa_n,\ \Delta^\kappa_n \mathbb{1}_{\Delta^\kappa_n>0},\ \ell^\kappa_n \mathbb{1}_{\Delta^\kappa_n>0}\,\mathrm{e}_\kappa\right).$$

By last, some properties for the extremal vectors associated to X will be established in the line of the corollaries of Theorem 2.3 in [8], which correspond to the generalization of several fluctuation theory identities for Lévy Processes for the multiparameter case of study.

Proposition 2.6. *The process $X = X^1 \oplus \cdots \oplus X^p$ satisfies the following properties for fixed $T \in (0,\infty)^p$:*

- *The following equalities hold*

$$\mathbb{E}\left[\underline{\tau}_T(X)\right] = \sum_{\kappa=1}^{p}\left[\int_0^{T_\kappa} \mathbb{P}\left(X_u^\kappa < 0\right)\mathrm{d}u\right] \mathrm{e}_\kappa \quad \text{and} \quad \mathbb{E}\left[\underline{X}_T\right] = \sum_{\kappa=1}^{p}\int_0^{T_\kappa} \mathbb{E}\left[X_u^\kappa \mathbb{1}_{X_u^\kappa<0}\right] u^{-1}\mathrm{d}u.$$

- *If $X^1,\dots,X^p$ are not compound Poisson processes, then $\left(\overline{X}_T, \overline{\tau}_T(X)\right) \overset{\mathrm{d}}{=} \left(X_T - \underline{X}_T, T - \underline{\tau}_T(X)\right)$.*

***Proof*.** By (2.2), for every $n \geq 1$ and $\kappa \in [p]$,

$$-\log\left(\ell_n^\kappa / T_\kappa\right) = -\log\left[U_n^\kappa \prod_{k=1}^{n-1}\left(1-U_k^\kappa\right)\right] = -\log\left(U_n^\kappa\right) - \sum_{k=1}^{n-1}\log\left(1-U_k^\kappa\right) \sim \Gamma(n,1), \tag{2.7}$$

since $\left(U_n^\kappa\right)_1^\infty$ are i.i.d. uniformly distributed random variables. Then, following [8], for every non-negative measurable function f,

$$\sum_{n\geq 1}\mathbb{E}\left[f\left(\ell_n^\kappa\right)\right] = \sum_{n\geq 1}\int_0^\infty f\left(T_\kappa e^{-g}\right)\frac{g^{n-1}e^{-g}}{\Gamma(n)}\mathrm{d}g = \sum_{n\geq 1}\int_0^{T_\kappa}\frac{f(u)}{T_\kappa}\frac{\left[-\log(u/T_\kappa)\right]^{n-1}}{(n-1)!}\mathrm{d}u = \int_0^{T_\kappa} f(u)\, u^{-1}\mathrm{d}u, \tag{2.8}$$

where the last equality holds according to Fubini's Theorem.

Then, due to the fact that ℓ^κ and X^κ are independent, conditioning with respect to L_{n-1}^κ and L_n^κ and taking into account independence between X^κ and $\sigma\left(\ell^\kappa, L^\kappa\right)$, it is easy to see that

$$\mathbb{E}\left[\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa<0}\right] = \mathbb{E}\left[\left(L_{n-1}^\kappa - L_n^\kappa\right)\mathbb{P}\left(X_{L_{n-1}^\kappa}^\kappa - X_{L_n^\kappa}^\kappa < 0 \,\middle|\, L_{n-1}^\kappa, L_n^\kappa\right)\right] = \mathbb{E}\left[\ell_n^\kappa \mathbb{P}\left(X_{\ell_n^\kappa}^\kappa < 0\right)\right]$$

for every $n \geq 1$. So, by Corollary 2.4 and the Monotone Convergence Theorem, (2.8) implies that

$$\mathbb{E}\left[\underline{\tau}_T\right] = \mathbb{E}\left[\sum_{n\geq 1}\sum_{\kappa=1}^{p}\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa<0}\,\mathrm{e}_\kappa\right] = \sum_{\kappa=1}^{p}\left(\sum_{n\geq 1}\mathbb{E}\left[\ell_n^\kappa \mathbb{P}\left(X_{\ell_n^\kappa}^\kappa < 0\right)\right]\right)\mathrm{e}_\kappa = \sum_{\kappa=1}^{p}\left[\left(\int_0^{T_\kappa}\mathbb{P}\left(X_u^\kappa < 0\right)\mathrm{d}u\right)\mathrm{e}_\kappa\right].$$

And, in a similar way,

$$\mathbb{E}\left[\underline{X}_T\right] = \mathbb{E}\left[\sum_{n\geq 1}\sum_{\kappa=1}^{p}\Delta_n^\kappa \mathbb{1}_{\Delta_n^\kappa<0}\right] = \sum_{\kappa=1}^{p}\left(\sum_{n\geq 1}\mathbb{E}\left[X_{\ell_n^\kappa}^\kappa \mathbb{1}_{X_{\ell_n^\kappa}^\kappa<0}\right]\right) = \sum_{\kappa=1}^{p}\left(\int_0^{T_\kappa}\mathbb{E}\left[X_u^\kappa \mathbb{1}_{X_u^\kappa<0}\right]u^{-1}\mathrm{d}u\right).$$

On the other hand, by the Duality Lemma [14, Lemma 3.4, p. 77], if $Y_s^\kappa = -X_{(T_\kappa-s)-}^\kappa - \left(-X_{T_\kappa}^\kappa\right)$ for $s \in [0, T_\kappa]$ then $Y \overset{\mathrm{def}}{=} Y^1 \oplus \cdots \oplus Y^p \overset{\mathrm{d}}{=} X$ on $[0,T]$. Now, since $\sum_{n\geq 1}\ell_n^\kappa = T_\kappa$ from (2.2), adding the zero

$\pm\ell_n^\kappa$ gives

$$\sum_{n\geq 1}\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa=0}=\sum_{n\geq 1}\left[-\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa\neq 0}\right]+T_\kappa=T_\kappa-\sum_{n\geq 1}\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa>0}-\sum_{n\geq 1}\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa<0}=T_\kappa-\overline{\tau}_{T_\kappa}\left(X^\kappa\right)-\underline{\tau}_{T_\kappa}\left(X^\kappa\right). \tag{2.9}$$

As before,

$$\mathbb{E}\left[\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa=0}\right]=\mathbb{E}\left[\left(L_{n-1}^\kappa-L_n^\kappa\right)\mathbb{P}\left(X_{L_{n-1}^\kappa}^\kappa-X_{L_n^\kappa}^\kappa=0\,\middle|\,L_{n-1}^\kappa,L_n^\kappa\right)\right]=\mathbb{E}\left[\ell_n^\kappa\right]\mathbb{P}\left(X_{\ell_n^\kappa}^\kappa=0\right).$$

So from (2.8) and (2.9),

$$0\leq \mathbb{E}\left[T_\kappa-\underline{\tau}_{T_\kappa}\left(X^\kappa\right)-\overline{\tau}_{T_\kappa}\left(Y^\kappa\right)\right]=\sum_{n\geq 1}\mathbb{E}\left[\ell_n^\kappa \mathbb{1}_{\Delta_n^\kappa=0}\right]=\int_0^{T_\kappa}\mathbb{P}\left(X_u^\kappa=0\right)\mathrm{d}u=0,$$

for the fact

$$\int_0^{T_\kappa}\mathbb{P}\left(X_u^\kappa=0\right)\mathrm{d}u>0$$

would imply that X^κ is a compound Poisson process. Hence, $\overline{\tau}_T(Y)=T-\underline{\tau}_T(X)$ almost surely. Moreover, since X^κ attains its maximum at a unique point [4, Proposition VI.4, p. 159], by its definition $Y_{\underline{\tau}_T}=X_T-\overline{X}_T$, almost surely. Therefore,

$$\left(\overline{X}_T,\overline{\tau}_T(X)\right)\stackrel{\mathrm{d}}{=}\left(\overline{Y}_T,\overline{\tau}_T(Y)\right)=\left(T-\underline{\tau}_T(X),X_T-\overline{X}_T\right).$$

◆

3 | A Poisson point process associated to additive Lévy processes

The objective of this section is to give a natural extension of Corollary 2 and Lemma 1 in [17, pp. 4, 18] for the present case of study, additive Lévy processes. That is, to show that the point process with atoms at the lengths and heights of excursion rectangles of the convex minorant of X up to an exponential time $T_\theta=\left(T_{\theta_\kappa}\right)_1^p$ with $T_{\theta_\kappa}\sim \mathrm{Exp}(\theta_\kappa)$ is, in fact, Poisson. This will be the means to establish some properties about fluctuations of additive Lévy processes given in Section 4.

First of all, a property concerning the stick-breaking process up to an exponential time will be given, for it will be of use as a lemma for studying the nature of the point processes of interest in the present section.

Lemma 3.1. *If $T \sim \operatorname{Exp}(\theta)$ and ℓ is a stick-breaking process on $[0,T]$, then the point process with atoms at the random set $\{\ell_n\}_1^\infty$ is Poisson with mean measure*

$$\mu'_\theta(\mathrm{d}s) = \mathrm{e}^{-\theta s} s^{-1} \mathrm{d}s, \qquad s \in (0,\infty). \tag{3.1}$$

***Proof*.** Consider a Poisson point process N with mean measure $\lambda_+ \otimes \mu'_1$. Set

$$\Gamma_t = \int_{\mathbb{R}_+\times\mathbb{R}^+} z\,\mathbb{1}_{[0,t]}(s)\; N(\mathrm{d}s,\mathrm{d}z) \quad \text{ for } t \geq 0$$

and note that due to Campbell's Theorem [14, p. 43], $(\Gamma_t)_{t\geq 0}$ is a **Moran Gamma subordinator**. That is, a pure jump Lévy process with Laplace transform

$$\mathbb{E}\left[\mathrm{e}^{-q\Gamma_t}\right] = \exp\left\{-t\int_0^\infty \left(1-\mathrm{e}^{-qz}\right)\mathrm{e}^{-z} z^{-1}\mathrm{d}z\right\} = \left(1+q\right)^{-t}. \tag{3.2}$$

Consider a stick-breaking process ℓ' on $[0,1]$ independent of $(\Gamma_t)_{t\geq 0}$, as well as the sequence ℓ' in descending order $\left(\ell'_{(n)}\right)_{n\geq 1}$. Let us denote by $(J_n)_{n\geq 1}$ the normalized ordered jumps of $(\Gamma_t)_{t\geq 0}$ on $[0,1]$. The proof relies on the following assertions: that the sequences $(J_n/\Gamma_1)_{n\geq 1}$ and $\left(\ell'_{(n)}\right)_{n\geq 1}$ are both Poisson-Dirichlet(1) distributed (cf. [13, pp. 95, 99]) and the well known fact that $(J_n/\Gamma_1)_{n\geq 1}$ and Γ_1 are independent. That is because the latter imply that

$$\left(\frac{\Gamma_1}{\theta}\ell'_{(n)}\right)_1^\infty \stackrel{\mathrm{d}}{=} \left(\frac{J_n}{\theta}\right)_1^\infty.$$

Therefore, denoting by N' the point process with atoms at $\{J_n/\theta\}_{n\geq 1}$ and letting $u : \mathbb{R}^+ \to \mathbb{R}_+$ be measurable,

$$\int_{\mathbb{R}^+} u(x) N'(\mathrm{d}x) = \sum_{n\geq 1} u\left(\frac{J_n}{\theta}\right) \stackrel{\mathrm{d}}{=} \sum_{n\geq 1} u\left(\frac{\Gamma_1}{\theta}\ell'_{(n)}\right) = \sum_{n\geq 1} u\left(\frac{\Gamma_1}{\theta}\ell'_n\right) \stackrel{\mathrm{d}}{=} \sum_{n\geq 1} u(\ell_n), \tag{3.3}$$

if N' turned out to be Poisson. In that case, $\sum_{n\geq 1}\delta(\ell_n;\bullet)$ and N' would share the same Laplace functional, from where it would follow that the point process $\sum_{n\geq 1}\delta(\ell_n;\bullet)$ is Poisson according to Lemma 10.1 in [9, p. 177].

Thus, let us show that N' is a Poisson point process. By the construction of N and Γ, the point process N has atoms at the positions and heights of the jumps of Γ, say, at the random set $\Pi \stackrel{\mathrm{def}}{=} \left\{\left(p_n, \mathfrak{J}_n\right)\right\}_1^\infty$. Consider $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ defined as $f(s,z) = z/\theta$; note that f is measurable and $\left(\lambda_+\big|_{\mathbb{B}_{[0,1]}} \otimes \mu'_1\right)\circ f^{-1}$ is clearly non-atomic. Then, by the Restriction Theorem [13, p. 17] followed by the Mapping Theorem [13, p. 18], the point process N' with atoms at the random set

$$f\left(\Pi\cap\left([0,1]\times\mathbb{R}^{+}\right)\right)=\left\{f\left(p_n,\mathcal{J}_n\right): p_n\in[0,1]\right\}=\{J_n/\theta\}_1^{\infty}$$

is Poisson. Furthermore, the mean measure of N' is given below for $(a,b)\subset\mathbb{R}^{+}$:

$$\left(\lambda_{+}\big|_{\mathbb{B}_{[0,1]}}\otimes\mu_1'\right)\circ f^{-1}\big((a,b)\big)=\lambda_{+}([0,1])\;\mu_1'\big((\theta a,\theta b)\big)=\int_{\theta a}^{\theta b}\mathrm{e}^{-z}z^{-1}\mathrm{d}z=\mu_\theta'\big((a,b)\big).$$

Thus, the mean measure of N' coincides with μ_θ' by the Monotone Class Lemma and the statement follows. ◆

Let us proceed with a property concerning a related point process that will be the link, along with Lemma 3.1, between Poisson point processes and the analysis of the excursions of additive Lévy processes over the convex minorant within a fixed time horizon.

Proposition 3.2. *Consider an additive Lévy process $X=X^1\oplus\cdots\oplus X^p$ on $\mathbb{R}$. For $\kappa\in[p]$, let T_{θ_κ} be exponentially distributed independent random variables with parameters $\theta_\kappa>0$ and let $\ell^\kappa=\left(\ell_n^\kappa\right)_1^\infty$ be uniform stick-breaking processes on $\left[0,T_{\theta_\kappa}\right]$, respectively, independent of X. Denote by $L^\kappa=\left(L_n^\kappa\right)_1^\infty$ the remainder process associated to ℓ^κ and the increments $\Delta_n^\kappa=X^\kappa_{L^\kappa_{n-1}}-X^\kappa_{L^\kappa_n}$. Then,*

$$\tilde{\Xi}_\theta(\bullet)=\sum_{n\geq 1}\delta\left(\ell_n^1,\ldots,\ell_n^p,\Delta_n^1+\cdots+\Delta_n^p;\bullet\right) \tag{3.4}$$

is a Poisson Point Process on $(0,\infty)^p\times\mathbb{R}$ with mean measure given as follows

$$A\times B\mapsto\int_{A\times(0,\infty)^p}\mathbb{P}(X_l\in B)\,\theta_1\cdots\theta_p\exp\left\{-\left(\theta_1\mathrm{e}^{t_1},\ldots,\theta_p\mathrm{e}^{t_p}\right)\cdot l\right\}\mathcal{H}_p\left(t_1\cdots t_p\right)\,\mathrm{d}(l,t), \tag{3.5}$$

where $A\in\mathbb{B}_{(0,\infty)^p}$, $B\in\mathbb{B}_{\mathbb{R}}$ and $\mathcal{H}_p:(0,\infty)\to\mathbb{R}$ is defined as $\mathcal{H}_p(z)=\sum_{n\geq 0}z^n/n!^p$.

The result will be proven by means of the facts that $\tilde{\Xi}_\theta$ is a **$\mathbb{R}$-marked point process on** $(0,\infty)^p$ that has **independent increments**. I.e., that $\tilde{\Xi}_\theta$ is a locally finite integer valued kernel from $(\Omega,\mathcal{F})$ to $\left((0,\infty)^p\times\mathbb{R},\mathbb{B}_{(0,\infty)^p}\otimes\mathbb{B}_{\mathbb{R}}\right)$ such that [9, p. 184]:

- The expression $\tilde{\Xi}_\theta(\{t\}\times\mathbb{R})\leq 1$ holds identically for all $t\in(0,\infty)^p$,
- The point processes $\left\{\tilde{\Xi}_\theta(B_i\times\bullet)\right\}_1^m$ on $\mathbb{R}$ are independent, where $\{B_i\}_1^m\subset\mathbb{B}_{(0,\infty)^p}$ are disjoint and relatively compact.

From the latter, the statement in Proposition 3.2 follows according to the following theorem, which can be found in much greater generality in [9, Theorem 10.11, p. 184].

Theorem 3.3. *Let Ξ be an $\mathbb{R}$-marked point process on $\mathbb{R}^n$ such that $\Xi(\{x\}\times\mathbb{R})=0$ a.s. for all $x\in\mathbb{R}^n$. Then Ξ is Poisson if and only if it has independent increments, in which case its mean measure is locally finite with diffuse projections onto $\mathbb{R}^n$.*

Proof of Proposition 3.2. On one hand, clearly $\tilde{\Xi}_\theta$ given in (3.4) is integer valued and measurable for fixed $A\in\mathbb{B}_{(0,\infty)^p\times\mathbb{R}}$, since it corresponds with the limit of a measurable sequence

$$\tilde{\Xi}_\theta(A)=\lim_{N\to\infty}\sum_{n=1}^{N}\mathbb{1}_A\left(\ell_n^1,\dots\ell_n^p,\Delta_n^1+\cdots+\Delta_n^p\right).$$

On the other hand, note that $\left(\Delta_n^\kappa\right)_1^\infty$ are independent for each $\kappa\in[p]$. Certainly, due to the fact that X^κ and ℓ^κ were chosen to be independent, conditioning with respect to L^κ, it follows that for every (distinct) $n_1,\dots,n_m\geq 1$ and $\{B_i\}_1^m\subset\mathbb{B}_\mathbb{R}$,

$$\mathbb{P}\left(\Delta_{n_1}^\kappa\in B_1,\dots,\Delta_{n_m}^\kappa\in B_m\right)=\mathbb{E}\left[\mathbb{E}\left[\prod_{i=1}^m\mathbb{1}_{B_i}\left(X^\kappa_{L^\kappa_{n_i-1}}-X^\kappa_{L^\kappa_{n_i}}\right)\middle|L^\kappa\right]\right]=\mathbb{E}\left[h\left(L^\kappa\right)\right],$$

where, since X^κ is a Lévy process,

$$h\left((l_n)_0^\infty\right)=\mathbb{E}\left[\prod_{i=1}^m\mathbb{1}_{B_i}\left(X^\kappa_{l_{n_i-1}}-X^\kappa_{l_{n_i}}\right)\right]=\prod_{i=1}^m\mathbb{P}\left(X^\kappa_{l_{n_i-1}}-X^\kappa_{l_{n_i}}\in B_i\right),\qquad\text{for } (l_n)_0^\infty\in\mathbb{R}_+^\infty.$$

Hence,

$$\mathbb{P}\left(\Delta_{n_1}^\kappa\in B_1,\dots,\Delta_{n_m}^\kappa\in B_m\right)=\prod_{i=1}^m\mathbb{P}\left(\Delta_{n_i}^\kappa\in B_i\right),$$

which implies that $\tilde{\Xi}^\kappa_{\theta_\kappa}(\bullet)=\sum_{n\geq 1}\delta\left(\ell_n^\kappa,\Delta_n^\kappa;\bullet\right)$ is a Poisson point process on $(0,\infty)\times\mathbb{R}$, according to Lemma 3.1 and the Marking Theorem [13, p. 55], which proves the case $p=1$.

Now, fix $\omega\in\Omega$ and $p\geq 2$. From the latter follows that $\tilde{\Xi}^1_{\theta_1}\otimes\cdots\otimes\tilde{\Xi}^p_{\theta_p}$, is an atomic measure which atoms $\left\{\left(\ell^1_{n_1},\Delta^1_{n_1},\ldots,\ell^p_{n_p},\Delta^p_{n_p}\right)\right\}_{n_1,\ldots,n_p\geq 1}$ have multiplicity 1 (for $\tilde{\Xi}^\kappa_{\theta_\kappa}$ are simple point processes). Then,

$$\begin{aligned}\tilde{\Xi}_\theta(A) &= \sum_{n\geq 1}\mathbb{1}_A\left(\ell^1_n,\ldots,\ell^p_n,\Delta^1_n+\cdots+\Delta^p_n\right)\quad \tilde{\Xi}^1_{\theta_1}\otimes\cdots\otimes\tilde{\Xi}^p_{\theta_p}\left(\left\{\left(\ell^1_n,\Delta^1_n,\ldots,\ell^p_n,\Delta^p_n\right)\right\}\right)\\ &= \sum_{n\geq 1}\int_{\left\{\left(\ell^1_n,\Delta^1_n,\ldots,\ell^p_n,\Delta^p_n\right)\right\}}\mathbb{1}_A\left(l_1,\ldots,l_p,z_1+\cdots+z_p\right)\,\mathrm{d}\tilde{\Xi}^1_{\theta_1}\otimes\cdots\otimes\tilde{\Xi}^p_{\theta_p}\left(l_1,z_1,\ldots,l_p,z_p\right)\\ &= \int_{\left\{\left(\ell^1_n,\Delta^1_n,\ldots,\ell^p_n,\Delta^p_n\right)\right\}_{n\geq 1}}\mathbb{1}_A\left(l_1,\ldots,l_p,z_1+\cdots+z_p\right)\,\mathrm{d}\tilde{\Xi}^1_{\theta_1}\otimes\cdots\otimes\tilde{\Xi}^p_{\theta_p}\left(l_1,z_1,\ldots,l_p,z_p\right)\end{aligned}$$

holds for each $A\in\mathbb{B}_{(0,\infty)^p}$; which shows $\tilde{\Xi}_\theta(A)$ is a measure on $\mathbb{B}_{(0,\infty)^p\times\mathbb{R}}$, for its σ-additivity follows from the Monotone Convergence Theorem. Hence, $\tilde{\Xi}_\theta$ is an integer valued kernel from the probability measure space onto $\left((0,\infty)^p\times\mathbb{R},\mathbb{B}_{(0,\infty)^p}\otimes\mathbb{B}_\mathbb{R}\right)$.

In order to verify that $\tilde{\Xi}_\theta$ is locally finite, note that

$$\tilde{\Xi}_\theta\left(B_1\times\cdots\times B_{p+1}\right)=\sum_{n\geq 1}\delta_{\ell^1_n}(B_1)\cdots\delta_{\ell^p_n}\left(B_p\right)\delta_{\Delta_{\boldsymbol{n}}}\left(B_{p+1}\right)\leq\sum_{n\geq 1}\delta_{\ell^1_n}(B_1), \tag{3.6}$$

for $B_1\times\cdots\times B_{p+1}\in\mathbb{B}_{(0,\infty)^p}\otimes\mathbb{B}_\mathbb{R}$. Now, since the field of relatively compact sets of $(0,\infty)^p\times\mathbb{R}$ coincides with that of its bounded borel sets, if A belongs to such class, it can be covered with a bounded rectangle $R_1\times\cdots\times R_{p+1}\supset A$. Then (3.6) implies that $\tilde{\Xi}_\theta(A)<\infty$, since $\sum_{n\geq 1}\delta_{\ell^1_n}$ is locally finite by Lemma 3.1 and sub-additivity of $\tilde{\Xi}_\theta$ for fixed $\omega\in\Omega$.

Furthermore, from (3.6) follows that $\tilde{\Xi}_\theta(\{t\}\times\mathbb{R})\leq 1$ since $\sum_{n\geq 1}\delta_{\ell^1_n}$ is simple as well, so $\tilde{\Xi}_\theta$ is a $\mathbb{R}$-marked point process on $(0,\infty)^p$. In addition, $\ell^1,\ldots,\ell^p$ are continuous random variables by (2.7) and almost surely $\delta_{\ell^\kappa_n}(\{t_\kappa\})=0$. Hence, $\tilde{\Xi}_\theta(\{t\}\times\mathbb{R})=0$ (a.s.), letting $\delta_{\Delta_{\boldsymbol{n}}}(\mathbb{R})=1$ in (3.6).

The only fact left to check in order to apply Theorem 3.3, from which the result follows, is that $\tilde{\Xi}_\theta$ has independent increments. Beforehand, since the components of X, ℓ and T_θ were chosen to be independent from each other, from the fact that $\left(\Delta^\kappa_n\right)_1^\infty$ are jointly independent ($\kappa\in[p]$) follows that the increments $\left(\Delta^1_n+\cdots+\Delta^p_n\right)_1^\infty$ are independent. Now, consider some disjoint relatively compact sets $\{B_i\}_1^m\subset\mathbb{B}_{(0,\infty)^p}$ –that is, bounded borel sets– and $\{s_i\}_1^m\subseteq\overline{\mathbb{N}}$. From the disjointness of $\{B_i\}_1^m$ and independence of $\left\{\Delta^1_n+\cdots+\Delta^p_n\right\}_1^\infty$,

$$\begin{aligned}H\left(l^1,\ldots,l^p\right) &\overset{\text{def}}{=}\mathbb{E}\left[\prod_{i=1}^m\mathbb{1}\left\{\sum_{n\geq 1}\delta\left(\left(l^\kappa_{n-1}-l^\kappa_n\right)^p_{\kappa=1};B_i\right)\delta\left(X_{\left(l^\kappa_{n-1}\right)^p_{\kappa=1}}-X_{\left(l^\kappa_n\right)^p_{\kappa=1}};\bullet\right)=s_i\right\}\right]\\ &=\prod_{i=1}^m\mathbb{P}\left(\sum_{n\geq 1}\delta\left(\left(l^\kappa_{n-1}-l^\kappa_n\right)^p_{\kappa=1};B_i\right)\delta\left(X_{\left(l^\kappa_{n-1}\right)^p_{\kappa=1}}-X_{\left(l^\kappa_n\right)^p_{\kappa=1}};\bullet\right)=s_i\right),\end{aligned}$$

where $l^\kappa = \left(l_n^\kappa\right)_0^\infty \subset \mathbb{R}_+$ for $\kappa \in [p]$. So, conditioning with respect to $L^1, \ldots, L^p$ gives

$$\mathbb{P}\left(\bigcap_{i=1}^m \left\{\sum_{n\geq 1} \delta\left(\ell_n^1, \ldots, \ell_n^p; B_i\right) \delta_{\Delta_n}(\bullet) = s_i\right\}\right) = \mathbb{E}\left[H\left(L^1, \ldots, L^p\right)\right] = \prod_{k=1}^m \mathbb{P}\left(\sum_{n\geq 1} \delta\left(\ell_n^1, \ldots, \ell_n^p; B_i\right) \delta_{\Delta_n}(\bullet) = s_i\right).$$

By last, the computation of the mean measure of $\tilde{\Xi}_\theta$ is given as follows. From (2.7) it is easy to see, conditioning with respect to T_{θ_κ}, that ℓ_n^κ has the following density function

$$\mathbb{R}^+ \ni l \mapsto \int_l^\infty \left(\frac{\left[-\log(l/t)\right]^{n-1}}{l\Gamma(n)} \mathrm{e}^{-\left[-\log(l/t)\right]}\right) \theta_\kappa \mathrm{e}^{-\theta_\kappa t} \mathrm{d}t = \int_0^\infty \frac{t^{n-1}}{\Gamma(n)} \theta_\kappa \mathrm{e}^{-\theta_\kappa l \mathrm{e}^t} \mathrm{d}t. \tag{3.7}$$

So, by the stationarity of the increments of $X^1, \ldots, X^p$ and the independence structure,

$$\begin{aligned}
&\mathbb{P}\Big(\left(\ell_n^1, \ldots, \ell_n^p\right) \in A,\ \Delta_n^1 + \cdots + \Delta_n^p \in B\Big) = \\
&\qquad = \int_A \mathbb{P}\Big(X_{l_1}^1 + \cdots + X_{l_p}^p \in B\Big)\left[\prod_{\kappa=1}^p \int_0^\infty \frac{t_\kappa^{n-1}}{\Gamma(n)} \theta_\kappa \mathrm{e}^{-\theta_\kappa l_\kappa \mathrm{e}^{t_\kappa}} \mathrm{d}t_\kappa\right] \mathrm{d}l \\
&\qquad = \int_{A\times(0,\infty)^p} \mathbb{P}(X_l \in B)\, \theta_1 \cdots \theta_p \mathrm{e}^{-\left(\theta_1 \mathrm{e}^{t_1}, \ldots, \theta_p \mathrm{e}^{t_p}\right)\cdot l} \frac{\left(t_1 \cdots t_p\right)^{n-1}}{(n-1)!^p}\, \mathrm{d}(l,t);
\end{aligned}$$

where the last equation holds due to Fubini's Theorem. Furthermore, applying it again twice yields

$$\begin{aligned}
\mathbb{E}\left[\tilde{\Xi}_\theta(A\times B)\right] &= \sum_{n\geq 1} \mathbb{E}\left[\mathbb{1}_{A\times B}\left(\ell_n^1, \ldots, \ell_n^p, \Delta_n^1 + \cdots + \Delta_n^p\right)\right] \\
&= \sum_{n\geq 1} \int_{A\times(0,\infty)^p} \mathbb{P}(X_l \in B)\, \theta_1 \cdots \theta_p \mathrm{e}^{-\left(\theta_1 \mathrm{e}^{t_1}, \ldots, \theta_p \mathrm{e}^{t_p}\right)\cdot l} \frac{\left(t_1 \cdots t_p\right)^{n-1}}{(n-1)!^p}\, \lambda_p \otimes \lambda_p\left(\mathrm{d}l, \mathrm{d}\left(t_1, \ldots, t_p\right)\right) \\
&= \int_{A\times(0,\infty)^p} \mathbb{P}(X_l \in B)\, \theta_1 \cdots \theta_p \exp\left\{-\left(\theta_\kappa \mathrm{e}^{t_\kappa}\right)_1^p \cdot l\right\} \mathcal{H}_p\left(t_1 \cdots t_p\right) \mathrm{d}(l,t).
\end{aligned}$$

◆

Lastly, let us properly state the main assertion of the section, which is directly met to be true due to Proposition 3.2 and the stick-breaking representation for the convex minorant of additive Lévy processes given in Corollary 2.4.

Proposition 3.4. *Consider an additive Lévy process $X = X^1 \oplus \cdots \oplus X^p$ on $\mathbb{R}$. For $\kappa \in [p]$, let T_{θ_κ} be exponentially distributed independent random variables with parameters $\theta_\kappa > 0$. The point process with atoms at lengths and heights of excursion rectangles of the convex minorant of X on $[0, T_\theta]$ is Poisson with mean measure given in (3.5).*

Remark 3.5. The function $\mathcal{H}_p$ in (3.5) coincides with the following Generalized Hypergeometric function

$$\mathcal{H}_p(z) = \sum_{n\geq 0} \frac{z^n}{n!^p} = \sum_{n\geq 0} \frac{1}{(1)_n \cdots (1)_n} \frac{z^n}{n!} = {}_0F_{p-1}(-; \underbrace{1,\ldots,1}_{p-1}; z), \quad \text{where } (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)},$$

which is an entire function of z since its radius of convergence is ∞ because $p-1 \geq 0$ [15, p. 275].

Furthermore, note that from Tonelli's theorem,

$$\begin{aligned}\int_{(0,\infty)^p} \mathcal{H}_p(t_1 \cdots t_p) \left(\prod_{\kappa=1}^{p} \theta_\kappa \mathrm{e}^{-\theta_\kappa l_\kappa \mathrm{e}^{t_\kappa}} \right) \mathrm{d}t &= \sum_{n\geq 0} \prod_{\kappa=1}^{p} \frac{1}{n!} \int_1^\infty \frac{[\log u]^n}{u} \theta_\kappa \mathrm{e}^{-\theta_\kappa l_\kappa u} \mathrm{d}u \\ &= \sum_{n\geq 0} \theta_1 E_1^n(\theta_1 l_1) \cdots \theta_p E_1^n(\theta_p l_p),\end{aligned}$$

where

$$E_s^0(z) \stackrel{\text{def}}{=} \int_1^\infty \frac{\mathrm{e}^{-zt}}{t^s} \mathrm{d}t \quad \text{and} \quad E_s^n(z) \stackrel{\text{def}}{=} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial s^n} E_s^0(z) = \frac{1}{n!} \int_1^\infty \frac{[\log t]^n}{t^s} \mathrm{e}^{-zt} \mathrm{d}t$$

are the so-called generalized integro-exponential functions [16, p. 443] and E_1^0 corresponds to the exponential integral [3, p. 229].

4 | Fluctuation theory for additive Lévy processes on $\mathbb{R}$

Given the stick-breaking representation established in Corollary 2.4, it is possible to describe the nature of the fluctuations of additive Lévy processes $X = X^1 \oplus \cdots \oplus X^p$ on a time horizon $[0, T_\theta]$, where $T_\theta = (T_{\theta_\kappa})_1^p$ and T_{θ_κ} are jointly independent Exponential random variables with parameters $\theta_\kappa > 0$, respectively, independent of X. The means to this analysis will be the Poisson point processes derived from the stick-breaking representation for the convex minorant of X studied in the last section.

It is worth mentioning that the properties in the present section constitute generalizations of well-known results about Lévy processes which can be found in [4], [18] and [14], to mention a few. In addition, while some of them can be obtained directly from the single parameter case of study, the proofs for general additive processes in $\mathbb{R}$ are included for sake of completeness. But, most of all, this was done in order to illustrate the fact that the analysis from the convex minorant perspective it is possible to shorten the approach to the fluctuations of Lévy processes, as shown in [17] and [8].

4.1 || Fluctuations up to an exponential time

Although the point processes in the Section 3 constitute a natural generalization of the single parameter case shown in [17], in order to establish the results given in this subsection it will be necessary to set up an ancillary point process associated to the excursions over the convex minorant as well.

Consider a p-parameter additive Lévy process $X = X^1 \oplus \cdots \oplus X^p$ on $\mathbb{R}$ and $f_\kappa : (0,\infty)\times\mathbb{R} \to \mathbb{R}_+^p \times \mathbb{R}$ defined as $f_\kappa(l,x) = (l\mathrm{e}_\kappa, x)$ for each $\kappa \in [p]$. In the light of Proposition 3.4, if $\tilde{\Xi}^\kappa_{\theta_\kappa}$ is the Poisson point process associated to X^κ given as in (3.4) for an$p=1$, then its mean measure is given as follows with an application of Fubini's Theorem and a proper change of variables in (3.7):

$$\tilde{\mu}^\kappa_{\theta_\kappa}(\mathrm{d}l \times \mathrm{d}x) \stackrel{\text{def}}{=} \int_{(0,\infty)} \mathbb{P}\left(X^\kappa_l \in \mathrm{d}x\right) \theta_\kappa \mathrm{e}^{-\theta_\kappa \mathrm{e}^t l} \mathrm{e}^t \,\mathrm{d}t\,\mathrm{d}l = \mathbb{P}\left(X^\kappa_l \in \mathrm{d}x\right) l^{-1} \mathrm{e}^{-\theta_\kappa l}\,\mathrm{d}l, \quad \text{for } l>0,\ x\in\mathbb{R}. \tag{4.1}$$

In addition, by definition of f_κ,

$$\left(\tilde{\mu}^\kappa_{\theta_\kappa} \circ f_\kappa^{-1}\right)\left(A' \times B\right) = \tilde{\mu}^\kappa_{\theta_\kappa}\left(\left\{l>0 : l\mathrm{e}_\kappa \in A'\right\} \times B\right) = \int_{\{l>0\,:\,l\mathrm{e}_\kappa \in A'\}} \mathbb{P}\left(X^\kappa_l \in B\right) l^{-1} \mathrm{e}^{-\theta_\kappa l}\,\mathrm{d}l, \tag{4.2}$$

for $A' \in \mathbb{B}_{[0,\infty)^p}$ and $B \in \mathbb{B}_\mathbb{R}$. Since f_κ is continuous and $\tilde{\mu}^\kappa_{\theta_\kappa} \circ f_\kappa^{-1}$ is clearly diffuse, the Mapping Theorem [13, p. 18] yields that the point process defined as

$$\Xi^\kappa_{\theta_\kappa}(\bullet) \stackrel{\text{def}}{=} \sum_{n\geq 1} \delta\left(f_\kappa\left(\ell^\kappa_n, \Delta^\kappa_n\right);\bullet\right) = \sum_{n\geq 1} \delta\left(\left(\ell^\kappa_n \mathrm{e}_\kappa, \Delta^\kappa_n\right);\bullet\right) \tag{4.3}$$

is Poisson as well and its mean measure say, $\mu^\kappa_{\theta_\kappa}$, corresponds with the one in (4.2). Furthermore, for $A \in \mathbb{B}_{(0,\infty)}$ the following expression holds

$$\mu^\kappa_{\theta_\kappa}\left(f_\kappa(A\times B)\right) = \mu^\kappa_{\theta_\kappa}\left(\left\{l'\mathrm{e}_\kappa : l' \in A\right\} \times B\right) = \int_A \mathbb{P}\left(X^\kappa_l \in B\right) l^{-1} \mathrm{e}^{-\theta_\kappa l}\,\mathrm{d}l.$$

Thus, by choosing the suitable independence structure for $\left(\tilde{\Xi}^\kappa_{\theta_\kappa}\right)_1^p$ of Proposition 3.4, from the Superposition Theorem [13, p. 16] follows that $\Xi_\theta \stackrel{\text{def}}{=} \Xi^1_{\theta_1} + \cdots + \Xi^p_{\theta_p}$ is a Poisson point process with mean measure $\mu_\theta \overset{\text{def}}{=} \mu^1_{\theta_1} + \cdots + \mu^p_{\theta_p}$. Moreover, if $\{A_\kappa\}_1^p \subset \mathbb{B}_{(0,\infty)}$ and $B \in \mathbb{B}_{\mathbb{R}}$, then

$$\Xi_\theta\left(\left[A_1 \cup \{0\}\right] \times \cdots \times \left[A_p \cup \{0\}\right] \times B\right) = \sum_{\kappa=1}^{p} \sum_{n \geq 1} \left[\delta_{\ell_n^\kappa}(A_\kappa)\delta_{\Delta_n^\kappa}(B) \prod_{j \neq \kappa} \delta_0\left(A_j \cup \{0\}\right)\right] = \sum_{\kappa=1}^{p} \tilde{\Xi}^\kappa_{\theta_\kappa}(A_\kappa \times B).$$

The latter evidences the fact that truly $\Xi_\theta \neq \tilde{\Xi}_\theta$ beyond structural matters. Clearly, as a generalization of Corollary 2 and Lemma 1 in [17], Ξ_θ is not as intuitive as the one in Section 3; nevertheless this point process will be the means of proving the following fluctuation identities for additive Lévy processes. Lastly, note that even though Ξ_θ is defined in such a way that its atoms are elements of $[0,\infty)^p \times \mathbb{R}$, the Mapping Theorem [13, p. 18] makes Ξ_θ equivalent to the superposition of the Poisson point processes

$$\sum_{n \geq 1} \delta\big((\underbrace{0,\ldots,0}_{\kappa-1}, \ell_n^\kappa, \underbrace{0,\ldots,0}_{p-\kappa}, \Delta_n^\kappa); \bullet\big);$$

so such a random element and Ξ_θ will be used indistinctly throughout the proofs below.

Theorem 4.1. *Let X be a p-parameter additive Lévy process and $T_\theta = \left(T_{\theta_\kappa}\right)_1^p$ where $T_{\theta_\kappa} \sim \mathrm{Exp}(\theta_\kappa)$ are independent. The $p+1$-dimensional random vectors*

$$\left(\overline{\tau}_{T_\theta}, \overline{X}_{T_\theta}\right) \quad \textit{and} \quad \left(T_\theta - \overline{\tau}_{T_\theta}, \overline{X}_{T_\theta} - X_{T_\theta}\right)$$

are independent and infinitely divisible. Moreover, the associated characteristic functions are given as follows for $\alpha \in \mathbb{R}^p$ and $\beta \in \mathbb{R}$:

$$\mathbb{E}\left[\mathrm{e}^{i\alpha\cdot\overline{\tau}_{T_\theta} + i\beta\overline{X}_{T_\theta}}\right] = \exp\left\{-\sum_{\kappa=1}^{p} \int_{(0,\infty)} \left[\int_{(0,\infty)} \left(1 - \mathrm{e}^{i\alpha_\kappa l + i\beta x}\right) l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right] \mathrm{d}l\right\} \tag{4.4}$$

and

$$\mathbb{E}\left[\mathrm{e}^{i\alpha\cdot\left(T_\theta - \overline{\tau}_{T_\theta}\right) + i\beta\left(\overline{X}_{T_\theta} - X_{T_\theta}\right)}\right] = \exp\left\{-\sum_{\kappa=1}^{p} \int_{(0,\infty)} \left[\int_{(-\infty,0]} \left(1 - \mathrm{e}^{i\alpha_\kappa l - i\beta x}\right) l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right] \mathrm{d}l\right\}. \tag{4.5}$$

***Proof*.** Let $\Xi^\kappa_{\theta_\kappa}$ as in (4.3) for $\kappa \in [p]$. Recall that $\Xi_\theta \overset{\text{def}}{=} \Xi^1_{\theta_1} + \cdots \Xi^p_{\theta_p}$ is a Poisson point process with mean measure $\mu_\theta = \mu^1_{\theta_1} + \cdots + \mu^p_{\theta_p}$, where the $\mu^\kappa_{\theta_\kappa}$ are given in (4.2).

Now note that, by its definition, $\Xi^\kappa_{\theta_\kappa}$ has atoms at $\left\{\left(\ell_n^\kappa \mathrm{e}_\kappa, \Delta_n^\kappa\right)\right\}_{n \geq 1}$. Thus, it is easy to see that the following identity holds due to the fact that $\Xi^{\kappa}_{\theta_\kappa}$ are simple point processes:

$$
\begin{aligned}
\int_{\mathbb{R}^p_+\times\mathbb{R}} &\left(t\mathbb{1}_{x>0}, x\mathbb{1}_{x>0}, t\mathbb{1}_{x\le 0}, x\mathbb{1}_{x\le 0}\right)\Xi_\theta(\mathrm{d}t,\mathrm{d}x)=\\
&=\sum_{\kappa=1}^{p}\int_{\mathbb{R}^p_+\times\mathbb{R}}\left(t\mathbb{1}_{x>0}, x\mathbb{1}_{x>0}, t\mathbb{1}_{x\le 0}, x\mathbb{1}_{x\le 0}\right)\Xi^{\kappa}_{\theta_\kappa}(\mathrm{d}t,\mathrm{d}x)\\
&=\sum_{\kappa=1}^{p}\sum_{n\ge 1}\left(\mathbb{1}_{\Delta^\kappa_n>0}\ell^\kappa_n\,\mathrm{e}_\kappa,\ \Delta^\kappa_n\mathbb{1}_{\Delta^\kappa_n>0},\ \mathbb{1}_{\Delta^\kappa_n\le 0}\ell^\kappa_n\,\mathrm{e}_\kappa,\ \Delta^\kappa_n\mathbb{1}_{\Delta^\kappa_n\le 0}\right)\\
&=\sum_{n\ge 1}\Big(\left(\ell^1_n\mathbb{1}_{\Delta^1_n>0},\dots,\ell^p_n\mathbb{1}_{\Delta^p_n>0}\right),\ \left(\Delta^1_n\right)^+ +\dots+\left(\Delta^p_n\right)^+,\\
&\qquad\left(\ell^1_n\mathbb{1}_{\Delta^1_n\le 0},\dots,\ell^p_n\mathbb{1}_{\Delta^p_n\le 0}\right),\ -\left(\Delta^1_n\right)^- -\dots-\left(\Delta^p_n\right)^-\Big).
\end{aligned}
\tag{4.6}
$$

Hence, from Corollary 2.4 follows that

$$
\int_{\mathbb{R}^p_+\times\mathbb{R}}\left(t\mathbb{1}_{x>0}, x\mathbb{1}_{x>0}, t\mathbb{1}_{x\le 0}, x\mathbb{1}_{x\le 0}\right)\Xi_\theta(\mathrm{d}t,\mathrm{d}x)\stackrel{\mathrm{d}}{=}\left(\overline{\tau}_{T_\theta},\overline{X}_{T_\theta},T_\theta-\overline{\tau}_{T_\theta},X_{T_\theta}-\overline{X}_{T_\theta}\right),
\tag{4.7}
$$

since $-x^- = x - x\mathbb{1}_{x>0}$ for $x\in\mathbb{R}$. Moreover,

$$
\int_{\mathbb{R}^p_+\times(0,\infty)}(t,x)\,\Xi_\theta(\mathrm{d}t,\mathrm{d}x)\stackrel{\mathrm{d}}{=}\left(\overline{\tau}_{T_\theta},\overline{X}_{T_\theta}\right)\quad\text{and}\quad\int_{\mathbb{R}^p_+\times(-\infty,0]}(t,x)\,\Xi_\theta(\mathrm{d}t,\mathrm{d}x)\stackrel{\mathrm{d}}{=}\left(T_\theta-\overline{\tau}_{T_\theta},X_{T_\theta}-\overline{X}_{T_\theta}\right);
$$

which implies the independence of $\left(\overline{\tau}_{T_\theta},\overline{X}_{T_\theta}\right)$ and $\left(T_\theta-\overline{\tau}_{T_\theta},\overline{X}_{T_\theta}-X_{T_\theta}\right)$, for $\Xi_\theta\left([0,\infty)^p\times(0,\infty)\right)$ is independent of $\Xi_\theta\left([0,\infty)^p\times(-\infty,0]\right)$ in virtue of Ξ_θ being Poisson.

Furthermore, the convergence in (4.7) is absolute almost surely thanks to the construction of the point processes $\Xi^{\kappa}_{\theta_\kappa}$ and Corollary 2.4. The latter holds due to the fact that if the second (fourth) entry were not absolutely convergent (a.s.), it would imply that the running supremum (infimum) of X^κ on $\left[0,T_{\theta_\kappa}\right]$ is not finite, which contradicts the fact that the paths of X^κ are càdlàg. Therefore, from Campbell's Theorem [14, Theorem 2.7, p. 43] follows that

$$
\int_{\mathbb{R}^p_+\times\mathbb{R}}\left(1\wedge|f|\right)\mathrm{d}\mu_\theta<\infty,
$$

when $f:\mathbb{R}^p_+\times\mathbb{R}\to\mathbb{R}_+$ is defined as $f(l,x)=\mathbb{1}_{x\in B}\,l\cdot\mathrm{e}_\kappa$ for $B=\mathbb{R}^+$ and $B=(-\infty,0]$, so as for $f(l,x)=x\mathbb{1}_{x\in B}$. Along with the latter, the well known expression of the characteristic functional of a Poisson point process(cf. Lemma 10.2 [9, p. 178]) yields

$$
\mathbb{E}\left[\exp\left\{i\alpha\cdot\overline{\tau}_{T_\theta}+i\beta\overline{X}_{T_\theta}\right\}\right]=\exp\left\{-\int_{\mathbb{R}^p_+\times\mathbb{R}}\left[1-\mathrm{e}^{i\mathbb{1}_{x>0}\alpha\cdot l+i\beta x\mathbb{1}_{x>0}}\right]\mu_\theta(\mathrm{d}l,\mathrm{d}x)\right\},
\tag{4.8}
$$

for $\alpha \in \mathbb{R}^p$ and $\beta \in \mathbb{R}$. Now, recall that for any measurable function $f' : [0,\infty)^p \times \mathbb{R} \to \mathbb{R}$, the change of variable identity

$$\int_{\mathbb{R}_+^p \times \mathbb{R}} f' \, \mathrm{d}\mu_\theta = \sum_{\kappa=1}^{p} \int_{\mathbb{R}_+^p \times \mathbb{R}} f' \, \mathrm{d}\left(\tilde{\mu}_{\theta_\kappa}^{\kappa} \circ f_\kappa^{-1}\right) = \sum_{\kappa=1}^{p} \int_{f_\kappa^{-1}(\mathbb{R}_+^p \times \mathbb{R})} f' \circ f_\kappa \, \mathrm{d}\tilde{\mu}_{\theta_\kappa}^{\kappa} = \sum_{\kappa=1}^{p} \int_{(0,\infty)\times\mathbb{R}} f' \circ f_\kappa \, \mathrm{d}\tilde{\mu}_{\theta_\kappa}^{\kappa}.$$

holds [9, Lemma 1.22, p. 12]. Then, from (4.1) follows that

$$\int_{\mathbb{R}_+^p \times \mathbb{R}} f' \, \mathrm{d}\mu_\theta = \sum_{\kappa=1}^{p} \int_{(0,\infty)\times\mathbb{R}} f'(l\mathrm{e}_\kappa, x)\, l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right) \mathrm{d}l. \tag{4.9}$$

Note that the latter can be extended substituting f' with $\mathrm{e}^{if'}$ with the definition of exp on $\mathbb{C}$. So, putting

$$f'(l,x) = 1 - \exp\left\{i \mathbb{1}_{x>0} \alpha \cdot l + i\beta x \mathbb{1}_{x>0}\right\}, \quad l \in \mathbb{R}_+^p,\ x \in \mathbb{R}$$

in (4.9) gives, along with (4.8),

$$\begin{aligned} -\log \mathbb{E}\left[\exp\left\{i\alpha \cdot \overline{\tau}_{T_\theta} + i\beta \overline{X}_{T_\theta}\right\}\right] &= \sum_{i=1}^{p} \int_{(0,\infty)\times\mathbb{R}} \left[1 - \mathrm{e}^{i\mathbb{1}_{x>0}\alpha\cdot(l\mathrm{e}_\kappa) + i\beta x \mathbb{1}_{x>0}}\right] l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right) \mathrm{d}l \\ &= \sum_{\kappa=1}^{p} \int_{(0,\infty)\times(0,\infty)} \left[1 - \mathrm{e}^{i\alpha_\kappa l + i\beta x}\right] l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right) \mathrm{d}l; \end{aligned}$$

from which (4.4) follows. By last, (4.5) can be derived in a completely analogous way as (4.4) with

$$f'(l,x) = 1 - \exp\left\{i \mathbb{1}_{x\le 0} \alpha \cdot l - i\beta x \mathbb{1}_{x\le 0}\right\}, \quad l \in \mathbb{R}_+^p,\ x \in \mathbb{R}.$$

◆

Remark 4.2. According to Proposition 2.6, if $X^1, \dots, X^p$ are not compound Poisson processes, then

$$\left(\underline{\tau}_T(X),\ -\underline{X}_T\right) = \left(\overline{\tau}_T(-X),\ \overline{(-X)}_T\right) \overset{\mathrm{d}}{=} \left(T - \underline{\tau}_T(-X),\ -X_T - \underline{(-X)}_T\right) = \left(T - \overline{\tau}_T(X),\ \overline{X}_T - X_T\right). \tag{4.10}$$

So, since T_θ, ℓ and X are independent, equation (4.5) of Theorem 4.1 gives

$$\mathbb{E}\left[\exp\left\{i\alpha \cdot \underline{\tau}_{T_\theta}(X) + i\beta \underline{X}_{T_\theta}\right\}\right] = \exp\left\{-\sum_{\kappa=1}^{p} \int_{(0,\infty)} \left[\int_{(-\infty,0]} \left(1 - \mathrm{e}^{i\alpha_\kappa l + i\beta x}\right) l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right] \mathrm{d}l\right\},$$

for $\alpha \in \mathbb{R}^p$ and $\beta \in \mathbb{R}$. Nevertheless, in fact (4.6) gives that (4.10) holds for any additive Lévy process on $\mathbb{R}$ if T is exponentially distributed.

Remark 4.3. Rephrasing the proof of Theorem 4.1 for $\alpha \in \mathbb{R}_+^p$ and $\beta > 0$, with

$$f'(l,x) = 1 - \exp\left\{-\alpha \cdot l\, \mathbb{1}_{x>0} - \beta\, x\, \mathbb{1}_{x>0}\right\}, \quad l \in \mathbb{R}_+^p,\ x \in \mathbb{R}$$

gives –with the first part of Lemma 10.2 in [9, p. 178] instead of the second–,

$$\mathbb{E}\left[\exp\left\{-\alpha \cdot \overline{\tau}_{T_\theta} - \beta \overline{X}_{T_\theta}\right\}\right] = \exp\left\{-\sum_{\kappa=1}^{p} \int_{(0,\infty)} \left[\int_{(0,\infty)} \left(1 - \mathrm{e}^{-\alpha_\kappa l - \beta x}\right) l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right] \mathrm{d}l\right\}. \quad \textbf{(4.11)}$$

So as for the next identity, in conjunction with Remark 4.2,

$$\mathbb{E}\left[\exp\left\{-\alpha \cdot \underline{\tau}_{T_\theta} - \beta \underline{X}_{T_\theta}\right\}\right] = \exp\left\{-\sum_{\kappa=1}^{p} \int_{(0,\infty)} \left[\int_{(-\infty,0]} \left(1 - \mathrm{e}^{-\alpha_\kappa l + \beta x}\right) l^{-1} \mathrm{e}^{-\theta_\kappa l} \mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right] \mathrm{d}l\right\}. \quad \textbf{(4.12)}$$

It should be highlighted that Theorem 4.1 can be found in [4, Theorem 5, p. 160] for Laplace transforms of the single-parameter case of study, so as that the arguments given above are based on those of [8, Theorem 8, p. 5] for Lévy processes, which made possible the extension to p-parameter additive Lévy processes. Next, a generalization for additive processes on $\mathbb{R}$ of the so called Wiener-Hopf factorisation [14, Theorem 6.15, p. 171] is stated.

Theorem 4.4. *Let X be a p-parameter additive Lévy process and an independent time horizon $T_\theta = \left(T_{\theta_\kappa}\right)_1^p$ where $T_{\theta_\kappa} \sim \mathrm{Exp}(\theta_\kappa)$ are independent. The functions*

$$\Psi_\theta^+(\alpha,\beta) = \mathbb{E}\left[\exp\left\{i\alpha \cdot \overline{\tau}_{T_\theta} + i\beta \overline{X}_{T_\theta}\right\}\right] \quad \textit{and} \quad \Psi_\theta^-(\alpha,\beta) = \mathbb{E}\left[\exp\left\{i\alpha \cdot \underline{\tau}_{T_\theta} + i\beta \underline{X}_{T_\theta}\right\}\right],$$

where $\alpha \in \mathbb{R}^p$ and $\beta \in \mathbb{R}$, yield the factorisation

$$\Psi_\theta^+(\alpha,\beta)\, \Psi_\theta^-(\alpha,\beta) = \prod_{\kappa=1}^{p} \frac{\theta_\kappa}{\theta_\kappa - i\alpha_\kappa + \Psi^\kappa(\beta_\kappa)},$$

where $\Psi^\kappa(\beta) = -\log \mathbb{E}\left[\exp\left\{i\beta X_1^\kappa\right\}\right]$ denotes the characteristic exponent associated to X^κ for $\kappa \in [p]$.

***Proof*.** On one hand, it is well known that a direct consequence of the distributional infinite divisibility of Lévy processes is that $\mathbb{E}\left[\exp\left\{i\beta X_t\right\}\right] = \exp\left\{-t\Psi(\beta)\right\}$ (cf. [14, Eq. 1.3]) if Ψ is the characteristic exponent associated to X. Then, since the components $X^1, \ldots, X^p$ of X are independent,

$$\mathbb{E}\left[\exp\left\{i\beta X_T\right\}\right] = \exp\left\{-T \cdot \left(\Psi^\kappa(\beta)\right)_1^p\right\},$$

for $T \in (0,\infty)^p$ and $\beta \in \mathbb{R}$. So, conditioning with respect to T_θ yields

$$\mathbb{E}\left[\mathrm{e}^{i\alpha\cdot T_\theta + i\beta X_{T_\theta}}\right] = \mathbb{E}\left[\mathrm{e}^{i\alpha\cdot T_\theta}\mathbb{E}\left[\mathrm{e}^{i\beta X_T}\right]\Big|_{T=T_\theta}\right] = \mathbb{E}\left[\exp\left\{\left[i\alpha - \left(\Psi^\kappa(\beta)\right)_1^p\right]\cdot T_\theta\right\}\right] = \prod_{\kappa=1}^{p}\left[1-\frac{i\alpha_\kappa - \Psi^\kappa(\beta)}{\theta_\kappa}\right]^{-1}, \tag{4.13}$$

since $T^\kappa_{\theta_\kappa}$ is exponentially distributed ($\kappa \in [p]$) and independent of X. Hence, the result follows from the fact that, by Remark 4.2 and the independence shown in Theorem 4.1, the latter implies that

$$\Psi^+_\theta(\alpha,\beta)\ \Psi^-_\theta(\alpha,\beta) = \mathbb{E}\left[\mathrm{e}^{i\alpha\cdot\overline{\tau}_{T_\theta} + i\beta\overline{X}_{T_\theta}}\right]\mathbb{E}\left[\mathrm{e}^{i\alpha\cdot\left(T_\theta - \overline{\tau}_{T_\theta}\right) - i\beta\left(\overline{X}_{T_\theta} - X_{T_\theta}\right)}\right] = \mathbb{E}\left[\mathrm{e}^{i\alpha\cdot T_\theta + i\beta X_{T_\theta}}\right].$$

◆

Take notice of the fact that equation (4.13) holds because the characteristic function of T_{θ_κ} can be extended analytically to $\mathbb{C}\setminus\{\theta_\kappa\}$, for $z \mapsto \theta_\kappa/(\theta_\kappa - z)$ is analytic on such a domain. Moreover, in order for it to make sense, $\alpha \in \mathbb{R}^p$ and $\beta \in \mathbb{R}$ must be such that $\theta \neq i\alpha - \left(\Psi^\kappa(\beta)\right)_1^p$, i.e., such that

$$\theta_\kappa \neq -\mathrm{Re}\Psi^\kappa(\beta) \quad \text{and} \quad \alpha_\kappa \neq \mathrm{Im}\Psi^\kappa(\beta) \quad \text{for every} \quad \kappa \in [p].$$

4.2 || Regularity of the origin

One of the properties studied in the analysis of fluctuations of Lévy processes is that of the regularity of the origin for some open or closed set B, that is, whether $\inf\{t>0 : X_t \in B\} = 0$ almost surely (cf. Definition 6.4 in [14, p. 155]). Such a notion is equivalent to asserting that the process visits B at arbitrarily small times almost surely [4, p. 167], so it can be extended to additive processes $X^1 \oplus \cdots \oplus X^p$ by establishing that the point 0 is **regular** for $B \in \mathbb{B}_\mathbb{R}$ if and only if the following set is $\mathbb{P}$-null:

$$\bigcup_{t>0}\ \bigcap_{\substack{0<s_\kappa\leq t\\ \kappa\in[p]}} \left\{X^1_{s_1} + \cdots + X^p_{s_p} \notin B\right\}.$$

Due to the right-continuity of the paths of X^κ ($\kappa = 1,\ldots,p$), the latter condition is equivalent to

$$P^*(B) \stackrel{\text{def}}{=} \mathbb{P}\left(\bigcap_{t\in\mathbb{Q}^+}\ \bigcup_{\substack{s_\kappa\in(0,t]\cap\mathbb{Q}\\ \kappa\in[p]}}\left\{X^1_{s_1} + \cdots + X^p_{s_p} \in B\right\}\right) = 1.$$

Take notice of the fact that the proposed notion of regularity for additive Lévy processes coincides with that of the one-parameter case of study if $p=1$. Moreover, it is resembled for such a notion for $p\geq 2$ in the sense that the origin is regular for B if there exists some *temporal direction* in which X visits B instantaneously.

A first condition for regularity of the origin for $(0,\infty)$ is given as follows. Let $T_{\mathbf{1}}=\left(T_1^{\kappa}\right)_{\kappa=1}^{p}$ be independent Exponential(1) random variables independent of $X^1\oplus\cdots\oplus X^p$, as well. On one hand, if $P^*(\mathbb{R}^+)<1$ almost surely there exists $t\in\mathbb{Q}^+$ such that

$$\begin{aligned}
0<\left[1-\mathrm{e}^{-t}\right]^p\ \mathbb{P}\left(\bigcap_{\substack{s_\kappa\in(0,t]\cap\mathbb{Q}\\ \kappa\in[p]}}\left\{X_{(s_1,\ldots,s_p)}\leq 0\right\}\right)\\
=\mathbb{P}\left(\left\{T_{\mathbf{1}}\in(0,t]^p\right\}\bigcap\bigcap_{\substack{s_\kappa\in(0,t]\cap\mathbb{Q}\\ \kappa\in[p]}}\left\{X_{(s_1,\ldots,s_p)}\leq 0\right\}\right)\\
\leq\mathbb{P}\left(\overline{X}_{T_{\mathbf{1}}}=0\right).
\end{aligned}$$

On the other hand, the definition of $\overline{X}_{T_{\mathbf{1}}}$ yields

$$\left\{\overline{X}_{T_{\mathbf{1}}}=0\right\}\subset\bigcup_{t>0}\bigcap_{\substack{0<s_\kappa\leq t_\kappa\\ \kappa\in[p]}}\left\{X^1_{s_1}+\cdots+X^p_{s_p}\leq 0\right\}. \tag{4.14}$$

Thus, $P^*(\mathbb{R}^+)=1$ implies that $\mathbb{P}\left(\overline{X}_{T_{\mathbf{1}}}=0\right)=0$. Hence,

$$P^*\left(\mathbb{R}^+\right)=1\quad\text{if and only if}\quad\mathbb{P}\left(\overline{X}_{T_{\mathbf{1}}}>0\right)=1. \tag{4.15}$$

Now, let us state the following, which arguments are inspired by those of the proof of Theorem 6 in [8, p. 4].

Proposition 4.5. *Consider a p-parameter additive Lévy process X on $\mathbb{R}$. The origin is regular for $(0,\infty)$ if and only if*

$$\int_{(0,1)^p\times(0,\infty)^p}\mathbb{P}(X_l>0)\,\mathcal{H}_p\left(t_1\cdots t_p\right)\exp\left\{-\left(\mathrm{e}^{t_\kappa}\right)_1^p\cdot l\right\}\,\mathrm{d}(l,t)=\infty.$$

***Proof*.** According to Proposition 3.2 and Proposition 3.4,

$$\mathbb{P}\left(\overline{X}_{T_1}=0\right)=\mathbb{P}\left(\tilde{\Xi}_{\mathbf{1}}\left((0,\infty)^p\times\mathbb{R}^+\right)=0\right)=\exp\left\{-\mathbb{E}\left[\tilde{\Xi}_{\mathbf{1}}\left((0,\infty)^p\times\mathbb{R}^+\right)\right]=\exp\left\{-\tilde{\mu}_{\mathbf{1}}^p\left((0,\infty)^p\times\mathbb{R}^+\right)\right\}\right\}. \tag{4.16}$$

So, in the light of (4.15), the origin is regular for $(0,\infty)$ if and only if

$$\infty=\tilde{\mu}_{\mathbf{1}}^p\left((0,\infty)^p\times\mathbb{R}^+\right)=\int_{(0,\infty)^p\times(0,\infty)^p}\mathbb{P}\left(X_{l_1}^1+\cdots+X_{l_p}^p>0\right)\mathcal{H}_p\left(t_1\cdots t_p\right)\exp\left\{-\sum_{\kappa=1}^p l_\kappa e^{t_\kappa}\right\}\mathrm{d}(l,t) \tag{4.17}$$

by means of (3.5).

Now, for the case $p=1$,

$$\tilde{\mu}_{\mathbf{1}}^1\left([1,\infty)\times\mathbb{R}^+\right)=\int_{[1,\infty)\times\mathbb{R}^+}\mathbb{P}(X_l>0)\mathcal{H}_1(t)\exp\left\{-l e^t\right\}\mathrm{d}(t,l)=\int_1^\infty\mathbb{P}(X_l>0)\frac{e^{-l}}{l}\mathrm{d}l\le e^{-1}.$$

Furthermore, by the convergence of the series $\mathcal{H}_q$ on $\mathbb{R}_+$ it is immediate to prove inductively that $\mathcal{H}_{p+1}(ab)\le\mathcal{H}_p(a)\mathcal{H}_1(b)$. Hence, assuming by induction that $\tilde{\mu}_{\mathbf{1}}^p\left([1,\infty)^p\times\mathbb{R}^+\right)$ is finite,

$$\tilde{\mu}_{\mathbf{1}}^{p+1}\left([1,\infty)^{p+1}\times\mathbb{R}^+\right)\le\tilde{\mu}_{\mathbf{1}}^p\left([1,\infty)^p\times\mathbb{R}^+\right)\,\tilde{\mu}_{\mathbf{1}}^1\left([1,\infty)\times\mathbb{R}^+\right)<\infty,$$

according to Tonelli's Theorem. Thus, (4.17) is equivalent to $\tilde{\mu}_{\mathbf{1}}^p\left((0,1)^p\times\mathbb{R}^+\right)=\infty$. ◆

Remark 4.6. From the last proposition follows the well-known result referred to as Rogozin's criterion (cf. Proposition VI.11 [4, p. 167]) for $p=1$. Equation (4.2) yields

$$\tilde{\mu}_1^1\left((0,1)\times\mathbb{R}^+\right)=\int_{(0,1)}\mathbb{P}(X_l>0)\frac{e^{-l}}{l}\,\mathrm{d}l.$$

Hence, since $e^{-l}\in\left(e^{-1},1\right)$ for $l\in(0,1)$, Proposition 4.5 gives that the origin is regular for $(0,\infty)$ if and only if

$$\int_{(0,1)}\mathbb{P}(X_l>0)\,\frac{1}{l}\,\mathrm{d}l=\infty.$$

In the light of Remark 4.6 it would intuitively appear that a criterion for the regularity of the origin for a p-parameter additive Lévy process with $p\ge 2$ could be

$$\int_{(0,1)^p}\mathbb{P}\left(X_{l_1}^1+\cdots+X_{l_p}^p>0\right)\,\frac{1}{l_1\cdots l_p}\,\mathrm{d}l=\infty, \tag{4.18}$$

but that is not the case. In order to show it, consider the independent processes $X^\kappa=Z^\kappa-\Gamma^\kappa$ for

$\kappa = 1,2$ where Z^κ are Poisson processes with arrival rate 1 and Γ^κ are Moran Gamma subordinators[3], independent from each other. Clearly, for each $\kappa \in \{1,2\}$ the origin is not regular for $(0,\infty)$ with respect to X^κ, since if τ^κ denotes the time of the first jump of Z^κ and T_1^κ is an independent Exponential(1) random variable,

$$\mathbb{P}\left(\overline{X}^\kappa_{T_1^\kappa} = 0\right) = \mathbb{P}\left(T_1^\kappa < \tau^\kappa\right) > 0.$$

Nevertheless, it will be seen that (4.18) holds anyway. Note that from the independence structure and infinite divisibility of Poisson and Gamma distributions follows that

$$\mathbb{P}\left(X^1_{l_1} + X^2_{l_2} > 0\right) = \sum_{z\geq 0}\left(\int_0^z \frac{\gamma^{l_1+l_2-1}}{\Gamma(l_1+l_2)}\mathrm{e}^{-\gamma}\,\mathrm{d}\gamma\right)\mathrm{e}^{-l_1-l_2}\frac{(l_1+l_2)^z}{z!} = \sum_{z\geq 1}\left[1-\frac{\Gamma(l_1+l_2,z)}{\Gamma(l_1+l_2)}\right]\mathrm{e}^{-l_1-l_2}\frac{(l_1+l_2)^z}{z!},$$

where

$$\Gamma(x,z) \stackrel{\text{def}}{=} \int_z^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

denotes the incomplete Gamma function [3, 6.5.3, p. 260]. Then, by Lemma A.2 in Appendix A,

$$\begin{aligned}\int_{(0,1)^2}\mathbb{P}\left(X^1_{l_1}+X^2_{l_2}>0\right)\frac{1}{l_1 l_2}\,\mathrm{d}l &\geq \int_{(0,1/2)^2}\frac{\mathrm{e}^{-l_1-l_2}}{l_1 l_2}\left[\mathrm{e}^{l_1+l_2}-1\right]\left[1-\mathrm{e}^{-1}\right]\mathrm{d}l\\ &\geq \left[1-\mathrm{e}^{-1}\right]\left(\int_0^{1/2}\frac{1-\mathrm{e}^{-l_1}}{l_1}\,\mathrm{d}l_1\right)\left(\int_0^{1/2}\frac{1}{l_2}\,\mathrm{d}l_2\right) = \infty.\end{aligned}$$

That is, (4.18) is not a sufficient condition for the origin to be regular for $(0,\infty)$ if $p\geq 2$, in general. However, it is easily shown to be necessary.

Proposition 4.7. *Let X be a p-parameter additive Lévy process on $\mathbb{R}$, if*

$$\int_{(0,1)^p}\mathbb{P}\left(X^1_{l_1}+\cdots+X^p_{l_p}>0\right)\frac{1}{l_1\cdots l_p}\,\mathrm{d}l < \infty,$$

then the origin is not regular for $(0,\infty)$.

***Proof*.** The assertion follows from Proposition 4.5 and the facts that $\mathrm{e}^{-l}/l \leq 1/l$ for $l\in(0,1)$ and

$$\tilde{\mu}^p_{\mathbf{1}}\left((0,1)^p\times(0,\infty)\right) \leq \int_{(0,1)^p}\mathbb{P}(X_l>0)\left(\prod_{\kappa=1}^p\int_0^\infty \mathrm{e}^t\,\mathrm{e}^{-l_\kappa \mathrm{e}^t}\,\mathrm{d}t\right)\mathrm{d}l = \int_{(0,1)^p}\mathbb{P}(X_l>0)\frac{\mathrm{e}^{-l_1}\cdots\mathrm{e}^{-l_p}}{l_1\cdots l_p}\,\mathrm{d}l,$$

[3] A pure jump Lévy process with Laplace transform given by (3.2).

since

$$\mathrm{e}^{t_1}\cdots\mathrm{e}^{t_p}=\sum_{n_1,\dots,n_p\geq 0}\frac{t_1^{n_1}\cdots t_p^{n_p}}{n_1!\cdots n_p!}\geq \mathcal{H}_p\big(t_1\cdots t_p\big)\quad\text{for } t_1,\dots,t_p>0.$$

◆

Let us move towards the impact of the regularity of the origin for the components $X^1,\dots,X^p$ in the additive process. Clearly, if for every $\kappa\in[p]$ the origin is regular for $(0,\infty)$ with respect to X^κ, then the origin is regular for $(0,\infty)$ respect to $X^1\oplus\cdots\oplus X^p$. Nevertheless, the latter is not a necessary condition for the origin to be regular for $(0,\infty)$ for X. For example, if X^1 is a Brownian Motion and X^2 is an independent compound Poisson process, clearly the origin is regular for $(0,\infty)$ for X since it is for X^1, even though it is not for X^2. As a matter of fact, the following result is turns out to be true.

Proposition 4.8. *The origin is regular for* $(0,\infty)$ *for* $X^1\oplus\cdots\oplus X^p$ *if and only if there exists* $\kappa\in[p]$ *for which the origin is regular for* $(0,\infty)$ *for* X^κ.

***Proof*.** Recall from (4.15) that $P^*(\mathbb{R}^+)=1$ if and only if $\mathbb{P}\big(\overline{X}_{T_1}=0\big)=0$. Furthermore, since either $\tilde{\mu}_{\mathbf{1}}^p((0,\infty)^p\times\mathbb{R}^+)$ is finite or infinite, from equation (4.16) follows that $\mathbb{P}\big(\overline{X}_{T_1}=0\big)\in\{0,1\}$. Thus, if $P^*(\mathbb{R}^+)<1$, expression (4.14) yields

$$P^*\big(\mathbb{R}^+\big)\leq\mathbb{P}\big(\overline{X}_{T_1}>0\big)=0;$$

which proves that $P^*(\mathbb{R}^+)\in\{0,1\}$ as well.

Lastly, since the running suprema of X and X^κ ($\kappa\in[p]$) are (not strictly) increasing and positive, by equality of the sets and independence of $X^1,\dots,X^p$,

$$\mathbb{P}\big(\overline{X}_{T_1}=0\big)=\mathbb{P}\Big(\overline{X}^1_{T_1^1}=0\Big)\cdots\mathbb{P}\Big(\overline{X}^p_{T_1^p}=0\Big).$$

Then, the result follows via (4.15). ◆

It should be highlighted that the characterization of the regularity of the origin for the additive process in terms of that of its components given in Proposition 4.8, in conjugation with Rogozin's criterion (cf. Remark 4.6), yields what could be thought of such a criterion for additive Lévy processes on $\mathbb{R}$ and is given as follows.

Theorem 4.9. *The origin is regular for* $(0,\infty)$ *for* $X^1 \oplus \cdots \oplus X^p$ *if and only if*

$$\int_{(0,1)} \mathbb{P}\left(X_l^{\kappa} > 0\right) \frac{1}{l} \, \mathrm{d}l = \infty \quad \textit{for some } \kappa \in [p].$$

Furthermore, Theorem 4.9 does not only give a criterion for the regularity of the origin for $(0,\infty)$, but is also gives one for the regularity for $(-\infty,0)$ due to the fact that the origin is regular for $(-\infty,0)$ for X if and only if it is regular for $(0,\infty)$ for $-X$. By last, the next proposition states a relation between the regularity for $(0,\infty)$ and $[0,\infty)$; which is equivalent for $(-\infty,0)$ and $(-\infty,0]$ in an analogous way.

Remark 4.10. *Consider a p-parameter additive Lévy process* $X = X^1 \oplus \cdots \oplus X^p$. *The origin is regular for* $(0,\infty)$ *for* X *if and only if the origin is regular for* $[0,\infty)$ *for* X, *unless every component* $X^1,\ldots,X^p$ *is a compound Poisson process.*

***Proof*.** To begin with, set $p=1$ and note that if X is a compound Poisson process, then the origin is regular for $[0,\infty)$ but not for $(0,\infty)$, since X stays in zero for a positive amount of time almost surely. Now, if the origin is regular for $[0,\infty)$ but not for $(0,\infty)$, (4.15) yields

$$1 = \mathbb{P}\left(\left\{\overline{X}_{T_1} = 0\right\} \bigcap \left[\bigcap_{t\in\mathbb{Q}^+} \bigcup_{s\in(0,t]\cap\mathbb{Q}} \{X_s \geq 0\}\right]\right) \leq \mathbb{P}\left(\bigcup_{s\in(0,T_1]\cap\mathbb{Q}} \{X_s = 0\}\right).$$

That is, X reaches the maximum value of 0 at the origin and at some point later on almost surely. Hence, by contrapositive of Proposition VI.4 in [4, p. 159] it follows that X is a compound Poisson process and the statement for $p=1$ is proven.

Moving on to the general case, let us assume that 0 is regular for $[0,\infty)$ and not regular for $(0,\infty)$ for X. From Proposition 4.8 follows that for every $\kappa \in [p]$ the origin is not regular for $(0,\infty)$ for X^{κ}. Moreover, there must be some $\kappa \in [p]$ for which the origin is regular for $[0,\infty)$ for X^{κ}, since the contrary would imply that the origin is not regular for $[0,\infty)$ for X by its definition. Therefore, at least one of the components X^{κ} of X is a compound Poisson process.

Lastly let us show that, given the latter, the origin cannot be irregular for $[0,\infty)$ for the rest of the components of X. In order to do so, consider $p=2$ (the result follows inductively for any $p>2$) and without loss of generality assume that X^1 is a compound Poisson process. By counterposition of the arguments, suppose that X^2 is not regular for $[0,\infty)$. Note that a consequence of the hypotheses is that

$$1 = \mathbb{P}\left(\left[\bigcup_{t>0}\bigcap_{0<s\le t}\{X_s^1 = 0\}\right]\bigcap\left[\bigcup_{t>0}\bigcap_{0<s\le t}\{X_s^2 < 0\}\right]\right) \le \mathbb{P}\left(\bigcup_{t>0}\bigcap_{0<s_1,s_2\le t}\left\{X_{s_1}^1 + X_{s_2}^2 < 0\right\}\right).$$

Thus, the origin being regular for $[0,\infty)$ for X implies that it is for $[0,\infty)$ for X^2 as well. Then, X^2 is a compound Poisson process, which finishes the proof since the other side of the equivalence asserted holds by definition of regularity for 0. ◆

4.3 || On the long time behavior

In this subsection a couple of well-known results about the long time behavior of Lévy processes will be stated in terms of additive processes. In order to do so, consider a p-parameter additive Lévy process $X = X_1 \oplus \cdots \oplus X_p$ and the $\overline{\mathbb{R}}$ valued random variables

$$\overline{X}_\infty = \sup_{t\in[0,\infty)^p}\{X_t\} \stackrel{\text{def}}{=} \sum_{\kappa=1}^{p}\overline{X}_\infty^\kappa \quad \text{and} \quad \overline{\tau}_\infty = \left(\inf\left\{t\ge 0 : X_{t-}^\kappa \wedge X_t^\kappa = \overline{X}_\infty^\kappa\right\}\right)_1^p \stackrel{\text{def}}{=} \left(\overline{\tau}_\infty^\kappa\right)_1^p.$$

In the same way, one can consider the following

$$\underline{X}_\infty = \inf_{t\in[0,\infty)^p}\{X_t\} \stackrel{\text{def}}{=} \sum_{\kappa=1}^{p}\underline{X}_\infty^\kappa \quad \text{and} \quad \underline{\tau}_\infty = \left(\inf\left\{t\ge 0 : X_{t-}^\kappa \wedge X_t^\kappa = \underline{X}_\infty^\kappa\right\}\right)_1^p \stackrel{\text{def}}{=} \left(\underline{\tau}_\infty^\kappa\right)_1^p.$$

An immediate property for such variables is the following and it is derived from Theorem 4.1.

Proposition 4.11. *Let X be p-parameter additive Lévy process. Then, for every $\alpha \in \mathbb{R}_+^p$ and $\beta \in \mathbb{R}_+$ the following identities hold:*

$$\mathbb{E}\left[\exp\left\{-\alpha\cdot\overline{\tau}_\infty - \beta\overline{X}_\infty\right\}\right] = \exp\left\{-\sum_{\kappa=1}^{p}\int_{(0,\infty)}\left[\int_{(0,\infty)}\left(1-\mathrm{e}^{-\alpha_\kappa l - \beta x}\right)l^{-1}\mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right]\mathrm{d}l\right\} \tag{4.19}$$

and

$$\mathbb{E}\left[\exp\left\{-\alpha\cdot\underline{\tau}_\infty - \beta\underline{X}_\infty\right\}\right] = \exp\left\{-\sum_{\kappa=1}^{p}\int_{(0,\infty)}\left[\int_{(-\infty,0]}\left(1-\mathrm{e}^{-\alpha_\kappa l + \beta x}\right)l^{-1}\mathbb{P}\left(X_l^\kappa \in \mathrm{d}x\right)\right]\mathrm{d}l\right\}. \tag{4.20}$$

***Proof*.** Let $T_{\mathbf{1}} = (T_1^\kappa)_1^p$ independent and exponentially distributed random variables of parameter 1. As in the proof of Theorem 48.1 in [18, p. 364], note that nT_1^κ ~Exp$(1/n)$ and that, $n \to \infty$, almost surely $\left(\overline{\tau}_{nT_1}, \overline{X}_{nT_1}\right) \to \left(\overline{\tau}_\infty, \overline{X}_\infty\right)$. Then, Lévy's continuity theorem [9, Theorem 4.3, p. 63] and identity (4.11) in Remark 4.3 yield

$$\exp\left\{-\sum_{\kappa=1}^{p}\int_{(0,\infty)}\left[\int_{(0,\infty)}\left(1-\mathrm{e}^{-\alpha_\kappa l-\beta x}\right)l^{-1}\mathrm{e}^{-l/n}\mathbb{P}\left(X_l^\kappa\in \mathrm{d}x\right)\right]\mathrm{d}l\right\}\to\mathbb{E}\left[\exp\left\{-\alpha\cdot\overline{\tau}_\infty-\beta\overline{X}_\infty\right\}\right],$$

as $n\to\infty$. By last, applying the Monotone Convergence Theorem to the left-hand side of the latter expression gives (4.19). The identity in (4.20) is obtained immediately with the same arguments and equation (4.12) in Remark 4.3. ◆

As Rogozin's criterion gives a characterization of the behavior of a Lévy process at *small* times in terms of the integrability of $l\mapsto\mathbb{P}(X_l>0)/l$ on $(0,1)$, Theorem 4.13 below for $p=1$ gives that of long time behavior of the process in terms of the integrability of such function on $(1,\infty)$. With that idea in mind, let us introduce the following notation, which will be used throughout the rest of the section:

$$\mathcal{I}_+^\kappa \stackrel{\text{def}}{=} \int_1^\infty \frac{1}{l}\,\mathbb{P}(X_l^\kappa>0)\,\mathrm{d}l \quad \text{and} \quad \mathcal{I}_-^\kappa \stackrel{\text{def}}{=} \int_1^\infty \frac{1}{l}\,\mathbb{P}(X_l^\kappa<0)\,\mathrm{d}l,$$

where $\kappa\in[p]$ for $p\geq 1$.

Lemma 4.12. *Let X be a Lévy process on $\mathbb{R}$. The following hold:*

- *Almost surely $\overline{X}_\infty=\infty$ if and only if $\mathcal{I}_+^1=\infty$.*
- *Almost surely $\underline{X}_\infty=-\infty$ if and only if $\mathcal{I}_-^1=\infty$.*

Proof. To begin with, note that for every $\alpha>0$,

$$\int_0^1 \frac{1-\mathrm{e}^{-\alpha l}}{l}\,\mathbb{P}(X_l>0)\,\mathrm{d}l \leq \int_0^\alpha \frac{1-\mathrm{e}^{-u}}{u}\,\mathrm{d}u = E_1^0(\alpha)+\log\alpha+\gamma<\infty, \tag{4.21}$$

where γ is Euler's constant and the last identity can be found in [3, 5.1.39, p. 230].

Moreover, since there exists $l'=l'(\alpha)>0$ such that $1-\mathrm{e}^{-\alpha l}>1/2$ for every $l\geq l'$, the expression

$$\mathcal{I}_+^1 \geq \int_1^\infty \frac{1-\mathrm{e}^{-\alpha l}}{l}\,\mathbb{P}(X_l>0)\,\mathrm{d}l \geq \frac{1}{2}\int_{1\vee l'}^\infty \frac{1}{l}\,\mathbb{P}(X_l>0)\,\mathrm{d}l = \frac{c(\alpha)}{2}\mathcal{I}_+^1, \tag{4.22}$$

holds, where

$$c(\alpha)\stackrel{\text{def}}{=}1-\frac{1}{\mathcal{I}_+^1}\int_1^{1\vee l'}\frac{\mathbb{P}(X_l>0)}{l}\,\mathrm{d}l\in(0,1]$$

and the equation is meant in the limiting sense, since $\int_1^{1\wedge l'} \mathbb{P}(X_l > 0)/l \, \mathrm{d}l < \infty$. Thus, in the light of Proposition 4.11, expressions (4.21) and (4.22) show that:

- If $\mathcal{I}_+^1 = \infty$ then $\mathbb{E}[\exp\{-\alpha\overline{\tau}_\infty\}] = 0$, and
- If $\mathcal{I}_+^1 < \infty$, the Dominated Convergence Theorem yields $\mathbb{E}[\exp\{-\alpha\overline{\tau}_\infty\}] \to 1$ as $\alpha \downarrow 0$.

Therefore, $\mathcal{I}_+^1 < \infty$ if and only if $\overline{X}_\infty < \infty$ (a.s.), due to the fact that the right continuity of the paths gives that $\overline{\tau}_\infty < \infty$ (a.s.) if and only if $\overline{X}_\infty < \infty$ (a.s.).

The second assertion follows from applying the first one to $-X$. ◆

It is now possible to state the main assertion of the subsection, which gives a generalization of Theorem 48.1 in [18, p. 363], since it characterizes the circumstances in which an additive Lévy process on $\mathbb{R}$ drifts to ∞, drifts to $-\infty$ or oscillates.

Theorem 4.13. *Consider a p-parameter additive Lévy process $X = X^1 \oplus \cdots \oplus X^p$. The following assertions are met to be true:*

- *Almost surely $X_{(t_1,\ldots,t_p)} \to \infty$ as $t_1,\ldots,t_p \to \infty$ if and only if $\mathcal{I}_-^\kappa < \infty$ for every $\kappa \in [p]$.*
- *Almost surely $X_{(t_1,\ldots,t_p)} \to -\infty$ as $t_1,\ldots,t_p \to \infty$ if and only if $\mathcal{I}_+^\kappa < \infty$ for every $\kappa \in [p]$.*
- *Neither of the latter holds if and only if*

$$\limsup_{t_1,\ldots,t_p\to\infty} X_{(t_1,\ldots,t_p)} = -\liminf_{t_1,\ldots,t_p\to\infty} X_{(t_1,\ldots,t_p)} = \infty.$$

***Proof*.** In the first place, from Lemma 4.12 follows that the existence of some $\kappa \in [p]$ such that $\mathcal{I}_+^\kappa = \infty$ is equivalent to $\overline{X}_\infty = \infty$ almost surely, which is easily seen to be equivalent to the fact that $\limsup_{t_1,\ldots,t_p} X_{(t_1,\ldots,t_p)} = \infty$ almost surely. Applying this argument to $-X$ and X simultaneously, gives the third assertion.

On one hand, the fact $X_{(t_1,\ldots,t_p)} \to \infty$ as $t_1,\ldots,t_p \to \infty$ implies that $\mathcal{I}_-^\kappa < \infty$ for every $\kappa \in [p]$ by contrapositive. That is, if there exists $\kappa \in [p]$ such that $\mathcal{I}_-^\kappa = \infty$, Lemma 4.12 gives that $\underline{X}_\infty^\kappa = -\infty$ almost surely. Thus, $\liminf_{t\to\infty} X_t^\kappa = -\infty$ and clearly $X_{(t_1,\ldots,t_p)} \not\to \infty$ as $t_1,\ldots,t_p \to \infty$.

On the other hand, if $\mathcal{I}_-^\kappa < \infty$ for every $\kappa \in [p]$, from Lemma 4.12 follows that $\underline{X}_\infty^\kappa > -\infty$ ($\kappa \in [p]$).

Now, note that $\mathcal{I}_+^\kappa = \infty$ for every $\kappa \in [p]$ as well, since $\mathcal{I}_-^\kappa < \infty$ and

$$\mathcal{I}_+^\kappa + \mathcal{I}_-^\kappa = \int_1^\infty \frac{1}{l}\,\mathrm{d}l = \infty.$$

Hence, $\overline{X}_\infty = \infty$ and the only fact left to prove is that $\liminf_{t_1,\dots,t_p} X_{(t_1,\dots,t_p)} = \infty$ almost surely. In order to do so, it will be proven that $\liminf_{t\to\infty} X_t^\kappa = \infty$ (a.s.) for every $\kappa \in [p]$ following an argument given in [14, p. 199].

Since $\underline{X}_\infty^\kappa > -\infty$ almost surely, the continuity of probability measures yields $\mathbb{P}\big(\underline{X}_\infty^\kappa \le -n\big) \to 0$ as $n \to \infty$, i.e.,

$$\text{for every } \varepsilon \in (0,1) \text{ there exists } N \ge 1 \text{ such that } \mathbb{P}\big(\underline{X}_\infty^\kappa \le -n\big) < \varepsilon \text{ for every } n \ge N. \tag{4.23}$$

Now, consider the stopping times given as $\tau_n \overset{\text{def}}{=} \inf\{t \ge 0 : X_t^\kappa > n\}$ and note that $\tau_n < \infty$ almost surely, for $\{\overline{X}_\infty^\kappa = \infty\} \subset \{\tau_n < \infty\}$. Consider the filtration associated to the coordinate process X^κ and note that since Lévy processes are (strongly) markovian (cf. [4, p. 18-20]),

$$\mathbb{P}\big(X_t^\kappa < n \text{ for some } t > \tau_{2n}\big) \le \mathbb{E}\Big[\mathbb{P}\Big(X_{\tau_{2n}+t}^\kappa - X_{\tau_{2n}}^\kappa < -n \text{ for some } t > 0 \Big| \mathcal{F}_{\tau_{2n}}\Big)\Big] = \mathbb{E}[h(\tau_{2n})],$$

where

$$h(\tau) = \mathbb{P}\big(X_{\tau+t}^\kappa - X_\tau^\kappa < -n \text{ for some } t > 0\big) = \mathbb{P}\big(X_t^\kappa < -n \text{ for some } t > 0\big) \le \mathbb{P}\big(\underline{X}_\infty^\kappa \le -n\big).$$

Recalling (4.23), it is possible to conclude that for every $\varepsilon \in (0,1)$ there exists $N \ge 1$ such that

$$1 - \varepsilon < \mathbb{P}\big(\underline{X}_\infty^\kappa > -n\big) \le \mathbb{P}\big(X_t^\kappa \ge n \text{ for every } t > \tau_{2n}\big) \le \mathbb{P}\Big(\liminf_{t\to\infty} X_t^\kappa \ge n\Big)$$

for every $n \ge N$. That is, again by continuity property of probability measures,

$$\mathbb{P}\Big(\liminf_{t\to\infty} X_t^\kappa = \infty\Big) = \lim_{n\to\infty} \mathbb{P}\Big(\liminf_{t\to\infty} X_t^\kappa \ge n\Big) = 1.$$

The second assertion in given by the application of the first one to $-X$. ◆

5 | Conclusion

The main purpose of this work was to derive some fluctuation properties for p-parameter additive Lévy processes on $\mathbb{R}$ through the perspective of the greatest convex minorant of the paths. First of all, a stick-breaking representation for additive Lévy processes was found based on that of Lévy processes, for the convex minorant associated to $X^1 \oplus \cdots \oplus X^p$ turned out to coincide with the sum of the convex minorants of X^κ ($\kappa \in [p]$). This is provided a stick-breaking representation of $X^1 \oplus \cdots \oplus X^p$ through the concave majorant as well.

Later on, the relationship between stick-breaking processes and the Dirichlet-Poisson distribution was proven to make the point process with atoms at the points of the stick-breaking process, ℓ^κ, Poisson. Along with the poissonian nature of the markings of Poisson point processes, the latter led to the fact that the point process with atoms at the vector formed by the stick-breaking processes $\ell^1, \ldots, \ell^p$ and the increments of the additive process sampled at the associated remainder processes is Poisson, as well. Next, due to the stick-breaking representation of additive Lévy processes found in Section 2, the main result of the section was stated as an immediate consequence of the latter: the point process with atoms at lengths and heights of excursion rectangles of the convex minorant of an additive Lévy process on an exponential time horizon is Poisson.

Having developed the aforementioned result, it was possible to obtain some fluctuation theoretical properties for additive processes. Firstly, on Section 4.1 were obtained the Fourier and Laplace transforms for the extremal vectors up to an exponential time $(\overline{\tau}_{T_\theta}, \overline{X}_{T_\theta})$ and $(\underline{\tau}_{T_\theta}, \underline{X}_{T_\theta})$. Furthermore, it was proven that $(\overline{\tau}_{T_\theta}, \overline{X}_{T_\theta})$ and $(T_\theta - \overline{\tau}_{T_\theta}, \overline{X}_{T_\theta} - X_{T_\theta})$ are independent and a Wiener-Hopf factorisation for additive processes was given. The latter was achieved with an ancillary point process based on that of Section 3.

Finally, some asymptotic properties were found for additive Lévy processes on $\mathbb{R}$. A notion of regularity for the origin for additive processes $X^1 \oplus \cdots \oplus X^p$ was proposed and it was characterized through the regularity of the origin of the components X^κ, which provided a way to establish a Rogozin-like criterion for regularity based on the classical criterion. Finally, it was possible to give the Laplace transform for $(\overline{\tau}_\infty, \overline{X}_\infty)$ and $(\underline{\tau}_\infty, \underline{X}_\infty)$ and an integral criterion for the condition $\overline{X}_\infty = \infty$ and $\underline{X}_\infty = -\infty$ was asserted. To conclude, the latter led to an integral characterization for drifting to $\pm\infty$ or oscillating for additive Lévy processes.

A | Appendix

Let us prove two results needed to give the counterexample in Section 4.2 to the assertion that the condition (4.18) is sufficient for the origin to be regular for $(0,\infty)$ for an additive Lévy process.

Lemma A.1. *The mapping $z \mapsto \Gamma(z,1)/\Gamma(z)$ is non decreasing on $(0,1)$.*

***Proof*.** On one hand, clearly $\Gamma(z_1,1) \le \Gamma(z_2,1)$ from the definition as long as $0 \le z_1 \le z_2$. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\Gamma(z)} = -\frac{1}{\Gamma(z)}\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{\Gamma(z)}\psi(z), \tag{A.1}$$

where ψ denotes the Digamma function [3, Eq. 6.3.1, p. 258] and satisfies the following [3, Eq. 6.3.21, p. 259] for $z > 0$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left[\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}\right]\mathrm{d}t \le \int_0^\infty \mathrm{e}^{-t}\left[\frac{1}{t} - \frac{1}{1-\mathrm{e}^{-t}}\right]\mathrm{d}t = \int_0^\infty \frac{\mathrm{e}^{-t}}{t(1-\mathrm{e}^{-t})}\left[1-\mathrm{e}^{-t}-t\right]\mathrm{d}t \le 0, \tag{A.2}$$

since $t \mapsto 1-\mathrm{e}^{-t}-t$ is decreasing on $[0,\infty)$. Thus, from (A.1) and (A.2) follows that $\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\Gamma(z)} > 0$ for $z \in (0,1)$. ◆

Lemma A.2. *For every $x, y \in (0,1/2)$ the following inequality holds*

$$\sum_{n\ge1}\frac{(x+y)^n}{n!}\left[1-\frac{\Gamma(x+y,n)}{\Gamma(x+y)}\right] \ge \left[\mathrm{e}^{x+y}-1\right]\left[1-\mathrm{e}^{-1}\right].$$

***Proof*.** First of all,

$$\sum_{n\ge1}\frac{(x+y)^n}{n!}\left[1-\frac{\Gamma(x+y,n)}{\Gamma(x+y)}\right] \ge \mathrm{e}^{x+y}-1-\sum_{n\ge1}\frac{(x+y)^n}{n!}\frac{\Gamma(x+y,1)}{\Gamma(x+y)} = \left[\mathrm{e}^{x+y}-1\right]\left[1-\frac{\Gamma(x+y,1)}{\Gamma(x+y)}\right], \tag{A.3}$$

since, by its definition, $\Gamma(x+y,n+1) \le \Gamma(x+y,1)$ for every $n \ge 1$. Thus, from Lemma A.1 follows that $z \mapsto 1-\Gamma(z,1)/\Gamma(z)$ is non increasing on $(0,1)$ and, if $x, y \in (0,1/2)$, then

$$1-\frac{\Gamma(x+y,1)}{\Gamma(x+y)} \ge 1-\frac{\Gamma(1/2+1/2,1)}{\Gamma(1/2+1/2)} = 1-\int_1^\infty \mathrm{e}^{-\gamma}\mathrm{d}\gamma = 1-\mathrm{e}^{-1}.$$

Plugging the latter in (A.3) gives the result. ◆

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