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PRESENTA:
ALFREDO ZARAGOZA CORDERO

DIRECTOR DE TESIS:
DR. RODRIGO JESÚS HERNÁNDEZ GUTIÉRREZ, DEPARTAMENTO DE MATEMÁTICAS, UAM-I

MIEMBROS DEL COMITÉ TUTOR:
DR. SALVADOR GARCÍA FERREIRA CENTRO DE CIENCIAS MATEMÁTICAS, UNAM.
DR. ÁNGEL TAMARIZ MASCARÚA, FACULTAD DE CIENCIAS UNAM

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Alfredo Zaragoza Cordero
Director de Tesis: Dr. Rodrigo Jesús Hernández Gutiérrez

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Hyperspaces of Almost Zero-Dimensional Spaces

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Introduction

Let P be a topological property, let X be a topological space and let $\mathcal{H}(X)$ be a hyperspace over X. A natural question to ask is

$$\text{does } \mathcal{H}(X) \text{ satisfy } P \text{ when } X \text{ satisfies } P?$$

In this work we study the properties of cohesion and almost zero-dimensionality on X and their relations to the hyperspace of non-empty compact subsets of X with the Vietoris topology.

This work is divided into five chapters: Chapter 1 consists of preliminaries. In this chapter the most important concepts and results are presented in order to explain the development of this work. This chapter is constituted by three sections. In the first, we talk about some basic concepts of topology. In the second section we analyze the property of being cohesive and the property of almost zero-dimensionality, we introduce Erdős space $\mathfrak{E}$, complete Erdős space $\mathfrak{E}_c$, and stable Erdős space $\mathfrak{E}_c^\omega$, and we study some of their properties. Also, in this section we present the topological characterizations of Erdős spaces, we introduce the concept of factor, and we give the characterization of the factors of Erdős spaces. In the third section of Chapter 1, we present the hyperspaces that we study throughout the thesis, and the basic theory of hyperspaces endowed with the Vietoris topology. In particular, we analyze when these hyperspaces are compact, connected, zero-dimensional and metrizable.

Chapter 2 is devoted to studying the relations between the properties of being cohesive and being almost zero-dimensional in the hyperspace of compact subsets of X, $\mathcal{K}(X)$, the hyperspace of finite subsets of X, $\mathcal{F}(X)$, and the symmetric products $\mathcal{F}_n(X)$ of X.

The first important result in Chapter 2 (Propositon 2.2) says that a space X is almost zero-dimensional if and only if $\mathcal{K}(X)$ is almost zero-dimensional. A natural question arises from the above result: Is there a space X that is not almost zero-dimensional such that $dim(X) = dim(\mathcal{K}(X)) = 1$? The last section of Chapter 2 gives us an affirmative answer to this question. Moreover, several examples are given.

Another important result in Chapter 2 (Proposition 2.8) says that if X is cohesive, then $\mathcal{F}_n(X)$, $\mathcal{F}(X)$ and $\mathcal{K}(X)$ are cohesive as well.

In Chapter 3 we study the hyperspaces of Erdős space and complete Erdős space. We show that for any natural number n, $\mathcal{F}_n(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$ (Theorem 3.7), $\mathcal{F}_n(\mathfrak{E}_\mathrm{c})$ is homeomorphic to $\mathfrak{E}_\mathrm{c}$ (Theorem 3.21) and that $\mathcal{F}(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$ (Theorem 3.10). In this chapter we also prove that the hyperspace of compact subsets of Erdős space is not homeomorphic to Erdős space. We also present a short analysis of the space $\mathcal{F}(\mathfrak{E}_c)$, and we decide why it is not homeomorphic to either $\mathfrak{E}_c$, $\mathfrak{E}$ or $\mathfrak{E}_c^\omega$.

From the above results, we note that $\mathcal{F}(\mathfrak{E}_c)$ has a topological structure similar to that of $\mathbb{Q} \times \mathfrak{E}_c$. This leads us to wonder if $\mathcal{F}(\mathfrak{E}_c)$ is homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$. The answer to the previous question is in the affirmative. We present the proof in Chapter 4.

In Chapter 4 we introduce the classes of spaces $\sigma\mathcal{L}$ and $\sigma\mathcal{E}$; these two classes are inspired by the classes SLC and E from [3]. We use these classes to give a characterization of the space $\mathbb{Q} \times \mathfrak{E}_c$. In this chapter we show that the class $\sigma\mathcal{L}$ is equal to $\sigma\mathcal{E}$, and that if $X \in \sigma\mathcal{E}$, then X is homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$. We also prove that $\mathcal{F}(\mathfrak{E}_c) \in \sigma\mathcal{E}$, and so we conclude that $\mathcal{F}(\mathfrak{E}_c)$ is homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$. Moreover, in this chapter we study the hyperspaces of $\mathbb{Q} \times \mathfrak{E}_c$, and we give a characterization of the factors of $\mathbb{Q} \times \mathfrak{E}_c$.

In Chapter 5 we present several results related to continuous images, extensions of continuous functions, and compactifications of almost zero-dimensional spaces.

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Chapter 1

Preliminaries

1.1 Basic topology

It will be assumed that all spaces are separable and metrizable. That is, all spaces X have a countable dense subset and there is a metric on X that generates the topology of X. We write $X \approx Y$ to denote the fact that X is homeomorphic to the space Y. By ω we denote the set of natural numbers including zero, $\mathbb{N}$ is the set $\omega \setminus \{0\}$, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{P}$ is the set of irrational numbers, 2^ω is the Cantor set, and $\mathbb{R}$ is the set of real numbers. We say that a space X is **crowded** if X does not have isolated points. Let A be a subset of a topological space X. We will write $int_X(A)$ for the interior, $cl_X(A)$ for the closure and $bd_X(A)$ for the boundary of A in X. Let d be an admissible metric for a space X. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is a **Cauchy sequence** if for every $\epsilon > 0$ there is an $M \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $n, m \geq M$. We call the metric space (X, d) complete if every Cauchy sequence has a limit in X. In this case we say that the space X is completely metrizable. A separable completely metrizable space is called **Polish**. A space X is a **first category space** if $X = \bigcup_{n\in\mathbb{N}} X_i$ where each X_i is nowhere dense in X; X is a**Baire space** if the intersection of countably many dense open subsets of X is still dense. A space X is **zero-dimensional** if it has a base of clopen sets, and X is a **totally disconnected** space if for any two distinct $x, y \in X$, there is a clopen subset U of X such that $x \in U$ and $y \notin U$. The following result gives some examples of Baire spaces.

Theorem 1.1 ([21, Theorem A.6.6])**.** *Every Polish is a Baire space.*

A *Borel set* is any set B in a topological space X that can be formed from open sets through the operations of countable union, countable intersection, and complement. Some types of sets that are Borel that we use in this thesis are the following.

Definition 1.2. *Let X be a space and A a subset of X.*

1. *We say that A is a G_δ-subset of X if it is a countable intersection of open subsets of X, and A is an F_σ-subset of X if it is a countable union of closed subsets of X.*

2. *We say that A is a $G_{\delta\sigma}$-subset of X if it is a countable union of G_δ-subsets of X, and A is an $F_{\sigma\delta}$-subset of X if it is a countable intersection of F_σ-subsets of X.*

Note that if A is a G_δ-subset of a space X, then $X \setminus A$ is an F_σ-subset of X, and if A is a $G_{\delta\sigma}$-subset of X, then $X \setminus A$ is an $F_{\sigma\delta}$-subset of X. A known fact is that every closed subset of a metric space X is a G_δ-subset of X. Therefore every open subset of a metric space is an F_σ-subset of X. An important property of complete spaces is the following.

Theorem 1.3 ([21, Theorem A.6.3])**.** *X is a Polish space if and only if X is a G_δ-subset of any space Y containing X.*

A space X is called an **absolute** G_δ if it is a G_δ-subset of every space it is embedded in. We define the notions **absolute** F_σ, **absolute** $G_{\delta\sigma}$ and **absolute** $F_{\sigma\delta}$ similarly. The following Theorem characterizes the sets that are absolute G_δ, absolute F_σ and absolute $G_{\delta\sigma}$, and absolute $F_{\sigma\delta}$.

Theorem 1.4 ([21, Theorem A.13.2])**.** *Let X be a space.*

1. *X is an absolute G_δ if and only if X is complete.*

2. *X is an absolute F_σ if and only if X is σ-compact.*

3. *X is an absolute $G_{\delta\sigma}$ if and only if X is the union of countably many complete spaces.*

4. *X is an absolute $F_{\sigma\delta}$ if and only if X can be imbedded in some complete space as an $F_{\sigma\delta}$-subset.*

Example 1.5. 1. *Since $\mathbb{Q}$ is a σ-compact space, by 2 of Theorem 1.4, it is an absolute F_σ.*

2. *By Theorem 1.3 every complete space is an absolute G_δ.*

3. *$\mathbb{Q}^\omega$ is an absolute $F_{\sigma\delta}$ space (see [[21], Corolallry A.13.4]).*

4. *Erdős space is an absolute $F_{\sigma\delta}$ but is not an absolute $G_{\delta\sigma}$ (see [[3], Remark 5.5]])*

5. *the product of the rational numbers and complete Erdős space that is an absolute $G_{\delta\sigma}$ and absolute $F_{\sigma\delta}$ (see [[3], Remark 4.12]]).*

To end this section we will talk a little about Stone's space. Let X be a zero-dimensional space, and let $\mathcal{B}(X) = \{A \subset X : A \textit{ is clopen}\}$. Note that $\mathcal{B}(X)$ satisfies the following:

1. If $A, B \in \mathcal{B}(X)$, then $A \cap B \in \mathcal{B}(X)$, and $A \cup B \in \mathcal{B}(X)$;

2. For each $A, B, C \in \mathcal{B}(X)$, we have that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, and;

3. For any $A \in \mathcal{B}(X)$, we have that $X \setminus A \in \mathcal{B}(X)$.

$\mathcal{B}(X)$ is known as the Boolean algebra of clopens. There are other types of more general Boolean algebras but in this thesis we only use the Boolean algebra of clopen sets. Recall that a filter p on X is a subset of $\wp(X)$ that satisfies the following conditins:

1. $X \in p$

2. if $A, B \in p$, then $A \cap B \in p$

3. if $A \in p$ and $A \subset B$, then $B \in p$.

An ultrafilter is a filter that is not proper contained in any other filter. We define S as the set $\{p \subset \mathcal{B}(X) : p \textit{ is an ultrafilter}\}$ and $\lambda(A) = \{p \in S : A \in p\}$. It is known that $\{\lambda(A) : A \in \mathcal{B}\}$ is a base for a Hausdorff topology in S. This space with this topology is known as **Stone space** associated with $\mathcal{B}(X)$. Some of the properties of S with the topology whose base is $\{\lambda(A) : A \in \mathcal{B}\}$ are the following:

1. S is a compact space and X is a dense subset of S.

2. S is a zero-dimensional space.

3. If $U \in \mathcal{B}(X)$, then $cl_S(U)$ is an open subset of S.

For more information about the properties of Stone's space consult ([24, pag. 155]).

1.2 Erdős spaces

The two most important spaces in this thesis are Erdős space and complete Erdős space. These spaces are subsets of

$$\ell^2 = \left\{ (x_n)_{n\in\omega} \in \mathbb{R}^\omega : \sum_{n\in\omega} x_n^2 < \infty \right\}$$

The topology on ℓ^2 is generated by the norm $\|z\| = (\sum_{n=0}^{\infty} z_n^2)^{1/2}$ where $z \in \ell^2$. Erdős space is defined to be the space

$$\mathfrak{E} = \{(x_n)_{n\in\omega} \in \ell^2 : \forall i \in \omega, x_i \in \mathbb{Q}\},$$

and complete Erdős space is the space

$$\mathfrak{E}_c = \{(x_n)_{n\in\omega} \in \ell^2 : \forall i \in \omega, x_i \in \{0\} \cup \{1/n : n \in \mathbb{N}\}\}$$

These two spaces were introduced by Erdős in 1940 in [9] as examples of totally disconnected and non-zero-dimensional spaces.

For every $z \in \ell^2 \setminus \mathfrak{E}_c$, there exists $n \in \omega$ such that $z_n \notin \{0\} \cup \{1/n : n \in \mathbb{N}\}$. So there exists an open subset U of $\mathbb{R}$ such that $z_n \in U$ and $U \cap (\{0\} \cup \{1/n : n \in \mathbb{N}\}) = \emptyset$. Then $W = \{x \in \ell^2 : x_n \in U\}$ is an open subset of ℓ^2 such that $z \in W \subset \ell^2 \setminus \mathfrak{E}_c$. Therefore $\mathfrak{E}_c$ is a closed subset of ℓ^2, and $\mathfrak{E}$ is an $F_{\sigma\delta}$ subset of ℓ^2 (see Remark 4.12 in [3]). Since ℓ^2 is a complete space by Theorem 1.4, $\mathfrak{E}_c$ is an absolute G_δ and $\mathfrak{E}$ is an absolute $F_{\sigma\delta}$ but $\mathfrak{E}$ is not an absolute $G_{\delta\sigma}$ (see Remark 4.12 in [3]).
We have the following important result about convergence in ℓ^2.

Proposition 1.6 ([21, Lemma 1.1.12]). *Suppose that $(x(n))_{n\in\omega}$ is a sequence in ℓ^2 and $x \in \ell^2$. Then the following statements are equivalent:*

1. $\lim_{n\to\infty} x(n) = x$ *in* ℓ^2

2. $\lim_{n\to\infty} \|x(n)\| = \|x\|$ *and for every* $i \in \omega$ $\lim_{n\to\infty} x(n)_i = x_i$.

This proposition shows that the norm topology on ℓ^2 is the weakest topology that contains the product topology inherited from $\mathbb{R}^\omega$ and makes the norm continuous.
Let $\varphi\colon X \to [-\infty,\infty]$ be a function. We say that φ is **upper semi-continuous** (USC) if for every $t \in \mathbb{R}$ the set $f^{\leftarrow}[[-\infty,t)]$ is an open subset of X. Similarly, f is called **lower semi-continuous** (LSC) if for every $t \in \mathbb{R}$ the set $f^{\leftarrow}[(t,\infty)]$ is an open subset of X.

Lemma 1.7. *Every closed ball $\{x \in \ell^2 : \|x\| \le t\}$ for $t > 0$ is a closed subset of $\mathbb{R}^\omega$.*

Proof. Let $t > 0$ and suppose that $x \in \mathbb{R}^\omega$ is such that $\|x\| > t$. Then we can find an $m \in \mathbb{N}$ such that $\sum_{i \le m} |x_i| > t^2$. Since this sum is a continuous function of the vector $(x_0,\ldots,x_m)$ we can find a $\delta > 0$ such that $\sum_{i \le m} |y_i| > t^2$ whenever $|x_i - y_i| < \delta$ for $0 \le i \le m$. Consider the basic open neighbourhood U of x with respect to the product topology on $\mathbb{R}^\omega$ given by

$$U = \{y \in \mathbb{R}^\omega : |x_i - y_i| < \delta \textit{ for } 0 \le i \le m\}.$$

Note that $\|y\| > t$ for all $y \in U$. □

Observe that the norm as a function from ℓ^2 to $[0,\infty]$ is not continuous with the product topology, because the sequence $x_n = (y_1,\ldots,y_{n^2},0\ldots) \to (0,\ldots,0\ldots)$ in ℓ^2 (where $y_k = 1/n$ for all $n \in \mathbb{N}$ and $k \in \{1,\ldots,n^2\}$) but $\|x_n\| = (\sum_{n=1} 1/n^2)^{1/2}$ does not converge to 0. Note that for any $t \in (0,1)$ and $x \in \{y \in \mathbb{R}^\omega : \|y\| \in (t,1]\}$ we have that $x \in U \subset \{y \in \mathbb{R}^\omega : \|y\| \in (t,1]\}$ (where U is as in the Lemma 1.7). This implies that the norm is an LSC function.

On the other hand with Proposition 1.6 and Lemma 1.7 we see that we can also describe the norm topology on ℓ^2 as the topology that is generated by the product topology together with the sets $\{z \in \ell^2 : \|z\| < t\}$ for $t > 0$. We point out the following connection between the two topologies on $\mathfrak{E}$ and $\mathfrak{E}_c$. From Lemma 1.7 that every closed ϵ-ball in $\mathfrak{E}$ is also a closed subset in $\mathbb{Q}^\omega$. This means that every point in $\mathfrak{E}$ has arbitrarily small neighbourhoods which are closed sets in $\mathbb{Q}^\omega$. Clearly, Lemma 1.7 also implies that every closed ϵ-ball in $\mathfrak{E}_c$ is also a closed subset in $(\{0\} \cup \{1/n : n \in \mathbb{N}\})^\omega$.

Definition 1.8. *A space (X,τ) is **almost zero-dimensional** (AZD) if there is a set Z that contains X and a topology $\mathcal{T}$ on Z, such that $(Z,\mathcal{T})$ is a zero-dimensional space, $O \cap X$ is an open subset in X for each $O \in \mathcal{T}$, and* *every point of X has a neighbourhood base in X consisting of sets that are closed in $(Z, \mathcal{T})$*

In Definition 1.8 we will also say that the space $(Z, \mathcal{T})$ is a **witness** to the almost zero-dimensionality of X.
Thus $\mathfrak{E}$ and $\mathfrak{E}_c$ are almost zero-dimensional spaces. The space $\mathbb{Q}^\omega$ is a witness to the almost zero-dimensionality of $\mathfrak{E}$ and the space $(\{0\} \cup \{1/n : n \in \mathbb{N}\})^\omega$ is a witness to the almost zero-dimensionality of $\mathfrak{E}_c$. From the Definition 1.8 the following is immediate.

Remark 1.9. *A space X is almost zero-dimensional if and only if there is a topology on X witnessing this fact.*

A set in a space is a C**-set** if it is the intersection of clopen sets. Observe that every C-set is closed and that finite unions and finite intersections of C-sets are also C-sets. The following Proposition gives us an important equivalence of almost zero-dimension using C-sets.

Proposition 1.10 ([3, Remark 2.4])**.** *A space is AZD if and only if it has a base of C-sets.*

A separable metric space X is one-dimensional if it is not zero-dimensional and has a base β of neighborhoods such that $bd_X(U) \neq \emptyset$ and is zero-dimensional for any $U \in \beta$. If X has dimension one we write $dim(X) = 1$. In general we can define the dimension of a space X for any $n \in \mathbb{N}$ but in this thesis we will only use the definition of dimension 0 and 1.
Erdős in [9] proved that both $\mathfrak{E}$ and $\mathfrak{E}_c$ are one-dimensional. This result make these spaces important examples in Dimension Theory.

Some properties of almost zero-dimensional spaces are the following ones.:

Proposition 1.11.

1. *All zero-dimensional spaces are almost zero-dimensional.*
2. *Any subset of a almost zero-dimensional space is almost zero-dimensional.*
3. *The countable product of almost zero-dimensional spaces is almost zero-dimensional.*
4. *All almost zero-dimensional spaces are totally disconnected.*

Proof. (1) Let $(X, \mathcal{W})$ be a zero-dimensional space. By Remark 1.9 it is clear that $\mathcal{W}$ is witness to the almost zero-dimensionality of X. Therefore X is an

AZD space.
(2) Let X be an almost zero-dimensional space, and A be a subset of X. By Remark 1.9 there is a witness topology $\mathcal{W}$ of X. Let $\mathcal{W} \restriction A = \{U \cap A : U \in \mathcal{W}\}$, let us note that $\mathcal{W} \restriction A$ is witness to the almost zero-dimensionality of A. Therefore A is an AZD space.
(3) Let $\{X_n : n \in \mathbb{N}\}$ be a family of almost zero-dimensional spaces and $X = \prod\{X_n : n \in \mathbb{N}\}$. By remark 1.9 there are witness topologies $\mathcal{W}_n$ of X_n for every $n \in \mathbb{N}$. Let $\mathcal{W}$ be the topology of $\prod\{(X_n, \mathcal{W}_n) : n \in \mathbb{N}\}$. We claim that $\mathcal{W}$ witness to the almost zero-dimensionality of X. It is clear that $(X, \mathcal{W})$ is a zero-dimensional space. Also if U is a basic open subset of $(X, \mathcal{W})$, then there are open subsets $U_1, \ldots U_n$ of $(X_1, \mathcal{W}_1), \ldots, (X_n, \mathcal{W}_n)$ respectively such that $U = \bigcap_{k \leq n} \pi_k^{\leftarrow}[U_k]$ (where $\pi_k : (X, \mathcal{W}) \to (X_k, \mathcal{W}_k)$ is the projection). Since $\mathcal{W}_k$ is witness the almost zero-dimensionality of X_k, then U_k is an open subset of X_k for each $k \leq n$. Therefore U is an open subset in X. Let $(x_n)_{n \in \mathbb{N}} \in X$ and let U be a basic open subset X such that $x \in U$ then there are $U_1, \ldots, U_n$ open subsets of $X_1, \ldots X_n$ respectively such that $U = \bigcap_{k \leq n} \pi_k^{\leftarrow}[U_k]$ (where $\pi_k : X \to X_k$ is the projection). Since $x \in U$, then $x_i \in U_i$, since $\mathcal{W}_i$ is a witness topology of X_i, then there exists a neighborhood V_i of x_i in X_i that is a closed in $\mathcal{W}_i$ and $V_i \subset U_i$. Let $V = \bigcap_{k \leq n} \pi_k^{\leftarrow}[V_k]$, then $x \in V \subset U$. Moreover since V_i is a closed subset of $(X, \mathcal{W}_i)$ for each $i \leq n$, then V is a closed subset of $(X, \mathcal{W})$. Since $V_1, \ldots, V_n$ are neighborhoods of $x_1, \ldots, x_n$ respectively then V is neighborhood of x in X. Therefore X is an AZD space
(4) Let X be an AZD space. By remark 1.9 there is a witness topology $\mathcal{W}$ of X. Let $x, y \in X$, since $(X, \mathcal{W})$ is a zero-dimensional space there exists a clopen subset in $(X, \mathcal{W})$, such that $x \in U$ and $y \notin U$. Since U is a clopen subset of X, the space X is totally disconnected. $\square$

Note that if X is a locally compact and AZD space, then X is zero-dimensional by item 4 from Proposition 1.11, and from the fact that all totally-disconnected and compact spaces are zero-dimensional (see Theorem 6.2.9 in [7]). This implies that an AZD space that is not a zero-dimensional space is not locally compact.

Let $\varphi \colon X \to [0, \infty)$ be a USC function. We define

$$\begin{aligned} G_0^{\varphi} &= \{\langle x, \varphi(x)\rangle \colon x \in X,\, \varphi(x) > 0\}, \text{ and} \\ L_0^{\varphi} &= \{\langle x, t\rangle \colon x \in X,\, 0 \leq t \leq \varphi(x)\}. \end{aligned}$$

The following Lemma tells us that every almost zero-dimensional space is homeomorphic to G_0^φ where φ is an USC function. This lemma is also of great importance in Chapter 4.

Lemma 1.12 ([3, Lemma 4.11])**.** *Let X be a space and let Z be a zero-dimensional space that contains X as a subset (but not necessarily as a subspace). Then the following statements are equivalent:*

1. *Z is a witness to the almost zero-dimensionality of X.*

2. *there exists a USC function $\varphi : Z \to [0,1]$ such that $\varphi^{\leftarrow}[0] = Z \setminus X$ and the map $h : X \to G_0^\varphi$ that is defined by the rule $h(x) = (x, \varphi(x))$ is a homeomorphism.*

Oversteegen and Tymchatyn proved that every almost zero-dimensional space is at most one-dimensional. Since $G_0^\varphi \subset [0,1]\times Z$, and $dim(Z\times[0,1]) = 1$ (see Theorem 7.3.17 in [7]), we have $dim(G_0^\varphi) \leq 1$ (see Theorem 7.1.1 in [7]). Therefore, by the Lemma 1.12, every almost zero-dimensional space has a dimension less than or equal to 1.

A subset A of ℓ^2 is called **bounded** if it is bounded in norm, that is, if there is an $M \in \mathbb{N}$ such that $\|a\| \leq M$ for all $a \in A$. If A is not bounded we call it an unbounded set.

Lemma 1.13 ([9])**.** *Every clopen subset of $\mathfrak{E}$ and $\mathfrak{E}_c$ is unbounded.*

From Lemma 1.13 follows that every point in $\mathfrak{E}$ has a neighborhood that does not contain (nonempty) clopen sets. Later in [3] J. Dijkstra and J. van Mill formalized this concept as follows.

Definition 1.14 ([3, Definition 5.1])**.** *Let X be a space and let $\mathcal{A} \subset \wp(X) \setminus \{\emptyset\}$. The space X is called **$\mathcal{A}$-cohesive space** if every point of the space has a neighborhood that does not contain nonempty proper clopen subsets of any element of $\mathcal{A}$. If a space X is $\{X\}$-cohesive then we simply call X cohesive.*

Definition 1.14 is of importance for the topological characterizations of $\mathfrak{E}$, $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$. Note that all connected spaces are cohesive and that the dimension of a cohesive space is greater than or equal to one.

Remark 1.15 ([3, Remark 5.2])**.** *If Y is any space and X is a space that is $\mathcal{A}$-cohesive then $X \times Y$ is $\{A \times B : A \in \mathcal{A}$ and $B \subset Y\}$-cohesive. In particular, if X is $\{A_s : s \in S\}$-cohesive, then X^n is $\{A_{s_1} \times \ldots \times A_{s_n} : s_j \in S\}$-cohesive for any $n \in \mathbb{N}$.*

As a generalization of the construction of $\mathfrak{E}$ and $\mathfrak{E}_c$, consider a fixed sequence $E_0, E_1, E_2, \ldots$ of subsets of $\mathbb{R}$ and let

$$\mathcal{E} = \{z \in \ell^2 : z_n \in E_n \textit{ for every } n \in \omega\}.$$

If we consider $E_n = \mathbb{Q}$ for all $n \in \mathbb{N}$ we obtain Erdős space and if we consider $E_n = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ for all $n \in \mathbb{N}$ we obtain complete Erdős space.

The following theorem tells us when space $\mathcal{E}$ is cohesive.

Theorem 1.16 ([6, Theorem 1]). *Assume that $\mathcal{E}$ is not empty and that every E_n is zero-dimensional. For each $\epsilon > 0$ we let $\eta(\epsilon) \in \mathbb{R}^\omega$ be given by*

$$\eta(\epsilon)_n = sup\{|a| : a \in E_n \cap [-\epsilon, \epsilon]\}$$

where $sup(\emptyset) = 0$. The following statements are equivalent:

1. *$\|\eta(\epsilon)\| = \infty$ for each $\epsilon > 0$*
2. *there exists $x \in \prod_{n\in\mathbb{N}} E_n$ with $\|x\| = \infty$ and $\lim_{n\to\infty} x_n = 0$;*
3. *every nonempty clopen subset of $\mathcal{E}$ is unbounded;*
4. *$\mathcal{E}$ is cohesive ; and*
5. *$dim(\mathcal{E}) > 0$*

Note that under the conditions of this theorem the space $\mathcal{E}$ is almost zero-dimensional: the product space $\prod_{n\in\mathbb{N}} E_n$ is a witness to the almost zero-dimensionality of $\mathcal{E}$ Since every almost zero-dimensional space is at most one-dimensional, the condition (5) is equivalent to $dim(\mathcal{E}) = 1$.

Recall that if $A_0, A_1, \ldots$ is a sequence of subsets of a space X then

$$\limsup_{n\to\infty} A_n = \bigcap_{n=0}^{\infty} cl_X(\bigcup_{n=k}^{\infty} A_k).$$

A point x in a topological space X is called a **cluster point** of a set $A \subset X$ if $x \in cl_X(A \setminus \{x\})$.

Corollary 1.17. *If 0 is a cluster point of $\limsup_{n\to\infty} E_n$ then every nonempty clopen subset of $\mathcal{E}$ is unbounded.*

Proof. If $\mathcal{E}$ is empty, then the conclusion is void. Let $\mathcal{E} \neq \emptyset$ and let $n \in \mathbb{N}$. Select a $t \in \limsup_{k\to\infty} E_n$ such that $0 < |t| < 1/n$. Choose a sequence of natural numbers $k_0 < k_1 < k_2 < \ldots$ such that there is, for each $j \in \mathbb{N}$, a $t_j \in E_{k_j}$ with $\lim_{j\to\infty} t_j = t$. We may assume that for every j, $\frac{1}{2}|t| < |t_j| < 1/n$. Thus $\eta(1/n)_{k_j} \geq |t_j| > \frac{1}{2}|t|$ for each j and hence $\|\eta(n)\| = \infty$, proving statement (1) of Theorem 1.16. □

We will use Corollary 1.17 to show that $\mathfrak{E}$ is cohesive. In this case $E_n = \mathbb{Q}$ for each $n \in \mathbb{N}$, then $\limsup_{n\to\infty} E_n = \mathbb{R}$. Clearly 0 is a cluster point $\mathbb{R}$. On the other hand $E_n = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ for each $n \in \mathbb{N}$ for the case of $\mathfrak{E}_c$, then $\limsup_{n\to\infty}(\{0\} \cup \{1/n : n \in \mathbb{N}\}) = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Note $0 \in cl_{\mathbb{R}}(\{1/n : n \in \mathbb{N}\})$, that is, 0 is a cluster point $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. By Colollary 1.17 every nonempty clopen subset of $\mathfrak{E}$ or $\mathfrak{E}_c$ is unbounded. By Theorem 1.16 $\mathfrak{E}$ and $\mathfrak{E}_c$ are cohesive spaces.

1.2.1 Characterization of $\mathfrak{E}$

The spaces $\mathfrak{E}$, $\mathfrak{E}_c$, and also $\mathfrak{E}_c^\omega$ were characterized by Dijkstra and van Mill in [3] [4], and [5] respectively. The characterizations of Erdős space and complete Erdős space are of great importance in this thesis. Before we can formulate these characterizations we need to introduce some notions.

Definition 1.18. *If A is a nonempty set then $A^{<\omega}$ denotes the set of all finite strings of elements of A, including the null string $\emptyset$. If $s \in A^{<\omega}$ then $|s|$ denotes its length. In this context the set A is called alphabet. Let A^ω denote the set of all infinite strings of elements of A.*

Note that if $s \in A^{<\omega}$ and $t \in A^{<\omega} \cup A^\omega$, then we put $s \prec t$ if s is an initial substring of t; that is, there is an $r \in A^{<\omega} \cup A^\omega$ with $s^\frown r = t$, where $\frown$ denotes concatenation of strings. Also if $t \in A^{<\omega} \cup A^\omega$ and $k \in \omega$, $t \restriction k \in A^{<\omega}$ is the element of $A^{<\omega}$ characterized by $t \restriction k \prec t$ and $|t \restriction k| = k$.

Definition 1.19. *A tree T on A is a subset of $A^{<\omega}$ that is closed under initial segments, i.e., if $s \in T$ and $t \prec s$ then $t \in T$.*

Elements of tree T are called nodes. An infinite branch of T is an element r of A^ω such that $r \restriction k \in T$ for every $k \in \omega$. The body of T, written as $[T]$, is the set of all infinite branches of T. If $s, t \in T$ are such that $s \prec t$ and $|t| = |s| + 1$, then we say that t is an immediate successor of s and $succ(s)$ denotes the set of immediate successors of s in T.

The trees considered in Definition 1.19 are said to have height ω. But there are also trees with heights greater than ω. In this thesis we will not talk about those types of trees, only countable trees are considered.

Definition 1.20. *Let $n \geq 2$. If $S_1, \ldots, S_n$ are trees over $A_1, \ldots, A_n$, respectively, and if $s_1 = a_1^1, \ldots, a_k^1 \in S_1, \ldots, s_n = a_1^n, \ldots, a_k^n \in S_n$, are strings of equal length, then we define the string $s_1 * \ldots * s_n$ over $A_1 \times \ldots \times A_n$ by $s_1 * \ldots * s_n = (a_1^1, \ldots, a_1^n), \ldots, (a_k^1, \ldots, a_k^n)$. We define the product tree $S_1 * \ldots * S_n$ over $A_1 \times \ldots \times A_n$ as the partially ordered subset $\{s_1 * \ldots * s_n : s_i \in S_i$ for all $i \in \{1, \ldots, n\}$ and $|s_1| = \ldots = |s_n|\}$ of the alphabet $(A_1 \times \ldots \times A_n)^{<\omega}$.*

Remark 1.21. *Let $n \geq 2$, and let $S_1, \ldots, S_n$ trees over $A_1, \ldots, A_n$, respectively. Then $S_1 * \ldots * S_n$ is indeed a tree over $A_1 \times \ldots \times A_n$. The following statements hold.*

1. *Let $s_1, t_1 \in S_1, \ldots s_n, t_n \in S_n$ with $|s_1| = \ldots = |s_n|$ and $|t_1| = \ldots = |t_n|$. Then $s_1 * \ldots * s_n \prec_{S_1 * \ldots * S_n} t_1 * \ldots * t_n$ if and only if $s_i \prec_{S_i} t_i$ for all $i \in \{1, \ldots, n\}$.*

2. *Let $s_1, t_1 \in S_1, \ldots s_n, t_n \in S_n$ with $|s_1| = \ldots = |s_n|$ and $|t_1| = \ldots = |t_n|$. Then $t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)$ in $S_1 * \ldots * S_n$ if and only if $t_i \in succ(s_i)$ in S_i for all $i \in \{1, \ldots, n\}$.*

3. *The body of $S_1 * \ldots * S_n$, $[S_1 * \ldots * S_n]$, is equal to the set $\{(\widehat{s_1}, \ldots, \widehat{s_n}) : \widehat{s_k} \in [S_k]$ for all $k \in \{1, \ldots, n\}\}$.*

4. *Let $\widehat{t_1} \in [S_1], \ldots, \widehat{t_n} \in [S_n]$, and let $k \in \omega$. Then*

$$(\widehat{t_1}, \ldots, \widehat{t_n}) \restriction k = (\widehat{t_1} \restriction k, \ldots, \widehat{t_n} \restriction k) = \widehat{t_1} \restriction k * \ldots * \widehat{t_n} \restriction k.$$

Let X be a space. Let $(A_n)_{n \in \omega}$ a sequence of sets of X. We say that $(A_n)_{n \in \omega}$ converges to x if for each open subset U such that $x \in U$ there exists $m \in \omega$ such that $A_k \subset U$ if $m \leq k$.

Definition 1.22. *Let T be a tree and let $(X_s)_{s \in T}$ be a system of subsets of a space X (called a scheme) such that $X_t \subset X_s$ whenever $s \prec t$. A subset A of X is called an anchor for $(X_s)_{s \in T}$ in X if either for every $t \in [T]$ we have $X_{t \restriction k} \cap A = \emptyset$ for some $k \in \omega$ or the sequence $X_{t \restriction k_0}, \ldots, X_{t \restriction n} \ldots$ converges to a point in X.*

Example 1.23. *As the space $\mathbb{Q}^\omega$ is a witness to the almost zero-dimensionality of $\mathfrak{E}$; let $\mathcal{W}$ be the topology that $\mathfrak{E}$ inherits from $\mathbb{Q}^\omega$. Put $T = \mathbb{Q}^{<\omega}$ and for $s = q_0, \ldots, q_n \in T$, with $n \in \omega$, let the closed subset $\mathbb{Q}^\omega_s$ of $\mathbb{Q}^\omega$ be given by*

$$\mathbb{Q}^\omega_s = \{x \in \mathbb{Q}^\omega : x_i = q_i \text{ for } 0 \leq i \leq n\}$$

Put $\mathfrak{E}_s = \mathbb{Q}^\omega_s \cap \mathfrak{E}$ for $s \in T$ and let B be a bounded subset of $\mathfrak{E}$. We show that B is an anchor for $(\mathfrak{E}_s)_{s \in T}$ in $(\mathfrak{E}, \mathcal{W})$. Let $z = (q_0, q_1, \ldots) \in [T]$ be such that $\mathfrak{E}_{z\restriction k} \cap B \neq \emptyset$; for all $k \in \omega$. It is clear that $\mathfrak{E}_{z\restriction k}$ converges to the point $z \in \mathbb{Q}^\omega$ in the product topology of $\mathbb{Q}^\omega$, hence it suffices to show that $z \in \mathfrak{E}$. Since B is bounded there is an $M \in \mathbb{N}$ such that $B \subset \{x \in \mathbb{Q}^\omega : \|x\| \leq M\}$ and because $\mathfrak{E}_{z\restriction k} \cap B \neq \emptyset$; for all $k \in \omega$ this means that $\|(q_0, q_1, \ldots, q_k, 0, 0, \ldots)\| \leq M$ for all $k \in \omega$. Beacuse there exists $p \in \mathfrak{E}_{z\restriction k} \cap B$ such that $p = (q_0, q_1, \ldots q_k, x_{k+1}, 0, \ldots)$ and $p \in \{x \in \mathbb{Q}^\omega : \|x\| \leq M\}$, then

$$\|(q_0, q_1, \ldots, q_k, 0, 0, \ldots)\| \leq \|p\| \leq M.$$

Since the norm function is LSC on $\mathbb{Q}^\omega$ we have

$$\|z\| \leq \lim_{n \to \infty} \|(q_0, \ldots, q_n, \ldots)\| \leq M$$

so $z \in \mathfrak{E}$.

Dijkstra and van Mill proved the following characterization of $\mathfrak{E}$.

Theorem 1.24 ([3, Theorem 8.13])**.** *A nonempty space E is homeomorphic to $\mathfrak{E}$ if and only if there exists a topology $\mathcal{W}$ on E that witnesses the almost zero-dimensionality of E and there exists a nonempty tree T over a countable alphabet and subspaces E_s of E that are closed with respect to $\mathcal{W}$ for each $s \in T$ such that:*

1. *$E_\emptyset = E$ and $E_s = \bigcup\{E_t : t \in succ(s)\}$ whenever $s \in T$,*
2. *each $x \in E$ has a neighborhood U that is an anchor for $(E_s)_{s \in T}$ in $(E, \mathcal{W})$*
3. *for each $s \in T$ and $t \in succ(s)$, we have that E_t is nowhere dense in E_s, and*
4. *E is $\{E_s : s \in T\}$-cohesive.*

Theorem 1.24 is known as the intrinsic characterization of $\mathfrak{E}$. Another characterization of $\mathfrak{E}$ in terms of USC functions is known as the extrinsic characterization of $\mathfrak{E}$. We will not mention this last characterization here but it is found in [3].
As an illustration of Theorem 1.24 we show that $\mathfrak{E}$ satisfies the conditions of Theorem 1.24. Let the topology $\mathcal{W}$ on $\mathfrak{E}$, the tree T, and the subspaces $\mathfrak{E}_s \subset \mathfrak{E}$ for $s \in T$ be as in Example 1.23. It is clear that $\mathfrak{E}_s$ is closed in $(E, \mathcal{W})$ for all $s \in T$ and conditions (1) and (3) are easily seen to be satisfied. Furthermore, it follows from Example 1.23 that every bounded neighbourhood of a point $x \in E$ is an anchor for $(\mathfrak{E}_s)_{s\in T}$ in $(E, \mathcal{W})$, so condition (2) is satisfied. Finally, as noted before, it follows from Corollary 1.17 that every nonempty clopen subset of $\mathfrak{E}_s$ is unbounded. This means that a bounded neighbourhood of a point $x \in \mathfrak{E}$ does not contain nonempty clopen subsets of any space $\mathfrak{E}_s$, hence condition (4) is satisfied as well.

A consequence of Theorem 1.24 is that every open subset of $\mathfrak{E}$ is homeomorphic to $\mathfrak{E}$. If U is an open subset of $\mathfrak{E}$, then the topology $\mathcal{W} \restriction U$, the tree $T' = \{s \in T : U \cap \mathfrak{E}_s \neq \emptyset\}$ and the sets $\{U \cap \mathfrak{E}_s : s \in T'\}$ satisfy the conditions of Theorem 1.24.

Definition 1.25. *Let X be a space. A space Y is called an X-factor, if there is a space Z such that $Y \times Z$ is homeomorphic to X.*

The following theorem characterizes the factors of $\mathfrak{E}$.

Theorem 1.26 ([3, Theorem 9.2 items (2) and (5)])**.** *E is an $\mathfrak{E}$-factor if and only if E admits a closed embedding into $\mathfrak{E}$.*

Since $\mathfrak{E} \times \mathfrak{E}$ is homeomorphic to $\mathfrak{E}$, then $\mathfrak{E}$ is a factor of itself. Also from Theorem 1.26 it can be shown that $\mathfrak{E}$ is homeomorphic to $\mathfrak{E}^\omega$(see [3, Corollary 9.4]) and that every complete AZD space is a factor of $\mathfrak{E}$ (see [3, Corollary 9.3]), with this we have $\mathfrak{E}$ is homeomorphic to $\mathfrak{E} \times \mathfrak{E}_c$ and to $\mathfrak{E} \times \mathfrak{E}_c^\omega$. Another consequence of Theorem 1.26 that is useful in this work is the following result.

Proposition 1.27 ([3, Proposition 9.1])**.** $\mathfrak{E}_c \times \mathbb{Q}^\omega \approx \mathfrak{E}$

1.2.2 Characterization of $\mathfrak{E}_c$

Now we will talk about the characterizations of complete Erdős space, in this case we will mention the intrinsic and extrinsic characterization. Both of these characterizations will be used.

Let $\varphi\colon X \to [0,\infty)$ be a USC function. We say that φ is a **Lelek function** if X is zero-dimensional, the set $\{x \in X\colon \varphi(x) > 0\}$ is dense in X and G_0^φ is dense in L_0^φ. The existence of Lelek functions with domain equal to the Cantor set 2^ω follows from Lelek's original construction [15] of what is now called the Lelek fan (Figure 1.1).

Example 1.28. *Let $\varphi_0 : \mathbb{R}^\omega \to [0,\infty]$ be the function given by $\varphi_0(x) = \sum_{n\in\mathbb{N}} |x_n|$. With an argument analogous to that of Lemma 1.7 it can be shown that φ_0 is an LSC function. Let $X = (\{0\} \cup \{1/n : n \in \mathbb{N}\})^\omega$ and let $\varphi = \varphi_0 \restriction X$, then φ is an LSC function. Let $\eta = f \circ \varphi$ where $f : [0,\infty] \to [0,1]$ given by $f(x) = \frac{1}{1+x}$ if $x \in [0,\infty)$ and $f(x) = 0$ if $x = \infty$. We are going to show that η is a Lelek function. Since f is a continuous and decreasing function, η is an USC function. To prove that η is a Lelek function it is enough to prove that $\{(x,\varphi(x)) : \varphi(x) < \infty\}$ is a dense subset in $\{(x,t) : \varphi(x) \le t\}$ because $g = id_{\mathbb{R}^\omega} \times f$ is a continuous function and $g[\{(x,\varphi(x)) : \varphi(x) < \infty\}] = G_0^\eta$ and $g[\{(x,t) : \varphi(x) \le t\}] = L_0^\eta$.*

Let $(x^0,t^0) \in \{(x,t) : \varphi(x) \le t\}$, $m \in \mathbb{N}$. If $x_n^0 > 0$, let's consider $W_n = \{x \in X : x_n = x_n^0\}$, and if $x_n^0 = 0$, let's consider $W_n = \{x \in X : x_n < 1/m\}$, then W_n is an open subset of X such that $x^0 \in W_n$. Let $U = \bigcap_{n\le m} W_n$, then $x^0 \in U$. Let $\delta > 0$, $k \in \omega$ such that $1/k < \delta$ and $d = t^0 - \sum_{n\le m} x_n^0$. Note that $d > 0$, because $0 < \sum_{n\in\mathbb{N}} x_n^0 \le t^0$. As $\sum_{n\in\mathbb{N}} 1/n$ does not converge, then the set $A = \{n \in \mathbb{N} : n/k > d\}$ is not empty. Let $r = \min A$ and $y = (x_1^0,\ldots,x_m^0,x_{m+1},\ldots,x_{m+r},0,\ldots)$ where $x_{m+1} = \ldots = x_{m+r} = 1/k$. Note that $y \in U$ and $\varphi(y) = \sum_{n\le m} x_n^0 + r/k$. For choice of r we have to

$$\frac{r-1}{k} < d < \frac{r+1}{k}$$

This implies that $|\frac{r}{k} - d| < \frac{1}{k} < \delta$. On the other hand

$$|t^0 - \varphi(y)| = |t^0 + d - d - \varphi(y)| = |t^0 + d + \sum_{n\le m} x_n^0 - t^0 - \sum_{n\le m} x_n^0 - \frac{r}{k}| =$$

$$|d - \frac{r}{k}| < \frac{1}{k} < \delta$$

Therefore $|t^0 - \varphi(y)| < \delta$. That is $\varphi(y) \in (t^0-\delta, t^0+\delta)$. Then $(y,\varphi(y)) \in \{(x,\varphi(x)) : \varphi(x) < \infty\}$ and $(y,\varphi(y)) \in U_m \times (t^0-\delta,t^0+\delta)$. Thus $\{(x,\varphi(x)) : \varphi(x) < \infty\}$ is dense in $\{(x,t) : \varphi(x) \le t\}$. So η is a Lelek function.

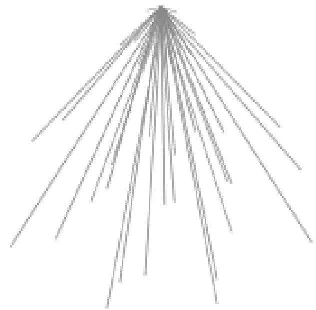

Figure 1.1: Lelek Fan

The following Lemma tells us under what conditions we can find a Lelek function when we have a USC function.

Lemma 1.29 ([3, Lemma 5.9])**.** *Let φ be a USC function from a zero-dimensional space X to $[0,\infty)$ and let $\mathcal{A}$ be a collection of subsets of X such that $\emptyset \notin \mathcal{A}$, G_0^φ is $\{G_0^\varphi \restriction A : A \in \mathcal{A}\}$-cohesive, and $A' = \{x \in A : \varphi(x) > 0\}$ is dense in A for each $A \in \mathcal{A}$. Then there exists a USC function $\psi : X \to [0,\infty)$ such that $\psi \leq \varphi$, the natural bijection h from the graph of φ to the graph of ψ is continuous, the restriction $h \restriction G_0^\varphi \to G_0^\psi$ is a homeomorphism, and for every $A \in \mathcal{A}$ we have that $\psi \restriction A$ is a Lelek function.*

The following Theorem was proved by Kawamura, Oversteegen, and Tymchatyn in [11].

Theorem 1.30. *If $\varphi\colon 2^\omega \to [0,1]$ is a Lelek function, then G_0^φ is homeomorphic to $\mathfrak{E}_c$.*

Theorem 1.30 is an extrinsic characterization of $\mathfrak{E}_c$. Since the space X in example 1.28 is homeomorphic to 2^ω, then by Theorem 1.30 $G_0^\eta \approx \mathfrak{E}_c$. By Corollary 1.17, G_0^η is cohesive and by Lemma 1.12, X is witness to the almost zero-dimension of G_0^η. By Definition 1.9 every point of G_0^η has a neighbourhood base β in G_0^η consisting of sets that are closed in X. Since X is a compact space, then every $U \in \beta$ is a compact subset in X. By Lemma 1.12 $G_0^\eta \subset X$ (but not as subspace). Let $Y = G_0^\eta$ be seen as a subspace of X, then Y is witness to the almost zero-dimension of G_0^η. Since $U \subset Y$, then U is a compact subset in Y. We conclude that G_0^η is cohesive and every point in G_0^η has a neighborhood which is compact subset of Y. Theorem 1.31 tells us that these two properties characterize the space $\mathfrak{E}_c$. Theorem 1.31 is the intrinsic characterization of $\mathfrak{E}_c$.

Theorem 1.31 ([2, Theorem 3.1, items (1), (2) and (3)]). *Let $(\mathcal{E}, \tau)$ be a topological space. The following statements are equivalent.*

1. *$\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$.*

2. *$\mathcal{E}$ is cohesive and there exists a zero-dimensional topology $\mathcal{W}$ in $\mathcal{E}$ such that every point in $\mathcal{E}$ has a neighborhood in τ which is compact with respect to $\mathcal{W}$.*

3. *$\mathcal{E}$ is cohesive and there exists a zero-dimensional topology $\mathcal{W}$ in $\mathcal{E}$ such that every point in $\mathcal{E}$ has a neighborhood in τ which is complete with respect to $\mathcal{W}$.*

Theorem 1.31 was proved by Dijkstra and van Mill. Also, in the case of $\mathfrak{E}_c$ each open subset of $\mathfrak{E}_c$ is homeomorphic to $\mathfrak{E}_c$. And we have the following Theorem that characterizes its factors.

Theorem 1.32 ([2, Theorem 3.2]). *E is an $\mathfrak{E}_c$-factor if and only if E admits a closed embedding into $\mathfrak{E}_c$.*

An important consequence of Theorems 1.31 and 1.32 is the following Theorem.

Theorem 1.33 ([2, Theorem 3.5]). *A nonempty space is homeomorphic to $\mathfrak{E}_c$ if and only if it is cohesive and $\mathfrak{E}_c$-factor.*

The following Proposition tells us that the spaces $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$ are different.

Proposition 1.34. *Every subset of $\mathfrak{E}_c^\omega$ with a nonempty interior contains closed copies of the space itself.*

Proof. Let A be a subset of $\mathfrak{E}_c^\omega$ such that $int_{\mathfrak{E}_c^\omega}(A) \neq \emptyset$. Then A contains a subset of the form $\{(x_1, \ldots, x_n)\} \times \mathfrak{E}_c^\omega$. $\square$

Corollary 1.35 ([5, Corollary 3.6]). *$\mathfrak{E}_c$ is not homeomorphic to $\mathfrak{E}_c^\omega$.*

The space $\mathfrak{E}_c^\omega$ also has its own characterization but we will not mention it (see [5]), we will only mention the characterization of its factors.

Theorem 1.36 ([5, Theorem 6.5]). *E is an $\mathfrak{E}_c^\omega$-factor if and only if E is an AZD complete space.*

1.3 Hyperspaces

In this section we will study some properties of hyperspaces such as metrizability, compactness and dimension zero.

1.3.1 Vietoris topology

Let X be a topological space. We define $\mathcal{CL}(X) \subset \wp(X)$ as the set of all closed nonempty subsets of X.

For $n \in \mathbb{N}$ and subsets $U_1, \ldots, U_n$ of a topological space X, we denote by $\langle U_1, \ldots, U_n \rangle$ the collection

$$\left\{ F \in \mathcal{CL}(X) : F \subset \bigcup_{k=1}^{n} U_k \text{ and } F \cap U_k \neq \emptyset \text{ for } k \leq n \right\}.$$

If $n = 1$ we have

$$\langle U_1 \rangle = \{F \in \mathcal{CL}(X) : F \subset U_1\} \text{ and}$$

If $n = 2$ and $U_1 = X$ we have

$$\langle X, U_2 \rangle = \{F \in \mathcal{CL}(X) : F \cap U_2 \neq \emptyset\}.$$

Usually $\mathcal{CL}(X)$ is endowed with the topology known as the **Vietoris Topology**, having as its canonical base all the sets of the form $\langle U_1, \ldots, U_n \rangle$ where U_k is a non-empty open subset of X for each $k \leq n$.

Remark 1.37. *If U is a closed subset of X, then $\langle U \rangle$ and $\langle U, X \rangle$ are closed subsets of $\mathcal{CL}(X)$. This is because $\langle U \rangle = \mathcal{CL}(X) \setminus \langle X, X \setminus U \rangle$ and $\langle X, U \rangle = \mathcal{CL}(X) \setminus \langle X \setminus U \rangle$.*

It follows that if $U_1 \ldots U_n$ are closed subsets of X, then $\langle U_1, \ldots U_n \rangle$ is a closed subset in $\mathcal{CL}(X)$ since that $\langle U_1, \ldots U_n \rangle = \langle \bigcup_{k \leq n} U_k \rangle \cap \langle U_1, X \rangle \cap \ldots \cap \langle U_n, X \rangle$. The following subsets of $\mathcal{CL}(X)$ are what we will study in this thesis.

Definition 1.38. *Let X be a topological space. We define*

- $\mathcal{F}_n(X) = \{A \in \mathcal{CL}(X) : |A| \leq n\}$ *for each* $n \in \mathbb{N}$,
- $\mathcal{F}(X) = \{A \in \mathcal{CL}(X) : |A| < \omega\}$,

- $\mathcal{K}(X) = \{F \in \mathcal{CL}(X) : F \text{ is a compact subset of } X\}$.

The hyperspace $\mathcal{F}_n(X)$ is also known as n-th **symmetric product**. Notice that for each $n \in \mathbb{N}$, $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$, and $\mathcal{F}(X) = \bigcup_{n\in\omega} \mathcal{F}_n(X)$.

Proposition 1.39 ([18, Proposition 2.4])**.** *Let $n \in \mathbb{N}$, and let X be a crowded space. Then $\mathcal{F}_n(X)$ is a closed and nowhere dense subset of $\mathcal{F}_{n+1}(X)$, of $\mathcal{F}(X)$ and $\mathcal{K}(X)$*

Proposition 1.40. *Let U be a proper open subset of X, then*

1. *$\mathcal{K}(U)$ is a proper open subset of $\mathcal{K}(X)$.*

2. *$\mathcal{F}(U)$ is a proper open subset of $\mathcal{F}(X)$.*

3. *$\mathcal{F}_n(U)$ is a proper open subset of $\mathcal{F}_n(X)$.*

Proof. We are going to show 1; the other items are shown in an analogous way. Let $F \in \mathcal{K}(U)$, then $F \subset U$. Since F is a compact and X is metrizable, there exists an open subset V of X such that $F \subset V \subset cl_X(V) \subset U$. We claim that $\langle V\rangle \subset \mathcal{K}(U)$. Let $H \in \langle V\rangle$, then $H \subset V \subset U$. Therefore $\mathcal{K}(U)$ is an open subset of $\mathcal{K}(X)$. On the other hand, given that $X \setminus U \neq \emptyset$, then $\{x\} \in \mathcal{K}(X) \setminus \mathcal{K}(U)$ (where $x \in X \setminus U$). That is, $\mathcal{K}(U)$ is a proper open subset of $\mathcal{K}(X)$. $\square$

Corollary 1.41. *Let B be a closed subset of X, then*

1. *$\mathcal{K}(B)$ is a closed subset of $\mathcal{K}(X)$.*

2. *$\mathcal{F}(B)$ is a closed subset of $\mathcal{F}(X)$.*

3. *$\mathcal{F}_n(B)$ is a closed subset of $\mathcal{F}_n(X)$.*

Proof. We are going to show 1; the other items are shown in an analogous way. Let B be a closed subset of X. We are going to show that $\mathcal{K}(X)\setminus\mathcal{K}(B)$, is an open subsets of $\mathcal{K}(X)$. Let $K \in \mathcal{K}(X) \setminus \mathcal{K}(B)$, then $K \cap B = \emptyset$. Since X is a metric space, there exists an open subset U of X such that $K \subset U \subset X \setminus B$. Let $\langle U\rangle$, then $\langle U\rangle$ is an open subset of $\mathcal{K}(X)$ such that $K \in \langle U\rangle$ and $\langle U\rangle \cap \mathcal{K}(B) = \emptyset$. This implies that $\mathcal{K}(X) \setminus \mathcal{K}(B)$ is an open subset of $\mathcal{K}(X)$. $\square$

Proposition 1.42. *If A admits a closed embedding into X, then for each $n \in \mathbb{N}$, $\mathcal{F}_n(A)$ admits a closed embedding into $\mathcal{F}_n(X)$, and $\mathcal{F}(A)$ admits a closed embedding into $\mathcal{F}(X)$.*

Proof. We will only show that $\mathcal{F}_n(A)$ admits a closed embedding into $\mathcal{F}_n(X)$, the proof that $\mathcal{F}(A)$ admits a closed embedding into $\mathcal{F}(X)$ is done in an analogous way. Let $e\colon A \to X$ be a closed embedding. Let $e_1 : \mathcal{F}_n(A) \to \mathcal{F}_n(X)$ given by $e_1(\{x_1, \dots, x_k\}) = \{e(x_1), \dots, e(x_k)\}$, by the continuity of e we have that $e_1^{\leftarrow}[\langle U_1, \dots, U_k\rangle] = \langle e^{\leftarrow}[U_1], \dots, e^{\leftarrow}[U_k]\rangle$. Therefore e_1 is a continuous function. On the other hand if $E, F \in \mathcal{F}_n(A)$ and $E \neq F$, then there exists $x \in E$ and $y \in F$ such that $x \neq y$ and $e(x) \neq e(y)$. Therefore $e_1(F) \neq e_1(E)$. This implies that e_1 is an injective function. Since $e[A]$ is a closed subset of X then by 1.41 $\mathcal{F}_n(e[A])$ is a closed subset of $\mathcal{F}_n(X)$. On the other hand, let us note that $e_1[\mathcal{F}_n(A)] = \mathcal{F}_n(e[A])$. Therefore $e_1[\mathcal{F}_n(A)]$ is closed subset of $\mathcal{F}_n(X)$. Thus e_1 is a closed embedding.

□

Proposition 1.43 ([18, Proposition 2.4.3])**.** *For each $n \in \mathbb{N}$ the function $q_n : X^n \to \mathcal{F}_n(X)$ defined by $q_n(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$ is continuous and perfect.*

From now on, in this thesis the symbol q_n will denote the function defined in Proposition 1.43. Note that q_1 is a homeomorphism, thus $X \approx \mathcal{F}_1(X)$. On the other hand with Proposition 1.43 we obtain that if X is a compact space, then $\mathcal{F}_n(X)$ is a compact space for all $n \in \mathbb{N}$. Later we will see that if X is a compact space, then $\mathcal{K}(X)$ is a compact space (Theorem 1.48). The next result tells us when the spaces $\mathcal{F}_n(X)$, $\mathcal{F}(X)$ and $\mathcal{K}(X)$ are connected. This result is useful in section 2 of Chapter 2

Theorem 1.44 ([18, Theorem 4.10])**.** *For a topological space X the following statements are equivalent.*

1. *X is a connected space;*
2. *$\mathcal{F}(X)$ is a connected space;*
3. *For all $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is a connected space;*
4. *There exists $n \in \mathbb{N}$ such that $\mathcal{F}_n(X)$ is a connected space;*
5. *$\mathcal{K}(X)$ is a connected space.*

Remember that in this thesis all spaces are considered metric, so it is natural to know when $\mathcal{CL}(X)$ is metric. The following Theorem tells us when $\mathcal{CL}(X)$ is normal.

Theorem 1.45 ([25, Velicko]). *$\mathcal{CL}(X)$ is normal if and only if X is a compact space.*

Theorem 1.45 implies that $\mathcal{CL}(X)$ is a metric space if and only if X is a compact metric space. This implies that $\mathcal{CL}(\mathfrak{E})$ and $\mathcal{CL}(\mathfrak{E}_c)$ are not a metric spaces since $\mathfrak{E}$ and $\mathfrak{E}_c$ are not compact spaces. But for the case of $\mathcal{K}(X)$ we have the following result:

Theorem 1.46 ([18, Proposition 4.1]). *Let X be a separable and metrizable space. Then*

- *$\mathcal{K}(X)$ is a separable and metrizable space.*
- •• *If X is a Polish space, then $\mathcal{K}(X)$ is a Polish space.*

By Theorem 1.46 $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are metrizable spaces and in the case that X is a Polish space then $\mathcal{F}_n(X)$ is a Polish space. Therefore in this thesis we will only study the hyperspaces $\mathcal{F}_n(X)$, $\mathcal{F}(X)$ and $\mathcal{K}(X)$. Another important property that we must know is when $\mathcal{K}(X)$ is a zero-dimensional space, since every AZD space has a zero-dimensional space associated with it.(Definition 1.8)

Theorem 1.47 ([18, Proposition 4.13.1]). *X is a zero-dimensional space if and only if $\mathcal{K}(X)$ is a zero-dimensional space.*

Since all zero-dimensional spaces are almost zero-dimensional, by Theorem 1.36 and Proposition 1.46 we have that if X is a complete and zero-dimensional space, then $\mathcal{K}(X)$, and $\mathcal{F}_n(X)$ are factors of $\mathfrak{E}_c^\omega$. In the next chapter we will show that this result is true for every complete AZD space.

Finally we present a Theorem about compactness in $\mathcal{K}(X)$. It is important in Chapter 3 of this work.

Theorem 1.48 ([18, Theorem 4.2]). *X is a compact space if and only if $\mathcal{K}(X)$ is a compact space.*

The following result is immediate from the previous Theorem.

Corollary 1.49. *Let $V_1, \ldots, V_n$ be compact subsets of X, then $\langle V_1, \ldots, V_n\rangle$ is a compact subset of $\mathcal{K}(X)$, and therefore $\langle V_1, \ldots, V_n\rangle \cap \mathcal{F}_n(X)$ is a compact subset of $\mathcal{F}_n(X)$ for each $n \in \mathbb{N}$.*

Chapter 2

Almost Zero-Dimesion and Cohesion in Hyperspaces

In this chapter we will talk about the properties of almost zero-dimensionality and cohesion in hyperspaces.

2.1 AZD Hyperspaces.

We begin by studying the property of almost zero-dimensionality in the different hyperspaces defined in Chapter 1.

Proposition 2.1. *Let X be a metric AZD space. Then $\mathcal{CL}(X)$ is a metric and AZD space if and only if X is a compact and zero-dimensional space.*

Proof. If $\mathcal{CL}(X)$ is a metric space by Theorem 1.45 X is a compact. By Theorem 1.45 $\mathcal{CL}(X)$ is a compact and AZD space. By Proposition 1.11, $\mathcal{CL}(X)$ is a zero-dimensional space. Therefore X is a zero-dimensional space, by Theorem 1.47. On the other hand if X is a compact, then $\mathcal{CL}(X) = \mathcal{K}(X)$. By Theorem 1.47 and Theorem 1.48, $\mathcal{CL}(X)$ is a compact and zero-dimensional space. □

This result tells us that the class of spaces such that $\mathcal{CL}(X)$ is an AZD metric space coincides with the class of zero-dimensional compact metric spaces. For the case of $\mathcal{K}(X)$ we have the following result. This result is fundamental in the remainder of the thesis.

Proposition 2.2. *For a topological space X the following statements are equivalent.*

1. *X is an AZD space.*

2. *$\mathcal{K}(X)$ is an AZD space.*

3. *If $\mathcal{F}_1(X) \subset \mathcal{A} \subset \mathcal{K}(X)$, then $\mathcal{A}$ is an AZD space.*

Proof. The implication (2) $\Rightarrow$ (3) is obvious, and (3) $\Rightarrow$ (1) follows from the fact that $\mathcal{F}_1(X)$ is homeomorphic to X.
(1) $\Rightarrow$ (2) : We are going to prove that $\mathcal{K}(X)$ satisfies the conditions of Remark 1.9. Let $\mathcal{W}$ be a topology which witnesses the almost zero-dimensionality of X. Consider the space $Y = (X, \mathcal{W})$. As $\mathcal{W}$ is coarser than the topology of X, then $\mathcal{K}(X) \subset \mathcal{K}(Y)$. Let $(Z, \mathcal{W}_0)$ be the space $\mathcal{K}(X)$ considered with the topology inherited as a subspace of $\mathcal{K}(Y)$. Since $\langle V_1, \ldots, V_n\rangle \cap Z$ is an open subset of $\mathcal{K}(X)$, when $V_1, \ldots, V_n$ are elements of $\mathcal{W}$, we have that the topology $\mathcal{W}_0$ of Z is coarser than the topology of the $\mathcal{K}(X)$. Moreover, by Proposition 1.47, $(Z, \mathcal{W}_0)$ is zero-dimensional. Now, we are going to prove that each element in $\mathcal{K}(X)$ has a local neighborhood base consisting of subsets that are closed in $(Z, \mathcal{W}_0)$. Let $F \in \mathcal{K}(X)$ and let $\mathcal{U} = \langle U_1, \ldots, U_m\rangle$ be a canonical open subset of $\mathcal{K}(X)$ such that $F \in \mathcal{U}$. For each $x \in F$ there is a neighborhood V_x of x in X such that $x \in V_x \subset \bigcap\{U_j : x \in U_j\}$ and V_x is closed in Y. Then $\{int_X(V_x) : x \in F\}$ is an open cover of F in X. As F is a compact subset of X, there exists $x_1, \ldots, x_k \in F$ such that $F \subset \bigcup_{i=1}^k V_{x_i}$. For each $i \leq m$, let $y_i \in F \cap U_i$. Note that $F \in \langle V_{x_1}, \ldots, V_{x_k}, V_{y_1}, \ldots, V_{y_m}\rangle$. Let us see that $\mathcal{V}_{\mathcal{U}} := \langle V_{x_1}, \ldots, V_{x_k}, V_{y_1}, \ldots, V_{y_m}\rangle \cap \mathcal{K}(X) \subset \mathcal{U}$. Indeed, let $H \in \mathcal{V}_{\mathcal{U}}$. Then $H \subset \bigcup_{i=1}^k V_{x_i} \cup \bigcup_{j=1}^m V_{y_j}$ and $H \cap V_z \neq \emptyset$ for $z \in \{x_1, \ldots, x_k, y_1, \ldots, y_m\}$. By the choice of V_{x_i} and V_{y_j}, we have $\bigcup_{i=1}^k V_{x_i} \cup \bigcup_{j=1}^m V_{y_j} \subset \bigcup_{i \leq m} U_i$, and for each $j \leq m$ there exists a $z \in \{x_1, \ldots, x_k, y_1, \ldots, y_m\}$ such that $V_z \subset U_j$. Then $H \subset \bigcup_{i \leq m} U_i$ and $H \cap U_i \neq \emptyset$ for $i \leq m$. Thus $\mathcal{V}_{\mathcal{U}} \subset \mathcal{U}$, moreover $\langle\langle V_{x_1}, \ldots, V_{x_k}, V_{y_1}, \ldots, V_{y_m}\rangle$ is a closed subset in $\mathcal{K}(Y)$ by Remark 1.37 so $\langle V_{x_1}, \ldots, V_{x_k}, V_{y_1}, \ldots, V_{y_m}\rangle \cap Z$ is a closed subset of Z. Therefore, the collection of all the sets $\mathcal{V}_{\mathcal{U}}$ where $\mathcal{U}$ is a canonical open set of $\mathcal{K}(X)$ containing F, form a local neighborhood base of F consisting of closed sets in $(Z, \mathcal{W}_0)$. Hence, by Remark 1.9, $\mathcal{K}(X)$ is an AZD space.

□

Corollary 2.3. *Let X be an AZD space. Suppose that $\mathcal{W}$ is a witness topology of the almost zero-dimensionality of X. If $U_1, \dots, U_n$ are closed sets of $(X, \mathcal{W})$, then $\langle U_1, \dots, U_n \rangle \cap \mathcal{K}(X)$ is a C-set of $\mathcal{K}(X)$.*

Proof. Let Z as in the proof of $2 \Rightarrow 1$ in Proposition 2.2 and $U_1, \dots, U_k$ closed subsets of $(X, \mathcal{W})$. Since $\langle U_1, \dots U_k \rangle \cap Z$ is a closed subset in Z and Z is a zero-dimensional space, then $\langle U_1, \dots, U_k \rangle \cap Z$ is a C-set of Z. By Remark 1.9 $\langle U_1, \dots, U_k \rangle \cap Z$ is a C-set of $\mathcal{K}(X)$. □

Corollary 2.4. *Let X be an AZD space. Suppose that $\mathcal{W}$ is a witness topology of the almost zero-dimensionality of X, then the topology of $\mathcal{F}_n(X, \mathcal{W})$, $(\mathcal{F}(X, \mathcal{W}))$, witnesses the almost zero-dimensionality of $\mathcal{F}_n(X)$, $(\mathcal{F}(X)$ respectively).*

Proof. Let Z be as in the proof of $2 \Rightarrow 1$ in Proposition 2.2. By item 2 of Proposition 1.11, $\mathcal{F}_n(X) \cap Z$ (and $\mathcal{F}(X) \cap Z$) witnesses the almost zero-dimensionality of $\mathcal{F}_n(X)$, ($\mathcal{F}(X)$ respectively). Note that $\mathcal{F}_n(X, \mathcal{W}) \approx \mathcal{F}_n(X) \cap Z$ and $\mathcal{F}(X, \mathcal{W}) \approx \mathcal{F}(X) \cap Z$ by the proof of Proposition 2.2. □

Since every AZD space has dimension less than or equal that one, by Proposition 2.2, $dim(\mathcal{K}(X)) \leq 1$. By Theorem 1.47 we have that if $dim(\mathcal{K}(X)) = 0$ if and only if $dim(X) = 0$. This implies that $dim(\mathcal{K}(X)) = 1$ if and only if $dim(X) = 1$. This argument implies the following Corollary.

Corollary 2.5. *Let X be an AZD space, then $dim(\mathcal{K}(X)) \leq 1$ and $dim(\mathcal{K}(X)) = 1$ if and only if $dim(X) = 1$.*

The omission of the hypothesis of almost zero-dimensionality on X in the previous corollary produces the following natural question.

Question 2.6. *Is there a space X that is not AZD such that $dim(X) = dim(\mathcal{K}(X)) = 1$?*

The answer to this question is affirmative, and was given Roman Pol in a personal communication. In the next section of this chapter we present an example. We finish this section with the following result.

Proposition 2.7. *Let X be an almost zero-dimensional space, then for each $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is a C-set of $\mathcal{CL}(X)$, $\mathcal{K}(X)$, $\mathcal{F}(X)$, and $\mathcal{F}_{n+1}(X)$.*

Proof. Let $m \in \mathbb{N}$ be fixed, we will show that $\mathcal{F}_m(X)$ is a C-set of $\mathcal{CL}(X)$. Let $\mathcal{W}$ be a witness topology of X and d a metric for X. There exists a dense embedding $e : (X, \mathcal{W}) \to 2^\omega$. Since 2^ω is a compact space for each $\epsilon > 0$, there exists a finite partition $\mathcal{U}_\epsilon$ of clopen subsets of 2^ω whose diameter is less than ϵ. Let $\mathcal{C}_n^k = \{\langle X \cap e^{\leftarrow}[U_1], \dots, X \cap e^{\leftarrow}[U_k]\rangle : U_1, \dots, U_k \in \mathcal{U}_{1/n}, \textit{and } k \leq m\}$, and let $\mathcal{C}_n = \bigcup_{k \leq m} \mathcal{C}_n^k$. Note that $X \cap e^{\leftarrow}[U_j]$ is a clopen subset of X for each $U_j \in \mathcal{U}_{1/n}$ and $X = \bigcup_{U_j \in \mathcal{U}_{1/n}} (X \cap e^{\leftarrow}[U_j])$, so that $\mathcal{C}_n$ is a cover of $\mathcal{F}_m(X)$ for each $n \in \mathbb{N}$. Since $\langle X \cap e^{\leftarrow}[U_1], \dots, X \cap e^{\leftarrow}[U_k]\rangle$ is a clopen subset of $\mathcal{CL}(X)$ for each $U_1, \dots, U_k \in \mathcal{U}_{1/n}$ and $\mathcal{U}_{1/n}$ is finite. We see that $\mathcal{D}_n = \bigcup \mathcal{C}_n$ is a clopen subset of $\mathcal{CL}(X)$ such that $\mathcal{F}_m(X) \subset \mathcal{D}_n$ for every $n \in \mathbb{N}$. We claim that $\mathcal{F}_m(X) = \bigcap_{n \in \mathbb{N}} \mathcal{D}_n$. Since for each $n \in \mathbb{N}$, we have $\mathcal{F}_m(X) \subset \mathcal{D}_n$, we conclude that $\mathcal{F}_m(X) \subset \bigcap_{n \in \mathbb{N}} \mathcal{D}_n$. On the other hand, if $H \in \mathcal{CL}(X) \setminus \mathcal{F}_m(X)$, then $|H| > m$. Let $x_1, \dots, x_{m+1} \in H$ such that $x_j \neq x_i$ if $i \neq j$ and $\epsilon = (\min\{d(x_i, x_j) : x_i, x_j \in \{x_1, \dots, x_{m+1}\}\})/2$. Then if $V \in \mathcal{U}_\epsilon$ we have $|V \cap \{x_1, \dots, x_{m+1}\}| \leq 1$. Let $i \in \mathbb{N}$ such that $1/i < \epsilon$, then if $\langle U_1, \dots, U_l\rangle \in \mathcal{C}_{1/i}$, we have that $H \notin \langle U_1, \dots, U_l\rangle$, for each $l \leq m$, and so $H \notin \mathcal{D}_{1/i}$. Therefore $H \notin \bigcap \mathcal{D}_n$. It follows that $\mathcal{F}_m(X) = \bigcap_{n \in \mathbb{N}} \mathcal{D}_n$.
To show that $\mathcal{F}_n(X)$ is a C-set of $\mathcal{F}_{n+1}(X)$, $\mathcal{F}(X)$ and $\mathcal{K}(X)$, note that $\mathcal{F}_{n+1}(X) \cap \mathcal{D}_m$, $\mathcal{F}(X) \cap \mathcal{D}_m$, and $\mathcal{K}(X) \cap \mathcal{D}_m$ are clopen subsets of $\mathcal{F}_{n+1}(X)$, $\mathcal{F}(X)$ and $\mathcal{K}(X)$ respectively and $\mathcal{F}_n(X) = \bigcap_{m \in \mathbb{N}} (\mathcal{D}_m \cap \mathcal{F}_{n+1}(X))$, $\mathcal{F}_n(X) = \bigcap_{m \in \mathbb{N}} (\mathcal{D}_m \cap \mathcal{F}(X))$, and $\mathcal{F}_n(X) = \bigcap_{m \in \mathbb{N}} (\mathcal{D}_m \cap \mathcal{K}(X))$. □

Since the C-sets are closed, then $\mathcal{F}(X)$ cannot be C-set of $\mathcal{K}(X)$ or $\mathcal{CL}(X)$. Also $\mathcal{K}(X)$ cannot be a C-set of $\mathcal{CL}(X)$, since $\mathcal{F}(X)$ is a dense subset of $\mathcal{K}(X)$ and of $\mathcal{CL}(X)$, and $\mathcal{K}(X)$ is a dense subset of $\mathcal{CL}(X)$.

2.2 Cohesive Hyperspaces

In this section we will see how cohesion behaves in the different hyperspaces defined in Chapter 1. The following result relates the cohesion property between a space X and its symmetric products.

Proposition 2.8. *Let $n \in \mathbb{N}$ and let X be a space that is $\{A_s : s \in S\}$-cohesive. Then $\mathcal{F}_n(X)$ is $\{q_n[A_{s_1} \times \ldots \times A_{s_n}] : s_1, \dots, s_n \in S\}$-cohesive.*

Proof. Suppose that $\mathcal{F}_n(X)$ is not $\{q_n[A_{s_1} \times \ldots \times A_{s_n}] : s_1, \dots, s_n \in S\}$-cohesive. Then there exists $F \in \mathcal{F}_n(X)$ and a local base β of F, such that any $\mathcal{U} \in \beta$ contains a non-empty proper clopen subset of some element of

$\{q_n[A_{s_1} \times \ldots \times A_{s_n}] : s_1, \ldots, s_n \in S\}$. Let us suppose that $F = \{x_1, \ldots, x_k\}$ with $x_j \neq x_i$ for each $i, j \in \{1, \ldots, k\}$. Let $\beta_{x_1}, \ldots, \beta_{x_k}$ be local bases of the points $x_1, \ldots, x_k$ respectively such that if $i \neq j$, $U_j \in \beta_{x_j}$ and $U_i \in \beta_{x_i}$, then $U_j \cap U_i = \emptyset$. Let $x = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$ in X^n, where $x_k = x_{k+1} = \ldots = x_n$. Note that $\beta_0 = \{U_1 \times \ldots \times U_{k+1} \ldots \times U_n : U_j \in \beta_{x_j}\ j \leq k \textit{ and } U_k = U_{k+1} = \ldots = U_n\}$ is a local base at x. Let $V_j \in \beta_{x_j}$ be fixed for each $j \in \{i, \ldots, k\}$. Note that $F \in \mathcal{N} = \langle V_1, \ldots, V_k \rangle$ and that $x \in V_1 \times \ldots \times V_n$. By our assumption there are $s_1, \ldots, s_n \in S$, and an open subset $\mathcal{U} \in \beta$, and a non-empty proper clopen subset $\mathcal{V}$ of $q_n[A_{s_1} \times \ldots \times A_{s_n}]$ such that $\mathcal{V} \subset \mathcal{U} \subset \mathcal{N}$.

Let $g = q_n \restriction A_{s_1} \times \ldots \times A_{s_n}$. As g is continuous, we have that $g^{\leftarrow}[\mathcal{V}]$ is a clopen subset of $A_{s_1} \times \ldots \times A_{s_n}$. Let $C = g^{\leftarrow}[\mathcal{V}] \cap (V_1 \times \ldots \times V_n)$.

It is clear that C is an open subset of $A_{s_1} \times \ldots \times A_{s_n}$, because $g^{\leftarrow}[\mathcal{V}]$ and $V_1 \times \ldots \times V_n$ are open. To see that is closed let us consider a sequence $\{(y_1^m, \ldots, y_n^m) : m \in \mathbb{N}\}$ of points of C such that $(y_1^m, \ldots, y_n^m) \to (y_1, \ldots, y_n)$. Since q_n is continuous, the sequence $\{q_n((y_1^m, \ldots, y_n^m)) = g((y_1^m, \ldots, y_n^m)) : m \in \omega\}$ converges to $q_n(y_1, \ldots, y_n)$. Note that for every $m \in \omega$, we have that $q_n((y_1^m, \ldots, y_n^m)) \in \mathcal{V}$. Since $\mathcal{V}$ is closed in $q_n[A_{s_1} \times \ldots \times A_{s_n}]$, we have that $q_n(y_1 \ldots, y_n) \in \mathcal{V}$. Hence $(y_1, \ldots, y_n) \in q_n^{\leftarrow}[\mathcal{V}]$. On the other hand, as $\mathcal{V} \subset N$, $y_j \in V_j$ as $y_j^m \in V_j$ for $m \in \mathbb{N}$. Thus C is clopen in $A_{s_1} \times \ldots \times A_{s_n}$. This is a contradiction, to remark 1.15, so X^n is $\{A_{s_1} \times \ldots \times A_{s_n} : s_1, \ldots, s_n \in S\}$-cohesive.

□

Lemma 2.9. *Let X be a space that is $\{A_s : s \in S\}$-cohesive, witnessed by a base $\mathcal{B}$ of open sets. Consider the following collection of subsets of $\mathcal{F}(X)$:*

$$\mathcal{A} = \{q_n[A_{s_1} \times \ldots \times A_{s_n}] : n \in \mathbb{N}, \forall i \in \{1, \ldots, n\}\, (s_i \in S)\}$$

Then $\mathcal{F}(X)$ is $\mathcal{A}$-cohesive, and the open sets that witness this may be taken from the collection $\mathcal{C} = \{\langle U_1, \ldots, U_n \rangle : \forall i \in \{1, \ldots, n\}\, (U_i \in \mathcal{B})\}$.

Proof. Let $F \in \mathcal{F}(X)$, suppose that $F = \{x_1, \ldots, x_k\}$ with $x_j \neq x_i$ if $i \neq j$. For each $j \in \{1, \ldots, n\}$, let $V_j \in \mathcal{B}$ with $x_j \in V_j$. We can assume that if $i \neq j$ then $V_i \cap V_j = \emptyset$. Let $V = \langle V_1, \ldots, V_k \rangle$, note that $F \in V$. We claim that V does not contain clopen subsets of any element of $\mathcal{A}$. Suppose there are $s_1, \ldots, s_m \in S$ such that V contains a non-empty proper clopen subset O of $q_m[A_{s_1} \times \ldots \times A_{s_m}]$. As $V \cap \mathcal{F}_{k-1}(X) = \emptyset$, it follows that $m \geq k$. If $i \in (k, m]$, we define $V_i = V_k$. In this way, $V = \langle V_1, \ldots, V_m \rangle$. Thus $O \cap \mathcal{F}_m(X)$

is a clopen subset of $\mathcal{F}_m(X)$. Let $x = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_m)$ where $x_k = x_{k+1} = \ldots = x_m$. Note that $x \in V_1 \times \ldots \times V_m$. Let $g_m = q_m \restriction A_{s_1} \times \ldots \times A_{s_m}$ and $C = g_m^{\leftarrow}[O \cap \mathcal{F}_m(X)] \cap (V_1 \times \ldots \times V_m)$. By Proposition 2.8, C is a clopen subset of $A_{s_1} \times \ldots \times A_{s_m}$ such that $C \subset V_1 \times \ldots \times V_m$; this is a contradiction by remark 1.15. $\square$

With the previous Lemma we can prove the following

Proposition 2.10. *If X a cohesive space, then $\mathcal{K}(X)$ is a cohesive space.*

Proof. Let $F \in \mathcal{K}(X)$, then there exists $x_1, \ldots, x_n \in F$ and $W_1, \ldots, W_n$ neighborhoods such that $x_i \in W_i$ and W_i does not contain clopen non-empty subsets of X. Let $\mathcal{W} = \langle W_1, \ldots, W_n \rangle$, then $F \in \mathcal{W}$. We claim that $\mathcal{W}$ does not contain clopen non-empty subsets of $\mathcal{K}(X)$. Suppose there exists a clopen subset O of $\mathcal{K}(X)$ such that $O \subset \mathcal{W}$. Then $O \cap \mathcal{F}(X)$ is a clopen subset and $\mathcal{W} \cap \mathcal{F}(X)$ is a neighborhood of $H = \{x_1, \ldots, x_n\}$ in $\mathcal{F}(X)$ such that $O \cap \mathcal{F}(X) \subset \mathcal{W} \cap \mathcal{F}(X)$. Furthermore, for each $n \in \mathbb{N}$, $O \cap \mathcal{F}_n(X) \subset O \cap \mathcal{F}(X) \subset \mathcal{W} \cap \mathcal{F}(X)$. For each $n \in \mathbb{N}$ we have the equality$q_n[X^n] = \mathcal{F}_n(X)$, let $\mathcal{A} = \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$, by Lemma 2.9 $\mathcal{F}(X)$ is $\{\mathcal{F}_n(X) : n \in \mathbb{N}\}$-cohesive, and $O \cap \mathcal{F}(X) \in \mathcal{C}$. So for every $n \in \mathbb{N}$, $O \cap \mathcal{F}_n(X) = \emptyset$, this implies that $O \cap \mathcal{F}(X) = \emptyset$, which is a contradiction. Therefore $\mathcal{W}$ does not contain clopens subsets of $\mathcal{K}(X)$. $\square$

We present an alternative method to show that the spaces $\mathcal{K}(X)$, $\mathcal{F}(X)$, and $\mathcal{F}_n(X)$ are cohesive using the concept of one-point connectification. The concept of one-point connectification relates the properties of cohesion and almost zero-dimensionality.

Definition 2.11. *A **one-point connectification** of a space X is a connected extension Y of the space such that the remainder $Y \setminus X$ is a singleton.*

Example 2.12. *Let p be a point outside $\mathfrak{E}_c$, consider $\mathfrak{E}_c^+ = \mathfrak{E}_c \cup \{p\}$ whose neighbourhoods of $\{p\}$ are the complements of closed bounded sets of $\mathfrak{E}_c$. We claim that $\mathfrak{E}_c^+$ is metric separable connected space.*

We will prove that $\mathfrak{E}_c^+$ is a metric space. It's enough prove that $\mathfrak{E}_c^+$ is regular second countable. Consider the following sets $B_m = \{x \in \mathfrak{E}_c : \|x\| \leq m\}$, then B_m is a closed bounded of $\mathfrak{E}_c$ for each $m \in \mathbb{N}$. On the other hand if U is an open subset of $\mathfrak{E}_c^+$ such that $p \in U$, then there exists a subset B that is closed and bounded such that $U = (\mathfrak{E}_c \setminus B) \cup \{p\}$. Since B is bounded there exists $m \in \mathbb{N}$ such that $B \subset B_m$, this implies that $(\mathfrak{E}_c \setminus B_m) \cup \{p\} \subset U$.

Therefore $\beta_0 = \{(\mathfrak{E}_c \setminus B_n) \cup \{p\} : n \in \mathbb{N}\}$ *is a local countable base of* p. *Therefore if* $\beta_1 = \{B(x, 1/n) : x \in \mathfrak{E}_c, n \in \mathbb{N}\}$ *where* $B(x, 1/n) = \{p \in \mathfrak{E}_c : d(x,p) < 1/n\}$, *then* $\beta = \beta_0 \cup \beta_1$ *is a countable base of* $\mathfrak{E}_c^+$. *We will now prove that* X *is regular. Let* $x \in \mathfrak{E}_c^+$, *and let* U *be an open subset of* $\mathfrak{E}_c^+$ *such that* $x \in U$. *Without loss of generality we can assume that* $U \in \beta$. *Note that if* $x \in \mathfrak{E}_c$, *then there exists an open subset* W *of* $\mathfrak{E}_c^+$ *such that* $x \in W \subset cl_{\mathfrak{E}_c^+}(W) \subset U$. *If* $x = p$, *then* $U = (\mathfrak{E}_c \setminus B_n) \cup \{p\}$ *for some* $n \in \mathbb{N}$. *Note that* $B_n \subset \{x \in \mathfrak{E}_c : \|x\| < n+1\} \subset B_{n+1}$, *then*

$$(\mathfrak{E}_c \setminus B_{n+1}) \cup \{p\} \subset (\mathfrak{E}_c \setminus \{x \in \mathfrak{E}_c : \|x\| < n+1\}) \cup \{p\} \subset (\mathfrak{E}_c \setminus B_n) \cup \{p\}.$$

Note that $(\mathfrak{E}_c \setminus \{x \in \mathfrak{E}_c : \|x\| < n+1\}) \cup \{p\}$ *is a closed subset of* $\mathfrak{E}_c^+$. *Therefore* $cl_{\mathfrak{E}_c^+}(\mathfrak{E}_c \setminus B_{n+1}) \cup \{p\} \subset (\mathfrak{E}_c \setminus \{x \in \mathfrak{E}_c : \|x\| < n+1\}) \cup \{p\}$. *Thus* $\mathfrak{E}_c^+$ *is a second countable and regular space.*

Finally, we will show that $\mathfrak{E}_c^+$ *is connected. Let* U *be a clopen subset of* $\mathfrak{E}_c^+$, *then* $V = \mathfrak{E}_c^+ \setminus U$ *is a clopen subset of* $\mathfrak{E}_c^+$. *Since* $\mathfrak{E}_c^+ = U \cup V$, *then* $p \in U$ *or* $p \in V$. *Suppose that* $p \in U$, *then* V *is a clopen subset of* $\mathfrak{E}_c$, *and by Lemma 1.13* V *is an unbounded set of* $\mathfrak{E}_c$. *On the other hand as* $p \in U$, *then there exists a bounded set* W *such that* $p \in (X \setminus W) \cup \{p\} \subset U$. *This implies that* $V \subset W$. *Therefore* V *is a bounded set of* $\mathfrak{E}_c$, *which is a contradiction.*

The following result gives necessary and sufficient conditions for a metric separable space X to have a metric and separable one-point connectification.

Theorem 2.13 (Knaster [14])**.** *Let* X *be a separable metric space. Then* X *has a one-point connectification* Y *which is metrizable and separable if and only if* X *is embeddable in a separable metric connected space* Z *as proper open subset of* Z.

Proposition 2.14. *If* X *has a metrizable and separable one-point connectification, then* $\mathcal{K}(X)$, $\mathcal{F}_n(X)$ *and* $\mathcal{F}(X)$ *each have a metric and separable one-point connectification.*

Proof. If Y is a metric and separable one-point connectification of X, then by Theorem 1.44 and Proposition 1.46 $\mathcal{K}(Y)$, $\mathcal{F}_n(Y)$ and $\mathcal{F}(Y)$ are metric and separable connected spaces. As X is a proper open subset of Y, then by Proposition 1.40 $\mathcal{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are proper open subsets of $\mathcal{K}(Y)$, $\mathcal{F}_n(Y)$ and $\mathcal{F}(Y)$, respectively. By Theorem 2.13 the spaces $\mathcal{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ each have, metric and separable one-point connectifications.

□

It is known that if a space admits a one-point connectification, then it is cohesive. Moreover if an almost zero-dimensional space is cohesive, then it admits a one point connectification (see [3, Proposition 5.4])

Corollary 2.15. *If X is a cohesive AZD space, then $\mathcal{F}_n(X), \mathcal{F}(X), \mathcal{K}(X)$ are cohesive AZD spaces.*

Proof. As X is a cohesive AZD space, it has a one-point connectification. By Proposition 2.14 $\mathcal{F}_n(X), \mathcal{F}(X), \mathcal{K}(X)$ have a one-point connectification. Thus $\mathcal{F}_n(X), \mathcal{F}(X), \mathcal{K}(X)$ are cohesive. Furthermore, by Proposition, 2.2 $\mathcal{F}_n(X), \mathcal{F}(X)$, and $\mathcal{K}(X)$ are AZD. □

Corollary 2.16. *$\mathcal{F}_n(\mathfrak{E}_{\mathrm{c}})$, $\mathcal{F}_n(\mathfrak{E})$, $\mathcal{F}(\mathfrak{E}_{\mathrm{c}})$, $\mathcal{F}(\mathfrak{E})$, $\mathcal{K}(\mathfrak{E}_{\mathrm{c}})$ and $\mathcal{K}(\mathfrak{E})$ are cohesive AZD spaces.*

Proof. This result follows from Corollary 2.15 and from the fact that $\mathfrak{E}_{\mathrm{c}}$ and $\mathfrak{E}$ are cohesive AZD spaces. □

The following result holds for any Hausdorff topological space.

Proposition 2.17. *If X has a one-point connectification, then $\mathcal{CL}(X)$, $\mathcal{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ each have a one-point connectification.*

Proof. Let $Y = \{p\} \cup X$ be a one-point connectification of X. Since Y is connected then by Theorem 1.44 $\mathcal{H}$ is connected if $\mathcal{H} \in \{\mathcal{CL}(Y), \mathcal{K}(Y), \mathcal{F}_n(Y)$ $\mathcal{F}(Y)\}$. Let $\mathcal{A} = \{F \in \mathcal{H} : p \in F\}$, and consider the space $\mathcal{Z} = \mathcal{H}/\mathcal{A}$. Note that $\mathcal{Z}$ is a connected spaces, since it is a continuous image of $\mathcal{H}$ and $\mathcal{Z} \setminus \{\mathcal{A}\}$ is homeomorphic to $\mathcal{CL}(X), \mathcal{K}(X), \mathcal{F}_n(X), \mathcal{F}(X)$ respectively. Therefore $\mathcal{Z}$ is a connected extension by a point of $\mathcal{CL}(X), \mathcal{K}(X), \mathcal{F}_n(X), \mathcal{F}(X)$ respectively. □

Note that the space $\mathcal{Z}$ in the proof of the Proposition 2.17 is not necessarily Hausdorff or metrizable, the behavior of the separation axioms and metrizability depend on the properties of the subset $\mathcal{A}$ in $\mathcal{H}$. The following two Corollaries follow from the previous Proposition.

Corollary 2.18. *If X has a one-point connectification, then $\mathcal{CL}(X)$, $\mathcal{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are cohesive spaces.*

Corollary 2.19. *$\mathcal{CL}(\mathfrak{E})$ and $\mathcal{CL}(\mathfrak{E}_c)$ are cohesive spaces.*

2.3 An Example by Roman Pol

In this part of the thesis we will study an example of a space Y that is not almost zero-dimensional but $dim(\mathcal{K}(Y)) = dim(Y) = 1$ This example answers question 2.6 asked in Section 1 of this chapter. This example was given by Roman Pol in personal communication with the author.

Let X be an AZD and cohesive space (for example $\mathfrak{E}$, $\mathfrak{E}_c$), then X has a one-point connectification (see Proposition in 5.4 [3]). We will give two examples of this type of spaces: the first one we will call Y and the other one P. Suppose that $Y = \{p\} \cup X$ where $p \notin X$. Since Y is connected is connected, it is not an AZD space.

Now let us consider $\mathfrak{E}_c^+$ of example 2.12 and $N = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Let

$$P = [\mathfrak{E}_c \times \{1/n : n \in \mathbb{N}\}] \cup \{(p, 0)\}$$

with the topology inherited from $\mathfrak{E}_c^+ \times N$.

Note that every $\mathfrak{E}_c \times \{1/n\}$ s clopen in P, hence $(p, 0)$ is a C-set in P. By item (3) from Proposition 1.11 $P \setminus \{(p, 0)\}$ is an AZD space, and P is a totally disconnected because if $x, y \in P$ such that $x \neq (p, 0)$ and $y \neq (p, 0)$ then there exists U and a clopen subset of P such that $x \in U$ and $y \notin U$, since $P \setminus \{(p, 0)\}$ is an AZD. On the other hand if $x = (p, 0)$, and $y = (x, 1/m)$ when $x \in \mathfrak{E}_c$, $U = \mathfrak{E}_c^+ \times \{1/m\}$ is a clopen subset of P, $y \in U$ and $x \notin U$. Let a be a fixed point in $\mathfrak{E}_c$ and consider the closed subset $A = \{(a, 1/n) : n \in \mathbb{N}\}$ of P. We claim that for every C-set neighbourhood U of $(p, 0)$ in P, the set $A \setminus U$ is finite (thus, P is not an AZD space and A is not a C-set). Let U be a C-set neighbourhood of $(p, 0)$ in P. Then there is a neighbourhood V of p in $\mathfrak{E}_c^+$ and $n \in \mathbb{N}$ such that $V \times \{1/k : k \geq n\} \subset U$. Assume that $(a, 1/k) \notin U$ for $k \geq n$. Select a clopen set C such that $(a, 1/k) \in C \subset P \setminus U$. Note that $C_0 = \{x \in \mathfrak{E}_c : (x, 1/k) \in C\}$ is a clopen subset of $\mathfrak{E}_c$ that is disjoint from V, and hence C_0 is bounded. Since $a \in C_0$ we have a contradiction by Lemma 1.13.

Let $Z \in \{P, Y\}$ and d a metric for Z. For each $n \in \mathbb{N}$, let $B_n = \{z \in Z : d(z, q) < 1/n\}$ and let $E_n = Z \setminus B_n$, where $q = p$ if $Z = Y$ or $q = (p, 0)$ if $Z = P$. Note that for any $n \in \mathbb{N}$, E_n is an AZD space and, by Proposition 2.2, $\mathcal{K}(E_n)$ is an AZD space. By item (3) from Proposition 1.11 $\mathcal{N} = \prod_{n \in \mathbb{N}} [\mathcal{K}(E_n) \cup \{\emptyset\}]$ is an AZD space (let's consider the set $\{\emptyset\}$ as an isolated point of $\mathcal{K}(E_n) \cup \{\emptyset\}$).

Let

$$\mathcal{L} = \{(K_1, K_2, \ldots) \in \mathcal{N} : \textit{for } m \geq n, K_m \cap E_n = K_n\}, \textit{and}$$

$$\mathcal{S} = \{H \in \mathcal{K}(Z) : q \in H\}.$$

Let's consider the following functions $\mathcal{G} : \mathcal{L} \to \mathcal{S}$ and $\mathcal{G}_n : \mathcal{S} \to \pi_n[\mathcal{L}]$ (where π_n is the projection to the n-th coordinate) such that

$$\mathcal{G}(K_1, K_2, \ldots) = \{q\} \cup \bigcup_{n\in\mathbb{N}} K_n, and$$

$$\mathcal{G}_n(K) = K \cap E_n$$

Theorem 2.20 (Roman Pol-2020)**.** *$\mathcal{G}$ is well defined and is a homeomorphism.*

Proof. To prove that $\mathcal{G}$ is well defined, let $\mathcal{C}$ be an open cover of $\{q\}\cup\bigcup_{n\in\mathbb{N}} K_n$ in Z. Since $q \in \{q\}\cup\bigcup_{n\in\omega} K_n$, we can suppose that $B_m \in \mathcal{C}$ for some m. Note that $(\{q\} \cup \bigcup_{n\in\mathbb{N}} K_n) \setminus K_m \subset B_m$, since $Z \setminus K_m \subset B_m$. As $\{U \cap E_m : U \in \mathcal{C}\}$ is an open cover of K_m and K_m is a compact subset of E_m, then there exists $U_1, \ldots, U_k$, such that $K_m \subset \bigcup_{i\leq k}(U_i \cap E_m)$. Therefore $\{B_m, U_1, \ldots, U_k\}$ is a finte subcover of $\mathcal{C}$, that is $\{q\}\cup\bigcup_{n\in\mathbb{N}} K_n$ is a compact subset of Z. Therefore $\mathcal{G}$ is well defined.

Let's prove that $\mathcal{G}$ is injective, let $\widehat{K} = (K_1, K_2, \ldots), \widehat{H} = (H_1, H_2, \ldots) \in \mathcal{L}$ such that $\widehat{H} \neq \widehat{K}$, then there exists $k \in \mathbb{N}$ so that $H_k \neq K_k$. Therefore there exists $x \in H_k \setminus K_k$, then $x \in \mathcal{G}(\widehat{H})$ and $x \notin \mathcal{G}(\widehat{K})$. That is, $\mathcal{G}$ is injective. Now let's see that $\mathcal{G}$ is surjective, let $K \in \mathcal{S}$. We define $K_n = K \cap E_n$, since E_n is a closed subset in Z, then K_n is empty or is a compact subset of E_n. Then $K_n \in \mathcal{K}(E_n) \cup \{\emptyset\}$ for each $n \in \mathbb{N}$, therefore $(K \cap E_1, \ldots) \in \mathcal{L}$ and $\mathcal{G}((K \cap E_1, \ldots)) = K$. That is, $\mathcal{G}$ is surjective. Before proving that $\mathcal{G}$ is a homeomorphism, let's show that

$$\mathcal{B} = \{\langle U_1, \ldots, U_n, B_k\rangle \cap \mathcal{S} : n, k \in \mathbb{N} \textit{ and } U_1, \ldots, U_n \textit{ are}$$

$$\textit{open subsets of } Z \setminus \{p\}\} \cup \{\langle B_k\rangle : k \in \mathbb{N}\}$$

is a base for $\mathcal{S}$.

Let $K \in \mathcal{S}$ and $\mathcal{W} = \langle W_1, \ldots, W_n\rangle$ an open subset of $\mathcal{K}(Z)$ such that $K \in \mathcal{W}$. If $K_n \neq \emptyset$ for some $n \in \mathbb{N}$, then $H_r \neq \emptyset$ for $r \geq n$, without loss of generality we can assume that $n = 1$. As $p \in K \in \mathcal{W}$, then there exists $j \leq n$ such that $p \in \bigcap\{W_j : j \leq n, p \in W_j\}$, and $k \in \mathbb{N}$ such that $p \in B_k \subset \bigcap\{W_j : j \leq n, p \in W_j\}$. To find an element $\mathcal{V}$ of the base $\mathcal{B}$ such that $K \in \mathcal{V} \subset \mathcal{W}$, we consider two cases. If $K \setminus B_k = \emptyset$ or if $K \setminus B_k \neq \emptyset$. If

$K \setminus B_k = \emptyset$ then $K \subset B_k$. Therefore $K \in \langle B_k \rangle \subset \mathcal{W}$. If $K \setminus B_k \neq \emptyset$ then for each $x \in K \setminus B_k$ there exists U_x such that $x \in U_x \subset \bigcap\{W_j : j \leq n, x \in W_j\}$, as $K \setminus B_k$ is compact and $\{U_x : x \in K \setminus B_k\}$ is an open cover of $K \setminus B_k$, there exists $x_1, \ldots, x_l \in K \setminus B_k$ such that $K \setminus B_k \in \langle U_{x_1}, \ldots, U_{x_l} \rangle$. Let $\mathcal{V} = \langle U_{x_1}, \ldots, U_{x_l}, B_k \rangle \cap \mathcal{S}$, note that $K \in \mathcal{V}$, and $\mathcal{V} \subset \mathcal{W} \cap \mathcal{S}$. On the other hand if $K = \{p\}$, there exists $k \in \mathbb{N}$ such that $p \in B_k$ and $p \in B_k \subset \bigcap\{W_j : j \leq n, p \in W_j\}$ this implies that $K \in \langle B_k \rangle \subset \mathcal{W}$. Therefore $\mathcal{B}$ is a base for $\mathcal{S}$.

Let $K = (H_1, \ldots, H_n, \ldots) \in \mathcal{L}$, and $\mathcal{U} \in \mathcal{B}$ such that $H = \mathcal{G}(K) \in \mathcal{U}$. If $H = \{p\}$, then $\mathcal{U} = \langle B_k \rangle$ for some $k \in \mathbb{N}$ and $H_n = \emptyset$ for each $n \in \mathbb{N}$. Let $W = \{\emptyset\}^k \times [\langle B_k \rangle \cap (\mathcal{K}(E_{k+1}) \cup \{\emptyset\})] \times \prod_{m > k+1}[\mathcal{K}(E_m) \cup \{\emptyset\}]$, note that $K \in W$, and $\mathcal{G}[W] \subset \mathcal{U}$. If $H_i \neq \emptyset$ for some $i \in \mathbb{N}$, then $H_r \neq \emptyset$ for $r \geq n$, without loss of generality we can assume that $i = 1$, and that $\mathcal{U} = \langle U_1, \ldots, U_n, B_k \rangle$ for some $n, k \in \mathbb{N}$. Let $A = \{j \in \mathbb{N} : U_l \cap H_j \neq \emptyset \textit{ for all } l \leq n\}$ and as $\{H_k : k \in \mathbb{N}\}$, is not finite, then $A \neq \emptyset$. Let $r = \min A$, if $r < k$, then $F_k \cap U_j = \emptyset$ for some $j \leq n$, so $F_r \cap U_j \neq \emptyset$ and $F_r \cap U_j \subset B_k$. Let

$$N = \{(F_1, F_2, \ldots) \in \mathcal{L} : F_r \in \langle U_1, \ldots, U_n, B_k \rangle\}.$$

Note that $K \in N$, if $F = (F_1, F_2, \ldots) \in N$ and $\mathcal{G}(F_1, F_2, \ldots) = F$, then $p \in F \setminus F_k \subset B_r \subset B_k$, so $F \in \mathcal{U}$. If $r \leq k$, then $H_k \setminus H_r \subset \bigcup_{j \leq n} U_j$ and $H \setminus H_k \subset B_k$. Let

$$N = \{(F_1, F_2, \ldots) \in \mathcal{L} : F_k \in \langle U_1, \ldots, U_n \rangle\}.$$

Note that $K \in N$. If $(F_1, \ldots, F_k, \ldots) \in N$, and $\mathcal{G}(F_1, \ldots, F_k, \ldots) = F$, then $p \in F \setminus F_k \subset B_k$, so $F \in \mathcal{U}$. This implies that $\mathcal{G}$ is a continuous function.
Finally we will show that $\mathcal{G}^{-1}$ is a continuous function, if U is a basic open subset of $\mathcal{L}$, then $U = (\bigcap_{j \in F} \pi_j^{\leftarrow}[W_j]) \cap \mathcal{L}$, where W_j is an open subset of $\mathcal{K}(E_j) \cup \{\emptyset\}$ and F is a finite subset of $\mathbb{N}$. Hence

$$(\mathcal{G}^{-1})^{\leftarrow}[U] = \bigcap_{j \in F} (\mathcal{G}^{-1})^{\leftarrow}[\pi_j^{\leftarrow}[W_j]] \cap \mathcal{S} = \bigcap_{j \in F} \mathcal{G}_n^{\leftarrow}[W_j].$$

So it is sufficient to show the continuity of $\mathcal{G}_n$ for any n. To prove that $\mathcal{G}_n$ is continuous, it is sufficient to show that $\mathcal{G}_n^{\leftarrow}[\langle U_1, \ldots, U_k \rangle \cap \mathcal{K}(E_n)]$ and $\mathcal{G}_n^{\leftarrow}[\{\emptyset\}]$ are open subsets of $\mathcal{S}$, where $U_1, \ldots, U_k$ are open subsets of $Z \setminus \{p\}$ such that $E_n \cap U_j \neq \emptyset$ for each $j \leq k$. We will show that

$$\mathcal{G}_n^{\leftarrow}[\langle U_1, \ldots U_k \rangle \cap \mathcal{K}(E_n)] = \mathcal{S} \cap \langle U_1, \ldots, U_k, B_n \rangle$$

and that

$$\mathcal{G}_n^{\leftarrow}[\{\emptyset\}] = \mathcal{S} \cap \langle B_n \rangle$$

Let $H \in \mathcal{S} \cap \langle U_1, \ldots, U_k, B_n \rangle$, then $H_n \neq \emptyset$, $H \setminus H_n \subset B_n$, and $H_n \in [\langle U_1, \ldots, U_k \rangle \cup \{\emptyset\}] \cap \mathcal{K}(E_n)$, thus $H \in \mathcal{G}_n^{\leftarrow}[\langle U_1, \ldots, U_k \rangle \cap \mathcal{K}(E_n)]$. Let $F \in \mathcal{G}_n^{\leftarrow}[\langle U_1, \ldots, U_k \rangle \cap \mathcal{K}(E_n)]$, then $F_n = F \cap E_n \in \langle U_1, \ldots, U_k \rangle \cap \mathcal{K}(E_n)$ and $F \setminus F_n \subset B_n$ then $F \in \langle U_1 \ldots, U_k, B_n \rangle$. Let $H \in \mathcal{S} \cap \langle B_n \rangle$, then $H_n = \emptyset$, therefore, $H_n \in \{\emptyset\}$, thus $H \in \mathcal{G}_n^{\leftarrow}[\{\emptyset\}]$. Let $F \in \mathcal{G}_n^{\leftarrow}[\{\emptyset\}]$, then $F_n = F \cap E_n = \emptyset$, thus $F \subset B_n$ then $F \in \langle B_n \rangle$. This implies that $\mathcal{G}_n$ is a continuous function. Therefore $\mathcal{G}$ is a homeomorphism.

□

Corollary 2.21. *$dim(Z) = dim(\mathcal{K}(Z)) = 1$.*

Proof. Note that $\mathcal{K}(Z) = \mathcal{K}(Z \setminus \{q\}) \cup \mathcal{S}$. As $\mathcal{K}(Z \setminus \{q\})$ is an AZD cohesive space, then $dim(\mathcal{K}(Z \setminus \{q\})) = 1$. By Theorem 2.20, S is homeomorphic to $\mathcal{L}$ and $dim(\mathcal{L}) = 1$ because $\mathcal{L}$ is an AZD, but $\mathcal{L}$ is not a zero-dimensinal space. This implies that $dim(\mathcal{S}) = 1$. Thus $dim(\mathcal{K}(Z)) = 1$. □

Corollary 2.22. *(Roman Pol-2020) There exists a connected space X such that $dim(X) = dim(\mathcal{K}(X)) = 1$.*

Proof. Consider the space Y at the beginning of the section. By Corollary 2.21 we have the result. □

Corollary 2.23. *There exists a totally disconnected space X which is not AZD such that $dim(X) = dim(\mathcal{K}(X)) = 1$.*

Proof. Consider the space P at the beginning of the section. By Corollary 2.21 we have the result. □

Note that the spaces given in the Corollaries 2.22 and 2.23 are unions of AZD spaces. A natural question is:
Does every space Z of dimension 1 that is not AZD and is a finite the union of subspaces AZD satisfies that $dim(\mathcal{K}(Z)) = 1$? The answer to this question is negative because $[0, 1]$ is not an AZD space, but is a union of $\mathbb{Q} \cap [0, 1]$ and $\mathbb{P} \cap [0, 1]$ which are AZD spaces, and $dim(\mathcal{K}([0, 1]))$ is not 1.

On the other hand it is known that if X is a compact space of dimension 1, then the dimension of $\mathcal{K}(X)$ is not finite (see [23, pag 123]). This implies that if a space X has a compact subset of dimension 1 then dimension of $\mathcal{K}(X)$ is not finite. Then for $\mathcal{K}(X)$ to have finite dimension each $A \in \mathcal{K}(X)$

must have dimension zero. From this the following question arises: If X is a non-compact space of dimension 1 such that every compact subset of X has dimension zero, then $\mathcal{K}(X)$ has dimension 1? With the following Theorem, we will show that it is not enough that the compact subsets of a space X of dimension 1 have dimension 0 for that hyperspace of compact subsets of X have to dimension 1.

Theorem 2.24. *[22, Theorem 4.1] There exists a space X of dimension 1 such that all its compacta have dimension 0 and $dim(X^2) = 2$.*

Example 2.25. *Let $Y = X \times \{0, 1\}$ where X is as in Theorem 2.24, then $dim(Y) = 1$ and for each compact subset F of X we have that $dim(F) = 0$. Let $f : X^2 \to \mathcal{K}(X)$ given by $f(x, y) = \{(x, 0), (y, 1)\}$. Note that f is an embedding. This implies that $dim(\mathcal{K}(Y)) \geq 2$.*

Question 2.26. *Let X be a space of dimension 1, such that $dim(X^{\omega}) = 1$ and for each $A \in \mathcal{K}(X)$, $dim(A) = 0$. Does $\mathcal{K}(X)$ have dimension 1?.*

Chapter 3

Hyperspaces of $\mathfrak{E}_c$ and $\mathfrak{E}$

In this chapter we will study the different hyperspaces of the spaces $\mathfrak{E}_c$ and $\mathfrak{E}$. The main objective of this chapter is to show that for any $n \in \mathbb{N}$, $\mathcal{F}_n(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$, $\mathcal{F}_n(\mathfrak{E}_c)$ is homeomorphic to $\mathfrak{E}_c$ and $\mathcal{F}(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$. Also we show why $\mathcal{K}(\mathfrak{E})$ is not homeomorphic to $\mathfrak{E}$ and that $\mathcal{F}(\mathfrak{E}_c)$ is not homeomorphic to $\mathfrak{E}_c$ or $\mathfrak{E}$.

3.1 Hyperspaces of $\mathfrak{E}$

In a personal communication Professor Jan van Mill explained to us that $\mathcal{K}(\mathfrak{E})$ is not Borel. Since $\mathfrak{E}$ contains a closed copy of $\mathbb{Q}$ (see [13]), the space $\mathcal{K}(\mathfrak{E})$ contains a closed copy of $\mathcal{K}(\mathbb{Q})$. But it is known that $\mathcal{K}(\mathbb{Q})$ is not a Borel set (see [13]), thus $\mathcal{K}(\mathfrak{E})$ is not a Borel set. Therefore $\mathcal{K}(\mathfrak{E})$ is not homeomorphic to $\mathfrak{E}$ or $\mathfrak{E}_c$. Furthermore, $\mathcal{K}(\mathfrak{E})$ cannot be a factor of any of these spaces. Therefore, in this section we only study the hyperspaces $\mathcal{F}(\mathfrak{E})$ and $\mathcal{F}_n(\mathfrak{E})$. However, there is another direction that is worth exploring. Michalewski proved in [19] that $\mathcal{K}(\mathbb{Q})$ is a topological group. Thus, the following is a natural question.

Question 3.1. *Is $\mathcal{K}(\mathfrak{E})$ homogeneous?*

3.1.1 Symmetric products of $\mathfrak{E}$

In this section we are going to show that $\mathcal{F}_n(\mathfrak{E})$ satisfies conditions of Theorem 1.24. First we present preliminary results.

Lemma 3.2. *Let $f : X \to Y$ be a continuous and surjective function and let $(A_n)_{n\in\omega}$ be a sequence of sets of X converging to x in X, then $(f[A_n])_{n\in\omega}$ converges to $f(x)$.*

Proof. Let U be an open subset of Y, such that $f(x) \in U$. Since f is a continuous function, $f^{\leftarrow}[U]$ is an open subset of X such that $x \in f^{\leftarrow}[U]$. Since $(A_n)_{n\in\mathbb{N}}$ converges to x, then there exists $m \in \mathbb{N}$ such that if $n \geq m$, then $A_n \subset f^{\leftarrow}[U]$. This implies that $f[A_n] \subset U$ if $n \geq m$, therefore $(f[A_n])_{n\in\mathbb{N}}$ converges to $f(x)$.

□

Lemma 3.3. *Let $f : X \to Y$ be a continuous and surjective function. Let A and B be subsets of Y, such that $f^{\leftarrow}[A]$ has empty interior in $f^{\leftarrow}[B]$. Then A has empty interior in B.*

Proof. Let us suppose A has non-empty interior in B. Then there exists $y \in A$ and an open subset V of Y such that $y \in B \cap V \subset A$. Let $x \in X$ such that $f(x) = y$, then $x \in f^{\leftarrow}[B] \cap f^{\leftarrow}[V] \subset f^{\leftarrow}[A]$, which contradicts the hypothesis. □

Lemma 3.4. *Let $A_1, \ldots, A_n$ be subsets of $B_1, \ldots, B_n$ respectively, if A_i has empty interior for some $i \in \{1, \ldots, n\}$, then $A_1 \times \ldots \times A_n$ has empty interior in $B_1 \times \ldots \times B_n$.*

Proof. Suppose that $A_1 \times \ldots \times A_n$ has non-empty interior in $B_1 \times \ldots \times B_n$, then there are open subsets $U_1, \ldots, U_n$ of $B_1, \ldots, B_n$ respectively, such that $U_1 \times \ldots \times U_n \subset A_1 \times \ldots \times A_n$. This implies that $U_i \subset A_i$ for each $i \in \{1, \ldots, n\}$; this contradicts that A_i has empty interior for some $i \in \{1, \ldots, n\}$. □

By Theorem 1.24 there is a topology $\mathcal{W}$ for $\mathfrak{E}$ which is a witness to the almost zero-dimensionality of $\mathfrak{E}$, a countable tree T and a family of sets $\mathcal{E} = \{E_s : s \in T\}$ which are closed with respect to $\mathcal{W}$ which satisfy the conditions of Theorem 1.24 for $\mathfrak{E}$. Let $\mathcal{W}^n$ the topology of $(\mathfrak{E}, \mathcal{W})^n$, $T^n = \{s_1 * \ldots * s_n : s_1, \ldots, s_n \in T \text{ and } |s_1| = \ldots = |s_n|\}$ and for each $s_1 * \ldots * s_n \in T^n$ let $E_{s_1 * \ldots * s_n}$ be the subset $E_{s_1} \times \ldots \times E_{s_n}$ of $\mathfrak{E}^n$.

Lemma 3.5. *The collections $\mathcal{W}^n$, T^n and $\mathcal{E}^n = \{E_{s_1 * \ldots * s_n} : s_1 * \ldots * s_n \in T^n\}$ satisfy the conditions of Theorem 1.24 for $\mathfrak{E}^n$.*

Proof. Since T is a tree over the countable alphabet A, T^n is a tree over the countable alphabet A^n (Definition 1.19). By item 3 of Proposition 1.11

$\mathcal{W}^n$ is witness to the almost zero-dimensionality of $\mathfrak{E}^n$. Moreover, it is clear that each $E_{s_1*\ldots*s_n}$ is closed in $\mathfrak{E}^n$. On the other hand $E_\emptyset = \mathfrak{E}$, so $E_{\emptyset*\ldots*\emptyset} = E_\emptyset \times \ldots \times E_\emptyset = \mathfrak{E}^n$. Furthermore,

$$E_{s_1} \times \ldots \times E_{s_n} = \bigcup\{E_{t_1} : t_1 \in succ(s_1)\} \times \ldots \times \bigcup\{E_{t_n} : t_1 \in succ(s_n)\} =$$

$$\bigcup\{E_{t_1} \times \ldots \times E_{t_n} : t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)\} =$$

$$\bigcup\{E_{t_1*\ldots*t_n} : t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)\}$$

(see Remark 1.21).
Now we are going to prove that $\mathcal{W}^n$, T^n and $\mathfrak{E}^n$ satisfy condition (2) of Theorem 1.24. Let $(x_1, \ldots, x_n) \in \mathfrak{E}^n$ and, for each $i \in \{1, \ldots, n\}$, let U_i be a neighborhood of x_i which is an anchor for $\mathcal{E}$. Let $\widehat{t} \in T^n$; then $\widehat{t} = (\widehat{t_1}, \ldots, \widehat{t_n})$ with $\widehat{t_i} \in [T]$ for each $i \in \{1, \ldots, n\}$ (Remark 1.21). We define

$$J = \{i \in \{1, \ldots, n\} : \textit{ there exists } m \in \mathbb{N} \textit{ such that } E_{\widehat{t_i}\restriction m} \cap U_i = \emptyset\}.$$

First assume that $J \neq \emptyset$; then $E_{(\widehat{t_1}, \ldots, \widehat{t_n})\restriction m} \cap (U_1 \times \ldots \times U_n) = (E_{\widehat{t_1}\restriction m} \times \ldots \times E_{\widehat{t_n}\restriction m})) \cap (U_1 \times \ldots \times U_n) = (E_{\widehat{t_1}\restriction m} \cap U_1) \times \ldots \times (E_{\widehat{t_n}\restriction m} \cap U_n) = \emptyset$. Now assume that $J = \emptyset$. Since U_i is an anchor for $\mathcal{E}$ for each $i \in \{1, \ldots, n\}$, the sequence $(E_{\widehat{t_i}\restriction j})_{j<\omega}$ converges to a point z_i in $\mathfrak{E}$. Therefore, the sequence $(E_{\widehat{t_1}\restriction j} \times \ldots \times E_{\widehat{t_n}\restriction j})_{j<\omega}$ converges to $(z_1, \ldots, z_n) \in \mathfrak{E}^n$. But $E_{\widehat{t_1}\restriction j} \times \ldots \times E_{\widehat{t_n}\restriction j} = E_{\widehat{t_1}\restriction j*\ldots*\widehat{t_n}\restriction j} = E_{(\widehat{t_1}\restriction j, \ldots, \widehat{t_n}\restriction j)} = E_{\widehat{t}\restriction j}$ (see Remark 1.21). Hence, the sequence $(E_{\widehat{t}\restriction j})_{j<\omega}$ converges to $(z_1, \ldots, z_n)$.

We now verify condition (3) of Theorem 1.24. Suppose $t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)$, then for each $i \in \{1, \ldots, n\}$ $t_i \in succ(s_i)$ (Remark 1.21). Thus, E_{t_i} is nowhere dense in E_{s_i} for each $i \in \{1, \ldots, n\}$. By Lemma 3.4 $E_{t_1} \times \ldots \times E_{t_n}$ is nowhere dense in $E_{s_1} \times \ldots \times E_{s_n}$. Condition (4) of Theorem 1.24 follows from Remark 1.15. □

We need to define the neighborhoods that will work as anchors for the symmetric products.

Let $F = \{x_1, \ldots, x_k\} \in \mathcal{F}(\mathfrak{E})$. For each $j \leq k$, let U_{x_j} be a neighborhood of x_j which is anchor in $(\mathfrak{E}, \mathcal{W})$. Let $\mathcal{U}_F = \langle U_{x_1}, \ldots, U_{x_k}\rangle$.

Lemma 3.6. *If $F \in \mathcal{F}_n(X)$, the set $\mathcal{U}_F \cap \mathcal{F}_n(X)$ is an anchor for $(\mathcal{F}_n(\mathfrak{E}), \mathcal{W}_n)$.*

Proof. Let $F = \{x_1, \dots, x_k\} \in \mathcal{F}_n(\mathfrak{E})$, and $\mathcal{U}_F$ defined as above. Consider $t \in [T^n]$. If there exists $i \in \omega$ such that $\mathcal{U}_F \cap q_n[E_{t_1\restriction i} \times \dots \times E_{t_n\restriction i}] = \emptyset$ then we are finished. Now suppose that for each $i \in \omega$

$$(*)\ \mathcal{U}_F \cap q_n[E_{t_1\restriction i} \times \dots \times E_{t_n\restriction i}] \neq \emptyset.$$

We claim that for all $i \in \omega$ and $j \leq n$, there exists $l(i,j) \in \omega$ such that, $U_{l(i,j)} \cap E_{t_j\restriction i} \neq \emptyset$. By $(*)$ there exists $(y_1, \dots, y_n) \in E_{t_1\restriction i}, \times \dots \times E_{t_n\restriction i}$ such that

$$\{y_1, \dots, y_n\} = q_n(y_1, \dots, y_n) \in \mathcal{U}_F = \langle U_{x_1}, \dots, U_{x_k}\rangle.$$

So there is $l \leq n$ such that $y_j \in U_{x_l}$; this proves the claim.

If we fix j, this defines a function $i \mapsto l(i,j)$ with domain ω and codomain $\{1, \dots, k\}$. This implies that there exists an infinite $A \subset \omega$ and fixed $l_j \in \mathbb{N}$ such that $E_{t_j\restriction i} \cap U_{l_j} \neq \emptyset$ for each $i \in A$. As $E_{t_j\restriction s} \subset E_{t_j\restriction t}$ if $s < t$, then $E_{t_j\restriction i} \cap U_{l_j} \neq \emptyset$ for all $i \in \omega$. Therefore $\{E_{t_j\restriction i} : i \in \omega\}$ converges in $(\mathfrak{E}, \mathcal{W})$ to a point $p_j \in \mathfrak{E}$; this holds for all $j \in \{1, \dots, n\}$.

Therefore $\{E_{t_1\restriction i} \times \dots \times E_{t_n\restriction i} : i \in \omega\}$ converges in $(\mathfrak{E}^n, \mathcal{W}^n)$ to $(p_1, \dots, p_n)$. So $\{q_n[E_{t_1\restriction i} \times \dots \times E_{t_n\restriction i}] : i \in \omega\}$ converges in $\mathcal{F}_n(\mathfrak{E}, \mathcal{W})$ to $q_n((p_1, \dots, p_n))$. □

Theorem 3.7. *For any $n \in \mathbb{N}$, $\mathcal{F}_n(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. We are going to prove that $\mathcal{F}_n(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$ using Theorem 1.24. Because of Lemma 3.5 we know that if the topology $\mathcal{W}$, the tree T and the family $\{E_s : s \in T\}$ satisfy the conditions of Theorem 1.24 for $\mathfrak{E}$, then the product topology $(\mathfrak{E}, \mathcal{W})^n$, which we denote here as $\mathcal{W}^n$, $T^n = \{s_1 * \dots * s_n : s_1, \dots, s_n \in T \text{ and } |s_1| = \dots = |s_n|\}$ (Definition 1.19) and the family $\{E_{s_1*\dots*s_n} : s_1 * \dots * s_n \in T^n\}$ where $E_{s_1*\dots*s_n} = E_{s_1} \times \dots \times E_{s_n}$ for each $s_1, \dots, s_n \in T$ with $|s_1| = \dots = |s_n|$, satisfy all the conditions of Theorem 1.24.

Let $q_n : \mathfrak{E}^n \to \mathcal{F}_n(\mathfrak{E})$. Let $\mathcal{W}'$ be the Vietoris topology in $\mathcal{F}_n(\mathfrak{E}, \mathcal{W})$. The tree T' that we are going to consider is $T' = T^n$, and the family $\mathcal{S}'$ of subsets of $\mathcal{F}_n(\mathfrak{E})$ indexing by T^n that we are going to prove to be closed with respect to $\mathcal{W}^n$ is $\mathcal{S}' = \{H_{s_1*\dots*s_n} : s_1 * \dots * s_n \in T^n\}$ where $H_{s_1*\dots*s_n} := q_n[E_{s_1*\dots*s_n}]$ for each $s_1 * \dots * s_n \in T^n$. We will prove then that $\mathcal{W}'$, T' and $\mathcal{S}'$ satisfy the conditions required in Theorem 1.24 for $\mathcal{F}_n(\mathfrak{E})$.

Indeed, the fact that the Vietoris topology in $\mathcal{F}(\mathfrak{E}, \mathcal{W})$ witnesses that $\mathcal{F}_n(\mathfrak{E})$ is almost zero-dimensional follows from the proof of Proposition 2.2. By Proposition 2.8, $\mathcal{F}_n(\mathfrak{E})$ is $\{q_n[E_{s_1} \times \dots \times E_{s_n}] : s_1 * \dots * s_n \in T^n\}$-cohesive.

That is, $\mathcal{F}_n(\mathfrak{E})$ is $\{H_{s_1*\ldots*s_n} : s_1 * \ldots * s_n \in T^n\}$-cohesive. Moreover, $T' = T^n$ is a tree over a countable alphabet. On the other hand, for each $s \in T$, E_s is a closed subset of $(\mathfrak{E}, \mathcal{W})$, hence for $s_1, \ldots, s_n \in T$ satisfying $|s_1| = \ldots = |s_n|$, $E_{s_1} \times \ldots \times E_{s_n}$ is closed in $(\mathfrak{E}, \mathcal{W})^n$. Additionaly, since function g is closed, $q_n[E_{s_1} \times \ldots \times E_{s_n}]$ is closed in $(\mathcal{F}_n(\mathfrak{E}), \mathcal{W}')$.

Now we are going to prove that $\mathcal{W}'$, T' and $\mathcal{S}'$ satisfy conditions (1), (2) and (3) of Theorem 1.24. For $\emptyset = \emptyset * \ldots * \emptyset \in T^n$, $H_{\emptyset*\ldots*\emptyset} = q_n[E_\emptyset \times \ldots \times E_\emptyset] = q_n[\mathfrak{E}] = \mathcal{F}_n(\mathfrak{E})$. Let $s_1 * \ldots * s_n$ be an element in T^n. Using Lemma 3.5, we have that $H_{s_1*\ldots*s_n} := q_n[E_{s_1*\ldots*s_n}] = q_n[\bigcup\{E_{t_1*\ldots*t_n} : t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)\}] = \bigcup\{q_n[E_{t_1*\ldots*t_n}] : t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)\} = \bigcup\{H_{t_1*\ldots*t_n} : t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)\}$. This proves that $\mathcal{W}'$, T' and $\mathcal{S}'$ satisfy condition (1) of Theorem 1.24. In order to prove that $\mathcal{W}'$, T' and $\mathcal{S}'$ satisfy condition (2) of Theorem 1.24, we take $F \in \mathcal{F}_n(\mathfrak{E})$. Assume that $F = \{x_1, \ldots, x_n\}$. For each $j \leq n$ there exists a neighborhood U_j of x_j which is an anchor for $\mathfrak{E}$. By Lemma 3.6 $\mathcal{U} = \langle U_1, \ldots, U_n\rangle$ is an anchor. Finally, we will prove that $\mathcal{W}'$, T' and $\mathcal{S}'$ satisfy condition (3) of Theorem 1.24. If $t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)$, then for each $i \in \{1, \ldots, n\}$, $t_i \in succ(s_i)$ (Remark 1.21). Thus, E_{t_i} is nowhere dense in E_{s_i} for each $i \in \{1, \ldots, n\}$. This implies that for each permutation $h : \{1, \ldots, n\} \to \{1, \ldots, n\}$, $E_{t_{h(1)}} \times \ldots \times E_{t_{h(n)}}$ is nowhere dense in $E_{s_{h(1)}} \times \ldots \times E_{s_{h(n)}}$. Therefore, $\bigcup_{h\in P}(E_{t_{h(1)}} \times \ldots \times E_{t_{h(n)}})$ is nowhere dense in $\bigcup_{h\in P}(E_{s_{h(1)}} \times \ldots \times E_{s_{h(n)}})$ where P is the set of permutations of $\{1, \ldots, n\}$. Note that

$$q_n^{\leftarrow}[q_n[E_{t_1} \times \ldots \times E_{t_n}]] = \bigcup_{h\in P}(E_{t_{h(1)}} \times \ldots \times E_{t_{h(n)}}) \subset q_n^{\leftarrow}[q_n[E_{s_1} \times \ldots \times E_{s_n}]]$$

and

$$q_n^{\leftarrow}[q_n[E_{s_1} \times \ldots \times E_{s_n}]] = \bigcup_{h\in P}(E_{s_{h(1)}} \times \ldots \times E_{s_{h(n)}}).$$

As the finite union of nowhere dense sets is nowhere dense, by Lemma 3.3 $q_n[E_{t_1} \times \ldots \times E_{t_n}]$ is a nowhere dense subset of $q_n[E_{s_1} \times \ldots \times E_{s_n}]$. By Theorem 1.24, all the above proves that $\mathcal{F}_n(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$. □

Corollary 3.8. *Let $n \in \mathbb{N}$, if X is an $\mathfrak{E}$-factor, then $\mathcal{F}_n(X)$ is an $\mathfrak{E}$-factor.*

Proof. If X is an $\mathfrak{E}$-factor. By Theorem 1.26 X can be embedded as a closed subset of $\mathfrak{E}$. By Proposition 1.42 $\mathcal{F}_n(X)$ can be embedded as a closed subset of $\mathcal{F}_n(\mathfrak{E})$. By Theorem 3.7, we have that $\mathcal{F}_n(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$. Thus $\mathcal{F}_n(X)$ is an $\mathfrak{E}$-factor by Theorem 1.26. □

Corollary 3.9. *For each $k < n$ the space $\mathcal{F}_n(\mathfrak{E}) \setminus \mathcal{F}_k(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$.*

Proof. This follows from Theorem 3.5 and the fact that $\mathcal{F}_n(\mathfrak{E}) \setminus \mathcal{F}_k(\mathfrak{E})$ is an open subset of $\mathcal{F}_n(\mathfrak{E})$. □

3.1.2 The hyperspace of finite sets of $\mathfrak{E}$

David S. Lipham in [16] has shown that if X is an Erdős space factor then the Vietoris hyperspace $\mathcal{F}(X)$ of finite subsets of X is an Erdős factor. The objective of this section is to wrap up this topic with the following result.

Theorem 3.10. *$\mathcal{F}(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$.*

Proof. By Theorem 3.7 for each $n \in \mathbb{N}$ the topology $\mathcal{W}_n$, the tree T^n and collection $\mathcal{E}_n = \{q_n(E_{s_1} \times \ldots \times E_{s_n}) : s_1 * \ldots * s_n \in T^n\}$ satisfy conditions (1), (3) and (4) of Theorem 1.24 for $\mathcal{F}_n(\mathfrak{E})$. Consider the tree

$$T = \{\emptyset\} \cup \{\langle n\rangle^\frown s_1 * \ldots * s_n : n \in \omega,\, s_1 * \ldots * s_n \in T^n\}.$$

Let $X_\emptyset = \mathcal{F}(\mathfrak{E})$ and $X_{\langle n\rangle^\frown s_1 * \ldots * s_n} = q_n[E_{s_1} \times \ldots \times E_{s_n}]$ for each $n \in \omega$ and $s_1 * \ldots * s_n \in T^n$. Let $\mathcal{W}'$ be the Vietoris topology on $\mathcal{F}(\mathfrak{E}, \mathcal{W})$. Then we will prove that $\mathcal{W}'$, T, $\mathcal{S} = \{X_s : s \in T\}$ and $\{\mathcal{U}_F : F \in \mathcal{F}(\mathfrak{E})\}$ satisfy the conditions required in Theorem 1.24 for $\mathcal{F}(\mathfrak{E})$. Indeed, the fact that the Vietoris topology on $\mathcal{F}(\mathfrak{E}, \mathcal{W})$ witnesses that $\mathcal{F}(\mathfrak{E})$ is almost zero-dimensional follows from the proof of Proposition 2.2. On the other hand, for each $s_1 * \ldots * s_n \in T_n$, $q_n[E_{s_1} \times \ldots \times E_{s_n}]$ is a closed subset of $\mathcal{F}_n(\mathfrak{E}, \mathcal{W})$, hence $q_n[E_{s_1} \times \ldots \times E_{s_n}]$ is closed in $\mathcal{F}(\mathfrak{E}, \mathcal{W})$, because $\mathcal{F}_n(\mathfrak{E}, \mathcal{W})$ is closed in $\mathcal{F}(\mathfrak{E}, \mathcal{W})$ (by Proposition1.39). For $\emptyset \in T$, we have $succ_T(\emptyset) = \{\langle n\rangle^\frown \emptyset_1 * \ldots * \emptyset_n : n \in \omega, \emptyset_1 * \ldots * \emptyset_n \in T_n\}$ so $X_{\langle n\rangle^\frown \emptyset_1 * \ldots * \emptyset_n} = \mathcal{F}_n(\mathfrak{E})$. Hence $X_{\langle n\rangle^\frown \emptyset_1 * \ldots * \emptyset_n}$ is a nowhere dense subset of $X_\emptyset$ and $X_\emptyset = \bigcup_{t \in succ_T(\emptyset)} X_t$. On the other hand, if $\langle n\rangle^\frown s_1 * \ldots * s_n \in T \setminus \{\emptyset\}$, $succ_T(\langle n\rangle^\frown s_1 * \ldots * s_n) = \{\langle n\rangle^\frown t : t \in succ_{T^n}(s_1 * \ldots * s_n)\}$. Then for each $\langle n\rangle^\frown s_1 * \ldots * s_n \in T \setminus \{\emptyset\}$, we have $X_{\langle n\rangle^\frown s_1 * \ldots * s_n} = \bigcup\{X_{\langle n\rangle^\frown t} : t \in succ_{T^n}(s_1 * \ldots * s_n)\}$ and $X_{\langle n\rangle^\frown t}$ is a nowhere dense subset of $X_{\langle n\rangle^\frown s_1 * \ldots * s_n}$ if $t \in succ_T(s_1 * \ldots * s_n)$. Thus, conditions (1) and (3) of Theorem 1.24 for $\mathcal{F}(\mathfrak{E})$ are satisfied. Now we prove condition (2). Let $F \in \mathcal{F}(\mathfrak{E})$. Suppose that $F = \{x_1, \ldots, x_k\}$ with $x_j \neq x_i$ if $i \neq j$ so $\mathcal{U}_F = \langle U_{x_1}, \ldots, U_{x_k}\rangle$. We claim that $\mathcal{U}_F$ is an anchor in $\mathcal{W}'$. Let $\widehat{t} \in [T]$, then there exists $n \in \omega$ such that $\widehat{t} = \langle n\rangle^\frown \widehat{t}_n$ and $\widehat{t}_n \in [T^n]$. Also, there exists

$t_i \in [T]$ such that $\widehat{t}_n = (t_1, \ldots, t_n)$.

Case 1: $k \leq n$. By Lemma 3.6, $\mathcal{U}_F \cap \mathcal{F}_n(\mathfrak{E})$ is a anchor of F in $(\mathcal{F}_n(\mathfrak{E}), \mathcal{W}_n)$. We will show that $\mathcal{U}_F \cap X_{\widehat{t}\restriction i} = \emptyset$ for some $i \in \omega$, or that $(X_{\widehat{t}\restriction i})_{i\in\omega}$ converges in $(\mathcal{F}(\mathfrak{E}), \mathcal{W}')$. If $\mathcal{U}_F \cap X_{\widehat{t}\restriction i} = \emptyset$ for some $i \in \omega$, we are finished. If $\mathcal{U}_F \cap X_{\widehat{t}\restriction i} \neq \emptyset$ for each $i \in \omega$, as $X_{\widehat{t}\restriction i+1} = X_{\langle n\rangle^\frown \widehat{t}_n\restriction i} = q_n[E_{t_1\restriction i} \times \ldots \times E_{t_n\restriction i}] \subset \mathcal{F}_n(\mathfrak{E})$ then $(\mathcal{U}_F \cap \mathcal{F}_n(\mathfrak{E})) \cap X_{\widehat{t}\restriction i} \neq \emptyset$ for each $i \in \omega$. Hence $(X_{\widehat{t}\restriction i})_{i\in\omega}$ converges in $(\mathcal{F}(\mathfrak{E}), \mathcal{W}')$ because $\mathcal{U}_F \cap \mathcal{F}_n(\mathfrak{E})$ is a anchor of F in $(\mathcal{F}_n(\mathfrak{E}), \mathcal{W}_n)$. So $\mathcal{U}_F$ is anchor of F in $(\mathcal{F}(\mathfrak{E}), \mathcal{W}')$.

Case 2: $k > n$. For each $j \in (n, k]$, let's consider $t_j \in [T]$ such that $t_j = t_n$. Let $\widehat{s} = (t_1, \ldots, t_n, t_{n+1}, \ldots, t_k)$, then $\widehat{s} \in [T^k]$. By Lemma 3.6, $\mathcal{U}_F \cap \mathcal{F}_k(\mathfrak{E})$ is a anchor of F in $(\mathcal{F}_k(\mathfrak{E}), \mathcal{W}_k)$, i.e. $X_{\langle k\rangle^\frown \widehat{s}\restriction j} \cap \mathcal{U}_F = \emptyset$ for some $j \in \omega$, or $(X_{\langle k\rangle^\frown \widehat{s}\restriction j})_{j\in\omega}$ converges. We claim that either $X_{\widehat{t}\restriction j} \cap \mathcal{U}_F = \emptyset$ for some $j \in \omega$, or $(X_{\widehat{t}\restriction j})_{j\in\omega}$ converges. Note that for all $j \in \omega$

$$(*)\ q_n[E_{t_1\restriction j} \times \ldots \times E_{t_n\restriction j}] \subset \varphi_k[E_{t_1\restriction j} \times \ldots \times E_{t_n\restriction j} \times E_{t_{n+1}\restriction j} \times \ldots \times E_{t_k\restriction j}].$$

Let's suppose that $X_{\widehat{t}\restriction j} \cap \mathcal{U}_F \neq \emptyset$ for each $j \in \omega$. By $(*)$ we have $X_{\langle k\rangle^\frown \widehat{s}\restriction j} \cap \mathcal{U}_F \neq \emptyset$, then $(X_{\langle k\rangle^\frown \widehat{s}\restriction j})_{j\in\omega}$ converges to some $A \in \mathcal{F}_k(\mathfrak{E})$. Hence $(q_n[E_{t_1\restriction j} \times \ldots \times E_{t_n\restriction j}])_{j\in\omega}$ converges to A. Then $(X_{\widehat{t}\restriction j})_{j\in\omega}$ converges to A.

Thus, condition (2) of Theorem 1.24 for $\mathcal{F}(\mathfrak{E})$ is satisfied. By Lemma 2.9, $\mathcal{F}(\mathfrak{E})$ is $\{X_s : s \in T\}$-cohesive. Hence, condition (4) of Theorem 1.24 for $\mathcal{F}(\mathfrak{E})$ is satisfied. Then by Theorem 1.24 the space $\mathcal{F}(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$. □

With the previous theorem we can give another proof of David S. Lipham's result mentioned at the beginning of this section.

Corollary 3.11. *If X is an $\mathfrak{E}$-factor, then $\mathcal{F}(X)$ is an $\mathfrak{E}$-factor.*

Proof. If X is an $\mathfrak{E}$-factor. By Theorem 1.26 X can be embedded as a closed subset of $\mathfrak{E}$. By Proposition 1.42 $\mathcal{F}(X)$ can be embedded as a closed subset of $\mathcal{F}(\mathfrak{E})$. By Theorem 3.10, the space $\mathcal{F}(\mathfrak{E})$ is homeomorphic to $\mathfrak{E}$. Thus $\mathcal{F}(X)$ is an $\mathfrak{E}$-factor. □

Corollary 3.12. *For each $n > 1$ we have $\mathcal{F}(\mathfrak{E}) \setminus \mathcal{F}_n(\mathfrak{E}) \approx \mathfrak{E}$.*

Proof. This follows from Theorem 3.10 and the fact that $\mathcal{F}(\mathfrak{E}) \setminus \mathcal{F}_k(\mathfrak{E})$ is an open subset of $\mathcal{F}(\mathfrak{E})$. □

3.1.3 Sierpiński stratifications

In this section we give some indirect applications of Theorems 3.7 and 3.10. A system of sets $(X_s)_{s\in T}$ is a Sierpiński stratification ([3, Definition 7.1, p. 31]) of a space X if:

1. T is a non-empty tree over a countable alphabet,
2. each X_s is a closed subset of X,
3. $X_\emptyset = X$, $X_s = \bigcup\{X_t : t \in succ(s)\}$,
4. if $\sigma \in [T]$ then the sequence $X_{\sigma\restriction 1}, \ldots, X_{\sigma\restriction n}, \ldots$ converges to a point in X.

Note that in Theorem 3.7 it is implicitly proved that if $(X_s)_{s\in T}$ is a Sierpiński stratification of a space X, then $\mathcal{C}_n = \{q_n[X_{s_1} \times \ldots \times X_{s_n}] : s_1 * \ldots * s_n \in T^n\}$ is a Sierpiński stratification of $\mathcal{F}_n(X)$. Define $X_\emptyset = X$ and $X_{\langle n\rangle^\frown s_1*\ldots*s_n} = q_n[X_{s_1} \times \ldots \times X_{s_n}]$ for $n \in \mathbb{N}$ and $s_1 * \ldots * s_n \in T^n$. Then by an argument similar to the proof of Theorem 3.10 we conclude that $\mathcal{C} = \{X_\emptyset\} \cup \{X_{\langle n\rangle^\frown s_1*\ldots*s_n} : n \in \mathbb{N}, s_1 * \ldots * s_n \in T^n\}$ is a Sierpiński stratification of $\mathcal{F}(X)$. So we have the following Corollary.

Corollary 3.13. *Let $(X_s)_{s\in T}$ be Sierpiński stratification of a space X. Then*

1. *For each $n \in \mathbb{N}$, $\mathcal{C}_n$ is a Sierpiński stratification of $\mathcal{F}_n(X)$.*
2. *$\mathcal{C}$ is a Sierpiński stratification of $\mathcal{F}(X)$.*

Van Engelen proved in ([8, Theorem A.1.6]) that a zero-dimensional X space is homeomorphic to $\mathbb{Q}^\omega$ if X has a Sierpiński stratification $(X_s)_{s\in T}$ such that X_t is nowhere dense in X_s, if $t \in succ(s)$. Using this fact, the following corollary can be proved.

Corollary 3.14. *The spaces $\mathcal{F}_n(\mathbb{Q}^\omega)$, $\mathcal{F}(\mathbb{Q}^\omega)$ are homeomorphic to $\mathbb{Q}^\omega$.*

Proof. Let $(X_s)_{s\in T}$ be Sierpiński stratification of a space $\mathbb{Q}^\omega$ such that X_t is nowhere dense in X_s, if $t \in succ(s)$. By Corollary 3.13 for each $n \in \mathbb{N}$, $\mathcal{C}_n$ and $\mathcal{C}$ are Sierpiński stratifications of $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$, respectively. By Lemma 3.3 we have that $q_n[X_{t_1} \times \ldots \times X_{t_n}]$ is nowhere dense in $q_n[X_{s_1} \times \ldots \times X_{s_n}]$ if $t_1 * \ldots * t_n \in succ(s_1 * \ldots * s_n)$ and that $X_{\langle n\rangle^\frown t_1*\ldots*t_n}$ is nowhere dense in $X_{\langle n\rangle^\frown s_1*\ldots*s_n}$ if $\langle n\rangle^\frown t_1 * \ldots * t_n \in succ(\langle n\rangle^\frown s_1 * \ldots * s_n)$. □

David Lipham proved in [16] that if X is an $\mathfrak{E}$-factor, then it admits a Sierpiński stratification $(B_s)_{s\in T}$. We can consider the sets B_s as subsets of $(X, \mathcal{W})$, where $\mathcal{W}$ is a topology witness to the almost zero-dimensionality of X. A natural question is this.

Question 3.15. *If X is a $\mathfrak{E}$-factor. Under what conditions is $(X, \mathcal{W})$ homeomorphic to $\mathbb{Q}^\omega$?*

3.2 Hyperspaces of $\mathfrak{E}_c$

We will begin by studying $\mathcal{K}(\mathfrak{E}_c)$. In this case $\mathcal{K}(\mathfrak{E}_c)$ is a Borel set, moreover by Proposition 1.46 and Theorem 1.4, $\mathcal{K}(\mathfrak{E}_c)$ is an absolute G_δ. An immediate result we have is the fallowing.

Corollary 3.16. *$\mathcal{K}(\mathfrak{E}_c)$ is an $\mathfrak{E}_c^\omega$-factor, and an $\mathfrak{E}$-factor.*

Proof. By the Proposition 1.46 and Proposition 2.2 $\mathcal{K}(\mathfrak{E}_c)$ is a Polish AZD space. Then by Theorem 1.36 $\mathcal{K}(\mathfrak{E}_c)$ is an $\mathfrak{E}_c^\omega$-factor, and by Corollary 9.3 of [3], $\mathcal{K}(\mathfrak{E}_c)$ is an $\mathfrak{E}$-factor.

$\square$

By Proposition 2.2 and 1.46 $\mathcal{K}(\mathfrak{E}_c)$ is a complete and AZD space. On the other hand let us note that the only AZD and complete spaces that have a characterization are $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$, then a natural question is:

Question 3.17. *Is $\mathcal{K}(\mathfrak{E}_c)$ homeomorphic to $\mathfrak{E}_\mathrm{c}$ or $\mathfrak{E}_\mathrm{c}^\omega$?*

3.2.1 Symmetric products of $\mathfrak{E}_c$

In this section we are going to show that $\mathcal{F}_n(\mathfrak{E}_c) \approx \mathfrak{E}_c$ and some of the consequences of this result. For a topological space (X, τ), the symbol $\tau_{\mathcal{F}_n(X)}$ denotes the Vietoris topology in $\mathcal{F}_n(X)$. We introduce the following definition to simplify the notation.

Definition 3.18. *Let X be a set and let τ_1 and τ_2 two topologies in X, if (X, τ_1, τ_2) satisfies that:*

1. *$\tau_1 \subset \tau_2$ and τ_1 is a zero-dimensional topology such that every point in X has a neighborhood in τ_2 which is compact with respect to τ_1, we say that (X, τ_1, τ_2) has property C_1.*

2. *$\tau_1 \subset \tau_2$ and τ_1 is a zero-dimensional topology such that every point in X has a neighborhood in τ_2 which is complete with respect to τ_1, we say that (X, τ_1, τ_2) has property C_2.*

Note that property C_1 is inherited by closed sets with respect to τ_1 and the property C_2 is inherited by G_δ subsets with respect to τ_1.

Using Definition 3.18 we can state Theorem 1.31 as follows:

Theorem 3.19. *Let $(\mathcal{E}, \tau)$ be a topological space. The following statements are equivalent.*

1. *$\mathcal{E}$ is homeomorphic to $\mathfrak{E}_c$.*

2. *$\mathcal{E}$ is cohesive and there exists a topology $\mathcal{W}$ in $\mathcal{E}$ such that $(\mathcal{E}, \tau, \mathcal{W})$ has property C_1.*

3. *$\mathcal{E}$ is cohesive and there exists a topology $\mathcal{W}$ in $\mathcal{E}$ such that $(\mathcal{E}, \tau, \mathcal{W})$ has property C_2.*

Proposition 3.20. *Let X be a set and let τ_1 and τ_2 be two topologies on X. If (X, τ_1, τ_2) has the property C_1 or C_2, then for any $n \in \mathbb{N}$, we have that $(\mathcal{F}_n(X), \tau_{1\mathcal{F}_n(X)}, \tau_{2\mathcal{F}_n(X)})$ has the property C_1 or C_2, respectively.*

Proof. First of all, since τ_1 is a zero-dimensional topology on X, then by Theorem 1.47 $\tau_{1\mathcal{F}_n(X)}$ is a zero-dimensional topology on $\mathcal{F}_n(X)$. Now, let $F \in \mathcal{F}_n(X)$ and $\mathcal{U} \in \tau_{2\mathcal{F}_n(X)}$ such that $F \in \mathcal{U}$. Let us suppose that $\mathcal{U} = \langle U_1, \ldots, U_m \rangle$ and that $F = \{x_1, \ldots, x_l\}$. For each $x_k \in F$ there exists $V_k \in \tau_2$ and a compact subset (resp., complete subset) K_k of X with respect to τ_1 such that $x_k \in V_k \subset K_k \subset \bigcap\{U_j : x_k \in U_j\}$. Then $F \in \langle V_1, \ldots, V_l \rangle \subset \langle K_1, \ldots, K_l \rangle \subset \mathcal{U}$. Note that $Z = \bigcup_{j=1}^{l} K_j$ is a compact subset (resp., complete subset) with respect to τ_1. Then by Theorem 1.48 and Proposition 1.46 $\mathcal{F}_n(Z)$ is a compact(resp., complete subset) with respect to $\tau_{1\mathcal{F}_n(X)}$. Moreover $\langle K_1, \ldots, K_l \rangle \subset \mathcal{F}_n(Z)$. By Remark 1.37 $\langle K_1, \ldots, K_m \rangle$ is a closed subset with respect to $\tau_{1\mathcal{F}_n(X)}$, then $\langle K_1, \ldots, K_m \rangle$ is a compact subset (resp., complete subset) with respect that $\tau_{1\mathcal{F}_n(X)}$. This proves that $(\mathcal{F}_n(X), \tau_{1\mathcal{F}_n(X)}, \tau_{2\mathcal{F}_n(X)})$ has property C_1 (resp.,C_2).

□

Theorem 3.21. *For any $n \in \mathbb{N}$, $\mathcal{F}_n(\mathfrak{E}_c)$ is homeomorphic to $\mathfrak{E}_c$.*

Proof. Let $\mathcal{W}$ be a topology in $\mathfrak{E}_{\mathrm{c}}$ which satisfies the conditions in item (2) of Theorem 3.19. By Corolary 2.4, $\mathcal{W}_{\mathcal{F}_n(\mathfrak{E}_{\mathrm{c}})}$ is coaser than the Vietoris topology on $\mathcal{F}_n(\mathfrak{E}_{\mathrm{c}})$. By Proposition 3.20 and Corollary 2.15, $\mathfrak{E}_{\mathrm{c}}$ satisfies all conditions in item (2) of Theorem 3.19. Thus, by Theorem 3.19 $\mathcal{F}_n(\mathfrak{E}_{\mathrm{c}})$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$.

□

Corollary 3.22. *Let $n \in \mathbb{N}$, if X is an $\mathfrak{E}_{\mathrm{c}}$-factor, then $\mathcal{F}_n(X)$ is an $\mathfrak{E}_{\mathrm{c}}$-factor.*

Proof. If X is an $\mathfrak{E}_{\mathrm{c}}$-factor. By Theorem 1.32 X can be embedded as a closed subset of $\mathfrak{E}_{\mathrm{c}}$. By 1.42 $\mathcal{F}_n(X)$ can be embedded as a closed subset of $\mathcal{F}_n(\mathfrak{E}_{\mathrm{c}})$. By Theorem 3.21, the space $\mathcal{F}_n(\mathfrak{E}_{\mathrm{c}})$ is homeomorphic to $\mathfrak{E}_{\mathrm{c}}$. Thus $\mathcal{F}_n(X)$ is an $\mathfrak{E}_{\mathrm{c}}$-factor. □

Corollary 3.23. *For each $k < n$ we have that $\mathcal{F}_n(\mathfrak{E}_c) \setminus \mathcal{F}_k(\mathfrak{E}_c)$ is homeomorphic to $\mathfrak{E}_c$.*

Proof. This follows from Theorem 3.21 and the fact that $\mathcal{F}_n(\mathfrak{E}_c) \setminus \mathcal{F}_k(\mathfrak{E}_c)$ is an open subset of $\mathcal{F}_n(\mathfrak{E}_c)$. □

3.2.2 The hyperspace of finite sets of $\mathfrak{E}_c$

In this section we are going to talk about why $\mathcal{F}(\mathfrak{E}_c)$ is not homeomorphic to $\mathfrak{E}$, $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$. This part of the work is the motivation for the next chapter.

By Proposition 1.39 $\mathcal{F}_n(\mathfrak{E}_c)$ is nowhere dense subset of $\mathcal{F}(\mathfrak{E}_c)$, then $\mathcal{F}(\mathfrak{E}_c)$ is of the first category. Since $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$ are Polish spaces, then by Theorem 1.1 $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$ are not of the first category. This implies that $\mathcal{F}(\mathfrak{E}_{\mathrm{c}})$ cannot be homeomorphic to $\mathfrak{E}_{\mathrm{c}}$ or $\mathfrak{E}_{\mathrm{c}}^\omega$.

On the other hand as $\mathfrak{E}_c$ is a complete space then by Proposition 1.46 and $\mathcal{F}_n(\mathfrak{E}_c)$ is a Polish space for any $n \in \mathbb{N}$, then by Theorem 1.3 and Theorem 1.4 $\mathcal{F}_n(\mathfrak{E}_c)$ is an absolute G_δ. Therefore $\mathcal{F}(\mathfrak{E}_c)$ is an absolute $G_{\delta\sigma}$ by Theorem 1.4. This implies that $\mathcal{F}(\mathfrak{E}_c)$ is not homeomorphic to $\mathfrak{E}$ since $\mathfrak{E}$ is an absolute $F_{\sigma\delta}$ but it is not a $G_{\delta\sigma}$ (see Remark 4.12 [3]).

Furthermore, the space $\mathcal{F}(\mathfrak{E}_{\mathrm{c}}))$ is not a factor of $\mathfrak{E}_c$ (or $\mathfrak{E}_c^\omega$ because otherwise by Theorems 1.32 and 1.36 there would be closed sets of $\mathfrak{E}_c$ (or $\mathfrak{E}_c^\omega$) homeomorphic to $\mathcal{F}(\mathfrak{E}_{\mathrm{c}})$. This would imply that $\mathcal{F}(\mathfrak{E}_{\mathrm{c}})$ is a Polish space which is false. But the space $\mathcal{F}(\mathfrak{E}_{\mathrm{c}})$ is a factor of $\mathfrak{E}$ by Theorem 1.26 and Corollary 3.11.

Proposition 3.24. *$\mathcal{F}(\mathfrak{E}_c)$ is an absolute $G_{\delta\sigma}$, an absolute $F_{\sigma\delta}$, and it is a countable union of nowhere dense copies of $\mathfrak{E}_c$.*

We already know that $\mathcal{F}(\mathfrak{E}_c)$ is not homeomorphic to $\mathfrak{E}$, $\mathfrak{E}_c$ or $\mathfrak{E}_c^\omega$ and by Theorem 3.21 and Proposition 1.39 $\mathcal{F}(\mathfrak{E}_c)$ is a countable union of nowhere dense copies of $\mathfrak{E}_c$. Next we will see that the space $\mathbb{Q} \times \mathfrak{E}_c$ has properties similar to those of $\mathcal{F}(\mathfrak{E}_c)$.

Remark 3.25. *$\mathbb{Q} \times \mathfrak{E}_c$ is an absolute $G_{\delta\sigma}$ and an absolute $F_{\sigma\delta}$*

Proof. To see that $\mathbb{Q} \times \mathfrak{E}_c$ is an absolute $G_{\delta\sigma}$, it is suficient to notice that $\mathbb{Q} \times \mathfrak{E}_c$ is a countable union of Polish spaces. Next, assume that $\mathbb{Q} \times \mathfrak{E}_c \subset X$ where X is any separable metrizable space. For each $q \in \mathbb{Q}$, let $F_q = \{q\} \times \mathfrak{E}_c$. Then $G = X \setminus \bigcup\{cl_X(F_q) : q \in \mathbb{Q}\}$ is a G_δ in X. Fix $q \in \mathbb{Q}$. Since F_q is Polish we know that $cl_X(F_q) \setminus F_q$ is a countable union of sets that are closed in F_q, and thus, in X. But closed sets in separable metrizable spaces are G_δ. Thus, $cl_X(F_q) \setminus F_q$ is $G_{\delta\sigma}$ in X. Since $X \setminus (\mathbb{Q} \times \mathfrak{E}_c) = G \cup (\bigcup\{cl_X(F_q) : q \in \mathbb{Q}\})$ we conclude that the complement of $\mathbb{Q} \times \mathfrak{E}_c$ is a $G_{\delta\sigma}$ so $\mathbb{Q} \times \mathfrak{E}_c$ itself is $F_{\sigma\delta}$ in X.

□

Proposition 3.26. *$\mathbb{Q} \times \mathfrak{E}_c$ is countable union of nowhere dense copies of $\mathfrak{E}_c$.*

Proof. Let d be the metric in $\mathbb{Q}$ that inherited from $\mathbb{R}$ with the usual metric. Since $\mathbb{Q}$ is countable we can list $\mathbb{Q}$ as $\{q_n : n \in \mathbb{N}\}$. We will construct a sequence $\{F_n : n \in \mathbb{N}\}$ of compact subspaces of $\mathbb{Q}$ such that

(a) $F_1 = \{q_1\}$ and $q_n \in F_n$

(b) $F_n \subset F_{n+1}$

(c) $F_{n+1} \setminus F_n$ is countable, discrete, and dense in F_{n+1}.

To construct F_2, let's consider a sequence $\{p_k : k \in \mathbb{N}\}$ that converges to q_1 and add the point q_2 to it. Note that F_2 is a compact space and satisfies the required conditions. In general, for any $n \in \mathbb{N}$ to construct F_{n+1} we do the following. By item (c) we have that $F_n \setminus F_{n-1}$ is a discrete space if $n \geq 2$, then for each $x \in F_n \setminus F_{n-1}$, there exists an $m \in \mathbb{N}$ such that $(x-(1/m), x+(1/m)) \cap (F_n \setminus F_{n-1}) = \{x\}$. Therefore $(x, x+(1/m)) \cap (F_n \setminus F_{n-1}) = \emptyset$. Let $A_x = \{p_k^x : k \in \mathbb{N}\}$ be a sequence such that converges to x and $A_x \subset (x, x+1/k)$ and $p_x^{n+1} > p_x^n$. On the other hand if $x \in F_{n-1}$, by item (c) there exists a sequence $\{p_x^k : k \in \mathbb{N}\} \subset F_n \setminus F_{n-1}$ such that and $p_x^{n+1} > p_x^n$ and $\{p_x^k : k \in \mathbb{N}\}$ converges to x. Let $A_x = \{p_x^k + (1/k) : k \in \mathbb{N}\}$, then A_x converges to x. We define $F_{n+1} = (\bigcup_{x \in F_n} A_x \cup \{q_{n+1}\}) \cup F_n$. Clearly $q_{n+1} \in F_{n+1}$, $F_n \subset F_{n+1}$ and F_{n+1} is countable and closed. Note that if $y \in F_{n+1} \setminus F_n$, then $y = q_x^k + (1/k)$ where $x \in F_{n-1}$. Therefore $(q_x^k, q_x^k + (2/k)) \cap (F_{n+1} \setminus F_n) = \{q_x^k + (1/k)\}$, if $y = q_x^k$ where $x \in F_n \setminus F_{n-1}$ then $(q_x^{k+1}, q_x^{k-1}) \cap (F_{n+1} \setminus F_n) = \{q_x^k\}$. If $r \in \mathbb{Q} \setminus F_{n+1}$, and as F_n is closed there exists $x, y \in \mathbb{Q}$ such that $r \in (x, y)$ and $(x, y) \cap F_n = \emptyset$. If $x \in F_n$ then $A_x \cap (x, y)$ is infinite and if $t \in F_n \setminus \{x\}$, then $A_t \cap (x, y) = \emptyset$. Then there exists $l \in \mathbb{N}$ such that $r \in U = (t - (1/l), t + (1/l))$ and $U \cap F_{n+1} = \emptyset$. If $y \in F_n$ then there exists $x_1 \in F_n$ such that $r \in (x_1, y)$ or not. If there exists $x_1 \in F_n$ such that $r \in (x_1, y)$ then we do the same as the previous case and if there is no exists $x_1 \in F_n$ such that $r \in (x_1, y)$, then $(x, y) \cap F_{n+1} = \emptyset$. Therefore $F_{n+1} \setminus F_n$ is a discrete space and F_{n+1} is closed. To prove that F_{n+1} is compact it is sufficient to prove that it is bounded. Since F_n is compact the set F_n is bounded, therefore there exists $N \in \mathbb{N}$ such that $F_n \subset (-N, N)$, by construction of F_{n+1}, we conclude that $F_{n+1} \subset (-N - (1/m), N + (1/r))$ for some $r, m \in \mathbb{N}$. So F_{n+1} is compact. With this we finish the construction.

Since F_n is a compact and zero-dimensional space, then F_n can be embedded as a closed subset of $\mathfrak{E}_c$. By Theorem 1.32 $F_n \times \mathfrak{E}_c \approx \mathfrak{E}_c$.

□

With Proposition 3.26 and the Remark 3.25 we have that $\mathbb{Q} \times \mathfrak{E}_c$ and $\mathcal{F}(\mathfrak{E}_c)$ has similar properties. This leads us to the next conjecture.

Conjecture 3.27. $\mathbb{Q} \times \mathfrak{E}_c \approx \mathcal{F}(\mathfrak{E}_c)$

In Chapter 4 we will show that this conjecture is true.

Chapter 4

A Characterization of the product $\mathbb{Q} \times \mathfrak{E}_c$

In this chapter we are going to prove an intrinsic and an extrinsic characterization of space $\mathbb{Q} \times \mathfrak{E}_c$ with these characterizations an answer is given to the conjecture 3.27 made in section 3 of Chapter 3. A characterization of the factors of space $\mathbb{Q} \times \mathfrak{E}_c$ will also be given.

4.1 Classes σL and $\sigma \mathcal{E}$

In this section we define the classes of spaces σL and $\sigma \mathcal{E}$ that we will use to characterize $\mathbb{Q} \times \mathfrak{E}_c$. The choice of these symbols is made in the spirit of classes SLC and $\mathcal{E}$ from [3].

Definition 4.1. *We define $\sigma \mathcal{L}$ to be the class of all triples (C, X, φ) such that C is a compact, zero-dimensional, crowded metrizable space, $\varphi\colon C \to [0, 1)$ is an USC function and $X = \bigcup\{X_n : n \in \omega\}$ is a dense subset of C such that for each $n \in \omega$ the following hold*

1. *X_n is a closed, crowded subset of C,*
2. *$X_n \subset X_{n+1}$,*
3. *$\varphi \restriction X_n$ is a Lelek function, and*
4. *$G_0^{\varphi \restriction X_n}$ is nowhere dense in $G_0^{\varphi \restriction X_{n+1}}$.*

We will say that a space E is generated by (C, X, φ) if E is homeomorphic to $G_0^{\varphi \restriction X}$.

As mentioned in the Chapter 1, by the extrinsic characterization of $\mathfrak{E}_c$ (Theorem 1.30), in Definition 4.1 the space $G_0^{\varphi \restriction X_n}$ is homeomorphic to $\mathfrak{E}_c$ for each $n \in \omega$. So indeed, E is a countable increasing union of nowhere dense subsets, each homeomorphic to complete Erdős space.

Definition 4.2. *We define $\sigma\mathcal{E}$ to be the class of all separable metrizable spaces E such that there exists a topology $\mathcal{W}$ on E that is witness to the almost zero-dimensionality of E, a collection $\{E_n \colon n \in \omega\}$ of subsets of E and a base β of neighborhoods of E such that*

(a) $E = \bigcup\{E_n \colon n \in \omega\}$,

(b) for each $n \in \omega$, E_n is a crowded nowhere dense subset of E_{n+1},

(c) for each $n \in \omega$, E_n is closed in $\mathcal{W}$,

(d) E is $\{E_n \colon n \in \omega\}$-cohesive, and

(e) for each $V \in \beta$, $V \cap E_n$ is compact in $\mathcal{W} \restriction E_n$ for each $n \in \omega$.

By the intrinsic characterization of $\mathfrak{E}_c$ (Theorem 1.31), in Definition 4.2, E_n is homemorphic to $\mathfrak{E}_c$ for every $n \in \omega$. So again E is a countable increasing union of nowhere dense subsets, each homeomorphic to complete Erdős space.

We first prove that the space that we want to characterize is an element of $\sigma\mathcal{E}$ and then, that spaces from $\sigma\mathcal{E}$ can be generated by triples from $\sigma\mathcal{L}$.

Lemma 4.3. *$\mathbb{Q} \times \mathfrak{E}_c \in \sigma\mathcal{E}$.*

Proof. By Theorem 1.31, there exists a topology $\mathcal{W}_1$ on $\mathfrak{E}_c$, witness of the almost zero-dimensionality of $\mathfrak{E}_c$, such that $\mathfrak{E}_c$ has a neighborhood base β_0 of subsets that are compact in $\mathcal{W}_1$. Let $\mathcal{W}$ be the product topology of $\mathbb{Q} \times (\mathfrak{E}_c, \mathcal{W}_1)$. Let β be the collection of all sets of the form $V \times B$, where V is non-empty and clopen in $\mathbb{Q}$, and $B \in \beta_0$. Choose a sequence $\{F_n \colon n \in \omega\}$ of compact subsets of $\mathbb{Q}$ such that (i) $F_n \subset F_{n+1}$ for every $n \in \omega$, (ii) $F_{n+1} \setminus F_n$ is countable discrete, and dense in F_{n+1} for every $n \in \omega$, and (iii) $\mathbb{Q} = \bigcup\{F_n \colon n \in \omega\}$ as in proof Proposition 3.26. Let $E_n = F_n \times \mathfrak{E}_c$ for every $n \in \omega$. We claim that the topology $\mathcal{W}$, the collection $\{E_n \colon n \in \omega\}$ and β satisfsy the conditions in Definition 4.2 for $\mathbb{Q} \times \mathfrak{E}_c$. First, notice that $\mathcal{W}$

witnesses that $\mathbb{Q} \times \mathfrak{E}_c$ is almost zero-dimensional. Conditions (a), (b) and (c) follow directly from our choices.

Next, we prove (d). Let $(x, y) \in \mathbb{Q} \times \mathfrak{E}_c$ and let $m = \min\{k \in \omega \colon x \in F_k\}$. Since $\mathfrak{E}_c$ is cohesive, there exists an open set U of $\mathfrak{E}_c$ such that $x \in U$ and U contains no non-empty clopen subsets. Let V be open in $\mathbb{Q}$ such that $x \in V$ and $V \cap F_k = \emptyset$ if $k < m$. Define $W = V \times U$. Let $n \in \omega$, we argue that $W \cap E_n$ contains no non-empty clopen sets. This is clear if $n < m$ since $W \cap E_n = \emptyset$ so consider the case when $n \geq m$. Assume that $O \subset W \cap E_n$ is clopen and non-empty, and consider $(a, b) \in O$. Then $(\{a\} \times \mathfrak{E}_c) \cap O$ is a non-empty clopen subset of $\{a\} \times \mathfrak{E}_c$ such that $(\{a\} \times \mathfrak{E}_c) \cap O \subset \{a\} \times U$. This is a contradiction to our choice of U. We conclude that (d) holds.

Finally, let us prove (e). Let $V \times B \in \beta$ and $n \in \omega$. Then $(V \times B) \cap E_n = (V \cap F_n) \times B$, which is compact. Also, it is clear that β is a base for the topology of $\mathbb{Q} \times \mathfrak{E}_c$. This completes the proof of this result. $\square$

For the next result we will need to state the following definitions.

If $\varphi : X \to \mathbb{R}$ then we define

$$M(\varphi) = \sup\{|\varphi(x)| : x \in X\} \in [0, \infty].$$

If $X = \emptyset$ then we use the convention $M(\varphi) = \sup \emptyset = 0$.

Definition 4.4. *Let $\varphi : X \to \mathbb{R}$ be a function and let X be a subset of a metric space (Y, d). We define $ext_Y \varphi : Y \to [0, \infty]$ by*

$$(ext_Y \varphi)(y) = \lim_{\epsilon \to 0} M(\varphi \restriction (X \cap U_\epsilon(y))); \ \textit{for } y \in Y,$$

where $U_\epsilon(y) = \{x \in Y : d(x, y) < \epsilon\}$.

Proposition 4.5. *If $E \in \sigma\mathcal{E}$ then there exists $(C, X, \varphi) \in \sigma\mathcal{L}$ that generates E.*

Proof. From Definition 4.2, let us consider for E: the witness topology $\mathcal{W}$, the base β of neighborhoods, and the collection $\{E_n \colon n \in \omega\}$.

We may assume that β is countable. For every $B \in \beta$, let $\mathcal{B}_B$ be a countable collection of clopen subsets of $(E, \mathcal{W})$ such that $B = \bigcap \mathcal{B}_B$. Let the boolean algebra clopens that is generated by $\{\mathcal{B}_B : B \in \beta\}$, then there exists a compact, zero-dimensional and metric space C containing $(E, \mathcal{W})$ as a dense subspace and such that $\mathrm{cl}_C(O)$ is clopen in C for every $O \in \bigcup\{\mathcal{B}_B \colon B \in \beta\}$

(see page 4). For every $n\in\omega$ let $X_n=\mathrm{cl}_C(E_n)$; notice that $X_n\cap E=E_n$ since E_n is closed in $\mathcal{W}$. Define $X=\bigcup\{X_n\colon n\in\omega\}$.

We claim that X is a witness to the almost zero-dimensionality; we will prove that B is closed in X for every $B\in\beta$. It is enough to prove that if $m\in\omega$ and $B\in\beta$ are fixed then

$$\left(\bigcap\{\mathrm{cl}_C(O):O\in\mathcal{B}_B\}\right)\cap X_m=B\cap X_m.\ (*)$$

The right side of $(*)$ is contained in the left side by the definition of $\mathcal{B}_B$. So take $z\in C$ that is not on the right side of $(*)$, we will prove that it is not on the left side.

We may assume that $z\in X_m$. By the choice of β, we know that $B\cap X_m$ is compact. So there is an open set U of C such that $z\in U$ and $\mathrm{cl}_C(U)\cap(B\cap X_m)=\emptyset$. Let $F=\mathrm{cl}_C(U)\cap E_m$. Notice that F is closed in $(E_m,\mathcal{W}\restriction E_m)$, and thus, in $(E,\mathcal{W})$. Also, since $U\cap X_n$ is open in X_n, E_n is dense in X_n and $z\in U\cap X_n$, then it easily follows that $z\in\mathrm{cl}_C(F)$. Finally, F is disjoint from B because $F\cap B=(\mathrm{cl}_C(U)\cap E_m)\cap B=\mathrm{cl}_C(U)\cap(B\cap E_m)=\mathrm{cl}_C(U)\cap(B\cap X_m)=\emptyset$. Then F and B are two disjoint closed subsets in $(E,\mathcal{W})$ so there exists $O\in\mathcal{B}_B$ such that $O\cap F=\emptyset$. Since $\mathrm{cl}_C(O)$ is open in K and disjoint from F, it is also disjoint from $\mathrm{cl}_C(F)$. But $z\in\mathrm{cl}_C(F)$, so $z\notin\mathrm{cl}_C(O)$. This shows that z is not on the left side of $(*)$.

We have proved that X is a witness to the almost zero-dimensionality of $\langle E,\mathcal{W}\rangle$. By Lemma 1.12 there exists a USC function $\psi_0\colon X\to[0,1)$ such that $\psi_0^{\leftarrow}(0)=X\setminus E$ and the function $h_0\colon E\to G_0^{\psi_0}$ defined by $h_0(x)=(x,\psi_0(x))$ is a homeomorphism. By condition (d) in Definition 4.2 we know that $G_0^{\psi_0}$ is $\{G_0^{\psi_0\restriction X_n}\colon n\in\omega\}$-cohesive. Moreover, $\{x\in X_n\colon\psi_0(x)>0\}=E_n$ is dense in X_n for every $n\in\omega$. Lemma 1.29 tells us that we can find a USC function $\psi_1\colon X\to[0,1)$ such that $\psi_1\restriction X_n$ is a Lelek function for each $n\in\omega$, and the function $h_1\colon G_0^{\psi_0}\to G_0^{\psi_1}$ given by $h_1((x,\psi_0(x)))=(x,\psi_1(x))$ is a homeomorphism. Now, let $\varphi=\mathrm{ext}_C(\psi_1)\colon C\to[0,1)$. Then (C,X,φ) can be easily seen to be an element of $\sigma\mathcal{L}$ and $h_1\circ h_0\colon E\to G_0^{\varphi\restriction X}$ is a homeomorphism.This completes the proof of this result. $\square$

Our main result will be the following.

Theorem 4.6. *Let E be a space. Then the following are equivalent:*

1. $E\in\sigma\mathcal{E}$,
2. *there exists $(C,X,\varphi)\in\sigma\mathcal{L}$ that generates E, and*

3. E is homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$.

The proof of Theorem 4.6 will be given as follows. First, notice that by Proposition 4.5, (i) implies (ii). That (iii) imples (i) is Lemma 4.3. Also, by Lemma 4.3, $\sigma\mathcal{E}$ is non-empty so $\sigma\mathcal{L}$ is non-empty as well. Thus, in order to prove that (ii) implies (iii) it is enough to show that any two spaces generated by triples of $\sigma\mathcal{L}$ are homeomorphic. This will be the content of Section 4.2.

Given a separable metrizable space X, in [20] $\mathsf{CAP}(X)$ is defined to be the class of separable metrizable spaces $Y = \bigcup\{X_n : n \in \omega\}$ such that X_n is closed in X, X_n is a nowhere dense subset of X_{n+1} and $X_n \approx X$ for each $n \in \omega$. So $\sigma\mathcal{E} \subset \mathsf{CAP}(\mathfrak{E}_c)$ but we do not know whether the other inclusion holds.

Question 4.7. *Is $\sigma\mathcal{E} = \mathsf{CAP}(\mathfrak{E}_c)$?*

4.2 Uniqueness theorem

In this section we present the proof of Theorem 4.6.

Definition 4.8. *Let $\varphi\colon X \to [0,\infty)$, $\psi\colon Y \to [0,\infty)$ be USC functions, φ and ψ are m-*equivalent *if there is a homeomorphism $h\colon X \to Y$ and a continuous function $\alpha\colon X \to (0,\infty)$ such that $\psi \circ h = \alpha \cdot \varphi$.*

It follows that when φ and ψ are m-equivalent then G_0^{φ} is homeomorphic to G_0^{ψ}. So, according to the discussion at the end of the previous section, in order to prove Theorem 4.6, it is sufficient to prove the following statement.

Proposition 4.9. *Let $(C, X, \varphi), (D, Y, \psi) \in \sigma\mathcal{L}$. Then there exists a homeomorphism $h\colon C \to D$ and a continuous function $\alpha\colon C \to (0,\infty)$ such that $f[X] = Y$ and $\psi \circ h = \alpha \cdot \varphi$.*

The rest of this section will consist on a proof of Proposition 4.9. The construction of the homeomorphism h will require us to use two different techniques and mix them. First, we need the tools used in [3] to extend homeomorphisms using Lelek functions.

Theorem 4.10 ([3, Theorem 6.2, p. 26])**.** *If $\varphi\colon C \to [0,\infty)$ and $\psi\colon D \to [0,\infty)$ are Lelek functions with C and D compact, and $t > |\log(M(\varphi)/M(\psi))|$, then there exists a homeomorphism $h\colon C \to D$ and a continuous function $\alpha\colon C \to (0,\infty)$ such that $\psi \circ h = \alpha \cdot \phi$ and $M(\log \circ \alpha) < t$.*

Theorem 4.11 ([3, Theorem 6.4, p. 28])**.** *Let $\varphi\colon C \to [0,\infty)$ and $\psi\colon D \to [0,\infty)$ be Lelek functions with C and D compact. Let $A \subset C$ and $B \subset D$ be closed such that $G_0^{\varphi\restriction A}$ and $G_0^{\psi\restriction B}$ are nowhere dense in G_0^{φ} and G_0^{ψ}, respectively. Let $h\colon A \to B$ be a homeomorphism and $\alpha\colon A \to (0,\infty)$ a continuous function such that $\psi \circ h = \alpha \cdot (\varphi \restriction A)$. If $t \in \mathbb{R}$ is such that $t > |\log(M(\psi)/M(\varphi))|$ and $M(\log\circ\alpha) < t$ then there is a homeomorphism $H\colon C \to D$ and a continuous function $\beta\colon C \to (0,\infty)$ such that $H \restriction A = h$, $\beta \restriction C = \alpha$, $\psi \circ H = \beta \cdot \varphi$ and $M(\log\circ\beta) < t$.*

Theorem 4.10 is called the Uniqueness Theorem for Lelek functions; Theorem 4.11 is the Homeomorphism Extension Theorem for Lelek funcions.

The second tool we will need is that of *Knaster-Reichbach covers*.

Definition 4.12. *Let X and Y be zero-dimensional spaces, $A \subset X$ and $B \subset Y$ be closed and nowhere dense in X and Y, respectively. Moreover, let $h\colon A \to B$ be a homeomorphism. A triple $\langle \mathcal{U}, \mathcal{V}, \eta\rangle$ is called a* Knaster-Reichbach cover*, or* KR-*cover, for $\langle X\setminus A, Y\setminus B, h\rangle$ if the following conditions are satisfied:*

1. *$\mathcal{U}$ is a partition of $X \setminus A$ into non-empty clopen subsets of X,*
2. *$\mathcal{V}$ is a partition of $Y \setminus B$ into non-empty clopen subsets of Y,*
3. *$\eta\colon \mathcal{U} \to \mathcal{V}$ is a bijection,*
4. *if for every $U \in \mathcal{U}$, $g_U\colon U \to \eta(U)$ is a bijection, then the function $H = h \cup \left(\bigcup_{U\in\mathcal{U}} g_U\right)$ is continuous at all points of A, and its inverse H^{-1} is continuous at all points of B.*

KR-covers were used by Knaster and Reichbach [12] to prove homeomorphism extension results in the class of all zero-dimensional spaces. The term KR-cover was first used by van Engelen [8] who proved their existence in a general setting. However, in this thesis we will not need the existence of KR-covers in general. We will only need the following straightforward result which is a specific case of KR-covers.

Lemma 4.13. *Fix a metric on 2^ω. Let $F \subset 2^\omega$ be closed and assume that $\mathcal{U} = \{U_n : n \in \omega\}$ is a partition of $2^\omega \setminus F$ into clopen sets such that for every $\epsilon > 0$ the set $\{n \in \omega : \operatorname{diam}(U_n) \geq \epsilon\}$ is finite. Assume that $h\colon 2^\omega \to 2^\omega$ has the following properties*

1. *h is a bijection,*

2. *$h \restriction F = \mathsf{id}_F$,*

3. *for each $n \in \omega$, $h[U_n] = U_n$, and*

4. *for each $n \in \omega$, $h \restriction U_n \colon U_n \to U_n$ is a homeomorphism.*

Then h is a homeomorphism.

Proof. Let $A = \{x_n \in X : n \in \mathbb{N}\}$ be a sequence of X such that $x_n \to x$. We have the following cases.

1. $A \subset X \setminus F$ and $x \in X \setminus F$

2. $A \subset F$ and $x \in F$,

3. $A \subset X \setminus F$ and $x \in F$.

Note that in cases 1, 2 by hypothesis we have that $\lim_{i\to\infty} h(x_i) = h(x)$. Then is enough prove item 3. As $A \subset X \setminus F$, then for any $x_n \in A$, there exists $k(n) \in \mathbb{N}$ such that $x_n \in V_{k(n)}$. Given that $h[V_{k(n)}] = V_{k(n)}$, then $\{x_n, h(x_n)\} \subset V_{k(n)}$. This implies that $d(x_n, h(x_n)) \leq diam(V_{k(n)})$. Given that $diam(V_{k(n)}) \to 0$, then $d(x_n, h(x_n)) \to 0$. On the other hand, note that,

$$d(h(x_n), x) \leq d(x_n, x) + d(x_n, h(x_n))$$

This implies that $h(x_n) \to x$. □

We then remark that the proof of Proposition 4.9 will be an amalgamation of the Dijkstra-van Mill proof of Theorem 7.5 from [3] and the van Engelen proof of Theorem 3.2.6 from [8]. The functions h and α in the statement of Proposition 4.9 will be uniform limits of functions. The following discussion can be found in [21].

Let X and Y be compact metrizable spaces and let ρ be a metric on Y. In the set $C(X, Y) = \{f \in Y^X \colon f \text{ is continuous}\}$ we define the **uniform metric** ρ by $\rho(f, g) = \sup\{\rho(f(x), g(x)) \colon x \in X\}$, when $f, g \in C(X, Y)$. It is known that this metric is complete so we may construct complicated continuous functions using Cauchy sequences of simpler continuous functions.

For a compact space X, $\mathcal{H}(X)$ denotes the subset of $C(X, X)$ consisting of homeomorphisms. However, even though Cauchy sequences of homeomorphisms will converge to continuous functions, they will not necessarily converge to a homeomorphism. In order to achieve this, we will use the *Inductive Convergence Criterion*. We present the statement of this criterion as it appears in [8].

Theorem 4.14 ([8, Lemma 3.2.5]). *Let X be a zero-dimensional compact metric space with metric ρ and for each $n \in \omega$, let $h_n \colon X \to X$ be a homeomorphism. If for every $n \in \mathbb{N}$ we have that $\rho(h_{n+1}, h_n) < \epsilon_n$, where*

$$\epsilon_n = \min\{2^{-n}, 3^{-n} \cdot min\{min\{\rho(h_i(x), h_i(y)) \colon x, y \in X, \\ \rho(x, y) \geq 1/n\} : i \leq n\}\},$$

then the uniform limit $h = lim_{n\to\infty} h_n$ is a homeomorphism.

The exact values of the numbers ϵ_n in the statement of Theorem 4.14 are not important. What we will use is that ϵ_n is a positive number than can be calculated once the first $n+1$ homeomorphisms $h_0, \dots, h_n$ have been defined.

Before we continue with the proof of Proposition 4.9, we stop to give two final ingredients in the proof.

Lemma 4.15 ([3, Lemma 4.8]). *Let X be a zero-dimensional space, $Y \subset X$, let $\psi : Y \to [0, \infty)$ be a USC function and $\varphi = ext_X(\psi)$. Then φ is USC, $\psi \subset \varphi$ and the graph of ψ is dense in the graph of φ.*

Lemma 4.16. *If $(C, X, \psi) \in \sigma\mathcal{L}$ then there exists a Lelek function $\varphi \colon C \to [0, 1]$ such that $(C, X, \varphi) \in \sigma\mathcal{L}$, $\varphi \restriction X = \psi \restriction X$ and the graph of $\varphi \restriction X$ is dense in the graph of φ.*

Proof. Let d_0 be a metric for C and consider the metric $d(\langle x, y\rangle, \langle z, w\rangle) = d_0(x, z) + |y - w|$ defined on $C \times [0, 1]$. Define $\varphi = \mathrm{ext}_C(\psi \restriction X)$.

We show that φ is a Lelek function. Let $p \in C$ with $\varphi(p) > 0$, $t \in (0, \varphi(p))$ and $\epsilon > 0$, we want to find $q \in G_0^\varphi$ such that $d(q, \langle p, t\rangle) < \epsilon$. By Lemma 4.15 we know that the graph of $\psi \restriction X$ is dense in the graph of φ so there exists $k \in \omega$ and $x \in X_k$ such that $d(\langle x, \psi(x)\rangle, \langle p, \varphi(p)\rangle) < \epsilon/2$. We may also assume that $\psi(x) > t$. Since $\psi \restriction X_k$ is a Lelek function, there is $z \in X_k$ such that $d(\langle z, \psi(z)\rangle, \langle x, t\rangle) < \epsilon/2$. So let $q = \langle z, \psi(z)\rangle$. By Lemma 4.15 we know that $\psi(z) = \varphi(z)$ so $q \in G_0^\varphi$. Then

$$\begin{aligned}
d(q, \langle p, t\rangle) &= d_0(z, p) + |\psi(z) - t| \\
&\leq d_0(z, x) + d_0(x, p) + |\psi(z) - t| \\
&= d(\langle z, \psi(z)\rangle, \langle x, t\rangle) + d_0(x, p) \\
&\leq d(\langle z, \psi(z)\rangle, \langle x, t\rangle) + d(\langle x, \psi(x)\rangle, \langle p, \varphi(p)\rangle) \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

This shows that φ is a Lelek function. The remaining condition holds directly from Lemma 4.15. $\square$

The constant function with value 1 will be denoted by $\mathbf{1}$.

Lemma 4.17. *Let $F \subset 2^\omega$ be closed and let $\{V_n : n \in \omega\}$ be a partition of $2^\omega \setminus F$ into clopen non-empty subsets. Assume that $\alpha\colon 2^\omega \to (0, \infty)$ has the following properties*

1. $\alpha \restriction F = \mathbf{1} \restriction F$,
2. $lim_{n\to\infty} M(\log \circ (\alpha \restriction V_n)) = 0$, *and*
3. $\alpha \restriction V_n$ *is continuous for each* $n \in \omega$.

Then α is continuous.

Proof. Let $A = \{x_n \in X : n \in \mathbb{N}\}$ be a sequence of X such that $x_n \to x$. We have the following cases.

1. $A \subset X \setminus F$ and $x \in X \setminus F$
2. $A \subset F$ and $x \in F$,
3. $A \subset X \setminus F$ and $x \in F$.

Note that in cases 1, 2 by hypothesis $\lim_{i\to\infty} \alpha(x_i) = \alpha(x)$. Then is enough prove item 3.

Let $\epsilon > 0$. By the continuity of the exponential function there is $\delta > 0$ be such that if $t \in (-\delta, \delta)$ then $e^t \in (1 - \epsilon, 1 + \epsilon)$. By condition (2) there exists $N \in \omega$ such that if $n \geq N$, then $|M(\log \circ (\alpha \restriction V_n))| < \delta$. On the other hand, there exists $k \in \omega$ such that if $i > k$ then $x_i \in \bigcup\{V_n : n \geq N\}$. If $i \geq k$ we obtain that $|\log(\alpha(x_i))| < \delta$ so $\log(\alpha(x_i)) \in (-\delta, \delta)$. Thus, $\alpha(x_i) \in (1 - \epsilon, 1 + \epsilon)$ so $|\alpha(x_i) - 1| < \epsilon$. $\square$

Proof of Proposition 4.9. Without loss of generality we assume that $C = D = 2^\omega$, and we fix some metric ρ on 2^ω. By an application of Lemma 4.16 we can assume that φ and ψ are Lelek functions, that the graph of $\varphi \restriction X$ is dense in the graph of φ, and that the graph of $\psi \restriction Y$ is dense in the graph of ψ. After this, apply Theorem 4.10, so we may assume that $\varphi = \psi$. Then $(2^\omega, X, \varphi), (2^\omega, Y, \varphi) \in \sigma\mathcal{L}$ so there are collections $\{X_n : n \in \omega\}$ and $\{Y_n : n \in \omega\}$ that satisfy the conditions in Definition 4.1. Notice that since the graphs of $\varphi \restriction X$ and $\varphi \restriction Y$ are dense in the graph of φ it is easy to see that

($*$) if $U \subset X$ is open then

$$M(\varphi \restriction U) = \sup\{M(\varphi \restriction U \cap X_i)\colon i \in \omega\} = \sup\{M(\varphi \restriction U \cap Y_i)\colon i \in \omega\}.$$

Given $s \in \omega^{<\omega}$, we construct clopen sets U_s and V_s of 2^ω, closed nowhere dense sets D_s and E_s of X and Y, respectively, and for every $m \in \omega$ a continuous function $\beta_m\colon 2^\omega \to (0,1)$ and a homeomorphism $h_m\colon 2^\omega \to 2^\omega$. We abreviate the composition $h_n \circ \ldots \circ h_0 = f_n$ for all $n \in \omega$. We will use the Inductive Convergence Criterion (Theorem 4.14) to make the homeomorphisms converge, so at step n we may calculate the corresponding $\epsilon_n > 0$. Our construction will have the following properties.

(a) $U_\emptyset = V_\emptyset = 2^\omega$.

(b) For each $s \in \omega^{<\omega}$, $D_s \subset U_s$ and $E_s \subset V_s$.

(c) For every $n \in \omega$ and $s \in \omega^n$, $\{U_{s^\frown i}\colon i \in \omega\}$ is a partition of $U_s \setminus D_s$ and $\{V_{s^\frown i}\colon i \in \omega\}$ is a partition of $V_s \setminus E_s$.

(d) For every $n \in \omega$, $X_n \subset \bigcup\{D_s\colon s \in \omega^{\leq n}\}$ and $Y_n \subset \bigcup\{E_s\colon s \in \omega^{\leq n}\}$.

(e) For every $n \in \omega$ and $s \in \omega^{n+1}$, $\mathrm{diam}(U_s) \leq 2^{-n}$ and $\mathrm{diam}(V_s) \leq \min\{2^{-n}, \epsilon_n\}$.

(f) For every $n \in \omega$ and $s \in \omega^n$, $f_n[D_s] = E_s$.

(g) For every $n \in \omega$ and $s \in \omega^n$, $h_{n+1} \restriction E_s = \mathsf{id}_{E_s}$.

(h) For every $n \in \omega$ and $s \in \omega^n$, $f_{n+1}[U_s] = V_s$.

(i) For every $n, k \in \omega$, $\{s \in \omega^n\colon \mathrm{diam}(U_s) \geq 2^{-k}\}$ is finite.

(j) For every $n \in \omega$ and $x \in 2^\omega$, $|\log(\beta_{n+1}(x)/\beta_n(x))| < 2^{-n}$.

(k) For every $n \in \omega$, $\varphi = (\beta_n \cdot \varphi) \circ f_n^{-1}$.

Let us assume that we have finished this construction, we claim that $f = \lim_{n\to\infty} f_n$ exists, is a homeomorphism and $f[X] = Y$.

First, let $x \in 2^\omega$ and $n \in \omega$. If $x \in \bigcup_{s\in\omega^n} D_s$, then $f_n(x) = f_{n+1}(x)$ by conditions (f) and (g). Thus, $\rho(f_n(x), f_{n+1}(x)) = 0$. Otherwise, by (c) there exists $s \in \omega^{n+1}$ with $x \in U_s$. By (h), $f_n(x) \in V_s$. Moreover, applying

(c) and (h) we conclude that $f_{n+1}(x) \in V_s$. So $\rho(f_n(x), f_{n+1}(x)) < \epsilon_n$ by the second part of (e). Thus, $\rho(f_n, f_{n+1}) < \epsilon_n$ and we can apply the Inductive Convergence Criterion to conclude that f is well-defined and in fact, a homeomorphism.

Next, let $x \in X$ so $x \in X_m$ for some $m \in \omega$. Thus, by (b) there exists $s \in \omega^{\leq m}$ such that $x \in D_s$. Then $f_{|s|}(x) \in E_s \subset Y$ by (f). By (g) it inductively follows that $f_n(x) = f_{|s|}(x)$ for every $n \geq |s|$. This implies that $f(x) \in Y$. A completely analogous argument shows that if $y \in Y$ then there is $x \in X$ such that $f(x) = y$. This shows that $f[X] = Y$.

By (j) we know that $\{\beta_n : n \in \omega\}$ is a Cauchy sequence with the uniform metric so $\beta = \lim_{n\to\infty} \beta_n$ exists and is a continuous function. Using the first part of (e) it is possible to prove that $\{f_n^{-1} : n \in \omega\}$ is also a Cauchy sequence and converges to f^{-1}; this proof is completely analogous to the proof that $f = \lim_{n\to\infty} f_n$ so we omit it. Then, by uniform continuity we infer that $\lim_{n\to\infty} \beta_n \circ f_n^{-1} = \beta \circ f$. So using that φ is USC and (k) we obtain the following

$$\begin{aligned}
\beta(x) \cdot \varphi(x) &= \lim_{n\to\infty} \beta_n(x) \cdot \varphi(x) \\
&= \lim_{n\to\infty} \varphi(f_n(x)) \\
&\leq \varphi(f(x)) \\
&= \lim_{n\to\infty} \varphi(f_n(f_n^{-1}(f(x)))) \\
&= \lim_{n\to\infty} \beta_n(f_n^{-1}(f(x))) \cdot \varphi(f_n^{-1}(f(x))) \\
&\leq \beta(x) \cdot \varphi(x)
\end{aligned}$$

Thus, $\varphi \circ f = \beta \cdot \varphi$. This argument is completely analogous to the one in [3, Theorem 7.5].

Now we carry out the construction. Let $\gamma : \omega^{<\omega} \setminus \{\emptyset\} \to \omega$ be any function such that $\gamma \restriction \omega^{m+1}$ is injective for all $m \in \omega$.

Step 0. Let $U_\emptyset = V_\emptyset = 2^\omega$, as in condition (a). From $(*)$ we infer that there exists $k_\emptyset \in \omega$ such that

$$\begin{aligned}
&\log(M(\varphi)) - \log(M(\varphi \restriction X_{k_\emptyset})) < 1/2, \quad \text{and} \\
&\log(M(\varphi)) - \log(M(\varphi \restriction Y_{k_\emptyset})) < 1/2.
\end{aligned}$$

Define $D_\emptyset = X_{k_\emptyset}$ and $E_\emptyset = Y_{k_\emptyset}$. Then $\varphi \restriction D_\emptyset$ and $\varphi \restriction E_\emptyset$ are Lelek functions, and $|\log(M(\varphi \restriction D_\emptyset)/M(\varphi \restriction E_\emptyset))| < 1$ so we may apply Theorem 4.10 to obtain a homeomorphism $\widehat{h}_\emptyset : D_\emptyset \to E_\emptyset$ and a continuous function $\alpha_\emptyset : D_\emptyset \to (0, \infty)$ such that $\varphi \circ \widehat{h}_\emptyset = (\varphi \restriction D_\emptyset) \cdot \alpha_\emptyset$ and $M(\log \circ \alpha_\emptyset) < t$. After this, apply Theorem 4.11 to find a homeomorphism $h_0 : 2^\omega \to 2^\omega$ and a continuous function $\beta_0 \colon 2^\omega \to (0, \infty)$ such that $h_0 \restriction D_\emptyset = \widehat{h}_\emptyset$, $\beta_0 \restriction D_\emptyset = \alpha_\emptyset$, $\varphi \circ h_0 = \varphi \cdot \beta_0$ and $M(\log \circ \alpha_0) < 1$.

Notice that since $h_0 = f_0$ this implies (k) for $n = 0$. Let $\{V_n \colon n \in \omega\}$ be a partition of $E_\emptyset$ into clopen sets with their diameters converging to 0. We may assume that $\mathrm{diam}(V_n) < \min\{\epsilon_0, 1\}$ for every $n \in \omega$. We define $U_n = h_0^{\leftarrow}[V_n]$ for each $n \in \omega$. Without loss of generality we may assume that for all $n \in \omega$, $\mathrm{diam}(U_n) < 1$. With this we have finished step 0 in the construction.

Inductive step: Assume that we have constructed the sets D_s, E_s for $s \in \omega^{\leq m}$, the sets U_s, V_s for $s \in \omega^{\leq m+1}$ the homeomorphisms h_i for $i \leq m$, and the continuous functions β_i for $i \leq m$. Notice that by condition (c) it inductively follows that $\bigcup\{D_s \colon s \in \omega^{\leq m}\}$ and $\bigcup\{E_s \colon s \in \omega^{\leq m}\}$ are closed because their complement is $\bigcup\{U_s : s \in \omega^{m+1}\}$, and $\bigcup\{V_s : s \in \omega^{m+1}\}$, respectively.

Fix $t \in \omega^{m+1}$. First, notice that by $(*)$ we have that there exists $k_t \in \omega$ such that

$$\log\left(M(\varphi \restriction V_t)\right) - \log\left(M(\varphi \restriction V_t \cap Y_{k_t})\right) < 2^{-(m+1+\gamma(t))}.$$

Notice that $\varphi \restriction V_t \cap Y_{k_t}$ is a Lelek function.

Recall that (k) says that $\varphi = (\beta_m \cdot \varphi) \circ f_n^{-1}$. In particular this implies that $\varphi \restriction V_t = (\beta_m \cdot \varphi) \restriction U_t \circ f_n^{-1} \restriction V_t$; from this we infer the following. First, using $(*)$ we may assume that $k_t \in \omega$ is such that

$$\log\left(M(\varphi \restriction V_t)\right) - \log\left(M(\varphi \restriction V_t \cap f_m[X_{k_t}])\right) < 2^{-(m+1+\gamma(t))}.$$

Also, $\varphi \restriction V_t \cap f_m[X_{k_t}]$ is a Lelek function.

So define $D_t = V_t \cap f_m[X_{k_t}]$ and $E_t = V_t \cap Y_{k_t}$. Then $\varphi \restriction D_t$ and $\varphi \restriction E_t$ are Lelek functions, and $|\log(M(\varphi \restriction D_t)/M(\varphi \restriction E_t))| < 2^{-(m+\gamma(t))}$ so we may apply Theorem 4.10 to obtain a homeomorphism $\widehat{h}_t \colon D_t \to E_t$ and a continuous function $\widehat{\alpha}_t \colon D_t \to (0, \infty)$ such that $\varphi \circ \widehat{h}_t = \varphi \cdot \widehat{\alpha}_t$ and $M(\log \circ \widehat{\alpha}_t) < 2^{-(m+\gamma(t))}$. Then apply Theorem 4.11 to find a homeomorphism $h_t \colon V_t \to V_t$ and a continuous function $\alpha_t \colon V_t \to (0, \infty)$ such that $h_t \restriction D_t = \widehat{h}_t$, $\alpha_t \restriction D_t = \widehat{t}_t$, $\varphi \circ h_t = \varphi \cdot \alpha_t$ and $M(\log \circ \alpha_t) < 2^{-(m+\gamma(t))}$.

Let $E_m = \bigcup\{E_s \colon s \in \omega^{\leq m}\}$. Then define

$$h_{m+1} = \mathsf{id}_{E_m} \cup \bigcup\{h_s \colon s \in \omega^{m+1}\},$$

by Lemma 4.13 it follows that h_{m+1} is a homeomorphism. Also, define

$$\alpha_{m+1} = 1 \restriction E_m \cup \bigcup\{\alpha_s \colon s \in \omega^{m+1}\},$$

and $\beta_{m+1}(x) = \alpha_{m+1}(f_m(x)) \cdot \beta_m(x)$ for all $x \in 2^\omega$. By Lemma 4.17, α_{m+1} is continuous so β_{m+1} is continuous.

Now, fix $t \in \omega^{m+1}$ again. Write $V_t \setminus E_t$ as a union of a countable, pairwise disjoint collection of clopen sets, all diameters of which are smaller than $\min\{\epsilon_m, 2^{-m}\}$ and converge to 0. Let $\{V_{t^\frown i} : i \in \omega\}$ be such partition and for each $i \in \omega$, let $U_{t^\frown i} = f_{m+1}^{-1}[V_{t^\frown i}]$. Without loss of generality we may assume that for $i \in \omega$, $\operatorname{diam}(U_{t^\frown i}) < 2^{-m}$.

We leave the verification that all conditions (a) to (k) hold in this step of the induction to the reader. This concludes the inductive step, and the proof of this result. $\square$

4.3 The hyperspace of finite sets of $\mathfrak{E}_c$

In this section we will apply the Theorem 4.6 to prove that $\mathcal{F}(\mathfrak{E}_c) \approx \mathbb{Q} \times \mathfrak{E}_c$.

Proposition 4.18. $\mathcal{F}(\mathfrak{E}_c) \in \sigma\mathcal{E}$

Proof. According to (2) in Theorem 1.31 there is a witness topology $\mathcal{W}_0$ for $\mathfrak{E}_c$ and a base β_0 for $\mathfrak{E}_c$ of sets that are compact in $\mathcal{W}_0$. Let $\mathcal{W}_1$ the Vietoris topology in $\mathcal{K}(\mathfrak{E}_c, \mathcal{W}_0)$ and define $\mathcal{W} = \mathcal{W}_1 \restriction \mathcal{F}(\mathfrak{E}_c)$. Let β be the collection of all sets of the form $\langle U_0, \dots, U_n \rangle \cap \mathcal{F}(\mathfrak{E}_c)$ where $n \in \omega$ and $U_j \in \beta_0$ for each $j \leq n$. Also, for every $n \in \omega$ let $E_n = \mathcal{F}_{n+1}(\mathfrak{E}_c)$. We will now check that these choices satisfy the conditions in Definition 4.2.

By 1.47 we know that $\mathcal{W}_1$ is zero-dimensional so $\mathcal{W}$ is also zero-dimensional. By Proposition 2.2 $\mathcal{W}$ witnesses that $\mathcal{F}(\mathfrak{E}_c)$ is almost zero-dimensional. Condition (a) clearly holds.

For (b), fix $n \in \omega$. Since $\mathfrak{E}_c$ is crowded and $\mathcal{F}_{n+1}(\mathfrak{E}_c)$ is a continuous image of $\mathfrak{E}_c^{n+1}$ (under the function q_n defined in Proposition 1.43), then $\mathcal{F}_{n+1}(\mathfrak{E}_c)$ is crowded. By Proposition 1.39 $\mathcal{F}_n(X)$ is closed in $\mathcal{K}(X)$ for any topological space X and all $n \in \mathbb{N}$. By Proposition 1.39 $\mathcal{F}_{n+2}(\mathfrak{E}_c) \setminus \mathcal{F}_{n+1}(\mathfrak{E}_c)$ is dense in $\mathcal{F}_{n+2}(\mathfrak{E}_c)$. Since $\mathfrak{E}_c$ has no isolated points then the set D of all $x \in \mathfrak{E}_c^{n+2}$ such that if $i, j \leq n+2$ and $i \neq j$, then $x(i) \neq x(j)$ is easily seen to be dense in $\mathfrak{E}_c^{n+2}$. Then $q_{n+2}[D] = \mathcal{F}_{n+2}(\mathfrak{E}_c) \setminus \mathcal{F}_{n+1}(\mathfrak{E}_c)$ is dense in $\mathcal{F}_{n+2}(\mathfrak{E}_c)$. This proves (b).

Also, $\mathcal{F}_{n+1}(\mathfrak{E}_c)$ is $\mathcal{W}$-closed in $\mathcal{F}(\mathfrak{E}_c)$ for all $n \in \omega$, which implies (c). Let $S = \{0\}$ and $A_0 = \mathfrak{E}_c$. The collection $\mathcal{A}$ from Lemma 2.9 is equal to $\{\mathcal{F}_{n+1}(\mathfrak{E}_c) : n \in \omega\}$. Thus, by Lemma 2.9 we obtain (d). Finally, by

Proposition 3.20 that if $\mathcal{U}\in\beta$ and $n\in\omega$, then $\mathcal{U}\cap\mathcal{F}_{n+1}(\mathfrak{E}_c)$ is compact in $\mathcal{W}\restriction\mathcal{F}_{n+1}(\mathfrak{E}_c)$, which implies (e). □

Corollary 4.19. $\mathcal{F}(\mathfrak{E}_c)\approx\mathbb{Q}\times\mathfrak{E}_c$.

Here it is natural to ask about $\mathcal{F}(\mathbb{Q}\times\mathfrak{E}_c)$, we will prove that this space is homemorphic to $\mathbb{Q}\times\mathfrak{E}_c$ as well.

Proposition 4.20. *Let $E\in\sigma\mathcal{E}$. If $n\in\mathbb{N}$ then $\mathcal{F}_n(E)\in\sigma\mathcal{E}$.*

Proof. Let $\mathcal{W}$, $\{E_n\colon n\in\mathbb{N}\}$ and β be witnesses of $E\in\sigma\mathcal{E}$. By Proposition 2.2, the Vietoris topology $\mathcal{W}_0$ of $\mathcal{F}_n(E,\mathcal{W})$ witnesses the almost zero-dimensionality of $\mathcal{F}(E)$. For each $m\in\mathbb{N}$, let $Z_m=p_m[E^n_m]$. We define β_0 to be the collection of the sets of the form $\langle U_0,\dots,U_k\rangle$ where $k\in\mathbb{N}$ and $U_i\in\beta$ for every $i\le k$. We claim that $\mathcal{W}_0$, $\{Z_m : m\in\mathbb{N}\}$ and β_0 witness that $\mathcal{F}_n(E)\in\sigma\mathcal{E}$.

Conditions (a), (b) and (c) are easily seen to follow. By Lemma 2.9, we infer that $\mathcal{F}_n(E)$ is $\{\mathcal{F}_n(E_m)\colon m\in\mathbb{N}\}$-cohesive, which is (d). Now, let $U=\langle U_0,\dots,U_k\rangle\in\beta_0$ and $m\in\mathbb{N}$. Notice that $U\cap Z_m\subset\langle U_0\cap E_m,\dots,U_k\cap E_m\rangle$. Now, by the choice of β we know that $U_i\cap E_m$ is compact in $\mathcal{W}$ for every $i\le k$. Thus, the set $\langle U_0\cap E_m,\dots,U_k\cap E_m\rangle$ is compact in $\mathcal{W}_0$. Since $U\cap Z_m$ is closed in $\mathcal{W}_0$, it is also compact. This proves (e) and completes the proof. □

Proposition 4.21. *If $E\in\sigma\mathcal{E}$, then $\mathcal{F}(E)\in\sigma\mathcal{E}$.*

Proof. Let $\mathcal{W}$, $\{E_n\colon n\in\mathbb{N}\}$ and β be witnesses of $E\in\sigma\mathcal{E}$. Let $\mathcal{W}_0$ be the Vietoris topology of $\mathcal{F}_n(E,\mathcal{W})$. For each $m\in\mathbb{N}$, let $Z_n=q_n[E^n_n]$. We define β_0 to be the collection of the sets of the form $\langle U_0,\dots,U_k\rangle$ where $k\in\mathbb{N}$ and $U_i\in\beta$ for every $i\le k$. The proof that $\mathcal{W}_0$, $\{Z_m : m\in\mathbb{N}\}$ and β_0 witness that $\mathcal{F}(E)\in\sigma\mathcal{E}$ is completely analogous to the proof of Proposition 4.20 and we will leave it to the reader. □

Corollary 4.22. *If $n\in\mathbb{N}$, then $\mathcal{F}_n(\mathbb{Q}\times\mathfrak{E}_c)\approx\mathbb{Q}\times\mathfrak{E}_c$. Also, $\mathcal{F}(\mathbb{Q}\times\mathfrak{E}_c)\approx\mathbb{Q}\times\mathfrak{E}_c$.*

4.4 The σ-product of $\mathfrak{E}_c$

Given a space X, a cardinal κ and $e\in X$, the *support* of x with respect to e is the set $\mathrm{supp}_e(x)=\{\alpha\in\kappa\colon x(\alpha)\neq e\}$. Then the σ-product of κ copies of

X with basic point e is $\sigma(X,e)^\kappa = \{x \in X^\kappa \colon |\mathrm{supp}_e(x)| < \omega\}$ as a subspace of X^κ. It is known that $\sigma(X,e)^\kappa$ is dense in X^κ.

Now, consider $X = \mathfrak{E}_c$. Since $\mathfrak{E}_c$ is homogeneous, the choice of the point e is irrelevant. Denote $\sigma(\mathfrak{E}_c^\omega, e) = \sigma\mathfrak{E}_c^\omega$. Since $\sigma\mathfrak{E}_c^\omega$ is separable and metrizable, it is natural to ask the following.

Question 4.23. *Is $\sigma{\mathfrak{E}_c}^\omega$ homeomorphic to $\mathbb{Q}\times\mathfrak{E}_c$?*

In [17] D. Lipham prove that the question 4.23 is affirmative. We use the following stratification. Given $n \in \omega$, define $\sigma_n\mathfrak{E}_c = \{x \in \mathfrak{E}_c^\omega \colon \mathrm{supp}_e(x) \subset n\}$. But with this stratification we were not able to answer question 4.23. In the following we will show some properties of this stratification.

Proposition 4.24. *$\sigma_n\mathfrak{E}_c = \{x \in \mathfrak{E}_c^\omega \colon \mathrm{supp}_e(x) \subset n\}$ is a closed subset of $\sigma{\mathfrak{E}_c}^\omega$ for each $n \in \mathbb{N}$.*

Proof. We will show that $\sigma{\mathfrak{E}_c}^\omega \setminus \sigma_n\mathfrak{E}_c$ is a open subset of $\sigma{\mathfrak{E}_c}^\omega$ for each $n \in \mathbb{N}$. Let $x \in \sigma{\mathfrak{E}_c}^\omega\setminus\sigma_n\mathfrak{E}_c$, then there exists $k_1,\dots,k_n \in \mathbb{N}$ such that $\pi_{k_j}(e) \neq \pi_{k_j}(x)$ for each $j \in \{1,\dots,n\}$. For each $i \in \{1,\dots,n\}$, let U_i a open subset of $\mathfrak{E}_c$ such that $\pi_{k_i}(x) \in U_i$ and $\pi_{k_j}(z) \notin U_i$ if $j \neq i$. Then $x \in \bigcap_{i=1}^n \pi_{k_i}^{\leftarrow}(U_i)$ and $\bigcap_{k=1}^n \pi_{k_i}^{\leftarrow}(U_i) \cap \sigma_n\mathfrak{E}_c = \emptyset$ thus $\sigma_n\mathfrak{E}_c$ is a closed subset of $\sigma{\mathfrak{E}_c}^\omega$. □

By Proposition 4.24 $\sigma_n\mathfrak{E}_c$ is closed in $\mathfrak{E}_c^\omega$. and homeomorphic to $\mathfrak{E}_c^n$ for each $n \in \omega$ since the function $f_n : \sigma_n\mathfrak{E}_c \to \mathfrak{E}_c^n$ defined by $f_n(x) = (x_i)_{i\in\mathbb{N}}$ is a homeomorphism; so in fact it is a closed copy of $\mathfrak{E}_c$ if $n \neq 0$. In fact, using an argument similar to the one in Remark 1.15 it is possible to prove the following.

Lemma 4.25. *$\sigma{\mathfrak{E}_c}^\omega$ is $\{\sigma_n\mathfrak{E}_c \colon n \in \mathbb{N}\}$-cohesive.*

Proof. Let $z \in \sigma{\mathfrak{E}_c}^\omega$, we need to find an open set V of $\sigma{\mathfrak{E}_c}^\omega$ containing z but no non-empty clopen subset of any element of $\{\sigma_n\mathfrak{E}_c \colon n \in \mathbb{N}\}$. Since $\mathfrak{E}_c$ is cohesive there exists an open set $U \subset \mathfrak{E}_c$ such that $z_0 \in U$ and U contains no non-empty clopen subsets of $\mathfrak{E}_c$. We claim that $V = U \times {\mathfrak{E}_c}^{\omega\setminus\{0\}}$ is the open set we are looking for.

Clearly, $z \in V$. Let $n \in \mathbb{N}$ and assume that O is a non-empty clopen set of $\sigma_n\mathfrak{E}_c$ with $O \subset V$. Notice that $V \cap \sigma_n\mathfrak{E}_c = U \times {\mathfrak{E}_c}^{n\setminus\{0\}} \times \{e\}^{\omega\setminus n}$. Let $p \in O$ and let $q = p \restriction \omega\setminus\{0\}$; notice that $q \in {\mathfrak{E}_c}^{n\setminus\{0\}} \times \{e\}^{\omega\setminus n}$. Then $O \cap (\mathfrak{E}_c \times \{q\})$ is a non-empty clopen subset of $V \cap \sigma_n\mathfrak{E}_c \cap (\mathfrak{E}_c \times \{p\}) = U \times \{q\}$. This contradicts our choice of U and concludes the proof. □

Also, a natural witness topology for $\sigma\mathfrak{E}_c^\omega$ can be obtained by using the restriction of the product topology of the witness topology for $\mathfrak{E}_c$. The reader will not find it difficult to prove that properties (a) to (d) of Definition 4.2 hold but property (e) does not hold.

4.5 Factors of $\mathbb{Q} \times \mathfrak{E}_c$

Recall that in section 2 of Chapter 1, we mentioned the characterizations of the factors of $\mathfrak{E}$, $\mathfrak{E}_c$ and $\mathfrak{E}_c^\omega$. Then we found it natural to try to characterize the factors of $\mathbb{Q} \times \mathfrak{E}_c$.

Lemma 4.26. *1. $\mathbb{Q}\times\mathfrak{E}_c$ does not contain any closed subspace homeomorphic to $\mathfrak{E}_c^\omega$.*

2. $\mathbb{Q} \times \mathfrak{E}_c$ does not contain any closed subspace homeomorphic to $\mathfrak{E}$.

Proof. Let $e\colon \mathfrak{E}_c^\omega \to \mathbb{Q}\times\mathfrak{E}_c$ be a closed embedding. Choose some enumeration $\mathbb{Q} = \{q_n : n \in \omega\}$. Notice that $F_n = e^{\leftarrow}[\{q_n\} \times \mathfrak{E}_c]$ is a closed subset of $\mathfrak{E}_c^\omega$ for every $n \in \omega$. By Theorem 1.1 there exists $m \in \omega$ such that F_m has non-empty interior in $\mathfrak{E}_c^\omega$. By Proposition 1.35 every open subset of $\mathfrak{E}_c^\omega$ has a closed copy of itself. Thus, this implies that there is a closed copy of $\mathfrak{E}_c^\omega$ in $\{q_m\}\times\mathfrak{E}_c$. However, $\mathfrak{E}_c^\omega$ is cohesive by Remark 1.15 and every closed cohesive subset of $\mathfrak{E}_c$ is homeomorphic to $\mathfrak{E}_c$ by Theorem 1.33. This is a contradiction to Corolary 1.35. Thus, (a) holds.

Now, let $e\colon \mathfrak{E} \to \mathbb{Q}\times\mathfrak{E}_c$ be a closed embedding. Again, let $\mathbb{Q} = \{q_n : n \in \omega\}$ be an enumeration and let $F_n = e^{\leftarrow}[\{q_n\} \times \mathfrak{E}_c]$ for every $n \in \omega$. Since e is a closed embedding, for every $n \in \omega$, F_n is homeomorphic to a closed subset of $\mathfrak{E}_c$ so it is completely metrizable. This implies that $\mathfrak{E}$ is an absolute $G_{\delta\sigma}$, and this contradicts [3, Remark 5.5]. This completes the proof of (b). □

Lemma 4.27. *1. Every $\mathfrak{E}_c$-factor is a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor.*

2. The space $\mathbb{Q}$ is a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor but is not a $\mathfrak{E}_c$-factor.

3. Every $(\mathbb{Q} \times \mathfrak{E}_c)$-factor is a $\mathfrak{E}$-factor.

4. The space $\mathfrak{E}$ is a $\mathfrak{E}$-factor that is not a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor.

5. The space $\mathfrak{E}_c^\omega$ is a $\mathfrak{E}$-factor that is not a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor.

Proof. For (i), let X be a $\mathfrak{E}_c$-factor. By Therem 1.32, $X \times \mathfrak{E}_c \approx \mathfrak{E}_c$. Thus, $X \times (\mathbb{Q} \times \mathfrak{E}_c) \approx \mathbb{Q} \times (X \times \mathfrak{E}_c) \approx \mathbb{Q} \times \mathfrak{E}_c$. For (2), notice that since $\mathbb{Q} \times \mathbb{Q} \approx \mathbb{Q}$ then $\mathbb{Q}$ is a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor but it is not a $\mathfrak{E}_c$-factor because it is not Polish. For (3), let X be a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor. By Proposition 1.27, $\mathfrak{E}_c \times \mathbb{Q}^\omega \approx \mathfrak{E}$. Thus, $X \times \mathfrak{E} \approx X \times (\mathfrak{E}_c \times \mathbb{Q}^\omega) \approx X \times (\mathbb{Q} \times \mathfrak{E}_c) \times \mathbb{Q}^\omega \approx (\mathbb{Q} \times \mathfrak{E}_c) \times \mathbb{Q}^\omega \approx \mathfrak{E}_c \times \mathbb{Q}^\omega \approx \mathfrak{E}$. For (4), it is clear that $\mathfrak{E}$ is a $\mathfrak{E}$-factor. However, $\mathfrak{E}$ is not a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor because in that case $\mathbb{Q} \times \mathfrak{E}_c$ would have a closed copy of $\mathfrak{E}$ and we have proved that this is impossible in Lemma 4.26. For (5), recall that $\mathfrak{E}_c^\omega$ is an $\mathfrak{E}$-factor by Corolary 9.3 of [3] and it cannot be a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor, again by Lemma 4.26. $\square$

Theorem 4.28. *For a non-empty space E the following are equivalent:*

(i) $E \times (\mathbb{Q} \times \mathfrak{E}_c)$ *is homeomorphic to* $\mathbb{Q} \times \mathfrak{E}_c$*,*

(ii) E *is a* $(\mathbb{Q} \times \mathfrak{E}_c)$*-factor,*

(iii) there are a topology $\mathcal{W}$ on E witnessing that E is almost zero-dimensional, a collection of $\mathcal{W}$-closed non-empty subsets $\{E_n : n \in \omega\}$ and a base of neighborhoods β such that

  (1) $E = \bigcup\{E_n \colon n \in \omega\}$*,*

  (2) for every $n \in \omega$, $E_n \subset E_{n+1}$, and

  (3) for every $U \in \beta$ and $n \in \omega$, $U \cap E_n$ is compact in $\mathcal{W}$.

Proof. Condition (i) clearly implies (ii).

Next, we prove that (ii) implies (iii). Since E is a $\mathbb{Q} \times \mathfrak{E}_c$-factor, there is a space Z such that $E \times Z \approx \mathbb{Q} \times \mathfrak{E}_c$. Let $\mathcal{W}$, $\{X_n \colon n \in \omega\}$ and β be witnesses of $E \times Z \in \sigma\mathcal{E}$ as in Definition 4.2. Fix $a \in Z$ and let $A = E \times \{a\}$; we may choose a in such a way that $A \cap E_0 \neq \emptyset$. We define $E_n = X_n \cap A$ for every $n \in \omega$, $\mathcal{W}_0 = \mathcal{W} \restriction A$ and $\beta_0 = \{U \cap A \colon U \in \beta\}$. It is not hard to prove that these sets have the corresponding properties (1), (ii) and (iii) replacing E for A.

Finally, we prove that (iii) implies (i). Let $\mathcal{W}_0$, $\{E_n \colon n \in \omega\}$ and β_0 as in item (iii) for E. Let $\mathcal{W}$, $\{X_n \colon n \in \omega\}$ and β witnessing that $\mathbb{Q} \times \mathfrak{E}_c$, as in Lemma 4.3. Let $\mathcal{W}_1$ be the product topology of $\langle E, \mathcal{W}_0\rangle \times \langle \mathbb{Q} \times \mathfrak{E}_c, \mathcal{W}\rangle$. Notice that $E_n \times X_n$ is $\mathcal{W}_1$-closed for every $n \in \omega$. Thus, $\mathcal{W}_1$ clearly witnesses that $E \times (\mathbb{Q} \times \mathfrak{E}_c)$ is almost zero-dimensional. Finally, let $\beta_1 = \{U \times V \colon U \in \beta_0, V \in \beta_1\}$.

We claim that $\mathcal{W}_1$, $\{E_n \times X_n : n \in \omega\}$ and β_1 witness that $E \times (\mathbb{Q} \times \mathfrak{E}_c) \in \sigma\mathcal{E}$. Conditions (a), (b) and (c) from Definition 4.2 are easily checked. By Remark 1.15 we obtain that $E \times (\mathbb{Q} \times \mathfrak{E}_c)$ is $\{E_n \times X_n : n \in \omega\}$-cohesive. Finally, given $U \times V \in \beta_1$ and $n \in \omega$, since $U \cap E_n$ is compact in $\mathcal{W}_0$ and $V \cap X_n$ is compact in $\mathcal{W}$, then $(U \times V) \cap (E_n \times X_n)$ is compact in $\mathcal{W}_1$. This concludes the proof. □

Corollary 4.29. *If X is a $(\mathbb{Q} \times \mathfrak{E}_c)$-factor, then $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are $(\mathbb{Q} \times \mathfrak{E}_c)$-factors.*

Proof. From the proof of Proposition 4.20 and Proposition 4.21 it follows that $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ satisfy the conditions (1),(2) and (3) of item (iii) of the Theorem 4.28. Therefore $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are $(\mathbb{Q} \times \mathfrak{E}_c)$-factors. □

Question 4.30. *Can we remove mention of the zero-dimensional witness topology in Theorem 4.28 by adding the following statement?*

(4) E is a union of a countable collection of C-sets, each of which is a $\mathfrak{E}_c$-factor.

David Lipham has informed us that, however, if we change "C-sets" to "closed sets" in (4) of Question 4.30, the resulting statement is not equivalent to E being an $(\mathbb{Q} \times \mathfrak{E}_c)$-factor. This is because in [16] he gave an example of an F_σ subset of $\mathfrak{E}_c$ that is not an $\mathfrak{E}$-factor.

4.6 Dense embeddings of $\mathbb{Q} \times \mathfrak{E}_c$

In this section we consider when $\mathbb{Q} \times \mathfrak{E}_c$ can be embedded in almost zero-dimensional spaces as a dense subset. The following theorem characterizes $\mathbb{Q}$.

Theorem 4.31 ([8, Theorem 2.4.1 (Sierpiński)]). *The space of rationals $\mathbb{Q}$ is the only countable crowded space.*

By Theorem 4.31 every countable crowded dense subset of $\mathfrak{E}_c$ is homeomorphic to $\mathbb{Q}$ and $\mathfrak{E}_c^n \approx \mathfrak{E}_c$ for each $n \in \mathbb{N}$, then we obtain the following.

Example 4.32. *There is a dense F_σ subset of $\mathfrak{E}_c$ that is homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$.*

Since question 4.23 is affirmative and $\sigma\mathfrak{E}_c^\omega$ is a dense and an F_σ subset of $\mathfrak{E}_c^\omega$, we have the following example.

Example 4.33. *There is a dense F_σ subset of ${\mathfrak{E}_c}^\omega$ that is homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$.*

We recall that it is still unkown whether the hyperspace $\mathcal{K}(\mathfrak{E}_c)$ is homeomorphic to $\mathfrak{E}_c$ (see Question 3.17) but now we know that it has a dense copy of $\mathbb{Q} \times \mathfrak{E}_c$ by Corollary 4.19. Thus we make the following question.

Question 4.34. *Let $X \subset \mathfrak{E}_c$ be dense and a countable union of nowhere dense C-sets. If X is cohesive, is it homeomorphic to $\mathbb{Q} \times \mathfrak{E}_c$?*

Notice that Question 4.34 is related to Question 4.7

Chapter 5

Miscellanea on AZD spaces

In this chapter we will talk about various topics related to almost zero-dimensional spaces that have little to do with the previous chapters. We will talk about extension of functions and compactificafions in the class of almost zero-dimensional spaces.

The following theorem is important in the theory of almost zero- dimensional spaces. We have not mentioned it before because it is only used in this part of the thesis.

Theorem 5.1 ([3, Theorem 4.19])**.** *A nonempty subset of an almost zero-dimensional space X is a retract of X if and only if it is a C-set in X.*

Corollary 5.2. *Let X be a space AZD, let A be a C-set of X, and $f : A \to X$ a continuous function, then there exists a continuous function $F : X \to X$ such that $F \restriction A = f$.*

Proof. Since A is a C-set of X, by Theorem 5.1 there is a retraction $r : X \to A$. Then $F = f \circ r$ satisfies that $F \restriction A = f$. □

Corollary 5.3. *Any continuous function $f : \mathcal{F}_n(X) \to \mathcal{H}$ has a continuous extension where $\mathcal{H} \in \{\mathcal{F}_{n+1}(X), \mathcal{F}(X), \mathcal{K}(X)\}$.*

Proof. It is immediate from Corollary 5.2 and Proposition 2.7. □

Corollary 5.4. *Let A and B be a C-sets of X and Y respectively. Let $f : A \to B$ be a continuous function. Then there exists a continuous function $F : X \to Y$ such that $F \restriction A = f$.*

Proof. From Theorem 5.1 it follows that there is continuous function $r : X \to A$ such that $r \restriction A = id_A$. Therefore it implies that $H = id_X \circ f \circ r$ satisfies what is requested. □

Proposition 5.5. *Let X be an AZD cohesive space. Let $f : \mathfrak{E}_c \to X$ be an open surjective function. Let $\mathcal{W}$ be a topology as in item 2 of Theorem 1.31. If $f_0 : (\mathfrak{E}_c, \mathcal{W}) \to X$, given by $f_0(x) = f(x)$ is continuous, then X is homeomorphic to $\mathfrak{E}_c$*

Proof. By Remark 1.9 $id_{\mathfrak{E}_c} : \mathfrak{E}_c \to (\mathfrak{E}_c, \mathcal{W})$ and $id_X : X \to (X, \mathcal{W}_1)$ are continuous functions, therefore $f\colon \mathfrak{E}_c \to X$ is a continuous function, since $f = f_0 \circ id$; also the function $f_1 : (\mathfrak{E}_c, \mathcal{W}) \to (X, \mathcal{W}_1)$ given by $f_1(x) = f_0(x)$ is a continuous function, because $f_1 = id_X \circ f_0$. Let $x \in X$ and let U be an open subset of X such that $x \in U$. Since f is a continuous and surjective function, there exists $w \in \mathfrak{E}_c$ such that $f(w) = x$ and V a neighborhood that is a compact in $\mathcal{W}$ such that $x \in f[V] \subset U$. Since f is a open function, then $x \in f[int_{\mathfrak{E}_c}(V)] \subset int_X f[V]$. That is, $f[V]$ is a neighborhood of x in X. Since f_1 is a continuous function and $f[V] = f_1[V]$ then $f[V]$ is a compact in $\mathcal{W}_1$. Therefore X is homeomorphic to $\mathfrak{E}_c$ by item 2 of Theorem 1.31. □

Proposition 5.6. *Let X be a Polish AZD, and cohesive space such that there exists a witness topology $\mathcal{W}$ such that $(X, \mathcal{W}) \approx \mathbb{Q} \times 2^\omega$, then there exists an open dense subset U of X such that $U \approx \mathfrak{E}_c$.*

Proof. By Theorem 5.2 in [20] we have that $(X, \mathcal{W}) = \bigcup_{n\in\mathbb{N}} F_n$ such that F_n is a compact nowhere dense in $(X, \mathcal{W})$ and $F_n \subset F_{n+1}$. On the other hand, X is also the union of the $\{F_n : n \in \mathbb{N}\}$. Since X is a Baire space, then there exists $n \in \mathbb{N}$ such that $int_X(F_n) \neq \emptyset$. Without loss of generality we can assume that $int_X(F_1) \neq \emptyset$. Let $n \in \mathbb{N}$, such thar $E_n = int_X(F_n) \neq \emptyset$, and $E = \bigcup_{n\in\mathbb{N}} E_n$. Note that E is an open dense subset of X. Let's prove that $E \approx \mathfrak{E}_c$. Since X is cohesive and E is an open subset of X, then E is cohesive. On the other hand, let $x \in E$, then $x \in E_n$ for some $n \in \mathbb{N}$. Since E_n is an open subset of X, then there exists a neighborhood W_x of x such that $W_x \subset E_n$ and W_x is a closed subset of $(X, \mathcal{W})$. Note that $W_x \subset (F_n, \mathcal{W} \restriction F_n)$, this implies that W_x is a compact subset of $(E_n, \mathcal{W} \restriction E_n)$. Let $\mathcal{A} = \{W_x : x \in E\}$, then $\mathcal{A}$ is base of E and satisfies the conditions of Theorem 1.31 Therefore $E \approx \mathfrak{E}_c$. □

Proposition 5.7. *$\mathfrak{E}_c^\omega$, is not the union of a space E homeomorphic to $\mathfrak{E}_c$ and a nowhere dense subset F such that $F \cap \mathfrak{E}_c = \emptyset$.*

Proof. Suppose that $\mathfrak{E}_c^\omega = E \cup F$, where F is a nowhere dense subset of $\mathfrak{E}_c^\omega$ and E homeomorphic to is $\mathfrak{E}_c$. Then $X = \mathfrak{E}_c^\omega \setminus F$ is homeomorphic to $\mathfrak{E}_c$. Since $int_{\mathfrak{E}_c^\omega}(X) \neq \emptyset$, then there exists a basic open subset U of $\mathfrak{E}_c^\omega$, such that U is subset a of X. This implies that U has a closed copy of $\mathfrak{E}_c^\omega$. On the other hand, since U is homeomorphic to an open subset V of $\mathfrak{E}_c$, then $U \approx V \approx \mathfrak{E}_c$ (see Proposition 3.4 in [2]). Therefore $\mathfrak{E}_c$ has a closed copy of $\mathfrak{E}_c^\omega$, by Theorem 1.32 $\mathfrak{E}_c^\omega \approx \mathfrak{E}_c$, which is a contradiction. □

From this result the following question arises

Question 5.8. *If X is a space as in Proposition 5.6, is X homeomorphic to $\mathfrak{E}_c$?*

Corollary 5.9. *If X is a space as in Proposition 5.6, then $\mathfrak{E}_c^\omega$ is not homeomorphic to X*

Proof. By Proposition 5.6 we have that X, is the union of $\mathfrak{E}_c$ and a nowhere dense subset F. Then by Proposition 5.7 $\mathfrak{E}_c^\omega$ is not homeomorphic to X. □

A **compactification** is a pair (Y, e) where Y is a compact space and $e : X \to Y$ is an embedding such that $e[X]$ is a dense in Y.

Corollary 5.10. *Let X be an AZD space, and Y be a metric compactification of X. If Y is an AZD space, then X is zero-dimensional.*

Proof. By item 4 from Proposition 1.11, Y is a compact totally disconnected space. This implies that Y is zero-dimensional. Therefore X is zero-dimensional. □

Note that in the case that we have an almost zero-dimensional space of dimension 1 by Corollary 5.10 we have that any metric compactification Y of X is not an AZD space. Even more, Y is also not a totally disconnected space. This implies there is a connected component of Y with more than one point. In the case of $\mathfrak{E}_c$ we have that there is a compactification Y of $\mathfrak{E}_c$ that is connected. This compactification is the Lelek fan (see [15]). The following statement gives us a necessary condition for a space X to have a connected compactification.

Proposition 5.11. *If X is a cohesive and almost zero-dimensional space, then X has a connected compactification.*

Proof. Since X is a cohesive and almost zero-dimensional space, then it has a connected extension by a point Y. Since Y is a metric, separable space, then it has a countable base, therefore Y admits a embedding e into $[0,1]^\omega$. Let $Z = cl_{[0,1]^\omega}(e[Y])$, then Z is a connected and compact space. Since X is a dense subset of Y, we conclude that X is a dense subset of Z. $\square$

Corollary 5.12. *Connected compactifications exist for the following spaces: $\mathfrak{E}_c^\omega$, $\mathfrak{E}$ and $\mathbb{Q} \times \mathfrak{E}_c$.*

Proof. It follows from Proposition 5.11. $\square$

Corollary 5.13. *Connected compactifications exist for the following spaces: $\mathcal{K}(\mathfrak{E})$, $\mathcal{K}(\mathfrak{E}_c)$, $\mathcal{K}(\mathfrak{E}_c^\omega)$, and $\mathcal{K}(\mathbb{Q} \times \mathfrak{E}_c)$.*

Proof. It follows from Proposition 5.11 and Corollary 2.15. $\square$

Note that $\mathbb{Q}$ is an AZD space that is not cohesive, but the space $[0,1]$ is a connected compactification of $\mathbb{Q}$. Let us observe that Lelek fan is a connected compactification of $\mathfrak{E}_c$ of dimension 1 (see [15]). Another question that arises from the proof of the Proposition 5.11 is the following:

Question 5.14. *Let Z be space of the proof of Proposition 5.11. Is Z finite-dimensional?*

From now on we will consider the AZD spaces as Hausdorff spaces that have a base of neighborhoods of C-sets.

Remark 5.15. *All Hausdorff AZD space X, is regular. Let $x \in X$ and U be an open subset of X such that $x \in U$. Since X is an AZD space, then there exists B a C-set of X such that $x \in int_X(B) \subset B \subset U$. Then $cl_X(int(B)) \subset U$, since B is a closed subset in X.*

Given a space X, its **weight** $w(x)$ is defined as the smallest infinite cardinal κ such that X has a base of cardinality κ.

Proposition 5.16. *Let X be an AZD space with $w(X) = \kappa$. Then there exists a zero-dimensional topology $\mathcal{W}$ coarser than the topology in X such that $w(X, \mathcal{W}) \leq \kappa$.*

Proof. Let β be a base of C-sets of X such that $|\beta| = \kappa$. Suppose that $\beta = \{B_\alpha : \alpha \in \kappa\}$. Since $w(X) = \kappa$, then for every B_α, there exists $\mathcal{F}_\alpha = \{U^\alpha_\gamma : \gamma \in \kappa\}$ such that for each $\gamma \in \kappa$, U^α_γ is a clopen subset of X and $B_\alpha = \bigcap \mathcal{F}_\alpha$. Let $\mathcal{F}^1_\alpha = \{X \setminus U^\alpha_\gamma : U^\alpha_\gamma \in \mathcal{F}_\alpha\}$, $\mathcal{C} = \bigcup_{\alpha\in\kappa} \mathcal{F}_\alpha \cup \bigcup_{\alpha\in\kappa} \mathcal{F}^1_\alpha$ and $\mathcal{W}$ be the topology whose subbase is $\mathcal{C}$, by construction $w(X, \mathcal{W}) \leq \kappa$ and $\mathcal{W}$ is coarser topology than topology on X. $\square$

Proposition 5.17. *Let X be an AZD space with $w(X) = \kappa$, then X can be condensed into $\mathfrak{E}^\kappa$ and $\mathfrak{E}^\kappa_c$.*

Proof. Note that 2^κ admits a embedding into $\mathfrak{E}^\kappa$ and $\mathfrak{E}^\kappa_c$. By Proposition 5.16 there exists a zero-dimensional topology $\mathcal{W}$, such that $w(X, \mathcal{W}) \leq \kappa$. So $(X, \mathcal{W})$ admits a embedding into 2^κ. Let $e_1 : 2^\kappa \to \mathfrak{E}^\kappa_c$, $e_2 : (X, \mathcal{W}) \to 2^\kappa$ be embeddings and $id : X \to (X, \mathcal{W})$, then $f = e_1 \circ e_2 \circ id$ is a condensation. $\square$

The following question arises from proposition

Question 5.18. *Let X be an AZD space with $w(X) = \kappa$. Does X admit a embedding into $\mathfrak{E}^\kappa$?*

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Symbols used

$\wp(X)$———- power set of X.

$f^{\leftarrow}(A)$ —— inverse image of A, i.e $f^{\leftarrow}(A)$ is the set $\{x : f(x) \in A\}$.

$int_X(A)$ —— interior of A in the topological space X.

$cl_X(A)$ —– clausure of A in the topological space X.

$bd_X(A)$ —– boundary of A in the topological space X.

$dim(X)$——– dimension of a space X.

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