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A new discretized model for Classical General Relativity

Tesis
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Resumen

El problema de encontrar una teoría cuántica de la gravedad es uno de los problemas abiertos más importantes en la física teórica actual. Pese a la gran cantidad de pistas y al número de incógnitas en física teórica que apuntan a una solución a través del marco de una teoría que combine los principios de la relatividad general y la mecánica cuántica, la formulación de una tal teoría que sea completa, consistente y que lleve a predicciones que puedan corroborarse mediante la observación es aún una cuestión abierta.

La gravedad cuántica de lazos es una de varias propuestas actuales que trata de aventurarse en el estudio de las propiedades cuánticas del campo gravitacional. En esta tesis pretendemos esbozar las principales virtudes y defectos de esta propuesta teórica que aspira a convertirse en una descripción cuántica del campo gravitacional. Después de esbozar una serie de motivaciones generales en el capítulo 1, en los capítulos 2 y 3 se exponen las principales características y los problemas aún sin resolver del enfoque de gravedad cuántica de lazos, tanto en su versión canónica como en su versión covariante. Se presentan algunos de los resultados que distinguen a esta propuesta teórica (como la discretitud de los operadores de área y de volumen) y se habla de algunos de los problemas abiertos que enfrenta. En el capítulo 4 exponemos la construcción teórica de un nuevo modelo discretizado de la relatividad general que pretender constituir un modelo alternativo a la discretización usual usada en gravedad cuántica de lazos covariante. En un trabajo de investigación futura esperamos que la cuantización de este modelo arroje luz sobre problemas aún sin resolver en la teoría covariante, en particular, en el límite semiclásico.

Agradecimientos

Nada se hace solo. Y no podemos hacer nada solos. Así pues este trabajo no podría haberse hecho sin el amable apoyo y colaboración de otras personas, además de mí.

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Finalmente, quiero dedicarle este humilde trabajo a mi madre Rosaura Montes de Oca. Ella, como nadie más, da su vida por mí; día con día procura siempre dar todo de sí, para que yo pueda así mismo dar todo de mi. Ella soporta mis enojos y desdenes, y mis frecuentes momentos de impaciencia. Aunque la burocracia diga otra cosa, esta tesis no es solo mía, es de ambos, y ambos la construimos juntos. No existen palabras para decir cuánto te quiero, pues es imposible expresar lo inexpresable. Hoy como nunca antes debo decirte: Gracias.

Contents

La buena noticia es que no está afuera, así que está aquí en algún lugar. Lo encontraré. ¿Alguién vió un ehhh... chummm? ... Nop.
Patrick Jane. El mentalista.

1 Introduction: The unfinished revolution

General relativity and quantum theory are the two fundamental pillars of contemporary physics. Both offer descriptions of the world that are deeper from those offered by classical theories. Both frameworks are the result of the hard work and the brilliance of some of the most prominent minds of all time in theoretical and experimental physics .

On one hand, quantum theory is not only a description of microscopic objects, but it is our currently more successful theory for the description of the evolution of dynamical systems. In its most sophisticated version, quantum field theory, also incorporates the fundamentals of the special theory of relativity. This allows describing most of the fundamental matter fields and interactions of the universe. This framework actually provides some of the most impressive agreements between theoretical predictions and experimental measurements [25].

On the other hand, general relativity is our modern theory of the gravitational interaction. Driven, among other things, by the conceptual incompatibility between special relativity and the Newtonian description of gravity, Albert Einstein formulated a new theory of the gravitational interaction. In this framework, gravity is described using the language of semi-Riemannian geometry. General relativity has taught us that gravity is encoded in the geometry of space and time. Spacetime is now a dynamical entity that reacts to the matter fields on it, and matter fields are also influenced by the geometry of the spacetime surrounding them. [1]

Quantum theory has helped to understand many before unknown aspects of nature, such as nuclear and molecular physics, three of the fundamental forces in nature, and many aspects and characteristics of the fundamental particles. All of this has led to most of the new technological developments of the modern era, including semiconductors, quantum information, and quantum optics. For its part, general relativity has also unmasked very exotic aspects of nature that involve the gravitational interaction. Among them are cosmology, black holes and, with the help of very surprising technological milestones, the development a new way to observe the universe out there through the gravitational observational astronomy using gravitational waves [26].

In general relativity, the description of all the physical fields, including the the gravitational field, is given over a C^∞ differentiable manifold. This is the mathematical structure under-

[1] This is the origin of the John Archibald Wheeler's quote: "Space tells matter how to move. Matter tells spacetime how to curve"[55].

lying the physical world. In particular, the gravitational field determines the causal relations and therefore the way in which the other fields interact with each other. The invariance under Lorentz transformations, which is the cornerstone of the special theory of relativity, is restricted to small spacetime regions. In general relativity, the dynamical evolution is completely deterministic, and this is a reminiscent of the Newtonian formulation of dynamics [1, 4, 11].

In the quantum theory, the world is described using a differentiable manifold with a fixed metric, usually a Minkowskian metric. Invariance under the Poincaré group is a global symmetry of the theory. Furthermore, the description of some physically measurable quantities is probabilistic and is subject to the Heisenberg's inequalities. This breaks with the old deterministic classical framework of the description of the world.

Nevertheless, from the quantum theory we have learned that every physical field has quantum properties. It is then reasonable to ask what the quantum properties of the gravitational field are, as well as what are the properties, if any, that determine the quantum aspects of the gravitational field.

Quantum theory and general relativity both emerged based on the need for descriptions of new phenomena for which the classical theories of Newtonian mechanics and classical electrodynamics did not offer satisfactory, or even coherent, answers. Each one broke, in its own way, the preconceptions that we had about the working of the cosmos. Both offered revolutionary ways of understanding the vast sea of physical phenomena that surround us. This hectic physical and philosophical revolution, however, also brought a problem of colossal proportions: both theories are incompatible. They describe the world with very different mathematical and conceptual structures. It looks as if the world were made of two fragmented pieces that are governed by contradictory laws. To solve this problem, we need a new theory: a theory of quantum gravity.

Beyond these conceptual and motivational reasons, there are more precise arguments that indicate the need to search for a quantum theory of gravitation.

- **Singularities in general relativity**
  In the framework of general relativity, some physical situations may lead to the creation of spacetimes with singularities: geodesically incomplete Lorentzian manifolds [4]. In this situations, the concentration of matter is so high that some components of the gravitational field actually diverge. The generic conditions for the occurrence of spacetimes with singularities is given by the so called singularity theorems [4, 11]. The existence of singularities, such as those of black holes and the initial singularity of the Big Bang, is physically interpreted as an indication of the existence of situations where the classical Einstein theory of gravity is no longer valid. It is expected that the quantum properties of gravitational field play an important role in the correct description of singularities. We hope that this will help to remove the divergences, providing a physically reasonable description of these situations.

- **The phenomenological motivation**
  The description of the initial cosmological singularity is also directly connected to the

correct description of the early instants of the evolution of our universe. There are a strikingly large number of arguments pointing to the Planck scale as the characteristic scale of quantum-gravity effects [2, 24]. Although clearly these arguments are not all independent, their overall weight must certainly be judged as substantial [24]. It is believed that the elementary particles of the early universe could have reached scattering energies of the order of the Planck energy with respect of their center-of-momentum frame. If this is true, then the correct description of such scattering processes must take into account the quantum properties of the gravitational field. Different scenarios for the description of the quantum gravity realm could be compared on the basis of their description of the early universe. That could help us to establish different predictions for some manifestations of those early instants of the evolution of the universe in our present observations [24].

- **The need for coherence**
  More precisely, the incompatibility between quantum theory and general relativity shows two different images of the world, as if theoretical physics were fragmented. In addition, the argument that we do not need a quantum theory of gravity because the energies needed to reach the quantum gravity realm are inaccessible for our present achievable experimental capacities is something both simplistic and against the spirit of theoretical physics itself. Moreover, the argument is unsustainable because of the situation described in the previous point). The fragmented nature of our present descriptions of the world becomes clear if we think of, for example, a way to describe the spacetime metric of an atomic nucleus in a quantum superposition of two eigenstates of the position. To do this, we need a mix between quantum theory and general relativity. Currently, however, we do not have theoretical tools to describe such apparently simple situation [2, 5].

- **Black holes thermodynamics**
  Classically, stationary black holes are characterized only by three parameters the mass, the electric charge and the angular momentum. It is, however, very surprising that by using only these quantities together with the surface gravity and the surface area of the event horizon, it is possible to construct four laws of the black holes' mechanics that are very similar to the four laws of thermodynamics. It is then possible to associate a temperature and a entropy with every black hole [2]. The physical meaning of these quantities is better understood when we study the behavior of quantum fields on the background of an object collapsing to form a black hole [2]. The dynamical background produces an initial vacuum that does not remain as a vacuum. Conversely, it becomes a thermal state with respect to late-time observers, and we can associate a temperature with the black hole, called the "Hawking temperature" [2]. Once we have calculated the temperature, the entropy of a black hole is also given. With this, however, a central problem arises: can the entropy of a black hole be derived from quantum-statistical considerations? Furthermore, what is the microscopical origin of such entropy? Does it correspond to microscopic degrees of freedom? It is believed that this entropy comes from no-trivial degrees of freedom of quantum gravity associated with the black hole configuration [2].

- **Thermal evaporation of black holes and the information paradox**
  The thermal production of particles because of the dynamic background that is responsible

for the Hawking temperature also produces the loss of mass and the subsequent evaporation of the black hole. When the black hole reaches the Planck scale, it is assumed that the semiclassical approximation where the non-gravitational fields are quantum but the gravitational field remains classical is no longer valid. In this scenario, the possible quantum properties of the gravitational field must be taken into account. The evaporation of a black hole, however, causes another conceptual problem: if the black hole evaporated completely and left only thermal radiation behind, it would mean that any initial quantum state, in particular a pure state, would evolve into a mixed state. Such evolution would violate the unitary evolution of the total closed system predicted by the quantum theory. It is expected that a quantum theory of gravity gives a definite answer to whether the unitarity is preserved or not [2].

There exist many other motivations and problems in theoretical physics with a possible solution that points to a quantum theory of gravity [1, 2, 3, 5, 6]. Here we only mentioned some of them to show that the search for quantum gravity is a truly important open problem in current theoretical physics.

Even though much intellectual work has been done, a complete and consistent quantum theory of gravity is still missing. The main problem [3] seems to be that *we only know how to do quantum physics using a fixed background metric*. Ordinary quantum field theory relies on the use of a fixed background metric, and the absence of a such structure leaves us with just a few tools for constructing a consistent theoretical framework. All of the above, together with the difficulty of experimentally testing the quantum properties of spacetime, leaves us with a serious theoretical problem. The revolution that began with quantum theory and general relativity is still unfinished.

There are very different approaches for trying to solve the problem of quantum gravity from different perspectives. Loop quantum gravity, the main subject of the present work, is one of those approaches that tries to adventure upon the research of the quantum properties of space and time. Such an approach takes the central lesson of general relativity seriously: gravity is geometry , and in a fundamental quantum gravity theory, there should be no background metric. All the physical fields, matter and gravity, should born quantum mechanically, and the smooth metric described by general relativity should arise in an appropriate limit.

Loop quantum gravity (LQG) is based on the implementation of the canonical quantization method applied to general relativity [7]. The latter is reformulated in such a way that its phase space closely resembles the phase space of a gauge theory [3]. The quantization is then based on a non-perturbative and background-independent procedure, without postulating the existence of a fixed background metric [27].

Among the most important successes of LQG are the mathematically rigorous framework, its manifestly non-perturbative and background independent language, the existence of certain clues about the ultraviolet finiteness for some theoretical results, and its inherent notion of a quantum discreteness of spacetime that is derived rather than postulated [1, 3]. This discreteness is associated with certain physical quantities related to the spacetime metric. Being more specific, the theory predicts a discrete spectra of the area and volume operators [1, 2, 3].

Some important issues that remain unsolved within the LQG framework are the following; the loss of four-dimensional diffeomorphism covariance, the non-linear structure of the Wheeler-DeWitt equation (the equation that governs the dynamics of the theory) and the subsequently poor understanding of the physics underlying it. There is also the difficulty in finding a complete set of gauge-invariant observables that determines the physically measurable quantities, and a still missing classical limit showing that the LQG has general relativity as its classical limit [1, 3, 6, 7].

The nonlinear structure of the Wheeler-Dewitt equation is strongly connected to the absence of a definition of the dynamics of the theory and the calculation of transition amplitudes. The spinfoam framework was introduced to remedy this problem and to express the quantum dynamics of spacetime in a clear and useful way.

The spinfoam framework intends to be, broadly speaking, an implementation of the functional integral methods to the quantization of the gravitational field see it as a field theory, but taking into account the discreteness of spacetime proposed in loop quantum gravity. The spinfoam framework defines transition amplitudes between quantum states of the geometry in loop quantum gravity [6, 7].

1.1 Main purpose of this work

The purpose of this written work is twofold. On one hand we intend to present a general overview of the LQG framework, trying to emphasize their merits and difficulties. On the other hand, we present the results of an original research program undertaken under the supervision of José A. Zapata. In such a program, we introduce a new discretization for classical general relativity. Our construction relies on the use of a cellular decomposition of the spacetime that was previously introduced by M. Reisenberger in [17] and [18]. Such decomposition has the advantage of leading to a clean separation of the boundary and bulk degrees of freedom. We were able to construct two discrete models of the gravitational theory that are ultimately equivalent. In the first model, we consider a tetrad and elements of the Lorentz group $SO(3,1)$ associated with some special structures of the cellular decomposition. In the second model, we consider variables that are natural for the discretization of a BF theory, and only after a set of constraints are solved, it becomes equivalent to the first model. We hope that our discretization will help us to study some issues of the covariant LQG models that are currently difficult to explore. Particularly, we hope to be able to produce a quantum version of this discrete model that makes the study of the semiclassical limit of the spin foam models more approachable.

In order to accomplish the two goals of this written work, we have organized it as follows. The first two chapters introduce the main characteristics of canonical and covariant loop quantum gravity frameworks.

Chapter one starts introducing the 3+1 formulation of general relativity. Next, we present the canonical quantization program and the result of its application to general relativity. We will see that it is possible to construct a more or less satisfactory kinematical states space. We will find several problems, however, when we try to define the dynamics of the theory. Motivated by this, chapter three discusses the covariant LQG approach. Such approach, also called the spin foam approach, tries to be an implementation of the path integral quantization method for gravity. We will see that this framework allows on to recover some features of the canonical framework. Nevertheless, we will find other important issues that this approach needs to solve.

Finally, in chapter four we introduce the results of our current research work. We introduce a way to discretize general relativity using some structures that were already used in [17]. We use such structures, however, to study a model that has not been considered yet. We discretize the action of a BF theory with $SO(3,1)$ as its gauge group, and we introduce a constraint that is intended to reduce this theory to general relativity in the continuum limit. This constraint introduces an element that is not present in the original formulation given in [17]. Such new element is a tetrad associated with each corner of a four-simplex. We introduce the main variables defining the model, and the classical dynamical equations that result from the variation of the action. Finally, we describe the boundary variables and their interpretation. We hope that the quantization of this model, as a project for future research, will allow keeping the main features and advantages that characterize the new spin foam models. In addition, we entail the hope that their use will allow us to study some currently unexplored situations.

2
Canonical loop quantum gravity in a nutshell

In this chapter, we give a brief introduction to the canonical quantization of Einstein's general relativity and its main results. This chapter serves as a preamble for the introduction of the spin foam models mainly because, as emphasized in [6], it is in the context of the canonical quantization where the spin foam models find their more natural interpretation as tools for understanding the dynamical content of quantum gravity [6].

The content of this chapter is mainly based on [1, 3, 6, 7] and is not exhaustive. Therefore we will not present the detailed calculations for arriving to the different results presented here, but refer the reader to the aforementioned references.

2.1 Canonical formulation of classical gravity

The procedure of canonical quantization is based on the hamiltonian formalism. For its implementation in the theory of gravity, we need to define the configuration variables and their corresponding canonical momenta and then write the Hamiltonian of the theory. We have therefore to introduce a split in the spacetime manifold $\mathcal{M}$ into space and time. Although it may initially appear that this breaks the diffeomorphism invariance of the theory, that is not the case because such division is arbitrary, and this arbitrariness exhausts the full diffeomorphism group [10].

Our starting point is a four-dimensional differentiable manifold $\mathcal{M}$ with a metric g with euclidean or lorentzian signature (the lorentzian signature will be chosen as $(-,+,+,+)$, so the timelike vectors will have negative norm). The dynamics of the theory is given by the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{\kappa} \int_M R dV, \tag{2.1}$$

where $\kappa = 16\pi G/c^3$, R is the Ricci scalar and dV is the volume form in $\mathcal{M}$. So as to rewrite (2.1) into canonical form, we will suppose that $\mathcal{M}$ has the special topology $\mathbb{R} \times \sigma$, where σ is a fixed three-dimensional manifold with arbitrary topology[2] and without boundaries. A

[2]This assumption has some subtleties; see [3].

condition for this to be fulfilled is that (M, g) is globally hyperbolic, that is, it possesses a Cauchy surface (an 'instant of time') on which initial data can be described to determine uniquely the whole space–time[2, 11]. In such cases, the classical initial value formulation makes sense, and the Hamiltonian form of GR can be constructed [11]. Thus, we establish a diffeomorphism $\mathcal{X} : \mathbb{R} \times \sigma \to M$ such that for every $t_0 \in \mathbb{R}$ fixed the function $\mathcal{X}_{t_0} : \sigma \to M$ defined by

$$\mathcal{X}_{t_0}(p) := \mathcal{X}(t_0, p) \qquad\qquad p \in \sigma,$$

is an embedding [10]. This defines a foliation of $\mathcal{M}$ into spacelike hypersurfaces $\Sigma_{t_0} := \mathcal{X}_{t_0}(\sigma)$.

The ten independent components of the metric $g_{\mu\nu}$ are replaced by the six components of the Riemanninan metric q_{ab} induced on Σ_{t_0}, plus the three components of the shift vector N^a, and the lapse function N. The lapse function is related to the separation between hypersurfaces, and the components of the shift vector are related to the displacement of a point when moving from one hypersurface to another.

In order to rewrite the gravitational action in terms of the new variables, we must to express the Ricci scalar R and the volume form $dV = \sqrt{-g}d^4x$ as functions of geometrical objects in the hypersurface and on the new variables N and N^a. To do this, it is important to consider the extrinsic curvature K_{ab} of the hypersurface Σ. With the help of this object, we can rewrite the Riemann curvature tensor and, therefore ,also the Ricci scalar R, as functions of q_{ab}, $\dot{q}_{ab}$, N and N^a, where the upper dot $\dot{q}_{ab}$ denotes the derivative with respect to the parameter t defined by $\mathcal{X}$. With all this, the action can be written in the form

$$S[q_{ab}, N, N_a] = \frac{1}{\kappa}\int dt \int_\Sigma d^3x\sqrt{q}N\left(K^{ab}K_{ab} - K^2 + R^{(3)}\right), \tag{2.2}$$

where q is the determinant of q_{ab}, $K := K^a_a$, and $R^{(3)}$ is the Ricci scalar corresponding to the metric q_{ab}[3].

It is important to remark that written in this form the lagrangian density does not depend on the derivatives of the shift and lapse functions with respect to t. This means that the canonical momenta associated to N and N^a are zero. This fact allows to consider such variables as no-relevant dynamical variables. Variations with respect to lapse and shift will not produce dynamical equations but constraint equations, relating the dynamical variables (q_{ab}, π_{ab}) to each other.

In order to construct the hamiltonian density, we must define the canonical momenta associated to the dynamical variables q_{ab} in the form

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}}$$

and perform the corresponding Legendre transformation only over the relevant dynamical variables q_{ab} and its canonical momenta:

$$\mathcal{H} = \pi^{ab}\dot{q}_{ab} - \mathcal{L}.$$

[3]The indices $a,\ b$ are raised and lowered with the three-dimensional metric q_{ab}.

Using the Hamiltonian density, we can rewrite the lagrangian density and the Einstein-Hilbert action in a form that the dynamical variables will be q_{ab} and the corresponding canonical momenta π_{ab}. The action of General Relativity then becomes

$$S_{EH}[q_{ab}, \pi^{ab}, N_a, N] = \frac{1}{\kappa}\int dt \int_\Sigma d^3x$$
$$\left[\pi^{ab}\dot{q}_{ab} + 2N_a\nabla^{(3)}_b(q^{-(1/2)}\pi^{ba}) + N\left(q^{1/2}\left[R^{(3)} - q^{-1}\pi_{ab}\pi^{ab} + \frac{1}{2}q^{-1}\pi^2\right]\right)\right], \quad (2.3)$$

where $\nabla^{(3)}_b$ is the Levi-Civita connection over Σ with respect to the metric q_{ab}. The canonical momenta π^{ab} are related to the extrinsic curvature K_{ab} of Σ by

$$\pi^{ab} = q^{-1/2}\left(K^{ab} - Kq^{ab}\right). \quad (2.4)$$

Variation with respect to the shift vector produces the constraint equations

$$V^a(q_{ab}, \pi^{ab}) = 2\nabla^{(3)}_b\left(q^{-1/2}\pi^{ba}\right) = 0, \quad (2.5)$$

whereas variation with respect to the lapse function produces

$$S(q_{ab}, \pi^{ab}) = \left(q^{1/2}\left[R^{(3)} - q^{-1}\pi_{ab}\pi^{ab} + (1/2)q^{-1}\pi^2\right]\right) = 0. \quad (2.6)$$

$V^i(q_{ab}, \pi^{ab})$ is called the vector constraint, and $S(q_{ab}, \pi^{ab})$ is called the scalar (or hamiltonian) constraint. The action can then be written as

$$S[q_{ab}, \pi^{ab}, N_i, N] = \frac{1}{\kappa}\int dt \int_\Sigma d^3x\left[\pi^{ab}\dot{q}_{ab} - N_aV^a(q_{ab}, \pi^{ab}) - NS(q_{ab}, \pi^{ab})\right], \quad (2.7)$$

where the hamiltonian density can be identified as

$$\mathcal{H}(q_{ab}, \pi^{ab}, N_a, N) = N_aV^a(q_{ab}, \pi^{ab}) + NS(q_{ab}, \pi^{ab}).$$

We can see that the hamiltonian density is a linear combination of first class constraints, which means that it vanishes on solutions of the dynamical equations[2, 3, 6]. The Poisson brackets between the dynamical variables can be calculated using the previous equations, and we obtain [2, 3, 6]:

$$\left\{\pi^{ab}(x), q_{cd}(y)\right\} = \kappa\delta^a_{(c}\delta^b_{d)}\delta(x-y) \qquad \left\{\pi^{ab}(x), \pi^{cd}(y)\right\} = \{q_{ab}(x), q_{cd}(y)\} = 0. \quad (2.8)$$

We then have six configuration variables $q_{ij}(x)$ and four constraint equations (2.5) and (2.6), which implies the two physical degrees of freedom of gravity.

2.2 The new variables

The canonical formulation presented above has the three-dimensional metric q_{ab} and its canonical momenta π^{ab} as dynamical variables. It is well known that the quantization of the theory using such variables presents several issues [2, 29]. For this reason, we will make a change of variables. The idea is to use a triad (also called "Repère Mobile" (see [30])) or moving frame. The triad is a set of three one-forms in terms of which the metric q_{ab} can be obtained as

$$q_{ab} = e_a^i e_b^j \delta_{ij}, \tag{2.9}$$

where $i, j = 1, 2, 3$ are called "internal indices". In terms of these new variables we can define the densitized triad:

$$E_i^a := \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}e_b^j e_c^k. \tag{2.10}$$

We also define:

$$K_a^i := \frac{1}{\sqrt{det(E)}} K_{ab} E_j^b \delta^{ij}. \tag{2.11}$$

The set (E_i^a, K_a^i) represents a new set of phase space variables, and we can rewrite the constraints $V^a(q_{ab}, \pi^{ab})$ and $S(q_{ab}, \pi^{ab})$ as functions of E and K as $V^a(E_i^a, K_a^i)$ and $S(E_a^a, K_a^i)$. The new variables, however, are redundant because we are using the nine components of E_i^a to describe the six components of q_{ab}. The geometrical interpretation of such redundancy is the following: the extra three degrees of freedom in the triad correspond to our freedom to choose between different local frames e_a^i. Such local frames are related to each other through an $SO(3)$ rotation acting in the internal indices $a = 1, 2, 3$. This means there must be a constraint between the new variables that expresses this redundancy. This new constraint actually comes from equation (2.11) because we did not take into account that $K_{ab} = K_{ba}$. Inverting the expressions (2.10) and (2.11) in order to write K_{ab} in terms of E_i^a and K_a^i, it can be shown that the condition $K_{ab} = K_{ba}$ reduces to

$$G_i(E_j^a, K_a^j) := \epsilon_{ijk}E^{aj}K_a^k = 0. \tag{2.12}$$

This is called the "Gauss constraint", and it is a direct consequence of the introduction of the variables (E_a^i, K_i^a) to parameterize the phase space of general relativity. Therefore, to use the new variables we must to include this additional constraint in the action of the theory. With all this, the action of General Relativity becomes

$$S[E_j^a, K_a^j, N_a, N, N^j] = \frac{1}{\kappa}\int dt \int_\Sigma d^3x \left[E_i^a \dot{K}_a^i - N_b V^b(E_i^a, K_a^i) - NS(E_i^a, K_a^i) - N^i G_i(E_j^a, K_a^j)\right], \tag{2.13}$$

where $N_j := e^b_j N_b$ and the internal indices i, j, k are raised and lowered using the three-dimensional metric δ_{ij}. The explicit form of the scalar and vector constraints in terms of the new variables can be obtained using the definitions presented above, but such expressions are not necessary for us and can be found in [31]. The Poisson brackets between the new variables are

$$\left\{E^a_j(x), K^i_b(y)\right\} = \kappa\delta^a_b\delta^i_j\delta(x-y) \qquad \left\{E^a_j(x), E^b_i(y)\right\} = \left\{K^j_a(x), K^i_b(y)\right\} = 0. \tag{2.14}$$

2.2.1 The Ashtekar-Barbero connection variables

Both the densitized triad (2.10) and its conjugate momentum K^a_i transform in the vector representation of $SO(3)$. There exists a $so(3)$-connection that is compatible with the triad. This connection is called the *spin connection* Γ^i_a and is characterized as the solution of Cartan's structure equations:

$$\partial_{[a}e^i_{b]} + \epsilon^i{}_{jk}\Gamma^j_{[a}e^k_{b]} = 0. \tag{2.15}$$

We can write the solution to the previous equation in terms of the components of the triad as follows:

$$\Gamma^i_a = -\frac{1}{2}\epsilon_k{}^{ij} e^b_j \left(\partial_{[a}e^k_{b]} + \delta^{kl}\delta_{ms}e^c_l e^m_a \partial_b e^s_c\right), \tag{2.16}$$

where e^a_i is the inverse triad ($e^a_i e^j_a = \delta^j_i$). With the aim of quantization in mind, we introduce another set of variables; we define a new connection A^i_a given by:

$$A^i_a := \Gamma^i_a + \gamma K^i_a, \tag{2.17}$$

where $\gamma \neq 0$ is a real number called the *Immirzi parameter*. The most important fact about this variable is that it is canonically conjugate to E^i_a. The Poisson brackets between the variables (E^i_a, A^i_a) (called the *Ashtekar-Barbero* variables) are

$$\left\{E^a_j(x), A^i_b(y)\right\} = \kappa\gamma\delta^a_b\delta^i_j\delta(x-y) \qquad \left\{E^a_j(x), E^b_i(y)\right\} = \left\{A^j_a(x), A^i_b(y)\right\} = 0, \tag{2.18}$$

Using the connection variables, the action takes the form

$$S[E^a_j, A^j_a, N_a, N, N^j] = \frac{1}{\kappa}\int dt \int_\Sigma d^3x \left[E^a_i \dot{A}^i_a - N^b V_b(E^a_j, A^j_a) - NS(E^a_j, A^j_a) - N^i G_i(E^a_j, A^j_a)\right], \tag{2.19}$$

and the constraints, in terms of the new variables, are explicitly given by

$$V_b(E^a_j, A^j_a) = E^a_j F^j_{ab} - \left(1+\gamma^2\right) K^i_b G_i, \tag{2.20}$$

$$S(E^a_j, A^j_a) = \frac{E^a_i E^b_j}{\sqrt{\det(E)}}\left(\epsilon^{ij}{}_k F^k_{ab} - 2\left(1+\gamma^2\right) K^i_{[a}K^j_{b]}\right), \tag{2.21}$$

$$G_i(E^a_j, A^j_a) = D_a E^a_i, \tag{2.22}$$

where $F^i_{ab} := \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^j_a A^k_b$ is the curvature of the connection A^i_a, and $D_a E^a_i = \partial_a E^a_i + \epsilon_{ij}{}^k A^j_a E^a_k$ is the covariant divergence of the densitized triad. These constraints are first class constraints [32] that not only impose conditions among the canonical variables but also generate infinitesimal gauge transformations. The phase space variables (A^i_a, E^b_j) constitute a set of 18 variables, but we also have seven constraints among them. Therefore we end up with 11 variables necessary to coordinatize the surface on the phase space where the above seven conditions hold. On that surface, the above constraints generate a seven-parameter family of gauge transformations, and this reduces the amount of independent canonical variables to only four. Therefore the resulting number of physical degrees of freedom is two.

The constraint (2.22) is formally equal to the Gauss constraint of a Yang-Mills theory [1, 2, 6, 7, 29]. If we ignore the constraints (2.20) and (2.21), the phase space variables (A^i_a, E^b_j) together with the Equation (2.22) characterize the physical phase space of a $SU(2)$ Yang-Mills theory [6, 7]. The gauge field is given by the connection A^i_a and its canonical momentum is E^b_j. Standard Yang-Mills theory is defined over a fixed background metric, and its dynamics is encoded in a non-vanishing hamiltonian density. On the other hand, General Relativity is a background-independent field theory, and its hamiltonian density, according to the canonical formalism sketched out here, is on-shell vanishing because is a linear combination of constraints. The dynamics is encoded in the constraint equations (2.20) to (2.22). In this sense general relativity can be regarded in the new variables as a kind of background-independent close relative of a $SU(2)$ Yang-Mills theory [2, 6, 7]. This similarity will allow us the implementation of quantization techniques that are natural in the context of Yang-Mills theories.

Before jumping to the geometric interpretation of the Ashtekar-Barbero variables, we will write the gauge transformations generated by the constraints and the algebra that they satisfy. More details can be found in [1, 2, 3, 6, 7].

The Gauss law (2.22) generates local $SU(2)$ transformations over the fields A^i_a and E^a_i. To see it, let us define the smeared version of (2.22) as [1, 3, 7]:

$$G(\alpha) := \int_\Sigma \alpha^i(x) G_i(E^a_j, A^j_a) d^3x, \tag{2.23}$$

where α^i are three arbitrary smooth smearing functions defined on Σ, such that the integral on the right-hand side is well defined. Thus, we find that [1, 3, 6, 7]

$$\delta_G A^i_a = \left\{A^i_a, G(\alpha)\right\} = -D_a \alpha^i \qquad\qquad \delta_G E^a_i = \{E^a_i, G(\alpha)\} = [E, \alpha]_i\,, \tag{2.24}$$

If we write $A_a = A^i_a \tau_i \in su(2)$ and $E^a = E^a_i \tau^i \in su(2)$ where τ_i are generators of $SU(2)$, the finite version of the previous transformations would be

$$\tilde{A}_a = g A_a g^{-1} + g \partial_a g^{-1} \qquad\qquad \tilde{E}^a = g E^a g^{-1}, \tag{2.25}$$

which is how the connection and the electric field transform under gauge transformations in a Yang-Mills theory.

The vector constraint (2.20) generates three-dimensional diffeomorphisms on Σ. To prove it we define the smeared vector constraint:

$$V(N^a) := \int_\Sigma N^a V_a(A^i_a, E^a_i) d^3x, \tag{2.26}$$

where N^a is a test vector field. The action of such smeared constraint on the canonical variables is given by

$$\delta_V A^i_a = \left\{A^i_a, V(N^a)\right\} = \mathcal{L}_N A^i_a \qquad \delta_V E^a_i = \{E^a_i, V(N^a)\} = \mathcal{L}_N E^a_i, \tag{2.27}$$

where $\mathcal{L}_N$ denotes the Lie derivative in the N^a direction. The finite version of these transformations becomes the action of finite diffeomorphisms acting on Σ [7].

Finally, the scalar constraint (2.21) generates coordinate time evolution. The total hamiltonian $H[\alpha, N^a, N]$ of general relativity can be written as

$$H(\alpha, N^a, N) = G(\alpha) + V(N^a) + S(N), \tag{2.28}$$

where

$$S(N) = \int_\Sigma N S(A^i_a, E^a_i) d^3x, \tag{2.29}$$

and $N \in C^\infty(\Sigma)$ is an arbitrary smooth real density defined on Σ.

With the help of the smeared constraints, we can find the constraint algebra of General Relativity in the new variables, that is given by [2, 3, 6, 7, 33]

$$\{G(\alpha), G(\beta)\} = G([\alpha, \beta]), \tag{2.30}$$

where $\alpha = \alpha^i \tau_i \in su(2)$, $\beta = \beta^i \tau_i \in su(2)$ and $[\alpha, \beta]$ is the Lie bracket in $su(2)$.

In addition, we have

$$\{G(\alpha), V(N^a)\} = -G(\mathcal{L}_N \alpha), \tag{2.31}$$

$$\{G(\alpha), S(N)\} = 0, \tag{2.32}$$

$$\{V(N^a, V(M^a))\} = V([N, M]^a), \tag{2.33}$$

where $[N, M]^a = N^b \partial_b M^a - M^b \partial_b N^a$ is the vector field commutator.

Additionally, we have

$$\{S(N), V(N^a)\} = -S(\mathcal{L}_N N), \tag{2.34}$$

and finally

$$\{S(N), S(M)\} = V(S)^a + \text{terms proportional to the Gauss constraint}, \tag{2.35}$$

where we are omitting terms proportional to the Gauss constraint (the complete expression can be found in [33]), and

$$S^a := \frac{E_i^a E_j^b \delta^{ij}}{|\det(E)|} \left(N\partial_b M - M\partial_b N\right). \tag{2.36}$$

2.2.2 The geometry of the three-surface Σ

Having defined a new set of variables to parametrize the phase space of General Relativity, we can give a geometrical interpretation of such variables that will be very important in the definition of the quantum theory. The connection A_a^i provides a definition for the parallel transport of $SU(2)$ spinors defined on the manifold Σ. If $l : J \to \Sigma$, with $J \subseteq \mathbb{R}$ an interval, is a curve on Σ, the holonomy of A_a^i along l is defined as [7, 29]

$$U[A, l] = Pexp\left(\int_l A\right) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P\left(\int_0^t A(l^{'}(s))ds\right). \tag{2.37}$$

In this way, $U[A, l] \in SU(2)$.

The densitized triad (also called the *gravitational electric field* because of its similarity with the electric field of Yang-Mills theories [1, 7]) encodes the Riemannian geometry of Σ. That means that any geometrical quantity in such hypersurface can be written as a functional of E_i^a. For example, consider a two-dimensional surface $S \subset \Sigma$ parametrized as $x^a = x^a(\sigma^1, \sigma^2)$, where (σ^1, σ^2) are local coordinates on S. The normal to such hypersurface is given by

$$n_a = \epsilon_{abc} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2}. \tag{2.38}$$

Hence, the area of S is given by

$$A_S[E_i^a] = \int_S \sqrt{E_i^a E_j^b \delta^{ij} n_a n_b}\, d\sigma^1 d\sigma^2. \tag{2.39}$$

Because of this, we can say that "*the area of a surface is the flux of (the norm of) the gravitational field across the surface*", see [1].

Analogously, the volume of a three-dimensional region R on Σ parametrized as $x^a = x^a(\sigma^1, \sigma^2, \sigma^3)$, where $(\sigma^1, \sigma^2, \sigma^3)$ are local coordinates on R, is given by

$$V_R[E_i^a] = \int_R \sqrt{|\det E|} d\sigma^1 d\sigma^2 d\sigma^3. \tag{2.40}$$

These formulas for the area and volume will be very important in the quantum theory.

2.3 Canonical quantization of gravity

As we have seen, the Hamiltonian formulation of general relativity shows that gravity is a totally constrained field theory and, with the help of the Ashtekar-Barbero variables, we could write it as a constrained field theory of connections over a three-dimensional manifold. In order to proceed with the quantization of the theory, we will use the standard quantization methods employed in other constrained theories, such as those used to quantize Yang-Mills theories [2, 3, 6, 29, 32] (even though Yang-Mills theories have nonzero Hamiltonians). This way of working assumes that general relativity and quantum theory still remain valid in scales that are beyond the scales where they have been tested and that a mix of both is what is necessary for describing the quantum gravity regimes. In LQG is assumed that, as mentioned in [34], extrapolation is the most effective tool in science. In this way, up to contrary empirical indications, that are always possible, a good bet is that what we have learned so far may well continue to hold. General relativity and quantum theory are therefore the well-established physical ground of LQG [34].

2.3.1 Non-perturbative quantization strategy

The canonical quantization of a constrained field theory, written in the Hamiltonian formalism, can be sketched as follows [2, 6, 7, 32]

a) Find a representation of the phase space variables of the theory as operators in an kinematical Hilbert space $\mathcal{H}_{kin}$, satisfying the commutation relations that are obtained with the replacement $\{\ ,\ \} \to -i/\hbar[\ ,\ ]$.

b) Promote the constraints to self-adjoint operators acting in $\mathcal{H}_{kin}$. In the case of gravity, we must quantize the seven constraints $G_i(A,E)$, $V_a(A,E)$, and $S(A,E)$.

c) Characterize the space of solutions of all the quantized constraints, and define the inner product that defines a notion of physical probability. This defines the physical Hilbert space $\mathcal{H}_{phys}$.

d) Find a complete set of gauge invariant observables, that is, a complete set of operators commuting with all the constraints. They will represent the physical measurable quantities of the theory.

2.3.2 The kinematical Hilbert space

In order to quantize the canonical formulation of General Relativity, we must first define the kinematical Hilbert space. The canonical variables associated with the classical theory are the connection A^i_a and the electric field E^a_i. We choose the connection A as the configuration variables. This choice has important advantages as we will see. The kinematical Hilbert space consists of a suitable set of functionals of the connection $\psi[A]$, which are square integrable with respect to a suitable measure $\mu[A]$.

In order to define the kinematical Hilbert space, we begin considering the space of smooth three-dimensional $su(2)$ connections A defined everywhere on Σ except possibly at isolated points. We will denote such space as $\mathcal{G}$. Let $\{\tau_i = -(i/2)\sigma_i\}$ be a basis for $su(2)$ with σ_i the Pauli matrices. We will then write:

$$A(x) = A^i_a(x)\tau_i dx^a. \tag{2.41}$$

Let γ be an oriented, piecewise smooth curve on Σ, and let $U[A,\gamma] \in SU(2)$ be the holonomy of A along γ. For a given γ, the holonomy is a functional defined on $\mathcal{G}$. Consider an ordered collection Γ of n smooth oriented curves $\{\gamma_k\}_{k=1}^n$ in Σ and a smooth real function

$$f : SU(2) \times SU(2) \times \cdots \times SU(2) \to \mathbb{R}$$

of n group elements. The couple (Γ, f) defines a functional on $\mathcal{G}$ given by

$$\Psi_{\Gamma,f}[A] := f\left(U(A,\gamma_1), \ldots, U(A,\gamma_n)\right). \tag{2.42}$$

We define $\mathcal{S}$ as the space of all functionals $\Psi_{\Gamma,f}[A]$, for all Γ and f. This is called the space of "cylindrical functions". Changing the order or the orientation of a graph is the same as changing the order of the arguments of the function f or replacing arguments with their inverse.

If two functionals $\Psi_{\Gamma,f}[A]$ and $\Psi_{\Gamma,g}[A]$ are defined in the same oriented graph Γ, we define

$$\langle \Psi_{\Gamma,f} | \Psi_{\Gamma,g} \rangle := \int_{SU(2)} \overline{f(U_1, \ldots, U_n)} g(U_1, \ldots, U_n) dU_1 \ldots dU_n, \tag{2.43}$$

where dU is the Haar measure on $SU(2)$ [35].

We can extend the scalar product defined above to functionals defined on the same graph but with different order or orientation [1, 3]. We can also extend it to functionals defined on different graphs. If we have two couples (Γ, f) and $(\tilde{\Gamma}, \tilde{f})$, if Γ is the union of the $\tilde{n}$ curves of $\tilde{\Gamma}$ and other m curves, and if $f(U_1 \ldots U_{\tilde{n}}, U_{\tilde{n}+1}, \ldots U_{\tilde{n}+m}) = \tilde{f}(U_1, \ldots, U_{\tilde{n}})$, then $\Psi_{\Gamma,f} = \Psi_{\tilde{\Gamma},\tilde{f}}$. Using this fact, we can rewrite any two functionals Ψ_{Γ_1,f_1} and Ψ_{Γ_1,f_2} as functional $\Psi_{\Gamma,f}$, and $\Psi_{\Gamma,g}$ having the same graph, where $\Gamma = \Gamma_1 \cup \Gamma_2$ [1]. Thus, (2.43) is a valid definition for any two functionals in $\mathcal{S}$

$$\langle \Psi_{\Gamma_1,f_2} | \Psi_{\Gamma_2,f_2} \rangle := \langle \Psi_{\Gamma,f} | \Psi_{\Gamma,g} \rangle. \tag{2.44}$$

In this way, we define the kinematical Hilbert space $\mathcal{H}_{kin}$ of LQG as the completion of $\mathcal{S}$ [36] in the norm defined by the scalar product (2.43)[4].

The main reason for this definition is that the scalar product defined in (2.43) is invariant under diffeomorphisms and local gauge transformations.

The kinematical Hilbert space $\mathcal{H}_{kin}$ can be seen as a space of square integrable functionals [3]. As is shown in [3], $\mathcal{H}_{kin} \simeq L^2[\mathcal{A}, \mu_o]$, where $\mathcal{A}$ is an extension of the space of smooth connections on Σ that includes distributional connections [1]. The measure μ_o is defined in this space and is called the Ashtekar-Lewandowski measure [3].

[4]It is, however, crucial that the space of cylindrical functions be invariant under diffeomorphisms on Σ, and that it is closed under multiplication (for 2.43 to be defined). Generally, this is not true for the piecewise smooth graphs. See [47], [48].

The space $\mathcal{H}_{kin}$ has some special characteristics that are very important in Loop Quantum Gravity. We will mention some of these characteristics below, and more details can be found in [1, 3].

- The set of cylindrical functions with support in a given graph Γ is a subspace $\tilde{\mathcal{H}}_\Gamma$ of $\mathcal{H}_{kin}$. In fact, we have that $\tilde{\mathcal{H}}_\Gamma = L^2[SU(2)^n]$, where n is the number of paths in Γ. If $\Gamma_1 \subseteq \Gamma_2$, then the Hilbert space $\tilde{\mathcal{H}}_{\Gamma_1}$ is a proper subspace of $\tilde{\mathcal{H}}_{\Gamma_2}$ [1, 3].

- We can find an orthonormal basis for $\mathcal{H}_{kin}$ with the help of the Peter-Weyl theorem. The irreducible representations of $SU(2)$ are labeled by a half-integer number j. We write the matrix elements of the representation j as

$$R^{(j)\alpha}{}_{\beta}(U) = \langle U|j, \alpha, \beta\rangle. \tag{2.45}$$

  For each graph Γ, choose an ordering and an orientation of its edges. By the Peter Weyl-theorem, if Γ has n paths, an orthonormal basis for $\tilde{\mathcal{H}}_\Gamma$

$$|\Gamma, j_k, \alpha_k, \beta_k\rangle := |\Gamma, j_1, \ldots, j_n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\rangle \tag{2.46}$$

  is given by

$$\langle A|\Gamma, j_k, \alpha_k, \beta_k\rangle = R^{(j_1)\alpha_1}{}_{\beta_1}(U[A, \gamma_1]) \ldots R^{(j_n)\alpha_n}{}_{\beta_n}(U[A, \gamma_n]), \tag{2.47}$$

  where $U[A, \gamma]$ is the holonomy of the connection A along the path γ. Note that in the definition of the basis we are assigning a $SU(2)$ irreducible representation to each edge in Γ.

  An orthonormal basis for the space $\mathcal{H}_{kin}$ is given by the states $|\Gamma, j_k, \alpha_k, \beta_k\rangle$, for all the ordered and oriented paths Γ, where the spins j_k are never zero.

- As we have already mentioned, $\mathcal{H}_{kin}$ can be seen as a space of square integrable functions, as is usual for the Hilbert spaces in the quantum theory, because there exist a Hilbert spaces isomorphism $\mathcal{H}_{kin} \simeq L^2[\mathcal{A}, \mu_o]$.

Before presenting the action of the quantum constraints on the space $\mathcal{H}_{kin}$, it is important to emphasize here that the scalar product defined in (2.43) is invariant under the action of $SU(2)$ gauge transformations and three-dimensional diffeomorphisms acting on Σ. We define an extended diffeomorphism $f : \Sigma \to \Sigma$ as a continuous invertible function that both it and its inverse are everywhere C^∞, except possibly at a finite number of isolated points on Σ. We denote as *Diff* $({}^*\Sigma)$ the group of extended three-dimensional diffeomorphisms over Σ. Thus, as is explicitly shown in [1, 3], $\mathcal{H}_{kin}$ carries a unitary representation of local $SU(2)$ gauge transformations and *Diff* $({}^*\Sigma)$. The action of the elements of *Diff* $({}^*\Sigma)$ on the members of the form $\Psi[A]$ is given by $U_\phi \Psi[A] = \Psi((\phi^*)^{-1}\, A)$.

The set of functions $\Psi_{\Gamma,f}[A]$ transforms under $SU(2)$ gauge transformations and extended diffeomorphisms in the following way.

Let $\lambda : \Sigma \to SU(2)$ be a smooth local $SU(2)$ gauge transformation defined on Σ. The connection A then transforms inhomogeneously under such a transformation [1] as follows

$$A \to A_\lambda = \lambda A \lambda^{-1} + \lambda d\lambda^{-1}.$$

For a couple (Γ, f), define

$$f_\lambda(U_1, \dots, U_L) := f\left(\lambda(x_f^{\gamma_1})U_1\lambda^{-1}(x_i^{\gamma_1}), \dots, \lambda(x_f^{\gamma_L})U_L\lambda^{-1}(x_i^{\gamma_L})\right)$$

where x_i^γ and x_f^γ are the initial and final points of γ, respectively. The transformation of the quantum states is given by [1]

$$\Psi_{\Gamma,f}(A) \to \Psi_{\Gamma,f}(A_{\lambda^{-1}}) = \Psi_{\Gamma,f_{\lambda^{-1}}}(A).$$

Under an extended diffeomorphism $\phi \in Diff^*(\Sigma)$, a cylindrical function $\Psi_{\Gamma,f}[A]$ is sent to a cylindrical function $\Psi_{\phi\Gamma,f}[A]$, that is to say one that is based on the shifted graph.

2.3.3 Operators on $\mathcal{H}_{kin}$

The next step to quantize the gravitational theory is to define the operators corresponding to the canonical variables acting on $\mathcal{H}_{kin}$. The two canonical variables in the classical theory are the $SU(2)$ connection A_a^i and its conjugate momenta E_i^a. These are the basic building blocks for constructing the rest of the important quantities of the theory. We have chosen the connection A_a^i as our generalized coordinate, so the conjugate momenta is $(1/8\pi G)E_i^a$, and the quantum states on $\mathcal{H}_{kin}$ are functionals of A. We then define the two operators

$$\begin{aligned} \widehat{A}_a^i(x)\left(\Psi[A]\right) &:= A_a^i(x)\Psi[A], \\ \frac{1}{8\pi G}\widehat{E}_i^a\left(\Psi[A]\right) &:= -i\hbar\frac{\delta}{\delta A_a^i(x)}\Psi[A] \end{aligned} \tag{2.48}$$

acting on the elements of $\mathcal{H}_{kin}$. We will choose units so that $8\pi G = 1$. The first is a multiplicative operator, and the second is a functional derivative. We must indicate that these operators send the elements $\Psi[A]$ out of the space $\mathcal{H}_{kin}$. In particular, according to [1, 2], they are not well defined on $\mathcal{H}_{kin}$. To fix these problems we will not take A and E but some functions of them.

First, let us consider the connection $A_a^i(x)$. The holonomy $U[A,\gamma]$ can be transformed into a well-defined operator acting on $\mathcal{S}$ (the space of cylindrical functions). Let ${U^A}_B(A,\gamma)$ be the matrix elements of $U(A,\gamma)$. Thus, according to [1], we define $\widehat{U}(A,\gamma)$ as

$$\left(\widehat{U}(A,\gamma)\Psi\right)[A] = {U^A}_B(A,\gamma)\Psi[A]. \tag{2.49}$$

If $\Psi[A] \in \mathcal{S}$, then the right-hand side is also in $\mathcal{S}$ [1]. Moreover, as is mentioned in [1, 2], any cylindrical function of the connection is immediately well-defined as a multiplicative operator in $\mathcal{H}_{kin}$.

Now, to define an operator corresponding to E_i^a, let us consider a two-dimensional surface σ embedded on the three-dimensional manifold Σ. If $\vec{\sigma} = (\sigma^1, \sigma^2)$ are coordinates on the

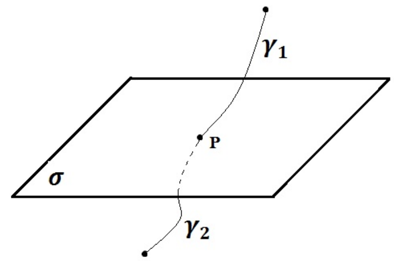

Figure 2.1: A curve that intersects the surface at a point P

surface σ, then the surface is defined by $(\sigma^1, \sigma^2) \to x^a(\sigma^1, \sigma^2)$. Now consider the operator

$$\widehat{E}_i(\sigma) := -i\hbar \int_\sigma d\sigma^1 d\sigma^2 n_a(\vec{\sigma}) \frac{\delta}{\delta A_a^i(x(\vec{\sigma}))}, \tag{2.50}$$

where

$$n_a(\vec{\sigma}) = \epsilon_{abc} \frac{\partial x^b(\vec{\sigma})}{\partial \sigma^1} \frac{\partial x^c(\vec{\sigma})}{\partial \sigma^2} \tag{2.51}$$

is the normal one-form on σ. To show how the operator $E_i(\sigma)$ acts on the holonomy, $U[A,\gamma]$ let us first consider that the curve γ along which we calculate the holonomy has two end points that do not lie on the surface σ. Let us also consider that the curve γ crosses the surface at just one point, and let P be the intersection point (if any), see Figure 2.1. The curve is then separated into two parts by P, and we can write $\gamma = \gamma_1 \cup \gamma_2$.

The action of the operator $\widehat{E}_i(\sigma)$ on $U(A,\gamma)$ is, according to [1] and [2], given by

$$\widehat{E}_i(\sigma)U(A,\gamma) = \pm i\hbar U(A,\gamma_1)\tau_i U(A,\gamma_2). \tag{2.52}$$

Therefore, the operator simply inserts the matrix $\pm i\hbar\tau_i$ at the point of intersection. The sign is dictated by the relative orientation of the surface with respect to the curve.

The previous results can be generalized to multiple intersections. Using P to label the intersection points, we have [1]

$$\widehat{E}_i(\sigma)U(A,\gamma) = \sum_{P\in(\sigma\cap\gamma)} \pm i\hbar U(A,\gamma_1^P)\tau_i U(A,\gamma_2^P), \tag{2.53}$$

and the action of the operator on the holonomy in an arbitrary $SU(2)$ representation j is given by

$$\widehat{E}_i(\sigma)R^j(U(A,\gamma)) = \pm i\hbar R^j(U(A,\gamma_1))\tau_i^{(j)} R^j(U(A,\gamma_2)), \tag{2.54}$$

where $\tau_i^{(j)}$ is the $SU(2)$ generator in the spin-j representation.

The special case where the curve γ or part of it lies on the surface σ is discussed in [1].

In summary, instead of the operators given in (2.48), we consider the couple of operators

$\widehat{U}(A,\gamma)$ and $\widehat{E}_i(\sigma)$. $\widehat{U}(A,\gamma)$ corresponds to the holonomy of the connection A along the curve γ. $\widehat{E}_i(\sigma)$ corresponds, according to [1, 2], to the flux of E^a_i through a two-dimensional surface. These operators, both defined on the kinematical Hilbert space $\mathcal{H}_{kin}$, form a representation (called the holonomy-flux representation) of the classical Poisson algebra defined by the holonomy of A and the flux of E [1, 2, 3]. A very important result in loop quantum gravity is the proof of a unicity theorem for this representation; under some assumptions, this holonomy-flux representation is unique. See [1, 2, 3].

2.3.4 Solutions of the kinematical constraints

Now that we have defined the kinematical Hilbert space $\mathcal{H}_{kin}$ and have found a representation of the phase space variables as operators acting in $\mathcal{H}_{kin}$, we must to promote the constraints to self-adjoint operators in $\mathcal{H}_{kin}$. Specifically, we must quantize the seven constraints $G_i(A,E)$, $V_a(A,E)$, and $S(A,E)$, then characterize the space of solutions of all the quantized constraints, and finally define the inner product that defines a notion of physical probability. This will define the physical Hilbert space $\mathcal{H}_{phys}$.

Let us first consider the set of *kinematical constraints*, which consist only of the Gauss and the vector constraint. The kinematical state space is a space of functionals of the connection $\Psi[A]$, but the constraints generate transformations over Σ. Therefore we need a space of states that are functionals invariant under these transformations. We need functionals invariant under local $SU(2)$ gauge transformations and three-dimensional extended diffeomorphisms.

Gauge invariant states

Let us start with the Gauss constraint. Call $\mathcal{H}_0$ the space of states in $\mathcal{H}_{kin}$ that are invariant under local $SU(2)$ transformations. In order to characterize the vector space $\mathcal{H}_0$, it is enough to find a basis, in this case an orthonormal basis, of such a space [14]. A basis of $\mathcal{H}_0$ is given by the *spin network states*.

Consider a graph Γ in Σ. We will call nodes to the end points of the oriented curves in Γ. Let us assume that Γ is a set of curves that, if they intersect, do it only on their nodes. Thus Γ can be considered a set of points in Σ joined by smooth curves γ, and these curves will be called links. The outgoing multiplicity of a node, denoted as m_{out}, will be the number of links on Γ that begin in that node, and the ongoing multiplicity, denoted by m_{in}, will be the number of links that end in that node. See [1]. The valence of the node will be the sum $m_{in}+m_{out}$.

Given a graph Γ in Σ with an ordering and orientation, let j_l be an assignment of an SU(2) irreducible representation, different from the trivial one, to each link l, and let i_n be an assignment of an intertwiner to each node. The intertwiner i_n associated with a node is between the representations associated with the links adjacent to the node. The triplet S= (Γ, j_l, i_n) is called a *spin network* in Σ. The choice of (j_l, i_n) is called a *coloring* of the links and nodes.

In order to define the spin network states, let us consider a spin network S= (Γ, j_l, i_n) with M links and N nodes. The state $|\Gamma, j_k, \alpha_k, \beta_k\rangle$ defined on (2.46) has M indices α_k and M indices β_k. The N intertwiners v_{i_k} have precisely a set of indices dual to these, so we can contract the indices of the state $|\Gamma, j_k, \alpha_k, \beta_k\rangle$ with the indices of the intertwiners to obtain a gauge-invariant state in the form

$$|S\rangle := \sum_{\alpha_k,\beta_k} v_{i_1}{}^{\beta_1\ldots\beta_{n_1}}{}_{\alpha_1\ldots\alpha_{n_1}} v_{i_2}{}^{\beta_{n_1+1}\ldots\beta_{n_2}}{}_{\alpha_{n_1+1}\ldots\alpha_{n_2}} \ldots v_{i_N}{}^{\beta_{n_{N-1}+1}\ldots\beta_M}{}_{\alpha_{n_{N-1}+1}\ldots\alpha_M} |\Gamma, j_k, \alpha_k, \beta_k\rangle. \tag{2.55}$$

The expression (2.55) defines the spin network state corresponding to the spin network S= (Γ, j_l, i_n), and we denote it by $|\mathrm{S}\rangle$. The pattern of contraction for the indices is dictated by the connectivity of the graph in the following way: the index α_k of the link k is contracted with the corresponding index of the intertwiner v_{i_s} of the node s where the link k starts. Similarly, the index β_k of the link k is contracted with the corresponding index of the intertwiner v_{i_s} of the node s where the link k ends. The states defined in (2.55) are $SU(2)$ gauge invariant. As a functional of the connection A, the spin network state can be written in the following way:

$$\Psi_S[A] = \langle A|S\rangle = \left(\bigotimes_k R^{(j_k)}\left(U[A, \gamma_k]\right)\right) \cdot \left(\bigotimes_s i_s\right), \tag{2.56}$$

where the dot on the right-hand side indicates contraction between dual spaces. Furthermore, on the left of the dot, the tensor product has indices in the space $\otimes_k \left(\mathcal{H}^*_{j_k} \otimes \mathcal{H}_{j_k}\right)$, and on the right side of the dot, the tensor product of the intertwiner has indices on the dual of this space.

When we choose the intertwiner i_s as belonging to an orthonormal basis in the space of intertwiners associated with a node, the set of spin network states $|S\rangle$ is an orthonormal basis of the space $\mathcal{H}_0$ with respect to the inner product defined in (2.43) (as is indicated in [1, 2, 3]). This basis is labeled by spin networks S= (Γ, j_l, i_n).

It is important to make some observations about the spin network states; the first is that we have assumed that the spins j_l assigned to the links of Γ are all different from zero. A spin network containing a link with $j_l = 0$ is identified with the spin network that does not contain the link l. The second important observation is that the spin network basis is highly non-unique because it depends on the (otherwise arbitrary) choice of a basis in the space of intertwiners at each node. Finally, notice that, in the basis $|S\rangle = |\Gamma, j_l, i_n\rangle$, the label Γ runs over all unoriented and unordered graphs, but in the definition of the coloring, Γ is an oriented and ordered graph [1, 2].

The space $\mathcal{S}_0$ is generated by finite linear combinations of spin network states, and such space, according to [1], is dense ([36]) in $\mathcal{H}_0$.

To be more, concrete we will give an example of a spin network state.

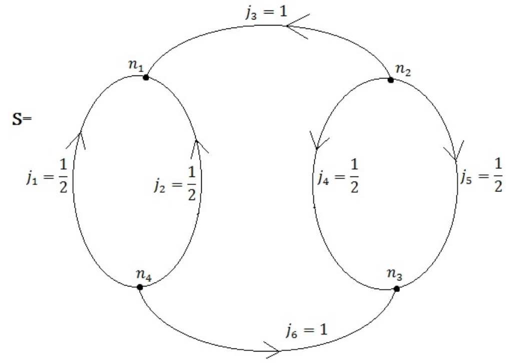

Figure 2.2: A somewhat complicated spin network with four trivalent nodes.

Let us say that Γ has four nodes n_1, n_2, n_3 and n_4, as well as six links. Some of the links will be colored by $j = 1/2$, and others will be colored by $j = 1$. The specific coloring, ordering and orientation of the spin network can be seen in Figure 2.2. At each node we must consider the tensor product of two fundamental and one adjoint representation of $SU(2)$. As shown in [1], the tensor product of these representations contains a single copy of the trivial representation; therefore there is only one intertwiner. That intertwiner is given by the triple of Pauli matrices:

$$v^{i,AB} = \frac{1}{\sqrt{3}}\sigma^{i,AB}$$

since these have precisely the invariance property

$$(R(U))^i{}_j U^A{}_D U^B{}_D \sigma^{j,CD} = \sigma^{i,AB}.$$

Here $(R(U))^i{}_j$ is the adjoint representation, with $i, j = 1, 2, 3$ vector indices and $U^A{}_C$ the fundamental representation, with $A, B = 0, 1$ spinor indices. Following the prescription given above, the corresponding spin network state is

$$\Psi_S[A] = \frac{1}{6}\sigma_{k,AC}\ (U[A,\gamma_1])^A{}_B\ (U[A,\gamma_2])^C{}_D\ \sigma^{j,BD}\ \left(R^{(1)}(U[A,\gamma_3])\right)^i{}_j$$
$$\sigma_{i,EG}\ (U[A,\gamma_4])^E{}_F\ (U[A,\gamma_5])^G{}_H\ \sigma^{l,FH}\ \left(R^1(U[A,\gamma_6])\right)^k{}_l.$$

Diffeomorphism invariance

Now we examine the invariance under three-dimensional diffeomorphisms. It is important to observe that the spin network states, as defined, are not invariant under diffeomorphisms. A diffeomorphism moves the graph around on the manifold but may also leave the graph Γ invariant and change the orientation and/or the ordering of the links.

An important point that must be considered is that, as indicated in [1] and [7], the diffeomorphism invariant states are not contained in the space $\mathcal{H}_{kin}$ (and consequently they are not contained in the space $\mathcal{H}_0$). Instead, they belong to $\mathcal{S}_0^*$. This is the dual space of $\mathcal{S}_0$. The elements of $\mathcal{S}_0$ are linear functionals Φ with domain on $\mathcal{S}_0$. The action of the extended diffeomorphisms group is defined in the elements of $\mathcal{S}_0^*$ by the following expression:

$$\left(U_\phi \Phi\right)(\Psi) := \Phi(U_{\phi^{-1}}\Psi). \tag{2.57}$$

This means that an element Φ of $\mathcal{S}_0^*$ invariant under diffeomorphisms is a linear functional defined on $\mathcal{S}_0$ such that

$$\Phi\left(U_\phi \Psi\right) := \Phi(\Psi). \tag{2.58}$$

We will call $\mathcal{H}_{diff}$ the space of such functionals.

Therefore, to characterize and understand the structure of $\mathcal{H}_{diff}$, we will follow the path indicated in [1] and [7] and define a map $P_D : \mathcal{S}_0 \to \mathcal{S}_0^*$ given by

$$\left(P_D \Psi\right)\left(\Psi^{'}\right) := \sum_{\Psi^{''}=U_\phi \Psi} \langle \Psi^{''} | \Psi^{'} \rangle, \tag{2.59}$$

where the sum is performed over all the states $\Psi^{''} \in \mathcal{S}_0$ for which there exists an element $\phi \in \mathit{Diff}^*$ such that $\Psi^{''} = U_\phi \Psi$. As is proved in [1], the sum in (2.59) is always finite, and $P_D \Psi$ is an element in $\mathcal{S}_0^*$ invariant under extended diffeomorphisms. If we write

$$([\Psi]| := P_D \Psi = \sum_{\Psi^{''}=U_\phi \Psi} \langle \Psi^{''} | = \sum_{\phi \in \mathit{Diff}^*} \langle U_\phi \Psi |, \tag{2.60}$$

it can be shown [1, 7] that this expression characterizes all the solutions of the equation (2.58). Then, the closure in the norm of the image of P_D is the space $\mathcal{H}_{diff}$. States in $\mathcal{S}_0$ that are related by a diffeomorphism are sent by P_D to the same element of $\mathcal{H}_{diff}$

$$P_D \Psi = P_D (U_\phi \Psi). \tag{2.61}$$

The inner product needed to make the space of invariant states under diffeomorphisms a Hilbert space is defined as

$$\langle P_D \Psi | P_D \Psi^{'} \rangle_{diff} := \left(P_D \Psi\right)\left(\Psi^{'}\right). \tag{2.62}$$

As mentioned in [1] and [7], the previous expression is well defined among diffeomorphism equivalent classes of states under the action of extended diffeomorphisms.

We can find an orthonormal basis for the space $\mathcal{H}_{diff}$ and, at the same time, understand the role of the spin networks in all this, if we consider that a diffeomorphism sends a spin network state $|S\rangle$ to an orthogonal state or to a state obtained by a change in the order of the orientation of the links, as is done in [1]. We denote $g_k|S\rangle$ as the states obtained from $|S\rangle$ by changes in the orientation or the ordering of the graph Γ that originally defines $|S\rangle$ and that can be obtained from a diffeomorphism. It can be shown [1] that the maps g_k form a discrete group, that we will be called G_Γ, and therefore the range of the index k is finite. We then have

$$\langle S | P_D | S^{'} \rangle = \begin{cases} 0 & \text{if } \Gamma \neq \phi \Gamma^{'}, \\ \sum_k \langle S | g_k | S^{'} \rangle & \text{if } \Gamma = \phi \Gamma^{'}. \end{cases} \tag{2.63}$$

An equivalent class K of graphs Γ under diffeomorphisms is called a "knot class" [1, 6, 7, 13]. Expression (2.63) shows that two spin networks S and $S^{'}$ define orthogonal states in $\mathcal{H}_{diff}$, unless they are defined on graphs Γ and $\Gamma^{'}$ belonging to the same knot class K. Therefore, the basis states in $\mathcal{H}_{diff}$ are labeled by knot classes K. Let $\mathcal{H}_K$ be the subspace of $\mathcal{H}_{diff}$ spanned by the basis elements labeled by the same knot classes K. The states in $\mathcal{H}_K$ are then distinguished by only the coloring of their links and nodes, but the colorings are not necessarily orthonormal [1]. Therefore, to find an orthonormal basis in $\mathcal{H}_K$, we must also diagonalize the quadratic form defined in the expression (2.63). Let $|s\rangle = |K, c\rangle$ be the resulting states. The label c is discrete and, except for certain complications with the discrete structure of the group G_Γ (as in [1, 2]), it corresponds to the coloring of the links and the nodes of Γ. The states $|s\rangle = |K, c\rangle$ are called *spin knot* states, or just s-knot states, and they represent an orthonormal basis for the space $\mathcal{H}_{diff}$. They generate all the solutions for the vector constraint.

The set of s-knot states is a discrete (countable) orthonormal basis [1, 7], and therefore the space $\mathcal{H}_{diff}$ is a separable Hilbert space. The definition of separable Hilbert space can be found for example in [36]. As mentioned in [1] *The "excessive size" of the kinematical Hilbert space* $\mathcal{H}_{kin}$ (that was originally non-separable) *turns out to be just a gauge artifact.*

With the definition of the space $\mathcal{H}_{diff}$, we have finished with the formal characterization of the solutions of all the quantum kinematical constraints.

2.4 Quantum geometry of spacetime

In the previous section, we showed how to define the kinematical Hilbert space $\mathcal{H}_{kin}$ by choosing the connection A_a^i as our configuration variable. At the classical level, we replaced the algebra generated by (A_i^a, E_i^a) by another called the holonomy-flux algebra. We introduced suitable operators acting on the elements of $\mathcal{H}_{kin}$, which constitute a representation of such classical algebra. In addition, we characterized the space of solutions of the Gauss and vector constraints.

We have not completely defined the quantum theory of the gravitational field because have not quantized the Hamiltonian constraint (2.21). However we can extract some interesting features of the theory already constructed at its kinematical level.

In this section, we will show how the quantization of the triad leads to the possibility of the introduction of a set of geometrical operators that contain one of the main physical predictions of loop quantum gravity: a discreteness of the spacetime geometry.

The area operator

The simplest geometric operator corresponds to the area of a two-dimensional surface $S \subset \Sigma$, which classically depends on the triad E_i^a as shown in the expression (2.39). We can introduce a decomposition of the surface S into two-cells and write the integral defining the area as the limit of a Riemann sum[5]

$$\mathcal{A}_S = \lim_{N\to\infty} \mathcal{A}_S^N, \tag{2.64}$$

where the Riemann sum can be expressed as

[5]As mentioned in [37]: *The moral is that anything which looks like a good approximation to an integral really is, provided that all the lengths $t_i - t_{i-1}$ of the intervals in the partition are small enough.*

$$\mathcal{A}_S^N := \sum_{I=1}^{N} \sqrt{E_i(S_I)E^i(S_I)}, \tag{2.65}$$

where N is the number of cells and $E_i(S_I)$ is the flux of the classical field E_i^a through the cell S_I. Equation (2.64) represents a "regularization" of the corresponding classical area operator. In fact, as mentioned in [1] and [7], the previous limit defines the area of a surface in classical geometry. The quantum area operator then becomes

$$\widehat{A}_S := \lim_{N\to} \widehat{A}_S^N \tag{2.66}$$

where, to define $\widehat{A}_S^N$, we replace the classical quantity $E_i(S_I)$ in (2.65) with $\widehat{E}_i(S_I)$ given in (2.50).

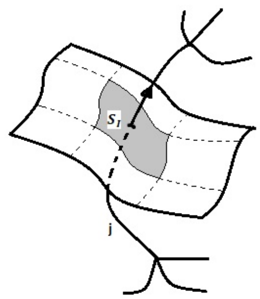

Figure 2.3: The regularization given in the expression (2.66) is defined so that in the limit where $N \to \infty$ each cell is punctured by at most one edge. The simplest eigenstate of the area of S is shown.

To evaluate the action of (2.66) on a spin network state, we will choose the cellular decomposition so that in the limit $N \to \infty$ each S_I is punctured at most at a single point by either an edge or a node (see Figure 2.3). It is also important to consider that the action of $\widehat{E}_i(S_I)\widehat{E}^i(S_I)$ on $R^{(j)}\left(U[A,\gamma]\right)$ is given by [1, 2, 6, 7]

$$\widehat{E}_i(S_I)\widehat{E}^i(S_I)\left(R^j\left(U[A,e]\right)\right)^m{}_n = \left(8\pi l_p^2\gamma\right)^2\left(j(j+1)\right)\left(R^j\left(U[A,e]\right)\right)^m{}_n \tag{2.67}$$

where l_p is the Planck length, γ is the Immirzi parameter, and the edge e is of an spin network that punctures S_I. In (2.67) it is considered that there is only one intersection P between the surface S and the spin network and that the intersection is performed by the edge e that carries a $SU(2)$ representation j. It is also assumed that the intersection point does not coincide with a node of the spin network. The remaining important case is when a spin network node is on S_I. As is explicitly shown in [2], the action of the operator $\widehat{E}_i(S_I)\widehat{E}^i(S_I)$ is still diagonal in this case.

Thus, the action of the area operator can be diagonalized by the spin network states. Spin network states are eigenvectors of the quantum area operator, and we have that

$$\widehat{A}_S|\tilde{S}\rangle = 8\pi l_p^2\gamma\sqrt{j(j+1)}|\tilde{S}\rangle \tag{2.68}$$

for a single puncture, where $|\tilde{S}\rangle$ denotes a spin network state. More generally, we have that

$$\widehat{A}_S|\tilde{S}\rangle = 8\pi l_p^2 \gamma \sum_p \sqrt{j_p(j_p+1)}|\tilde{S}\rangle, \tag{2.69}$$

where the sum is over the intersection points p of the given spin network state $|\tilde{S}\rangle$ with the surface S.

When some of the nodes lay on S the expression for the eigenvalues of $\widehat{A}_S$ is also known; see for example [1] or [2]. It is important to notice that the spectrum of the area operator depends on the value of the Immirzi parameter γ. This, in fact, is a general property of geometric operators in loop quantum gravity.

The smallest non-vanishing eigenvalue in (2.69), considering that the Immirzi parameter is equal to 1, is:

$$\mathbf{A}_0 = 4\sqrt{3}\pi\hbar G c^{-3} \sim 10^{-66}\mathrm{cm}^2 \tag{2.70}$$

This is a sort of quanta of area, the elementary and smallest unit of area in space, of the order of the Planck area. It is the quanta of area carried by a link in the fundamental representation $j = 1/2$. This means that, in the loop quantum gravity framework, there is a sort of minimal size of physical space at the Planck scale.

The volume operator

A second operator that plays a key role in the geometrical description of spacetime given in loop quantum gravity is the operator $\mathbf{V}(\mathcal{R})$ corresponding to the volume of a region $\mathcal{R}$. As for the area operator, this quantity requires a bit of work to be defined in the quantum theory because of the care that must be taken in the definition of the operator products involved in the expression of $\det E$ and in the square root.

As was indicated in 2.2.2, the volume of a three-dimensional region $R \subset \Gamma$ is classically given by equation (2.40), that can be rewritten as

$$V_R[E_i^a] = \int_R \sqrt{\left|\frac{1}{3!}\epsilon_{abc}E_i^a E_j^b E_k^c \epsilon^{ijk}\right|}\, d\sigma^1 d\sigma^2 d\sigma^3. \tag{2.71}$$

Following a similar regularization technique as in the case of the area operator, we can write the previous equation as the limit of Riemann sums defined in terms of a decomposition of the region R using three-cells. Thus, we can quantize the regularizated version using the flux operators given in (2.50) associated to infinitesimal cells

$$\widehat{V}_R = \lim_{N\to\infty} \widehat{V}_R^N, \tag{2.72}$$

where

$$\widehat{V}_R^N = \sum_{I=1}^N \sqrt{\left|\frac{1}{3!}\epsilon_{abc}\widehat{E}_i(S_I^a)\widehat{E}_j(S_I^b)\widehat{E}_k(S_I^c)\epsilon^{ijk}\right|}. \tag{2.73}$$

The limit $N \to \infty$ is taken by keeping spin network nodes inside a single three-cell. We will not present the detailed process to arrive to the expression of the volume operator. Instead, we will mention below some general properties of the volume operator that are important to consider.

- There are at least two consistent quantizations of the volume operator; see [1, 2, 3, 7]. See also [38]. The differences are the following. i)The constants in front of each operator, ii)the way the operator sums up the variables for each link, and iii)the absence of the sign factor in one of the operators. The differences are the result of a different regularization technique applied to each one, even when both start from the same classical quantity (equation (2.71)).

- When the volume operator acts on spin network states, the operator does not change the graph, nor the coloring of the links. The operator only acts on the intertwiners in each node. In this way, only the nodes contribute to the volume of a region; the volume is concentrated at the nodes of a spin network [1, 2, 7].

- Nodes with valence three or fewer do not contribute to the volume. A node must be at least quadrivalent to have a non-vanishing volume [1, 2, 7].

- The volume operators are always well-defined self-adjoint operators with discrete spectrum. For each given graph and labeling, it is always possible to choose a basis i_n of interwiners that diagonalize the volume operator [1].

With the construction of the area and volume operators in loop quantum gravity, we arrive at an intrinsic discreteness of physical space at the Planck scale. Such discreteness was long expected in the framework of quantum gravity. In the context of loop quantum gravity, the discreteness is not imposed or postulated but is a direct consequence of a straightforward quantization of General Relativity. As mentioned in [1]: "*Space geometry is quantized in the same manner in which the energy of an harmonic oscillator is quantized*". (Or in the same way as the energy of an hydrogen atom).

As mentioned in [1] and [2], there is some degree of disagreement as to whether the area and volume operators truly represent physically measurable quantities because they are not invariant under three-dimensional diffeomorphisms [2]. Some people even mention (as is done in [2]) that it is far from clear whether the area and volume spectrum are of any operational significance in the sense of a measurement analysis with rods and clocks. On the other hand, some people consider (as is done in [1]) the physical significance of the quantities represented by the area and volume operators as truly measurable quantities. Moreover, they call them a precise and quantitative prediction of loop quantum gravity that could have indirect effects or even be verified. In some quantum gravity research works, these results are used as a guide in the searching for experimental signals of a quantization of the gravitational field [24].

2.4.1 Quantum geometry

The area and volume operators lead to a geometric interpretation of the spin network states as representing a quantization of the geometry of space.

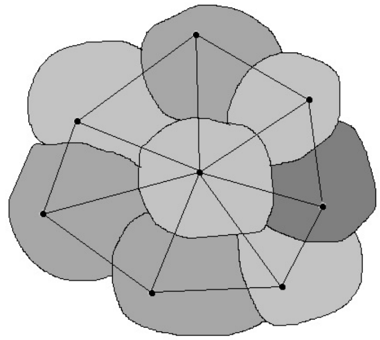

Figure 2.4: The graph of a spin network and the ensemble of chunks of space that represents.

We can interpret a spin network with N nodes as a set of N quanta of volume or N chunks of space. The chunks of space are separated from each other by two-dimensional surfaces. The area of those surfaces is given by the area operator, which has contributions from each link that punctures the surface. Two chunks of space are contiguous if the corresponding nodes are connected by a link of the spin network. If this is the case, then there is an elementary surface separating them, and the area of such a surface is determinated by the coloring j_l of the link l through the formula (2.68). We obtain an image of the space at the Planck scale, represented by a spin network, similar to Fig. 2.4.

The intertwiners associated with the nodes are the quantum numbers of the volume, and the spins associated with the links are the quantum numbers of the area. The volume is concentrated in the nodes and the area in the links. The graph Γ establishes the adjacency relations among the chunks of space. This exotic picture of the space at the Planck scale is in agreement with the direction in which a background independent formulation would be set up. As mentioned in [7], all the information about the degrees of freedom of the geometry (hence the gravitational field) is contained in the combinatorial aspects of the graph and in the discrete quantum numbers labeling area and volume quanta.

We can give a similar but more impressive interpretation of the s-knot states. Such interpretation can be found greatly detailed in [1].

The discreteness of space described by the area and volume operators is compatible with the smooth geometry picture of the classical theory. The spectrum of the operators grows very rapidly when one gets to larger geometries because the spacing between eigenvalues decreases exponentially for large eigenvalues [7].

2.5 Quantization of the Hamiltonian constraint

With the characterization of the solutions of the vector and Gauss constraint, the only remaining constraint to be considered is the Hamiltonian constraint given by (2.21) or its smeared version given by (2.29).

In contrast to the Gauss and vector constraints, the Hamiltonian constraint does not have a simple three-dimensional geometrical meaning, and this makes its quantization more complicated. In the quantization of such a constraint, one needs the volume operator, which has a quantization that presents some degree of ambiguity.

Even with this difficulties, it has been possible to define some versions of the quantum Hamiltonian constraints that are finite and well defined; see for example [1] or[3]. The successful definition of the quantum Hamiltonian constraint including its coupling with matter fields is a remarkable achievement of LQG [7]. There are, however, some issues that is important to emphasize.

There is, in fact, a large degree of ambiguity in the definition of the quantum Hamiltonian constraint. The nature of the solutions for this constraints and the dynamics critically depends on these ambiguities [7]. There are factor ordering ambiguities as well. Therefore, instead of a single theory, we have infinitely many theories that are mathematically consistent. Another important unsolved issue is whether any of these theories can reproduce general relativity, our modern theory of classical gravity, in the classical limit [7].

The issues and ambiguities in the definition of the Hamiltonian constraint are also related with the definition of the observable quantities of the theory. The structure of the problem suggests the possibility of defining a large variety of physical observables. There are actually many physical observables that one can construct for a given quantization of the Hamiltonian constraint [7]. At present, however, it is not clear what could be the physical interpretation of these observables.

Different avenues of research are being explored to address the issues that arise in the quantization of the Hamiltonian constraint. The difficulties seem to arise because the 3+1 splitting breaks the manifest four-dimensional covariance of the theory. Because of this, there has been a growing interest in approaching the problem of the dynamics by defining a covariant formulation of quantum gravity thorough a path integral quantization approach. The idea is that one can keep manifest four-dimensional covariance in the path integral formulation. The spin foam approach is an attempt to define the path integral quantization of gravity using what we have learned from canonical loop quantum gravity. That will be the subject of the next chapter.

Before finishing this chapter, we should mention an important general characteristic of the Hamiltonian constraint that will play an important role in the following. The scalar constraint modifies spin networks by creating (or destroying) new links around nodes (as shown in Figure 2.5), with an that amplitude depends on the details of the action of the volume operator, the local spin labels and other local features at the nodes [6, 7]

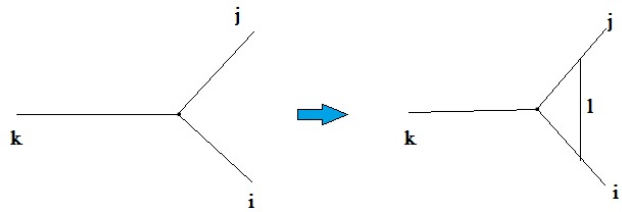

Figure 2.5: A typical transition generated by the action of the Hamiltonian constraint. The letters denote the spins of the links.

An important property of the definition of the quantum Hamiltonian constraint is that the new edges added (or annihilated) are of a very special kind. Not only do the new nodes carry zero volume, but also they are invisible to subsequent actions of the Hamiltonian constraint [7]. Hence these edges are called *exceptional edges*.

3
Covariant loop quantum gravity

In the last chapter, we presented an attempt to implement the canonical quantization method to general relativity. We found that, if we rewrite the theory as a theory of connections over a three-dimensional submanifold, we can advance in the implementation of some steps of the quantization method. We can define the kinematical Hilbert space and characterize the solutions of two of three constraints that characterize general relativity. On the other hand, we found several issues in the quantization and implementation of the Hamiltonian constraint and the definition of the physical inner product.

In this chapter, we present an approach that tries to address the problem of the dynamics of loop quantum gravity through a path integral formalism. Such approach is called the *spin foam formalism*. Because the standard formulation of spin foam models is based on using the Lorentz connection ω instead of the Ashtekar-Barbero connection A, and in this formalism we can consider arbitrary regions of space-time surrounded by three-dimensional boundaries, the formalism is also called *covariant loop quantum gravity*.

It is important to remark that, as mentioned in [28], the spin foam approach is not only an attempt to derive the physical Hilbert space of loop quantum gravity; it is also an attempt to derive a full quantum gravity theory from a path integral approach perspective.

As in the last chapter, the material that we present here is mainly based on [1, 3, 6, 7, 8, 9, 16, 19, 23] and is not exhaustive. Therefore, we will not show the detailed calculations for arriving at the different results presented here, but we refer the reader to the aforementioned references.

3.1 The path integral approach in loop quantum gravity

The idea of using a sum-over-paths formalism in quantum gravity was previously introduced by Charles Misner and Stephen Hawking [1]. The idea was to attempt defining a path integral over four-dimensional metrics

$$\int \mathrm{D}[g_{\mu\nu}(x)]e^{iS_{GR}[g]}, \tag{3.1}$$

where S_{GR} denotes the classical Einstein-Hilbert action. In particular, if we consider initial and final three-dimensional metrics g and $g^{'}$ respectively, then the transition amplitude between these metrics would be given by

$$W[g,g^{'}] = \int_{g}^{g^{'}} \mathrm{D}[g_{\mu\nu}(x)]e^{iS_{GR}[g]}. \tag{3.2}$$

As mentioned in [1], however, it is very difficult to give a reasonable definition of the functional integral in the context of gravity. There are some attempts that try to define the functional integral in different ways, for example [49] and [50], which lead to important and interesting results. Within the framework of loop quantum gravity, an attempt is made to define the path integral by taking the spin network states as the kinematical quantum states of the gravitational field.

The spin network states are the quantum states of the three-dimensional geometry in canonical loop quantum gravity. They carry the geometrical information of the hypersurface Σ in a diffeomorphism-invariant way. The remarkable characteristic of these states is that they constitute a discrete set, and that suggests the possibility of constructing the path integral using sums over spin network world sheet amplitudes. Speaking heuristically, four-dimensional geometries are represented by histories of quantum states of three-dimensional geometries, and those quantum states are given by spin network states. The corresponding transition amplitudes W will be between quantum states of the three-dimensional geometry, or spin network states $W(s,s')$, and will be given as a sum over discrete sequences, or histories, of the spin networks states. We can then write the amplitude $W(s,s^{'})$ as a sum over paths of spin networks. The paths are generated by individual steps, and the amplitude of the history is the product of the individual amplitudes of the steps.

A history of spin networks $\sigma = (s, s_N, \ldots, s_1, s^{'})$ is called a spin foam. We can imagine the structure generated by the evolution of a spin network if we consider an initial three-dimensional manifold Σ_i and an initial spin network s_i defined on Σ_i. This initial spin network state evolves without intersections along a "time" coordinate until a final spin network s_f. The final spin network is defined in another three-dimensional manifold Σ_f. In each step, the spin network changes under the action of the Hamiltonian constraint. We will call faces (f) the worldsurfaces of the links of the graph and "edges" e the worldlines of the nodes of the graph.

Now it is time to consider the general property of the Hamiltonian constraint that was mentioned at the end of the last chapter. The Hamiltonian acts on nodes of the spin network, so at each step there is a probability that a node will branch out and that would change the number of nodes. We will call "vertices" v the points where the edges branch.

In this way, we obtain a collection of faces f, meeting at edges e, which in turn meet at vertices v. The resulting structure, which we will denote as K, is characterized by its combinatorial properties and their adjacency relations. This structure is called a simplicial complex.

The spin networks are not only characterized by their graph but also by the coloring of their links and nodes. The simplicial complex K corresponding to a sequence of spin networks will inherit the characteristic of being colored. Its faces f will be colored by $SU(2)$ irreducible representations j_f, and its edges will be colored by the intertwiners i_e, that correspond to the tensor product among the representations of the faces that meet at the particular edge. A spin foam $\sigma = (K, j_f, i_e)$ is a simplicial complex K with colored faces and edges.

3.1.1 The spin foam framework

A sum over paths formulation of quantum gravity can then be defined as a sum over spin foams of individual amplitudes associated with histories of spin networks, with such amplitudes given by products of individual vertex amplitudes.

As we have constructed them, the boundary of a spin foam is a spin network. If the spin-foam σ is bounded by the spin network s, we will write $\partial\sigma = s$.

Let us consider a given simplicial complex K with boundary ∂K, and a spin network state Ψ_s that defines a boundary state on ∂K. The spinfoam amplitude associated to K and Ψ_s is written as a sum over all possible representations and intertwiners living in the bulk and that are consistent with the given boundary spin network [1, 28]:

$$A[K, \Psi_s] = \sum_{j_f, i_e} \left[\prod_f A_f[j_f] \prod_e A_e[j_f, i_e] \prod_v A_v[j_f, i_e] \right], \tag{3.3}$$

where the representations and intertwiners j_f and i_e for faces and edges on the boundary are fixed and given by our choice of the boundary state Ψ_s. The expression (3.3) is called the local ansatz for spinfoam amplitudes [28].

The functions A_f and A_e are amplitudes associated to faces and edges . As mentioned in [28], they are considered as kinematical and can be introduced in a redefinition of the vertex amplitude A_v. The face amplitude A_f is usually chosen as the dimension of the representation j_f of the given face.

Finally, the transition amplitude for the given spin network Ψ_s is obtained eliminating the discretization i.e summing over all possible simplicial complexes compatible with the boundary data [28]:

$$W(s) = \sum_{K|\partial K = \Gamma} w[K] A[K, \Psi_s] \tag{3.4}$$

where $K|\partial K = \Gamma$ indicates that we are summing over all the simplicial complexes with boundary given by the graph Γ associated with the given spin network state and $w[K]$ is a statistical weight depending only on the simplicial complex K. Unfortunately, this sum is much less controlled than the previous expression given in (3.3) [1, 28]. One way to define it is through a formalism that is called group field theory. We will not touch this topic here, and we refer the reader to the references cited at the beginning of the chapter for more details.

If we choose a connected boundary, the expression given in (3.4) can be interpreted as the transition amplitude associated with the gravitational field represented be the spin network Ψ_s in the connected boundary of a spacetime region. On the other hand, if we choose a disconnected boundary $\partial K = \Gamma_{in} \cup \Gamma_{out}$, and consequently the boundary spin network state is composed of two connected components, then the amplitude is interpreted as a transition amplitude between the initial state $\Psi_{\Gamma_{in}}$ and the final state $\Psi_{\Gamma_{out}}$.

The current situation in the spinfoam framework is not that we can compute the transition amplitude $W(s)$. There is uncertainty in the definition of the model and the corresponding definition of the vertex amplitude A_v. In addition, the way to eliminate the discretization produces some important issues. Perhaps one of the most important open problems is the relation between the spinfoam framework and the canonical approach. For more difficulties related to the spinfoam framework, see [6].

3.2 Discretization theory

One way to define the path integral in quantum theory is to use some form of discretization of the underlying structure and then take the limit when this discretization is refined. The same is true when we try to define a spin foam model. In this section, we present the way to discretize a differentiable manifold, which is the most commonly used in the definition of spinfoam models. The presentation is based on [3, 19].

To discretize the differentiable manifold $\mathcal{M}$, we will replace it with a collection of discrete objects known as simplices, glued together properly. To this end, we give some general definitions, which were taken from [3].

Definition 3.2.1. *Let $D \in \mathbb{N}$. If $0 \leq p \leq D$ is an integer, we define a p-simplex $\sigma^p = [\vec{v}_0, \ldots, \vec{v}_p] \subset \mathbb{R}^D$ as the convex hull of $p+1$ vectors, that is*

$$\sigma^p := \left\{ \sum_{k=0}^{p} a_k \vec{v}_k \ ; \quad a_k \geq 0, \ \sum_{k=0}^{p} a_k = 1 \right\}. \tag{3.5}$$

We call the vectors $\vec{v}_k$ the vertices of the p-simplex.

It is important to add some observations to the last definition:

- As mentioned in [2], we can solve the equation $\sum_{k=0}^{p} a_k = 1$ and write $0 \leq a_0 = 1 - \sum_{k=1}^{p} a_k$. Thus, we can describe equivalently a p-simplex as the convex hull of the p vectors $\vec{v}_k^{\,\prime} = \vec{v}_k - \vec{v}_0$, $k = 1, \ldots, p$, which are linearly independent.

- The orientation of the p-simplices is given by the order in which the vertices $\vec{v}_k$ appear in the list $[\vec{v}_0, \ldots, \vec{v}_p]$.

- The boundary $\partial\sigma^p$ of a p-simplex is defined as the set of points for which $a_k = 0$ for $k = 0, \ldots, p$. In this way, we obtain $p+1$ different $(p-1)$-simplices given by $\sigma_k^{p-1} = [\vec{v}_0, \ldots, \tilde{v}_k, \ldots, \vec{v}_p]$, where the tilde over a vertex denotes that such a vertex is omitted. We say that these $(p-1)$-simplices are oriented equally relative to $[\vec{v}_0, \ldots, \vec{v}_p]$ if k is even and that they have the opposite orientation if k is odd. This defines the induced orientation of the faces σ^p. We then obtain that $\partial\sigma^p = \cup_k \sigma_k^{p-1}$. By repeating this process, we obtain all the subsimplices of σ^p.

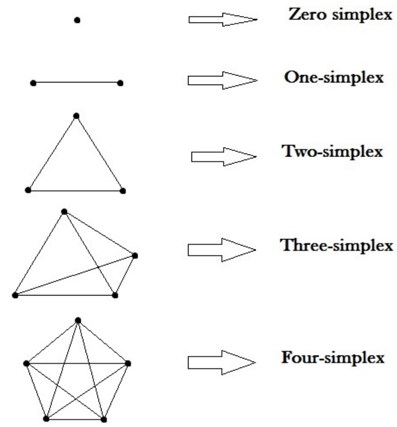

Figure 3.1: Simplices of different dimensions. The case of four dimensions is usually drawn as a set of five points joined by ten lines. Remember that in this last case we are trying to draw a four-dimensional object on a two-dimensional piece of paper.

More intuitively we can say that in $\mathbb{R}^D$ a p-simplex is the set of points inside a convex hull constructed with $p+1$ points. Figure 3.1 shows some examples of simplices, going from zero to four-simplices.

A zero-simplex is just a point. A one-simplex is a line, whereas a two-simplex looks like a triangle, a three-simplex like a tetrahedron, and so on. Notice that a one-simplex is constructed using two zero-simplices joined by a straight line. In a similar way, a two-simplex is constructed by three zero-simplices joined by three lines, that is, three one-simplices, and the points inside the area that these lines surround, and so on.

Now we will give another important definition.

Definition 3.2.2. *Given a p-simplex $\sigma^p = [\vec{v}_0, \ldots, \vec{v}_p]$, the barycenter of σ^p is defined as the point:*

$$C_{\sigma^p} := \frac{\sum_{k=0}^{p} \vec{v}_k}{p+1}. \tag{3.6}$$

The barycenter of a p-simplex is a point inside the simplex that is equidistant from the vertices, with respect to the euclidean metric in $\mathbb{R}^D$. As an example, the barycenter of a one-simplex $\sigma^1 = [\vec{v}_0, \vec{v}_1]$ is the midpoint of the line segment that joins the vertices $\vec{v}_0$ and $\vec{v}_1$.

To discretize a differentiable manifold using simplices, we must glue together simplices of different dimensions appropriately. The way to achieve the latter is given by the concept of *simplicial complex.*

Definition 3.2.3. *A simplicial complex K of dimension D is a collection of simplices σ_i^p with $p = 0, \ldots, D$, $i = 1, \ldots, N_p$, of different dimensions, with the following properties:*

a) All the subsimplices of each σ_i^p also belong to K.

b) Two simplices σ_i^p and σ_j^q intersect at most in a common subsimplex, which has the opposite orientation in both.

Thus, a simplicial complex is defined as a set of simplices so that they intersect at most in a common subsimplex. For example, two triangles joined along one of their edges form a two-dimensional simplicial complex if we consider, in the list of simplices, all their edges and vertices.

It is important to remark that a simplicial complex K is not a topological space; it is only a set of geometrical simplices [46].

Definition 3.2.4. *Let K a simplicial complex of dimension D. The polyhedron $|K|$ of K is defined as the set of points of $\mathbb{R}^D$ that lie in at least one of the simplices of K.*

The polyhedron $|K|$ of a simplicial complex is topologized as a topological subspace of $\mathbb{R}^D$.

Definition 3.2.5. *Let $\mathcal{M}$ be a differentiable manifold. A triangulation of $\mathcal{M}$ is a pair (K, f), where K is a simplicial complex and $f : |K| \to \mathcal{M}$ is an homeomorphism between $|K|$ and $\mathcal{M}$.*

We have the following important result [40, 41].

Theorem 3.2.1. *Every differentiable manifold has a triangulation.*

It is important to notice that the triangulation is highly no-unique. As emphasized in [3], a triangulation need not be simplicial, and it can consist of other types of cells. For example oriented cubes.

What is useful about triangulations is that they yield to a combinatorial description of differentiable manifolds. In this way, $\mathcal{M}$ can be replaced by a simplicial complex K, which is topologically equivalent to $\mathcal{M}$, and can be described by a purely combinatorial structure.

Before defining our first spin foam model in the next section, we must to introduce a very important concept that will be used in the rest of this work: the dual K^* of a simplicial complex K.

Definition 3.2.6. *Let $K = \{\sigma_i^p;\ p = 0, \ldots, D,\ i = 1, \ldots, N_p\}$ be a simplicial complex. Let $\sigma_{j_0}^p \in K$ be any p-simplex on K, and consider all possible $(D-p)$-tuples of simplices $\sigma_{j_k}^{(p+k)} \in K$ with $k = 1, \ldots, D-p$ and $1 \leq j_k \leq N_{p+k}$ subject to the following condition.*

For all $l = 0, \ldots, D-p-1$, the simplex $\sigma_{j_l}^{(p+l)}$ is a face of $\sigma_{j_{l+1}}^{(p+l+1)}$ with the induced orientation.

Therefore, for each such $(D-p)$-tuple of simplices, construct the $(D-p)$-simplex $[C_{\sigma_{j_0}^{(p)}}, C_{\sigma_{j_1}^{(p+1)}}, \ldots, C_{\sigma_{j_{D-p}}^{(D)}}]$ constructed by using the barycenters of those simplices.

The cell dual to $\sigma_{j_0}^p$ is defined as

$$*_K\left[\sigma_{j_0}^p\right] := \bigcup_{(\sigma_{j_1}^{(p+1)}, \sigma_{j_2}^{(p+2)}, \ldots, \sigma_{j_{D-p}}^{(D)});\ \sigma_{j_l}^{(p+l)} \subset \partial\sigma_{j_{l+1}}^{(p+l+1)}:\ l=0,\ldots,D-p-1} [C_{\sigma_{j_0}^{(p)}}, C_{\sigma_{j_1}^{(p+1)}}, \ldots, C_{\sigma_{j_{D-p}}^{(D)}}] \tag{3.7}$$

where the union is taken over all the $(D-p)$ tuples of simplices $\sigma_{j_k}^{(p+k)} \in K$ subject to the condition mentioned above.

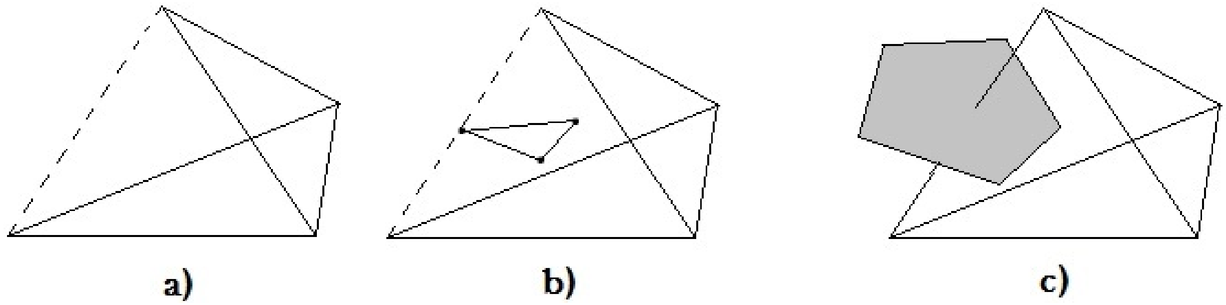

Figure 3.2: Construction of the cell dual to a one-simplex in three dimensions. In Figure a), we show the one-simplex considered. Figure b) shows the simplex constructed by using the barycenters of the triangle and the tetrahedron, as is dictated by Definition 3.2.6. In Figure c), we show the cell dual to the line.

The complex K^* dual to K is obtained by gluing all the dual cells along common subcells.

Behind this forbidding definition is hidden a very simple construction. To illustrate it, let us consider the triangulation of a three-dimensional manifold using three-simplices (tetrahedra). Let us pick only one line: the dotted line showed in Figure 3.2 a). Now, by following the definition 3.2.6, let us consider one couple (triangle, tetrahedron) such that

a) the line considered is an edge of the triangle,

b) and the triangle is a face of the tetrahedron.

Next, we draw the two-simplex that results from taking the barycenter of the line, the triangle and tetrahedron as the vertices of such a two-simplex. The resulting structure for a given couple (triangle, tetrahedron) is shown in Figure 3.2 b). We now consider all the couples (triangle, tetrahedron) that meet the two previous conditions and repeat the process. Finally, we take the union of all the two-simplices constructed in this way. We obtain a two-dimensional object that is punctured by the line initially considered with its vertices in the barycenter of the tetrahedron that surrounds the line. This object is the cell dual to the initially considered dotted line.

Generally, in a three-dimensional simplicial complex, the dual of a line is a two-cell called a face f. The dual of a tetrahedron is just a point. Such point is the barycenter of the tetrahedron, and is called a vertex ν. The cell dual to a triangle is the line joining the barycenters of the two tetrahedra that meet at the triangle considered and is called an edge. Finally, the cell dual to a point is a bubble constructed by gluing the faces that surround such point.

Now we have all the ingredients for presenting the first spin foam model constructed for one of the most simple theories: the spin foam model for three-dimensional Euclidean general relativity.

3.3 The Ponzano-Regge model

The Ponzano-Regge model was the first spin foam model introduced in quantum gravity. It appeared in the 1960's, 30 years before the term “spin foam” was used for the first time, and almost 20 years before the first works about loop quantum gravity [22]. It is a model for three-dimensional Euclidean General Relativity and illustrates how the path integral approach in loop quantum gravity takes the form of a spin foam model.

In order to describe the Ponzano-Regge model, we need to take a small detour and to present the corresponding classical theory in which the model is based, emphasizing its most important features. That is what we will do in the next section.

3.3.1 Three-dimensional Euclidean general relativity

Although the real world is described by four-dimensional Lorentzian General Relativity , the Euclidean model in three dimensions is simpler and useful for studying some general features that a quantum theory of gravity may have. As we will also see, the path integral approach is well defined in the three-dimensional case, as well as the relation between such an approach and the canonical formulation.

Euclidean General Relativity in three dimensions can be defined in a very similar way to four-dimensional gravity. In the three-dimensional case, instead of a tetrad, we have a triad field:

$$e^i(x) = e^i_a(x)dx^a \tag{3.8}$$

with values in $\mathbb{R}^3$. The indices $a, b, \dots = 1, 2, 3$ are spacetime indices, and $i, j, k, \dots = 1, 2, 3$ are internal indices. This triad field is similar to that defined in the last chapter, but the triad field here is defined over all the spacetime $\mathcal{M}$, not only over a hypersurface Σ; remember that we are dealing here with a three dimensional spacetime $\mathcal{M}$. Since the theory is Euclidean, this triad field transforms as a vector under the group $SO(3)$:

$$e^i_a \to U^i{}_j(x)e^j_a(x), \tag{3.9}$$

with $U^i{}_j(x) \in SO(3)$. The connection ω is a one-form with values in the Lie algebra $so(3)$

$$\omega^i{}_j(x) = \omega^i_{aj}(x)dx^a \tag{3.10}$$

and the corresponding curvature two-form will be

$$R^i{}_j(x) = R^i_{abj}(x)dx^a dx^b = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \tag{3.11}$$

The action defining the theory is a functional of the triad field and the connection ω and is given by

$$S[e, \omega] = \frac{1}{16\pi G} \int \epsilon_{ijk} e^i \wedge R^{jk}[\omega]. \tag{3.12}$$

The variation of the action (3.12) with respect to the connection ω gives the Cartan structure equation $de^i + \omega^i_j \wedge e^j = 0$. This implies that the connection ω^i_j is the spin connection. The variation with respect to the triad field e^i gives the equation $R^i{}_j = 0$, which implies that the spacetime is flat. The latter does not imply that the theory is completely trivial for two reasons:

- If the spacetime has a nontrivial topology, it can have global degrees of freedom and therefore a global dynamics, even though is locally flat. For example a three-dimensional hyperplane that changes in size.

- If we consider a compact region with a boundary, there will be relations between boundary partial observables [19].

Let us be a little more specific about the last point. If we consider a compact region of spacetime $\mathcal{R}$ with a two-dimensional boundary $\partial\mathcal{R} = \Sigma$, the action in metric variables is given by [19]

$$S[g] = \frac{1}{16\pi G}\int_{\mathcal{R}} \sqrt{g}\ R\ dx^3 + \frac{1}{8\pi G}\int_{\Sigma} d\sigma^2 k^{ab} q_{ab}\sqrt{q}, \tag{3.13}$$

where q_{ab} is the metric of Σ and k^{ab} is its extrinsic curvature. On-shell, the bulk term in the action does not contribute because the Ricci tensor vanishes and the action takes the form

$$S[q] = \frac{1}{8\pi G}\int_{\Sigma} d\sigma^2 k^{ab}[q] q_{ab}\sqrt{q}. \tag{3.14}$$

In other words, it depends only on the metric q_{ab} on Σ. The extrinsic curvature k_{ab} is a function of q_{ab} and is what encodes the dynamics of the theory. The dependence of k_{ab} with q_{ab} is non-local. Therefore the no-triviality of the theory is in the global dependence of the extrinsic curvature of the boundary on its intrinsic metric.

As we can see, even though the vacuum solutions of the theory give us a flat spacetime, the theory can be highly no-trivial.

3.3.2 Discretization

We have presented the most important characteristics of Euclidean General Relativity in three dimensions, and we have shown that the theory can possess a non-trivial dynamics. Our next task is to implement a path-integral formulation of the theory, taking into account the lessons that we learned from canonical loop quantum gravity. In order to do this, we will begin discretizing the theory.

Let us consider a compact spacetime region $\mathcal{R}$ with boundary $\partial\mathcal{R} = \Sigma$. We fix a triangulation $\triangle$ of $\mathcal{R}$ as defined in definition 3.2.5 using a three-dimensional simplicial complex K, and we also consider the dual $\triangle^*$ of the given triangulation (as given in definition. 3.2.6). Let us call here vertices ν the points in $\triangle^*$, edges l the lines of $\triangle^*$ and faces f the 2-simplices in $\triangle^*$. It is important to remember that the vertices $\nu \in \triangle^*$ are dual to tetrahedra $\tau \in \triangle$, the edges $e \in \triangle^*$ are dual to triangles $t \in \triangle$, and the faces $f \in \triangle^*$ are dual to the segments (or lines) $s \in \triangle$.

The discretization of the bulk of $\mathcal{R}$ induces a discretization of the boundary Σ. With respect to the original triangulation $\triangle$, the boundary Σ is discretized by the boundary triangles of $\triangle$ separated by the boundary segments of $\triangle$. The end points of the edges that are dual to these triangles will be called *nodes* n. The nodes are actually just boundary vertices. The boundary of the faces $f \in \triangle^*$ that are dual to the boundary segments are called links l. The links are just boundary edges. The set of links and nodes forms the graph of the boundary, which we denote as Γ; see Figure 3.3. It can be shown that the boundary graph is the boundary of $\triangle^*$ and also the dual of the boundary of $\triangle$ [19]:

$$\Gamma = \partial(\triangle^*) = (\partial\triangle)^*. \tag{3.15}$$

We discretize general relativity discretizing the main variables that describe the theory, and defining the discretized action that defines the dynamics of the model. This is done in such a way that the discretized action converges to the continuum action when the simplicial complex is refined.

The variables that we use to describe General Relativity are the connection ω, which is a one form with values in the Lie algebra $su(2)$ and the triad field e. Let us define

$$\begin{aligned} \omega^i &= \frac{1}{2}\epsilon^i{}_{jk}\omega^{jk} \\ \omega &= \omega^i\tau_i, \end{aligned} \tag{3.16}$$

where $\tau_i = -\frac{i}{2}\sigma_i$ are the generators of the Lie algebra $su(2)$ and σ_i are the Pauli matrices.

We discretize the connection assigning a $SU(2)$ group element U_e with each edge $e \in \triangle^*$, and we discretize the triad associating a $su(2)$ Lie algebra element L_f with each face $f \in \triangle^*$. The associations are given in the following way: the discretization of the connection is given by taking the holonomy of the connection along the edges of the triangulation:

$$U_e = Pexp \int_e \omega. \tag{3.17}$$

Thus, U_e is the matrix of the parallel transport generated by the connection along the edge e.

The discretization of the triad is a little different. Let

$$L_s^i := \int_s e^i \tag{3.18}$$

be the line integral of the triad e^i along the segment $s \in \triangle$. Defined in this way, L_s^i are three real numbers (one per value of the index i) associated with each segment s, but each segment $s \in \triangle$ is dual to a face $f \in \triangle^*$. Therefore we can consider L_s^i as associated with the face f dual to s: $L_f^i = L_{s_f}^i$. If we define

$$L_f := L_f^i\tau_i, \tag{3.19}$$

then L_f becomes an element of the $su(2)$ Lie algebra. Therefore, the variables of the discretized theory are

- An $SU(2)$ group element U_e associated with each edge $e \in \triangle^*$,
- An $su(2)$ Lie algebra element L_f associated with each face $f \in \triangle^*$

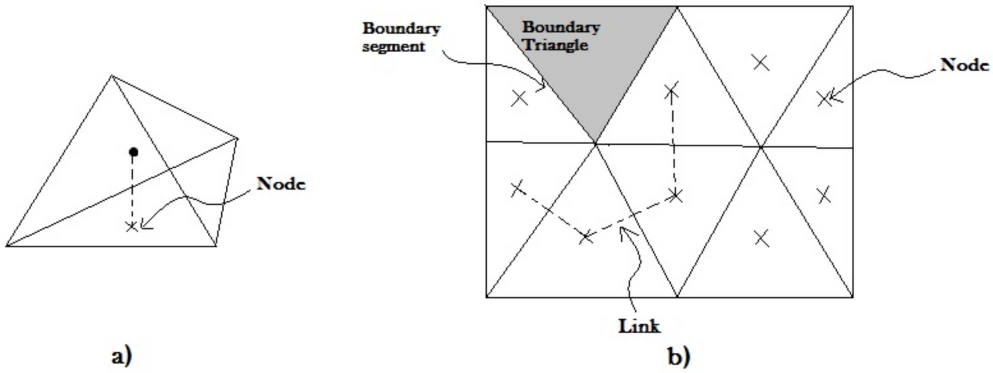

Figure 3.3: Schematic diagram showing the discretization of a two-dimensional boundary. A tetrahedron with a lower face on the boundary is shown in a). The end point of the edge dual to that face is a node and is indicated with a cross. In b), we show a drawing of a two-dimensional boundary and its corresponding discretization. Only some nodes and links are drawn, and the rectangular shape is chosen for simplicity.

The discretized action of the model is given by

$$S_{\triangle} = \frac{1}{8\pi G} \sum_{f} \mathrm{Tr}(L_f U_f), \tag{3.20}$$

where the sum includes all the faces $f \in \triangle^*$, and U_f is the holonomy of the connection around the face f. This quantity is obtained by the product of the group elements U_{e_k} associated with the edges e that surround f:

$$U_f := U_{e_1} U_{e_2} \ldots U_{e_n}. \tag{3.21}$$

It is important to consider that if we have a compact region with a boundary in that boundary we must close the perimeter of the faces to write the quantity U_f for the faces that end in the boundary. This implies that we also have group quantities U_l associated with the links of the boundary. If $\lambda : \Sigma \to SU(2)$ is a smooth local $SU(2)$ gauge transformation, the holonomy transforms as

$$U_e \to \lambda(x_f^e) U_e \lambda^{-1}(x_i^e). \tag{3.22}$$

The boundary variables play an important role in the construction of the boundary Hilbert space in the quantum theory, and for that reason we will study them in a little more detail. There are two kinds of variables in the boundary: the group elements U_l associated with the boundary edges (the links) and the algebra elements L_s associated with the boundary segments s. Notice that there is one boundary segment s per each link l, and the two cross each other. For this reason, we can rename L_s as L_l, where l is the link crossing the boundary segment s. To have an idea of what is going on, we recommend checking Figure 3.3.

We have one couple $(L_l, U_l) \in su(2) \times SU(2)$ of variables per link l. The boundary phase space of the discretized theory is $T[SU(2)^L]^*$ [19], where L is the number of links of the boundary graph. Finally, the Poisson brackets between the variables in the boundary are [19]

$$\begin{aligned}
\{U_l, U_{l'}\} &= 0 \\
\{U_{l'}, L_l^i\} &= (8\pi G)\delta_{ll'} U_l \tau^i \\
\{L_l^i, L_{l'}^j\} &= (8\pi G)\delta_{ll'} \epsilon^{ij}{}_k L_l^k.
\end{aligned} \tag{3.23}$$

3.3.3 Quantum kinematics

To quantize the theory, we start by considering a compact three-dimensional region with a two-dimensional boundary Σ. We triangulate the region using a simplicial complex K and its dual K^*. The corresponding triangulation and its dual are denoted again by $\triangle$ and $\triangle^*$, respectively. The triangulation induces a discretization of Σ, and the dual simplicial complex will be bounded by a graph Γ where the boundary variables are a group element and an algebra element;

$$(U_l, L_l)$$

associated with each link l of the boundary graph Γ. Considering the Poisson brackets given in (3.23), we search for operators $\widehat{U}_l$ and $\widehat{L}_l$ that meet the condition

$$\left[\widehat{U}_l, \widehat{L}_{l'}\right] = (8\pi G\hbar) i \delta_{ll'} \widehat{U}_l \tau^i \tag{3.24}$$

on a given Hilbert space. Let us consider the group elements U_l as the configuration variables and the algebra elements L_l as the canonical momenta. The Hilbert space is the space of square integrable functions of the configuration variables

$$\mathcal{H}_\Gamma = L^2\left[SU(2)^L\right]. \tag{3.25}$$

The quantum states are wavefunctions $\Psi(U_l) = \Psi(U_1, \ldots, U_L)$ of L group elements U_l associated with the links of the graph Γ. The scalar product with respect to which $\mathcal{H}_\Gamma = L^2\left[SU(2)^L\right]$ is a Hilbert space is given by the Haar measure μ [35] on L copies of $SU(2)$:

$$\langle\Psi|\Phi\rangle = \int_{SU(2)^L} \Psi(\bar{U}_l)\Phi(U_l) dU_l. \tag{3.26}$$

As is usual in the Schrödinger representation, the operator corresponding to the canonical momentum must be a derivative operator with respect to the configuration variable. The natural $SU(2)$ derivative operator on the functions $\Psi(U_l)$ is the left-invariant vector field, which is defined as [19, 34]

$$\left(J^i\Psi\right)(U) := -i\frac{d}{dt}\left[\Psi(Ue^{t\tau_i})\right]\Big|_{t=0}. \tag{3.27}$$

To obtain an operator $\widehat{L}_l$ that satisfies (3.24), it is sufficient to scale the operator defined in (3.27) with the appropriate factor:

$$\widehat{L}_l := (8\pi G\hbar)\vec{J}_l, \tag{3.28}$$

where $\vec{J}_l$ is the left-invariant vector field acting on the argument U_l. Now we have the two operators that satisfy the commutation relation (3.24) :D.

As shown in [19], the length L_s of a segment $s \in \triangle$ is given by the $\mathbb{R}^3$ norm $L_s = \parallel \vec{L}_f \parallel$, where f is the face dual to the segment s. In this way, the length of a boundary segment s is given by the operator $L_l = (L_l^1)^2 + (L_l^2) + (L_l^3)$, where l is the link crossing s. Since $\vec{J}$ is a generator of $SU(2)$, its square J^2 is the $SU(2)$ Casimir operator. Its eigenvalues are given in exactly the same way as the eigenvalues of the total angular momentum in quantum mechanics $j(j+1)$, where j is a half-integer. Therefore, the spectrum of the length of a link l is

$$L_{j_l} = 8\pi G\hbar\sqrt{j_l(j_l+1)}. \tag{3.29}$$

Thus, in the three-dimensional theory, the length is quantized.

The quantization of the length in the three-dimensional theory is analogous to the quantization of the area in the four-dimensional case[6]. Here the length has a discrete spectrum at the Planck scale. In this way, the covariant theory starts to show similarities and contact points with the canonical theory. Moreover, the granularity of space will reappear in the four-dimensional spin-foam models in such a way that will match exactly to that predicted by the four-dimensional canonical theory.

By paying attention to the definition of $\mathcal{H}_\Gamma$, we will see that this Hilbert space is very similar to $\widetilde{\mathcal{H}}_\Gamma$ given in Section 2.3.2. The difference here is that all the nodes of the graph Γ are trivalent; only three lines emerge and/or end on each node. In order to define the kinematical Hilbert space of boundary states, we must consider that the theory must be invariant under the $SU(2)$ gauge transformations defined in (3.22). The gauge invariant states must satisfy

$$\Psi(U_l) = \Psi(\Lambda_{xi_l} U_l \Lambda_{xf_l}^{-1}), \tag{3.30}$$

where $\Lambda_n \in SU(2)$, and xi_l and xf_l are the initial and final points of l, respectively.

By taking into account this last condition, we find that the kinematical Hilbert space of boundary states has a basis given by the spin network states [19]. The basis is given by the same spin network states defined in the canonical theory but now defined only over a two-dimensional surface Σ and with the difference that all the nodes of the graph Γ are trivalent. The length operators L_l are gauge invariant, and they form a complete commuting set [19]. The spin network basis defined on the graph Γ diagonalizes those length operators.

It is important to observe that the boundary graph Γ only has three-valent nodes. By considering some facts about the representation theory of the $SU(2)$ group and following the recipe given in Section 2.3.4, we find that the boundary spin network states all have the same structure. If $S = (\Gamma, j_l, i_n)$ is a spin network defined on Σ with Γ as its graph and Γ as L links l and N nodes n, then the corresponding spin network state is given by

$$\Psi_{j_l}(U_l) = i_1^{m_1,m_2,m_3} \dots i_N^{m_{L-2},m_{L-1},m_L} D^{j_1}_{m_1,n_1}(U_{l_1}) \dots D^{j_L}_{m_L,n_L}(U_{l_L}), \tag{3.31}$$

[6] A length operator with a discrete spectrum can also be constructed in the four-dimensional theory.

where $D^{j_l}_{m,n}(U)$ are the Wigner matrices in the representation j_l and i^{m_1,m_2,m_3} are, in each case $1,\ldots,N$, the Wigner 3j symbol between the representations of the links that converge in each node. The indices n of the Wigner matrices are not the same that the nodes n. Note that all the indices are contracted between the intertwiners i and the Wigner D matrices in a way that is determined by the connectivity of the graph Γ. Therefore, a gauge-invariant state must have the form

$$\Psi(U_l) = \sum_{j_1,\ldots j_l} \mathcal{C}_{j_1,\ldots j_l} i_1^{m_1,m_2,m_3} \ldots i_N^{m_{L-2},m_{L-1},m_L} D^{j_1}_{m_1,n_1}(U_{l_1}) \ldots D^{j_L}_{m_L,n_L}(U_{l_L}). \tag{3.32}$$

These are the boundary states of three-dimensional covariant quantum gravity. They match those that we would find if we quantize the three-dimensional theory presented in Section 3.3.1 following the canonical approach.

3.3.4 Quantum dynamics

To calculate quantum amplitudes, we consider a compact three-dimensional region $\mathcal{R}$ with two-dimensional boundary $\partial\mathcal{R} = \Sigma$. The boundary can be formed by two disconnected pieces, and we can consider these pieces as the past and future boundaries. Conversely, we can consider the boundary as one connected region and define the amplitude associated with any state on this boundary. This last point of view is considered more covariant [34], and it will be used here to describe the dynamics.

We fix a triangulation $\triangle$ of $\mathcal{R}$, and we construct the corresponding dual triangulation $\triangle^*$. The transition amplitude is a function of the states defined on the boundary graph $\Gamma = (\partial\triangle)^*$ and will be denoted as $W_\triangle$. Here the subscript $_\triangle$ indicates that the amplitude is computed (and depends) on the discretization $\triangle$.

To compute $W_\triangle$ for the theory discretized on the dual triangulation $\triangle^*$, we use the path integral approach. As is standard in quantum physics, the amplitude is given by the integral over all classical configurations weighted by the exponential of the classical action

$$W_\triangle(U_l) = \mathcal{N} \int dU_e \int dL_f \; exp\left\{ \frac{i}{8\pi G\hbar} \sum_f \mathrm{Tr}[U_f L_f] \right\}, \tag{3.33}$$

where we have an integral over $SU(2)$ per edge $e \in \triangle^*$ and an integral over the $su(2)$ Lie algebra per face $f \in \triangle^*$. Furthermore, $\mathcal{N}$ is a constant where we absorb various constant contributions.

As shown in [1] and [19], the integral over the momenta can be performed since it is an integral of an exponential, which gives a Dirac delta function over the $SU(2)$ group. Thus, we obtain

$$W_\triangle(U_l) = \mathcal{N} \int dU_e \prod_f \delta(U_f). \tag{3.34}$$

Remember that $U_f := \vec{\prod}_{e\in\partial f} U_e$ is the oriented product of the elements U_e around the face f. This defines the closed holonomy around f, or, equivalently, the product around the corresponding dual edge of the triangulation.

The last expression is actually a compact way to write spinfoam path integrals. This is called the *connection representation* [28] of spin foam amplitudes. As mentioned in [28], it is dual to the local spinfoam ansatz defined in (3.3) in the sense that the amplitude is expressed in terms of the group elements in the connection representation (3.34), whereas the same amplitude is expressed in terms of the group representations in the local ansatz (3.3). For the local spinfoam ansatz, we can write $W_\triangle(j_l) = \langle W_\triangle | j_l\rangle$, and, for the connection representation, we can write $W_\triangle(U_l) = \langle W_\triangle | U_l\rangle$.

It is important to consider that, as emphasized in [28], [19] (and somewhere in [6]), the path integral (3.34) has some delta functions that are redundant, and the path integral formally has some divergences caused by those redundancies. It has been shown [28], however, that the redundancies can be removed and that the path integral can be defined with the minimally necessary number of delta distributions.

To compute the remaining integral in equation (3.34), we expand the delta distribution as a sum over the group representations in the form [1, 6, 19, 28]

$$\delta(U) = \sum_j d_j \mathrm{Tr}(D^j(U)), \tag{3.35}$$

where $d_j := dim(j) = 2j+1$ is the dimension of the vector space of the carriers of the representation j.

To perform the integrations over the $SU(2)$ elements, let us consider one edge e and its corresponding integral $\int dU_e$. Note that each edge precisely bounds three faces because the edge is dual to a triangle, which is bounded by three segments s, and each of those segments is dual to a face. Therefore, each integral $\int dU_e$ is of the form [19, 28]

$$\int_{SU(2)} dU \; D^{j_{l_1}}_{m_1 n_1}(U) D^{j_{l_2}}_{m_2 n_2}(U) D^{j_{l_3}}_{m_3 n_3}(U),$$

and each of these integrals has the value [1, 19, 28]

$$\int_{SU(2)} dU \; D^{j_{l_1}}_{m_1 n_1}(U) D^{j_{l_2}}_{m_2 n_2}(U) D^{j_{l_3}}_{m_3 n_3}(U) = i^{m_1 m_2 m_3} i^{n_1 n_2 n_3}, \tag{3.36}$$

where $i^{m_1 m_2 m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3j symbol between the representations j_1, j_2, j_3 of the faces bounded by the particular edge e.

Thus, the result of the integrals in (3.34) is a bunch of 3j symbols contracted among themselves. We will not give the precise pattern of contraction but details can be found in [1], [19] and [28]. What is important to mention is that the contraction pattern produces a 6j symbol per vertex [1, 19, 28]. The final result, carefully keeping track of the signs, is [19]

$$W_\triangle(j_l) = N_\triangle \sum_{j_f} \prod_f (-1)^{j_f} d_{j_f} \prod_v (-1)^{J_v} \{6j\}_v, \tag{3.37}$$

where the sum is over all the representations associated with each face $f \in \triangle^*$, $J_v = \sum_{k=1}^{6} j_k$ and j_k are the spins of the faces adjacent to the vertex v. $N_\triangle$ is a normalization factor that can depend on the triangulation $\triangle$, and we have a 6j symbol per vertex v.

The expression (3.37) is the transition amplitude expressed in the form of the spinfoam ansatz with the choices detailed below.

- The set of simplicial complexes summed over is formed by a single simplicial complex, chosen as the skeleton of the dual of a three-dimensional triangulation.
- The representations are the unitary representations of $SU(2)$ assigned to each face f of the dual triangulation.
- The intertwiners are associated with each edge and are the 3j symbols between the representation of the faces bounded by such edge.
- The vertex amplitude is given by the 6j symbol.

The expression (3.37) is the defining formula for the Ponzano-Regge model, and it has the correct form for a local spinfoam amplitude. It was written by Ponzano and Regge in the 1960s [42]. The model has some special characteristics that are important to mention.

- Consider a flat geometrical tetrahedron with sides of lengths $L_1, \ldots, L_6$. We associate six spins $j_{1,\ldots,j_6}$ with these lengths so that $L_i = j_i + \frac{1}{2}$. Let V be the volume of the tetrahedron and S the Regge action of the discretization given by this tetrahedron. Ponzano and Regge [42] provided evidence that in the large spin limit $j \to \infty$ the 6j symbol associated with the vertex in the baricentre of the tetrahedron has the following behavior:

$$\lim_{j\to\infty} \{6j\} \sim \frac{1}{2\sqrt{-12i\pi V}} e^{iS} + \frac{1}{2\sqrt{12i\pi V}} e^{-iS}. \tag{3.38}$$

  Such a result was proven in 1998 by Roberts [43], and in [1] it is called *the miracle of the dynamics of General Relativity in a symbol*. As mentioned in [19], this shows that if we consider only large spins we can disregard quantum discreteness, and the sum over spins is approximated by an integral over the lengths in the Regge geometry represented by the tetrahedron. In this way, the integrand is given by a function of exponentials of the action. This represents a discretization of a path integral over geometries of the exponential of the Einstein-Hilbert action, and therefore the expression (3.37) represents an implementation of the formal expression:

$$\int D[g] e^{\frac{i}{\hbar}\int \sqrt{-g}R}.$$

  The watchful reader must have already seen that the expression (3.38) has two terms with opposite signs. The reason for this is very interesting. It has to be with the fact that we are actually quantizing the triad, not the metric, and at each vertex, there are two triad configurations for each metric configuration. For more details, see [1] and [19].

- The other remarkable result of the Ponzano and Regge theory is the triangulation independence: the expression (3.37) depends on the global topology of the three-dimensional region $\mathcal{R}$ but not on the triangulation $\triangle$. Moreover, if we refine the triangulation $\triangle$ while keeping the boundary graph Γ fixed, the expression (3.37) does not change. On the other hand, if we refine the boundary graph, the physical Hilbert space of the theory becomes independent of the boundary graph.

  As mentioned in [1], the triangulation independence of the Ponzano-Regge theory is a consequence of the fact that three-dimensional Euclidean general relativity lacks local degrees of freedom. For this reason, the triangulation does not produce the loss of degrees of freedom, and the triangulation independence is not expected to be a characteristic of the four-dimensional Lorentzian general relativity, where we indeed have local degrees of freedom.

- The Ponzano-Regge model suffers from infrared divergences. These divergences lead to some behaviors that are called *bubbles and spikes*. We will not touch this interesting topic here. For details see [1], [6] and [19].

3.4 BF theory and general relativity as a constrained theory (again)

After having presented the quantization of Euclidean general relativity in three dimensions, we next attempt to apply the same (or at least a similar) procedure to quantize General Relativity in four dimensions. We hope to find some of the same surprises that we found in the Ponzano-Regge model: that the area and volume are quantizing in exactly the same way that they are in the canonical theory, that the boundary state space exactly matches the one of canonical loop quantum gravity, and that the transition amplitude could be written using simple functions from the group representation theory.

To the disappointment of many quantum gravity theorists, the project turned out to be much more difficult that expected, and until now we do not have a satisfactory and complete formulation of the path integral approach for quantum gravity. The reason is that general relativity in four dimensions is a theory with local degrees of freedom, and that makes its quantization much more difficult. Thus, an important question emerges: what could be the relation between a theory without local degrees of freedom and a theory with local degrees of freedom?. The answer is that classical general relativity can be written as a theory without local degrees of freedom plus some constraints over one of its fields. These constraints allows recovering the degrees of freedom of general relativity. We will see how to do this.

3.4.1 BF theory

BF theory is a field theory that lacks local degrees of freedom, and for this reason it can be discretized without affecting its physical content. The theory can be defined using whatever Lie group and in various spacetime dimensions.

As our starting point, we will consider a differentiable manifold $\mathcal{M}$ of n-dimensions and a Lie group G. BF theory is defined by two fields: a 2-form B^{IJ} with values in the Lie algebra $\mathfrak{g}$ of G and a G connection ω^{IJ} on a G principal bundle over $\mathcal{M}$. The action is given by the following expression:

$$S[B,\omega] = \int_{\mathcal{M}} B_{IJ} \wedge F^{IJ}[\omega], \tag{3.39}$$

where $F[\omega]$ is the curvature two-form of ω. The name of the theory comes from the fields B and F. The dynamical equations impose that the connection is flat $F[\omega] = 0$ and that the B-field has a trivial parallel transport $dB + \omega \wedge B = 0$ [28]. The theory lacks local degrees of freedom and constitutes an extension to four dimensions of three-dimensional Euclidean general relativity. In fact, the latter is precisely a BF theory in three dimensions with $G = SU(2)$ as the gauge group .

We can discretize BF theory and define a path integral corresponding to it following the same steps we took in defining the Ponzano-Regge model. Let us consider an arbitrary Lie group G as the gauge group. We consider a compact n-dimensional spacetime region $\mathcal{M}$ with $(n-1)$-dimensional boundary $\partial\mathcal{M} = \Sigma$. By discretizing $\mathcal{M}$ using a simplicial complex K and its dual K^*, we obtain a triangulation $\triangle$, constructed by using n-simplices and its corresponding dual triangulation $\triangle^*$. The discretization of the connection is given by its holonomy along the edges $e \in \triangle^*$. The discretization of the B-field is, in this case, given by the integral of B over each $n-2$ simplex $t \in \triangle$:

$$B_f := \int_t B, \tag{3.40}$$

where we use the subscript f because we consider the discretized field as a field defined on each face $f \in \triangle^*$ using the one-to-one correspondence between faces $f \in \triangle^*$ and $n-2$ simplices $t \in \triangle$.

Our aim is to give meaning to the formal expression [6, 28]

$$Z = \int [dB][d\omega] e^{i\int_{\mathcal{M}} B_{IJ}\wedge F^{IJ}} \tag{3.41}$$

that corresponds to the BF path integral.

In the discrete setting, the discretized version of the path integral (3.41) is given by [6, 28]

$$Z(\triangle) = \int_G \prod_{e\in\triangle^*} dU_e \int_{\mathfrak{g}} \prod_{f\in\triangle^*} dB_f e^{iB_f U_f}, \tag{3.42}$$

where $U_f := \vec{\prod}_{e\in\partial f} U_e$ is again the oriented product of the elements U_e around f. Such a quantity defines the holonomy around f or equivalently the product around the corresponding dual edge of the triangulation. In this way, U_f is also a function of U_e.

We can integrate over the B field, and we obtain

$$Z(\triangle) = \int_G \prod_{e\in\triangle^*} dU_e \prod_{f\in\triangle^*} \delta(U_f), \tag{3.43}$$

where we have an integral per edge $e \in \triangle^*$ and the integration in the group variables is performed in terms of the invariant measure in G.

As mentioned in [6], it is important to know that the integration over the B-field does not give the group delta distribution. For example, when $G = SU(2)$ and then $B \in su(2)$ integration leads to the $SO(3)$ delta distribution, which only contains integer spin representations in the expansion given at (3.35) [6]. This fact is, unfortunately, usually ignored in the construction of spin foam models.

Expression (3.43) is usually the starting point for the construction of spinfoam models of different BF theories. We illustrated the construction of one of such model in the last section, where we constructed the spin foam model of three-dimensional Euclidean general relativity. We can repeat the procedure in the four-dimensional case in a more or less similar way. Let us consider that $\dim(\mathcal{M}) = 4$ and $G = SU(2)$. We can again expand the delta distribution using the expression (3.35). The difference with the three-dimensional case is that now $\triangle^*$ is dual to a four-dimensional cellular decomposition. Examining at the integration over the element U_e, we can see that an edge e will be part of four different faces f, and the integral over U_e will consist in the product of four representation matrices. Instead of (3.36) we now have

$$\int_{SU(2)} dU \; D^{j_1}_{m_1 n_1}(U) D^{j_2}_{m_2 n_2}(U) D^{j_3}_{m_3 n_3}(U) D^{j_4}_{m_4 n_4}(U) = \sum_i v^i_{m_1 m_2 m_3 m_4} v^i_{n_1 n_2 n_3 n_4}, \tag{3.44}$$

where the index i labels the orthonormal basis $v^i_{n_1 n_2 n_3 n_4}$ in the space of intertwiners between the representations of spin j_1, j_2, j_3 and j_4 [1, 28]. We assign a representation j_f to each face or equivalently to each triangle. Thus, we have a sum over intertwiners for each edge in addition to the sums over representations for each face. Notice that each vertex of $\triangle^*$ bounds 10 faces $f \in \triangle^*$, and because of this, we have now 10 representations and five intertwiners in each vertex. The contraction of the five intertwiners defines a function called the 15j symbol [1, 28] and is denoted as $\{15j\}$. In this way, we obtain the following expression for the partition function of four-dimensional BF theory with gauge group $G = SU(2)$

$$Z(\triangle) = \sum_{j_f, i_e} \prod_f dim(j_f) \; \prod_v \{15j\}_v. \tag{3.45}$$

This is all very good[7], however we are looking for a quantum theory of gravity, not a quantization a strange theory that does not describe any physical interaction.

3.4.2 Gravity as a constrained BF theory

In order to establish the connection between BF theory and general relativity, we must consider the standard Palatini action, which can be written in terms of the tetrad field e and the $sl(2, \mathbb{C})$-connection ω in the form

$$S[e, \omega] := \frac{1}{16\pi G} \int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}[\omega] \tag{3.46}$$

[7]Well, not so good. The model suffers from infrared divergences, and there are subtitles in its construction. See [6] and especially [3]

where $F[\omega]$ is the curvature of ω. In terms of the metric g and the connection Γ, this expression can be rewritten exactly as the expression for the Einstein-Hilbert action (2.1), except that both the connection and metric are dynamical variables.

As has been indicated in many places (for example [1], [6], [7]. [8], [19], [28]), we can add an additional term to the action given in (3.46) without affecting the dynamical equations. For this reason added term is called a *topological term.* The new action that we will consider is given by

$$S[e,\omega] = \frac{1}{16\pi G}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}[\omega] + \frac{1}{16\pi G\gamma}\int e^I \wedge e^J \wedge F_{IJ}[\omega], \tag{3.47}$$

where γ is the Immirzi parameter that we introduced in the canonical theory. The second term has no effect on the dynamical equations. The variation with respect to the connection again yields the torsion-less condition, and when this is used the second term takes the form

$$\int e^I \wedge e^J \wedge F_{IJ}[\omega] = \int R_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} dx^4 = 0, \tag{3.48}$$

which is zero because of the identity $R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$.

Expression (3.47) defines the *Holst action.* What is remarkable about this action is that its canonical analysis leads directly to the connection variables that we previously defined Chapter 2 [6]. This fact shows that the topological term added to the Palatini action affects the quantum theory, even though it does not affect the classical theory.

If we define

$$B^{IJ} := \epsilon_{IJKL} e^K \wedge e^L, \tag{3.49}$$

the Holst action takes the form

$$S[e,\omega] = \frac{1}{16\pi G}\int \left[B_{IJ} \wedge F^{IJ} + \frac{1}{\gamma}({}^*B)_{IJ} \wedge F^{IJ} \right]. \tag{3.50}$$

If we consider a BF theory defined in four dimensions with the gauge group $G = SL(2,\mathbb{C})$ and with action given by (3.50), the condition (3.49) over the B field transforms such a BF theory into general relativity.

There exists a way to enforce the B field to be, sometimes, of the form (3.49), and it is given by the following theorem [1, 3, 6, 9].

Theorem 3.4.1. *Suppose that B^{IJ} is a 2-form with values in the lie algebra $SL(2,\mathbb{C})$ that meets the condition*

$$B^{IJ} \wedge B^{KL} = e\epsilon^{IJKL}, \tag{3.51}$$

where

$$e := \frac{1}{4!}\epsilon_{MNPQ} B^{MN} \wedge B^{PQ} \neq 0. \tag{3.52}$$

Then, there exist a real co-tetrad field e^I such that either

$$\begin{aligned} B^{IJ} &= \pm e^I \wedge e^J \\ &\textit{or} \\ B^{IJ} &= \pm \frac{1}{2}\epsilon_{IJKL} e^K \wedge e^L. \end{aligned} \tag{3.53}$$

The proof of the theorem is given with much detail in [3].

The condition (3.51) is sometimes called the *simplicity constraint*. Here we will call it the *geometric constraint*. This name is not standard but reflects the fact that the co-tetrad field allows to construct a Lorentzian metric over the differentiable manifold.

Theorem 3.4.1 shows that if the B field satisfies the constraint equation (3.51), then there exist four different possibilities for the form of B. These four possibilities can be joined into two sets called sectors: sector I where $B = \pm e \wedge e$ and sector II where $B = \pm^*(e \wedge e)$. As remarked in [16], if the Immirzi parameter has finite non-trivial values, then both sectors in fact yield general relativity, but the value of the Newton constant and Immirzi parameter are different in each sector. Therefore, it is important to distinguish between these two sectors, especially in the possible physical applications of the theory.

Summarizing, general relativity can be considered a BF theory in four dimensions, with gauge group $G = SL(2,\mathbb{C})$ and with action principle given by equation (3.50), with the B field of the form (3.49). A way to enforce the last is given by the equation (3.51).

Having constructed successful quantum theories for topological BF theories, the next challenge is to implement the constraint equation (3.51) into the quantum theory. The latter has led to some new proposals for defining discretized path integrals in quantum gravity. The description of the currently most accepted model is the subject of the next section.

We must make two additional remarks in order to advance to the presentation of the new spin foam models. The first is that when $e := (1/4!)\epsilon_{MNPQ}B^{MN} \wedge B^{PQ} \neq 0$ there is an equivalent formulation of (3.51) that is given by

$$\epsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = \tilde{e}\epsilon_{\mu\nu\rho\sigma}, \tag{3.54}$$

where

$$\tilde{e} := \frac{1}{4!}\epsilon_{MNPQ}\epsilon^{\alpha\beta\gamma\delta} B^{MN}_{\alpha\beta} B^{PQ}_{\gamma\delta}. \tag{3.55}$$

The proof of the equivalence between (3.54) and (3.51) can be found for example in [3] and [9]. We will use this form of the geometric constraint in what follows.

The second remark is that there exists another solution for the B field that is called the *degenerate sector*. When the tetrad field (or equivalently the B-field) is degenerate, the four solutions given in (3.53) coincide. As indicated in [3], the degenerate sector does not have an interpretation as a classical theory of gravity.

3.5 Implementation of the geometric constraints: The EPRL model

The formulation of General Relativity presented in the previous section is the starting point of the new spin foam models. To implement the geometric constraints (3.51) or its equivalent version (3.54) in the quantum framework, we need to find a way to discretize them in the classical theory and transform such discretized version into quantum operators acting as quantum constraint equations.

3.5.1 Discretizing the geometric constraints

We consider a four-dimensional manifold and a Lie group G as the gauge group. For the Riemannian theory $G = SO(4)$, and for the Lorentzian theory $G = SL(2,\mathbb{C})$. We discretize a compact region $\mathcal{M}$ with three-dimensional boundary $\partial\mathcal{M} = \Sigma$ as in the previous section. The field $B^{IJ}_{\mu\nu}$ that appears in (3.54) is replaced by the discrete quantities B^{IJ}_f. The constraints (3.54) are local constraints that are valid at every spacetime point. In the discrete setting, spacetime points are represented by four-simplices or equivalently vertices $v \in \triangle^*$. The discretized version of the constraints (3.54) is given by the following expression [6, 9]:

$$\epsilon_{IJKL} B^{IJ}_f B^{KL}_{\tilde{f}} = V(f, \tilde{f}), \tag{3.56}$$

where $V(f,\tilde{f}) = \int_{x\in f, x\in\tilde{f}} \tilde{e}\epsilon_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dy^\rho \wedge dy^\sigma$ is the four-volume spanned by f and $\tilde{f}$. The last equation can be translated into three different conditions as follows [6]:

$$\text{Diagonal constraints} \qquad \epsilon_{IJKL} B^{IJ}_f B^{KL}_f = 0, \tag{3.57}$$

for all $f \in v$, that is to say, for each and every face of the 10 possible faces touching the vertex v.

$$\text{Off-diagonal constraints} \qquad \epsilon_{IJKL} B^{IJ}_f B^{KL}_{\tilde{f}} = 0, \tag{3.58}$$

for all $f, \tilde{f} \in v$ such that they are dual to triangles sharing a one-simplex, that is to say, belonging to the same tetrahedron out of the possible five. Remember that a four-simplex is bounded by five tetrahedra. Finally,

$$\text{4-simplex constraints} \qquad \epsilon_{IJKL} B^{IJ}_f B^{KL}_{\tilde{f}} = V_v, \tag{3.59}$$

for any pair of faces $f, \tilde{f} \in v$ that are dual to triangles that share a single point. The constraint (3.59) is actually taken as the definition of the four-volume V_v of the four-simplex. The true constraint demands that the volume is equal when is calculated using different possible pairs of f and $\tilde{f}$ in a four simplex, with the condition that their dual triangles share a single point and that the pairs $f - \tilde{f}$ are ordered sharing the same orientation of the complex $\triangle^*$ [6].

The implementation of the geometric constraints in the form given the equations (3.57)-(3.59) is the basis of the definition of some of the most studied spinfoam models for quantum gravity. In the case where $G = SO(4)$, the implementation of the constraints in the quantum theory leads to the well-known Barrett-Crane model (BC model) [1, 2, 6, 9, 60]. The interest in the model resides in its remarkable simplicity [6] and its finiteness properties [8]. Soon it as found, however, that the BC-model suffered from serious limitations. First, its boundary state space does not exactly match that of canonical loop quantum gravity [8]. Second, the volume operator is ill-defined in such a model. In addition, some studies indicated that may not yield to the correct tensorial structure of the graviton propagator [8]. Finally, if we take into account Theorem 3.4.1, the BC model is a quantization that mixes all the sectors involved in the solution of the simplicity constraints [9], and, for that reason, the model seems not to be a quantization of general relativity as such.

All the mentioned limitations of the BC model motivated the search for new ways to impose the simplicity constraints at the quantum level. All these efforts led to a new spin foam model that tries to ameliorate the properties of the BC model and is usually called the EPRL-model[8]. We will proceed to roughly describe the construction of the model. For more details, the reader can see the references cited at the beginning of the chapter.

As we mentioned at the beginning of this subsection, we start considering a four-dimensional manifold and the Lie group $G = SL(2,\mathbb{C})$ as the gauge group. We discretize a compact region $\mathcal{M}$ with a three-dimensional boundary $\partial\mathcal{M} = \Sigma$ as we performed in the previous section. As already mentioned in Theorem 3.4.1, the constraint (3.51), or its equivalent expression (3.54), has two different sectors of solutions. As proven in [8], the corresponding discretized version given in Equation (3.56) also has two different sectors as its solutions. To really quantize general relativity, we somehow need to choose one of the sectors. As was proven in [22] and subsequently applied in [8] and [16], if we replace the condition (3.56) with the requirement

- For each tetrahedron $\tau \in \triangle$, there exists a vector $n^I \in \mathbb{R}^4$ such that for all the faces $f \in \triangle^*$ that are dual to the triangles $t \in \tau$ we have that

$$C_f^J := n_I({}^*B_f)^{IJ} = 0, \tag{3.60}$$

we correctly choose only the sector II that corresponds to general relativity. Thus, B is necessarily of the form $B = \pm^*(e \wedge e)$. Geometrically n_I represents the normal one-form to the tetrahedron τ. This condition is called the *linear simplicity constraint.* We will call it the *linear geometric constraint.* The discovery of the condition (3.60) as implying the choice of the right sector prompted the development of the new spin foam models.

3.5.2 Quantum kinematics

As mentioned before, we consider the triangulation of a compact four-dimensional region $\mathcal{M}$ with boundary Σ. The simplicial complex dual to the triangulation in the boundary Σ defines a graph Γ, with vertices v dual to boundary tetrahedra τ and links l dual to boundary triangles. A face that touches the boundary is dual to a boundary triangle and therefore corresponds to a boundary link l. This link is the intersection of the face with the boundary. Therefore, a boundary link l is a boundary edge, but it is also associated with a face f that touches the boundary.

[8]The model has this name because of the initials of the surnames of four famous characters in the LQG community.

Since the variables U_e and B_f are associated with edges and faces, respectively, the boundary variables are respectively associated with boundary edges: that is, the links l and the boundary of the faces, which are also links. Therefore, the boundary variables are $B_l \in sl(2,\mathbb{C})$ and $U_l \in SL(2,\mathbb{C})$ on each link of the boundary graph Γ. These variables will become operators in the quantum theory.

Let us define

$$J_f := \frac{1}{\kappa}\left(B_f + \frac{1}{\gamma}\,{}^*B_f\right). \tag{3.61}$$

Inverting the equation gives

$$B_f := \left(\frac{\kappa\gamma^2}{\gamma^2+1}\right)\left(J_f - \frac{1}{\gamma}\,{}^*J_f\right). \tag{3.62}$$

The constraint (3.60) can be expressed in terms of J_f as follows:

$$C_f^J := n_I\left(({}^*J_f)^{IJ} - \frac{1}{\gamma}J_f^{IJ}\right) = 0. \tag{3.63}$$

Following [8], [16] and [23], let us fix $n_I = \delta_I^0$. This choice restricts all tetrahedra to be spacelike [16]. With this choice, the linear geometric constraint takes the following form:

$$C_f^j = \frac{1}{2}\epsilon^j{}_{kl}J_f^{kl} - \frac{1}{\gamma}J_f^{0j} = L_f^j - \frac{1}{\gamma}K_f^j = 0, \tag{3.64}$$

where $\epsilon^j{}_{kl} := \epsilon^{0j}{}_{kl}$, $L_f^j := \epsilon^j{}_{kl}J_f^{kl}$ are the generators of the $SU(2)$ subgroup that leaves n_I invariant and $K_f^j := J_f^{0j}$ are the generators of the corresponding boosts. In the above expression, f refers to a face $f \in \triangle^*$, and the mentioned $SU(2)$ subgroup is chosen arbitrarily at each tetrahedron or, equivalently, on each edge $e \in \triangle$. We will take (3.64) as our basic geometric constraint. The commutation relations of the previous restrictions are [6]

$$\left[C_f^i, C_{f'}^j\right] = 2\delta_{e,e'}\epsilon^{ij}{}_k C_f^k - \delta_{e,e'}\frac{\gamma^2+1}{\gamma^2}\epsilon^{ij}{}_k L_f^k. \tag{3.65}$$

This relation implies that the constraint algebra is not closed. The latter means that the commutator of two restrictions is not a restriction. For that reason, we cannot impose the closure relation (3.64) as an operator equation of the states summed over in the BF partition function. The EPRL model is obtained by restricting the representations appearing in the expression of the BF partition function so that at each tetrahedron the linear geometric constraint (3.64) is satisfied in the strongest way possible [6]. We will see how this is done.
The two Casimir operators of $SL(2,\mathbb{C})$ are given by $\vec{K}_f \cdot \vec{L}_f$ and $|\vec{K}_f|^2 - |\vec{L}_f|^2$.

The variable L_f^j has an important geometric interpretation. Consider a triangle in the boundary. If f is the face dual to such triangle, in the approximation in which the metric is constant in the triangle, it can be proven that the norm of L_f^j is proportional to the area of the triangle [19]:

$$|L_f| = \frac{1}{\gamma}A_{t_f}. \tag{3.66}$$

The quantization of the theory leads to states $\psi(U_l)$, which are functions on $SL(2,\mathbb{C})^L$, where the operator B_l is realized as the generator of $SL(2,\mathbb{C})$ transformations [19]. However, we must impose the constraint (3.64) in order to obtain the boundary state space and define the corresponding transition amplitudes.

Boundary state space

As emphasized in [19], functions in $SL(2,\mathbb{C})$ can be expanded into irreducible representations of the group $SL(2,\mathbb{C})$. The unitary representations of $SL(2,\mathbb{C})$ are of infinite dimension, and they are labeled by two numbers: a positive real number p and a no-negative half-integer k. The Hilbert space $H^{(p,k)}$ of the (p,k) representation can be written as a direct sum of spaces of irreducible representations of the subgroup $SU(2) \subset SL(2,\mathbb{C})$ in the form

$$H^{(p,k)} = \bigoplus_{j=k}^{\infty} \mathcal{H}_j, \tag{3.67}$$

where $\mathcal{H}_|$ is the $2j+1$–dimensional space of the carriers of the spin j irreducible representation of $SU(2)$. In the (p,k) representation, we can choose a basis $|p,k,j,m\rangle$, with $j = k, k+1, \ldots$ and $|m| \leq j$. The numbers (p,k) are related to the two Casimir operators of $SL(2,\mathbb{C})$ by the equations [19, 34]:

$$|\vec{K}|^2 - |\vec{L}|^2 = p^2 - k^2 + 1, \tag{3.68}$$

$$\vec{K} \cdot \vec{L} = pk, \tag{3.69}$$

whereas the numbers j, m are the numbers indexing the basis of eigenstates of $|\vec{L}|^2$ and L^z.

If we demand the equation (3.64) be satisfied in the limit of large (p,k), then the Casimir must satisfy [19]

$$|\vec{K}|^2 - |\vec{L}|^2 = (\gamma^2 - 1)|\vec{L}|^2, \tag{3.70}$$

$$\vec{K} \cdot \vec{L} = \gamma |\vec{L}|^2. \tag{3.71}$$

By combining (3.68) and (3.69) with (3.70) and (3.71) and by taking the limit of large p, k, j, we obtain the following equations: [19]

$$p^2 - k^2 = (\gamma^2 - 1)j^2, \tag{3.72}$$

$$pk = \gamma j^2, \tag{3.73}$$

which have the solutions

$$p = \gamma k, \tag{3.74}$$

$$k = j. \tag{3.75}$$

The first relation establishes a restriction on the set of the unitary representations of $SL(2,\mathbb{C})$. The second one selects, from all the $SU(2)$ subspaces involved in the sum (3.67), the subspace with the lowest spin. The states that satisfy these relations have then the form

$$|p,k,j,m\rangle = |\gamma j, j, j, m\rangle. \tag{3.76}$$

The vector space formed by the states given in (3.76) is isomorphic to the space of $SU(2)$ states given by $|j.m\rangle$. We can introduce an isomorphism $Y_\gamma : \mathcal{H}_j \to H^{(p=\gamma j, k=j)}$ between them in the most obvious way:

$$Y_\gamma |j.m\rangle := |\gamma j, j, j, m\rangle. \tag{3.77}$$

All the vectors in the image of Y_γ satisfy the linear geometric constraint in the weak sense:

$$\langle Y_\gamma \psi_1 | \vec{L} - \frac{1}{\gamma}\vec{K} | \psi_2 \rangle = 0, \tag{3.78}$$

in the limit of j large [19, 34].

The map Y_γ can be extended to a map from functions over $SU(2)$ to functions over $SL(2,\mathbb{C})$ in the following way:

$$Y_\gamma(\psi)(U) = Y_\gamma \left(\sum_{jmn} c_{jmn} D^{(j)}_{mn}(\_) \right)(U) := \sum_{jmn} c_{jmn} D^{(\gamma j, j)}_{jmjn}(U). \tag{3.79}$$

Now consider a graph Γ and a function $\psi(h_l)$ of the $SU(2)$ group elements associated to the links. We can use the extension defined above to send ψ to a function $Y_\gamma \psi$ of $SL(2,\mathbb{C})$ group elements on the links of Γ. As emphasized in [34], this function is not $SL(2,\mathbb{C})$ invariant at the nodes, but we can make it gauge invariant by integrating over the gauge group. We define

$$(P_{SL(2,\mathbb{C})}\phi)(U_l) := \int_{SL(2,\mathbb{C})} dU'_n \phi \left(U_{x^l_i} U_l U_{x^l_f} \right), \tag{3.80}$$

where the prime in dg_n indicates that one of the edge integrals is dropped because it is redundant [34] and x^l_i, x^l_f are the initial and final points of l. We can then define the linear map

$$f_\gamma := P_{SL(2,\mathbb{C})} \circ Y_\gamma \tag{3.81}$$

that sends $SU(2)$ spin networks into $SL(2,\mathbb{C})$ spin networks. This function allows connecting the boundary state space of the EPRL model with the standard spin network states of canonical loop quantum gravity.

The boundary Hilbert space of the EPRL model is given by the image of $SU(2)$ spin networks under the map Y_γ. The boundary states of the spin foam model defined by the EPRL model are in correspondence to $SU(2)$ spin network states. As emphasized in [6], this suggests that the spin foam amplitudes can be interpreted as dynamical transition amplitudes of the canonical theory described in Chapter 2.

Finally, recall that the norm of the variable L^j_f, that is $|L^j_f|$, is proportional to the area of the triangle dual to the face f, as is given in the expression (3.66). Let us consider a face in the boundary and the boundary triangle dual to that face. If we remember that a boundary face corresponds to a boundary link l, and that the link l is normal to the corresponding dual triangle in the boundary, and if we restore the units, then the area of the triangle normal to the link l is given by [19]

$$A_l = 8\pi\gamma\hbar G|\vec{L}|. \tag{3.82}$$

This expression gives the eigenvalues of the area of a single triangle:

$$A_l = 8\pi\gamma\hbar G\sqrt{j(j+1)}. \tag{3.83}$$

The eigenvalues of the area of an arbitrary surface punctured N times $n_1, n_2, \ldots, N$ are [19]

$$A_{l_n} = 8\pi\gamma\hbar G\sum_n \sqrt{j_n(j_n+1)}, \tag{3.84}$$

where j and j_n are no-negative half integers and correspond to the representations of the links that puncture the triangle or surface considered.

The above expression is the same expression that we found in the canonical theory for the area of a two-dimensional surface punctured by the lines of a spin network graph. Therefore, the area operator in the EPRL model exactly matches that of the canonical theory. The same occurs with the volume operator, but the construction is more complicated. Details can be found in [44].

The match, however, between the eigenstates of the area and volume operator in the canonical and covariant theories and it is important to remember that in (3.64) we considered all the faces as spacelike. The latter means that the model only considers spacelike faces. It is possible to generalize the model, however, to include timelike faces, but timelike faces are not compatible with the Hilbert space of the canonical theory. For more details see [52] and [53].

3.5.3 Quantum dynamics

In order to complete the construction of the model, we will write the expression for the transition. Instead of describing the detailed way to derive the form of the transition amplitude in the EPRL model, we will give only the final result. More details can be found in [6, 8, 16, 19, 23, 34]

We know that the transition amplitude must be a function of $SU(2)$ group elements living on the boundary graph Γ, but it must involve the $SL(2,\mathbb{C})$ group elements and be invariant under $SL(2,\mathbb{C})$ transformations. To obtain this kind of amplitude, we take $SL(2,\mathbb{C})$ integrals instead of $SU(2)$ integrals. We must also map the $SU(2)$ group elements into $SL(2,\mathbb{C})$ ones. For the latter, we use the map Y_γ or more specifically the map f_γ defined in (3.81). We then define

$$A_v(\psi) := (f_\gamma\psi)(\mathbb{I}), \tag{3.85}$$

where ψ is a $SU(2)$ spin network function and $\mathbb{I}$ represents the identity element in each link. It is important to notice that $(f_\gamma\psi)(\mathbb{I})$ is a linear functional on the space of $SU(2)$ spin networks and is called the *vertex amplitude* [19]. If we draw a small sphere surrounding a vertex in $\triangle^*$, then the intersection between this sphere and the simplicial complex is a graph Γ_v. The vertex amplitude corresponds to the transition amplitude of the boundary spin network states defined over the sphere using Γ_v as their boundary graph. In the group representation, the vertex amplitude can be written in the following way [34]:

$$A_v(h_l) = \int_{SL(2,\mathbb{C})} dg_n' \prod_l K\left(h_l, g_{x_i^l} g_{x_f^l}^{-1}\right), \tag{3.86}$$

where l and n are, respectively, the links nodes of Γ_v and K is given by

$$K(h,g) = \sum_j \int_{SU(2)} dk\ d_j^2\ \overline{\chi^j(hk)}\ \chi^{\gamma j,j}(kg), \tag{3.87}$$

where $h,k \in SU(2)$, $g \in SL(2,\mathbb{C})$ and $\chi^{p,k}(g) := \mathrm{Tr}\left[D^{pk}(g)\right]$ is the $SL(2,\mathbb{C})$ character in the (p,k) unitary representation. See [34].

The transition amplitude in the connection representation, for $SU(2)$ spin network states defined on Σ with boundary graph Γ is given by

$$W(U_l) = \int_{SU(2)} dU_{vf} \prod_f \delta(U_f) \prod_v A_v(U_{vf}), \tag{3.88}$$

where U_f is the holonomy of the connection around the face f and $U_{vf} = U_{lv}U_{l'v}^{-1}$, where l and l' are the two links originating from the vertex v and bounding the face f.

The transition amplitude can also be written in the form of a spin foam expansion. Such expression can be found in [34].

As mentioned in [34], the expression (3.88) defines truncations of the full transition amplitude. The full physical transition amplitude is formally given by

$$W(U_l) := \lim_{\triangle\to\infty} W_\triangle(U_l), \tag{3.89}$$

where the limit means that we take a refining sequence of simplicial decompositions such that we recover the continuous manifold when the simplicial complex is infinitely refined.

This completes the construction of the EPRL model for four-dimensional Lorentzian general relativity. According to [19], it can be proven that the theory is ultraviolet finite and is related to general relativity in the classical limit. By adding the cosmological constant, it is also infrared finite and leads to the n-point correlation functions for the gravitons of perturbative general relativity, to the Friedmann equation of cosmology, and to the correct expression for the Bekenstein-Hawking black hole entropy. As mentioned in [19], however, there are open questions in the theory, and maybe it needs to be adjusted in order to describe nature. Here we will mention some of the open questions concerning this model. A more detailed discussion about it can be found in [19] and [34].

Some of the open issues and conceptual problems about this model are:

- There is still no way to connect the EPRL model with canonical loop quantum gravity. There is still no Hamiltonian constraint defined in such a way that their transition amplitudes coincide with those calculated using the EPRL model.

- The theory is defined on 2-complexes and graphs dual to a simplicial complex.More general complexes, however, can be used, and these might be relevant. A generalization considering arbitrary 2-complexes not necessarily dual to a simplicial complex, where vertices and edges have an arbitrary valence, is presented in [54].

- The issue of the coupling to matter remains as an unexplored territory [6].
- If we define the map Y_γ by the equations $p = \gamma(j+1)$, $k = j$, it can be proven [19] that the condition (3.78) is satisfied exactly and not just in the limit of large j. We still do not know if there is something that favors one of the two theories over the other [19].
- It is still difficult to extract physically relevant results from the model. There are, however, some calculations involving black holes and cosmology [19].

4
A new discretization of four dimensional general relativity

The EPRL model is the most widely accepted spin foam model in the loop quantum gravity community. As emphasized in [6], the weak imposition of the linear geometric constraint allows correcting some issues in the Barrett-Crane model. The EPRL model has been used with some degree of success to study some aspects of the thermodynamics of black holes and in certain studies of quantum cosmology [6, 19].

The EPRL model can be thought of as resulting from a quantization of a discretization of classical general relativity. Such discretization considers general relativity as a field theory whose dynamics is given by the BF action plus geometric constraints applied to the B field. When the EPRL model is studied in the semiclassical regime, it is compared with Regge calculus.

In this fourth and final chapter, we introduce the main topic that concerns our work. Broadly speaking, we introduce a new way to discretize four dimensional general relativity. This new discretization has three main advantages. First, it allows distinguishing between boundary degrees of freedom and degrees of freedom in the bulk. Such a distinction is, in some sense, obscure in the classical discretization on which the EPRL model is based, even though the distinction is important in constructing the quantum spin foam model. Second, in our discretization the study of only one four-simplex already contains information about curvature. That does not happen in the standard EPRL model, where the four simplices are flat. Third, the model has as variables a gauge field and a variable that contains the information about the tetrad. This helps to establish a closer analogy with the classical continuum theory than that of the Regge discretizations, where the variables are directly related to the metric.

Our hope is that our construction allows the study of some aspects of the EPRL model that are currently difficult to attack. Particularly, some calculations involving the semiclassical limit of the new spin foam models have been performed in the EPRL model while only considering a single four-simplex and in situations where the curvature is small. The introduction of a new discretization to define a spin foam model could help to perform such computations considering more than one four-simplex and in situations where the curvature is arbitrary.

Unlike the previous chapters, the material presented here is the product of original research and calculations made in a collaboration between the author of this thesis and José A. Zapata, who actively supervised the project. The basic structures used here to discretize the underlying differentiable manifold were previously used by M. Reisenberger in [17]. In [17], Reisenberger discretized Euclidean general relativity formulated in terms of left-handed fields. In our exposition, we present a new model where we consider four-dimensional Lorentzian general relativity, with the introduction of a tetrad field. The existence of the tetrad field is a consequence of imposing the linear geometric constraint on the B-field in a suitable way, adapted to the special characteristics of the discretization.

The material in the present chapter is organized as follows. Section one presents an introduction about the recovering of the semiclassical limit in the EPRL model. We emphasize the importance of the research of the behavior of the transition amplitude in this regime and its connection with the calculation of correlation functions. In section two we define the discretization of the spacetime manifold in which the model is based and the dynamical variables associated with it. In section three we adapt the linear geometric constraint (3.60) to our discretization, and we will mention that such a constraint implies the existence of a discrete tetrad field associated with our model[9]. The action for the model is presented in section four, together with the dynamical equations and their interpretation. Section five presents an analysis of the boundary variables associated with the model. Finally, in section six we establish some possible avenues for future work.

Throughout this chapter, we will denote the Minkowski space $(\mathbb{R}^4, \eta)$ as $\mathcal{M}$.

4.1 Motivation

The EPRL model presented in the last chapter is currently the most accepted spin foam model in quantum gravity. Unlike the previous Barrett-Crane model, it imposes not the quadratic geometric constraint given in (3.56) but the linear geometric constraint (3.60). Moreover, it imposes such a constraint only weakly, in the limit of large quantum numbers. This produces a model with very good characteristics, which were described in Chapter 3. This does not mean that this new spin foam model is fully studied and is free of some defects. In particular in its physical applications, in spite of the good agreement with some previously known results, there are still open issues that need to be investigated. One of these issues is the semiclassical limit of the model and its connection to classical general relativity. Strongly related is the issue of computing correlation functions in the context of background independence.

In [56], Rovelli discussed the general framework for the definition of correlation functions in the context of background-independent theories. The application of such a framework to the Barrett-Crane model showed that two-point correlation functions did not yield the expected results in the semiclassical limit. This was one of the main motivations for the construction of the EPRL model[8] ,[16].

After the introduction of the EPRL model, the two-point correlation function was calculated in [57] by using the new vertex amplitude (3.85) and (3.86). Such a calculation showed a result that exactly matches that obtained from lorentzian Regge calculus in the limit $\gamma \to 0$. This is

[9] The proof of such an affirmation is given in Appendix B.

interpreted in [19] as an indication that the semiclassical limit of the EPRL model, defined by using the limits $\gamma \to 0$, $j \to \infty$ with γj = constant, gives back the usual linearized theory of gravity.

As emphasized in [6], however, all the calculations involved in arriving to such conclusions have been performed using a simplicial complex with a single four-simplex. Some computations have been performed involving more than one four-simplex for the case of the Barrett-Crane model in [58]. Certain peculiar properties were found, and it is not clear if this same issues remain in the EPRL model, or if other kind of peculiarities appear.

The lack of a clear definition between boundary and bulk degrees of freedom in the classical discretization is one of the reasons that makes the study of the semiclassical limit of the EPRL model difficult to perform for more that only one four-simplex. One motivation for our model is to try to correct this issue. We hope that the future quantization of this discrete model will produce a spin foam model that preserves characteristics of the EPRL model that make it particularly attractive. More specifically, we hope to be able to recover the $SU(2)$ spin networks states as boundary states of the theory. We also expect that the corresponding area and volume operator will match those defined in the canonical theory. This apparently new spin foam model tries to be quite similar to the EPRL model, or perhaps an alternative route for constructing the EPRL model itself, but with the advantage of allowing a better investigation of the semiclassical limit in cases involving a single four-simplex, because of the inclusion of curvature.

4.2 Discretization

The discrete model presented here uses two structures that were already defined in [17]. Their definitions can be given in any number of spacetime dimensions greater than one, but we will give them in the special case of four dimensions. To introduce such structures, we will consider a four-dimensional simplicial complex $\triangle$ and its dual triangulation, as were defined in Section 3.2.

In the rest of this work, simplices will be denoted using Greek letters. We will denote a four-simplex in $\triangle$ as $\nu < \triangle$. Triangles will be denoted as σ, whereas tetrahedra will be denoted as τ. The notation $\rho < \triangle$ will mean that ρ is a simplex of $\triangle$, and $\lambda < \rho$ will mean that λ is a subsimplex of ρ.

Definition 4.2.1. *Given a four-dimensional triangulation $\triangle$ and its corresponding dual triangulation $\triangle^*$, let us consider one four-simplex $\nu < \triangle$ and one triangle σ of that four-simplex. We define the wedge $s(\sigma\nu)$ associated with σ and ν as the intersection of the four-simplex ν and the two-cell of $\triangle^*$, which is dual to the triangle σ.*

Each wedge is associated with a four-simplex ν and a two-simplex σ of ν. It is a piece of plane formed by the barycenters of the triangle σ and the four-simplex ν, as well as the two tetrahedra τ_1, τ_2 of ν that share σ.

Now we will subdivide each four-simplex of $\triangle$ into new structures that are called *corner cells*; each of them is associated with a vertex of the simplex, as follows.

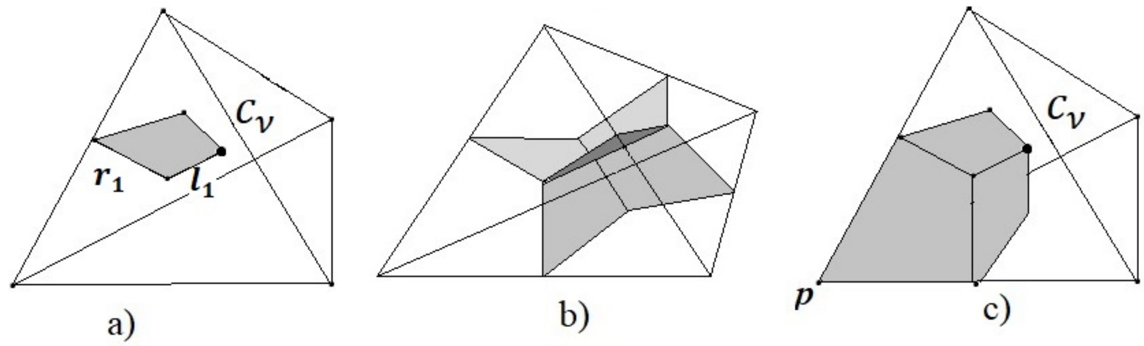

Figure 4.1: This figure shows the structures introduced in Definitions 4.2.1 and 4.2.2 in a three dimensional simplicial complex. Considering a tetrahedron, in a) it is shown the wedge corresponding to the line l and the tetrahedra ν. In b), it is shown all the wedges in ν associated with all the lines l in the tetrahedron. In c) the corner cell associated with the vertex p is shown.

Definition 4.2.2. *Let us consider a four-simplex ν in $\triangle$. Let p be a vertex in ν, and let us consider all the triangles σ in ν that have p as one of their vertices. The set of points in ν enclosed by the wedges dual to those triangles, the triangles themselves, and all the tetrahedra in ν that have p as one of their vertex will be called the corner cell c_p associated with p.*

As emphasized in [17], c_p is topologically a hypercube. It has one vertex in the interior of the four-simplex ν, which is the barycenter of ν. The other vertices of c_p are in the barycenter of the subsimplices of ν incident on p. It can be shown that the intersection of c_p with a subsimplex $\mu < \nu$ is the corner cell of p in μ. The new structures defined in 4.2.1 and 4.2.2 are illustrated in Fig.4.1 in the three-dimensional case.

The wedges and the corner cells defined above have some properties that are important to mention.

- Given a fixed four-simplex ν, the wedges $s(\sigma\nu)$ in ν are in one-to-one correspondence with the triangles σ of ν.
- The boundary of the wedge $s(\sigma\nu)$ can be constructed by drawing four lines. Two such lines travel from the barycenter of ν to the barycenters of the two tetrahedra τ_1 and τ_2 in ν that share the triangle σ. We will denote such lines as l_1 and l_2, respectively. One such line was indicated in panel a) of Fig.4.1. The other two lines travel from the barycenters of these tetrahedra to the barycenter of σ. These lines will be denoted by r_1 and r_2. As previously mentioned, one of these lines was indicated in panel a) of Fig.4.1. The lines l_1, l_2, r_1, r_2 will be called the edges of the wedge $s(\sigma\nu)$.
- If we consider a fixed triangle $\sigma < \triangle$, the union of all the wedges $s(\sigma\nu)$ for all the four simplices ν that share σ is the face in $\triangle^*$ dual to the triangle σ.
- A fixed corner cell in ν has six wedges. These wedges belong to the boundary of the corner cell. This means that a fixed corner cell intersects with the six triangles of the four-simplex ν.
- A fixed corner cell in ν intersects with four of the five tetrahedra in ν.

- Two different corner cells c_p and c_q in the same four-simplex share three different wedges that also belong to ν.

- Consider a fixed four-simplex $\nu < \triangle$ and a corner cell c_p in ν. Any two tetrahedra in ν that intersect c_p share a triangle with dual wedge that also belongs to c_p.

- Any two different corner cells c_p and c_q in ν share three different wedges. These wedges are dual to three triangles in ν. Each couple of these triangles belongs to the same tetrahedra.

All these properties will be very important, particularly in the appendices of this thesis.

Now we will define the variables of the model. To do this, we will consider a four-dimensional BF theory in the same way as in Section 3.4.1. The gauge group will be $SO(3,1)$. We discretize the differentiable manifold using a four-dimensional simplicial complex $\triangle$ and its dual $\triangle^*$, and we will also consider all the wedges and corner cells that can be constructed in $\triangle$. Let us consider a four-simplex $\nu < \triangle$ and a corner cell $c_p \in \nu$. If we take a wedge $s \in c_p$, we associate an element of the Lie algebra $so(3,1)$ to s. We will denote such an element as $B(s)$. This variable will represent the discretization of the B field.

The discretization of the connection will be given by its holonomy along the edges of each wedge $s(\sigma\nu)$. More specifically, we associate an element $h_l \in SO(3,1)$ to each edge $l(\nu\tau)$ that goes from the barycenter of the four-simplex ν to the barycentre of the tetrahedron τ. In addition, we associate an element $k_r \in SO(3,1)$ with each edge $r(\tau\sigma)$ that goes from the barycenter of the tetrahedron τ to the barycenter of the triangle σ. The elements h_l here correspond to those that were denoted as V_{vt} in[8], where v denotes a four-simplex and t is a tetrahedron in the boundary of v.

The discretization of the connection deserves some important comments. In the EPRL model introduced in Section 3.5, we considered a four-dimensional triangulation using as basis the Regge calculus. Therefore, the four-simplices were considered flat, and the information about the curvature was concentrated in the holonomy around the face $f \in \triangle^*$ dual to a triangle $\sigma \in \triangle$. This means that to obtain information about the curvature we need to consider more than one four-simplex. On the other hand, in the discretization that we are introducing here, a single four-simplex already has information about the curvature. This will be one of the facts that will help us to separate the bulk degrees of freedom from those in the boundary.

Regarding the last point, we associate to the boundary of each wedge an element $g_{\partial s} \in SO(3,1)$, defined as

$$g_{\partial s} := h_{l_2}^{-1} k_{r_2}^{-1} k_{r_1} h_{l_1}, \tag{4.1}$$

which is the holonomy of the continuous connection around the boundary $\partial s(\sigma\nu)$ of the wedge. Notice that the definition of $g_{\partial s}$ requires the definition of an orientation for the wedge s. As in [59], we will leave the orientation free and keep in mind that if we denote as $\bar{s}$ the same wedge but with opposite orientation, we will then have that $g_{\partial \bar{s}} = g_{\partial s}^{-1}$. The quantity defined in equation 4.1 represents the discretization of the curvature.

4.3 Constraint on the B field

The discrete field $B(s)$ is an independent variable in our model. It is an element of the Lie algebra $so(3,1)$ and can be considered an element attached to the barycenter of every four-simplex. Thus, in the barycenter of every four-simplex we have three spaces: a tangent space $T_{C_\nu}M$ that is isometric to the Minkowski space (denoted as $\mathcal{M}$), a copy of the Lie group $SO(3,1)$ and its corresponding Lie algebra $so(3,1)$[10].

The discrete field $B(s)$ represents approximately the discretization of the continuous field $B(x)$. As already mentioned in Subsection 3.4.2, in order to recover general relativity from a BF theory with $SO(3,1)$ as the gauge group we must impose a restriction over B to reduce the action to that of general relativity.

In Subsection 3.4.2, particularly in Theorem 3.4.1, we saw that a necessary and sufficient condition that can be imposed over the field B to recover general relativity in some cases is given by the geometric constraint (3.51). This constraint, however, has two different sectors of solutions. Moreover, in the EPRL model, it is not the quadratic constraint but the linear geometric constraint, equation (3.60), that is imposed. We repeat the statement that defines how this constraint is imposed in the discretization used in the that model:

- For each tetrahedron $\tau \in \triangle$, there exists a vector $n^I \in \mathbb{R}^4$ such that for all the faces $f \in \triangle^*$ that are dual to the triangles $t \in \tau$ we have

$$C_f^J := n_I({}^*B_f)^{IJ} = 0. \tag{4.2}$$

As was proven in [8], at the classical level the previous statement implies that for each tetrahedron τ there exists a tetrad $\{e_k(\tau)\}$ such that the quantities B_f associated with each face $f \in \triangle^*$ with a dual triangle that belongs to that tetrahedron are of the form $B_f = {}^*(e_{i(f)}(\tau) \wedge e_{j(f)}(\tau))$. In a second step, it is asked these five tetrads to be identified by internal parallel transport.

In the model we are constructing here, we need to impose a constraint over the B field again. It is clear that if we want to stay as close as possible to the standard EPRL model we need to impose the linear constraint. We can not impose it, however, using the same statement given above by the following reason.

If we impose equation (4.2) for every tetrahedron, as in the previous statement, we will again obtain a basis for the Minkowski space $\mathcal{M}$ that generates, through the wedge product, six elements of $so(3,1)$. The problem is that such elements are associated with wedges. In the discretization presented here, if we consider the entire discretization, we do not have a one-to-one correspondence between wedges and triangles. There will be more than six wedges associated with one triangle, and it will not possible to use just one tetrad to generate all of the $B(s(\sigma\nu))$ associated with a tetrahedron.

[10]In order to perform the calculation of the dynamical equations, we will consider only the connected component of the group.

To solve this problem, we will impose equation (4.2), but by using a statement that adapts it to the particular discretization that we are considering here. The whole statement will receive the name of *Adapted Linear Geometric Constraint*, and it will be given in the following way:

Adapted Linear Geometric Constraint(ALGC): Let $\nu < \triangle$ be a four simplex, c_p a corner cell in ν, τ a tetrahedron in ν that intersects c_p and $l(\nu\tau)$ the line that goes from the barycenter of ν to the barycenter of τ. There exists a vector $\vec{n}(\tau, c_p) \in \mathcal{M}$ different from zero, such that for all the wedges $s < c_p$ in c_p such that $l(\nu\tau) \in \partial s$, we have that

$$n(\tau, c_p)_I B(s)^{IJ} = 0.$$

This constraint intends to be analogous to such imposed in the standard EPRL model, in the sense that we expect that it will allows to recover the correct action for general relativity in the continuum limit.

Here it is important to mention that the term "linear" given to the restriction is given only to follow the convention. The restriction is actually not linear, and the sum of two solutions is not a solution. Despite this, we will follow the standard convention and we will call it linear.

We said that the linear geometric constraint, as imposed in the EPRL model, implies the existence of a tetrad for each tetrahedron and that these tetrads are identified by parallel transport. In the model that we present here the ALGC is not enough to uniquely define a tetrad. In order to fix this, we need to impose another condition over the quantities $B(s)$ associated with a corner cell. This additional constraint receives the name of the *four-volume* constraint, and is given by the following statement:

Four-volume constraint:
Given a four-simplex ν and a corner cell $c_p < \nu$ we have that

$$\mathrm{sgn}(s, s')\epsilon_{IJKL} B(s)^{IJ} \wedge B(s')^{KL} = \mathrm{sgn}(s'', s''')\epsilon_{IJKL} B(s'')^{IJ} \wedge B(s''')^{KL}, \tag{4.3}$$

for every couple of wedges (s, s') and (s'', s''') in c_p such that s and s' only share the point C_ν (the barycenter of ν), and the same for s'' and s'''.

The quantity $\mathrm{sgn}(s, s') := \mathrm{sgn}(\sigma, \sigma')$ is the sign of the oriented four-volume spanned by the two-simplices σ, σ' associated with s and s' in c_p. If σ and σ' share only one vertex, then the orientations of σ and σ' define an orientation for ν. If this orientation matches that already chosen for ν, then $\mathrm{sgn}(s, s') = +1$. If it is the opposite, $\mathrm{sgn}(s, s') = -1$. If σ and σ' share more than one point, then $\mathrm{sgn}(s, s') = 0$.

The name of the condition given above comes from the fact that the quantity:

$$V(\nu) := \mathrm{sgn}(s, s')\epsilon_{IJKL} B(s)^{IJ} \wedge B(s')^{KL}$$

is the four-volume of the four-simplex ν [8, 17]. In this way, the four-volume constraint essentially says that the four-volume, calculated using different couples of wedges with the characteristic that the two wedges of the couple share only one point, must be the same.

Together, the ALGC and the four-volume constraint imply the existence of a tetrad, but instead of a tetrad for each tetrahedron, we have a tetrad for each corner cell c_p in a four-simplex ν. This tetrad generates, through the wedge product, all the elements $B_{s(\sigma\nu)} \in so(3,1)$ such that $s < c_p$. Moreover, we do not obtain 20 vectors, but instead we have only five vectors associated with every four-simplex. This set of vectors, grouped into sets of four, are the five bases for each of the corner cells associated with the four-simplex. More specifically, the volume constraint and the ALGC imply the following two results.

Theorem 4.3.1. *Let $\nu < \triangle$ be a four-simplex in $\triangle$ and c_p a corner cell in ν. Consider that the six $B(s)$ associated with c_p are linearly independent. If the adapted linear geometric constraint is true and the four-volume constraint is satisfied, then for each corner cell $c_p < \nu$ there exist four linearly independent vectors $\vec{e}_{l0}(c_p), \ldots, \vec{e}_3(c_p) \in \mathcal{M}$ such that, for every $B(s) \in so(3,1)$ associated with a wedge s belonging to c_p we have that*

$$B(s) =^* \left(\vec{e}_{l_i(s)}(c_p) \wedge \vec{e}_{l_j(s)}(c_p)\right),$$

where $\partial s = l_i + r_i - r_j - l_j$.

Theorem 4.3.2. *Let $\nu < \triangle$ be a fixed four simplex. If the conditions in theorem 4.3.1 hold, then, for each four-simplex $\nu < \triangle$ there exist five vectors $\vec{e}_0, \ldots, \vec{e}_4 \in \mathcal{M}$ such that the five sets that can be constructed using four of such vectors, with all of them different, are the basis associated with each corner cell mentioned in theorem 4.3.1.*

These two affirmations are proven in Appendix B.

4.3.1 The tetrad for each corner cell

The five vectors associated with each four-simplex and the five bases that they produce deserve some comments.

We have said that there are five vectors, $\vec{e}_1, \ldots, \vec{e}_5$, associated with each four-simplex. These vectors are elements of a copy of $\mathcal{M}$ attached at the barycenter of the four-simplex. It can be seen that inside each four-simplex there are five lines starting at C_ν and finishing at the barycenters of the five tetrahedra in the boundary of ν. Those are the lines that we denoted as l_k. Thus, we can associate each of the five vectors $\vec{e}_k$ to one of the lines l_k inside the four-simplex. Considering this, given one four-simplex $\nu < \triangle$, we will write $\vec{e}_{l_1}, \ldots, \vec{e}_{l_5}$ to denote the set of five vectors associated with ν. The base associated with a corner cell c_p in ν is the set $\{\vec{e}_{l_i}, \vec{e}_{l_j}, \vec{e}_{l_k}, \vec{e}_{l_m}\}$, where $l_i, \ldots, l_m$ are the four lines l that belong to c_p.

The elements $B(s)$ that are a solution of the ALGC and the four-volume constraint are given in the form

$$B(s) =^* \left(\vec{e}_{l_i(s)} \wedge \vec{e}_{l_j(s)}\right) \tag{4.4}$$

where $\partial s = l_i + s_i - s_j - l_j$. In words, the element $B(s)$ is the Hodge dual of the wedge product of the two vectors associated with the lines l_i, l_j in the boundary of s.

There exists a way to understand the connection between the continuous and the discretized tetrads. In the continuum theory, we have a tetrad field $e_I(x)$ and its corresponding cotetrad field $e^I(x)$. The cotetrad field is a set of four one-forms valued in the real numbers. One one-form per value of the index I. These four one-forms can be joined to form a single one-form valued in the Minkowski space $\mathcal{M}$. In this way, we can say that the cotetrad field is a one-form with values in the Minkowski space, as defined in [1]. If we choose a basis $\{\vec{v}_I\}_{I=0}^3$ for $\mathcal{M}$, we can write

$$e = \vec{v}_I e^I_\mu dx^\mu.$$

In the discrete theory presented here, we have five elements of the Minkowski space associated with each four-simplex. We can consider that to obtain such vectors we first take the continuous cotetrad field e, and we evaluate it in the point of M (the spacetime manifold) correspoding to the barycentre of the four-simplex. Thus, we obtain a linear function

$$e(C_\nu) : T_{C_\nu}M \to \mathbb{R}^4.$$

This function, evaluated in the five vectors $\vec{v}_{l_k}$ associated with the lines l_k, give the five elements $\vec{e}_k$.

We can compare our tetrad with the tetrads defined in the standard EPRL model. In the EPRL model, there is a tetrad associated with the barycenter of each tetrahedron. In a given four-simplex, two of such tetrads are related through the parallel transport of the connection along the lines that join the barycenters of such tetrahedra and the barycenter of the four-simplex. Thus, we obtain a matrix belonging to the gauge group ($SO(4)$ in the Euclidean theory and $SL(2,\mathbb{C})$ in the Lorentzian theory). In the model considered here, we have five vectors $\vec{e}_k$ attached to the barycentre of every four-simplex. The point to which each tetrad is attached is different. Here we do not associate a basis to each tetrahedron of a four-simplex.

4.4 Dynamics

4.4.1 Discrete action

The dynamics of the model is given by a discrete action. There are, however, two options that we can consider here. On one hand, we can write the discrete action in terms of the variables $B(s)$, h_l and k_r. Thus, we define our model as given by such a dynamical action but with the variables $B(s)$ constrained by the adapted linear geometric constraint. On the other hand, we can start considering, from the beginning the existence of the set of five vectors $e_{l_1}(\nu), \ldots, e_{l_5}(\nu)$ for each four-simplex, as well as considering the action as a function of the variables e_{l_i}, h_l and k_r. The first would be a discrete BF theory with a restriction over B. The second intends to be a direct discretization of the Holst action for general relativity. Theorems 4.3.1 and 4.3.2 ensure that these two models are indeed equivalent.

Here we choose the second option. We write the action in terms of $B(s)$, h_l and k_r but considering that $B(s)$ is given by equation (4.4). The action for the model is given by the following expression:

$$S_{\triangle} = \sum_{\nu < \triangle} \left[\sum_{c_p \in \nu} \left(\sum_{s, s' \in c_p} \mathrm{sgn}(s, s') \mathrm{tr} \left(B(s') g_{\partial s} + \frac{1}{\gamma} \, {}^{*}B(s') g_{\partial s} \right) \right) \right], \tag{4.5}$$

where $\gamma \in \mathbb{R}$.

If the region of spacetime considered has a three-dimensional boundary, the addition of boundary terms to the action must be considered in some cases. If the connection is kept fixed at the boundary, no boundary terms are necessary. For a complete discussion, see [17].

It is important to add some comments.

a) The action given in (4.5) can be rewritten as

$$S_{\triangle} = \sum_{\nu < \triangle} \left[\sum_{s, s' < \nu} \left(\sum_{\substack{c_p \in \nu \\ s, s' \in c_p}} \mathrm{sgn}(s, s') \mathrm{tr} \left(B(s') g_{\partial s} + \frac{1}{\gamma} \, {}^{*}B(s') g_{\partial s} \right) \right) \right]. \tag{4.6}$$

The sum $\sum_{\substack{c_p \in \nu \\ s, s' \in c_p}}$ is interpreted as the sum over all the corner cells in ν such that the wedges s, s' belong to c_p.

b) We can isolate the action that corresponds to a single corner cell, which is given by

$$S_{c_p} := \sum_{s, s' \in c_p} \mathrm{sgn}(s, s') \mathrm{tr} \left(B(s') g_{\partial s} + \frac{1}{\gamma} \, {}^{*}B(s') g_{\partial s} \right). \tag{4.7}$$

In the same way, we can easily isolate the action of a single four-simplex ν, which is given as the sum of the actions of every corner cell in ν:

$$S_{\nu} := \sum_{c_p < \nu} S_{c_p}. \tag{4.8}$$

The action of our model is similar to the action considered in [17]. There are some differences that are worth mentioning, however.

The action of our model allows isolating the action corresponding to a single corner cell.

In [17], the gauge group is $SU(2)$ because the discretization intends to be a discrete model for a version of Euclidean general relativity where the dynamical fields can be represented entirely by left-handed fields. In our model, we consider $SO(3,1)$ as the group defining the theory. This means that in our case we are considering the construction of a discrete model for Lorentzian general relativity.

In addition, in the action considered here we have introduced an extra term given by

$$\sum_{\nu < \triangle} \left[\sum_{c_p < \nu} \left(\sum_{s, s' \in c_p} \frac{1}{\gamma} \mathrm{sgn}(s, s') \mathrm{tr} \left({}^{*}B(s') g_{\partial s} \right) \right) \right].$$

This tries to be a discrete version of the Holst term introduced in Subsection 3.4.2, specifically in equation (3.47). We hope that the addition of this term allows establishing a closer contact with canonical loop quantum gravity, once the model is quantized.

4.4.2 Field equations

To define the dynamics of the model, we will calculate the dynamical equations by taking the variations of the action with respect to the independent variables.

To calculate the variation of the action, we will denote as $\{T_i\}_{i=1}^{6}$ the six generators of the Lie algebra $so(3,1)$. The elements $\{T_i\}_{i=1}^{3}$ will be the generators of rotations and $\{T_i\}_{i=4}^{6}$ the generators of boosts.

First, let us consider the variation of the action with respect to the variable h_l. Let us consider a single line l_1 in $\triangle$. This choice singles out one single four-simplex $\nu < \triangle$. Thus, we can consider the action of only one four-simplex ν such that $l_1 \in \nu$. We parametrize the variation of h_{l_1} as $\tilde{h}_{l_1} := h_{l_1}\exp(\lambda\alpha^i_{l_1}T_i)$, where $\lambda \in \mathbb{R}$, and we consider a sum over i. Thus, this variation on h_{l_1} induces

$$\tilde{g}_{\partial s}(\lambda) := h^{-1}_{l_2}k^{-1}_{r_2}k_{r_1}\tilde{h}_{l_1}(\lambda) = g_{\partial s}\exp(\lambda\alpha^i_{l_1}T_i). \tag{4.9}$$

By substituting this quantity in the action, we obtain

$$\begin{aligned}\tilde{S}_\nu(\lambda) := S_\nu(\tilde{g}_{\partial s}(\lambda)) = &\sum_{\substack{c_p\in\nu\\ l_1\notin c_p}}\left[\sum_{s,s'\in c_p}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')g_{\partial s} + \frac{1}{\gamma}\,{}^*B(s')g_{\partial s}\right)\right] + \\ &+\sum_{\substack{c_p\in\nu\\ l_1\in c_p}}\left[\sum_{\substack{s,s'\in c_p\\ l_1\notin s}}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')g_{\partial s} + \frac{1}{\gamma}\,{}^*B(s')g_{\partial s}\right)\right] + \\ &+\sum_{\substack{c_p\in\nu\\ l_1\in c_p}}\left[\sum_{\substack{s,s'\in c_p\\ l_1\in s}}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')g_{\partial s}\exp(\lambda\alpha^i_{l_1}T_i) + \frac{1}{\gamma}\,{}^*B(s')g_{\partial s}\exp(\lambda\alpha^i_{l_1}T_i)\right)\right].\end{aligned}$$

Let us define the variation of the action, denoted as $\delta S_\triangle$, as

$$\delta S_\triangle := \left.\frac{d}{d\lambda}\right|_{\lambda=0}\tilde{S}_\nu(\lambda). \tag{4.10}$$

If we consider that $l(\nu,\tau) \in \partial s(\sigma\nu)$ if and only if $\sigma < \tau$, we obtain that the condition $\delta S_\triangle = 0$ implies that

$$\sum_{\substack{c_p\in\nu\\ \tau\cap c_p\neq\emptyset}}\left[\sum_{\substack{s(\sigma\nu),s'\in c_p\\ \sigma<\tau}}\tilde{w}(s(\sigma\nu),s')_i\right] = 0, \tag{4.11}$$

where

$$\tilde{w}_i(s(\sigma\nu),s') := \mathrm{sgn}(s,s')\mathrm{tr}\left[T_i\left(B(s') + \frac{1}{\gamma}\,{}^*B(s')\right)g_{\partial s}\right]. \tag{4.12}$$

The variation with respect to the variable k_r proceeds in a similar way. Let us consider only one line $r_1(\tau,\sigma)$. Notice that this line belongs to the boundary of two wedges $s_1(\sigma\nu_1)$ and $s_2(\sigma\nu_2)$, where ν_1 and ν_2 are the two four-simplices that share the tetrahedron τ. Let us define $\tilde{k}_{r_1}(\lambda) := \exp(\lambda\alpha^i_{l_1}T_i)k_{r_1}$. Considering that $\tilde{g}_{\partial s}(\lambda) := g_{\partial s}(\tilde{k}_{r_1})$, the substitution of $\tilde{k}_{r_1}$ in the expression for the action gives

$$\tilde{S}_\triangle(\lambda) = \sum_{\substack{\nu<\triangle\\ r_1\notin\nu}}\left[\sum_{c_p\in\nu}\left(\sum_{s,s'\in c_p}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')g_{\partial s}+\frac{1}{\gamma}\,{}^*B(s')g_{\partial s}\right)\right)\right]+$$
$$+\sum_{\substack{\nu<\triangle\\ r_1\in\nu}}\left[\sum_{\substack{c_p\in\nu\\ r_1\notin c_p}}\left(\sum_{s,s'\in c_p}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')g_{\partial s}+\frac{1}{\gamma}\,{}^*B(s')g_{\partial s}\right)\right)\right]+$$
$$+\sum_{\substack{\nu<\triangle\\ r_1\in\nu}}\left[\sum_{\substack{c_p\in\nu\\ r_1\in c_p}}\left(\sum_{\substack{s,s'\in c_p\\ r_1\notin\partial s}}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')g_{\partial s}+\frac{1}{\gamma}\,{}^*B(s')g_{\partial s}\right)\right)\right]+$$
$$+\sum_{\substack{\nu<\triangle\\ r_1\in\nu}}\left[\sum_{\substack{c_p\in\nu\\ r_1\in c_p}}\left(\sum_{\substack{s,s'\in c_p\\ r_1\in\partial s}}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')\tilde{g}_{\partial s}(\lambda)+\frac{1}{\gamma}\,{}^*B(s')\tilde{g}_{\partial s}(\lambda)\right)\right)\right].$$

Notice that we have $\tilde{g}_{\partial s}(\lambda) = h^{-1}_{l_2}k^{-1}_{r_2}\exp(\lambda\alpha^i_{l_1}T_i)k_{r_1}h_{l_1}$. and

$$\frac{d}{d\lambda}\tilde{g}_{\partial s}(\lambda) = \alpha^i_{l_1}h^{-1}_{l_2}k^{-1}_{r_2}T_i\exp(\lambda\alpha^i_{l_1}T_i)k_{r_1}h_{l_1}.$$

Now we need to take into account that $r_1(\tau,\sigma)$ belongs to the boundary of two wedges $s_1(\sigma\nu_1)$ and $s_2(\sigma\nu_2)$, where ν_1 and ν_2 are the two four-simplices that share the tetrahedron τ. Considering this and defining $\delta S_\triangle$ as in (4.10), we obtain

$$\delta S_\triangle = \sum_{\substack{\nu<\triangle\\ r_1\in\nu}}\left[\sum_{\substack{c_p\in\nu\\ r_1\in c_p}}\left(\sum_{\substack{s,s'\in c_p\\ r_1\in\partial s}}\mathrm{sgn}(s,s')\mathrm{tr}\left(B(s')\alpha^i_{l_1}h^{-1}_{l_2}k^{-1}_{r_2}T_ik_{r_1}h_{l_1}+\frac{1}{\gamma}\,{}^*B(s')\alpha^i_{l_1}h^{-1}_{l_2}k^{-1}_{r_2}T_ik_{r_1}h_{l_1}\right)\right)\right].$$

Let us define $\hat{\alpha}^i_{r_1} := h^{-1}_{l_1}k^{-1}_{r_1}\alpha^i_{r_1}k_{r_1}h_{l_1}$. Therefore, we have that $\alpha^i_{l_1}T_i = k_{r_1}h_{l_1}\hat{\alpha}^i_{r_1}T_ih^{-1}_{l_1}k^{-1}_{r_1}$ and $B(s')\alpha^i_{l_1}h^{-1}_{l_2}k^{-1}_{r_2}T_ik_{r_1}h_{l_1} = B(s')g_{\partial s}\hat{\alpha}^i_{r_1}T_i$, where we have a sum over i. By using this, we obtain

$$\mathrm{tr}\left(B(s')\alpha^i_{l_1}h^{-1}_{l_2}k^{-1}_{r_2}T_ik_{r_1}h_{l_1}+\frac{1}{\gamma}\,{}^*B(s')\alpha^i_{l_1}h^{-1}_{l_2}k^{-1}_{r_2}T_ik_{r_1}h_{l_1}\right) = \alpha^i_{r_1}\tilde{u}(s(\sigma\nu),s')_i,$$

with

$$\tilde{u}_i(s(\sigma\nu),s') := h^{-1}_{l_1}k^{-1}_{r_1}\tilde{w}_i(s(\sigma\nu),s')k_{r_1}h_{l_1}. \tag{4.13}$$

By using all of the above, the condition $\delta S_\triangle = 0$ implies that

$$\sum_{\substack{\nu<\triangle\\ r\in\nu}}\left[\sum_{\substack{c_p\in\nu\\ r\in c_p}}\left(\sum_{\substack{s,s'\in c_p\\ r\in\partial s}}\tilde{u}_i(s(\sigma\nu),s')\right)\right]=0. \tag{4.12a}$$

In order to rewrite this equation, it is important to consider the following two facts.

- Given a line r, there exist two four-simplices ν_1, $\nu_2 < \triangle$ such that $r\in\nu_1,\nu_2$.
- Given a line r and a four-simplex $\nu<\triangle$ containing that line, there exists only one wedge $s\in\nu$ such that $r\in\partial s$. This wedge belongs to three different corner cells in ν that we will call $c_{p_1}, c_{p_2}, c_{p_3}$.

Let us define

$$\tilde{u}_i(s(\sigma\nu),c_p):=\sum_{s'\in c_p}\tilde{u}_i(s(\sigma\nu),s'),$$

and

$$u_i(\sigma\nu):=\tilde{u}_i(s(\sigma\nu),c_{p_1})+\tilde{u}_i(s(\sigma\nu),c_{p_2})+\tilde{u}_i(s(\sigma\nu),c_{p_3}). \tag{4.12b}$$

By considering the two facts previously mentioned, as well as the definition (4.12b), equation (4.12a) can be written in the following way:

$$u_i(\sigma\nu_1)=u_i(\sigma\nu_2). \tag{4.14}$$

The last dynamical equation can be obtained by taking variations of the action with respect to the variable $\vec{e}_k$. In order to calculate such a variation, we need to write $B(s)$ explicitly as given in equation (4.4). Let us consider just one vector e that we will denote as $\vec{e}_{l_1(s,c_p)}$. This choice singles out only one four-simplex $\nu<\triangle$, only one corner cell c_p in ν, and only one wedge $s\in c_p$. The action of the corresponding four-simplex can be written in the form

$$S_\nu=\sum_{c_p\in\nu}\left[\sum_{s,s'\in c_p}\left(\mathrm{sgn}(s,s')\mathrm{tr}\left({}^*\left(\vec{e}_{l_i(s')}\wedge\vec{e}_{l_j(s')}\right)g_{\partial s}-\frac{1}{\gamma}\left(\vec{e}_{l_i(s')}\wedge\vec{e}_{l_j(s')}\right)g_{\partial s}\right)\right)\right].$$

In the last expression, we used that ${}^*\left({}^*\left(\vec{e}_{l_i}\wedge\vec{e}_{l_j}\right)\right)=-\vec{e}_{l_i}\wedge\vec{e}_{l_j}$.

The elements $\vec{e}$ belong to a vector space. This means that we can parametrize the variation of such quantities in the form $\tilde{e}_l=\vec{e}_l+\lambda\vec{v}_l$, where $\lambda\in\mathbb{R}$ and $\vec{v}_l$ is an element of $\mathcal{M}$ different from zero. Notice that our choice of taking the variation with respect to $\vec{e}_{l_1(s)}$ also singles out one line $l_1(\nu,\tau)$, but $l_1(\nu,\tau)\in c_p$ if and only if $\tau\cap c_p\neq\emptyset$. By substituting $\vec{e}_{l_1}$ by $\tilde{e}_{l_i}$ and by defining $\delta S_\triangle$ as in equation (4.10), we obtain

$$\delta S_\triangle=-\sum_{\substack{c_p\in\nu\\ \tau\cap c_p\neq\emptyset}}\left[\sum_{\substack{s,s'\in c_p\\ l_b\in\partial s'}}\left(\mathrm{sgn}(s,s')\mathrm{tr}\left({}^*\left(\vec{e}_{l_b(s')}\wedge\vec{v}_{l_1(s')}\right)g_{\partial s}-\frac{1}{\gamma}\left(\vec{e}_{l_b(s')}\wedge\vec{v}_{l_1(s')}\right)g_{\partial s}\right)\right)\right],$$

where $\sum_{\substack{c_p\in\nu \\ \tau\cap c_p\neq\emptyset}}$ indicates that we are considering the sum over all the corner cell c_p in ν such that $\tau\cap c_p\neq\emptyset$. We can expand the definition of the trace of a matrix to rewrite the last expression. By doing this and by imposing $\delta S_\triangle=0$, we obtain that the sought field equation is

$$\sum_{\substack{c_p\in\nu \\ \tau\cap c_p\neq\emptyset}}\left[\sum_{\substack{s,s'\in c_p \\ l\in\partial s'}}\operatorname{sgn}(s,s')\left(\epsilon^{I}{}_{JKL}e^{K}_{l(s')}\left(g_{\partial s}\right)^{J}{}_{I}-\frac{1}{\gamma}e^{I}_{l(s')}\left(\eta_{LK}\left(g_{\partial s}\right)^{K}{}_{I}-\eta_{KI}\left(g_{\partial s}\right)^{K}{}_{L}\right)\right)\right]=0. \quad (4.15)$$

In this last expression, η_{LK} are the components of the Minkowski metric η, $\left(g_{\partial s}\right)^{K}{}_{I}$ are the components of $g_{\partial s}$ as a matrix of $SO(3,1)$ in the fundamental representation, and e^K_l is the k-component of the vector $\vec{e}_l$.

If $\vec{e}_{l_1}\neq 0$, equation (4.15) is equivalent to the following expression:

$$\sum_{\substack{c_p\in\nu \\ \tau\cap c_p\neq\emptyset}}\left[\sum_{s\in c_p}\operatorname{sgn}(s,s')\operatorname{tr}\left(B(s')g_{\partial s}+\frac{1}{\gamma}\,{}^{*}B(s')g_{\partial s}\right)\right]=0, \quad (4.16)$$

which makes no reference to any particular representation for the elements $B(s)$ and $g_{\partial s}$.

In summary, we have obtained three dynamical equations. They correspond to the variation of the action $S_\triangle$ with respect to the independent variables h_l, k_r and $\vec{e}_l$. They can be written respectively as

$$\sum_{\substack{c_p\in\nu \\ c_p\cap\tau\neq\emptyset}}\left[\sum_{\substack{s(\sigma\nu),s'\in c_p \\ \sigma<\tau}}\operatorname{sgn}(s,s')\operatorname{tr}\left(T_iB(s')g_{\partial s}+\frac{1}{\gamma}T_i\,{}^{*}B(s')g_{\partial s}\right)\right]=0, \quad (4.17)$$

$$u_i(\sigma\nu_1)=u_i(\sigma\nu_2), \quad (4.18)$$

with $u_i(\sigma\nu):=\tilde{u}_i(s(\sigma\nu),c_{p_1})+\tilde{u}_i(s(\sigma\nu),c_{p_2})+\tilde{u}_i(s(\sigma\nu),c_{p_3})$, $\tilde{u}_i(s(\sigma\nu),c_p):=\sum_{s'\in c_p}\tilde{u}_i(s(\sigma\nu),s')$ and $\tilde{u}_i(s(\sigma\nu),s'):=h_{l_1}^{-1}k_{r_1}^{-1}\tilde{w}_i(s(\sigma\nu),s')k_{r_1}h_{l_1}$, and

$$\sum_{\substack{c_p\in\nu \\ \tau\cap c_p\neq\emptyset}}\left[\sum_{s\in c_p}\operatorname{sgn}(s,s')\operatorname{tr}\left(B(s')g_{\partial s}+\frac{1}{\gamma}\,{}^{*}B(s')g_{\partial s}\right)\right]=0. \quad (4.19)$$

These equations must be supplemented with the requirement of $B(s)$ given by expression (4.4).

Equations (4.17), (4.18) and (4.19) define the dynamics of our model.

4.4.3 Physical interpretation

The physical interpretation of the dynamical equations presented above is an important topic.

Let us start with the equation generated through the variation of the variables h_l: equation (4.17). The expressions (4.11) and (4.17) are obviously two equivalent forms writing the same expression.

The quantity $\tilde{w}_i(s,s')$ given in equation (4.12) represents an element of the Lie algebra $so(3,1)$ attached to the barycenter of every four-simplex. It depends on two wedges s and s'. Moreover, equation (4.11) can be rewritten in the following way:

$$\sum_{\substack{s(\sigma\nu)\in\nu \\ \sigma<\tau}} \left[\sum_{\substack{c_p\in\nu \\ s\in c_p \\ \tau\cap c_p\neq\emptyset}} \left(\sum_{s'\in c_p} \tilde{w}_i(s,s') \right) \right] = 0. \tag{4.20}$$

We can define another quantity $w_i(s(\sigma\nu))$ as

$$w_i(s(\sigma\nu)) := \sum_{\substack{c_p\in\nu \\ s\in c_p \\ \tau\cap c_p\neq\emptyset}} \left(\sum_{s'\in c_p} \tilde{w}_i(s,s') \right).$$

Thus, $w_i(s)$ is again an element of the Lie algebra but that depends only on the wedges s. Therefore, equation (4.20) can be written as

$$\sum_{\substack{s(\sigma\nu) \\ \sigma<\tau}} w_i(s) = 0 \tag{4.21}$$

This expression says that the sum of the $w_i(s)$ associated with the wedges that are dual to the triangles of a tetrahedron is zero. This condition is similar to the closure constraint found in the standard discretized models, such as those in [8] and [16]. The main difference is that each $w_i(s)$ is a sum of contributions of every corner cell c_p in ν. Each quantity $\sum_{s'\in c_p} \tilde{w}_i(s,s')$ represents the contribution of a single corner cell in ν. The total contribution of the four-simplex ν is the sum

$$\sum_{\substack{c_p\in\nu \\ s\in c_p \\ \tau\cap c_p\neq\emptyset}} \left(\sum_{s'\in c_p} \tilde{w}_i(s,s') \right)$$

of the contributions of every corner cell in ν.

Let us define

$$J(s,s') := \operatorname{sgn}(s,s') \left[B(s') + \frac{1}{\gamma}\, {}^{*}B(s') \right]. \tag{4.22}$$

In this way, $w_i(s,s')$ are essentially, the components of $J(s,s')$ in the basis determined by the generators T_i but multiplied by a factor that goes to one as $g_{\partial s}$ approaches the identity $\mathbb{I}$.

Equation (4.17) represents the discretization of the equation of the continuous theory that gives the condition of zero torsion: equation (4.17) is the discretized version of the equation

$$\epsilon_{KLIJ}T^K \wedge e^L + \frac{1}{\gamma}T_I \wedge e_J = 0, \tag{4.23}$$

where $T^I := de^I + \omega^I{}_J \wedge e^J$ is the torsion two-form of the continuous theory. Equation (4.23) is obtained from the variation of the Holst action (equation (3.47)) with respect to the connection ω. In the theory in the continuum, equation (4.20) implies that $T^I = 0$, that is, torsion is zero in the vacuum. On the other hand, its corresponding discretized version (4.17) does not allow, at least in principle, isolating a quantity that permits to be interpreted as the discretized version of the torsion T^I so that such a quantity is also annulled in the discrete setting.

This result is not something disturbing. After all, as is strongly emphasized in the spin foam framework, the discretization of the theory and the introduction of a triangulation of the space-time manifold are only auxiliary structures. For example in [6], it is mentioned that the cellular decomposition $\triangle$ has no physical meaning. $\triangle$ is only a subsidiary regulating structure that has to be removed when computing physical quantities. Thus, the important thing is to be able to recover equation (4.23) in the continuum limit of the classical theory.

There exist, however, some special cases in which we can define a quantity that resembles something similar to torsion in the continuum theory. Let

$$g_{\tau,i} := \pm\frac{1}{2}\sum_{\substack{c_p\in\nu \\ c_p\cap\tau\neq\emptyset}}\left[\sum_{\substack{s(\sigma\nu),s'\in c_p \\ \sigma<\tau}} \mathrm{sgn}(s,s')\mathrm{tr}\left(T_i\, {}^*B(s')g_{\partial s}\right)\right],$$

where we take the sign minus when $i = 1,2,3$, and the positive sign if $i = 4,5,6$. Now let us define

$$\mathcal{G}_\tau := \sum_{i=1}^{6} T_i g_{\tau,i}.$$

When the simplices are flat or the holonomy around the boundary of the wedges is close to the identity, equation (4.17) can be written in the following way:

$$\mathcal{G}_\tau + \frac{1}{\gamma}\,{}^*\mathcal{G}_\tau = 0. \tag{4.24}$$

This expression implies that $\mathcal{G}_\tau = 0$. Notice that $\mathcal{G}_\tau$ is an element of the Lie algebra $so(3,1)$ associated with each tetrahedron $\tau < \triangle$. In the theory in the continuum, the quantity $T^I \wedge e^J$ is a three-form with values in the lie algebra $so(3,1)$. This suggests that when the simplices are flat or when the curvature is small the quantity $\mathcal{G}_\tau$ represents a discretized version of $T^I \wedge e^J$.

Notice that in the case in which $\gamma \to \infty$ equation (4.17) directly implies that $\mathcal{G}_\tau = 0$. This case corresponds to a discrete model of the Palatini action, where the Holst term is zero.

Now let us discuss the meaning of equation(4.18). This equation is in fact a constraint over

$u_i(\sigma\nu)$, and tells that $u_i(\sigma\nu)$ does not depend on the four-simplex ν but only on the triangle σ.

The quantity $\tilde{u}_i(s(\sigma\nu), s')$ is the variable $\tilde{w}_i(s, s')$ parallel transported along a couple of lines l and r in ∂s, from the barycenter of the four-simplex ν to the barycenter of one of the triangles σ in the boundary of ν. Thus, $\tilde{u}_i(s(\sigma\nu), s')$ is an element of the Lie algebra $so(3,1)$ attached to the barycenter of one triangle $\sigma < \nu$. Moreover, as emphasized in [17], $\tilde{u}_i(s(\sigma\nu), s')$ can be seen, like $\tilde{w}_i(s, s')$, essentially as the components of the variable $J(s(\sigma\nu), s')$ but multiplied by a factor which goes to one as the group elements h_l and k_r approach the identity $\mathbb{I}$. In this way, when the simplices are flat, the quantities $\tilde{w}_i(s, s')$, $\tilde{u}_i(s(\sigma\nu), s')$ both define the components of $J(s, s')$ in the basis given by T_i. The only difference is that J and $\tilde{w}_i$ are attached to the barycenter of ν, whereas $\tilde{u}_i$ is attached to the barycentre of one triangle in the boundary of ν.

Now let us analyze equation (4.19). Such an expression is produced by taking the variation of the action with respect to the tetrad $\vec{e}_l$ associated with each four-simplex ν.

In the continuous theory, the dynamical equation that is produced when taking the variation of the Holst action with respect to the tetrad field is given by

$$\epsilon_{IJKL} e^I \wedge F^{KL} + \frac{1}{\gamma} e^I \wedge F_{IJ} = 0, \tag{4.25}$$

where F^{KL} is the curvature two-form. The first term is in fact the Einstein's field equation. The term proportional to $1/\gamma$ cancels on-shell because of (4.20).

Therefore, equation (4.19) together with the condition given in (4.4) is the discretized version of (4.25).

It is important to notice that as with equation (4.17) (4.19) has an additional term proportional to $1/\gamma$ that is different from zero. In the discrete setting, none of the other dynamical equations leads to the cancellation of this second term, as occurs in the continuum. As with what happens with equation (4.17), we expect that the term proportional to $1/\gamma$ is canceled when we take the continuum limit and we take into account the rest of the dynamical equations. From the expression (4.15), however, we can see that such a second term is canceled in the special cases in which the curvature of the wedges is small, when the four-simplices are flat or when the constant γ becomes large.

4.5 Boundary variables

The boundary variables of the theory are of fundamental importance in the quantization of the model. We will analyze them in more detail.

The three-dimensional boundary of $\triangle$ is discretized by a three dimensional cellular decomposition. That boundary cellular decomposition is formed by tetrahedra, triangles and points, and their corresponding dual structures. Some wedges in the bulk intersect the boundary through their lines $r(\tau, \sigma)$. Thus, for every boundary tetrahedron, we have four lines $r(\tau, \sigma)$ corresponding to the four triangles in the boundary of the tetrahedron. In addition, every tetrahedron in the boundary belongs to only one four-simplex in the bulk, and every triangle in the boundary belongs to only two boundary tetrahedra. All this means that the dual of the boundary triangulation forms a boundary graph $\Gamma_{\partial\triangle}$ with four-valent nodes. The nodes are located in the barycenter of the boundary tetrahedra, and the links are the lines $r(\tau\sigma)$ that come from the intersection of a wedge in the bulk and the boundary.

The boundary variables are given by $k_{r(\tau,\sigma)} \in SO(3,1)$ and $u_i(\sigma)$. Such variables are associated with the elements of the boundary graph $\Gamma_{\partial\triangle}$. More specifically, both the quantities $k_{r(\tau,\sigma)}$ and $u_i(\sigma)$ are associated with the links of $\Gamma_{\partial\triangle}$. Note that for a given link in $\Gamma_{\partial\triangle}$ connecting two nodes there is one element $u_i(\sigma)$ and two elements $k_{r_1(\tau_1,\sigma)}$, $k_{r_2(\tau_2,\sigma)}$ associated with it. These last two correspond to the two tetrahedra τ_1 and τ_2 that share the triangle σ.

All of the above means that we have a discrete theory with a phase space that is of a discretized $SO(3,1)$ Yang-Mills theory. The configuration space is $\mathcal{C} = SO(3,1)^L$, where L is the number of links of the boundary graph $\Gamma_{\partial\triangle}$. The corresponding phase space is the cotangent bundle $[T^*SO(3,1)]^L$, with $(k_{r_1}, k_{r_2}, u_i(\sigma))$ a point in such a space.

It is important to notice that in our model there is a clear distinction between boundary and bulk degrees of freedom. Moreover, if we consider just one four-simplex, we can easily isolate and separate both classes of variables and give a description of the theory defined only in such four-simplex. Furthermore, equation (4.18), which is a condition on $u_i(\sigma)$, is a gluing equation that allows gluing together two four-simplices. This condition requires that the boundary variable $u_i(\sigma)$ of one four-simplex coincides with that of the adjacent four-simplex. These characteristics are already present in the model given in [17].

There exists another description of the boundary graph $\tilde{\Gamma}_{\partial\triangle}$. In this description, the nodes and the links of $\tilde{\Gamma}_{\partial\triangle}$ are the same as those of $\Gamma_{\partial\triangle}$ however, the variables assigned to the different elements of the graph are different. In this description, the configuration variables are defined as $M_{ij} := k_{r_j}^{-1} k_{r_i}$, and they describe the parallel transport from C_{τ_i} to C_{τ_j}. The momentum variable is $u_i(\sigma)$ with the appropriate orientation, but parallel transported from C_σ to either C_{τ_i} or to C_{τ_j}. In the first case, we obtain the variable E_{ij}. In the second case we have E_{ij}. They are related by the expression $E_{ij} = -k_{r_j}^{-1} k_{r_i} E_{ji} k_{r_i}^{-1} k_{r_j}$. Which of the two descriptions, either with $\Gamma_{\partial\triangle}$ or $\tilde{\Gamma}_{\partial\triangle}$, is more appropriate in order to quantize the model is still an open issue.

4.6 Open issues and future research

The model that we presented intends to be a new way to discretize general relativity. It is a model with a clear separation between boundary variables and variables belonging to the bulk. This is a consequence of the use of the discretization originally introduced by Reisenberger in [17]. The variables of he model are a gauge field and a variable that contains the information about the tetrad. This helps to establish a closer analogy with the classical Holst formulation of general relativity than that of Regge calculus. There are, however, some open issues and a wide field for future research. Here we will mention some of them.

We cannot be completely sure that we really have a discretization of general relativity without an study of the continuum limit. The calculation of the continuum limit of the action to be sure that we can recover the Holst action is an important open issue. It requires the use of some tools previously used in [17].

Strongly connected with the last point is the issue of the role of the torsion in the discretized model presented here. In our framework we did not find a discretized version of the torsion. If there is a discrete analog of torsion, it does not seem to be a quantity that vanishes as a consequence of the dynamical equations in vacuum. The crucial point is, however, to be able to recover the correct dynamical equations in the continuum limit. This would make the non-vanishing of the torsion in the discrete model only a discretization artifact, without physical consequences.

The study of the multisymplectic structure and the Peierls brackets of the boundary variables is an important topic for the quantization of the model. In the EPRL model, the simplectic structure is calculated without imposing the geometric constraint [8]. We hope to be able to study the effect of the geometric constraint on the multysimplectic structure and the dynamic compatible with that structure.

The study of gauge symmetries in discrete quantum gravity, particularly in spin foam models, is an important area that has been explored in some particular cases [61]. The discretization $\triangle$ is seen only as a regulator introduced to define the spin foam model. Even when the regulator (or the discretization dependence) eventually has to be removed, the theory is presumed to remain discrete at the fundamental level. This raises the question of whether the discretization procedures used in the derivation of the spin foams are compatible with the expectation that one is approximating a diffeomorphism invariant theory [6]. This question is important and has been one of the central concerns of some works in the last few years [62, 63]. We expect to be able to study the role of the diffeomorphism-invariance and the gauge symmetries in the new discretized model that we presented here.

Another important field for future work, and maybe the most obvious, is the quantization of the model and the construction of the corresponding spin foam model. We hope that the quantization of our model allows maintaining the characteristics that make the EPRL model especially attractive and helps in the study of the semiclassical limit of spin foam models. We expect to be able to study some scenarios that are problematic when the current spin foam models are used, particularly those involving situations with more than one four-simplex and regions with non-vanishing curvature.

Which of the boundary graphs, $\Gamma_{\partial\triangle}$ or $\tilde{\Gamma}_{\partial\triangle}$, is the most appropriate for the quantization is another open issue.

The present model, which uses $SO(3,1)$ as its gauge group, allows the coupling to bosons. It could be also very useful to have an $SL(2,\mathbb{C})$ model that allows the coupling to fermions. The definition of such a model seems to be straightforward.

5
General review and conclusions

5.1 General review of this thesis

In this work, we tried to introduce the most important features and issues of loop quantum gravity.

Chapter one introduced the problem of quantizing the gravitational interaction. We offered some motivations, both theoretical and phenomenological, pointing toward a description of the gravitational field involving the principles of quantum theory. The motivations range from black holes and singularities to the Big Bang theory and the need to reconcile two contradictory theories. All of this points to a quantum description of the gravitational field as being something necessary for solving many open problems in theoretical physics[11].

In chapter two, we introduced the canonical quantization of the gravitational field. The main motivation for canonically quantizing gravity is that general relativity is a theory explicitly formulated in a background-independent way. The canonical quantization is a method that allows maintaining such background independence as a fundamental characteristic also in the quantum framework.

In order to quantize the theory, it was necessary to isolate the dynamical content, writing the theory as a Cauchy problem. To do this, we sketched the 3+1 formulation of general relativity. As a result, it is found that there exists a set of relations among the phase space variables (defined on a spacelike Cauchy hypersurface) known as constraints. These constraints define the Poisson algebra of infinitesimal generators of gauge transformations. The constraints are usually called the vector constraint, the Gauss constraint, and the Hamiltonian constraint.

The vector constraint is responsible for generating three-dimensional diffeomorphisms on every hypersurface. The Gauss constraint generates $SU(2)$ gauge transformations. Finally, the Hamiltonian constraint is related to the remaining gauge symmetry related to the four-diffeomorphism symmetry of the Lagrangian formulation of the theory.

Motivated by the objective to quantize the theory, we presented the way in which the theory is written in terms of variables that are different from the standard metric variables. Such

[11]For a more complete overview of the reasons that motivate the quantization of the gravitational field, see [2]. In this reference you will find another point of view that affirms that general relativity could not be able to be quantized because gravity could not be a fundamental interaction.

variables are called *Ashtekar-Barbero variables.* With them, it was possible to cast the gravitational theory in the form of a background-independent $SU(2)$ gauge theory with an extended phase space that is that of a $SU(2)$ Yang-Mills theory.

The canonical framework of loop quantum gravity is constructed by quantizing a phase space formulation of general relativity in terms of $SU(2)$ connection variables. It is possible to advance in some steps in the canonical quantization of the theory. The kinematical Hilbert space satisfying the commutation relations that follow from the substitution $\{\ \}, \rightarrow -i/\hbar\,[\ ,]$ can successfully be defined. The algebra of phase space smeared variables is replaced by the holonomy-flux algebra, which is represented by operators in a kinematical Hilbert space composed of functionals of the generalized connection. These functionals are square-integrable with respect to the *Ashtekar-Lewandowski* measure. We can also define suitable operators acting in the kinematical Hilbert space for both the Gauss and vector constraint. The kernel of the Gauss constraint allows introducing the famous *spin network* basis. The quantization of the Hamiltonian constraint presents serious difficulties. Some concrete quantizations producing well-defined operators have been encountered. The space of solutions of the quantum Hamiltonian constraint, however, remains as an open issue.

Already at the kinematical level, loop quantum gravity can produce some concrete predictions. Some of them are the fundamental discreteness of spacetime at the level of the Planck scale. This phenomena is produced by the discreteness of the spectrum of the area and volume operators. This discreteness was used in some calculations involving black holes and cosmology. The calculations have been able to reproduce some already known results but this time from a more fundamental point of view.

The spin foam framework was initially considered as a means to tackle the question of the dynamics and the definition of observable quantities in loop quantum gravity. It is, however, a broader framework, which is intended to construct a quantum gravity theory from the path integral perspective, while remaining compatible with the canonical approach.

In Chapter three we introduced the main characteristics of the spin foam framework. We start by motivating the general form of the partition function in the spin foam framework. Next, we introduce how to discretize differentiable manifolds. We do this to give meaning to the path-integral in the context of gravity. After that, we introduced the spin foam model for three-dimensional general relativity, which is called the Ponzano-Regge model. This model recovers some characteristics of the quantum theory found in the canonical framework, such as the spin network states and the discreteness of the length operator. Moreover ,such a model is independent of the discretization introduced to define it.

As a second step, we introduced the concept of BF theory. Moreover, general relativity can be considered a constrained BF theory through what we called the geometric constraints. We presented a spin foam model for a four-dimensional BF theory. Next, we mentioned the introduction a new geometric constraint that was called linear geometric constraint. This linear constraint was better in the sense that it allows recovering general relativity by leaving aside some ambiguities that were present with the quadratic constraints. The use of the linear geometric constraint allows the introduction of the currently most accepted spin foam model, which is called the EPRL model.

Finally, in Chapter four, we presented a new way to discretize general relativity with a quantization that is expected to provide a way to tackle some open issues of the EPRL model.

Our new discrete model has three main advantages. First, it allows distinguishing between boundary degrees of freedom and degrees of freedom in the bulk. Such a distinction is in some sense obscure in the discretization on which the EPRL model is based. Second, in our discretization the study of only one four-simplex already contains information about curvature. That does not happen in the standard EPRL model, where the four-simplices are flat. Third, the model has as variables a gauge field and a variable that contains the information about the tetrad. This helps to establish a closer analogy with the classical continuum theory than that of the Regge discretizations.

Our hope is that our construction allows the study of some aspects of the EPRL model that are currently difficult to attack. Particularly, some calculations involving the semiclassical limit of the new spin foam models, and computations of correlation functions, have been done in the EPRL model by only considering a single four-simplex. Therefore, the introduction of a new discretization for defining the EPRL model could help to perform such computations by considering more than one four-simplex.

5.2 Conclusions

LQG is a prominent candidate for a satisfactory quantum theory of gravity. It allows considering the background independence of the theory from the beginning and has produced some predictions that, unfortunately, have not been test experimentally yet. Contrary to what many people think, the canonical quantization of gravity is still a rich and active area of research [64].

The covariant formulation of LQG emerges as an alternative route. It intends to shed light into the dynamics of quantum gravity, trying to give a discretized and regularized version of the path integral applied to gravity. The emergence of different spin foam models is a clear indication that there is still very much to do if we can say that we really have a quantum theory of the gravitational interaction.

The particular route taken by LQG, where we try to construct a quantum theory of gravity and then try to extract physically experimental results and probe their physically and mathematical consistency is sometimes called a “bottom up inquiry” perspective[12].

Our current line of research intends to produce an alternative route towards the construction of spin foam models. We have defined a discrete classical theory that tries to be a new discretization of general relativity. We still need to take the continuum limit to be sure that we have a discretization of the gravitational theory. Based on the results given in [17], we hope to find that we can recover the Holst action in the continuum limit of our model.

[12]See Daniel Sudarsky’s personal webpage.
http://epistemia.nucleares.unam.mx/web?name=Daniel+Sudarsky_en

The clear separation between bulk and boundary degrees of freedom allows the construction of regions with more than one four-simplex in a clear way. This is an important improvement over the conventional classical discretization used to define the current spin foam models.

One important characteristic of our discretization is the way in what the action is written. The action allows isolating the contribution of a single corner cell, being the contribution of a four-simplex the sum of the contributions of its corner cells. In this way, the corner cells act as a kind of “smaller” or more fundamental structures than the four-simplices themselves. This is a novel shift over the conventional discretizations, and has the potential to introduce interesting characteristics in the quantum theory.

The non-vanishing of the term proportional to the Immirzi parameter, both in the action and in the dynamical equations, is an interesting surprise. In the theory in the continuum, this second term vanishes in vacuum as a consequence of the dynamical equations themselves. In the discretization presented here, this term vanishes only in certain special cases. The non-vanishing in the general case seems to be only a consequence of the discretization, but we need to do more research about this point.

Connected with the last point, the possible appearance of torsion in vacuum in our discretized model is an interesting open issue. If we are able to recover the correct dynamical equations in the continuum limit, we will be sure that the appearance of torsion at the discretized level is only a consequence of the discretization without physical meaning.

We hope to be able to successfully quantize the classical model and obtain a model closely related to the EPRL model. The research of the semiclassical limit of a quantum theory is something that needs to be checked in order to ensure that we are proceeding correctly. In this way, we hope to make some studies concerning the propagation of gravitational waves in discretized scenarios, and to perform the calculation of the graviton propagator using the quantum theory emerging from our model. We expect that our research will help understanding some issues that have not been properly understood in the spin foam framework, particularly in the semiclassical limit of the theory [6, 28].

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Appendices

A
Elementary results in linear algebra

In this seemingly innocent appendix we give some definitions and results in linear algebra that are, however, of fundamental importance. In particular, the proof of the existence of a tetrad field for each corner cell, which is given in the next appendix, uses the majority of the results presented here. We will also introduce some definitions and results that can be considered pretty elementary. We do this only for completeness, and with the aim of to be clear about our notation (which is largely taken from [14])

In the rest of the appendix, V will denote a vector space of finite dimension. We will denote the dual space as V^*. Also, $\bigwedge^k V$ will denote to the set of alternating k tensors defined on V^*. In this way, $T \in \bigwedge^k V$ will denote an alternating tensor $T : V^* \times \cdots \times V^* \to \mathbb{R}$. In particular, $\bigwedge^2 V$ is the space of the so called bivectors.

Two remarks are necessary:

- We will say that a bivector $B \in \bigwedge^2 V$ is simple if there exist two vectors $\vec{v}, \vec{w} \in V$ such that $B = \vec{v} \wedge \vec{w}$.
- If W is a vector subspace of V we will write $W \leq V$.
- The Minkowski space $(\mathbb{R}^4, \eta)$ will be denoted by $\mathcal{M}$

Definition. *Let V be a vector space, and $X \subseteq V$ a subset of V. We define:*

$$\mathcal{L}(X) := \bigcap \{W \leq V / X \subseteq W\}.$$

$\mathcal{L}(X)$ is called the subspace of V generated by X.[14]

Definition. *Let $\{W_\alpha\}_{\alpha \in S}$ be a family of subspaces of V. We define*

$$\sum_{\alpha \in S} W_\alpha := \mathcal{L}\left(\bigcup_{\alpha \in S} \{W_\alpha\} \right).$$

That is, the sum of a family of vector subspaces is the subspace of V generated by the union of the family of subspaces.

Proposition 1. *$\sum_{\alpha \in S} W_\alpha$ is the smallest subspace of V containing $\bigcup_{\alpha \in S}\{W_\alpha\}$.*

By *the smallest* we mean that, if there exist another subspace of Vcontaining $\bigcup_{\alpha \in S}\{W_\alpha\}$, Z for example, then necessarily $\sum_{\alpha \in S} W_\alpha \leq Z$. The proof of this proposition can be found in [14].

Definition. *Let W_1, $W_2 \leq V$ subspaces of V. It is said that V is the direct sum of W_1 and W_2 when:*

a) $W_1 \cap W_2 = \{\vec{0}\}$ *and*

b) $W_1 + W_2 = V$.

In this case we write $V = W_1 \oplus W_2$. When $W_1 \cap W_2 = \{\vec{0}\}$, but not necessarily $V = W_1 + W_2$, we will write $W_1 + W_2 = W_1 \oplus W_2$.

Proposition 2. *Let W_1, W_2 and $W_3 \leq V$ subspaces of V. Then:*

a) $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$,

b) $dim(W_1 + W_2 + W_2) \leq dim(W_1) + dim(W_2) + dim(W_3) - dim(W_1 \cap W_2) - dim(W_2 \cap W_3) - dim(W_1 \cap W_3) + dim(W_1 \cap W_2 \cap W_3)$.

The proof is given in [14].

Lemma 1. *Let W_1, $W_2 \leq V$. Then, $dim(W_1 \cap W_2) = dim(W_1)$ if and only if $W_1 \subseteq W_2$.*

Proof. The result is clear from the fact that $W_1 \cap W_2 = W_1$ if and only if $W_1 \subseteq W_2$.

Lemma 2. *If $W \leq V$ and $dim(W) = dim(V)$, then $V = W$.*

Proof. It is enough to prove that $V \leq W$. However, this last affirmation is clear from lemma 1 because $W \cap V = W$ and then $\dim(W \cap V) = \dim(W) = \dim(V)$.

Lemma 3. *Let V be a vector space such that $dim(V) \geq 3$. Let $U, W \leq V$ subspaces of V such that $dim(U) = dim(W) = n - 1$, and $U \neq W$. Then $dim(U \cap W) = n - 2$.*

Proof. We know that $U, W \leq U + W \leq V$, which means that $n - 1 \leq \dim(U + W) \leq n$. Then, by lemma 2 we will have that, either U, or $W = U + W$, or $U + W = V$.

If $U + W = U$, then $W \leq U$, and by lemma 2 we will have that $W = U$. This contradicts the fact that $W \neq U$. Then $U \neq U + W$. Analogously we can prove that $W \neq U + W$.

All of the above implies that $U + W = V$, but by Proposition 2 a) we will have that $\dim(U + W) = n = \dim(U) + \dim(W) - \dim(U \cap W) = 2n - 2 - \dim(U \cap W)$. Then $\dim(U \cap W) = n - 2$.

Lemma 4. *Let V a vector space such that $dim(V) \geq 4$. Let $U, W, X \leq V$ subspaces of V such that $dim(U) = dim(W) = dim(X) = n - 1$. Then $dim(U \cap W \cap X) = n - 3$.*

Proof. Let $k := n - 1$. Then $\dim(U) = \dim(W) = \dim(X) = k \geq 3$. Note that $U \cap W \leq W$. By lemma 3 we have that $\dim(U \cap W) = n - 2 = k - 1$. In the same way we have that $\dim(W \cap X) = k - 1$. Then, applying lemma 3 again, we obtain that

$$\dim(U \cap W \cap X) = k - 2 = (n - 1) - 2 = n - 3.$$

Proposition 3. *Let V be a vector space. Let V_1, V_2, $V_3 \leq V$ such that $dim(V_1) = dim(V_2) = dim(V_3) = 2$, and $dim(V_i \cap V_j) = 1$ if $i \neq j$. Let us define $W_1 := V_2 \cap V_3$, $W_2 := V_1 \cap V_3$, and $W_3 := V_1 \cap V_2$. Then only one of the following two possibilities is fulfilled:*

a) dim$(V \cap V_2 \cap V_3) = 1$*, which implies that dim*$(W_1 + W_2 + W_3) = 1$

b) dim$(V \cap V_2 \cap V_3) = 0$*, which implies that dim*$(W_1 + W_2 + W_3) =$*dim*$(V_1 + V_2 + V_3) = 3$.

Proof. By definition, ir follows that $W_1 + W_2 + W_3 = \mathcal{L}(W_1 \cup W_2 \cup W_3)$. Also, by proposition 2b), we have that

$$\dim(W_1 + W_2 + W_2) \leq \dim(W_1) + \dim(W_2) + \dim(W_3) - \dim(W_1 \cap W_2) - \dim(W_2 \cap W_3) - \dim(W_1 \cap W_3) + \dim(W_1 \cap W_2 \cap W_3)$$

Particularly, in this case we have that $\dim(W_1) = \dim(W_2) = \dim(W_3) = 1$. Also, we have that $W_1 \cap W_2 = W_2 \cap W_3 = W_1 \cap W_3 = W_1 \cap W_2 \cap W_3 = V_1 \cap V_2 \cap V_3$. All of this implies that $\dim(W_1 + W_2 + W_3) \leq 3 - 2\dim(V_1 \cap V_2 \cap V_3)$.

Let us note that $V_1 \cap V_2 \cap V_2 \leq V_1 \cap V_2$, so that $\dim(V_1 \cap V_2 \cap V_3) = 0, 1$. Also, as $W_1 \subseteq W_1 \cup W_2 \cup W_3 \subseteq W_1 + W2 + W_3$, and $\dim(W_1) = 1$, then $\dim(W_1 + W_2 + W_3) \neq 0$.

In short, we have that $\dim(W_1 + W_2 + W_3) \neq 0$ and also or $\dim(V_1 \cap V_2 \cap V_3) = 0$, or $\dim(V_1 \cap V_2 \cap V_3) = 1$.

If $\dim(V_1 \cap V_2 \cap V_3) = 1$, then $0 < \dim(W_1 + W_2 + W_3) \leq 1$, that is, $\dim(W_1 + W_2 + W_3) = 1$.

On the other hand, if $\dim(V_1 \cap V_2 \cap V_3) = 0$, using lemma 2, proposition 2 and the fact that $\dim(V_j \cap V_j) = 1$ if $i \neq j$, we have that $V_1 = (V_1 \cap V_2) \oplus (V_1 \cap V_3)$, $V_2 = (V_2 \cap V_3) \oplus (V_2 \cap V_1)$, and $V_3 = (V_3 \cap V_1) \oplus (V_3 \cap V_2)$. In this way, we obtain that $V_1 + V_2 + V_3 = (V_1 \cap V_2) + (V_1 \cap V_3) + (V_2 \cap V_3) = W_1 + W_2 + W_3$, and then $\dim(V_1 + V_2 + V_3) = \dim(W_1 + W_2 + W_3) \leq 3$. However, by proposition 2 we have that $\dim(V_1 + V_3) = 3$. Using this and the fact that $V_1 + V_2 \leq V_1 + V_2 + V_3$, we obtain that $3 \leq \dim(V_1 + V_2 + V_3)$. This implies that $\dim(V_1 + V_2 + V_3) = \dim(W_1 + W_2 + W_3) = 3$ ■

Proposition 4. *Let V be a vector subspace such that dim$(V) \geq 4$. Let $E_1, E_2, E_3, E_4 \leq V$ subspaces of V such that*

a) dim$(E_k) = 3$ *for* $k = 1, 2, 3, 4$.

b) dim$(E_i \cap E_j) = 2$ *if* $i \neq j$,

c) dim$(E_i \cap E_j \cap E_k) = 1$ *if i, j and k are all distinct*

d) dim$(E_1 \cap E_2 \cap E_3 \cap E_4) = 0$

Let us define:

$$S_1 := E_1 \cap E_2$$

$$S_2 := E_2 \cap E_3$$

$$S_3 := E_3 \cap E_4$$

$S_4 := E_4 \cap E_1$

and:

$W_1 := S_1 \cap S_2$

$W_2 := S_2 \cap S_3$

$W_3 := S_3 \cap S_4$

$W_4 := S_4 \cap S_1$

Then dim$(W_i+W_j+W_k) = 3$ for any i, j, k distinct between them, and dim$(W_1+W_2+W_3+W_4) = 4$

Proof. First it is important to notice that $S_1 + S_2 = (E_1 \cap E_2) + (E_3 \cap E_4) \leq V$. Then, by proposition 2a) and the fact that $\dim(S_1 \cap S_3) = 0$, we have that $\dim(S_1 + S_3) = 4$. In this way, by lemma 2, we have that $S_1 + S_3 = V$. Analogously we can prove that $S_2 + S_4 = V$.

Let us define $A := (S_1 \cap S_2) + (S_2 \cap S_3)$, and $B := (S_1 + S_3) \cap S_2$. Let us note that $A = \mathcal{L}((S_1 \cap S_2) \cup (S_2 \cap S_3)) = \mathcal{L}(S_2 \cap (S_1 \cup S_3))$. We know that, by definition, $S_2 + S_3 = \mathcal{L}(S_1 \cup S_3)$. Then $S_2 \cap (S_1 \cup S_3) \subseteq B$, but by proposition 1 it follows that $A \leq B$. Since $B \subseteq S_2$, then $0 \leq \dim(B) \leq 2$. However $A \leq B$, and then $\dim(A) = \dim(W_1 + W_2) = 2 \leq \dim(B) \leq 2$. In this way, we have that $dim(A) = \dim(B)$, and $A \leq B$. Then, by lemma 2 we obtain that $A = B$, that is, $(S_1 \cap S_2) + (S_2 \cap S_3) = (S_1 + S_3) \cap S_2$.

In the same way it can be proved that $(S_3 \cap S_4) + (S_4 \cap S_1) = (S_1 + S_3) \cap S_4$.

Using all of the above we have that

$$\begin{aligned} W_1 + W_2 + W_3 + W_4 &= (S_1 \cap S_1) + (S_2 \cap S_3) + (S_3 \cap S_4) + (S_4 \cap S_1) = \\ &= [(S_1 + S_3) \cap S_2] + [(S_1 + S_3) \cap S_4] = [V \cap S_2] + [V \cap S_4] = S_2 + S_4 = V. \end{aligned}$$

In a similar way, using all of the above, and proposition 2a), it can be proved that $\dim(W_i + W_j + W_k) = 3$ for any i, j, k distinct between them. ■

Now we present some facts involving the wedge product of two or more vectors. This results, together with those the presented above will be of fundamental importance in the proof of the proposition given in the next appendix.

Proposition 5. *An element $B \in \bigwedge^2 \mathcal{M}$ is simple if and only if there exist a vector $\vec{n} \in \mathcal{M}$ such that $n_I({}^*B)^{IJ} = 0$, where $({}^*B)^{IJ} := \frac{1}{2}\epsilon^{IJKL}B_{KL}$*

This is lemma II.3 of [22].

Proposition 6. *Two simple elements B_1, $B_2 \in \bigwedge^2 \mathcal{M}$ span a three-dimensional subspace of $\mathcal{M}$ if and only if there exist a vector $\vec{n} \in \mathcal{M}$ such that $n_I({}^*B_1)^{IJ} = n_I({}^*B_2)^{IJ} = 0$*

This is lemma II.4 of [23]. The proof of proposition 5 and 6 can be found in this same article.

Proposition 7. *An element $B \in \bigwedge^2 \mathcal{M}$ is simple if and only if $^*B$ is simple*

The proof of this proposition can be found in [20].

Next two results demand elementary methods for their proofs. However, the proof itself is a little difficult to find in the literature.

Lemma 5. *Let V be a vector space. The elements $\vec{v}_1, \dots, \vec{v}_m \in V$ are linearly independent if and only if $\vec{v}_1 \wedge \dots \wedge \vec{v}_m \neq 0$*

Proof. Suppose that $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent, and let us consider that $\vec{v}_m = \sum_{i<m} a_i \vec{v}_i$. Then:

$$\vec{v}_1 \wedge \dots \wedge \vec{v}_m = \sum_{i<m} a_i (\vec{v}_1 \wedge \dots \wedge \vec{v}_{m-1} \wedge \vec{v}_i) = 0.$$

If $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent, there exist a basis $b := \{\vec{w}_k\}_{k=1}^n$ of V such that $\vec{v}_i \in b$ for each $i = 1, \dots, m$ [14]. Then, $\vec{v}_1 \wedge \dots \wedge \vec{v}_m$ is an elements of a basis of the space $\bigwedge^m V$ (proposition 3.23 in [10]), and therefore it is not zero.

Lemma 6. *Let $\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2 \in V$. Suppose that that all of them are distinct from zero, and that they are different from each other. If $\vec{v}_1 \wedge \vec{v}_2 = \vec{w}_1 \wedge \vec{w}_2 \neq 0$, then $\mathcal{L}(\{\vec{v}_1, \vec{v}_2\}) = \mathcal{L}(\{\vec{w}_1, \vec{w}_2\})$*

Proof. By hypothesis we know that $\vec{v}_1 \wedge \vec{v}_2 \neq 0$ and $\vec{w}_1 \wedge \vec{w}_2 \neq 0$, which means, by lemma 5, that $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{w}_1, \vec{w}_2\}$ are linearly independent sets. But we also know that $\vec{v}_1 \wedge \vec{v}_2 = \vec{w}_1 \wedge \vec{w}_2$, which means, by lemma 5, that $\{\vec{v}_1, \vec{v}_2, \vec{w}_1\}$ and $\{\vec{w}_1, \vec{w}_2\, \vec{w}_2\}$ are linearly dependent sets.

The linear dependence of $\{\vec{w}_1, \vec{w}_2\, \vec{w}_2\}$ implies that there exist $a, b, c \in \mathbb{R}$ with at least one of them different from zero, such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{w}_2 = 0$. Using the fact that $\vec{v}_1, \vec{v}_2, \vec{w}_2$ are different from zero and that $\vec{v}_1, \vec{v}_2$ are linearly independent, it follows that or $c \neq 0$, or $a \neq 0$ or $b \neq 0$. In any case, we will have that $\vec{w}_2 \in \mathcal{L}(\{\vec{v}_1, \vec{v}_2\})$. From this also follows that $\vec{v}_1, \vec{v}_2 \in \mathcal{L}(\{\vec{w}_1, \vec{w}_2\})$.

All of the above means that $\mathcal{L}(\{\vec{v}_1, \vec{v}_2\}) = \mathcal{L}(\{\vec{w}_1, \vec{w}_2\})$.

B
Existence of a tetrad for each corner cell

In this optional reading appendix[13], we embark on the task of proving that the linear geometric restriction given in chapter 4 implies the existence of a tetrad (a set of four linearly independent vectors in $\mathcal{M}$) for each corner cell of a four-simplex. Such assignment is done in such a way that each tetrad generates all of the $B(s) \in so(3,1)$ associated with the wedges s that belong to the corner cell c_p.

For completeness, we start by stating the form of the linear geometric constraint adapted to our model.

Adapted linear geometric constraint(ALGC): Let $\nu < \triangle$ be a four simplex, c_p a corner cell in ν, τ a tetrahedron in ν that intersects c_p and $l(\nu\tau)$ the line that goes from the barycenter of ν to the barycenter of τ. There exists a vector $\vec{n}(\tau, c_p) \in \mathcal{M}$ different from zero, such that for all the wedges $s < c_p$ in c_p such that $l(\nu\tau) \in \partial s$, we have that

$$n(\tau, c_p)_I B(s)^{IJ} = 0.$$

As we mentioned in chapter four, this condition is used in conjunction with what we call the four-volume constraint:

Four-volume constraint:
Given a corner cell $c_p < \nu$ we have that

$$\mathrm{sgn}(s, s')\epsilon_{IJKL} B(s)^{IJ} \wedge B(s')^{KL} = \mathrm{sgn}(s'', s''')\epsilon_{IJKL} B(s'')^{IJ} \wedge B(s''')^{KL}, \tag{B.1}$$

for every couple of wedges (s, s') and (s'', s''') in c_p such that s and s' only share the point C_ν (the barycenter of ν), and the same for s'' and s'''.

Now we enunciate and prove that, as mentioned in chapter 4, this two constraints together imply the existence of a tetrad for each corner cell generating the B's associated with the wedges of it. In order to be able to follow the proof you need to have read appendix A. All the propositions and lemmas mentioned in the proof refer to those of appendix A.

Theorem B.0.1. *Let $\nu < \triangle$ be a four-simplex in $\triangle$ and c_p a corner cell in ν. Consider that the six $B(s)$ associated with c_p are linearly independent. If the adapted linear geometric constraint is true and the four-volume constraint is satisfied, then for each corner cell $c_p < \nu$ there exist*

[13] The style of this appendix is largely inspired by Chapter 20 of [37]

four linearly independent vectors $\vec{e}_{l0}(c_p), \ldots, \vec{e}_3(c_p) \in \mathcal{M}$ *such that, for every* $B(s) \in so(3,1)$ *associated with a wedge* s *belonging to* c_p *we have that*

$$B(s) =^* \left(\vec{e}_{l_i(s)}(c_p) \wedge \vec{e}_{l_j(s)}(c_p)\right),$$

where $\partial s = l_i + r_i - r_j - l_j$.

Proof. Let $\nu < \triangle$ be a four simplex and $c_p < \nu$ a corner cell in ν. Let $\{\tau_1, \tau_2, \tau_3, \tau_4\}$ be the four tetrahedra intersecting c_p. Let us consider only one of such tetrahedron, τ_1 for example. Then τ_1 has three triangles $\sigma_1^1, \sigma_2^1, \sigma_3^1$ intersecting c_p, and whose wedges are s_1, s_2, s_3 respectively. Let B_1, B_2, B_3 be the elements of $so(3,1)$ associated to those wedges in c_p.

By proposition 5 for each B_k there exist two vectors $\vec{a}_k, \vec{b}_k \in \mathcal{M}$ such that $B_k = \vec{a}_k \wedge \vec{b}_k$, and by proposition 7 there exist two vectors $\vec{\alpha}_k, \vec{\beta}_k \in \mathcal{M}$ such that ${}^*B_k = \vec{\alpha}_k \wedge \vec{\beta}_k$.

By proposition 6, if $i \neq j$ then the sets $\{\alpha_i, \beta_i, \alpha_j, \beta_j\}$ are linearly dependent. This means that if we define $V_i := \mathcal{L}\left(\{\alpha_i, \beta_i\}\right)$, then $\dim(V_i \cap V_j) \neq 0$. But the three elements B_{s_k} are linearly independent, and that implies that $\dim(V_i \cap V_j) = 1$ if $i \neq j$. Also we will have that $\dim(V_i) = 2$ for $i = 1, 2, 3$.

Let us define $W_1 := V_2 \cap V_3$, $W_2 := V_1 \cap V_3$, and $W_3 := V_1 \cap V_2$. Then by proposition 3 it follows that only one of the following two possibilities is fulfilled:

a) $\dim(V \cap V_2 \cap V_3) = 1$, which implies that $\dim(W_1 + W_2 + W_3) = 1$

b) $\dim(V \cap V_2 \cap V_3) = 0$, which implies that $\dim(W_1 + W_2 + W_3) = \dim(V_1 + V_2 + V_3) = 3$.

Suppose that that $\dim(V_1 \cap V_2 \cap V_3) = 0$ and $\dim(W_1 + W_2 + W_3) = 3$. This last implies that the sum $W_1 + W_2 + W_3$ is a direct sum, that is:

$$W_1 \oplus W_2 \oplus W_3 = (V_2 \cap V_3) \oplus (V_1 \cap V_3) \oplus (V_1 \cap V_2).$$

Let us take $\vec{f}_1 \in W_1$, $\vec{f}_2 \in W_2$ and $\vec{f}_3 \in W_3$ such that the set $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ is linearly independent.

Notice that $\vec{f}_2, \vec{f}_3 \in V_1$. This means that there exist $a, b, c, d \in \mathbb{R}$ such that $\vec{\alpha}_1 = a\vec{f}_2 + b\vec{f}_3$ and $\vec{\beta}_1 = c\vec{f}_2 + d\vec{f}_3$. Then, ${}^*B_1 = q_1 \vec{f}_2 \wedge \vec{f}_2$ with $q_1 = \left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right|$.

In the same way, there exist $e, g, h, i \in \mathbb{R}$ such that $\vec{\alpha}_2 = h\vec{f}_1 + i\vec{f}_3$, and $\vec{\beta}_2 = e\vec{f}_1 + g\vec{f}_3$. Then ${}^*B_2 = q_2 \vec{f}_3 \wedge \vec{f}_1$ with $q_2 = \left|\begin{smallmatrix} e & g \\ h & i \end{smallmatrix}\right|$.

Analogously, there exist $j, k, l, m \in \mathbb{R}$ such that $\vec{\alpha}_3 = j\vec{f}_1 + k\vec{f}_2$, and $\vec{\beta}_2 = l\vec{f}_1 + m\vec{f}_2$. Then ${}^*B_3 = q_3 \vec{f}_1 \wedge \vec{f}_2$ with $q_3 = \left|\begin{smallmatrix} j & k \\ l & m \end{smallmatrix}\right|$.

Let us define $\vec{e}_k := \lambda_k \vec{f}_k$ for $k = 1, 2, 3$, with $\lambda_k \in \mathbb{R} - \{0\}$. If we ask that ${}^*B_1 = \vec{e}_2 \wedge \vec{e}_3$, ${}^*B_2 = \vec{e}_3 \wedge \vec{e}_1$ and ${}^*B_3 = \vec{e}_1 \wedge \vec{e}_2$, we will obtain the system of equations:

$$\lambda_2 \lambda_3 = k_1$$
$$\lambda_3 \lambda_1 = k_2$$
$$\lambda_1 \lambda_2 = k_3,$$

with solution:

$$\lambda_1 = \pm\sqrt{\left|\frac{k_3k_2}{k_1}\right|}$$
$$\lambda_2 = \pm k_3\sqrt{\left|\frac{k_1}{k_2k_3}\right|}$$
$$\lambda_3 = \pm k_2\sqrt{\left|\frac{k_1}{k_2k_3}\right|}.$$

This proves that there exist three linearly independent vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathcal{M}$ such that ${}^*B_1 = \vec{e}_2 \wedge \vec{e}_3$, ${}^*B_2 = \vec{e}_3 \wedge \vec{e}_1$ and ${}^*B_3 = \vec{e}_1 \wedge \vec{e}_2$. However this contradicts the ALGC. Then we have that $\dim(V \cap V_2 \cap V_3) = 1$ and $\dim(W_1 + W_2 + W_3) = 1$.

We can repeat the process given above for each tetrahedron intersecting c_p. Let $\alpha_k(\tau_i), \beta_k(\tau_i)$ be the vectors α and β associated with τ_i, $V_1(\tau_i), V_2(\tau_i), V_3(\tau_i)$ be the three sets V corresponding to the tetrahedron τ_i and $W_1(\tau_i), \dots, W_3(\tau_i)$ be the sets W corresponding to τ_i. We know that $\dim(V_1(\tau_i) \cap V_2(\tau_i) \cap V_3(\tau_i)) = 1$ for all τ_i in c_p. Let us take a vector $\vec{e}_i(c_p) \in (V_1(\tau_i) \cap V_2(\tau_i) \cap V_3(\tau_i))$ for every τ_i. In this way we obtain a set of four vectors $\vec{e}_1(c_p), \dots, \vec{e}_4(c_p)$. Moreover, since we know that the six $B(s)$ with $s \in c_p$ are linearly independent, and $\dim(V_1(\tau_i) \cap V_2(\tau_i) \cap V_3(\tau_i)) = 1)$, then the set of vectors $\vec{e}_1(c_p), \dots, \vec{e}_4(c_p)$ is linearly independent.

Let $s \in c_p$ be a wedge in c_p and $B(s) \in so(3,1)$ its associated element B. The triangle $\sigma(s)$ in ν associated with s belongs to two different tetrahedra. Without lose of generality suppose that $\sigma(s) \in \tau_1 \cap \tau_2$. Then ${}^*B(s) = t\alpha(\tau_1) \wedge \beta(\tau_1)$ with $\alpha(\tau_1), \beta(\tau_1) \in V_k(\tau_1)$, $t \in \mathbb{R}$, and also ${}^*B(s) = \alpha(\tau_2) \wedge \beta(\tau_2)$ with $\alpha(\tau_2), \beta(\tau_2) \in V_k(\tau_2)$, for some k. However, $\vec{e}_1(c_p) \in V_k(\tau_1)$ and $\vec{e}_2(c_p) \in V_k(\tau_2)$. Moreover, since $\sigma(s) \in \tau_1 \cap \tau_2$ then, by lemma 6 of Appendix A we have that $V_k(\tau_1) = V_k(\tau_2)$. All this implies that $\vec{e}_1 = a\alpha_1(\tau_1) + b\beta_1(\tau_1)$ with $a, b \in \mathbb{R}$ with at least one of them different from zero, and $\vec{e}_2 = c\alpha_1(\tau_1) + d\beta_1(\tau_1)$ with $c, d \in \mathbb{R}$ with at least one different from zero. Then, we have that ${}^*B(s) = \lambda\vec{e}_1(c_p) \wedge \vec{e}_2(c_p)$ with $\lambda \neq 0$.

Let $B_1, \dots, B_6$ the six elements B associated with the six wedges $s \in c_p$. Repeating the process given above we find that

$$\begin{aligned}
{}^*B_1 &= \lambda_1 (e_1 \wedge e_2) \\
{}^*B_2 &= \lambda_2 (e_1 \wedge e_3) \\
{}^*B_3 &= \lambda_3 (e_1 \wedge e_4) \\
{}^*B_4 &= \lambda_4 (e_2 \wedge e_3) \\
{}^*B_5 &= \lambda_5 (e_2 \wedge e_4) \\
{}^*B_6 &= \lambda_6 (e_3 \wedge e_4).
\end{aligned}$$

with $\lambda_k \in \mathbb{R} - \{0\}$, $k = 1, \dots 6$. Let us take $\lambda_k > 0$ for every k. Let us consider that the four-volume constraint takes the form:

$$\begin{aligned}
\epsilon_{IJKL} B_1^{IJ} \wedge B_6^{KL} &= \epsilon_{IJKL} B_2^{IJ} \wedge B_5^{KL} \\
\epsilon_{IJKL} B_1^{IJ} \wedge B_6^{KL} &= \epsilon_{IJKL} B_3^{IJ} \wedge B_4^{KL} \\
\epsilon_{IJKL} B_2^{IJ} \wedge B_5^{KL} &= \epsilon_{IJKL} B_3^{IJ} \wedge B_4^{KL}.
\end{aligned} \tag{B.2}$$

where we have supposed that the wedges $1, 6$ in c_p share only the point C_ν and the same for $2, 5$. The set of equations (B.2) implies the following set of relations between the numbers λ_k:

$$\begin{aligned} \lambda_1\lambda_6 &= \lambda_2\lambda_5 \\ \lambda_1\lambda_6 &= \lambda_3\lambda_4 \\ \lambda_2\lambda_5 &= \lambda_3\lambda_4. \end{aligned} \tag{B.3}$$

Using this system of equations we will find that

$$\begin{aligned} {}^*B_1 &= \frac{\lambda_3\lambda_4}{\lambda_6}\left(\vec{e}_1 \wedge \vec{e}_2\right) \\ {}^*B_2 &= \frac{\lambda_3\lambda_4}{\lambda_5}\left(\vec{e}_1 \wedge \vec{e}_3\right) \\ {}^*B_3 &= \lambda_3\left(\vec{e}_1 \wedge \vec{e}_4\right) \\ {}^*B_4 &= \lambda_4\left(\vec{e}_2 \wedge \vec{e}_3\right) \\ {}^*B_5 &= \lambda_5\left(\vec{e}_2 \wedge \vec{e}_4\right) \\ {}^*B_6 &= \lambda_6\left(\vec{e}_3 \wedge \vec{e}_4\right). \end{aligned} \tag{B.4}$$

Now, let us define four vectors $\tilde{e}_1, \ldots, \tilde{e}_4$ such that $\vec{e}_1 = a\tilde{e}_1$, $\vec{e}_2 = b\tilde{e}_2$, $\vec{e}_3 = c\tilde{e}_3$ and $\vec{e}_4 = d\tilde{e}_4$ for some real numbers a, b, c, d different from zero. If we choose

$$\begin{aligned} a &= \lambda_3\sqrt{\frac{\lambda_4}{\lambda_5\lambda_6}} \\ b &= \sqrt{\frac{\lambda_4\lambda_5}{\lambda_6}} \\ c &= \sqrt{\frac{\lambda_4\lambda_6}{\lambda_5}} \\ d &= \sqrt{\frac{\lambda_5\lambda_6}{\lambda_4}} \end{aligned}$$

then we will have that

$$\begin{aligned} {}^*B_1 &= \frac{\lambda_1\lambda_3\lambda_4}{\lambda_6}\left(\tilde{e}_1 \wedge \tilde{e}_2\right) \\ {}^*B_2 &= \frac{\lambda_3\lambda_4}{\lambda_5}\left(\tilde{e}_1 \wedge \tilde{e}_3\right) \\ {}^*B_3 &= \tilde{e}_1 \wedge \tilde{e}_4 \\ {}^*B_4 &= \tilde{e}_2 \wedge \tilde{e}_3 \\ {}^*B_5 &= \tilde{e}_2 \wedge \tilde{e}_4 \\ {}^*B_6 &= \tilde{e}_3 \wedge \tilde{e}_4. \end{aligned} \tag{B.5}$$

Let us define

$$\begin{aligned} k_1 &:= \frac{\lambda_1\lambda_3\lambda_4}{\lambda_6} \\ k_2 &:= \frac{\lambda_3\lambda_4}{\lambda_5} \end{aligned}$$

In this way, we can say that there exist two real numbers k_1 and k_2 different from zero, such that ${}^*B_1 = k_1\left(\tilde{e}_1 \wedge \tilde{e}_2\right)$ and ${}^*B_2 = k_2\left(\tilde{e}_1 \wedge \tilde{e}_3\right)$.

Let us use the four-volume constraint again. Let us write ${}^*B_1 = k_1\left(\tilde{e}_1 \wedge \tilde{e}_2\right)$ and ${}^*B_2 = k_2\left(\tilde{e}_1 \wedge \tilde{e}_3\right)$, and let us use for B_3, B_4, B_5 y B_6 the expressions given in (B.5). If we use the first of the equations given in (B.2) we obtain that $k_1 = k_2$. Using the second of the equations given in (B.2) we obtain that $k_1 = 1$. In this way we obtain that

$$\begin{aligned} {}^*B_1 &= \tilde{e}_1 \wedge \tilde{e}_2 \\ {}^*B_2 &= \tilde{e}_1 \wedge \tilde{e}_3 \\ {}^*B_3 &= \tilde{e}_1 \wedge \tilde{e}_4 \\ {}^*B_4 &= \tilde{e}_2 \wedge \tilde{e}_3 \\ {}^*B_5 &= \tilde{e}_2 \wedge \tilde{e}_4 \\ {}^*B_6 &= \tilde{e}_3 \wedge \tilde{e}_4 \end{aligned} \tag{B.6}$$

■

In spite of everything, the previous proof only proves the existence of a tetrad associated with each corner cell. That proof was already quite complicated. However, we actually have said something more. In chapter four we said we don't actually have 20 vectors associated with a four simplex, but only five. And we said that those five vectors, grouped in sets of four, produced the five tetrads associated with a four simplex.

This second affirmation requires an additional proof. Such proof is nothing more that a continuation of the previous proof. We have decided to put it apart, otherwise the proof would have become extremely large, and difficult to read. This second proof, as you will see, uses some of the facts about corner cells and wedges mentioned in chapter 4. We will start from where we left off, using the same notation, terminology and the same sets and subspaces that we have already defined previously.

Theorem B.0.2. *Let $\nu < \triangle$ be a fixed four simplex. If the conditions in theorem B.0.1 hold, then, for each four-simplex $\nu < \triangle$ there exist five vectors $\vec{e}_0, \ldots, \vec{e}_4 \in \mathcal{M}$ such that the five sets that can be constructed using four of such vectors, with all of them different, are the basis associated with each corner cell mentioned in theorem B.0.1.*

Proof. Let us consider a corner cell $c_p < \nu$. If we use lemma 6 of appendix A we can easily find that $W_1(\tau_k) = W_2(\tau_k) = W_3(\tau_k) = V_1(\tau_k) \cap V_2(\tau_k) \cap V_3(\tau_k)$ for every tetrahedron τ_k in ν that intersects c_p. Also, if we use lemma 6 of Appendix A and the fact that every triangle belongs to two tetrahedra, then we will find that there are only six different sets $V_1, \ldots, V_6$ for c_p. Taking into account all this, and the proof of theorem B.0.1, we will denote as $V_i^p, i = 1, 2, 3, 4, 5, 6$ to the subspaces V_k^p associated with $c_p$and $W_k^p, k = 1, 2, 3, 4$ to the subspaces W.

Let us we take two wedges $s_1, s_2 < c_p$ such that their respective dual triangles σ_1, σ_2 belong to the same tetrahedron. Then, we will have that ${}^*B(s_1) = \vec{e}_i(c_p) \wedge \vec{e}_j(c_p)$ and, ${}^*B(s_2) = \vec{e}_i(c_p) \wedge \vec{e}_k(c_p)$ or ${}^*B(s_2) = \vec{e}_k(c_p) \wedge \vec{e}_i(c_p)$. That is to say, ${}^*B(s_1)$ and ${}^*B(s_2)$ share one vector $e_i(c_p)$.

It is also important to consider that, taking two corner cells c_u, c_v in the same four simplex ν they share three different wedges. Such wedges have dual triangles that in groups of two belong to the same tetrahedron.

Let us take another corner cell $c_q < \nu$ different from c_p. All of the above means that there exist three subspaces V_i^p, V_j^p, V_k^p associated with c_p and three subspaces V_l^q, V_m^q, V_n^q associated with c_q, such that $V_i^p = V_l^q$, $V_j^p = V_m^q$ $V_k^p = V_n^q$. Moreover, the intersections of such subspaces are different from the zero vector. This in turn implies that there exist three subspaces W_i^p, W_j^p, W_k^p associated with c_p and three subspaces W_l^q, W_m^q, W_n^q associated with c_q, such that $W_i^p = W_l^q$, $W_j^p = W_m^q$ and $W_k^p = W_n^q$. That is to say, any two corner cell $c_p, c_q < \nu$ share three subspaces W, and therefore three vectors $\vec{e}_i, \vec{e}_j, \vec{e}_k$.

In this way, given two corner cells $c_p, c_q < \nu$, instead of six we have five one-dimensional subspaces associated with them. We will denote to such subspaces as $\tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4, \tilde{W}_5$. Four of these subspaces are associated with c_p. Of those four, three are associated with both c_p and c_q. Remains a fifth set, let us say W_5, that is associated with c_q but not with c_p.

Let us consider a third corner cell $c_r < \nu$ such that $c_r \neq c_p$ and $c_r \neq c_q$. We know that c_r shares three wedges with c_p and three wedges with c_q. However, there remains a wedge $\tilde{s} < c_r$ such that $\tilde{s} \nless c_p$ and $\tilde{s} \nless c_q$. This means that five of the wedges of c_r belong also, three of them to c_p and three of them to c_q. This implies that, of the six two-dimensional subspaces V_k^r, $k = 1, \ldots, 6$ associated with c_r, only five of them are shared with c_p and c_q.

Considering the previous observations, we can see that, even with only five two dimensional-subspaces V_k in a corner cell, it is possible to generate the four one-dimensional subspaces $\tilde{W}_k$ associated with it. This means that the four one-dimensional subspaces $\tilde{W}_k^r$, $k = 1, \ldots, 4$ associated with c_r coincide with four of the five subspaces W_i, $i = 1, \ldots, 5$ associated with the set of two corner cells c_p and c_q.

It follows that for any three different corner cells c_p, c_q, c_r in ν, there exist five vectors $\vec{e}_0, \ldots, \vec{e}_4 \in \mathcal{M}$ such that three sets formed with four different of such vectors are the tetrads associated with c_p, c_q and c_r. But this affirmation is valid for any three corner cells in the same four simplex ν. Then the theorem follows from this. ■

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