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A PRICING METHOD IN A CONSTRAINED MARKET WITH DIFFERENTIAL INFORMATIONAL FRAMEWORKS

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Contents

1 Introduction

One of the most important problems in financial mathematics as well as in practice is how to price financial instruments such as options, futures, etc. It is well known that under certain assumptions, such as market completeness (a market without arbitrage opportunities where every contingent claim can be replicated), it is possible to find the price of financial instruments in a mathematically consistent way. However, in practice those hypotheses do not hold, and therefore, problems such as uniqueness of the price, arbitrage opportunities, hedging problems, etc., appear.

In the real world, there is always a group of investors that has some hedging constraints on their portfolios since they do not have access to information and proprietary technology that institutional investors have. For this type of investors perfect replication of a derivative is usually not possible, and the market shows properties of incompleteness relative to these agents. Moreover, almost any market has institutional investors who combine technical analysis, fundamental analysis, and economics to assess potential investments on stock derivatives. Their decisions depend on their informational framework, and their perception of the probabilities of future macroeconomic events. When this happens investors face the problem of what measure to use to price financial instruments. This type of environments has been a topic of interest for researchers in the field of mathematical finance, and some have created methods to deal with this problem. Some of the most common characterizations have been the existence of many martingale measures and hedging restrictions. When this happens, quantitative analysts face the problem of what criteria they have to use to pick up a probability measure to price the financial instruments they are working with. There are some works that have been addressing this problem, but some of them offer a mathematical answer rather than a financial solution of how to price derivatives. For example, the minimal martingale measure proposed by (Föllmer and Schweizer, 1991); the minimax measure by (Belini and Fritelli, 2002) or the minimal distance martingale by (Goll and Rüschendorf, 2001). Most recently and as it is explained by (Cheridito et al., 2016) there have been several works giving more financial sense about pricing derivatives under utility indifference arguments; however, utility based prices are personal and reflecting the preferences of a single agent. Moreover the structure of the function and the optimization process make the estimation of prices difficult under this methodology.

In this work we propose the following method to compute the price of a derivative in this market. Let C be the payoff of a derivative with maturity at time T. Let P^M be the price of the derivative (at time zero) in this micro-market. Let $P^{\bar{\lambda}}(C)$ be the discounted expected value of the payoff of the derivative under the martingale measure $Q^{\bar{\lambda}}$ such that the expected value of the financial indicators of the companies related with the derivative match their respective forecasts. More specifically

$$P^{\bar{\lambda}}(C) := E_{Q^{\bar{\lambda}}}\left[\tilde{C}\right] \tag{1.1}$$

with $\bar{\lambda}$ computed from

$$\max_{\substack{\bar{\lambda} \\ \|\bar{\lambda}\|=1}} H(\lambda_1, ..., \lambda_m) \tag{1.2}$$

subject to

$$E_{Q^{\bar{\lambda}}}[X_i] = \bar{X}_i \quad \text{for} \quad i = 1, ..., LM, \tag{1.3}$$

where $\tilde{C}$ is the discounted value of C under the numéraire; H is the Shannon's information entropy; X_i is the i-th macroeconomic indicator that includes part of the information of the financial health of the stocks related with the derivative, and $\bar{X}_i$ is the respective forecast. Let $p^{\alpha_j}(v_0^{(j)}, C)$ be the price for the α_j-stereotyped (characterized by a utility function $\hat{U}_{\alpha_j}(v_0^{(j)}, C)$) agent with an initial endowment $v_0^{(j)}$ with $j = 1, ..., N$. If we compute these prices for each $t = 0, 1, ..., T$, we can define $P_t^{\bar{\lambda}}(C)$, and $p_t^{\alpha_j}(v_0^{(j)}, C)$. What we propose is that the micro-market price should be of the form

$$P_t^M = \mathbf{Y}_t'\mathbf{P}_t + \varepsilon_t, \quad \text{and} \quad \|\mathbf{Y}_t\| \leq 1, \tag{1.4}$$

where

$$\mathbf{P}_t = \begin{bmatrix} P_t^{\bar{\lambda}}(C) \\ p_t^{\alpha_1}(v_0^{(1)}, C) \\ \vdots \\ p_t^{\alpha_N}(v_0^{(N)}, C) \end{bmatrix}, \tag{1.5}$$

and the errors ε_t must be studied using time series analysis to know what is the most accurate model for calibration and forecasting. Here $\|\mathbf{Y}_t\|$ represents the coefficient of participation of each agent in the market. In theory, we should have $\|\mathbf{Y}_t\| = 1$, but since there are errors that cannot be captured by ε_t, we use the inequality instead.

This approach can serve as a bridge between the theoretical economic models of market prices and the technical methods of how to find market prices of derivatives using black boxes. There have been some approaches that provide a good approximation. Abedinia et al. (2019) propose a model to find optimal offering and bidding strategies for large consumers in specific markets using stochastic hybrid approaches. Saeedi et al. (2019*b*) find a way through which decision makers can select as a risk-neutral strategy the most robust decision via robust optimization approach. Saeedi et al. (2019*a*) show a multiblock-neural network (NN) that is optimized by an algorithm to increase the training and forecasting capabilities for price and load prediction. Xu (2006) proposes new methods with financial intuition about how to price and hedge financial instruments through super-replication strategies. Consiglio and Giovanni (2008) show a mathematical model to determine the fair price of bonus and default options using constraints and super-replication via stochastic programming. Sirignano and Cont (2018) provide a Deep Learning approach to uncover evidence for exitence of a universal and stationary price formation mechanism relating the dynamics of supply and demand for a stock. Most of these models provide either a numerical approach or a theoretical financial intuition. However, we need to find a holistic approach where we not only know how different agents price derivatives, but how the macroeconomic events and other economic sectors can influence the price of financial instruments, and the a posteriori probability of occurrence of those interactions. This information will be useful for decision takers and organizations that base their strategies in the expectations on future financial events.

We know that the prices of stocks options are not only affected by the underlying assets, but also by the diversity of informational frameworks, and the other stocks that are correlated with the underlying assets of the stock option. We use, expand, and create algorithms using the theory developed by El Karoui and Rouge (2000) and Brigo et al. (2004) to describe the different types of agents in this micro-market. To know how other stocks affect the price of stock options, we break down the volatility of the underlying asset into what we call *components of volatility*, parts of the volatility of the underlying that depend on other assets. In addition to that, we need to consider all this information together in a mathematical structure that let us understand where the weights $\mathbf{Y}_t$ come from, and how they change when agents choose prices by relations of preference, and macroeconomic information. For simplicity, we will assume that the function of the expected value of the market price is a linear function of the prices proposed by each one of the agents. This allow researchers to use many approximation methods. In particular, by using linear approximations under very small intervals (Taylor approximations) of time, we can assume 1.4 should hold. In practice, because of the lack of accuracy, forecasts are never made over long periods. Therefore, we can use the estimator of $\mathbf{Y}_t$ to make predictions of the price of the derivative over short periods.

In this work, we first explore the continuous time pricing method for institutional investors. We will not only create the formula for derivatives, but also provide information about the distribution of the derivative and the algorithm used to compute the optimization problem in 1.1. Next, we study the behavior and two ways small investors price derivatives and how they intertwine. We begin exploring a continuous time pricing technique under relations of preference and the connection with *superreplicating prices*, a way of computing the price of derivatives when the investor has some market constraints that do not let her find perfect hedging strategies. After that, we set the theoretical way to compute the price of the derivative as in (El Karoui and Rouge, 2000). In the next section we explore the discrete time version for the theoretical formulas and limit theorems to recover the time continuous pricing formula. Finally in the last section, we provide new methods to estimate the parameters of each one of the pricing formulas for the institutional and retail investors. In this process we will create new maximum likelihood estimation methods combining economic restrictions, and a new method using semigroup theory. For the readability purposes most of the theory and background is located in the appendix section of this work.

2 Continuous Time Setting for Big Agents

To introduce the continuous time setting, we proceed as follows. In the most basic discrete-time financial scenario we consider a market with two assets $S_t^{(0)}$ and $S_t^{(1)}$. The asset $S_t^{(0)}$ is the riskless asset(*numéraire*), satisfying $S_t^{(0)} = (1+r)S_{t-1}^{(0)}$, and $S_t^{(1)}$ is the risky asset. We talk about an attainable contingent claim (*pay-off* of an option), C_T, if we can find a *self-financing* portfolio strategy $\xi_t := (\xi_t^0, \xi_t^1)$ (predictable process) such that the portfolio value $V_t := \xi_t \cdot (S_t^{(0)}, S_t^{(1)})$ satisfies $V_T = C_T$, and the difference between they values at time $t+1$ and t equals

$$\begin{aligned} V_{t+1} - V_t &= \xi_{t+1} \cdot (S_{t+1}^{(0)}, S_{t+1}^{(1)}) - \xi_t \cdot (S_t^{(0)}, S_t^{(1)}) \\ &= \xi_{t+1} \cdot (S_{t+1}^{(0)}, S_{t+1}^{(1)}) - \xi_{t+1} \cdot (S_t^{(0)}, S_t^{(1)}) \\ &= \xi_{t+1}^0 (S_{t+1}^{(0)} - S_t^{(0)}) + \xi_{t+1}^1 (S_{t+1}^{(1)} - S_t^{(1)}), \end{aligned} \tag{2.1}$$

where the second line in 2.1 means that the value of the portfolio investment remains the same under portfolio re-allocations at the end of each period. These portfolio processes are useful and one of the most common ways to price financial instruments. Even in a continuous time framework, under a complete market without arbitrage opportunities, we can find a unique martingale measure (risk-neutral measure), Q, that allows us to calculate the unique *arbitrage-free price*, $\pi_{t=0}(V) = E_Q[(1+r)^{-T} C_T]$ at time $t = 0$ of the contingent claim, C_T, with maturity T.

As we might expect from 2.1 the continuous time scenario should be

$$dV_t = \xi_t^0 dS_t^{(0)} + \xi_t^1 dS_t^{(1)},$$

where $(S_t^{(0)})_{t=0}^T$ and $(S_t^{(1)})_{t=0}^T$ should satisfy in some sense dynamics of the type

$$dS_t^{(0)} = r(t)S_t^{(0)}, \qquad \text{and} \qquad dS_t^{(1)} = \gamma(t)S_t^{(1)}dt + v_t S_t^{(1)}dW_t, \tag{2.2}$$

where $(W_t)_{t=0}^T$ is a standard Brownian motion, the process $(\xi_t)_{t=0}^T$ is predictable with respect to the filtration $\mathcal{F}_t = \sigma\{S_l^{(1)}, l \leq t\} = \sigma\{W_l, l \leq t\}$, and therefore the dynamics of $S^{(1)}$ splits into deterministic and random dynamics.

Fortunately, the continuous-time scenario shares similar results with the discrete-time model. In fact, a sufficient condition that assures that a continuous-time market does not have arbitrage opportunities is the existence of a risk-neutral measure Q and under which the discounted process $(S_t^{(1)}/S_t^{(0)})_{t=0}^T$ is a martingale. Moreover, in complete markets, it can be established that the unique arbitrage-free price for a contingent claim $C_T \in L^2(Q)$ is given by $\Pi_t(C_T) = S_t^{(0)} E_Q[C_T/S_T^{(0)}|\mathcal{F}_t]$, where $\mathcal{F}_t = \sigma\{S_l^{(1)}, l \leq t\} = \sigma\{W_l, l \leq t\}$ (Harrison and Pliska, 1981).

In the context of the Black-Scholes-Merton model, we can deduce from the results given in appendix A and B that the dynamics of S, under the risk neutral measure, can be assumed to be of the form

$$dS_t^{(1)} = r(t)S_t^{(1)}dt + v(t)S_t^{(1)}dW_t, \qquad S_0^{(1)} = s_0, \qquad t \in [0, T], \tag{2.3}$$

where s_0 is a positive constant value; r and v are well behaved strictly positive functions.

By using Itô's lemma with $\ln(S_t^{(1)})$, we get

$$\ln(S_t^{(1)}) = \ln(S_0^{(1)}) + \int_0^t \left(r(s) - \frac{v^2(s)}{2} \right) ds + \int_0^t v(s) dW_s,$$

and using $\int_0^t v(s)dW_s \sim \mathcal{N}\left(0, \int_0^t v^2(s)ds\right)$, we get

$$\ln\left(\frac{S_t^{(1)}}{S_0^{(1)}}\right) \sim \mathcal{N}\left(R(0,t) - \frac{1}{2}V^2(t), V^2(t)\right), \qquad R(b,t) := \int_b^t r(s)ds,$$
$$V^2(t) := \int_0^t v^2(s)ds.$$

Now, from these results and the remarks in the appendix B, we have all the tools to compute the *unique free-arbitrage price*

$$\Pi_0(C_T) = S_0^{(0)} E_Q[C_T/S_T^{(0)}|\mathcal{F}_0] = E_Q[C_T/S_T^{(0)}],$$

for any contingent claim $C_T \in L^2(Q)$ under complete and free-arbitrage markets.

Now with all these results, we can analyze the type of models that "deviate the least" from the standard financial models. This approach can give us a better understanding of how to price derivatives. There are some proposals for this approach, but perhaps the first natural step (Brigo, 2002) is via a mixture of diffusion processes. Institutional traders do not have complete deterministic information about the markets. In fact, their knowledge (that can be described by the interest rate, drift and diffusion coefficients) changes and depends on macroeconomic scenarios. This problem gives rise to different ways to compute interest rates, volatilities, etc. Because big companies usually have advanced technological platforms, qualified staff, and state-of-the-art technology, we assume that financial derivatives are *attainable* for these types of firms, i.e. there exists a self-financing portfolio strategy that is a hedge for the derivative.

Suppose that we have a financial market with riskless and risky assets $(S_t^{(0)})_{t=0}^T$, and $(S_t^{(1)})_{t=0}^T$ respectively. The dynamics of $S^{(0)}$ and $S^{(1)}$ are of the form

$$dS_t^{(0)} = r_t dS_t^{(0)} \qquad \text{and} \qquad dS_t^{(1)} = S_t^{(1)}\mu(t)dt + S_t^{(1)}\xi_t dW_t, \tag{2.4}$$

where the processes $(\mu_t), (r_t)$, and (ξ_t) depend on the macroeconomic events $\Lambda_1, \Lambda_2, ..., \Lambda_n$. The following proposition, a consequence of the results presented by Fabio Mercurio in (Brigo et al., 2004), proves that such type of markets have infinitely many martingale measures, and therefore, they are incomplete.

Proposition 2.1. *Let $(\tilde{W}_t)_{t\in[0,T]}$ be a one dimensional Brownian motion on a filtered probability space $(\Omega^{\tilde{W}}, \mathcal{F}_T^{\tilde{W}}, (\mathcal{F}_t^{\tilde{W}})_{t\in[0,T]}, P^{\tilde{W}})$; and let $(\Omega^\Lambda, \mathcal{F}^\Lambda, P^\Lambda)$ be a probability space with $\Omega^\Lambda = \{\Lambda_1, ..., \Lambda_m\}$ and probability measure satisfying $P^\Lambda(\Lambda_j) := p_j > 0$ for $j = 1, ..., m$. Set the process $(\tilde{\xi}_t)_{t\in[0,T]}$, $(\tilde{\mu}_t)_{t\in[0,T]}$, and $(\tilde{r}_t)_{t\in[0,T]}$ such that:*

1. *$\tilde{\xi}_t(\Lambda_j) = v_j(t)$ for each j and $t \geq 0$, where $v_1, ..., v_m$ are strictly positive, continuous, and bounded away from zero,*
2. *$\tilde{\mu}_t(\Lambda_j) = \mu_j(t)$ for each j and $t \geq 0$, where $\mu_1, ..., \mu_m$ are continuous,*
3. *$\tilde{r}_t(\Lambda_j) = r_j(t)$ for each j and $t \geq 0$, where $r_1, ..., r_m$ are continuous.*

Set $\Omega := \Omega^\Lambda \times \Omega^{\tilde{W}}$, $\mathcal{F}_t := \mathcal{F}^\Lambda \otimes \mathcal{F}_t^{\tilde{W}}$, $P := P^\Lambda \otimes P^{\tilde{W}}$; and let $\xi_t(\Lambda, y) := \tilde{\xi}_t(\Lambda)$, $\mu_t(\Lambda, y) := \tilde{\mu}_t(\Lambda)$, $r_t(\Lambda, y) := \tilde{r}_t(\Lambda)$, and $W_t(\Lambda, y) := \tilde{W}_t(y)$ be defined on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, P)$. If we define a measure Q^Λ in $(\Omega, \mathcal{F}_T)$ by

$$\frac{dQ^\Lambda}{dP}(\Lambda_j, y) := \frac{\lambda_j}{p_j} \qquad \textit{for all} \quad y \in \Omega^{\tilde{W}}, \tag{2.5}$$

where λ_j is strictly positive for each $j = 1, ..., m$ with $\sum_{j=1}^m \lambda_j = 1$; and the process $(S_t^{(1)})_{t\in[0,T]}$ satisfies

$$dS_t^{(1)} = S_t^{(1)}\mu(t)dt + S_t^{(1)}\xi_t dW_t;$$

then there exist a risk-neutral measure $Q^{\bar{\lambda}}$ in $(\Omega, \mathcal{F}_T)$ associated with $(S_t)_{t\in[0,T]}$ and the numéraire $S_t^{(0)} = e^{\int_0^t r_u(\cdot,\cdot)du}$ that satisfy

$$\frac{dQ^{\bar{\lambda}}}{dP} = \frac{dQ^\Lambda}{dP} \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{\mu(t) - r(t)}{\xi_t}\right)^2 dt - \int_0^T \left(\frac{\mu(t) - r(t)}{\xi_t}\right) dW_t\right\}. \tag{2.6}$$

Proof. To prove this result, it is enough to show that $(S_t^{(1)}/S_t^{(0)})_{t\in[0,T]}$ is an $(\mathcal{F}_t, Q^{\bar{\lambda}})$-martingale (see for instance Harrison and Pliska, 1981; Shreve, 2004; Privault, 2020). We observe that $(W_t)_{t\in[0,T]}$ and $(\xi_t)_{t\in[0,T]}$ are independent on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$. Moreover, $(W_t)_{t\in[0,T]}$ is an $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$-Brownian motion, and $(\xi_t)_{t\in[0,T]}$ has the same law as $(\tilde{\xi}_t)_{t\in[0,T]}$.

To prove that $Q^{\bar{\lambda}}$ is a measure on $\mathcal{F}_T$, it is enough to verify $E_P\left[dQ^{\bar{\lambda}}/dP\right] = 1$. Let us then set

$$L_t := -\int_0^t \left(\frac{\mu(l) - r(l)}{\xi_l}\right) dW_l, \ \ L_t^i := -\int_0^t \left(\frac{\mu_i(l) - r_i(l)}{v_i(l)}\right) dW_l, \ \ Z_t^i := \exp\left\{L_t^i - \frac{1}{2}\langle L^i\rangle_t\right\}. \quad (2.7)$$

The process $(Z_t^i)_{t\in[0,T]}$ is such that $Z_t^i = 1 + \int_0^t Z_s^i dL_s^i$ and $\langle L^i\rangle$ is deterministic and bounded on $[0,T]$; therefore $(Z_t^i)_{t\in[0,T]}$ is a martingale (see results and remarks in A.3) and then $E[Z_l^i] = 1$ for all $l \in [0,T]$. Finally, we use a linear combination of the indicator functions $1_{\{\Lambda_i\}\times\Omega^{\tilde{W}}}$ and independence to conclude $E_P\left[dQ/dP\right] = 1$.

Now, we show that $S_T^{(1)} e^{-\int_0^T r_u(\cdot,\cdot)du}$ is a $Q^{\bar{\lambda}}$- martingale. By Bayes' formula (see appendix C), we have

$$E_{Q^{\bar{\lambda}}}\left[S_T^{(1)} e^{-\int_0^T r_u(\cdot,\cdot)du}\middle|\, \mathcal{F}_t\right] = \frac{E_P\left[S_T^{(1)} e^{-\int_0^T r_u(\cdot,\cdot)du} \dfrac{dQ^\Lambda}{dP} e^{L_T - \frac{1}{2}\langle L\rangle_T} \middle|\, \mathcal{F}_t\right]}{E_P\left[\dfrac{dQ^\Lambda}{dP} e^{L_T - \frac{1}{2}\langle L\rangle_T}\middle|\, \mathcal{F}_t\right]}$$

$$= \frac{E_P\left[S_T^{(1)} e^{-\int_0^T r_u(\cdot,\cdot)du} \dfrac{dQ^\Lambda}{dP} e^{L_t - \frac{1}{2}\langle L\rangle_t} e^{(L_T - L_t) - \frac{1}{2}\langle L\rangle_t^T} \middle|\, \mathcal{F}_t\right]}{E_P\left[\dfrac{dQ^\Lambda}{dP} e^{L_T - \frac{1}{2}\langle L\rangle_T}\middle|\, \mathcal{F}_t\right]}.$$

Using the notation in 2.7, the fact that each $1_{\Lambda_j\times\Omega^{\tilde{W}}}$ is $\mathcal{F}_t$-measurable, and all of the terms are positive, the last formula equals

$$\frac{S_t^{(1)} e^{-\int_0^t r_u(\cdot,\cdot)du} \frac{dQ^\Lambda}{dP} e^{L_t - \frac{1}{2}\langle L\rangle_t} E_P\left[\frac{S_T^{(1)}}{S_t^{(1)}} e^{-\int_t^T r_u(\cdot,\cdot)du} e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T} \middle| \mathcal{F}_t\right]}{\frac{dQ^\Lambda}{dP} e^{L_t - \frac{1}{2}\langle L\rangle_t} E_P\left[e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T} \middle| \mathcal{F}_t\right]}$$

$$= \frac{S_t^{(1)} e^{-\int_0^t r_u(\cdot,\cdot)du} \left\{\sum_{j=1}^m 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} E_P\left[e^{\int_t^T \left(\mu_j(u) - \frac{v_j^2(u)}{2} - r_j(u)\right)du + \int_t^T v_j(u)dW_u} e^{(L_T^j - L_t^j) - \frac{1}{2}\langle L^j\rangle_t^T} \middle| \mathcal{F}_t\right]\right\}}{\sum_{j=1}^m 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} E_P\left[e^{(L_T^j - L_t^j)-\frac{1}{2}\langle L^j\rangle_t^T} \middle| \mathcal{F}_t\right]}.$$

By proposition A.3, and the fact that $(W_t)_{t\in[0,T]}$ have independent and stationary increments (by construction), the last part equals

$$\frac{S_t^{(1)} e^{-\int_0^t r_u(\cdot,\cdot)du} \sum_{j=1}^m 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} E_P\left[e^{\int_t^T \left(\frac{v_j^2(u) - (\mu_j(u) - r_j(u))}{v_j(u)}\right)dW_u - \frac{1}{2}\int_t^T \left(\frac{v_j^2(u) - (\mu_j(u) - r_j(u))}{v_j(u)}\right)^2 du} \middle| \mathcal{F}_t\right]}{\sum_{j=1}^m 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} E_P\left[e^{(L_T^j - L_t^j)-\frac{1}{2}\langle L^j\rangle_t^T}\right]}$$

$$= S_t^{(1)} e^{-\int_0^t r_u(\cdot,\cdot)du} \sum_{j=1}^m 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} E_P\left[e^{\int_t^T \left(\frac{v_j^2(u) - (\mu_j(u) - r_j(u))}{v_j(u)}\right)dW_u - \frac{1}{2}\int_t^T \left(\frac{v_j^2(u) - (\mu_j(u) - r_j(u))}{v_j(u)}\right)^2 du}\right]$$

$$= S_t^{(1)} e^{-\int_0^t r_u(\cdot,\cdot)du}.$$

which ends the proof. □

This result shows that for each set of positive numbers $\lambda_1, ..., \lambda_m$ such that $\sum_{j=1}^m \lambda_j = 1$, we get a martingale measure of the form 2.6. Moreover, we have information about the density of $S^{(1)}$ under P in Proposition 2.1 because for each $0 \le u \le t \le T$ and $A \in \mathcal{B}_{\mathbb{R}_+}$

$$E_P[S_t^{(1)} \in A] = E_P\left[\sum_{j=1}^m 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} 1_{\{S_t^{j,(1)}\in A\}}\right] = \sum_{j=1}^m E_P\left[1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}}\right] E_P\left[1_{S_t^{j,(1)}\in A}\right]$$

$$= \sum_{j=1}^m p_j \int_A P(S_t^{j,(1)} \in dl) = \sum_{j=1}^m p_j \int_A \frac{\exp\left\{\frac{-\left(\ln(l) - \int_0^t \left(\mu_j(z) - \frac{v_j^2(z)}{2}\right) dz - \ln(S_0^{j,(1)})\right)^2}{2\int_0^t v_j^2(z)dz}\right\}}{l\left(\int_0^t v_j^2(z)dz\right)^{1/2}\sqrt{2\pi}} dl, \tag{2.8}$$

where $S_t^{j,(1)}$ satisfies

$$dS_t^{j,(1)} = S_t^{j,(1)}\mu_j(t)dt + S_t^{j,(1)}v_j(t)dW_t,$$

under P. However, the density of $S^{(1)}$ is slightly different under $Q^{\bar{\lambda}}$ as is showed in the next corollary.

Corollary 2.2. *Let $S^{(0)}$ and $S^{(1)}$ be as in Proposition 2.1 in the investment period $[0,T]$. Then the density of $S^{(1)}$ under $Q^{\bar{\lambda}}$ is of the form*

$$Q^{\bar{\lambda}}\left(S_t^{(1)} \in dx\right) = \sum_{i=1}^m \frac{\lambda_i \exp\left\{\frac{-\left(\ln(x) - \int_0^t \left(r_i(z) - \frac{v_i^2(z)}{2}\right) dz - \ln(S_0^{i,(1)})\right)^2}{2\int_0^t v_i^2(z)dz}\right\}}{x\left(\int_0^t v_i^2(z)dz\right)^{1/2}\sqrt{2\pi}}. \tag{2.9}$$

Proof. Under each $\{\Lambda_j\}\times\Omega^{\tilde{W}}$ the dynamics of $S^{(1)}$ is of the form

$$dS_t^{(1)} = S_t^{(1)}\mu_j(t)dt + S_t^{(1)}v_j(t)dW_t. \tag{2.10}$$

If we denote by $S^{j,(1)}$ the process that satisfies the previous dynamics, and let

$$\frac{dQ^{j,\bar{\lambda}}}{dP} := \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{\mu_j(t)-r_j(t)}{v_j(t)}\right)^2 dt - \int_0^T \left(\frac{\mu_j(t)-r_j(t)}{v_j(t)}\right) dW_t\right\}, \tag{2.11}$$

we have, by Girsanov's theorem, that

$$W_t^j := W_t - \left\langle W_\cdot, -\int_0^\cdot \left(\frac{\mu_j(l)-r_j(l)}{v_j(l)}\right) dW_l\right\rangle_t = W_t + \int_0^t \left(\frac{\mu_j(l)-r_j(l)}{v_j(l)}\right) dl \tag{2.12}$$

is a $Q^{j,\bar{\lambda}}-$Brownian motion, and $S^{j,(1)}$ satisfies

$$dS_t^{j,(1)} = S_t^{j,(1)}r_j(t)dt + S_t^{j,(1)}v_j(t)dW_t^j, \tag{2.13}$$

under $Q^{j,\bar{\lambda}}$. Therefore, for each $A \in \mathcal{B}(\mathbb{R}_+)$, we have

$$
\begin{aligned}
&E_{Q^{\bar{\lambda}}}[S_t^{(1)} \in A] \\
&= E_P\left[\sum_{j=1}^{m} 1_{\{\Lambda_j\}\times\Omega^{\tilde{W}}} 1_{\{S_t^{j,(1)}\in A\}} \left(\frac{\lambda_j}{p_j}\right) \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{\mu_j(t)-r_j(t)}{v_j(t)}\right)^2 dt - \int_0^T \left(\frac{\mu_j(t)-r_j(t)}{v_j(t)}\right) dW_t\right\}\right] \\
&= \sum_{j=1}^{m} \lambda_j E_P\left[1_{\{S_t^{j,(1)}\in A\}} \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{\mu_j(t)-r_j(t)}{v_j(t)}\right)^2 dt - \int_0^T \left(\frac{\mu_j(t)-r_j(t)}{v_j(t)}\right) dW_t\right\}\right] \\
&= \sum_{j=1}^{m} \lambda_j E_{Q^{j,\bar{\lambda}}}\left[1_{\{S_t^{j,(1)}\in A\}}\right] \\
&= \sum_{j=1}^{m} \lambda_j \int_A \frac{\exp\left\{\dfrac{-\left(\ln(l) - \int_0^t \left(r_j(z) - \dfrac{v_j^2(z)}{2}\right) dz - \ln(S_0^{j,(1)})\right)^2}{2\int_0^t v_j^2(z)dz}\right\}}{l\left(\int_0^t v_j^2(z)dz\right)^{1/2}\sqrt{2\pi}} dl.
\end{aligned}
\tag{2.14}
$$

□

As it is expected, it is possible to expand this result to some risky assets. However, we need to add some tools to make it work.

Suppose that we have a financial market with riskless asset $(S_t^{(0)})_{t\in[0,T]}$ and risky assets $(S_t^{(i)})_{t\in[0,T]}$ with $i = 1, ..., n$. The dynamics of $(S_t^{(0)})$ and $(S_t^{(i)})$ are of the form

$$
dS_t^{(0)} = r_t dS_t^{(0)} \quad \text{and} \quad dS_t^{(i)} = \mu_t^i S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} \sigma_t^{(ij)} dW_t^{(j)}\right), \tag{2.15}
$$

for each $i = 1, ..., n$. The processes r_t, $\mu_t^{(i)}$, and $\sigma_t^{(ij)}$ depend on the macroeconomic events $\Lambda_1, ..., \Lambda_m$. The following proposition, the multiprice version of the model presented by (Brigo et al., 2004), proves that such type of markets have infinitely many martingale measures, and therefore, they are incomplete.

Intuitively, the system of prices 2.15, under the macroeconomic event Λ_k, is of the form

$$
dS_t^{(0)} = {}^k r_t dS_t^{(0)} \quad \text{and} \quad dS_t^{(i)} = {}^k\mu_t^{(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} {}^k\sigma_t^{(ij)} dW_t^{(j)}\right). \tag{2.16}
$$

Using Ito's formula, we get

$$
d\left(\frac{S_t^{(i)}}{S_t^{(0)}}\right) = \left({}^k\mu_t^{(i)} - {}^k r_t\right) \frac{S_t^{(i)}}{S_t^{(0)}} dt + \frac{S_t^{(i)}}{S_t^{(0)}} \sum_{j=1}^{n} {}^k\sigma_t^{(ij)} dW_t^{(j)}. \tag{2.17}
$$

Now, we will make some assumptions about the matrix ${}^k\sigma_t$ for all $t \in [0,T]$.

Assumption 2.3. *For each $k = 1, ..., m$, the matrix ${}^{(k)}\sigma_t\, {}^{(k)}\sigma_t^*$ is non-degenerate in the sense that $\theta^* \left({}^k\sigma_t\, {}^k\sigma_t^*\right) \theta > \epsilon\theta^*\theta$ for all $(\theta, t) \in \mathbb{R}^n \setminus \{0\} \times [0,T]$ and for some $\epsilon > 0$. Here the symbol $*$ means transpose.*

With this assumption we can find a process ${}^k\theta_t = ({}^k\theta_t^{(1)}, ..., {}^k\theta_t^{(n)})$ such that

$$ {}^k\mu_t^{(i)} - {}^kr_t = \sum_{j=1}^{n} {}^k\sigma_t^{(ij)}\, {}^k\theta_t^{(j)}, \tag{2.18}$$

for each $i = 1, ..., n$ and $k = 1, ..., m$. Then using ${}^k\theta_t = ({}^k\theta_t^{(1)}, ..., {}^k\theta_t^{(n)})$ the system for prices 2.17 "under Λ_k" becomes

$$ d\left(\frac{S_t^{(i)}}{S_t^{(0)}}\right) = \frac{S_t^{(i)}}{S_t^{(0)}} \sum_{j=1}^{n} {}^k\sigma_t^{(ij)} \left({}^k\theta_t^{(j)}dt + dW_t^{(j)}\right). \tag{2.19}$$

Now we define

$$ \left.\frac{dQ^{k\theta}}{dP}\right|_{\mathcal{F}_t} = \exp\left\{-\int_0^t {}^k\theta_l \cdot dW_l - \frac{1}{2}\int_0^t |{}^k\theta_l|^2 dl\right\}. \tag{2.20}$$

Using the vector form of Girsanov's Theorem, we get that the process $({}^kW_t)$ with dynamics

$$ d({}^kW_t) = {}^k\theta_t dt + dW_t \tag{2.21}$$

is an $n-$dimensional Brownian motion under $Q^{k\theta}$. As a result, by using Ito's formula and the expression 2.19, we get that $S_t^{(i)}/S_t^{(0)}$ "under Λ_k" is of the form

$$ \frac{S_t^{(i)}}{S_t^{(0)}} = \frac{S_0^{(i)}}{S_0^{(0)}} \exp\left\{\sum_{j=1}^{n}\int_0^t {}^k\sigma_t^{(ij)} d({}^kW_t^{(j)}) - \frac{1}{2}\sum_{j=1}^{n}\int_0^t \left|{}^k\sigma_t^{(ij)}\right|^2 dl\right\}. \tag{2.22}$$

Assuming that the coefficients ${}^k\sigma^{(ij)}$, ${}^k\mu$, ${}^kr$, and ${}^k\theta$ are uniformly bounded on $\Omega \times [0,T]$, we can use Novikov condition to get that $S_t^{(i)}/S_t^{(0)}$ is a $Q^{k\theta}$-martingale "under Λ_k." Now, we are ready to show these remarks in a formal way.

Proposition 2.4. *Let $(\tilde{W}_t) = (\tilde{W}_t^{(1)}, ..., \tilde{W}_t^{(n)})_{t\in[0,T]}$ be an n-dimensional Brownian motion on a filtered probability space $(\Omega^{\tilde{W}}, \mathcal{F}_T^{\tilde{W}}, (\mathcal{F}_t^{\tilde{W}})_{t\in[0,T]}, P^{\tilde{W}})$; and let $(\Omega^\Lambda, \mathcal{F}^\Lambda, P^\Lambda)$ be a probability space with $\Omega^\Lambda = \{\Lambda_1, ..., \Lambda_m\}$ and probability measure satisfying $P^\Lambda(\Lambda_k) := p_k > 0$ for $k = 1, ..., m$. Set the process $(\tilde{\sigma}_t^{(ij)})_{t\in[0,T]}$, $(\tilde{\mu}_t^{(i)})_{t\in[0,T]}$, and $(\tilde{r}_t^{(i)})_{t\in[0,T]}$ such that:*

1. *$\tilde{\sigma}_t^{(ij)}(\Lambda_k) = {}^k\sigma_t^{(ij)}$ for each $k = 1, ..., m$; $i, j = 1, ..., n$; and $t \in [0,T]$, where each ${}^k\sigma_t^{(ij)}$ is deterministic, strictly positive, continuous, and bounded away from zero and satisfies assumption 2.3.*

2. *$\tilde{\mu}_t^{(i)}(\Lambda_k) = {}^k\mu_t^i$ for each $k = 1, ..., m$; $i = 1, ..., n$; and $t \in [0,T]$, where each ${}^k\mu_t^i$ is deterministic, and continuous.*

3. $\tilde{r}_t(\Lambda_k) = {}^k r_t$ *for each* $k = 1, ..., m$*; and* $t \in [0, T]$*, where each* ${}^k r_t$ *is deterministic and continuous.*

Set $\Omega := \Omega^{\Lambda} \times \Omega^{\tilde{W}}$*,* $\mathcal{F}_t := \mathcal{F}^{\Lambda} \otimes \mathcal{F}_t^{\tilde{W}}$*,* $P := P^{\Lambda} \otimes P^{\tilde{W}}$*; and let* $\sigma_t^{(ij)}(\Lambda, y) := \tilde{\sigma}_t^{(ij)}(\Lambda)$*,* $\mu_t^{(i)}(\Lambda, y) := \tilde{\mu}_t^{(i)}(\Lambda)$*,* $r_t(\Lambda, y) := \tilde{r}_t(\Lambda)$*, and* $W_t(\Lambda, y) := \tilde{W}_t(y)$ *be defined on* $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, P)$*. If we define a measure* Q^{Λ} *in* $(\Omega, \mathcal{F}_T)$ *by*

$$\frac{dQ^{\Lambda}}{dP}(\Lambda_k, y) := \frac{\lambda_k}{p_k} \qquad \textit{for all} \quad y \in \Omega^{\tilde{W}}, \tag{2.23}$$

where λ_k *is strictly positive for each* $k = 1, ..., m$ *with* $\sum_{k=1}^m \lambda_k = 1$*; and the processes* $(S_t^{(0)})_{t\in[0,T]}$ *and* $(S_t^{(i)})_{t\in[0,T]}$ *satisfy*

$$dS_t^{(0)} = r_t S_t^{(0)} \quad \textit{and} \quad dS_t^{(i)} = \mu_t^{(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^n \sigma_t^{(ij)} dW_t^{(j)} \right), \tag{2.24}$$

for each $i = 1, ..., n$*, respectively, then there exist a risk-neutral measure* $Q^{\bar{\lambda}}$ *in* $(\Omega, \mathcal{F}_T)$ *associated with* $(S_t)_{t\in[0,T]} = (S_t^{(1)}, ..., S_t^{(n)})_{t\in[0,T]}$ *and the numéraire* $S_t^{(0)} = e^{\int_0^t r_u(\cdot,\cdot)du}$ *that satisfy*

$$\frac{dQ^{\bar{\lambda}}}{dP} = \frac{dQ^{\Lambda}}{dP} \exp\left\{ -\int_0^T \theta_l \cdot dW_l - \frac{1}{2}\int_0^T |\theta_l|^2 dl \right\}, \tag{2.25}$$

where $\theta_t = (\theta_t^{(1)}, ..., \theta_t^{(n)})^*_{t\in[0,T]}$ *is the* $\mathcal{F}_t$*-measurable process that satisfies 2.18, i.e.*

$$\begin{bmatrix} \sigma_t^{(11)} & \dots & \sigma_t^{(1n)} \\ \vdots & \ddots & \\ \sigma_t^{(n1)} & \dots & \sigma_t^{(nn)} \end{bmatrix} \begin{bmatrix} \theta_t^{(1)} \\ \vdots \\ \theta_t^{(n)} \end{bmatrix} = \begin{bmatrix} \mu_t^{(1)} - r_t \\ \vdots \\ \mu_t^{(n)} - r_t \end{bmatrix}, \tag{2.26}$$

and $\theta_t(\Lambda_k, y) = (\theta_t^{(1)}(\Lambda_k, y), ..., \theta_t^{(n)}(\Lambda_k, y))_{t\in[0,T]}$ *is the process* ${}^k\theta_t = ({}^k\theta_t^{(1)}, ..., {}^k\theta_t^{(n)})$ *as in 2.18.*

Proof. Abusing the notation in the statement, we use $r_t, \mu_t^{(i)}, \sigma_t^{(ij)}$, and θ_t instead of $r_t(\cdot,\cdot), \mu_t^{(i)}(\cdot,\cdot), \sigma_t^{(ij)}(\cdot,\cdot)$, and $\theta_t(\cdot,\cdot)$ respectively. This is to make the notation less awkward.

We only need to show that $(S_t^{(i)}/S_t^{(0)})_{t\in[0,T]}$ is an $(\mathcal{F}_t, Q^{\bar{\lambda}})$-martingale (see for instance Harrison and Pliska, 1981; Shreve, 2004; Privault, 2020). We point out that $(W_t)_{t\in[0,T]}$ is independent of each $(\sigma_t^{(ij)})_{t\in[0,T]}$, $(r_t^{(i)})_{t\in[0,T]}$, and $(\mu_t^{(i)})_{t\in[0,T]}$ on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$. Moreover, $(W_t)_{t\in[0,T]}$ is an n-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$, and $(\sigma_t^{(ij)})_{t\in[0,T]}$, $(\mu_t^{(i)})_{t\in[0,T]}$, and $(r_t)_{t\in[0,T]}$ have the same laws as $(\tilde{\sigma}_t^{(ij)})_{t\in[0,T]}$, $(\tilde{\mu}_t^{(i)})_{t\in[0,T]}$, and $(\tilde{r}_t)_{t\in[0,T]}$ respectively.

To prove that $Q^{\bar{\lambda}}$ is a measure on $\mathcal{F}_T$ it is enough to show that $E_P[dQ^{\bar{\lambda}}/dP] = 1$. Let us then set

$$L_t := -\int_0^t \theta_l \cdot dW_l \quad {}^kL_t := -\int_0^t {}^k\theta_l \cdot dW_l \quad \text{and} \quad {}^kZ_t := \exp\{{}^kL_t - \frac{1}{2}\langle {}^kL\rangle_t\}. \tag{2.27}$$

The process $({}^kZ_t)_{t\in[0,T]}$ satisfies the SDE ${}^kZ_t = 1 + \int_0^t {}^kZ_l d({}^kL_l)$. Since the quadratic variation $\langle {}^kL\rangle$ is deterministic and bounded in $[0,T]$; therefore, $({}^kZ_t)_{t\in[0,T]}$ is a martingale (see for instance Meyer, 2000) and then $E[{}^kZ_l] = 1$ for all $l \in [0,T]$. Finally, we use a linear combination of the indicator functions $1_{\Lambda_i\times\Omega^{\tilde{W}}}$ and independence to conclude $E_P\left[dQ^{\tilde{\lambda}}/dP\right] = 1$.

We now show that $S_t^{(i)}e^{-\int_0^t r_l dl}$ is an $(\mathcal{F}_t, Q^{\tilde{\lambda}})$-martingale. By Bayes' formula, we have

$$E_{Q^{\tilde{\lambda}}}\left[S_T^{(i)}e^{-\int_0^T r_u du}\middle|\mathcal{F}_t\right] = \frac{E_P\left[S_T^{(i)}e^{-\int_0^T r_u du}\dfrac{dQ^\Lambda}{dP}e^{L_T-\frac{1}{2}\langle L\rangle_T}\middle|\mathcal{F}_t\right]}{E_P\left[\dfrac{dQ^\Lambda}{dP}e^{L_T-\frac{1}{2}\langle L\rangle_T}\middle|\mathcal{F}_t\right]}$$

$$= \frac{E_P\left[S_T^{(i)}e^{-\int_0^T r_u du}\dfrac{dQ^\Lambda}{dP}e^{L_t-\frac{1}{2}\langle L\rangle_t}e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]}{E_P\left[\dfrac{dQ^\Lambda}{dP}e^{L_T-\frac{1}{2}\langle L\rangle_T}\middle|\mathcal{F}_t\right]}.$$

Using the notation in 2.27, the fact that each $1_{\Lambda_k\times\Omega^{\tilde{W}}}$ is $\mathcal{F}_t$-measurable, ${}^kZ_t$ is a martingale that has stationary and independent increments, and all of the terms are positive, the last formula equals

$$\frac{S_t^{(i)}e^{-\int_0^t r_u du}\dfrac{dQ^\Lambda}{dP}e^{L_t-\frac{1}{2}\langle L\rangle_t}E_P\left[\dfrac{S_T^{(i)}}{S_t^{(i)}}e^{-\int_t^T r_u du}e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]}{\dfrac{dQ^\Lambda}{dP}e^{L_t-\frac{1}{2}\langle L\rangle_t}E_P\left[e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]}$$

$$= \frac{S_t^{(i)}e^{-\int_0^t r_u du}\left\{\sum_{k=1}^m 1_{\Lambda_k\times\Omega^{\tilde{W}}}E_P\left[\dfrac{S_T^{(i)}}{S_t^{(i)}}e^{-\int_t^T r_u du}e^{({}^kL_T-{}^kL_t)-\frac{1}{2}\langle {}^kL\rangle_t^T}\middle|\mathcal{F}_t\right]\right\}}{\sum_{k=1}^m 1_{\Lambda_k\times\Omega^{\tilde{W}}}E_P\left[e^{({}^kL_T-{}^kL_t)-\frac{1}{2}\langle {}^kL\rangle_t^T}\middle|\mathcal{F}_t\right]}$$

$$= S_t^{(i)}e^{-\int_0^t r_u du}\left\{\sum_{k=1}^m 1_{\Lambda_k\times\Omega^{\tilde{W}}}E_P\left[\frac{S_T^{(i)}}{S_t^{(i)}}e^{-\int_t^T r_u du}e^{({}^kL_T-{}^kL_t)-\frac{1}{2}\langle {}^kL\rangle_t^T}\middle|\mathcal{F}_t\right]\right\}.$$

We know that under $\Lambda_k\times\Omega^{\tilde{W}}$ and using Ito's formula on 2.16

$$S_t^{(i)} = S_0^{(i)}\exp\left\{\int_0^t\left({}^k\mu_u^{(i)} - \frac{1}{2}\sum_{j=1}^n\left({}^k\sigma_u^{(ij)}\right)^2\right)du + \sum_{j=1}^n\int_0^t {}^k\sigma_u^{(ij)}dW_u^{(j)}\right\}, \tag{2.28}$$

and

$$\sum_{j=1}^{n} {}^k\sigma_u^{(ij)}\, {}^k\theta_u^{(j)} = {}^k\mu_u^{(i)} - {}^k r_u, \tag{2.29}$$

for $i = 1, ..., n$ and $k = 1, ..., m$. With these remarks, we conclude that

$$\begin{aligned}
&S_t^{(i)} e^{-\int_0^t r_u du} \left\{ \sum_{k=1}^{m} 1_{\Lambda_k \times \Omega^{\tilde W}} E_P \left[\frac{S_T^{(i)}}{S_t^{(i)}} e^{-\int_t^T r_u du} e^{({}^kL_T - {}^kL_t) - \frac{1}{2}\langle {}^kL\rangle_t^T} \middle| \mathcal{F}_t \right] \right\} \\
&= S_t^{(i)} e^{-\int_0^t r_u du} \sum_{k=1}^{m} 1_{\Lambda_k \times \Omega^{\tilde W}} E_P \left[e^{\int_t^T \left({}^k\mu_u^{(i)} - {}^kr_u - \frac{1}{2} \sum_{j=1}^n \left({}^k\sigma_u^{(ij)} \right)^2 + \left({}^k\theta_u^{(j)} \right)^2 \right) du + \sum_{j=1}^n \int_t^T ({}^k\sigma_u^{(ij)} - {}^k\theta_u^{(j)}) dW_u^{(j)}} \middle| \mathcal{F}_t \right] \\
&= S_t^{(i)} e^{-\int_0^t r_u du} \sum_{k=1}^{m} 1_{\Lambda_k \times \Omega^{\tilde W}} E_P \left[e^{\sum_{j=1}^n \int_t^T ({}^k\sigma_u^{(ij)} - {}^k\theta_u^{(j)}) dW_u^{(j)} - \frac{1}{2} \left\langle \sum_{j=1}^n \int_0^\cdot ({}^k\sigma_u^{(ij)} - {}^k\theta_u^{(j)}) dW_u^{(j)} \right\rangle_t^T} \middle| \mathcal{F}_t \right] \\
&= S_t^{(i)} e^{-\int_0^t r_u du},
\end{aligned}$$

which ends the proof. □

This result shows that for each set of positive numbers $\lambda_1, ..., \lambda_m$ such that $\sum_{j=1}^m \lambda_j = 1$, we get a martingale measure of the form 2.25. This rises the question, What criterion should be used to choose the martingale measure, and therefore, the price of the derivative? Before solving this question, we will give the characterization of the distribution of the prices under the measure $Q^{\bar\lambda}$.

Corollary 2.5. *Let $(S_t^{(0)})_{t\in[0,T]}$, and $(S_t^{(1)}, ..., S_t^{(n)})_{t\in[0,T]}$ be as in Proposition 2.4. Then the density of $S^{(i)}$ under $Q^{\bar\lambda}$ is of the form*

$$Q^{\bar\lambda}\left(S_t^{(i)} \in dx\right) = \sum_{k=1}^{m} \frac{\lambda_k \exp\left\{ \dfrac{-\left(\ln(x) - \int_0^t \left({}^kr_z - \frac{1}{2} \sum_{j=1}^n \left({}^k\sigma_z^{(ij)} \right)^2 \right) dz - \ln({}^kS_0^{(i)}) \right)^2}{2 \sum_{j=1}^n \int_0^t \left({}^k\sigma_z^{(ij)} \right)^2 dz} \right\}}{x \left(\sum_{j=1}^n \int_0^t \left({}^k\sigma_z^{(ij)} \right)^2 dz \right)^{1/2} \sqrt{2\pi}}. \tag{2.30}$$

Proof. Under each $\{\Lambda_k\} \times \Omega^{\tilde W}$ the dynamics of $S^{(i)}$ is of the form

$$dS_t^{(i)} = {}^k\mu_t^{(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} {}^k\sigma_t^{(ij)} dW_t^{(j)} \right). \tag{2.31}$$

If we denote by ${}^kS^{(i)}$ the process that satisfies the previous dynamics, and letting

$$\frac{d({}^kQ^{\bar\lambda})}{dP} := \exp\left\{ -\int_0^T {}^k\theta_l \cdot dW_l - \frac{1}{2} \int_0^T |\,{}^k\theta_l|^2 dl \right\}, \tag{2.32}$$

we have, by the vector form of Girsanov's theorem, that ${}^kW_t := \int_0^t {}^k\theta_l dl + W_t$ is an $n-$dimensional Brownian motion under ${}^kQ^{\bar{\lambda}}$. Moreover, ${}^kS^{(i)}$ satisfies

$$\begin{aligned} d\,{}^kS_t^{(i)} &= {}^kS_t^{(i)}\,{}^kr_t dt + {}^kS_t^{(i)} \left(\sum_{j=1}^{n} \left({}^k\sigma_t^{(ij)}\,{}^k\theta_t^{(j)} \right) dt + {}^k\sigma_t^{(ij)} dW_t^{(j)} \right) \\ &= {}^kS_t^{(i)}\,{}^kr_t dt + {}^kS_t^{(i)} \sum_{j=1}^{n} {}^k\sigma_t^{(ij)} d\left({}^kW_t^{(j)} \right) \end{aligned} \tag{2.33}$$

under ${}^kQ^{\bar{\lambda}}$. Therefore, for each $A \in \mathcal{B}(\mathbb{R}_+)$ we have

$$\begin{aligned} &E_{Q^{\bar{\lambda}}}[S_t^{(i)} \in A] \\ &= E_P\left[\sum_{k=1}^{m} 1_{\Lambda_k \times \Omega^{\tilde{W}}} 1_{\{{}^kS_t^{(i)} \in A\}} \left(\frac{\lambda_k}{p_k} \right) \exp\left\{ -\int_0^T {}^k\theta_l \cdot dW_l - \frac{1}{2} \int_0^T |{}^k\theta_l|^2 dl \right\} \right] \\ &= \sum_{k=1}^{m} \lambda_k E_P\left[1_{\{{}^kS_t^{(i)} \in A\}} \exp\left\{ -\int_0^T {}^k\theta_l \cdot dW_l - \frac{1}{2} \int_0^T |{}^k\theta_l|^2 dl \right\} \right] \\ &= \sum_{k=1}^{m} \lambda_k E_{\left({}^kQ^{\bar{\lambda}}\right)} \left[1_{\{{}^kS_t^{(i)} \in A\}} \right] \\ &= \sum_{k=1}^{m} \lambda_k \int_A \frac{\exp\left\{ \dfrac{-\left(\ln(l) - \int_0^t \left({}^kr_z - \frac{1}{2} \sum_{j=1}^{n} \left({}^k\sigma_z^{(ij)} \right)^2 \right) dz - \ln({}^kS_0^{(i)}) \right)^2}{2 \sum_{j=1}^{n} \int_0^t \left({}^k\sigma_z^{(ij)} \right)^2 dz} \right\}}{l \left(\sum_{j=1}^{n} \int_0^t \left({}^k\sigma_z^{(ij)} \right)^2 dz \right)^{1/2} \sqrt{2\pi}} dl. \end{aligned} \tag{2.34}$$

□

In this kind of markets pricing financial derivatives is not immediate as in the Black-Scholes-Merton setting for complete markets. If we assume that the only martingale measures are of the form 2.25, and we denote them by $Q^{\bar{\lambda}}$ for each vector $\bar{\lambda} \in \mathbb{R}_+^m$ such that $\sum_{j=1}^m \lambda_j = 1$, then the problem of pricing an *attainable* contingent claim, C, is picking up an arbitrage-free price in

$$I = \left(\inf_{\substack{\bar{\lambda} \in \mathbb{R}_+^m \\ \sum_{i=1}^m \lambda_i = 1}} E_{Q^{\bar{\lambda}}}[\tilde{C}], \sup_{\substack{\bar{\lambda} \in \mathbb{R}_+^m \\ \sum_{i=1}^m \lambda_i = 1}} E_{Q^{\bar{\lambda}}}[\tilde{C}] \right), \tag{2.35}$$

such that it maximizes a specific criterion where $\tilde{C} = e^{-\int_0^T r_t} C$. This problem is equivalent to maximize a function of the finite distributions, $\bar{\lambda}$, such that some restrictions hold. We propose that a possible criterion is the maximization of Shannon's information entropy under some macroeconomic restrictions that include information about the underlying stocks in the derivative in question.

Nowadays, there are many indicators in the financial markets that help to see how financially well a company is. As can be seen in (Investopedia, 2018a; NASDAQ, 2018), Profit Margin is sometimes considered the best single indicator of a company's financial health and long-term viability. However, there are some aspects of a company that cannot be measured by this indicator. As an example, if we pay attention to the annual income statement in 2017 for Apple Inc. (Ticket AAPL) and Amazon.com Inc. (Ticket AMZN) provided by NASDAQ, we see that

$$\text{Apple Inc. Profit margin} = \frac{\$48,351,000}{\$229,234,000} = 0.2109$$

and

$$\text{Amazon.com Inc. Profit Margin} = \frac{\$3,033,000}{\$177,866,000} = 0.01705$$

Therefore, according to this criterion Apple Inc. should be more evaluated than Amazon.com Inc. in the American financial markets. However, this is not true. The prices of Amazon.com Inc. and Apple Inc. in 2017 increased about 55%,and 44% respectively. Moreover, the price of Amazon.com Inc. closed at $1630.04 and the price of Apple Inc. closed at $186.97 on May 31, 2018.

Annual Income Statement (values in 000's)					Get Quarterly Data
Period Ending:	**Trend**	**9/30/2017**	**9/24/2016**	**9/26/2015**	**9/27/2014**
Total Revenue		$229,234,000	$215,639,000	$233,715,000	$182,795,000
Cost of Revenue		$141,048,000	$131,376,000	$140,089,000	$112,258,000
Gross Profit		**$88,186,000**	**$84,263,000**	**$93,626,000**	**$70,537,000**
Operating Expenses					
Research and Development		$11,581,000	$10,045,000	$8,067,000	$6,041,000
Sales, General and Admin.		$15,261,000	$14,194,000	$14,329,000	$11,993,000
Non-Recurring Items	----	$0	$0	$0	$0
Other Operating Items	----	$0	$0	$0	$0
Operating Income		**$61,344,000**	**$60,024,000**	**$71,230,000**	**$52,503,000**
Add'l income/expense items		$2,745,000	$1,348,000	$1,285,000	$980,000
Earnings Before Interest and Tax		$64,089,000	$61,372,000	$72,515,000	$53,483,000
Interest Expense	----	$0	$0	$0	$0
Earnings Before Tax		$64,089,000	$61,372,000	$72,515,000	$53,483,000
Income Tax		$15,738,000	$15,685,000	$19,121,000	$13,973,000
Minority Interest	----	$0	$0	$0	$0
Equity Earnings/Loss Unconsolidated Subsidiary	----	$0	$0	$0	$0
Net Income-Cont. Operations		$48,351,000	$45,687,000	$53,394,000	$39,510,000
Net Income		**$48,351,000**	**$45,687,000**	**$53,394,000**	**$39,510,000**
Net Income Applicable to Common Shareholders		**$48,351,000**	**$45,687,000**	**$53,394,000**	**$39,510,000**

Figure 1: Annual income statement of Apple Inc.

Annual Income Statement (values in 000's)					Get Quarterly Data
Period Ending:	**Trend**	**12/31/2017**	**12/31/2016**	**12/31/2015**	**12/31/2014**
Total Revenue		$177,866,000	$135,987,000	$107,006,000	$88,988,000
Cost of Revenue		$111,934,000	$88,265,000	$71,651,000	$62,752,000
Gross Profit		**$65,932,000**	**$47,722,000**	**$35,355,000**	**$26,236,000**
Operating Expenses					
Research and Development		$0	$0	$0	$0
Sales, General and Admin.		$61,826,000	$43,536,000	$33,122,000	$26,058,000
Non-Recurring Items		$0	$0	$0	$0
Other Operating Items		$0	$0	$0	$0
Operating Income		**$4,106,000**	**$4,186,000**	**$2,233,000**	**$178,000**
Add'l income/expense items		$548,000	$190,000	($206,000)	($79,000)
Earnings Before Interest and Tax		$4,654,000	$4,376,000	$2,027,000	$99,000
Interest Expense		$848,000	$484,000	$459,000	$210,000
Earnings Before Tax		$3,806,000	$3,892,000	$1,568,000	($111,000)
Income Tax		$769,000	$1,425,000	$950,000	$167,000
Minority Interest		$0	$0	$0	$0
Equity Earnings/Loss Unconsolidated Subsidiary		($4,000)	($96,000)	($22,000)	$37,000
Net Income-Cont. Operations		$3,033,000	$2,371,000	$596,000	($241,000)
Net Income		**$3,033,000**	**$2,371,000**	**$596,000**	**($241,000)**
Net Income Applicable to Common Shareholders		**$3,033,000**	**$2,371,000**	**$596,000**	**($241,000)**

Figure 2: Annual income statement of Amazon.com Inc.

This problem and some others force us to look for indicators that can reflect the financial health and long-term viability of the companies more accurately. Fortunately, as can be seen in (Investopedia, 2018*a*; NASDAQ, 2018) a better financial approach to a company's financial health can be obtained by looking at their Liquidity, Solvency, Operating Efficiency, and Profitability. These criteria have important indicators that are used by the largest American exchanges such as NASDAQ and NYSE to see how strong the stocks are, and by public companies to forecast their performance in each quarter. In fact, this is a well known area of finance known as *financial statement forecasting*.

Liquidity (Investopedia, 2018*a*) refers to the amount of assets that are easily convertible to cash. A company owns this type of assets to manage short-term debts and obligations. The *quick ratio* is the most precise index to see how liquid a company is because it excludes inventory from assets, and also excludes the current part from long-term debts from liabilities. It is computed by the formula

$$\text{Quick Ratio} = \frac{\text{Current Assets} - \text{Inventories}}{\text{Current Liabilities}}. \tag{2.36}$$

Solvency (Investopedia, 2018*a*) is the company's ability to face its debt obligations on the long-term. Its best indicator is the *debt/equity ratio*. This measures how much debts a company is using to finance its assets based on shareholders' equity

$$\text{Debt/Equity Ratio} = \frac{\text{Total Liabilities}}{\text{Shareholders' Equity}}. \tag{2.37}$$

Operating Efficiency (Investopedia, 2018a) is the capacity of a company to deliver products and services in the most cost-effective possible way without reducing the quality of its products and services. *operating margin* is the best indicator of operating efficiency. This measure not only indicates a company's basic operational profit margin after deducing the production costs, but also gives a signal of how well the company manages their costs

$$\text{Operating Profit Margin} = \frac{\text{Operating Income}}{\text{Operating Revenue}}. \tag{2.38}$$

Profitability (Investopedia, 2018a). We have seen that liquidity, solvency and operating efficiency are important factors for a company. However, perhaps the most important factor is *profitability.* A company can survive for some years without being profitable based on creditors and investors, but in the long-term a company needs to be profitable to survive.

The best indicator of profitability is the *net margin* that is the ratio of profits to revenues. This indicator is important because a measure given by any currency is not adequate to assess a company's financial health. For example, if a company shows a net income of many millions of dollars, and the net margin is very small, then a very slight increase in production costs may take the company out of business

$$\text{Net Margin} = \frac{\text{Net Income}}{\text{Total Revenues}}. \tag{2.39}$$

In addition to the previous ratios, we have the *earnings per share* and *price-earnings ratio* given by the formulas

$$\text{Earnings Per Share} = \frac{\text{Net Income} - \text{Dividends On Preferred Stock}}{\text{Average Outstanding Shares}}, \tag{2.40}$$

$$\text{Price-Earnings Ratio} = \frac{\text{Market Value Per Share}}{\text{Earnings Per Share}}, \tag{2.41}$$

and the DuPont framework

$$\begin{array}{ccccccc} & & \text{Profitability} & & \text{Efficiency} & & \text{Laverage} \\ \text{Return on Equity} & = & \text{Profit Margin} & \times & \text{Asset Turnover} & \times & \text{Equity Multiplier} \\ \dfrac{\text{Net Income}}{\text{Equity}} & = & \dfrac{\text{Net Income}}{\text{Sales}} & \times & \dfrac{\text{Sales}}{\text{Assets}} & \times & \dfrac{\text{Assets}}{\text{Equity}}, \end{array} \tag{2.42}$$

a formula commonly used by investors to take investing decisions.

We were talking about macroeconomic restrictions; however, these indexes are microeconomic indicators because they only refer to specific companies. This problem is fixed when we consider the aggregate accounts: net income, sales, assets, equity, etc. of the biggest companies of a country that are related to the underlying stocks of the derivative in question. Financial analysts can include more financial restrictions, but for the sake of simplicity, we restrict ourselves to these financial accounts.

As it is expected, we will equal these numeric restrictions of the accounts, already known under the measure P^{Λ}, with their respective expected values under the measure $Q^{\bar{\lambda}}$ with $\bar{\lambda}$ as in 2.35. As an example, we can consider $X_{\text{Apple},1}, X_{\text{Apple},2}, \ldots$ as the random variables that represent the net income, sales, assets, equity,.., etc., of Apple Inc. which are functions of the possible macroeconomic events $\Lambda_1, \ldots, \Lambda_m$. Therefore, some of our restrictions will be of the form

$$E_{Q^{\bar{\lambda}}}[X_{\text{Apple},j}] = E_{P^\Lambda}[X_{\text{Apple},j}] = \sum_{i=1}^{m} P^\Lambda(\Lambda_i) X_{\text{Apple},j}(\Lambda_i) \quad \text{for} \quad j = 1, \ldots, M, \tag{2.43}$$

where the right hand side of this formula is already known under the knowledge described by the measure P^Λ, and M is the number of microeconomic characteristics of Apple Inc. such as net income, sales, etc.

When we consider the S&P 500 companies that are correlated with the underlying, we get equations of the form

$$E_{Q^{\bar{\lambda}}}[X_{i,j}] = E_{P^\Lambda}[X_{i,j}] \quad \text{for} \quad i = 1, \ldots, l; \quad j = 1, \ldots, M, \tag{2.44}$$

where j is the microeconomic feature, and i is the company. Now, we can rewrite these microeconomic restrictions into macroeconomic ones by considering the sum on each one of the L economic sectors, $(I_k)_{k=1}^{L}$, of the economy

$$E_{Q^{\bar{\lambda}}}\left[\sum_{i \in I_k} X_{i,j}\right] = E_{P^\Lambda}\left[\sum_{i \in I_k} X_{i,j}\right] = \sum_{i \in I_k} E_{P^\Lambda}\left[X_{i,j}\right] \qquad \text{for} \quad j = 1, \ldots, M, \quad \text{and} \quad k = 1, \ldots, L. \tag{2.45}$$

Finally, if we let $X_1, \ldots, X_{LM}$ denote these new random variables, we have that our complete set of macroeconomic restrictions will be

$$E_{Q^{\bar{\lambda}}}[X_i] = E_{P^\Lambda}[X_i] \quad \text{for} \quad i = 1, \ldots, LM. \tag{2.46}$$

Since the institutional trader, under the informational framework, $(\mathcal{F}_t)_{t \in [0,T]}$, does not have any intuitive idea about the nature and the impact of the distributions $\lambda_1, \ldots, \lambda_n$ on its investment performance, it should try to find a distribution that maximizes the company's state of uncertainty under some macroeconomic restrictions that take into account information of derivative in question. In practice one of the most common ways to measure uncertainty is using the Shannon's information entropy[1]. Here, we denote this function by H, and it is defined as $H(\lambda_1, \ldots, \lambda_m) = -\sum_{i=1}^{m} \lambda_i \log(\lambda_i)$, on any finite probability distribution $(\lambda_1, \ldots, \lambda_m)$.

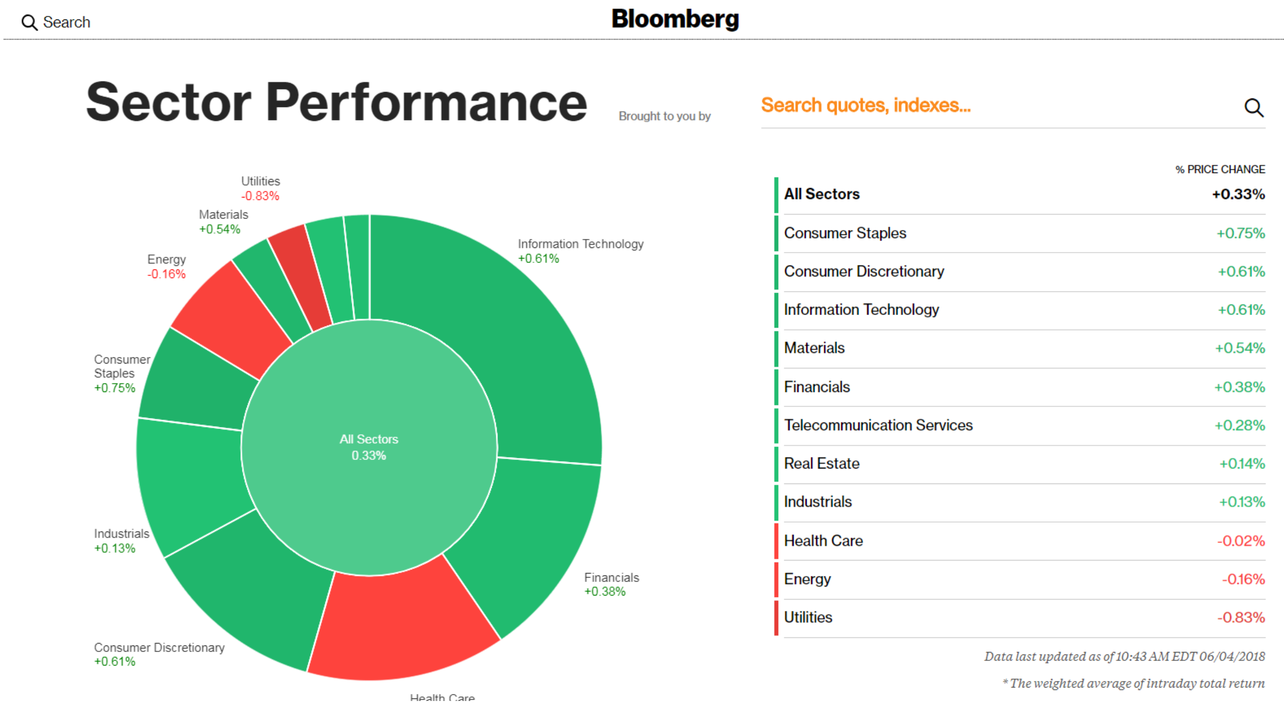
From now on, unless otherwise stated, we will consider all of the random variables X_i as they appear in 2.46. This is done for the sake of simplicity of restrictions and computations in the optimization processes that will be mentioned later. Therefore, if we let $\bar{X}_i$ be the expected value of X_i under P^Λ, the problem of computing

$$\max_{\substack{\bar{\lambda} \\ \|\bar{\lambda}\|=1}} H(\lambda_1, \ldots, \lambda_m) \tag{2.47}$$

subject to

$$E_{Q^{\bar{\lambda}}}[X_i] = \bar{X}_i \quad \text{for} \quad i = 1, \ldots, LM, \quad \text{and} \quad m > LM \tag{2.48}$$

[1] More references and applications can be found in the work created by (Jaynes et al., 2003)



Bloomberg's sector performance

is equivalent to the problem of picking up a $Q^{\bar{\lambda}}$ such that it reflects the maximum state of uncertainty of the choice of the arbitrage-free price in

$$I = \left(\inf_{\substack{\bar{\lambda} \\ ||\bar{\lambda}||=1}} E_{Q^{\bar{\lambda}}}[\tilde{C}], \ \sup_{\substack{\bar{\lambda} \\ ||\bar{\lambda}||=1}} E_{Q^{\bar{\lambda}}}[\tilde{C}] \right), \tag{2.49}$$

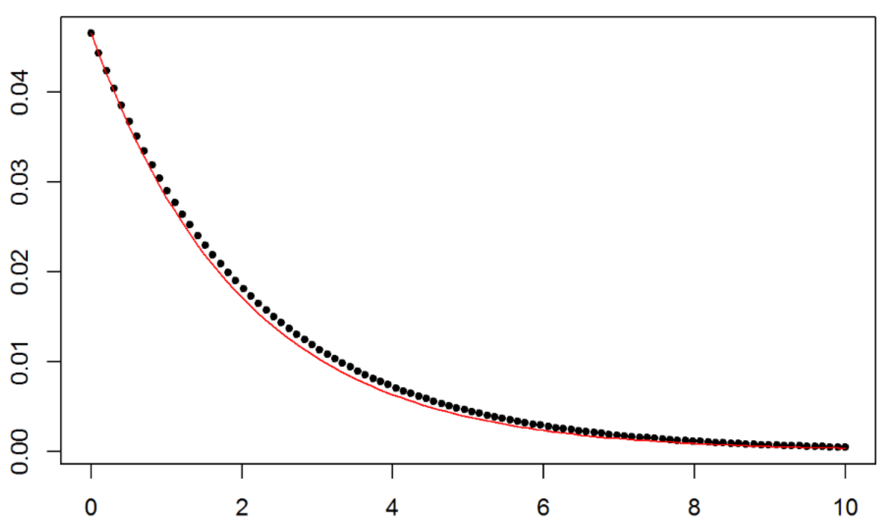
and at the same time satisfies the macroeconomic restrictions previously known under the information represented by the measure P^{Λ}. In other words, the measure $Q^{\bar{\lambda}}$ resulting from 2.47 and 2.48 will be a distribution as spread-out as it can be while agreeing with available information in 2.48 . This kind of reasoning is completely desirable, because we do not want to favor any macroeconomical event over any other. On the contrary, we want to make a decision based exclusively on the entire data we have at the moment of picking up $Q^{\bar{\lambda}}$. It is straightforward to solve a maximization problem of this kind since there are available packages such as CVXfromR in R programming language for convex optimization. This is a tool that not only has a comprehensive documentation,[2] but it also makes easy to solve general convex problems, and the syntax to input problems is simple. This package was created by Michael Grant[3] and Stephen Boyd[4] . In addition to that, Jacob Bien[5] is the person is in charge of upgrading the package for R.

[2] Theoretical and technical details of the package can be found at http://cvxr.com/cvx/.
[3] http://cvxr.com/cvx/.
[4] https://web.stanford.edu/~boyd/.
[5] http://faculty.marshall.usc.edu/jacob-bien/cvxfromr.html

Here is an example of the use of the package CVXfromR in practice taken from http://www.di.fc.ul.pt/~jpn/r/maxent/maxent.html.

```
library(CVXfromR)

n <- 100
a <- seq(0,10,len=n) # theoretical support [0,+oo) but we assume a light tail
lambda <- .5

A <- matrix(a, ncol=n)
b <- 1/lambda           # f_1(n) = n, ie, E[f_1(n)] = E[n] = 1/lambda

# ref: web.cvxr.com/cvx/examples/cvxbook/Ch07_statistical_estim/html/maxent.html
# entr(x)=-x*log(x), elementwise entropy function [cvxr.com/cvx/doc/funcref.html]
cvxcode <- "
    variables pmaxent(n)
    maximize( sum(entr(pmaxent)) )
    sum(pmaxent) == 1;
    A * pmaxent == b;
"

# it takes sometime to run a matlab session
opt.vals <- CallCVX(cvxcode, const.vars=list(n=n, A=A, b=b),
                    opt.var.names="pmaxent",
                    setup.dir="C:\\Users\\jpn.INFORMATICA\\Software\\_Langs\\cvx")

plot(a,opt.vals$pmaxent, pch=20, ylab="")
diff <- dexp(0,rate=lambda) / opt.vals$pmaxent[1] # scale back to maxent approx
curve(dexp(x,rate=lambda)/diff, col="red", add=T)
```

Figure 4: Shannon's information entropy code using CVXfromR package



As it is expected in this example, if we use as constraint the expected value of an exponential random variable, the solution to the Shannon's entropy maximization problem will be the distribution of the exponential random variable.

3 Continuous Time Setting for Small Agents

3.1 Informational framework of smaller buyers and short sellers

In this part we will be using some of the ideas by (Duffie and Huang, 1986) and (El Karoui and Rouge, 2000) to analyze how agents with differential information in a constrained market take decisions, and price financial derivatives. With these tools, we can have a deeper look at how small agents are taking decisions. This machinery will help us to track the parameters that make the prices of small agents different from the prices computed by institutional investors, and therefore, we can have more insights about the impact of small investors in the price of the market.

First we consider a countable set of agents $\mathcal{A}$ endowed with the characteristics $\{\succeq_\alpha, V_\alpha, \mathcal{F}^\alpha\}$, where for each α, $\succeq_\alpha$ is the relation of preference of the α-stereotype agent; $V_\alpha = \mathbb{R} \times L^1(\Omega, \mathcal{F}^\alpha, P)$ is the space of pairs (x, C) where x is the initial investment in a strategy related to the payoff of the financial instrument that the α-stereotyped agent sold or bought and whose payoff is $-C$ and C respectively.

We have that an α-stereotyped agent with initial endowment x who wishes to sell or buy at time $t = 0$ a claim $C \in L^1_+$ have two possibilities:

1. In the case of the seller, delivering the claim C at the maturity date T in exchange for an additional endowment of y at time zero. That is, she chooses the pair $(x + y, -C)$. In the case of the buyer, she will exercise the security with payoff C in exchange for an additional payment of y at time zero. That is, she chooses the pair $(x - y, C)$.
2. Not selling or buying the security. That is $(x, 0)$.

The price to sell the security with payoff $-C$ under an $\alpha-$preference is defined as the minimum additional endowment such that a seller prefers to sell the security with payoff C rather than do nothing:

$$p^\alpha(x, C) = \inf\{y \geq 0 : (x + y, -C) \succeq_\alpha (x, 0)\}. \tag{3.1}$$

In a similar way, *the price to buy* the security with payoff C under an $\alpha-$preference is defined as the maximum additional endowment such that a buyer prefers to buy the security with payoff C rather than do nothing:

$$-\,p^\alpha(x, -C) = \inf\{y \geq 0 : (x - y, C) \succeq_\alpha (x, 0)\}. \tag{3.2}$$

We conveniently take non-positive values for the price to buy in order to summarize the two formulas in a single one

$$p^\alpha(x, C) = \inf\{y \in \mathbb{R} : (x + y, -C) \succeq_\alpha (x, 0)\}. \tag{3.3}$$

Now we consider the price system $(S_t)_{t\in[0,T]} = (S_t^{(0)}, ..., S_t^{(n)})_{t\in[0,T]}$ satisfying that S_t is a vector of non-negative semimartingales adapted with respect to the filtered space of the institutional trader, $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, P)$, as in Proposition 2.4 such that

$$\mathcal{F}^\alpha \subseteq \mathcal{F} \text{ for all } \alpha \in \mathcal{A}, \text{ and } E[{}^{\mathcal{F}}\langle S, S\rangle_T^{1/2}] < \infty, \tag{3.4}$$

where $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$; $\mathcal{F}^\alpha = (\mathcal{F}_t^\alpha)_{t\in[0,T]}$ represents the information of the α-stereotyped agent, and the symbol $\subseteq$ means that $\mathcal{F}^\alpha$ is a subfiltration of $\mathcal{F}$, i.e., $\mathcal{F}_t^\alpha \subset \mathcal{F}_t$ for all $t \in [0, T]$.

We will use some results and notations that appear in (Duffie and Huang, 1986) that will help us to understand some essential features that a market with differential informational framework has.

We assume that the agents learn from the price system to refine their information. The information filtration of an α-stereotyped agent after seeing a price system $(S_t)_{t\in[0,T]}$ is denoted by $\mathcal{F}^{\alpha,S} := (\mathcal{F}_t^{\alpha,S})_{t\in[0,T]}$ where $F_t^{\alpha,S} := \mathcal{F}_t^\alpha \vee \mathcal{F}_t^S$, and $\mathcal{F}^S := (\mathcal{F}_t^S)_{t\in[0,T]}$ with $\mathcal{F}_t^S := \sigma\{S_l : l \leq t\}$.

Assumption 3.1. *We assume that $(\mathcal{F}^{\alpha,S})_{t\in[0,T]}$ is right-continuous for each $\alpha \in \mathcal{A}$.*

Lemma 3.2. *The price system, $(S_t)_{t\in[0,T]}$, under assumption 3.1, is a vector of $\mathcal{F}^{\alpha,S}$ semimartingales for each $\alpha \in \mathcal{A}$.*

Proof. By construction (S_t) is $\mathcal{F}^{\alpha,S}-$adapted and right-continuous, then (S_t) is $\mathcal{F}^{\alpha,S}-$optional (see for instance Chung and Williams, 2013). Therefore, (S_t) is an $\mathcal{F}-$semimartingale, and $\mathcal{F}^{\alpha,S}-$optional. It follows from (Theorem 9.19 (a) Jacod, 1979) that (S_t) is an $\mathcal{F}^{\alpha,S}-$semimartingale since $\mathcal{F}^{\alpha,S}$ is a subfiltration of $\mathcal{F}$. □

In order to apply Ito's formula, we need information about the quadratic covariation between prices computed under different filtrations $\mathcal{F}^\alpha$.

Lemma 3.3. *There exists a process $(\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ that is a common version of the processes $({}^{\mathcal{F}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ and $({}^{\mathcal{F}^{\alpha,S}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ for each $\alpha \in \mathcal{A}$.*

Proof. We know that (S_t^m) and (S_t^n) are semimartingales with respect to the filtrations $\mathcal{F}$, and $\mathcal{F}^{\alpha,S}$ for each $\alpha \in \mathcal{A}$, then the processes $({}^{\mathcal{F}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ and $({}^{\mathcal{F}^{\alpha,S}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ for each $\alpha \in \mathcal{A}$ are well-defined.

Now using the identities

$$ {}^{\mathcal{F}}\langle S^{(m)}, S^{(n)}\rangle_t = \frac{1}{4}\left({}^{\mathcal{F}}\langle S^{(n)} + S^{(m)}, S^{(n)} + S^{(m)}\rangle_t - {}^{\mathcal{F}}\langle S^{(n)} + S^{(m)}, S^{(n)} + S^{(m)}\rangle_t\right), \tag{3.5} $$

$$ {}^{\mathcal{F}^{\alpha,S}}\langle S^{(m)}, S^{(n)}\rangle_t = \frac{1}{4}\left({}^{\mathcal{F}^{\alpha,S}}\langle S^{(n)} + S^{(m)}, S^{(n)} + S^{(m)}\rangle_t - {}^{\mathcal{F}^{\alpha,S}}\langle S^{(n)} + S^{(m)}, S^{(n)} + S^{(m)}\rangle_t\right), \tag{3.6} $$

and (Theorem 9.19 (b) Jacod, 1979), we have that there exists a common version of ${}^{\mathcal{F}}\langle S^{(n)}+S^{(m)}, S^{(n)}+S^{(m)}\rangle_{t\in[0,T]}$ and ${}^{\mathcal{F}^{\alpha,S}}\langle S^{(n)} + S^{(m)}, S^{(n)} + S^{(m)}\rangle_{t\in[0,T]}$, and similarly, a common version of ${}^{\mathcal{F}}\langle S^{(n)} - S^{(m)}, S^{(n)} - S^{(m)}\rangle_{t\in[0,T]}$ and ${}^{\mathcal{F}^{\alpha,S}}\langle S^{(n)} - S^{(m)}, S^{(n)} - S^{(m)}\rangle_{t\in[0,T]}$. This last implies that we can find a common version of $({}^{\mathcal{F}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ and $({}^{\mathcal{F}^{\alpha,S}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ for each $\alpha \in \mathcal{A}$ since the set $\mathcal{A}$ is countable. □

With this result and the right-continuity assumption, we can go a little bit further. We know that $(\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ is a common version of the processes $({}^{\mathcal{F}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ and $({}^{\mathcal{F}^{\alpha,S}}\langle S^{(m)}, S^{(n)}\rangle_t)_{t\in[0,T]}$ for each $\alpha \in \mathcal{A}$. Since $\mathcal{A}$ is countable and these processes are right-continuous, they are indistinguishable.

Let $\mathcal{P}^{\alpha,S}$ define the $\sigma-$algebra of predictable subsets of $\Omega \times [0,T]$ with respect to the filtration $\mathcal{F}^{\alpha,S}$ (see for details Meyer, 2000, Chapter 3). This can also be expressed as the $\sigma-$algebra generated by the left-continuous and $\mathcal{F}^{\alpha,S}-$adapted processes (see for instance Bass, 2011, Chapter 16). We then say that a process (Z_t) from $\Omega \times [0,T]$ to $\mathbb{R}$ is $\mathcal{F}^{\alpha,S}-$predictable if it is $\mathcal{P}^{\alpha,S}-$measurable.

We will now make some assumptions on the set of admissible strategies for an α-stereotyped agent. Some of these assumptions are made to define the portfolio value as a stochastic integral, and some others based on financial intuition.

Definition 3.4. *Given a price system $(S_t)_{t\in[0,T]}$, an admissible set of strategies for an α-stereotyped agent is an $n+1-$vector of $\mathcal{F}^{\alpha,S}-$predictable process $(\xi_t)_{t\in[0,T]} = (\xi_t^{(0)}, ..., \xi_t^{(n)})_{t\in[0,T]}$ such that*

1. *The stochastic integral $\int_0^t \xi_u \cdot dS_u$ exists for each $t \in [0,T]$.*
2. *The strategy is self-financing:*

$$V_t = \xi_t \cdot S_t = \xi_0 \cdot S_0 + \int_0^t \xi_u \cdot dS_u \text{ for each } t \in [0,T] \tag{3.7}$$

3. $E_P \left[\left(\int_0^T (\xi_t^{(i)})^2 d\langle S^{(i)}, S^{(i)} \rangle_t \right) \right]^{1/2} < \infty$ *for each $i \in \{0, 1, ..., n\}$.*

Now we will define the set of admissible strategies for each type of agent.

Definition 3.5. *Let $Z^{\alpha,S}$ denote the space of admissible strategies of an α-stereotyped agent when the price system is $(S_t)_{t\in[0,T]}$.*

We can easily check that $Z^{\alpha,S}$ is a vector space by using linearity of the stochastic integral and Kunita-Watanabe inequality.

In an complete market, the problem of pricing and hedging a given contingent claim is well known. In fact, in a scenario of no arbitrage and perfect hedging, the price of a contingent claim at time $t = 0$ with maturity T equals the discounted expected value under the unique martingale measure (see Harrison and Pliska, 1981). However, in an constraint market where perfect replication is not possible some ways to find a price have been proposed. One way to find a coherent price is via a concept known as *superreplicating prices.*

The superreplicating price is the minimum amount of money that has to be invested in a strategy such that its portfolio value process at the maturity date dominates the payoff of the contingent claim. (El Karoui and Quenez, 1995) showed this price is characterized as the essential supremum on the set of equivalent measures of the expected value of the discounted payoff, and this result was later generalized by (Cvitanić and Karatzas, 1993). These authors also showed that any price between the superreplicating price for sellers $h^{\alpha,\text{up}}$, and the superreplicating price for buyers $h_{\alpha,\text{low}}$ does not lead to arbitrage opportunities. Although this result is important, it is unsatisfactory because the prices they set are too high (El Karoui and Rouge, 2000).

In a constrained market, the sellers naturally want to find an initial endowment v_0 and a portfolio strategy ξ such that the portfolio value at the maturity date, V_T, dominates the payoff, C, of the contingent claim. The same idea is used for buyers but with $-C$ instead. This idea will allow us to build an *arbitrage free interval* for a claim for an α-stereotype seller and buyer.

We define the *α-seller's superreplicating price* of a claim with payoff $-C$, denoted by $h^{\alpha,\mathrm{up}}$, as the smallest amount of money invested in a strategy for which its portfolio value superreplicate the payoff C for the buyer.

$$h^{\alpha,\mathrm{up}} = \inf\left\{v_0 : \quad \exists\xi \in Z^{\alpha,S}, \quad V_T^{v_0,\xi} := v_0 + \int_0^T \xi_u \cdot dS_u \geq C\right\}. \tag{3.8}$$

Using the same idea, we define the *α-buyer's superreplicating price* as

$$h_{\alpha,\mathrm{low}} = \sup\left\{v_0 : \quad \exists\xi \in Z^{\alpha,S}, \quad V_T^{-v_0,\xi} := -v_0 + \int_0^T \xi_u \cdot dS_u \geq -C\right\}. \tag{3.9}$$

By the results of (Cvitanić and Karatzas, 1993), we have that under the informational framework $\mathcal{F}^{\alpha,S}$, the price for a contingent claim may vary between the superreplicating price for buyers $h_{\alpha,\mathrm{low}}$, and the superreplicating price for sellers $h^{\alpha,\mathrm{up}}$. They also showed that any price in $(h_{\alpha,\mathrm{low}}, h^{\alpha,\mathrm{up}})$ does not lead to arbitrage opportunities.

Although we have for each agent $\alpha \in \mathcal{A}$ a relation of preference $\succeq_\alpha$, we need to find a way to quantify it. A natural way to do this is via utility functions U (concave and strictly increasing). Here, we will use the maximal expected utility $\hat{U}$.

The maximal utility $\hat{U}_\alpha(v_0, C)$ of $(v_0, C) \in \mathbb{R} \times L$ under the relation of preference of agent α is defined as

$$\hat{U}_\alpha(v_0, C) = \max_{\xi \in Z^{\alpha,S}} E_P\left[U_\alpha(V_T^{v_0,\xi} + C)\right], \tag{3.10}$$

and then we define the relation of preference, $\succeq_\alpha$, of the α-stereotyped agent by $(v_0, C) \succeq_\alpha (v_0^{'}, C^{'})$ if and only if $\hat{U}_\alpha(v_0, C) \geq \hat{U}_\alpha(v_0^{'}, C^{'})$[6].

We now show a result by (El Karoui and Rouge, 2000) that sets a relationship between the price for an α-stereotyped agent 3.3, and the *α-arbitrage free interval* $(h_{\alpha,\mathrm{low}}, h^{\alpha,\mathrm{up}})$.

Proposition 3.6. *Let $C \in L^1$. The price $p^\alpha(v_0, C)$ in 3.3 derived from the utility maximization 3.10 for an agent α is consistent with arbitrage:*

$$h_{\alpha,low} \leq p^\alpha(v_0, C) \leq h^{\alpha,up}. \tag{3.11}$$

Proof. Let ξ^α be the portfolio strategy that satisfies $h^{\alpha,\mathrm{up}}$, then $V_T^{h^{\alpha,\mathrm{up}},\xi^\alpha} \geq C$ P^α-a.s. By the definition of $Z^{\alpha,S}$ we know that $\xi, \xi^{'} \in Z^{\alpha,S}$ implies $\xi + \xi^{'} \in Z^{\alpha,S}$, and then $V_t^{v_0,\xi} + V_t^{v_0^{'},\xi^{'}} = V_t^{v_0+v_0^{'},\xi+\xi^{'}}$. Therefore,

$$\begin{aligned} \max_{\xi \in Z^{\alpha,S}} E_{P^\alpha}\left[U_\alpha(V_T^{v+h^{\alpha,\mathrm{up}},\xi} - C)\right] &\geq \max_{\xi^{'} \in Z^{\alpha,S}} E_{P^\alpha}\left[U_\alpha(V_T^{v,\xi^{'}} + V_T^{h^{\alpha,\mathrm{up}},\xi} - C)\right] \\ &\geq \max_{\xi^{'} \in Z^{\alpha,S}} E_{P^\alpha}\left[U_\alpha(V_T^{v,\xi^{'}})\right]. \end{aligned} \tag{3.12}$$

[6] The use of "Max" instead of "Sup" comes form the results of (El Karoui and Rouge, 2000)

The last inequality comes from the fact that $V_T^{h^{\alpha,\mathrm{up}},\xi} - C \geq 0$ and U is strictly increasing. With this inequality we get $\hat{U}_\alpha(v + h^{\alpha,\mathrm{up}}, -C) \geq \hat{U}_\alpha(v, 0)$. This means $h^{\alpha,\mathrm{up}} \geq p^\alpha(v, C)$. Using a similar reasoning, we can prove that $\hat{U}_\alpha(v - h_{\alpha,\mathrm{low}}, C) \geq \hat{U}_\alpha(v, 0)$ which means $h_{\alpha,\mathrm{low}} \leq -p^\alpha(x, -C)$. Now with these two inequalities we obtain $h_{\alpha,\mathrm{low}} \leq p^\alpha(x, C) \leq h^{\alpha,\mathrm{up}}$ □

3.1.1 Consequences of more information

In this part, we show some general results of the consequences of having access to more information. We show that a more informed agent is better in the sense that he or she has access to more admissible strategies, and his or her *set of potential arbitrage opportunities* is bigger. Next, we give other results about what happens when an agent has a bigger set of admissible strategies than another. We begin by giving a definition of what means having more information.

First, we provide a definition of what arbitrage opportunity means, and what being more informed means.

Definition 3.7. *We say that a strategy $\xi^\alpha \in Z^{\alpha,S}$ with initial investment v_0, provides an* ***arbitrage opportunity*** *for an α-stereotyped agent if*

$$V_0^{v_0,\xi} \leq 0 \quad P^\alpha\text{-a.s.} \quad \text{and} \quad P^\alpha[V_T^{v_0,\xi} > 0] > 0, \tag{3.13}$$

where $P^\alpha = P|_{\mathcal{F}_T^{\alpha,S}}$.

Definition 3.8. *We say that an α-stereotyped agent* ***is more informed*** *than a β-stereotyped agent after seeing the price system $(S_t)_{t\in[0,T]}$ if $\mathcal{F}^{\beta,S} \subseteq \mathcal{F}^{\alpha,S}$, i.e.*

$$\mathcal{F}_t^{\beta,S} \subseteq \mathcal{F}_t^{\alpha,S} \quad \text{for all} \quad t \in [0, T]. \tag{3.14}$$

The next remark is an extension of the results by (Duffie and Huang, 1986) about arbitrage opportunities.

Theorem 3.9. *If an agent α is more informed than an agent β, then*

1. *Agent α has a bigger set of admissible strategies than agent β, i.e. $Z^{\beta,S} \subseteq Z^{\alpha,S}$.*
2. *The* ***set of potential arbitrage opportunities*** *for agent α is bigger than agent β.*

Proof. We first start proving that more information implies a bigger set of admissible strategies. Let $\xi^\beta \in Z^{\beta,S}$, so we have that ξ^β is $\mathcal{F}^{\beta,S}$-predictable, and then $\mathcal{F}^{\alpha,S}$-predictable by hypothesis. Since $(S_t)_{t\in[0,T]}$ is an $\mathcal{F}^{\alpha,S}$ and $\mathcal{F}^{\beta,S}$ semimartingale (see Lemma 3.2), we can use (Theorem 9.26 of Jacod, 1979) to show that $V_t^{v_0,\xi^\beta} := \xi_t^\beta \cdot S_t = \xi_0^\beta \cdot S_0 + \int_0^t \xi_u^\beta \cdot dS_u$ is well-defined with respect to $\mathcal{F}^{\alpha,S}$ as well. Therefore, $\xi^\beta \in Z^{\alpha,S}$.

For the second part, we know that if $\xi^\beta \in Z^{\beta,S}$ such that $V_0^{v_0,\xi^\beta} = \xi_0^\beta \cdot S_0 \leq 0$ P^β-a.s., and $P^\beta[\xi_T^\beta \cdot S_T \geq 0] > 0$, then $\xi^\beta \in Z^{\alpha,S}$ with $V_0^{v_0,\xi^\beta} = \xi_0^\beta \cdot S_0 \leq 0$ P^α-a.s., and $P^\alpha[\xi_T^\beta \cdot S_T \geq 0] > 0$. □

(Duffie and Huang, 1986) showed that if an agent has a bigger set of admissible strategies it does not imply that he or she has more information. This can be illustrated in the following proposition.

Proposition 3.10. *Let $\mathcal{F}_t^{\beta,S} \subseteq \mathcal{F}_t^{\alpha,S}$ for all $t \in [0,T)$. Then $\mathcal{P}^{\beta,S} \subseteq \mathcal{P}^{\alpha,S}$ and $Z^{\beta,S} \subseteq Z^{\alpha,S}$.*

Proof. We know that $\mathcal{P}^{\beta,S}$ is generated by the sets of the form $\{0\} \times B_0$ and $(s,t] \times B_s$ with $B_0 \in \mathcal{F}_0^{\beta,S}$ and $B_s \in \mathcal{F}_s^{\beta,S}$ respectively. By hypothesis $B_0 \in \mathcal{F}_0^{\alpha,S}$, and $B_s \in \mathcal{F}_s^{\alpha,S}$. Consequently, $\mathcal{P}^{\beta,S} \subseteq \mathcal{P}^{\alpha,S}$

Now with this result we know that each $\mathcal{F}^{\beta,S}$-predictable process is $\mathcal{F}^{\alpha,S}$-predictable. Using (Theorem 9.26 Jacod, 1979) and the same reasoning as in Theorem 3.9 we get that $Z^{\beta,S} \subseteq Z^{\alpha,S}$.

□

The intuition of the previous result is obvious. In a finite horizon setting the information revealed at the maturity date does not provide any kind of advantage.

Now we draw another conclusion from another theorem by (Duffie and Huang, 1986).

Theorem 3.11. *If $\mathcal{F}_T^{\beta,S} \subseteq \mathcal{F}_T^{\alpha,S}$ with S a system of prices strictly positive, then $Z^{\beta,S} \subseteq Z^{\alpha,S}$ implies $\mathcal{F}^{\beta,S} \subseteq \mathcal{F}^{\alpha,S}$.*

Proof. Suppose that there exists a $t \in [0,T)$ such that $\mathcal{F}_t^{\beta,S} \not\subseteq \mathcal{F}_t^{\alpha,S}$. Without loss of generality we can assume that $t \in (0,T)$. Then there exists an $A \in \mathcal{F}_t^{\beta,S}$, but $A \notin \mathcal{F}_t^{\alpha,S}$. We know that $1_{(t,T]\times A}$ is $\mathcal{F}^{\beta,S}$-predictable since $(t,T] \times A \in \mathcal{P}^{\beta,S}$. Now, we construct our self-financing strategy $\xi_t = (\xi_t^{(0)}, ..., \xi_t^{(n)})$ defined as

$$\xi_u^{(k)}(w) = \begin{cases} \theta_t^{(i)} 1_{(t,T]\times A}(u,w), & \text{if } k = i, \\ \theta_t^{(j)} 1_{(t,T]\times A}(u,w), & \text{if } k = j, \\ 0, & \text{otherwise,} \end{cases} \tag{3.15}$$

with

$$\theta_t^{(k)}(w) = \begin{cases} \theta_t^{(i)}(w), & \text{if } k = i, \\ \dfrac{-\theta_t^{(i)}(w) S_t^{(i)}(w)}{S_t^{(j)}(w)}, & \text{if } k = j \text{ and } S_t^{(j)}(w) > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{3.16}$$

where $i \neq j$, and $\theta_t^{(i)}$ is any strictly positive $\mathcal{F}_t^{\beta,S}$-measurable random variable.

We have that $\xi_u \cdot S_u = 0$ and $\xi_0 \cdot S_0 + \int_0^u \xi_l \cdot dS_l = 0$ for $u \in [0,t]$. And for $u \in (t,T]$ we obtain

$$\begin{aligned} \xi_u \cdot S_u &= \theta_t^{(i)} S_u^{(i)} + \theta_t^j S_u^{(j)} \\ &= \theta_t^{(i)} (S_u^{(i)} - S_t^{(i)}) + \sum_{k \neq i} \theta_t^{(k)} (S_u^{(k)} - S_t^{(k)}) \\ &= \xi_0 \cdot S_0 + \int_0^u \xi_l \cdot dS_l. \end{aligned} \tag{3.17}$$

Therefore, ξ is an $\mathcal{F}^{\beta,S}$-predictable self-financing strategy, and $\xi \in Z^{\beta,S}$. However, ξ is not $\mathcal{F}^{\alpha,S}$-predictable since $(\xi^{(i)})^{-1}(\mathbb{R}_+ \setminus \{0\}) = (t,T] \times A \notin \mathcal{P}^{\alpha,S}$. This contradicts the hypothesis $Z^{\beta,S} \subseteq Z^{\alpha,S}$. Consequently, $\mathcal{F}^{\beta,S} \subseteq \mathcal{F}^{\alpha,S}$. □

Corollary 3.12. *If* $\mathcal{F}_T^{\alpha,S} = \mathcal{F}_T^{\beta,S}$ *and* $Z^{\alpha,S} = Z^{\beta,S}$, *then* $\mathcal{F}^{\alpha,S} = \mathcal{F}^{\beta,S}$.

3.2 The market model and the price of a contingent claim from a small investor's perspective

In the previous section we described how an institutional investor computes the price of a derivative. We assume that the small investors and the big investor share similar data to some extent. From this perspective we can assume that each one of the α-stereotyped agents, with $\alpha \in \mathcal{A}$, perceives the prices as in 2.24, i.e.

$$dS_t^{(0)} = r_t^\alpha S_t^{(0)} \quad \text{and} \quad dS_t^{(i)} = \mu_t^{\alpha,(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} \sigma_t^{\alpha,(ij)} dW_t^{(j)} \right), \tag{3.18}$$

adapted to the filtered space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t\in[0,T]}, P)$ as in Proposition 2.4 for each $i = 1, ..., n$. However, since they do not have the information, equipment, and appropriate staff that allow them to make accurate and prudent forecasts of $\mu_t^{\alpha,(i)}, r_t^\alpha$, and $\sigma_t^{\alpha,(ij)}$ for each macroeconomic event $\Lambda_1, ..., \Lambda_m$., the estimation process has to be different. In the algorithm used to compute these prices, we will consider these coefficients as deterministic functions for each agent α. The results of Subsection 3.1 allow us to define the system of prices, and make all of the analysis from each informational framework $(\Omega, \mathcal{F}_T^{\alpha,S}, (\mathcal{F}_t^{\alpha,S})_{t\in[0,T]}, P^\alpha)$.

Another important feature that we need to take into account is the heterogeneity of the preferences and informational framework. In a market we cannot assume that all of the individuals are the same and that have the same quantity of information. For this reason, we will use the ideas developed in Subsection 3.1. In other words, we consider a countable set of agents $\mathcal{A}$ endowed with the characteristics $\{\succeq_\alpha, V_\alpha, \mathcal{F}^\alpha\}$, where for each $\alpha \in \mathcal{A}$, $\succeq_\alpha$ is the relation of preference of the α-stereotyped agent, $V_\alpha = \mathbb{R} \times L^1(\Omega, \mathcal{F}^\alpha, P)$ is the space of pairs (x, C) where x is the initial investment in a strategy related to the payoff of the financial instrument that the α-stereotyped agent sold or bought and whose payoff is $-C$ and C respectively. Additionally, we will quantify these relations of preference using different types of utility functions. In this model for each agent $\alpha \in \mathcal{A}$, we will use the utility function given by the negative exponential:

$$U_\alpha(x) := \exp(-\gamma_\alpha x), \tag{3.19}$$

where γ_α represents the coefficient of risk aversion of agent α. Using 3.10 we have that the maximal utility $\hat{U}_\alpha(v_0, C)$ of $(v_0, C) \in \mathbb{R} \times L$ under the relation of preference of agent α is

$$\hat{U}_\alpha(v_0, C) = \max_{\xi^\alpha \in Z^{\alpha,S}} E_{P^\alpha} \left[-\exp\left\{ -\gamma_\alpha (V_T^{v_0,\xi^\alpha} + C) \right\} \right], \tag{3.20}$$

where $P^\alpha = P|_{\mathcal{F}_T^{\alpha,S}}$. Therefore, we set $(v_0, C) \preceq_\alpha (v_0', C')$ if and only if $\hat{U}_\alpha(v_0, C) \leq \hat{U}_\alpha(v_0', C')$.

3.2.1 Entropy and price

The concept of entropy has different interpretations and is used in different contexts. We used it in the case of a big company that wants to maximize its state of uncertainty (not being biased in favor of some event) under some macroeconomic restrictions. However, entropy can also be used to express the *cost* of using some information instead of another. Here, we use *the relative entropy*, $h(Q|P^\alpha) = E[(dQ/dP^\alpha)\ln(dQ/dP^\alpha)]$, of Q with respect to P^α as the cost of using Q instead of P^α for an α-stereotyped agent.

We need to find a coherent price that a small agent will compute under his or her circumstances. One way to do that is by the use of the *free energy*, $\ln E[\exp(X)]$, of a random variable X. For a bounded from below random variable, there exists a relation between *free energy* and *relative entropy* also known as the *duality for the Kullback-Leibler divergence*

$$\ln E[\exp(X)]) = \sup_{Q \ll P} [E_P[X] - h(Q|P)]. \tag{3.21}$$

Using this property we can find the stochastic game between an α-stereotyped agent and the market denoted by $\Psi_\alpha(v_0, C)$:

$$\begin{aligned}
\Psi_\alpha(v_0, C) &:= \frac{1}{\gamma_\alpha} \ln[-\hat{U}_\alpha(v_0, -C)] \\
&= \inf_{\xi^\alpha \in Z^{\alpha,S}} \frac{1}{\gamma_\alpha} \ln E_{P^\alpha}[\exp(-\gamma_\alpha(V_T^{v_0,\xi^\alpha} - C))] \\
&= \inf_{\xi^\alpha \in Z^{\alpha,S}} \sup_{Q \ll P^\alpha} \left\{ E_Q[-V_T^{v_0,\xi^\alpha} + C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\}.
\end{aligned} \tag{3.22}$$

Assumption 3.13. *As it is mentioned in (El Karoui and Rouge, 2000), each α-stereotyped agent considers that the accumulation of behaviors in the market makes it risk neutral, and then they should behave in the same way. Therefore, the α-stereotyped agent should consider $Q \sim P^\alpha$, and then*

$$\Psi_\alpha(v_0, C) = \inf_{\xi^\alpha \in Z^{\alpha,S}} \sup_{Q \sim P^\alpha} \left\{ E_Q[-V_T^{v_0,\xi^\alpha} + C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\}, \tag{3.23}$$

where each probability measure Q satisfies

$$E_Q[V_T^{v_0,\xi^\alpha}] \leq v_0 / B_{0,T}. \tag{3.24}$$

The process $(B^\alpha_{t,T})_{t \in [0,T]}$ is the price of a zero-coupon bond with maturity T. Therefore, $v_0/B^\alpha_{0,T}$ can be interpreted as the ***forward price*** *of v_0.*

With assumption 3.13, and $(B^\alpha_{t,T})_{t \in [0,T]}$ as the numéraire for the small agents, we get that

$$\Psi_\alpha(v_0, C) \geq -\frac{v_0}{B^\alpha_{0,T}} + \sup_{Q \sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\} \tag{3.25}$$

from the perspective of an α-stereotyped agent.

We know that under these circumstances, it is intuitive that the agent α wants to compute the price of the claim as in 3.3, i.e., the smallest p such that

$$\max_{\xi^\alpha \in Z^{\alpha,S}} E_{P^\alpha} \left[-\exp\left\{ -\gamma_\alpha (V_T^{v_0+p,\xi^\alpha} + C) \right\} \right] \geq \max_{\xi^\alpha \in Z^{\alpha,S}} E_{P^\alpha} \left[-\exp\left\{ -\gamma_\alpha (V_T^{v_0,\xi^\alpha}) \right\} \right]. \tag{3.26}$$

(Frittelli, 2000) showed that under a general setting where portfolios are bounded from below it is possible to find this price. (El Karoui and Rouge, 2000) proposed a Brownian model using dynamic programming. We will show that it is possible to find an estimate of this price proposed by (El Karoui and Rouge, 2000) using semi-group theory. For this purpose, we need to make some changes on this model in order to use the formulas we built in 3.18 and in 2.24. First, we will show how the price is computed when

$$\Psi_\alpha(v_0, C) = -\frac{v_0}{B^\alpha_{0,T}} + \sup_{Q\sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\}, \tag{3.27}$$

and next we will show how we can prove 3.27 under the assumption that the system of prices follows the model 3.18.

Proposition 3.14. *If the value function of 3.23 can be written as*

$$\Psi_\alpha(v_0, C) = -\frac{v_0}{B^\alpha_{0,T}} + \sup_{Q\sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\}, \tag{3.28}$$

then the price 3.3 for the α-stereotyped agent is

$$p^\alpha(v_0, C) = B^\alpha_{0,T} \left(\sup_{Q\sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\} - \sup_{Q\sim P^\alpha} \left\{ -\frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\} \right). \tag{3.29}$$

Proof. Using the definition of Ψ_α and the usual properties of inequalities for increasing and decreasing functions, we get

$$\begin{aligned}
&\min\left\{p : \Psi_\alpha(v_0 + p, C) \le \Psi_\alpha(v_0, 0)\right\} \\
&= \min\left\{p : \frac{1}{\gamma_\alpha} \ln\left[-\hat{U}(v_0 + p, -C)\right] \le \frac{1}{\gamma_\alpha} \ln\left[-\hat{U}(v_0, 0)\right]\right\} \\
&= \min\left\{p : \exp\left\{\gamma_\alpha\left(\frac{1}{\gamma_\alpha} \ln\left[-\hat{U}(v_0 + p, -C)\right]\right)\right\} \le \exp\left\{\gamma_\alpha\left(\frac{1}{\gamma_\alpha} \ln\left[-\hat{U}(v_0, 0)\right]\right)\right\}\right\} \\
&= \min\left\{p : -\hat{U}(v_0 + p, -C) \le -\hat{U}(v_0, 0)\right\} \\
&= \min\left\{p : \hat{U}(v_0 + p, -C) \ge \hat{U}(v_0, 0)\right\} \\
&= \min\left\{p : (v_0 + p, -C) \succeq_\alpha (v_0, 0)\right\}.
\end{aligned} \tag{3.30}$$

On the other hand, we know that

$$\Psi_\alpha(v_0 + p, C) - \Psi_\alpha(v_0, 0) = \frac{-p}{B^\alpha_{0,T}} + \sup_{Q\sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\} - \sup_{Q\sim P^\alpha} \left\{ -\frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\}. \tag{3.31}$$

Therefore, $p^\alpha(v_0, C)$ must satisfy

$$p^\alpha(v_0, C) = B^\alpha_{0,T} \left(\sup_{Q\sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\} - \sup_{Q\sim P^\alpha} \left\{ -\frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\} \right). \tag{3.32}$$

□

As we said earlier, each one of the α-stereotyped agents, with $\alpha \in \mathcal{A}$, perceives the prices as in 3.18, i.e.

$$dS_t^{(0)} = r_t^\alpha S_t^{(0)} \quad \text{and} \quad dS_t^{(i)} = \mu_t^{\alpha,(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} \sigma_t^{\alpha,(ij)} dW_t^{(j)} \right), \tag{3.33}$$

for each $i = 1, ..., n$. We want to show that the value function of each agent $\alpha \in \mathcal{A}$, under assumption 3.13, is of the form

$$\Psi_\alpha(v_0, C) = -\frac{v_0}{B_{0,T}^\alpha} + \sup_{Q \sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha) \right\}, \tag{3.34}$$

and by using Proposition 3.14, we get the price that a small investor "intuitively" should compute under his or her position.

Since we are using the machinery employed by (El Karoui and Rouge, 2000), we need to make some adjustments to our price system 3.18. Using

$$V_t^{v_0,\xi^\alpha} = \sum_{i=0}^{n} \xi_t^{\alpha,(i)} S_t^{(i)} \tag{3.35}$$

we can rewrite the dynamics of $(V_t^{v_0,\xi^\alpha})$ as

$$\begin{aligned} dV_t^{v_0,\xi^\alpha} &= r_t^\alpha \xi_t^{\alpha,(0)} S_t^{(0)} dt + \sum_{i=1}^{n} \xi_t^{\alpha,(i)} dS_t^{(i)} \\ &= r_t^\alpha \left(V_t^{v_0,\xi^\alpha} - \sum_{i=1}^{n} \xi_t^{\alpha,(i)} S_t^{(i)} \right) dt + \sum_{i=1}^{n} \xi_t^{\alpha,(i)} S_t^{(i)} \left(\mu_t^{\alpha,(i)} dt + \sum_{j=1}^{n} \sigma_t^{\alpha,(ij)} dW_t^{(j)} \right) \end{aligned} \tag{3.36}$$

If we set $\eta_t := (\sigma_t^\alpha)^{-1}(\mu_t^\alpha - r_t^\alpha J_{n,1})$ with

$$J_{n,1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}, \quad \sigma_t^\alpha = \begin{bmatrix} \sigma_t^{\alpha,(11)} & \dots & \sigma_t^{\alpha,(1n)} \\ \vdots & \ddots & \\ \sigma_t^{\alpha,(n1)} & \dots & \sigma_t^{\alpha,(nn)} \end{bmatrix}_{n\times n}, \quad \text{and} \quad \mu_t^\alpha = \begin{bmatrix} \mu_t^{\alpha,(1)} \\ \vdots \\ \mu_t^{\alpha,(n)} \end{bmatrix}_{n\times 1} \tag{3.37}$$

the equation 3.36 becomes

$$dV_t^{v_0,\xi^\alpha} = r_t^\alpha V_t^{v_0,\xi^\alpha} dt + (\pi_t^\alpha)^* \sigma_t^\alpha (dW_t + \eta_t dt) \quad \text{with} \quad V_0^{v_0,\xi^\alpha} = v_0 \tag{3.38}$$

where

$$\pi_t^\alpha := \begin{bmatrix} \pi_t^{\alpha,(1)} \\ \vdots \\ \pi_t^{\alpha,(n)} \end{bmatrix}_{n\times 1} = \begin{bmatrix} \xi_t^{\alpha,(1)} S_t^{(1)} \\ \vdots \\ \xi_t^{\alpha,(n)} S_t^{(n)} \end{bmatrix}_{n\times 1} \tag{3.39}$$

represents the amount of money invested in the risky assets.

Assumption 3.15. *We need to make some assumptions about how each agent α perceives the price model.*

1. *The coefficients r_t^α, μ_t^α, and σ_t^α are $\mathcal{F}^{\alpha,S}$ progressively measurable, and uniformly bounded on $[0,T]\times\Omega$.*

2. *The matrix σ_t^α also satisfies the assumption 2.3.*

3. *We assume that each portfolio strategy ξ^α is in $Z^{\alpha,S}$ as in Definition 3.4.*

Under the assumption in Definition 3.4, we know that $Z^{\alpha,S}$ is a vector space, and then we can assume that the process π_t^α remains in a vector space, $\mathcal{V}_\alpha$, for all $t\in[0,T]$. Moreover, these assumptions set that

$$E_{P^\alpha}\left[\left(\int_0^T (\xi_t^{\alpha,(i)})^2 d\langle S^{(i)}, S^{(i)}\rangle_t\right)\right]^{1/2} < \infty \quad \text{for each} \quad i=0,1,...,n, \tag{3.40}$$

and therefore, we obtain

$$E_{P^\alpha}\left[\int_0^T \|\pi_s^\alpha\|^2 ds\right] < \infty. \tag{3.41}$$

As a result, with either 3.40 or 3.41 and using the first lemmas of Section 3.1, we conclude that 3.36 and 3.38 are well-defined for each filtration $\mathcal{F}^{\alpha,S}$.

From now on, and for the sake of simplicity we will use V_t, π_t and ξ_t instead of $V_t^{v_0,\xi^\alpha}, \pi_t^\alpha$ and ξ_t^α respectively.

Proposition 3.16. *Under assumption 3.15 we have that the process $V_t^{v_0,\xi}$ is bounded in $L^2(\Omega, P)$*

Proof. We can rewrite the formula 3.38 as

$$\begin{aligned} dV_t &= r_t^\alpha V_t dt + \pi_t^* \sigma_t^\alpha (dW_t + \eta_t dt) \\ &= (\pi_t^*\sigma_t^\alpha \eta_t + r_t^\alpha V_t)dt + \pi_t^*\sigma_t^\alpha dW_t \\ &= (\pi_t^*\sigma_t^\alpha \eta_t + r_t^\alpha V_t)dt + \sum_{j=1}^n (\pi_t^*\sigma_t^\alpha)^{(j)} dW_t^{(j)} \end{aligned} \tag{3.42}$$

Let us use $g(t,V_t) = V_t e^{-\int_0^t r_s^\alpha ds}$. We have that

$$\begin{cases} g_t(t,V_t) = -r_t^\alpha V_t e^{-\int_0^t r_s^\alpha ds} \\ g_x(t,V_t) = e^{-\int_0^t r_s^\alpha ds} \\ g_{xx}(t,V_t) = 0, \end{cases} \tag{3.43}$$

and using Ito's formula we get

$$\begin{aligned} dg(t,V_t) &= -r_t^\alpha V_t e^{-\int_0^t r_s^\alpha ds} dt + \left(e^{-\int_0^t r_s^\alpha ds}\right)\left[(\pi_t^*\sigma_t^\alpha\eta_t + r_t^\alpha V_t)dt + \sum_{j=1}^n (\pi_t^*\sigma_t^\alpha)^{(j)} dW_t^{(j)}\right] \\ &= e^{-\int_0^t r_s^\alpha ds}\pi_t^*\sigma_t^\alpha \eta_t dt + \sum_{j=1}^n e^{-\int_0^t r_s^\alpha ds}(\pi_t^*\sigma_t^\alpha)^{(j)} dW_t^{(j)}, \end{aligned} \tag{3.44}$$

and integrating yields

$$V_t e^{-\int_0^t r_s^\alpha ds} = V_0 + \int_0^t e^{-\int_0^s r_u^\alpha du} \pi_s^* \sigma_s^\alpha \eta_s ds + \sum_{j=1}^n \int_0^t e^{-\int_0^s r_u^\alpha du} (\pi_s^* \sigma_s^\alpha)^{(j)} dW_s^{(j)}. \tag{3.45}$$

Thus,

$$V_t = e^{\int_0^t r_s^\alpha ds} \left\{ V_0 + \int_0^t e^{-\int_0^s r_u^\alpha du} \pi_s^* \sigma_s^\alpha \eta_s ds + \sum_{j=1}^n \int_0^t e^{-\int_0^s r_u^\alpha du} (\pi_s^* \sigma_s^\alpha)^{(j)} dW_s^{(j)} \right\}. \tag{3.46}$$

Since r^α, σ^α, and μ_α are uniformly bounded on $\Omega \times [0, T]$; π_t satisfies inequality 3.41, and the terms

$$\int_0^t e^{-\int_0^s r_u^\alpha du} (\pi_s^* \sigma_s^\alpha)^{(j)} dW_s^{(j)} \tag{3.47}$$

are independent martingales for $j = 1, ..., n.$, by Doob's L^p-inequality, and Ito's isometry, we obtain

$$\begin{aligned} E\left[\sup_{t \le T} \left| \int_0^t e^{-\int_0^s r_u^\alpha du} (\pi_s^* \sigma_s^\alpha)^{(j)} dW_s^{(j)} \right|^2 \right] &\le 4E\left[\left| \int_0^T e^{-\int_0^s r_u^\alpha du} (\pi_s^* \sigma_s^\alpha)^{(j)} dW_s^{(j)} \right|^2 \right] \\ &\le 4E\left[\int_0^T \left(e^{-\int_0^s r_u^\alpha du} (\pi_s^* \sigma_s^\alpha)^{(j)} \right)^2 ds \right] \\ &< \infty. \end{aligned} \tag{3.48}$$

Therefore, we conclude that

$$E\left[\sup_{t \le T} \left(V_t^{v_0, \xi}\right)^2\right] < \infty. \tag{3.49}$$

□

Since $P^\alpha = P|_{\mathcal{F}_T^{\alpha,S}}$, the previous result holds for either P^α or P.

3.2.2 Change of Numéraire

Every investor does not necessarily quote the price of each one of the financial assets under the same numéraire. In fact, in a market where investors may have different informational frameworks, it is natural to think that each investor should have its own numéraire. In Proposition 2.4, we assumed that the big company has $S_t^{(0)} = S_0^{(0)} e^{\int_0^t r_u^\alpha du}$ as numéraire. However, here we assume that each agent $\alpha \in \mathcal{A}$, has a numéraire that is the *zero-coupon bond*, $(B_{t,T}^\alpha)_{t \in [0,T]}$, with maturity date T. Using again the same tools as in Subsection 3.1, we can assume that there exists an $\mathbb{R}^n$-valued and $\mathcal{F}^{\alpha,S}$ progressively measurable and uniformly bounded process $(\varrho_t^\alpha)_{t \in [0,T]}$ such that the dynamics of $B_{t,T}^\alpha$ is of the form

$$dB_{t,T}^\alpha = B_{t,T}^\alpha [(r_t^\alpha + (\varrho_t^\alpha)^* \sigma_t^\alpha \eta_t) dt + (\varrho_t^\alpha)^* \sigma_t^\alpha dW_t] \quad \text{and} \quad B_{0,T}^\alpha = 1. \tag{3.50}$$

In most cases a numéraire is a "riskless" financial asset and unit of reference. Therefore, we should have that the discounted prices of the financial assets in 3.33 with respect to any numéraire from the informational framework of an α-stereotyped agent should have some similar properties.

We know that our model 3.33 has a martingale measure. Let us recall some details. If we define

$$\frac{dQ^{\alpha,\theta^\alpha}}{dP^\alpha} = \exp\left\{-\int_0^t \theta_u^\alpha \cdot dW_u - \frac{1}{2}\int_0^t \|\theta_u^\alpha\|^2 du\right\}, \tag{3.51}$$

where $\theta_t^\alpha = (\theta_t^{\alpha,(1)}, ..., \theta_t^{\alpha,(n)})^*$ is the $\mathcal{F}^{\alpha,S}$ progressively measurable process that satisfies

$$\begin{bmatrix} \sigma_t^{\alpha,(11)} & \dots & \sigma_t^{\alpha,(1n)} \\ \vdots & \ddots & \\ \sigma_t^{\alpha,(n1)} & \dots & \sigma_t^{\alpha,(nn)} \end{bmatrix} \begin{bmatrix} \theta_t^{\alpha,(1)} \\ \vdots \\ \theta_t^{\alpha,(n)} \end{bmatrix} = \begin{bmatrix} \mu_t^{\alpha,(1)} - r_t^\alpha \\ \vdots \\ \mu_t^{\alpha,(n)} - r_t^\alpha \end{bmatrix}. \tag{3.52}$$

Therefore, under Q^{α,θ^α} we have that

$$dW_t^{\alpha,\theta^\alpha} = \theta_t^\alpha dt + dW_t \tag{3.53}$$

is an n-dimensional Brownian motion. Now using Ito's formula and 3.53 we get

$$\begin{aligned} d\left(\frac{S_t^{(i)}}{S_t^{(0)}}\right) &= \left(\mu_t^{\alpha,(i)} - r_t^\alpha\right)\frac{S_t^{(i)}}{S_t^{(0)}}dt + \frac{S_t^{(i)}}{S_t^{(0)}}\sum_{j=1}^n \sigma_t^{\alpha,(ij)} dW_t^{(j)} \\ &= \frac{S_t^{(i)}}{S_t^{(0)}}\sum_{j=1}^n \sigma_t^{\alpha,(ij)}\left(\theta_t^{\alpha,(j)}dt + dW_t^{(j)}\right) \\ &= \frac{S_t^{(i)}}{S_t^{(0)}}\sum_{j=1}^n \sigma_t^{\alpha,(ij)}\left(dW_t^{\alpha,\theta^\alpha,(j)}\right), \end{aligned} \tag{3.54}$$

and by Ito's formula on $\ln\left(S_t^{(i)}/S_t^{(0)}\right)$, we obtain

$$\frac{S_t^{(i)}}{S_t^{(0)}} = \frac{S_0^{(i)}}{S_0^{(0)}}\exp\left\{\sum_{j=1}^n \int_0^t \sigma_u^{\sigma,(ij)} d(W_u^{\alpha,\theta^\alpha,(j)}) - \frac{1}{2}\sum_{j=1}^n \int_0^t \left|\sigma_u^{\sigma,(ij)}\right|^2 du\right\}. \tag{3.55}$$

Since σ_t^α is uniformly bounded on $\Omega \times [0,T]$, we can conclude that $(S_t^{(i)}/S_t^{(0)})$ is an $\mathcal{F}^{\alpha,S}$-martingale under Q^{α,θ^α}.

As it is expected, we will show that the forward prices $S_t^{(i)}/B_{t,T}^\alpha$, are martingales with respect to some equivalent measure with respect to P^α. For this purpose we will provide some results that allow us to conclude this remark.

Proposition 3.17. *The discounted zero-coupon bond* $(B_{t,T}^\alpha/e^{\int_0^t r_u^\alpha du})$ *is an* $\mathcal{F}^{\alpha,S}$*-martingale under* Q^{α,θ^α}.

Proof. Using Ito's formula, we get

$$\begin{aligned}
d\left(\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}}\right) &= \left(\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}}\right)[((\varrho^{\alpha}_t)^{*}\sigma^{\alpha}_t\eta_t)dt + (\varrho^{\alpha}_t)^{*}\sigma^{\alpha}_t dW_t] \\
&= \left(\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}}\right)[(\varrho^{\alpha}_t)^{*}(\mu^{\alpha}_t - r^{\alpha}_t J_{n\times 1})dt + (\varrho^{\alpha}_t)^{*}\sigma^{\alpha}_t dW_t] \\
&= \left(\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}}\right)(\varrho^{\alpha}_t)^{*}\,[(\mu^{\alpha}_t - r^{\alpha}_t J_{n\times 1})dt + \sigma^{\alpha}_t dW_t] \\
&= \left(\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}}\right)(\varrho^{\alpha}_t)^{*}\sigma^{\alpha}_t\,[\theta^{\alpha}_t dt + dW_t] \\
&= \left(\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}}\right)(\varrho^{\alpha}_t)^{*}\sigma^{\alpha}_t dW^{\alpha,\theta^{\alpha}}_t.
\end{aligned} \tag{3.56}$$

By the same reasoning as in 3.55, we have

$$\frac{B^{\alpha}_{t,T}}{e^{\int_0^t r^{\alpha}_u du}} = \exp\left\{\sum_{j=1}^{n}\int_0^t ((\varrho^{\alpha}_u)^{*}\sigma^{\alpha}_u)^{(j)}\,d(W^{\alpha,\theta^{\alpha},(j)}_u) - \frac{1}{2}\sum_{j=1}^{n}\int_0^t \left|((\varrho^{\alpha}_u)^{*}\sigma^{\alpha}_u)^{(j)}\right|^2 du\right\}. \tag{3.57}$$

Since $(\varrho^{\alpha})^{*}\sigma^{\alpha}_t$ is uniformly bounded on $\Omega\times[0,T]$, we conclude that $(B^{\alpha}_{t,T}/S^{(0)}_t)$ is an $\mathcal{F}^{\alpha,S}$-martingale under $Q^{\alpha,\theta^{\alpha}}$. □

Now we will use some results about *change of numéraire* by (Geman et al., 1995) that can also be found in (Privault, 2020). This result states that if $(N_t)_{t\in[0,T]}$ is a numèraire, and $e^{-\int_0^t r^{\alpha}_s ds}N_t$ is an $\mathcal{F}^{\alpha,S}$-martingale under $Q^{\alpha,\theta^{\alpha}}$, we can define the probability measure

$$\frac{d\tilde{P}^{\alpha}}{dQ^{\alpha,\theta^{\alpha}}} := e^{-\int_0^T r^{\alpha}_s ds}\frac{N_T}{N_0}, \tag{3.58}$$

such that the following proposition holds

Proposition 3.18. *If $(Y_t)_{t\in[0,T]}$ is a continuous and $\mathcal{F}^{\alpha,S}$-adapted process such that*

$$e^{-\int_0^t r^{\alpha}_s ds}Y_t \quad \text{with} \quad t\in[0,T], \tag{3.59}$$

is an $\mathcal{F}^{\alpha,S}$-martingale under $Q^{\alpha,\theta^{\alpha}}$, then the discounted price of Y_t with respect to N_t

$$\frac{Y_t}{N_t} \quad \text{with} \quad t\in[0,T], \tag{3.60}$$

is an $\mathcal{F}^{\alpha,S}$-martingale under $\tilde{P}^{\alpha}$.

Proof. We want to show

$$E_{\tilde{P}^{\alpha}}\left[\left.\frac{Y_t}{N_t}\right|\mathcal{F}^{\alpha,S}_s\right] = \frac{Y_s}{N_s}, \tag{3.61}$$

for $s \leq t$. For this purpose, we will use the formal definition of conditional expectation. Let X be a bounded $\mathcal{F}_s^{\alpha,S}$-measurable random variable. Using 3.58 we get

$$
\begin{aligned}
E_{\tilde{P}^\alpha}\left[X\frac{Y_t}{N_t}\right] &= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_t}{N_t}\frac{d\tilde{P}^\alpha}{dQ^{\alpha,\theta^\alpha}}\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_t}{N_t}E_{Q^{\alpha,\theta^\alpha}}\left[\frac{d\tilde{P}^\alpha}{dQ^{\alpha,\theta^\alpha}}\middle|\mathcal{F}_t^{\alpha,S}\right]\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_t}{N_t}e^{-\int_0^t r_u^\alpha du}\frac{N_t}{N_0}\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_t}{N_0}e^{-\int_0^t r_u^\alpha du}\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[XE_{Q^{\alpha,\theta^\alpha}}\left[\frac{Y_t}{N_0}e^{-\int_0^t r_u^\alpha du}\middle|\mathcal{F}_s^{\alpha,S}\right]\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_s}{N_0}e^{-\int_0^s r_u^\alpha du}\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_s}{N_s}e^{-\int_0^s r_u^\alpha du}\frac{N_s}{N_0}\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_s}{N_s}E_{Q^{\alpha,\theta^\alpha}}\left[\frac{d\tilde{P}^\alpha}{dQ^{\alpha,\theta^\alpha}}\middle|\mathcal{F}_s^{\alpha,S}\right]\right] \\
&= E_{Q^{\alpha,\theta^\alpha}}\left[X\frac{Y_s}{N_s}\frac{d\tilde{P}^\alpha}{dQ^{\alpha,\theta^\alpha}}\right] \\
&= E_{\tilde{P}^\alpha}\left[X\frac{Y_s}{N_s}\right]
\end{aligned}
\tag{3.62}
$$

□

The following corollary is a consequence of these remarks

Corollary 3.19. *The forward prices using the zero-coupon bond process*

$$\frac{S_t^{(i)}}{B_{t,T}^\alpha} \quad \text{for} \quad i = 0, ..., n \quad \text{and} \quad t \in [0, T] \tag{3.63}$$

are $\mathcal{F}^{\alpha,S}$-martingales with respect to $\tilde{P}^\alpha$, with

$$\frac{d\tilde{P}^\alpha}{dQ^{\alpha,\theta^\alpha}} = e^{-\int_0^T r_s^\alpha ds}\frac{B_{T,T}^\alpha}{B_{0,T}^\alpha} = e^{-\int_0^T r_s^\alpha ds}\frac{1}{B_{0,T}^\alpha} \tag{3.64}$$

Proof. We already showed that $(S_t^{(i)}/S_t^{(0)})_{t\in[0,T]}$ is a martingale under Q^{α,θ^α} for each $i = 0, ..., n$. Now, using Proposition 3.18, $Y_t = S_t^{(i)}$, $N_t = B_{t,T}^\alpha$ as in 3.58 and Proposition 3.17 respectively, we obtain the result. □

Assumption 3.20. *We assume that the process ϱ^α is uniformly bounded in $[0,T]\times\Omega$ and remains in $\mathcal{V}_\alpha$.*

Let $\beta^\alpha_{t,T} = (B^\alpha_{t,T})^{-1}$. Since our goal is to compute the price $p^\alpha(v_0, C)$, we have to make use of Ψ_α. Therefore, the formula 3.27 tells us that we have to employ the process $\beta^\alpha_{t,T}V_t$ in the procedure. We are ready to show the next proposition by (El Karoui and Rouge, 2000) that allow us to rewrite $\beta^\alpha_{t,T}V_t$ in a more convenient way.

Proposition 3.21. *The dynamics of the forward wealth $\beta^\alpha_{t,T}V_t$ is of the form*

$$d\beta^\alpha_{t,T}V_t = [-\beta^\alpha_{t,T}V_t\varrho^\alpha_t + \beta^\alpha_{t,T}\pi]^*\sigma^\alpha_t dW^{\rho^\alpha}_t \tag{3.65}$$

with $W^{\rho^\alpha}_t = W_t + \int_0^t \rho^\alpha_s ds$, and $\rho^\alpha_t = (\sigma^\alpha_t)^{-1}(\mu^\alpha_t - r^\alpha_t J_{n\times 1} - \sigma^\alpha_t(\sigma^\alpha_t)^*\varrho^\alpha_t)$ for $t \in [0,T]$.

Proof. Using Ito's formula in $\beta^\alpha_{t,T} = (B^\alpha_{t,T})^{-1}$ we deduce the formula

$$d\beta^\alpha_{t,T} = -\beta^\alpha_{t,T}[(r^\alpha_t + (\varrho^\alpha_t)^*\sigma^\alpha_t\eta_t)dt + (\varrho^\alpha_t)^*\sigma^\alpha_t dW_t] + \beta^\alpha_{t,T}\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt \tag{3.66}$$

Now, with the use of $\eta_t := (\sigma^\alpha_t)^{-1}(\mu^\alpha_t - r^\alpha_t J_{n,1})$ and

$$dV_t = r^\alpha_t V_t dt + \pi^*_t\sigma^\alpha_t(dW_t + \eta_t dt), \tag{3.67}$$

we can conclude

$$\begin{aligned}
d\beta^\alpha_{t,T}V_t &= V_t d\beta^\alpha_{t,T} + \beta^\alpha_{t,T}dV_t + d\langle V_\cdot, \beta^\alpha_{\cdot,T}\rangle_t \\
&= -V_t\beta^\alpha_{t,T}r^\alpha_t dt - V_t\beta^\alpha_{t,T}(\varrho^\alpha_t)^*\sigma^\alpha_t\eta_t dt - V_t\beta^\alpha_{t,T}(\varrho^\alpha_t)^*\sigma^\alpha_t dW_t + V_t\beta^\alpha_{t,T}\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt \\
&\quad + \beta^\alpha_{t,T}r^\alpha_t V_t dt + \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t dW_t + \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t\eta_t dt - \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt \\
&= [-\beta^\alpha_{t,T}V_t\varrho^\alpha_r + \beta^\alpha_{t,T}\pi_t]^*\sigma^\alpha_t dW_t - V_t\beta^\alpha_{t,T}r^\alpha_t dt - V_t\beta^\alpha_{t,T}(\varrho^\alpha_t)^*\sigma^\alpha_t\eta_t dt \\
&\quad + V_t\beta^\alpha_{t,T}\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt + \beta^\alpha_{t,T}r^\alpha_t V_t dt + \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t\eta_t dt - \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt \\
&= [-\beta^\alpha_{t,T}V_t\varrho^\alpha_r + \beta^\alpha_{t,T}\pi_t]^*\sigma^\alpha_t dW_t - V_t\beta^\alpha_{t,T}(\varrho^\alpha_t)^*\sigma^\alpha_t\eta_t dt + V_t\beta^\alpha_{t,T}\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt \\
&\quad + \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t\eta_t dt - \beta^\alpha_{t,T}\pi^*_t\sigma^\alpha_t\left((\varrho^\alpha_t)^*\sigma^\alpha_t\right)^* dt \\
&= [-\beta^\alpha_{t,T}V_t\varrho^\alpha_r + \beta^\alpha_{t,T}\pi_t]^*\sigma^\alpha_t dW_t + [-\beta^\alpha_{t,T}V_t(\varrho^\alpha_r)^* + \beta^\alpha_{t,T}\pi_t]^*\sigma^\alpha_t\rho^\alpha_t dt \\
&= [-\beta^\alpha_{t,T}V_t\varrho^\alpha_r + \beta^\alpha_{t,T}\pi_t]^*\sigma^\alpha_t dW^{\rho^\alpha}_t.
\end{aligned} \tag{3.68}$$

□

By assumption 3.15 and 3.20, we know that ρ^α is bounded and we can define the probability measure

$$\frac{dP^{\rho^\alpha}}{dP^\alpha} = H^{\rho^\alpha}_T = \exp\left\{-\int_0^T \rho^\alpha_t\cdot dW_t - \frac{1}{2}\int_0^T \|\rho^\alpha_t\|^2 dt\right\}. \tag{3.69}$$

By the vector form of the Girsanov's theorem, $W^{\rho^\alpha}$is an n-dimensional Brownian motion under P^{ρ^α}.

Proposition 3.22. *If $\xi \in Z^{\alpha,S}$ then the process $\beta^\alpha_{t,T}V^{v_0,\xi}_t$ is an $\mathcal{F}^{\alpha,S}$-martingale under P^{ρ^α}.*

Proof. Let $\xi \in Z^{\alpha,S}$. Using the remark of assumption 3.15, we have that

$$E_{P^\alpha}\left[\int_0^T \|\pi_s\|^2 ds\right] < \infty. \tag{3.70}$$

We know that $\beta^\alpha_{t,T} V_t^{v_0,\xi}$ is a local martingale under P^{ρ^α}. Since ϱ^α_t is uniformly bounded on $\Omega \times [0,T]$, and $V_t^{v_0,\xi}$ is bounded on $L^2(\Omega, P^\alpha)$, we have

$$E_{P^{\rho^\alpha}}\left[\left\langle \beta^\alpha_{\cdot,T} V_\cdot^{v_0,\xi}\right\rangle_T\right] < \infty. \tag{3.71}$$

Therefore, $\beta^\alpha_{t,T} V_t^{v_0,\xi}$ is an $\mathcal{F}^{\alpha,S}$-martingale under P^{ρ^α}. □

3.2.3 Optimal portfolio

As pointed out by (El Karoui and Rouge, 2000), the superreplicating price is characterized as the essential supremum on the set of equivalent measures of the expected value of the discounted payoff. Under Assumption 3.13, the value function used to compute the price of the payoff of the contingent claim satisfies

$$\Psi_\alpha(v_0, C) \geq -\frac{v_0}{B^\alpha_{0,T}} + \sup_{Q \sim P^\alpha}\left\{E_Q[C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha)\right\}, \tag{3.72}$$

and if the previous result is an equality, we can use Proposition 3.14 to compute the price.

Since

$$\Psi_\alpha(v_0, C) = \inf_{\xi \in Z^{\alpha,S}} \sup_{Q \sim P^\alpha}\left\{E_Q[-V_T^{v_0,\xi} + C] - \frac{1}{\gamma_\alpha} h(Q|P^\alpha)\right\}, \tag{3.73}$$

(El Karoui and Rouge, 2000) proposed a "dual" optimal problem that can give a solution to 3.27. This method was based on the idea that, if we consider 3.73 with respect the collection of measures $Q^{\rho^\alpha,u}$ as in 3.76 such that $\beta^\alpha_{t,T} V_t^{v_0,\xi}$ is a martingale , then

$$\begin{aligned}\Psi_\alpha(v_0, C) &= \inf_{\xi \in Z^{\alpha,S}} \sup_{Q^{\rho^\alpha,u} \sim P^\alpha}\left\{E_{Q^{\rho^\alpha,u}}[-V_T^{v_0,\xi} + C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u}|P^\alpha)\right\} \\ &= -\frac{v_0}{B^\alpha_{0,T}} + \sup_{Q^{\rho^\alpha,u} \sim P^\alpha}\left\{E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u}|P^\alpha)\right\}.\end{aligned} \tag{3.74}$$

By Proposition 3.14, we can compute the price. Moreover, we can also have the same results using a weaker hypothesis about $\mathcal{V}_\alpha$. In fact, if $\mathcal{V}_\alpha$ is a closed cone then by 3.87, we have that $\beta^\alpha_{t,T} V_t^{v_0,\xi}$ is a supermartingale and properties 3.23 and 3.25 hold. Finally we get the result 3.74 from Theorem 4.2. by (El Karoui and Rouge, 2000).

Having mentioned the ideas about how to compute the price, we build the tools to find it. Let $\mathcal{V}_\alpha^\perp$ be the orthogonal complement of $\mathcal{V}_\alpha$, and $\tilde{\mathcal{V}}^\perp_{\alpha,t} = (\sigma^\alpha_t)^{-1}(\mathcal{V}^\perp_\alpha)$, and take u in

$$\mathcal{U}_\alpha = \{u \;\text{ progressively measurable, and uniformly bounded, such that }\; u_t \in \tilde{\mathcal{V}}^\perp_{\alpha,t}\}. \tag{3.75}$$

We define $Q^{\rho^{\alpha},u}$ as

$$\frac{dQ^{\rho^{\alpha},u}}{dP^{\alpha}} = H_T^{\rho^{\alpha,u}} = \exp\left\{-\int_0^T \rho_t^{\alpha,u}\cdot dW_t - \frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right\}, \tag{3.76}$$

where $\rho_t^{\alpha,u} = \rho_t^{\alpha} + u_t$. Using the vector form of Girsanov's theorem, $W_t^{\alpha,u} = W_t + \int_0^t \rho_s^{\alpha,u} ds$ is an n-dimensional $\mathcal{F}^{\alpha,S}$-Brownian motion under $Q^{\rho^{\alpha},u}$.

Let V_t be the portfolio value process of agent α. We know that $\sigma_t^{\alpha} u_t \in \mathcal{V}_{\alpha,t}^{\perp}$, and by assumption 3.20, $(\varrho_t^{\alpha})^* \sigma_t^{\alpha} u_t = 0$. We can rewrite the discounted wealth with respect to the zero-coupon bond as

$$\begin{aligned} d\beta_{t,T}^{\alpha} V_t &= [-\beta_{t,T}^{\alpha} V_t \varrho_t^{\alpha} + \beta_{t,T}^{\alpha}\pi_t]^* \sigma_t^{\alpha} dW_t^{\rho^{\alpha}} \\ &= [-\beta_{t,T}^{\alpha} V_t \varrho_t^{\alpha} + \beta_{t,T}^{\alpha}\pi_t]^* \sigma_t^{\alpha} dW_t^{\alpha,u} - \beta_{t,T}^{\alpha}\pi_t^* \sigma_t^{\alpha} u_t dt \\ &= [-\beta_{t,T}^{\alpha} V_t \varrho_t^{\alpha} + \beta_{t,T}^{\alpha}\pi_t]^* \sigma_t^{\alpha} dW_t^{\alpha,u}, \end{aligned} \tag{3.77}$$

where the last part comes from $\pi_t^* \sigma_t^{\alpha} u_t = 0$ since $\pi_t \in \mathcal{V}_{\alpha}$. Therefore, the process $\beta_{t,T}^{\alpha} V_t$ is a martingale under $Q^{\rho^{\alpha},u}$ for each $u \in \mathcal{U}_{\alpha}$, and the *forward neutral point of view* property 3.23 holds.

Now we are ready to introduce the dual static value function problem proposed by (El Karoui and Rouge, 2000)

$$V_0^{\alpha,C} := \sup_{Q^{\rho^{\alpha},u}} V_0^{\alpha,C,u} \qquad \text{with} \qquad V_0^{\alpha,C,u} := E_{Q^{\rho^{\alpha},u}}[C] - \frac{1}{\gamma_{\alpha}} h(Q^{\rho^{\alpha},u}|P^{\alpha}). \tag{3.78}$$

Since the entropy of $Q^{\rho^{\alpha},u}$ with respect to P^{α} is

$$\begin{aligned} h(Q^{\rho^{\alpha},u}|P^{\alpha}) &= E_{P^{\alpha}}\left[\exp\left\{-\int_0^T \rho_t^{\alpha,u}\cdot dW_t - \frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right\}\left(-\int_0^T \rho_t^{\alpha,u}\cdot dW_t - \frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right)\right] \\ &= E_{Q^{\rho^{\alpha},u}}\left[-\int_0^T \rho_t^{\alpha,u}\cdot dW_t - \frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right] \\ &= E_{Q^{\rho^{\alpha},u}}\left[\int_0^T \|\rho_t^{\alpha,u}\|^2 dt - \int_0^T \rho_t^{\alpha,u}\cdot dW_t^{\alpha,u} - \frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right] \\ &= E_{Q^{\rho^{\alpha},u}}\left[\frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right], \end{aligned} \tag{3.79}$$

we have

$$V_0^{\alpha,C,u} = E_{Q^{\rho^{\alpha},u}}\left[C - \frac{1}{2\gamma_{\alpha}}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right], \tag{3.80}$$

and we take

$$V_t^{\alpha,C,u} = E_{Q^{\rho^{\alpha},u}}\left[C - \frac{1}{2\gamma_{\alpha}}\int_t^T \|\rho_t^{\alpha,u}\|^2 dt \,\middle|\, \mathcal{F}_t\right]. \tag{3.81}$$

$V_t^{\alpha,C,u}$ can be written as

$$\begin{aligned} V_t^{\alpha,C,u} &= E_{Q^{\rho^{\alpha},u}}\left[C - \frac{1}{2\gamma_{\alpha}}\int_t^T \|\rho_s^{\alpha,u}\|^2 ds \,\middle|\, \mathcal{F}_t\right] \\ &= \frac{1}{2\gamma_{\alpha}}\int_0^t \|\rho_s^{\alpha,u}\|^2 ds + E_{Q^{\rho^{\alpha},u}}\left[C - \frac{1}{2\gamma_{\alpha}}\int_0^T \|\rho_s^{\alpha,u}\|^2 ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{3.82}$$

By the martingale representation theorem, we can find a predictable portfolio process $\pi_t^{\alpha,u}$ such that

$$V_t^{\alpha,C,u} = V_0^{\alpha,C,u} + \frac{1}{2\gamma_\alpha}\int_0^t \|\rho_s^{\alpha,u}\|^2 ds + \int_0^t (\pi_s^{\alpha,u})^* \sigma_s^\alpha \cdot dW_s^{\alpha.u}. \tag{3.83}$$

Following (El Karoui and Rouge, 2000), we have to make some changes in 3.82 in order to compute the essential supremum of $V_t^{\alpha,C,u}$. To this end, it will be convenient to take $z_t^{\alpha,u} = (\sigma_t^\alpha)^* \pi_t^{\alpha,u}$ and $f_{\rho^\alpha,t}(z_t, u_t) = -1/2\gamma_\alpha \|\rho_t^{\alpha,u}\|^2 - (\rho_t^{\alpha,u})^* z$. Using this notation we can rewrite 3.82 as a linear backward stochastic differential equation (BSDE)

$$-dV_t^{\alpha,C,u} = f_{\rho^\alpha,t}(z_t^{\alpha,u}, u_t)dt - (z_t^{\alpha,u})^* dW_t \quad \text{and} \quad V_T^{\alpha,C,u} = C. \tag{3.84}$$

By the comparison principle for BSDE's (see for instace El Karoui and Rouge, 2000; Liu and Ren, 2002; Cohen et al., 2010) we only need to maximize in u for a given z the driver $f_{\rho^\alpha,t}(z,u)$. The driver can be written as

$$\begin{aligned} f_{\rho^\alpha,t}(z,u) &= -\frac{1}{2\gamma_\alpha}\|\rho_t^{\alpha,u}\|^2 - (\rho_t^{\alpha,u})^* z \\ &= -\frac{1}{2\gamma_\alpha}(\rho_t^{\alpha,u})^* \rho_t^{\alpha,u} - \frac{1}{2\gamma_\alpha}\left(2(\rho_t^{\alpha,u})^* \gamma_\alpha z\right) - \frac{1}{2\gamma_\alpha}\gamma_\alpha^2 z^* z + \frac{\gamma_\alpha}{2} z^* z \\ &= -\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + u_t + \gamma_\alpha z\|^2 + \frac{\gamma_\alpha}{2}\|z\|^2. \end{aligned} \tag{3.85}$$

So for fixed z, we only have to maximize in $u \in \tilde{\mathcal{V}}_{\alpha,t}^\perp$ the function $-\|\rho_t^\alpha + u_t + \gamma_\alpha z\|^2$. We know that this function is maximized by only one u that is characterized by

$$\tilde{u}_t = \arg \sup_{u \in \tilde{\mathcal{V}}_{\alpha,t}^\perp} f_{\rho^\alpha,t}(z,u) = \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z). \tag{3.86}$$

Here the function $\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(y)$ is the orthogonal projection of $y \in \mathbb{R}^n$ on $\tilde{\mathcal{V}}_{\alpha,t}^\perp$.

All of the results mentioned before could have been obtained by using a weaker assumption, namely $\mathcal{V}_\alpha$ be a closed convex set. Following the ideas by (El Karoui and Rouge, 2000), the only tools that would have changed are

1. Let $\delta(x) = \sup_{\pi \in \mathcal{V}_\alpha} -\pi^* x$ be the support function of the convex set $-\mathcal{V}_\alpha$
2. Let $\tilde{\mathcal{V}}_\alpha := \{x \in \mathbb{R}^n : \delta(x) < \infty\}$ the effective domain of δ, and $\tilde{\mathcal{V}}_{\alpha,t}^\sigma = (\sigma_t^\alpha)^{-1}(\tilde{\mathcal{V}}_\alpha)$.

Using them together with the assumption $\varrho_t^\alpha \in -\mathcal{V}_\alpha \cap \mathcal{V}_\alpha$, we would have obtained $(\varrho_t^\alpha)^* \sigma_t^\alpha u_t = 0$ and

$$\begin{aligned} d\beta_{t,T}^\alpha V_t &= [-\beta_{t,T}^\alpha V_t \varrho_t^\alpha + \beta_{t,T}^\alpha \pi_t]^* \sigma_t^\alpha dW_t^{\rho^\alpha} \\ &= [-\beta_{t,T}^\alpha V_t \varrho_t^\alpha + \beta_{t,T}^\alpha \pi_t]^* \sigma_t^\alpha dW_t^{\alpha,u} - \beta_{t,T}^\alpha \pi_t^* \sigma_t^\alpha u_t dt. \end{aligned} \tag{3.87}$$

Since $-y^* \sigma_t^\alpha u_t \leq 0$ for all $y \in \mathcal{V}_\alpha$ we have that 3.87 is a supermartingale and the desired property 3.23 is satisfied.

We know that the function $\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}$ satisfies the following inequalities

- $\langle \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}} y - y, x - \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}} y\rangle = 0$ for all $x \in \tilde{\mathcal{V}}^{\perp}_{\alpha,t}$ and $y \in \mathbb{R}^n$

- $\left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}} y - y\right)^* \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}} y = 0$ for all $y \in \mathbb{R}^n$,

and for the case $\mathcal{V}_\alpha$ a closed convex set, the properties are

- $\langle \Pi_{\tilde{\mathcal{V}}^{\sigma}_{\alpha,t}} y - y, x - \Pi_{\tilde{\mathcal{V}}^{\sigma}_{\alpha,t}} y\rangle \geq 0$ for all $x \in \tilde{\mathcal{V}}^{\sigma}_{\alpha,t}$ and $y \in \mathbb{R}^n$

- $\left(\Pi_{\tilde{\mathcal{V}}^{\sigma}_{\alpha,t}} y - y\right)^* \Pi_{\tilde{\mathcal{V}}^{\sigma}_{\alpha,t}} y = 0$ for all $y \in \mathbb{R}^n$.

In these two scenarios we get

$$\rho^\alpha_t + \gamma_\alpha z + \tilde{u} \in (\sigma^\alpha_t)^* \mathcal{V}_\alpha \quad \text{and} \quad (\varrho^\alpha_t + \gamma_\alpha z + \tilde{u})^* \tilde{u} = 0. \tag{3.88}$$

We now need to get an expression for the dynamics of

$$V^{\alpha,C}_t = \operatorname*{ess\,sup}_{u \in \mathcal{U}_\alpha} V^{\alpha,C,u}_t. \tag{3.89}$$

By a result of (El Karoui and Rouge, 2000) we will show that the BSDE with driver

$$\tilde{f}_{\rho^\alpha,t}(z) := \operatorname*{ess\,sup}_{u \in \tilde{\mathcal{V}}^{\perp}_{\alpha,t}} f_{\rho,t}(z,u) = f_{\rho,t}(z, \tilde{u}(t,\rho^\alpha,z)), \tag{3.90}$$

and terminal condition C admits a solution that dominates the BSDEs with driver f in 3.84. As it is pointed out by (El Karoui and Rouge, 2000) the driver $\tilde{f}$ is quadratic and the results about quadratic BSDEs are not easy to obtain.

Theorem 3.23. *Let* $V^{\alpha,C}_t = \operatorname{ess\,sup}_{u \in \mathcal{U}_\alpha} V^{\alpha,C,u}_t$. *Then there exists a process* $z \in \mathbb{H}^{2,d}_T$ *such that* (V^C_t, Z_t) *satisfies the BSDE*

$$\begin{cases} -dV^{\alpha,C}_t = \left[-\frac{1}{2\gamma_\alpha} \|\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t)\|^2 - (\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t))^* z_t \right] dt - z^*_t dW_t \\ V^{\alpha,C}_T = C \end{cases}. \tag{3.91}$$

Proof. The formula 3.85 can be written as

$$\begin{aligned}
&-\frac{1}{2\gamma_\alpha}\|\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t) + \gamma_\alpha z\|^2 + \frac{\gamma_\alpha}{2}\|z_t\|^2 \\
&= -\frac{1}{2\gamma_\alpha}\left\{ \|\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t)\|^2 + 2\left(\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t)\right)^* \gamma_\alpha z_t + \gamma^2_\alpha \|z_t\|^2 \right\} + \frac{\gamma_\alpha}{2}\|z_t\|^2 \\
&= -\frac{1}{2\gamma_\alpha}\|\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t)\|^2 - (\rho^\alpha_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^\alpha_t - \gamma_\alpha z_t))^* z_t
\end{aligned} \tag{3.92}$$

in 3.91.

Now, we need to show that the driver $f(x, z)$ in 3.91 is majorized by $k(1 + |z|^2)$ for some positive constant k. Then, by using (Theorem B.1 El Karoui and Rouge, 2000) we get the existence of the solution.

Using the properties of inner products of orthogonal projections with $x = 0$, we have that $\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}} y\|^2 \leq \langle \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}} y, y\rangle \leq \|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}} y\|\|y\|$, and dividing by $\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}} y\|$ we get $\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}} y\| \leq \|y\|$. Now

$$
\begin{aligned}
&-\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)\|^2 - (\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t))^* z_t \\
&\leq -\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)\|^2 + \|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)\|\|z_t\| \\
&\leq -\frac{1}{2\gamma_\alpha}\left(\|\rho_t^\alpha\| - \|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)\|\right)^2 + \left(2\|\rho_t^\alpha\| + \gamma_\alpha\|z_t\|\right)\|z_t\| \\
&\leq -\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha\|^2 + \frac{1}{\gamma_\alpha}\|\rho_t^\alpha\|\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)\| - \frac{1}{2\gamma_\alpha}\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)\|^2 + 2\|\rho_t^\alpha\|\|z_t\| + \gamma_\alpha\|z_t\|^2 \\
&\leq \frac{1}{2\gamma_\alpha}\|\rho_t^\alpha\|^2 + \|\rho_t^\alpha\|\|z_t\| + \frac{1}{2\gamma_\alpha}\left(\|\rho_t^\alpha\| + \gamma_\alpha\|z_t\|\right)^2 + 2\|\rho_t^\alpha\|\|z_t\| + \gamma_\alpha\|z_t\|^2 \\
&\leq \frac{1}{\gamma_\alpha}\|\rho_t^\alpha\|^2 + 4\|\rho_t^\alpha\|\|z_t\| + \frac{3}{2}\gamma_\alpha\|z_t\|^2 \\
&\leq k(1 + \|z_t\|^2)
\end{aligned}
\tag{3.93}
$$

for some constant $k > 0$. The last part of these inequalities comes from the fact that ρ^α is uniformly bounded on $\Omega \times [0, T]$. As it is pointed by (El Karoui and Rouge, 2000) after some lengthy work the conditions by (Theorem B.2 El Karoui and Rouge, 2000) holds, and then we get the domination and uniqueness. □

Optimal portfolio

We worked on the properties of the dual problem 3.78. With these tools we are ready to show the solution to the problem 3.22 due to (El Karoui and Rouge, 2000).

Theorem 3.24. *The value function of 3.91 is given by*

$$
\begin{aligned}
\Psi_\alpha(v_0, C) &= \inf_{\pi \in \mathcal{V}_\alpha} \frac{1}{\gamma_\alpha} \ln E_{P^\alpha}\left[\exp(-\gamma_\alpha(V_T^{v_0,\xi} - C))\right] \\
&= -\frac{v_0}{\beta_{0,T}^\alpha} + \sup_{u \in \mathcal{U}_\alpha}\left\{E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u}|P^\alpha)\right\}
\end{aligned}
\tag{3.94}
$$

The forward wealth is given by

$$
\tilde{V}_t^\alpha = -\Psi_\alpha(v_0, C) - \frac{1}{\gamma_\alpha}\ln H_t^{\rho^\alpha,\tilde{u}} + V_t^{\alpha,C} \tag{3.95}
$$

and its dynamics is given by

$$
d\tilde{V}_t^\alpha = \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^*(dW_t + \rho_t^\alpha dt) \tag{3.96}
$$

where the process z_t *is determined by the BSDE* 3.91 *and* $\rho_t^{\alpha,\tilde{u}} = \rho_t + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^{\perp}}(-\rho_t^\alpha - \gamma_\alpha z_t)$. *Moreover, the optimal portfolio is*

$$\pi_t = B_{t,T}^\alpha \left[((\sigma_t^\alpha)^{-1})^* \left(\frac{1}{\gamma_\alpha} \rho_t^{\alpha,\tilde{u}} + z_t \right) + \varrho_t^\alpha \tilde{V}_t \right]. \tag{3.97}$$

Proof. First, we prove the value function. We know that each $\xi \in Z^{\alpha,S}$ characterizes a particular π_t, and each π_t is characterized by a $\xi \in Z^{\alpha,S}$. So we can use

$$\Psi_\alpha(v_0, C) = \inf_{\pi \in \mathcal{V}_\alpha} \frac{1}{\gamma_\alpha} \ln E_{P^\alpha} \left[\exp(-\gamma_\alpha (V_T^{v_0,\pi} - C)) \right] \tag{3.98}$$

instead of

$$\Psi_\alpha(v_0, C) = \inf_{\xi \in Z^{\alpha,S}} \frac{1}{\gamma_\alpha} \ln E_{P^\alpha}[\exp(-\gamma_\alpha (V_T^{v_0,\xi} - C))]. \tag{3.99}$$

Under the assumption $\mathcal{V}_\alpha$ being a vector space, we have that

$$d\beta_{t,T}^\alpha V_t^{\alpha,\pi} = [-\beta_{t,T}^\alpha V_t^{\alpha,\pi} \varrho_t^\alpha + \beta_{t,T}^\alpha \pi]^* \sigma_t^\alpha dW_t^{\alpha,u} \tag{3.100}$$

is a martingale under $Q^{\rho^\alpha,u}$ for each $u \in \mathcal{U}_\alpha$ as it was proved in 3.77 by using Proposition 3.22. Taking into account that $B_{T,T}^\alpha = 1$, we get

$$\begin{aligned}
\Psi_\alpha(v_0, C) &= \inf_{\pi \in \mathcal{V}_\alpha} \frac{1}{\gamma_\alpha} \ln E_{P^\alpha} \left[\exp(-\gamma_\alpha (V_T^{v_0,\pi} - C)) \right] \\
&= \inf_{\pi \in \mathcal{V}_\alpha} \sup_{Q^{\rho^\alpha,u} \sim P^\alpha} \left\{ E_{Q^{\rho^\alpha,u}}[-V_T^{v_0,\pi} + C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u} | P^\alpha) \right\} \\
&= \inf_{\pi \in \mathcal{V}_\alpha} \sup_{Q^{\rho^\alpha,u} \sim P^\alpha} \left\{ -\frac{v_0}{B_{0,T}^\alpha} + E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u} | P^\alpha) \right\} \\
&= -\frac{v_0}{B_{0,T}^\alpha} + \sup_{Q^{\rho^\alpha,u} \sim P^\alpha} \left\{ E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u} | P^\alpha) \right\}.
\end{aligned} \tag{3.101}$$

To prove Theorem 3.24 we go the other way around. We define V_t^α as

$$V_t^\alpha = -\Psi_\alpha(v_0, C) - \frac{1}{\gamma_\alpha} \ln H_t^{\rho^{\alpha,\tilde{u}}} + V_t^{\alpha,C}, \tag{3.102}$$

and we verify that its dynamics, portfolio and value function are 3.96, 3.97, and 3.94 respectively.

From the equation 3.102, we know that the dynamics of V_t^α satisfies

$$
\begin{aligned}
d\tilde{V}_t^\alpha &= -d\Psi_\alpha(v_0, C) - \frac{1}{\gamma_\alpha} d\ln H_t^{\rho^\alpha,\tilde{u}} + dV_t^{\alpha,C} \\
&= -\frac{1}{\gamma_\alpha} d\ln H_t^{\rho^\alpha,\tilde{u}} + dV_t^{\alpha,C} \\
&= -\frac{1}{\gamma_\alpha}\left\{-(\rho_t^{\alpha,\tilde{u}})^* dW_t - \frac{1}{2}\|\rho_t^{\alpha,\tilde{u}}\|^2 dt\right\} + \frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t)\|^2 dt \\
&\quad + (\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t))^* z_t dt + z_t^* dW_t \\
&= \left(\frac{1}{2\gamma_\alpha}\|\rho_t^{\alpha,\tilde{u}}\|^2 + \frac{1}{2\gamma_\alpha}\|\rho_t^{\alpha,\tilde{u}}\|^2 + (\rho_t^{\alpha,\tilde{u}})^* z_t\right) dt + \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^* dW_t \\
&= \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^* \left(dW_t + \rho_t^{\alpha,\tilde{u}} dt\right),
\end{aligned}
\tag{3.103}
$$

and using the fact that

$$
\left(\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t) + \gamma_\alpha z_t\right)^* \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t) = 0,
\tag{3.104}
$$

we get

$$
\begin{aligned}
d\tilde{V}_t^\alpha &= \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^* \left(dW_t + \rho_t^{\alpha,\tilde{u}} dt\right) \\
&= \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^* (dW_t + \rho_t^\alpha dt) + \frac{1}{\gamma_\alpha}\left(\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t) + \gamma_\alpha z_t\right)^* \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t) dt \\
&= \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^* (dW_t + \rho_t^\alpha dt).
\end{aligned}
\tag{3.105}
$$

By using 3.88, we know that $((\sigma_t^\alpha)^{-1})^* \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)$ is in $\mathcal{V}_\alpha$. In addition, we notice that $\varrho_t^\alpha V_t^\alpha B_{t,T}^\alpha$ belongs to $\mathcal{V}_\alpha$. Therefore, the expression 3.97 for π_t belongs to $\mathcal{V}_\alpha$ as it was stated in the consequences of assumption 3.15.

We have to prove that by using 3.97, the formula 3.105 can be expressed in a closed form as in 3.65. It is showed as follows

$$
\begin{aligned}
d\tilde{V}_t^\alpha &= \left(\frac{1}{\gamma_\alpha}\rho_t^{\alpha,\tilde{u}} + z_t\right)^* (dW_t + \rho_t^\alpha dt) \\
&= \left((\sigma_t^\alpha)^* \pi_t \beta_{t,T}^\alpha - (\sigma_t^\alpha)^* (\varrho_t^\alpha) \tilde{V}_t^\alpha\right)^* dW_t^{\rho^\alpha} \\
&= \left[-\beta_{t,T}^\alpha V_t^\alpha \varrho_t^\alpha + \beta_{t,T}^\alpha \pi_t\right]^* \sigma_t^\alpha dW_t^{\rho^\alpha}.
\end{aligned}
\tag{3.106}
$$

Finally, by definition of $\tilde{V}_t^\alpha$, we have $\tilde{V}_T^\alpha = V_T^\alpha$ and $V_T^\alpha - C = -\Psi_\alpha(v_0, C) - \frac{1}{\gamma_\alpha}\ln H_T^{\rho^{\alpha,\tilde{u}}}$. Therefore, we get

$$
\Psi_\alpha(v_0, C) = \frac{1}{\gamma_\alpha}\ln E_{P^\alpha}\left[\exp\{-\gamma_\alpha(V_T^\alpha - C)\}\right],
\tag{3.107}
$$

which completes the proof. $\square$

A similar proof can be given for the case where $\mathcal{V}_\alpha$ is a closed cone.

4 Discrete Time Setting

In this part we will provide some of the basic tools that can be used to discretize the time-continuous models mentioned in the previous sections. We start by showing the relation of the CRR model with Proposition 2.2 and Corollary 2.5. Next, we show the standard procedure of how to discretize the time-continuous models for the big agent, and then we proceed to show the discretization of the time-continuous model for small agents.

4.1 Convergence to the single price model under different macroeconomic events

Throughout this part we consider the time interval $[0, T]$ with N equally-spaced times $T/N, 2T/N, ..., T$ that we identify with indices $1, 2, ..., N$ respectively. As in the CRR model let us consider $S_t^{N,j,(0)}$, $S_t^{N,j,(1)}$, $R_t^{(N),j}$ the riskless asset, the risky asset and the return of the risky asset when the macroeconomic event Λ_j occurs. The return of the risky asset is given by the formula

$$R_t^{(N),j} = \frac{S_t^{N,j,(1)} - S_{t-1}^{N,j,(1)}}{S_{t-1}^{N,j,(1)}}. \tag{4.1}$$

Our goal in this subsection is to rigorously show that Corollary 2.2 can be obtained as the limit of the CRR binomial tree model.

We first need to build the returns of the risky assets under different macroeconomic events. For this purpose let $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega\right)$ be a measurable space where $\Omega = \{u, d\}$ with $u =$ up and $d =$ down is the space of states for the risky assets. We define, $R_t^{(N),j}$, the rate of the risky asset under the macroeconomic event Λ_j by

$$R_t^{(N),j} = \begin{cases} b_j^{(N)}, & \text{on } \Pi_t^{-1}(\{u\}), \\ a_j^{(N)}, & \text{on } \Pi_t^{-1}(\{d\}), \end{cases} \tag{4.2}$$

where

$$a_j^{(N)} := e^{-v_j\sqrt{T/N}} - 1, \qquad b_j^{(N)} := e^{v_j\sqrt{T/N}} - 1, \tag{4.3}$$

and

$$\Pi_t : \Omega^N \to \Omega^N \qquad \text{with} \qquad \Pi_t(w_1, ..., w_N) = w_t \tag{4.4}$$

for $(w_1, ..., w_N) \in \Omega^N$ and distinct positive numbers $v_1, ..., v_m$.

Under Λ_j, the risky and riskless assets are described by the formulas

$$S_t^{N,j,(1)} := S_0^{N,j,(1)} \prod_{k=1}^{t} (1 + R_k^{(N),j}) \qquad \text{and} \qquad S_t^{N,j,(0)} = (1 + r_j^{(N)})^t, \tag{4.5}$$

where $r_j^{(N)} = r_j T/N$.

A consequence of this construction is that $\mathcal{F}_t^{N,(j)} := \sigma\{R_1^{(N),j}, ..., R_t^{(N),j}\} = \sigma\{S_1^{(N),j,(1)}, ..., S_t^{(N),j,(1)}\}$ for $j = 1, ..., m$. In fact, we can easily see that

$$\mathcal{F}_t^{N,(i)} = \mathcal{F}_t^{N,(j)} \quad \text{for} \quad i \neq j \quad \text{and} \quad t \in \{1, 2, ..., N\}. \tag{4.6}$$

Therefore, from now on, we use $\mathcal{F}_t^N$ to denote $\mathcal{F}_t^{N,j}$ for any $j \in \{1, ..., m\}$.

Now, we are interested in finding a probability measure $P^{N,j}$ with respect to Λ_j and under which the equations

$$P^{N,j}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] + P^{N,j}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] = 1 \tag{4.7}$$

and

$$E_{P^{N,j}}\left[\left.\frac{S_t^{(N),j,(1)}}{1 + r_j^{(N)}}\right| \mathcal{F}_{t-1}^N\right] = S_{t-1}^{(N),j,(1)} \tag{4.8}$$

hold.

By using equation 4.5, we can rewrite the formula 4.8 as

$$E_{P^{N,j}}\left[\left. R_t^{(N),j} \right| \mathcal{F}_{t-1}\right] = r_j^{(N)} \tag{4.9}$$

that is equivalent to

$$b_j^{(N)} P^{N,j}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] + a_j^{(N)} P^{N,j}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] = r_j^{(N)}. \tag{4.10}$$

Therefore, the problem of finding the probability measure becomes the following linear problem

$$\begin{bmatrix} b_j^{(N)} & a_j^{(N)} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} P^{N,j}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] \\ P^{N,j}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] \end{bmatrix} = \begin{bmatrix} r_j^{(N)} \\ 1 \end{bmatrix}. \tag{4.11}$$

Using Cramer's rule, we have that

$$P^{N,j}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] = \frac{r_j^{(N)} - a_j^{(N)}}{b_j^{(N)} - a_j^{(N)}} \quad \text{and} \quad P^{N,j}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] = \frac{b_j^{(N)} - r_j^{(N)}}{b_j^{(N)} - a_j^{(N)}}. \tag{4.12}$$

Thus

$$P^{N,j}\left[\Pi_t^{-1}(\{u\})\right] = \frac{r_j^{(N)} - a_j^{(N)}}{b_j^{(N)} - a_j^{(N)}} \quad \text{and} \quad P^{N,j}\left[\Pi_t^{-1}(\{d\})\right] = \frac{b_j^{(N)} - r_j^{(N)}}{b_j^{(N)} - a_j^{(N)}}, \tag{4.13}$$

for any $t \in \{1, 2, ..., N\}$. Since these conditional probabilities do not depend on t, the conclusion that $R_1^{(N),j}, ..., R_N^{(N),j}$ are i.i.d. under $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega, \mathcal{F}_t^N, P^{N,j}\right)$ is straightforward.

Corollary 4.1. *Let $S_t^{(N),j,(0)}$, $S_t^{(N),j,(1)}$ and $R_t^{(N),j}$ be the riskless asset, the risky asset, and the return of the risky asset defined on $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega, \mathcal{F}_t^N, P^{N,j}\right)$ under the macroeconomic event Λ_j, then $S_t^{(N),j,(1)}/S_t^{(N),j,(0)}$ is a martingale, and $R_1^{(N),j}, ..., R_N^{(N),j}$ are i.i.d. under $P^{N,j}$.*

Proof. This is a consequence of 4.7, 4.8 and 4.13. □

Now, we will use some results by (Föllmer and Schweizer, 1991) to show that $S_N^{(N),j,(1)}$ under $P^{N,j}$ converges weakly to a log-normal distribution.

Remark 4.2. $S_N^{(N),j,(1)}$ *under* $P^{N,j}$ *converges weakly to the distribution of*

$$S_0^{(N),j,(1)} \exp\left\{v_j N(0,T) + \left(r_j - \frac{1}{2}v_j^2\right) T\right\}, \tag{4.14}$$

where $N(0,T)$ *is the normal distribution with mean* 0*, and variance* T*.*

Proof. By Taylor formula, we get

$$\ln(1+x) = x - \frac{1}{2}x^2 + q(x)x^2, \tag{4.15}$$

where the last term q satisfies

$$|q(x)| \leq \delta(a,b) \quad \text{for} \quad -1 < a \leq x \leq b, \tag{4.16}$$

and $\delta(a,b) \to 0$ as $a, b \to 0$. Thus, by applying logarithm to

$$S_t^{N,j,(1)} = S_0^{N,j,(1)} \prod_{k=1}^{t} (1 + R_k^{(N),j}), \tag{4.17}$$

we get

$$\ln\left(S_N^{N,j,(1)}\right) = \sum_{k=1}^{N}\left(R_k^{(N),j} - \frac{1}{2}\left(R_k^{(N),j}\right)^2\right) + \varepsilon_j^{(N)}, \tag{4.18}$$

where

$$|\varepsilon_j^N| \leq \delta(a_j^N, b_j^N) \sum_{k=1}^{N}\left(R_k^{(N),j}\right)^2. \tag{4.19}$$

We know from the definition of $a_j^{(N)}$ and $b_j^{(N)}$ that

$$\sqrt{N}a_j^{(N)} \to -v_j\sqrt{T} \qquad \text{and} \qquad \sqrt{N}b_j^{(N)} \to v_j\sqrt{T}. \tag{4.20}$$

and by using $E_{P^{N,j}}\left[R_t^{(N),j}\right] = r_j^{(N)}$, we get

$$\begin{aligned} E_{P^{N,j}}\left[|\varepsilon_j^{(N)}|\right] &\leq \delta(a_j^{(N)}, b_j^{(N)}) N E_{P^{N,j}}\left[\left(R_1^{(N),j}\right)^2\right] \\ &\leq \delta(a_j^{(N)}, b_j^{(N)}) N \left(\left(b_j^{(N)}\right)^2 \frac{r_j^{(N)} - a_j^{(N)}}{b_j^{(N)} - a_j^{(N)}} + \left(a_j^{(N)}\right)^2 \frac{b_j^{(N)} - r_j^{(N)}}{b_j^{(N)} - a_j^{(N)}}\right) \to 0, \end{aligned} \tag{4.21}$$

since

$$\lim_{N\to\infty} N\left(\left(b_j^{(N)}\right)^2 \frac{r_j^{(N)} - a_j^{(N)}}{b_j^{(N)} - a_j^{(N)}} + \left(a_j^{(N)}\right)^2 \frac{b_j^{(N)} - r_j^{(N)}}{b_j^{(N)} - a_j^{(N)}}\right) = \left(v_j^2 T\frac{1}{2} + v_j^2 T\frac{1}{2}\right) = v_j^2 T, \tag{4.22}$$

By using the fact that L^1 convergence implies convergence in probability, we can use Slutsky's theorem (see for instance Jacod and Protter, 2004) to conclude that the corresponding laws of $\varepsilon_j^{(N)}$ converge weakly to the Dirac measure δ_0. As a consequence, we only need to focus on the weak convergence of

$$Z^{(N)} = \sum_{k=1}^{N} \left(R_k^{(N),j} - \frac{1}{2} \left(R_k^{(N),j} \right)^2 \right). \tag{4.23}$$

We will prove that this process converges weakly to the normal distribution $N(r_j T - \frac{1}{2} v_j^2 T, v_j^2 T)$ by using the results that appear in Theorem A.37 by (Föllmer and Schweizer, 1991). Therefore, by the use of that theorem, we only have to show that the mean and variance of $Z^{(N)}$ converges to $r_j T - \frac{1}{2} v_j^2 T$ and $v_j^2 T$ respectively.

For the mean we have that

$$\begin{aligned} E_{P^{N,j}} \left[\sum_{k=1}^{N} \left(R_k^{(N),j} - \frac{1}{2} \left(R_k^{(N),j} \right)^2 \right) \right] &= N r_j^{(N)} - \frac{N}{2} E_{P^{N,j}} \left[\left(R_1^{(N),j} \right)^2 \right] \\ &= N r_j^{(N)} - \frac{N}{2} \left(\left(b_j^{(N)} \right)^2 \frac{r_j^{(N)} - a_j^{(N)}}{b_j^{(N)} - a_j^{(N)}} + \left(a_j^{(N)} \right)^2 \frac{b_j^{(N)} - r_j^{(N)}}{b_j^{(N)} - a_j^{(N)}} \right) \\ &\to r_j T - \frac{1}{2} v_j^2 T. \end{aligned} \tag{4.24}$$

On the other hand, for the variance we have that

$$\begin{aligned} &\mathrm{Var}_{P^{N,j}} \left[\sum_{k}^{N} R_k^{(N),j} - \frac{1}{2} \left(R_k^{(N),j} \right)^2 \right] \\ &= N \mathrm{Var}_{P^{N,j}} \left[R_1^{(N),j} - \frac{1}{2} \left(R_1^{(N),j} \right)^2 \right] \\ &= N E_{P^{N,j}} \left[\left(R_1^{(N),j} - \frac{1}{2} \left(R_1^{(N),j} \right)^2 \right)^2 \right] - N E_{P^{N,j}} \left[R_1^{(N),j} - \frac{1}{2} \left(R_1^{(N),j} \right)^2 \right]^2 \\ &= N E_{P^{N,j}} \left[\left(R_1^{(N),j} \right)^2 - \left(R_1^{(N),j} \right)^3 + \frac{1}{4} \left(R_1^{(N),j} \right)^4 \right] - N \left(\frac{r_j T}{N} - \frac{1}{2} E_{P^{N,j}} \left[\left(R_1^{(N),j} \right)^2 \right] \right)^2. \end{aligned} \tag{4.25}$$

Now using the fact that for $p > 2$

$$\sum_{k=1}^{N} E_{P^{N,j}} \left[|R_k^{(N),j}|^p \right] \leq \max \left\{ |a_j^{(N)}|, |b_j^{(N)}| \right\}^{p-2} \sum_{k=1}^{N} E_{P^{N,j}} \left[\left(R_k^{(N),j} \right)^2 \right] \to 0, \tag{4.26}$$

we get that

$$\mathrm{Var}_{P^{N,j}} \left[\sum_{k}^{N} R_k^{(N),j} - \frac{1}{2} \left(R_k^{(N),j} \right)^2 \right] \to v_j^2 T. \tag{4.27}$$

Therefore, the Theorem A.37 by (Föllmer and Schweizer, 1991) holds. □

We have proved so far that each $S_t^{(N),j,(1)}/S_t^{(N),j,(0)}$ for $j = 1, ..., m.$, is a martingale, and $R_1^{(N),j}, ..., R_N^{(N),j}$ are i.i.d. on $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega, \mathcal{F}_t, P^{N,j}\right)$. However, these properties don't hold on a single filtered probability space for all $j = 1, ..., m$. This problem can be addressed to some extent if we consider the product space and the projection processes of the prices that will preserve the original distributions. This idea is described by the following result.

Proposition 4.3. *Let $X_t^{(1)}, ..., X_t^{(m)}$ be processes that are martingales under the filtered probability spaces $(\Omega, \mathcal{F}_t, P^{(1)}), ..., (\Omega, \mathcal{F}_t, P^{(m)})$ respectively, then there exist processes $\tilde{X}_t^{(1)}, ..., \tilde{X}^{(m)}$ and a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{P})$ such that $\tilde{X}_t^{(1)}, ..., \tilde{X}_t^{(m)}$ are pairwise independent and martingales under $\tilde{P}$ such that*

$$\tilde{P}\left[\tilde{X}_t^{(j)} \in A\right] = P^{(j)}\left[X_t^{(j)} \in A\right] \tag{4.28}$$

for every $j = 1, ..., m.$, $t = 0, 1, ..., N.$, and any A measurable set.

Proof. It suffices to show this result for the case $m = 2$. For this purpose let us define $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$ on the filtered probability space $(\Omega \otimes \Omega, \mathcal{F}_t \otimes \mathcal{F}_t, P^{(1)} \otimes P^{(2)})$ that are given by

$$\tilde{X}_t^{(1)} = X_t^{(1)}(w) \quad \text{on} \quad \Pi_1^{-1}(\{w\}), \tag{4.29}$$

and

$$\tilde{X}_t^{(2)} = X_t^{(2)}(w) \quad \text{on} \quad \Pi_2^{-1}(\{w\}), \tag{4.30}$$

where Π_1 and Π_2 are the coordinate projections from $\Omega \times \Omega$ into Ω i.e., $\Pi_1(w_1, w_2) = w_1$ and $\Pi_2(w_1, w_2) = w_2$ for $(w_1, w_2) \in \Omega \times \Omega$.

By construction and the definition of product measure, we know that $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$ are independent and also

$$P^{(i)}\left[X_t^{(i)} \in A\right] = P^{(1)} \otimes P^{(2)}\left[\tilde{X}_t^{(i)} \in A\right] \tag{4.31}$$

for A any measurable set, and $i = 1, 2$.

Now we only need to prove that $\tilde{X}_t^{(1)}$ and $\tilde{X}_t^{(2)}$ are martingales on $(\Omega \otimes \Omega, \mathcal{F}_t \otimes \mathcal{F}_t, P^{(1)} \otimes P^{(2)})$. To this end we will use the monotone class theorem for λ-π systems (see for instance Gut, 2013), and the formal definition of conditional expectation.

Let

$$\mathcal{C}_{t-1} = \left\{A \times B \in \mathcal{F}_{t-1} \times \mathcal{F}_{t-1} : E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(1)} 1_{A \times B}\right] = E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(1)} 1_{A \times B}\right] \text{ for } i = 1, 2\right\}. \tag{4.32}$$

We want to show $\mathcal{C}_{t-1}$ is a π-system that generates $\mathcal{F}_{t-1} \otimes \mathcal{F}_{t-1}$. To this end, it suffices to show that $\mathcal{C}_{t-1} = \mathcal{F}_{t-1} \times \mathcal{F}_{t-1}$.

For each $A, B \in \mathcal{F}_{t-1}$ we have

$$\begin{aligned}
E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(1)} 1_{A \times B}\right] &= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(1)} 1_{A \times \Omega} 1_{\Omega \times B}\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(1)} 1_{A \times \Omega}\right] E_{P^{(1)} \otimes P^{(2)}}\left[1_{\Omega \times B}\right] \\
&= E_{P^{(1)}}\left[X_t^{(1)} 1_{\{A\}}\right] E_{P^{(1)} \otimes P^{(2)}}\left[1_{\Omega \times B}\right] \\
&= E_{P^{(1)}}\left[X_{t-1}^{(1)} 1_{\{A\}}\right] E_{P^{(1)} \otimes P^{(2)}}\left[1_{\Omega \times B}\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[X_{t-1}^{(1)} 1_{A \times \Omega}\right] E_{P^{(1)} \otimes P^{(2)}}\left[1_{\Omega \times B}\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[X_{t-1}^{(1)} 1_{A \times \Omega} 1_{\Omega \times B}\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[X_{t-1}^{(1)} 1_{A \times B}\right].
\end{aligned} \tag{4.33}$$

Using an analogous reasoning, we can deduce the same result for $\tilde{X}_t^{(2)}$. This proves that in fact $\mathcal{C}_{t-1} = \mathcal{F}_{t-1} \times \mathcal{F}_{t-1}$.

Let

$$\mathcal{D}_{t-1} = \left\{ D \in \mathcal{F}_{t-1} \otimes \mathcal{F}_{t-1} : E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_D\right] = E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(i)} 1_D\right] \text{ for } i = 1, 2 \right\}. \tag{4.34}$$

We know that $\mathcal{C}_{t-1} \subseteq \mathcal{D}_{t-1}$, so we only need to show that $\mathcal{D}_{t-1}$ is a λ-system to finish the proof that $\tilde{X}_t^{(i)}$ is a martingale under $P^{(1)} \otimes P^{(2)}$ for $i = 1, 2$.

Let $D, E \in \mathcal{D}_{t-1}$ with $D \subseteq E$, then

$$\begin{aligned}
E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_{E \setminus D}\right] &= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} (1_E - 1_D)\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_E\right] - E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_D\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(i)} 1_E\right] - E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(i)} 1_D\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(i)} (1_E - 1_D)\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(i)} 1_{E \setminus D}\right].
\end{aligned} \tag{4.35}$$

for $i = 1, 2$. Moreover, for any $(D_n)_{n=1}^{\infty}$ increasing sequence of elements in $\mathcal{D}_{t-1}$, we have that

$$\begin{aligned}
E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_{\bigcup_n D_n}\right] &= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} \left(\lim_n 1_{D_n}\right)\right] \\
&= \lim_n E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_{D_n}\right] \\
&= \lim_n E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_{t-1}^{(i)} 1_{D_n}\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} \left(\lim_n 1_{D_n}\right)\right] \\
&= E_{P^{(1)} \otimes P^{(2)}}\left[\tilde{X}_t^{(i)} 1_{\bigcup_n D_n}\right],
\end{aligned} \tag{4.36}$$

and this finishes the proof. $\square$

From the previous results the following corollary is derived

Corollary 4.4. *Let $S_t^{(N),j,(0)}$, $S_t^{(N),j,(1)}$ and $R_t^{(N),j}$ the riskless asset, the risky asset, and the return of the risky asset defined on $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega, \mathcal{F}_t, P^{N,j}\right)$ under the macroeconomic event Λ_j for $j = 1, ..., m$. If we define the processes $\tilde{S}_t^{(N),j,(0)}$, $\tilde{S}_t^{(N),j,(1)}$ and $\tilde{R}_t^{(N),j}$ on $\left(\Pi_{i=1}^m \Omega^N, \otimes_{i=1}^m \left(\otimes_{j=1}^N 2^\Omega\right), \otimes_{i=1}^m \mathcal{F}_t^N, \otimes_{i=1}^m P^{N,i}\right)$ by*

$$\tilde{S}_t^{N,j,(1)} := \tilde{S}_0^{N,j,(1)} \prod_{k=1}^t (1 + \tilde{R}_k^{(N),j}), \qquad \tilde{R}_t^{(N),j} = \begin{cases} b_j^{(N)}, & \text{on } \Omega^N \times \cdots \times \underbrace{\Pi_t^{-1}(\{u\})}_{j\text{-th position}} \times \cdots \times \Omega^N, \\ a_j^{(N)}, & \text{on } \Omega^N \times \cdots \times \underbrace{\Pi_t^{-1}(\{d\})}_{j\text{-th position}} \times \cdots \times \Omega^N, \end{cases} \tag{4.37}$$

and

$$\tilde{S}_t^{N,j,(0)} = (1 + r_j^{(N)})^t, \tag{4.38}$$

then each $\tilde{S}_t^{(N),j,(1)}/\tilde{S}_t^{(N),j,(0)}$ is a martingale, and $\tilde{R}_1^{(N),j}, ..., \tilde{R}_N^{(N),j}$ are i.i.d. under $\otimes_{i=1}^m P^{N,i}$.

Proof. It is a direct consequence of remark 4.2, and Proposition 4.3. $\square$

From corollary 4.4 and by using the definition of product measure, we conclude that for any function f continuous and bounded

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{(1 + r_j^{(N)})^N} E_{\otimes_{i=1}^m P^{N,i}} \left[f\left(\tilde{S}_N^{N,j,(1)} \right) \right] &= \lim_{N\to\infty} \frac{1}{(1 + r_j^{(N)})^N} E_{P^{N,j}} \left[f\left(S_N^{N,j,(1)} \right) \right] \\ &= e^{-r_j T} E\left[f\left(S_0^{j,(1)} e^{v_j N(0,T) + r_j T - \frac{1}{2} v_j^2 T} \right) \right] \end{aligned} \tag{4.39}$$

where the last part comes from using the remark 4.2.

By using $(\Omega^\Lambda, \mathcal{F}^\Lambda, Q^\Lambda)$ as in Proposition 2.1 i.e., $Q^\Lambda(\{\Lambda_j\}) = \lambda_j$ with $\sum_{j=1}^m \lambda_j = 1$, we can define on the filtered probability space $\left(\Omega^\Lambda \times \Pi_{i=1}^m \Omega^N, \mathcal{F}^\Lambda \otimes \left(\otimes_{i=1}^m \left(\otimes_{j=1}^N 2^\Omega\right)\right), \mathcal{F}^\Lambda \otimes (\otimes_{i=1}^m \mathcal{F}_t), Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})\right)$ the price process of the risky asset, $S_t^{N,(1)}$, and the riskless asset, $S_t^{N,(0)}$, by

$$S_t^{N,(1)} = \tilde{S}_t^{N,j,(1)} \text{ and } S_t^{N,(0)} = \tilde{S}_t^{N,j,(0)}, \text{ on } \{\Lambda_j\} \times \Pi_{i=1}^m \Omega^N \tag{4.40}$$

Now we are ready to show the limit version of Corollary 2.2.

Proposition 4.5. *Let f be a continuous and bounded function on $\mathbb{R}$. The price at time $t = 0$ of a* *contingent claim with payoff* $C = f(S_N^{N,(1)})$ *converges as follows*

$$\lim_{N\to\infty} E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[\frac{1}{S_N^{N,(0)}} f\left(S_N^{N,(1)}\right)\right]$$
$$= \sum_{i=1}^m \lambda_i e^{-r_i T} \int f(x) \frac{\exp\left\{\dfrac{-\left(\ln(x) - \int_0^t \left(r_i(z) - \dfrac{v_i^2(z)}{2}\right) dz - \ln(\tilde{S}_0^{N,i,(1)})\right)^2}{2\int_0^t v_i^2(z)dz}\right\}}{x\left(\int_0^t v_i^2(z)dz\right)^{1/2}\sqrt{2\pi}} dx \tag{4.41}$$

Proof. By using the definition of product measure, $S_t^{N,(0)}$ and $S_t^{N,(1)}$ as in 4.40, we have that

$$\begin{aligned}
&\lim_{N\to\infty} E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[\frac{1}{S_N^{N,(0)}} f\left(S_N^{N,(1)}\right)\right] \\
&= \lim_{N\to\infty} E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[\frac{1}{S_N^{N,(0)}} f\left(S_N^{N,(1)}\right)\left(\sum_{j=1}^m 1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N}\right)\right] \\
&= \lim_{N\to\infty} E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[\sum_{j=1}^m \frac{1}{\tilde{S}_N^{N,j,(0)}} f\left(\tilde{S}_N^{N,j,(1)}\right) 1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N}\right].
\end{aligned} \tag{4.42}$$

Now, by using the independence of $1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N}$ with respect to $\tilde{S}_N^{N,j,(0)}$ and $\tilde{S}_N^{N,j,(1)}$ we can use the result in 4.39 to get

$$\begin{aligned}
&\lim_{N\to\infty} E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[\sum_{j=1}^m \frac{1}{\tilde{S}_N^{N,j,(0)}} f\left(\tilde{S}_N^{N,j,(1)}\right) 1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N}\right] \\
&= \lim_{N\to\infty} \sum_{j=1}^m E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[\frac{1}{\tilde{S}_N^{N,j,(0)}} f\left(\tilde{S}_N^{N,j,(1)}\right)\right] E_{Q^\Lambda \otimes (\otimes_{i=1}^m P^{N,i})}\left[1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N}\right] \\
&= \sum_{j=1}^m \lim_{N\to\infty} \lambda_j E_{\otimes_{i=1}^m P^{N,i}}\left[\frac{1}{\tilde{S}_N^{N,j,(0)}} f\left(\tilde{S}_N^{N,j,(1)}\right)\right] \\
&= \sum_{j=1}^m \lambda_j e^{-r_j T} \int f(x) \frac{\exp\left\{\dfrac{-\left(\ln(x) - \int_0^t \left(r_j(z) - \dfrac{v_j^2(z)}{2}\right) dz - \ln(\tilde{S}_0^{N,j,(1)})\right)^2}{2\int_0^t v_j^2(z)dz}\right\}}{x\left(\int_0^t v_j^2(z)dz\right)^{1/2}\sqrt{2\pi}} dx
\end{aligned} \tag{4.43}$$

where in this case $v_i(t) = v_i$ for all $t \in [0, T]$, and $i = 1, ..., m$. This result coincides with the formula in Corollary 2.2. $\square$

4.2 Convergence to the multi-price model under different macroeconomic events

As in the previous subsection we consider the time interval $[0, T]$ with N equally-spaced times $T/N, 2T/N, ..., NT/N$ that we identify with indices $1, 2, ..., N$ respectively. As in the CRR model let us consider ${}^kS_t^{N,(0)}, {}^kS_t^{N,(1)}, ..., {}^kS_t^{N,(n)}$,and ${}^kR_t^{N,(1)}, ..., {}^kR_t^{N,(n)}$ the riskless asset, the risky assets and the returns of the risky assets when the macroeconomic event Λ_k occurs. The return of the j-th risky asset under the macroeconomic event Λ_k is given by the formula

$$ {}^kR_t^{N,(j)} = \frac{{}^kS_t^{N,(j)} - {}^kS_{t-1}^{N,(j)}}{{}^kS_{t-1}^{N,(j)}}. \tag{4.44} $$

Now our goal is to show that it is possible to prove that the Corollary 2.5 can be expressed as a limit of CRR binomial tree model. Since our task is to prove this statement, we need to do an analogous procedure as with the model with only one risky asset.

We first need to build the returns of the risky assets under different macroeconomic events. First notice that under each Λ_k the dynamics of ${}^kS_t^{(i)}$ is of the form

$$ {}^kS_t^{(i)} = {}^kS_0^{(i)} \exp\left\{ \int_0^t \left({}^k\mu_u^{(i)} - \frac{1}{2}\sum_{j=1}^n \left({}^k\sigma_u^{(ij)}\right)^2 \right) du + \sum_{j=1}^n \int_0^t {}^k\sigma_u^{(ij)} dW_u^{(j)} \right\}. \tag{4.45} $$

Since we are using the same tools as in the previous subsection, we are assuming that all of the coefficients are constant and then ${}^k\sigma_t = {}^k\sigma$ for all $t \in [0, T]$ where ${}^k\sigma$ is non-degenerate as in assumption 2.3. Thus, we have

$$ \sum_{j=1}^n \int_0^t {}^k\sigma_u^{(ij)} dW_u^{(j)} \sim N\left(0, t\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2\right) \tag{4.46} $$

Therefore, we can consider the dynamics of ${}^kS_t^{(i)}$ to be of the form

$$ {}^kS_t^{(i)} = {}^kS_0^{(i)} \exp\left\{ t\left({}^k\mu^{(i)} - \frac{1}{2}\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2 \right) + N\left(0, t\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2\right) \right\}. \tag{4.47} $$

As we noticed, in the limit of the CRR model for a single risky price, the notation turned out to be cumbersome. Since we are interested in pricing European options for several risky assets, we can follow a similar approach to Subsection 5.1. All what we need to do is create the model for each i-th risky asset $S_t^{(i)}$.

For this purpose let $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega\right)$ be a measurable space where $\Omega = \{u, d\}$ with $u =$ up and $d =$ down is the space of states for the risky assets. We define, ${}^kR_t^{N,(i)}$, the rate of the i-th risky asset under the macroeconomic event Λ_k by

$$ {}^kR_t^{N,(i)} = \begin{cases} b_k^{N,(i)}, & \text{on } \Pi_t^{-1}(\{u\}), \\ a_k^{N,(i)}, & \text{on } \Pi_t^{-1}(\{d\}), \end{cases} \tag{4.48} $$

where

$$a_k^{N,(i)} := e^{-\sqrt{\sum_{j=1}^{n}\left({}^k\sigma^{(ij)}\right)^2}\sqrt{T/N}} - 1, \qquad b_k^{N,(i)} := e^{\sqrt{\sum_{j=1}^{n}\left({}^k\sigma^{(ij)}\right)^2}\sqrt{T/N}} - 1, \tag{4.49}$$

and

$$\Pi_t : \Omega^N \to \Omega \qquad \text{with} \qquad \Pi_t(w_1, ..., w_N) = w_t \tag{4.50}$$

for $(w_1, ..., w_N) \in \Omega^N$ with positive numbers $\sigma_{i1}, ..., \sigma_{in}$ that are estimated in the Subsection 5.1.

Under Λ_k, the risky and riskless assets are described by the formulas

$${}^kS_t^{N,(i)} := {}^kS_0^{N,(i)} \prod_{l=1}^{t}(1 + {}^kR_l^{N,(i)}) \qquad \text{and} \qquad {}^kS_t^{N,(0)} = (1 + r_k^{(N)})^t \tag{4.51}$$

where $r_k^{(N)} = r_k T/N$ as in the CRR model for only one risky asset in the market. A consequence of this construction is that ${}^k\mathcal{F}_t^{N,(i)} := \sigma\{{}^kR_1^{N,(i)}, ..., {}^kR_t^{N,(i)}\} = \sigma\{{}^kS_1^{N,(i)}, ..., {}^kS_t^{N,(i)}\}$ for $k = 1, ..., m$. In fact, we can easily see that

$${}^j\mathcal{F}_t^{N,(i)} = {}^k\mathcal{F}_t^{N,(i)} \quad \text{for} \quad j \neq k \quad \text{and} \quad t \in \{1, 2, ..., N\}. \tag{4.52}$$

Therefore, we can use $\mathcal{F}_t^{N,(i)}$ to denote ${}^k\mathcal{F}_t^{N,(i)}$ for any $k \in \{1, ..., m\}$. Moreover, since each one of the risky asset only take two different values, we have that

$$\mathcal{F}_t^{N,(i)} = \mathcal{F}_t^{N,(j)} \tag{4.53}$$

for any risky asset $i, j = 1, ..., n$. As a result, we will use $\mathcal{F}_t^N$ instead of $\mathcal{F}_t^{N,(i)}$.

Now we are interested in finding a probability measure ${}^kP^{N,(i)}$ with respect to Λ_k and under which the equations

$${}^kP^{N,(i)}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] + {}^kP^{N,(i)}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] = 1 \tag{4.54}$$

and

$$E_{{}^kP^{N,(i)}}\left[\left.\frac{{}^kS_t^{(N),(i)}}{1 + r_k^{(N)}}\right| \mathcal{F}_{t-1}^N\right] = {}^kS_{t-1}^{(N),(i)} \tag{4.55}$$

hold.

By using equation 4.51, we can rewrite the formula 4.55 as

$$E_{{}^kP^{N,(i)}}\left[\left.{}^kR_t^{N,(i)}\right| \mathcal{F}_{t-1}^N\right] = r_k^{(N)} \tag{4.56}$$

that is equivalent to

$$b_k^{N,(i)}\,{}^kP^{N,(i)}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] + a_k^{N,(i)}\,{}^kP^{N,(i)}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] = r_k^{(N)}. \tag{4.57}$$

Therefore, the problem of finding the probability measure becomes the following linear problem

$$\begin{bmatrix} b_k^{N,(i)} & a_k^{N,(i)} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} {}^kP^{N,(i)}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] \\ {}^kP^{N,(i)}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] \end{bmatrix} = \begin{bmatrix} r_k^{(N)} \\ 1 \end{bmatrix}, \tag{4.58}$$

Using Cramer's rule we have that

$$ {}^{k}P^{N,(i)}\left[\Pi_t^{-1}(\{u\})|\mathcal{F}_{t-1}^N\right] = \frac{r_k^{(N)} - a_k^{N,(i)}}{b_k^{N,(i)} - a_k^{N,(i)}} \quad \text{and} \quad {}^{k}P^{N,(i)}\left[\Pi_t^{-1}(\{d\})|\mathcal{F}_{t-1}^N\right] = \frac{b_k^{N,(i)} - r_k^{(N)}}{b_k^{N,(i)} - a_k^{N,(i)}} \tag{4.59} $$

Thus

$$ {}^{k}P^{N,(i)}\left[\Pi_t^{-1}(\{u\})\right] = \frac{r_k^{(N)} - a_k^{N,(i)}}{b_k^{N,(i)} - a_k^{N,(i)}} \quad \text{and} \quad {}^{k}P^{N,(i)}\left[\Pi_t^{-1}(\{d\})\right] = \frac{b_k^{N,(i)} - r_k^{(N)}}{b_k^{N,(i)} - a_k^{N,(i)}} \tag{4.60} $$

for any $t \in \{1, 2, ..., N\}$. Since these conditional probabilities do not depend on t, the conclusion that ${}^{k}R_1^{N,(i)}, ..., {}^{k}R_N^{N,(i)}$ under $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega, \mathcal{F}_t^N, {}^{k}P^{N,(i)}\right)$ are i.i.d. is straightforward.

Corollary 4.6. *Let ${}^{k}S_t^{N,(0)}$, ${}^{k}S_t^{N,(i)}$ and ${}^{k}R_t^{N,(i)}$ the riskless asset, the i-th risky asset, and the return of the i-th risky asset defined on $\left(\Omega^N, \otimes_{i=1}^N 2^\Omega, \mathcal{F}_t^N, {}^{k}P^{N,(i)}\right)$ under the macroeconomic event Λ_k, then ${}^{k}S_t^{N,(i)} / {}^{k}S_t^{N,(0)}$ is martingale, and ${}^{k}R_1^{N,(i)}, ..., {}^{k}R_N^{N,(i)}$ are i.i.d. under ${}^{k}P^{N,(i)}$.*

Proof. This is a consequence of 4.54, 4.55 and 4.60. □

Now we will use some results by (Föllmer and Schweizer, 1991) to show that ${}^{k}S_N^{N,(i)}$ under ${}^{k}P^{N,(i)}$ converges weakly to a log-normal distribution.

Remark 4.7. *Under the previous assumptions ${}^{k}S_N^{N,(i)}$ under ${}^{k}P^{N,(i)}$ converges weakly to the distribution of*

$$ {}^{k}S_0^{N,(i)} \exp\left\{ \sqrt{\sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2} N(0,T) + \left(r_j - \frac{1}{2} \sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2 \right) T \right\}, \tag{4.61} $$

where $N(0,T)$ is the normal distribution with mean 0, and variance T.

Proof. By Taylor formula, we get

$$ \ln(1+x) = x - \frac{1}{2}x^2 + q(x)x^2 \tag{4.62} $$

where the last term q satisfies

$$ |q(x)| \leq \delta(a,b) \quad \text{for} \quad -1 < a \leq x \leq b \tag{4.63} $$

where $\delta(a,b) \to 0$ as $a, b \to 0$. Thus, by applying logarithm to

$$ {}^{k}S_t^{N,(i)} = {}^{k}S_0^{N,(i)} \prod_{l=1}^{t} (1 + {}^{k}R_l^{N,(i)}), \tag{4.64} $$

we get

$$ \ln\left({}^{k}S_N^{N,(i)}\right) = \sum_{l=1}^{N} \left({}^{k}R_l^{N,(i)} - \frac{1}{2}\left({}^{k}R_l^{N,(i)}\right)^2 \right) + \varepsilon_k^{N,(i)} \tag{4.65} $$

where

$$|\varepsilon_k^{N,(i)}| \leq \delta(a_k^{N,(i)}, b_k^{N,(i)}) \sum_{l=1}^{N} \left({}^{k}R_l^{N,(i)}\right)^2 \tag{4.66}$$

We know from the definition of $a_k^{N,(i)}$ and $b_k^{N,(i)}$ that

$$\sqrt{N} a_k^{N,(i)} \to -\sqrt{T \sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2} \qquad \text{and} \qquad \sqrt{N} b_k^{N,(i)} \to \sqrt{T \sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2}, \tag{4.67}$$

and by using $E_{({}^{k}P^{N,(i)})}\left[{}^{k}R_t^{N,(i)}\right] = r_k^{(N)}$, we get

$$\begin{aligned} E_{({}^{k}P^{N,(i)})}\left[|\varepsilon_k^{N,(i)}|\right] &\leq \delta(a_k^{N,(i)}, b_k^{N,(i)}) N E_{({}^{k}P^{N,(i)})}\left[\left({}^{k}R_1^{N,(i)}\right)^2\right] \\ &\leq \delta(a_k^{N,(i)}, b_k^{N,(i)}) N \left(\left(b_k^{N,(i)}\right)^2 \frac{r_k^{(N)} - a_k^{N,(i)}}{b_k^{N,(i)} - a_k^{N,(i)}} + \left(a_k^{N,(i)}\right)^2 \frac{b_k^{N,(i)} - r_k^{(N)}}{b_k^{N,(i)} - a_k^{N,(i)}} \right) \xrightarrow[N\to\infty]{} 0, \end{aligned} \tag{4.68}$$

since

$$\begin{aligned} \lim_{N\to\infty} N \left(\left(b_k^{N,(i)}\right)^2 \frac{r_k^{(N)} - a_k^{N,(i)}}{b_k^{N,(i)} - a_k^{N,(i)}} + \left(a_k^{N,(i)}\right)^2 \frac{b_k^{N,(i)} - r_k^{(N)}}{b_k^{N,(i)} - a_k^{N,(i)}} \right) &= \left(\sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2 T \frac{1}{2} + \sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2 T \frac{1}{2} \right) \\ &= \sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2 T, \end{aligned} \tag{4.69}$$

By using the fact that L^1 convergence implies convergence in probability, we can use Slutsky's theorem to conclude that the corresponding laws of $\varepsilon_j^{(N)}$converge weakly to the Dirac measure δ_0. As a consequence, we only need to focus on the weak convergence of

$$Z^{(N)} = \sum_{l=1}^{N} \left({}^{k}R_l^{N,(i)} - \frac{1}{2}\left({}^{k}R_l^{N,(i)}\right)^2 \right). \tag{4.70}$$

We will prove that this process converges weakly to the normal distribution

$$N\left(r_j T - \frac{1}{2}\sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2 T, \sum_{j=1}^{n} \left({}^{k}\sigma^{(ij)}\right)^2 T \right) \tag{4.71}$$

by using the results that appear in Theorem A.37 by (Föllmer and Schweizer, 1991). For this purpose we only have to show that the mean and variance of $Z^{(N)}$ converges to $r_j T - \frac{1}{2}\sum_{j=1}^{n}\left({}^{k}\sigma^{(ij)}\right)^2 T$ and $\sum_{j=1}^{n}\left({}^{k}\sigma^{(ij)}\right)^2 T$ respectively.

For the mean we have that

$$\begin{aligned}
E_{({}^kP^{N,(i)})}\left[\sum_{l=1}^{N}\left({}^kR_l^{N,(i)}-\frac{1}{2}\left({}^kR_l^{N,(i)}\right)^2\right)\right] &= Nr_k^{(N)}-\frac{N}{2}E_{({}^kP^{N,(i)})}\left[\left({}^kR_l^{N,(i)}\right)^2\right]\\
&= Nr_k^{(N)}-\frac{N}{2}\left(\left(b_k^{N,(i)}\right)^2\frac{r_k^{(N)}-a_k^{N,(i)}}{b_k^{N,(i)}-a_k^{N,(i)}}+\left(a_k^{N,(i)}\right)^2\frac{b_k^{N,(i)}-r_k^{(N)}}{b_k^{N,(i)}-a_k^{N,(i)}}\right)\\
&\to r_jT-\frac{1}{2}\sum_{j=1}^{n}\left({}^k\sigma^{(ij)}\right)^2T.
\end{aligned}\tag{4.72}$$

On the other hand, for the variance we have that

$$\begin{aligned}
&\mathrm{Var}_{({}^kP^{N,(i)})}\left[\sum_{l}^{N}{}^kR_l^{N,(i)}-\frac{1}{2}\left({}^kR_l^{N,(i)}\right)^2\right]\\
&= N\mathrm{Var}_{({}^kP^{N,(i)})}\left[{}^kR_1^{N,(i)}-\frac{1}{2}\left({}^kR_1^{N,(i)}\right)^2\right]\\
&= NE_{({}^kP^{N,(i)})}\left[\left({}^kR_1^{N,(i)}-\frac{1}{2}\left({}^kR_1^{N,(i)}\right)^2\right)^2\right]-NE_{({}^kP^{N,(i)})}\left[{}^kR_1^{N,(i)}-\frac{1}{2}\left({}^kR_1^{N,(i)}\right)^2\right]^2\\
&= NE_{({}^kP^{N,(i)})}\left[\left({}^kR_1^{N,(i)}\right)^2-\left({}^kR_1^{N,(i)}\right)^3+\frac{1}{4}\left({}^kR_1^{N,(i)}\right)^4\right]-N\left(\frac{r_jT}{N}-\frac{1}{2}E_{({}^kP^{N,(i)})}\left[\left({}^kR_1^{N,(i)}\right)^2\right]\right)^2
\end{aligned}\tag{4.73}$$

Now using the fact that for $p>2$

$$\sum_{l=1}^{N}E_{({}^kP^{N,(i)})}\left[|{}^kR_l^{N,(i)}|^p\right]\le\max\left\{|a_k^{N,(i)}|,|b_k^{N,(i)}|\right\}^{p-2}\sum_{l=1}^{N}E_{({}^kP^{N,(i)})}\left[\left({}^kR_l^{N,(i)}\right)^2\right]\to 0,\tag{4.74}$$

we get that

$$\mathrm{Var}_{({}^kP^{N,(i)})}\left[\sum_{l=1}^{N}{}^kR_l^{N,(i)}-\frac{1}{2}\left({}^kR_l^{N,(i)}\right)^2\right]\to\sum_{j=1}^{n}\left({}^k\sigma^{(ij)}\right)^2T.\tag{4.75}$$

Therefore, the Theorem A.37 by (Föllmer and Schweizer, 1991) holds. □

We have proved so far that each ${}^kS_t^{N,(i)}/{}^kS_t^{N,(0)}$ for $k=1,...,m.$, is a martingale, and ${}^kR_1^{N,(i)},...,{}^kR_N^{N,(i)}$ are i.i.d. on $\left(\Omega^N,\otimes_{i=1}^N2^\Omega,\mathcal{F}_t^N,{}^kP^{N,(i)}\right)$. However, these properties don't hold on a single filtered probability space for all $k=1,...,m$. This problem can be addressed using Proposition 4.3 as it is described in the following corollary.

Corollary 4.8. *Let ${}^kS_t^{(N),(0)}$, ${}^kS_t^{(N),(i)}$ and ${}^kR_t^{N,(i)}$ be the riskless asset, the i-th risky asset, and the return of the i-th risky asset defined on $\left(\Omega^N,\otimes_{i=1}^N2^\Omega,\mathcal{F}_t^N,{}^kP^{N,(i)}\right)$ under the macroeconomic event Λ_k*

for $k = 1, ..., m$. *If we define the processes* ${}^{k}\tilde{S}_t^{(N),(0)}$, ${}^{k}\tilde{S}_t^{(N),(i)}$ *and* ${}^{k}\tilde{R}_t^{(N),(i)}$ *on the filtered probability space* $\left(\Pi_{j=1}^{m}\Omega^N, \otimes_{j=1}^{m}\left(\otimes_{j=1}^{N}2^{\Omega}\right), \otimes_{j=1}^{m}\mathcal{F}_t^N, \otimes_{j=1}^{m}{}^{j}P^{N,(i)}\right)$ *by*

$$
{}^{k}\tilde{S}_t^{N,(i)} := {}^{k}\tilde{S}_0^{N,(i)}\prod_{l=1}^{t}(1 + {}^{k}\tilde{R}_l^{N,(i)}), \qquad {}^{k}\tilde{R}_t^{N,(i)} = \begin{cases} b_k^{N,(i)}, & \text{on } \Omega^N \times \cdots \times \underbrace{\Pi_t^{-1}(\{u\})}_{k\ position} \times \cdots \times \Omega^N, \\ a_k^{N,(i)}, & \text{on } \Omega^N \times \cdots \times \underbrace{\Pi_t^{-1}(\{d\})}_{k\ position} \times \cdots \times \Omega^N, \end{cases} \tag{4.76}
$$

and

$$
{}^{k}\tilde{S}_t^{N,(0)} = (1 + r_k^{(N)})^t, \tag{4.77}
$$

then each ${}^{k}\tilde{S}_t^{N,(i)} / {}^{k}\tilde{S}_t^{N,(0)}$ *for* $k = 1, ..., m.$, *is a martingale, and* ${}^{k}\tilde{R}_1^{N,(i)}, ..., {}^{k}\tilde{R}_N^{N,(i)}$ *are i.i.d. under* $\otimes_{j=1}^{m}{}^{j}P^{N,(i)}$.

Proof. It is a direct consequence of remark 4.7, Proposition 4.3, and Corollary 4.4. □

From corollary 4.8 and by using the definition of product measure, we conclude that for any continuous and bounded function, f, we get

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{(1 + r_k^{(N)})^N} E_{\otimes_{j=1}^{m}{}^{j}P^{N,(i)}}\left[f\left({}^{k}\tilde{S}_N^{N,(i)}\right)\right] \\
&= \lim_{N\to\infty}\frac{1}{(1 + r_k^{(N)})^N} E_{{}^{k}P^{N,(i)}}\left[f\left({}^{k}S_N^{(N),(i)}\right)\right] \\
&= e^{-r_k T} E\left[f\left({}^{k}\tilde{S}_0^{(i)} e^{\sqrt{\sum_{j=1}^{n}\left({}^{k}\sigma^{(ij)}\right)^2} N(0,T) + r_j T - \frac{1}{2}\sum_{j=1}^{n}\left({}^{k}\sigma^{(ij)}\right)^2 T}\right)\right],
\end{aligned} \tag{4.78}
$$

where the last part comes from using the remark 4.7.

By using $(\Omega^{\Lambda}, \mathcal{F}^{\Lambda}, Q^{\Lambda})$ as in Proposition 2.1 i.e., $Q^{\Lambda}(\{\Lambda_j\}) = \lambda_j$ with $\sum_{j=1}^{m}\lambda_j = 1$, we can define on the space $\left(\Omega^{\Lambda} \times \Pi_{j=1}^{m}\Omega^N, \mathcal{F}^{\Lambda} \otimes \left(\otimes_{k=1}^{m}\left(\otimes_{j=1}^{N}2^{\Omega}\right)\right), \mathcal{F}^{\Lambda} \otimes \left(\otimes_{j=1}^{m}\mathcal{F}_t\right), Q^{\Lambda} \otimes \left(\otimes_{j=1}^{m}{}^{j}P^{N,(i)}\right)\right)$ the price process of the i-th risky asset, $S_t^{N,(i)}$, and the riskless asset, $S_t^{N,(0)}$, by

$$
S_t^{N,(i)} = {}^{k}\tilde{S}_N^{N,(i)} \text{ and } S_t^{N,(0)} = {}^{k}\tilde{S}_N^{N,(0)}, \text{ on } \{\Lambda_k\} \times \Pi_{i=1}^{m}\Omega^N. \tag{4.79}
$$

Now, we are ready to show the limit version of Corollary 2.5.

Proposition 4.9. *Let* f *be a continuous and bounded function on* $\mathbb{R}$. *The price at time* $t = 0$ *of a* *contingent claim with payoff $C = f(S_N^{N,(i)})$ converges as follows*

$$\lim_{N\to\infty} E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[\frac{1}{S_N^{N,(0)}} f\left(S_N^{N,(i)}\right) \right]$$
$$= \sum_{k=1}^m \lambda_k e^{-r_k T} \int f(x) \frac{\exp\left\{ \dfrac{-\left(\ln(x) - T\left(r_k - \frac{1}{2}\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2\right) - \ln({}^k\tilde{S}_0^{(i)})\right)^2}{2T\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2} \right\}}{x\sqrt{T 2\pi \sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2}} dx. \tag{4.80}$$

Proof. By using the definition of product measure, $S_t^{N,(0)}$ and $S_t^{N,(i)}$ as in 4.79, we have that

$$\lim_{N\to\infty} E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[\frac{1}{S_N^{N,(0)}} f\left(S_N^{N,(i)}\right) \right]$$
$$= \lim_{N\to\infty} E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[\frac{1}{S_N^{N,(0)}} f\left(S_N^{N,(i)}\right) \left(\sum_{j=1}^m 1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N} \right) \right] \tag{4.81}$$
$$= \lim_{N\to\infty} E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[\sum_{j=1}^m \frac{1}{{}^j\tilde{S}_N^{N,(0)}} f\left({}^j\tilde{S}_N^{N,(i)}\right) 1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N} \right].$$

Now, by using the independence of $1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N}$ with respect to ${}^j\tilde{S}_N^{N,(0)}$ and ${}^j\tilde{S}_N^{N,(i)}$ we can use the result in 4.39 to get

$$\lim_{N\to\infty} E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[\sum_{j=1}^m \frac{1}{{}^j\tilde{S}_N^{N,(0)}} f\left({}^j\tilde{S}_N^{N,(i)}\right) 1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N} \right]$$
$$= \lim_{N\to\infty} \sum_{j=1}^m E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[\frac{1}{{}^j\tilde{S}_N^{N,(0)}} f\left({}^j\tilde{S}_N^{N,(i)}\right) \right] E_{Q^\Lambda \otimes \left(\otimes_{j=1}^m {}^j P^{N,(i)}\right)} \left[1_{\{\Lambda_j\}\times \Pi_{i=1}^m \Omega^N} \right]$$
$$= \sum_{j=1}^m \lim_{N\to\infty} \lambda_j E_{\otimes_{j=1}^m {}^j P^{N,(i)}} \left[\frac{1}{{}^j\tilde{S}_N^{N,(0)}} f\left({}^j\tilde{S}_N^{N,(i)}\right) \right] \tag{4.82}$$
$$= \sum_{i=1}^m \lambda_i e^{-r_j T} \int f(x) \frac{\exp\left\{ \dfrac{-\left(\ln(x) - T\left(r_k - \frac{1}{2}\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2\right) - \ln({}^k\tilde{S}_0^{(i)})\right)^2}{2T\sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2} \right\}}{x\sqrt{T \sum_{j=1}^n \left({}^k\sigma^{(ij)}\right)^2}\sqrt{2\pi}} dx$$

□

Notice that by the use of the Proposition 4.3 we could have created a bigger probablility space under which Proposition 4.9 holds for each $i = 1, ..., n$. However, for the sake of simplicity, we decided to work with the i-th risky asset since in practice the most active derivatives usually work with only one underlying asset. For rainbow options, financial instruments with more than one underlying asset, we can use the same reasoning mentioned before by using Proposition 4.3; however, the notation can become cumbersome if the researcher or investor has many underlying assets.

4.3 Discretization of the continuous time stochastic models

In this part we will provide some numerical methods for the discretization of the continuous-time models seen so far. We will focus especially on the Euler method since the price for small investors involves BSDEs that are not easy discretized by binomial methods such as CRR. In addition to that, the Euler method provides reasonable accurate predictions in small periods of time, and this is the type of predictions that we are looking for because of the difficulty in finding several derivatives of different stocks with the same issue and maturity over long periods of time. Since most of the methods that we will use to estimate parameters on Section 5 are based on the fact that the risk-rate, diffusion and drift coefficients are constant we will use the same assumptions throughout this subsection.

4.3.1 Euler discretization of continuous time model for big agents in a market with one risky asset

In the case of a market with only one single risky asset $S^{(1)}$, we know from Proposition 2.1 that its dynamics under the macroeconomic event Λ_k is of the form

$$dS_t^{k,(1)} = S_t^{k,(1)} \mu_k dt + S_t^{k,(1)} \upsilon_k dW_t. \tag{4.83}$$

In order to simulate this continuous-time model, the instances of time have to be specified. Following the ideas of the CRR limit model let us denote the time points by

$$0 = t_0, \quad t_1 = \frac{T}{N}, \quad t_2 = \frac{2T}{N}, \quad ..., \quad t_N = T \tag{4.84}$$

with $N \in \mathbb{Z}^+$. We assume the *simulation step size*, Δ_t, to be constant and equal to T/N. Therefore, the sequence of $(S_{t_i}^{k,(1)})_{i=0}^N$ via the Euler discretization scheme for the stochastic deferential equation 4.83 is given by

$$\begin{aligned} S_{t_0}^{k,(1)} &= S_0^{k,(1)} \\ S_{t_{i+1}}^{k,(1)} &= S_{t_i}^{k,(1)} + S_{t_i}^{k,(1)} \mu_k \Delta_t + S_{t_i}^{k,(1)} \upsilon_k \Delta W_{t_i} \end{aligned} \tag{4.85}$$

with $\Delta W_{t_i} \sim N(0, T/N)$ and $i = 0, 1, ..., N-1$.

4.3.2 Euler discretization of the continuous time model for big agents in a market with some risky assets

As in the case of the market with only one risky asset, we will create an analogous procedure for a market with several risky assets. We know from Corollary 2.2 that the dynamics of the price of the

i-th risky asset under the macroeconomic event Λ_k is of the form

$$d\,{}^kS_t^{(i)} = {}^k\mu_t^{(i)}\,{}^kS_t^{(i)}dt + {}^kS_t^{(i)}\left(\sum_{j=1}^{n} {}^k\sigma_t^{(ij)} dW_t^{(j)}\right). \tag{4.86}$$

where the processes $W_t^1, ..., W_t^n$ are independent Brownian motions, and $({}^k\sigma_t^{(ij)})^2$ is the component of the volatility of the i-th stock explained by the j-th equity, see Subsection 5.1 for more details. Because of the fact that ${}^k\sigma_u^{(ij)} \equiv {}^k\sigma^{(ij)}$ for all $u \in [0, T]$ and

$$\sum_{j=1}^{n} \int_0^t {}^k\sigma_u^{(ij)} dW_u^{(j)} \sim N\left(0, t\sum_{j=1}^{n}\left({}^k\sigma^{(ij)}\right)^2\right), \tag{4.87}$$

the discretization method using the Euler scheme for the price of the i-th risky asset under the macroeconomic event Λ_k is of the form

$${}^kS_{t_0}^{(i)} = {}^kS_0^{(i)}, \quad \text{and} \quad {}^kS_{t_{j+1}}^{(i)} = {}^kS_{t_j}^{(i)} + {}^kS_{t_j}^{(i)}\mu_k\Delta_t + {}^kS_{t_j}^{(i)}\sum_{j=1}^{n} {}^k\sigma_t^{(ij)}\Delta W_{t_j}, \tag{4.88}$$

with $\Delta W_{t_j} \sim N(0, T/N)$ and $j = 0, 1, ..., N-1$.

4.3.3 Euler discretization of continuous time model for small agents in a market with some risky assets

To use the Euler method in the price formula for small agents, we need to find a BSDE that describes the evolution of the prices $p^\alpha(v_0, C)$ with respect to the time for each one the stereotyped agents.

From formulas

$$\frac{p^\alpha(v_0, C)}{B_{0,T}^\alpha} = \sup_{Q\sim P^\alpha}\left\{E_Q[C] - \frac{1}{\gamma_\alpha}h(Q|P^\alpha)\right\} - \sup_{Q\sim P^\alpha}\left\{-\frac{1}{\gamma_\alpha}h(Q|P^\alpha)\right\}, \tag{4.89}$$

and

$$h(Q^{\rho^\alpha,u}|P^\alpha) = E_{Q^{\rho^\alpha,u}}\left[\frac{1}{2}\int_0^T \|\rho_t^{\alpha,u}\|^2 dt\right], \quad \text{and} \quad V_t^{\alpha,C,u} = E_{Q^{\rho^\alpha,u}}\left[C - \frac{1}{2\gamma_\alpha}\int_t^T \|\rho_t^{\alpha,u}\|^2 dt \middle| \mathcal{F}_t\right], \tag{4.90}$$

we have

$$V_t^{\alpha,C,u} - V_0^{\alpha,0,u} = E_{Q^{\rho^\alpha,u}}\left[C - \frac{1}{2\gamma_\alpha}\int_t^T \|\rho_t^{\alpha,u}\|^2 dt \middle| \mathcal{F}_t\right] - \left(-h(Q^{\rho^\alpha,u}|P^\alpha)\right). \tag{4.91}$$

Therefore, by the use of Theorem 3.91 and the remark made at that part we conclude that the dynamics of $p_t^\alpha(v_0, C)/B_{t,T}^\alpha$ is of the form

$$d\left(p_t^\alpha(v_0, C)/B_{t,T}^\alpha\right) = d(V_t^{\alpha,C} - V_0^{\alpha,0}). \tag{4.92}$$

We can rewrite the previous formula as a shorter one by using the next theorem due to (El Karoui and Rouge, 2000).

Theorem 4.10. *The forward price, $\beta_{t,T}p_t^\alpha(v_0,C)$, of the derivative with payoff C for the α-stereotyped agent is described by the BSDE*

$$-d(\beta_{t,T}p_t^\alpha(v_0,C)) = \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp} z_t\|dt - z_t^* dW_t^{\alpha,\tilde{u}_t} \tag{4.93}$$

where $\tilde{u}_t^0 = \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^0)$. The processes $(x_t^{0,\alpha}, z_t^{0,\alpha})$ and $(x_t^{C,\alpha}, z_t^{C,\alpha})$ are the solutions in Theorem 3.91 with conditions $x_T^{0,\alpha} = 0$ and $x_T^{C,\alpha} = C$ respectively. The process $W_t^{\alpha,\tilde{u}}$ is an n-dimensional Brownian motion under $Q^{\rho^\alpha,\tilde{u}^0}$.

Core inflation usually refers to the inflation rate calculated based on a price index of goods and services except food and energy.

Proof. From Theorem 3.23, we have that

$$\begin{aligned}
&-d(V_t^{\alpha,C} - V_0^{\alpha,C})\\
&= \left[-\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{C,\alpha})\|^2 - (\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{C,\alpha}))^* z_t^{C,\alpha}\right] dt - (z_t^{C,\alpha})^* dW_t\\
&- \left[-\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{0,\alpha})\|^2 - (\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{0,\alpha}))^* z_t^{0,\alpha}\right] dt + (z_t^{0,\alpha})^* dW_t
\end{aligned} \tag{4.94}$$

our task is to transform this equation into 4.93.

First, we will work with the coefficients of dt. Rearranging the second part of the coefficients of dt we get

$$\begin{aligned}
&-(\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{C,\alpha}))^* z_t^{C,\alpha} dt + (\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{0,\alpha}))^* z_t^{0,\alpha} dt\\
&= (\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(\rho_t^\alpha) - \rho_t^\alpha)^*(z_t^{C,\alpha} - z_t^{0,\alpha})dt + \left[-\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\gamma_\alpha z_t^{C,\alpha}))^* z_t^{C,\alpha} + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\gamma_\alpha z_t^{0,\alpha}))^* z_t^{0,\alpha}\right] dt.
\end{aligned} \tag{4.95}$$

The remaining part of the coefficients of dt is equal to

$$\begin{aligned}
&-\frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{C,\alpha})\|^2 dt + \frac{1}{2\gamma_\alpha}\|\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha - \gamma_\alpha z_t^{0,\alpha})\|^2 dt\\
&= -\frac{1}{2\gamma_\alpha}\|\left(\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha)\right) + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\gamma_\alpha z_t^{C,\alpha})\|^2 dt + \frac{1}{2\gamma_\alpha}\|\left(\rho_t^\alpha + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\rho_t^\alpha)\right) + \Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(-\gamma_\alpha z_t^{0,\alpha})\|^2 dt\\
&= -\frac{1}{2\gamma_\alpha}\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(\gamma_\alpha z_t^{C,\alpha})\|^2 dt + \frac{1}{2\gamma_\alpha}\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(\gamma_\alpha z_t^{0,\alpha})\|^2 dt\\
&= \frac{\gamma_\alpha}{2}\left(-\|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(z_t^{C,\alpha})\|^2 + \|\Pi_{\tilde{\mathcal{V}}_{\alpha,t}^\perp}(z_t^{0,\alpha})\|^2\right) dt
\end{aligned} \tag{4.96}$$

We know that the coefficient of dt in formula 4.94 is equal to the sum of the formulas 4.95 and

4.96 and this is equal to

$$\begin{aligned}
&(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\rho^{\alpha}_t) - \rho^{\alpha}_t)^*(z^{C,\alpha}_t - z^{0,\alpha}_t)dt + \frac{\gamma_\alpha}{2}\left(\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t)\|^2 - \|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)\|^2\right)dt \\
&+ \left[-\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\gamma_\alpha z^{C,\alpha}_t))^* z^{C,\alpha}_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\gamma_\alpha z^{0,\alpha}_t))^* z^{0,\alpha}_t\right]dt + -\gamma_\alpha\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t)\|^2 dt + \gamma_\alpha\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)\|^2 dt \\
&= (\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\rho^{\alpha}_t) - \rho^{\alpha}_t)^*(z^{C,\alpha}_t - z^{0,\alpha}_t)dt + \frac{\gamma_\alpha}{2}\left(\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t)\|^2 - \|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)\|^2\right)dt \\
&\quad + \gamma_\alpha\left\{-\left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t) - z^{C,\alpha}_t\right)^*\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t) + \left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t) - z^{0,\alpha}_t\right)^*\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)\right\} \\
&= (\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\rho^{\alpha}_t) - \rho^{\alpha}_t)^*(z^{C,\alpha}_t - z^{0,\alpha}_t)dt + \frac{\gamma_\alpha}{2}\left(\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t)\|^2 - \|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)\|^2\right)dt.
\end{aligned} \tag{4.97}$$

The last formula leads us to

$$\begin{aligned}
&-d(V^{\alpha,C}_t - V^{\alpha,C}_0) \\
&= \left[(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\rho^{\alpha}_t) - \rho^{\alpha}_t)^*(z^{C,\alpha}_t - z^{0,\alpha}_t) + \frac{\gamma_\alpha}{2}\left(\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t)\|^2 - \|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)\|^2\right)\right]dt - (z^{C,\alpha}_t - z^{0,\alpha}_t)^* dW_t.
\end{aligned} \tag{4.98}$$

Now, by using the formula $a^2 - b^2 = (a-b)^2 + 2(a-b)b$ and the change of variable $z^{\alpha}_t = z^{C,\alpha}_t - z^{0,\alpha}_t$ we write the coefficient of dt as

$$\begin{aligned}
&-(\rho^{\alpha}_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^{\alpha}_t))^* z^{\alpha}_t dt + \gamma_\alpha\left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\right)^*\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)dt + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\|^2 dt \\
&= -\left(\rho^{\alpha}_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^{\alpha}_t - \gamma_\alpha z^{0,\alpha}_t) + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\gamma_\alpha z^{0,\alpha}_t)\right)^* z^{\alpha}_t dt + \gamma_\alpha\left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\right)^*\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)dt \\
&\quad + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\|^2 dt.
\end{aligned} \tag{4.99}$$

From the definition of $\tilde{u}^0_t$ the previous equation can be expressed as

$$\begin{aligned}
&= -\left(\rho^{\alpha}_t + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(-\rho^{\alpha}_t - \gamma_\alpha z^{0,\alpha}_t) + \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\gamma_\alpha z^{0,\alpha}_t)\right)^* z^{\alpha}_t dt + \gamma_\alpha\left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\right)^*\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{0,\alpha}_t)dt \\
&\quad + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\|^2 dt \\
&= -\left(\rho^{\alpha}_t + \tilde{u}^0_t\right)^* z^{\alpha}_t dt - \left(\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(\gamma_\alpha z^{0,\alpha}_t)\right)^*\left(z^{\alpha}_t - \Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{\alpha}_t)\right)dt + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{C,\alpha}_t - z^{0,\alpha}_t)\|^2 dt \\
&= -\left(\rho^{\alpha}_t + \tilde{u}^0_t\right)^* z^{\alpha}_t dt + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{\alpha}_t)\|^2 dt,
\end{aligned} \tag{4.100}$$

where the last line comes from basic properties of orthogonal projections. Therefore, by the use of $\rho^{\alpha,\tilde{u}_t}_t = \rho^{\alpha}_t + \tilde{u}_t$ and $W^{\alpha,\tilde{u}}_t = W_t + \int_0^t \rho^{\alpha,\tilde{u}_t}_s ds$, formula 4.98 becomes

$$\begin{aligned}
-d(\beta_{t,T}p^{\alpha}_t(v_0, C)) &= -d(V^{\alpha,C}_t - V^{\alpha,C}_0) \\
&= -\left(\rho^{\alpha}_t + \tilde{u}^0_t\right)^* z^{\alpha}_t dt + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{\alpha}_t)\|^2 dt - (z^{C,\alpha}_t - z^{0,\alpha}_t)^* dW_t \\
&= -\left(\rho^{\alpha,\tilde{u}^0_t}_t\right)^* z^{\alpha}_t dt + \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}}(z^{\alpha}_t)\|^2 dt - (z^{C,\alpha}_t - z^{0,\alpha}_t)^* dW_t \\
&= \frac{\gamma_\alpha}{2}\|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t}} z_t\| dt - z^*_t dW^{\alpha,\tilde{u}^0}_t.
\end{aligned} \tag{4.101}$$

$\square$

As in the case of the Euler discretization method for big agents we can do the same process here. We choose the same simulation time points with the same *simulation step size* Δ_t, i.e.,

$$0 = t_0, \quad t_1 = \frac{T}{N}, \quad t_2 = \frac{2T}{N}, \quad ..., \quad t_N = T \qquad \text{and} \qquad \Delta_t = T/N, \tag{4.102}$$

with $N \in \mathbb{Z}^+$. From 4.101 we have the Euler discretization for small agents

$$\begin{aligned} &\beta_{t_0,T} p^{\alpha}_{t_0}(v_0, C) = \beta_{0,T} p^{\alpha}_0(v_0, C) \\ &\beta_{t_{j+1},T} p^{\alpha}_{t_{j+1}}(v_0, C) = \beta_{t_j,T} p^{\alpha}_{t_j}(v_0, C) + \frac{\gamma_\alpha}{2} \|\Pi_{\tilde{\mathcal{V}}^{\perp}_{\alpha,t_j}} z_{t_j}\| \Delta_t - z^*_{t_j} \Delta W^{\alpha,\tilde{u}}_t, \end{aligned} \tag{4.103}$$

with $\Delta W^{\alpha,\tilde{u}}_t \sim N(0, T/N)$ under $Q^{\rho^\alpha,\tilde{u}^0}$ and $j = 0, 1, ..., N-1$.

5 Numerical Methods

This work involves the estimation and forecast of the parameters and prices proposed by each agent. In particular, to estimate the parameters of the big agent as functions of the macroeconomic events, we first estimate the parameters of the SDEs of the small agent using standard methods such as maximum likelihood with restrictions, and then we use a new method to add macroeconomic factors that modify the previous estimate such that the resulting estimate reflects to some extent the effects of the events. We will use the following diagram to have an idea of the order at which each parameter will be estimated and the methods that will be used [7].

$$\text{Big Agent Price} \begin{cases} \hat{\sigma} = \hat{\sigma}(S_t, \sigma^{\text{imp}}) \to \text{Maximum likelihood, components of volatility} \\ \hat{\mu} = \hat{\mu}(S_t, \hat{\sigma}) \to \text{Maximum likelihood} \end{cases}$$

$$\text{Small Agent Price} \begin{cases} \hat{r} \to \text{Vasicek model, OLS estimation} \\ \text{Price Bond} \begin{cases} \hat{\eta} = \hat{\eta}(\hat{\mu}, \hat{r}, \hat{\sigma}) \to \text{Maximum likelihood under restrictions} \\ \hat{\lambda}^{\text{bond}} = \hat{\lambda}^{\text{bond}}(\hat{r}, \hat{\eta}) \to \text{Maximum likelihood under restrictions} \\ \hat{\varrho} = \hat{\varrho}(\hat{\sigma}, \hat{\lambda}^{\text{bond}}) \to \text{Maximum likelihood under restrictions} \\ \hat{\hat{\varrho}} = \pi_{1,2}(\hat{\varrho}) \to \text{Projection on space } \mathcal{V}_\alpha \end{cases} \\ \hat{\rho} = \hat{\rho}(\hat{\sigma}, \hat{\mu}, \hat{r}, \hat{\hat{\varrho}}) \\ \hat{u} = \hat{u}(\hat{\sigma}, \hat{\rho}) \to \text{Semigroup theory, and representation theorem} \end{cases}$$

The formulas developed so far can be applied to any number of assets. However, for the sake of simplicity, we will make all of the numerical computations for a financial market with 3 risky assets. In this study we will price an MSFT call option, and we will use information of ORCL and GOOG call options that belong to the same economic sector as MSFT[8].

[7] The description of the data used for the application, and the code of each algorithm are written in R, and included in the supplementary information.

[8] The data that will be used to make the statistical methods was purchased in the Chicago Board Options Exchange, https://datashop.cboe.com/

The data is an End-of-Day Option Quotes with the following characteristics:

- **Underlying Symbols:** GOOG,MSFT,ORCL
- **Dates:** 08/01/2019 - 09/30/2019
- **Files Grouping:** Per day
- **Columns**
  - Underlying Symbol
  - Quote Date
  - Root
  - Expiration
  - Strike
  - Option Type
  - Open
  - High
  - Low
  - Close
  - Trade Volume
  - Bid Size 1545
  - Bid 1545
  - Ask Size 1545
  - Ask 1545
  - Underlying Bid 1545
  - Underlying Ask 1545
  - Implied Underlying Price 1545
  - Active Underlying Price 1545
  - Implied Volatility 1545
  - Delta 1545
  - Gamma 1545
  - Theta 1545
  - Vega 1545
  - Rho 1545
  - Bid Size Eod
  - Bid Eod
  - Ask Size Eod
  - Ask Eod

– Underlying Bid Eod
– Underlying Ask Eod
– VWAP
– Open Interest

Characteristic	**Call option MSFT**	**Call option ORCL**	**Call option GOOG**
Issue date	2019-08-01	2019-08-01	2019-08-01
Maturity date	2019-10-18	2019-10-18	2019-10-18
Strike	140	55	1200

The difficulty in finding derivatives of different stocks with the same issue and maturity date, forces us to work with call options over periods of time of over 2 months at most. For the maximum likelihood estimation of volatility and interest rates we did not find accurate estimation over the same short periods of time. As a result of that and for illustrative purposes, we decided to consider longer periods for the first order conditions that involve the estimates of those parameters.

5.1 Estimation of parameters for big agents

5.1.1 Estimate of $\hat{\sigma}$

In this part we will estimate the parameter $\mu^{(i)}$ and $\sigma^{(ij)}$ using maximum likelihood estimation. We start by building the estimation in a market where there is only one risky asset. Next, we show the importance of extending these ideas to a market with several risky assets. Finally, we show the estimation of the parameters in a market with many financial assets.

In a market with only one risky asset, namely $S_t^{(1)}$, we use the dynamics

$$dS_t^{(1)} = dS_t^{(1)}(\mu dt + \sigma dW_t) \tag{5.1}$$

where $K, T, S_0^{(1)}$, and r are known. We find the implied volatility by matching the market price, M, to the Black-Scholes price, $\text{BSCall}(S_0^{(1)}, K, T, r, \sigma)$, and solving for σ where T is the maturity date, K is the strike price, and r is the risk-free rate. The solution is usually denoted by $\sigma^{\text{imp}}(K, T)$.

For the maximum likelihood estimator of the volatility, we proceed in the following way. Let us assume that $(S_{t_k})_{k=0}^M$ is a set of prices observed at different times with $t_k - t_{k-1} = T/M$, for all $k = 1, ..., M$.

From the differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

we know that the solution S_{t_n} with initial condition $S_{t_{n-1}}$, is given by

$$S_{t_n} = S_{t_{n-1}} \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma(W_{t_n} - W_{t_{n-1}}) \right\},$$

for each $n \in \{1, 2, ..., M\}$ where $\Delta t = t_j - t_{j-1}$.

We know that S_t is a function of the P-Brownian motion W_t; therefore

$$\begin{aligned} P[S_{t_n} \leq s_{t_n} | \mathcal{F}_{t_{n-1}}] &= P\left[\left(\frac{S_{t_n}}{S_{t_{n-1}}} \right) S_{t_{n-1}} \leq s_{t_n} \middle| \mathcal{F}_{t_{n-1}} \right] \\ &= P\left[\left(\frac{S_{t_n}}{S_{t_{n-1}}} \right) x \leq s_{t_n} \right]_{x = S_{t_{n-1}}} \\ &= P_{t_n - t_{n-1}}(S_{t_{n-1}}, (-\infty, s_{t_n}]), \end{aligned} \tag{5.2}$$

where the second and third lines come from the independent increments of the Brownian motion and the fact that $S_{t_{n-1}}$ is $\mathcal{F}_{n-1}$-measurable. The function $P_{t_n - t_{n-1}}(s_{t_{n-1}}, (-\infty, s_{t_n}])$ is the transition probability of the Markov process $(S_t)_{t \in [0,T]}$ given by the formula

$$\begin{aligned} &P_{t_n - t_{n-1}}(s_{t_{n-1}}, s_{t_n}) \\ &= \frac{1}{s_{t_n}\sqrt{\sigma^2(t_n - t_{n-1})2\pi}} \exp\left\{ -\frac{\left(\ln(s_{t_n}) - \ln(s_{t_{n-1}}) - \left(\mu - \frac{\sigma^2}{2} \right)(t_n - t_{n-1}) \right)^2}{2\sigma^2(t_n - t_{n-1})} \right\}. \end{aligned} \tag{5.3}$$

We know that the finite-dimensional distributions of a Markov process are characterized by the transition functions of the process itself (see Revuz and Yor, 1999, Section III.1), and satisfy the following formula

$$\begin{aligned} P[S_{t_0} \in A_0, ..., S_{t_M} \in A_M] &= \int_{A_0} \delta_{S_0}(dx_0) \int_{A_1} P_{t_1 - 0}(x_0, dx_1) \cdots \int_{A_M} P_{t_M - t_{M-1}}(x_{M-1}, dx_M) \\ &= \int_{A_0} \int_{A_1} \cdots \int_{A_M} \delta_{S_0}(dx_0) P_{t_1 - 0}(x_0, dx_1) \cdots P_{t_M - t_{M-1}}(x_{M-1}, dx_M), \end{aligned} \tag{5.4}$$

where δ_{S_0} is the Dirac delta distribution with point mass at S_0. Therefore, the density of this finite-dimensional distribution is of the form

$$f(\mu, \sigma; \underline{S}) := \prod_{j=1}^{M} \frac{1}{s_{t_j}\sqrt{\sigma^2(\Delta t)2\pi}} \exp\left\{ -\frac{\left(\ln(s_{t_j}) - \ln(s_{t_{j-1}}) - \left(\mu - \frac{\sigma^2}{2} \right) \Delta t \right)^2}{2\sigma^2 \Delta t} \right\}. \tag{5.5}$$

To find the maximum likelihood estimators of μ and σ, we first notice that our parameter space is

$$\Theta = \mathbb{R} \times \mathbb{R}_+ = \{(\mu, \sigma) : \mu \in \mathbb{R} \quad \text{y} \quad \sigma > 0\},$$

and the function $(\lambda_1(\mu, \sigma), \lambda_2(\mu, \sigma))$ from Θ on itself given by the formulas $\lambda_1 = \left(\mu - \frac{\sigma^2}{2} \right) \Delta t$ and $\lambda_2 = \sigma^2 \Delta t$ is an homeomorphism. Since maximum likelihood estimators are invariant under homeomorphisms, we can consider the likelihood function

$$f(\lambda_1, \lambda_2; \underline{S}) = \prod_{j=1}^{M} \frac{1}{s_{t_j}\sqrt{2\lambda_2\pi}} \exp\left\{ -\frac{\left(\ln(s_{t_j}) - \ln(s_{t_{j-1}}) - \lambda_1 \right)^2}{2\lambda_2} \right\},$$

with respect to the parameters (λ_1, λ_2).

Now, our maximum likelihood estimators come from the usual maximization process. From

$$\frac{\partial \ln f(\lambda_1, \lambda_2; \underline{S})}{\partial \lambda_1} = -\frac{\partial}{\partial \lambda_1} \sum_{j=1}^{M} \left\{ -\frac{\left(\ln(s_{t_j}) - \ln(s_{t_{j-1}}) - \lambda_1\right)^2}{2\sigma^2 \Delta t} \right\} = 0,$$

we get

$$\hat{\lambda_1} = \frac{1}{M} \sum_{j=1}^{M} (\ln(s_{t_j}) - \ln(s_{t_{j-1}})) = \frac{\ln(s_{t_M}) - \ln(s_{t_0})}{M}.$$

On the other hand, from the equation

$$\begin{aligned}
&\frac{\partial \ln f(\lambda_1, \lambda_2; \underline{S})}{\partial \lambda_2} \\
&= \frac{\partial}{\partial \lambda_2} \left(\frac{-M}{2} \ln(\lambda_2) + \ln \left(\prod_{j=1}^{M} \frac{1}{s_{t_j} \sqrt{2\pi}} \right) - \sum_{j=1}^{M} \frac{\left(\ln(s_{t_j}) - \ln(s_{t_{j-1}}) - \lambda_1\right)^2}{2\lambda_2} \right) \\
&= 0
\end{aligned}$$

we get that

$$\frac{-M}{2\lambda_2} + \frac{1}{2\lambda_2^2} \sum_{j=1}^{M} \left(\ln(s_{t_j}) - \ln(s_{t_{j-1}}) - \lambda_1\right)^2 = 0,$$

and solving for λ_2, we obtain

$$\hat{\lambda_2} = \frac{1}{M} \sum_{j=1}^{M} \left(\ln(s_{t_j}) - \ln(s_{t_{j-1}}) - \hat{\lambda_1}\right)^2.$$

By using $\lambda_2 = \sigma^2 \Delta t$ and $\lambda_1 = \left(\mu - \frac{\sigma^2}{2}\right) \Delta t$, we get our maximum likelihood estimators

$$\lambda_1 = \left(\mu - \frac{\sigma^2}{2}\right) \Delta t \quad \text{and} \quad \hat{\mu} = \frac{\hat{\lambda_1}}{\Delta t} + \frac{\hat{\sigma}^2}{2}. \tag{5.6}$$

There are some drawbacks of using any of these estimators in this model. One of the most obvious is that the maximum likelihood property and the matching market price property of the implied volatility cannot hold simultaneously in this model. Another problem is that these models do not split the volatility of the i-th stock as a sum of different components $\sigma^{(ij)}$ that depend on the correlation of the stock with other equities. This is something that is useful for investments, and should be included in the models. As an example, when an investor goes long on SPY or QQQ, some of the most traded ETFs in the United States, his or her shares are allocated into several stocks of the S&P 500 or QQQ 100 index. (Investopedia, 2018*b*).

Financial sector	**SPY's percent allocation**
Information technology	26.2%
Health care	15%
Financial	13.46%
Consumer discretionary	13.12%
Industrials	9.72%
Consumer staples	6.69%
Energy	6.01%
Utilities	2.78%
Real state	2.61%
Materials	2.44%
Telecommunication services	1.97%

Table 1: SPY's percent allocation

Financial sector	**QQQ's percent allocation**
Information technology	60.44%
Health care	9.85%
Consumer discretionary	22.38%
Industrials	2.1%
Consumer staples	4.42%
Telecommunication services	0.81%

Table 2: Nasdaq 100 QQQ's percent allocation

Since ETFs are highly traded, and traders use them as benchmarks, this fact makes the correlations of stocks in the SPY or QQQ category even stronger. One way to prove this fact is by computing correlations between stocks in the same ETF group. As an example the Pearson's correlation of daily close prices of MSFT and ORCL between 2016-01-04 and 2019-09-30 was about 0.863552[9]. However, there are other methods that can be used in practice to notice this pattern. One of the most practical is by using the Beta indicator (measure of a stock's volatility in relation to the market) of each stock as can be seen in figure 6 provided by (Yahoo!, 2018). The Beta indicator of a stock is given by the formula

$$\text{Beta}_{\text{stock}} = \frac{\text{Sample covariance of stock's daily returns and SPY's daily retuns}}{\text{Sample variance of SPY's daily returns}}. \tag{5.7}$$

[9]Computed using the R programming language. Source: Yahoo! Finance.

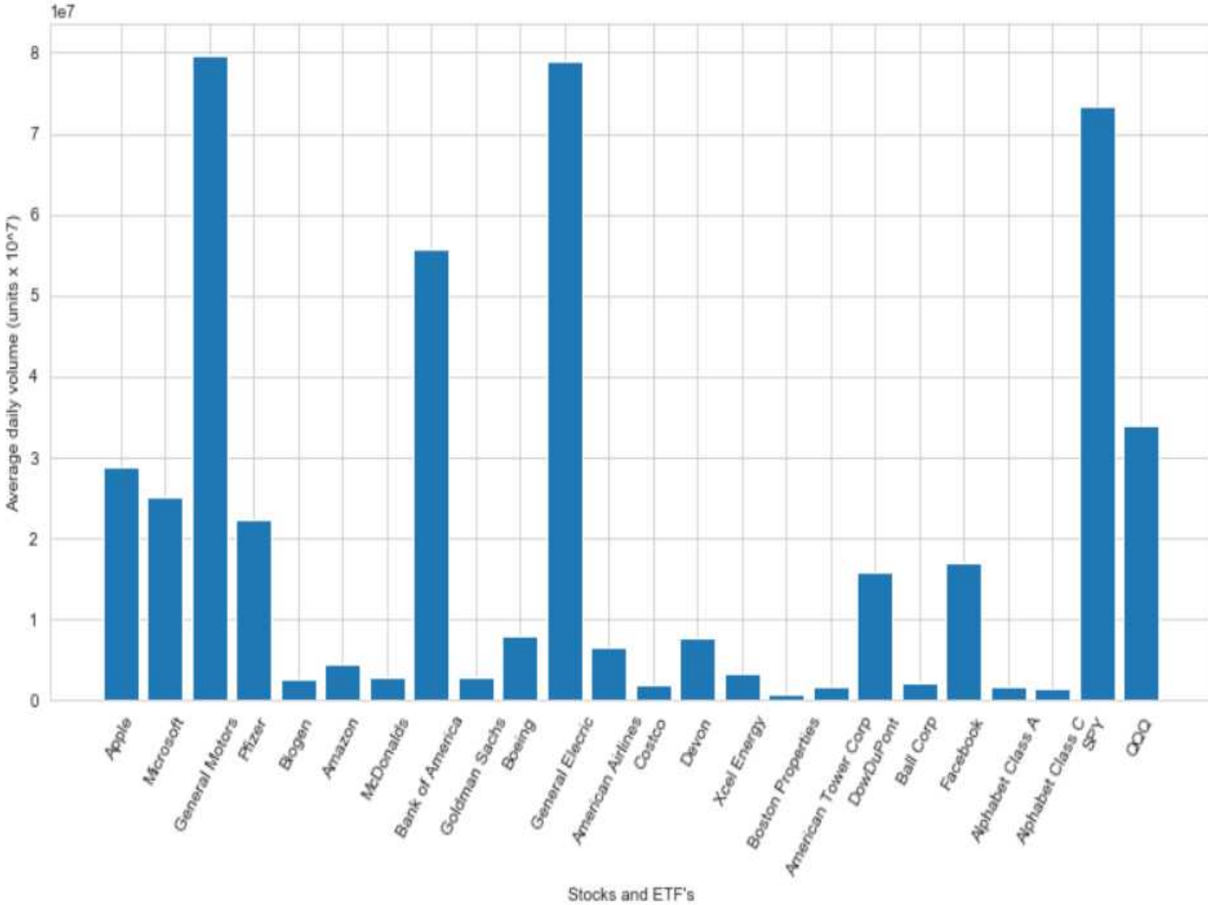


Figure 5: Average daily volume of stocks of the S&P 500 index, and ETF's, August 2018

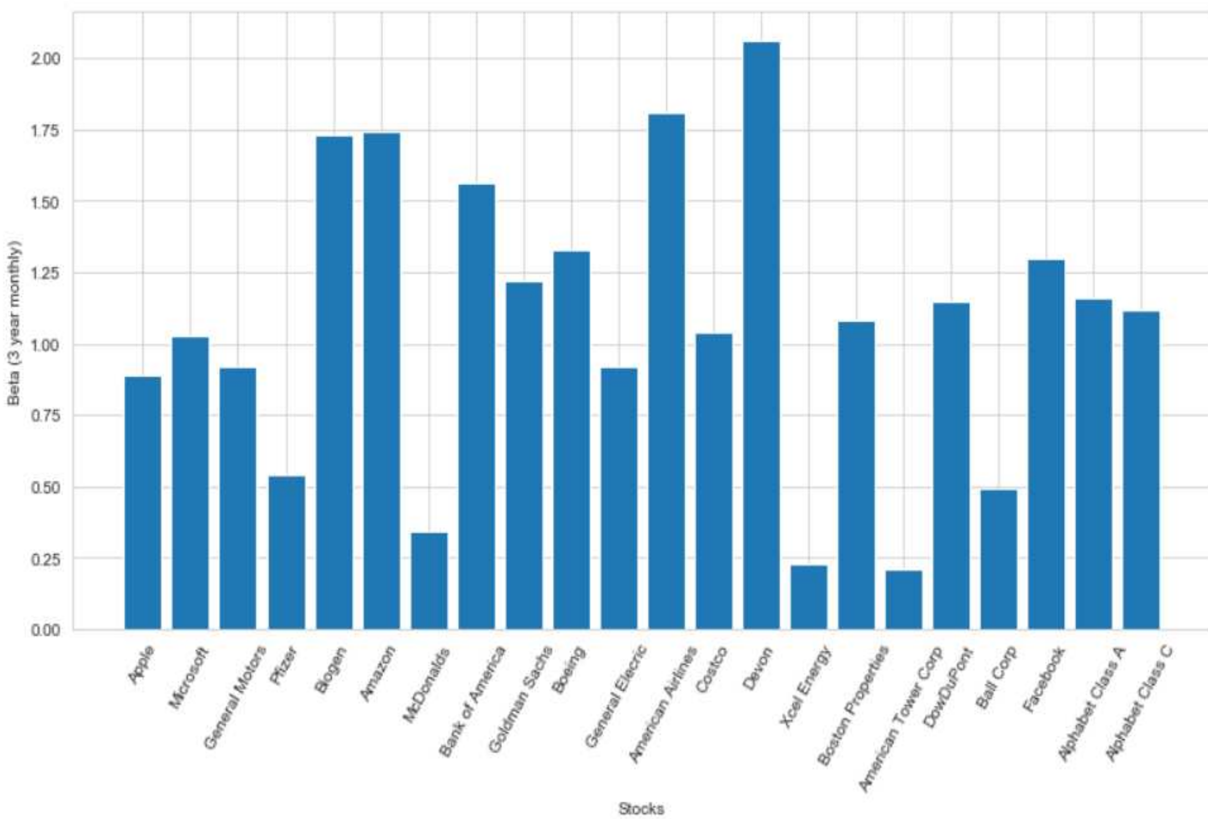


Figure 6: Beta of stocks of the S&P 500 index, August 2018

With the formula 5.7, we can go a little further. If we use a for a stock, b for SPY, $r_{\mathrm{a,b}}$ for the sample Pearson correlation between the stock and the SPY, we have

$$r_{\mathrm{a,b}} = \frac{Q_{\mathrm{a,\ b}}}{S_{\mathrm{a}} S_{\mathrm{b}}} = \frac{\mathrm{Beta}_{\mathrm{a}} S_{\mathrm{b}}}{S_{\mathrm{a}}}, \tag{5.8}$$

where $Q_{\mathrm{a,\ b}}$ is the sample covariance of the returns of the stock and the market; S_{b} is the sample standard deviation of the market's returns; and S_{a} is the sample standard deviation of the stock's returns.

This result helps us to understand why some stocks go up in a specific sector, they create a chain reaction on the other stocks in the same sector. This effect is not explicitly shown in any of the parameters of the usual Black-Scholes-Merton formula for one risky asset 5.1.

For the multiprice model , we need a different statistical approach. Let us consider the dynamics of $(S^{(i)})_{t\in[0,T]}$ as in Proposition 2.4, i.e.

$$dS_t^{(i)} = \mu_t^{(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} \sigma_t^{(ij)} dW_t^{(j)} \right). \tag{5.9}$$

We know that under the macroeconomic event, Λ_k, the drift, and diffusion coefficients associated with $S^{(i)}$ on 5.9 are ${}^k\mu_t^{(i)}$ and ${}^k\sigma_t^{(i,\cdot)}$. As it usually happens with most of the statistical methods in practice we assume that ${}^k\mu_t^{(i)}$ and ${}^k\sigma_t^{(i,\cdot)}$ are constants. To estimate the parameters, we need to make some changes to formula 5.5.

Let us assume that $(S_{t_k}^{(i)})_{k=0}^M$ is a set of prices observed at different times with $t_M = T$, $t_0 = 0$, and $t_k - t_{k-1} = T/M$, for all $k = 1, ..., M$. From the dynamics 5.9, we know that the price satisfies

$$S_{t_l}^{(i)} = S_{t_{l-1}}^{(i)} \exp\left\{ \left(\mu^{(i)} - \frac{1}{2}\sum_{j=1}^{n} (\sigma^{(ij)})^2 \right) \Delta t + \sum_{j=1}^{n} \sigma^{(ij)} (W_{t_l} - W_{t_{l-1}}) \right\},$$

for each $l \in \{1, 2, ..., M\}$ where $\Delta t = t_l - t_{l-1}$. Moreover, since $S_t^{(i)}$ is a function of the n-dimensional Brownian motion W_t, then

$$\begin{aligned} P[S_{t_n}^{(i)} \leq s_{t_n}^{(i)} | \mathcal{F}_{t_{n-1}}] &= P\left[\left(\frac{S_{t_n}^{(i)}}{S_{t_{n-1}}^{(i)}} \right) S_{t_{n-1}}^{(i)} \leq s_{t_n}^{(i)} \middle| \mathcal{F}_{t_{n-1}} \right] \\ &= P\left[\left(\frac{S_{t_n}^{(i)}}{S_{t_{n-1}}^{(i)}} \right) x \leq s_{t_n}^{(i)} \right]_{x = S_{t_{n-1}}^{(i)}} \\ &= P_{t_n - t_{n-1}}^{(i)} \left(S_{t_{n-1}^{(i)}}, (-\infty, s_{t_n}^{(i)}] \right), \end{aligned} \tag{5.10}$$

where the second and third line comes from the independent increments of the Brownian motion and the fact that $S_{t_{n-1}}^{(i)}$ is $\mathcal{F}_{n-1}$-measurable. The function $P_{t_n - t_{n-1}}^{(i)}(s_{t_{n-1}}, (-\infty, s_{t_n}])$ is the transition probability of the Markov process $(S_t^{(i)})_{t \in [0,T]}$ given by the formula

$$P_{t_n - t_{n-1}}^{(i)}(s_{t_{n-1}}^{(i)}, s_{t_n}^{(i)}) = \frac{\exp\left\{ -\frac{\left(\ln(s_{t_n}^{(i)}) - \ln(s_{t_{n-1}}^{(i)}) - \left(\mu^{(i)} - \frac{1}{2}\sum_{j=1}^n (\sigma^{(ij)})^2 \right) \Delta t \right)^2}{2\Delta t \sum_{j=1}^n (\sigma^{(ij)})^2} \right\}}{s_{t_n^{(i)}} \sqrt{2\pi \left(\sum_{j=1}^n (\sigma^{(ij)})^2 \right) \Delta t}}. \tag{5.11}$$

With these remarks we are ready to obtain the finite-dimensional distributions that are described by

$$
\begin{aligned}
&P\left[S_{t_M}^{(i)} \in A_M, ..., S_{t_0}^{(i)} \in A_0\right] \\
&= E\left[\prod_{l=0}^{M-1} 1_{\{S_{t_l}^{(i)} \in A_l\}} E\left[1_{\{S_{t_M}^{(i)} \in A_M\}} \middle| \mathcal{F}_{t_{M-1}}\right]\right] \\
&= E\left[\prod_{l=0}^{M-1} 1_{\{S_{t_l}^{(i)} \in A_l\}} \int_{A_M} P_{t_M - t_{M-1}}^{(i)}(S_{t_{M-1}}^{(i)}, ds_M^{(i)})\right] \\
&= E\left[\prod_{l=0}^{M-2} 1_{\{S_{t_l}^{(i)} \in A_l\}} E\left[1_{\{S_{t_{M-1}}^{(i)} \in A_{M-1}\}} \int_{A_M} P_{t_M - t_{M-1}}^{(i)}(S_{t_{M-1}}^{(i)}, ds_M^{(i)}) \middle| \mathcal{F}_{t_{M-2}}\right]\right] \\
&= E\left[\prod_{l=0}^{M-2} 1_{\{S_{t_l}^{(i)} \in A_l\}} \int_{A_{M-1}} P_{t_{M-1} - t_{M-2}}^{(i)}(S_{t_{M-2}}^{(i)}, ds_{M-1}^{(i)}) \int_{A_M} P_{t_M - t_{M-1}}^{(i)}(s_{M-1}^{(i)}, ds_M^{(i)})\right],
\end{aligned} \tag{5.12}
$$

and by repeating the same reasoning, we get

$$
\begin{aligned}
&= P\left[S_{t_M}^{(i)} \in A_M, ..., S_{t_0}^{(i)} \in A_0\right] \\
&= \int_{A_0} \delta_{S_0^{(i)}}(ds_0^{(i)}) \int_{A_1} P_{t_1 - 0}^{(i)}(s_0^{(i)}, ds_1^{(i)}) \cdots \int_{A_M} P_{t_M - t_{M-1}}^{(i)}(s_{M-1}^{(i)}, ds_M^{(i)}) \\
&= \int_{A_0} \int_{A_1} \cdots \int_{A_M} \delta_{S_0^{(i)}}(ds_0^{(i)}) P_{t_1 - 0}^{(i)}(s_0^{(i)}, ds_1^{(i)}) \cdots P_{t_M - t_{M-1}}^{(i)}(s_{M-1}^{(i)}, ds_M^{(i)}).
\end{aligned} \tag{5.13}
$$

Consequently, our likelihood function is of the form

$$
f(\mu^{(i)}, \sigma^{(i\cdot)}; \underline{S}) = \prod_{k=1}^{M} \frac{\exp\left\{-\dfrac{\left(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \left(\mu^{(i)} - \frac{1}{2}\sum_{j=1}^n (\sigma^{(ij)})^2\right)\Delta t\right)^2}{2\Delta t \sum_{j=1}^n (\sigma^{(ij)})^2}\right\}}{s_{t_k^{(i)}} \sqrt{2\pi \left(\sum_{j=1}^n (\sigma^{(ij)})^2\right)\Delta t}}. \tag{5.14}
$$

We know that our parametric space is

$$
\Theta = \mathbb{R} \times \mathbb{R}^n = \{(\mu^{(i)}, \sigma^{(i1)}, ..., \sigma^{(in)}) : \mu^{(i)} \in \mathbb{R}, \quad \sigma^{(i1)}, ..., \sigma^{(in)} \in \mathbb{R}_+\}. \tag{5.15}
$$

By using the homeomorphism, λ, from Λ onto itself with $\lambda^{(i)}(\mu^{(i)}, \sigma^{(i\cdot)}) = (\lambda_1^{(i)}(\mu^{(i)}, \sigma^{(i\cdot)}), ..., \lambda_{n+1}^{(i)}(\mu^{(i)}, \sigma^{(i\cdot)}))$ where

$$
\lambda_1^{(i)}(\mu^{(i)}, \sigma^{(i\cdot)}) = \Delta t \left(\mu^{(i)} - \frac{1}{2}\sum_{j=1}^n (\sigma^{(ij)})^2)\right) \quad \text{and} \quad \lambda_k^{(i)}(\mu^{(i)}, \sigma^{(i\cdot)}) = (\sigma^{(i,k-1)})^2 \Delta t, \tag{5.16}
$$

for $\quad k = 2, ..., n+1$, we have that our likelihood function becomes

$$
f(\lambda_1^{(i)}, ..., \lambda_{n+1}^{(i)}; \underline{S}^{(i)}) = \prod_{k=1}^{M} \frac{\exp\left\{-\dfrac{\left(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \lambda_1^{(i)}\right)^2}{2\sum_{j=1}^n \lambda_{j+1}^{(i)}}\right\}}{s_{t_k^{(i)}} \sqrt{2\pi \sum_{j=1}^n \lambda_{j+1}^{(i)}}}. \tag{5.17}
$$

Therefore our first-order conditions are of the form

$$\frac{\partial \ln f(\lambda_1^{(i)}, ..., \lambda_{n+1}^{(i)}; \underline{S}^{(i)})}{\partial \lambda_1^{(i)}} = -\frac{\partial}{\partial \lambda_1^{(i)}} \sum_{k=1}^{M} \frac{(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \lambda_1^{(i)})^2}{2\sum_{j=1}^{n} \lambda_{j+1}^{(i)}} = 0, \tag{5.18}$$

and

$$\frac{\partial \ln f(\lambda_1^{(i)}, ..., \lambda_{n+1}^{(i)}; \underline{S})}{\partial \lambda_k^{(i)}} = 0 \tag{5.19}$$

for $k = 2, ..., n+1$.

From the equation 5.18, we conclude that the maximum likelihood estimator, $\hat{\lambda}_1^{(i)}$ is of the form

$$\hat{\lambda}_1^{(i)} = \frac{1}{M} \sum_{k=1}^{M} \ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) = \frac{1}{M}\left(\ln(s_{t_M}^{(i)}) - \ln(s_{t_0}^{(i)})\right). \tag{5.20}$$

For 5.19 we have the following computations

$$\begin{aligned}
&\frac{\partial \ln f(\lambda_1^{(i)}, ..., \lambda_{n+1}^{(i)}; \underline{S}^{(i)})}{\partial \lambda_k^{(i)}} \\
&= \frac{\partial}{\partial \lambda_k^{(i)}} \ln\left(\prod_{k=1}^{M} \frac{1}{s_{t_k^{(i)}}\sqrt{2\pi\sum_{j=1}^{n}\lambda_{j+1}^{(i)}}} \exp\left\{-\frac{\left(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \lambda_1^{(i)}\right)^2}{2\sum_{j=1}^{n}\lambda_{j+1}^{(i)}}\right\}\right) \\
&= \frac{\partial}{\partial \lambda_k^{(i)}}\left(-\frac{M}{2}\ln\left(\sum_{j=1}^{n}\lambda_{j+1}^{(i)}\right) + \ln\left(\prod_{k=1}^{M}\frac{1}{s_{t_k^{(i)}}\sqrt{2\pi}}\right) - \sum_{k=1}^{M}\frac{(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \hat{\lambda}_1^{(i)})^2}{2\sum_{j=1}^{n}\lambda_{j+1}^{(i)}}\right).
\end{aligned} \tag{5.21}$$

Therefore, the first order condition 5.19 for $\lambda_k^{(i)}$ can be rewritten as

$$-\frac{M}{2}\left(\frac{1}{\sum_{j=1}^{n}\lambda_{j+1}^{(i)}}\right) + \frac{1}{2\left(\sum_{j=1}^{n}\lambda_{j+1}^{(i)}\right)^2}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \hat{\lambda}_1^{(i)}\right)^2 = 0. \tag{5.22}$$

Thus, we get

$$\sum_{j=1}^{n}\hat{\lambda}_{j+1}^{(i)} = \frac{1}{M}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \hat{\lambda}_1^{(i)}\right)^2. \tag{5.23}$$

By using the definition of $\hat{\lambda}_{j+1}^{(i)}$, we get

$$\sum_{j=1}^{n}(\hat{\sigma}^{(i,j)})^2 = \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(i)}) - \ln(s_{t_{k-1}}^{(i)}) - \hat{\lambda}_1^{(i)}\right)^2. \tag{5.24}$$

Assumption 5.1. *Following the assumption 2.3, we assume that*

$$(\hat{\sigma}^{(i,j)})^2 = (\hat{\sigma}^{(j,i)})^2 \tag{5.25}$$

for $i, j = 1, ..., n$.

By using the previous assumption, in the case of $n = 3$, we have the following linear system

$$
\begin{aligned}
(\hat{\sigma}_t^{(11)})^2 + (\hat{\sigma}_t^{(12)})^2 + (\hat{\sigma}_t^{(13)})^2 &= \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(1)}) - \ln(s_{t_{k-1}}^{(1)}) - \hat{\lambda}_1^{(1)}\right)^2 \\
(\hat{\sigma}_t^{(12)})^2 + (\hat{\sigma}_t^{(22)})^2 + (\hat{\sigma}_t^{(23)})^2 &= \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(2)}) - \ln(s_{t_{k-1}}^{(2)}) - \hat{\lambda}_1^{(2)}\right)^2 \\
(\hat{\sigma}_t^{(13)})^2 + (\hat{\sigma}_t^{(23)})^2 + (\hat{\sigma}_t^{(33)})^2 &= \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(3)}) - \ln(s_{t_{k-1}}^{(3)}) - \hat{\lambda}_1^{(3)}\right)^2
\end{aligned}
\tag{5.26}
$$

This system can be written in matrix form

$$
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} (\hat{\sigma}_t^{(11)})^2 \\ (\hat{\sigma}_t^{(12)})^2 \\ (\hat{\sigma}_t^{(13)})^2 \\ (\hat{\sigma}_t^{(22)})^2 \\ (\hat{\sigma}_t^{(23)})^2 \\ (\hat{\sigma}_t^{(33)})^2 \end{bmatrix}
=
\begin{bmatrix} \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(1)}) - \ln(s_{t_{k-1}}^{(1)}) - \hat{\lambda}_1^{(1)}\right)^2 \\ \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(2)}) - \ln(s_{t_{k-1}}^{(2)}) - \hat{\lambda}_1^{(2)}\right)^2 \\ \frac{1}{M\Delta t}\sum_{k=1}^{M}\left(\ln(s_{t_k}^{(3)}) - \ln(s_{t_{k-1}}^{(3)}) - \hat{\lambda}_1^{(3)}\right)^2 \\ c_1(\sigma^{(1),\text{imp}})^2 \\ c_2(\sigma^{(2),\text{imp}})^2 \\ c_3(\sigma^{(3),\text{imp}})^2 \end{bmatrix},
\tag{5.27}
$$

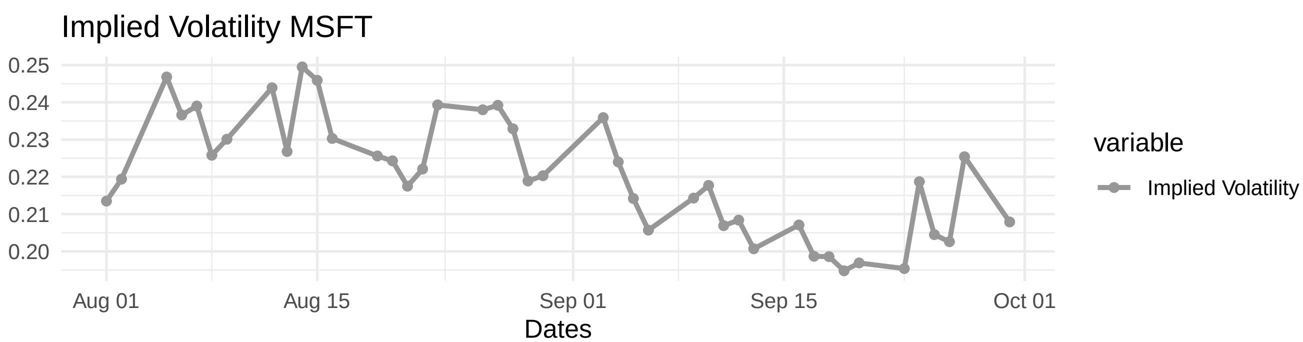
where $(\hat{\sigma}^{(ii)})$ is considered as the *intrinsic volatility* of the $i-$th stock , and the other components, $\hat{\sigma}^{(ij)}$ with $i \neq j$, are the *volatility components* of the $i-$th stock affected by the $j-$th stock. The element $\sigma^{(i),\text{imp}}$ in the linear equation is the implied volatility of the stock i. The choice of the constants c_1, c_2 and c_3 depends on the analysis made by investors about how much of the volatility of the company is due to itself (operating activities, management,etc.) and not to external factors such as a boom or a crisis in the whole sector. This choice also has to satisfy assumption 2.3 what is a drawback of this approach. However, as M gets bigger the values of 5.26 offer a good approximation to $\sigma^{(1),\text{imp}}, \sigma^{(2),\text{imp}}$ and $\sigma^{(3),\text{imp}}$; and the choice of the constants c_1, c_2 and c_3 can be made such that assumption 2.3 holds. Finally, the estimate of $\hat{\mu}$ follows immediately from the formulas 5.16.

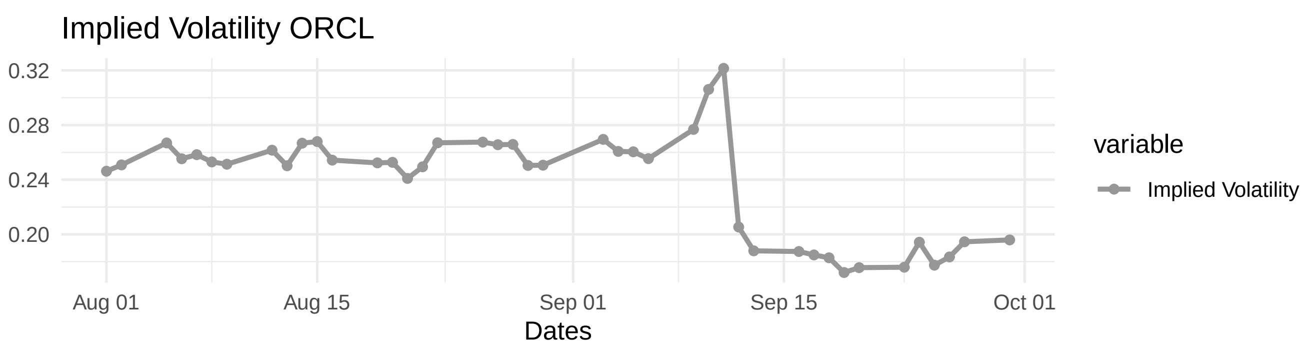
5.1.2 Implied Volatility Time Series Analysis, and Forecasting

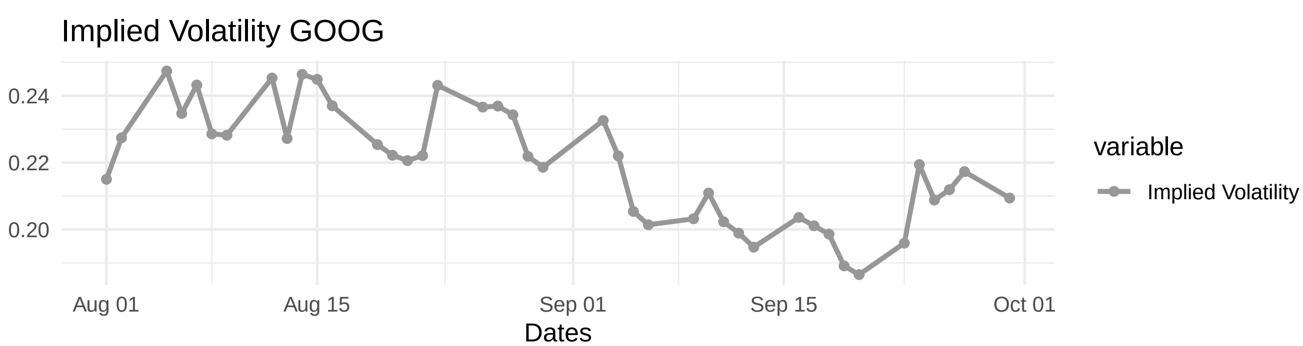
As it was mentioned earlier, we need to make forecasts of the inputs for the small and big agent pricing models. In this part, we make use of time series analysis on the implied volatility of the three stocks. The structure of the data and the model force us to work with small samples, something that happens in the study of macroeconomic measures such as GDP. We are aware that there are other techniques such as Bayesian analysis for the study of small samples. Since our purpose is to explore new estimation methods involving economic restrictions, we will consider the tools mentioned so far. The Bayesian analysis may be another good topic of research in this study. For the sake of simplicity, we will omit some steps in this analysis.

First, we analyze the time series of the implied volatility. Next, we evaluate if it is necessary to apply some transformation to the time series. Finally, we choose the model that seems to be most accurate to the pattern of the transformed time series.

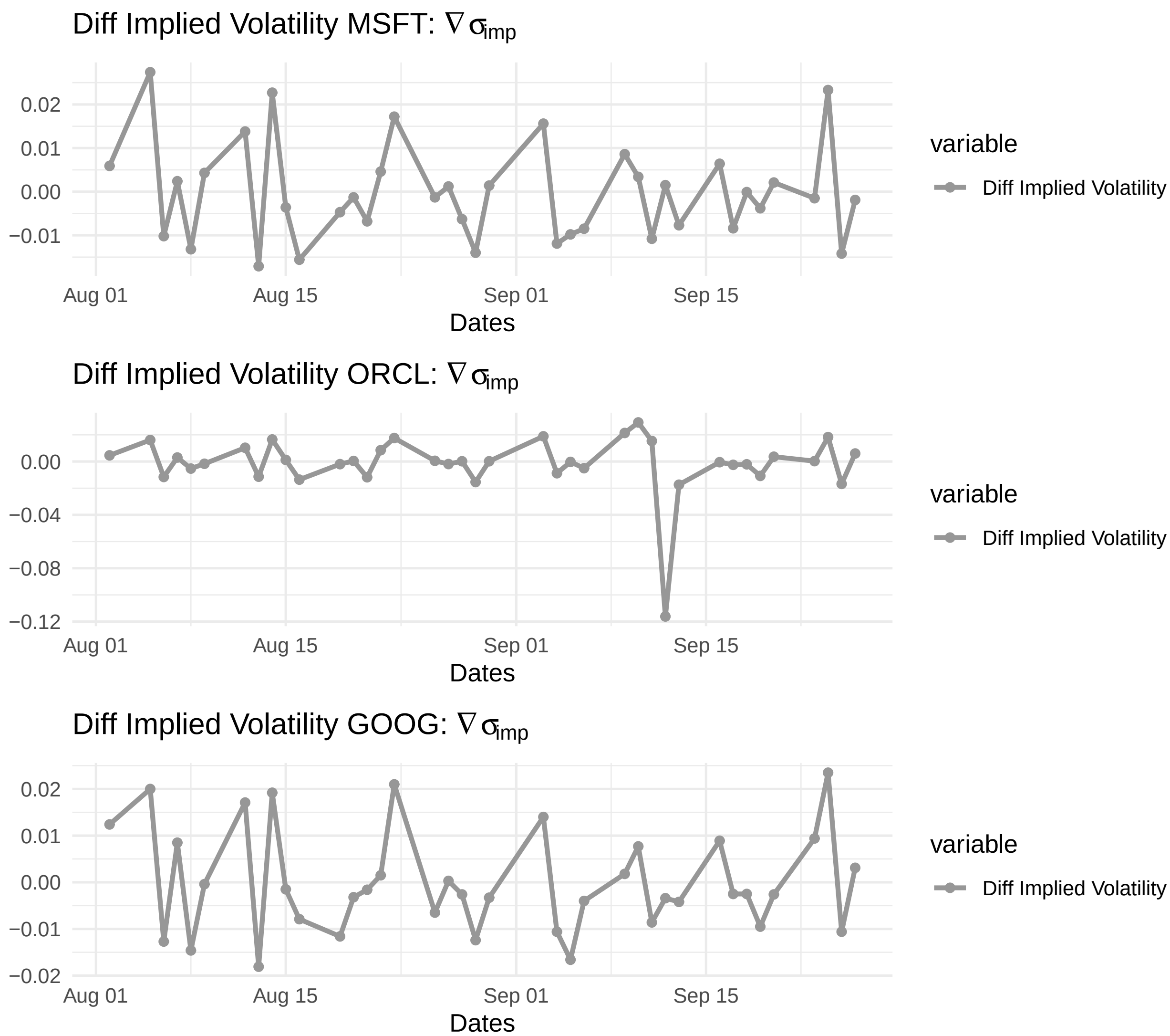
The following time series show linear trends, and their variance seems to be constant. Therefore, we will apply the difference operator on the time series.







Once applied to the series there are signs of stationarity except for the spike on the corresponding time series of ORCL. However, this spike is not followed by an substantial increase in volatility.

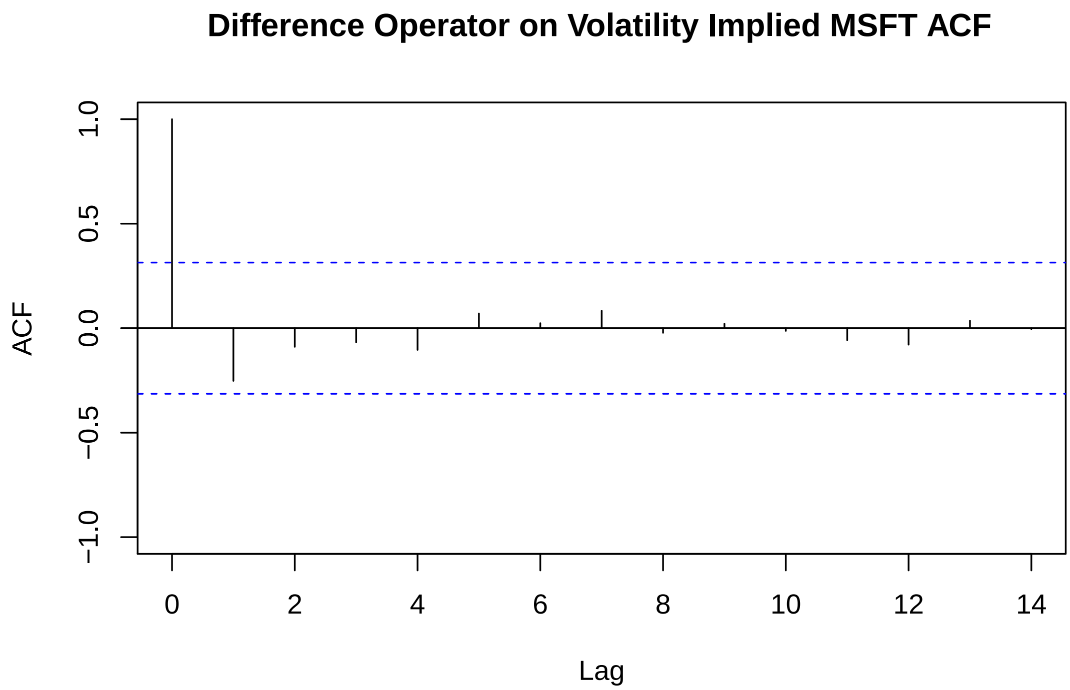


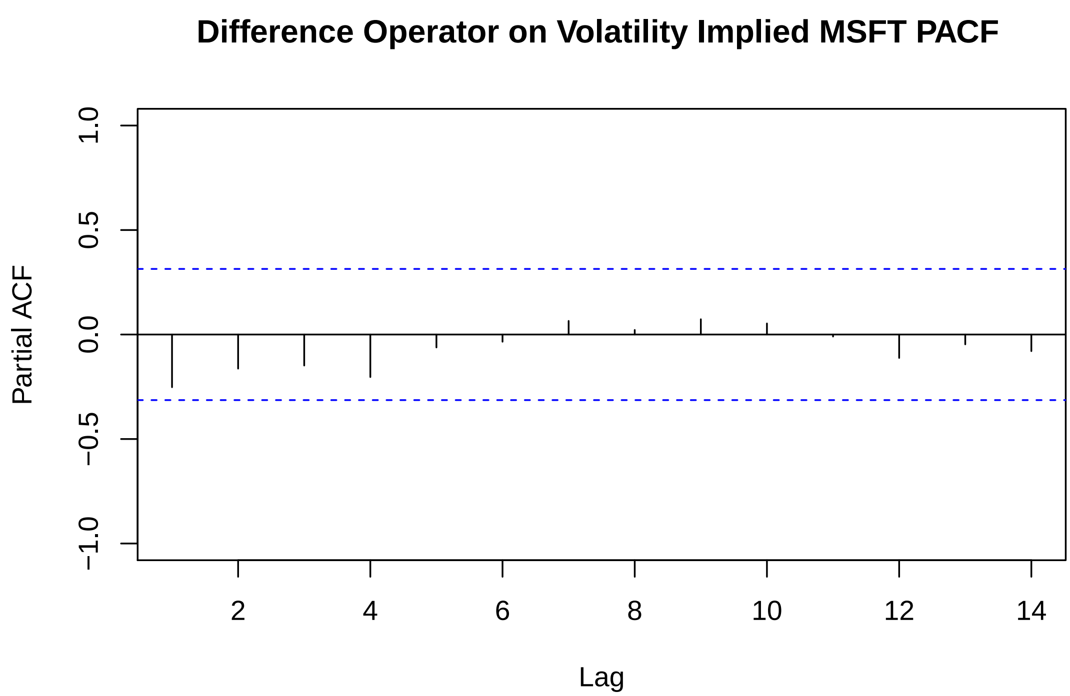
We check those findings using the Augmented Dickey-Fuller test

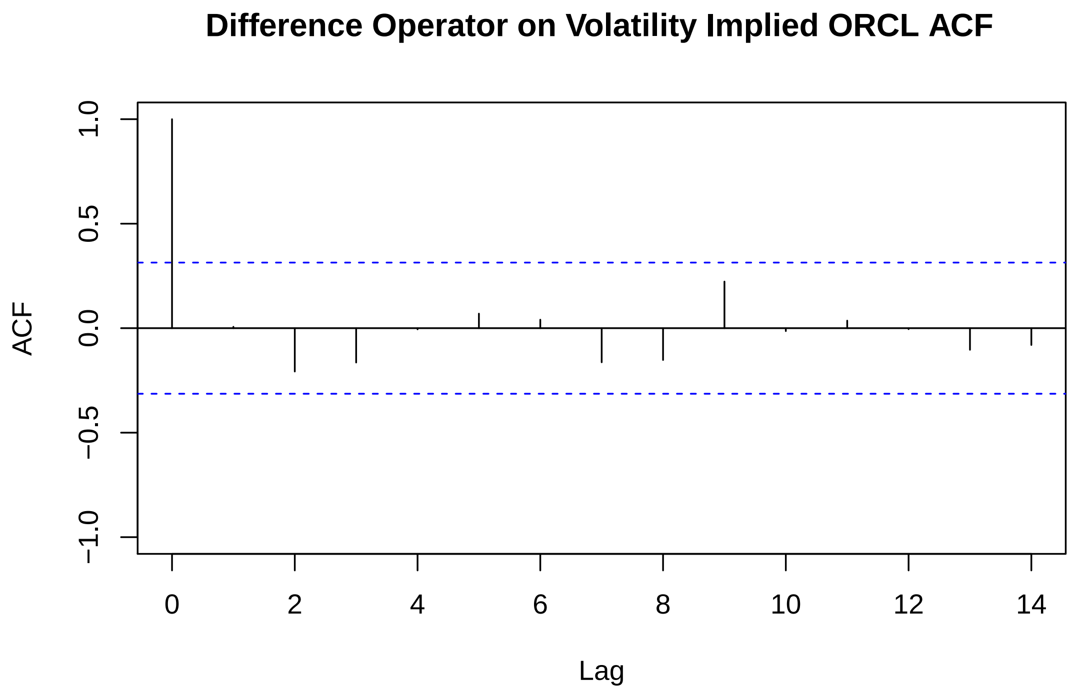
```
        Augmented Dickey-Fuller Test

data:  diff(implied_volatility_msft[1:(length(implied_volatility_msft) -     2)])
Dickey-Fuller = -4.8637, Lag order = 3, p-value = 0.01
alternative hypothesis: stationary


        Augmented Dickey-Fuller Test

data:  diff(implied_volatility_orcl[1:(length(implied_volatility_orcl) -     2)])
Dickey-Fuller = -3.6302, Lag order = 3, p-value = 0.04366
alternative hypothesis: stationary


        Augmented Dickey-Fuller Test

data:  diff(implied_volatility_goog[1:(length(implied_volatility_goog) -     2)])
Dickey-Fuller = -4.8924, Lag order = 3, p-value = 0.01
alternative hypothesis: stationary
```

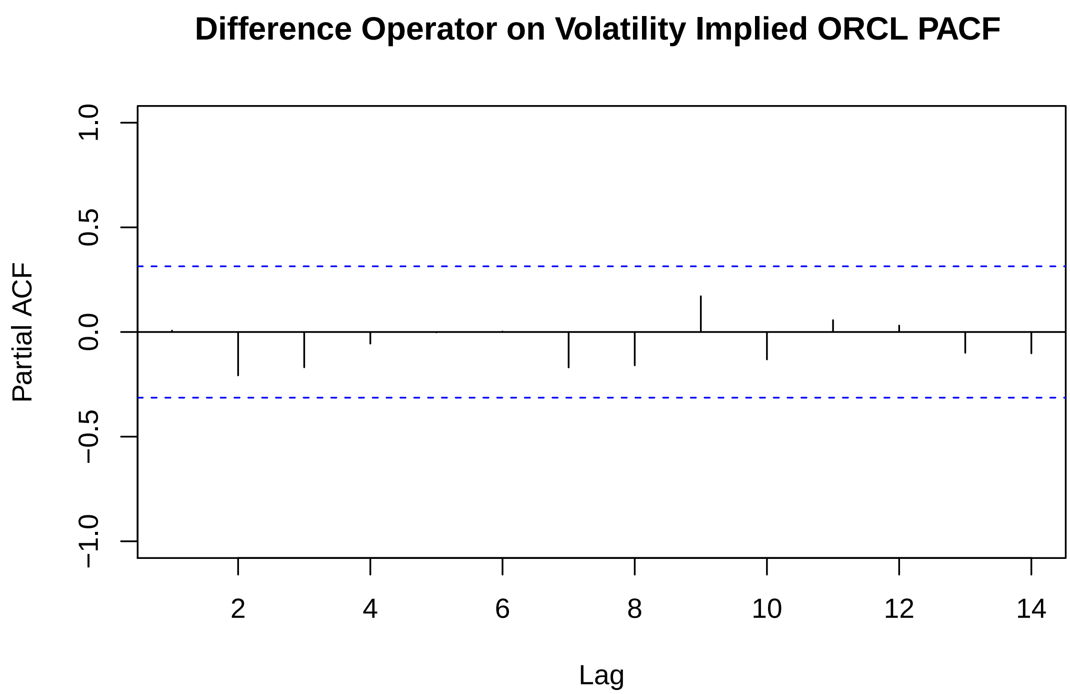
Since the p-value is less than 0.05, the data does not have a unit root and is stationary[10]. Therefore, we proceed with the ACF, and PACF analysis.

[10]To use the Augmented Dicky-Fuller test, we make use of the library tseries in the R programming language.

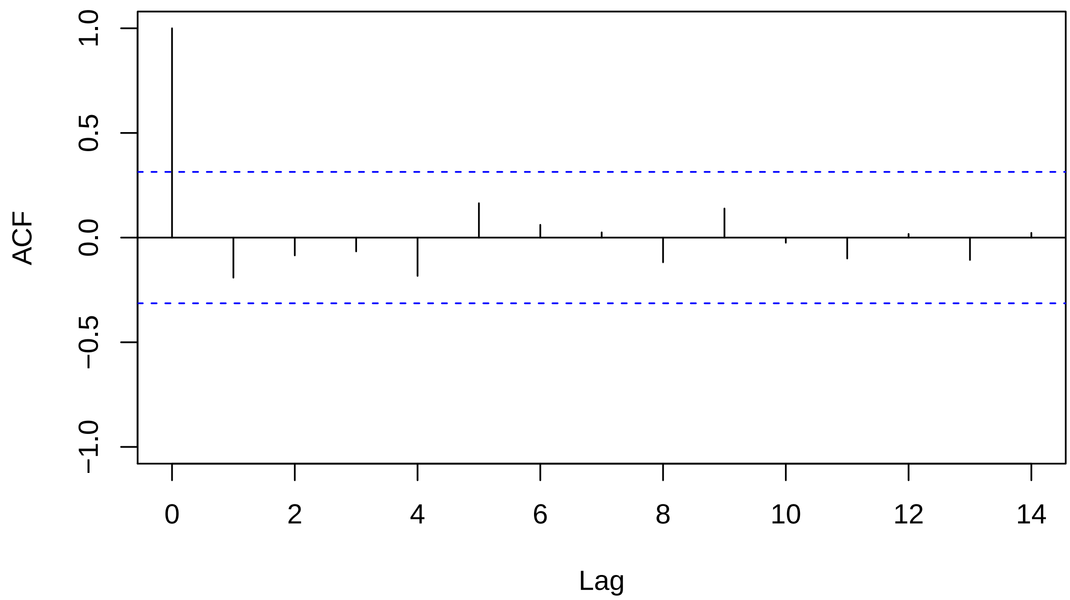

Difference Operator on Volatility Implied MSFT ACF
ACF
1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
14
Lag


Difference Operator on Volatility Implied MSFT PACF
Partial ACF
1.0
0.5
0.0
−0.5
−1.0
2
4
6
8
10
12
14
Lag

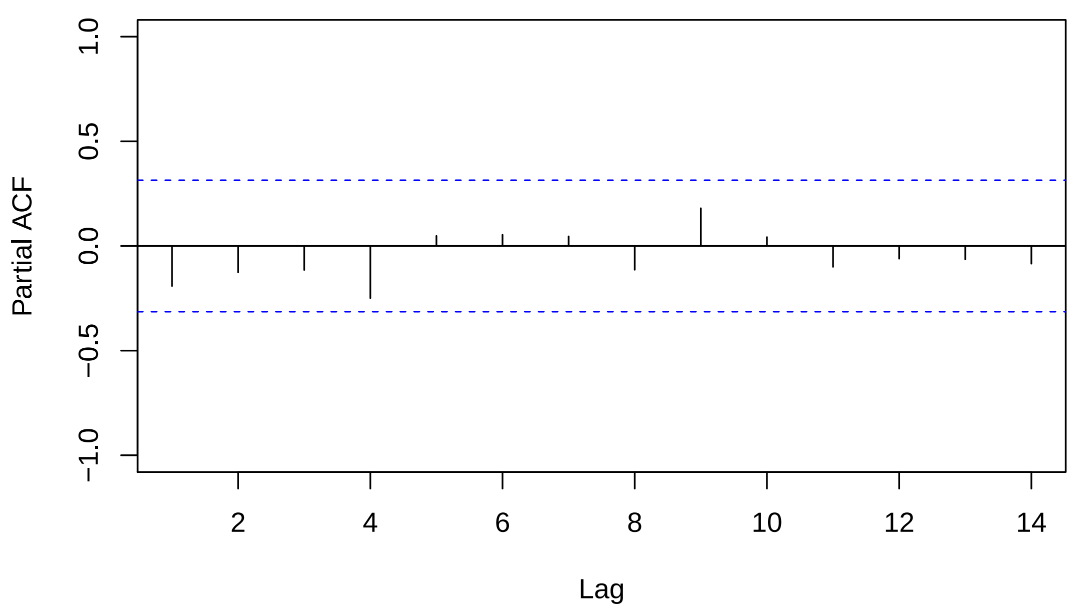

Difference Operator on Volatility Implied ORCL ACF
ACF
1.0
0.5
0.0
−0.5
−1.0
0
2
4
6
8
10
12
14
Lag


Difference Operator on Volatility Implied ORCL PACF
Partial ACF
1.0
0.5
0.0
−0.5
−1.0
2
4
6
8
10
12
14
Lag

Difference Operator on Volatility Implied GOOG ACF



Difference Operator on Volatility Implied GOOG PACF



Since the ACF, and PACF tails off after lag$(1-1)$ (they are in the region where the sample is significantly small), we will use the ARIMA(1,1,1) model for the implied volatility of each one of the stocks. The following is the code to estimate the ARIMA(1,1,1) parameters, and make the forecasts 1 and 2 steps ahead for illustrative purposes.

```
# ACF and PACF of implied volatility of ORCL
acf_diff_imp_volatility_orcl <-
  acf(diff(implied_volatility_orcl[1:(length(implied_volatility_orcl)-2)]),
      ylim=range(-1,1),lag.max = 14)
plot(acf_diff_imp_volatility_orcl, main="Diffence Operator on Volatility Implied ORCL ACF",
     ylim=range(-1,1))
pacf_diff_imp_volatility_orcl <-
  pacf(diff(implied_volatility_orcl[1:(length(implied_volatility_orcl)-2)]),
       ylim=range(-1,1), lag.max =  14)
plot(pacf_diff_imp_volatility_orcl, main="Diffence Operator on Volatility Implied ORCL PACF",
     ylim=range(-1,1))

# ACF and PACF of implied volatility of GOOG
acf_diff_imp_volatility_goog <-
  acf(diff(implied_volatility_goog[1:(length(implied_volatility_goog)-2)]),
      ylim=range(-1,1),lag.max = 14)
plot(acf_diff_imp_volatility_goog, main="Diffence Operator on Volatility Implied GOOG ACF",
     ylim=range(-1,1))
pacf_diff_imp_volatility_goog <-
  pacf(diff(implied_volatility_goog[1:(length(implied_volatility_goog)-2)]),
       ylim=range(-1,1), lag.max =  14)
plot(pacf_diff_imp_volatility_goog, main="Diffence Operator on Volatility Implied GOOG PACF",
     ylim=range(-1,1))



# ARIMA parameters of the implied volatility of MSFT
implied_volatility_arima_parameters_msft <-
  arima(implied_volatility_msft[1: ( length(implied_volatility_msft) -2 )], order=c(1,1,1))

# ARIMA parameters of the implied volatility of ORCL
implied_volatility_arima_parameters_orcl <-
  arima(implied_volatility_orcl[1: ( length(implied_volatility_orcl) -2 )], order=c(1,1,1))

# ARIMA parameters of the implied volatility of GOOG
implied_volatility_arima_parameters_goog <- arima(implied_volatility_goog, order=c(1,1,1))


# ARIMA forecast implied volatility MSFT
implied_volatility_msft_2_ahead <-
                 predict(implied_volatility_arima_parameters_msft, n.ahead =2)
# ARIMA forecast implied volatility MSFT
implied_volatility_orcl_2_ahead <-
                 predict(implied_volatility_arima_parameters_orcl, n.ahead =2)
# ARIMA forecast implied volatility GOOG
implied_volatility_goog_2_ahead <-
                 predict(implied_volatility_arima_parameters_goog, n.ahead =2)
```

Now, we need to add the forecasts to the the object `linear_restrictions`

```
# Inclusion of the forecast of the implied volatility
# We assume that the level /mu will not change drastically withint the forecast of 2 days

t_forecast <- length(MSFT_Close)-start_date_time_s_analysis -2
i <- 0
for (i in c(1:2)){
  linear_restrictions[[t_forecast + 1 + i]] <-
                                    c(list_restrictions_l_equations[[1]][t_forecast + 1 + i],
                                     list_restrictions_l_equations[[2]][t_forecast + 1 + i],
                                     list_restrictions_l_equations[[3]][t_forecast + 1 + i],
                                     0.5*(implied_volatility_msft_2_ahead$pred[i]^2),
                                     0.5*(implied_volatility_orcl_2_ahead$pred[i]^2),
                                     0.5*(implied_volatility_goog_2_ahead$pred[i]^2))

}
```

We use the following lines of code to check that the matrix σ_t is invertible, and its entries are all positive

```
# check the solutions of the equation are all positive
b <- c()
for(i in c(1:(length(MSFT_Close)-start_date_time_s_analysis ))){
  a <- (ginv(linear_equation_volatility) %*% linear_restrictions[[i]] > 0)
  a <- all(a, na.rm = FALSE)
  b <- c(b,a)
}
all(b, na.rm = FALSE)

# list of matrices of sigma (square)
list_sigma <- list()
for (i in c(0:(length(MSFT_Close)-start_date_time_s_analysis ))){
  a <- ginv(linear_equation_volatility) %*% linear_restrictions[[i+1]]
  list_sigma[[i+1]] <- sqrt(t(matrix(c(a[1],a[2],a[3],
                                       a[2],a[4],a[5],
                                       a[3],a[5],a[6]), ncol = 3)))
}

# check the sigma is invertible for each time t
b <-c()
for (i in c(0:(length(MSFT_Close)-start_date_time_s_analysis))){
  a <- det(list_sigma[[i+1]]) != 0
  b <- c(b,a)
}
all(b, na.rm = FALSE)
```

Now, we check how accurate the approximation is

```
# Actual error of the square of the sum of the components of
# volatility and the implied volatility
error_estimation_volatility <- list()
comparison_implied_volatility_components_of_volatility <- c()
a<- 0
b<- 0
c<- 0
error_volatility_estimation_msft <-c()
error_volatility_estimation_orcl <- c()
error_volatility_estimation_goog <- c()
for (i in c(0: (length(MSFT_Close)-start_date_time_s_analysis-2)) ){
  a <- abs(sqrt(list_sigma[[i+1]][1,1]^(2) + list_sigma[[i+1]][1,2]^(2) +
                        list_sigma[[i+1]][1,3]^(2)) - implied_volatility_msft[[i+1]] )
```

```
  b <- abs(sqrt(list_sigma[[i+1]][2,1]^(2) + list_sigma[[i+1]][2,2]^(2) +
                 list_sigma[[i+1]][2,3]^(2)) - implied_volatility_orcl[[i+1]] )
  c <- abs(sqrt(list_sigma[[i+1]][3,1]^(2) + list_sigma[[i+1]][3,2]^(2) +
                 list_sigma[[i+1]][3,3]^(2)) - implied_volatility_goog[[i+1]] )
  error_estimation_volatility[[i+1]] <- c( a, b, c)
  error_volatility_estimation_msft <- c(error_volatility_estimation_msft, a)
  error_volatility_estimation_orcl <- c(error_volatility_estimation_orcl, b)
  error_volatility_estimation_goog <- c(error_volatility_estimation_goog, c)

}

# Root Mean Squared Error(RMSE)
sqrt(product(error_volatility_estimation_msft))/sqrt(length(error_volatility_estimation_msft))
sqrt(product(error_volatility_estimation_orcl))/sqrt(length(error_volatility_estimation_orcl))
sqrt(product(error_volatility_estimation_goog))/sqrt(length(error_volatility_estimation_goog))
```

As an example the `error_volatility_estimation_msft` is

```
Console  Terminal
~/
> error_volatility_estimation_msft
 [1] 0.05489847 0.04905653 0.02234566 0.03177650 0.02938149 0.04306188 0.03878198 0.02475850 0.04159281
[10] 0.01849671 0.02203513 0.03784693 0.04268903 0.04317448 0.04996369 0.04538765 0.02874949 0.03002782
[19] 0.02648465 0.03274084 0.04615811 0.04454207 0.02908698 0.04001547 0.04981777 0.05835521 0.04983279
[28] 0.04653518 0.05716151 0.05564074 0.06287992 0.05651002 0.06489266 0.06486279 0.06822269 0.06624321
[37] 0.06761687 0.04425270 0.05852005 0.05964089
```

Figure 7: Error estimation of MSFT implied volatility using components of volatility method

The root mean square error of those estimations(RMSE) are

```
Console  Terminal
~/
> # Root Mean Squared Error(RMSE)
> sqrt(product(error_volatility_estimation_msft))/sqrt(length(error_volatility_estimation_msft))
[1] 0.04719101
> sqrt(product(error_volatility_estimation_orcl))/sqrt(length(error_volatility_estimation_orcl))
[1] 0.03927289
> sqrt(product(error_volatility_estimation_goog))/sqrt(length(error_volatility_estimation_goog))
[1] 0.05517172
```

Figure 8: Error estimation of MSFT implied volatility using components of volatility method

This suggests that the estimation is relatively good since RMSE if close to zero.

5.1.3 Estimate of $\hat{r}$

To estimate r we will use the Vasicek model on the 1-treasure bill. There are different types of estimation. In the application we will use the OLS estimation of the SDE's parameters.

We have to estimate the parameters ϱ^{α} and ρ^{α}. However, their computation depends on the type of zero-coupon bond that we choose. We suppose that the zero-coupon bond for the small investors will be the 1-year treasury bill[11]. Since stock options are not quoted on holidays, but 1-year treasury bills do, we have to synchronize the stock data set with the 1-year treasury bill data. One way to do that is by merging the stocks' dataset with the 1-year treasury bill. Once we finish this step, we need to remove the rows containing NA values. Next, we split the data again and work with the cleaned 1-year treasury bill data. The algorithm is provided below

```
# 2019-10-18 the expiration date of the European Call options

# 1-Year Treasury Bill: Secondary Market Rate
t_bill_1year_daily <- getSymbols("DTB1YR", src = "FRED", auto.assign = FALSE)

# Filling missing values using linear interpolation
t_bill_1year_daily <- na.approx(t_bill_1year_daily)

# Check 1 year treasury bill and stocks have the same indeces
# MSFT, ORCL, GOOG has the same indeces, but 1-Year Treasury Bill not
merged_data <- merge.xts(t_bill_1year_daily, MSFT_Close, by = "ID")
merged_data <- subset(merged_data, select = c(1,2))

# Remove values that does not share the same index
start_date_merged_data <-
  min(which(format(index(merged_data), "%Y-%m-%d") == start_likelihood_estimation_date))
end_date_merged_data <-
  min(which(format(index(merged_data), "%Y-%m-%d") == end_time_series_date))
merged_data <- merged_data[start_date_merged_data:end_date_merged_data]
merged_data <- merged_data[!is.na(merged_data[,2]),]

# clean data of the 1-Year Treasury Bill
t_bill_1year_daily <- merged_data[, c(1)]
```

Most of the treasury notes follow a mean reversion property. In this case, we will assume the same for the U.S. 1-year Treasury Bill. The model that we will use for estimation, and forecasting will be the Vasicek model (see for instance Privault, 2020)

$$dr_t = (a - br_t)dt + \sigma dW_t \tag{5.28}$$

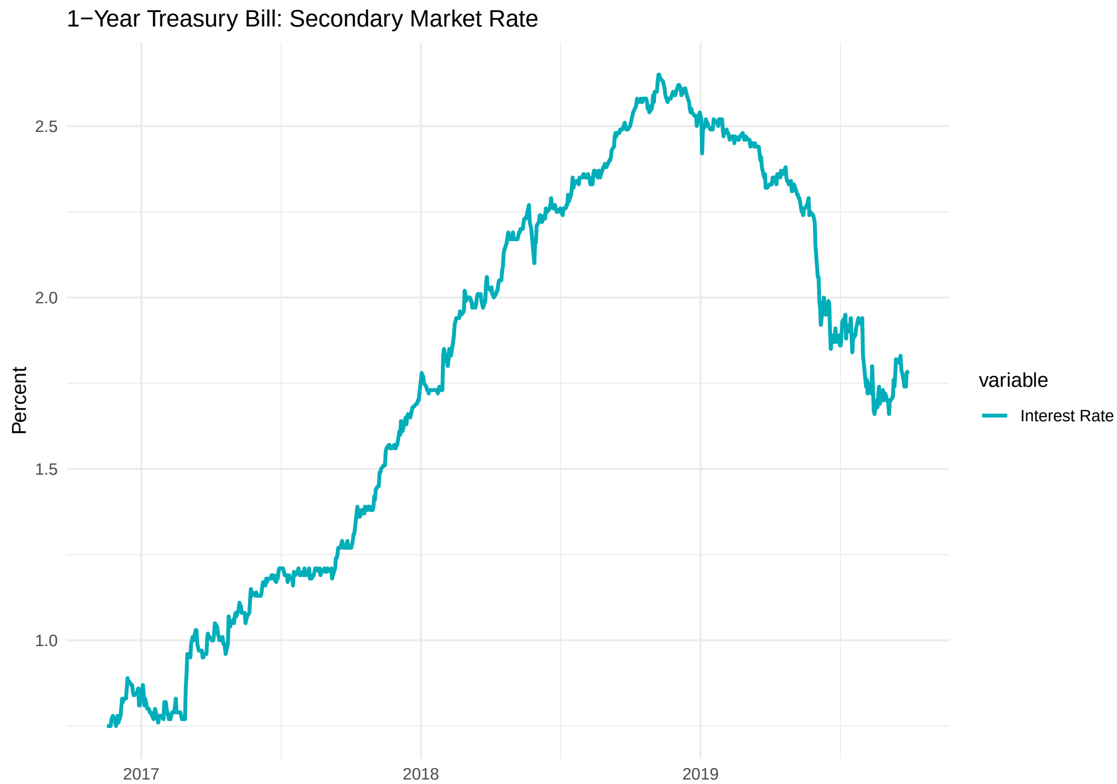
We can discretize this model using a sequence of times $(t_k)_{k=0}^{N+1}$, obtaining

$$r_{t_{k+1}} - r_{t_k} = (a - br_{tk})\Delta t + \sigma W_t. \tag{5.29}$$

Instead of using the maximum likelihood estimator, we will minimize the quadratic residual

$$\sum_{k=0}^{N+1} (r_{t_{k+1}} - r_{t_k} - (a - br_{tk})\Delta t)^2 \tag{5.30}$$

[11] The data was taken from https://fred.stlouisfed.org/



Source: Board of Governors of the Federal Reserve System (US)

Using partial differentiation with respect to a and b, and the following code, we get the estimators.

```
# Vasicek model calibration, and forecast

# Term b of the Vasicek model
sample_t_bill_1year_daily <-
  t_bill_1year_daily[(length(t_bill_1year_daily)-720): (length(t_bill_1year_daily) -2)]
sample_t_bill_1year_daily <- na.approx( as.numeric(sample_t_bill_1year_daily))

numerator_1_vasicek_b <-
  sum(sample_t_bill_1year_daily[-1]*
                 sample_t_bill_1year_daily[-length(sample_t_bill_1year_daily)])
numerator_2_vasicek_b <-
  sum(sample_t_bill_1year_daily[-length(sample_t_bill_1year_daily)])*
                                       sum(sample_t_bill_1year_daily[-1])
denominator_1_vasicek_b <- sum(sample_t_bill_1year_daily**2)
denominator_2_vasicek_b <-
                 sum(sample_t_bill_1year_daily[-length(sample_t_bill_1year_daily)])*2

# Term b of the Vasicek Model
term_1_vasicek_b <-
  (numerator_1_vasicek_b - (numerator_2_vasicek_b/length(sample_t_bill_1year_daily)) ) /
  (denominator_1_vasicek_b - (denominator_2_vasicek_b/ length(sample_t_bill_1year_daily)))
vasicek_b <- (term_1_vasicek_b -1)*(-365)

# Term a of the Vasicek model
term_1_vasicek_a <- sample_t_bill_1year_daily[-1] -
  (1- (vasicek_b*(1/365)) )*sample_t_bill_1year_daily[-length(sample_t_bill_1year_daily)]
```

```
vasicek_a <- ((1/length(sample_t_bill_1year_daily))*sum(term_1_vasicek_a))*365
```

Now, with this estimation, we are ready to use Euler-Maruyama method and include the forecast two steps ahead

```
# Estimation of the 1 and 2 step ahead values of the Vasicek model
# Euler Maruyama Method for SDE
tail(t_bill_1year_daily)
tail(sample_t_bill_1year_daily)

rate_estimation_1_ahead <- sample_t_bill_1year_daily[length(sample_t_bill_1year_daily)] +
    (vasicek_a -
     (vasicek_b*sample_t_bill_1year_daily[length(sample_t_bill_1year_daily)]) )*(1/365)

rate_estimation_2_ahead <- rate_estimation_1_ahead +
            (vasicek_a - (vasicek_b*rate_estimation_1_ahead) )*(1/365)

# Inclusion of the last two element in the forecast
t_bill_1year_daily[length(t_bill_1year_daily)-1] <- rate_estimation_1_ahead
t_bill_1year_daily[length(t_bill_1year_daily)] <- rate_estimation_2_ahead
```

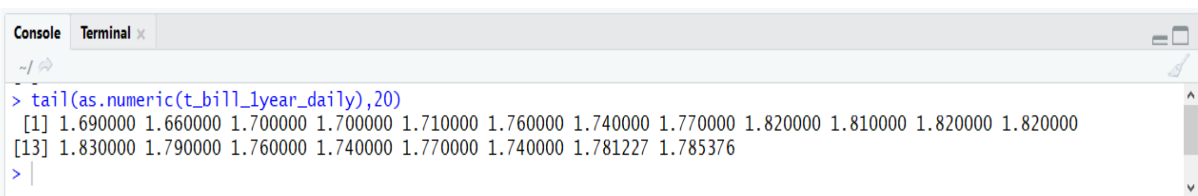


Figure 9: Interest rate values and forecast.

where the last two values are the two forecasts.

5.1.4 Relative entropy algorithm under macroeconomic restrictions

We have to build the most significative macroeconomic events that can change the dynamics of the prices of MSFT, ORCL, and GOOG. For the sake of simplicity we will restrict ourselves to the events: A_1 = China-U.S trade deal, A_2 = New NAFTA or USMCA trade deal , and A_3 = Hong Kong Resolution. We can build our probability space as

$$\Omega^{\Lambda} = \{C_1 \times C_2 \times C_3 : C_i = A_i \text{ or } C_i = A_i^{\complement}\}, \tag{5.31}$$

where the probability measure P^{Λ} will be the a priori distribution as in Proposition 2.4. The variables that we will use for the restriction are: net income, revenue, assets, equity, current assets, and current liabilities.

In financial statement forecasting it is not clear how each account depends on macroeconomic events. This forces us to express each accounting variable as a function of past information, and variables that are really sensitive to macroeconomic events. In this work, we will focus on revenue as the variable that is sensitive to macroeconomic events. This idea can be written as follows

$$\text{Account}_{t+1} \approx \text{Account}_t + f(\text{Revenue}_t, \text{Other Accounts}_t, \mathbf{r}), \tag{5.32}$$

where $\mathbf{r}$ is a vector of growth rates that contains the expected rate of growth (from t to $t+1$) of revenue, and the known rates of growth of other accounts such as cost of good sold, tax rate, etc. Here we list some of the notation that will be used throughout this part.

Definition	**Notation**
Revenue growth rate	r_{rev}
COGS as percentage of revenue	r_{COGS}
Tax rate	r_{tax}
Net Cash flow from financing activities	CFF
Net Cash flow from investing activities	CFI
Net Cash flow from operating activities	CFO
Earnings before taxes	EBT
Property, plant and equipment	PPE
Net increase(cash)	CFF + CFI + CFO
Accounting Variable$_{t+1}$ − Accounting Variable$_t$	ΔAccounting Variable$_{t+1}$

Table 3: Accounting notations

The formulas that we will use are the following[12]. For revenue we use

$$\text{Revenue}_{t+1} \approx \text{Revenue}_t(1 + r_{\text{rev}}). \tag{5.33}$$

For cost of goods sold

$$\text{COGS}_{t+1} \approx r_{\text{COGS}}\text{Revenue}_{t+1}. \tag{5.34}$$

For net income

$$\text{Net Income}_{t+1} \approx \text{EBT}_{t+1} - \text{Taxes}_{t+1} \approx \text{Net Income}_t + (1 - r_{\text{tax}})r_{\text{rev}}\text{Gross Profit}_t. \tag{5.35}$$

For current liabilities

$$\text{Current Liabilities}_{t+1} \approx \text{Accounts Payable}_{t+1} \approx \text{Current Liabilities}_t + r_{\text{rev}}\text{Accounts Payable}_t. \tag{5.36}$$

For equity

$$\text{Equity}_{t+1} \approx \text{Equity Capital}_{t+1} + \text{Retained Earnings}_{t+1} \approx \text{Equity}_t + \Delta\text{Retained Earnings}_t. \tag{5.37}$$

For MSFT and GOOG, the asset account will be of the form

$$\begin{aligned}
&\text{Assets}_{t+1} \\
&\approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Inventory}_{t+1} + \text{Marketable Securities}_{t+1} + \text{PPE}_{t+1} \\
&\approx \text{Assets}_t + r_{\text{rev}}(\text{Accounts Receivable}_t + \text{Inventory}_t) + \text{Net Increase(cash)}_t \\
&\quad + \Delta\text{Net Increase(cash)}_t + \Delta\text{Marketable Securities}_t
\end{aligned} \tag{5.38}$$

[12]Proof of the formulas can be found in Appendix E.

However, for ORCL, we use the following formula for assets

$$\begin{aligned}\text{Assets}_{t+1} &\approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Marketable Securities}_{t+1} + \text{PPE}_{t+1} \\ &\approx \text{Assets}_t + r_{\text{rev}}\text{Accounts Receivable}_t + \text{Net Increase(cash)}_t \\ &\quad + \Delta\text{Net Increase(cash)}_t + \Delta\text{Marketable Securities}_t\end{aligned} \tag{5.39}$$

because it does not have inventories. Using the same concept, for MSFT, and GOOG, the current asset account is

$$\begin{aligned}\text{Current Assets}_{t+1} &\approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Inventory}_{t+1} + \text{Marketable Securities}_{t+1} \\ &\approx \text{Current Assets}_t + r_{\text{rev}}(\text{Accounts Receivable}_t + \text{Inventory}_t) + \text{Net Increase(cash)}_t \\ &\quad + \Delta\text{Net Increase(cash)}_t + \Delta\text{Marketable Securities}_t.\end{aligned} \tag{5.40}$$

For ORCL the corresponding account is

$$\begin{aligned}\text{Current Assets}_{t+1} &\approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Marketable Securities}_{t+1} \\ &\approx \text{Current Assets}_t + r_{\text{rev}}\text{Accounts Receivable}_t + \text{Net Increase(cash)}_t \\ &\quad + \Delta\text{Net Increase(cash)}_t + \Delta\text{Marketable Securities}_t.\end{aligned} \tag{5.41}$$

Notice ORCL uses trade receivables account, but MSFT, and GOOG use accounts receivable instead. Moreover, MSFT uses the account short term investments, but ORCL, and GOOG use the account marketable securities instead. However, the name of the accounts does not affect the accounting meaning in these formulas. For this reason, we use the names accounts receivable, and marketable securities in the formulas above.

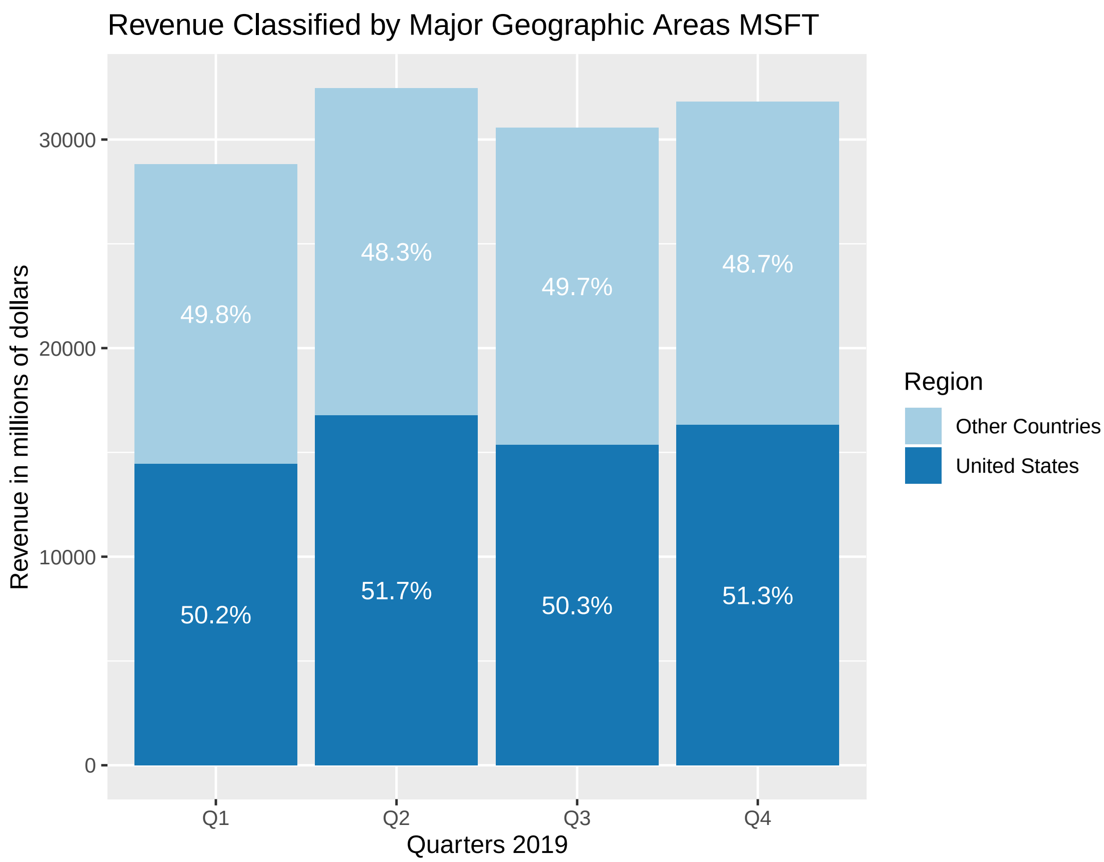
Now, we need to know how sensible revenue is with respect to the each event. One way to do this is by decomposing the revenue rate of growth in market share growth, and market growth. First, we start with the approximation

$$\text{Company's Revenue}_t \approx (\text{Company's Market Share}_t)(\text{Company's Market}_t). \tag{5.42}$$

Therefore, using $\Delta \ln x \approx \%\Delta x$, we get the formula

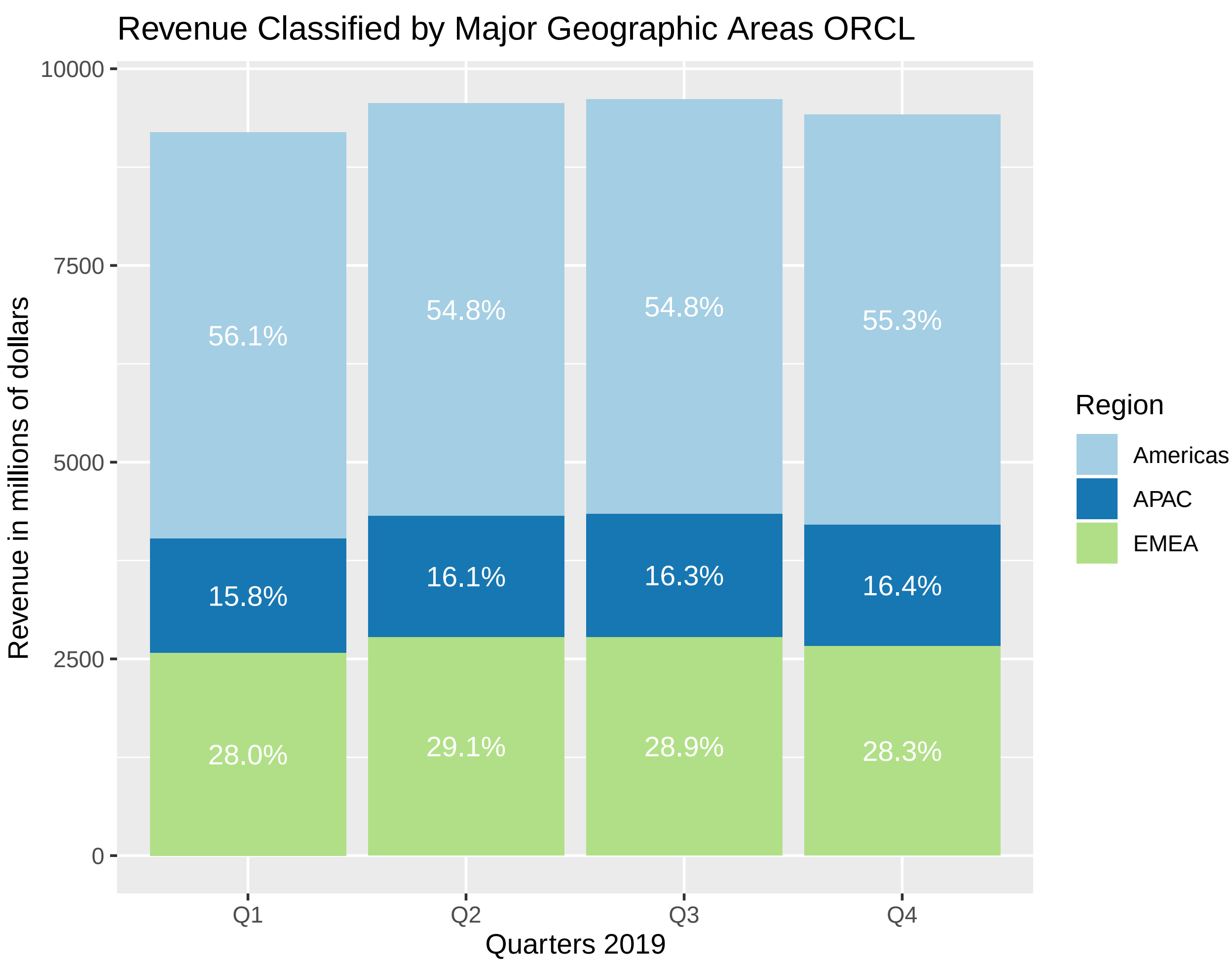
$$\text{Company's Revenue Growth}_t \approx \text{Company's Market Share Growth}_t + \text{Company's Market Growth}_t. \tag{5.43}$$

We can check in https://ustr.gov/countries-regions that more than 55% of the exports of the US to China, México, Canada, and Hong Kong come from electrical machinery, machinery, mineral fuels, plastics, vehicles, medical instruments, and aircraft. In addition, the U.S exports, and imports of goods by F.A.S basis with the same countries, taken from https://fred.stlouisfed.org/, do not show a significant change due to the macroeconomic events that occurred prior to August 01, 2019. Therefore, the macroeconomic events may only have a possible small impact on revenues, drift and diffusion coefficients. The information is provided below.



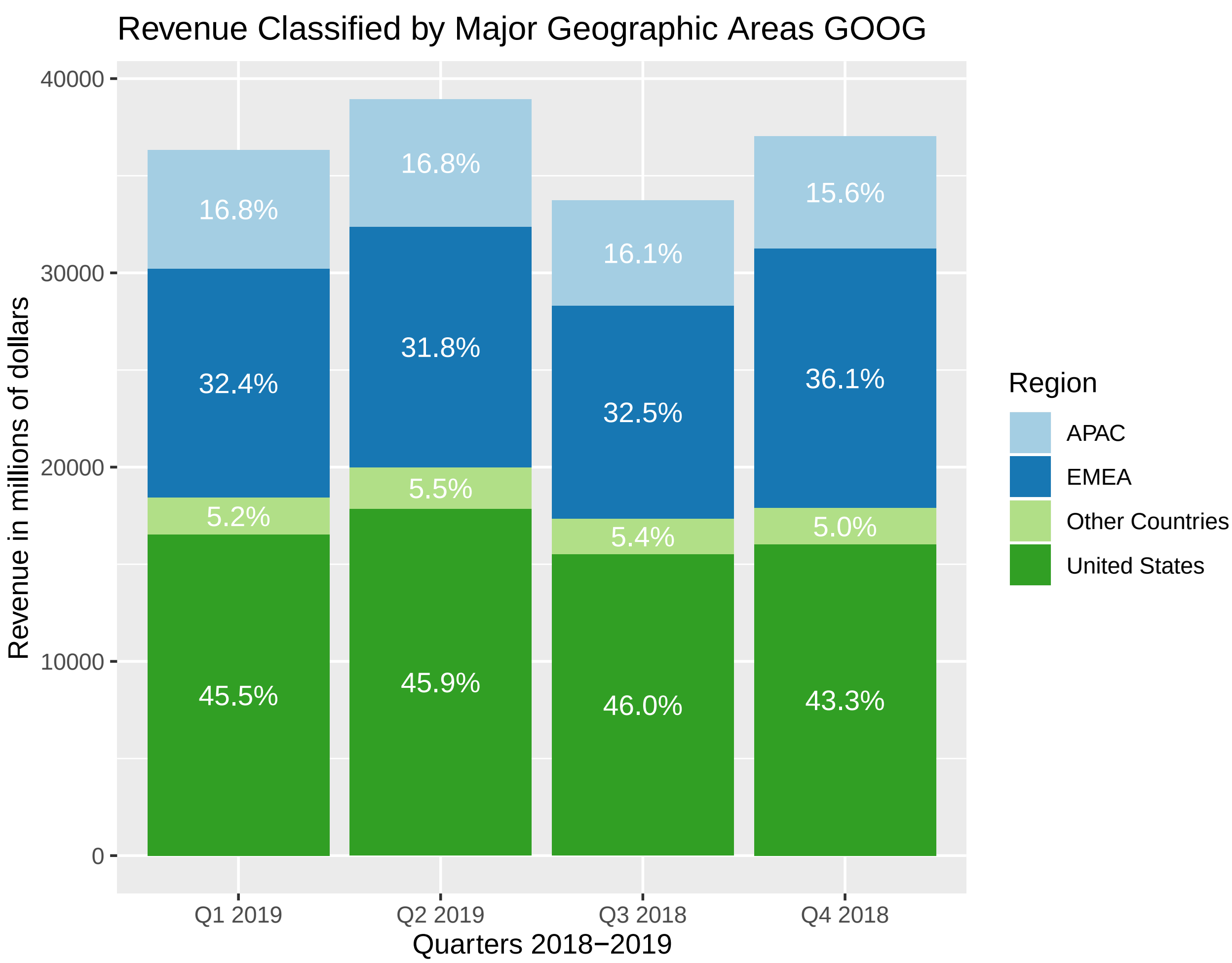
Source https://www.sec.gov/[13].

[13]Programmed in ggplot2, https://www.sec.gov/. Source code in Appendix F.



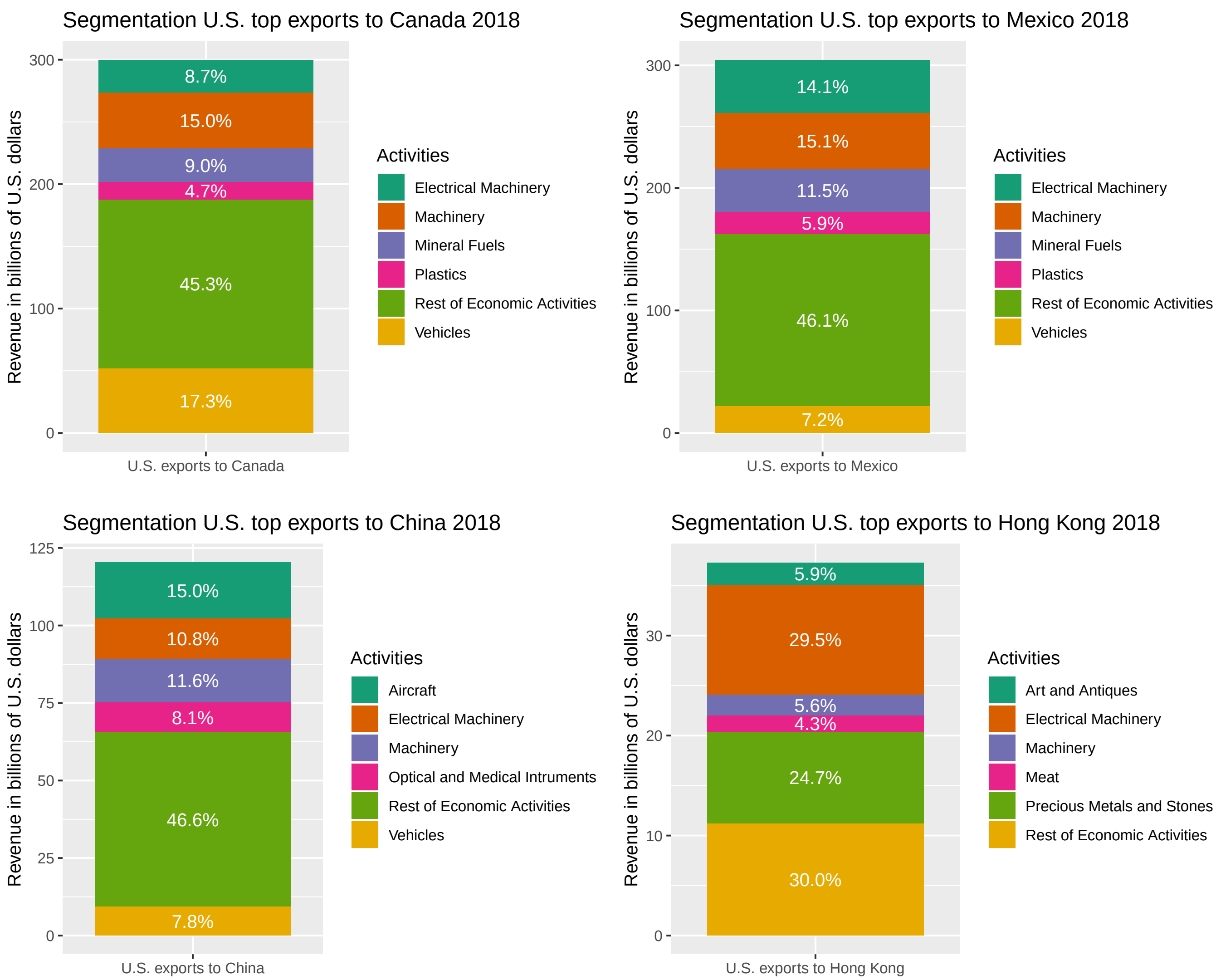
Regions represent Europe, the Middle East, and Africa (EMEA); Asia-Pacific (APAC); and Americas. Source https://www.sec.gov/.[14]

[14] Programmed in ggplot2, https://ggplot2.tidyverse.org/. Source code in Appendix F.



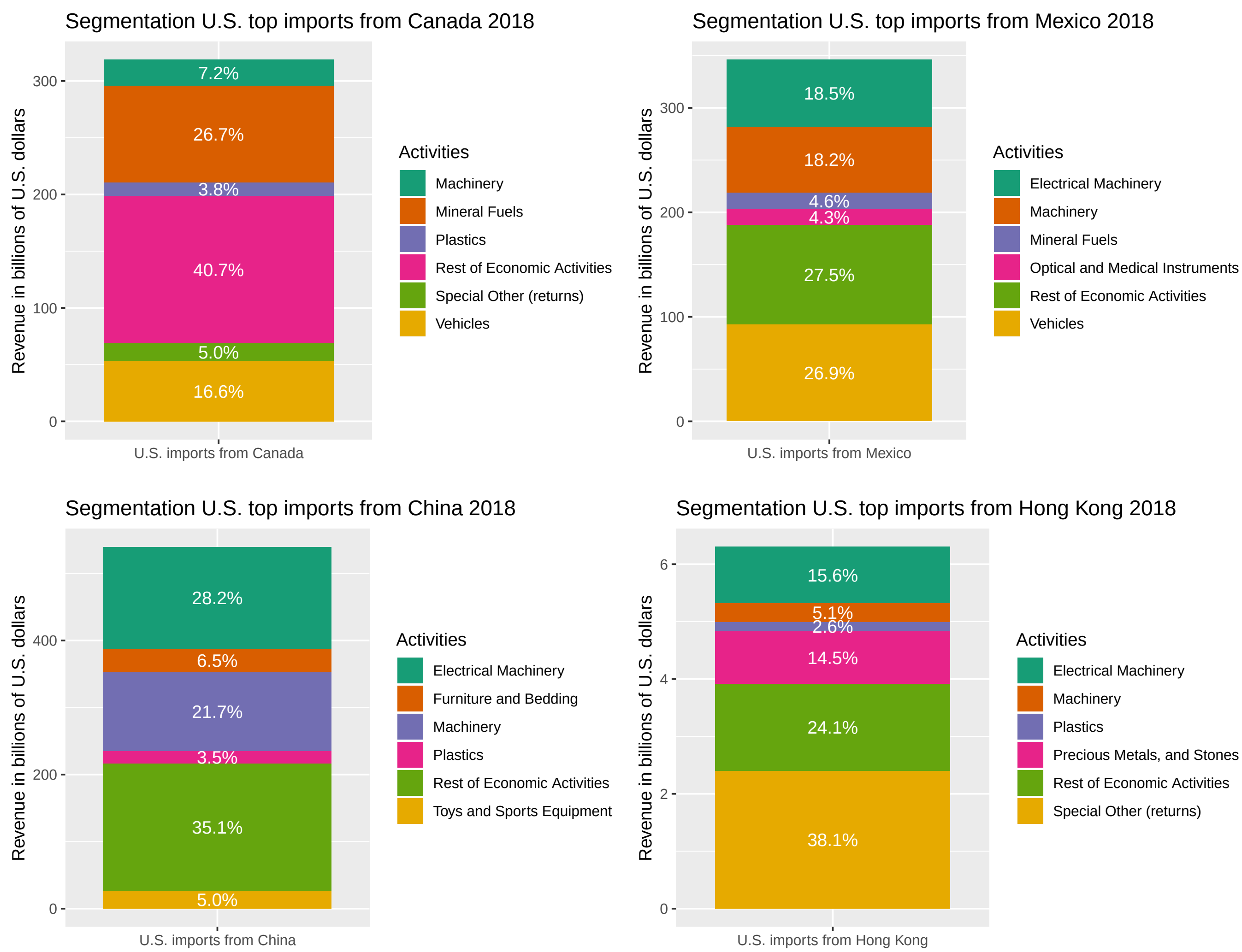
Regions represent Europe, the Middle East, and Africa (EMEA); Asia-Pacific (APAC); and Canada and Latin America (Other Countries). Source https://www.sec.gov/. [15]

[15] Programmed in ggplot2, https://ggplot2.tidyverse.org/. Source code in Appendix F.



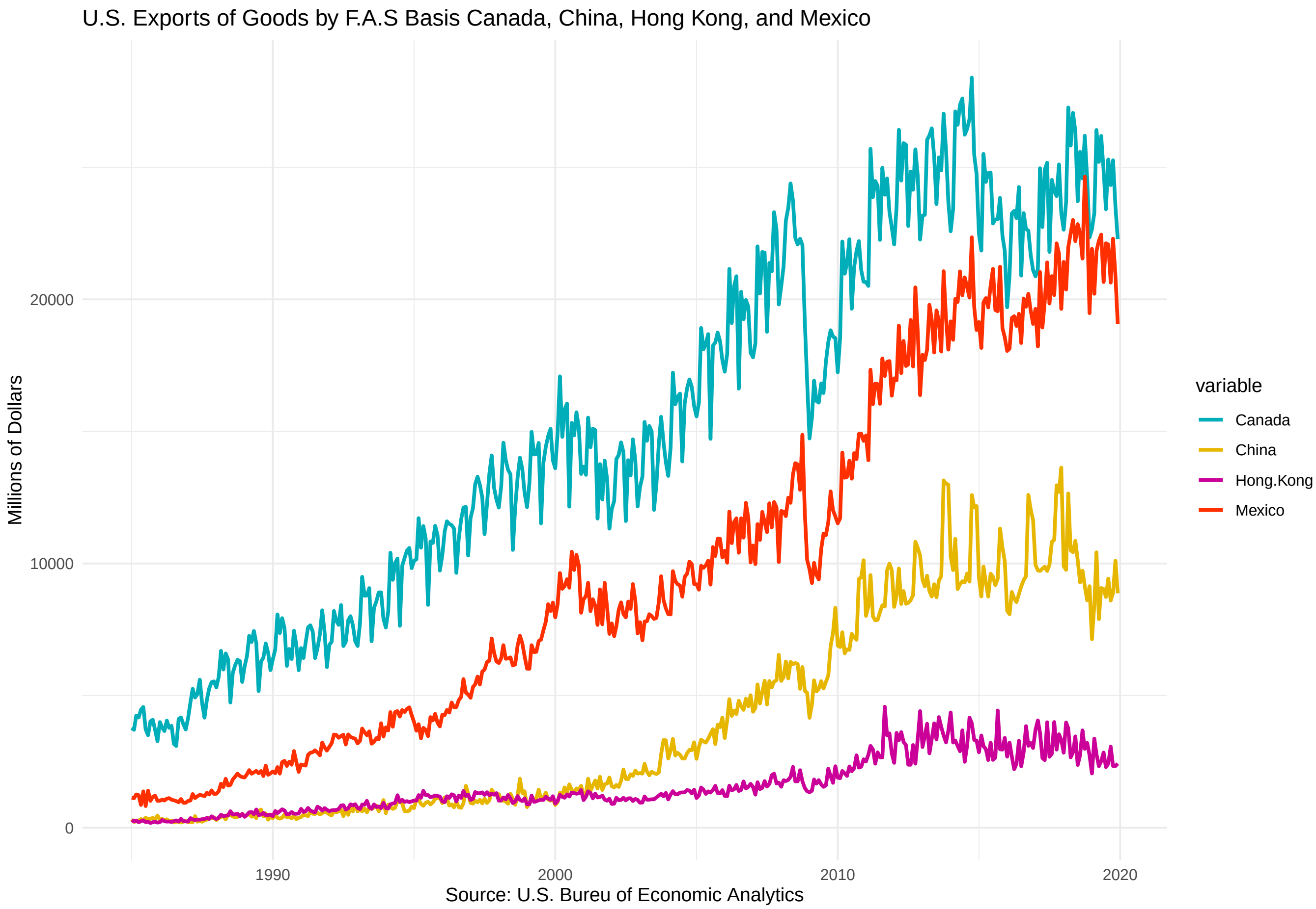
Segmentation of U.S. top exports to Canada, China, Hong Kong, and México. Source https://ustr.gov/countries-regions. [16]

[16] Programmed in ggplot2, https://ggplot2.tidyverse.org/. Source code in Appendix F.



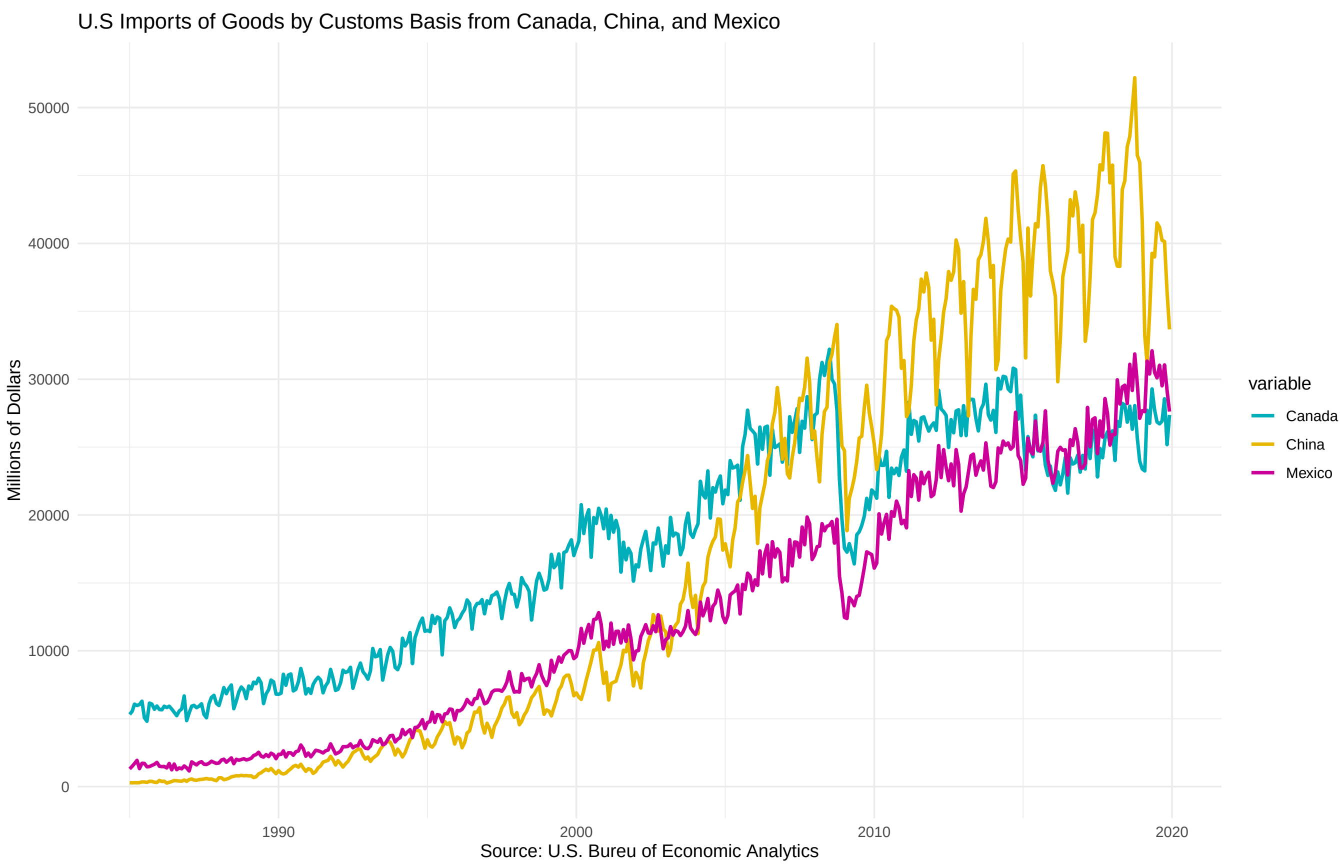
Segmentation of U.S. top imports from Canada, China, Hong Kong, and México. Source https://ustr.gov/countries-regions. [17]

[17] Programmed in ggplot2, https://ggplot2.tidyverse.org/. Source code in Appendix F.



U.S. exports of goods by F.A.S basis to the Hong Kong and the top 3 importers: Canada, China, and México. Source https://fred.stlouisfed.org/. [18]

[18] Programmed in ggplot2, https://ggplot2.tidyverse.org/. Source code in Appendix F.

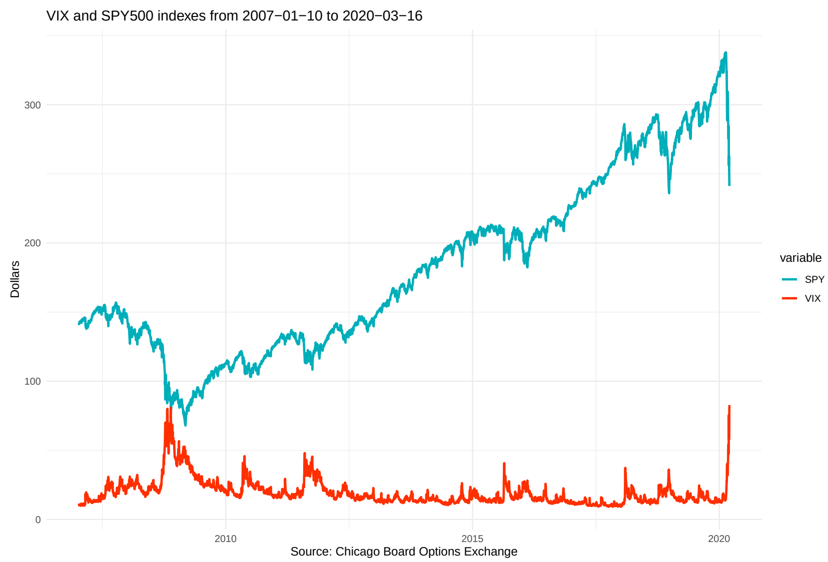


U.S. imports of goods by F.A.S basis from their top 3 exporters: Canada, China, and México.[19] Source https://fred.stlouisfed.org/. [20]

[19]U.S. imports of goods by F.A.S basis from Hong Kong not available in https://fred.stlouisfed.org/

[20]Programmed in ggplot2, https://ggplot2.tidyverse.org/. Source code in Appendix F.

There has been some research about the correlation between volatility and economic growth such as the works by (Dabusinskas et al., 2013), and (Mobarak, 2005) that support this hypothesis. The following chart displaying Volatility Index (VIX), and the SPY demonstrates a potential negative autocorrelation between volatility and economic growth. In fact, we have that during the period from 2007-01-10 to 2020-03-16 the Pearson's autocorrelation of these variables was about -0.5222381.



With the previous information, we have the following data about the impact of the macroeconomic events on the direction of each one of the variables.

Event	**Effect on σ**	**Effect on μ**	**Effect on sales**	**Effect on r**
$A_1 \times A_2 \times A_3$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$
$A_1^c \times A_2 \times A_3$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A_1 \times A_2^c \times A_3$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A_1^c \times A_2^c \times A_3$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A_1 \times A_2 \times A_3^c$	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$
$A_1^c \times A_2 \times A_3^c$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A_1 \times A_2^c \times A_3^c$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$
$A_1^c \times A_2^c \times A_3^c$	$\uparrow$	$\downarrow$	$\downarrow$	$\downarrow$

Table 4: Macroeconomic effects on coefficients of the SDE for big agents: $\uparrow$ (increase), and $\downarrow$ (decrease)

Now, we need to quantify the values of $\uparrow$ and $\downarrow$ for each one of the macroeconomic events. For this step, we must follow a conservative and cautious approach considering small changes since the data do not give reasons to consider big changes.

Event	Change factor on σ	Change factor on μ	Change factor on sales	Change factor on r
$A_1 \times A_2 \times A_3$	$\sigma_{\downarrow}$	$\mu_{\uparrow}$	$r_{\text{rev},\uparrow}$	$r_{\uparrow}$
$A_1^c \times A_2 \times A_3$	$0.5\sigma_{\uparrow}$	$0.5\mu_{\downarrow}$	$0.5r_{\text{rev}\downarrow}$	$0.5r_{\downarrow}$
$A_1 \times A_2^c \times A_3$	$0.7\sigma_{\uparrow}$	$0.7\mu_{\downarrow}$	$0.7r_{\text{rev},\downarrow}$	$0.7r_{\downarrow}$
$A_1^c \times A_2^c \times A_3$	$0.9\sigma_{\uparrow}$	$0.9\mu_{\downarrow}$	$0.9r_{\text{rev},\downarrow}$	$0.9r_{\downarrow}$
$A_1 \times A_2 \times A_3^c$	$0.2\sigma_{\downarrow}$	$0.2\mu_{\uparrow}$	$0.2r_{\text{rev},\uparrow}$	$0.2r_{\uparrow}$
$A_1^c \times A_2 \times A_3^c$	$0.55\sigma_{\uparrow}$	$0.55\mu_{\downarrow}$	$0.55r_{\text{rev},\downarrow}$	$0.55r_{\downarrow}$
$A_1 \times A_2^c \times A_3^c$	$0.75\sigma_{\uparrow}$	$0.75\mu_{\downarrow}$	$0.75r_{\text{rev},\downarrow}$	$0.75r_{\downarrow}$
$A_1^c \times A_2^c \times A_3^c$	$\sigma_{\uparrow}$	$\mu_{\downarrow}$	$r_{\text{rev},\downarrow}$	$r_{\downarrow}$

Table 5: Quantification macroeconomic effects

Here the notation is self-explanatory. Following the notation in table 5, for each one of the columns, we consider the highest and lowest values for the change factors (that correspond to the best or worst macroeconomic events) and multiply them by some factors that depend on the impact of the macroeconomic events and the methods the big agent used to estimate them. Here, we build those factors based on the top exporters and importers. As an example, for the column of diffusion coefficient, we denote by $\sigma_{\uparrow}$ and $\sigma_{\downarrow}$ the highest and lowest possible numerical values for the stocks' volatility that correspond to the worst and best macroeconomic event respectively. Following the context, if $A_1^c \times A_2^c \times A_3^c$ happens, the worst case scenario is happening, then it is expected that the highest volatility value, $\sigma_{\uparrow}$, must occur and the net effect will be capture through the estimators

$$\hat{r}_{\text{rev}} = \frac{\text{Revenue}_t - \text{Revenue}_{t-1}}{\text{Revenue}_{t-1}} + r_{\text{rev},\downarrow} \qquad \hat{\hat{\sigma}} = \hat{\sigma}(1+\sigma_{\uparrow}) \qquad \text{and} \qquad \hat{\hat{r}} = \hat{r}(1+r_{\downarrow}), \tag{5.44}$$

that will appear in the formula 5.50. For the sake of simplicity, and because the macroeconomic information does not provide any clue to take a more aggressive assumptions, we will adopt a more conservative approach by using small factors:

$$\mathrm{r}_{\text{rev},\uparrow} = 0.05 \quad \mathrm{r}_{\text{rev},\downarrow} = -0.05 \quad \sigma_{\uparrow} = 0.005, \;\; \sigma_{\downarrow} = -0.005, \;\; r_{\uparrow} = 0.005, \;\; \text{and} \;\; r_{\downarrow} = -0.005. \tag{5.45}$$

For the optimization process of 2.47 and 2.48, we require the a prior distribution P^{Λ}. It will be given by the following table

Macroeconomic Event	P^{Λ}	Q^{λ}
$A_1 \times A_2 \times A_3$	0.5/7	0.0276
$A_1^c \times A_2 \times A_3$	0.5/7	0.1192
$A_1 \times A_2^c \times A_3$	0.5/7	0.1449
$A_1^c \times A_2^c \times A_3$	0.5/7	0.1761
$A_1 \times A_2 \times A_3^c$	0.5/7	0.0603
$A_1^c \times A_2 \times A_3^c$	0.5/7	0.1252
$A_1 \times A_2^c \times A_3^c$	0.5/7	0.1521
$A_1^c \times A_2^c \times A_3^c$	0.5	0.1941

Table 6: A priori distribution.

This means that around August 01, 2019, we are assuming the investor's negative expectations about the macroeconomic events were above 50%. However, after using the CVXfromR package tools, and the information of the last four financial quarters (prior to 2019-08-01) of MSFT, ORCL, and GOOG to build the constraints, we get that the a posteriori distribution $Q^{\bar{\lambda}}$ is more coherent with what happened later: the macroeconomic environment was not as severe as expected. In fact, 2019 turned out to be a good year for the stock market.

The code for these computations is the following[21]

```
# 15 accounting factors MSFT, ORCL, GOOG

# Revenue
revenue_t <- c(33717,11136,38944)
revenue_rate <- c((33717/30571)-1,(11136/9614)-1,(38944/36339)-1)


# Net Income
net_income_t <- c(13187,3740,9947)
gross_profit_t <- c(23305,9072,21648)
tax_rate <- c(0.1020,0.0970,0.1330)

# Current Liabilities
current_liabilities_t <- c(69420,18630,37000)
accounts_payable_t <- c(9382,580,3925)

# Equity
equity_t <- c(102330,22363,192192)
retained_earnings_t <- c(24150,-3496,145346)
retained_earnings_t_m_1 <- c(18338,-1287,138720)

# Assets
assets_t <- c(286556,108709,257101)
accounts_receivable_t <- c(29524,5134,20965)
inventory_t <- c(2063,0,964)
net_increase_cash_t <- c(144,5794,-2561)
net_increase_cash_t_m_1 <- c(4574,3896,2447)
marketable_sec_t <- c(133819,17313,104469)
marketable_sec_t_m_1 <- c(131618,25310,94340)

# Current Assets
current_assets_t <- c(175552,46386,147437)


# Accounting Forecasting Formulas

#

accounting_function <- function(macro_factor, rev_up_down){

  # Forecast Revenue
  revenue_forecast <- revenue_t*(1 + revenue_rate + (macro_factor*rev_up_down)  )
  # Forecast Net Income
  net_income_forecast <- net_income_t +
    ((1- tax_rate)*(revenue_rate +
                    (macro_factor*rev_up_down)) *gross_profit_t)
  # Forecast Current Liabilities
  current_liabilities_forecast <- current_liabilities_t +
    ( (revenue_rate + (macro_factor*rev_up_down))
```

[21]Code using CVXfromR package created by professor Jacob Bien.

```
        *accounts_payable_t)
    # Forecast Equity
    equity_forecast <- equity_t + (retained_earnings_t - retained_earnings_t_m_1)

    # Forecast Assets
    assets_forecast <- assets_t + ((revenue_rate + (macro_factor*rev_up_down))*
                                  (accounts_receivable_t + inventory_t)) +
      net_increase_cash_t + (net_increase_cash_t- net_increase_cash_t_m_1) +
      (marketable_sec_t - marketable_sec_t_m_1)

    # Forecast Current Assets
    current_assets_forecast <- current_assets_t +
      ((revenue_rate + (macro_factor*rev_up_down))*(accounts_receivable_t + inventory_t)) +
      net_increase_cash_t + (net_increase_cash_t- net_increase_cash_t_m_1) +
      (marketable_sec_t - marketable_sec_t_m_1)
    vector_accounting <- c(sum(revenue_forecast), sum(net_income_forecast),
                           sum(current_liabilities_forecast),sum(equity_forecast),
                           sum(assets_forecast), sum(current_assets_forecast))
    return(vector_accounting)

}

# Size of the distribution
size_p_distribution <- 8
size_q_distribution <- 8

# A_accounting
A_accounting <- matrix( c(accounting_function(1,0.05),
                           accounting_function(0.5, -0.05),
                           accounting_function(0.7, -0.05),
                           accounting_function(0.9, -0.05),
                           accounting_function(0.2, 0.05),
                           accounting_function(0.55, -0.05),
                           accounting_function(0.75, -0.05),
                           accounting_function(1,-0.05)),
                        ncol=size_q_distribution, byrow = F)


#  Optimization Shannon's entropy using CVXfromR
n   <- size_q_distribution
A <- A_accounting
b <- A%*% c(0.5/7,0.5/7,0.5/7,0.5/7,0.5/7,0.5/7,0.5/7,0.5)

opt.vals <- CallCVX(cvxcode, const.vars=list(n=n, A=A, b=b),
                    opt.var.names="pmaxent")

priori_distribution <- c(0.5/7,0.5/7,0.5/7,0.5/7,0.5/7,0.5/7,0.5/7,0.5)
posteriori_distribution <- opt.vals$pmaxent
```

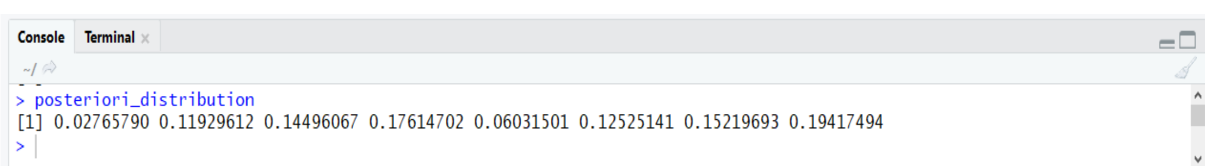


Figure 10: A posteriori distribution.

5.1.5 Estimation of the price from a big agent perspective

Now, we are ready to compute the price of the MSFT call option from the perspective of the big agent. From Corollary 2.5, we remember that the formula for the price of a big agent is given by

$$
\begin{aligned}
P(C) &= E_{Q^{\bar{\lambda}}}\left[e^{-\int_0^T r_s ds}\left(S_T-K\right)_+\right] \\
&= E_P\left[\sum_{k=1}^m 1_{\Lambda_k\times\Omega^{\bar{W}}} e^{-\int_0^T {}^k r_s ds}\left({}^k S_T-K\right)_+\left(\frac{\lambda_k}{p_k}\right)\exp\left\{-\int_0^T {}^k\theta_l\cdot dW_l-\frac{1}{2}\int_0^T|{}^k\theta_l|^2 dl\right\}\right] \\
&= \sum_{k=1}^m \lambda_k E_{\left({}^k Q^{\bar{\lambda}}\right)}\left[e^{-\int_0^T {}^k r_s ds}\left({}^k S_T-K\right)_+\right]
\end{aligned}
\tag{5.46}
$$

where ${}^k S^{(i)}$ satisfies the dynamics of the form

$$
\begin{aligned}
d({}^k S_t^{(i)}) &= {}^k\mu_t^{(i)}\,{}^k S_t^{(i)}dt+{}^k S_t^{(i)}\left(\sum_{j=1}^n {}^k\sigma_t^{(ij)}dW_t^{(j)}\right) \\
&= {}^k S_t^{(i)}\,{}^k r_t dt+{}^k S_t^{(i)}\left(\sum_{j=1}^n\left({}^k\sigma_t^{(ij)}\,{}^k\theta_t^{(j)}\right)dt+{}^k\sigma_t^{(ij)}dW_t^{(j)}\right) \\
&= {}^k S_t^{(i)}\,{}^k r_t dt+{}^k S_t^{(i)}\sum_{j=1}^n {}^k\sigma_t^{(ij)}d\left({}^k W_t^{(j)}\right),
\end{aligned}
\tag{5.47}
$$

${}^k Q^{\bar{\lambda}}$ is given by

$$
\frac{d({}^k Q^{\bar{\lambda}})}{dP} := \frac{dQ^\Lambda}{dP}\exp\left\{-\int_0^T {}^k\theta_l\cdot dW_l-\frac{1}{2}\int_0^T|{}^k\theta_l|^2 dl\right\},
\tag{5.48}
$$

and the process

$$
{}^k W_t := \int_0^t {}^k\theta_l dl+W_t
\tag{5.49}
$$

is an $n-$dimensional Brownian motion under ${}^k Q^{\bar{\lambda}}$. Since we are going to make the estimation of prices of call options on short periods of time, we can assume the parameters are constants. Therefore, we can use all the machinery we have developed so far. Using the standard procedure to price Black-

Scholes-Merton formula for call options, we get

$$
\begin{aligned}
&P(C) \\
&= \sum_{k=1}^{m} \lambda_k \frac{e^{-({}^k r)T}}{\sqrt{T \sum_{j=1}^{3} ({}^k\sigma^{(ij)})^2}} \int_{\log(K) - f_k({}^k S_0^{(i)})}^{\infty} (e^{f_k({}^k S_0^{(i)}) + x} - K) e^{-x^2/2T\sum_{j=1}^{3}({}^k\sigma^{(ij)})^2} dx \\
&= \sum_{k=1}^{m} \lambda_k e^{-({}^k r)T} \left(\frac{e^{f_k({}^k S_0^{(i)})} e^{v_k^2/2}}{\sqrt{2\pi v_k^2}} \int_{\log(K) - f_k({}^k S_0^{(i)})}^{\infty} e^{-(x - v_k^2)^2/2v_k^2} dx - K\Phi\left(\frac{f_k({}^k S_0^{(i)}) - \log(K)}{\sqrt{T\sum_{j=1}^{3}({}^k\sigma^{(ij)})^2}} \right) \right) \\
&= \sum_{k=1}^{m} \lambda_k e^{-({}^k r)T} \left(e^{f_k(S_0^{(i)})} e^{v_k^2/2} \Phi\left(\frac{f_k({}^k S_0^{(i)}) - \log(K)}{v_k} + v_k \right) - K\Phi\left(\frac{f_k({}^k S_0^{(i)}) - \log(K)}{v_k} \right) \right)
\end{aligned} \tag{5.50}
$$

where

$$
v_k = T \sum_{j=1}^{3} \left({}^k\sigma^{(ij)} \right)^2 \tag{5.51}
$$

and

$$
f_k({}^k S_0^{(i)}) = \log(S_0^{(i)}) + \left(-\frac{1}{2} \sum_{j=1}^{3} \left({}^k\sigma^{(ij)} \right)^2 + {}^k r \right) T. \tag{5.52}
$$

The code is provided below

```
# price of big agent
vector_of_strikes <- c(strike_msft, strike_orcl, strike_goog)
list_prices_big_agent <- list()
vector_prices_big_agent <- c()
j <- 0
p_big_agent <- function(s_0,sigma_agent,t, K, r){
  c1 <- dot_product(sigma_agent, sigma_agent)
  f <- log(s_0) + ((-c1/2) + (r/100))*(t/365)
  first_term <- exp(-(r/100)*(t/365))*exp(f)*exp((t/365)*c1*0.5)*
    pnorm( ((f - log(K))/sqrt((t/365)*c1)) + sqrt((t/365)*c1))
  second_term <- exp(-(r/100)*(t/365))*K*pnorm(((f - log(K))/sqrt((t/365)*c1)))
  price_formula <- first_term - second_term
  return(price_formula)
}
for (stock  in list_of_stocks){
  j <- j + 1
  for (i in c(0: ( length(t_bill_1year_daily) - start_date_bond_analysis))){
    t_maturity <-
      as.numeric(difftime(maturity_1t_bill,
           index(t_bill_1year_daily[i + start_date_bond_analysis]), units = "days"))
    if (i > t_forecast){
      vector_prices_big_agent <- c(vector_prices_big_agent,
          p_big_agent(as.numeric(stock_step_ahead[[j]][i - t_forecast]),
               list_sigma[[1+ i]][j,], t_maturity, vector_of_strikes[j],
             as.numeric(t_bill_1year_daily[start_date_bond_analysis + t_forecast])) )
    }
    else {
      vector_prices_big_agent <- c(vector_prices_big_agent,
            p_big_agent(as.numeric(stock[start_date_bond_analysis + i]),
              list_sigma[[1+i]][j,], t_maturity,
```

```
                    vector_of_strikes[j],
                  as.numeric(t_bill_1year_daily[start_date_bond_analysis + i])) )
    }

  }
  list_prices_big_agent[[j]] <- vector_prices_big_agent
  vector_prices_big_agent <- c()
}
```

5.2 Estimation of parameters for small agents

In the application of the theory just described, we can assume that the retail investors are interested in 2 stocks of a set of 3 correlated stocks. In other words, for each α-stereotyped agent, $\mathcal{V}_\alpha = \mathbb{R}^2 \times \{0\}$. This hypothesis makes sense since small investors cannot keep track and trade many stocks at the same time. In addition to that, and since we want to forecast prices within short periods of time, we also assume that the processes u_t in $\mathcal{U}_\alpha$ are deterministic. The idea behind this assumption is based on how the stock option models are applied in the financial markets: the random variables of a model such as interest rates, volatility, etc., are upgraded every day and considered as constant inputs in the model to price the option.

Estimate of $\hat{\sigma}^\alpha$ and $\hat{\mu}^\alpha$

We assume the estimation $\hat{\sigma}^\alpha$ and $\hat{\mu}^\alpha$ comes from the solution to the linear equations 5.27.

5.2.1 Estimate of $\hat{\eta}^\alpha$

The estimation of η^α is straightforward using the formula $\hat{\eta}^\alpha = (\hat{\sigma}^\alpha)^{-1}(\hat{\mu}^\alpha - \hat{r}J_{n,1})$. The code is provided below

```
# Eta estimation
j <- 0
list_eta_stocks <-list()
length_mu <- length(list_mu[[1]])
for (i in c(0:(length(MSFT_Close)-start_date_time_s_analysis))){
    list_eta_stocks[[i+1]] <-
      ginv(list_sigma[[i+1]]) %*%(list_mu[[i+1]] -
                   rep(t_bill_1year_daily[start_date_time_s_analysis +i],length_mu))
}
```

5.2.2 Estimate of $\hat{\varrho}^\alpha$ and λ^α for the bond model

To estimate the ϱ^α, we will use 1-year treasury bills. The formula used in practice [22] to compute the price of a 1-year treasury bill with expiration date T; time (in days) that has passed since quote date t; and interest rate r is given by

$$b_{t,T} = \left[1 - \left(\frac{T-t}{360}\right) r\right]. \tag{5.53}$$

[22] https://www.nasdaq.com/articles/how-calculate-price-treasury-bills-2016-01-05

Now, to estimate ϱ^α, we will use Lagrange multipliers together with the maximum likelihood approach as with $\hat{\sigma}$. We start by describing the dynamics, and likelihood function of the zero-coupon bond $B^\alpha_{t,T}$. We will assume $B^\alpha_{t,T}$ is the same for each agent $\alpha \in \mathcal{A}$. To estimate the parameters using maximum likelihood method on transition functions, we will assume the parameters $r^\alpha, \varrho^\alpha, \eta^\alpha$ and σ^α as constants. Then, using Ito's formula, and $dB^\alpha_{t,T} = B^\alpha_{t,T}[(r^\alpha_t + (\varrho^\alpha_t)^*\sigma^\alpha_t\eta^\alpha_t)dt + (\varrho^\alpha_t)^*\sigma^\alpha_t dW_t]$. we get

$$d\ln(B^\alpha_{t,T}) = (r^\alpha + (\varrho^\alpha)^*\sigma^\alpha\eta^\alpha)dt - \frac{1}{2}((\varrho^\alpha)^*\sigma^\alpha)((\varrho^\alpha)^*\sigma^\alpha)^*dt + (\varrho^\alpha)^*\sigma^\alpha dW_t. \tag{5.54}$$

Thus,

$$B^\alpha_{t,T} = B^\alpha_{0,T}\exp\left\{\left(r^\alpha + (\varrho^\alpha)^*\sigma^\alpha\eta^\alpha - \frac{1}{2}((\varrho^\alpha)^*\sigma^\alpha)^*((\varrho^\alpha)^*\sigma^\alpha)\right)t + (\varrho^\alpha)^*\sigma^\alpha W_t\right\}. \tag{5.55}$$

Following the same ideas to construct likelihood function as in 5.18, we have that the transition functions of the zero-coupon bond are given by

$$\begin{aligned}
&P_{t_n - t_{n-1}}(b^\alpha_{t_{n-1}}, b^\alpha_{t_n}) \\
&= \frac{1}{b_{t_n}\sqrt{2\pi\|(\varrho^\alpha)^*\sigma^\alpha\|^2\Delta t}}\exp\left\{\frac{-\left(\log(b^\alpha_{t_n}) - \log(b^\alpha_{t_{n-1}}) - \left[r^\alpha + (\varrho^\alpha)^*\sigma^\alpha\eta^\alpha - \frac{1}{2}\|(\varrho^\alpha)^*\sigma^\alpha\|^2\right]\Delta t\right)}{2\Delta t\|(\varrho^\alpha)^*\sigma^\alpha\|^2}\right\}.
\end{aligned} \tag{5.56}$$

We make some transformations on the transition probabilities to compute the maximum likelihood estimators. Let λ be a function defined as $\lambda(\varrho^\alpha) = \sigma^\alpha\varrho^\alpha$. We can see that λ is a homeomorphism of $\{(\varrho^{\alpha,(1)}, ..., \varrho^{\alpha,(n)}) : \varrho^{\alpha,(1)}, ..., \varrho^{\alpha,(n)} \in \mathbb{R}\}$ onto itself. Therefore, by using this transformation, our likelihood function can be written as

$$\begin{aligned}
&f(\lambda^{\alpha,(1)}, ..., \lambda^{\alpha,(n)}; b^\alpha_{t_1,T}, ..., b^\alpha_{t_M,T}) \\
&= \prod_{k=1}^{M}\frac{1}{b_{t_k,T}\sqrt{2\pi\|\lambda^\alpha\|^2\Delta t}}\exp\left\{\frac{-\left(\log(b^\alpha_{t_k,T}) - \log(b^\alpha_{t_{k-1},T}) - \left[r^\alpha + (\lambda^\alpha)^*\eta^\alpha - \frac{1}{2}\|\lambda^\alpha\|^2\right]\Delta t\right)^2}{2\Delta t\|\lambda^\alpha\|^2}\right\}.
\end{aligned} \tag{5.57}$$

After some computations, we get

$$\begin{aligned}
&\frac{\partial \log f(\lambda^{\alpha,(1)}, ..., \lambda^{\alpha,(n)}; b^\alpha_{t_1,T}, ..., b^\alpha_{t_M,T})}{\partial\lambda^{\alpha,(i)}} \\
&= \frac{-M}{2}\left[\frac{2\lambda^{\alpha,(i)}\Delta t}{\Delta t\|\lambda^\alpha\|^2}\right] + \frac{4\lambda^{\alpha,(i)}\Delta t}{(2(\lambda^\alpha)^*\lambda^\alpha\Delta t)^2}\sum_{k=1}^{M}\left(\log(b^\alpha_{t_k,T}) - \log(b^\alpha_{t_{k-1},T}) - \left[r^\alpha + (\lambda^\alpha)^*\eta^\alpha - \frac{1}{2}\|\lambda^\alpha\|^2\right]\Delta t\right)^2 \\
&\quad + \frac{2(\eta^{\alpha,(i)}\Delta t - \lambda^{\alpha,(i)}\Delta t)\sum_{k=1}^{M}\left(\log(b^\alpha_{t_k,T}) - \log(b^\alpha_{t_{k-1},T}) - \left[r^\alpha + (\lambda^\alpha)^*\eta^\alpha - \frac{1}{2}\|\lambda^\alpha\|^2\right]\Delta t\right)}{2\Delta t\|\lambda^\alpha\|^2} \\
&= \lambda^{\alpha,(i)}g_1(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}) + \eta^{\alpha,(i)}g_2(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}),
\end{aligned} \tag{5.58}$$

where

$$
\begin{aligned}
&g_1(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}) \\
&= \frac{-M}{\|\lambda^\alpha\|^2} + \frac{1}{\|\lambda^\alpha\|^4 \Delta t} \sum_{k=1}^{M} \left(\log(b^\alpha_{t_k,T}) - \log(b^\alpha_{t_{k-1},T}) - \left[r^\alpha + (\lambda^\alpha)^* \eta^\alpha - \frac{1}{2}\|\lambda^\alpha\|^2 \right] \Delta t \right)^2 \\
&- \frac{\sum_{k=1}^{M} \left(\log(b^\alpha_{t_k,T}) - \log(b^\alpha_{t_{k-1},T}) - \left[r^\alpha + (\lambda^\alpha)^* \eta^\alpha - \frac{1}{2}\|\lambda^\alpha\|^2 \right] \Delta t \right)}{\|\lambda^\alpha\|^2},
\end{aligned} \tag{5.59}
$$

and

$$
\begin{aligned}
&g_2(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}) \\
&= \frac{\sum_{k=1}^{M} \left(\log(b^\alpha_{t_k,T}) - \log(b^\alpha_{t_{k-1},T}) - \left[r^\alpha + (\lambda^\alpha)^* \eta^\alpha - \frac{1}{2}\|\lambda^\alpha\|^2 \right] \Delta t \right)}{\|\lambda^\alpha\|^2}.
\end{aligned} \tag{5.60}
$$

However, we need some restriction on λ^α to get a solution. We begin by making the expected values equal to the past values of the bond.[23] Assume that $b_{t,T}$ is the market value of the zero-coupon bond, then by solving the equation

$$
E\left[B^\alpha_{0,T} \exp\left\{ \left(r^\alpha + (\varrho^\alpha)^* \sigma^\alpha \eta^\alpha - \frac{1}{2}((\varrho^\alpha)^* \sigma^\alpha)^*((\varrho^\alpha)^* \sigma^\alpha) \right) t + ((\varrho^\alpha)^* \sigma^\alpha)^* W_t \right\} \right] = b_{t,T}, \tag{5.61}
$$

and using the expected value formula for log-normal distributions, we get

$$
B^\alpha_{0,T} \exp\left\{ (r^\alpha + (\lambda^\alpha)^* \eta^\alpha) t - \frac{1}{2}\|\lambda^\alpha\|^2 t + \frac{1}{2}\|\lambda^\alpha\|^2 t \right\} = b_{t,T}. \tag{5.62}
$$

Solving this equation for $(\lambda^\alpha)^* \lambda^\alpha$, we obtain

$$
(\lambda^\alpha)^* \lambda^\alpha = \frac{\left(\log\left(b_{t,T} / B^\alpha_{0,T} \right) / t - r^\alpha \right)^2}{(\eta^\alpha)^* \eta^\alpha}. \tag{5.63}
$$

Now, we will use Lagrange multipliers with target function 5.57 and restriction 5.63, this lead us to

$$
\mathcal{L}(\lambda^\alpha, k) = \log f(\lambda^{\alpha,(1)}, ..., \lambda^{\alpha,(n)}; b^\alpha_{t_1,T}, ..., b^\alpha_{t_M,T}) - k \left(\|\lambda^\alpha\|^2 - \frac{\left(\log\left(b_{t,T} / B^\alpha_{0,T} \right) / t - r^\alpha \right)^2}{(\eta^\alpha)^* \eta^\alpha} \right). \tag{5.64}
$$

Using 5.58, we get that the conditions of the first order conditions of the Lagrangian function are of the form

$$
\begin{aligned}
&\frac{\partial \log \mathcal{L}(\lambda^\alpha, k)}{\partial \lambda^{\alpha,(i)}} \\
&= \lambda^{\alpha,(i)} g_1(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}) + \eta^{\alpha,(i)} g_2(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}) - 2k\lambda^{\alpha,(i)} \\
&= 0
\end{aligned} \tag{5.65}
$$

[23] This method is similar to the implied volatility method used in the Black-Scholes-Merton formula.

for $i = 1, 2, 3.$, and

$$\frac{\partial \mathcal{L}(\lambda^\alpha, k)}{\partial k} = -\left(\|\lambda^\alpha\|^2 - \frac{\left(\log\left(b_{t,T}/B^\alpha_{0,T}\right)/t - r^\alpha\right)^2}{(\eta^\alpha)^*\eta^\alpha} \right) = 0. \tag{5.66}$$

From equation 5.65, we get that

$$\frac{(\lambda^{\alpha,(i)})^2}{(\eta^{\alpha,(i)})^2} = \frac{\left(g_2(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T})\right)^2}{\left(g_1(\Delta t, r^\alpha, \eta^\alpha, \|\lambda^\alpha\|, b_{t_1,T}, ..., b_{t_M,T}) - 2k\right)^2}. \tag{5.67}$$

Since the right-hand side of the equation does not depend on i, we get that

$$\frac{(\lambda^{\alpha,(i)})^2}{(\eta^{\alpha,(i)})^2} = \frac{(\lambda^{\alpha,(j)})^2}{(\eta^{\alpha,(j)})^2} \tag{5.68}$$

for any $i, j = 1, 2, 3$. Now, using the previous equation, we get

$$\sum_{i=1}^{3} (\lambda^{\alpha,(i)})^2 = \frac{(\lambda^{\alpha,(j)})^2}{(\eta^{\alpha,(j)})^2} \sum_{i=1}^{3} (\eta^{\alpha,(i)})^2 \tag{5.69}$$

for $j = 1, 2, 3$. By the use of the condition 5.66, these equations can be rewritten as

$$\frac{\left(\log\left(b_{t,T}/B^\alpha_{0,T}\right)/t - r^\alpha\right)^2}{(\eta^\alpha)^*\eta^\alpha} = \frac{(\lambda^{\alpha,(j)})^2}{(\eta^{\alpha,(j)})^2} \sum_{i=1}^{3} (\eta^{\alpha,(i)})^2. \tag{5.70}$$

Therefore,

$$\hat{\lambda}^{\alpha,(j)} = \frac{\left(\log\left(b_{t,T}/B^\alpha_{0,T}\right)/t - r^\alpha\right)\hat{\eta}^{\alpha,(j)}}{\|\hat{\eta}^\alpha\|^2}, \qquad \hat{\varrho}^{\alpha,(j)} = (\hat{\sigma}^{-1}\hat{\lambda}^\alpha)_{(j)} \qquad \text{for} \qquad j = 1, 2, 3. \tag{5.71}$$

The code for $\hat{\lambda}^{\alpha,(j)}$ is the following

```
# Estimation bond price

# start date maximum likelihood estimation bond
start_date_maximum_likelihood_bond <-
  min(which(format(index(t_bill_1year_daily), "%Y-%m-%d") == '2016-01-06'))

# start date 1-Year Treasure Bill analysis
start_date_bond_analysis <-
  min(which(format(index(t_bill_1year_daily), "%Y-%m-%d") == start_time_series_date))

# maturity bond
maturity_1t_bill <-as.Date("2019-10-18", tz = "UTC")

# How to price Treasury Bills
# https://www.nasdaq.com/articles/how-calculate-price-treasury-bills-2016-01-05
bond_price_function<-  function(r){
  a <- as.numeric(difftime(maturity_1t_bill,index(r), units = "days"))
  decimal_rate <- as.numeric(r)/100
```

```
  b <- (a* decimal_rate)/365
  return( 1- b )
}

# Real 1 year T-Bill prices
bond_prices <- lapply(t_bill_1year_daily,bond_price_function)
bond_prices <- bond_prices$DTB1YR


# restrictions of the maximum likelihood estimation of /rho for bonds
product <- function(x){
  a <- sum(x*x)
  return(a)
}

t <- 0
numerator_formula <- 0
denominator_formula <- 0
log_b_B <- 0
eta_inner_product <- lapply(list_eta_stocks, product)
list_bond_lambdas <- list()
for (i in c(0: ( length(t_bill_1year_daily) - start_date_bond_analysis))){
  t <- as.numeric(difftime(maturity_1t_bill,
      index(t_bill_1year_daily[i + start_date_bond_analysis]), units = "days"))
  log_b_B <- log(bond_prices[i + start_date_bond_analysis]/
                   bond_prices[start_date_bond_analysis])
  numerator_formula <- (log_b_B/(t/365) -
           (as.numeric(t_bill_1year_daily[i+start_date_bond_analysis]/100)) )*
                                               list_eta_stocks[[i+1]]
  denominator_formula <-  eta_inner_product[[i+1]]
  list_bond_lambdas[[i+1]] <- numerator_formula/denominator_formula
}
```

Now, by the use of the definition of the homeomorphism

$$\varrho^{\alpha,(j)} = (\sigma^{-1}\lambda^{\alpha})_{(j)} \tag{5.72}$$

for $j = 1, 2, 3.$, we get

```
# list of var_rho parameters of the bond model
list_var_rho <- list()
for (i in c(0:(length(t_bill_1year_daily)-start_date_bond_analysis))){
  list_var_rho[[i+1]] <- ginv(list_sigma[[i+1]])%*%list_bond_lambdas[[i+1]]
}
```

5.2.3 Estimate of $\hat{\hat{\varrho}}$

Since ϱ^{α} must lie in $\mathcal{V}_{\alpha} = \mathbb{R}^2 \times \{0\}$, we must ensure that any estimation of $\hat{\varrho}^{\alpha}$ must lie in $\mathcal{V}_{\alpha}$ too. Given the fact that the stocks, and the bond are close to maturity, it is expected that the values of $\hat{\varrho}$ and $\hat{\lambda}^{\alpha}$ should be close to zero. Therefore, we should use the estimator $\hat{\hat{\varrho}} = \pi_{1,2}(\hat{\varrho}^{\alpha})$ where $\pi_{1,2} : \mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$ is defined by $\pi_{1,2}(x, y, z) = (x, y, 0)$. We provide the code for the estimation of $\hat{\hat{\varrho}}$ together with the error of estimation of the bond prices under the estimation $\hat{\hat{\varrho}}$

```
# Bond price estimation using the SDE
a<- 0 # variable used to build formula_bond_estimation variable
formula_estimation_bond <- 0
```

```
# abs value of difference between formula_estimation_bond and bond_prices
error_bond_estimation <- list()
# estimation of bond prices using projection_var_rho instead of list_var_rho
list_estimation_bond <- list()
# estimation of list_var_rho using its projection on R^{2} \times {0}
projection_var_rho <- list()
for (i in c(0:(length(t_bill_1year_daily)-start_date_bond_analysis))){
  t <-
    as.numeric(difftime(maturity_1t_bill,
                        index(t_bill_1year_daily[i + start_date_bond_analysis]),
                        units = "days"))
  projection_var_rho[[i+1]] <- c(list_var_rho[[i+1]][1],list_var_rho[[i+1]][2], 0)
  a <-( t(list_sigma[[i+1]]%*%projection_var_rho[[i+1]])%*%list_eta_stocks[[i+1]])
  a <- (a + as.numeric(t_bill_1year_daily[i+start_date_bond_analysis]/100))*(t/365)
  formula_estimation_bond <- exp(a)*bond_prices[start_date_bond_analysis]
  list_estimation_bond[[i+1]] <- formula_estimation_bond
  error_bond_estimation[[i+1]] <- abs(list_estimation_bond[[i+1]] -
                                      bond_prices[i + start_date_bond_analysis])
}
```

The errors of the estimation are the following

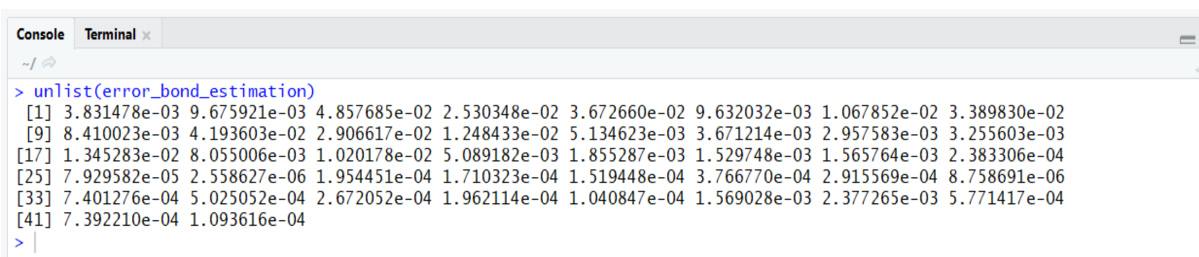

```
Console  Terminal
~/
> unlist(error_bond_estimation)
 [1] 3.831478e-03 9.675921e-03 4.857685e-02 2.530348e-02 3.672660e-02 9.632032e-03 1.067852e-02 3.389830e-02
 [9] 8.410023e-03 4.193603e-02 2.906617e-02 1.248433e-02 5.134623e-03 3.671214e-03 2.957583e-03 3.255603e-03
[17] 1.345283e-02 8.055006e-03 1.020178e-02 5.089182e-03 1.855287e-03 1.529748e-03 1.565764e-03 2.383306e-04
[25] 7.929582e-05 2.558627e-06 1.954451e-04 1.710323e-04 1.519448e-04 3.766770e-04 2.915569e-04 8.758691e-06
[33] 7.401276e-04 5.025052e-04 2.672052e-04 1.962114e-04 1.040847e-04 1.569028e-03 2.377265e-03 5.771417e-04
[41] 7.392210e-04 1.093616e-04
>
```

Figure 11: Bond estimation error using $\hat{\hat{\varrho}}^{\alpha}$

5.2.4 Estimate of $\hat{\rho}^{\alpha}$

The estimation of ρ^{α} comes from the formula $\hat{\rho}^{\alpha} = (\hat{\sigma}^{\alpha})^{-1}(\hat{\mu}^{\alpha} - \hat{r}^{\alpha} J_{n\times 1} - \hat{\sigma}^{\alpha}(\hat{\sigma}^{\alpha})^{*}\hat{\hat{\varrho}}^{\alpha})$.

```
# Estimation \rho^{\alpha}
i <- 0
list_rho <- list()
for (i in c(0:(length(MSFT_Close)-start_date_time_s_analysis))){
  a <- list_sigma[[i+1]]%*%projection_var_rho[[i+1]]
  list_rho[[i+1]] <- list_eta_stocks[[i+1]] - a
}
```

5.2.5 Estimate of u

Since ρ^α and u^α are deterministic functions, we have

$$\begin{aligned} V_t^{\alpha,C,u} &= E_{Q^{\rho^\alpha,u}}\left[C - \frac{1}{2\gamma_\alpha}\int_t^T \|\rho_s^{\alpha,u}\|^2 ds \middle| \mathcal{F}_t \right] \\ &= \frac{1}{2\gamma_\alpha}\int_0^t \|\rho_s^{\alpha,u}\|^2 ds + E_{Q^{\rho^\alpha,u}}\left[C - \frac{1}{2\gamma_\alpha}\int_0^T \|\rho_s^{\alpha,u}\|^2 ds \middle| \mathcal{F}_t \right] \\ &= V_0^{\alpha,C,u} + \frac{1}{2\gamma_\alpha}\int_0^t \|\rho_s^{\alpha,u}\|^2 ds + \int_0^t (\pi_s^{\alpha,u})^* \sigma_s^\alpha \cdot dW_s^{\alpha.u}, \end{aligned} \tag{5.73}$$

and by the representation theorem of martingales (see for instance Revuz and Yor, 1999, Section V.4)

$$E_{Q^{\rho^\alpha,u}}\left[C | \mathcal{F}_t \right] = E_{Q^{\rho^\alpha,u}}\left[C \right] + \int_0^t (\pi_s^{\alpha,u})^* \sigma_s^\alpha \cdot dW_s^{\alpha.u}. \tag{5.74}$$

In the case of a vanilla option i.e. options whose payoffs depends on the terminal value of the underlying asset we have

$$C = f\left(S_T\right) = \left(f \circ g\right)\left(W_T^{\alpha,u}\right), \tag{5.75}$$

and as a result

$$E_{Q^{\rho^\alpha,u}}\left[\left(f \circ g\right)\left(W_T^{\alpha,u}\right) \middle| \mathcal{F}_t \right] = E_{Q^{\rho^\alpha,u}}\left[\left(f \circ g\right)\left(W_T^{\alpha,u}\right) \right] + \int_0^t (\pi_s^{\alpha,u})^* \sigma_s^\alpha \cdot dW_s^{\alpha.u}. \tag{5.76}$$

for some measurable functions f and g.

Let us assume that f and g are $\mathcal{C}^2$, and that $(P_t)_{t=0}^\infty$ is the transition probabilities of the $Q^{\rho^\alpha,u}$-Brownian motion $(W_t^{\alpha,u})_{t=0}^T$, then using the definition of Markov property via transition probabilities, and Ito's formula, we get

$$\begin{aligned} E_{Q^{\rho^\alpha,u}}\left[\left(f \circ g\right)\left(W_T^{\alpha,u}\right) \middle| \mathcal{F}_t \right] &= P_{T-t}(f \circ g)\left(W_t^{\alpha,u}\right) \\ &= P_T(f \circ g)\left(W_0^{\alpha,u}\right) + \sum_{j=1}^n \int_0^t \frac{\partial}{\partial x_j} P_{T-l}\left(f \circ g\right)\left(W_l^{\alpha,u}\right) dW_l^{\alpha,u,(j)}, \end{aligned} \tag{5.77}$$

where the other terms accompanying dt are zero because $E_{Q^{\rho^\alpha,u}}\left[\left(f \circ g\right)\left(W_T^{\alpha,u}\right) \middle| \mathcal{F}_t \right]$ is a $Q^{\rho^\alpha,u}$-martingale. By using 5.76 and 5.77 we have that

$$((\pi_t^{\alpha,u})^* \sigma_t^\alpha)^{(j)} = \frac{\partial}{\partial x_j} P_{T-l}\left(f \circ g\right)\left(W_l^{\alpha,u}\right) \tag{5.78}$$

for each $j = 1, ..., n$ with

$$(\pi_t^{\alpha,u})^* \sigma_t^\alpha = \left[((\pi_t^{\alpha,u})^* \sigma_t^\alpha)^{(1)} \quad \ldots \quad ((\pi_t^{\alpha,u})^* \sigma_t^\alpha)^{(n)} \right]. \tag{5.79}$$

Since we want to estimate the u over short periods of time, we will consider it constant to be estimated. Following these ideas, the dynamics of $S^{(i)}$ can be rewritten as

$$
\begin{aligned}
dS_t^{(i)} &= \mu^{(i)} S_t^{(i)} dt + S_t^{(i)} \left(\sum_{j=1}^{n} \sigma^{(ij)} dW_t^{(j)} \right) \\
&= S_t^{(i)} \left[\mu^{(i)} dt + \sum_{j=1}^{n} \sigma^{(ij)} \left(dW_t^{\alpha,u,(j)} - \rho^{\alpha,u,(j)} dt \right) \right].
\end{aligned} \tag{5.80}
$$

Therefore, we get

$$
S_t^{(i)} = S_0^{(i)} \exp\left\{ t \left(\mu^{(i)} - \sum_{j=1}^{n} \sigma^{(ij)} \rho^{\alpha,u,(j)} - \frac{1}{2} \sum_{j=1}^{n} (\sigma^{(ij)})^2 \right) + \sum_{j=1}^{n} \sigma^{(ij)} W_t^{\alpha,u,(j)} \right\}. \tag{5.81}
$$

Let us use the function $g_t(\cdot)$ of class $\mathcal{C}^{1,2}$ as

$$
g_s(W_l^{\alpha,u}) = S_0^{(i)} \exp\left\{ s \left(\mu^{(i)} - \sum_{j=1}^{n} \sigma^{(ij)} \rho^{\alpha,u,(j)} - \frac{1}{2} \sum_{j=1}^{n} (\sigma^{(ij)})^2 \right) + \sum_{j=1}^{n} \sigma^{(ij)} W_l^{\alpha,u,(j)} \right\}, \tag{5.82}
$$

where $g_t(W_t^{\alpha,u}) = S_t^{(i)}$. Thus, by the previous notation we have

$$
\begin{aligned}
&P_{T-t}(f \circ g_T)\left(W_t^{\alpha,u}\right) \\
&= E_{Q^{\rho^{\alpha,u}}} \left[(f \circ g_T)\left(W_T^{\alpha,u}\right) \middle| \mathcal{F}_t \right] \\
&= E_{Q^{\rho^{\alpha,u}}} \left[f \left(S_0^{(i)} \exp\left\{ T \left(\mu^{(i)} - \sum_{j=1}^{n} \sigma^{(ij)} \rho^{\alpha,u,(j)} - \frac{1}{2} \sum_{j=1}^{n} (\sigma^{(ij)})^2 \right) + \sum_{j=1}^{n} \sigma^{(ij)} dW_T^{\alpha,u,(j)} \right\} \right) \middle| \mathcal{F}_t \right] \\
&= E_{Q^{\rho^{\alpha,u}}} \left[f \left(\frac{g_T(W_T^{\alpha,u})}{g_t(W_t^{\alpha,u})} g_t(x) \right) \right]_{x = W_t^{\alpha,u}}.
\end{aligned} \tag{5.83}
$$

By using chain rule and 5.82, we get

$$
\begin{aligned}
\frac{\partial}{\partial x_j} E_{Q^{\rho^{\alpha,u}}} \left[f \left(\frac{g_T(W_T^{\alpha,u})}{g_t(W_t^{\alpha,u})} g_t(x) \right) \right]_{x = W_t^{\alpha,u}} &= \left(\frac{\partial g_t(x)}{\partial x_j} \right) \Bigg|_{x = W_t^{\alpha,u}} \frac{d}{dy} E_{Q^{\rho^{\alpha,u}}} \left[f \left(\frac{g_T(W_T^{\alpha,u})}{g_t(W_t^{\alpha,u})} y \right) \right]_{y = g_t(W_t^{\alpha,u})} \\
&= \sigma^{(ij)} S_t^{(i)} \frac{d}{dy} E_{Q^{\rho^{\alpha,u}}} \left[f \left(\frac{g_T(W_T^{\alpha,u})}{g_t(W_t^{\alpha,u})} y \right) \right]_{y = g_t(W_t^{\alpha,u})}.
\end{aligned} \tag{5.84}
$$

Now we are ready for the following lemma

Lemma 5.2. *Let $f(z) = (z - K)^+$. Then*

$$
((\pi_t^{\alpha,u})^* \sigma_t^{\alpha})^{(j)} = \sigma^{(ij)} S_t^{(i)} E_{Q^{\rho^{\alpha,u}}} \left[\frac{S_T^{(i)}}{S_t^{(i)}} 1_{[K,+\infty)} \left(\frac{S_T^{(i)}}{S_t^{(i)}} y \right) \right]_{y = g_t(W_t^{\alpha,u})}. \tag{5.85}
$$

Proof. By using approximations to $f(x) = (x-K)^+$ by $\mathcal{C}^2$ functions, and the relation

$$P_{T-t}(h \circ g_T)\,(W_t^{\alpha,u}) = E_{Q^{\rho^\alpha,u}}\left[h\left(\frac{g_T(W_T^{\alpha,u})}{g_t(W_t^{\alpha,u})}g_t(x)\right)\right]_{x=W_t^{\alpha,u}}. \tag{5.86}$$

with h of class $\mathcal{C}^2$ as in the equation 5.84 we get the result. $\square$

Thanks to Lemma 5.2 the problem of pricing can be solved by maximizing with respect to u the function $f_{\rho^\alpha,t}(z^{\alpha,u},u)$ given by

$$\begin{aligned}
f_{\rho^\alpha,t}(z^{\alpha,u},u) &= -\frac{1}{2\gamma_\alpha}\|\rho_t^{\alpha,u}\|^2 - (\rho_t^{\alpha,u})^* z^{\alpha,u} \\
&= -\frac{1}{2\gamma_\alpha}\|\rho_t^{\alpha,u}\|^2 - (\rho_t^{\alpha,u})^*((\pi_t^{\alpha,u})^*\sigma_t^\alpha) \\
&= -\frac{1}{2\gamma_\alpha}\|\rho_t^{\alpha,u}\|^2 - \sum_{j=1}^n(\rho_t^{\alpha,u})^{(j)}\sigma^{(ij)}S_t^{(i)}E_{Q^{\rho^\alpha,u}}\left[\frac{S_T^{(i)}}{S_t^{(i)}}1_{[K,+\infty)}\left(\frac{S_T^{(i)}}{S_t^{(i)}}y\right)\right]_{y=g_t(W_t^{\alpha,u})}.
\end{aligned} \tag{5.87}$$

In order to make the computations easier, we need to rewrite the previous formula to be applied in an algorithm. We know that

$$\begin{aligned}
&\sigma^{(ij)}S_t^{(i)}E_{Q^{\rho^\alpha,u}}\left[\frac{S_T^{(i)}}{S_t^{(i)}}1_{[K,+\infty)}\left(\frac{S_T^{(i)}}{S_t^{(i)}}y\right)\right]_{y=g_t(W_t^{\alpha,u})} \\
&= \sigma^{(ij)}S_t^{(i)}E_{Q^{\rho^\alpha,u}}\left[e^{h(W^{\alpha,u},t,T,\sigma,u,\alpha,\mu)}1_{[k,\infty)}\left(ye^{h(W^{\alpha,u},t,T,\sigma,u,\alpha,\mu)}\right)\right]_{y=g_t(W_t^{\alpha,u})},
\end{aligned} \tag{5.88}$$

where

$$\begin{aligned}
&h(W^{\alpha,u},t,T,\sigma,u,\alpha,\mu) \\
&= \exp\left\{\sum_{j=1}^n\sigma^{(ij)}(W_T^{\alpha,u,(j)} - W_t^{\alpha,u,(j)}) + \left(-\frac{1}{2}\sum_{j=1}^n(\sigma^{(ij)})^2 - \sum_{j=1}^n\sigma^{(ij)}\rho_t^{\alpha,u,(j)} + \mu^{(i)}\right)(T-t)\right\}.
\end{aligned} \tag{5.89}$$

Using the fact that $W^{\alpha,u}$ is an n-dimensional Brownian motion under $Q^{\rho^\alpha,u}$, we get that

$$E_{Q^{\rho^\alpha,u}}\left[\frac{S_T^{(i)}}{S_t^{(i)}}1_{[K,+\infty)}\left(\frac{S_T^{(i)}}{S_t^{(i)}}y\right)\right]_{y=g_t(W_t^{\alpha,u})} \tag{5.90}$$

can be written as

$$\frac{1}{\sqrt{2\pi(T-t)}}\int_{l(\sigma,u,t,T,S_t^{(i)},K,\alpha)}^{\infty} e^{-y^2/2(T-t)}e^{\left(\sum_{j=1}^n(\sigma^{(ij)})^2\right)^{1/2}y+\left(-\sum_{j=1}^n(\sigma^{(ij)})^2/2-\sum_{j=1}^n\sigma^{(ij)}\rho^{\alpha,u,(j)}+\mu^{(i)}\right)(T-t)}dy, \tag{5.91}$$

where

$$l(\sigma, u, t, T, S_t^{(i)}, K, \alpha) = \frac{\log(k/S_t^{(i)}) + \left(\frac{1}{2}\sum_{j=1}^n(\sigma^{(ij)})^2 + \sum_{j=1}^n \sigma^{(ij)}\rho_t^{\alpha,u,(j)} - \mu^{(i)}\right)(T-t)}{\sqrt{\sum_{j=1}^n(\sigma^{(ij)})^2}}. \quad (5.92)$$

Now, by the binomial formula, we get that 5.88 can be expressed as

$$\frac{\sigma^{(ij)}S_t^{(i)}e^{(-\sum_{j=1}^n \sigma^{(ij)}\rho^{\alpha,u,(j)}+\mu^{(i)})(T-t)}}{\sqrt{2\pi(T-t)}} \int_{l(\sigma,u,t,T,S_t^{(i)},K,\alpha)}^{\infty} e^{\frac{-\left(y-(T-t)\left(\sum_{j=1}^n(\sigma^{(ij)})^2\right)^{1/2}\right)^2}{2(T-t)}} dy. \quad (5.93)$$

By substitution $z = y/\sqrt{T-t}$, the previous formula can be rewritten as

$$\frac{\sigma^{(ij)}S_t^{(i)}e^{(-\sum_{j=1}^n \sigma^{(ij)}\rho^{\alpha,u,(j)}+\mu^{(i)})(T-t)}}{\sqrt{2\pi}} \int_{l(\sigma,u,t,T,S_t^{(i)},K,\alpha)/\sqrt{T-t}}^{\infty} e^{\frac{\left(y-\sqrt{(T-t)\sum_{j=1}^n(\sigma^{(ij)})^2}\right)^2}{2}} dy. \quad (5.94)$$

Using substitution $z = y - \sqrt{(T-t)\sum_{j=1}^n(\sigma^{(ij)})^2}$, the integral

$$\int_{l(\sigma,u,t,T,S_t^{(i)},K,\alpha)/\sqrt{T-t}}^{\infty} e^{\frac{\left(y-\sqrt{(T-t)\sum_{j=1}^n(\sigma^{(ij)})^2}\right)^2}{2}} dy \quad (5.95)$$

equals

$$\int_{\left(-\log(S_t^{(i)}/k)+\left(-\frac{1}{2}\sum_{j=1}^n(\sigma^{(ij)})^2+\sum_{j=1}^n \sigma^{(ij)}\rho_t^{\alpha,u,(j)}-\mu^{(i)}\right)(T-t)\right)/\sqrt{(T-t)\sum_{j=1}^n(\sigma^{(ij)})^2}}^{\infty} e^{\frac{y^2}{2}} dy. \quad (5.96)$$

Therefore, by symmetry property of the normal distribution, we get

$$E_{Q^{\rho^{\alpha,u}}}\left[\frac{S_T^{(i)}}{S_t^{(i)}} 1_{[K,+\infty)}\left(\frac{S_T^{(i)}}{S_t^{(i)}}y\right)\right]_{y=g_t(W_t^{\alpha,u})}$$
$$= e^{(-\sum_{j=1}^n \sigma^{(ij)}\rho^{\alpha,u,(j)}+\mu^{(i)})(T-t)}\Phi\left(\frac{\log(S_t^{(i)}/k) + \left(\frac{1}{2}\sum_{j=1}^n(\sigma^{(ij)})^2 - \sum_{j=1}^n \sigma^{(ij)}\rho_t^{\alpha,u,(j)} + \mu^{(i)}\right)(T-t)}{\sqrt{(T-t)\sum_{j=1}^n(\sigma^{(ij)})^2}}\right). \quad (5.97)$$

In the application of the theory just described, we can assume that the retail investors are interested in 2 stocks out of a set of 3 correlated stocks. In other words, for each α-stereotyped agent $\mathcal{V}_\alpha = \mathbb{R}^2 \times \{0\}$. This hypothesis makes sense since small investors cannot keep track and trade many stocks at the same time. In addition to that, and since we want to forecast prices within short periods of time, we also assume that the processes u_t in $\mathcal{U}_\alpha$ are deterministic. The idea behind this assumption is based on how the stock option models are applied in the financial markets: the random variables of a model such as interest rates, volatility, etc., are upgraded every day and considered as constant inputs in the model to price the option.

We can go a little bit further to write 5.97 into a easier formula. Following the hypotheses mentioned before, and using the definition of $\mathcal{U}_\alpha$, we get

$$\begin{aligned}\mathcal{U}_\alpha &= \{(\sigma^\alpha)^{-1}(x) : x \in \{0\} \times \{0\} \times \mathbb{R}\} \\ &= \left\{ (\sigma^\alpha)^{-1} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}.\end{aligned} \tag{5.98}$$

Now, using the definition $\rho^{\alpha,u} = \rho^\alpha + u$, the function $f_{\rho^\alpha,t}(z^{\alpha,u}, u)$ can be restated as follows

$$\begin{aligned}
&f_{\rho^\alpha,t}(z^{\alpha,u}, u) \\
&= -\frac{1}{2\gamma_\alpha}\|\rho^{\alpha,u}\|^2 - \sum_{j=1}^{n} (\rho^{\alpha,u})^{(j)} (\sigma^\alpha)^{(ij)} S_t^{(i)} E_{Q^{\rho^{\alpha,u}}} \left[\frac{S_T^{(i)}}{S_t^{(i)}} 1_{[K,+\infty)} \left(\frac{S_T^{(i)}}{S_t^{(i)}} y \right) \right]_{y=g_t(W_t^{\alpha,u})} \\
&= -\frac{1}{2\gamma_\alpha} \left\| \rho^\alpha + (\sigma^\alpha)^{-1} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right\|^2 \\
&- \sigma^\alpha \left(\rho^\alpha + (\sigma^\alpha)^{-1} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right)_{(i)} S_t^{(i)} \exp\left\{ \left(-\sum_{j=1}^{n} (\sigma^\alpha)^{(ij)} \rho^{\alpha,u,(j)} + \mu^{(i)} \right) (T-t) \right\} \Phi\left(\psi(\sigma^\alpha, u, t, T, S_t^{(i)}, K, \alpha) \right) \\
&= -\frac{1}{2\gamma_\alpha} \left\| \rho^\alpha + (\sigma^\alpha)^{-1} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right\|^2 \\
&- (\sigma^\alpha \rho^\alpha)_{(i)} S_t^{(i)} \exp\left\{ \left(-\sum_{j=1}^{n} (\sigma^\alpha)^{(ij)} \rho^{\alpha,(j)} + \mu^{(i)} \right) (T-t) \right\} \Phi\left(\psi(\sigma^\alpha, u, t, T, S_t^{(i)}, K, \alpha) \right),
\end{aligned} \tag{5.99}$$

where

$$\psi(\sigma, u, t, T, y, K, \alpha) = \frac{\log(y/k) + \left(\frac{1}{2} \sum_{j=1}^{n} (\sigma^{(ij)})^2 - \sum_{j=1}^{n} \sigma^{(ij)} \rho_t^{\alpha,(j)} + \mu^{(i)} \right) (T-t)}{\sqrt{(T-t) \sum_{j=1}^{n} (\sigma^{(ij)})^2}}. \tag{5.100}$$

Therefore,

$$\tilde{u} = \arg \sup_{u \in \tilde{\mathcal{V}}_\alpha^\perp} f_{\rho^\alpha,t}(z,u) = \arg \sup_{(0,0,z)^\top \in \mathbb{R}^3} - \left\| \rho^\alpha + (\sigma^\alpha)^{-1} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right\|^2 = \arg \inf_{(0,0,z)^\top \in \mathbb{R}^3} \left\| \rho^\alpha + (\sigma^\alpha)^{-1} \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \right\|^2. \tag{5.101}$$

```
# Estimation hat u
list_hat_u <- list()
list_optimization_u <- list() # list of the arg and target value of the opt process of u
values <- c()
for (j in c(0:(length(MSFT_Close)-start_date_time_s_analysis))){
  values <- c()
  target_func_opt_u <- function(z){
    a <- list_rho[[j+1]] + (ginv(list_sigma[[j+1]])%*%c(0,0,z))
    return(product(a))
  }
  values <- c(optimize(target_func_opt_u, c(-10000,10000) )$minimum,
              optimize(target_func_opt_u, c(-10000,10000) )$objective)
  names(values) <- c("value min", "value function")
  list_optimization_u[[j+1]] <- values
  list_hat_u[[j+1]] <- c(0,0,as.numeric(values[1]))
}
```

5.2.6 Estimation of the price under a small agent perspective

In this part we will generate a more practical formula to compute the price of call options for any $\alpha-$stereotyped agent. First, we use the formula for small agents developed in Proposition 3.14 together with the equation 5.97. Next, we use the estimation of u as in 5.101. Finally, we connect these results to create the equation we will need to price call options using R code.

We start by explaining what is the role of u in 5.101 in the process of pricing $p^\alpha(v_0, C)$. As we observed earlier, the formula of the price of a derivative C from a $\alpha-$stereotyped agent perspective is given by the formula

$$p^\alpha(v_0, C) = B^\alpha_{0,T} \left(\sup_{Q^{\rho^\alpha,u} \sim P^\alpha} \left\{ E_Q[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u} | P^\alpha) \right\} - \sup_{Q^{\rho^\alpha,u} \sim P^\alpha} \left\{ -\frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u} | P^\alpha) \right\} \right). \tag{5.102}$$

By the use of the comparison theorem for BSDEs (see for instance El Karoui and Rouge, 2000, Theorem B.2.), and the fact that every expected value $E_{Q^{\rho^\alpha,u}}[X]$, can be written in the form

$$E_{P^\alpha}\left[X \frac{dQ^{\rho^\alpha,u}}{dP^\alpha} \right], \tag{5.103}$$

the maximization problem

$$\sup_{Q^{\rho^\alpha,u} \sim P^\alpha} \left\{ E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u} | P^\alpha) \right\} \tag{5.104}$$

is equivalent to the maximization of the driver $f_{\rho^\alpha,t}(z_t,u_t) = -1/2\gamma_\alpha \|\rho_t^{\alpha,u}\|^2 - (\rho_t^{\alpha,u})^* z$ of the dynamics $-dV_t^{\alpha,C,u} = f_{\rho^\alpha,t}(z_t^{\alpha,u},u_t)dt - (z_t^{\alpha,u})^* dW_t$ of the process

$$V_t^{\alpha,C,u} = E_{Q^{\rho^\alpha,u}}\left[C - \frac{1}{2\gamma_\alpha}\int_t^T \|\rho_t^{\alpha,u}\|^2 dt \middle| \mathcal{F}_t \right] \tag{5.105}$$

with initial value satisfying

$$V_0^{\alpha,C,u} = E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u}|P^\alpha). \tag{5.106}$$

Therefore using the expression 5.97, the definition of $\tilde{u}$ as in 5.101, and the fact that the estimators of all the parameters of the models are constants, we get

$$\begin{aligned} \sup_{Q^{\rho^\alpha,u}\sim P^\alpha} \left\{ E_{Q^{\rho^\alpha,u}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,u}|P^\alpha) \right\} &= E_{Q^{\rho^\alpha,\tilde{u}}}[C] - \frac{1}{\gamma_\alpha} h(Q^{\rho^\alpha,\tilde{u}}|P^\alpha) \\ &= E_{Q^{\rho^\alpha,\tilde{u}}}\left[\left(S_T^{(i)} - K\right)_+ \right] - \frac{1}{2\gamma_\alpha}\|\rho^{\alpha,\tilde{u}}\|^2 T. \end{aligned} \tag{5.107}$$

We will rewrite the formula

$$E_{Q^{\rho^\alpha,\tilde{u}}}\left[\left(S_T^{(i)} - K\right)_+ \right] \tag{5.108}$$

into a better equation to compute in practice. We have that

$$\begin{aligned} &E_{Q^{\rho^\alpha,\tilde{u}}}\left[\left(S_T^{(i)} - K\right)_+ \right] \\ &= E_{Q^{\rho^\alpha,\tilde{u}}}\left[\left(e^{\log(y)} e^{\sum_{j=1}^n (\sigma^\alpha)^{(ij)} W_T^{\alpha,u,(j)} + \left(-\sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2/2 - \sum_{j=1}^n (\sigma^\alpha)^{(ij)} \rho_t^{\alpha,u,(j)} + \mu^{(i)}\right)T} - K \right)_+ \right] \\ &= E_{Q^{\rho^\alpha,\tilde{u}}}\left[\left(e^{f(y)+X} - K \right)_+ \right]_{y=S_0^{(i)}} \end{aligned} \tag{5.109}$$

where

$$\begin{aligned} f(y) &= \log(y) + \left(-\sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2/2 - \sum_{j=1}^n (\sigma^\alpha)^{(ij)} \rho_t^{\alpha,u,(j)} + \mu^{(i)} \right) T \\ X &= \sum_{j=1}^n (\sigma^\alpha)^{(ij)} W_T^{\alpha,u,(j)}. \end{aligned} \tag{5.110}$$

Then, using the basic rules of normal distributions and substitution in the integral, we get that

$$\begin{aligned} &E_{Q^{\rho^\alpha,\tilde{u}}}\left[\left(e^{f(y)+X} - K \right)_+ \right]_{y=S_0^{(i)}} \\ &= \frac{1}{\sqrt{2\pi T \sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2}} \int_{\log(K)-f(S_0^{(i)})}^\infty \left(e^{f(S_0^{(ij)})+x} - K \right) e^{-x^2/2T\sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2} dx \\ &= \frac{1}{\sqrt{2\pi T \sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2}} \int_{\log(K)-f(S_0^{(ij)})}^\infty e^{f(S_0^{(i)})+x} e^{-x^2/2T\sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2} dx - K\Phi\left(\frac{f(S_0^{(i)}) - \log(K)}{\sqrt{T\sum_{j=1}^n ((\sigma^\alpha)^{(ij)})^2}} \right). \end{aligned} \tag{5.111}$$

If we use the notation

$$v = \sqrt{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2}, \tag{5.112}$$

we have that formula 5.111 can be written as

$$\begin{aligned}
&E_{Q^{\rho^{\alpha,\tilde{u}}}}\left[\left(e^{f(y)+X}-K\right)_+\right]_{y=S_0^{(i)}}\\
&=\frac{e^{f(S_0^{(i)})}}{\sqrt{2\pi v^2}}\int_{\log(K)-f(S_0^{(i)})}^{\infty}e^{x-x^2/2v}dx-K\Phi\left(\frac{f(S_0^{(i)})-\log(K)}{\sqrt{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2}}\right)\\
&=\frac{e^{f(S_0^{(i)})}e^{v^2/2}}{\sqrt{2\pi v^2}}\int_{\log(K)-f(S_0^{(i)})}^{\infty}e^{-(x-v^2)^2/2v^2}dx-K\Phi\left(\frac{f(S_0^{(i)})-\log(K)}{\sqrt{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2}}\right)\\
&=e^{f(S_0^{(i)})}e^{v^2/2}\Phi\left(\frac{f(S_0^{(i)})-\log(K)}{v}+v\right)-K\Phi\left(\frac{f(S_0^{(i)})-\log(K)}{v}\right).
\end{aligned} \tag{5.113}$$

Therefore,

$$\begin{aligned}
&\sup_{Q^{\rho^{\alpha},u}\sim P^{\alpha}}\left\{E_{Q^{\rho^{\alpha},u}}[C]-\frac{1}{\gamma_\alpha}h(Q^{\rho^{\alpha},u}|P^{\alpha})\right\}\\
&=E_{Q^{\rho^{\alpha,\tilde{u}}}}\left[\left(S_T^{(i)}-K\right)_+\right]-\frac{1}{2\gamma_\alpha}\|\rho^{\alpha,\tilde{u}}\|^2T\\
&=e^{f(S_0^{(i)})}e^{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2/2}\Phi\left(\frac{f(S_0^{(i)})-\log(K)}{\sqrt{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2}}+\sqrt{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2}\right)-K\Phi\left(\frac{f(S_0^{(i)})-\log(K)}{\sqrt{T\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2}}\right)\\
&\quad-\frac{1}{2\gamma_\alpha}\|\rho^{\alpha,\tilde{u}}\|^2T,
\end{aligned} \tag{5.114}$$

where in this case

$$f(S_0^{(i)})=\log(S_0^{(i)})+\left(-\sum_{j=1}^{n}((\sigma^{\alpha})^{(ij)})^2/2-\sum_{j=1}^{n}(\sigma^{\alpha})^{(ij)}\rho_t^{\alpha,\tilde{u},(j)}+\mu^{(i)}\right)T \tag{5.115}$$

Now, we have to solve

$$\sup_{Q^{\rho^{\alpha},u}\sim P^{\alpha}}\left\{-\frac{1}{\gamma_\alpha}h(Q^{\rho^{\alpha},u}|P^{\alpha})\right\}=\sup_{Q^{\rho^{\alpha},u}\sim P^{\alpha}}\left\{-\frac{1}{\gamma^{\alpha}}E_{Q^{\rho^{\alpha},u}}\left[\frac{1}{2}\int_0^T\|\rho_t^{\alpha,u}\|^2dt\right]\right\}. \tag{5.116}$$

Since

$$\sup_{Q^{\rho^{\alpha},u}\sim P^{\alpha}}\left\{-\frac{1}{\gamma_\alpha}h(Q^{\rho^{\alpha},u}|P^{\alpha})\right\}=\sup_{u\in\mathcal{U}_\alpha}-\frac{1}{2\gamma_\alpha}\|\rho^{\alpha,u}\|^2T=\sup_{(0,0,z)^\top\in\mathbb{R}^3}-\left\|\rho^{\alpha}+(\sigma^{\alpha})^{-1}\begin{pmatrix}0\\0\\z\end{pmatrix}\right\|^2 \tag{5.117}$$

the solution to the previous maximization problem is the same as $\tilde{u}$ in 5.101. Therefore, the estimation of price of the derivative C, with underlying $S^{(i)}$ with $i = 1, 2$, from an $\alpha-$stereotyped agent perspective is given by

$$
\begin{aligned}
&p^{\alpha}(v_0, C)/B^{\alpha}_{0,T} \\
&= \left(\sup_{Q^{\rho^{\alpha},u} \sim P^{\alpha}} \left\{ E_Q[C] - \frac{1}{\gamma_{\alpha}} h(Q^{\rho^{\alpha},u} | P^{\alpha}) \right\} - \sup_{Q^{\rho^{\alpha},u} \sim P^{\alpha}} \left\{ -\frac{1}{\gamma_{\alpha}} h(Q^{\rho^{\alpha},u} | P^{\alpha}) \right\} \right) \\
&= e^{f(S_0^{(i)})} e^{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2/2} \Phi \left(\frac{f(S_0^{(i)}) - \log(K)}{\sqrt{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2}} + \sqrt{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2} \right) - K \Phi \left(\frac{f(S_0^{(i)}) - \log(K)}{\sqrt{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2}} \right) \\
&\quad - \frac{1}{2\gamma_{\alpha}} \|\rho^{\alpha,\tilde{u}}\|^2 T + \frac{1}{2\gamma_{\alpha}} \|\rho^{\alpha,\tilde{u}}\|^2 T \\
&= e^{f(S_0^{(i)})} e^{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2/2} \Phi \left(\frac{f(S_0^{(i)}) - \log(K)}{\sqrt{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2}} + \sqrt{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2} \right) - K \Phi \left(\frac{f(S_0^{(i)}) - \log(K)}{\sqrt{T \sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2}} \right),
\end{aligned}
\tag{5.118}
$$

where

$$
f(S_0^{(i)}) = \log(S_0^{(i)}) + \left(-\sum_{j=1}^{n} ((\sigma^{\alpha})^{(ij)})^2/2 - \sum_{j=1}^{n} (\sigma^{\alpha})^{(ij)} \rho_t^{\alpha,(j)} + \mu^{(i)} \right) T. \tag{5.119}
$$

From this estimation method of the prices of small agents, we can conclude that the prices seem to not be affected by the level of risk aversion. These findings do not offer a contradiction, but they agree to some extent with economists such as Eugene Fama. The following is the code to compute the price of the big agent.

```
# price of small agents
t_forecast <- length(MSFT_Close)-start_date_time_s_analysis -2
length(t_bill_1year_daily) - start_date_bond_analysis -2
vector_of_strikes <- c(strike_msft, strike_orcl, strike_goog)
list_prices_small_agents <- list()
vector_prices_small_agents <- c()
j <- 0

p_small_agent <- function(s_0,sigma_agent, rho_agent,t, K, mu){
  c1 <- dot_product(sigma_agent, sigma_agent)
  c2 <- dot_product(sigma_agent, rho_agent)
  f <- log(s_0) + ((-c1/2) - (c2) + mu)*(t/365)
  first_term <- exp(f)*exp((t/365)*c1*0.5)*pnorm( ((f - log(K))/sqrt((t/365)*c1)) +
   sqrt((t/365)*c1))
  second_term <- K*pnorm(((f - log(K))/sqrt((t/365)*c1)))
  price_formula <- (first_term - second_term)*as.numeric(list_estimation_bond[[i+1]])
  return(price_formula)
}

for (stock  in list_of_stocks){
  j <- j + 1
  for (i in c(0: ( length(t_bill_1year_daily) - start_date_bond_analysis))){
    t_maturity <-
      as.numeric(difftime(maturity_1t_bill,
```

```
          index(t_bill_1year_daily[i + start_date_bond_analysis]), units = "days"))
    if (i > t_forecast){
      vector_prices_small_agents <-
        c(vector_prices_small_agents,
          p_small_agent(as.numeric(stock_step_ahead[[j]][i - t_forecast]),
                list_sigma[[1+ t_forecast]][j,], list_rho[[1+ i]], t_maturity,
                                  vector_of_strikes[j], list_mu[[1+ i]][j]) )
    }
    else{
      vector_prices_small_agents <- c(vector_prices_small_agents,
              p_small_agent(as.numeric(stock[start_date_bond_analysis + i]),
                 list_sigma[[1+i]][j,], list_rho[[1+i]], t_maturity,
                                 vector_of_strikes[j], list_mu[[1+i]][j]) )
    }

  }
  list_prices_small_agents[[j]] <- vector_prices_small_agents
  vector_prices_small_agents <- c()
}

#Mixture Diffusion Pricing
sigma_up <- 0.005
sigma_down <- -0.005
drift_up <- 0.005
drift_down <- -0.005
vector_macro_factors <- c(1,0.5,0.7,0.9,0.2,0.55,0.75,1)
risk_f_rate_factors <- c(drift_up, drift_down, drift_down, drift_down,
                         drift_up, drift_down, drift_down, drift_down)*
                        vector_macro_factors
sigma_factors <- c(sigma_down, sigma_up, sigma_up, sigma_up,
                   sigma_down, sigma_up, sigma_up, sigma_up)*
                  vector_macro_factors

vector_of_strikes <- c(strike_msft, strike_orcl, strike_goog)
list_prices_big_agent <- list()
vector_prices_big_agent <- c()
j <- 0
p_big_agent <- function(s_0,sigma_agent,t, K, r){
  c1 <- dot_product(sigma_agent, sigma_agent)
  f <- log(s_0) + ((-c1/2) + (r/100))*(t/365)
  first_term <- exp(-(r/100)*(t/365))*exp(f)*exp((t/365)*c1*0.5)*
    pnorm( ((f - log(K))/sqrt((t/365)*c1)) + sqrt((t/365)*c1))
  second_term <- exp(-(r/100)*(t/365))*K*pnorm(((f - log(K))/sqrt((t/365)*c1)))
  price_formula <- first_term - second_term
  return(price_formula)
}
for (stock  in list_of_stocks){
  j <- j + 1
  for (i in c(0: ( length(t_bill_1year_daily) - start_date_bond_analysis))){
    t_maturity <- as.numeric(difftime(maturity_1t_bill,
            index(t_bill_1year_daily[i + start_date_bond_analysis]), units = "days"))
    if (i > t_forecast){
      mix_price <- 0
      for (k in c(1:size_q_distribution)){
        mix_price <- mix_price + (posteriori_distribution[k]*
                        p_big_agent(as.numeric(stock_step_ahead[[j]][i - t_forecast]),
                            list_sigma[[1+ i]][j,]*(1 + sigma_factors[k]), t_maturity,
                            vector_of_strikes[j],
              as.numeric(t_bill_1year_daily[start_date_bond_analysis + t_forecast])*
                (1 + risk_f_rate_factors[k])))
      }
```

```
      vector_prices_big_agent <- c(vector_prices_big_agent,mix_price )
    }
    else {
      mix_price <- 0
      for (k in c(1:size_q_distribution)){
        mix_price <- mix_price + (posteriori_distribution[k]*
                              p_big_agent(as.numeric(stock[start_date_bond_analysis + i]),
                                  list_sigma[[1+i]][j,]*(1 + sigma_factors[k]), t_maturity,
                                               vector_of_strikes[j],
                                 as.numeric(t_bill_1year_daily[start_date_bond_analysis + i])*
                                   (1+risk_f_rate_factors[k]) ))
      }
      vector_prices_big_agent <- c(vector_prices_big_agent, mix_price)
    }

  }
  list_prices_big_agent[[j]] <- vector_prices_big_agent
  vector_prices_big_agent <- c()
}
```

5.2.7 Coefficients of market participation of small and big agents

In this part, we will use the previous information of the prices of the agents in the market to build the market price. We recall that we will price an MSFT call option using the information of three derivatives in the market

Characteristic	Call option MSFT	Call option ORCL	Call option GOOG
Issue date	2019-08-01	2019-08-01	2019-08-01
Maturity date	2019-10-18	2019-10-18	2019-10-18
Strike	140	55	1200

Table 7: Stock options under analysis.

For this purpose, we will use the idea mentioned at the beginning of this work. In other words, we will compute the OLS coefficient to known what is the impact of the the prices that propose each agent on the market price. After fitting the OLS model, we will make time series analysis on the OLS errors. Finally, we will add the values of the OLS model and the estimation of the error to get an estimation of the market price of the call options of MSFT.

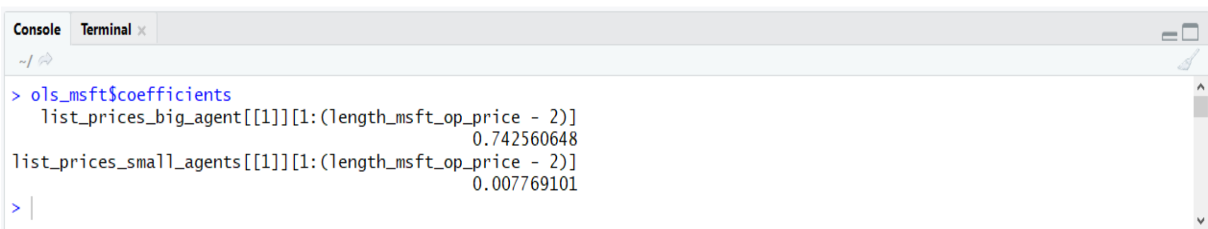
The following is the code for the call option prices of MSFT, ORCL, and GOOG.

```
# get number of the column of bids of call options
# for list of quotes of MSFT
number_column_bid_1545_msft <-
  which(colnames(at_common_maturity_msft[[1]]) == "bid_1545")
# for list of quotes of ORCL
number_column_bid_1545_orcl <-
  which(colnames(at_common_maturity_orcl[[1]]) == "bid_1545")
# for list of quotes of GOOG
number_column_bid_1545_goog <-
  which(colnames(at_common_maturity_goog[[1]]) == "bid_1545")
```

```
# get number of the column of asks of call options
# for list of quotes of MSFT
number_column_ask_1545_msft <-
  which(colnames(at_common_maturity_msft[[1]]) == "ask_1545")
# for list of quotes of ORCL
number_column_ask_1545_orcl <-
  which(colnames(at_common_maturity_orcl[[1]]) == "ask_1545")
# for list of quotes of GOOG
number_column_ask_1545_goog <-
  which(colnames(at_common_maturity_goog[[1]]) == "ask_1545")


# take the bid prices of of the list of quotes
#for MSFT
bid_1545_msft <-
  lapply(at_common_maturity_msft,"[[",number_column_bid_1545_msft)
#for ORCL
bid_1545_orcl <-
  lapply(at_common_maturity_orcl,"[[",number_column_bid_1545_orcl)
#for GOOG
bid_1545_goog <-
  lapply(at_common_maturity_goog,"[[",number_column_bid_1545_goog)

# take the ask prices of of the list of quotes
#for MSFT
ask_1545_msft <-
  lapply(at_common_maturity_msft,"[[",number_column_ask_1545_msft)
#for ORCL
ask_1545_orcl <-
  lapply(at_common_maturity_orcl,"[[",number_column_ask_1545_orcl)
#for GOOG
ask_1545_goog <-
  lapply(at_common_maturity_goog,"[[",number_column_ask_1545_goog)



# get list of bid prices and convert them into a vector
# for MSFT
bid_1545_msft <- unlist(bid_1545_msft, use.name=FALSE)
# for ORCL
bid_1545_orcl <- unlist(bid_1545_orcl, use.name=FALSE)
# for GOOG
bid_1545_goog <- unlist(bid_1545_goog, use.name=FALSE)

# get list of ask prices and convert them into a vector
# for MSFT
ask_1545_msft <- unlist(ask_1545_msft, use.name=FALSE)
# for ORCL
ask_1545_orcl <- unlist(ask_1545_orcl, use.name=FALSE)
# for GOOG
ask_1545_goog <- unlist(ask_1545_goog, use.name=FALSE)

# market prices
# for MSFT
market_call_option_price_msft <- (bid_1545_msft + ask_1545_msft)/2
# for ORCL
market_call_option_price_orcl <- (bid_1545_orcl + ask_1545_orcl)/2
# for GOOG
market_call_option_price_goog <- (bid_1545_goog + ask_1545_goog)/2
```

With this block of code we can compute the OLS coeffcients without intercept as follows

```
# OLS errors for the P = /beta*prices + error model
list_error_ols <- list()
length(MSFT_Close) - start_date_time_s_analysis
length_msft_op_price <- length(market_call_option_price_msft)
ols_msft <- lm(formula = market_call_option_price_msft[1:(length_msft_op_price -2)] ~
                  list_prices_big_agent[[1]][1:(length_msft_op_price - 2)] +
                  list_prices_small_agents[[1]][1:(length_msft_op_price - 2)] + 0)

for (i in c(0: (length(MSFT_Close) - start_date_time_s_analysis -2 )   )) {
  list_error_ols[[i+1]] <- market_call_option_price_msft[i+1] -
    as.vector(ols_msft$coefficients) %*% c(list_prices_big_agent[[1]][i+1],
                                           list_prices_small_agents[[1]][i+1])
}
```



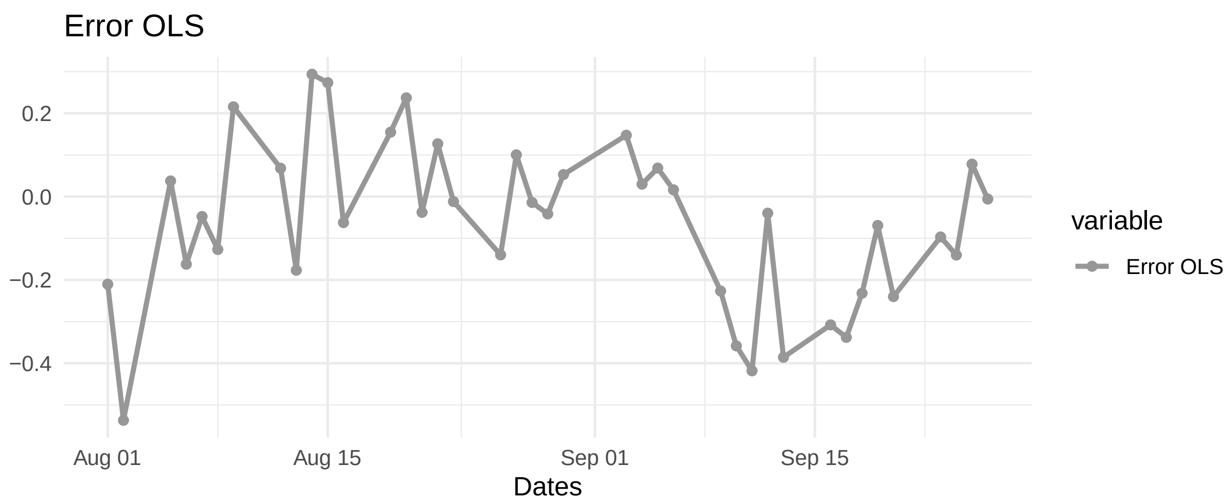
```
> ols_msft$coefficients
  list_prices_big_agent[[1]][1:(length_msft_op_price - 2)]
                                                0.742560648
list_prices_small_agents[[1]][1:(length_msft_op_price - 2)]
                                                0.007769101
>
```

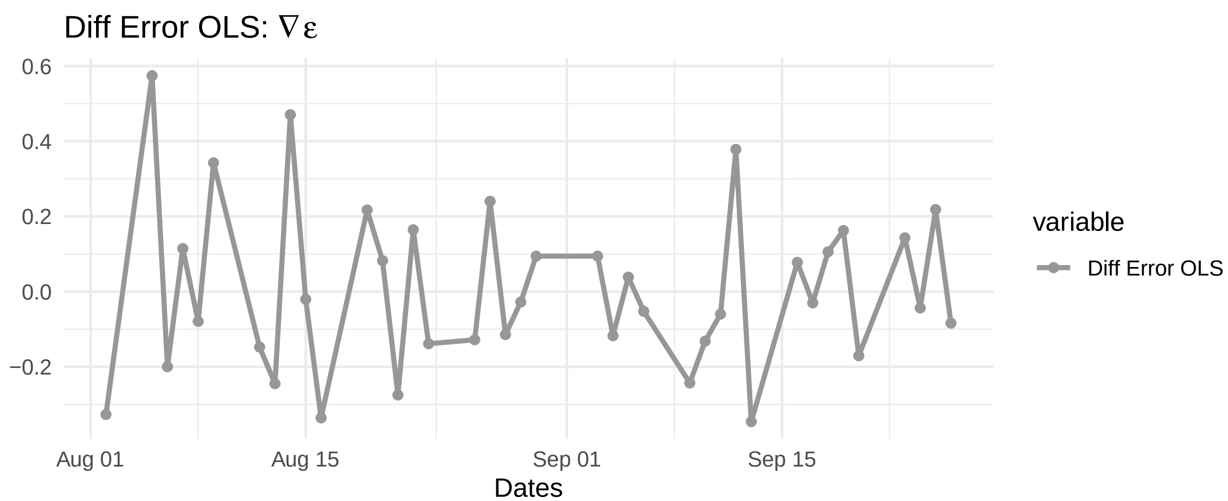
Figure 12: OLS coefficients.

There are some remarks that have to be pointed out about the OLS estimation. Since the data we are using in this work is not big enough, we are assuming $\mathbf{Y}_t$ constant for every t. Moreover, this part is not intended to analyze the impact that trading volume has on the price of financial instruments. Instead, the idea is to give an estimation of the impact that each one of the agents has on the prices based on the amount of information they have. In other words, the more information an agent has, the more accurate its price estimation will be, and therefore, the more supply and demand will be around that price level. This idea is captured by the model since $\|\mathbf{Y}_t\| \leq 1$, and most of the weight of $\mathbf{Y}_t$ belongs to the big agent.

5.2.8 Time series analysis of the errors

In this part, we analyze the errors in the model. In the following picture we notice that the series is not stationary. Therefore, we apply the difference operator once to check if the property holds.



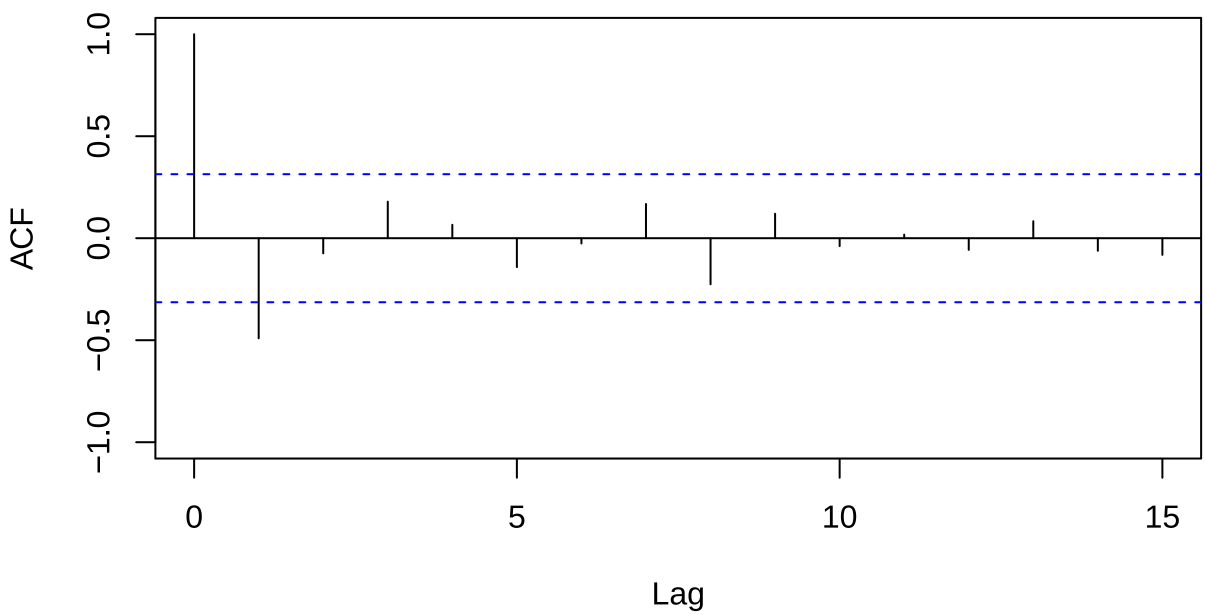



```
        Augmented Dickey-Fuller Test

data:  diff(cleaned_error_ols)
Dickey-Fuller = -6.46, Lag order = 3, p-value = 0.01
alternative hypothesis: stationary
```

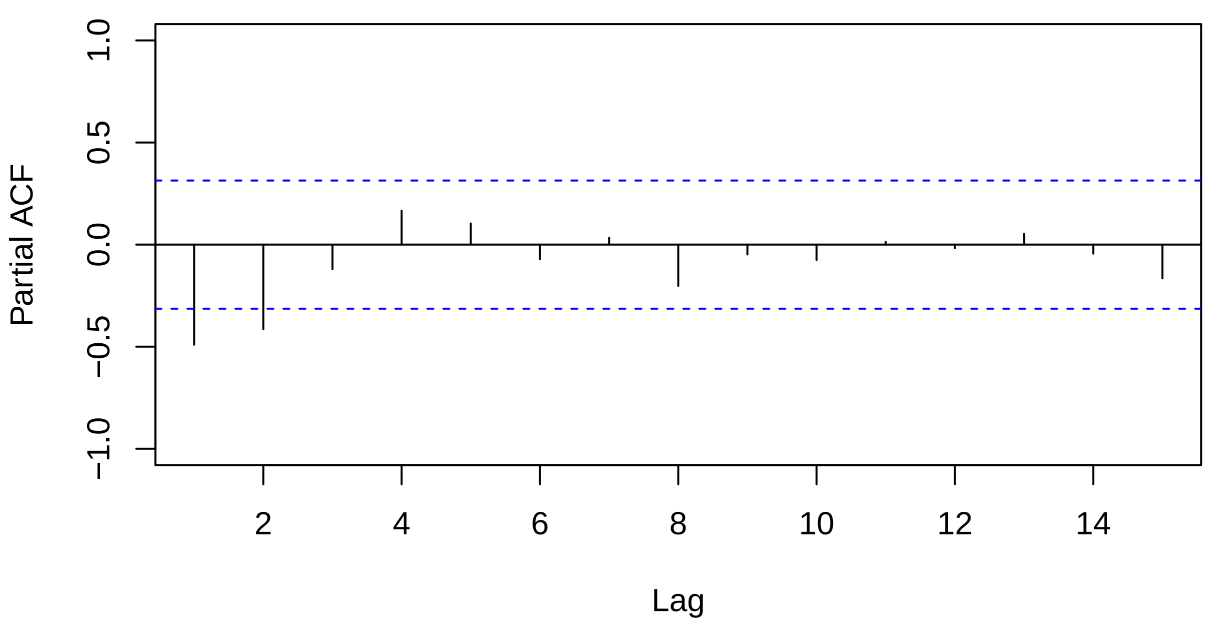
As it was showed in the image above, the first differentiation shows signs stationarity characteristic. We make the Augmented Dickey-Fuller test. Since the p-value is less than 0.05, the data does not have a unit root and is stationary[24]. Now, we make ACF and PACF analysis on the transformed data.

[24]Source code in Appendix F

Difference Operator on Error OLS ACF



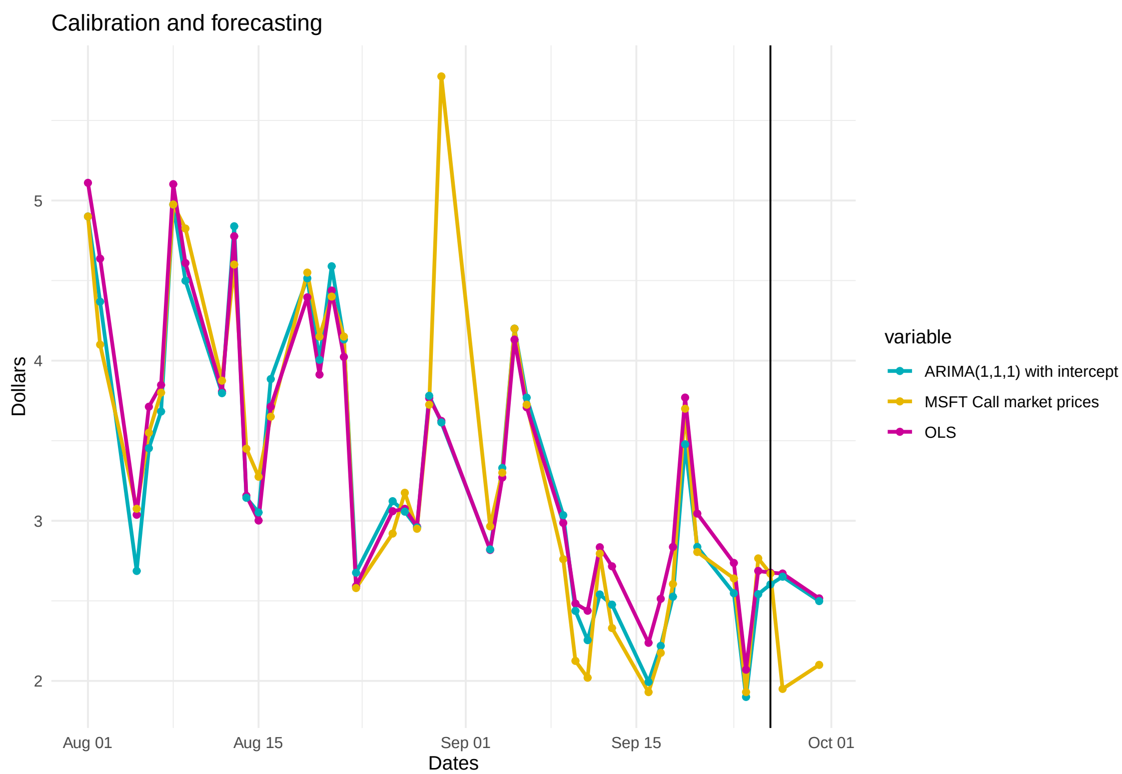
Difference Operator on Error OLS PACF



The functions show patterns of a possible ARIMA(1,1,1) model as it was explained for the implied volatility case. We will use the model ARIMA(1,1,1) for the errors that come from OLS in this example.

5.2.9 Calibration and forecasting

In the following example, we will assume that big and small investors want to invest in an MSFT call option issued at 2019-08-01 with maturity date at 2019-10-18, and strike 140.



In the previous chart, the blue line shows the price of the micro market; $P_t^M = \mathbf{Y}_t'\mathbf{P}_t + \varepsilon_t$ as in 1.4 and 1.5 where the prices of each agent are given by the formulas 5.50, 5.118; and the error follows an ARIMA(1,1,1). The other two lines correspond to the market price of the MSFT call option, and the OLS model. The values that come after the vertical black line are the forecasts 1 and 2 steps ahead. The lack of accuracy of the prediction is due to the SDE model and the Euler-Maruyama discretization method used to estimate the prices of the stocks, and the sensitivity of the OLS term to small changes in prices.

For the purpose of comparability, we show that the model proposed in this work is slightly more accurate than the OLS. However, we need to give a more accurate measure of this idea. We will use the root mean square deviation (RMSD) for this purpose.

```
Console  Terminal ×
~/
> # Compute the euclidean measure to know the accuracy
> sqrt(product(market_call_option_price_msft- estimation_price_ols))/
+                                       sqrt(length(estimation_price_ols))
[1] 0.4073394
> sqrt(product(market_call_option_price_msft - estimation_price_with_arima_intercept))/
+                       sqrt(length(estimation_price_with_arima_intercept))
[1] 0.394128
```

5.2.10 Conclusions

The pricing method that we propose uses the pricing methods by (El Karoui and Rouge, 2000), and (Brigo, 2002) to compute the price of the micro-market where there are several risky assets, small agents, and one big agent. We extend the results by (Brigo, 2002) to a multiprice model, and create a method for computing the prices of derivatives based on macroeconomic information of the stocks. In this process, we also develop what we call *volatility components*, the parts of the volatility that depends on other stocks that are correlated with the underlying. This result lets analysts have an idea of where the volatility comes from, and a better understanding of systematic risk. With this model we capture to some extent how a big firm invest in stock options: technical analysis, fundamental analysis, and economics. For the small agents we create algorithms to give an estimation of the price proposed by (El Karoui and Rouge, 2000). We conclude that under this new method the price of small agents has a similar structure to the Black-Scholes-Merton model, and under this estimation method the price of small investors does not depend on the level of risk aversion when the relations of preference are assumed to be given by the negative exponential utility function. The results of this estimation coincide to some extent with the point of view of some economists such as Eugene Fama. We use these prices in a linear regression model $P_t^M = \mathbf{Y}_t^{'}\mathbf{P}_t + \varepsilon_t$ to determine the price of this micro-market. The estimation of the OLS coefficients, without the use of any restriction, coincide with the intuition, $\|\mathbf{Y}_t\| \leq 1$, where most of the weight of this coefficient pertains to the big agent. This conclusion coincides with the purpose of this part of the work that is to give an estimation of the impact that each one of the agents has on the prices based on the amount of information they have. In other words, the more information an agent has, the more accurate its price estimation will be, and therefore, the more supply and demand will be around that price level. The errors were studied using the difference operator once, and PACF, ACF. We conclude that it follows an ARIMA(1,1,1) pattern. The model proposed in this work outperforms other standard models such as the OLS. However, the forecasting capabilities of this model are not as good as expected. The supplementary material of this work contains the code of each one of the algorithms, and supporting evidence of the macroeconomic hypotheses and the magnitudes we used in this study.

Appendix A

Throughout this part, all the processes considered are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions(see Karatzas and Shreve, 2014). The following results have been derived from (Meyer, 2000, Section I.9 and III.4) and (Shreve, 2004, Chapter 4).

Proposition A.1. *If M is a $(\mathcal{F}_t, Q)$-continuous local martingale with $M_0 \in L^2$ and $\langle M\rangle_T \in L^1$, then*

M is a L^2-martingale in $[0,T]$.

Proof. Let (T_n) be a sequence of increasing stopping times that simultaneously reduces $M^2 - \langle M \rangle$ and M. Since $M_0^2 - \langle M \rangle_0 = M_0^2$ is integrable, we get $M^2_{t\wedge T_n} - \langle M \rangle_{t\wedge T_n}$ is a $(\mathcal{F}_t, Q)$-martingale (for example see Meyer, 2000, Section I.8). Thus, $E[M^2_{t\wedge T_n}] \leq E[M_0^2] + E[\langle M \rangle_T] < \infty$ for each $n \in \mathbb{N}$ and as result $(M_{t\wedge T_n})_{n\in\mathbb{N}}$ is uniformly integrable.

Since $M_{t\wedge T_n} \xrightarrow[n\to\infty]{a.s} M_t$ and $(M_{t\wedge T_n})_{n\in\mathbb{N}}$ is uniformly integrable, we have $M_{t\wedge T_n} \xrightarrow[n\to\infty]{L^1} M_t$ and therefore

$$M_s = \lim_{n\to\infty} M_{s\wedge T_n} = \lim_{n\to\infty} E[M_{t\wedge T_n}|\mathcal{F}_s] = E[M_t|\mathcal{F}_s]$$

for all $0 \leq s \leq t \leq T$ □

Proposition A.2. *Let M be a $(\mathcal{F}_t, Q)$-continuous local martingale. If $E[\sup_{l\leq t}|M_l|] < \infty$ for each $t \geq 0$, then M is martingale.*

Proof. Since M satisfies $E[\sup_{l\leq t}|M_l|] < \infty$ for each $t \geq 0$, we have for each sequence of stopping times (T_n) that reduces M,

$$M_{t\wedge T_n} \xrightarrow[n\to\infty]{a.s} M_t \qquad \text{and} \qquad M_{t\wedge T_m} \leq \sup_{l\leq t}|M_l|$$

for each $m \in \mathbb{N}$. Therefore, $M_{t\wedge T_n} \xrightarrow[n\to\infty]{L^1} M_t$, and hence $M_s = \lim_{n\to\infty} M_{s\wedge T_n} = \lim_{n\to\infty} E[M_{t\wedge T_n}|\mathcal{F}_s] = E[M_t|\mathcal{F}_s]$ for each $0 \leq s \leq t$. □

Proposition A.3. *Let $W = (W_t)_{t\geq 0}$ be a $(\mathcal{F}_t, Q)$-Brownian motion and v a well-behaving, real valued function. If $M_t = \int_0^t v(s)dW_s$, then $M_t \sim \mathcal{N}\left(0, \int_0^t v^2(s)ds\right)$ for each $t \geq 0$.*

Proof. We define the Doleans exponential of a process X as

$$\mathcal{E}_t(X) := \exp\left\{X_t - \frac{1}{2}\langle X \rangle_t\right\}.$$

Let us now set $M_t := \int_0^t v(s)dW_s$ and $Y_t := \mathcal{E}_t(2M)$. We know from basic results for L^2_{Loc} processes (see Revuz and Yor, 1999, Section IV.2) and Itô's lemma that Y and $\mathcal{E}^2(M)$ are local martingale and semimartingale respectively. Therefore, we can get an increasing sequence of stopping times (T_n) that reduces Y. Now, by the equality $\mathcal{E}^2_t(M) = Y_t \exp(\langle M \rangle_t)$, we get $\mathcal{E}^2_{t\wedge T_n}(M) = Y_{t\wedge T_n}\exp(\langle M \rangle_{t\wedge T_n}) \leq Y_{t\wedge T_n}\exp(\langle M \rangle_t)$ and $\mathcal{E}^4_{t\wedge T_n}(M) \leq Y^2_{t\wedge T_n}\exp(2\langle M \rangle_t)$.

Since $\langle \int_0^\cdot v(s)dW_s \rangle_t = \int_0^t v^2(s)ds < \infty$, we get, by using L^2-Doob's maximal inequality, that

$$\begin{aligned} E\left[\sup_{s\in[0,t]} \mathcal{E}^4_{s\wedge T_n}(M)\right] &\leq E\left[\sup_{s\in[0,t]} Y^2_{s\wedge T_n}\exp(2\langle M \rangle_t)\right] \\ &\leq 4E[Y_{t\wedge T_n}]\exp(2\langle M \rangle_t) \\ &= 4E[Y_0]\exp(2\langle M \rangle_t) \\ &= 4\exp(2\langle M \rangle_t) < \infty. \end{aligned}$$

Because $\sup_{s\in[0,t]} \mathcal{E}^4_{s\wedge T_n}(M) \uparrow \sup_{s\in[0,t]} \mathcal{E}^4_s(M)$ as $n \to \infty$, we obtain

$$E\left[\sup_{s\in[0,t]} \mathcal{E}^4_s(M)\right] < 4\exp(2\langle M\rangle_t) < \infty,$$

and therefore $E\left[\sup_{s\in[0,t]} \mathcal{E}_s(M)\right] < \infty$ for each $t \geq 0$. By Proposition A.2, we conclude $\mathcal{E}(M)$ is martingale, and then $E[\mathcal{E}_t(M)] = 1$ for all $t \geq 0$.

By similar arguments to Proposition A.1 and A.2, we get M is a martingale that, by Itô's isometry, has mean 0 and variance $\int_0^t v^2(s)ds$ for each $t \geq 0$. From these results we conclude

$$E\left[\exp\left\{\lambda M_t - \frac{\lambda^2}{2}\int_0^t v^2(s)ds\right\}\right] = 1,$$

for all λ, and therefore M_t has the same moment-generating function of a normal random variable with mean 0 and variance $\int_0^t v^2(s)ds$. $\square$

Appendix B

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$ be a filtered probability space where $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, and consider the dynamics of the well-behaving, strictly positive risky and riskless assets respectively $(S_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$ being given by

$$dS_t = \gamma(t)S_t dt + v(t)S_t dW_t, \qquad dB_t = r(t)B_t dt,$$

where $W = (W_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t, P)$-Brownian motion. If we set $\tilde{S}_t = S_t/B_t$, then, by applying Itô's lemma to the function $f(x,y) = xy$ evaluated on the vector semimartingale $(S_t, e^{-\int_0^t r(s)ds})$ we get

$$d\tilde{S}_t = -r(t)e^{-\int_0^t r(s)ds}S_t dt + e^{-\int_0^t r(s)ds}dS_t = (\gamma(t) - r(t))\tilde{S}_t dt + \tilde{S}_t v(t)dW_t. \tag{B.1}$$

Now, we will find the risk-neutral measure by using Girsanov's theorem. Let us then set

$$D_t := \frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}} = \exp\left\{-\int_0^t \lambda(s)dW_s - \frac{1}{2}\int_0^t \lambda^2(s)ds\right\},$$

where $\lambda(s) = \dfrac{\gamma(s) - r(s)}{v(s)}$ and $L_t = -\int_0^t \lambda(s)dW_s$. Since D_t satisfies $D_t = 1 + \int_0^t D_s dL_s$, we get that D is continuous local martingale. Moreover, by reasoning showed in the proof Proposition A.3, we get $E[\langle L\rangle_T] < \infty$, where $D_T = \mathcal{E}_T(L)$, and hence D is a martingale in $[0,T]$. Now, by Girsanov's theorem we have that

$$\tilde{W}_t = W_t - \langle W, L\rangle_t = W_t + \int_0^t \frac{\gamma(s) - r(s)}{v(s)}ds$$

is a $(\mathcal{F}_t, Q)$-Brownian motion. Then we get by B.1:

$$d\tilde{S}_t = \tilde{S}_t v(t)d\tilde{W}_t, \qquad \tilde{S}_t = \tilde{S}_0 \exp\left\{\int_0^t v(s)d\tilde{W}_s - \frac{1}{2}\int_0^t v^2(s)ds\right\},$$

and therefore $\tilde{S}$ is a $(\mathcal{F}_t, Q)$-martingale in $[0,T]$ (see proof of Proposition A.3). Finally, if S satisfies $dS_t = S_t\gamma(t)dt + v(t)S_t dW_t$ under the measure P, we get S also satisfies the $dS_t = S_t r(t)dt + v(t)S_t d\tilde{W}_t$ under the measure Q, and the reverse is also true. Moreover the arbitrage-free price does not depend on γ for the case of European options (see for instance Elliott and Kopp, 2006, Section 7.5). This shows that dynamics of our risky asset $(S_t)_{t\in[0,T]}$ under the risk-neutral measure Q can be assumed to be of the form 2.3 with numéraire $(B_t)_{t\in[0,T]}$ satisfying dynamics of the type $dB_t = r(t)B_t dt$.

Appendix C

Proposition C.1. *Let $(\Omega, \mathcal{F})$ a measurable space endowed with two probability measures P and Q such that $Q \ll P$. Then for any sub σ-algebra $\mathcal{G}$ and non-negative random variable X on $(\Omega, \mathcal{F})$ we get*

$$E_Q[X|\mathcal{G}]E_P\left[\frac{dQ}{dP}\middle|\,\mathcal{G}\right] = E_P\left[X\frac{dQ}{dP}\middle|\,\mathcal{G}\right].$$

Proof. By definition of conditional expectation, it is enough to prove

$$\int_A E_Q[X|\mathcal{G}]E_P\left[\frac{dQ}{dP}\middle|\,\mathcal{G}\right]dP = \int_A E_P\left[X\frac{dQ}{dP}\middle|\,\mathcal{G}\right]dP,$$

for each $A \in \mathcal{G}$. By using Radon-Nikodym's theorem and basic conditional expectation properties, we get

$$\begin{aligned}\int_A E_P\left[X\frac{dQ}{dP}\middle|\,\mathcal{G}\right]dP = \int_A X\,dQ = \int_A \frac{dQ}{dP}E_Q[X|\mathcal{G}]dP &= \int_A E_P\left[\frac{dQ}{dP}E_Q[X|\mathcal{G}]\middle|\,\mathcal{G}\right]dP \\ &= \int_A E_Q[X|\mathcal{G}]E_P\left[\frac{dQ}{dP}\middle|\,\mathcal{G}\right]dP,\end{aligned}$$

which ends the proof. □

Appendix D

Throughout this part all information will be taken from (Jaynes et al., 2003). As it was mention at the beginning of this work, we are facing the problem of assigning a specific probability distribution to the macroeconomic model. Assigning probabilities under the standard point of view of statisticians and mathematicians makes use of the principle of indifference. This process can be done if we can break the phenomenon that we want to analyze into mutually exclusive, exhaustive possibilities where no one of them is preferred more than another. However, in many cases such as risk analysis, pricing derivatives, etc., researchers have initial information that does not change the set of possibilities, but gives reasons for preferring one possibility to another, and the question is what can they do in this case?

The traditional methods deviates from this problem by ignoring fixed information for fixed parameters and maintaining the idea that sampling probabilities are known frequencies. So far many researchers have never seen a real problem in which they have prior information about sampling frequencies. Statisticians, and mathematics often starts sampling probabilities by assigning them from standard mathematical models such as binomial distributions, etc. If we want to be above such false ideas, we have to give more principles for assigning initial probabilities by logical analysis of the prior information.

We have defined so far two different problems: estimating a frequency distribution, and assigning a probability distribution. However, these two problems are almost identical. As a researchers we want to take into account all of the information we have, and we don't want conclude things that are not warranted by the evidence we have. We know that a uniform probability represents an impartial state of mind, that is, it no favors one over any other possibility. When a researcher has information about average values of an specific phenomenon, she has reasons for preferring some possibilities over others, but she would like to assign a probability distribution as uniform as possible and agreeing with the available information.

The previous problem can be written into a mathematical problem where we first need to find a measure that quantifies how uniform a distribution is, and then we will need to maximize this function subjected to some specific restrictions based on our prior information. In other words, our problem will be a variational problem where we have to maximize this new measure over probability distributions subjected to some restrictions that reflect our initial information and the properties these distributions must obey. It is clear that this new measure will quantify the level uncertainty that we have about the phenomenon in question via a probability distribution.

The following approach that can be found in (Jaynes et al., 2003) is the most cited work of Shannon's information theory (Shannon, 1948), and this shows what are the characteristic that this measure of uncertainty in probability distributions needs to satisfy.

The characteristic of this measure must obey the following assumptions

1. For each n, there exist a numerical measure $H_n(\lambda_1...., \lambda_n)$ on $\mathcal{P}_n$ that quantifies the "amount of uncertainty" expressed by the probability distribution $\bar{\lambda} = (\lambda_1, ..., \lambda_n)$.
2. H_n has to be a continuous function. If this were not true, we would assume that small changes in the probability distribution would generate big changes in the amount of the uncertainty.
3. This function has to quantify the idea: the more possibilities considered, the more uncertainty the researcher faces. This concept can be represented in the scenario where the λ_i are all equal, and the function
$$h(n) := H_n\left(\frac{1}{n}, ..., \frac{1}{n}\right) \tag{D.1}$$
is increasing with respect to n.
4. We require that the measure H_n be consistent, i.e., if there is more than one way of expressing this value, we must have the same answer for each alternative expression.

Additional to the previous assumption we need to consider another important characteristic of this function that will allow us to find an algebraic expression for H_n. Suppose that you have a distribution

$\bar{\lambda} = (\lambda_1, q)$, but now you know that the second alternative can be split into two new alternatives each one with probabilities λ_2 and λ_3 respectively, that is $\lambda_2 + \lambda_3 = q$., then our amount of uncertainty $H_2(\lambda_1, q)$ should satisfy the following equation

$$H_3(\lambda_1, \lambda_2, \lambda_3) = H_2(\lambda_1, q) + qH_2\left(\frac{\lambda_2}{q} + \frac{\lambda_3}{q}\right). \tag{D.2}$$

That is, the amount of uncertainty that represent $(\lambda_1, \lambda_2, \lambda_3)$ can be split as the sum of two expressions. The first part expresses the uncertainty when we remove the possibility that the second event can be decomposed as two mutually exclusive events with probabilities λ_2 and λ_3 respectively. The second term expresses the fact that with probability q the researchers encounters the additional uncertainty that the events 2 and 3 happens.

For the sake of simplicity, we will drop the index n on H_n because the full proof of shannon make use of different indeces.

As it is explained in (Jaynes et al., 2003), we can find a most general form for D.2. Suppose that we start with n propositions $(A_1, ..., A_n)$and to each one we assign the probabilities $(\lambda_1, ..., \lambda_n)$. Suppose that for some reasons we want to group them to form different large groups, we might group the first k_1 of them as proposition $A_1 + \cdots + A_{k_1}$ and we assign probability $v_1 = \lambda_1 + \cdots + \lambda_{k_1}$; then the next k_2 propositions are grouped into $A_{k_1+1} + \cdots + A_{k_1+k_2}$ to which we assign the probability $v_2 = \lambda_{k_1+1} + \cdots + \lambda_{k_1+k_2}$, etc.

Now we assign the conditional probabilities $\lambda_1/v_1, ..., \lambda_{k_1}/v_1$ to the propositions $A_1, ..., A_{k_1}$ given the composite proposition $A_1 + \cdots + A_{k_1}$. The additional uncertainty with probability v_1, is then $H_k(\lambda_1/v_1, ..., \lambda_k/v_1)$. We use the same reasoning for the propositions $A_{k_1+1} + \cdots + A_{k_1+k_2}$, etc. By the consistency hypothesis mentioned before the state of uncertainty of $(\lambda_1, ..., \lambda_n)$ yield the same uncertainty to the case where the choices were broken down. Therefore, we get

$$H(\lambda_1, ..., \lambda_n) = H(v_1, ..., v_r) + v_1 H\left(\frac{\lambda_1}{v_1}, ..., \frac{\lambda_{k_1}}{v_1}\right) + \cdots + v_2 H\left(\frac{\lambda_{k_1+1}}{v_2}, ..., \frac{\lambda_{k_1+k_2}}{v_2}\right) + \cdots \tag{D.3}$$

that is the general form of D.2.

Since H_n is continuous, it will be enough to determine its formula for all rational number of the form

$$\lambda_i = \frac{n_i}{\sum_{j=1}^n n_j}.$$

So using D.1 and D.3 we get

$$h\left(\sum_{j=1}^n n_j\right) = H(\lambda_1, ..., \lambda_n) + \sum_{j=1}^n \lambda_j h(n_j). \tag{D.4}$$

Now letting $n_i = m$ for all $i = 1, ..., n$ the previous formula becomes

$$h(mn) = h(m) + h(n). \tag{D.5}$$

Because m and n are integer we cannot immediately conclude that the solution to D.5 is $k \ln(\cdot)$ where k is a specific constant satisfying initial conditions. Fortunately, this problem can be solved using a standard procedure. By induction we can extend D.5 to

$$h(m_1 \cdots m_k) = h(m_1) + \cdots + h(m_k) \tag{D.6}$$

and if the factors are the same, we get

$$h(n^k) = kh(n). \tag{D.7}$$

Now let t and s be two integers greater than or equal to 2. We can then find for any large m an integer n such that

$$\frac{n}{m} \leq \frac{\ln(s)}{\ln(t)} < \frac{n+1}{m}, \quad \text{or} \quad t^n \leq s^m < t^{n+1} \tag{D.8}$$

Since h is increasing, $h(t^n) \leq h(s^m) \leq h(s^{n+1})$; and by D.7 this is equivalent to

$$nh(t) \leq mh(s) \leq (n+1)h(t), \tag{D.9}$$

and can be rewritten as

$$\frac{n}{m} \leq \frac{h(s)}{h(t)} \leq \frac{n+1}{m}. \tag{D.10}$$

Taking the difference between D.8 and D.10, we obtain

$$\left| \frac{h(s)}{h(t)} - \frac{\ln(s)}{\ln(t)} \right| \leq \frac{1}{m} \tag{D.11}$$

that is equivalent to

$$\left| \frac{h(s)}{\ln(s)} - \frac{h(t)}{\ln(t)} \right| \leq \epsilon \tag{D.12}$$

where

$$\epsilon := \frac{h(t)}{m \ln(s)} \tag{D.13}$$

is arbitrarily small. Thus $h(t)/\ln(t)$ must be constant. In other words, $h(t)$ has to be of the form $k \ln(t)$ where k is constant and depend on the logarithm bases. We can leave the base arbitrary for a moment and just write $h(t)$ as $h(t) = \ln(t)$.

Since H is continuous, from D.4 we get that H can be expressed in terms of h and even more

$$H(\lambda_1, ..., \lambda_n) = -\sum_{i=1}^{n} \lambda_i \ln(\lambda_i) \tag{D.14}$$

Appendix E

In this part of the appendix, we will explain how to come up with the accounting equations for revenue, net income, current liabilities, assets, equity, and current assets used in the implementation of the model in this work.

We start making some assumptions that are commonly used in financial forecasting

$$\text{Revenue}_{t+1} \approx (1 + r_{\text{rev}})\text{Revenue}_t, \tag{E.1}$$

$$\text{COGS}_{t+1} \approx r_{\text{COGS}}\text{Revenue}_{t+1}, \tag{E.2}$$

$$\text{Inventory}_{t+1} \approx \left(\frac{\text{Inventory(Days)}}{365}\right)\text{COGS}_{t+1}, \tag{E.3}$$

$$\text{Accounts Receivable}_{t+1} \approx \left(\frac{\text{Days of sales outstanding}}{365}\right)\text{Revenue}_{t+1}, \tag{E.4}$$

$$\text{Accounts Payable}_{t+1} \approx \left(\frac{\text{Number of days payable}}{365}\right)\text{COGS}_{t+1}, \tag{E.5}$$

$$\text{Marketable Securities}_{t+1} \approx \text{Marketable Securities}_t + \Delta\text{Marketable Securities}_t, \tag{E.6}$$

$$\text{Net Increase(Cash)}_{t+1} \approx \text{Net Increase(Cash)}_t + \Delta\text{Net Increase(Cash)}_t, \tag{E.7}$$

$$\text{Retained Earnings}_{t+1} \approx \text{Retained Earnings}_t + \Delta\text{Retained Earnings}_t, \tag{E.8}$$

$$\text{Taxes}_{t+1} \approx r_{\text{tax}}\text{EBT}_{t+1} \tag{E.9}$$

where r_{rev} is the growth rate of revenue, r_{COGS} is the percentage of COGS to revenue, and r_{tax} is the tax rate on revenue. In addition to the formulas mentioned above, we include other assumptions based on practical purposes and the comparatively small changes on the accounts equity capital, PPE, and debt in short periods of time.

$$\text{PPE}_{t+1} \approx \text{PPE}_t \qquad \text{and} \qquad \text{Equity Capital}_{t+1} \approx \text{Equity Capital}_t. \tag{E.10}$$

From these assumptions, we get the following equations for forecasting. For inventories, we get

$$\begin{aligned}
\text{Inventory}_{t+1} &\approx \left(\frac{\text{Inventory(Days)}}{365}\right)\text{COGS}_{t+1} \\
&\approx \left(\frac{\text{Inventory(Days)}}{365}\right) r_{\text{COGS}}\text{Revenue}_{t+1} \\
&\approx \left(\frac{\text{Inventory(Days)}}{365}\right) r_{\text{COGS}}(1+r_{\text{rev}})\text{Revenue}_t \\
&\approx \left(\frac{\text{Inventory(Days)}}{365}\right)(1+r_{\text{rev}})\text{COGS}_t \\
&\approx (1+r_{\text{rev}})\text{Inventory}_t.
\end{aligned} \tag{E.11}$$

For accounts receivable

$$\begin{aligned}
\text{Accounts Receivable}_{t+1} &\approx \left(\frac{\text{Days of sales outstanding}}{365}\right)\text{Revenue}_{t+1} \\
&\approx \left(\frac{\text{Days of sales oustanding}}{365}\right)(1+r_{rev})\text{Revenue}_t \\
&\approx (1+r_{\text{rev}})\text{Accounts Receivable}_t.
\end{aligned} \tag{E.12}$$

For accounts payable

$$\begin{aligned}
\text{Accounts Payable}_{t+1} &\approx \left(\frac{\text{Number of days payable}}{365}\right)\text{COGS}_{t+1} \\
&\approx \left(\frac{\text{Number of days payable}}{365}\right) r_{\text{COGS}}\text{Revenue}_{t+1} \\
&\approx \left(\frac{\text{Number of days payable}}{365}\right) r_{\text{COGS}}(1+r_{\text{rev}})\text{Revenue}_{t} \\
&\approx \left(\frac{\text{Number of days payable}}{365}\right)(1+r_{\text{rev}})\text{COGS}_{t} \\
&\approx (1+r_{\text{rev}})\text{Accounts Payable}_{t}.
\end{aligned} \tag{E.13}$$

For net income, we assume it is of the form

$$\text{Net Income}_{t+1} \approx \text{EBT}_{t+1} - \text{Taxes}_{t+1}. \tag{E.14}$$

Using the formula for taxes E.9, we get

$$\begin{aligned}
\text{Net Income}_{t+1} &\approx \text{EBT}_{t+1} - \text{Taxes}_{t+1} \\
&\approx \text{EBT}_{t+1} - r_{\text{tax}}\text{EBT}_{t+1} \\
&\approx (1-r_{\text{tax}})\text{EBT}_{t+1} \\
&\approx (1-r_{\text{tax}})[\text{Gross Profit}_{t+1} - \text{Total Expenses}_{t+1}]
\end{aligned} \tag{E.15}$$

Now, we have to rewrite Net Income$_{t+1}$ as a function of the its past observation plus other variables that are sensible to macroeconomic events. This can be done by the formulas

$$\begin{aligned}
\text{Gross Profit}_{t+1} &\approx \text{Revenue}_{t+1} - \text{COGS}_{t+1} \\
&\approx (1+r_{\text{rev}})\text{Revenue}_{t} - r_{\text{COGS}}\text{Revenue}_{t+1} \\
&\approx (1+r_{\text{rev}})\text{Revenue}_{t} - r_{\text{COGS}}(1+r_{\text{rev}})\text{Revenue}_{t} \\
&\approx (1+r_{\text{rev}})\text{Gross Profit}_{t},
\end{aligned} \tag{E.16}$$

and

$$\begin{aligned}
\text{Total Expenses}_{t+1} \approx\ & \text{Wages}_{t+1} + \text{Rent and Overhead}_{t+1} + \delta_{\text{dep}}\text{PPE}_{t} \\
&+ \text{Debt Opening}_{t+1}\text{Interest}_{t+1},
\end{aligned} \tag{E.17}$$

where Interest$_{t+1}$ $\approx$ Interest Expense$_{t+1}$ and it is computed as a percentage of Debt Opening$_{t+1}$. The account PPE$_{t+1}$ stands for property, plant and equipment at time $t+1$, and δ_{dep} is the depreciation rate. Since Microsoft, Alphabet, and Oracle exhibit characteristics of economies of scale, we will assume Total Expense$_{t+1}$ $\approx$ Total Expense$_{t}$ in short periods of time for this forecast. Therefore, by formulas E.16 and E.17, we can rewrite Net Income$_{t+1}$ as follows

$$\begin{aligned}
\text{Net Income}_{t+1} &\approx (1-r_{\text{tax}})(\text{Gross Profit}_{t+1} - \text{Total Expenses}_{t+1}) \\
&\approx (1-r_{\text{tax}})((1+r_{\text{rev}})\text{Gross Profit}_{t} - \text{Total Expenses}_{t}) \\
&\approx \text{Net Income}_{t} + (1-r_{\text{tax}})r_{\text{rev}}\text{Gross Profit}_{t}
\end{aligned} \tag{E.18}$$

where the last line comes from equation E.15.

For the sake of simplicity, we approximate current liabilities by accounts receivable

$$\begin{aligned}\text{Current Liabilities}_{t+1} &\approx \text{Accounts Payable}_{t+1} \\ &\approx (1 + r_{\text{rev}})\text{Accounts Payable}_t \\ &\approx \text{Current Liabilities}_t + r_{\text{rev}}\text{Accounts Payable}_t.\end{aligned} \tag{E.19}$$

For equity, using the assumptions E.8 and E.10, we obtain

$$\begin{aligned}\text{Equity}_{t+1} &\approx \text{Equity Capital}_{t+1} + \text{Retained Earnings}_{t+1} \\ &\approx \text{Equity Capital}_t + \text{Retained Earnings}_t + \Delta\text{Retained Earnings}_t \\ &\approx \text{Equity}_t + \Delta\text{Retained Earnings}_t\end{aligned} \tag{E.20}$$

For MSFT, and ORCL the assets account is of the form

$$\text{Assets}_{t+1} \approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Inventory}_{t+1} + \text{Marketable Securities}_{t+1} + \text{PPE}_{t+1}. \tag{E.21}$$

On the other hand, the formula for ORCL is of the form

$$\text{Assets}_{t+1} \approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Marketable Securities}_{t+1} + \text{PPE}_{t+1}. \tag{E.22}$$

To come up with a formula for assets that can be used for forecasting, we write the following approximation for cash

$$\begin{aligned}\text{Cash}_{t+1} &\approx \text{Closing Cash Balance}_{t+1} \\ &\approx \text{Opening Cash Balance}_{t+1} + (\text{CFF}_{t+1} + \text{CFI}_{t+1} + \text{CFO}_{t+1}) \\ &\approx \text{Cash}_t + (\text{CFF}_{t+1} + \text{CFI}_{t+1} + \text{CFO}_{t+1}) \\ &\approx \text{Cash}_t + \text{Net Increse(Cash)}_{t+1} \\ &\approx \text{Cash}_t + \text{Net Increse(Cash)}_t + \Delta\text{Net Increse(Cash)}_t\end{aligned} \tag{E.23}$$

Using the previous expression, we can rewrite the assets account for MSFT, and ORCL as follows

$$\begin{aligned}\text{Assets}_{t+1} \approx\ &\text{Cash}_t + \text{Net Increase(Cash)}_t + \Delta\text{Net Increase(Cash)}_t + (1 + r_{\text{rev}})\text{Accounts Receivable}_t \\ &+ (1 + r_{\text{rev}})\text{Inventory}_t + \text{Marketable Securities}_t + \Delta\text{Marketable Securities}_t + \text{PPE}_t \\ \approx\ &\text{Assets}_t + r_{\text{rev}}(\text{Accounts Receivable}_t + \text{Inventory}_t) + \text{Net Increase(Cash)}_t \\ &+ \Delta\text{Net Increase(Cash)}_t + \Delta\text{Marketable Securities}_t.\end{aligned} \tag{E.24}$$

Using the same procedure, the asset account for ORCL is

$$\begin{aligned}\text{Assets}_{t+1} \approx\ &\text{Cash}_t + \text{Net Increase(Cash)}_t + \Delta\text{Net Increase(Cash)}_t + (1 + r_{\text{rev}})\text{Accounts Receivable}_t \\ &+ \text{Marketable Securities}_t + \Delta\text{Marketable Securities}_t + \text{PPE}_t \\ \approx\ &\text{Assets}_t + r_{\text{rev}}\text{Accounts Receivable}_t + \text{Net Increase(Cash)}_t + \Delta\text{Net Increase(Cash)}_t \\ &+ \Delta\text{Marketable Securities}_t.\end{aligned} \tag{E.25}$$

For MSFT and GOOG current assets accounts, the formula is given by

$$\begin{aligned}\text{Currents Assets}_{t+1} &\approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Inventory}_{t+1} + \text{Marketable Securities}_{t+1} \\ &\approx \text{Current Assets}_t + r_{\text{rev}}(\text{Accounts Receivable}_t + \text{Inventory}_t) + \text{Net Increase(Cash)}_t \\ &\quad + \Delta\text{Net Increase(Cash)}_t + \Delta\text{Marketable Securities}_t,\end{aligned} \tag{E.26}$$

and for ORCL the formula is

$$\begin{aligned}\text{Currents Assets}_{t+1} &\approx \text{Cash}_{t+1} + \text{Accounts Receivable}_{t+1} + \text{Marketable Securities}_{t+1} \\ &\approx \text{Current Assets}_t + r_{\text{rev}}\text{Accounts Receivable}_t + \text{Net Increase(Cash)}_t \\ &\quad + \Delta\text{Net Increase(Cash)}_t + \Delta\text{Marketable Securities}_t,\end{aligned} \tag{E.27}$$

Appendix F

Bar plot Microsoft's revenue classified by major geographic areas

```
market_geo_msft <- data.frame(Region=rep(c('United States', 'Other Countries'),
 each = 4), Quarter=rep(c('Q1','Q2', 'Q3', 'Q4'), 2),
          Revenue = c(14470,16787,15372,16321,14344,15684,15199,15492))
market_geo_msft
market_geo_msft  %>% group_by(Quarter, Region) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue / sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x=Quarter, y=sum_revenue, fill=Region))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Paired")+
  ggtitle("Revenue Classified by Major Geographic Areas MSFT") +
  xlab("Quarters 2019") +
  ylab("Revenue in millions of dollars")
```

Bar plot Oracle's revenue classified by major geographic areas

```
market_geo_goog <- data.frame(Region=rep(c('United States', 'EMEA', 'APAC',
 'Other Countries'), each = 4),
    Quarter=rep(c('Q3 2018','Q4 2018', 'Q1 2019', 'Q2 2019'), 4),
Revenue = c(15523,16027,16532,17863, 10958,13374,11791,12401,
                      5424,5768,6112,6551,1835,1869,1904,2129))
market_geo_goog
market_geo_goog %>% group_by(Quarter, Region) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue / sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x=Quarter, y=sum_revenue, fill=Region))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Paired")+
  ggtitle("Revenue Classified by Major Geographic Areas GOOG") +
  xlab("Quarters 2018-2019") +
  ylab("Revenue in millions of dollars")
```

Bar plot Alphabet's revenue classified by major geographic areas

```
market_geo_orcl <- data.frame(Region=rep(c('Americas', 'EMEA', 'APAC'),
each = 4), Quarter=rep(c('Q1','Q2', 'Q3', 'Q4'), 3),
  Revenue = c(5161,5243,5266,5208,2576,2782,2781,2667,1456,1537,1567,1541))
market_geo_orcl
market_geo_orcl %>% group_by(Quarter, Region) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue / sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x=Quarter, y=sum_revenue, fill=Region))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Paired")+
  ggtitle("Revenue Classified by Major Geographic Areas ORCL") +
  xlab("Quarters 2019") +
  ylab("Revenue in millions of dollars")
```

Bar plot top U.S. exports to Canada

```
us_canada_export <- data.frame(Activities=c('Vehicles','Machinery',
'Mineral Fuels','Electrical Machinery',
'Plastics', 'Rest of Economic Activities'),
                                     Revenue=c(52,45,27,26,14, 135.8))
us_canada_export
us_canada_export_chart <- us_canada_export %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  arrange(sum_revenue) %>%
  ggplot(aes(x='U.S. exports to Canada', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top exports to Canada 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Bar plot top U.S. exports to México

```
us_mexico_export <- data.frame(Activities=c('Machinery','Electrical Machinery',
                            'Mineral Fuels','Vehicles',
                                'Plastics', 'Rest of Economic Activities'),
                                     Revenue=c(46,43,35,22,18,140.4))

us_mexico_export_chart <- us_mexico_export %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x='U.S. exports to Mexico', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top exports to Mexico 2018") +
```

```
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Bar plot top U.S. exports to China

```
us_china_export <- data.frame(Activities=c('Aircraft','Machinery',
              'Electrical Machinery', 'Optical and Medical Intruments',
                     'Vehicles', 'Rest of Economic Activities'),
                               Revenue=c(18,14,13,9.8,9.4,56.1))

us_china_export_chart <-  us_china_export %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x='U.S. exports to China', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top exports to China 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Bar plot top U.S. exports to Hong Kong

```
us_hongkong_export <- data.frame(Activities=c('Electrical Machinery',
'Precious Metals and Stones','Art and Antiques',
            'Machinery','Meat', 'Rest of Economic Activities'),
                               Revenue=c(11,9.2,2.2,2.1,1.6,11.2))

us_hongkong_export_chart <- us_hongkong_export %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x='U.S. exports to Hong Kong', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top exports to Hong Kong 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Merging U.S. exports bar plots

```
ggarrange(us_canada_export_chart, us_mexico_export_chart, us_china_export_chart,
          us_hongkong_export_chart, ncol=2, nrow=2)
```

Bar plot Top U.S. imports from Canada

```
us_canada_import <- data.frame(Activities=c('Mineral Fuels',
'Vehicles','Machinery','Special Other (returns)',
                        'Plastics', 'Rest of Economic Activities'),
                                Revenue=c(85,53,23,16,12,129.8))

us_canada_import_chart <- us_canada_import %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  arrange(sum_revenue) %>%
  ggplot(aes(x='U.S. imports from Canada', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top imports from Canada 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Bar plot Top U.S. imports from México

```
us_mexico_import <- data.frame(Activities=c('Vehicles','Electrical Machinery',
      'Machinery','Mineral Fuels',
         'Optical and Medical Instruments',
            'Rest of Economic Activities'),
                    Revenue=c(93,64,63,16,15,95.1))

us_mexico_import_chart <- us_mexico_import %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x='U.S. imports from Mexico', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top imports from Mexico 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Bar plot top U.S. imports from China

```
us_china_import <- data.frame(Activities=c('Electrical Machinery',
    'Machinery','Furniture and Bedding',
          'Toys and Sports Equipment', 'Plastics',
                             'Rest of Economic Activities'),
                                 Revenue=c(152,117,35,27,19,189.5))

us_china_import_chart <-  us_china_import %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x='U.S. imports from China', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
```

```
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top imports from China 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Bar plot top U.S. imports from Hong Kong

```
us_hongkong_import <- data.frame(Activities=c('Special Other (returns)',
'Electrical Machinery','Precious Metals, and Stones',
         'Machinery', 'Plastics', 'Rest of Economic Activities'),
                      Revenue=c(2.4,0.98,0.916, 0.321, 0.166, 1.517))

us_hongkong_import_chart <- us_hongkong_import %>%
  group_by(Activities) %>%
  summarize(sum_revenue = sum(Revenue)) %>%
  mutate(percent = sum_revenue/sum(sum_revenue), cum_sum = cumsum(sum_revenue)) %>%
  ggplot(aes(x='U.S. imports from Hong Kong', y=sum_revenue, fill=Activities))+
  geom_bar(stat='identity') +
  geom_text(aes(label=paste0(sprintf("%1.1f", percent*100),"%")),
            position=position_stack(vjust=0.5), colour="white") +
  scale_fill_brewer(palette="Dark2")+
  ggtitle("Segmentation U.S. top imports from Hong Kong 2018") +
  xlab("") +
  ylab("Revenue in billions of U.S. dollars")
```

Merging U.S. import bar plots

```
ggarrange(us_canada_import_chart, us_mexico_import_chart, us_china_import_chart,
          us_hongkong_import_chart, ncol=2, nrow=2)
```

Graph U.S. exports to Canada, China, Hong Kong, and México, Canada

```
us_china_export_ts <- getSymbols("EXPCH", src = "FRED", auto.assign = FALSE)
us_mexico_export_ts <- getSymbols("EXPMX", src = "FRED", auto.assign = FALSE)
us_canada_export_ts <- getSymbols("EXPCA", src = "FRED", auto.assign = FALSE)
us_hongkong_export_ts <- getSymbols("EXP5820", src = "FRED", auto.assign = FALSE)

merged_exports <- merge.xts(us_china_export_ts, us_mexico_export_ts,
                            us_canada_export_ts, us_hongkong_export_ts, by = "ID")
names(merged_exports) <- c('China', 'Mexico', 'Canada', 'Hong Kong', 'ID')
merged_exports <- subset(merged_exports, select = c(1,2,3,4))
merged_exports <- data.frame(merged_exports)
merged_exports$Dates <- index(us_canada_export_ts)
head(merged_exports)
df1 <- merged_exports %>%
  select(Dates , China, Mexico, Canada, Hong.Kong)  %>%
  gather(key = "variable", value = "value", -Dates)

ggplot(df1, aes(x = Dates, y = value)) +
  geom_line(aes(color = variable), size = 1) +
  scale_color_manual(values = c("#00AFBB", "#E7B800", "#CC0099", "#FF3300")) +
  theme_minimal() +
  ggtitle("U.S. Exports of Goods by F.A.S Basis Canada, China, Hong Kong, and Mexico") +
  xlab("Source: U.S. Bureu of Economic Analytics") +
  ylab("Millions of Dollars")
```

Graph U.S. imports from Canada, China, and México

```
us_china_import_ts <- getSymbols("IMPCH", src = "FRED", auto.assign = FALSE)
us_mexico_import_ts <- getSymbols("IMPMX", src = "FRED", auto.assign = FALSE)
us_canada_import_ts <- getSymbols("IMPCA", src = "FRED", auto.assign = FALSE)

merged_imports <- merge.xts(us_china_import_ts, us_mexico_import_ts,
                            us_canada_import_ts, by = "ID")
names(merged_imports) <- c('China', 'Mexico', 'Canada', 'ID')
merged_imports <- subset(merged_imports, select = c(1,2,3))
merged_imports <- data.frame(merged_imports)
merged_imports$Dates <- index(us_canada_import_ts)
head(merged_imports)
  df2 <- merged_imports %>%
  select(Dates , China, Mexico, Canada)  %>%
  gather(key = "variable", value = "value", -Dates)

ggplot(df2, aes(x = Dates, y = value)) +
  geom_line(aes(color = variable), size = 1) +
  scale_color_manual(values = c("#00AFBB", "#E7B800", "#CC0099")) +
  theme_minimal() +
  ggtitle("U.S Imports of Goods by Customs Basis from Canada, China, and Mexico") +
  xlab("Source: U.S. Bureu of Economic Analytics") +
  ylab("Millions of Dollars")
```

Graph of VIX vs SPY, and their correlation

```
vix <- getSymbols("VIXCLS", src = "FRED", auto.assign = FALSE)
spy <- getSymbols("SPY", src = "yahoo", auto.assign = FALSE)
spy_vix<- merge.xts(spy, vix, by = "ID")

# start date spy and vix
start_date_spy_vix <-
  min(which(format(index(spy_vix), "%Y-%m-%d") == '2007-01-10'))

# end date SPY and VIX
end_date_spy_vix <-
  min(which(format(index(spy_vix), "%Y-%m-%d") == '2020-03-16'))

spy_vix <- spy_vix[start_date_spy_vix: end_date_spy_vix]
spy_vix <- na.approx(spy_vix)
spy_vix <- subset(spy_vix, select = c(1,7))
names(spy_vix) <- c('SPY', 'VIX')
dates_spy_vix <- index(spy_vix)
spy_vix <- data.frame(spy_vix)
spy_vix$Dates <- dates_spy_vix


df3 <- spy_vix %>%
  select(Dates , SPY, VIX)  %>%
  gather(key = "variable", value = "value", -Dates)

ggplot(df3, aes(x = Dates, y = value)) +
  geom_line(aes(color = variable), size = 1) +
  scale_color_manual(values = c("#00AFBB", "#FF3300")) +
  theme_minimal() +
  ggtitle("VIX and SPY500 indexes from 2007-01-10 to 2020-03-16 ") +
  xlab("Source: Chicago Board Options Exchange") +
```

```r
  ylab("Dollars")


cor(spy_vix$SPY, spy_vix$VIX)
```

Error OLS and difference operator applied to Error OLS plots

```r
# Time series with outliers
error_ols <- unlist(list_error_ols, use.name=FALSE)

# Cleaned time series errors
cleaned_error_ols <- tsclean(error_ols)

# Error OLS
dates_cleaned_errors_ols <-
                index(MSFT_Close[start_date_time_s_analysis  :(length(MSFT_Close)-2)])
df_cleaned_error_ols <- data.frame(cleaned_error_ols)
df_cleaned_error_ols$Dates <- dates_cleaned_errors_ols
colnames(df_cleaned_error_ols) <- c( "Error OLS", "Dates")
df_cleaned_error_ols <- df_cleaned_error_ols %>%
  select(Dates, "Error OLS") %>%
  gather(key="variable", value = "value", -Dates)

plot_error_ols <- ggplot(df_cleaned_error_ols, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle(TeX("Error OLS")) +
  xlab("Dates") +
  ylab("")


# Diff Error OLS
dates_diff_cleaned_errors_ols <-
        index(MSFT_Close[(start_date_time_s_analysis+1)  :(length(MSFT_Close)-2)])
df_diff_cleaned_error_ols <- data.frame(diff(cleaned_error_ols))
df_diff_cleaned_error_ols$Dates <- dates_diff_cleaned_errors_ols
colnames(df_diff_cleaned_error_ols) <- c( "Diff Error OLS", "Dates")
df_diff_cleaned_error_ols <- df_diff_cleaned_error_ols %>%
  select(Dates, "Diff Error OLS") %>%
  gather(key="variable", value = "value", -Dates)

plot_diff_error_ols <- ggplot(df_diff_cleaned_error_ols, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle(TeX("Diff Error OLS: $\\nabla \\epsilon$")) +
  xlab("Dates") +
  ylab("")

# Plotting Error OLS and diff Error OLS
ggarrange(plot_error_ols, plot_diff_error_ols, ncol=1, nrow=2)
```

Implied volatility MSFT, ORCL, and GOOG plots

```r
# Implied Volatility MSFT
dates_prices_msft <- index(MSFT_Close[start_date_time_s_analysis  :length(MSFT_Close)])
df_impl_volatility_msft <- data.frame(implied_volatility_msft)
df_impl_volatility_msft$Dates <- dates_prices_msft
colnames(df_impl_volatility_msft) <- c( "Implied Volatility", "Dates")

df_impl_volatility_msft <- df_impl_volatility_msft %>%
  select(Dates, "Implied Volatility") %>%
  gather(key="variable", value = "value", -Dates)

plot_imp_volatility_msft <- ggplot(df_impl_volatility_msft, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle("Implied Volatility MSFT") +
  xlab("Dates") +
  ylab("")

# Implied Volatility ORCL
dates_prices_orcl <- index(ORCL_Close[start_date_time_s_analysis  :length(ORCL_Close)])
df_impl_volatility_orcl <- data.frame(implied_volatility_orcl)
df_impl_volatility_orcl$Dates <- dates_prices_msft
colnames(df_impl_volatility_orcl) <- c( "Implied Volatility", "Dates")

df_impl_volatility_orcl <- df_impl_volatility_orcl %>%
  select(Dates, "Implied Volatility") %>%
  gather(key="variable", value = "value", -Dates)

plot_imp_volatility_orcl <- ggplot(df_impl_volatility_orcl, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle("Implied Volatility ORCL") +
  xlab("Dates") +
  ylab("")


# Implied Volatility GOOG
dates_prices_goog <- index(GOOG_Close[start_date_time_s_analysis  :length(GOOG_Close)])
df_impl_volatility_goog <- data.frame(implied_volatility_goog)
df_impl_volatility_goog$Dates <- dates_prices_goog
colnames(df_impl_volatility_goog) <- c( "Implied Volatility", "Dates")

df_impl_volatility_goog <- df_impl_volatility_goog %>%
  select(Dates, "Implied Volatility") %>%
  gather(key="variable", value = "value", -Dates)

plot_imp_volatility_goog <- ggplot(df_impl_volatility_goog, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle("Implied Volatility GOOG") +
  xlab("Dates") +
  ylab("")



ggarrange(plot_imp_volatility_msft, plot_imp_volatility_orcl,
```

```
         plot_imp_volatility_goog, ncol=1, nrow=3)
```

Difference operator on implied volatility of MSFT, ORCL, and GOOG plots

```
# Diff Implied Volatility MSFT
dates_prices_msft <- index(MSFT_Close[start_date_time_s_analysis  :length(MSFT_Close)])
df_diff_impl_volatility_msft <-
      data.frame(diff(implied_volatility_msft[1:(length(implied_volatility_msft)-2)]))
id_remove_msft_dates <- which(dates_prices_msft %in%
         c(dates_prices_msft[1],dates_prices_msft[41],dates_prices_msft[42]))
dates_prices_msft <- dates_prices_msft[-id_remove_msft_dates]
df_diff_impl_volatility_msft$Dates <- dates_prices_msft
colnames(df_diff_impl_volatility_msft) <- c( "Diff Implied Volatility", "Dates")
df_diff_impl_volatility_msft
df_diff_impl_volatility_msft <- df_diff_impl_volatility_msft %>%
  select(Dates, "Diff Implied Volatility") %>%
  gather(key="variable", value = "value", -Dates)

plot_diff_volatility_msft <- ggplot(df_diff_impl_volatility_msft, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle(TeX("Diff Implied Volatility MSFT: $\\nabla \\sigma_{imp}")) +
  xlab("Dates") +
  ylab("")



# Diff Implied Volatility ORCL
dates_prices_orcl <- index(ORCL_Close[start_date_time_s_analysis  :length(ORCL_Close)])
df_diff_impl_volatility_orcl <-
  data.frame(diff(implied_volatility_orcl[1:(length(implied_volatility_orcl)-2)]))
id_remove_orcl_dates <- which(dates_prices_orcl %in%
          c(dates_prices_orcl[1],dates_prices_orcl[41],dates_prices_orcl[42]))
dates_prices_orcl <- dates_prices_orcl[-id_remove_orcl_dates]
df_diff_impl_volatility_orcl$Dates <- dates_prices_orcl
colnames(df_diff_impl_volatility_orcl) <- c( "Diff Implied Volatility", "Dates")
df_diff_impl_volatility_orcl <- df_diff_impl_volatility_orcl %>%
  select(Dates, "Diff Implied Volatility") %>%
  gather(key="variable", value = "value", -Dates)

plot_diff_volatility_orcl<- ggplot(df_diff_impl_volatility_orcl, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle(TeX("Diff Implied Volatility ORCL: $\\nabla \\sigma_{imp}")) +
  xlab("Dates") +
  ylab("")


# Diff Implied Volatility GOOG
dates_prices_goog <- index(GOOG_Close[start_date_time_s_analysis  :length(GOOG_Close)])
df_diff_impl_volatility_goog <-
  data.frame(diff(implied_volatility_goog[1:(length(implied_volatility_goog)-2)]))
id_remove_goog_dates <- which(dates_prices_goog %in%
         c(dates_prices_goog[1],dates_prices_goog[41],dates_prices_goog[42]))
dates_prices_goog <- dates_prices_goog[-id_remove_goog_dates]
```

```r
df_diff_impl_volatility_goog$Dates <- dates_prices_goog
colnames(df_diff_impl_volatility_goog) <- c( "Diff Implied Volatility", "Dates")
df_diff_impl_volatility_goog
df_diff_impl_volatility_goog <- df_diff_impl_volatility_goog %>%
  select(Dates, "Diff Implied Volatility") %>%
  gather(key="variable", value = "value", -Dates)

plot_diff_volatility_goog<- ggplot(df_diff_impl_volatility_goog, aes(x = Dates, y = value)) +
  geom_line(aes( color = variable), size = 1) +
  scale_color_manual(values = c("#999999")) +
  theme_minimal() +
  geom_point(aes(color=variable))+
  ggtitle(TeX("Diff Implied Volatility GOOG: $\\nabla \\sigma_{imp}")) +
  xlab("Dates") +
  ylab("")


ggarrange(plot_diff_volatility_msft,plot_diff_volatility_orcl,
          plot_diff_volatility_goog, ncol=1, nrow=3)
```

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