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Combinatorial Properties of Filters and Ideals

TESIS
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Introduction.

0.1 Filters and ideals.

It is not clear when the term *filter* was first introduced. Presumably, Henri Cartan was the first one in using the term *filter*, in his articles *Théorie des filtres*(1937, see (19)) and *Filtres et ultrafiltres*(1937, see (18)). However, there is previous work by H. Stone in 1936 (see (63)), where he was working with filters in fact, by mean of the notion *dual ideal*, which are filters whenever they do not contain the empty set (see (27), page 76). Moreover, the existence of ultrafilters was proved by A. Tarski in 1930, in his article *Une contribution à la théorie de la mesure* (see (65)), although he never used the term filter neither ultrafilter (or ideal and maximal ideal). Neverless, there is prior work related to the notion of filters, which can be tracked up to C. Carathéodory in the year of 1913 (see (17)), making use of decreasing sequence of non-empty sets, which are essentially filter bases (see also (27), page 76).

Filters and its dual notion, ideals, have played a prominent role along Topology and Set Theory. Some topological notions can be characterized in terms of filters, such as that of being a Hausdorff space, which is equivalent to saying that any filter converges to at most one point, and the compactness of a topological space X can be stated as the property that any ultrafilter on X converges to at least one point. Of course, their importance goes beyond these simple characterizations. With major degree of importance, they have been very useful in providing suitable representation theorems for some kind of objects. Maybe one of the most earlier and relevant results of this kind is the Stone's representation theorem:

Theorem 1 (H. Stone, see (63)). *Each Boolean space is homeomorphic to the Stone space of its characteristic algebra.*

This theorem has a very interesting consequence when talking about the boolean space $\beta\omega$, since in this case the characteristic algebra of $\beta\omega$ is $\mathcal{P}(\omega)$, whose Stone space is the family of all ultrafilters on ω, with the topology induced by the discrete topology on ω, that is, the basics of the topology are defined as $A^* = \{\mathcal{F} \in \beta\omega : A \in \mathcal{F}\}$, for $A \in \mathcal{P}(\omega)$. This formulation provides a nice way to study the topology of $\beta\omega$, and more interestingly, its residual $\beta\omega^* = \beta\omega \setminus \omega$, with the topology as subspace of $\beta\omega$.

The topological properties of $\beta\omega^*$ have been extensively studied, finding inside Set Theory a great leverage point, since its topological properties can be reformulated in terms of combinatorial properties, providing theorems in ZFC as well as independence results. There are several examples of how combinatorics of $\mathcal{P}(\omega)$ or $\mathcal{P}(\omega)/\mathsf{fin}$ help in the study of

$\beta\omega^*$, but we stick to the classical result of W. Rudin's theorem of the existence of p-points in $\beta\omega^*$ under CH.

Theorem 2 (W. Rudin, see (58)). *Assuming the Continuum Hypothesis, there are p-points in $\beta\omega^*$.*

This result implies that consistently $\beta\omega^*$ is not homogeneous, since there are points in $\beta\omega^*$ which are not p-points. The set theoretic proof of the previous theorem goes over an induction of length ω_1, and provides an ultrafilter having the following combinatorial property (recall that $A \subseteq^* B$ means that $A \setminus B$ is finite):

Definition 1 (W. Rudin, G. Choquet). *An ultrafilter $\mathcal{U}$ on ω is a p-point if for any countable collection $\{A_n : n \in \omega\} \subseteq \mathcal{U}$ there is $B \in \mathcal{U}$ such that for all $n \in \omega$, $B \subseteq^* A_n$.*

It is easy to see that the previous definition is equivalent to say that for any function $f : \omega \to \omega$, there is $A \in \mathcal{U}$ such that $f \restriction A$ is constant or finite to one. In this way, one can see that combinatorial properties of ultrafilters begin to arise in the study of $\beta\omega^*$. It was later stablished by K. Kunen, in ZFC alone, that $\beta\omega^*$ is not homogeneous, by constructing some special kind of ultrafilter having strong combinatorial properties (see (44)). However, such ultrafilters are not p-points, but weak p-points, that is, they do not live in the clousure of any countable subset of $\beta\omega^*$. Regarding the question about the existence of p-points in ZFC, it was proved by S. Shelah that it is relatively consistent with ZFC the non existence of p-points in $\beta\omega^*$:

Theorem 3 (S. Shelah, see (67), see also (20)). *It is relatively consistent with* ZFC *that there is no p-point.*

Another context where ultrafilters with special properties arise is in measure theory. In the year of 1968, G. Mokobodzki introduced in (54) the notion of rapid ultrafilters, and proved that under the Continuum Hypothesis they exist. He used these ultrafilters to construct a special kind of measurable function. In the same year, G. Choquet in his article *Deux classes remarquables d'ultrafiltres sur ω*, introduced, in modern terminology the p-points, q-points, and selective ultrafilters.

Definition 2 (G. Choquet, 1968, see (21)). *An ultrafilter $\mathcal{U}$ is a q-point if for any partition $\langle I_n : n \in \omega\rangle$ of ω into finite sets, there is $A \in \mathcal{U}$ such that $A \cap I_n$ has at most one element.*

Definition 3 (G. Choquet, 1968, see (21)). *An ultrafilter is a selective ultrafilter if it is a p-point and a q-point at the same time.*

Definition 4 (G. Mokobodzki, 1968, see (54)). *An ultrafilter $\mathcal{U}$ is rapid if for any function $f : \omega \to \omega$, there is $A \in \mathcal{U}$ such that $f \leq^* e_A$, where e_A is the increasing enumeration of A.*

Choquet also defined the notion of p-point, although he used the terminology δ-stable. Also, selective ultrafilters were originally named by him as *absolute* ultrafilters. All the previous notions of ultrafiters can be proved to exist under the Continuum Hypothesis. However, as in the case of p-points, their existence can not be proved from ZFC:

Theorem 4 (K. Kunen, (46)). *It is relatively consistent with* ZFC *that there is no selective ultrafilter.*

Theorem 5 (A. W. Miller, (51)). *It is relatively consistent with* ZFC *that there is no q-point, as well as there is no rapid ultrafilter.*

Another classical combinatorial notion of ultrafilter is that of *Hausdorff* ultrafilter. These ultrafilters have come to be quite elusive. The definition of Hausdorff ultrafilters is usually atributted to M. Daguenet-Tessier in her article *Ultrafiltres a la Façon de Ramsey*, in 1979 (24), but actually, they were previously defined by R. A. Pitt in his doctoral thesis in the year of 1971 (see (57)). Recall that given a function $f : \omega \to \omega$ and an ultrafilter $\mathcal{U}$, the ultrafilter $f(\mathcal{U})$ is defined as $f(\mathcal{U}) = \{A \in \mathcal{P}(\omega) : f^{-1}[A] \in \mathcal{U}\}$. R. A. Pitt, in his doctoral thesis, mentioned correspondence between his advisor R. O. Davies and G. Choquet, were G. Choquet makes the remark that if for any two functions $f, g \in \omega^\omega$ and any ultrafilter $\mathcal{U}$, if there is $A \in \mathcal{U}$ such that $f \restriction A = g \restriction A$, then $f(\mathcal{U}) = g(\mathcal{U})$, and considered the converse implication, that is, whether the equality $f(\mathcal{U}) = g(\mathcal{U})$ implies the existence of a set $A \in \mathcal{U}$ such that $f \restriction A = g \restriction A$. Then Choquet remarks that A. Connes proved that selective ultrafilters satisfies this implication, result which appeared in (23)[1], and Choquet asked whether this implication is a characterization of selective ultrafilters. Then, R. A. Pitt defines ultrafilters with property (C):

Definition 5 (R. A. Pitt, 1971, see (57), pages 102-103). *An ultrafilter $\mathcal{U}$ has property (C) if for any pair of functions $f, g \in \omega^\omega$ such that $f(\mathcal{U}) = g(\mathcal{U})$, there exists $A \in \mathcal{U}$ such that $f \restriction A = g \restriction A$.*

R. A. Pitt answered Choquet's question in the negative, that is, assuming the Continuum Hypothesis there is an ultrafilter with property (C) which is not a selective ultrafilter.

Ultrafilters with property (C) were also studied by M. Daguenet-Tessier, in (24). They have been extensively studied by M. Di Nasso and M. Forti in (25; 56; 26; 31), and in their article *Hausdorff ultrafilters*, they changed the name from ultrafilters with property (C) to Hausdorff ultrafilters, given the fact that this ultrafilters provides ultrapowers of ω which are Hausdorff topological spaces with the topology induced by the discrete topology on ω, that is, the basic open sets are of the form $A^* = \{[f]_\mathcal{U} \in \omega^\omega/\mathcal{U} : (\exists B \in \mathcal{U})(f[B] \subseteq A)\}$. This terminology has prevailed since then, and we stick to it. So, an equivalent formulation of Hausdorff ultrafilters is the following:

Definition 6. *An ultrafilter is Hausdorff if and only if for any pair of functions $f, g \in \omega^\omega$, there is $A \in \mathcal{U}$ such that either: $f[A] \cap g[A] = \emptyset$ or $f \restriction A = g \restriction A$.*

They mention in (26) that the question about the existence of Hausdorff ultrafilters in ZFC had been answered by T. Bartoszyński and S. Shelah, by proving the consistency of the assertion that there is no Hausdorff ultrafilter. However, later it was seen that the proof they presented had some mistakes, so the problem remained open. What Bartoszyński and Shelah actually proved, was the fact that in the Rational Perfect set model, Hausdorff ultrafilters are dense in the Rudin-Blass ordering (see (5)).

[1]The author thanks M. Hrušák by providing this reference.

0.2 $\mathscr{I}$-ultrafilters and the Katětov order.

As we saw in the previous section, there are several combinatorial notions of ultrafilters, so an attempt to classify their combinatorial notions may be of interest. There are two frameworks which are useful in such task: the notion of $\mathscr{I}$-ultrafilters, due to J. Baumgartner (see (6)), and the Katětov order on definable ideals, due to M. Katětov in 1968 (see (42)).

Definition 7 (J. Baumgartner, 1995)**.** *Let $\mathscr{I}$ be an ideal and $\mathcal{U}$ and ultrafilter, both of them on ω. We say that $\mathcal{U}$ is a $\mathscr{I}$-ultrafilter if for any function $f:\omega\to\omega$ there is $A\in\mathcal{U}$ such that $f[A]\in\mathscr{I}$. If we required the functions of be finite to one, then we say that $\mathcal{U}$ is a weak $\mathscr{I}$-ultrafilter.*

Definition 8 (M. Katětov, 1968)**.** *Let $\mathscr{I}$ and $\mathscr{J}$ be two ideals on ω. We say that $\mathscr{I}$ is Katětov below $\mathscr{J}$ if there is $f:\omega\to\omega$ such that for all $A\in\mathscr{I}$, $f^{-1}[A]\in\mathscr{J}$, and write $\mathscr{I}\leq_K\mathscr{J}$. If we require the function f to be finite to one, then we say that $\mathscr{I}$ is Katětov-Blass below $\mathscr{J}$, and write $\mathscr{I}\leq_{KB}\mathscr{J}$.*

It can be easily seen that the notion of $\mathscr{I}$-ultrafilter is codified in the Katětov order, since a given ultrafilter $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter if and only if $\mathscr{I}\nleq_K\mathcal{U}^*$, where $\mathcal{U}^*$ is the dual ideal to the ultrafilter $\mathcal{U}$. A similar remark holds between weak $\mathscr{I}$-ultrafilters and the Katětov-Blass order. Based on these remarks, we will write $\mathscr{I}\leq_{K(B)}\mathcal{U}^*$ when talking about (weak) $\mathscr{I}$-ultrafilters.

The strength of the Katětov order as a tool to classify the combinatorial properties of ultrafilters can be seen by the following proposition:

Proposition 1. *Let $\mathcal{U}$ be an ultrafilter on ω. The the following holds:*

1. *$\mathcal{U}$ is a p-point if and only if $\mathsf{Fin}\times\mathsf{Fin}\nleq_K\mathcal{U}^*$.*
2. *$\mathcal{U}$ is a q-point if and only if $\mathcal{ED}_{fin}\nleq_{KB}\mathcal{U}^*$.*
3. *$\mathcal{U}$ is a selective ultrafilter if and only if $\mathcal{ED}\nleq_K\mathcal{U}^*$.*
4. *$\mathcal{U}$ is a rapid ultrafilter if and only if for any summable ideal $\mathscr{I}$, $\mathscr{I}\nleq_{KB}\mathcal{U}^*$.*
5. *$\mathcal{U}$ is a Hausdorff ultrafilter if and only if $\mathcal{G}_{fc}\nleq_K\mathcal{U}^*$.*

We refer the reader to the Preliminaries section for the definition of the ideals in the previous proposition.

Katětov order has been extensively studied by M. Hrušák, D. Meza-Alcántara, H. Minami, H. Sakai and S. Solecki. In recent years, many interesting results have been proved. Concerning the structure of the Katětov order in Borel ideals, D. Meza-Alcántara proved (50) that $\mathcal{P}(\omega)/\mathsf{fin}$ is embeddable in the Katětov order, giving insight of how complex the Katětov order is. J. Grebík and M. Hrušák have proved (33) that there is no Katětov-minimal element among Borel ideals. H. Minami and H. Sakai (53) have as well proved that the Katětov and Katětov-Blass orders restricted to F_σ ideals are upward directed with cofinal type $(\omega^\omega,\leq^*)$. Sakai (59) has also proved that Katětov-Blass order on Borel ideals is countably upward directed and that all F_σ ideals are KB-below a common analytic p-ideal. On the other hand, considering the relations of Borel ideals and ultrafilters, the classical results about the non-existence of certain kind of ultrafilters, such as Ramsey ultrafilters (46), p-points (67), q-points (51), nwd-ultrafilters (60) and rapid ultrafilters (51), all of these are equivalent to saying that all ultrafilters are above some(or many) critical ideal, gives a measure of the strength of the Katětov order to classify ultrafilters.

However, the existence of $\mathscr{I}$-ultrafilters for some Borel ideal $\mathscr{I}$ was open (39). Recently, O. Guzmán González and M. Hrušák proved (see (32)) in ZFC the existence of an $F_{\sigma\delta\sigma}$ ideal for which the generic existence of $\mathscr{I}$-ultrafilters holds.

Theorem 6 (O. Guzmán González, M. Hrušák, 2019)**.** *There is an $F_{\sigma\delta\sigma}$ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist generically.*

They also proved that the complexity can not be lowered, since it is consistent that for all $F_{\sigma\delta}$ ideals generic existence does not hold, and raised the question about the existence of an F_σ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist.

Question 1 (O. Guzmán González, M. Hrušák)**.** *Is there an F_σ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist?*

0.3 The work of this thesis.

The main topic of this thesis is the existence of $\mathscr{I}$-ultrafilters.

Chapter 1 is devoted to introduce some preliminary notions and results that will be used in later chapters.

In Chapter 2, we consider parametrized diamond principles in order to produce $\mathscr{I}$-ultrafilters for definable ideals $\mathscr{I}$. For this purpose we define a new cardinal invariant associated to a given ideal. Then we proceed to give characterizations of this cardinal invariants for the most known Borel ideals in terms of most well known cardinal invariants of the continuum.

In Chapter 3, we answer a question of J. Blobner concerning the optimal cardinality of a family of summable ideals with the property that such family has enough information to characterize rapid ultrafilters, by proving that the minimum possible cardinality of such family is $\mathfrak{d}$, the dominating number. On the other hand, we answer another question of J. Blobner and generalize the theorem of T. Bartoszyński and S. Shelah about Hausdorff ultrafilters in the Rational Perfect set forcing, by proving that, in the Rational Perfect set model, for any analytic tall p-ideal on ω, $\mathscr{I}$-ultrafilters are dense in the Rudin-Blass ordering.

In Chapter 4, we answer Question 1 from the previous section, by proving that it is relatively consistent with ZFC that $\mathscr{I}$-ultrafilters does not exist for any F_σ ideal. In particular, this result implies the consistency of the assertion that there is no Hausdorff ultrafilter, answering the question raised by M. Di Nasso and M. Forti. The main theorem of this chapter also implies an answer to several other questions by J. Blobner, and gives a new model where there is no ultrafilter with property M, question that was originally solved by S. Shelah in his model for no nwd-ultrafilters.

In Chapter 5, we make use of parametrized diamond principles to answer some questions of D. Monk, concerning the existence of certain substructures of the boolean algebra $\mathcal{P}(\omega)/\mathsf{fin}$ with cardinality smaller than the continuum.

Introducción

0.4 Filtros e ideales.

No es claro cuándo se introdujo por primera vez el término de *filtro*. Probablemente, Henri Cartan fue el primero en usar el término *filtro*, en sus artículos *Théorie des filtres* (1937, ver (19)) y *Filtres et ultrafiltres* (1937, ver (18)). Sin embargo, existe trabajo previo de H. Stone en 1936 (ver (63)), en el cual de hecho trabajaba con filtros, mediante la noción de *ideal dual*, que son filtros siempre que no contienen el conjunto vacío (ver (27), página 76). Además, la existencia de ultrafiltros fue probada por A. Tarski en 1930, en su artículo *Une contributión a la théorie de la mesure* (ver (65)), aunque nunca usó el término filtro ni ultrafiltro (o ideal o ideal maximal). Sin embargo, existe un trabajo previo relacionado con la noción de filtros, que se puede rastrear hasta C. Carathéodory en el año de 1913 (ver (17)), haciendo uso de una sucesión decreciente de conjuntos no vacíos, que son esencialmente bases de filtros (ver también (27), página 76).

Los filtros y su noción dual, los ideales, han jugado un papel prominente a lo largo de la Topología y la Teoría de Conjuntos. Algunas nociones topológicas se pueden caracterizar en términos de filtros, como el de ser un espacio Hausdorff, lo que equivale a decir que cualquier filtro converge a lo sumo a un punto, y la compacidad de un espacio topológico X puede expresarse como la propiedad de que cualquier ultrafiltro sobre X converge al menos en un punto. Por supuesto, su importancia va más allá de estas simples caracterizaciones. Con mayor grado de importancia, han sido muy útiles para proporcionar teoremas de representaciones adecuadas para algún tipo de objetos. Quizás uno de los resultados más tempranos y relevantes de este tipo es el teorema de representación de Stone:

Teorema 1 (H. Stone, see (63))**.** *Cada espacio booleano es homeomorfo al espacio de Stone de su álgebra característica.*

Este teorema tiene una consecuencia muy interesante cuando se habla del espacio booleano $\beta\omega$, ya que en este caso el álgebra característica de $\beta\omega$ es $\mathcal{P}(\omega)$, cuyo espacio de Stone es la familia de todos los ultrafiltros sobre ω, con la topología inducida por la topología discreta en ω, es decir, los abiertos básicos de la topología se definen como $A^* = \{\mathcal{F} \in \beta\omega : A \in \mathcal{F}\}$, para $A \in \mathcal{P}(\omega)$. Esta formulación proporciona una buena manera de estudiar la topología de $\beta\omega$ y, lo que es más interesante, el residuo $\beta\omega^* = \beta\omega \setminus \omega$, con la topología como subespacio de $\beta\omega$.

Las propiedades topológicas de $\beta\omega^*$ han sido ampliamente estudiadas, encontrando dentro de la Teoría de Conjuntos un gran punto de apalancamiento, ya que sus propiedades topológicas pueden reformularse en términos de propiedades combinatorias, proporcionando teoremas en ZFC así como resultados de independencia. Hay varios ejemplos de cómo la combinatoria de $\mathcal{P}(\omega)$ o $\mathcal{P}(\omega)/\mathsf{fin}$ ayuda en el estudio de $\beta\omega^*$. Aquí presentamos el resultado clásico del teorema de W. Rudin de la existencia de p-puntos en $\beta\omega^*$ bajo CH.

Teorema 2 (W. Rudin, ver (58))**.** *Asumiendo la Hipótesis del Continuo, existen p-puntos en $\beta\omega$.*

Este resultado implica que $\beta\omega^*$ no es homogéneo, ya que hay puntos en $\beta\omega^*$ que no son p-puntos. La prueba conjuntista del teorema anterior es una inducción de longitud ω_1, y proporciona un ultrafiltro que tiene la siguiente propiedad combinatoria (recordemos que $A \subseteq^* B$ significa que $A \setminus B$ es finito) :

Definición 1 (W. Rudin, G. Choquet)**.** *Un ultrafiltro $\mathcal{U}$ sobre ω es un p-punto si para cada colección numerable $\{A_n : n \in \omega\} \subseteq \mathcal{U}$ hay $B \in \mathcal{U}$ tal que para todo $n \in \omega$, $B \subseteq^* A$.*

Es fácil ver que la definición anterior es equivalente a decir que para cualquier función $f : \omega \to \omega$, hay $A \in \mathcal{U}$ tal que $f \restriction A$ es constante o finito a uno. De esta forma, se puede ver que las propiedades combinatorias de los ultrafiltros comienzan a surgir en el estudio de $\beta\omega^*$. Más tarde, K. Kunen estableció, en ZFC, que $\beta\omega^*$ no es homogéneo, al construir un tipo especial de ultrafiltro con propiedades combinatorias fuertes (ver (44)). Sin embargo, estos ultrafiltros no son p-puntos , sino p-puntos débiles, es decir, no se encuentran en la cerradura de ningún subconjunto numerable de $\beta\omega^*$. Con respecto a la pregunta sobre la existencia de p-puntos en ZFC, S. Shelah demostró que es relativamente consistente con ZFC la no existencia de p-puntos en $\beta\omega^*$:

Teorema 3 (S. Shelah, ver (67), también ver (20))**.** *Es relativamente consistente con* ZFC *que no existen los p-puntos en $\beta\omega^*$.*

Otro contexto donde surgen ultrafiltros con propiedades especiales es en la Teoría de la Medida. En el año de 1968, G. Mokobodzki introdujo en (54) la noción de ultrafiltros rápidos, y demostró que bajo la Hipótesis Continuo existen. Mokobodzki usó estos ultrafiltros para construir un tipo especial de función medible. En el mismo año, G. Choquet en su artículo *Deux classes remarquables d'ultrafiltres sur omega*, definió, en terminología moderna, los p-puntos, q-puntos y los ultrafiltros selectivos.

Definición 2 (G. Choquet, 1968, ver (21))**.** *Un ultrafiltro $\mathcal{U}$ es un q-punto si para cualquier partición $\langle I_n : n \in \omega\rangle$ de ω en conjuntos finitos, existe $A \in \mathcal{U}$ tal que $A \cap I_n$ tiene a lo más un elemento.*

Definición 3 (G. Choquet, 1968, ver (21))**.** *Un ultrafiltro es selectivo si es un p-punto y un q-punto al mismo tiempo.*

Definición 4 (G. Mokobodzki, 1968, ver (54))**.** *Un ultraafiltro $\mathcal{U}$ es rápido si para cualquier función $f : \omega \to \omega$, hay $A \in \mathcal{U}$ tal que $f \leq^* e_A$, donde e_A es la enumeración creciente de A.*

Choquet también definió la noción de p-punto, aunque usó la terminología δ-estable. Además, los ultrafiltros selectivos originalmente fueron nombrados por él como ultrafiltros *absolutos.* Se puede probar que todas las nociones anteriores de ultrafitros existen bajo la Hipótesis del Continuo. Sin embargo, como en el caso de p-puntos, su existencia no se puede probar a partir de ZFC:

Teorema 4 (K. Kunen, (46)). *Es relativamente consistente con* ZFC *que no hay ultrafiltros selectivos.*

Teorema 5 (A. W. Miller, (51)). *Es relativamente consistente con* ZFC *que no hay q-puntos ni ultrafiltros rápidos.*

Otra noción combinatoria clásica de ultrafiltro es la de ultrafiltro *Hausdorff*. Estos ultrafiltros has sido bastante esquivos. La definición de ultrafiltros Hausdorff se suele atribuir a M. Daguenet-Tessier en su artículo *Ultrafiltres a la Façon de Ramsey*, en 1979 (24), pero en realidad fueron definidos previamente por R. A. Pitt en su tesis doctoral en el año de 1971 (ver (57)). Recordemos que dada una función $f : \omega \to \omega$ y un ultrafiltro $\mathcal{U}$, el ultrafiltro $f(\mathcal{U})$ se define como $f(\mathcal{U}) = \{A \in \mathcal{P}(\omega) : f^{-1}[A] \in \mathcal{U}\}$. R. A. Pitt, en su tesis doctoral, menciona la correspondencia entre su asesor R. O. Davies y G. Choquet, donde G. Choquet hace la observación de que para dos funciones cualesquiera $f, g \in \omega^\omega$ y cualquier ultrafiltro $\mathcal{U}$, si hay $A \in \mathcal{U}$ tal que $f \upharpoonright A = g \upharpoonright A$, entonces $f(\mathcal{U}) = g(\mathcal{U})$, y consideró la implicación inversa, es decir, si la igualdad $f(\mathcal{U}) = g(\mathcal{U})$ implica la existencia de un conjunto $A \in \mathcal{U}$ tal que $f \upharpoonright A = g \upharpoonright A$. Entonces Choquet comenta que A. Connes demostró que los ultrafiltros selectivos satisfacen esta implicación, resultado que apareció en (23) [2]. Choquet preguntó si esta implicación es una caracterización de ultrafiltros selectivos. Entonces, R. A. Pitt define los ultrafiltros con propiedad (C):

Definición 5 (R. A. Pitt, 1971, see (57), pages 102-103). *Un ultrafiltro tiene la propiedad (C) si para cualquier par de funciones $f, g \in \omega^\omega$ tal que $f(\mathcal{U}) = g(\mathcal{U})$, existe $A \in \mathcal{U}$ tal que $f \upharpoonright A = g \upharpoonright A$.*

R. A. Pitt respondió negativamente a la pregunta de Choquet, es decir, asumiendo la Hipótesis del Continuo, hay un ultrafiltro con propiedad (C) que no es un ultrafiltro selectivo.

Los ultrafiltros con la propiedad (C) también fueron estudiados por M. Daguenet-Tessier, en (24). Han sido extensamente estudiados por M. Di Nasso y M. Forti en (25; 56; 26; 31), y en su artículo *Hausdorff Ultrafilters*, cambiaron el nombre de ultrafiltros con propiedad (C) a ultrafiltros Hausdorff, dado que estos ultrafiltros proporcionan ultrapotencias de ω que son espacios topológicos Hausdorff con la topología inducida por la topología discreta en ω, es decir, los conjuntos abiertos básicos son de la forma $A^* = \{[f]_\mathcal{U} \in \omega^\omega/\mathcal{U} : (\exists B \in \mathcal{U})(f[B] \subseteq A)\}$. Esta terminología ha prevalecido desde entonces y nos atenemos a ella. Una formulación equivalente de ultrafiltros Hausdorff es la siguiente:

Definición 6. *Un ultrafiltro es Hausdorff si y sólo si para cualquier par de funciones $f, g \in \omega^\omega$, hay $A \in \mathcal{U}$ tal que ocurre exactamente una de las siguientes: $f[A] \cap g[A] = \emptyset$ o $f \upharpoonright A = g \upharpoonright A$.*

Se menciona en (26) que la pregunta sobre la existencia de ultrafiltros Hausdorff en ZFC había sido respondida por T. Bartoszyński y S. Shelah, al demostrar la consistencia de la afirmación de que no hay ultrafiltros de Hausdorff. Sin embargo, posteriormente se vio que la prueba que presentaban tenía algunos errores, por lo que el problema quedó

[2]El autor agradece a M. Hrušák por proporcionar esta referencia.

abierto. Lo que Bartoszyński y Shelah realmente demostraron fue el hecho de que en el modelo de Miller, los ultrafiltros Hausdorff son densos en el orden de Rudin-Blass (ver (5)).

0.5 $\mathscr{I}$-ultrafiltros y el orden de Katětov.

Como vimos en la sección anterior, existen varias nociones combinatorias de ultrafiltros, por lo que puede ser de interés intentar clasificar sus nociones combinatorias. Hay dos marcos que son útiles en tal tarea: la noción de $\mathscr{I}$-ultrafiltros, debido a J. Baumgardner, y el orden Katětov sobre ideales definibles, debido a M. Katětov(ver (42)).

Definición 7 (J. Baumgartner, 1995)**.** *Sea $\mathscr{I}$ un ideal y $\mathcal{U}$ un ultrafiltro, ambos sobre ω. Decimos que $\mathcal{U}$ es un $\mathscr{I}$-ultrafiltro si para cualquier función $f : \omega \to \omega$ hay $A \in \mathcal{U}$ tal que $f[A] \in \mathscr{I}$. Si requerimos que la función sea finito a uno, entonces decimos que $\mathcal{U}$ es un $\mathscr{I}$-ultrafiltro débil.*

Definición 8 (M. Katětov, 1968)**.** *Sean $\mathscr{I}$ y $\mathscr{J}$ dos ideales sobre ω. Decimos que $\mathscr{I}$ está Katětov debajo de $\mathscr{J}$ si hay una función $f : \omega \to \omega$ tal que para todo $A \in \mathscr{I}$, $f^{-1}[A] \in \mathscr{J}$, y escribimos $\mathscr{I} \leq_K \mathscr{J}$. Si pedimos que la función f sea finito a uno, entonces decimos que $\mathscr{I}$ está Katětov-Blass debajo de $\mathscr{J}$, y escribimos $\mathscr{I} \leq_{KB} \mathscr{J}$.*

Puede verse fácilmente que la noción de $\mathscr{I}$-ultrafiltro está codificada en el orden de Katětov, ya que un ultrafiltro dado $\mathcal{U}$ es un $\mathscr{I}$-ultrafiltro si y sólo si $\mathscr{I} \nleq_K \mathcal{U}^*$, donde $\mathcal{U}^*$ es el ideal dual del ultrafiltro $\mathcal{U}$. La observación análoga entre los $\mathscr{I}$-ultrafiltros débiles y el orden Katětov-Blass también se cumple. Con base en estos comentarios, escribiremos $\mathscr{I} \leq_{K(B)} \mathcal{U}^*$ cuando hablemos de $\mathscr{I}$-ultrafiltros(débiles).

La fuerza del orden Katětov como herramienta para clasificar las propiedades combinatorias de los ultrafiltros se puede ver en la siguiente proposición:

Proposición 1. *Sea $\mathcal{U}$ un ultrafiltro sobre ω. Lo siguiente es verdadero:*

1. *$\mathcal{U}$ es un p-punto si y sólo si* $\mathsf{Fin} \times \mathsf{Fin} \nleq_K \mathcal{U}^*$.
2. *$\mathcal{U}$ es un q-punto si y sólo si $\mathcal{ED}_{fin} \nleq_{KB} \mathcal{U}^*$.*
3. *$\mathcal{U}$ es un ultrafiltro selectivo si y sólosi $\mathcal{ED} \nleq_K \mathcal{U}^*$.*
4. *$\mathcal{U}$ es un ultrafiltro rápido si y sólo si para cualquier ideal sumable $\mathscr{I}$, $\mathscr{I} \nleq_{KB} \mathcal{U}^*$.*
5. *$\mathcal{U}$ es un ultrafiltro Hausdorff si y sólo si $\mathcal{G}_{fc} \nleq_K \mathcal{U}^*$.*

Remitimos al lector a la sección de Preliminares para la definición de los ideales en la proposición anterior.

El orden Katětov ha sido estudiado extensamente por M. Hrušák, D. Meza-Alcántara, H. Minami, H. Sakai y S. Solecki. En los últimos años se han demostrado varios resultados interesantes. Con respecto a la estructura del orden Katětov en los ideales borelianos, D. Meza-Alcántara demostró (ver (50)) que $\mathcal{P}(\omega)/\mathsf{fin}$ se encaja en el orden de Katětov restringido a los ideales F_σ, dando una idea de su complejidad. J. Grebík y M. Hrušák han demostrado (ver (33)) que no existe un elemento mínimo entre los ideales borelianos. H. Minami y H. Sakai (ver (53)) también han demostrado que los órdenes de Katětov y Katětov-Blass restringidos a ideales F_σ son dirigidos hacia arriba con tipo cofinal $(\omega^\omega, \leq^*)$. Sakai (ver (59)) también ha demostrado que el orden de Katětov-Blass en los ideales de Borel es numerablemente dirigido hacia arriba y que todos los ideales F_σ están Katětov-Blass por debajo de un mismo p-ideal analítico. Por otro lado, considerando las relaciones de los ideales borelianos y ultrafiltros, los resultados clásicos sobre la inexistencia de cierto tipo de ultrafiltros, como los ultrafiltros Ramsey (46), p-puntos (67), q-puntos (51), nwd-ultrafiltros (60) y ultrafiltros rápidos (51), todos ellos equivalen a decir que todos los ultrafiltros están por encima de alguno (o varios) ideal crítico, dando una idea de la fuerza del orden de Katětov para clasificar los ultrafiltros.

Sin embargo, la existencia de $\mathscr{I}$-ultrafiltros para algún ideal boreliano $\mathscr{I}$ permanecia abierta (ver (39)). Recientemente, O. Guzmán González y M. Hrušák demostraron (ver (32)) en ZFC la existencia de un $F_{\sigma\delta\sigma}$ ideal para el que se cumple la existencia genérica de $\mathscr{I}$-ultrafiltros.

Teorema 6 (O. Guzmán González, M. Hrušák, 2019)**.** *Existe un ideal $\mathscr{I}$ que es $F_{\sigma\delta\sigma}$ para el cual existen los $\mathscr{I}$-ultrafiltros genéricamente.*

También demostraron que la complejidad no se puede reducir, ya que es consistente que para todos los ideales $F_{\sigma\delta}$ la existencia genérica no se cumple, y plantearon la pregunta sobre la existencia de un ideal $\mathscr{I}$ F_σ para el cual $\mathscr{I}$-existen ultrafiltros.

Pregunta 1 (O. Guzmán González, M. Hrušák)**.** *¿Existe un ideal F_σ $\mathscr{I}$ para el cual existe un $\mathscr{I}$-ultrafiltro?*

0.6 El trabajo de esta tesis.

El tema principal de esta tesis es la existencia de $\mathscr{I}$-ultrafiltros.

El Capítulo 1 está dedicado a introducir algunas nociones y resultados preliminares que se utilizarán en capítulos posteriores.

En el Capítulo 2, consideramos los principios de diamantes parametrizados para producir $\mathscr{I}$-ultrafiltros para ideales definibles $\mathscr{I}$. Para ello definimos un nuevo invariante cardinal asociado a un ideal dado. Luego procedemos a dar caracterizaciones de estos invariantes cardinales para los ideales borelianos más conocidos en términos de los invariantes cardinales del continuo más conocidos.

En el Capítulo 3, respondemos a una pregunta de J. Blobner sobre la cardinalidad óptima de una familia de ideales sumables con la propiedad de que dicha familia tiene suficiente información para caracterizar ultrafiltros rápidos, demostrando que la cardinalidad mínima posible de dicha familia es $\mathfrak{d}$, el número dominante. Por otro lado, respondemos a otra pregunta de J. Blobner y generalizamos el teorema de T. Bartoszyński y S. Shelah sobre los ultrafiltros de Hausdorff en el modelo de Miller, demostrando que, en el modelo de Miller, para cualquier p-ideal analítico alto sobre ω, $\mathscr{I}$-los ultrafiltros son densos en el orden de Rudin-Blass.

En el Capítulo 4, respondemos a la última pregunta de la sección anterior, demostrando que es relativamente consistente con ZFC que para ningún ideal $\mathscr{I}$ que es F_σ, existen $\mathscr{I}$-ultrafiltros. En particular, este resultado implica la consistencia de la afirmación que no existen los ultrafiltros de Hausdorff, respondiendo a la pregunta planteada por M. Di Nasso y M. Forti. El teorema principal de este capítulo también implica una respuesta a varias otras preguntas de J. Blobner, y da un nuevo modelo donde no hay ultrafiltro con propiedad M, problema que originalmente fue resuelto por S. Shelah en su modelo donde no existen los nwd-ultrafiltros.

En el capítulo 5, utilizamos los principios del diamante parametrizado para responder algunas preguntas de D. Monk, relativas a la existencia de ciertas subestructuras del álgebra booleana $\mathcal{P}(\omega)/fin$ con cardinalidad menor a la del continuo.

Chapter 1

Preliminaries.

Cardinal invariants have played a prominent role in the study of combinatorial properties of the continuum. Since the development of the forcing technique by P. Cohen, it has been a long road which has lead to a very broad field of research about relative consistency and independence results. It turns out that many properties of the continuum have a combinatorial formulation. In this chapter we settle the set theoretic background in order to proceed with the following chapters. Our basic framework will be the Zermelo-Fraenkel set theory with Axiom of Choice, usually abbreviated as ZFC in the standard literature. We assume the reader is familiar with the fundamentals of ZFC, as well as with the forcing technique up to the Countable Support Iteration machinery developed by Shelah, and with some basics of Descriptive Set Theory.

This chapter is intended as a reference section for the following chapters. Here we present the notions, concepts and results that will be used in the rest of the work.

1.1 Basics of notation.

In this section we define the basic notions and notation that will serve as a starting point for the upcoming chapters.

As usual, $\omega^{<\omega}$ denotes the set of all finite sequences of natural numbers. For $s \in \omega^{<\omega}$ and $k \in \omega$, denote by $s^\frown k$ the sequence obtained by adjoining the natural number k at the end of s. For $s \in \omega^{<\omega}$, $|s|$ denotes the length of the sequence s. A *tree* $T \subseteq \omega^{<\omega}$ is a family of sequences such that for all $s \in T$ and $n \in \omega$, $s \restriction n \in T$, and the set $\{s \restriction i : i < |s|\}$ is well ordered by inclusion. The elements of a tree will be called *nodes*. In what follows, T denotes a tree on $\omega^{<\omega}$. If $s \in T$ and $k \in \omega$ is such that $s^\frown k \in T$, we say that k is an *immediate successor* of s in T, and denote by $succ_T(s) = \{k \in \omega : s^\frown k \in T\}$ the set of all immediate successors of s in T. If $s \in T$ has more than one immediate successor, we say that s is an *splitting node* in T, and denote by $split(T)$ the set of all splitting nodes in T. For any $s \in T$, $T \restriction s$ denotes the tree of all nodes in T which are $\subseteq$-comparable with s. Also, define $(T)_n$ as the set of all nodes in T with length exactly n,

$$(T)_n = \{s \in T : |s| = n\}$$

And for $F \subseteq \omega$,

$$(T)_F = \bigcup_{k \in F} (T)_k$$

A tree $T \subseteq \omega^{<\omega}$ is a *superperfect tree* if it satisfies the following:

1. For all $s \in T$, s is a strictly increasing sequence.

2. For all $s \in T$, there is $t \in split(T)$ extending s.

3. For all $s \in split(T)$, s has infinitely many immediate successors.

The *Rational Perfect set forcing* is the partial order whose members are all the superperfect trees, ordered by inclusion, that is, $S \leq T$ if and only if $S \subseteq T$. This partial order will be denoted by $\mathbf{PT}$. The reader may consult details of this forcing in (52).

For $T \in \mathbf{PT}$, the *stem* of T, denoted by $st(T)$, is defined as the unique node in $split(T)$ of minimal length.

We denote by φ_T the order isomorphism between $\omega^{<\omega}$ and $split(T)$ given as follows: $\varphi_T(\emptyset) = st(T)$; suppose $\varphi_T(s)$ is defined, let $\{i_k : k \in \omega\}$ be the increasing enumeration of $succ_T(\varphi_T(s))$, and for each $k \in \omega$, let $r_{s,k} \in split(T)$ be the splitting node in T of minimal length extending $\varphi_T(s)^\frown i_k$, and define $\varphi_T(s^\frown k) = r_{s,k}$.

A forcing $\mathbf{P}$ satisfies the Laver property if for all $g \in \omega^\omega$, whenever $p \in \mathbf{P}$ is a condition, $\dot{x}$ is a name for a function from ω to ω which is forced by p to be bounded by g, there are $q \leq p$ and $h \in ([\omega]^{<\omega})^\omega$ such that for all $n \in \omega$, $|h(n)| \leq n$, and $q \Vdash$ "$(\forall n \in \omega)(\dot{x}(n) \in h(n))$". We obtain an equivalent formulation if we replace $|h(n)| \leq n$ by $|h(n)| \leq f(n)$, whit a fixed increasing function $f \in \omega$ that serves for all the conditions in the forcing. We refer the reader to (4) for a deep study of the Laver property.

We say that $\mathbf{P}$ does not add splitting reals if every infinite subset of ω in the generic extension by $\mathbf{P}$ contains some $A \in [\omega]^\omega \cap V$ or is disjoint from some $A \in [\omega]^\omega \cap V$.

Given two sets A and B, we have say that A is almost contained in B whenever $A \setminus B$ is finite, and write $A \subseteq^* B$.

A family $\mathscr{I} \subseteq \mathcal{P}(\omega)$ is an *ideal on* X if it satisfies the following conditions:

1. $X \notin \mathscr{I}$.

2. For any $A, B \in \mathscr{I}$, $A \cup B \in \mathscr{I}$.

3. For any $A \in \mathscr{I}$ and $B \in \mathscr{P}(X)$ such that $B \subseteq^* A$, it holds that $B \in \mathscr{I}$.

We are interested in ideals on ω or any suitable countable set. Also, we assume that all the ideals $\mathscr{I}$ we work with are *tall*, which means that for any $A \in [\omega]^\omega$, there is an infinite set $B \in \mathscr{I}$ such that $B \subseteq A$.

The dual notion of an ideal is called a filter, which is a family $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following:

1) $\emptyset \notin \mathcal{F}$.

2) For any $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$.

3) For any $A \in \mathcal{F}$ and $B \in \mathcal{P}(X)$, if $A \subseteq^* B$ then $B \in \mathcal{F}$.

As it is customary, a maximal ultrafilter is called an *ultrafilter*. Note that condition 3) above implies that all the filters we work with are *free* filters, that is, $\bigcap \mathcal{F} = \emptyset$.

Given a family $\mathscr{I}$ on ω, the dual of $\mathscr{I}$ is defined as $\mathscr{I}^* = \{\omega \setminus A : A \in \mathscr{I}\}$. In this way we define the dual of an ideal $\mathscr{I}$ as $\mathscr{I}^*$, which is a filter, and we say that $\mathscr{I}^*$ is the dual filter of $\mathscr{I}$. Similarly, for a given filter $\mathcal{F}$ on ω, its dual ideal is defined as $\mathcal{F}^*$. For an ideal $\mathscr{I}$ on ω, the family of $\mathscr{I}$-positive sets is defined as $\mathscr{I}^+ = \mathcal{P}(\omega) \setminus \mathscr{I}$.

A *base* for an ultrafilter $\mathcal{U}$ is a set $\mathcal{B} \subseteq \mathcal{U}$ such that each $A \in \mathcal{U}$ contains some $B \in \mathcal{B}$, and the character of $\mathcal{U}$, denoted by $\chi(\mathcal{U})$, is the minimum cardinality of a base for $\mathcal{U}$.

A family $\mathcal{R} \subseteq [\omega]^\omega$ is a reaping family provided that for any $A \in [\omega]^\omega$, there is $B \in \mathcal{R}$ such that either $B \subseteq^* A$ or $A \cap B$ is finite, and we say that B reaps A. In a similar way, we say that the family $\mathcal{R}$ is a σ-reaping family if for any countable family $\{A_n : n \in \omega\} \subseteq [\omega]^\omega$, there is $A \in \mathcal{R}$ that reaps each A_n.

Given two functions $f, g \in \omega^\omega$, we say that f is dominated by g or that g dominates f, if the set $\{n \in \omega : g(n) < f(n)\}$ is finite, and we write $f \leq^* g$. It is easy to see that this relation is in fact a preorder. A family of functions $\mathcal{B} \subseteq \omega^\omega$ is an *unbounded* family if for any $f \in \omega^\omega$, there is $g \in \mathcal{B}$ such that $g \not\leq^* f$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating* family if for any $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$.

1.2 Cardinal invariants of the continuum.

Cardinal invariants have been a rich source of independence results, and they can be seen as a synthetization of combinatorial properties of certain kind of objects. Here we introduce those which are basic for the subsequent sections. We follow the notation from (9) and refer the reader to it for a deeper study of the subject.

The most basic cardinal invariant is the continuum itself, that is, the cardinality of the real line, denoted usually by $\mathfrak{c}$. It is customary to write 2^ω to denote the cardinality of $\mathfrak{c}$.

Two of the most commonly known cardinal invariants are the dominating number and the unbounding number. The unbounding number is defined as follows:

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \text{ is unbounded}\}$$

And the dominating number is defined as

$$\mathfrak{d} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \omega^\omega \text{is dominating}\}$$

It is immediate that $\mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$, and via a diagonalization argument it follows easily that $\aleph_0 < \mathfrak{b}$. So we have the following proposition:

Proposition 2. $\aleph_0 < \mathfrak{b} \leq \mathfrak{d} \leq 2^\omega$.

Other cardinal invariant of common use are the ultrafilter number and the reaping number. The ultrafilter number is defined as follows:

$$\mathfrak{u} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq [\omega]^\omega \text{ and } \mathcal{B} \text{ is a base for an ultrafilter}\}$$

The reaping number, has the following definition:

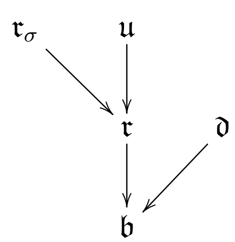
$$\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega \text{ and } \mathcal{R} \text{ is a reaping family}\}$$

Regarding σ-reaping families, there is the σ-reaping number:

$$\mathfrak{r}_\sigma = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega \text{ and } \mathcal{R} \text{ is a } \sigma\text{-reaping family}\}$$

It is clear that every σ-reaping family is a reaping family, so it holds that $\mathfrak{r} \leq \mathfrak{r}_\sigma$. The question of whether the strict inequality $\mathfrak{r} < \mathfrak{r}_\sigma$ is consistent is a long standing open question.

The following diagram summarizes the relations between the cardinal invariants we have defined. An arrow from a to b means that the inequality $b \leq a$ is a theorem of ZFC, meanwhile the aussence of an arrow means that no relations can be established in ZFC (except for $\mathfrak{r}_\sigma$ and $\mathfrak{u}$).



There is actually a general framework that captures the notion of cardinal invariants in an abstract form, and enable us to establish relations between them. This framework was introduced by Vojtáš in (66), and is extensively developed in (9).

We say that a triple $\mathbb{A} = (A, B, \to)$ is an invariant if the following holds:

1. $\to \subseteq A \times B$.
2. $(\forall a \in A)(\exists b \in B)(a \to b)$.
3. There is no single $b \in B$ such that for all $a \in A$, $a \to b$

If in addition it happens that A, B, $\to$ are Borel subsets of Polish spaces, we say that $(A, B, \to)$ is a Borel invariant. A family $D \subseteq B$ such that for all $a \in A$ there is $d \in D$ such that $a \to d$ is called a dominating family. Then, the evaluation of an invariant $(A, B, \to)$ is defined as

$$\langle A, B, \to \rangle = \min\{|D| : D \subseteq B \text{ and } D \text{ is dominating}\}$$

Several of the previous cardinal characteristics can be reformulated into this framework as follows:

$$\mathfrak{d} = \langle \omega^\omega, \omega^\omega, \leq^* \rangle$$
$$\mathfrak{b} = \langle \omega^\omega, \omega^\omega, \not\geq^* \rangle$$
$$\mathfrak{r} = \langle [\omega]^\omega, [\omega]^\omega, \textit{is reaped by} \rangle$$
$$\mathfrak{r}_\sigma = \langle ([\omega]^\omega)^\omega, [\omega]^\omega, \textit{is } \sigma\textit{-reaped by} \rangle$$

It is easy to check that the previous expresions are in fact Borel invariants and are exactly the same as previously defined.

Borel invariants are very useful, but at some point we will need a little bit more of complexity in the applications we are going to develop, so we extend the class of invariants in the previously defined fashion to the $L(\mathbb{R})$-*invariants*, which are those such that each one of A, B and $\rightarrow$ are subsets of a Polish space and all of them belong to $L(\mathbb{R})$, meaning that each one of them has a definition which requires only a subset of ω as a parameter.

$L(\mathbb{R})$-invariants will be necessary in order to appropriately work with some of the cardinal invariants we consider, to say, the sequential composition of two cardinal invariants. Given two cardinal invariants $\mathbb{A} = (A_-, A_+, \rightarrow_\mathbb{A})$ and $\mathbb{B} = (B_-, B_+, \rightarrow_\mathbb{B})$, the sequential composition is defined as follows:

$$\mathbb{A}; \mathbb{B} = (A_- \times Borel(B_-^{A_+}), A_+ \times B_+, \rightarrow)$$

Where $Borel(B_-^{A_+})$ denotes the set of all Borel functions from A_+ to B_-, and the relation $\rightarrow$ is defined as $(a_-, f) \rightarrow (a_+, b_+)$ if and only if $a_- \rightarrow_\mathbb{A} a_+$ and $f(a_+) \rightarrow_\mathbb{B} b_+$. It turns out that $\mathbb{A}; \mathbb{B}$ is an $L(\mathbb{R})$-invariant, and its evaluation is given by $\langle \mathbb{A}; \mathbb{B} \rangle = \max\{\langle \mathbb{A} \rangle, \langle \mathbb{B} \rangle\}$

1.3 Filters and ideals.

In this section we define all the filters and ideals we will be working with. For a deeper understanding of these ideals, their properties and relations, we refer the reader to (50). Although we usually say "*an ideal on ω*", we make an abuse of language and use this words to refer to an ideal on a suitable countable set.

An ideal on ω is said to be a *p-ideal* if whenever we have a sequence $\{A_n : n \in \omega\} \subseteq \mathscr{I}$, there is $B \in \mathscr{I}$ which almost contains each A_n. Definible p-ideals have a very nice and useful characterization in terms of measure functions. A submeasure is a function $\varphi : \mathcal{P}(\omega) \rightarrow [0, \infty]$ which satisfies the following properties:

a) $\varphi(\emptyset) = 0$.

b) (Monotonicity) If $A \subseteq B$, then $\varphi(A) \leq \varphi(B)$.

c) (Subadditivity) For all $A, B \subseteq \omega$, $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

If in addition φ satisfies the following, we say that φ is a *lower semicontinuous submeasure*, which will be abbreviated by lscsm:

d) (Lower semicontinuity) For all $A \in \mathcal{P}(\omega)$, $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$.

Given φ a lscsm, there are two ideals associated to it:

1. $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$.

2. $Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$.

As mentioned, this measures give a characterization of definable p-ideals, which is stated in the following two classical theorems from Descriptive Set Theory:

Theorem 7 (see (49))**.** *An ideal $\mathscr{I}$ is an F_σ ideal if and only if there is a lscsm φ such that $\mathscr{I} = Fin(\varphi)$.*

Theorem 8 (see (61))**.** *An ideal $\mathscr{I}$ is an analytic p-ideal if and only if there is a lscsm φ such that $\mathscr{I} = Exh(\varphi)$.*

As can be seen in the previous theorems, the complexity of an ideal plays an important role when talking about its combinatorial properties. In a more general setting, when talking about meager ideals the following theorem is very useful. A filter $\mathcal{F}$ is unbounded if the family $\{e_A : A \in \mathcal{F}\}$ is an unbounded family, where e_A is the increasing enumeration of the set A; otherwise it is said to be bounded.

Theorem 9 (see (4),(64),(41))**.** *Let $\mathcal{F}$ be a filter on ω. The following conditions are equivalent:*

1. *$\mathcal{F}$ has the Baire property.*

2. *$\mathcal{F}$ is meager.*

3. *$\mathcal{F}$ is bounded.*

4. *There is a partition of ω into intervales $\langle I_n : n \in \omega\rangle$ such that for any $F \in \mathcal{I}$, for all but finitely many $n \in \omega$, $F \cap I_n \neq \emptyset$.*

5. *There is a finite to one function $f \in \omega^\omega$ such that $\{X \in \mathcal{P}(\omega) : f^{-1}[X] \in \mathcal{F}\}$ is the cofinite filter.*

Since every Borel ideal is a meager ideal, it turns out that the previous theorem applies to all Borel ideals in particular.

Now we proceed to the definitions of the ideals we are working with.

Summable ideals.

Let $g : \omega \to [0, \infty)$ be a function such that $\sum_{n\in\omega} g(n)$ diverges to infinity. Then g defines a lscsm on ω by setting $\varphi_g(A) = \sum_{n\in A} g(n)$, which provides us with the ideal $\mathscr{I}_g = Fin(\varphi_g)$. Ideals of this form are called summable ideals, and will be treated in

Chapter 3. Usually it is assumed that the function g is non-increasing and converges to 0 when $n \to \infty$. This last requirement is necessary in order to have a tall ideal.

The Random Graph ideal.

This ideal is defined on $[\omega]^2$ and is defined as follows: let $\{I_n : n \in \omega\}$ be a countable independent family. We can assume that this family satisfies the condition that for any $n, m \in \omega$, $n \in I_m$ if and only if $m \in I_n$. Otherwise, just make finite changes in the sets I_n via a recursion over $n \in \omega$. Define a set of edges as $E = \{\{n, m\} : n \in I_m\}$. The Random Graph is the graph $\langle \omega, E\rangle$.

Now, define a coloring $\varphi_E : [\omega]^2 \to 2$ as $\varphi_E(\{n, m\}) = 0$ if and only if $\{n, m\} \in E$. Finally, we define the Random Graph ideal, denoted by $\mathcal{R}$, as the ideal generated by the family of subsets of ω which are homogeneous for the coloring φ_E. The complexity of this ideal is F_σ, and the following function is a lscsm for it:

$$\psi(A) = \min\{|\mathcal{F}| : (\forall X \in \mathcal{F})(X \text{ is } \varphi_E \text{ homogeneous and } A \subseteq \bigcup \mathcal{F})\}$$

Eventually different ideal.

This ideal is defined on $\omega \times \omega$ as follows:

$$\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists n, m \in \omega)(\forall k \geq n)(|(A)_k| \leq m)\}$$

It can easily be seen that this ideal has as a basis the set of all functions from ω to ω together with the sets of the form $\{n\} \times \omega$ for $n \in \omega$. $\mathcal{ED}$ is in fact an F_σ ideal and its corresponding lscsm is given by the minimum $n \in \omega$ such that for all $k \geq n$ $|(A)_k| \leq n$, whenever such natural number exists, otherwise, A has infinite measure.

The ideal $\mathcal{ED}_{fin}$.

Let Δ be the set $\{(n, m) \in \omega \times \omega : m \leq n\}$. The ideal $\mathcal{ED}_{fin}$ is defined as the restriction of $\mathcal{ED}$ to Δ,

$$\mathcal{ED}_{fin} = \mathcal{ED} \restriction \Delta$$

It turns out that this ideal is an F_σ ideal, since it is the restriction of an F_σ ideal to a positive set relative to $\mathcal{ED}$.

The ideal $\mathcal{G}_{fc}$.

The ideal $\mathcal{G}_{fc}$ is the ideal of countable graphs with finite chromatic number, that is,

$$\mathcal{G}_{fc} = \{A \subseteq [\omega]^2 : \chi(A) < \infty\}$$

where $\chi(A)$ is the chromatic number of the graph (ω, A). It turns out the function χ is a lscsm, so the ideal $\mathcal{G}_{fc}$ is an F_σ ideal. The reader may consult (36; 50) for details.

The ideal $\mathcal{S}$.

This is the Solecki ideal, and was defined by S. Solecki in (62) when dealing with Fatou's property. It is defined on the set Ω of clopen subsets of 2^ω with Lebesgue measure 1/2. This ideal has as basis the following family of sets

$$\{A \subseteq \Omega : \bigcap A \neq \emptyset\}$$

This ideal is an F_σ ideal, which can be read in Proposition 1.6.1 from (50). In (36) it was proved that $\mathcal{S}$ is critical respect to the Fubini property.

The ideal conv.

This ideal is defined on the rational numbers in the interval $[0, 1]$, that is $\mathbb{Q} \cap [0, 1]$, and has as a basis the set of all convergent sequences.

The ideal scattered.

This is the ideal of scattered subsets of the rationals.

The ideal disc.

This is the ideal of discrete subsets of the rationals.

The asymptotical density zero ideal.

This is a particular case of a broad class of ideals. Let $f : \omega \to [0, \infty)$ and define a function $\varphi_f : \mathcal{P}(\omega) \to [0, \infty)$ as follows,

$$\varphi_f(A) = \sup_{n \in \omega} \frac{\sum_{k \in A \cap n} f(k)}{\sum_{i<n} f(i)}$$

It follows that φ_f is a lscsm, and its associated exhaustive measure zero ideal is known as the Erdös-Ulam ideal for f, which is denoted by $\mathcal{EU}_f$. Of particular importance is the case when f is constant 1, which gives place to the ideal known as *the asymptotical density zero ideal*, denoted by $\mathcal{Z}$, which has the following equivalent definition

$$\mathcal{Z} = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\right\}$$

Or equivalently,

$$\mathcal{Z} = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap [2^n, 2^{n+1})|}{2^n} = 0\right\}$$

Fin × Fin

This is an ideal on $\omega \times \omega$, and it is defined as $A \in \mathbf{Fin} \times \mathbf{Fin}$ if and only if there is $n \in \omega$ such that for all $k \geq n$, $(A)_k$ is finite.

The ideal $\mathcal{SC}$.

This ideal is generated by sets $\{a_n : n \in \omega\}$ such that $lim_{n\to\infty}(a_{n+1} - a_n) = \infty$.

Definable ideals play a big role in the classification of combinatorial properties of ultrafilters. M. Katětov and J. Baumgartner provided two general frameworks that allow to investigate this classification.

Definition 9. *Given an ideal $\mathscr{I}$ on ω, an ultrafilter $\mathcal{U}$ is an $\mathscr{I}$-*ultrafilter *if for any $f : \omega \to \omega$ there is $A \in \mathcal{U}$ such that $f[A] \in \mathscr{I}$. If the function f is required to be finite to one, we talk about weak $\mathscr{I}$-ultrafilters.*

Definition 10. *Given two ideals $\mathscr{I}$ and $\mathscr{J}$ on ω, we say that $\mathscr{I}$ is Katětov below $\mathscr{J}$ if there is a function $f : \omega \to \omega$ such that f-preimages of elements from $\mathscr{I}$ are elements of $\mathscr{J}$, and write $\mathscr{I} \leq_K \mathscr{J}$. If the function f is required to be finite to one, we say that $\mathscr{I}$ is Katětov-Blass below $\mathscr{J}$, and we write $\mathscr{I} \leq_{KB} \mathscr{J}$.*

It is straightforward to see that for any ultrafilter $\mathcal{U}$, $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter if and only if $\mathscr{I} \nleq_K \mathcal{U}^*$. Similarly, an ultrafilter $\mathcal{U}$ is a weak $\mathscr{I}$-ultrafilter if and only if $\mathscr{I} \nleq_{KB} \mathcal{U}^*$.

Another classical order notions are the *Rudin-Keisler* order and the *Rudin-Blass* order. This orderings are defined on ultrafilters and they are less flexible than the Katětov order and the Katětov-Blass order.

Definition 11. *Let $\mathcal{U}$ and $\mathcal{V}$ be two filters or ideals on ω. Then:*

1. *$\mathcal{U} \leq_{RK} \mathcal{V}$ if there is a function $f : \omega \to \omega$ such that for all $A \subseteq \omega$, $A \in \mathcal{U}$ if and only if $f^{-1}[A] \in \mathcal{V}$.*

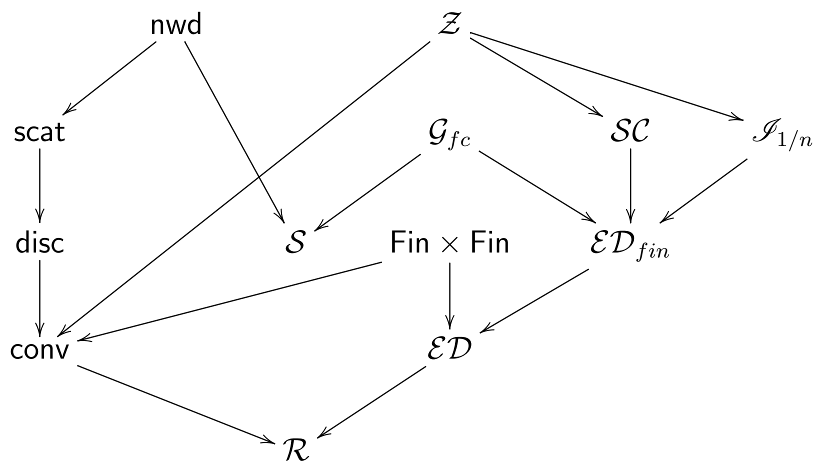
2. *$\mathcal{U} \leq_{RB} \mathcal{V}$ if there is a finite to one function $f : \omega \to \omega$ such that for all $A \subseteq \omega$, $A \in \mathcal{U}$ if and only if $f^{-1}[A] \in \mathcal{V}$.*

Where RK stands for the Rudin-Keisler order and RB for the Rudin-Blass order.

For two ideals $\mathscr{I}$ and $\mathscr{J}$, if it happens $\mathscr{I} \leq_K \mathscr{J}$ and $\mathscr{J} \leq_K \mathscr{I}$, we say that they are Katětov equivalent.

It is easy to check that whenever $X \in \mathscr{I}^+$, $\mathscr{I} \leq_K \mathscr{I} \restriction X$. The other inequality does not hold in general, an example of this is $\mathcal{ED}$: the restriction of $\mathcal{ED}$ to Δ, as defined before, is the ideal $\mathcal{ED}_{fin}$ which is not equivalent to $\mathcal{ED}$. For an ideal having the property of being Katětov equivalent to all its restrictions, we say it is K-uniform. Two typical examples of K-uniform ideals are $\mathcal{ED}_{fin}$ and $\mathcal{Z}$. Another example was provided by J.J. Pelayo Gómez, and more recently S. Navarro provided a new construction yielding an increasing sequence of ideals in the Katětov order, each one of them being K-uniform.

Between the previously defined ideals there are several relations concerning the Katětov order, all of them summarized by the following diagram, where an arrow from $\mathscr{J}$ to $\mathscr{I}$ means $\mathscr{I} \leq_K \mathscr{J}$:



Ultrafilters with combinatorial properties have been a subject of study for a long time. The most common combinatorial properties usually considered are the following:

Definition 12. *Let $\mathcal{U}$ be an ultrafilter. Then:*

1. *$\mathcal{U}$ is a Ramsey (or selective) ultrafilter if for any $c : [\omega]^2 \to 2$ there is $A \in \mathcal{U}$ such that $c \restriction [A]^2$ is constant.*
2. *$\mathcal{U}$ is a p-point if for any $\{A_n : n \in \omega\} \subseteq \mathcal{U}$ there is $A \in \mathcal{U}$ such that for all $n \in \omega$, $A \subseteq^* A_n$.*
3. *$\mathcal{U}$ is a q-point if for any partition of ω into finite sets $\langle F_n : n \in \omega \rangle$, there is $A \in \mathcal{U}$ such that for all $n \in \omega$, $A \cap F_n$ has at most one element.*
4. *$\mathcal{U}$ is a rapid ultrafilter if the family $\{e_A : A \in \mathcal{U}\}$ is a dominating family.*

All of these ultrafilters can be stated as relations between the ultrafilter and suitable definable ideals. Some well known examples are shown in the following proposition. The reader can find a good reference at (50):

Proposition 3 (see (50))**.** *Let $\mathcal{U}$ be an ultrafilter on ω. Then:*

1. *$\mathcal{U}$ is a Ramsey ultrafilter if and only if it has non empty intersection with every analytic ideal.*
2. *$\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{R} \not\leq_K \mathcal{U}^*$.*
3. *$\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{ED} \not\leq_K \mathcal{U}^*$.*
4. *$\mathcal{U}$ is a p-point if and only if $\mathbf{Fin} \times \mathbf{Fin} \not\leq_K \mathcal{U}^*$.*
5. *$\mathcal{U}$ is a p-point if and only if $\mathsf{conv} \not\leq_K \mathcal{U}^*$.*

6. *$\mathcal{U}$ is a rapid ultrafilter if it has non empty intersection with every summable ideal.*

7. *$\mathcal{U}$ is a rapid ultrafilter if and only if for any summable ideal $\mathscr{I}$, $\mathscr{I} \not\leq_{KB} \mathcal{U}$.*

8. *$\mathcal{U}$ is a q-point if and only if $\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{U}^*$*

1.4 Parametrized diamonds.

Guessing principles have proved to be very useful at the moment of establishing the existence of certain kinds of objets. Two of the most famous principles of this class are Jensen's Diamond ($\diamondsuit$) and the Club guessing principle ($\clubsuit$). The typical example of use for Jensen's Diamond is the construction of a Suslin tree and the construction of an Ostaszewski space. It is well known that $\diamondsuit$ is relatively consistent with ZFC, and in fact, it follows from the set theoretic assumption that every set is constructible. The $\clubsuit$ principle is actually a consequence of $\diamondsuit$, and they are not equivalent, but assuming $CH+\clubsuit$ one can prove $\diamondsuit$ as a consequence. Therefore, guessing principles are useful in providing examples of relative consistency.

In (55), the authors develop an extensive machinery regarding guessing principles which are weakenings of the Jensen's diamond. They prove that for every Borel cardinal invariant there is a corresponding guessing principle. Below we introduce their results which will be used later.

Definition 13. *Let A be a Borel subset of a Polish space. A function $f : 2^{<\omega_1} \to A$ is Borel provided that for every $\alpha < \omega_1$, $f \restriction 2^\alpha$ is Borel.*

The original definition of parametrized diamond principles makes use of Borel cardinal invariant, but it was noted that the same results can be achieved making use of $L(\mathbb{R})$-invariants. This results can be consulted in M. Hrušák's talk "Parametrized principles and canonical models", which is available at (34); it is also possible to consult the slides of the talk in (35). The authors develop a wide framework in which our applications of parametrized guessing principles are contained.

Definition 14. *Let $(A, B, \to)$ be an $L(\mathbb{R})$ cardinal invariant. $\diamondsuit(A, B, \to)$ denotes the following principle:*

For every $F : 2^{<\omega_1} \to A$ such that $F \restriction 2^\alpha \in L(\mathbb{R})$ for all $\alpha < \omega_1$, there is $g : \omega_1 \to B$ such that for all $f \in 2^{\omega_1}$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha) \to g(\alpha)\}$ is stationary.

The main theorem associated to this class of principles is then stated as follows:

Theorem 10. *Suppose that $\langle \mathbb{P}_\alpha : \alpha \in \omega_1 \rangle$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha$ is equivalent to $\mathcal{P}(2)^+ \times \mathbb{P}_\alpha$ as a forcing notion, and let $\mathbb{P}_{\omega_2}$ be countable support iteration of this sequence. If $\mathbb{P}_{\omega_2}$ is proper and $(A, B, \to)$ is an $L(\mathbb{R})$-invariant, then $\mathbb{P}_{\omega_2}$ forces $\langle A, B, \to \rangle \leq \omega_1$ if and only if $\mathbb{P}_{\omega_2}$ forces $\diamondsuit(A, B, \to)$.*

By a Borel partial order, we understand that there is a Borel code which allows to compute the forcing notion. Standard forcing notions such a the Perfect Set forcing, the Rational Set Forcing, Mathias, Laver, Cohen and Random are Borel forcing notions.

By the previous theorem, whenever we have a countable support iteration of Borel proper forcing notions leading to the evaluation of a cardinal invariant $(A, B, \rightarrow)$ to be equal to ω_1, then the corresponding parametrized diamond principle holds in the generic extension. In particular, the Perfect Set forcing model will be the standard model for us when refering to diamond principles.

1.5 Forcing.

We will be dealing with countable support iterations of proper forcing notions, and we will follow the framework developed in (1), with the only difference that the stronger conditions in the forcing are oriented downward, that is, for $q \leq p$ we consider q being stronger than p. Everything else remains the same as pointed in (1). For the basics of forcing we refer the reader to (45). In this section we introduce the basic results needed for the subsequent chapters.

Given a forcing $\mathbb{P}$ and an countable elementary submodel $\mathcal{M} \prec H(\theta)$ such that $\mathbb{P} \in \mathcal{M}$, a condition $p \in \mathbb{P}$ is $(\mathcal{M}, \mathbb{P})$-generic if $\mathbb{P} \cap \mathcal{M}$ is predense below p. The forcing $\mathbb{P}$ is proper if for all countable $\mathcal{M} \prec H(\theta)$ such that $\mathbb{P} \in \mathcal{M}$, every $p \in \mathbb{P} \cap \mathcal{M}$ can be extended to an $(\mathcal{M}, \mathbb{P})$-generic condition.

We will make use of three of the classical preservation theorems, which we state below. Recall that a forcing is ω^ω-bounding if the reals from the ground model are dominating in the forcing extensión, and a forcing $\mathbb{P}$ preserves an ultrafilter provided the ultrafilter generates an ultrafilter in the generic extension by $\mathbb{P}$.

Theorem 11 (Elementary submodels extension)**.** *Let $\mathbb{P}$ be a proper forcing and $\mathcal{M} \prec H(\theta)$ (for θ big enough) a countable elemenary submodel such that $\mathbb{P} \in \mathcal{M}$. Then for every generic filter $G \subseteq \mathbb{P}$, it holds that $\mathcal{M}[G] \prec H(\theta)^{V[G]}$*

Theorem 12 (Generic conditions for two step iteration.)**.** *Let $\mathbb{P}$ be a proper forcing and $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper forcing notion. Let $\mathcal{M} \prec H(\theta)$ be a countable elementary submodel such that $\mathbb{P} * \dot{\mathbb{Q}} \in \mathcal{M}$. Then a condition $(p, \dot{q})$ is $(\mathcal{M}, \mathbb{P} * \dot{\mathbb{Q}})$-generic if and only if p is $(M, \mathbb{P})$-generic and $p \Vdash$ "$\dot{q}$ is $(\mathcal{M}[\dot{G}], \dot{\mathbb{Q}}) - generic$".*

Theorem 13 (Preservation of properness.)**.** *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \lambda \rangle$ be a countable support iteration of forcings such that for all $\alpha \in \lambda$, $\mathbb{P}_\alpha$ forces $\dot{\mathbb{Q}}_\alpha$ to be proper. Then $\mathbb{P}_\lambda$ is proper.*

Theorem 14 (Preservation of p-points.)**.** *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \lambda \rangle$ be a countable support iteration of proper forcing notions such that for all $\alpha < \lambda$, $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ preserves p-points". Then $\mathbb{P}_\lambda$ preserves p-points.*

Chapter 2

Smashing number for ideals.

The notion of $\mathscr{I}$-ultrafilters was introduced by Baumgartner in (6). Several authors have studied the existence of $\mathscr{I}$-ultrafilters. Among them J. Baumgartner, M. Canjar, J. Ketonen and J. Brendle have obtained several results regarding the generic existence of certain classes of $\mathscr{I}$-ultrafilters. J. Blobner and J. Brendle have made a long study about $\mathscr{I}$-ultrafilters by introducing the cardinal invariant $\mathfrak{ge}(\mathscr{I})$, which stablish a bridge between generic existence and relations between more familiar cardinal invariants of the continuum. Here we adopt a different approach concerning parametrized guessing principles. In some sense, we start from an opposite viewpoint, which is regarding how many subsets of the natural numbers are needed in order to know the combinatorics of all functions from ω to ω (relative to a given ideal), rather than looking at the combinatorics of filters.

2.1 From big sets to small sets.

Looking at the definition of $\mathscr{I}$-ultrafilter, one gets the intuition that $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter if for every function $f \in \omega^\omega$ we can find a big set in $\mathcal{U}$ which after applying f becomes a small set in $\mathscr{I}$. So it is natural to ask about the cardinality of a family of infinite subsets of ω with the property that for every function we can find an element in the family whose image under the function is a small set. More exactly, we define the following cardinal invariant (recall that we assume that all our ideals are tall, so $\mathfrak{z}(\mathscr{I})$ below is well defined):

Definition 15. *Let $\mathscr{I}$ be an ideal on ω. We define the cardinal invariant $\mathfrak{z}(\mathscr{I})$ as the smallest cardinality of a family $\mathcal{D}$ of infinite subsets of ω such that for all $f \in \omega^\omega$ there is a set $A \in \mathcal{D}$ such that $f[A] \in \mathscr{I}$, that is,*

$$\mathfrak{z}(\mathscr{I}) = \min\{|\mathcal{D}| : (\forall f \in \omega^\omega)(\exists A \in \mathcal{D})(f[A] \in \mathscr{I})\}$$

In a similar way we define $\mathfrak{z}_{fin}(\mathscr{I})$, for which we only consider finite to one functions:

$$\mathfrak{z}_{fin}(\mathscr{I}) = \min\{|\mathcal{D}| : (\forall f \in \omega^\omega)(\text{ if } f \text{ is finite to one then } \exists A \in \mathcal{D} \text{ such that } f[A] \in \mathscr{I})\}$$

Using the Vojtáš framework, the previous cardinal invariants can be written as follows:

$$\mathfrak{z}(\mathscr{I}) = \langle \omega^\omega, [\omega]^\omega, \in_{\mathscr{I}} \rangle$$

$$\mathfrak{z}_{fin}(\mathscr{I}) = \langle \mathsf{FtO}, [\omega]^\omega, \in_{\mathscr{I}} \rangle$$

where $\mathsf{FtO} = \{f \in \omega^\omega : f \text{ is finite to one}\}$ and $f \in_{\mathscr{I}} A$ if and only if $f[A] \in \mathscr{I}$. So it can bee seen that whenever the ideal $\mathscr{I}$ is a Borel ideal, the cardinal invariants $\mathfrak{z}(\mathscr{I})$ and $\mathfrak{z}_{fin}(\mathscr{I})$ are Borel cardinal invariants. Thus, we can talk about the corresponding parametrized diamond principles.

Another two variations of this cardinal invariants are the following:

$$\mathfrak{z} = \min\{|\mathcal{D}| : (\forall f \in \omega^\omega)(\exists A \in \mathcal{D})(f[A] \neq^* \omega)\}$$

and the finite to one version:

$$\mathfrak{z}_{fin} = \min\{|\mathcal{D}| : (\forall f \in \omega^\omega)(\text{ if } f \text{ is finite to one then } \exists A \in \mathcal{D} \text{ such that } f[A] \neq^* \omega)\}$$

It is obvious that for any ideal $\mathscr{I}$, $\mathfrak{z}_{fin} \leq \mathfrak{z} \leq \mathfrak{z}(\mathscr{I})$, and also $\mathfrak{z}_{fin} \leq \mathfrak{z}_{fin}(\mathscr{I}) \leq \mathfrak{z}(\mathscr{I})$. It turns out that this cardinal invariants are related to the existence of $\mathscr{I}$-ultrafilters via its corresponding parametrized diamond principle, which is defined below:

Definition 16. *Let $\mathscr{I}$ a Borel ideal. The guessing principle $\diamondsuit(\mathfrak{z}(\mathscr{I}))$ is the following assertion, which will be denoted by $\diamondsuit(\mathfrak{z}(\mathscr{I}))$:*

For any Borel function $F : 2^{<\omega_1} \to \omega^\omega$, there is a function $g : \omega_1 \to [\omega]^\omega$ such that for all $f \in 2^{\omega_1}$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha)[g(\alpha)] \in \mathscr{I}\}$ is stationary.

The function g above is called a $\diamondsuit(\mathfrak{z}(\mathscr{I}))$-guessing sequence. In a similar way we define $\diamondsuit(\mathfrak{z}_{fin}(\mathscr{I}))$.

Theorem 15. *For any Borel ideal $\mathscr{I}$, $\diamondsuit(\mathfrak{z}(\mathscr{I}))$ implies the existence of $\mathscr{I}$-ultrafilters. Similarly, for any Borel ideal $\mathscr{I}$, $\diamondsuit(\mathfrak{z}_{fin}(\mathscr{I}))$ implies the existence of weak $\mathscr{I}$-ultrafilters.*

Proof. We prove only the first assertion, since the second one follows completely similar lines. By a suitable coding, we can assume that the domain of the function F consists of ordered paris $(f, \vec{A})$, where $f \in \omega^\omega$ and $\vec{A} = \langle A_\beta : \beta < \alpha \rangle$ is a sequence of countable length of infinite subsets of ω. Define F as follows:

i) If $\vec{A}$ is a centered family, let A_α be a pseudointersection of $\langle A_\beta : \beta < \alpha \rangle$ (defined in a recursively or Borel way), and $\varphi_{\vec{A}} : \omega \to A_\alpha$ be its increasing enumeration. Define $F(f, \vec{A}) = f \circ \varphi_{\vec{A}}$.

ii) If $\vec{A}$ is not a centered family, then $F(f, \vec{A}) = id$.

Let g be a $\diamondsuit(\mathfrak{z}(\mathscr{I}))$-guessing sequence for F. Recursively define a sequence $\vec{B} = \langle B_\alpha : \alpha \in \omega_1 \rangle$ of centered sets. Start with $B_n = \omega \setminus n$ and suppose $\langle B_\beta : \beta < \alpha \rangle$ has been defined. Then define $B_\alpha = \varphi_{\langle B_\beta : \beta < \alpha \rangle}[g(\alpha)]$. Then $\vec{B} = \langle B_\alpha : \alpha \in \omega_1 \rangle$ is a $\subseteq^*$-decreasing sequence of sets. Let us see that $\vec{B}$ is a witness for $\mathfrak{z}(\mathscr{I})$. Pick any $f \in \omega^\omega$ and consider $(f, \vec{B})$. Then the set $\{\alpha \in \omega_1 : F(f, \langle B_\beta : \beta < \alpha \rangle)[g(\alpha)] \in \mathscr{I}\}$ is stationary. Let $\alpha \in \omega_1$ be such that $F(f, \langle B_\beta : \beta < \alpha \rangle)[g(\alpha)] \in \mathscr{I}$. Since $\langle B_\beta : \beta < \alpha \rangle$ is a centered family, it follows from the definition of F that $F(f, \langle B_\beta : \beta < \alpha\rangle) = f \circ \varphi_{\langle B_\beta:\beta<\alpha\rangle}$, so the image of $g(\alpha)$ under this function is $f \circ \varphi_{\langle B_\beta:\beta<\alpha\rangle}[g(\alpha)] = f[\varphi_{\langle B_\beta:\beta<\alpha\rangle}[g(\alpha)]] = f[B_\alpha]$, and due to the choice of α, $f[B_\alpha] \in \mathscr{I}$. Finally, extend the family $\vec{B}$ to an ultrafilter, which is an $\mathscr{I}$-ultrafilter given that it contains $\vec{B}$. □

Lemma 1. *For any family $\mathcal{D} \subseteq [\omega]^\omega$ with cardinality smaller than $\mathfrak{r}$, there is a partition $\langle P_n : n \in \omega\rangle$ of ω into infinite sets such that for all $n \in \omega$ and $A \in \mathcal{D}$, the intersection $A \cap P_n$ is infinite.*

Proof. Recursively construct two sequences $\langle \mathcal{D}_n : n \in \omega\rangle$ and $\langle Z_n : n \in \omega\rangle$ as follows:

1. Since $\mathcal{D}$ has cardinality less than $\mathfrak{r}$, there is $X \in [\omega]^\omega$ such that for all $A \in \mathcal{D}$, $A \cap X$ and $A \cap (\omega \setminus X)$ are infinite. Let Z_0 be one of such sets X and consider $\mathcal{D}_1 = \{A \cap (\omega \setminus X) : A \in \mathcal{D}\}$.

2. Suppose $\mathcal{D}_n$ is defined. Then $\mathcal{D}_n$ has cardinality smaller than $\mathfrak{r}$, so there is $X \in [\omega \setminus \bigcup_{i\leq n} Z_n]^\omega$. such that for all $A \in \mathcal{D}_n$, the sets $A \cap X$ and $A \cap (\omega \setminus (\bigcup_{i\leq n} Z_i \cup X))$ are both infinite. Define $Z_{n+1} = X$ and $\mathcal{D}_{n+1} = \{A \cap (\omega \setminus \bigcup_{i\leq n+1} Z_i) : A \in \mathcal{D}_n\}$.

Suppose the recursion is done and let $Z = \omega \setminus \bigcup_{n\in\omega} Z_n$. Now define $P_0 = Z_0 \cup Z$ and $P_n = Z_n$ for $n > 0$. Then $\langle P_n : n \in \omega\rangle$ satisfies the requirement. □

Lemma 2. *For any family $\mathcal{D} \subseteq [\omega]^\omega$ of cardinality smaller than $\mathfrak{b}$, there is an interval partition $\langle I_n : n \in \omega\rangle$ of ω such that for each $A \in \mathcal{D}$ and for all but finitely many $n \in \omega$ $A \cap I_n$ is not empty.*

Proof. For each $A \in \mathcal{D}$, let $e_A : \omega \to \omega$ be the increasing enumeration of A, and define $f_A : \omega \to \omega$ as follows:

$$f_A(n) = min\{k \in \omega : (\exists m \in \omega)(n < e_A(m) < k)\} \tag{2.1}$$

Since the family $\{f_A : A \in \mathcal{D}\}$ has cardinality smaller than $\mathfrak{b}$, there is a strictly increasing function f dominating each one of the f_A. Let f be one of such functions and define another function h as $h(0) = f(0)$ and $h(n+1) = f(h(n))$. Define an interval partition as follows: $I_0 = [0, h(0))$ and $I_n = [h(n), h(n+1))$ for $n > 0$. The partition $\langle I_n : n \in \omega\rangle$ satisfies the requirement. Indeed, fix $A \in \mathcal{D}$ and let $n_0 \in \omega$ be such that for all $n \geq n_0$, $f_A(n) \leq f(n)$. Then for $n \geq n_0$, $h(n) < f_A(h(n)) \leq f(h(n)) = h(n+1)$, and by the definition of f_A, there is $k \in \omega$ such that $h(n) < e_A(k) < f_A(n)$, which implies $h(n) < e_A(k) < h(n+1)$, so $A \cap I_n \neq \emptyset$.

□

The following two lemmas appeared in (48). We include their proof for completeness.

Lemma 3. *There is a family $\mathcal{P}$ of interval partitions of ω with cardinality $\mathfrak{b}$, such that for every interval partition $\mathcal{I}$ there is $\mathcal{J} \in \mathcal{P}$ such that infinitely many intervals from $\mathcal{J}$ contain an interval from $\mathcal{I}$.*

Proof. Let $\mathcal{B}$ be an unbounded family of strictly increasing functions. For each $f \in \mathcal{B}$ define $h(0) = f(0)$ and for $n > 0$, $h_f(n+1) = f(h_f(n))$. Let $\mathcal{I} = \langle [i_n, i_{n+1}) : n \in \omega \rangle$ be an arbitrary interval partition, and define $g_{\mathcal{I}} \in \omega^\omega$ as $g_{\mathcal{I}}(k) = i_{n+4}$ if and only if $k \in [i_n, i_{n+1})$. Let $f \in \mathcal{B}$ which is not bounded by $g_{\mathcal{I}}$, and pick $n \in \omega$ such that $g_{\mathcal{I}}(n) < h_f(n)$, and let $l \in \omega$ be such that $n \in [h_f(l-1), h_f(l))$. Then $g_{\mathcal{I}}(n) < h_f(n) < f(h_f(l)) = h_f(l+1)$. Also, let $k \in \omega$ be such that $g_{\mathcal{I}}(n) = i_{k+4}$. There are two possibilities:

a) $h_f(l) \leq i_{k+3}$: since $h_f(l+1) > g_{\mathcal{I}}(n) = i_{k+4}$, it follows that $[i_{k+3}, i_{k+4}) \subseteq [h_f(l), h_f(l+1))$.

b) $h_f(l) > i_{k+3}$: then $h_f(l) > i_{k+3} > i_{k+2} > i_{k+1} > n \geq h_f(l-1)$, so $[i_{k+1}, i_{k+3}) \subseteq [h_f(l-1), h_f(l))$

Since there are infinitely many $n \in \omega$ such that $g_{\mathcal{I}}(n) < h_f(n)$, there are infinitely many intervals $[h_f(m), h_f(m+1))$ which contain an interval from $\mathcal{I}$. So the family $\mathcal{P} = \{\mathcal{J}_f : f \in \mathcal{B}\}$, where $\mathcal{J}_f = \langle [h_f(k), h_f(k+1)) : k \in \omega \rangle$, satisfies the lemma. □

Lemma 4. *Let $\mathcal{P}$ be a family of interval partitions with cardinality smaller than $\mathfrak{d}$. Then there is an interval partition $\mathcal{I} = \langle [i_n, i_{n+1}) : n \in \omega \rangle$ such that for any partition $\mathcal{J} \in \mathcal{P}$, there are infinitely many intervals from $\mathcal{I}$ which contain some interval from $\mathcal{J}$.*

Proof. The argument is essentially the same as in the previous lemma. For each interval partition $\mathcal{J} \in \mathcal{P}$, assume $\mathcal{J} = \langle [j_n, j_{n+1}) : n \in \omega \rangle$, and define $g_{\mathcal{J}}(k) = i_{n+4}$ if and only if $k \in [j_n, j_{n+1})$. Let $f \in \omega^\omega$ be a strictly increasing function which is not dominated by $\{g_{\mathcal{J}} : \mathcal{J} \in \mathcal{P}\}$, and consider the function h_f defined in the previous lemma and the interval partition given by $\mathcal{I}_{h_f} = \langle [h_f(n), h_f(n+1)) : n \in \omega \rangle$. Let $\mathcal{J} \in \mathcal{P}$ be an arbitrary partition, and note that the function f is not bounded by $g_{\mathcal{J}}$. Then apply the same argument as before to see that the interval partition $\mathcal{I}_{h_f}$ has infinitely many intervals that contain an interval from $\mathcal{J}$. □

Proposition 4. *The following holds in* ZFC*:*

a) $\mathfrak{b} = \mathfrak{z}_{fin}$.

b) $\mathfrak{r} = \mathfrak{z}$.

Proof. a) Let $\mathcal{D}$ be a family of subsets of ω of cardinality less than $\mathfrak{b}$. By an application of Lemma 2, we get an interval partition $\langle I_n : n \in \omega \rangle$, such that each element of $\mathcal{D}$ meets almost every interval I_n. Define a function $\varphi : \omega \to \omega$ as $\varphi(k) = n$ if and only if $k \in I_n$. It is clear that $\varphi[A] =^* \omega$ for all $A \in \mathcal{D}$. This proves that $\mathfrak{b} \leq \mathfrak{z}_{fin}$.

Now let $\mathcal{P}$ be the family of interval partitions from Lemma 3. For each $\mathcal{I} \in \mathcal{P}$, define $A_{\mathcal{I}} = \bigcup_{n \in \omega} I_{2n}$ and $B_{\mathcal{I}} = \bigcup_{n \in \omega} I_{2n+1}$. Let $f \in \omega^\omega$ be any finite to one function, and define an interval partition $\langle I_n : n \in \omega \rangle$ such that each I_n contains $f^{-1}(k)$ for some $k \in \omega$. Pick any $\mathcal{J} \in \mathcal{P}$ such that infinitely many J_n contain some $\mathcal{I}_k$. Then, for infinitely many $k \in \omega$, $f^{-1}(k) \subseteq A_{\mathcal{J}}$; or for infinitely many $k \in \omega$, $f^{-1}(k) \subseteq B_{\mathcal{J}}$. In the first case $\omega \setminus f[B_{\mathcal{J}}]$ is infinite, and in the second case $\omega \setminus f[A_{\mathcal{J}}]$ is infinite. Then the family $\{A_{\mathcal{J}}, B_{\mathcal{J}} : \mathcal{J} \in \mathcal{P}\}$ is a witness for $\mathfrak{z}_{fin}$. This proves the inequality $\mathfrak{z}_{fin} \leq \mathfrak{b}$.

b) Let $\mathcal{R}$ be a reaping family. Fix any $f \in \omega^\omega$. We can assume that $f[\omega] =^* \omega$. Let $A \subseteq \omega$ infinite and coinfinite. Then there is $B \in \mathcal{R}$ such that $B \subseteq^* f^{-1}[A]$ or $B \subseteq^* f^{-1}[\omega \setminus A]$. In either case it is clear that $f[B]$ is co-infinite. Then we have $\mathfrak{z} \leq \mathfrak{r}$.

Now let $\mathcal{D} \subseteq [\omega]^\omega$ be a family of cardinality less than $\mathfrak{r}$. By Lemma 1, there is a partition $\langle P_n : n \in \omega \rangle$ such that for all $A \in \mathcal{D}$ and for all $n \in \omega$, $A \cap P_n$ is infinite. Define a function $f(k) = n$ if and only if $k \in P_n$. It is clear that for all $A \in \mathcal{D}$ it holds that $f[A] = \omega$. This proves that $\mathfrak{r} \leq \mathfrak{z}$. □

Proposition 5. *Let $\mathscr{I}$, $\mathscr{J}$ be ideals on ω. If $\mathscr{I} \leq_K \mathscr{J}$, then $\mathfrak{z}(\mathscr{J}) \leq \mathfrak{z}(\mathscr{I})$. Similarly, if $\mathscr{I} \leq_{KB} \mathscr{J}$, then $\mathfrak{z}_{fin}(\mathscr{J}) \leq \mathfrak{z}_{fin}(\mathscr{I})$*

Proof. We only prove the first part of the proposition since the second one is completely analogous. Let $\mathcal{D}$ be a witness for $\mathfrak{z}(\mathscr{I})$, and let $\varphi \in \omega^\omega$ be a witness for $\mathscr{I} \leq_K \mathscr{J}$. We will see that $\mathcal{D}$ is also a witness for $\mathfrak{z}(\mathscr{J})$. Pick any $f \in \omega^\omega$. For $\varphi \circ f$ there is $A \in \mathcal{D}$ such that $\varphi \circ f[A] \in \mathscr{I}$, so $\varphi^{-1}[\varphi \circ f[A]] \in \mathscr{J}$, but $f[A] \subseteq \varphi^{-1}[\varphi \circ f[A]]$, which implies $f[A] \in \mathscr{J}$. □

Theorem 16. *If $\mathscr{I}$ is an ideal on ω such that for some $n, k \in \omega$ there exists a coloring $\varphi : [\omega]^n \to k$ whose homogeneous sets are in the ideal $\mathscr{I}$, then $\mathfrak{z}_{fin}(\mathscr{I}) \leq \max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$.*

Proof. Recall that $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\} = \mathfrak{hom}$, the minimum cardinality of a family $\mathcal{H} \subseteq [\omega]^\omega$ such that for any coloring $\varphi : [\omega]^n \to k$ there is a homogeneous set in $\mathcal{H}$ (see (9), Theorem 3.10). Fix $\mathcal{H}$ a family witnessing the minimality of $\mathfrak{hom}$. Moreover, we require the family $\mathcal{H}$ to be hereditarily a witness for $\mathfrak{hom}$, that is, for any $A \in \mathcal{H}$, $\mathcal{H}_A = \{B \in \mathcal{H} : B \subseteq A\}$ is a witness for $\mathfrak{hom}$ for colorings on $[A]^2$. Let $\varphi : [\omega]^n \to k$ be such that any homogeneous set belongs to the ideal. Let $f \in \omega^\omega$ be an arbitrary finite to one function. We can assume that $f[\omega] \in \mathscr{I}^+$. First, define a new coloring $\varphi^0_f : [\omega]^n \to 2$ as $\varphi^0_f(k_0, \ldots, k_{n-1}) = 0$ if $f(k_i) \neq f(k_j)$ for $i \neq j$, and 1 otherwise. Let $A \in \mathcal{H}$ a φ^0_f-homogeneous. Since f is finite to one, and A is infinite it can not be the case that for some $\{k_0, \ldots, k_{n-1}\} \in [A]^n$ there are $i \neq j$ such that $f(k_i) = f(k_j)$, so $f \restriction A$ is inyective. Now define a coloring $\varphi^1_f : [A]^2 \to k$ as $\varphi^1_f(k_0, \ldots, k_{n-1}) = \varphi(f(k_0) \ldots, f(k_{n-1}))$, and let $B \in \mathcal{H}_A$ an infinite φ^1_f-homogeneous set, which exists by our choice of $\mathcal{H}$. By definition of φ^1_f, we have that $f[B]$ is a φ-homogeneous set, so $f[B] \in \mathscr{I}$. □

Relative to general bounds on $\mathfrak{z}(\mathscr{I})$ there is the following.

Proposition 6. *For any ideal $\mathscr{I}$, $\mathfrak{z}(\mathscr{I}) \leq \max\{\mathfrak{r}_\sigma, \mathfrak{z}_{fin}(\mathscr{I})\}$.*

Proof. Let $\mathcal{R}$ be a σ-reaping family and $\mathcal{D}$ a witness for $\mathfrak{z}_{fin}(\mathscr{I})$, both of minimum possible cardinality. For each $X \in \mathcal{R}$, let $\varphi_X : \omega \to X$ be a biyection. Define $\mathcal{B} = \mathcal{R} \cup \{\varphi_X[A] : A \in \mathcal{D},\ X \in \mathcal{R}\}$. We claim that $\mathcal{B}$ is a witness for $\mathfrak{z}(\mathscr{I})$. Pick any $f \in \omega^\omega$. Then there is a set $X \in \mathcal{R}$ such that there is an $n \in \omega$ for which $X \subseteq f^{-1}[n]$, or for all $n \in \omega$ the intersection $X \cap f^{-1}[n]$ is finite. In the first case we are done. In the second case consider the finite to one function $f \circ \varphi_X : \omega \to \omega$. Then there is a set $A \in \mathcal{D}$ such that $f \circ \varphi_X[A] \in \mathscr{I}$, so $\varphi_X[A]$ is as required for the function f. Clearly $\mathcal{B}$ has cardinality at most $\max\{\mathfrak{r}_\sigma, \mathfrak{z}_{fin}(\mathscr{I})\}$. □

Theorem 17. *If $\mathscr{I}$ is a meager ideal, then $\min\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathscr{I})$.*

Proof. Let $\mathcal{D} \subseteq [\omega]^\omega$ be a family of cardinality less than $\min\{\mathfrak{r}, \mathfrak{d}\}$. By the Jalali-Naini-Talagrand theorem (see Theorem 9 from Chapter 1), there is a finite to one function $g \in \omega^\omega$ witnessing $\mathsf{fin} \leq_{RB} \mathscr{I}$. Let $X \in [\omega]^\omega$ be such that $g^{-1}[X]$ and $\omega \setminus g^{-1}[X]$ are both infinite. Let $\langle a_k : k \in \omega\rangle$ be the increasing enumeration of $g^{-1}[X]$. Let $\nu : \omega \to \omega$ be a function with the property that for all $n \in \omega$, there is a $i \in X$ such that $g^{-1}(i) \subseteq \{a_n, a_{n+1}, \dots, a_{\nu(n)}\}$. Since $|\mathcal{D}| < \mathfrak{r}$, by Lemma 1 there is a partition $\langle P_n : n \in \omega\rangle$ of ω into infinitely many infinite sets such that for all $A \in \mathcal{D}$ and $n \in \omega$, $A \cap P_n$ is infinite. Fix a partition $\langle P_n : n \in \omega\rangle$ having this property. For each $A \in \mathcal{D}$ define a function $f_A : \omega \to \omega$ as $f_A(0) = 0$ and:

$$f_A(n+1) = \min\{k :\in \omega : k > f_A(n) \wedge \forall i \in [n+1, \nu(n+1)] P_i \cap A \cap k \neq \emptyset\}$$

Since $|\mathcal{D}| < \mathfrak{d}$, there is a strictly increasing function $\varphi : \omega \to \omega$ which is not dominated by $\{f_A : A \in \mathcal{D}\}$. Now define a function $\phi : \omega \to \omega$ as follows:

$$\phi(m) = \begin{cases} a_k \text{ if for some } k \in \omega,\ m \in P_k \cap \varphi(k) \\ \eta(m) \text{ in other case.} \end{cases} \tag{2.2}$$

Where $\eta : \bigcup_m P_m \setminus \varphi(m) \to \omega \setminus g^{-1}[X]$ is a biyective function. It is clear that ϕ is a finite to one function. We claim that for all $A \in \mathcal{D}$, $\phi[A] \in \mathscr{I}^+$. Fix $A \in \mathcal{D}$. Note that for any $n \in \omega$ such that $f_A(n) \leq \varphi(n)$, it is the case that for all $i \in [n, \nu(n)]$, $A \cap P_i \cap \varphi(n) \neq \emptyset$, which implies that for all $i \in [n, \nu(n)]$ it happens that $A \cap P_i \cap \varphi(i) \neq \emptyset$, since φ is an increasing function, so for each $i \in [n, \nu(n)]$ there is $k \in A$ such that $\phi(k) = a_i$. More explicitly, for every n such that $f_A(n) \leq \varphi(n)$, we have that $\{a_n, \dots, a_{\nu(n)}\} \subseteq \phi[A]$. Since the inequality $f_A(n) \leq \varphi(n)$ occurs for infinitely many $n \in \omega$, there are infinitely many n such that $\{a_n, \dots, a_{\nu(n)}\} \subseteq \phi[A]$. By the choice of ν, this means that there are infinitely many i such that $g^{-1}(i) \subseteq \phi[A]$, which implies that $\phi[A]$ is $\mathscr{I}$-positive. □

Corollary 1. *For any meager ideal* $\mathscr{I}$, $\min\{\mathfrak{r}_\sigma, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathscr{I})$.

Proof. J. Aubrey has proved in (2) that $\min\{\mathfrak{r}, \mathfrak{d}\} = \min\{\mathfrak{r}_\sigma, \mathfrak{d}\}$, which implies the corollary. □

The above results give some general bounds for the Borel ideals, but in some cases it is possible to get some improvements.

Proposition 7. *The following holds in* ZFC*:*

a) $\mathfrak{z}(\mathsf{conv}) = \mathfrak{r}_\sigma$.

b) $\mathfrak{r} \leq \mathfrak{z}(\mathsf{nwd}) \leq \mathfrak{r}_\sigma$.

c) $\mathfrak{z}_{fin}(\mathcal{ED}_{fin}) = \mathfrak{d}$.

Proof. a) It is well known that $\mathfrak{r}_\sigma$ is the minimum cardinality of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for every bounded sequence of reals $\langle a_n : n \in \omega\rangle$ there is $X \in \mathcal{R}$ such that $\langle a_n : n \in A\rangle$ converges to some real (see (9), Theorem 3.7). This is equivalent if we replace sequence of reals by sequence of rationals.

b) The second inequality follows from the previous item, the fact that $\mathsf{conv} \leq_K \mathsf{nwd}$[1] and Proposition 5. For the first inequality, let us recall the cardinal invariant deq defined by Cichoń in (22), which is defined as the minimum cardinality of a family $\mathcal{S} \subseteq [\omega]^\omega$ such that there is no one to one function $f : \omega \to \mathbb{Q}$ such that for all $A \in \mathcal{S}$, $f[A]$ is dense. It was proved by J. Cichoń that $\mathsf{deq} \leq \mathfrak{r}$, and in (3) it was proved that in fact the equality holds, that is $\mathfrak{r} = \mathsf{deq}$. Then, the first inequality follows immediately from $\mathfrak{r} = \mathsf{deq}$.

c) To see that $\mathfrak{z}_{fin}(\mathcal{ED}_{fin}) \leq \mathfrak{d}$, let $\mathcal{D}$ be a dominating family of interval partitions of ω. Let us assume that each $\mathcal{P} \in \mathcal{D}$ is of the form $\mathcal{P} = \langle [i_n^{\mathcal{P}}, i_{n+1}^{\mathcal{P}}) : n \in \omega \rangle$. We claim that $\mathcal{F} = \{\{i_n^{\mathcal{P}} : n \in \omega\} : \mathcal{P} \in \mathcal{D}\}\}$ is a witness for $\mathfrak{z}_{fin}(\mathcal{ED}_{fin})$. Pick any $f : \omega \to \Delta$ finite to one function. Define recursively a function $\varphi : \omega \to \omega$ as follows:

1. $\varphi(0) = 0$.

2. Suppose $\varphi(i)$ is defined for all $i < n$. Let $a_n = \max\{i \in \omega : (\exists k \leq \varphi(n-1))(f(k) \in (\Delta)_i)\}$, and define $A_n = \bigcup_{l \leq a_n} (\Delta)_l$

$$\varphi(n+1) = \min\{k \in \omega : f^{-1}[A_n] \subseteq k\} + 1 \tag{2.3}$$

Consider the partition $\mathcal{I}_\varphi = \langle [\varphi(n), \varphi(n+1)) : n \in \omega \rangle$. Then for each $k \in \omega$, $\varphi[(\Delta)_k]$ is covered by at most two consecutive intervals from $\mathcal{I}_\varphi$. Indeed, for $k \in \omega$ let n_k be the minimum such that $f([\varphi(n_k), \varphi(n_k+1))) \cap (\Delta)_k \neq \emptyset$. Then by the definition of φ, $f^{-1}[(\Delta)_k)] \subseteq [\varphi(n_k), \varphi(n_k+2))$. Let $\mathcal{P} = \langle [i_n, i_{n+1}) : n \in \omega \rangle \in \mathcal{D}$ be an interval partition dominating $\mathcal{I}_\varphi$. We claim that $f[\{i_n : n \in \omega\}] \cap (\Delta)_k$ has at most two elements. Since each interval from $\mathcal{P}$ contains an interval from $\mathcal{I}_\varphi$, it follows that every two consecutive intervals from $\mathcal{I}_\varphi$ are contained in two consecutive intervals from $\mathcal{P}$. Then, since $f^{-1}[(\Delta)_k]$ is covered by two consecutive intervals from $\mathcal{I}_\varphi$, it is as well covered by two consecutive intervals from $\mathcal{P}$. That is, for all $k \in \omega$, there is $n_k \in \omega$ such that $f^{-1}[(\Delta)_k] \subseteq [i_{n_k}, i_{n_{k+2}})$. This implies that $(\Delta)_k \cap f[\{i_n : n \in \omega\}] \subseteq f[\{i_{n_k}, i_{n_{k+1}}\}]$. This proves that $\mathfrak{z}_{fin}(\mathcal{ED}) \leq \mathfrak{d}$.

Now consider a family $\mathcal{D} \subseteq [\omega]^\omega$ of cardinality less that $\mathfrak{d}$, and for each $A \in \mathcal{D}$ define a strictly increasing function $f_A : \omega \to \omega$ such that:

1) $f_A(0) = 0$.

2) $\lim_{n\to\infty} |A \cap [f_A(n), f_A(n+1))| = \infty$.

For each $A \in \mathcal{D}$, let $\mathcal{I}_A = \langle [f_A(n), f_A(n+1)) : n \in \omega \rangle$ be the partition defined by f_A. Since $\{\mathcal{I}_A : A \in \mathcal{D}\}$ has cardinality less than $\mathfrak{d}$ by Lemma 4, there is an interval partition $\mathcal{I} = \langle \mathcal{I}_n : n \in \omega \rangle$ such that for all $A \in \mathcal{D}$, there are infinitely many intervals from $\mathcal{I}$ containing an interval from $\mathcal{I}_A$. We can assume that for all $n \in \omega$, $|\mathcal{I}_n| < |\mathcal{I}_{n+1}|$. For each $n \in \omega$, let e_n be the increasing enumeration of $\mathcal{I}_n$, and define $\varphi(k) = (|\mathcal{I}_m|, e_m^{-1}(k))$, where m is such that $k \in \mathcal{I}_m$. We claim that for all $A \in \mathcal{D}$, $\varphi(A) \in \mathcal{ED}^+_{fin}$. Indeed, since there are infinitely many intervals from $\mathcal{I}$ containing an interval from $\mathcal{I}_A$, we have that $\limsup_{n\to\infty} |A \cap \mathcal{I}_n| = \infty$, which implies that $\limsup_{n\to\infty} |\varphi(A \cap \mathcal{I}_n)| = \infty$, and by the definition of φ it holds that $\varphi(A \cap \mathcal{I}_n) \subseteq (\Delta)_{|\mathcal{I}_n|}$, which in turn implies that $\varphi(A)$ is positive. This proves that $\mathfrak{d} \leq \mathfrak{z}_{fin}(\mathcal{ED}_{fin})$

□

[1]This Katětov relation follows from the fact that every convergent sequence is a nowhere dense subset, so the inclusion function is a witness for $\mathsf{conv} \leq_K \mathsf{nwd}$

Corollary 2. *For any analytic p-ideal $\mathscr{I}$, it holds that* $\min\{\mathfrak{r}_\sigma, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathscr{I}) \leq \mathfrak{d}$.

Proof. Let $\mathscr{I}$ be an analytic p-ideal. The first inequality follows from Corollary 1, since every analytic p-ideal is meager. The second inequality follows from the fact that every analytic p-ideal contains a copy of $\mathcal{ED}_{fin}$ and by Proposition 5. □

Note that the previous corollary implies that whenever $\mathfrak{d} < \mathfrak{r}$, then $\mathfrak{z}_{fin}(\mathscr{I}) = \mathfrak{d}$. Corollary 3 from Chapter 3, shows that for every analytic p-ideal $\mathscr{I}$ the inequality $\mathfrak{r}_\sigma = \mathfrak{z}_{fin}(\mathscr{I}) < \mathfrak{d}$ is consistent with ZFC, so we might conjecture that $\min\{\mathfrak{r}_\sigma, \mathfrak{d}\} = \mathfrak{z}_{fin}(\mathscr{I})$ whenever $\mathscr{I}$ is an analytic p-ideal.

Question 2. *Is it true that for any analytic p-ideal $\mathscr{I}$ it holds that* $\mathfrak{z}_{fin}(\mathscr{I}) = \min\{\mathfrak{r}_\sigma, \mathfrak{d}\}$*?*

However, the results from Chapter 4 imply that the inequality $\mathfrak{r} < \mathfrak{z}_{fin}(\mathscr{I}) = \mathfrak{d}$ is consistent with ZFC, for all F_σ ideal $\mathscr{I}$ at the same time, so in particular, this holds for summable ideals, which are analytic p-ideals. So the answer to Question 3 is that it is not true for all analytic p-ideals. We can still ask the following weakening of the previous question:

Question 3. *Is there an analytic p-ideal $\mathscr{I}$ for which it holds that* $\mathfrak{z}_{fin}(\mathscr{I}) = \min\{\mathfrak{r}_\sigma, \mathfrak{d}\}$*?*

The answer of this question is given in the positive by Proposition 9 of this chapter.

For the next proposition, we consider the cardinal invariant $\mathfrak{r}_{part}$ defined as the minimum cardinality of a family $\mathcal{RP} \subseteq [\omega]^\omega$ such that for any partition $\langle A_n : n \in \omega \rangle$ of ω into infinite sets there is $X \in \mathcal{RP}$ such that either: for all $n \in \omega$ $A_n \cap X$ is finite or there is $n \in \omega$ such that $X \subseteq^* A_n$. It is easy to see that $\mathfrak{r} \leq \mathfrak{r}_{part} \leq \mathfrak{r}_\sigma$.

Proposition 8. *The following holds in* ZFC*:*

1. $\mathfrak{z}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{r}_{part}$.
2. $\mathfrak{z}(\mathcal{ED}) = \mathfrak{z}_{fin}(\mathcal{ED}) = \max\{\mathfrak{r}_{part}, \mathfrak{d}\}$.
3. $\max\{\mathfrak{r}_{part}, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathcal{R}) \leq \mathfrak{z}(\mathcal{R}) \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$.

Proof. 1. Let us prove first that $\mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin}) \leq \mathfrak{r}_{part}$. Let $\mathcal{RP}$ be a witness for $\mathfrak{r}_{part}$. Let $f : \omega \to \omega \times \omega$ be a finite to one function, and consider the partition given by $A_n = f^{-1}[\{n\} \times \omega]$. Let $X \in \mathcal{RP}$ such that there is $n \in \omega$, $X \subseteq^* A_n$, or for all $n \in \omega$, $A_n \cap X$ is finite. Then $f[X] \in \mathsf{Fin} \times \mathsf{Fin}$. This proves that $\mathcal{RP}$ is a witness for $\mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin})$.

To prove $\mathfrak{r}_{part} \leq \mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin})$, let $\mathcal{F}'$ be a witness for $\mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin})$, and modify it to obtain a family $\mathcal{F}$ such that for any $A \in \mathcal{F}$, the family $\mathcal{F}_A = \{X \in \mathcal{F} : X \subseteq A\}$ is a reaping family relative to A. Note that this can be done in such a way that $|\mathcal{F}| \leq \max\{\mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin}), \mathfrak{r}\}$. Now pick any partition of ω into infinite sets, say $\vec{A} = \langle A_n : n \in \omega \rangle$, and for each $n \in \omega$, let $e_n : \omega \to A_n$ be the increasing enumeration of A_n. Define a function $f_{\vec{A}} : \omega \to \omega \times \omega$ as $f_{\vec{A}}(k) = (n, e_n^{-1}(k))$, where $n \in \omega$ is such that $k \in A_n$. Let $B \in \mathcal{F}$ be such that $f_{\vec{A}}[B] \in \mathsf{Fin} \times \mathsf{Fin}$. We have three cases: 1) $f_{\vec{A}}[B]$ is contained in finitely many verticals from $\omega \times \omega$; 2) $f_{\vec{A}}[B]$ intersects each vertical in finitely many points; 3) There is $n \in \omega$ such that $f_{\vec{A}}[B] \subseteq D \cup \bigcup_{k \leq n}\{n\} \times \omega$, where D is disjoint from $\bigcup_{k \leq n}\{k\} \times \omega$ and intersects each vertical in finitely many points. Let us deal with case 1) first. Since $\mathcal{F}_B$ is a reaping family relative to B, there is $C \in \mathcal{F}_B$ such that there is $k \leq n$ such that $C \subseteq f_{\vec{A}}^{-1}[\{n\} \times \omega] = A_n$, which finishes the case. For case 2) there is nothing to do, since B intersects each A_n in a finite set. For the third case, let $C \in \mathcal{F}_B$ be such that $C \subseteq f_{\vec{A}}^{-1}[D]$ or $C \subseteq f_{\vec{A}}^{-1}[\bigcup_{k \leq n}\{k\} \times \omega]$. Then proceed as in case 1) or case 2) taking C instead of B. Now, we have proved that $\mathfrak{r}_{part} \leq \max\{\mathfrak{r}, \mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin})\}$, but $\mathfrak{r} \leq \mathfrak{r}_{part}$, which implies that $\mathfrak{r}_{part} \leq \mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin})$. This finishes the proof of $\mathfrak{r}_{part} = \mathfrak{z}_{fin}(\mathsf{Fin} \times \mathsf{Fin})$. The case for the equality $\mathfrak{z}(\mathsf{Fin} \times \mathsf{Fin}) = \mathfrak{r}_{part}$ follows similar lines.

2. Since $\mathcal{ED} \leq_{KB} \mathsf{Fin} \times \mathsf{Fin}$ and $\mathcal{ED} \leq_{KB} \mathcal{ED}_{fin}$, by Proposition 5, item $c)$ from Proposition 7 and the previous item, it follows that $\max\{\mathfrak{r}_{part}, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathcal{ED})$. To prove the other inequality, let $\mathcal{RD}' \subseteq [\omega]^\omega$ be a witness for $\mathfrak{r}_{part}$, and modify it to obtain a family $\mathcal{RD}$ such that for each $A \in \mathcal{RD}$, the family $\mathcal{RD}_A = \{B \in \mathcal{RD} : B \subseteq A\}$ is a family of selectors for finite partitions of the set A, and also a reaping family relative to A. It is clear that the family $\mathcal{RD}$ can be find with cardinality $\max\{\mathfrak{r}_{part}, \mathfrak{d}\}$. Let $f : \omega \to \omega \times \omega$ be a finite to one function. We can assume that $f[\omega] \in \mathcal{ED}^+$. Consider the sets $A_n = f^{-1}[\{n\} \times \omega]$. There is $B \in \mathcal{RD}$ such that either: for all $n \in \omega$, $X \cap A_n$ is finite, or there is $n \in \omega$ such that $B \subseteq A_n$. In the second case we are done, since $f[B] \subseteq \{n\} \times \omega$. In the first case, consider the partition of B into finite sets given by $F_n = B \cap A_n$. Since $\mathcal{RD}_B$ is a family of selectors for partitions of B into finite sets, there is $C \in \mathcal{RD}_B$ which intersects each F_n in at most one element. Then $f[C]$ intersects each vertical from $\omega \times \omega$ in at most one element, so $f[C] \in \mathcal{ED}$. It follows that $\mathfrak{z}_{fin}(\mathcal{ED}) \leq \max\{\mathfrak{r}_{part}, \mathfrak{d}\}$. The same argument works to prove that $\mathfrak{z}(\mathcal{ED}) \leq \max\{\mathfrak{r}_{part}, \mathfrak{d}\}$, so we have the following sequence of inequalities:

$$\max\{\mathfrak{r}_{part}, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathcal{ED}) \leq \mathfrak{z}(\mathcal{ED}) \leq \max\{\mathfrak{r}_{part}, \mathfrak{d}\}$$

   It follows that $\mathfrak{z}(\mathcal{ED}) = \mathfrak{z}_{fin}(\mathcal{ED}) = \max\{\mathfrak{r}_{part}, \mathfrak{d}\}$.

3. For the inequality $\mathfrak{z}_{fin}(\mathcal{R}) \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ apply Theorem 16 with the coloring used for the definition of the Random Graph. To see the inequality $\max\{\mathfrak{r}_{part}, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathcal{R})$, note that $\mathcal{R} \leq_{KB} \mathcal{ED}$, then by Proposition 5 we have $\max\{\mathfrak{r}_{part}, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathcal{R})$. On the other hand, by Proposition 6, we have that $\mathfrak{z}(\mathcal{R}) \leq \max\{\mathfrak{r}_\sigma, \mathfrak{z}_{fin}(\mathcal{R})\} \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$. □

Proposition 9. $\mathfrak{z}_{fin}(\mathcal{Z}) = \mathfrak{z}_{fin}(\mathcal{SC}) = \min\{\mathfrak{r}_\sigma, \mathfrak{d}\}$.

Proof. The inequality $\min\{\mathfrak{d}, \mathfrak{r}_\sigma\} \leq \mathfrak{z}_{fin}(\mathcal{Z})$ follows from Corollary 1. Since $\mathcal{SC} \leq_{KB} \mathcal{Z}$, by Proposition 5, we have that $\mathfrak{z}_{fin}(\mathcal{Z}) \leq \mathfrak{z}_{fin}(\mathcal{SC})$. The inequality $\mathfrak{z}_{fin}(\mathcal{SC}) \leq \min\{\mathfrak{r}_\sigma, \mathfrak{d}\}$ follows from the facts that $\mathsf{conv} \leq_{KB} \mathcal{SC}$ and $\mathcal{ED}_{fin} \leq_{KB} \mathcal{SC}$, Proposition 7 and Proposition 5. □

Chapter 3

More about $\mathscr{I}$-ultrafilters.

3.1 Two questions of J. Flašková.

Let us recall the following theorem due to Vojtáš:

Theorem 18. *An ultrafilter $\mathcal{U}$ on ω is rapid if and only if $\mathcal{U}$ has non-empty intersection with every summable ideal.*

In (30), between several results, J. Flašková proves the following refinement of Vojtáš theorem:

Theorem 19. *There is a family $\mathcal{D}$ of summable ideals with cardinality $\mathfrak{d}$ such that an ultrafilter is rapid if and only if it has non-empty intersection with every ideal from $\mathcal{D}$.*

Then she asks the following two questions:

Question 4. *What is the minimal size of a family $\mathcal{D}$ of summable ideals such that an ultrafilter $\mathcal{U}$ is rapid if and only if it has non-empty intersection with each ideal in $\mathcal{D}$?*

Question 5. *Is it true that whenever the cardinality of $\mathcal{D}$* [a family of summable ideals] *is less than $\mathfrak{d}$, then there exist an ultrafilter on the natural numbers which is an $\mathscr{I}_g$-ultrafilter for every $\mathscr{I}_g \in \mathcal{D}$, but not a rapid ultrafilter?*

By theorem 19, we know that such cardinality is at most $\mathfrak{d}$. The first section of this chapter proves that in fact the equality holds. In sections 3.3 to 3.5, we establish the consistency in the positive way of Question 5, showing that in the Rational Perfect Set model, for any family $\mathcal{D}$ of summable ideals with cardinality less than $\mathfrak{d}$, there is an ultrafilter which is an $\mathscr{I}$-ultrafilter for every $\mathscr{I} \in \mathcal{D}$, although there is no rapid ultrafilter in such model. Results from Chapter 4 will show that Question 5 is in fact independent from ZFC.

3.2 $\mathfrak{d}$ is the best.

For the sake of simplicity, let us say that a family $\mathcal{D}$ of summable ideals is a Vojtáš family if an ultrafilter is rapid if and only if it has non-empty intersection with each member from $\mathcal{D}$. The main theorem of this section can be stated as follows:

Theorem 20. *$\mathfrak{d}$ is equal to the minimum cardinality of a Vojtáš family.*

Since Theorem 19 means that $\mathfrak{d}$ is an upper bound for the minimum cardinality of a Vojtáš family, we only have to prove the following:

Proposition 10. *No family $\mathcal{D}$ of summable ideals on ω of cardinality less than $\mathfrak{d}$ is a Vojtáš family.*

Proof. It is enough to prove that given any non-empty family of summable ideals of cardinality less than $\mathfrak{d}$, there is an ultrafilter meeting each ideal in the family, but still there is a tall summable ideal avoiding the ultrafilter. Fix $\mathcal{D} = \{\mathscr{I}_\alpha : \alpha < \lambda\}$ an arbitrary family of tall summable ideals such that $\lambda < \mathfrak{d}$, and for $\alpha \in \lambda$, let $g_\alpha : \omega \to \mathbb{R}$ be a function from which $\mathscr{I}_\alpha$ is defined. Let E be the set of even natural numbers. For each $F \in [\lambda]^{<\omega} \setminus \{\emptyset\}$, define a function ψ_F on E as follows:

$$\begin{gathered}\psi_F(0) = 0\\ \psi_F(n+2) = \min\{k \in \omega : k > \psi_F(n) \text{ and } (\forall \alpha \in F)(\forall i \geq k)\\ (g_\alpha(i) \leq 1/(2^{n+2}(n+2)^2))\}\end{gathered}$$

Since $\{\psi_F : F \in [\lambda]^{<\omega} \setminus \{\emptyset\}\}$ has cardinality less than $\mathfrak{d}$, there is a function $f : E \to \omega$ which is not dominated by $\{\psi_F : F \in [\lambda]^{<\omega} \setminus \{\emptyset\}\}$. We can assume that for all $n \in E$, $f(n+2) - f(n) > n^2$, and we can define the following function:

$$\tilde{f}(n) = \begin{cases} f(n) \text{ if } n \in E \\ f(n-1) + (n-1)^2 \text{ if } n \notin E \end{cases}$$

For each $F \in [\lambda]^{<\omega} \setminus \{\emptyset\}$, let A_F be the set where ψ_F is dominated by f, that is $A_F = \{n \in E : \psi_F(n) \leq f(n)\}$. Note that the family $\{A_F : F \in [\lambda]^{<\omega} \setminus \{\emptyset\}\}$ is a centered family. Indeed, pick $F_0, \ldots, F_n \in [\lambda]^{<\omega} \setminus \{\emptyset\}$. By the definition of the functions ψ_F we have that if $F \subseteq G$, then $\psi_F \leq \psi_G$, so in particular, if $\psi_{F_0 \cup \ldots \cup F_n}(k) \leq f(k)$, then we have $\psi_{F_i}(k) \leq f(k)$ for $i = 0, \ldots, n$, and so $k \in A_0 \cap \ldots \cap A_n$.

Now, for $F \in [\lambda]^{<\omega} \setminus \{\emptyset\}$ let B_F be the set $\bigcup_{k \in A_F} [\tilde{f}(k), \tilde{f}(k+1))$. Finaly, take the family $\mathcal{G} = \{B_F : F \in [\lambda]^{<\omega} \setminus \{\emptyset\}\}$. It follows that $\mathcal{G}$ is a centered family, by the previous paragraph.

Let us see that $\mathcal{G}$ has non-empty intersection with each ideal in $\mathcal{D}$. For $\alpha \in \lambda$, we claim that $B_{\{\alpha\}} \in \mathscr{I}_\alpha$. This follows easily from the definition of $\psi_{\{\alpha\}}$ and $A_{\{\alpha\}}$. We have that for all $n \in A_{\{\alpha\}}$, $\psi_{\{\alpha\}}(n) \leq f(n) = \tilde{f}(n)$, and by the definition of $\psi_{\{\alpha\}}$, for every $k \geq \psi_{\{\alpha\}}(n)$, $g_\alpha(k) \leq \frac{1}{2^n n^2}$, so in particular, for every $n \in A_{\{\alpha\}}$ and every $k \geq \tilde{f}(n)$, $g_\alpha(k) \leq \frac{1}{2^n n^2}$. We have the following:

$$\sum_{n\in B_{\{\alpha\}}} g_\alpha(n) = \sum_{n\in A_{\{\alpha\}}} \left[\sum_{m\in[\tilde f(n),\tilde f(n+1))} g_\alpha(m) \right]$$
$$\leq \sum_{n\in A_{\{\alpha\}}} \left[\sum_{m\in[\tilde f(n),\tilde f(n+1))} \frac{1}{2^n n^2} \right]$$
$$= \sum_{n\in A_{\{\alpha\}}} \frac{n^2}{2^n n^2} \leq \sum_{n\in A_{\{\alpha\}}} \frac{1}{2^n} \leq 1$$

What remains is to find a tall summable ideal $\mathscr{I}$ which has empty intersection with some ultrafilter extending $\mathcal{G}$. Consider the following function from ω to $\mathbb{R}$,

$$h(n) = \begin{cases} 1 \text{ if } n \in [0, \tilde f(0)) \\ \frac{1}{k+1} \text{ if } n \in [\tilde f(k), \tilde f(k+1)) \end{cases}$$

Let $\mathscr{I}_h$ be the corresponding summable ideal. We claim that $\mathcal{G} \subseteq \mathscr{I}_h^+$. Pick any $B_F \in \mathcal{G}$. Then,

$$\sum_{n\in B_F} h(n) = \sum_{n\in A_F} \left[\sum_{m\in[\tilde f(n),\tilde f(n+1))} h(m) \right]$$
$$= \sum_{n\in A_F} \left[\sum_{m\in[\tilde f(n),\tilde f(n+1))} \frac{1}{n+1} \right]$$
$$= \sum_{n\in A_F} \frac{n^2}{n+1} \longrightarrow \infty$$

Then we have $\mathcal{G} \subseteq \mathscr{I}_h^+$, so $\mathcal{G} \cup \mathscr{I}_h^*$ can be extended to an ultrafilter $\mathcal{U}$. Obviously $\mathscr{I}_h$ and $\mathcal{U}$ satisfy our requirements. □

Now, Theorem 20 follows immediately by Theorem 19 and Proposition 10.

3.3 Rational Perfect set forcing and p-ideals.

Definition 17. *Let $\mathscr{I}$ be an ideal on ω and let $\mathcal{U}$ be an ultrafilter on ω. We say that $\mathcal{U}$ is an $(\mathscr{I}, p)$-point provided $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter and a p-point at the same time.*

Recall from the Preliminaries the definition of the Rational Perfect set forcing, denoted by **PT**. The following lemmas are well known properties of the Rational Perfect set forcing. We refer the reader to (52) for details of the proofs. Also, recall the definition of the isomorphism $\varphi_T : \omega^{<\omega} \to \text{split}(T)$.

Lemma 5. *The Rational Perfect set forcing satisfies the Axiom A. In particular, it is proper.*

Lemma 6 (Continuous reading of names)**.** *Let $\dot{f}$ be a* **PT***-name for a function from ω to ω, $T \in \mathbf{PT}$ a condition and $g \in \omega^\omega$ such that $T \Vdash$ "$\dot{f} \leq g$". Then there is $T' \leq T$, and a family $\{h_s : s \in \mathrm{split}(T')\} \subseteq \omega^\omega$ such that for all $s \in \mathrm{split}(T')$ and for all but finitely many $k \in \mathrm{succ}_T(s)$, it holds that $T \upharpoonright s^\frown k \Vdash$ "$\dot{f} \upharpoonright (|s| + n) = h_s \upharpoonright (|s| + n)$".*

Lemma 7. *Let $T \in \mathbf{PT}$ be a condition, and A, B a partition of the splitting nodes of T. There is a stronger condition $T' \leq T$ such that $\mathrm{split}(T') \subseteq A$ or $\mathrm{split}(T') \subseteq B$.*

Theorem 21 (p-point preservation)**.** *Let $\mathcal{U}$ be p-point. Then $\mathcal{U}$ generates a p-point in the generic extension by* **PT**.

The following lemma will be useful in the proof of Theorem 22, since it will enable us to use Lemma 6.

Lemma 8. *Let $\mathscr{I}$ be an analytic p-ideal. Then there exists a function $g_{\mathscr{I}} : \omega \to \omega$ such that for any ultrafilter $\mathcal{U}$, $\mathcal{U}$ is a weak $\mathscr{I}$-ultrafilter if and only if for every finite to one $f \leq g_{\mathscr{I}}$ there is $A \in \mathcal{U}$ such that $f[A] \in \mathscr{I}$.*

Proof. We prove only the non-trivial direction. Let φ be a lscsm such that $\mathscr{I} = Exh(\varphi)$. Note that for any $n \in \omega$ there is $k \in \omega$ such that for all $i \geq k$, $\varphi(\{i\}) \leq \frac{1}{2^n}$ (recall that $\mathscr{I}$ is a tall ideal). Recursively construct a sequence $\langle k_n : n \in \omega\rangle$ such that for all $i \geq k_n$, $\varphi(\{i\}) \leq \frac{1}{2^n}$ and $k_n < k_{n+1}$. Then define $g_{\mathscr{I}}(n) = k_n$. Let us see that $g_{\mathscr{I}}$ is a witness for the lemma. Let $f \in \omega^\omega$ be an arbitrary finite to one function. If $g_{\mathscr{I}} \leq_{\mathcal{U}} f$, then it follows that $D_{g_{\mathscr{I}} \leq f} = \{i \in \omega : g_{\mathscr{I}}(i) \leq f(i)\} \in \mathcal{U}$, and note that for any $i \in D_{g_{\mathscr{I}} \leq f}$, $\varphi(\{f(i)\}) \leq 1/2^i$, so for any $n \in \omega$,

$$\varphi(f[D_{g_{\mathscr{I}} \leq f}] \cap n) \leq \sum_{j \in f[D_{g_{\mathscr{I}} \leq f}] \cap n} \varphi(\{j\}) \leq \sum_{i \in D_{g_{\mathscr{I}} \leq f}, f(i) < n} \varphi(\{f(i)\}) \leq \sum_{i \in D_{g_{\mathscr{I}} \leq f}, f(i) < n} \frac{1}{2^i} \leq \sum_{i \in D_{g_{\mathscr{I}} \leq f}} \frac{1}{2^i}$$

Then, by the lower semicontinuity of φ, it follows that $\varphi(f[D_{g_{\mathscr{I}} \leq f}]) \leq \sum_{i \in D_{g_{\mathscr{I}} \leq f}} \frac{1}{2^i}$. In the same way it can be proved that $\varphi(f[D_{g_{\mathscr{I}} \leq f}] \setminus n) \leq \sum_{i \in D_{g_{\mathscr{I}} \leq f}, f(i) \geq n} \frac{1}{2^i}$, which implies that $\lim_{n \to \infty} \varphi(f[D_{g_{\mathscr{I}} \leq f}] \setminus n) = 0$, so $f[D_{g_{\mathscr{I}} \leq f}] \in \mathscr{I}$. If $f \leq_{\mathcal{U}} g_{\mathscr{I}}$, then there is $h \in \omega^\omega$ such that $h \leq g_{\mathscr{I}}$ and $f =_{\mathcal{U}} h$. By hypothesis, there is $A \in \mathcal{U}$ such that $h[A] \in \mathscr{I}$. Define $B = A \cap \{n \in \omega : f(n) = h(n)\} \in \mathcal{U}$, then we have $f[B] = h[B] \in \mathscr{I}$. □

Finally, we will need to make use of the p-point game.

Definition 18 (p-point game)**.** *Given an ultrafilter $\mathcal{U}$, the two players game $G(\mathcal{U})$ is defined as a sequence of choices were, at round n, Player I chooses an element from the ultrafilter, $A_n \in \mathcal{U}$, and Player II chooses a finite subset $F_n \in [A_n]^{<\omega}$, and Player II wins if and only if $\bigcup_{n \in \omega} F_n \in \mathcal{U}$:*

Player I	$A_0 \in \mathcal{U}$		$A_1 \in \mathcal{U}$		$A_2 \in \mathcal{U}$	
Player II		$F_0 \in [A_0]^{<\omega}$		$F_1 \in [A_1]^{<\omega}$		$F_2 \in [A_2]^{<\omega}$

The following well known characterization of p-points will be useful in proving that the forcing preserves $(\mathscr{I}, p)$-points.

Lemma 9 (C. Laflamme, see (47))**.** *Let $\mathcal{U}$ be an ultrafilter on ω. Then $\mathcal{U}$ is a p-point if and only if Player I has no winning strategy in the p-point game.*

Proof. We ask the reader to consult C. Laflamme's article (47). Another good reference is W. Wohofsky master thesis (68), pages 53-59. $\square$

Now we are ready to prove the following. We want to remark that $(\mathscr{I}, p)$-points and weak $(\mathscr{I}, p)$-points are exactly the same.

Theorem 22. *Let $\mathscr{I}$ be an anlytic p-ideal and $\mathcal{U}$ an $(\mathscr{I}, p)$-point. Then in the generic extension by the Rational Perfect set forcing $\mathcal{U}$ generates an $(\mathscr{I}$,$p)$-point.*

Proof. Fix $\mathcal{U}$ an $(\mathscr{I}, p)$-point. Let $\dot{f}$ be a **PT**-name and $p \in$ **PT** a condition such that $p \Vdash$ "$\dot{f} \in \omega^\omega$". By Lemma 8, we can assume that $p \Vdash$ "$\dot{f} \leq g_{\mathscr{I}}$". By the remark right before this theorem, and by Theorem 21, $\mathcal{U}$ remains as a p-point in the generic extension, we can assume that $p \Vdash$ "$\dot{f}$ is finite to one"(the case when $\dot{f}$ has a restriction on which it is constant follows immediately). By using the continous reading of names, we can assume that for any $s \in \text{split}(p)$ there is a function $h_s \in \omega^\omega$ such that for all $n \in \omega$ and for all but finitely many $k \in \text{succ}_p(s)$, $p \restriction s^\frown k \Vdash \dot{f} \restriction (|s| + n) = h_s \restriction (|s| + n)$. Since $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter, for every $s \in \text{split}(p)$ we find $A_s \in \mathcal{U}$ such that $h_s[A_s] \in \mathscr{I}$. Note that h_s is not necesarily a finite to one function, but since $\mathcal{U}$ is a p-point, we can assume that $h_s \restriction A_s$ is either constant or finite to one. Now, by Lemma 7, we can as well assume that the condition p has one of the following properties:

a) For all $s \in \text{split}(p)$ the restriction $h_s \restriction A_s$ is finite to one, or

b) For all $s \in \text{split}(p)$ the restriction $h_s \restriction A_s$ is constant a_s.

Let us consider first the case a). Since $\mathscr{I}$ is a p-ideal and for all $s \in \text{split}(p)$ we have $h_s[A_s] \in \mathscr{I}$, then there is $Z \in \mathscr{I}$ such that for all $s \in \text{split}(p)$, $h_s[A_s] \subseteq^* Z$. By taking off finitely many elements of each A_s, we can assume that for all $s \in \text{split}(p)$ we have $h_s[A_s] \subseteq Z$. Making use of the p-point game we will construct two sequences $\{F_n : n \in \omega\}$ and $\{T_n : n \in \omega\}$ such that:

1. $T_0 \leq_0 p$.
2. For all $n \in \omega$, $T_{n+1} \Vdash$ "$\dot{f}[F_n] \subseteq Z$", and $T_{n+1} \leq_{n+1} T_n$.
3. $\bigcup_{n\in\omega} F_n \in \mathcal{U}$.
4. $T_\omega = \bigcap_{n\in\omega} T_n$ is a condition.

The construction is as follows:

i) Player I starts playing $A_0 = A_{\mathrm{st}(p)}$, and define $T_0 = p$.

ii) Suppose Player II has answered with a set $F_0 \subseteq A_0$. Let $k_0 \in \omega$ be big enough so that for all $i \in \mathrm{succ}_{T_0}(\mathrm{st}(T_0)) \setminus k_0$ it holds that $T \restriction \mathrm{st}(T_0)^\frown i \Vdash$ "$\dot{f} \restriction (\max(F_0)+1) = h_{\mathrm{st}(T_0)} \restriction (\max(F_0)+1)$. Note that this implies that for all $i \geq k_0$, $T_0 \restriction \mathrm{st}(T_0)^\frown i \Vdash$ "$\dot{f}[F_0] \subseteq Z$". Then define

$$T_1 = \bigcup_{i \in \mathrm{succ}_{T_0}(\mathrm{st}(T_0)). i \geq k_0} T_0 \restriction \mathrm{st}(T_0)^\frown i$$

And Player I responds with $A_1 = \bigcap_{s \in 1^{\leq 1}} A_s \setminus (\max(F_0) + 1)$.

iii) Suppose at move number n and Player I has played

$$A_n = \bigcap_{s \in n^{\leq n}} A_{\varphi_{T_n}(s)} \setminus (\max\left(\bigcup_{k<n} F_k\right) + 1)$$

Suppose Player II responds with a set $F_n \subseteq A_n$. Then, let $k_n \in \omega$ be big enough such that for all $r \in \varphi_{T_n}(n^{\leq n})$ and all $i \in \mathrm{succ}_{T_n}(r) \setminus k_n$, it holds that $T_n \restriction r^\frown i \Vdash$ "$\dot{f} \restriction (\max\left(\bigcup_{k \leq n} F_n\right) + 1) = h_r \restriction (\max\left(\bigcup_{k \leq n} F_n\right) + 1)$". Note that this implies that for each $r \in \varphi_{T_n}(n^{\leq n})$ and $i \in \mathrm{succ}_{T_n}(r) \setminus k_n$, $T_n \restriction r^\frown i \Vdash$ "$\dot{f}[F_n] \subseteq Z$". Then define T_{n+1} as follows:

$$T_{n+1} = \bigcup_{s \in n^{\leq n}} \bigcup_{i \in \mathrm{succ}_{T_n}(\varphi_{T_n}(s)), i \geq k_n} T \restriction \varphi_{T_n}(s)^\frown i$$

By construction it follows that $T_{n+1} \Vdash$ "$\dot{f}[F_n] \subseteq Z$". Let Player I play the set

$$A_{n+1} = \bigcap_{s \in (n+1)^{\leq n+1}} A_{\varphi_{T_{n+1}}(s)} \setminus (\max\left(\bigcup_{k \leq n} F_k\right) + 1)$$

By Lemma 9, this is not a winning strategy for Player I, so there is a play where Player II wins the game. Let $\{F_n : n \in \omega\}$ and $\{T_n : n \in \omega\}$ be the sequences obtained by Player I in this play, and $T_\omega = \bigcap_{n \in \omega} T_n$.

Claim. $T_\omega \Vdash$ "$\dot{f}[\bigcup_{n \in \omega} F_n] \subseteq Z$". Following the construction of the three sequences, it is not hard to see that $T_{n+1} \Vdash$ "$\dot{f}[F_n] \subseteq Z$". But for all n, $T_\omega \leq T_n$, so the claim is true for T_ω.

Now let us deal with case b). Fist note that there is no condition $q \leq p$ such that the set $\{a_s : s \in \mathrm{split}(q)\}$ is finite. To prove this assume it is false and apply Lemma 7 finitely many times, then recall that the function $\dot{f}$ is a finite to one function to arrive at a contradiction. By using this remark it is possible to find a stronger condition $q \leq p$ such that $Z = \{a_s : s \in \mathrm{split}(q)\} \in \mathscr{I}$. The rest follows the same lines of the previous case. Using the p-point game again we construct two sequnces $\{F_n : n \in \omega\}$ and $\{T_n : n \in \omega\}$ such that condition (2)-(5) of the construction for case a) hold, and condition (1) is changed to (1') $T_0 \leq q$. The strategy for player I is defined in the same way as it was in the previous case. Then the condition T_ω and the set $\bigcup_{n\in\omega} F_n$ obtained in this way satisfy $T \Vdash$ "$\dot{f}[\bigcup_{n\in\omega} F_n] \subseteq Z$".

$\square$

3.4 A preservation theorem for $(\mathscr{I}, p)$-points.

In this section we prove the following preservation theorem for $(\mathscr{I}, p)$-points.

Theorem 23. *Let $\mathscr{I}$ be an analytic p-ideal such that $\mathscr{I} = Fin(\varphi) = Exh(\varphi)$, for some lscsm defining $\mathscr{I}$. Let $\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta : \beta \leq \alpha\rangle$ be a countable support iteration of proper forcing notions such that for all $\beta < \alpha$, $\mathbb{P}_\beta$ is proper, preserves $(\mathscr{I}, p)$-points and also $\mathbb{P}_\beta \Vdash$ "$\dot{\mathbb{Q}}_\beta$ is proper and preserves $(\mathscr{I}, p)$-points". Then $\mathbb{P}_\alpha$ preserves $(\mathscr{I}, p)$-points.*

Recall that given a $\mathbb{P}$-name $\dot{f}$ for a real, a pair $(\langle p_n : n \in \omega\rangle, h)$ is an interpretation for $\dot{f}$ if for all $n \in \omega$, $p_n \Vdash$ "$\dot{f} \restriction n = h \restriction n$". We need a strengthening of this property.

Definition 19. *Let $\dot{f}$ be a $\mathbb{P}$-name for a finite to one function from ω to ω. A fine interpretation for $\dot{f}$ is a pair $(\langle p_n : n \in \omega\rangle, h)$ such that $\langle p_n : n \in \omega\rangle$ is a decreasing sequence, h is a finite to one function and for all $n \in \omega$, $p_n \Vdash$ "$\dot{f} \restriction n = h \restriction n$".*

It is easy to see that if $\dot{f}$ is forced to be a finite to one function, then given a condition p, we can find a fine interpretation for $\dot{f}$ below p. In order to prove Theorem 23, we need the following technical definition and the lemma following the definition.

Definition 20. *Let $\mathbb{P}*\dot{\mathbb{Q}}$ be a two step iteration, and let $\dot{f}$ be a $\mathbb{P}*\dot{\mathbb{Q}}$-name and $(p, \dot{q}) \in \mathbb{P}*\dot{\mathbb{Q}}$ a condition such that $(p, \dot{q}) \Vdash$ "$\dot{f} \in \omega^\omega$". Let $(\langle r_n : n \in \omega\rangle, h)$ be an interpretation of $\dot{f}$ below $(p, \dot{q})$, where $r_n = (p_n, \dot{q}_n)$. Let $\sqsubseteq$ be a well ordering of $\mathbb{P} * \dot{\mathbb{Q}}$. Suppose $G_\mathbb{P}$ is a generic filter for $\mathbb{P}$ over V. Then, in $V[G_\mathbb{P}]$, define a decreasing sequence $\langle t_n : n \in \omega\rangle$ of conditions in $\dot{\mathbb{Q}}[G_\mathbb{P}]$ which interprets $\dot{f}$ as follows:*

1. *Define $n^* = \sup(\{-1\} \cup \{n \in \omega : p_n \in G_\mathbb{P}\})$. If $n^* = -1$, define $t_{-1} = 1_{\dot{Q}[G_\mathbb{P}]}$.*
2. *For $n \in \omega$, if $n \leq n^*$, then $t_n = \dot{q}_n[\dot{G}_\mathbb{P}]$.*
3. *For $n \in \omega$, if $n > n^*$, then define t_n to be $\dot{s}_n[G_\mathbb{P}]$, where $\dot{s}_n$ is such that there is $u_n \in \mathbb{P}$ such that $(u_n, \dot{s}_n) \in \mathbb{P} * \dot{\mathbb{Q}}$ is the $\sqsubseteq$-first extension of $(u_{n-1}, \dot{s}_{n-1})$ such that $u_n \in G_\mathbb{P}$ and $\dot{s}_n[G_\mathbb{P}]$ decides the value of $\dot{f} \restriction n$.*

The sequence $\langle t_i : i \in \omega\rangle$ defined in $V[\dot{G}_\mathbb{P}]$ is said to be the derived sequence from $\langle r_n : n \in \omega\rangle$, $G_\mathbb{P}$ and $\dot{f}$, and we write $\delta_{G_\mathbb{P}}(\langle r_n : n \in \omega\rangle, \dot{f})$ to denote this sequence. The $\mathbb{P}$-name of this sequence will be denoted by $\tilde{\delta}(\langle r_n : n \in \omega\rangle, \dot{f})$. If $\tilde{\delta}(\langle r_n : n \in \omega\rangle, \dot{f})$ is a $\mathbb{P}$-name for a derived sequence in $V[G_\mathbb{P}]$, then define by $int(\tilde{\delta}(\langle r_n : n \in \omega\rangle, \dot{f}), \dot{f})$ a $\mathbb{P}$-name of the interpretation in $V[G_\mathbb{P}]$ of $\dot{f}$ by the sequence $\delta_{G_\mathbb{P}}(\langle r_n : n \in \omega\rangle, \dot{f})$

It is easy to see that if $\dot{g} = int(\delta(\bar{r}, \dot{f}), \dot{f})$, and $\bar{r}$ interprets $\dot{f}$ as h, then $\bar{p}$ interprets $\dot{g}$ as h.

Lemma 10. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a forcing iteration, $\dot{f}$ a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a finite to one function, and $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ a condition. Let $(\langle r_n : n \in \omega\rangle, h)$ be a fine interpretation for $\dot{f}$ below $(p, \dot{q})$, where $r_n = (p_n, \dot{q}_n)$. Then there exists a derived sequence $\tilde{\delta} = \tilde{\delta}(\langle r_n : n \in \omega, \dot{f}\rangle)$ such that the pair $(\tilde{\delta}, int(\tilde{\delta}, \dot{f}))$ is forced to be a fine interpretation for $\dot{f}$ in $V[G_{\mathbb{P}}]$, and moreover, $int(\tilde{\delta}, \dot{f})$ is interpreted by $\langle p_n : n \in \omega\rangle$ as h.*

Proof. We follow the same notation as in Definition 20. Let $G \subseteq \mathbb{P}$ be a generic filter over V, and define $n^* = \sup(\{-1\} \cup \{n \in \omega : p_n \in G\})$. Define $\tilde{\delta}$ as a $\mathbb{P}$-name for a sequence $\langle t_n : n \in \omega\rangle$ of conditions in $\dot{\mathbb{Q}}[G]$ such that the following holds:

> For all $n \in \omega$, if $n \leq n^*$, then $t_n = \dot{q}_n[G]$, and for all natural number $n > n^*$, define t_n to be $\dot{s}_n[G]$, where $\dot{s}_n$ is such that there is $u_n \in \mathbb{P}$ such that $(u_n, \dot{s}_n) \in \mathbb{P} * \dot{\mathbb{Q}}$ is the first $\sqsubseteq$-first extension of $(u_{n-1}, \dot{s}_{n-1})$ such that $u_n \in G$, and $\dot{s}_n[G]$ decides the value of $\dot{f}^{-1}(n)$ and $\dot{f} \restriction n$. Let $g \in \omega^\omega$ be the function such that $t_n \Vdash$ "$\dot{f} \restriction n = g \restriction n$". Define $int(\tilde{\delta}, \dot{f})$ as a $\mathbb{P}$-name for g.

By definition of $\tilde{\delta}$ and the definition of $int(\tilde{\delta}, \dot{f})$, it follows that the pair $(\tilde{\delta}, int(\tilde{\delta}, \dot{f}))$ is forced to a fine interpretation of $\dot{f}$ in $V[\dot{G}]$, since $int(\tilde{\delta}, \dot{f})$ is actually the name of the function g. It is easy to see that $\langle p_n : n \in \omega\rangle$ interprets $int(\tilde{\delta}, \dot{f})$ as h. □

The next lemma can be seen as an intermediate step in the proof of Theorem 23, but we consider that it is more readable to write it as a separate lemma.

Lemma 11. *Let $\mathscr{I}$ be an ideal defined by a lscsm φ such that $\mathscr{I} = Fin(\varphi) = Exh(\varphi)$. Let $\mathcal{U}$ be an $(\mathscr{I}, p)$-point. Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a two step iteration of proper forcings such that $\mathbb{P}$ preserves $(\mathscr{I}, p)$-points. Let $\dot{f}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a finite to one function from ω to ω. Let $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ be a condition and $(\langle r_n : n \in \omega\rangle, h_0)$ be a fine interpretation of $\dot{f}$ below the condition $(p, \dot{q})$. We can assume that $r_n = (p_n, \dot{q}_n)$. Let $\mathcal{M} \prec H(\theta)$ be a countable elementary submodel such that $p, \dot{f}, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{U}, (\langle r_n : n \in \omega\rangle, h_0) \in \mathcal{M}$, and let $Z \in \mathcal{U}$ be a pseudointersection of $\mathcal{U} \cap \mathcal{M}$. Let $m \in \omega$ be such that $\varphi(h_0[Z]) < m$. Then for all $n \in \omega$, there are the following:*

1) An $(\mathcal{M}, \mathbb{P})$-generic condition $p_{\mathcal{M}}$.

2) A $\mathbb{P}$-name $\tilde{s}_n$.

3) A $\mathbb{P}$-name $\tilde{h}_n$ for a finite to one function from ω to ω.

4) A $\mathbb{P}$-name $\underset{\sim}{\vec{t}_n}$ for a sequence of conditions in $\dot{\mathbb{Q}}[\dot{G}_{\mathbb{P}}]$.

Such that:

i) $p_{\mathcal{M}} \leq p_0$.

ii) $p_{\mathcal{M}} \Vdash$ "$\tilde{s}_n \leq \dot{q}$" and $p_{\mathcal{M}} \Vdash$ "$\tilde{s}_n \in \dot{\mathbb{Q}}[\dot{G}_{\mathbb{P}}] \cap \mathcal{M}[\dot{G}_{\mathbb{P}}]$".

iii) $p_{\mathcal{M}}$ forces that $(\underset{\sim}{\vec{t}_n}, \tilde{h}_n) \in \mathcal{M}[\dot{G}_{\mathbb{P}}]$ is a fine interpretation of $\dot{f}$ below $\tilde{s}_n$ and such that $\varphi(\tilde{h}_n[Z]) < m$.

iv) $p_{\mathcal{M}} \Vdash$ "$\tilde{s}_n \Vdash$ "$\dot{f} \restriction n = \dot{\dot{h}}_n \restriction n$"".

Proof. Fix a natural number $n \in \omega$. We need to find the objects in clauses 1)-4). First note that by Lemma 10, we can find inside $\mathcal{M}$ a $\mathbb{P}$-name for a fine derived sequence $\dot{\delta} = \dot{\delta}(\langle r_n : n \in \omega\rangle, \dot{f})$ such that $int(\dot{\delta}, \dot{f})$ is interpreted by $\langle p_n : n \in \omega\rangle$ as h_0. For short, denote by $\dot{h}$ the name $int(\dot{\delta}, \dot{f})$.

Let $G \subseteq \mathbb{P}$ be a generic filter over V. By hypothesis, $\mathcal{U}$ remains as an $(\mathscr{I}, p)$-point in $V[G]$, and by elementarity, $\mathcal{M}[G] \vDash \mathcal{U}$ is an $(\mathscr{I}, p)$-point. So for each $k \in \omega$, there are $u_k \leq p_k$ and $X_k \in \mathcal{U}$ such that $u_k \Vdash$ "$\varphi(\dot{h}[X_k]) < 1$", since $\mathscr{I} = Exh(\varphi)$. Clearly this can by done inside $\mathcal{M}$, so we assume these two sequences are in $\mathcal{M}$. Let $X \in \mathcal{M} \cap \mathcal{U}$ be a pseudointersection of $\langle X_k : k \in \omega\rangle$. Now, working in $\mathcal{M}$, define by recursion a sequence of natural numbers $\langle k_l : l \in \omega\rangle$ as follows:

1) $k_0 = 0$
2) For all $i \leq k_l$, $X \setminus k_{l+1} \subseteq X_i$.
3) For all $i \geq k_{l+1}$, $p_i \Vdash$ "$\dot{h} \restriction k_l = h_0 \restriction k_l$"
4) For all $i \leq k_l$, there is $u_i^{l+1} \leq u_i \leq p_i$ such that $u_i^{l+1} \Vdash$ "$\varphi(\dot{h}[X_i \setminus k_{l+1}]) < m - \max\{\varphi(h_0[F]) : F \subseteq k_l \wedge \varphi(h_0[F]) < m\}$".

Note that 4) can be achieved since $\dot{h}$ is forced to be finite to one and $\dot{h}[X_i] \in \mathscr{I} = Exh(\varphi)$ is forced by u_i. Conditions 1)-3) are trivial.

Assume the sequence $\langle k_j : j \in \omega\rangle$ has been constructed. Without loss of generality, we can assume that $Y = \bigcup_{j\in\omega}[k_{3j+1}, k_{3j+2}) \in \mathcal{U}$. Let j_0 be such that $Z \setminus k_{3j_0} \subseteq X \cap Y$ (recall that Z is a pseudointersection of $\mathcal{M} \cap \mathcal{U}$, and $X \cap Y \in \mathcal{U} \cap \mathcal{M}$). Note that for any $l > j_0$ we have $[k_{3l-1}, k_{3l+1}) \cap Z = \emptyset$. Fix one of such l. We claim that $u_{k_{3l}}^{3l+1} \Vdash$ "$\varphi(\dot{h}[Z]) < m$". Since $u_{k_{3l}}^{3l+1} \leq p_{k_{3l}}$, by 3) above we have,

$$u_{k_{3l}}^{3l+1} \Vdash \text{“}\dot{h} \restriction k_{3l-1} = h_0 \restriction k_{3l-1}\text{”}$$

Which in turn implies,

$$u_{k_{3l}}^{3l+1} \Vdash \text{“}\dot{h}[Z \cap k_{3l-1}] = h_0[Z \cap k_{3l-1}]\text{”}$$

and so,

$$u_{k_{3l}}^{3l+1} \Vdash \text{“}\varphi(\dot{h}[Z \cap k_{3l-1}]) \leq \max\{\varphi(h_0[F]) : F \subseteq k_{3l} \wedge \varphi(h_0[F]) < m\}\text{”} \tag{3.1}$$

Also, by 2) and the choice of l we have

$$Z \setminus k_{3l+1} \subseteq X \cap Y \setminus k_{3l+1} \subseteq X_{k_{3l}} \setminus k_{3l+1} \tag{3.2}$$

note that since $Z \cap [k_{3l-1}, k_{3l+1}) = \emptyset$, the following holds,

$$u_{k_{3l}}^{3l+1} \Vdash \text{“}\varphi(\dot{h}[Z]) \leq \varphi(\dot{h}[Z \setminus k_{3l}]) + \varphi(\dot{h}[Z \cap k_{3l}]) = \varphi(\dot{h}[Z \setminus k_{3l+1}]) + \varphi(\dot{h}[Z \cap k_{3l-1}])\text{”} \tag{3.3}$$

Note that (3.1) implies the following:

$$u^{3l+1}_{k_{3l}} \Vdash \text{“}m - \max\{\varphi(h_0[F]) : F \subseteq k_{3l} \wedge \varphi(h_0[F]) < m\} \leq m - \varphi(\dot{h}[Z \cap k_{3l-1}])\text{”}$$

This inequality, joint with 4), gives the following:

$$u^{3l+1}_{k_{3l}} \Vdash \text{“}\varphi(\dot{h}[X_{k_{3l}} \setminus k_{3l+1}]) < m - \varphi(\dot{h}[Z \cap k_{3l-1}])\text{”}$$

or equivalently,

$$u^{3l+1}_{k_{3l}} \Vdash \text{“}\varphi(\dot{h}[X_{k_{3l}} \setminus k_{3l+1}]) + \varphi(\dot{h}[Z \cap k_{3l-1}]) < m\text{”}$$

by (3.2) and the monotonicity of φ, we have that,

$$u^{3l+1}_{k_{3l}} \Vdash \text{“}\varphi(\dot{h}[Z \setminus k_{3l+1}]) + \varphi(\dot{h}[Z \cap k_{3l-1}]) < m\text{”}$$

Finally by (3.3) we get,

$$u^{3l+1}_{k_{3l}} \Vdash \text{“}\varphi(\dot{h}[Z]) < m\text{”}$$

Now let $l > j_0$ be a natural number big enough so that $n < k_{3l-1}$, and let $q \leq u^{3l+1}_{k_{3l}}$ be an $(\mathcal{M}, \mathbb{P})$-generic condition. Define the following:

a) In $\mathcal{M}$, let $\tilde{s}_n$ be the $\mathbb{P}$-name $\dot{q}_{k_{3l}}$.

b) In $\mathcal{M}$, let $\underset{\sim}{\vec{t}_n}$ be a $\mathbb{P}$-name for the fine derived sequence $\tilde{\delta}(\langle r'_i : i \in \omega\rangle, \dot{f})$, where $r'_i = r_{i+k_{3l}}$, for all $i \in \omega$.

c) In $\mathcal{M}$, let $\tilde{h}_n$ be the $\mathbb{P}$-name $\dot{h}$.

d) Define $p_{\mathcal{M}} = q$.

We claim that $p_{\mathcal{M}}$, $\tilde{s}_n$, $\tilde{h}_n$ and $\underset{\sim}{\vec{t}_n}$ are as required.

Conditions 1) - 4) are easily seen to be hold by the construction. Condition i), follows from the fact that $p_{\mathcal{M}} \leq p_{k_{3l}} \leq p_0$. The first part of ii) follows from a) and $(p_{k_{3l}}, \dot{q}_{k_{3l}}) \leq (p, \dot{q})$ and the choice of $\tilde{s}_n$. The second part of condition ii) follows from the fact that $\langle (p_k, \dot{q}_k) : k \in \omega\rangle \in \mathcal{M}$, $p_{\mathcal{M}} \leq p_{k_{3l}}$ and $\tilde{s}_n = \dot{q}_{k_{3l}}$.

Let us see condition iii). By definition $\tilde{\delta}(\langle \underset{\sim}{r'_i} : i \in \omega\rangle, \dot{f})$, is a fine interpretation of $\dot{f}$. To see that $\tilde{\delta}(\langle r'_i : i \in \omega\rangle, \dot{f})$ is forced to be below $\tilde{s}_n$ by $p_{\mathcal{M}}$, let $G \subseteq \mathbb{P}$ be any generic filter such that $p_{\mathcal{M}} \in G$. By definition $r'_0 = (p_{k_{3l}}, \dot{q}_{k_{3l}})$. Since $p_{\mathcal{M}} \leq p_{k_{3l}}$ and $p_{\mathcal{M}} \in G$, it follows that the first term of $\tilde{\delta}(\langle \underset{\sim}{r'_i} : i \in \omega\rangle)$ is $\dot{q}_{k_{3l}}[G] = \tilde{s}_n[G]$, and all the remaining terms are below the first one since $\tilde{\delta}(\langle r'_i : i \in \omega\rangle, \dot{f})$ is a decreasing sequence. To see that $p_{\mathcal{M}} \Vdash$ “$\varphi(\tilde{h}_n[Z]) < m$”, recall that $l > j_0$, $p_{\mathcal{M}} \leq u^{3l+1}_{k_{3l}}$ and $u^{3l+1}_{k_{kl}} \Vdash$ “$\varphi(\dot{h}[Z]) < m$”.

To see that condition iv) holds, just note that $n < k_{3l-1}$, $(p_{\mathcal{M}}, \tilde{s}_n) \leq (p_{k_{3l}}, \dot{q}_{k_{3l}})$ and

$$(p_{k_{3l}}, \dot{q}_{k_{3l}}) \Vdash \text{“}\dot{f} \restriction k_{3l} = h_0 \restriction k_{3l}\text{”}$$

which implies

$$p_{\mathcal{M}} \Vdash \text{“}\tilde{s}_n \Vdash \text{“}\dot{f} \restriction k_{3l} = h_0 \restriction k_{3l}\text{””}$$

and also,

$$p_{k_{3l}} \Vdash \text{“}\tilde{h}_n \restriction k_{3l} = \dot{h} \restriction k_{3l} = h_0 \restriction k_{3l}\text{”}$$

so it follows that

$$p_{\mathcal{M}} \Vdash \text{“}\tilde{s}_n \Vdash \text{“}\dot{f} \restriction k_{3l} = \tilde{h}_n \restriction k_{3l}\text{””}$$

since $n < k_{3l-1}$, the required property follows.

□

Before proving Theorem 23 we introduce some notation and conventions to be used in the proof. Given a countable support iteration $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n \in \omega \rangle$ which is factorized as $\mathbb{P} = \mathbb{P}_n * \tilde{\mathbb{P}}_{[n+1,\omega)}$, and a condition $p \in \mathbb{P}$, we denote by $(p_0, \dot{p}_{[1,\omega)})$ the factorization of the condition p relative to $\mathbb{P}_n * \tilde{\mathbb{P}}_{[n+1,\omega)}$. In a similar way, if $\langle r_n : n \in \omega \rangle$ is a sequence of conditions in $\mathbb{P}$, we denote by $(r_0^k, \dot{r}_{[1,\omega)}^k)$ the factorization of the condition r_k relative to $\mathbb{P}_n * \tilde{\mathbb{P}}_{[n+1,\omega)}$. On the other hand, consider $\mathbb{P}$ factorized as $\mathbb{P}_n * \dot{\mathbb{Q}}_{n+1} * \tilde{\mathbb{P}}_{[n+2,\omega)}$ and $G_n \subseteq \mathbb{P}_n$ a generic filter over V. We will be translating $\dot{\mathbb{Q}}_{n+1}[G_n]$-names in $V[G_n]$ to $\mathbb{P}_{n+1}$-names names in V, and vice versa. In this setting, when working in $V[G_n]$, we consider $\dot{\mathbb{Q}}_{n+1}[G_n]$ as the evaluation of the $\mathbb{P}_n$-name $\dot{\mathbb{Q}}_{n+1}$, and $\tilde{\mathbb{P}}_{[n+2,\omega)}[G_n]$ as a suitable $\dot{\mathbb{Q}}_{n+1}[G_n]$-name for the residual iteration after $\mathbb{P}_{n+1} = \mathbb{P}_n * \dot{\mathbb{Q}}_{n+1}$, or more exactly, for the quotient $\mathbb{P}_\omega / G_{n+1}$, where $G_{n+1} \subseteq \mathbb{P}_{n+1}$ is a generic filter over V. So, when working in $V[G_n]$, we will write $\dot{\mathbb{Q}}_{n+1}[G_n] * \tilde{\mathbb{P}}_{n+1}[G_n]$ for the quotient forcing $\mathbb{P}_\omega / G_n$ in $V[G_n]$. The point of this is the two step factorization of $\mathbb{P}_\omega / G_n$, so we can apply the previous lemma in $V[G_n]$, and then make a translation of elements from $V[G_n]$ to $\mathbb{P}_n$-names in V. Finally, we will be making an abuse of notation when referring to the $\mathbb{P}_\omega$-name $\dot{f}$: when talking about $\dot{f}$ in the extensions given by $\mathbb{P}_n$, we implicitly assume that we are taking about an appropiated $\mathbb{P}_{[n+1,\omega)}$-name which is equivalent to $\dot{f}$ after forcing with $\mathbb{P}_{[n+1,\omega)}$ on the generic extensión by $\mathbb{P}_n$.

Proof of Theorem 23. Let us assume all the hypothesis in the statement of Theorem 23. First note that we can assume that the iteration has length ω, since every real appears in a successor stage or in a countable cofinality stage, so we assume $\mathbb{P}_\omega = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_n : n \in \omega \rangle$. Let $\mathcal{U}$ be an $(\mathscr{I}, p)$-point, $\dot{f}$ a $\mathbb{P}_\omega$-name for a function from ω to ω and $p \in \mathbb{P}_\omega$ a condition. First note that by Theorem 14, we have that $\mathcal{U}$ generates a p-point after forcing with $\mathbb{P}_\omega$, so we can assume that $p \Vdash$ “$\mathcal{U}$ generates a p-point”. If there is a condition $q \leq p$ and $X \in \mathcal{U}$ such that $q \Vdash$ “$\dot{f} \restriction X$ *is constant*”, then we are done. Otherwise, there is $q \leq p$ and there exists $A \in \mathcal{U}$ such that $\dot{f} \restriction A$ is forced to be finite to one by q, so we can assume that $\dot{f}$ is finite to one. Let $\mathcal{M} \prec H(\theta)$ be a countable elementary submodel such that $\dot{f}, \mathcal{U}, p, \mathbb{P}_\omega \in \mathcal{M}$. Fix $Z \in \mathcal{U}$ a pseudointersection of $\mathcal{U} \cap \mathcal{M}$. In $\mathcal{M}$, let $(\langle r_n : n \in \omega \rangle, h_0)$ be a fine interpretation for $\dot{f}$ below p, and assume $\varphi(h_0[Z]) < m$ for some fixed $m \in \omega$. For each $n \in \omega$, we assume r_n has the form $\langle r_n^k : k \in \omega \rangle$. We will construct by recursion four sequences $\langle q_n : n \in \omega \rangle$, $\langle \dot{s}_n : n \in \omega \rangle$, $\langle \dot{h}_n : n \in \omega \rangle$ and $\langle \vec{t}_n : n \in \omega \rangle$ such that the following holds:

a) For $n = 0$, q_0 is a $(\mathcal{M}, \mathbb{P}_0)$-generic condition.

b) For all $n \in \omega$, q_n is $(\mathcal{M}, \mathbb{P}_n)$-generic and $q_n = (q_0, \ldots, \dot{q}_n)$, where $\dot{q}_n$ is a $\mathbb{P}_{n-1}$-name for $n > 0$.

c) For all positive $n \in \omega$, $\dot{h}_n$ is a $\mathbb{P}_{n-1}$-name and $q_{n-1} \Vdash$ "$\dot{h}_n \in \mathcal{M}[\dot{G}_{\mathbb{P}_{n-1}}]$".

d) For all positive $n \in \omega$, $\vec{\underset{\sim}{t}}_n$ is a $\mathbb{P}_{n-1}$-name for a decreasing sequence of conditions in $\tilde{\mathbb{P}}_{[n,\omega)}[\dot{G}_{\mathbb{P}_{n-1}}]$.

e) For all positive $n \in \omega$, $\dot{s}_n$ is a $\mathbb{P}_{n-1}$-name and q_{n-1} forces that $\dot{s}_n$ is a condition in $\tilde{\mathbb{P}}_{[n,\omega)}[\dot{G}_{\mathbb{P}_{n-1}}] \cap \mathcal{M}[\dot{G}_{\mathbb{P}_{n-1}}]$.

f) For all positive $n \in \omega$, q_{n-1} forces that $(\vec{\underset{\sim}{t}}_n, \dot{h}_n) \in \mathcal{M}[\dot{G}_{\mathbb{P}_{n-1}}]$ is a fine interpretation of $\dot{f}$ below $\dot{s}_n$.

g) For all positive $n \in \omega$, q_{n-1} forces that $\varphi(\dot{h}_n[Z]) < m$.

h) For all positive $n \in \omega$, $q_{n-1} \Vdash$ "$\dot{s}_n \Vdash$ "$\dot{f} \restriction n = \dot{h}_n \restriction n$"".

i) For all positive $n \in \omega$, $q_{n-1} \Vdash$ "$\dot{q}_n \leq \dot{s}_0^n$" and $q_0 \leq r_0^0$.

j) For all $n \in \omega$, $q_n \Vdash$ "$\dot{s}_{n+1} \leq \dot{s}^n_{[1,\omega)}$".

Suppose first we have succeeded in the above construction and consider the condition $q = \bigcup_{n\in\omega} q_n$. Note that by construction of q we have $q \leq q_0$. We claim that $q \Vdash$ "$\varphi(\dot{f}[Z]) \leq m$". It is enough to prove $q \Vdash$ "$(\forall n \in \omega)(\dot{f} \restriction n = \dot{h}_n \restriction n)$", since by g), this would imply that q forces the measure of the initial segments of $\dot{f}[Z]$ to be smaller than m, and by the lower semicontinuity of φ, this implies that $\varphi(\dot{f}[Z])$ is at most m. First note that for any positive $n \in \omega$, it holds that $q_n^\frown \dot{s}_{n+1} \leq q_{n-1}^\frown \dot{s}_n$. To see this fix a positive $n \in \omega$. Clause j is equivalent to write $q_{n-1}^\frown \dot{q}_n \Vdash$ "$\dot{s}_{n+1} \leq \dot{s}^n_{[1,\omega)}$". This, together with $q_{n-1} \Vdash$ "$\dot{q}_n \leq \dot{s}_0^n$"(see clause i)) and the factorization $\dot{s}_n = (\dot{s}_0^n, \dot{s}^n_{[1,\omega)})$ gives the conclusion that $q_n^\frown \dot{s}_{n+1} \leq q_{n-1}^\frown \dot{s}_n$. Then, note that this implies that for all $k \geq n$, it holds that $q_k^\frown \dot{s}_{k+1} \leq q_{n-1}^\frown \dot{s}_n$. This implies that for all $k \in \omega$, $q \restriction k = q_k \leq q_{n-1}^\frown \dot{s}_n \restriction k$, which means that $q \leq q_{n-1}^\frown \dot{s}_n$. Since clause h) is equivalent to $q_{n-1}^\frown \dot{s}_n \Vdash$ "$\dot{f} \restriction n = \dot{h}_n \restriction n$", we conclude that $q \Vdash$ "$\dot{f} \restriction n = \dot{h}_n \restriction n$". Finally, since n was an arbitrary positive natural number, we conclude that $q \Vdash$ "$(\forall n \in \omega)(\dot{f} \restriction n = \dot{h}_n \restriction n)$".

Let us construct the sequences. We start defining $\dot{s}_0 = p_0$, $\dot{h}_0 = h_0$ and $\vec{\underset{\sim}{t}}_0 = \langle r_n : n \in \omega\rangle$. Consider the factorization $\mathbb{P} = \mathbb{P}_0 * \tilde{\mathbb{P}}_{[1,\omega)}$. Regarding this factorization, any condition r in $\mathbb{P}_\omega$ is factored as $r = (r_0, \dot{r}_{[1,\omega)})$. For the sequence $\langle r_n : n \in \omega\rangle$ we will write $r_n = (r_0^n, \dot{r}^n_{[1,\omega)})$. Then note that all conditions from Lemma 11 are satisfied for $\mathscr{I}$, φ, $\mathbb{P}_0 * \tilde{\mathbb{P}}_{[1,\omega)}$, $\mathcal{U}$, $\dot{f}$, $p = (p_0, \dot{p}_{[1,\omega)})$, $(\langle r_n : n \in \omega\rangle, h_0)$, Z, m and $\mathcal{M}$. So for $n = 1$, by an application of Lemma 11 we get $q_{\mathcal{M}}$, $\tilde{s}_1$, $\tilde{h}_1$, $\vec{\underset{\sim}{t}}_1$ that satisfy conditions 1)-4) from Lemma 11, and such that conditions $i) - iv)$ are adapted as follows:

1) $q_{\mathcal{M}} \leq r_0^0$ is a $(\mathcal{M}, \mathbb{P}_0)$-generic condition.

2) $q_{\mathcal{M}} \Vdash$ "$\tilde{s}_1 \leq \dot{p}_{[1,\omega)}$" and $q_{\mathcal{M}} \Vdash$ "$\tilde{s}_1 \in \tilde{\mathbb{P}}_{[1,\omega)}[\dot{G}_{\mathbb{P}_0}] \cap \mathcal{M}[\dot{G}_{\mathbb{P}_0}]$".

3) $q_{\mathcal{M}}$ forces that $(\underset{\sim}{\vec{t}}_1, \tilde{h}_1) \in \mathcal{M}[\dot{G}_{\mathbb{P}_0}]$ is a fine interpretation of $\dot{f}$ below $\tilde{s}_1$, as well as $\varphi(\tilde{h}_1[Z]) < m$.

4) $q_{\mathcal{M}} \Vdash$ "$\tilde{s}_1 \Vdash$ "$\dot{f} \restriction 1 = \tilde{h}_1 \restriction 1$"".

Now define $\dot{q}_0 = q_{\mathcal{M}}$, $\dot{s}_1 = \tilde{s}_1$, $\dot{h}_1 = \tilde{h}_1$, and $\underset{\sim}{\vec{t}_1}$ just as was obtained. Let us see that conditions a) to j) are satisfied. Condition a) and b) follows trivially from 1) above. Condition c) follows from 3) above. Condition d) follows from 3) above. Condition e) follows from 2) above. Condition f) and g) follow from 3) above. Condition h) is condition 4) above. Condition i) follows from 1) and 2). Condition j) follows from 2).

Now assume that we have constructed q_{n-1}, $\dot{s}_n$, $\dot{h}_n$ and $\underset{\sim}{\vec{t}_n}$ with the required properties. Consider the factorization $\mathbb{P}_\omega = \mathbb{P}_{n-1} * \dot{\mathbb{Q}}_n * \tilde{\mathbb{P}}_{[n+1,\omega)}$. Let $G_{n-1} \subseteq \mathbb{P}_{n-1}$ be a generic filter over V such that $q_{n-1} \in G_{n-1}$. Then let $s_n = \dot{s}_n[G_{n-1}]$, $h_n = \dot{h}_n[G_{n-1}]$ and $\vec{t}_n$ be the evaluations of $\dot{s}_n$, $\dot{h}_n$ and $\underset{\sim}{\vec{t}_n}$, respectively, by the filter G_{n-1}. We work in $\mathcal{M}[G_{n-1}]$. Then we have that $\dot{\mathbb{Q}}_n[G_{n-1}] * \tilde{\mathbb{P}}_{[n+1,\omega)}[G_{n-1}] \in \mathcal{M}[G_{n-1}]$. Moreover,

$$\mathcal{M}[G_{n-1}] \vDash (\vec{t}_n, h_n) \text{ is a fine interpretation of } \dot{f} \text{ below } s_n$$

and

$$V[G_{n-1}] \vDash \varphi(h_n[Z]) < m$$

Summarizing, we have the following:

1. $s_n = (s_0^n, \dot{s}_{[1,\omega)}^n), \dot{f}, \mathscr{I}, \varphi, \mathcal{U}, \dot{\mathbb{Q}}_n[G_{n-1}] * \tilde{\mathbb{P}}_{[n+1,\omega)}[G_{n-1}] \in \mathcal{M}[G_{n-1}]$.

2. $(\vec{t}_n, h_n) \in \mathcal{M}[G_{n-1}]$ is a fine interpretation of $\dot{f}$ below s_n.

3. Z is a pseudointersection of $\mathcal{M}[G_{n-1}] \cap \mathcal{U}$ and $\varphi(h_n[Z]) < m$

Then we are in conditions to apply Lemma 11 for $n+1$ and s_n taking the place of the condition p in the lemma. We get the following:

1) A $(\mathcal{M}[G_{n-1}], \dot{\mathbb{Q}}_n[G_{n-1}])$-generic condition $q_{\mathcal{M}[G_{n-1}]}$.

2) A $\dot{\mathbb{Q}}_n[G_{n-1}]$-name $\tilde{s}_{n+1}$ for a condition in $\tilde{\mathbb{P}}_{[n+1,\omega)}[G_{n-1}][\dot{G}_{\dot{\mathbb{Q}}_n[G_{n-1}]}] \cap \mathcal{M}[G_{n-1}][\dot{G}_{\dot{\mathbb{Q}}_n[G_{n-1}]}]$.

3) A $\dot{\mathbb{Q}}_n[G_{n-1}]$-name $\tilde{h}_{n+1}$ for a finite to one function from ω to ω.

4) A $\dot{\mathbb{Q}}_n[G_{n-1}]$-name $\underset{\circ}{\vec{t}}_{n+1}$ for a sequence of conditions in $\tilde{\mathbb{P}}_{[n+1,\omega)}[G_{n-1}][\dot{G}_{\dot{\mathbb{Q}}_n[G_{n-1}]}]$.

And such that the following hold:

i) $q_{\mathcal{M}[G_{n-1}]} \leq s_n^0$ is a $(\mathcal{M}[G_{n-1}], \dot{\mathbb{Q}}_n[G_{n-1}])$-generic condition.

ii) $q_{\mathcal{M}[G_{n-1}]} \Vdash$ "$\tilde{s}_{n+1} \leq \dot{s}_{[1,\omega)}^n$" and $q_{\mathcal{M}} \Vdash$ "$\tilde{s}_{n+1} \in \tilde{\mathbb{P}}_{[n+1,\omega)}[G_{n-1}][\dot{G}_{\mathbb{Q}_n[G_{n-1}]}] \cap \mathcal{M}[\dot{G}_{n-1}][\dot{G}_{\mathbb{Q}_n[G_{n-1}]}]$".

iii) $q_{\mathcal{M}[G_{n-1}]}$ forces that $(\underset{\circ}{\vec{t}}_{n+1}, \tilde{h}_{n+1}) \in \mathcal{M}[\dot{G}_{n-1}][\dot{G}_{\dot{\mathbb{Q}}_n[G_{n-1}]}]$ is a fine interpretation of $\dot{f}$ below $\tilde{s}_{n+1}$, as well as $\varphi(\tilde{h}_{n+1}[Z]) < m$.

iv) $q_{\mathcal{M}[G_{n-1}]} \Vdash$ "$\tilde{s}_{n+1} \Vdash$ "$\dot{f} \restriction (n+1) = \tilde{h}_{n+1} \restriction (n+1)$"".

All of these happen regarding $V[G_{n-1}]$ and $\mathcal{M}[G_{n-1}]$, and the construction was achieved by only assuming that $q_{n-1} \in G_{n-1}$, and so everything above is forced by q_{n-1}. Now, going back to V, we define the corresponding $\mathbb{P}_n$-names:

α) $\dot{q}_n$ as a $\mathbb{P}_{n-1}$-name for $\dot{q}_{\mathcal{M}[G_{n-1}]}$.

β) $\dot{s}_{n+1}$ as a $\mathbb{P}_{n-1} * \dot{\mathbb{Q}}_n$-name for $\tilde{s}_{n+1}$.

γ) $\dot{h}_{n+1}$ as a $\mathbb{P}_{n-1} * \dot{\mathbb{Q}}_n$-name for $\tilde{h}_{n+1}$.

δ) $\underset{\sim}{\vec{t}}_{n+1}$ as a $\mathbb{P}_{n-1} * \dot{\mathbb{Q}}_n$-name for $\underset{\circ}{\vec{t}}_{n+1}$.

Finally, define $q_n = q_{n-1} {}^\frown \dot{q}_n$. Checking that these names are as required is routine as for case $n = 1$. This finishes the construction of the sequences, and therefore, the proof of Theorem 23.

□

3.5 Answer to Question 5.

Now we are in condition to answer Question 5 stated at the begining of the chaper. We only state a remark that help us to prove a slightly more general result.

Lemma 12. *Let $\mathscr{I}$ an analytic p-ideal. There is a summable ideal $\mathscr{J}$ such that $\mathscr{J} \subseteq \mathscr{I}$.*

Proof. Let φ be the a submeasure for $\mathscr{I}$. Define $Z = \{n \in \omega : \varphi(\{n\}) = 0\}$. Now, for $n \notin Z$, define $g(n) = \varphi(\{n\})$, and for $n \in Z$, define $g(n) = 1/2^n$. The lower semicontinuity of φ implies that $\mathscr{I}_g \subseteq \mathscr{I}$. □

It is clear that if $\mathscr{J} \subseteq \mathscr{I}$ are ideals, and $\mathcal{U}$ is a $\mathscr{J}$-ultrafilter, then $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter as well.

Theorem 24. *It is consistent that there is no rapid ultrafilter but given any family $\mathcal{D}$ of analytic p-ideals such that $|\mathcal{D}| < \mathfrak{d}$, there is an ultrafilter $\mathcal{U}$ which is an $\mathscr{I}$-ultrafilter for all $\mathscr{I} \in \mathcal{D}$.*

Proof. First note that for all summable ideal $\mathscr{I}_g$, it holds that $\mathscr{I} = Fin(g) = Exh(g)$. The forcing $\mathbb{P}_{\omega_2}$ is an ω_2-length countable support iteration of the Rational Perfect set forcing, over a model of $\mathsf{ZFC} + \mathsf{CH}$. Theorem 22 states that Rational Perfect forcing preserves $(\mathscr{I}, p)$-points whenever $\mathscr{I}$ in an analytic p-ideal, and Theorem 23 makes sure that these $(\mathscr{I}, p)$-ultrafilters are preserved along the iteration. Recall that in the Rational Perfect model the dominating number is equals to ω_2. So let $\mathcal{D}$ be a family of cardinality at most ω_1 of tall analytic p-ideals. Since every real appears before a stage of cofinality ω, it can be assumed that each ideal in $\mathcal{D}$ belongs to the ground model. By Lemma 12, for every $\mathscr{I} \in \mathcal{D}$

there is a summable ideal $\mathscr{J}(\mathscr{I})$ such that $\mathscr{J}(\mathscr{I}) \subseteq \mathscr{I}$. Define $\mathcal{D}' = \{\mathscr{J}(\mathscr{I}) : \mathscr{I} \in \mathcal{D}\}$. Since this ideals are from the ground model, there is an ultrafilter $\mathcal{U}$ that is an $(\mathscr{I}, p)$-point for every ideal $\mathscr{I} \in \mathcal{D}'$, for example, let $\mathcal{U}$ be a Ramsey ultrafilter. This ultrafilter remains as an $(\mathscr{I}, p)$-point for every $\mathscr{I} \in \mathcal{D}$ in the forcing extension by $\mathbb{P}_{\omega_2}$. □

As a consequence, we answer Question 2 from Chapter 2:

Corollary 3. *It is relatively consistent with* ZFC *that for every analytic p-ideal $\mathscr{I}$, $\mathfrak{r}_\sigma = \mathfrak{z}_{fin}(\mathscr{I}) < \mathfrak{d}$.*

3.6 Final remarks

Let us recall that T. Bartoszyński and S. Shelah have proved that in the Rational Perfect set model, the Hausdorff ultrafilters are dense in the Rudin-Blass ordering (see (5)).

Theorem 25 (T. Bartoszyński and S. Shelah, see (5))**.** *In the Rational Perfect set model, Hausdorff ultrafilters are dense in the Rudin-Blass ordering.*

Theorem 24 implies the same fact for a broader class of ideals. First let us recall the Near Coherence of Filters principle:

Definition 21 (Near Coherence of Filters principle(NCF), A. Blass, see (8))**.** *Let $\mathcal{U}$ and $\mathcal{V}$ be two ultrafilters on ω. Then there is a finite to one function $f : \omega \to \omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$.*

Theorem 26 (A. Blass, S. Shelah, see (11))**.** *The Near Coherence of Filters holds in the Rational Perfect set model.*

Let $\mathscr{I}$ be any analytic p-ideal, and apply Theorem 24 with the family $\mathcal{D} = \{\mathscr{I}\}$ to obtain an $\mathscr{I}$-ultrafilter $\mathcal{U}$. Now let $\mathcal{V}$ be an arbitrary ultrafilter on ω. By Theorem 26, there is a finite to one function $f \in \omega^\omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$. Since $f(\mathcal{U})$ is a $\mathscr{I}$-ultrafilter and $f(\mathcal{U}) = f(\mathcal{V}) \leq_{RB} \mathcal{V}$, we are done. We have proved the following:

Corollary 4. *It is relatively consistent with* ZFC *that for any analytic p-ideal, $\mathscr{I}$-ultrafilters are dense in the Rudin-Blass ordering. Actually, this holds in the Rational Perfect set model.*

Also, recall that C. Laflamme and J. Zhu, in their article *The Rudin Blass ordering of ultrafilters* from 1998 (see (48)), have proved in ZFC that there is an ultrafilter $\mathcal{U}$ with no Rudin-Blass predecessors which are rapid. In particular, no predecessor of $\mathcal{U}$ is a q-point, equivalently, a weak $\mathcal{ED}_{fin}$-ultrafilter. Corollary 4 implies that it is not possible to prove the same for $\mathscr{I}$-ultrafilters whenever $\mathscr{I}$ is an analytic p-ideal.

Recall that given two ideals $\mathscr{I}$ and $\mathscr{J}$ on ω, we say that $\mathscr{I}$ is Katětov below $\mathscr{J}$, denoted by $\mathscr{I} \leq_K \mathscr{J}$, if there is a function $f : \omega \to \omega$ such that for all $A \in \mathscr{I}$, $f^{-1}[A] \in \mathscr{J}$. If the function f is required to be finite to one, then we say that $\mathscr{I}$ is Katětov-Blass below $\mathscr{J}$, and it is denoted by $\mathscr{I} \leq_{KB} \mathscr{J}$. For a given ideal $\mathscr{I}$ and an ultrafilter $\mathcal{U}$, it is easily seen that $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter if and only if $\mathscr{I} \not\leq_K \mathcal{U}^*$. Similarly, $\mathcal{U}$ is a weak $\mathscr{I}$-ultrafilter if and only if $\mathscr{I} \not\leq_{KB} \mathcal{U}^*$. Using this terminology, Laflamme and Zhu's theorem can be restated as follows:

Theorem 27 (C. Laflamme, J. Zhu, see (48)). *There exists an ultrafilter $\mathcal{U}$ such that for all finite to one $f \in \omega^\omega$, $\mathcal{ED}_{fin} \leq_{KB} f(\mathcal{U})^*$.*

Thus, it follows trivially that for any Borel ideal $\mathscr{I} \leq_{KB} \mathcal{ED}_{fin}$, there is an ultrafilter for which any Rudin-Blass predecessor $\mathcal{V} \leq_{RB} \mathcal{U}$, its dual ideal is Katětov-Blass above $\mathscr{I}$, that is $\mathscr{I} \leq_{KB} \mathcal{V}^*$. However we do not know of a Borel ideal $\mathscr{I} \not\leq_{KB} \mathcal{ED}_{fin}$ having this property.

Definition 22. *Let $\mathscr{I}$ be a tall ideal on ω. We say that $\mathscr{I}$ is Laflamme-Zhu if there is an ultrafilter $\mathcal{U}$ all of whose Rudin-Blass predecessors are such that its dual ideal is Katětov-Blass above of $\mathscr{I}$. We say that $\mathscr{I}$ is trivially Laflamme-Zhu if $\mathscr{I} <_{KB} \mathcal{ED}_{fin}$.*

The previous remarks lead us to ask the following:

Question 6. *Does there exist a Borel or analytic tall ideal other than $\mathcal{ED}_{fin}$ which is non-trivially Laflamme-Zhu?*

Question 7. *Is there a critical ideal $\mathscr{I}$ for the Borel (analytic) Laflamme-Zhu ideals, i. e., such that any ideal $\mathscr{J}$ is Laflamme-Zhu if and only if $\mathscr{J} \leq_{KB} \mathscr{I}$?*

Question 8. *In case there is an analytic ideal $\mathscr{I} \not\leq_{KB} \mathcal{ED}_{fin}$ which is Laflamme-Zhu, let $\mathbf{LZ}_{Borel}$ and $\mathbf{LZ}_{Analytic}$ be the families of all Borel and analytic ideals which are Laflamme-Zhu, respectively. What is the structure of the Katětov-Blass order restricted to such classes?*

In the case of an affirmative answer to **Question 24**, such ideal can not be Katětov-Blass above the ideal conv generated by convergent sequences of rationals in $\mathbb{Q} \cap [0,1]$, since p-points are characterized as the conv-ultrafilters. Also, by Corollary 4, such an ideal can not be an analytic p-ideal. In particular it can not be a summable ideal. In (36) it has been proved that Hausdorff ultrafilters are exactly the $\mathcal{G}_{fc}$-ultrafilters, which together with Theorem 25 imply that any Lafflame-Zhu ideal can not be Katětov-Blass above the ideal $\mathcal{G}_{fc}$, the ideal on $[\omega]^2$ of graphs with finite chromatic number.

Chapter 4

Every maximal ideal may be Katětov above of all F_σ ideals

4.1 Introduction.

In the introduction it was mentioned how useful the Katětov order is when classifying the combinatorial properties of ultrafilters. In this chapter we prove that the combinatorics of ultrafilters can be so complicated that the class of F_σ ideals is not enough to provide combinatorial information of ultrafilters.

The classical results about the non-existence of certain kind of ultrafilters, such as Ramsey ultrafilters (46), p-points (67), q-points (51), nwd-ultrafilters (60) and rapid ultrafilters (51), all of these equivalent to saying that all ultrafilters are above of some(or many) critical ideal, give a measure of the strength of the Katětov order to classify ultrafilters.

However, the existence of $\mathscr{I}$-ultrafilters for some Borel ideal $\mathscr{I}$ was open (39). Recently, O. Guzmán González and M. Hrušák proved in ZFC the existence of an $F_{\sigma\delta\sigma}$ ideal for which the generic existence of $\mathscr{I}$-ultrafilters holds (see (32)). They also proved that the complexity can not be lowered, since it is consistent that for all $F_{\sigma\delta}$ ideals generic existence does not hold, and raised the question about the existence of an F_σ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist.

We will give an answer to the previous question by showing that every maximal ideal may be Katětov above of all F_σ ideals (i.e., there is no $\mathscr{I}$-ultrafilters for any F_σ-ideal):

Main Theorem. It is relatively consistent with ZFC that there is no $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$. Moreover, it is relatively consistent with ZFC that there is no weak $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$.

The main theorem also implies a solution to a widely known problem. Recall that Hausdorff ultrafilters were defined in the introduction. An ultrafilter is Hausdorff if the ultrapower of the natural numbers modulo the ultrafilter, when considering the topology induced by the discrete topology on ω, is a Hausdorff space. These ultrafilters have been extensively studied, for example in (24; 25; 56; 26; 31; 5). It is not hard to see that an ultrafilter $\mathcal{U}$ is Hausdorff if and only if for any pair of functions $f, g \in \omega^\omega$, there is $A \in \mathcal{U}$

such that either $f \restriction A = g \restriction A$ or $f[A] \cap g[A] = \emptyset$. Many results about their existence have been proved under additional assumptions to ZFC(see for example (25; 56; 26)). T. Bartoszyński and S. Shelah have proved (5) that in the Rational Perfect set model they are dense in the Rudin-Blass ordering. In recent work, these ultrafilters have been considered in (38) in the construction of a countably compact group without non-trivial convergent sequences. But the question about their existence remained elusive for many years. Results of D. Meza Alcántara and M. Hrušák (36) imply that in the model we construct there is no Hausdorff ultrafilter, giving an answer to the problem.

Additionally, the model we construct gives another solution to a question raised by M. Benedikt, giving another model where there is no ultrafiter with property M.

4.2 More preliminaries.

Recall that a family $\mathcal{G}$ is groupwise dense if it is open and dense in the ordering $([\omega]^\omega, \subseteq^*)$, and for every partition of ω into finite sets $\langle I_n : n \in \omega \rangle$, there is an infinite $A \subseteq \omega$ such that $\bigcup_{n\in A} I_n \in \mathcal{G}$. The groupwise density number, $\mathfrak{g}$, is defined as the minimum cardinality of a collection of groupwise dense families with empty intersection:

$$\mathfrak{g} = \min\{|\Gamma| : (\forall \mathcal{G} \in \Gamma)(\mathcal{G} \text{ is groupwise dense}) \ \& \ \bigcap \Gamma = \emptyset\}$$

It has been proved that the inequality $\mathfrak{u} < \mathfrak{g}$ implies that for any two nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$, there is a finite to one function $f \in \omega^\omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$. This last assertion is known as the Near Coherence of Filters:

Definition 23 (Near Coherence of Filters, NCF)**.** *For any two nonprincipal ultrafilters on ω, $\mathcal{U}$ and $\mathcal{V}$, there is a function $f \in \omega^\omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$*

Reformulating the remark before the definition, we have the following:

Theorem 28 (A. Blass, see (9))**.** *The inequality $\mathfrak{u} < \mathfrak{g}$ implies the NCF principle.*

Proof. We refer the reader to (9), from Definition 9.14 to Remark 9.19. □

This highly counterintuitive statement has been proved to be relatively consistent with ZFC, and in fact, it holds in the model obtained by a countable support iteration of the Rational Perfect set forcing (11) or the Blass-Shelah forcing (10). These two models are in fact models of the stronger inequality $\mathfrak{u} < \mathfrak{g}$. We will see in Section 5 that the model we present here is in fact a model for the inequality $\mathfrak{u} < \mathfrak{g}$. The NCF principle has many interesting consequences, among them we mention the following:

Theorem 29 (see (10; 11; 9))**.** *The Near Coherence of Filters principle implies the following:*

1. *The Rudin-Keisler (in fact, Rudin-Blass) order of ultrafilters is downward directed.*

2. $\mathfrak{u} < \mathfrak{d}$.[1]

[1]The dominating number, denoted by $\mathfrak{d}$, is defined as the minimum cardinality of a $\leq^*$-dominating family on ω^ω, where $g \leq^* f$ if there is $n \in \omega$ such that for all $k \geq n$, $g(k) \leq f(k)$.

3. *For any ultrafilter $\mathcal{U}$, there is a finite to one function $f \in \omega^\omega$ such that $f(\mathcal{U})$ is p-point (p-points are dense in the Rudin-Blass order).*

4. *There are no q-points.*

For more details the reader may consult (10; 11; 9).

A function $\varphi : \mathcal{P}(\omega) \to \mathbb{R}$ is a lower semicontinuous submeasure, *lscsm* for short, if φ satisfies the following:

1. For all $A \in \mathcal{P}(\omega)$, $\varphi(A) \geq 0$.

2. For all $A, B \in \mathcal{P}(\omega)$, $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

3. For all $A \in \mathcal{P}(\omega)$, $\lim_{n\to\infty} \varphi(A \cap n) = \varphi(A)$.

4. For all $A, B \in \mathcal{P}(\omega)$, if $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$.

It is a well known theorem of Mazur (49), that every F_σ ideal is the family of all subsets of ω with finite φ-measure, where φ is a suitable lower semicontinuous submeasure.

Returning to the Katětov order, as we mentioned in the introduction many combinatorial properties of ultrafilters can be stated as not being Katětov above of some ideal $\mathscr{I}$, for a suitable ideal $\mathscr{I}$ (equivalently, as being an $\mathscr{I}$-ultrafilter).

As mentioned earlier in the introduction, Hausdorff ultrafilters are those for which the discrete topology on ω induces a Hausdorff topology on the ultraproduct of ω modulo the ultrafilter. These ultrafilters can be characterized as the $\mathcal{G}_{fc}$-ultrafilters, where $\mathcal{G}_{fc}$ is the ideal on $[\omega]^2$ of graphs with finite chromatic number, which is defined as follows:

$$\mathcal{G}_{fc} = \{A \subseteq [\omega]^2 : \chi(A) < \infty\}$$

where $\chi(A)$ is the chromatic number of the graph (ω, A). It turns out the function *ch* is a lscsm, so the ideal $\mathcal{G}_{fc}$ is an F_σ ideal. The reader may consult (36; 50) for details.

We need the following proposition:

Proposition 11 (see (40))**.** *Let $\mathcal{U}$ be an ultrafilter on ω. The following are equivalent:*

1. *$\mathcal{U}$ is a Hausdorff ultrafilter.*

2. *For any two functions $f, g \in \omega^\omega$, there is $A \in \mathcal{U}$ such that either $f \restriction A = g \restriction A$ or $f[A] \cap g[A] = \emptyset$.*

3. *$\mathcal{G}_{fc} \not\leq_K \mathcal{U}^*$.*

Proof.

(1)$\Longrightarrow$(2) Assume $\mathcal{U}$ is a Hausdorff ultrafilter, and let $f, g \in \omega^\omega$ be two functions. Consider $[f]_\mathcal{U}$ and $[g]_\mathcal{U}$. If $[f]_\mathcal{U} = [g]_\mathcal{U}$, then there is $A \in \mathcal{U}$ such that $f \restriction A = g \restriction A$. If $[f]_\mathcal{U} \neq [g]_\mathcal{U}$, since $\omega^\omega/\mathcal{U}$ is Hausdorff, there are disjoint sets $A, B \in [\omega]^\omega$ such that $A^* \cap B^* = \emptyset$, $[f]_\mathcal{U} \in A^*$ and $[g]_\mathcal{U} \in B^*$. This means that there are $X, Y \in \mathcal{U}$ such that $f[X] \subseteq A$ and $g[Y] \subseteq B$. Define $Z = X \cap Y \in \mathcal{U}$. Then $f[Z] \subseteq A$ and $g[Z] \subseteq B$, which implies $f[Z] \cap g[Z] = \emptyset$.

(2)$\Longrightarrow$(3) Let $f : \omega \to [\omega]^2$ be a function. Define $g_1(n) = \min(f(n))$ and $g_2 = \max(f(n))$. Then apply hypothesis (2), and note that the first option is not possible, so there is $A \in \mathcal{U}$ such that $g_1[A] \cap g_2[A] = \emptyset$. This implies that $f[A] \subseteq \mathcal{B}_{g_1[A],g_2[A]}$, where $\mathcal{B}_{g_1[A],g_2[A]}$ is the complete bipartite graph defined by $g_1[A]$ and $g_2[A]$, which has chromatic number 2, so $\mathcal{B}_{g_1[A],g_2[A]} \in \mathcal{G}_{fc}$. Note that this implies that $A \subseteq f^{-1}[\mathcal{B}_{g_1[A],g_2[A]}] \notin \mathcal{U}^*$.

(3)$\Longrightarrow$(1) Let $f, g \in \omega^\omega$ be such that $[f]_\mathcal{U} \neq [g]_\mathcal{U}$. Then there is $A \in \mathcal{U}$ such that for all $n \in A$, $f(n) \neq g(n)$. Fix one of such sets $A \in \mathcal{U}$ and define $h : \omega \to [\omega]^2$ as $h(n) = \{f(n), g(n)\}$ if $n \in A$, and $h(n) = \{0, 1\}$ if $n \notin A$. Since $\mathcal{G}_{fc} \not\leq_K \mathcal{U}^*$, there is $X \in \mathcal{G}_{fc}$ such that $h^{-1}[X] \notin \mathcal{U}^*$, that is, $h^{-1}[X] \in \mathcal{U}$. Note that we can assume that X is a bipartite graph, since every graph with finite chromatic number can be covered by a finite number of bipartite graphs. Define $Z = h^{-1}[X] \cap A \in \mathcal{U}$. Since we assumed X to be bipartite, there are disjoint sets $B, C \in [\omega]^\omega$ such that $X \subseteq \mathcal{B}_{B,C}$, so in particular $h[Z] \subseteq \mathcal{B}_{B,C}$. This implies that $f[Z] \subseteq B$ and $g[Z] \subseteq C$. Coming back to the ultrapower, this means that $[f]_\mathcal{U} \in B^*$ and $[g]_\mathcal{U} \in C^*$, and $B^* \cap C^* = \emptyset$. □

Another ideal which is important for us is the Solecki ideal $\mathcal{S}$. This ideal was defined by S. Solecki in (62) when dealing with Fatou's property, and is defined on the set Ω of clopen subsets of 2^ω with Lebesgue measure 1/2. It is generated by the family of sets $\{A \subseteq \Omega : \bigcap A \neq \emptyset\}$. It was proved by S. Solecki in (62) that this ideal is in fact an F_σ ideal. In (36) it was proved that $\mathcal{S}$ is critical with respect to the Fubini property, which is an equivalent formulation of the property M when talking about ultrafilters. The following proposition summarizes all we need to know for our purposes, details can be found in (36):

Proposition 12. *Let $\mathcal{U}$ be an ultrafilter on ω. The following are equivalent:*

1. $\mathcal{S} \not\leq_K \mathcal{U}^*$.
2. *(Property M) For any $\epsilon > 0$ and any sequence $\langle A_n : n \in \omega\rangle$ of Borel subsets of 2^ω, if $\mu(A_n) > \epsilon$ for all $n \in \omega$, then there is $B \in \mathcal{U}$ such that $\bigcap_{n\in B} A_n \neq \emptyset$.*

The forcing notions we are working with are variations of the Rational Perfect set forcing, which is adapted to destroy the *$\mathscr{I}$-ultrafilterness* of ultrafilters from the ground model whenever $\mathscr{I}$ is an F_σ ideal, while keeping the main combinatorial properties of the standard Rational Perfect forcing, such as preserving p-points and destroying groupwise dense families from the ground model. In order to achieve this, we make use of the lower semicontinuous submeasure associated to a given F_σ ideal.

4.3 Results.

The main theorem is stated as follows:

Theorem 30 (Main theorem)**.** *It is relatively consistent with* ZFC *that for no F_σ ideal $\mathscr{I}$ there are $\mathscr{I}$-ultrafilters. Actually, it is consistent with* ZFC *that there is no weak $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$.*

Now we list the questions we are answering. All questions are answered in the negative, that is, their negation is relatively consistent with ZFC.

The first one that is answered directly by Main Theorem is the following one from (32):

Question 9 (see (32), Problem 18)**.** *Is there an F_σ ideal $\mathscr{I}$ for which $\mathscr{I}$-ultrafilters exist?*

By Proposition 11 of the previous section, we also answer the following question:

Question 10 (see (26))**.** *Do Hausdorff ultrafilters exist in* ZFC*?*

In close relation with this question is the following, which appears as an open problem in the introduction from (56). A positive semiring is a triple $(A, \oplus, \otimes)$ which satisfies the following conditions

1. $\oplus : A \times A \to A$ is associative and commutative.
2. $\otimes : A \times A \to A$ is associative and distributive with respect to $\oplus$.
3. For all $a, b \in A$, there is a unique c such that $a \oplus c = b$ or $a = b \oplus c$.
4. If $a \oplus c = b$ and $b \oplus c = a$, then $a = b$.

If $A \subseteq \beta\omega$ is a positive semiring and in addition the following holds, then we say that A is an ultrafilter semiring:

1. A is invariant, that is, for any $f \in \omega^\omega$ and any $\mathcal{U} \in A$, $f(\mathcal{U}) \in A$.
2. For all $\mathcal{U} \in A$ and $f, g \in \omega^\omega$, $f(\mathcal{U}) \oplus g(\mathcal{U}) = (f + g)(\mathcal{U})$.
3. For all $\mathcal{U} \in A$ and $f, g \in \omega^\omega$, $f(\mathcal{U}) \otimes g(\mathcal{U}) = (f \cdot g)(\mathcal{U})$

It is a nice theorem in (56) that any ultrafilter semiring is a non-standard model of the natural numbers, and also that a subset $A \subseteq \beta\omega$ which is an ultrafilter semiring should consist only of Hausdorff ultrafilters. Then, the non-existence of Hausdorff ultrafilters implies the non-existence of ultrafilter semirings, which anwers the following question:

Question 11 (see (56), Introduction)**.** *Does the existence of proper ultrafilter semirings follow from* ZFC*?*

The fourth question answered, although partially, appears in (28):

Question 12 (see (28), Open problem 8)**.** *Do (H)-ultrafilters (and (S)-ultrafilters) exist in ZFC?*

Here, (S)-ultrafilters refer to the $\mathscr{I}_{1/n}$-ultrafilters, where $\mathscr{I}_{1/n}$ is the ideal defined by the sets with finite φ-measure, where $\varphi(A) = \sum_{n \in A} \frac{1}{n+1}$, which is a lscsm, so the ideal $\mathscr{I}_{1/n}$ is an F_σ ideal. Meanwhile (H)-ultrafilters refer to $\mathscr{Z}$-ultrafilters, where $\mathscr{Z}$ is the density zero ideal, which is defined as $A \in \mathscr{Z}$ if and only if $\lim_{n\to\infty} |A \cap n|/n = 0$, for any $A \subseteq \omega$. The ideal $\mathscr{Z}$ is not an F_σ ideal, but an $F_{\sigma\delta}$ ideal. Our theorem does not apply for this ideal. There are in fact several reasons for this. One of them is that p-points are also characterized as conv-ultrafilters, where conv is the ideal generated by convergent sequences of rationals in 2^ω, and $\mathsf{conv} \leq_K \mathcal{Z}$, so any conv-ultrafilter is a $\mathcal{Z}$-ultrafilter (we refer the reader to (13) for details). Since the model we present here is obtained by a countable support iteration of proper forcings where each iterand preserves p-points, there are p-points in our model, therefore there are $\mathcal{Z}$-ultrafilters.

The following question appears as part of the text in section 3 from (29):

Question 13 (see (29), section 3)**.** *Do weak $\mathscr{I}$-ultrafilters exist for some summable ideal $\mathscr{I}$?*

Also, the following question from (30) is answered ($\mathcal{D}$ denotes a family of summable ideals):

Question 14 (see (30), question 5.2)**.** *Is it true that whenever the cardinality of $\mathcal{D}$ is less than $\mathfrak{d}$ then there exists an ultrafilter on the natural numbers which is an $\mathscr{I}_g$-ultrafilter for every $\mathscr{I}_g \in \mathcal{D}$, but not a rapid ultrafilter?*

In (7), it was asked if ultrafilters with property M exist in ZFC:

Question 15 (see(7; 60))**.** *Is there in ZFC an ultrafilter with property M?*

It was proved by Shelah that every ultrafilter with property M is a nowhere dense ultrafilter, so in his model for the non-existence of nowhere dense ultrafilters there are no ultrafilters with property M either, which anwers Benedikt's question. However, by Proposition 12, we have that an ultrafilter has property M if and only if it is an $\mathcal{S}$-ultrafilter. It was also proved in (37) that $\mathcal{S} \leq_K \mathsf{nwd}$. So in our model we have no ultrafilter with property M, yet nowhere dense ultrafilters exist (every p-point is a nowhere dense ultrafilter).

4.4 The forcing.

Definition 24. *Let $S \subseteq \omega^{<\omega}$. The tree generated by S, denoted by $gt(S)$, is defined as all the sequences in $\omega^{<\omega}$ contained in some element of S,*

$$gt(S) = \{t \in \omega^{<\omega} : (\exists s \in S)(t \subseteq s)\}$$

Definition 25. *Let $\mathscr{I}$ be an F_σ ideal and φ a lscsm defining $\mathscr{I}$. Let $T \in$ **PT** be a superperfect tree. We say that T has φ-block structure provided there is $S \subseteq split(T)$ such that the following holds:*

1. *$T = gt(S)$.*
2. *$st(T) \in S$.*
3. *There is $\{F^T_s : s \in S\} \subseteq [\omega]^{<\omega}$ such that the following holds:*
   (a) *For all $s \in S$, $|s| = \min(F^T_s)$.*
   (b) *For all $s \in S$, $(T \restriction s)_{F^T_s} \subseteq split(T)$.*

(c) *For all $s, t \in S$, if $s \subsetneq t$, then* $\max(F_s^T) < \min(F_t^T)$.

(d) *For all $s, t \in S$, if $s \subsetneq t$, then $\varphi(F_s^T) < \varphi(F_t^T)$.*

(e) *For all $t \in T$ and $n \in \omega$, there is $s \in S$ extending t such that $\varphi(F_s^T) > n$.*

(f) *For all $s, t \in S$ such that t properly extends s and t has minimal length, if $r \in (T \restriction s)_{[|s|,|t|)}$ is a splitting node, then $r \in (T \restriction s)_{F_s^T}$.*

For each $s \in S$, the set of nodes $(T \restriction s)_{F_s^T}$ will be called a φ-block of s in T. Also, for $s \in S$, define $n_1(s, T) = \max(F_s^T)$.

Strictly speaking, $n_1(s, T)$ should depend on the set S giving T a φ-block structure, but we omit it since we are dealing with only one φ-block structure on a condition at a time. Note that condition (3)(c) implies that for such $s, t \in S$, $(T \restriction s)_{F_s^T} \cap (T \restriction t)_{F_t^T} = \emptyset$. Also note that the previous definition may allow more than one $S \subseteq \textit{split}(T)$ giving a φ-block structure to T, but we are only interested in the existence of such structure. Condition (3)(f) joint with (1) imply that for any $t \in \textit{split}(T)$, there is $s \in S$ such that $t \in (T \restriction s)_{F_s^T}$. We want to point out that conditions (3)(c) and (3)(f) are useful in proving Lemma 21.

Definition 26. *Let $\mathscr{I}$ be an F_σ ideal and φ a lscsm defining $\mathscr{I}$. Define $\mathbf{PT}(\varphi)$ as the set of all trees in $\mathbf{PT}$ which have a φ-block structure, with the order given by the set inclusion, that is, for $T, S \in \mathbf{PT}(\varphi)$, it holds that $S \leq T$ if and only if $S \subseteq T$.*

The first fact we prove about this forcing is that it is proper. Recall that a forcing $\mathbf{P}$ satisfies the Axiom A provided it has the following properties:

1) There is a sequence of partial order relations $\langle \leq_n : n \in \omega \rangle$ such that for all $n \in \omega$, $\leq_{n+1} \subseteq \leq_n$ and $\leq_0 \subseteq \leq$.

2) For all $p \in \mathbf{P}$, $\mathcal{A} \subseteq \mathbf{P}$ a maximal antichain and $n \in \omega$, there is $q \leq_n p$ which is compatible with at most countably many elements from $\mathcal{A}$.

3) If $\langle p_n : n \in \omega \rangle$ is such that $p_{n+1} \leq_n p_n$, then there is p_ω such that $p_\omega \leq p_n$ for all $n \in \omega$.

Lemma 13. *The forcing $\mathbf{PT}(\varphi)$ satisfies Axiom A, therefore, it is proper.*

Proof. The proof is a standard fusion argument. For $n \in \omega$ and $T \in \mathbf{PT}(\varphi)$, with $S \subseteq \textit{split}(T)$ giving a φ-block structure to T, define φ-$\textit{split}_n(T)$ as follows:

$$\varphi\text{-}\textit{split}_n(T) = \{s \in S : s \text{ has minimal length and satisfies } \varphi(F_s^T) \geq n\}$$

Then, for $T_1, T_2 \in \mathbf{PT}(\varphi)$, define $T_2 \leq_n T_1$ if the following holds:

1. $T_2 \leq T_1$.

2. There are $S_1 \subseteq \textit{split}(T_1)$ and $S_2 \subseteq \textit{split}(T_2)$ giving φ-block structure to T_1 and T_2, respectively, such that conditions (3) and (4) below hold:

3. φ-$\textit{split}_n(T_1) = \varphi$-$\textit{split}_n(T_2)$.

4. For all $s \in \varphi\text{-}split_n(T_1)$, $F_s^{T_2 \restriction s} = F_s^{T_1 \restriction s}$.

It is not hard to see that these orderings satisfy the conditions of Axiom A. $\square$

The following definition provides two reals which will be useful. The generic real which serves to make $\mathfrak{g}$ big, and the function which destroys the $\mathscr{I}$-ultrafilterness for ultrafilters from the ground model. Recall that the definition of superperfect tree we are working with requires the nodes of any superperfect tree to be strictly increasing sequences of natural numbers, so the generic real $\dot{x}_{gen}$ defined below is a strictly increasing sequence, which allows the real $\dot{f}_{gen}$ to be well defined.

Definition 27. *Let $\dot{G} \subseteq \mathbf{PT}(\varphi)$ be a generic filter over the ground model. The generic real added by $\mathbf{PT}(\varphi)$ is defined as $\dot{x}_{gen} = \bigcup\bigcap \dot{G}$. Also, define $\dot{f}_{gen}(k) = n+1$ if and only if $k \in [\dot{x}_{gen}(n), \dot{x}_{gen}(n+1))$.*

Lemma 14. *For all $X \in [\omega]^\omega \cap V$, it holds that $\mathbf{PT}(\varphi) \Vdash$ "$\dot{f}_{gen}[X] \in \mathscr{I}^+$".*

Proof. Pick some $X \in [\omega]^\omega$ and $m \in \omega$. Let $T \in \mathbf{PT}(\varphi)$ be a condition with φ-block structure given by S. Let $n \in \omega$ be an arbitrary positive natural number, and $s \in S$ with $\varphi(F_s^T) > n$. It is easy to construct a condition $T' \leq_0 T \restriction s$ such that for all $k \in F_s^{T'}$, $f \in (T')_k$ and $m \in succ_{T'}(f)$, it happens that $X \cap [f(k-1), m) \neq \emptyset$. Then T' forces that for all $k \in F_s^{T'}$, $[\dot{x}_{gen}(k-1), \dot{x}_{gen}(k)) \cap X \neq \emptyset$, which implies that $F_s^{T'} \subseteq \dot{f}_{gen}[X]$. Since $F_s^{T'}$ has measure bigger than n, so does $\dot{f}_{gen}[X]$. $\square$

The following two lemmas are an adaptation of the construction used in Proposition 3.3 from (52) to the present forcing. The essence of the lemma is the same, it is only a little bit more involved. Recall the definition of the orderings $\leq_n$ from the proof of Lemma 13, and note that for the case of $n = 0$, given two conditions $T_1, T_2 \in \mathbf{PT}(\varphi)$, $T_2 \leq_0 T_1$ means that $st(T_1) = st(T_2)$ and $F_{st(T_1)}^{T_1} = F_{st(T_2)}^{T_2}$.

Lemma 15. *Let $\dot{x}$ be a $\mathbf{PT}(\varphi)$-name for an infinite subset of ω, and $T \in \mathbf{PT}(\varphi)$ be a condition. Then there is $T' \leq_0 T$ such that for each $f \in (T')_{F_{st(T')}^{T'}}$ there is a set $X_f \subseteq \omega$ such that for all $n \in \omega$, for all but finitely many $k \in succ_{T'}(f)$:*

$$T' \restriction f^\frown k \Vdash \text{“}\dot{x} \cap n = X_f \cap n\text{”}$$

Proof. We prove this lemma by induction on the size of $F_{st(T)}^T$. If $|F_{st(T)}^T| = 1$, the argument is exactly as for the usual Rational Perfect set forcing, which goes briefly as follows: if $F_{st(T)}^T$ has only one element, it must be $|st(T)|$, and $(T)_{F_{st(T)}^T} = \{st(T)\}$. For each $k \in succ_T(st(T))$, pick $a_k \leq T \restriction st(T)^\frown k$ which forces $\dot{x} \cap k$ to be some set $z_k \subseteq k$. By a compactness argument, there is $\{k_n : n \in \omega\} \subseteq succ_T(st(T))$ and $Y \subseteq \omega$ such that $\lim_{n\in\omega} z_{k_n} = Y$. Define $T' = \bigcup_{n\in\omega} a_{k_n}$ and $X_{st(T')} = Y$. The required condition follows trivially by construction of T'.

Now assume that $|F_{st(T)}^T| = n+1$, and the lemma is true for all conditions S with $F_{st(S)}^S$ of size at most n. For each $k \in succ_T(st(T))$, let $T_k \leq_0 T \restriction st(T)^\frown k$[2] which satisfies

[2]Note that $S \cap split(T \restriction st(T)^\frown k) \cup \{st(T \restriction st(T)^\frown k)\}$ gives φ-block structure to $T \restriction st(T)^\frown k$, with $F_{st(T\restriction st(T)^\frown k)}^{T\restriction st(T)^\frown k} = F_{st(T)}^T \setminus \{st(T)\}$.

the required condition for $T \restriction st(T)^\frown k$ from the lemma. Also, note that this can be done such that if $S_k \subseteq split(T_k)$ gives φ-block structure to T_k, then for all $r \in S_k \setminus \{st(T_k)\}$, it holds that $\varphi(F_r^{T_k}) > \varphi(F_{st(T)}^T)$. This is to make sure that the tree T' defined below is in fact a condition, since it has to satisfy point $3(d)$ from Definition 25. We can assume in addition that for all $k \in succ_T(st(T))$, T_k forces the value of $\dot{x} \cap k$ to be some $z_k \subseteq k$ (since T_k satisfies the required condition for $T \restriction st(T^\frown k)$, it is sufficient to prune finitely many successors from the stem of T_k). Making again a compactness argument, we find $Y \subseteq \omega$ and $\{k_n : n \in \omega\}$ such that $\lim_{n\in\omega} z_{k_n} = Y$. Define $T' = \bigcup_{n\in\omega} T_{k_n}$ and $X_{st(T)} = Y$. By induction hypothesis, it holds that $F_{st(T_k)}^{T_k} = F_{st(T\restriction st(T)^\frown k)}^{T\restriction st(T)^\frown k}$ for all $k \in succ_T(st(T))$, so $F_s^T \subseteq \{|s|\} \cup F_{st(T_k)}^{T_k}$, which implies that $F_s^T \subseteq F_s^{T'}$. In fact, we have $F_s^T = F_s^{T'}$. □

Lemma 16. *Let $\mathcal{U}$ be an ultrafilter, $\dot{x}$ a $\mathbf{PT}(\varphi)$-name for an infinite subset of ω, and $T \in \mathbf{PT}(\varphi)$ a condition. Then there is $T' \leq T$ such that:*

1. *Either: $T' \leq_0 T$, or there is $k \in succ_T(st(T))$ such that $T' \leq T \restriction st(T)^\frown k$*
2. *$\varphi(F_{st(T')}^{T'}) \geq \varphi(F_{st(T)}^T)/2$.*
3. *For each $f \in (T')_{F_{st(T')}^{T'}}$ there is a set $X_f \subseteq \omega$ such that for all $n \in \omega$, for all but finitely many $k \in succ_{T'}(f)$:*

$$T' \restriction f^\frown k \Vdash \text{“}\dot{x} \cap n = X_f \cap n\text{”}$$

4. *Exactly one of the following happens:*
   - *(a) For all $f \in (T')_{F_{st(T')}^{T'}}$, $X_f \in \mathcal{U}$.*
   - *(b) For all $f \in (T')_{F_{st(T')}^{T'}}$, $\omega \setminus X_f \in \mathcal{U}$.*

Proof. Let $T' \leq_0 T$ be a condition given by Lemma 15. Then we have that $F_{st(T')}^{T'} = F_{st(T)}^T$. We claim the following:

Claim. There are $T'' \leq_0 T'$ and A, B such that:

1. $F_{st(T'')}^{T''} = F_{st(T)}^T = A \cup B$ and $A \cap B = \emptyset$.
2. For all $f \in (T'')_A$, $X_f \in \mathcal{U}$.
3. For all $f \in (T'')_B$, $\omega \setminus X_f \in \mathcal{U}$.

Proof of Claim. We proceed by induction on the size of $F_{st(T')}^{T'}$(which is equal to $F_{st(T)}^T$). If $F_{st(T')}^{T'}$ has only one element, the result follows immediately from the construction of condition T' in the proof of Lemma 15.

Now assume $|F_{st(T')}^{T'}| = n+1$ and that claim is true for all conditions $S \in \mathbf{PT}(\varphi)$ such that $F_{st(S)}^S$ has size at most n. Then, for each $k \in succ_{T'}(st(T'))$, pick $T_k \leq_0 T' \restriction st(T')^\frown k$ such that there are A_k and B_k satisfying:

1. $F_{st(T_k)}^{T_k} = F_{st(T'\restriction st(T')^\frown k)}^{T'\restriction st(T')^\frown k} = A_k \cup B_k$, and $A_k \cap B_k = \emptyset$.

2. For all $f \in (T_k)_{A_k}$, $X_f \in \mathcal{U}$.

3. For all $f \in (T_k)_{B_k}$, $\omega \setminus X_f \in \mathcal{U}$.

This can be done by induction hypothesis. Then there are $\tilde{A}$ and $\tilde{B}$ such that for infinitely many $k \in succ_{T'}(st(T'))$, $A_k = \tilde{A}$ and $B_k = \tilde{B}$. Let us say that $H \subseteq succ_{T'}(st(T'))$ is infinite and such that for all $k \in H$, $A_k = \tilde{A}$ and $B_k = \tilde{B}$. Also, since $T' \leq_0 T$ satisfies the condition of Lemma 15, we can define $T'' = \bigcup_{k \in H} T_k$ and $X_{st(T'')} = X_{st(T')}$. If $X_{st(T'')} \in \mathcal{U}$, define $A = \tilde{A} \cup \{|s|\}$ and $B = \tilde{B}$, otherwise define $A = \tilde{A}$ and $B = \tilde{B} \cup \{|s|\}$. This finishes the proof of the claim.

Now, let $T'' \leq_0 T'$ and A, B as given by the claim. Then $\varphi(A) \geq \varphi(F^T_{st(T)})/2$ or $\varphi(B) \geq \varphi(F^T_{st(T)})/2$. Let C be the one with bigger measure. If $st(T'') \in C$, then we construct a condition $T''' \leq_0 T''$ which safisfies the conclusion recursively as follows: assume $C = \{|st(T'')| = m_0 < \ldots < m_j\}$, and for $i = 0$, define $S_0 = T'' \restriction st(T'') = T''$. Suppose S_i is defined, and for each $f \in (S_i)_{m_i}$ and $k \in succ_{S_i}(f)$, let $r_{f,k} \in (S_i)_{m_{i+1}}$ such that $f^\frown k \subseteq r_{f,k}$, and define $S_{i+1} = \bigcup_{f \in (S_i)_{m_i}} \bigcup_{k \in succ_{S_i}(f)} S_i \restriction r_{f,k}$. The condition S_j is the one we are looking for, that is, $T''' = S_j$.

If $st(T'') \notin C$, then pick $k \in succ_{T''}(st(T''))$ and construct $T''' \leq T'' \restriction st(T'')^\frown k$ in the same way as in the previous case, but starting with $S_0 = T'' \restriction r$ for some $r \in (T'')_{\min(C)}$ and $st(T'')^\frown k \subseteq r$. Everything else goes over as before. □

Lemma 17. *Let $T \in \mathbf{PT}(\varphi)$ be a condition, with φ-block structure given by S. If $c : S \to 2$ is a coloring, then there is $T' \leq T$ with φ-block structure given by $S' \subseteq S$ such that $c \restriction S'$ is constant.*

Proof. There are two cases:

Case 1. There are $t \in T$ and $n \in \omega$ for which all $r \in S$ extending t, with $\varphi(F^T_r)$ bigger than n, are all of the same color. Fix t_0 one of such nodes in T and let $r_0 \in S$ be a node extending t_0 such that $\varphi(F^T_{r_0}) > n$. Then define $T' = T \restriction r_0$.

Case 2. For all $t \in T$ and for all $n \in \omega$, there are $u, v \in S$ extending t of the two colors with $\varphi(F^T_u) > n$ and $\varphi(F^T_v) > n$. Then construct S' recursively adding nodes in S as follows: fix a color, fix a node of that color, and then start adding nodes $s \in S$ of the same color, joint with their corresponding $(T \restriction s)_{F^T_s}$, repeat infinitely many times taking care that you extend the maximal nodes of each $(T \restriction s)_{F^T_s}$ you added to an $r \in S$ with $\varphi(F^T_r)$ big enough and with the color you choose, and making every node from $(T \restriction s)_{n_1(s,T)}$ a splitting node. Then consider $T' = gt(S')$.

□

Lemma 18. *Let $\mathcal{U}$ be an ultrafilter, $T \in \mathbf{PT}(\varphi)$ be a condition, $\dot{x}$ be a $\mathbf{PT}(\varphi)$-name. Then there is $T' \leq T$ such that for all $s \in split(T')$, there is $X_s \subseteq \omega$ satisfying the following two conditions:*

1. *Exactly one of the following happens:*

   (a) *For all $s \in split(T')$, $X_s \in \mathcal{U}$.*

(b) *For all $s \in split(T')$, $\omega \setminus X_s \in \mathcal{U}$.*

2. *For all $s \in split(T')$, for all $n \in \omega$ and for all but finitely many $k \in succ_{T'}(s)$, $T' \upharpoonright s^\frown k \Vdash$ "$\dot{x} \cap n = X_s \cap n$"*

Proof. We make use of the orderings $\leq_n$ defined in Lemma 13. Let $T \in \mathbf{PT}(\varphi)$ be a condition. We define two sequences $\langle T_n : n \in \omega \rangle$ and $\langle S_n : n \in \omega \rangle$ such that:

1. S_n gives a φ-block structure to T_n.

2. $T_0 \leq T$ satisfies conditions (1)-(4) from Lemma 16.

3. Suppose T_n is defined. For $s \in \varphi\text{-}split_n(T_n)$, and $f \in (T_n \upharpoonright s)_{n_1(s,T_n)+1}$, pick $r(s,f) \in S_n$ such that $f \subseteq r(s,f)$ and $\varphi(F^{T_n}_{r(s,f)}) > 2 \cdot (\varphi(F^{T_n}_s) + 1)$. Then apply Lemma 16 to get a condition $T_{r(s,f)} \leq T_n \upharpoonright r(s,f)$ satisfying conditions (1)-(4) from Lemma 16. Note that $\varphi(F^{T_{r(s,f)}}_{st(T_{r(s,f)})}) \geq \varphi(F^{T_n}_s) + 1$. Then define

$$T_{n+1} = \bigcup_{s \in \varphi\text{-}split_n(T_n)} \bigcup_{f \in (T_n \upharpoonright s)_{n_1(s,T_n)+1}} T_{r(s,f)}$$

   Finally, define $S_{n+1} = S_n \cap split(T_{n+1})$.

Let $T' = \bigcap_{n \in \omega} T_n$ and $S' = \bigcup_{n \in \omega} \varphi\text{-}split_n(T_n)$. Note that step (3) of the previous construction guaranties that, regarding conditions (2)-(4) from Lemma 16, we are taking care of all splitting nodes in T', since each splitting node from T' belongs to $(T_{n+1} \upharpoonright r)_{F^{T_{n+1}}_r}$ for some $r \in \varphi\text{-}split_n(T_n)$ and $n \in \omega$, and all the splitting nodes in $(T')_{F^{T'}_{st(T')}}$ satisfy the conditions due to the choice of T_0. Then condition (2) from this lemma holds for $T' \upharpoonright s$ for all $s \in S'$, but for some nodes $s \in S$, the nodes from $(T' \upharpoonright s)_{F^{T'}_s}$ satisfy clause (1)(a) and for some other nodes $s \in S$, the nodes from $(T' \upharpoonright s)_{F^{T'}_s}$ satisfy clause (1)(b). Now consider the coloring c over S' given by $c(s) = 1$ if and only if $X_s \in \mathcal{U}$. Then apply the previous lemma to get a condition $T'' \leq T'$ with φ-block structure given by some $S'' \subseteq S'$ such that $c \upharpoonright S''$ is constant. The condition T'' satisfies the conclusion. □

Lemma 19. *Let $\dot{x}$ be a $\mathbf{PT}(\varphi)$-name for a function from ω to ω, and $T \in \mathbf{PT}(\varphi)$ be a condition which forces $\dot{x}$ to be bounded by $g \in \omega^\omega$. Then there are $T' \leq T$ and $S \subseteq split(T')$ which gives φ-block structure to T', such that for all $s \in S$:*

> *For each $r \in (T' \upharpoonright s)_{F^{T'}_s}$ there is a function $f_r \in \omega^\omega$ such that for all $n \in \omega$, for all but finitely many $k \in succ_{T'}(r)$:*
>
> $$T' \upharpoonright r^\frown k \Vdash \text{"}\dot{x} \upharpoonright (|r| + n) = f_r \upharpoonright (|r| + n)\text{"}$$

Proof. The proof of this lemma is essentially the same of Lemma 18, but with the omission of the ultrafilter. □

For the purpose of the Propositions 13, 14 and 15, we define other order relations $\sqsubseteq_n$. Recall the functions $\varphi_T : \omega^{<\omega} \to split(T)$ from section 2.

Definition 28. *Let $R, T \in \mathbf{PT}(\varphi)$. Define $R \sqsubseteq_n T$ if the following holds:*

1. *$R \leq T$.*
2. *For all $s \in n^{\leq n}$, $\varphi_T(s) = \varphi_R(s)$*
3. *For all $s \in split(T)$, if $s \in R$ then $s \in split(R)$.*

The only property that we need from these relations is the following:

Lemma 20. *Let $\langle T_n : n \in \omega \rangle$ be a sequence such that for all $n \in \omega$, $T_n \in \mathbf{PT}(\varphi)$, $T_{n+1} \sqsubseteq_n T_n$ and $S_n = T_n \cap S_0$ gives φ-block structure to T_n, where S_0 gives φ-block structure to T_0. Then there is $T_\omega \in \mathbf{PT}(\varphi)$ such that for all $n \in \omega$, $T_\omega \leq T_n$.*

Proof. Let us fix some notation: for $r \in split(T_m)$, denote by $\langle i_n^{r,m} : n \in \omega \rangle$ the increasing enumeration of $succ_{T_m}(r)$. Let $S_0 \subseteq T_0$ be such that S_0 gives φ-block structure to T_0. Define $T_\omega = \bigcap_{n \in \omega} T_n$ and $S_\omega = T_\omega \cap S_0$. We will prove that for all $s \in S_\omega$, all the nodes from $(T_\omega \restriction s)_{F_s^{T_0}}$ are splitting nodes in T_ω. For any $s \in split(T_\omega)$, define $n_s \in \omega$ as the minimum natural number such that $s \in \varphi_{T_\omega}(n_s^{\leq n_s})$. By clause (3) from Definition 4.12, we know that for all $m \in \omega$, $F_s^{T_m} = F_s^{T_0}$, so for all $m \in \omega$, the nodes from $(T_m \restriction s)_{F_s^{T_m}}$ are all splitting nodes in T_m. Pick an arbitrary node $s \in S_\omega$. Define $l_s = |F_s^{T_0}| - 1$ and let $\langle j_0 \dots, j_{l_s} \rangle$ be the increasing enumeration of $F_s^{T_0}$. Let $r \in (T_\omega \restriction s)_{F_s^{T_0}}$ be an arbitrary node. Then there is $k \in \{0, \dots, l_s\}$ such that $r \in (T_\omega \restriction s)_{j_k}$. Note that this means that for some $m \in \omega$, $r \in \varphi_{T_\omega}((n_s + m)^{\leq n_s + k})$, which implies that for $m \in \omega$ big enough, $r \in \varphi_{T_{n_s+m}}((n_s + m)^{\leq n_s + k})$. Fix m_0 one of such $m \in \omega$ and let $t_r \in (n_s + m_0)^{\leq n_s + k}$ be such that $r = \varphi_{T_{n_s+m_0}}(t_r)$. By the definition of the functions φ_{T_i}, this implies that for all $l \in \omega$, $\varphi_{T_{n_s+m_0+l+1}}(t_r^\frown l)$ is a splitting node extending r, and $\varphi_{T_{n_s+m_0+l+1}}(t_r^\frown l)$ extends the $(l+1)^{th}$ successor of r in $T_{n_s+m_0+l+1}$, that is, $\varphi_{T_{n_s+m_0+l+1}}(t_r^\frown l)$ extends $r^\frown i_l^{r, n_s+m_0+l+1}$. Also, by clause (2) from Definition 4.12, we have that for all $h \in \omega$, $\varphi_{T_{n_s+m_0+l+1}}(t_r^\frown l)$ belongs to $T_{n_s+m_0+l+h+1}$, which implies that $\varphi_{T_{n_s+m_0+l+1}}(t_r^\frown l)$ belongs to T_ω. Since this is true for all $l \in \omega$, r is an splitting node in T_ω. Given that $r \in (T_\omega \restriction s)_{F_s^{T_0}}$ was an arbitrary node, it follows that all the nodes from $(T_\omega \restriction s)_{F_s^{T_0}}$ are splitting nodes in T_ω.

Let us check clause (1) from Definition 25. Pick an arbitrary node $t \in T_\omega$, and let $r \in split(T_\omega)$ be such that $t \subsetneq r$. Then $r \in split(T_0)$, and by clause (3)(f), there is $s \in S_0$ such that $r \in (T_0 \restriction s)_{F_s^{T_0}}$. If $t \subseteq s$ we are done. Otherwise, following the same notation from the previous paragraph, consider the natural numbers n_s and l_s. Note that the inclusions $s \subsetneq t \subsetneq r$ hold, so $t \in (T_0 \restriction s)_{[|s|, n_1(s, T_0)]}$. Also note that $j_{l_s} = n_1(s, T_0)$. Then, the nodes from $\varphi_{T_{n_s+l_s}}((n_s + l_s)^{n_s+l_s})$ are nodes from $(T_{n_s+l_s} \restriction s)_{j_{l_s}}$. This implies that, for all natural number $m \geq n_s + l_s + 1$, the nodes from $\varphi_{T_m}(m^{n_s+l_s+1})$ are nodes from $S_m = T_m \cap S_0$, due to clause (2) from Definition 4.12 and clause (3)(f) from Definition 25. Let $k \geq n_s + l_s + 1$ big enough so $r \in \varphi_{T_k}(k^{\leq k})$. Then, for one node $t' \in k^{n_s+l_s+1}$, we have that $r \subseteq \varphi_{T_k}(t')$ and $\varphi_{T_k}(t') \in S_k$, which, due to condition (2) from Definition 4.12, appears in all the remaining conditions of the sequence $\langle T_n : n \in \omega \rangle$. This implies that t has an extension in S_ω, namely, the node $\varphi_{T_k}(t')$ (since $t \subseteq r \subseteq \varphi_{T_k}(t') \in S_\omega$).

Now, for $s \in S_\omega$, define $F_s^{T_\omega} = F_s^{T_0}$. Condition (2) from Definition 25 follows from the fact that $st(T_0) = st(T_n)$ for all $n \in \omega$ and the fact proved in the first paragraph of this lemma. By the same fact, conditions (3)(a)-(3)(f) are easily seen to be inherited to condition T_ω and S_ω. □

Proposition 13. *The forcing* $\mathbf{PT}(\varphi)$ *has the Laver property.*

Proof. Let $g \in \omega^\omega$ be any increasing function, $\dot{x}$ a $\mathbf{PT}(\varphi)$-name for a function from ω to ω and T a condition such that $T \Vdash$ "$(\forall n \in \omega)(\dot{x}(n) \leq g(n))$". We prove that there is $h \in ([\omega]^{<\omega})^\omega$ such that for all n, $|h(n)| \leq f(n)$, where $f(n) = |n^{\leq n}|$.

Let $T' \leq T$ be a condition given by Lemma 19. Define two sequences $\langle T_n : n \in \omega \rangle$, and $\langle B_n : n \in \omega \rangle$ as follows:

1. $T_0 \leq_0 T'$ and T_0 decides the value of $\dot{x}(0)$. This can be done by dropping finitely many successors from $st(T')$.

2. For all $n \in \omega$, $T_{n+1} \sqsubseteq_{n+1} T_n$.

3. For all $n \in \omega$, $|B_n| \leq f(n)$, and $T_n \Vdash$ "$\dot{x}(n) \in B_n$".

By the previous lemma, there is T_ω such that for all $n \in \omega$, $T_\omega \leq T_n$.

Suppose the sequences are defined up to the n^{th} element. Define T_{n+1} as follows. For each $t \in \varphi_{T_n}((n+1)^{\leq n+1})$, let $R_t \leq_0 T_n \restriction t$ which decides the value of $\dot{x}(n+1)$ to be m_t^{n+1}, which is found by dropping finitely many successors from $st(T_n \restriction t)$. Then define $T_{n+1} = \bigcup_{t \in \varphi_{T_n}((n+1)^{\leq n+1})} R_t$, and $B_{n+1} = \{m_s : s \in \varphi_{T_n}((n+1)^{\leq n+1})\}$.

Condition (2) follows from the fact that we are pruning the condition T_n in the horizontal direction, keeping all the splitting nodes extending the nodes which are not pruned. Condition (3) follows immediately from definition of T_{n+1}. Since $T_\omega \leq T_n$ for all $n \in \omega$, it follows from condition (3) that for all $n \in \omega$, $T_\omega \Vdash$ "$\dot{x}(n) \in B_n$". □

Proposition 14. *The forcing* $\mathbf{PT}(\varphi)$ *does not add splitting reals.*

Proof. Let $\dot{x}$ be a $\mathbf{PT}(\varphi)$-name for an infinite subset of ω, and $T \in \mathbf{PT}(\varphi)$ a condition. We follow the same lines as in the proof of the previous proposition, but we make some changes. We start with $T' \leq T$ given by Lemma 18, where $\mathcal{U}$ is a fixed ultrafilter. We can assume that the first option from condition 1 of Lemma 18 holds. Then we replace the sequence $\langle B_n : n \in \omega \rangle$ by a sequence of natural numbers $\langle m_n : n \in \omega \rangle$, and clause (3) from Lemma 13 by the following:

(3') For all $n \in \omega$, $T_n \Vdash$ "$m_n \in \dot{x}$".

The construction of the sequences is as follows. Assume the sequences are build up to the n^{th} term. Then for each $s \in \varphi_{T_n}((n+1)^{\leq n+1})$ we have that $X_s \in \mathcal{U}$, so there is $m_{n+1} \in \bigcap_{s \in \varphi_{T_n}((n+1)^{\leq n+1})} X_s$ and $m_n < m_{n+1}$. Let $R_s \leq_0 T_n \restriction s$ which forces $m_{n+1} \in \dot{x}$, which can be found by dropping finitely many successors from the stem of $T_n \restriction s$. Define $T_{n+1} = \bigcup_{s \in \varphi_{T_n}((n+1)^{\leq n+1})} R_s$. By applying Lemma 20, we get T_ω such that for all $n \in \omega$, $T_\omega \leq T_n$. Then $T_\omega \Vdash$ "$\{m_n : n \in \omega\} \subseteq \dot{x}$". □

Proposition 15. *The forcing* $\mathbf{PT}(\varphi)$ *preserves p-points.*

Proof. Let $\dot{x}$ be a $\mathbf{PT}(\varphi)$-name for an infinite subset of ω and $\mathcal{U}$ a p-point on ω. Let $T_0 \leq T$ be a condition given by Lemma 18. We can assume that point (a) from item (1) of Lemma 18 holds. We make use of the p-point game, Definition 18. We will give a strategy for Player I on which she and Player II will construct two sequences $\langle T_n : n \in \omega\rangle$ and $\langle B_n : n \in \omega\rangle$ such that T_n is a condition in $\mathbf{PT}(\varphi)$, B_n is a finite subset of ω, conditions (1) and (2) from the proof of Proposition 13 hold, and (3) is replaced by the following:

(3') $T_{n+1} \Vdash$ "$B_n \subseteq \dot{x}$"

The construction is as follows:

i) Player I starts playing the set $A_0 = X_{st(T_0)}$.

ii) Suppose Player II has answered with set $B_0 \subseteq X_{st(T_0)}$. Then let k_0 be big enough so that for all $i \in succ_{T_0}(st(T_0)) \setminus k_0$ it holds that $T_0 \restriction st(T_0)^\frown i \Vdash$ "$\dot{x} \cap (\max(B_0)+1) = X_{st(T_0)} \cap (\max(B_0)+1)$. Note that this implies that for all $i \in succ_{T_0}(st(T_0)) \setminus k_0$, $T_0 \restriction st(T_0)^\frown i \Vdash$ "$B_0 \subseteq \dot{x}$". Then define

$$T_1 = \bigcup_{i \in succ_{T_0}(st(T_0)), i \geq k_0} T_0 \restriction st(T_0)^\frown i$$

And Player I responds with $A_1 = \bigcap_{s \in 1^{\leq 1}} X_{\varphi_{T_1}(s)} \setminus B_0$.

iii) Suppose at move number n and Player I has played

$$A_n = \bigcap_{s \in n^{\leq n}} X_{\varphi_{T_n}(s)} \setminus \bigcup_{k<n} B_k$$

Suppose Player II responds with a set $B_n \subseteq A_n$. Then, let $k_n \in \omega$ be big enough such that for all $r \in \varphi_{T_n}(n^{\leq n})$ and all $i \in succ_{T_n}(r) \setminus k_n$, it holds that $T_n \restriction r^\frown i \Vdash$ "$\dot{x} \cap (\max(B_n)+1) = X_r \cap (\max(B_n)+1)$". Note that this implies that for each $r \in \varphi_{T_n}(n^{\leq n})$ and $i \in succ_{T_n}(r) \setminus k_n$, $T_n \restriction r^\frown i \Vdash$ "$B_n \subseteq \dot{x}$". Then define T_{n+1} as follows:

$$T_{n+1} = \bigcup_{s \in n^{\leq n}} \bigcup_{i \in succ_{T_n}(\varphi_{T_n}(s)), i \geq k_n} T \restriction \varphi_{T_n}(s)^\frown i$$

By construction it follows that $T_{n+1} \Vdash$ "$B_n \subseteq \dot{x}$". Let Player I play the set $A_{n+1} = \bigcap_{s \in (n+1)^{\leq n+1}} X_{\varphi_{T_{n+1}}(s)} \setminus \bigcup_{k \leq n} B_k$.

Since Player I can not have a winning strategy, there should be a play in which Player II wins. Let $\langle T_n : n \in \omega\rangle$ and $\langle B_n : n \in \omega\rangle$ be the sequences constructed by Player I and Player II along such play. Then $\bigcup_{n\in\omega} B_n \in \mathcal{U}$, and each T_n forces that $\bigcup_{k<n} B_n \subseteq \dot{x}$. By Lemma 20, let T_ω be such that for all $n \in \omega$, $T_\omega \leq T_n$. Then it follows that T_ω forces $\bigcup_{n\in\omega} B_n \subseteq \dot{x}$. $\square$

4.5 Forcing the Near Coherence of Filters Principle.

We prove the stronger fact $\mathfrak{u} < \mathfrak{g}$. Then, by Theorem 28, we have that NCF follows as a consequence. The proof follows the same lines as in the Rational Perfect forcing, we only take care that the trees we get in the construction are indeed conditions of our forcings.

The following definition was given in (11) for the Rational Perfect set forcing. Here we use it for the version we are working with.

Definition 29. *A condition $T \in \mathbf{PT}(\varphi)$ has interval structure if there is a partition of ω into intervals $\langle I_n : n \in \omega \rangle$ such that:*

1. *For all $n \in \omega$, $\max(I_n) < \min(I_{n+1})$.*
2. *If $s \in split(T)$, and $\max(s) \in I_n$, then for all $k > n$, $I_k \cap succ_T(s)$ contains exactly one element.*
3. *For all $s \in split(T)$, and $k \in succ_T(s)$, if $k \in I_m$, then there is $r \in split(T)$ such that $s^\frown k \subseteq r$ and $\max(r) \in I_m$.*

Lemma 21. *The conditions $T \in \mathbf{PT}(\varphi)$ with interval structure are dense.*

Proof. Let $T \in \mathbf{PT}(\varphi)$ be any condition, and let $S \subseteq split(T)$ be a set of nodes giving T a φ-block structure. Recursively construct a sequence $\langle i_k : k \in \omega \rangle$, and two partial functions $t : \omega \times split(T) \to T$ and $r : \omega \times split(T) \to split(T)$ as follows:

1. $i_0 = 0$ and $i_1 = \max(st(T)) + 1$.
2. Suppose the sequences are defined up to n. Then let i_{n+1} be big enough such that for all $s \in split(T)$, if $\max(s) < i_n$, then there are $t(n, s)$ an immediate successor of s such that $i_n < \max(t(n, s)) < i_{n+1}$ and $r(n, s) \in split(T)$ with minimal length extending $t(n, s)$ with $\max(r(n, s)) < i_{n+1}$.

Let us fix some notation: for $T \in \mathbf{PT}(\varphi)$, a non-empty set $H \in [split(T)]^{<\omega}$ and $k \in \omega$, define $T \restriction [H, k] = \bigcup_{s \in H} \bigcup_{n \in succ_T(s), n > k} T \restriction s^\frown n$. Now define two sequences $\langle T_n : n \in \omega \rangle$ and $\langle H_n : n \in \omega \rangle$ as follows:

1. Define $T_0 = T \restriction [\{st(T)\}, i_1]$ and $H_0 = \{st(T_0)\}$
2. Suppose T_n is defined. Then define $H_{n+1} = H_n \cup \{r(n+1, s) : s \in H_n\}$, and define $T_{n+1} = T_n[H_{n+1}, i_{n+2}]$.

Now, for each $n \in \omega$, let $k_n \in \omega$ be the minimum natural number such that $\varphi_{T_{k_n}}(n^{\leq n})$ is contained in H_{k_n}. To prove the existence of such natural number, first note that for all $n \in \omega$, every node from $\{r(n+1, s) : s \in H_n\}$ is a splitting node extending some node from H_n, and every node from H_n has an extension in $\{r(n+1, s) : s \in H_n\}$. So each set H_{n+1} has one more extension to all the nodes from H_n, besides those already included in H_n. This implies that at some point, H_k contains a subset isomorphic to $n^{\leq n}$, and for such k, each node from H_k is a splitting node in T_k, so $\varphi_{T_k} \restriction n^{\leq n}$ gives the isomorphism between $n^{\leq n}$ and a subset of H_k. Now note that for all $l \geq k_n$, $T_l \sqsubseteq_n T_{k_n}$. In particular, for all

$n \in \omega$, $T_{k_{n+1}} \sqsubseteq_n T_{k_n}$. Then apply Lemma 20 to $\langle T_{k_n} : n \in \omega \rangle$ to see that $T_\omega = \bigcap_{n\in\omega} T_{k_n}$ is a condition which is below all the conditions T_n. Condition T_ω is the one we are looking for, along with the interval partition $\{[i_n, i_{n+1}) : n \in \omega\}$. To see that this partition gives interval structure to T_ω, note that $split(T_\omega) = \bigcup_{n\in\omega} H_n$ and recall clause (2) from the definition of the functions t and r.

□

Lemma 22. *The ω_2-step iteration with countable support of* $\mathbf{PT}(\varphi)$ *forces the inequality* $\mathfrak{u} < \mathfrak{g}$.

Proof. Let us first prove that $\mathbf{PT}(\varphi)$ adds a real which is contained in every groupwise dense family from the ground model. Let us fix some notation: let $T \in \mathbf{PT}(\varphi)$ be a condition and $S \subseteq split(T)$ giving φ-block structure to T. For $s \in S$ define $l_s = |F^T_s| - 1$, and $F^T_s \setminus \{|s|\} = \{m^s_1 < \ldots < m^s_{l_s}\}$ be the increasing enumeration of $F^T_s \setminus \{s\}$.

We will prove that $\dot{x}_{gen}[\omega] \in \mathcal{G}$ for any groupwise dense family from the ground model. Fix a groupwise dense family $\mathcal{G}$ in the ground model, and let $T \in \mathbf{PT}(\varphi)$ be a condition. By the previous lemma, there is $T' \leq_0 T$ that has interval structure given by $\langle I_n : n \in \omega \rangle$. Let $S' \subseteq split(T')$ be a set of nodes giving T' a φ-block structure. Then there is an infinite $A \subseteq \omega$ such that $\bigcup_{n\in A} I_n \in \mathcal{G}$. We can assume that $0 \in A$, since $\mathcal{G}$ is closed under finite modifications of its elements. Also, note that $\max(st(T)) \in I_0$, according to the construction of T' in the proof of Lemma 21. Now consider the tree T'' given by all the nodes from T' whose image is contained in the union of finitely many consecutive intervals from $\langle I_n : n \in A \rangle$. Let us see that T'' is in fact a condition in $\mathbf{PT}(\varphi)$. First, note that the tree T'' has the same stem as condition T', since each interval I_n after the one containing $\max(st(T'))$ contains one successor from $st(T')$, and we add all the nodes whose maximum is in some I_n with $n \in A$. Denote by s the stem of T'. Now, recall the definition of l_s and note that for all $\{k_1 < \ldots < k_{l_s}\} \in [A]^{l_s}$, with $k_1 > 0$, there are two sequences $r_1 \subseteq r_2 \subseteq \ldots \subseteq r_{l_s}$ and $t_1 \subseteq t_2 \subseteq \ldots \subseteq t_{l_s}$ such that:

1. For all $i \in \{1, 2, \ldots, l_s\}$, $r_i, t_i \in T'$.
2. t_1 is an immediate successor of $s = st(T')$ in T'(in fact, in T'', by the following point).
3. For all $i \in \{1, 2, \ldots, l_s\}$, $t_i \subseteq r_i$ and $\max(t_i), \max(r_i) \in I_{k_i}$.
4. For all $i \in \{1, 2, \ldots, l_s\}$, t_{i+1} is an immediate successor of r_i.
5. For all $i \in \{1, 2, \ldots, l_s\}$, $r_i \in (T)_{m^s_i}$, where $s = st(T')$.

Then note that this implies that for all $m \in F^{T'}_{st(T')}$, the nodes from $(T'')_m$ are all splitting nodes, so we have $F^{T''}_{st(T'')} = F^{T'}_{st(T')}$. Now, for $r \in (T'')_{\max(F^{T''}_{st(T'')})}$ and $k \in succ_{T''}(r)$, if $k \in I_l$ with $l \in A$, then there is $t \in S'$ such that $r^\frown k \subseteq t$ and $\max(t) \in I_l$, by construction of T', so $t \in T''$. Now apply the same argument that works for the stem of T'' to see that $F^{T''}_t = F^{T'}_t$. It is clear how to extend the argument to the whole tree T''.

It follows that T'' is below T and forces $\dot{x}_{gen} \in [T'']$. By construction of T'', it follows that $T'' \Vdash$ "$\dot{x}_{gen}[\omega] \subseteq \bigcup_{n\in A} I_n$", so $T'' \Vdash$ "$\dot{x}_{gen}[\omega] \in \mathcal{G}$".

Finally, to prove that $\mathfrak{u} < \mathfrak{g}$ holds in the model, let $\{\mathcal{G}_\alpha : \alpha \in \omega_1\}$ be a collection of groupwise dense families in $V[G]$. We will find a set $X \in [\omega]^\omega$ which is contained in all of them. By a reflection argument, there is an ω_1-closed subset C of ω_2 such that for all $\alpha \in C$, each family $\mathcal{G}_\beta$ reflects as a groupwise dense family. Fix one of such $\alpha \in C$. Then for all $\beta \in \omega_1$, $\mathcal{G}_\beta \cap V[G_\alpha] \in V[G_\alpha]$ is groupwise dense, so by the previous argument $\dot{x}^\alpha_{gen}[\omega]$(the generic real added by $\dot{\mathbf{Q}}_\alpha$) is almost contained in some $X_\beta \in \mathcal{G}_\beta \cap V[G_\alpha]$, for all $\beta \in \omega_1$. Then $\dot{x}^\alpha_{gen}[\omega] \in \mathcal{G}_\beta$ for all $\beta \in \omega_1$. Then we have $\mathfrak{g} = \omega_2$. The equality $\mathfrak{u} = \omega_1$ follows from Proposition 15 and the p-point preservation theorem along countable support iterations, Theorem 14. □

The following corollary finishes the proof of the main theorem of this chapter.

Corollary 5. *Let $\mathbf{P} = \langle \mathbf{P}_\alpha, \dot{\mathbf{Q}}_\alpha : \alpha \in \omega_2 \rangle$ be a countable support iteration where each iterand is of the form $\mathbf{PT}(\varphi)$, where φ runs over all the lscsm that appear in the intermediate steps, and each lscsm is taken care of cofinally often. Then $\mathbf{P}$ forces the NCF principle, and the following statements hold in this model:*

i) There is no $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$.

ii) There is no weak $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$.

iii) In particular, there is no Hausdorff ultrafilter.

Proof. Denote by $\dot{f}^\alpha_{gen}$ the real $\dot{f}_{gen}$ added by the α-th step of the iteration(see Definition 27).

Since $\mathfrak{u} < \mathfrak{g}$ in the model, the NCF principle holds in it. Let $\mathcal{U}$ be an arbitrary ultrafilter. By item (3) of Theorem 29, there is a p-point $\mathcal{V}$ which is Rudin-Keisler(actually, Rudin-Blass below) below $\mathcal{U}$. Then it suffices to show that $\mathcal{V}$ is not an $\mathscr{I}$-ultrafilter for any F_σ ideal $\mathscr{I}$, that is, for any F_σ ideal $\mathscr{I}$, $\mathscr{I} \leq_{KB} \mathcal{V}^*$. This, togheter with $\mathcal{V}^* \leq_{RB} \mathcal{U}^*$, results in $\mathscr{I} \leq_{KB} \mathcal{U}^*$ for any F_σ ideal $\mathscr{I}$. Fix an F_σ ideal $\mathscr{I}$ and a lscsm φ defining $\mathscr{I}$. By a standard reflection argument, we find $\alpha \in \omega_2$ such that $\mathcal{V} \cap V[G_\alpha]$ is a p-point in $V[G_\alpha]$, where G_α is a generic filter for $\mathbf{P}_\alpha$. By the p-point preservation theorem, $\mathcal{V}$ remains as a p-point in $V[G_\beta]$ for all $\beta \geq \alpha$. Since the iterands repeat a given submeasure cofinally often, there is $\beta > \alpha$ such that $\dot{\mathbf{Q}}_\beta$ is the forcing $\mathbf{PT}(\varphi)$ for the previously fixed lscsm φ. Then we have that $\mathbf{P}_{\beta+1} = \mathbf{P}_\beta * \dot{\mathbf{Q}}_\beta$ forces that $\dot{f}^\beta_{gen}[\mathcal{V}] \cap \mathscr{I} = \emptyset$, so $\mathcal{V}$ is not an $\mathscr{I}$-ultrafilter in $V[G_{\beta+1}]$. Since $\mathcal{V}$ remains the same up to $V[G_{\omega_2}]$, $\mathcal{V}$ is not an $\mathscr{I}$-ultrafilter in $V[G_{\omega_2}]$. □

Regarding Question 2 from Chapter 3, we have the following corollary. Note that since our forcings preserve p-points, and any p-point in the forcing extension has character ω_1, we have that $\mathfrak{r}_\sigma = \omega_1$ holds in our model.

Corollary 6. *It is relatively consistent with* ZFC *that for any F_σ-ideal $\mathscr{I}$ it holds that $\min\{\mathfrak{r}_\sigma, \mathfrak{d}\} < \mathfrak{z}_{fin}(\mathscr{I}) = \mathfrak{d}$. In particular this holds for any summable ideal.*

4.6 Additional remarks.

As pointed out in the introduction, the existence of many classes of $\mathscr{I}$-ultrafilters has been studied by several people. In his paper, Baumgartner investigated the existence of some $\mathscr{I}$-ultrafilters, and provided sufficient conditions for their existence. He worked on p-points, discrete, scattered, measure zero, nowhere dense and ordinal ultrafilters, and established several consistency results about their existence. However, he left open the question about their existence in ZFC-alone.

Let us recall the classical theorem of Ketonen (43) stating that the equality $\mathfrak{d} = \mathfrak{c}$ is equivalent to the generic existence of p-points, and Canjar's theorem (16) establishing the equivalence between generic existence of Ramsey ultrafilters and the equality $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. J. Brendle has several results of the same flavor (12). He has proved that generic existence of nowhere dense ultrafilters is equivalent to the cardinal equality $\mathsf{cof}(\mathcal{M}) = \mathfrak{c}$, and that generic existence of measure zero ultrafilters is equivalent to the equality $\mathsf{cof}(\mathcal{E}, \mathcal{M}) = \mathfrak{c}$. These results were later extended by J. Blobner and J. Brendle in (13), providing a broad study about the generic existence by introducing the cardinal invariant $\mathfrak{ge}(\mathscr{I})$, which serves as a starting point to investigate the generic existence of $\mathscr{I}$-ultrafilters for several ideals $\mathscr{I}$, relating these cardinal invariants to more well known cardinal invariants of the continuum. All of these results provide ZFC-independent statements for the existence of $\mathscr{I}$-ultrafilters. The first example in ZFC was due to O. Guzmán González and M. Hrušák, providing an $F_{\sigma\delta\sigma}$ ideal for which generic existence is a theorem of ZFC. They also proved that for $F_{\sigma\delta}$ ideals, it is consistent that generic existence does not hold, so regarding generic existence, $F_{\sigma\delta\sigma}$ is the simplest complexity that can be obtained in ZFC. However, the question about the existence of an $F_{\sigma\delta}$ ideal $\mathscr{I}$ for which the simple existence of $\mathscr{I}$-ultrafilters is a theorem of ZFC is still unsolved.

J. Flašková has investigated $\mathscr{I}$-ultrafilters for $\mathscr{I}$ being a summable ideal, mainly under assumptions as Martin's Axiom for countable orders or σ-centered orders (28; 29; 30). Recall that $\mathscr{I}$ is a summable ideal if there is a function $g : \omega \to \mathbf{R}^+$ which induces a measure for subsets of ω by summing the values of g over the elements of $A \subseteq \omega$, such that the elements of $\mathscr{I}$ are those with finite measure, that is, $A \in \mathscr{I}$ if and only if $\sum_{n\in A} g(n) < \infty$. It is worth mentioning her ZFC theorem stating the existence of *friendly* $\mathscr{I}_{1/n}$-ultrafilters (29), where $\mathscr{I}_{1/n}$ is the summable ideal defined by $g(n) = 1/(n+1)$ ($\mathcal{U}$ is a *friendly* $\mathscr{I}$-ultrafilter if for any *one to one* $f \in \omega^\omega$, there is $A \in \mathcal{U}$ such that $f[A] \in \mathscr{I}$). Also, she points out a similar theorem of A. Gryzlov for the density zero ideal $\mathcal{Z}$: the existence of friendly $\mathcal{Z}$-ultrafilters is provable in ZFC. The main theorem of this chapter says that Flašková's theorem is optimal when considering the natural extension from friendly to weak $\mathscr{I}$-ultrafilters.

The non-existence of Hausdorff ultrafilters in our model constrasts with the theorem proved by T. Bartoszyński and S. Shelah in the Rational Perfect set model, where Hausdorff ultrafilters are dense in the Rudin-Blass order (5). This situation is not exclusive to the Hausdorff ultrafilters, since Corollary 4 from Chapter 3 says that $\mathscr{I}$-ultrafilters are dense in the Rational Perfect set model whenever $\mathscr{I}$ is an analytic p-ideal.

Chapter 5

Two applications of parametrized diamonds.

In this chapter we show two applications of parametrized diamond principles in order to answer two questions of D. Monk.

A family $\mathcal{B} \subseteq [\omega]^\omega$ is an ideal independent family if no set $X \in \mathcal{B}$ is almost contained in the union of finitely many elements from $\mathcal{B} \setminus \{X\}$.

The cardinal invariant $\mathfrak{s}_{mm}$ is defined as the minimum cardinality of a maximal ideal independent family:

$$\mathfrak{s}_{mm} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a maximal ideal independent family})\} \tag{5.1}$$

It is easy to see that an ideal independent family $\mathcal{B}$ is maximal if and only if for any $X \in [\omega]^\omega$, there is $F \in [\mathcal{B}]^{<\omega}$ such that $X \subseteq^* \bigcup F$ or there are $A \in \mathcal{B}$ and $F \in [\mathcal{B}]^{<\omega}$ such that $A \setminus \bigcup F \subseteq^* X$, so in particular this implies $\mathfrak{r} \leq \mathfrak{s}_{mm}$.

In May 2013, it was asked by Donald Monk in a conference at the Ben-Gurion University of Negev, if $\mathfrak{s}_{mm}$ was equal to $\mathfrak{u}$. Here we answer this question by showing that $\mathfrak{d} \leq \mathfrak{s}_{mm}$. Since in the Rational Perfect set model it holds that $\mathfrak{u} < \mathfrak{d}$, the consistency of the inequality $\mathfrak{u} < \mathfrak{s}_{mm}$ follows. Also, we prove that the diamond principle $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$[1] implies the existence of a maximal ideal independent family of cardinality ω_1. This results were joint work with O. Guzmán González and appeared in a joint paper together with A. W. Miller in (15).

Given a partial order $(\mathbf{P}, \leq)$ with maximal element $1_{\mathbf{P}}$, we say that a family $\mathcal{T}$ is a tree if $1_{\mathcal{P}} \in \mathcal{T}$ and for any $s \in \mathcal{T}$, the set $pred_{\mathcal{T}}(s) = \{r \in \mathcal{T} : r \geq s\}$ is well ordered by the inverse ordered defined by $\geq$. The family of all these trees can be ordered by end-extension, which is defined as $\mathcal{S} \leq \mathcal{T}$ if and only if $\mathcal{S} \subseteq \mathcal{T}$ and for all $s \in S$, $pred_{\mathcal{S}}(s) = pred_{\mathcal{T}}(s)$. It can be easily seen that this order has maximal elements, so we can talk about maximal trees without ambiguity.

[1]We refer the reader to the sections 1.2 and 1.4 Preliminaries chapter for the definition of sequential composition of two cardinal invariants and the parametrized diamond principles.

The cardinal invariant $\mathfrak{tr}$ is defined as the minimum cardinality of a maximal tree on $([\omega]^\omega, \subseteq^*)$,

$$\mathfrak{tr} = \min\{|\mathcal{T}| : \mathcal{T} \subseteq [\omega]^\omega \text{ is a maximal tree}\} \tag{5.2}$$

The following lemma gives a characterization of when a tree $\mathcal{T} \subseteq [\omega]^\omega$ is a maximal tree, and will be used in some sections below. We omit the proof since it is easy.

Lemma 23. *A tree $\mathcal{T} \subseteq [\omega]^\omega$ is a maximal tree if for any $A \in [\omega]^\omega$ one of the following happens:*

1. *There is $X \in \mathcal{T}$ such that $X \subseteq^* A$.*
2. *There are $X, Y \in \mathcal{T}$ such that $A \subseteq^* X \cap Y$.*

D. Monk asked if the size of maximal trees on the boolean algebra $\mathcal{P}(\omega)/\mathsf{fin}$ can be consistently be smaller than the continuum. We anwer this question in the positive by showing that $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ implies the existence of maximal trees with cardinality ω_1. We provide two constructions which differ notably in the shape of the tree constructed. These results are joint work with G. Campero Arena, M. Hrušák and F. E. Miranda Perea, and appeared in (14).

5.1 Ideal independent families.

In this section we answer the question of D. Monk of whether the equality $\mathfrak{u} = \mathfrak{s}_{mm}$ by proving that in fact the inequality $\mathfrak{d} \leq \mathfrak{s}_{mm}$ follows from ZFC.

Theorem 31 (J. C. M., O. Guzmán González, A. W. Miller)**.** $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{s}_{mm}$.

Proof. Given a maximal ideal independent family $\mathcal{I}$, it is easy to see that the following family of sets is a reaping family:

$$\{A \setminus \bigcup F : F \in [\mathcal{I}]^{<\omega} \wedge A \in \mathcal{I} \setminus F\}$$

It remains to prove that $\mathfrak{d} \leq \mathfrak{s}_{mm}$. Assume otherwise that $\mathfrak{s}_{mm} < \mathfrak{d}$, and let $\mathcal{A}$ be a witness for this. Note that $\omega =^* \bigcup \mathcal{A}$, so we can assume that indeed the equality holds. Let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ be such that its union is ω. Define $C_0 = A_0$ and $C_{n+1} = A_{n+1} \setminus \bigcup_{i \leq n} A_i$. For each $F \in [\mathcal{A}]^{<\omega}$ and $B \in \mathcal{A} \setminus (F \cup \{A_i : i < \omega\})$, define a function as follows:

$$\varphi_{F,B}(n) = \min\{k \in \omega : (\exists j \geq n)(C_j \cap B \cap k \setminus \bigcup F \neq \emptyset)\}$$

Since the family $\mathcal{A}$ is ideal independent, the functions $\varphi_{F,B}$ are always well defined. Let h_0 be an increasing function not dominated by

$$\{\varphi_{F,B} \ : \ F \in [\mathcal{A}]^{<\omega}, \ B \in \mathcal{A} \setminus (F \cup \{A_i : i < \omega\})\}.$$

Define $D_n = C_n \setminus h_0(n)$. Now for each $F \in [\mathcal{A}]^{<\omega}$, whenever it is possible, define a function as follows:

$$\tilde{\varphi}_F(n) = \min\{k \in \omega : (\exists j \geq n)(D_j \cap k \setminus \bigcup F \neq \emptyset)\}$$

This is defined for n, otherwise

$$\bigcup_{j\geq n} D_j = \bigcup_{j\geq n}(C_j \setminus h_0(j)) \subseteq \bigcup F$$

But then for some $j \geq n$ such that $A_j \notin F$ we would have

$$A_j \subseteq^* \bigcup_{i<j} A_i \cup \bigcup F$$

which contradicts that $\mathcal{A}$ is an ideal independent family.

Let $h_1 > h_0$ be an increasing function not dominated by any totally defined $\tilde{\varphi}_F$ for $F \in [\mathcal{A}]^{<\omega}$ and such that $C_n \cap [h_0(n), h_1(n))$ is nonempty for all n.

Let

$$Y = \bigcup_{n\in\omega}(C_n \cap [h_0(n), h_1(n))) = \bigcup_{n\in\omega} D_n \cap h_1(n)$$

Let's see that $\mathcal{A} \cup \{Y\}$ is an ideal independent family.

Claim 1. For all $F \in [\mathcal{A}]^{<\omega}$, $Y \not\subseteq^* \bigcup F$.

If the function $\tilde{\varphi}_F$ is not defined, then $Y \cap \bigcup F$ is finite. Otherwise, by the definition of the function $\tilde{\varphi}_F$, if $\tilde{\varphi}_F(n) \leq h_1(n)$, then for some $j \geq n$ we have $D_j \cap \tilde{\varphi}_F(n) \setminus \bigcup F \neq \emptyset$, which implies

$$\emptyset \neq D_j \cap h_1(n) \setminus \bigcup F \subseteq D_j \cap h_1(j) \setminus \bigcup F \subseteq Y.$$

Since this happens for infinitely many j and the family $\{D_j : j \in \omega\}$ is disjoint, we are done.

Claim 2. For any $F \in [\mathcal{A}]^{<\omega} \setminus \{\emptyset\}$ and $B \in \mathcal{A} \setminus F$, we have $B \not\subseteq^* Y \cup \bigcup F$.

If $B = A_n$ for some n this is clear. Otherwise, by the definition of $\varphi_{F,B}$ and the choice of h_0, we have that if $\varphi_{F,B}(n) \leq h_0(n)$, then for some $j \geq n$,

$$\emptyset \;\neq\; C_j \cap B \cap \varphi_{F,B}(n) \setminus \bigcup F \;\subseteq\; C_j \cap B \cap h_0(j) \setminus \bigcup F.$$

If $m \in C_j \cap B \cap h_0(j) \setminus \bigcup F$, then $m \notin Y \cup \bigcup F$. Since this happens infinitely many times, we are done. □

The following proposition shows that the existence of ideal independent families with cardinality smaller than the continuum is consistent. We use a parametrized diamond principle to produce an ideal independent family of cardinality ω_1. Since this principle holds in the Sack's model, in the Sack's model there is an ideal indepenedent family of cardinality smaller than the continuum.

Proposition 16 (O. Guzmán González, J. Cancino Manríquez). $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ *implies* $\mathfrak{s}_{mm} = \omega_1$.

Proof. We need to define a function F into $([\omega]^\omega)^\omega \times Bor((\omega^\omega)^{[\omega]^\omega})$ such that for all $\alpha \in \omega_1$, $F \restriction \alpha$ is in $L(\mathbb{R})$. For each $\alpha < \omega_1$, let $e_\alpha : \omega \to \alpha$ be an enumeration of α in $L(\mathbb{R})$[2]. By a suitable coding, we can assume that the domain of F is the set

$$\bigcup_{\alpha\in\omega_1} [\omega]^\omega \times ([\omega]^\omega)^\alpha$$

Given $(A, \vec{\mathcal{I}}) \in [\omega]^\omega \times ([\omega]^\omega)^\alpha$ proceed as follows. If $\vec{\mathcal{I}}$ is not an ideal independent family, define $F(A, \vec{\mathcal{I}}) = (\omega, e)$, where $e(X)$ for $X \in [\omega]^\omega$ is the enumeration of X. Otherwise, define $B_n^{\vec{\mathcal{I}}} = I_{e_\alpha(n)} \setminus \bigcup_{i<n} I_{e_\alpha(i)}$. For each n, let $Z_n^{\vec{\mathcal{I}}} \subseteq B_n^{\mathcal{I}}$ be an infinite subset such that for all $\beta \neq e_\alpha(n)$, $Z_n^{\mathcal{I}} \cap I_\beta$ is finite [3], and let $\varphi_{\vec{\mathcal{I}},n}$ be a recursive enumeration of $Z_n^{\vec{\mathcal{I}}}$. Then define $A_n = \varphi_{\vec{\mathcal{I}},n}^{-1}[Z_n^{\vec{\mathcal{I}}} \cap A]$. Now define a function $f_{A,\vec{\mathcal{I}}} : [\omega]^\omega \to \omega^\omega$ as follows: if $X \in [\omega]^\omega$ reaps A_n for all n, then define

$$f_{A,\vec{\mathcal{I}}}(X)(n) = \min\{k \in \omega : X \setminus k \subseteq A_n \vee (X \setminus k) \cap A_n = \emptyset\}$$

Otherwise define $f_{A,\vec{\mathcal{I}}}(X)$ to be the identity function. Finally, the value of F in $(A, \vec{\mathcal{I}})$ is given by $F(A, \vec{\mathcal{I}}) = (\langle A_n : n \in \omega\rangle, f_{A,\vec{\mathcal{I}}})$. Let $g : \omega_1 \to [\omega]^\omega \times \omega^\omega$ be a $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$-guessing sequence for F. We can assume that for all α the set A_α in $g(\alpha) = (A_\alpha, h_\alpha)$ is coinfinite. Recursively define an ideal independent family as follows:

1) Start with a partition of ω into infinitely many infinite sets $\vec{\mathcal{I}}_\omega = \langle I_n : n \in \omega\rangle$.

2) Suppose we have defined $\vec{\mathcal{I}}_\alpha = \langle I_\beta : \beta < \alpha\rangle$. Now define I_α as follows:

$$I_\alpha = \bigcup_{n\in\omega} B_n^{\vec{\mathcal{I}}_\alpha} \setminus \varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)]$$

Let $\vec{\mathcal{I}}_{\alpha+1}$ be the family $\langle I_\beta : \beta \leq \alpha\rangle$. Finally, let $\mathcal{I} = \langle I_\alpha : \alpha \in \omega_1\rangle$ be the family obtained by the above recursion. Let's see that $\mathcal{I}$ is a witness for $\mathfrak{s}_{mm}$.

Claim 1. $\mathcal{I}$ is an ideal independent family. We proceed by induction on $\alpha \in \omega_1$. Clearly $\mathcal{I}_\omega$ is ideal independent. Assume $\vec{\mathcal{I}}_\alpha$ is ideal independent. Then $\vec{\mathcal{I}}_{\alpha+1}$ is ideal independent:

a) For all $H \in [\alpha]^{<\omega}$, $I_\alpha \nsubseteq^* \bigcup H$. Let $n \in \omega$ be such that H is contained in $\{e_\alpha(0), \ldots, e_\alpha(n)\}$, so $\bigcup_{\beta\in H} I_\beta \subseteq \bigcup_{i\leq n} B_i^{\vec{\mathcal{I}}_\alpha}$. By the definition of I_α, $I_\alpha \setminus \bigcup_{i\leq n} B_i^{\vec{\mathcal{I}}_\alpha}$ is infinite.

b) For all $H \in [\alpha]^{<\omega}$ and $\beta \in \alpha \setminus H$, $I_\beta \nsubseteq^* I_\alpha \cup \bigcup_{\gamma\in H} I_\gamma$. Let n be such that $\beta = e_\alpha(n)$. By the choice of $Z_n^{\vec{\mathcal{I}}_\alpha}$, we have that for any $\gamma \in \alpha \setminus \{\beta\}$, $Z_n^{\vec{\mathcal{I}}_\alpha} \cap I_\gamma$ is finite, so in particular, $Z_n^{\vec{\mathcal{I}}_\alpha} \cap \bigcup_{\gamma\in H} I_\gamma$ is finite. Also by the construction of I_α, $B_n^{\vec{\mathcal{I}}_\alpha} \cap I_\alpha \cap$

[2] Recall that every countable ordinal can be embedded into $\mathbb{Q}$, so every countable ordinal can be seen as a subset of $\mathbb{Q}$, and by definition of $L(\mathbb{R})$ any of such subsets codifying a countable ordinal is an element of $L(\mathbb{R})$, so the function e_α can be found in $L(\mathbb{R})$.

[3] $Z_n^{\vec{\mathcal{I}}} \subseteq B_n^{\mathcal{I}}$ should be found in a recursive way and should depend only on $\vec{\mathcal{I}}$

$\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)]$ is finite. This both facts together give $\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)] \setminus I_\alpha \cup \bigcup_{\gamma\in H} I_\gamma$ is infinite. Since $\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)] \setminus \left(I_\alpha \cup \bigcup_{\gamma\in H} I_\gamma\right) \subseteq I_\beta \setminus \left(I_\alpha \cup \bigcup_{\gamma\in H} I_\gamma\right)$, we are done.

Claim 2. $\mathcal{I}$ is maximal. Pick any $X \in [\omega]^\omega$. If g guesses $(X, \langle I_\alpha : \alpha \in \omega_1\rangle)$ in γ, then we have that A_γ σ-reaps $\langle X_n : n \in \omega\rangle$ and h_γ almost dominates the function $l = f_{X,\mathcal{I}_\gamma}(A_\gamma)$. There are two cases:

i) There are infinitely many $n \in \omega$ such that $A_\gamma \subseteq^* X_n$. Pick one n such that $l(n) \leq h_\gamma(n)$. Then $A_\gamma \setminus h_\gamma(n) \subseteq X_n$, so $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq X \cap B_n^{\vec{\mathcal{I}}_\gamma}$. Then by the definition of I_γ, $B_n^{\vec{\mathcal{I}}_\gamma} \subseteq I_\gamma \cup \varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq I_\gamma \cup X$, which implies $I_{e_\gamma(n)} \subseteq X \cup I_\gamma \cup \bigcup_{i<n} I_{e_\gamma(n)}$.

ii) For almost all $n \in \omega$ $A_\gamma \subseteq^* \omega \setminus X_n$. Then for almost all n, $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq Z_n^{\vec{\mathcal{I}}_\gamma} \setminus X$, so for almost all n, $X \cap Z_n^{\vec{\mathcal{I}}_\gamma} \subseteq I_\gamma$, and for finitely many n, $A_\gamma \subseteq^* X_n$, so $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq^* Z_n^{\vec{\mathcal{I}}} \cap X \subseteq B_n^{\vec{\mathcal{I}}_\gamma} \cap X$, which implies $B_n^{\vec{\mathcal{I}}_\gamma} \setminus X \subseteq^* B_n^{\vec{\mathcal{I}}_\gamma} \setminus \varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq I_\gamma$. Putting all this together we have that $X \subseteq^* I_\gamma \cup \bigcup_{i\leq k} B_i$, for some $k \in \omega$.

□

5.2 Maximal trees.

In this section we addrress the second question of D. Monk by proving that $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ implies the existence of a maximal tree on $\mathcal{P}(\omega)/\mathsf{fin}$ of cardinality ω_1. The first construction was given by M. Hrušák, and it looks like a broom of height ω_1 and width ω_1. The second construction uses the same diamond principle, and gives a maximal tree of height ω_1 and width ω_1.

Theorem 1 (M. Hrušák). *$\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ implies that there is a maximal tree of size ω_1.*

Proof. First, for every $\alpha \in \omega_1$, fix a bijection $e_\alpha : \omega \to \alpha$, and for a set $A \in [\omega]^\omega$ and a countable $\subseteq^*$-decreasing sequence $\vec{X}$ of subsets of ω such that $X_0 \subseteq^* A$, denote by $P(A, \vec{X}) \subseteq A$ a pseudo-intersection found in a Borel way.

By a suitable coding, we can assume that the domain of the function F is $\bigcup_{\alpha\in\omega_1}(([\omega]^\omega)^\alpha)^\alpha \times [\omega]^\omega$. For $(\langle \vec{X}_\beta : \beta < \alpha\rangle, Z) \in (([\omega]^\omega)^\alpha)^\alpha \times [\omega]^\omega$, we define $F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Z)$ as follows (where for $\beta < \alpha, \vec{X}_\beta = \langle X_{\beta,\gamma} : \gamma < \alpha\rangle$):

1. If $\langle X_{\beta,0} : \beta < \alpha\rangle$ is not an AD family or does not cover ω, or if one of the sequences $\vec{X}_\beta$ is not a $\subseteq^*$-decreasing, let

$$F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Z) = \langle \bar{\omega}, \overline{Id}\rangle.$$

Here $\bar{\omega}$ denotes sequence which takes constant value ω and $\overline{Id}$ denotes a function from $[\omega]^\omega$ to ω^ω which takes every set to the identity function. In other words, this is the irrelevant case.

2. If $\langle X_{\beta,0} : \beta < \alpha\rangle$ is an AD family and covers ω, and every $\vec{X}_\beta$ is a $\subseteq^*$-decreasing sequence, define $A_0 = X_{e_\alpha(0),0}$, and for $n > 0$, $A_n = X_{\beta,0} \setminus \bigcup_{i<n} X_{e_\alpha(i),0}$. Let $H_n : \omega \to P(A_n, \vec{X}_{e_\alpha(n)})$ be the increasing enumeration of $P(A_n, \vec{X}_{e_\alpha(n)})$. Then let $\vec{Z} = \langle Z_n : n \in \omega\rangle$, where

$$Z_n = H_n^{-1}[P(A_n, \vec{X}_{e_\alpha(n)}) \cap Z].$$

Now, define a function $\varphi_{\vec{Z}} : [\omega]^\omega \to \omega^\omega$ as follows:

(a) If $A \in [\omega]^\omega$ does not σ-reap $\vec{Z}$, then define $\varphi_{\vec{Z}}(A) = Id$.

(b) If $A \in [\omega]^\omega$ σ-reaps $\vec{Z}$, then define $\varphi_{\vec{Z}}(A) = h_{\vec{Z},A}$, defined by $h_{\vec{Z},A}(n) = \min\{k \in \omega : Z_n \setminus A \subseteq k \text{ or } Z_n \cap A \subseteq k\}$.

Finally, define $F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Z) = (\vec{Z}, \varphi_{\vec{Z}})$.

Let $g : \omega_1 \to [\omega]^\omega \times \omega^\omega$ be a guessing function for the function F. Let $D_\alpha \in [\omega]^\omega$ and $h_\alpha \in \omega^\omega$ be such that $g(\alpha) = (D_\alpha, h_\alpha)$. We are going to construct sequences $\langle \vec{X}_\alpha : \alpha \in \omega_1\rangle$, where $\vec{X}_\alpha = \langle X_{\alpha,\gamma} : \gamma \in \omega_1\rangle$ is a $\subseteq^*$-decreasing sequence of infinite subsets of ω and such that $\{X_{\alpha,0} : \alpha \in \omega_1\}$ is an AD family. The construction is as follows:

1. Start with a sequence $\langle \vec{X}_n : n \in \omega\rangle$ such that $\vec{X}_n$ is a $\subseteq^*$-decreasing sequence, and $\{X_{n,0} : n \in \omega\}$ is a partition of ω into infinite sets.

2. Suppose $\vec{X}_\beta = \langle X_{\beta,\gamma} : \gamma < \alpha\rangle$ has been constructed for all $\beta < \alpha$. Define $A_0^\alpha = X_{e_\alpha(0),0}$, and for $n > 0$, $A_n^\alpha = X_{e_\alpha(n)} \setminus \bigcup_{i<n} X_{e_\alpha(i),0}$. Let H_n^α be the increasing enumeration of $P(A_n^\alpha, \vec{X}_{e_\alpha(n)})$. Then define $X_{e_\alpha(n),\alpha} = X_{e_\alpha(n),\alpha+1} = H_n^\alpha[D_\alpha]$. Now, for $n \in \omega$, let $a_n^0, a_n^1 \in H_n^\alpha[D_\alpha \setminus h_\alpha(n)]$ be distinct natural numbers and define $X_{\alpha,0} = \{a_n^0 : n \in \omega\}$ and $X_{\alpha+1,0} = \{a_n^1 : n \in \omega\}$. Finally, let $\vec{X}_\alpha$ and $\vec{X}_{\alpha+1}$ be $\subseteq^*$-decreasing sequences of length $\alpha + 2$ whose first element are $X_{\alpha,0}$ and $X_{\alpha+1,0}$, respectively.

Now for every infinite $\alpha \in \omega_1$, define two sets B_α^0, B_α^1 as follows:

$$B_\alpha^0 = X_{\alpha,0} \cup \bigcup_{n\in\omega} A_n^\alpha \setminus H_n^\alpha[D_\alpha \setminus h_\alpha(n)];$$
$$B_\alpha^1 = X_{\alpha+1,0} \cup \bigcup_{n\in\omega} A_n^\alpha \setminus H_n^\alpha[D_\alpha \setminus h_\alpha(n)].$$

Claim 1. *The family $\{B_\alpha^0, B_\alpha^1 : \alpha \in \omega_1\}$ is an incomparable family.* For a fixed α it is clear that B_α^0 and B_α^1 are incomparable. For $\beta < \alpha$, note that $X_{\beta+i,0} \subseteq^* B_\beta^i$, but $X_{\beta,0} \not\subseteq^* B_\alpha^j$. On the other hand, note that $X_{\alpha+i,0} \not\subseteq^* B_\beta^j$.

Now define the following tree $\mathcal{T}$:

$$\mathcal{T} = \bigcup_{\alpha\in\omega_1} \{X_{\alpha,\beta} : \beta \in \omega_1\} \cup \{B_\alpha^0, B_\alpha^1 : \alpha \in [\omega, \omega_1)\}$$

Claim 2. *$\mathcal{T}$ is a maximal tree.* Let $Z \in [\omega]^\omega$ be an arbitrary set. Let $\alpha \in \omega_1$ such that g guesses the branch $(\langle \vec{X}_\alpha : \alpha \in \omega_1\rangle, Z)$ in α. Then $F(\langle \vec{X}_\beta : \beta < \alpha\rangle, Z) = (\vec{Z}, \varphi_{\vec{Z}})$ is dominated by $g(\alpha) = (D_\alpha, h_\alpha)$, which means that for all $n \in \omega$, D_α σ-reaps $\vec{Z}$ and $\varphi_{\vec{Z}}(D_\alpha) \leq h_\alpha$. There are two cases:

Case 1. There is $n \in \omega$ such that $D_\alpha \subseteq^* Z_n$. Then

$$X_{e_\alpha(n),\alpha} = H^\alpha_n[D_\alpha] \subseteq^* H^\alpha_n[Z_n] = Z \cap P(A^\alpha_n, \vec{X}_{e_\alpha(n)} \restriction \alpha) \subseteq Z$$

Case 2. For all $n \in [\omega]$, $D_\alpha \cap Z_n$ is finite. Then for all $n \in \omega$,

$$Z_n \cap D_\alpha \setminus h_\alpha(n) = \emptyset$$

This implies that for all n,

$$H^\alpha_n[D_\alpha \setminus h_\alpha(n)] \cap Z \cap A^\alpha_n = \emptyset$$

Which in turn implies:

$$Z \subseteq \bigcup_{n\in\omega} A^\alpha_n \setminus H^\alpha_n[D_\alpha \setminus h_\alpha(n)]$$

This implies that $Z \subseteq B^0_\alpha \cap B^1_\alpha$.

□

Theorem 2. $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ *implies that there is a tree* $\mathcal{T} \subseteq [\omega]^\omega$ *of height* ω *and size* ω_1 *which is maximal both as a subtree of* $([\omega]^\omega, \subseteq^*)$*, and as a subtree of* $\mathcal{P}(\omega)$*.*

Proof. Recall that a tree $\mathcal{T} \subseteq [\omega]^\omega$ is an ideal-tree if for any $A \in \mathcal{T}$, the family $\{A \cap B : B \in \mathcal{T} \wedge A \not\subseteq^* B\}$ generates a proper ideal $\mathcal{I}_A$ on A. We shall, in fact, be constructing and ideal-tree.

Given $f \in 2^{<\omega_1}$, let us say that f *codes a family of sets* $\mathcal{F}$ if for all $X \in \mathcal{F}$ there is a limit ordinal $\alpha \in dom(f)$ such that for $n \in \omega$, $n \in X$ if and only if $f(\alpha + n) = 1$. For each limit $\alpha \in \omega_1$ fix a bijection $e_\alpha : \omega \to lim(\alpha)$. If α is limit and $f \in 2^\alpha$ codes an ideal-tree $\mathcal{T}$, let $\{A_n : n \in \omega\}$ be the enumeration of $\mathcal{T}$ given by

$$A_n = \{m : f(e_\alpha(n) + m) = 1\}.$$

Also, in this proof, for a given set $X \in [\omega]^\omega$, the symbol X plays two roles according to the context: it denotes the set X, and also denotes the increasing enumeration of X, that is, for $n \in \omega$, $X(n)$ is the nth element of X.

Now, it is easy, yet tedious, to show that there is a Borel function $H : 2^\alpha \to ([\omega]^\omega)^\omega$ such that if f codes an ideal-tree then $H(f) = \langle Z_n : n \in \omega\rangle$ is such that

1. $\{Z_n : n \in \omega\}$ is pairwise disjoint
2. $Z_n \subseteq A_n$, and
3. $Z_n \cap I$ is finite for every $I \in \mathcal{I}_{A_n}$.

Having fixed all that, define a function $F : [\omega]^\omega \times 2^{<\omega_1} \to [\omega]^\omega \times \omega^\omega$ as follows[4]:

[4] A very simple coding turns such a function into a function with domain 2^{ω_1} - use the first ω bits to code the first coordinate of F and the rest on the second coordinate.

1. If $f \in 2^\alpha$ does not code an ideal-tree, or if α is not a limit ordinal, let $F(X, f) = (\bar{\omega}, \overline{Id})$.

2. If α is limit and $f \in 2^\alpha$ codes an ideal-tree $\mathcal{T}$, let $\{A_n : n \in \omega\}$ be the enumeration of $\mathcal{T}$ given above and let $\{Z_n : n \in \omega\}$ be the pairwise disjoint refinement of $\{A_n : n \in \omega\}$ given by $H(f)$. Furthermore, let $Y_n = Z_n^{-1}[Z_n \cap X]$, and define a function $\varphi_{(X,f)} : [\omega]^\omega \to \omega^\omega$ by:

   (a) If $W \in [\omega]^\omega$ does not σ-reap $\langle Y_n : n \in \omega \rangle$, let $\varphi_{(X,f)}(W) = Id$.

   (b) If $W \in [\omega]^\omega$ does σ-reap $\langle Y_n : n \in \omega \rangle$, let

   $$\varphi_{(X,f)}(W)(n) = \min\{k \in \omega : \ W \setminus Y_n \subseteq k \text{ or } W \cap Y_n \subseteq k\}.$$

   Then define $F(X, f) = (\langle Y_n : n \in \omega \rangle, \varphi_{(X,f)})$.

Let $g : \omega_1 \to [\omega]^\omega \times \omega^\omega$ be a guessing function for F. For $\alpha \in \omega_1$, let $X_\alpha \in [\omega]^\omega$ and h_α be such that $g(\alpha) = (X_\alpha, h_\alpha)$. For every $\alpha \in \omega_1$, let $D_\alpha \subseteq X_\alpha$ be an infinite co-infinite subset in X_α. Recursively construct three sequences $\langle \mathcal{T}_\beta : \beta \in \omega_1 \rangle$, $\langle f_\beta : \beta \in \omega_1 \rangle$ and $\langle \alpha_\beta : \beta \in \omega_1 \rangle$ such that:

1. $\langle \mathcal{T}_\beta : \beta \in \omega_1 \rangle$ is a sequence of countable ideal-trees.

2. For all β, $f_\beta \in 2^{\alpha_\beta}$ and codes the tree $\mathcal{T}_\beta$.

3. $\{\alpha_\beta : \beta \in \omega_1\}$ is an increasing continuous sequence of countable ordinals.

4. For all $\beta \in \omega_1$, $f_\beta \subseteq f_{\beta+1}$.

The construction is as follows:

1. (*Base step*:) Start with a countable ideal-tree $\mathcal{T}_0$ of heigth ω, such that $\omega \in \mathcal{T}_0$, the successors of every $A \in \mathcal{T}$ form an almost disjoint family of infinite subsets of A such that for any finite $F \subseteq \omega$, there are incomparable $t_0, t_1 \in \mathcal{T}_0$ such that $F \subseteq t_0 \cap t_1$. Let $f_0 \in 2^{\alpha_0}$ code $\mathcal{T}_0$.

2. (*Successor step*:) Suppose that the ideal-tree $\mathcal{T}_\beta$ has been defined, enumerated as above as $\{A_n : n \in \omega\}$, and coded by an $f_\beta \in 2^{\alpha_\beta}$. Let

   $$B_\beta = \omega \setminus \bigcup_{n \in \omega} Z_n[X_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)].$$

   Let $m \in \omega$ be such that $A_m = \omega$ (in the fixed enumeration of $\mathcal{T}_\beta$). Let $C_0, C_1 \subseteq Z_m \setminus Z_m[D_{\alpha_\beta}]$ be disjoint sets such that $\omega \setminus (C_0 \cup C_1 \cup B_\beta \cup Z_m[D_{\alpha_\beta}])$ is infinite. Then define

   $$\mathcal{T}_{\beta+1} = \mathcal{T}_\beta \cup \{Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] : n \in \omega\} \cup \{B_\beta \cup C_0, B_\beta \cup C_1\},$$

   and let $f_{\beta+1} \in 2^{\alpha_{\beta+1}}$ be a sequence extending f_β coding $\mathcal{T}_{\beta+1}$.

3. (*Limit step*:) If β is a limit ordinal and the trees $\mathcal{T}_\gamma$ have been defined for all $\gamma < \beta$, let $\mathcal{T}_\beta = \bigcup_{\gamma<\beta} \mathcal{T}_\gamma$, $f_\beta = \bigcup_{\gamma<\beta} f_\gamma$ and $\alpha_\beta = \sup\{\alpha_\gamma : \gamma < \beta\}$. Note that this way f_β codes $\mathbb{T}_\beta$.

(The sets D_α are used to prove that in every step of the recursion the trees $\mathbb{T}_{\alpha+1}$ are ideal-trees).
Finally, let $\mathcal{T} = \bigcup_{\alpha<\omega_1} \mathcal{T}_\alpha$, and let $f = \bigcup_{\alpha\in\omega_1} f_\alpha \in 2^{\omega_1}$ be the branch that codes all of $\mathcal{T}$.

Obviously, $\mathcal{T}$ is a tree, being an increasing union of trees. Also no finite set can be added to $\mathcal{T}$ by the construction of $\mathcal{T}_0$.

Claim. *$\mathcal{T}$ is a maximal tree in $\mathcal{P}(\omega)$.* Let $X \in [\omega]^\omega$ be arbitrary. Since g guesses every branch stationarily often, there is β such that g guesses (X, f) in α_β. Then $\mathcal{T}_\beta$ is coded by $f \restriction \alpha_\beta = f_\beta$. Consequently, D_β σ-reaps $\langle Z_n^{-1}[Z_n \cap X] : n \in \omega\rangle$ and $\varphi_{(X,f\restriction\alpha_\beta)}(X_{\alpha_\beta})(n) \leq h_{\alpha_\beta}(n)$ for every $n \in \omega$.

If there is $n \in \omega$ such that $X_{\alpha_\beta} \subseteq^* Z_n^{-1}[Z_n \cap X]$, then

$$Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] \subseteq Z_n \cap X.$$

(Recall that $Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] \in \mathcal{T}$).

If for all $n \in \omega$, $X_{\alpha_\beta} \cap Z_n^{-1}[Z_n \cap X]$ is finite, then for all $n \in \omega$,

$$D_{\alpha_\beta} \cap Z_n^{-1}[Z_n \cap X] \subseteq h_{\alpha_\beta}(n).$$

This implies that for all $n \in \omega$,

$$Z_n[D_{\alpha_\beta} \setminus h_{\alpha_\beta}(n)] \cap X = \emptyset$$

which in turn implies,

$$X \subseteq \omega \setminus \bigcup_{n\in\omega} Z_n[D_\beta \setminus h_{\alpha_\beta}(n)] = B_\beta$$

By the construction of $\langle \mathcal{T}_\beta : \beta \in \omega_1\rangle$, $X \subseteq (B_\beta \cup C_0) \cap (B_\beta \cup C_1)$, both of which are elements of $\mathcal{T}$.

To finish the proof note that, by the construction, $\mathcal{T}$ has height ω. □

Open questions.

This chapter is a compilation of several questions that were raised on the research of the results previously presented.

In relation to $\mathscr{I}$-ultrafilters, the following question is of interest:

Question 16. *Is there, in* ZFC*, an* $F_{\sigma\delta}$ *ideal for which* $\mathscr{I}$*-ultrafilters exist?*

Another question that arises by the results of Chapter 4, is the following:

Question 17. *How can we characterized the pairs of* F_σ *ideals* $\mathscr{I}$ *and* $\mathscr{J}$ *such that it is consistent that* $\mathscr{I}$*-ultrafilters exist while* $\mathscr{J}$*-ultrafilters do not exist?*

Some examples of such pairs of ideals are the following:

1. $\mathcal{ED}$ and $\mathcal{G}_{fc}$.
2. $\mathcal{ED}$ and $\mathscr{I}_{1/(n+1)}$.
3. $\mathcal{ED}_{fin}$ and $\mathcal{G}_{fc}$.
4. $\mathcal{ED}_{fin}$ and $\mathscr{I}_{1/(n+1)}$.

We hope that a deeper understanding of the forcings $\mathbf{PT}(\varphi)$ and the kind of reals they add serve to answer the **Question 18.**

Also, there is a long standing classical question regarding rapid ultrafilters and q-points:

Question 18. *Is it consistent that rapid ultrafilters exist, but there is no q-point?*

We hope that results from Chapter 4 help in driving the answer of this question.

We have seen in the results of chapters 3 and 4 that for any F_σ p-ideal $\mathscr{I}$, it holds that $\min\{\mathfrak{r}_\sigma, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathscr{I}) \leq \mathfrak{d}$, with both strict inequalities being consistent for all F_σ p-ideals at the same time. In relation to this we have the following question:

Question 19. *Is it true that for any two* F_σ *p-ideals* $\mathscr{I}$ *and* $\mathscr{J}$*, it holds that* $\mathfrak{z}_{fin}(\mathscr{I}) = \mathfrak{z}_{fin}(\mathscr{J})$ *?*

Let us consider $\mathcal{C}$ a class of ideals and let us define the $\mathfrak{z}_{fin}$-spectrum of the class $\mathcal{C}$ as follows

$$\mathfrak{z}_{fin}\text{-}spec(\mathcal{C}) = \{\mathfrak{z}_{fin}(\mathscr{I}) : \mathscr{I} \in \mathcal{C}\}$$

Let us denote by $\mathbf{F}_\sigma$ the class of F_σ ideals, by $\mathbf{F}_\sigma\mathbf{p}$ the class of F_σ p-ideals, $\mathbf{An_p}$ the class of analytic p-ideals, and $\mathbf{Bo}$ the class of Borel ideals. Then we can ask the following:

Question 20. *How comlpex can be $\mathfrak{z}_{fin}$-spec$(\mathcal{C})$ for $\mathcal{C} \in \{\mathbf{F}_\sigma, \mathbf{F}_\sigma\mathbf{p}, \mathbf{An_p}, \mathbf{Bo}\}$?*

Note that the results presented in this thesis imply the following possibilities are consistent (besides the trivial case when CH holds):

1. $\mathfrak{z}_{fin}$-*spec*$(\mathbf{F}_\sigma) = \{\omega_1, \omega_2\}$.
2. $\mathfrak{z}_{fin}$-*spec*$(\mathbf{F}_\sigma) = \{\omega_2\}$.
3. $\mathfrak{z}_{fin}$-*spec*$(\mathbf{F}_\sigma\mathbf{p}) = \{\omega_1\}$.
4. $\mathfrak{z}_{fin}$-*spec*$(\mathbf{F}_\sigma\mathbf{p}) = \{\omega_2\}$.
5. $\mathfrak{z}_{fin}$-*spec*$(\mathbf{An}_p) = \{\omega_1, \omega_2\}$.
6. $\mathfrak{z}_{fin}$-*spec*$(\mathbf{An}_p) = \{\omega_1\}$.

In the same direction, we ask the following:

Question 21. *Is it consistent that there is a countable family $\langle \mathscr{I}_n : n \in \omega \rangle$ of Borel or analytic ideals such that for all $n \in \omega$ it holds that $\mathscr{I}_{n+1} \leq_K \mathscr{I}_n$, and $\mathfrak{z}_{fin}(\mathscr{I}_n) < \mathfrak{z}_{fin}(\mathscr{I}_{n+1})$?*

Let us recall that Theorem 16 says that whenever $\mathscr{I}$ is an ideal for which there is $\varphi : [\omega]^n \to k$ for some $n, k \in \omega$ such that all the φ-monochromatic sets belong to the ideal, then $\mathfrak{z}_{fin}(\mathscr{I}) \leq \max\{\mathfrak{r}_\sigma, \mathfrak{d}\}$. This lead us to ask the following:

Question 22. *Is there an ideal $\mathscr{I}$ such that* ZFC *proves* $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\} \leq \mathfrak{z}_{fin}(\mathscr{I})$*?*

Question 23. *Is it consistent that there is a Borel or analytic ideal $\mathscr{I}$ such that* $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\} < \mathfrak{z}_{fin}(\mathscr{I})$*?*

Next we reproduce the questions mentioned in the last section from Chapter 3.

Definition 30. *We say that $\mathscr{I}$ is Laflamme-Zhu if there is an ultrafilter $\mathcal{U}$ all of whose Rudin-Blass predecessors are such that its dual ideal is Katětov-Blass above of $\mathscr{I}$. We say that $\mathscr{I}$ is trivially Laflamme-Zhu if $\mathscr{I} <_{KB} \mathcal{ED}_{fin}$.*

Question 24. *Does there exist a Borel or analytic tall ideal other than $\mathcal{ED}_{fin}$ which is non-trivially Laflamme-Zhu?*

Question 25. *Is there a critical ideal $\mathscr{I}$ for the Borel (analytic) Laflamme-Zhu ideals, i. e., such that any ideal $\mathscr{J}$ is Laflamme-Zhu if and only if $\mathscr{J} \leq_{KB} \mathscr{I}$?*

Question 26. *In case there is an analytic ideal $\mathscr{I} \not\leq_{KB} \mathcal{ED}_{fin}$ which is Laflamme-Zhu, let $\mathbf{LZ}_{Borel}$ and $\mathbf{LZ}_{Analytic}$ be the families of all Borel and analytic ideals which are Laflamme-Zhu, respectively. What is the structure of the Katětov-Blass order restricted to such classes?*

Regarding the results from Chapter 5, the following questions remain open:

Question 27. *Is* $\mathfrak{d} \leq \mathfrak{tr}$*?*

Question 28. *Is* $\mathfrak{s}_{mm} < \mathfrak{u}$ *consistent? What about* $\mathfrak{i} < \mathfrak{s}_{mm}$*?*

Bibliography

[1] Uri Abraham. *Proper Forcing*, pages 333–394. 12 2009. `doi:10.1007/978-1-4020-5764-9_6`.

[2] Jason Aubrey. Combinatorics for the dominating and unsplitting numbers. *The Journal of Symbolic Logic*, 69(2):482–498, 2004. URL: `http://www.jstor.org/stable/30041739`.

[3] B. Balcar, F. Hernández, and M. Hrušák. Combinatorics of dense subsets of the rationals. *Fundamenta Mathematicae*, 183, 01 2004. `doi:10.4064/fm183-1-4`.

[4] T. Bartoszyński and Haim Judah. Set Theory: On the Structure of the Real Line. 1995.

[5] Tomek Bartoszyński and Saharon Shelah. On the density of Hausdorff ultrafilters. In *Logic Colloquium 2004*, volume 29 of *Lect. Notes Log.*, pages 18–32. Assoc. Symbol. Logic, Chicago, IL, 2008. arXiv: math/0311064.

[6] James E. Baumgartner. Ultrafilters on ω. *The Journal of Symbolic Logic*, 60(2):624–639, 1995.

[7] Michael Benedikt. Ultrafilters Which Extend Measures. *Journal of Symbolic Logic*, 63(2):638–662, 1998. `doi:http://projecteuclid.org/euclid.jsl/1183745526`.

[8] Andreas Blass. Near coherence of filters, I: Cofinal Equivalence of Models o Arithmetic. *Notre Dame Journal of Formal Logic*, 27(4):579–591, 1986. `doi:10.1305/ndjfl/1093636772`.

[9] Andreas Blass. *Combinatorial Cardinal Characteristics of the Continuum*, pages 395–489. Springer Netherlands, Dordrecht, 2010. `doi:10.1007/978-1-4020-5764-9_7`.

[10] Andreas Blass and Saharon Shelah. There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed. *Ann. Pure Appl. Logic*, 33(3):213–243, 1987. `doi:10.1016/0168-0072(87)90082-0`.

[11] Andreas Blass and Saharon Shelah. Near coherence of filters. III. A simplified consistency proof. *Notre Dame J. Formal Logic*, 30(4):530–538, 1989. `doi:10.1305/ndjfl/1093635236`.

[12] Jörg Brendle. Between p-points and nowhere dense ultrafilters. *Israel Journal of Mathematics*, 113:205–230, 1999. URL: `https://doi.org/10.1007/BF02780177`.

[13] Jörg Brendle and Jana Flašková. Generic existence of ultrafilters on the natural numbers. *Fundamenta Mathematicae*, 236:201–245, 2017.

[14] G. Campero-Arena, J. Cancino, Michael Hrušák, and Favio Miranda-Perea. Incomparable families and maximal trees. *Fundamenta Mathematicae*, 234:1–17, 01 2016. `doi:10.4064/fm125-1-2016`.

[15] Jonathan Cancino, Osvaldo Guzmán, and Arnold W. Miller. Ideal independent families and the ultrafilter number. *The Journal of Symbolic Logic*, page 1–12, 2021. `doi:10.1017/jsl.2019.14`.

[16] R. Michael Canjar. On the generic existence of special ultrafilters. *Proc. Am. Math. Soc.*, 110(1):233–241, 1990.

[17] C. Carathéodory. Über die Begrenzung einfach zusammenhängender Gebiete. *Mathematische Annalen*, 73:323–370.

[18] Henri Cartan. Filtres et ultrafiltres. *Comptes rendus de l'Académie des Sciences*, 205:777 – 779, 1937.

[19] Henri Cartan. Théorie des filtres. *Comptes rendus de l'Académie des Sciences*, 205:595 – 598, 1937.

[20] David Chodounský and Osvaldo Guzmán. There are no p-points in silver extensions. *Israel Journal of Mathematics*, 232, 03 2017. `doi:10.1007/s11856-019-1886-2`.

[21] G. Choquet. Deux classes remarquable d'ultrafiltres sur $\mathbb{N}$. *Bull. Sci. Math.*, 92:143–153, 1968.

[22] Jacek Cichoń. On dense subsets of rational numbers. *Acta Universitatis Carolinae. Mathematica et Physica*, 42, 01 2001.

[23] Alain Connes. Détermination de modèles minimaux en analyse non standard et application. *Comptes rendus de l'Académie des Sciences*, 271:969–971, 02 1970.

[24] Maryvonne Daguenet-Teissier. Ultrafiltres a la Façon de Ramsey. *Transactions of the American Mathematical Society*, 250:91–120, 1979. URL: `http://www.jstor.org/stable/1998980`.

[25] Mauro Di Nasso and Marco Forti. Topological and Nonstandard Extensions. *Monatshefte für Mathematik*, 144:89–112, 02 2005. `doi:10.1007/s00605-004-0255-2`.

[26] Mauro Di Nasso and Marco Forti. Hausdorff ultrafilters. *Proceedings of the American Mathematical Society*, 134(6):1809–1818, 2006. URL: `http://www.jstor.org/stable/4098398`.

[27] R. M. Dudley. *Real Analysis and Probabilty.* Cambridge University Press, 2002.

[28] Jana Flašková. Ultrafilters and two ideals on ω. *WDS'05 Proceedings of Contributed Papers: Part I - Mathematics and Computer Sciences*, pages 78–83, 2005. URL: https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.186.7591&rep=rep1&type=pdf.

[29] Jana Flašková. More than a 0-point. *Comment.Math.Univ.Carolin.*, 47(4):617–621, 2006.

[30] Jana Flašková. Rapid ultrafilters and summable ideals. eprint August 2012. URL: https://arxiv.org/abs/1208.3690.

[31] Marco Forti, Mauro Di Nasso, and Vieri Benci. Hausdorff Nonstandard Extensions. *Boletim da Sociedade Paranaense de Matemática*, 20:9–20, 11 2002. URL: https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/view/7519/4334, doi:10.5269/bspm.v20i1-2.7519.

[32] Osvaldo Guzmán González and Michael Hrušák. On Pospíšil ideals. *Topology and its Applications*, 259:242–250, 2019. URL: https://www.sciencedirect.com/science/article/pii/S0166864119300501, doi:https://doi.org/10.1016/j.topol.2019.02.032.

[33] Jan Grebík and Michael Hrušák. No minimal tall Borel ideal in the Katětov order. *Fundamenta Mathematicae*, 248(2):135–145, 2020. doi:10.4064/fm724-4-2019.

[34] Michael Hrušák. Parametrized principles and canonical models(online talk). URL: https://video-archive.fields.utoronto.ca/view/3589.

[35] Michael Hrušák. Parametrized principles and canonical models(slides). URL: https://www.fields.utoronto.ca/programs/scientific/14-15/forcing/slides/tuesday/Hrusak.pdf.

[36] Michael Hrušák and David Meza-Alcántara. Katětov order, Fubini property and Hausdorff ultrafilters. *Rend. Inst. Mat. Univ. Trieste*, 44:1–9, 2012.

[37] Michael Hrušák, David Meza-Alcántara, and Hiroaki Minami. Pair-splitting, pair-reaping and cardinal invariants of F_σ-ideals. *The Journal of Symbolic Logic*, 75(2):661–677, 2010. URL: http://www.jstor.org/stable/25676800.

[38] Michael Hrušák, Jan van Mill, Ulises Ariet Ramos-García, and Saharon Shelah. Countably compact groups without non-trivial convergent sequences. *Transactions of the American Mathematical Society*, 374:1277–1296, 02 2021. doi:https://doi.org/10.1090/tran/8222.

[39] Michael Hrušák. Combinatorics of filters and ideals. *Contemporary Mathematics*, 533(8):29–69, 2011. doi:10.1090/conm/533/10503.

[40] Michael Hrušák and David Meza-Alcántara. Katětov order, Fubini property and Hausdorff ultrafilters. *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, 44, 01 2012.

[41] S. A. Jalali-Naini. *IThe monotone subsets of Cantor Space,filters and descriptive set theory*. PhD thesis, Oxford, 1976.

[42] Miroslav Katětov. Products of filters. *Commentationes Mathematicae Universitatis Carolinae*, 009(1):173–189, 1968. URL: http://eudml.org/doc/16267.

[43] Jussi Ketonen. On the existence of P-points in the Stone-Čech compactification of integers. *Fundam. Math.*, 92:91–94, 1976.

[44] K. Kunen. Weak p-points in $\mathbb{N}^*$. 1993.

[45] K. Kunen. *Set Theory*. Studies in Logic: Mathematical. College Publications, 2011. URL: https://books.google.com.mx/books?id=Zn8ppwAACAAJ.

[46] Kenneth Kunen. Some points in $\beta\omega$. *Mathematical Proceedings of the Cambridge Philosophical Society*, 80(3):385–398, 1976. doi:10.1017/S0305004100053032.

[47] Claude Laflamme. Filter games and combinatorial properties of winning strategies. *Contemporary Mathematics*, 192:51–67, 1996.

[48] Claude Laflamme and Jian-Ping Zhu. The Rudin-Blass ordering of ultrafilters. *Journal of Symbolic Logic*, 63(2):584–592, 1998. doi:10.2307/2586852.

[49] Krzysztof Mazur. F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $\mathcal{P}(\omega)/\mathscr{I}$. *Fundamenta Mathematicae*, 138(2):103–111, 1991. URL: http://eudml.org/doc/211857.

[50] David Meza-Alcántara. *Ideals and filters on countable sets*. PhD thesis, Universidad Nacional Autónoma de México, 2009.

[51] Arnold W. Miller. There are no Q-Points in Laver's Model for the Borel Conjecture. *Proceedings of the American Mathematical Society*, 78(1):103–106, 1980. URL: http://www.jstor.org/stable/2043048.

[52] Arnold W. Miller. Rational perfect set forcing. *Contemporary Mathematics*, 31:143–159, 1984.

[53] Hiroaki Minami and Hiroshi Sakai. Katětov and Katětov-Blass orders on F_σ-ideals. *Archive for Mathematical Logic*, 55:883–898, 2016. doi:10.1007/s00153-016-0500-0.

[54] G. Mokobodzki. Ultrafiltres rapides sur ${N}$. construction d'une densité relative de deux potentiels comparables. 1968.

[55] Justin Tatch Moore, Michael Hrušák, and Mirna Džamonja. Parametrized $\diamondsuit$-principles. *Transactions of the American Mathematical Society*, 356(6):2281–2306, 2004. URL: http://www.jstor.org/stable/3844948.

[56] Mauro Di Nasso and Marco Forti. Ultrafilter semirings and nonstandard submodels of the Stone-čech compactification of the natural numbers. *Contemporary Mathematics*, 380:45–51, 2005.

[57] R. A. Pitt. *The classification of ultrafilters on* $\mathbb{N}$. PhD thesis, University of Leicester, 1971. URL: https://leicester.figshare.com/articles/thesis/The_classification_of_ultrafilters_on_N_/10155530.

[58] Walter Rudin. Homogeneity problems in the theory of Čech compactifications. *Duke Mathematical Journal*, 23(3):409 – 419, 1956. doi:10.1215/S0012-7094-56-02337-7.

[59] Hiroshi Sakai. On Katětov and Katětov–Blass orders on analytic P-ideals and Borel ideals. *Archive for Mathematical Logic*, 57:317–327, 2018. doi:10.1007/s00153-017-0572-5.

[60] Saharon Shelah. There may be no nowhere dense ultrafilter. In *Logic Colloquium '95 (Haifa)*, volume 11 of *Lecture Notes Logic*, pages 305–324. Springer, Berlin, 1998. arXiv: math/9611221. doi:10.1007/978-3-662-22108-2_17.

[61] Sławomir Solecki. Analytic ideals and their applications. *Annals of Pure and Applied Logic*, 99:51–72, 1999.

[62] Sławomir Solecki. Filters and sequences. *Fundamenta Mathematicae*, 163(3):215–228, 2000. URL: http://eudml.org/doc/212440.

[63] M. Stone. The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, 40:37–111, 1936.

[64] Michel Talagrand. Compacts de fonctions measurables et filtres non meaesurables. *Studia Mathematica*, 67:13–43, 1980.

[65] A. Tarski. Une contribution à la théorie de la mesure. *Fundamenta Mathematicae*, 15:42–50.

[66] P. Vojtáš. *Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis*. Freie Universität Berlin, Fachbereich Mathematik: Ser. A, Mathematik. Freie Univ., Fachbereich Mathematik, 1991. URL: https://books.google.com.mx/books?id=jyDjHAAACAAJ.

[67] Edward Wimmers. The Shelah P-point independence theorem. *Israel Journal of Mathematics*, 43:28–48, 1982. doi:10.1007/BF02761683.

[68] Wolfgang Wohofsky. On the existence of p-points and other ultrafilters in the Stone-Čech-compactification of $\mathbb{N}$, April 2008.