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RELATION BETWEEN TURNPIKES AND DISSIPATIVITY PROPERTIES OF OPTIMAL CONTROL PROBLEMS

TESIS
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PRESENTA:
DANIEL ENRIQUE MILANES PEREZ

DIRECTOR DE LA TESIS:
DR. ONESIMO HERNANDEZ LERMA
CINVESTAV-IPN

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Abstract

In this thesis we study the relation between two properties which are of importance in optimal control problems: dissipativity of the underlying dynamics with respect to a specific supply rate and the turnpike property. We present various results, in discrete and continuous time, providing necessary and sufficient conditions for dissipativity and strict dissipativity in terms of different types of turnpikes.

Resumen

En esta tesis estudiamos la relación entre dos propiedades que son de importancia en los problemas de control óptimo: la disipatividad de la dinámica subyacente con respecto a una razón de insumo específica y la propiedad de autopista. Presentamos varios resultados, en tiempo discreto y continuo, que brindan condiciones necesarias y suficientes para la disipatividad y la disipatividad estricta en términos de diferentes tipos de 'turnpikes'.

Contents

Introduction

Dissipativity and strict dissipativity have been recognized as important systems theoretic properties since their introduction by Willems in [16]. Dissipativity formalizes the fact that a system cannot store more energy than supplied from the outside. Strict dissipativity in addition requires that a certain amount of the stored energy is dissipated to the environment. Recent developments on model predictive control (MPC) rely heavily on dissipativity notions of optimal control problems ([1], [12], [9], [8]), where this property plays a crucial role for stability considerations. Our goal is to study the relation between dissipativity and turnpike properties in both discrete and continuous time.

The turnpike phenomenon is a property of trajectories of controlled systems that has long been observed in optimal control, even back to early work by von Neumann ([15]). The turnpike property describes the fact that a trajectory "most of the time" stays close to an equilibrium point. The name "turnpike property" was coined in 1958 in the book by Dorfman et. al. ([3]), who compared the phenomenon to the optimal way of driving by car from a point A to a point B using a turnpike or highway, which consists of three phases: driving to the highway (i.e., approaching the equilibrium), driving on the highway (i.e., staying near the equilibrium) and leaving the highway (i.e., moving away from the equilibrium).

While the turnpike properties had received considerable attention in economics ([2], [11]), it was not until ([8], [6], [5]) that it was of significant interest in MPC.

The relation between these two properties have been studied recently in [7] and [4]. We present those results and extend them by providing necessary and sufficient conditions, not presented in these papers, for dissipativity properties in terms of different turnpike properties. More precisely:

- We prove that dissipativity implies the steady state turnpike property.

(Theorem 1.13)

- We study the relation between the turnpike property and dissipativity properties in discrete time (Theorems 1.14, 1.15, 1.16 and 1.17).
- We show that strict dissipativity implies the exponential turnpike property (Theorem 1.20), and that this turnpike property implies dissipativity (Theorem 1.18).
- We study the relation between the steady state turnpike property and dissipativity properties in continuous time (Theorems 2.7, 2.8, 2.9 and 2.10).
- We show that dissipativity and the steady state turnpike property imply the state turnpike property (Theorem 2.12), and that the state turnpike property implies both dissipativity and strict dissipativity (Theorems 2.13 and 2.14).
- We prove that the input-state turnpike property implies dissipativity and strict dissipativity (Theorems 2.18 and 2.19).

The thesis is composed by two chapters. Chapter 1 concerns discrete time systems and Chapter 2 continuous time systems. Both are organized similarly. In section 1 we describe the optimal control problem we are considering. Next, in section 2, we introduce the various definitions that are used in the chapter. In section 3 we present auxiliary results necessary to prove the main results that are in section 4. Finally, in section 5 we present examples where we apply some of the results of the previous section.

Chapter 1

Discrete Time

1.1 Problem Statement

We consider discrete time nonlinear systems of the form:

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0, \tag{1.1}$$

for a continuous map $f : X \times U \to \mathcal{X}$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$. We impose the constraints $(x, u) \in \mathbb{Y} \subset X \times U$ on the state x and the input u and define $\mathcal{X} = \{x \in X |\ \exists u \in U : (x, u) \in \mathbb{Y}\}$ and $\mathcal{U} = \{u \in U |\ \exists x \in X : (x, u) \in \mathbb{Y}\}$. We assume both of these sets are compact.

A control sequence $u \in \mathcal{U}^N$ is called admissible for $x_0 \in \mathcal{X}$ if $(x(k), u(k)) \in \mathbb{Y}$ for $k = 0, ..., N-1$ and $x(N) \in \mathcal{X}$. The set of admissible control sequences is denoted by $\mathcal{U}^N(x_0)$. Likewise, we define $\mathcal{U}^\infty(x_0)$ as the set of all control sequences $u \in \mathcal{U}^\infty$ with $(x(k), u(k)) \in \mathbb{Y}$ for all $k \in \mathbb{N}_0$. We assume that $\mathcal{U}^\infty(x_0) \neq \emptyset$ for all $x_0 \in \mathcal{X}$.

Given a continuous stage cost $l : \mathbb{Y} \to \mathbb{R}$ and a time horizon $N \in \mathbb{N}$, we consider the optimal control problem

$$V_N(x) = \min_{u \in \mathcal{U}^N(x_0)} J_N(x, u) = \min_{u \in \mathcal{U}^N(x_0)} \sum_{k=0}^{N-1} l(x(k), u(k)) \tag{1.2}$$

subject to 1.1. The trajectories of 1.1 are denoted by $x_u(k, x_0)$ or simply by $x(k)$ if there is no ambiguity about x_0 and u, and the optimal trajectories are denoted by $x_{u^*}(k, x_0)$ or by $x^*(k)$, respectively.

1.2 Definitions

A pair $(x^e, u^e) \in \mathbb{Y}$ is said to be a steady state pair for the system 1.1 if $f(x^e, u^e) = x^e$. If, in addition, (x^e, u^e) solves the steady state problem

$$\textit{minimize } l(x,u) \;\; \textit{subject to} \;\; f(x,u) = x$$

then (x^e, u^e) is said to be an optimal steady state pair.

The next definition formalizes the dissipativity property. Let $w(x,u) : \mathbb{Y} \to \mathbb{R}$ be given by

$$w(x,u) = l(x,u) - l(x^e, u^e)$$

for l from 1.2 and a steady state pair (x^e, u^e) of 1.1.

Definition 1.1. *The system 1.1 is called dissipative with respect to* (x^e, u^e) *if there exists a bounded storage function* $\lambda : \mathcal{X} \to \mathbb{R}_0^+$ *such that*

$$w(x,u) + \lambda(x) - \lambda(f(x,u)) \geq 0 \tag{1.3}$$

holds for all $(x,u) \in \mathbb{Y}$ *with* $f(x,u) \in \mathcal{X}$*. If, in addition, for some* $\rho \in \mathcal{K}_\infty$[1] *and* $\xi \in \{x, (x,u)\}$,

$$w(x,u) + \lambda(x) - \lambda(f(x,u)) \geq \rho(||\xi - \xi^e||) \tag{1.4}$$

then, the system is called strictly dissipative with respect to the steady state x^e *if* $\xi = x$*, and strictly dissipative with respect to the steady state pair* (x^e, u^e) *if* $\xi = (x,u)$.

Remark 1.1. *Note that whenever necessary without loss of generality we can assume* $l(x^e, u^e) = 0$ *and* $\lambda(x^e) = 0$ *since adding constants to* λ *and* l *changes neither the optimal trajectories nor the validity of 1.3 and 1.4.*

Definition 1.2. *If for all* $N \in \mathbb{N}$ *and all* $x_0 \in \mathcal{X}$*, the dissipation inequality 1.3 or 1.4 hold along any optimal solution of 1.2, then the optimal control problem is said to be dissipative or strictly dissipative, respectively.*

Observe that in the non-strict case, Definition 1.1 and Definition 1.2 are equivalent. However, in the strict case Definition 1.2 is weaker than Definition 1.1.

[1] A function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to $\mathcal{K}_\infty$ if $\alpha(0) = 0$, it is strictly increasing and $\lim_{r\to\infty} \alpha(r) = \infty$

The following definition presents three variants of the turnpike property. The behavior of the trajectories described in the three definitions is essentially identical and in all cases demands that the trajectory stays in a neighborhood of a steady state most of the time. What distinguishes the different variants are the conditions on the trajectories under which we demand this property to hold and in case of c) the bound on the size of the neighborhood.

Definition 1.3. *Consider the optimal control problem 1.2 and let* (x^e, u^e) *be a steady state pair of 1.1.*

a) *The optimal control problem is said to have the steady state turnpike property with respect to* x^e *if there exist* $C_a > 0$ *and* $\rho \in \mathcal{K}_\infty$ *such that for each* $x \in \mathcal{X}$, $\delta > 0$ *and* $K \in \mathbb{N}$, *each control sequence* $u \in \mathcal{U}^K(x)$ *satisfying* $J_K(x,u) \leq Kl(x^e,u^e) + \delta$ *and each* $\epsilon > 0$ *the value* $Q_\epsilon = \#\{k \in \{0,...,K-1\} \mid ||x_u(k,x) - x^e|| \leq \epsilon\}$ *satisfy the inequality* $Q_\epsilon \geq K - (\delta + C_a)/\rho(\epsilon)$.

b) *The optimal control problem is said to have the turnpike property with respect to* x^e *if there exist* $C_b > 0$ *and* $\rho \in \mathcal{K}_\infty$ *such that for each* $x \in \mathcal{X}$, $K \in \mathbb{N}$ *and the corresponding optimal control sequence* $u^* \in \mathcal{U}^K(x)$, *for each* $\epsilon > 0$ *the value* $Q_\epsilon = \#\{k \in \{0,...,K-1\} \mid ||x_{u^*}(k,x) - x^e|| \leq \epsilon\}$ *satisfy the inequality* $Q_\epsilon \geq K - C_b/\rho(\epsilon)$.

c) *The optimal control problem is said to have the exponential turnpike property with respect to* (x^e,u^e) *if there exist* $C_c > 0$ *and* $\eta \in (0,1)$ *such that for each* $x \in \mathcal{X}$, $K \in \mathbb{N}$ *and the corresponding optimal control sequence* $u^* \in \mathcal{U}^K(x)$, *the inequality* $\max\{||x_{u^*}(k,x) - x^e||, ||u^*(k) - u^e||\} \leq C_c \max\{\eta^k, \eta^{K-k}\}$ *holds for all* $k \in \{0,...,K-1\}$ *but at most* C_c *times.*

The steady state turnpike property (a) ensures that each trajectory for which the associated cost is close to the optimal steady state value stays most of the time in a neighborhood of x^e. The turnpike property (b) demands that for all initial values the optimal trajectory shows this behavior. The exponential turnpike property (c) is stronger than the turnpike property (b) in the sense that the imposed inequality involves x and u and that the distance from the steady state is required to decrease exponentially fast.

Definition 1.4. *Consider the optimal control problem 1.2 and let* (x^e, u^e) *be a steady state pair of 1.1. We say that the system is non-averaged steady state optimal at* (x^e,u^e) *if there exists a constant* $E > 0$ *with* $V_K(x) \geq Kl(x^e,u^e) - E$ *for all* $x \in \mathcal{X}$ *and all* $K \in \mathbb{N}$.

This property formalizes that up to an additive constant the optimal value cannot be better than the optimal steady state value.

The next definition states that no feasible state-input sequence pair results in a better asymptotic average performance than the optimal steady state cost.

Definition 1.5. *Consider the optimal control problem 1.2 and let* (x^e, u^e) *be a steady state pair of 1.1. The system is called optimally operated at* (x^e, u^e) *if for all* $x_0 \in \mathcal{X}$ *and* $u \in \mathcal{U}^\infty(x_0)$ *the inequality*

$$\liminf_{K\to\infty} \frac{\sum_{k=0}^{K-1} l(x_u(k, x_0), u(k))}{K} \geq l(x^e, u^e)$$

holds.

Finally, some of the results presented here need a local controllability property near a steady state.

Definition 1.6. *We say that the system 1.1 is locally controllable around a steady state* (x^e, u^e) *if there exists* $\kappa > 0$ *such that for each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for any two points* $x, y \in B_\delta(x^e)$ *there is a control* $u \in \mathcal{U}^\kappa$ *with* $x_u(\kappa, x) = y$ *and* $\max\{||x_u(k, x) - x^e||, ||u(k) - u^e||\} \leq \epsilon$ *for all* $k \in \{0, ..., \kappa\}$.

1.3 Auxiliary Results

In order to verify strict dissipativity, one has two possibilities: compute a storage function or rely on converse dissipativity results. Next, we recall such a result.

Theorem 1.1 ([9])**.** *System 1.1 is dissipative on* $\mathbb{Y}$ *with respect to the supply rate* $w(x, u)$ *if and only if the available storage*

$$\lambda(x_0) = \sup_{K\geq 0, u\in\mathcal{U}^K(x_0)} \sum_{k=0}^{K-1} -(l(x(k), u(k)) - l(x^e, u^e)) \tag{1.5}$$

is bounded on $\mathcal{X}$*. Moreover,* $\lambda(x_0)$ *is a storage function.*

Remark 1.2. *Theorem 1.1 has another version in the case of strict dissipativity (see Proposition 3.3 of [7]). In this case the available storage is required to be finite and not necessarily bounded.*

The next three theorems show that the dissipativity property is equivalent to non-averaged steady state optimality and that both of these properties imply optimal operation at the steady state.

Theorem 1.2 ([7])**.** *The optimal control problem 1.2 is dissipative with respect to the steady state pair* (x^e, u^e) *if and only if it is non-averaged steady state optimal at* (x^e, u^e)*.*

Theorem 1.3 ([7])**.** *If the optimal control problem 1.2 is non-averaged steady state optimal at* (x^e, u^e)*, then it is optimally operated at the steady state pair* (x^e, u^e)*.*

Theorem 1.4. *Let* (x^e, u^e) *be a steady state pair of 1.1. If the optimal control problem 1.2 is dissipative with respect to* (x^e, u^e)*, the system is optimally operated at* (x^e, u^e)*.*

Proof. From Theorem 1.2 dissipativity implies non-averaged steady state optimality, and from Theorem 1.3, this implies optimal operation at the steady state. □

For the next theorem we need to define the following sets. For a given $N \in \mathbb{N}$, denote by $\mathcal{X}_N$ the set of states that can be steered to the optimal steady state x^e in N steps:

$$\mathcal{X}_N = \{x_0 \in X |\ \exists u \in U : (x(k), u(k)) \in \mathbb{Y}\ \forall k \in \{0, ..., N-1\},\ x(N) = x^e\}$$

Next, let $\mathcal{R}_N$ be the set of states that can be reached from the optimal steady state x^e in N steps:

$$\mathcal{R}_N = \{x \in X |\ \exists u \in U : x_0 = x^e,\ \ (x(k), u(k)) \in \mathbb{Y}\ \forall k \in \{0, ..., N-1\},\ x(N) = x\}$$

Note that $\mathcal{X}_N \cap \mathcal{R}_N \neq \emptyset$, as by definition x^e is contained in both $\mathcal{X}_N$ and $\mathcal{R}_N$. Now define the set $\mathbb{Y}_N$ as the set of state-input pairs such that $x(k) \in \mathcal{X}_N \cap \mathcal{R}_N$ for all times:

$$\mathbb{Y}_N = \{(x, u) \in \mathbb{Y} |\ (x(k), u(k)) \in \mathbb{Y},\ x(k) \in \mathcal{X}_N \cap \mathcal{R}_N\ \forall k \in \mathbb{N}_0\}$$

Finally, denote the projection of $\mathbb{Y}_N$ on X by X_N:

$$X_N = \{x \in X |\ \exists u \in U :\ (x, u) \in \mathbb{Y}_N\}$$

Theorem 1.5 ([9]). *Suppose that the system 1.1 is optimally operated at the steady state pair* (x^e, u^e). *Then, for each* $N \in \mathbb{N}$, *system 1.1 is dissipative on* $\mathbb{Y}_N$ *with respect to* (x^e, u^e).

Theorem 1.5 together with Theorem 1.4 show that under certain conditions, optimal operation and dissipativity are equivalent.

Theorem 1.6. *Consider the optimal control problem 1.2. Assume that* l *is Lipschitz on* $\mathbb{Y}$ *and let* (x^e, u^e) *be a steady state pair. If the optimal control problem has the turnpike property and* $\frac{\partial l}{\partial u} = 0$, *then, the system is optimally operated.*

Proof. Note that $\frac{\partial l}{\partial u} = 0$ implies that l does not depend on u. Fix $x_0 \in \mathcal{X}$ and, for contradiction, assume that there exist an infinite time admissible pair (x_∞, u_∞) and a sequence $\{N_k\}_{k=1}^{\infty}$ with $N_{k+1} \geq N_k$ and $N_k \to \infty$ such that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{k=0}^{N_k-1} l(x_\infty(k)) \leq l(x^e) - \sigma$$

for some $\sigma > 0$. Observe that the turnpike property and the Lipschitz continuity of $l(x)$ imply that:

$$\#(\Omega_N(\epsilon)) < \frac{C_b}{\rho(\epsilon)}$$

with $\Omega_N(\epsilon) = \{k \in \{0, ...N-1\} : |l(x_N^*) - l(x^e)| > \epsilon\}$. Set $m = \min l(x)$ and $\bar{\Omega} = \{0, ..., N_k\}/\Omega_{N_k}(\epsilon)$. Then, for arbitrary ϵ, we have

$$\sum_{k=0}^{N_k} l(x_{N_k}^*(k)) = \sum_{\Omega_N(\epsilon)} l(x_{N_k}^*(k)) + \sum_{\bar{\Omega}} l(x_{N_k}^*(k))$$

$$\geq m\#[\Omega_N(\epsilon)] + \sum_{\bar{\Omega}} l(x^e) - \epsilon$$

$$= m\#[\Omega_N(\epsilon)] + \sum_{k=0}^{N_k} l(x^e) - \epsilon - \sum_{\Omega_N} l(x^e) - \epsilon$$

$$= m\#[\Omega_N(\epsilon)] + N_k(l(x^e) - \epsilon) - \#\Omega_N(\epsilon)(l(x^e) - \epsilon)$$

Since $\#(\Omega_N(\epsilon)) < \frac{C_b}{\rho(\epsilon)}$, dividing by N_k and letting $k \to \infty$ gives

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{k=0}^{N_k} l(x_{N_k}^*(k)) \geq l(x^e) - \epsilon$$

Selecting $\epsilon < \sigma$ leads to a contradiction. $\square$

The next theorem provides a bound on the cost of trajectories staying near a steady state.

Theorem 1.7 ([7])**.** *Let (x^e, u^e) be an optimal steady state pair. Then for each $\delta > 0$ and $P \in \mathbb{N}$ there is $\epsilon = \epsilon(\delta, P) > 0$ such that for each admissible trajectory satisfying*

$$||x_u(k, x) - x^e|| < \epsilon \quad \forall k = 0, ..., P - 1$$

the inequality $J_P(x, u) > Pl(x^e, u^e) - \delta$ holds.

Theorem 1.8 below establishes a relation between the steady state turnpike property and optimal operation.

Theorem 1.8 ([7])**.** *Let $(x^e, u^e$ be an optimal steady state. Assume that the system has the steady state turnpike property at x^e. Then, the system is optimally operated at (x^e, u^e).*

1.4 Main Results

1.4.1 Steady State Turnpike Property

Theorem 1.9 ([7])**.** *Consider the optimal control problem 1.2 and let x^e be a steady state. If the optimal control problem is dissipative with supply rate $w(x, u)$ and has the steady state turnpike property, it is strictly dissipative with respect to the steady state x^e.*

Theorem 1.10 ([10])**.** *Consider the optimal control problem 1.2 and let (x^e, u^e) be a steady state pair. If the system is optimally operated at the steady state, has the steady state turnpike property and is locally controllable, is dissipative.*

Theorem 1.11 ([7])**.** *Consider the optimal control problem 1.2. Let (x^e, u^e) be a steady state with $u^e \in argmin\{l(x^e, u)| \ (x^e, u) \in \mathbb{Y}, f(x^e, u) = x^e\}$ around which the system is locally controllable. Then, if the optimal control problem has the steady state turnpike property, it is strictly dissipative with respect to x^e.*

Theorem 1.12 ([6])**.** *Consider the optimal control problem 1.2 and assume that it is strictly dissipativity with respect to the steady state x^e. Then, it has the steady state turnpike property.*

The next theorem is a consequence of Theorem 1.2 and Theorem 3.9 in [7].

Theorem 1.13. *Consider the optimal control problem 1.2 and let* (x^e, u^e) *be a steady state pair. Assume that the optimal control problem is dissipative with respect to* (x^e, u^e) *and there exists a constant* $C > 0$ *and* $\rho \in \mathcal{K}_\infty$ *such that for each* $x \in \mathcal{X}$, $\delta > 0$ *and* $K \in \mathbb{N}$, *each control sequence* $u \in \mathcal{U}^K(x)$ *satisfying* $J_K(x, u) \leq V_K(x) + \delta$ *and each* $\epsilon > 0$ *the value of* $Q_\epsilon = \#\{k \in \{0, ..., K-1\} | \, ||x_u(k, x) - x^e|| \leq \epsilon\}$ *satisfy the inequality* $Q_\epsilon \geq K - (\delta + C)/\rho(\epsilon)$. *Then, the optimal control problem has the steady state turnpike property.*

Proof. By Theorem 1.2 the optimal control problem is non-averaged steady state optimal. The inequalities

$$J_K(x, u) \leq Kl(x^e, u^e) + \delta$$

$$V_K(x) \geq Kl(x^e, u^e) - E$$

imply that

$$J_K(x, u) \leq V_K(x) + \delta + E$$

from which the steady state turnpike property follows with $C_a = C + E$

□

1.4.2 Turnpike Property

Assumption 1.1. *For all* $x_0 \in \mathcal{X}$, *there exists an infinite-horizon admissible input* $u(\cdot, x_0)$, $C > 0$, $\rho \in [0, 1)$, *such that*

$$||(x_u(k, x_0), u(k)) - (x^e, u^e)|| \leq C\rho^k$$

Theorem 1.14. *If the optimal control problem 1.2 is strictly dissipative with respect the steady state* x^e, l *is Lipschitz on* $\mathbb{Y}$ *and Assumption 1.1 holds, then it has the turnpike property.*

Proof. From the dissipation inequality we have:

$$\lambda(x^*(N)) - \lambda(x^*(0)) \leq -\sum_{k=0}^{N-1} \rho(||x^*(k) - x^e||) + \sum_{k=0}^{N-1} (l(x^*(k), u^*(k)) - l(x^e, u^e))$$

$$\leq -\sum_{k=0}^{N-1} \rho(||x^*(k) - x^e||) + \sum_{k=0}^{N-1} (l(x(k), u(k)) - l(x^e, u^e))$$

where (x, u) are from assumption 1.1. From the assumption and the fact that l is Lipschitz, the second sum is bounded from above by $L_l \frac{C}{1-p}$, where L_l is a Lipschitz constant of l and p, C are from the assumption. Since λ is bounded in absolute value by some constant S, we have:

$$\sum_{k=0}^{N-1} \rho(||x^*(k) - x^e||) \leq L_l \frac{C}{1-p} + 2S$$

Noting that $N - Q_\epsilon$ is the amount of time that the optimal trajectories stays outside an epsilon neighborhood of the steady state, and using that $\rho \in \mathcal{K}_\infty$, we also have:

$$\sum_{k=0}^{N-1} \rho(||x^*(k) - x^e||) \geq (N - Q_\epsilon)\rho(\epsilon)$$

Combining both inequalities we obtain

$$L_l \frac{C}{1-p} + 2S \geq (N - Q_\epsilon)\rho(\epsilon)$$

$$Q_\epsilon \geq N - \frac{L_l C(1-p)^{-1} + 2S}{\rho(\epsilon)}$$

The last inequality proves the turnpike property. □

Theorem 1.15. *Consider the optimal control problem 1.2 and let (x^e, u^e) be a steady state pair. If the optimal control problem is dissipative with supply rate $w(x, u)$, has the steady state turnpike property and satisfy Assumption 1.1, it has the turnpike property.*

Proof. By Theorem 1.9 the assumptions imply that the optimal control problem is strictly dissipative. The result follows from Theorem 1.14. □

Theorem 1.16. *Consider the optimal control problem 1.2 and let (x^e, u^e) be a steady state pair. If the optimal control problem has the turnpike property, l is Lipschitz and $\frac{\partial l}{\partial u} = 0$, it is dissipative on $\mathbb{Y}_N$.*

Proof. By Theorem 1.6 the optimal control problem is optimally operated at (x^e, u^e) and by Theorem 1.5 it is dissipative on $\mathbb{Y}_N$. □

Theorem 1.17. *Consider the optimal control problem 1.2 and let (x^e, u^e) be a steady state pair. If the optimal control problem has the turnpike property, the steady state turnpike property, l is Lipschitz and $\frac{\partial l}{\partial u} = 0$, it is strictly dissipative on $\mathbb{Y}_N$ with respect to x^e.*

Proof. By Theorem 1.6 the optimal control problem is optimally operated at (x^e, u^e) and by Theorem 1.5 the optimal control problem is dissipative on $\mathbb{Y}_N$. Using Theorem 1.9 we obtain the result. □

1.4.3 Exponential Turnpike Property

The next theorem is a consequence of Theorem 1.2 and Theorem 3.10 in [7].

Theorem 1.18. *If the optimal control problem 1.2 has the exponential turnpike property and l is Hölder continuous in a neighborhood of the steady state pair (x^e, u^e), it is dissipative.*

Proof. Let the ball $B_\delta((x^e, u^e))$, $\delta > 0$, be contained in the neighborhood on which l is Holder continuous. Then the exponential turnpike property implies that the optimal trajectory is outside $B_\delta((x^e, u^e))$ for at most $C_c + K_\delta$ time indices, where $K_\delta = 2\lceil log(\delta/C_c)/log\eta\rceil$. Denoting the bound on $|l|$ by M_l, this property together with the turnpike property yields

$$|V_K(x) - Kl(x^e, u^e)| = |\sum_{k=0}^{K-1} l(x_{u^*}(k, x), u^*(x)) - Kl(x^e, u^e)|$$

$$\leq \sum_{k=0}^{K-1} |l(x_{u^*}(k, x), u^*(x)) - l(x^e, u^e)|$$

$$\leq (C_c + K_\delta)M_l + H2^\gamma C_c^\gamma \sum_{k=0}^{K-1} \max\{\eta^k, \eta^{K-k}\}^\gamma$$

$$\leq (C_c + K_\delta)M_l + 2H2^\gamma C_c^\gamma/(1 - \eta^\gamma)$$

This shows non averaged steady state optimality which by Theorem 1.2 implies dissipativity. □

Theorem 1.19 ([7])**.** *Consider the optimal control problem 1.2 with Holder continuous stage cost l. Let (x^e, u^e) be a steady state pair and assume that the optimal control problem has the steady state and the exponential turnpike property at (x^e, u^e). Then the optimal control problem is strictly dissipative with respect to x^e.*

The next theorem is a slight modification of Theorem 6.5 of [14]. We assume strict dissipativity with respect to the steady state pair instead of with respect to the steady state. The proof remains the same and is thus omitted.

Theorem 1.20. *Consider the optimal control problem 1.2, assume it is strictly dissipative with respect to the steady state pair (x^e, u^e) and without loss of generality that $l(x^e, u^e) = 0$ and $\lambda(x^e) = 0$. Also, assume that λ is continuous, and that there are constants C_1, C_2, p, $\eta > 0$ such that the inequalities*

$$C_1(||x - x^e||^p) \leq \bar{l}(x,u)^2 \leq C_2(||x - x^e||^p + ||u - u^e||^p) \tag{1.6}$$

hold for all $x \in B_\eta(x^e)$ and $u \in B_\eta(u^e)$. Suppose that inequality 1.4 holds with $\alpha(r) \geq M\min(r^p, r^q)$, for positive constants $M, p, q \in \mathbb{R}$ and all $r \geq 0$. Assume, furthermore, that there exists $\epsilon > 0$ such that the following conditions hold

a) There exists a set $\mathbb{X}_0 \subset \mathbb{X}$ and a $K \in \mathbb{N}$ such that for each $x \in \mathbb{X}_0$ there exist $k_x \leq K$ and a control $u_x \in \mathcal{U}^{k_x}$ with $x_{u_x}(k_x, x) \in B_\epsilon(x^e)$.

b) There exists $\epsilon > 0$ and $N' \in \mathbb{N}$ such that the system is controllable to and from $x^e \in B_\epsilon(x^e)$ in N' steps: there is $C > 0$ such that for all $x \in B_\epsilon(x^e)$ there exists $u_1 \in \mathcal{U}^{N'}(x)$ and $u_2 \in \mathcal{U}^{N'}(x)$ with

$$x_{u_1}(N', x) = x^e \qquad x_{u_2}(N', x^e) = x$$

and

$$\max(||x_{u_1} - x^e||, ||x_{u_2} - x^e||, ||u_1 - u^e||, ||u_2 - u^e||) \leq C||x - x^e||$$

Then the system has the exponential turnpike property on $\mathbb{X}_0$.

Remark 1.3. *The assumption of strict dissipativity in Theorem 5.6 in [14] also needs to be with respect to the steady state pair (x^e, u^e) so that the bound for the exponential turnpike property holds.*

[2] $\bar{l}(x,u) = l(x,u) + \lambda(x) - \lambda(f(x,u))$

1.5 Examples

1.5.1 Steady State Turnpike Property

Consider a system on $\mathbb{Y} = [-1/2, 1/2] \times [-1, 1]$ with dynamics and stage cost

$$x(k+1) = \frac{1}{2}x(k)$$

$$l(x, u) = u^2 + \frac{\log(2)}{\log|x|}$$

for $x \neq 0$ and l continuously extended to $l(0, u^2) = 0$. The system has an optimal steady state at $(x^e, u^e) = (0, 0)$, and all the trajectories of the system converge to $(0, 0)$. This implies that the system has a steady state turnpike property at (x^e, u^e).

To see if the system is dissipative lets use Theorem 1.1.

$$\lambda(x_0) = \sup_{K \geq 0, u \in \mathcal{U}^K(x_0)} \sum_{k=0}^{K-1} -(l(x(k), u(k)) - l(x^e, u^e))$$

$$= \sup_{K \geq 0, u \in \mathcal{U}^K(x_0)} \sum_{k=0}^{K-1} -(u^2 + \frac{\log(2)}{\log|x|})$$

$$\geq \sup_{K \geq 0, u \in \mathcal{U}^K(x_0)} \sum_{k=0}^{K-1} \frac{\log(2)}{\log(2^{-k}x_0)}$$

$$= \sup_{K \geq 0, u \in \mathcal{U}^K(x_0)} \sum_{k=0}^{K-1} \frac{\log(2)}{-k\log(2) + \log(x_0)}$$

$$= \sup_{K \geq 0, u \in \mathcal{U}^K(x_0)} \sum_{k=0}^{K-1} \frac{1}{k - \log(x_0)/\log(2)} = \infty$$

Thus, the system is not dissipative. Since all the assumptions from Theorem 1.11 hold, except for the controllability of the system around the steady state, this is the reason why is not dissipative.

1.5.2 Turnpike Property

Consider a system on $\mathbb{Y} = [-1/2, 1/2] \times [-2, 2]$ with dynamics

$$x(k+1) = 2x(k) + u(k)$$

The optimal control problem reads

$$\min_{u\in\mathcal{U}^N} \sum_{k=0}^{N-1} x^2$$

subject to $x(k+1) = 2x(k) + u(k)$.

An optimal solution to this problem is to steer the system to the steady state $x^e = 0$ in a finite number of steps k' and set $u(k) = u^e = 0$ for $k \geq k'$, thus, the optimal control problem satisfies the turnpike property.

All the hypothesis from Theorem 1.16 hold, so we can say that the optimal control problem is dissipative on $\mathbb{Y}_N$.

1.5.3 Exponential Turnpike Property

Consider a system on $\mathbb{Y} = [-1/2, 1/2] \times [0, 1]$ with dynamics

$$x(k+1) = x^2(k)$$

The optimal control problem reads

$$\min_{u\in\mathcal{U}^N} \sum_{k=0}^{N-1} u(k) - x^2(k) + 1$$

subject to $x(k+1) = x^2(k)$.

Since every trajectory of the system converges to the steady state pair $(x^e, u^e) = (0, 0)$ exponentially fast, it has the exponential turnpike property at $(0, 0)$.

The partial derivatives of l are continuous:

$$\frac{\partial l}{\partial u} = 1 \qquad \frac{\partial l}{\partial x} = -2x$$

which implies that l is Lipschitz and thus, Hölder continuous. All hypothesis from Theorem 1.18 hold, so we can say that the optimal control problem is dissipative.

Chapter 2

Continuous Time

2.1 Problem Statement

We consider the non-linear system given by:

$$\dot{x} = f(x,u), \qquad x(0) = x_0 \tag{2.1}$$

where the states $x \in \mathbb{R}^n$ and the inputs $u \in \mathbb{R}^m$, are constrained to lie in the compact sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$. We assume that the vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz on $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$.

The set $\mathcal{L}([0,T],\mathcal{U})$ denotes the set of admissible controls, that is, $\mathcal{U}-$valued measurable functions on $[0,T]$. For $u(\cdot) \in \mathcal{L}([0,T],\mathcal{U})$, if there exists an absolutely continuous solution $x(\cdot,x_0,u(\cdot))$, which satisfies $x(\cdot,x_0,u(\cdot)) \in \mathcal{X}$ for all $t \in [0,T]$, the pair $z(\cdot,x_0,u(\cdot)) = (x(\cdot,x_0,u(\cdot)),u(\cdot))$ is called admissible. An optimal pair is denoted as $z^*(\cdot,x_0,u^*(\cdot)) = (x^*(\cdot,x_0,u^*(\cdot)),u^*(\cdot))$.

Consider the set

$$\mathcal{X}_0 = \{x_0 \in \mathcal{X} |\ \exists u(\cdot) \in \mathcal{L}([0,\infty),\mathcal{U}) :\ \forall t \geq 0\ x(t,x_0,u(\cdot)) \in \mathcal{X}\}$$

where $\mathcal{L}([0,\infty),\mathcal{U})$ denotes the class of measurable functions on $[0,\infty)$ taking values in $\mathcal{U}$. Here, we assume that $\mathcal{X}_0 = \mathcal{X}$. Furthermore, consider an optimal control problem that aims at minimizing the objective functional

$$J_T(x_0,u(\cdot)) = \int_0^T F(x(t),u(t))dt$$

where $F : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is the cost function. We assume that F is Lipschitz on $\mathcal{X} \times \mathcal{U}$. The optimal control problem reads

$$V_T(x_0) = \min_{u(\cdot)\in\mathcal{L}([0,T],\mathcal{U})} J_T(x_0, u(\cdot)) \tag{2.2}$$

subject to 2.1. A solution to 2.1, starting at x_0 at time 0, driven by the input $u : [0, \infty) \to \mathcal{U}$, is denoted as $x(\cdot, x_0, u(\cdot))$, and an optimal solution is denoted as $x^*(\cdot, x_0, u^*(\cdot))$.

2.2 Definitions

A pair $z^e = (x^e, u^e) \in \mathcal{Z}$ is said to be a steady state pair for the system 2.1 if $f(x^e, u^e) = 0$. If, in addition, (x^e, u^e) solves the steady state problem

$$\inf_{z^e} F(z^e) \qquad \textit{subject to} \quad z^e \in \mathcal{Z}^e$$

where $\mathcal{Z}^e = \{z^e \in \mathcal{Z} |\ f(z^e) = 0\}$, (x^e, u^e) is an optimal steady state pair.

Let $w : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ be given by

$$w(x, u) := F(x, u) - F(z^e)$$

with $z^e \in \mathcal{Z}^e$, and F is the cost function in 2.2. The function w will be called the supply rate.

Definition 2.1. *The system 2.1 is said to be dissipative on $\mathcal{Z}$ with respect to $z^e = (x^e, u^e) \in \mathcal{Z}^e$ if there exists a bounded storage function $S : \mathcal{X} \to \mathbb{R}_0^+$ such that, for all $x_0 \in \mathcal{X}$, all $T \geq 0$, $u(\cdot) \in \mathcal{L}([0, T), \mathcal{U})$, satisfying $x(t, x_0, u(\cdot)) \in \mathcal{X}$ for all $t \in [0, T]$, we have*

$$S(x_T) - S(x_0) \leq \int_0^T w(x(t), u(t))dt \tag{2.3}$$

where $x_T = x(T, x_0, u(\cdot))$

If, in addition, for some $\alpha \in \mathcal{K}_\infty$ and $\xi \in \{x, z\}$,

$$S(x_T) - S(x_0) \leq \int_0^T -\alpha(||\xi(t) - \xi^e||) + w(x(t), u(t))dt \tag{2.4}$$

then,

i) for $\xi = x$, the system 2.1 is said to be strictly dissipative on $\mathcal{Z}$ with respect to the steady state x^e;

ii) for $\xi = z$, the system 2.1 is said to be strictly dissipative on $\mathcal{Z}$ with respect to the steady state pair z^e.

Definition 2.2. *If, for all $x_0 \in \mathcal{X}$, 2.3 or 2.4, is satisfied along any optimal pair, we say that the optimal control problem is dissipative, respectively strictly dissipative, with respect to $\xi^e \in \{x^e, z^e\}$.*

Note that in the non-strict case, one can show that dissipativity of the system and dissipativity of the optimal control problem are equivalent. Furthermore, strict dissipativity of the system implies strict dissipativity of optimal control problem but not vice versa.

A classical characterization of dissipativity is given by the available storage ([16]). It is an essential function in determining whether or not a system is dissipative. This is shown in Theorem 2.1.

Definition 2.3. *The available storage, S_a, of the dynamical system 2.1 with supply rate w is the function defined by*

$$S_a(x_0) := \sup_{u(\cdot)\in\mathcal{L}([0,T],\mathcal{U}),T} - \int_0^T w(x(t), u(t))dt$$

subject to 2.1.

The next definition presents the three variants of the turnpike property we are going to study.

Definition 2.4. *Consider the optimal control problem 2.2 and let (x^e, u^e) be a steady state pair of 2.1.*

a) The optimal control problem is said to have the steady state turnpike property with respect to x^e if there exist $C_a > 0$ and $\rho \in \mathcal{K}_\infty$ such that for each $x_0 \in \mathcal{X}$ and $\delta > 0$, each control $u \in \mathcal{L}([0,T), \mathcal{U})$ satisfying $J_T(x_0, u) \leq TF(x^e, u^e) + \delta$ and each $\epsilon > 0$ the value $Q_\epsilon = \mu\{t \in [0,T]|\ ||x(t, x_0, u(\cdot)) - x^e|| \leq \epsilon\}$ satisfy the inequality $Q_\epsilon \geq T - (\delta + C_a)/\rho(\epsilon)$.

b) The optimal control problem is said to have the state turnpike property with respect to x^e if there exists a function $v_{x^e} : [0,\infty) \to [0,\infty]$ such that for all $x_0 \in \mathcal{X}$ and the corresponding optimal control $u^* \in \mathcal{L}([0,T),\mathcal{U})$, $\mu\{t \in [0,T]|\ ||x^*(t, x_0, u^*(\cdot)) - x^e|| > \epsilon\} < v_{x^e}(\epsilon) < \infty$ for all $\epsilon \geq 0$.

c) The optimal control problem is said to have the input-state turnpike property with respect to (x^e, u^e) if there exists a function $v_{z^e} : [0,\infty) \to [0,\infty]$

such that for all $x_0 \in \mathcal{X}$ *and the corresponding optimal control* $u^* \in \mathcal{L}([0,T),\mathcal{U})$, $\mu\{t \in [0,T] | \, ||(x^*(t,x_0,u^*(\cdot)),u^*) - (x^e,u^e)|| > \epsilon\} < v_{z^e}(\epsilon) < \infty$ *for all* $\epsilon \geq 0$.

As in discrete time, the steady state turnpike property ensures that each trajectory for which the associated cost is close to the optimal steady state value stays most of the time in a neighborhood of x^e. The state turnpike property states that, for any initial condition x_0 and any horizon length $T > 0$, the time that the optimal solutions of the optimal control problem spend outside an ϵ-neighborhood of x^e is bounded by $v_{x^e}(\epsilon)$, where $v_{x^e}(\epsilon)$ is independent of the horizon length T. The input-state turnpike property is stronger in the sense that the inequality involves x and u.

If the conditions in *b*) or *c*) hold for $\epsilon = 0$, then, the optimal solutions have to enter the turnpike exactly (exact turnpike) for some part of the horizon.

Definition 2.5. *System 2.1 is said to be optimally operated at the steady state pair* (x^e, u^e) *if for any initial condition* $x_0 \in \mathcal{X}$ *and any infinite-time admissible pair* $z(\cdot, x_0)$*, we have*

$$\liminf_{T\to\infty} \frac{J_T(x_0,u(\cdot))}{T} \geq F(x^e,u^e) \tag{2.5}$$

Definition 2.6. *We say that the system 2.1 is locally controllable around a steady state pair* (x^e, u^e) *if there exists* $\kappa > 0$ *such that for each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that for any two points* $x, y \in B_\delta(x^e)$ *there is a control* $u \in \mathcal{L}([0,\kappa],\mathcal{U})$ *with* $x(\kappa, x, u(\cdot)) = y$ *and* $\max\{||x(t,x,u(t)) - x^e||, ||u(t) - u^e||\} \leq \epsilon$ *for all* $t \in [0, \kappa]$.

2.3 Auxiliary Results

Theorem 2.1 ([16])**.** *The available storage,* S_a*, is finite for all* $x_0 \in \mathcal{X}$ *if and only if 2.1 is dissipative. Moreover, if the system is dissipative with respect to the supply rate* w*, then* $0 \leq S_a(x_0) \leq S(x_0)$ *for any storage function* S *and* S_a *is itself a possible storage function.*

Remark 2.1. *(Verifying Strict Dissipativity). Note that in order to verify strict dissipativity, one uses that Theorem 2.1 applies to generic supply rates* w*, i.e., one swaps* w *with* $-\alpha + w$ *in Definition 2.3 and shows that, for at least one* $\alpha \in \mathcal{K}_\infty$*, the available storage* S_a *is finite.*

Remark 2.2. *(Strict Dissipativity of $OCP_T(x_0)$). To show strict dissipativity of an optimal control problem, one swaps w with $-\alpha + w$ and "$u(\cdot) \in \mathcal{L}([0,T],\mathcal{U})$" in Definition 2.3 with "$u(\cdot)$ is optimal".*

Theorem 2.2 ([4])**.** *If the optimal control problem 2.2 is dissipative with respect to the steady state pair z^e, then the system 2.1 is optimally operated at z^e.*

Remark 2.3. *Note that Theorem 2.2 also hold for strict dissipativity since this property implies dissipativity.*

For the next theorem we need to define the following sets:

$$\mathcal{C}(x^e,T) = \{x_0 \in \mathcal{X} |\ \exists u(\cdot) \in \mathcal{L}([0,T],\mathcal{U}),\ \forall t \in [0,T] :$$
$$x(t,x_0,u(\cdot)) \in \mathcal{X},\ x(T,x_0,u(\cdot)) = x^e\}$$

which is the set of initial conditions $x_0 \in \mathcal{X}$ that can be steered, in some finite time $T \in (0,\infty)$, by means of an admissible input, to $x^e \in \mathcal{X}$. Likewise, define the set

$$\mathcal{R}(x^e,T) = \{x_T \in \mathcal{X} |\ \exists u(\cdot) \in \mathcal{L}([0,T],\mathcal{U}),\ \forall t \in [0,T] :$$

$$x(t,x^e,u(\cdot)) \in \mathcal{X},\ x(T,x^e,u(\cdot)) = x_T\}$$

which contains all the states that can be reached, in some finite time $T \in (0,\infty)$, by means of an admissible input, starting from x^e. Since, by construction, x^e is contained in both sets, we have $\mathcal{C}(x^e,T) \cap \mathcal{R}(x^e,T) \neq \emptyset$. Let

$$\mathcal{X}_T = \{x_0 \in \mathcal{X} |\ \forall t \in [0,T] : \exists u(\cdot) \in \mathcal{L}([0,T],\mathcal{U}),$$

$$x(t,x_0,u(\cdot)) \in \mathcal{C}(x^e,T) \cap \mathcal{R}(x^e,T)\}$$

be the set of initial conditions x_0, for which there exists a corresponding admissible pair $z(\cdot,x_0,u(\cdot))$ and the inclusion $x(t,x_0,u(\cdot)) \in \mathcal{C}(x^e,T) \cap \mathcal{R}(x^e,T)$ holds for all $T \in [0,T]$.

Theorem 2.3 ([4])**.** *If the system 2.1 is optimally operated at z^e, then the optimal control problem 2.2 is dissipative on $\mathcal{Z}_T = \mathcal{X}_T \times \mathcal{U}$.*

Theorem 2.4 ([4])**.** *Consider the optimal control problem* 2.2 *and let (x^e, u^e) be a steady state pair. If the optimal control problem has the state turnpike* *property and $\frac{\partial F}{\partial u} = 0$, or has the input-state turnpike property, the system is optimally operated.*

2.4 Main Results

2.4.1 Steady State Turnpike Property

Theorem 2.5. *Consider the optimal control problem 2.2 and let (x^e, u^e) be a steady state pair. If the optimal control problem is dissipative with supply rate $w(x, u)$ and has the steady state turnpike property, then it is strictly dissipative with respect to the steady state x^e.*

Proof. Consider a two sided strictly increasing sequence ϵ_i, $i \in \mathbb{Z}$, with $\epsilon_i \to \infty$ as $i \to \infty$, $\epsilon_i \to 0$, as $i \to -\infty$ and $\rho(\epsilon_0) = 1$ for ρ from the steady state turnpike property. Let $\overline{\rho} \in \mathcal{K}_\infty$ be linear on $[\epsilon_i, \epsilon_{i+1}]$ for all $i \in \mathbb{Z}$, then $\overline{\rho}$ is uniquely determined by its values $\overline{\rho}_i = \overline{\rho}(\epsilon_i)$ and it holds that $\overline{\rho}(r) \leq \overline{\rho}_{i+1}$ for all $r \in [\epsilon_i, \epsilon_{i+1}]$.

We now set $\overline{\rho}_i = \rho(\epsilon_{i-1})^2/8$ for $i \leq 1$ and $\overline{\rho}_i = \sqrt{\rho(\epsilon_{i-1})}/4$ for $i \geq 2$ and claim that the system is strictly dissipative with the resulting piecewise linear $\overline{\rho}$. In order to prove this, consider an arbitrary admissible trajectory $x(\cdot)$ of length T with control $u(\cdot)$. We define $\delta = \max\{J_T(x, u) - TF(x^e, u^e), 0\}$, implying that the condition in the steady state turnpike property is satisfied with this δ.

Consider the index sets $Q_i = \{t \in [0, T] |\ ||x(t, x_0, u(t)) - x^e|| \in (\epsilon_i, \epsilon_{i+1}]\}$. Then the definition of $\overline{\rho}$ implies:

$$\int_0^T \overline{\rho}(||x(t, x_0, u(t)) - x^e||)dt \leq \sum_{i=-\infty}^{\infty} \mu(Q_i)\overline{\rho}_{i+1}$$

After a certain i, the $\mu(Q_i)$-terms in the sum are going to be equal to 0. This is because $||x(t, x_0, u(t)) - x^e||$ is a continuous function in a compact set, hence $||x(t, x_0, u(t)) - x^e|| \leq M$, which implies that for $\epsilon_i > M$, $\mu(Q_i) = 0$. Now we can write that sum as

$$\sum_{i=-\infty}^{\infty} \mu(Q_i)\overline{\rho}_{i+1} = \sum_{i=-\infty}^{-m-1} \mu(Q_i)\overline{\rho}_{i+1} + \sum_{i=-m}^{m} \mu(Q_i)\overline{\rho}_{i+1}$$

for any $m > M$. Now the steady state turnpike implies the inequality

$$\kappa_j = \sum_{i=j}^{\infty} \mu(Q_i) \leq \frac{\delta + C_a}{\rho(\epsilon_j)} = P_{j,\delta}$$

for the constant C_a from the definition of steady state turnpike property. Since $\mu(Q_i) = \kappa_i - \kappa_{i+1}$ this implies

$$\sum_{i=-m}^{m} \mu(Q_i)\overline{\rho}_{i+1} = \sum_{i=-m}^{m} (\kappa_i - \kappa_{i+1})\overline{\rho}_{i+1}$$

$$= \kappa_{-m}\overline{\rho}_{-m+1} + \sum_{i=-m+1}^{m} \kappa_i(\overline{\rho}_{i+1} - \overline{\rho}_i) - \kappa_{m+1}\overline{\rho}_{m+1}$$

$$= \kappa_{-m}\overline{\rho}_{-m+1} + \sum_{i=-m+1}^{m} \kappa_i(\overline{\rho}_{i+1} - \overline{\rho}_i)$$

$$\leq P_{-m,\delta}\overline{\rho}_{-m+1} + \sum_{i=-m+1}^{m} P_{i,\delta}(\overline{\rho}_{i+1} - \overline{\rho}_i)$$

where in the third step we took into account that the choice of m implies $\kappa_{m+1} = 0$. For the first term we obtain the estimate

$$P_{-m,\delta}\overline{\rho}_{-m+1} \leq \frac{\delta + C_a}{\rho(\epsilon_{-m})}\frac{\rho(\epsilon_{-m})^2}{2} = \frac{\delta + C_a}{2}\rho(\epsilon_{-m})$$

and since m can be chosen arbitrarily large, we may choose m such that $\rho(\epsilon_{-m}) \leq 1/4$ implying

$$P_{-m,\delta}\overline{\rho}_{-m+1} \leq \frac{\delta + C_a}{8}$$

For the second term, using the definition of $P_{i,\delta}$ and that the definition of $\overline{\rho}_i$ implies $\rho(\epsilon_{i-1}) = \sqrt{8}\sqrt{\overline{\rho}_i}$ for $i \leq 1$ and $\rho(\epsilon_{i-1}) = 16\overline{\rho}^2$ for $i \geq 2$, we can estimate

$$\sum_{i=m+1}^{m} P_{i,\delta}(\overline{\rho}_{i+1} - \overline{\rho}_i) = (\delta + C_a)\sum_{i=-m+1}^{m} \frac{\overline{\rho}_{i+1} - \overline{\rho}_i}{\rho(\epsilon_i)} = (\delta + C_a)\sum_{i=-m+2}^{m+1} \frac{\overline{\rho}_i - \overline{\rho}_{i-1}}{\rho(\epsilon_{i-1})}$$

$$= (\delta + C_a)\sum_{i=-m+2}^{1} \frac{\overline{\rho}_i - \overline{\rho}_{i-1}}{\rho(\epsilon_{i-1})} + (\delta + C_a)\sum_{i=2}^{m+1} \frac{\overline{\rho}_i - \overline{\rho}_{i-1}}{\rho(\epsilon_{i-1})}$$

$$= (\delta + C_a)\sum_{i=-m+2}^{1} \frac{\overline{\rho}_i - \overline{\rho}_{i-1}}{\sqrt{8}\sqrt{\overline{\rho}_i}} + (\delta + C_a)\sum_{i=2}^{m+1} \frac{\overline{\rho}_i - \overline{\rho}_{i-1}}{16\overline{\rho}_i^2}$$

$$\leq (\delta + C_a) \int_0^{1/8} \frac{1}{\sqrt{8}\sqrt{x}} dx + (\delta + C_a) \int_{1/8}^{\infty} \frac{1}{16x^2} dx$$

$$\leq (\delta + C_a)(\frac{1}{4} + \frac{1}{2}) = \frac{3}{4}(\delta + C_a)$$

Here in the fourth step we used that the respective sums are lower Riemann sums for the respective integrals since the integrands $1/\sqrt{x}$ and $1/x^2$ are strictly decreasing. All in all we thus proved that we obtain

$$\int_0^T \overline{\rho}(||x(t, x_0, u(t)) - x^e||) \leq \delta + C_a + \sum_{i=-\infty}^{-m-1} \mu(Q_i)\overline{\rho}_{i+1} \leq \delta + C_a + T\overline{\rho}_{-m}$$

for all admissible trajectories of arbitrary length T, with $\delta = \max\{J_T(x, u) - TF(x^e, u^e), 0\}$. Now for any admissible trajectories with this definition of δ we obtain

$$\int_0^T -(F(x(t), u(t)) - F(x^e, u^e) - \overline{\rho}(||x(t, x_0, u(t)) - x^e||))dt$$

$$= -J_T(x, u) + TF(x^e, u^e) + \int_0^T \overline{\rho}(||x(t, x_0, u(t)) - x^e||)$$

$$\leq -(J_T(x, u) - TF(x^e, u^e)) + \delta + C_a + T\overline{\rho}_{-m}$$

$$= T\overline{\rho}_{-m} + C_a + \max\{0, - \inf_{T \in \mathbb{R}_+, u \in \mathcal{L}([0,T],\mathcal{U})} J_T(x, u) - TF(x^e, u^e)\} = C' < \infty$$

where C' is finite because the system is dissipative and hence the $-$ inf-term is bounded by Theorem 2.1. Using Theorem 2.1 with $\overline{\rho}$ shows strict dissipativity. □

Theorem 2.6. *Consider the optimal control problem 2.2 and let* $(x^e, u^e) \in$ *int*(Z) *be a steady state pair. If 2.1 has the steady state turnpike property, is optimally operated at* (x^e, u^e) *and is locally controllable around the steady state, the optimal control problem is dissipative.*

Proof. Assume without loss of generality that $F(x^e, u^e) = 0$. Now assume that the system is not dissipative on $\mathcal{Z}$. By Theorem 2.1, this is equivalent to the fact that the available storage is unbounded on $\mathcal{X}$, and hence for each $r \geq 0$ there exists some $y \in \mathcal{X}$ such that

$$\inf_{x(0)=y, T\geq 0} \int_0^T F(x(t), u(t))dt \leq -r \tag{2.6}$$

This mean that for each $r \geq 0$, there exist some $y \in \mathcal{X}$ and an admissible pair (x_r, u_r) together with a time instant $T_r \in \mathbb{R}_+$ such that $x_r(0) = y$ and the next inequality is satisfied

$$\int_0^{T_r} F(x_r(t), u_r(t))dt \leq -r \tag{2.7}$$

Note that since F is bounded, it follows that $T_r \to \infty$ as $r \to \infty$. Now, lets fixed $\hat{\epsilon} > 0$ such that $B_{\hat{\epsilon}}(x^e, u^e) \subset \mathcal{Z}$, and choose the corresponding δ according to Definition 2.6. Since the optimal control problem has the steady state turnpike property and is optimally operated at the steady state, for all admissible pairs at least one of the following two conditions is satisfied for some $\bar{t} \in \mathbb{R}_+$

$$\int_0^T \frac{F(x(t), u(t))}{T} dt \geq F(x^e, u^e) \quad \textit{for all } T \geq \bar{t} \tag{2.8}$$

$$|x(s) - x^e| \leq \delta \quad \textit{for all } s \in (0, \bar{t}]/A, \; \textit{with } \mu(A) = C_a/\rho(\delta) \tag{2.9}$$

Because of the steady state turnpike property, C_a and ρ are fixed. By choosing $\bar{t} > C_a/\rho(\delta)$ we can guarantee that at least one of the previous inequalities is going to be satisfied. Note that for any admissible pair we can have (for $T = \bar{t}$):

$$\int_0^T \frac{F(x(t), u(t))}{T} dt \geq F(x^e, u^e) \; or \; \int_0^T \frac{F(x(t), u(t))}{T} dt < F(x^e, u^e)$$

For the second inequality, the conditions for the turnpike property are satisfied and thus (2.9) holds for some $s \in (0, \bar{t}]$. If the first inequality is satisfied, it can either hold for all $T \geq \bar{t}$ or there exists a $T > \bar{t}$ for which the inequality does not hold. In the first case, we have that (2.8) is satisfied, and in the second we can choose $T = \bar{t}$ and do the same analysis.

Define c as

$$c = \max\{\kappa \max_{(x,u)\in B_{\hat{\epsilon}(x^e,u^e)}} F(x,u), -\bar{t} \min_{(x,u)\in \mathcal{Z}} F(x,u)\} \tag{2.10}$$

Now consider some $r \geq 1+3c$ and note that, in this case, $T_r \geq 3\bar{t}+a$, for some $a \in \mathbb{R}_+$, as $-r < 3\bar{t}\min_{(x,u)\in\mathcal{Z}} F(x,u)$. Hence, due to the steady state turnpike property and optimal operation at the steady state, we conclude that $|x_r(s_1)-x^e| \leq \delta$ for some $s_1 \in (0,\bar{t}]$. Furthermore, as $\int_0^{s_1} F(x_r(t),u_r(t)) \geq s_1 \min_{(x,u)\in\mathcal{Z}} F(x,u) \geq -c$ by definition of c, we have

$$\int_{s_1}^{T_r} F(x_r(t),u_r(t))dt \leq -(1+2c) \tag{2.11}$$

and $T_r - s_1 \geq 2\bar{t}+a$ as $s_1 \leq \bar{t}$. We can now apply the above argument to the shifted sequence $x_r^{'}(t) = x_r(t+s_1)$ and conclude by the steady state turnpike property and optimal operation at the steady state that $|x_r^{'}(s_2)-x^e| = |x_r(s_1+s_2)-x^e| \leq \delta$ for some $s_2 \in (0,\bar{t}]$. Furthermore,

$$\int_{s_1+s_2}^{T_r} F(x_r(t),u_r(t))dt \leq -(1+c) \tag{2.12}$$

by definition of c, and $T_r - s_1 - s_2 \geq \bar{t}+a$ as $s_2 \leq \bar{t}$. Repeating again the above argument, we conclude that $|x_r(s_1+s_2+s_3)-x^e| \leq \delta$ for some $s_3 \in (0,\bar{t}]$. We can now distinguish two different cases. Either we have

$$\int_{s_1+s_2+s_3}^{T_r} F(x_r(t),u_r(t))dt \geq -c \tag{2.13}$$

or 2.13 does not hold, in which case the definition of c implies that $T_r-(s_1+s_2+s_3) > \bar{t}$. In the latter case, we can apply the above argument recursively to obtain time instances s_i, $i > 4$, with $|x_r(s_1+...+s_i)-x^e| \leq \delta$ until

$$\int_{s_1+...s_j}^{T_r} F(x_r(t),u_r(t))dt \geq -c \tag{2.14}$$

for some $j \geq 4$ (the s_k can be selected in such a way that make j finite).

Summarizing the above, it has been proven that both $|x_r(s_1)-x^e| \leq \delta$ and $|x_r(s_1+...+s_j)-x^e| \leq \delta$, and

$$\int_{s_1}^{s_1+...+s_j} F(x_r(t),u_r(t))dt \leq -(1+c) \tag{2.15}$$

Hence, by local controllability at the optimal steady state (x^e,u^e), there exists an admissible pair $(x^{'},u^{'})$ satisfying $x^{'}(0) = x_r(s_1+...+s_j)$, $x^{'}(\kappa) =$

$x_r(s_1)$, and $(x^{'}(t), u^{'}(t)) \in B_{\hat{\epsilon}}(x^e, u^e) \cap \mathcal{Z}$ for all $t \in [0, \kappa]$. By definition of c we have

$$\int_0^{\kappa} F(x^{'}(t), u^{'}(t)) \leq c \tag{2.16}$$

Now define the input sequence:

$$\hat{u}(k(s_2+...+s_j+\kappa)+i) = \begin{cases} u_r(s_1 + i) & k \in \mathbb{N},\ i \in [0, s_2 + ... + s_j] \\ u^{'}(i) & k \in \mathbb{N},\ i \in (s_2 + ... + s_j, ..., s_2 + ... + s_j + \kappa] \end{cases} \tag{2.17}$$

which results in a cyclic state sequence with $\hat{x}(k(s_2+...+s_j+\kappa)) = x_r(s_1)$ for all $k \in \mathbb{N}_0$. By construction, this pair $(\hat{x}(t), \hat{u}(t))$ is admissible. Furthermore, we obtain for all $t \in \mathbb{R}_+ \cup \{0\}$:

$$\int_0^{s_2+...+s_j+\kappa} F(\hat{x}(k(s_2+...+s_j+\kappa)+i), \hat{u}(k(s_2+...+s_j+\kappa)+i))di \leq -1 \tag{2.18}$$

However, this implies that

$$\liminf_{T\to\infty} \int_0^T \frac{F(\hat{x}(t), \hat{u}(t))}{T} dt = \frac{1}{s_2 + ... + s_j + \kappa} \int_0^{s_2+...+s_j+\kappa} F(\hat{x}(i), \hat{u}(i))di \tag{2.19}$$

$$\leq \frac{-1}{s_2 + ... + s_j + \kappa} < 0$$

which contradicts optimal steady state operation. Therefore, we conclude that the system is dissipative on $\mathcal{Z}$.

□

Theorem 2.7. *Consider the optimal control problem 2.2. Let (x^e, u^e) be a steady state around which the system is locally controllable and assume it is optimally operated at that steady state. Then, if the optimal control problem has the steady state turnpike property, it is strictly dissipative with respect to x^e.*

Proof. From Theorem 2.6 we have that the system is dissipative. By theorem 2.5, this, together with the steady state turnpike property implies strict dissipativity. □

Theorem 2.8. *Consider the optimal control problem 2.2 and let* (x^e, u^e) *be a steady state pair. Assume that the optimal control problem is dissipative with respect to that steady state and there exists a constant* $C > 0$ *and* $\rho \in \mathcal{K}_\infty$ *such that for each* $x_0 \in \mathcal{X}$*,* $\delta > 0$ *and* $T \in \mathbb{R}_+$*, each control* $u \in \mathcal{L}([0,T],\mathcal{U})$ *satisfying* $J_T(x_0, u) \leq V_T(x_0) + \delta$ *and each* $\epsilon > 0$ *the value of* $Q_\epsilon = \mu\{t \in [0,T] \mid ||x(t, x_0, u(t)) - x^e|| \leq \epsilon\}$ *satisfy the inequality* $Q_\epsilon \geq T - (\delta + C)/\rho(\epsilon)$*. Then, the optimal control problem has the steady state turnpike property.*

Proof. For all $T \in \mathbb{R}_+$, $x_0 \in \mathcal{X}$ and $u \in \mathcal{L}([0,T],\mathcal{U})$, the dissipation inequality 2.3 implies

$$J_T(x_0, u) = \int_0^T F(x, u)$$

$$\geq S(x_T) - S(x_0) + TF(x^e, u^e) \geq TF(x^e, u^e) - M$$

where M is a bound on $S(x_T) - S(x_0)$. Since this holds for every $u \in \mathcal{L}([0,T],\mathcal{U})$, it holds for V_T. This inequality, together with

$$J_T(x_0, u) \leq KF(x^e, u^e) + \delta$$

imply that

$$J_T(x_0, u) \leq V_T(x_0) + \delta + M$$

from which the result follows with $C_a = C + M$

□

Theorem 2.8 is also valid in the case of strict dissipativity, since this property implies dissipativity.

2.4.2 State Turnpike Property

Assumption 2.1. *For all* $x_0 \in \mathcal{X}$*, there exist an optimal steady state* x^e*, an infinite admissible input* $u_\infty(\cdot, x_0) \in \mathcal{L}([0,\infty),\mathcal{U})$*, and constants* $c \in \mathbb{R}^+$ *and* $\rho \in [0, 1)$*, independent of* x_0*, such that*

$$||(x(t, x_0, u_\infty(\cdot, x_0)), u_\infty(t)) - (x^e, u^e)|| \leq c\rho^t$$

Theorem 2.9 ([4])**.** *For all* $x_0 \in \mathcal{X}$*, let the optimal control problem 2.2 be strictly dissipative with respect to* x^e*. Suppose that Assumption 2.1 hold, then, the optimal control problem has a state turnpike at* x^e*.*

Theorem 2.10. *Consider the optimal control problem 2.2 and let* (x^e, u^e) *be a steady state pair. If the optimal control problem is dissipative with supply rate* $w(x, u)$*, has the steady state turnpike property and satisfy Assumption 2.1, then it has the state turnpike property.*

Proof. By Theorem 2.5 the optimal control problem is strictly dissipative. The result follows from Theorem 2.9. □

Theorem 2.11. *Consider the optimal control problem 2.2 and let* (x^e, u^e) *be a steady state pair. If the optimal control problem has the state turnpike property and* $\frac{\partial F}{\partial u} = 0$*, it is dissipative on* $\mathcal{Z}_T$*.*

Proof. By Theorem 2.4 the optimal control problem is optimally operated at the steady state. The results follows from Theorem 2.3 □

Theorem 2.12. *Consider the optimal control problem 2.2 and let* (x^e, u^e) *be a steady state pair. If the optimal control problem has the state turnpike property, the steady state turnpike property and* $\frac{\partial F}{\partial u} = 0$*, it is strictly dissipative on* $\mathcal{Z}_T$ *with respect to* x^e*.*

Proof. By Theorem 2.4 the optimal control problem is optimally operated at the steady state. The results follows from Theorems 2.5 and 2.11. □

2.4.3 Input-State Turnpike Property

Theorem 2.13 ([4]). *Assume that for all* $x_0 \in \mathcal{X}$*, the optimal solutions of the control problem have an exact input-state turnpike at* z^e*. Then the optimal control problem is strictly dissipative with respect to* z^e*.*

Assumption 2.2. *There exists a constant* $\rho > 0$ *and* $\alpha_\rho \in \mathcal{K}_\infty$ *such that*

$$\alpha_\rho(||z - z^e||) \leq F(z) - F(z^e), \qquad \forall z \in B_\rho(\overline{z}) \cap Z.$$

Theorem 2.14 ([4]). *Suppose that for all* $x_0 \in \mathcal{X}$*, the optimal solutions of the control problem 2.2 have a turnpike at* z^e *and that Assumption 2.2 holds. Then there exists a storage function* S *depending on* α_ρ*, such that the optimal control problem is strictly dissipative with respect to* z^e*.*

Assumption 2.3. *For all* $x_0 \in \mathcal{X}$*, there exist an optimal steady state* z^e*, an infinite admissible input* $u_\infty(\cdot, x_0) \in \mathcal{L}([0, \infty), \mathcal{U})$*, and constants* $c, \lambda \in \mathbb{R}_+$*, independent of* x_0*, such that*

$$||(x(t, x_0, u_\infty(\cdot, x_0)), u_\infty(\cdot, x_0)) - z^e)|| \leq ce^{-\lambda t}$$

Theorem 2.15 ([4])**.** *For all* $x_0 \in \mathcal{X}$*, let the optimal control problem 2.2 be strictly dissipative with respect to* z^e*. Suppose that Assumption 2.3 holds. Then, the optimal control problem has a turnpike at* z^e*.*

Theorem 2.16. *Consider the optimal control problem 2.2 and let* (x^e, u^e) *be a steady state pair. If the optimal control problem has the input-state turnpike property, it is dissipative on* $\mathcal{Z}_T$*.*

Proof. By Theorem 2.4 the optimal control problem is optimally operated at the steady state. The results follows from Theorem 2.3 □

Theorem 2.17. *Consider the optimal control problem 2.2 and let* (x^e, u^e) *be a steady state pair. If the optimal control problem has the input-state turnpike property and the steady state turnpike property, it is strictly dissipative on* $\mathcal{Z}_T$ *with respect to* x^e*.*

Proof. By Theorem 2.4 the optimal control problem is optimally operated at the steady state. The results follows from Theorems 2.5 and 2.16. □

2.5 Examples

2.5.1 Steady State Turnpike Property

Consider a system on $\mathcal{Z} = [0, 4] \times [0, 1/2]$ with dynamics and stage cost

$$\dot{x}(t) = -2x(t)$$

$$F(x, u) = u^2(t) + x^2(t)$$

The system has an optimal steady state at $(x^e, u^e) = (0, 0)$, and all the trajectories of the system converge to $(0, 0)$, because of this, the system has a steady state turnpike at $(0, 0)$.

Using Theorem 2.1 we can see that the system is dissipative with the available supply as the storage function:

$$S_a(x_0) = \sup_{u(\cdot)\in\mathcal{L}([0,T],\mathcal{U}),T\geq 0} \int_0^T -(F(x, u) - F(x^e, u^e))dt$$

$$= \sup_{u(\cdot)\in\mathcal{L}([0,T],\mathcal{U}),T\geq 0} \int_0^T -x^2(t) - u^2(t)dt = 0$$

All hypothesis from Theorem 2.5 hold, thus the optimal control problem is strictly dissipative.

2.5.2 State Turnpike Property

Consider a system on $\mathcal{Z} = [-1/2, 1/2] \times [-2, 2]$ with dynamics

$$\dot{x}(t) = -\frac{1}{2}x(t) + u^2(t)$$

The optimal control problem reads

$$\min_{u \in \mathcal{L}([0,T],\mathcal{U})} \int_0^T x(t)dt$$

subject to $\dot{x}(t) = -\frac{1}{2}x(t) + u^2(t)$

The Hamiltonian and the adjoint equation of this optimal control problem is the following:

$$H(x, u, p) = -x(t) + p(t)(-\frac{1}{2}x(t) + u^2(t))$$

$$p^{'}(t) = 1 + \frac{p(t)}{2}$$

The solution of the adjoint equation is $p(t) = e^{\frac{t+c}{2}} - 2$. Since the Hamiltonian is convex in $u(t)$, finding its minimum will give the optimal control.

$$H_u(x, u, p) = p(t)u(t) = (e^{\frac{t+c}{2}} - 2)u(t)$$

this implies that $u^*(t) = 0$. The solution, $x^*(t) = ae^{-\frac{1}{2}t}$, of the differential equation $\dot{x}(t) = -\frac{1}{2}x(t)$ is the optimal trajectory of the problem.

Clearly, the optimal control problem has a steady state pair at $(0, 0)$ and the optimal solution converge to it, thus, the optimal control problem has a turnpike at the steady state.

All the assumptions from Theorem 2.13 hold, thus, the optimal control problem is dissipative on $\mathcal{Z}_N$

2.5.3 Input-State Turnpike Property

In a continuously stirred tank reactor, three endothermal chemical reactions $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ and $2A \xrightarrow{k_3} D$ take place. A partial model of the reactor, including the concentration of species A and B, c_A, c_B and the reactor temperature v as state variables, reads

$$\dot{c}_A = r_A(c_A, v) + (c_{in} - c_A)u_1$$

$$\dot{c}_B = r_B(c_A, c_B, v) - c_B u_1$$

$$\dot{v} = h(c_A, c_B, v) + \alpha(u_2 - v) + (v_{in} - v)u_1$$

where

$$r_A(c_A, v) = -k_1(v)c_A - 2k_3(v)c_A^2$$

$$r_B(c_A, c_B, v) = k_1(v)c_A - k_2(v)c_B$$

$$h(c_A, c_B, v) = -\delta(k_1(v)c_A\Delta H_{AB} + k_2(v)c_B\Delta_{BC})$$

$$+2k_3(v)c_A^2\Delta H_{AD})$$

$$k_i(v) = k_{i0}exp\frac{-E_i}{v + v_0}$$

All other symbols denote constant parameters and its values can be found in [13]. The inputs u_1 and u_2 are the normalized flow rate of A through the reactor and the temperature in the cooling jacket. The states and inputs are subject to the constraints

$$c_A \in [0, 6] \quad c_B \in [0, 4] \quad v \in [70, 200]$$

$$u_1 \in [3, 35] \quad u_2 \in [0, 200]$$

We consider the problem of maximizing the production rate of c_B. Thus we specify the cost function F as

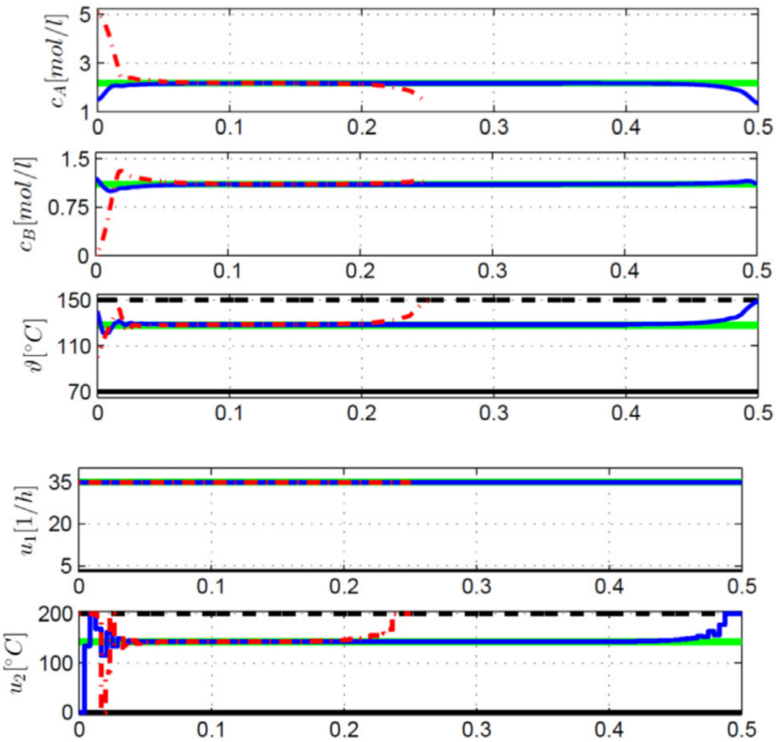
$$F(c_B, u_1) = -\beta c_B u_1 \qquad \beta > 0.$$

As shown in [4], the system has an exact turnpike property at the optimal steady state pair

$$x^e = [2.1756, 1.1049, 128.53] \qquad u^e = [35, 142.76]$$

Then, by Theorem 2.16, the system is strictly dissipative at (x^e, u^e).

The next figure show the exact turnpike property for the state and the input for different initial conditions and horizon length.



Conclusions

In this thesis we have studied the relationship between dissipativity properties and different versions of the turnpike property.

For discrete time optimal control problems, we have used previous results in this setting and extended continuous time results to show that dissipativity and strict dissipativity are equivalent to the turnpike property. Also, we have proven that strict dissipativity properties implies the exponential turnpike property.

For continuous time optimal control problems, we have extended discrete time results to show the equivalence of dissipativity properties and steady state turnpike properties. We have used this equivalences to prove that dissipativity implies the state turnpike property.

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