Universidad Nacional Autónoma de México

POSGRADO EN CIENCIAS FÍSICAS

GRAVITATIONAL WAVES IN TELEPARALLEL GRAVITY

T E S I S

QUE PARA OPTAR POR EL GRADO DE

MAESTRO EN FÍSICA

PRESENTA:
ARMANDO ARAM REYES AGUILAR

Tutor Principal:
Dra. Celia del Carmen Escamilla Rivera
Instituto de Ciencias Nucleares

Miembros del Comité Tutor:
Dra. Celia del Carmen Escamilla Rivera
Instituto de Ciencias Nucleares
Dr. Roberto Allan Sussman Livovsky
Instituto de Ciencias Nucleares
Dra. Myriam Mondragón Ceballos
Instituto de Física

México, CDMX. Marzo 2021

Fac me vere tecum flere,
Crucifixo condolere,
Donec ego vixero.
Juxta crucem tecum stare
Te libenter sociare
In planctu desidero.

Gratitude

I thank to my family and friends for all the support and patience during all this process. I would also like to thank to those people who helped me to understand CLASS; in particular José Alberto Vásquez, Nayeli Jiménez, Osvaldo Tellez, Carlos García, Rafael Nunes and Jurgen Mifsud. I thank to Jackson Levi for clearing me theoretical doubts with regard to this beautiful theory called Teleparallel Gravity. Finally, I extend my gratitude to Dr. Celia Escamilla for all her support and kindness from the beginning of this journey.

Abstract

Gravitational waves (GWs) have become an important topic during the last years. The description of them can be achieved using the so known General Relativity (GR). The GWs can be formed in different ways, for example a binary system of black holes rotating around each other. However, the primordial GWs, which are located at the largest redshifts, might be better described by a completely different theory of gravitation; Teleparallel Gravity (TG), rather than GR. In this thesis, I show that deviations can be reached using TG extensions and that for other several reasons it can be considered a more appropriate theory in the description of the gravitational interaction.

Contents

Introduction

In 1907 Albert Einstein started to work in a new project when he came up with a deep idea: For an observer in a free-falling cabin there is no gravitational field, and this would be because of the equality of inertial and gravitational masses; in the free-falling cabin the gravitational force and the force of inertia (which is the property common to all the bodies that remain in their state, either at rest or in motion) cancel each other. This idea would be known later as 'the equivalence principle', and Einstein thought that it was a necessity to generalize the postulate of relativity on non inertial coordinate systems [1]. From the equivalence principle Einstein concluded that the velocity of light depend on the gravitational potential. Years of work ended with the publication of the theory of General Relativity (GR) in 1915, a theory that is constructed over pseudo-Riemannian geometry and that is based on the idea that the physics should not depend on the reference systems, hence invariance of basic equations under transformations of coordinates in a general coordinate system is an important requirement. The curvature of space-time is the unique cause of gravity in GR; of course, this new vision broke up with the Newtonian vision of gravity.

At this point, scientists had little knowledge about the universe. The idea that it was static seemed reasonable, and with the intention to produce a universe with such feature, in 1917 Einstein included an additional term in his equations of GR, which was weighted by a constant Λ called the cosmological constant [2]. Later on, the idea would be rejected when in 1931 Edwin Hubble, after improving his 1929 work, reported in his observations of the redshift of galaxies that the universe was expanding. Einstein considered to have made a terrible mistake with the cosmological constant. Nonetheless, the use of this extra term would be considered, with other purposes, decades after.

In 1922, before the discovery of spin, the French mathematician Élie Cartan modified Einstein's relativity allowing space-time to have, in addition to curvature, torsion, and relating torsion to the density of intrinsic angular momentum [3]. Around those years, and independently, the Austrian mathematician Roland Weitzenböck, thanks to his work in differential geometry, developed a particular metric connection (that today is called after him) which nullifies curvature and allows torsion to be the only element that bends the manifold (space-time); indeed, torsion would be another natural space-time deformation [4]. Theories with active torsion sometimes are referred as teleparallel, independently whether curvature is active too. Nevertheless, what we might call Teleparallel Gravity (TG) is the theory in concordance with Weitzenböck's vision; torsion is the unique element responsible for gravity.

Following these important works on torsion, in 1928 Einstein included torsion in GR in his attempt of a unified field theory with teleparalellism. In fact, he received advise from both Cartan and Weitzenböck, and Cartan wrote an essay about the history of the relevant mathematical topic for Einstein's paper [5]. This modification of GR is known as the Einstein-Cartan theory. However, these kind of theories were left aside until the late 1950s, when further investigation on torsion appeared. Theories that consider important the inclusion of torsion, in addition to curvature, in the description of the gravitational interaction; such as Einstein-Cartan and gauge theories for the Poincaré and the affine groups, proposed curvature and torsion as independent degrees of freedom, and considered that torsion should become relevant only when spins are important.

The growing observational tensions across decades has motivated several physicist to explore the possible landscape of gravity theories beyond GR, and specifically to study the use of torsion to describe the gravitational interaction [6], independently whether it is related with spin or not. TG started to gain some attention since then.

In 1927 the Belgian cosmologist Catholic priest Georges Lemaître, convinced that the universe was expanding, followed the work of Einstein to propose a cosmological model. This solution, which built the base for the Big Bang theory, was a symmetrical spherical space that grew exponentially over time, showing that the static cosmogonies were unstable. Observations, as those captured by Hubble, gave popularity to Lemaître and the idea that the universe is actually in expansion. Later on, in 1946, Lemaître published a book that was made of a collection of five lectures delivered by him between 1929 and 1945 with regard to the primeval atom [7].

The properties of the expanding universe can also be investigated through the study of Gravitational Waves (GWs), whose existence had been predicted by GR. This is because a possible detection would lead to the understanding of not only the GWs properties, as speed and polarization modes, but also to the understanding of the background on which they propagate. Evidently, a GW detection offers a new spectrum of possibilities on which a better research of nature can be done at a fundamental level [8].

On September 14, 2015, almost one hundred years after the publication of GR, it was directly measured, for the very first time, a GW. It was achieved by LIGO (Laser Interferometer Gravitational-Wave Observatory), and corresponds to GWs that were emitted by two black holes colliding and merging into one. This system was located 1.3 billion light-years away from earth [9], and in that moment, GWs passed from having an indirect experimental confirmation to a direct experimental confirmation; GR turned out to be a successful and consistent theory. Many models assume GR as the correct theory to describe gravity as happens with the well known ΛCDM cosmological model, which is the simplest model that predicts some of the most important features of the universe and that is in concordance with the Big Bang theory. It was proposed in

1998 and substituted SCDM (Standard CDM) of 1982 [10]. The ΛCDM cosmological model is, in consequence, motivated by well-known observational successes in describing phenomenology at all scales to a very high precision [11, 12]. It does this through the proposal of cold dark matter to explain the dynamics of galaxies and their clusters [13, 14], and through a cosmological constant that models dark energy to explain the late-time accelerated expansion of the Universe [15, 16]. The recovery of the cosmological constant was an interesting fact; the initial purpose of using it was evidently wrong, as it was mentioned by Einstein himself, but it is seen that its introduction gives to gravitation the possibility to act repulsively over big spatial scales. In consequence, its use seems to be feasible.

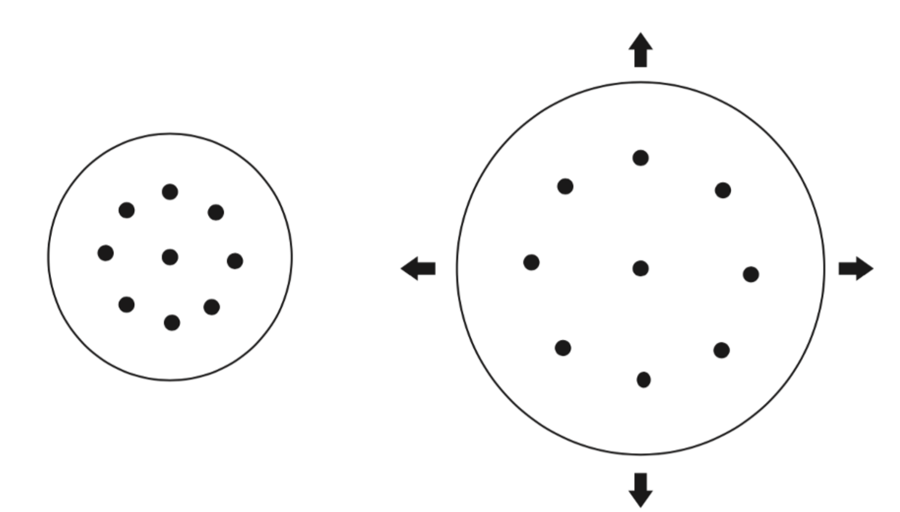

Figure 1: This figure, taken from [17], tries to explain the expansion of the universe. As the figure grows, all relative distances increase at a rate that is proportional to their magnitudes. This fact is stated in the Hubble-Lemaître law, which can be expressed as follows: *The redshift of a galaxy is directly proportional to the distance between it and the observer*. Therefore, the galaxies get apart from each other with a velocity that is directly proportional to the distance between them.

The Standard Big Bang model, with the consideration of dark matter and dark energy, has been proved on cosmological levels with high precision and currently accepted by most of cosmologists. It is raised over three assumptions [18]:

1. The physical laws at the present time can be considered as valid in the early universe. Consequently, GR is the correct theory to describe the gravitational interaction at all epochs of the universe.

2. The cosmological principle is correct, so the geometrical properties of the Universe, considering sufficiently large scales, are based on the homogeneity and isotropy. Figure 1, taken from [17], depicts these properties in a universe in expansion. This is manifested in the so called Friedmann-Lemaître-Robertson-Walker (FLRW) metric.

3. On small scales, where there is anisotropy, the universe is described by a linear expansion of the metric around the FLRW background.

Actual observations of the expansion of the universe indicate that around 70% of the total energy density is characterized by a negative pressure (dark energy component) and the remaining 30% correspond to non-relativistic matter (including baryons and dark matter). Figure 2 schematizes these observations. There have been an enormous amount of dynamical models for the dark energy that predict a dynamical equation of state; such as quintessence and barotropic models [19]. Nevertheless, contemplating a model with a cosmological constant and cold dark matter still seems to be consistent with current observations [12].

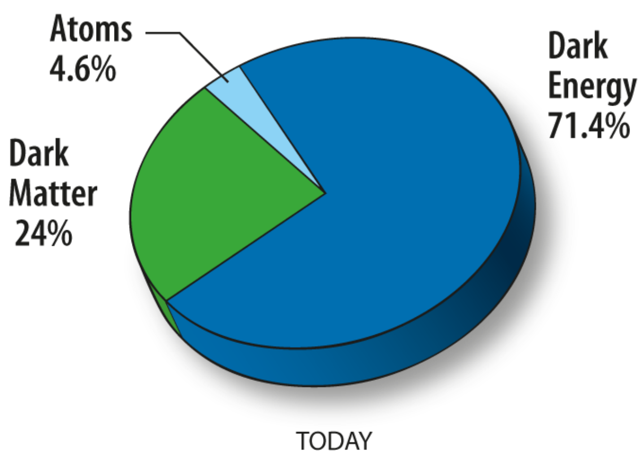

Figure 2: Actual quantitative disposition of the energy density of the universe.

By virtue of what has been said, it is not strange that ΛCDM has had an immense success. However, despite extraordinary efforts, dark matter remains to be directly detected, and the cosmological constant description of dark energy continues to have numerous problems. Moreover, the Standard Big Bang model cannot explain or solve naturally three issues [20]:

- The flatness problem: It is known that the total density parameter Ω is within a few percent of unity (the case $\Omega = 1$ implies a flat universe), so at early times it must have been extremely close to 1. However, it has been found that almost all initial conditions lead either to a closed universe that recollapses almost immediately, or to an open universe in which Ω quickly becomes smaller than is now allowed by observation [21]. See Figure 3.

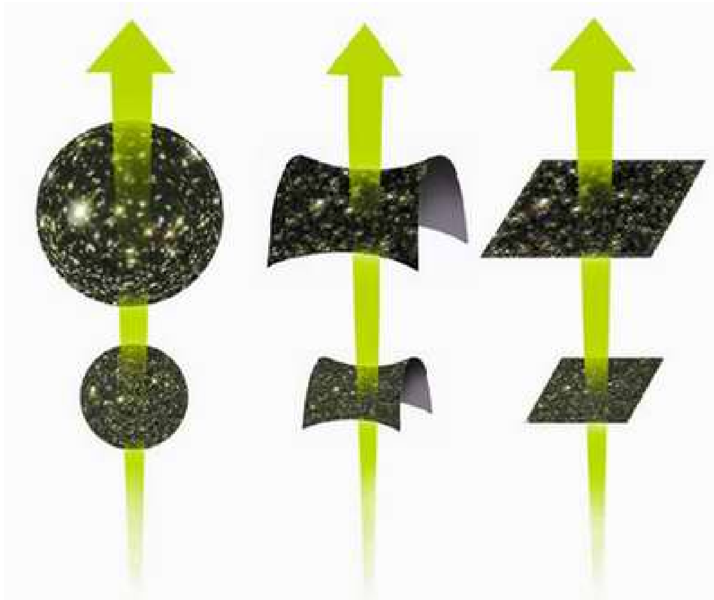

Figure 3: The universe can be closed (left figure) with $\Omega > 1$, open (middle figure) with $\Omega < 1$ and flat (right figure) with $\Omega = 1$. Observations strongly support the fact that the universe is practically flat and that the tendency for Planck power spectra to favour closed universes is caused by systematic errors [22]. Consequently, the right figure seems to be the correct description.

- The horizon problem: The temperature seen in different regions of the sky is approximately the same, but this should not be the case since at early times, the universe, from the point of view of the Standard Big Bang model, is greater than the horizon distance, so the influence between real separated components of the universe is not reached and, in consequence, thermal equilibrium would not suppose to be seen [23]. See Figure 4.

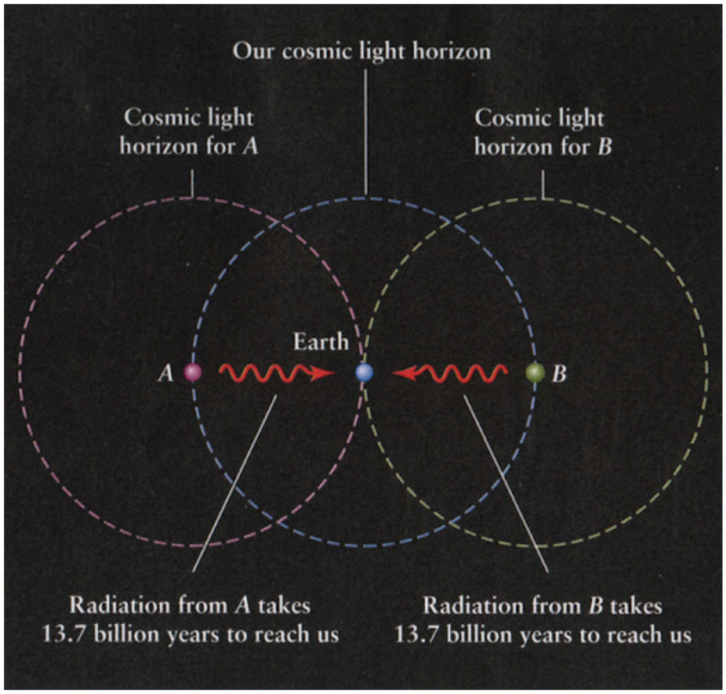

Figure 4: The Hubble horizon must have been greater than the universe at early epochs in order to allow causal influence between the components of the universe and thermal equilibrium could be reached, as it is now observed.

- The unwanted relics: Particles produced at early times in the universe, for example the magnetic monopoles (see Figure 5), are not observable in current observations [24].

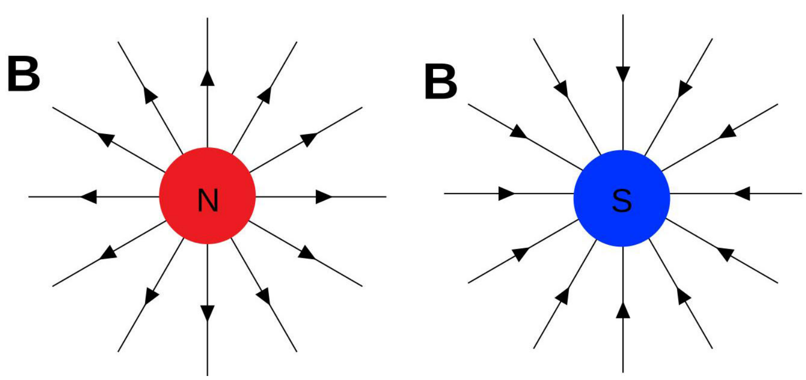

Figure 5: Why are we not able to detect magnetic monopoles? The answer might be in the theory known as 'inflation'.

These issues can be solved with inflation, a theory proposed formally in 1981 by the cosmologist Alan Guth that states that the early universe had an exponential growth epoch (Figure 6, taken from [25], depicts this behaviour), but evidently Einstein's GR must be modified in order to include it; the common way to do that is to include an scalar field called inflaton. As we will see later, this kind of change enters in the first point of possible forms in which one can modify the field equations of Einstein.

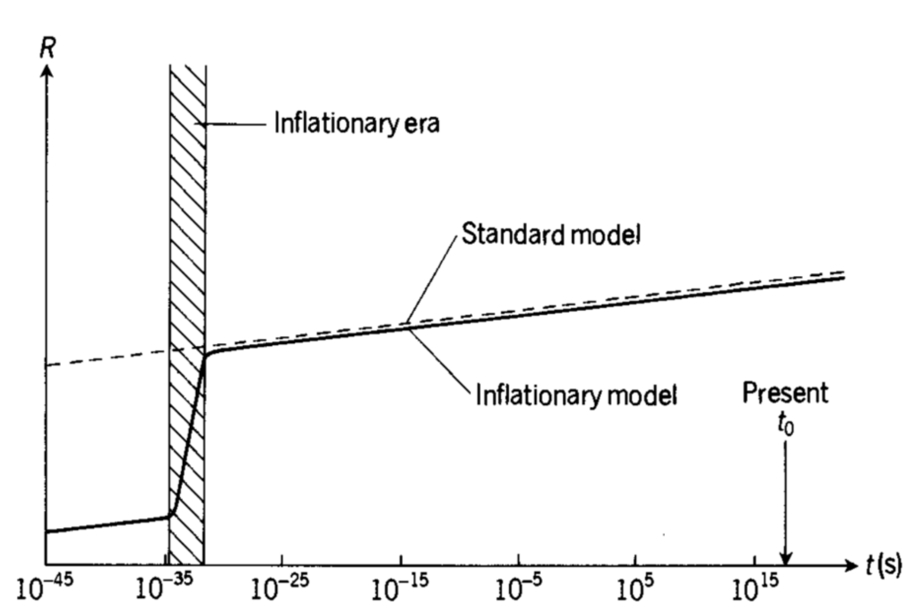

Figure 6: Inflation, which occurs between $t = 10^{-36}s$ and $t = 10^{-33}s$, is currently accepted by the majority of cosmologists. When working in GR one needs to add, for example, a scalar field in order to include inflation. However, some of TG models seem to include it naturally. The figure, which shows the behavior of the scale factor against the physical time, was taken from [25]. Notice how the scale factor R grows incredibly fast during inflation.

Recently, the effectiveness of the late-time explaining power of the ΛCDM model has been called into question. This has primarily taken the form of the so-called H_0 tension problem which characterises the disparity between late-time model-independent measurements of the expansion of the Universe and their corresponding model-dependent predictions from the early Universe (see Figure 7). This was first reported by the Planck collaboration [26, 27], but has since grown in statistical relevance due to strong lensing by the H0LiCOW (H_0 lenses in Cosmograil's wellspring) collaboration [28] and measurements from Cepheids variables[1] via SH0ES (Supernovae H_0 for equation of state). While measurements based on the tip of the red giant branch (TRGB, Carnegie-Chicago Hubble Program) have yielded a lower H_0 tension, new observations may be needed to shed light on the problem such as through the use of GW astronomy which may lead to more precise measurements of the possible discrepancy.

[1]A type of stars that change their brightness, through the variation of diameter and temperature, with stable periods.

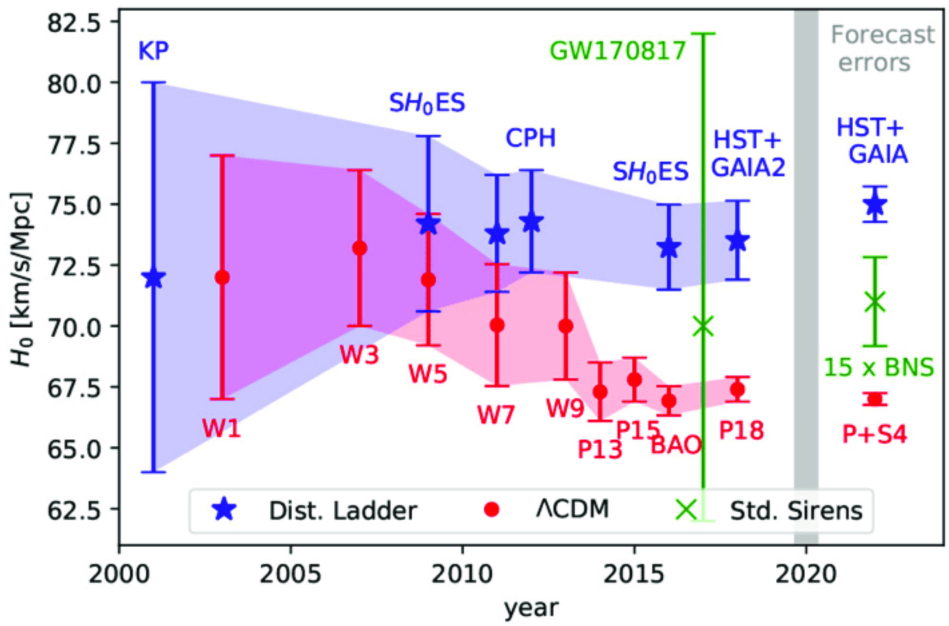

Figure 7: There are two kinds of groups which have measured the Hubble parameter at actual time. The first group, whose results are marked in blue in the figure, which uses Cepheids variables and Supernovae, are based on the Distance Ladder Method and predict, in recent years, $H_0 = 74.03 \pm 1.42\ km\ s^{-1}Mpc^{-1}$. The second group, whose results are marked in red in the figure, uses data of the early universe such as CMB, and extrapolate the value of H to actual times, using a ΛCDM model and predicting, in recent years, $H_0 = 67.4 \pm 0.5\ km\ s^{-1}Mpc^{-1}$. What is happening? The two error bars should superpose each other. One choice is to consider that the result of the first group is wrong and has to improve its observations. The other possibility is that the result of the second group is inaccurate due to the consideration of GR as the fundamental theory. For many reasons it is adequate to seek a modified theory of gravity, an alternative theory which refines the results.

The H_0 tension problem may point to a problem in standard gravity at early-times which is a rich area of observations. One such probe is the possible detection of primordial GWs, that is, the GWs produced by the Big Bang, and their imprint on the cosmic microwave background radiation (CMB) which is depicted in Figure 8 (this was taken from [29]). The effect of primordial GWs is on a non-vanishing CMB BB power spectrum; this would occur in the very low frequency band and would bring important cosmological information such as the tensor-to-scalar ratio, r, which is bounded to $r < 0.06$ (with 95% confident level, at the pivot scale $0.002Mpc^{-1}$) by the Planck collaboration [26]. Primordial GWs may serve as a probe of novel cosmological models in determining which are physically viable [30, 31].

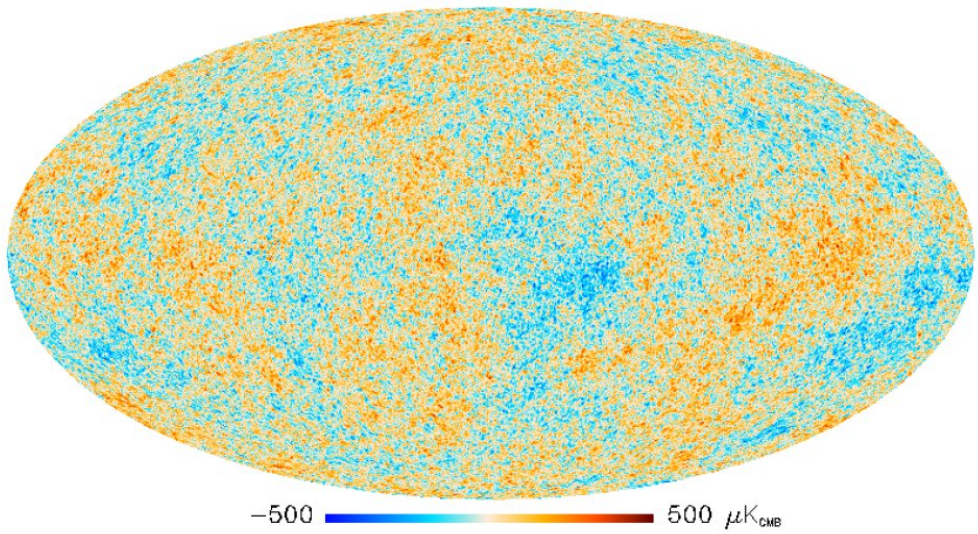

Figure 8: The Cosmic Microwave Background, or CMB, is electromagnetic radiation remaining from the early epoch of the universe. It was discovered in 1965 by A. Penzias and R. Wilson, and is one of the proves in favour of the Big Bang theory [29]. In most of the theoretical models it is produced by an isotropic Gaussian stochastic process. The CMB anisotropies, caused due to the interaction of radiation with hot gas or gravitational potentials, are quite important because we can extract all the information related with the underlying theoretical model via the angular power spectrum from them. This figure, taken from [29], shows an optimized image that filtered out any unwanted impurities.

Rather than modifying the matter content of cosmological models, as in the ΛCDM model, it may be the case that gravity needs to be modified. This supposition make us thing about the fact that perhaps GR is not the ultimate description of the gravitational interaction. There have been, in fact, a plethora of proposed theories of gravity in which GR is either extended or alternated [12, 32], and primordial GWs would act as an excellent probe of these models and may elucidate which models are physically consistent. With 'extended' we mean that GR is generalized using some of its elements (the Ricci scalar, for example) and, because of that, you can construct more complex Lagrangians, meanwhile with 'alternated' we refer to change the elements used in GR, and apply these other objects to describe gravity. One interesting approach that has recently started gaining momentum is, precisely, that of TG (which belongs to the modified-alternative theories of gravity). In TG, the curvature-based description of gravity is replaced with torsion by the replacement of the Levi-Civita connection with its Weitzenböck connection analog. TG is a gauge theory for the translations group, so the gravitational field is represented by a translational gauge potential that appears in the non-trivial part of the tetrad (and in fact this keeps the torsion different from zero). According to GR, spacetime is geometrized by the use of curvature. However, TG attributes gravitation to torsion not by geometrizing the interaction, but by acting as a force; in fact, as a gauge theory, the gravitational interaction is described by a force and the particles trajectories are not geodesic but force equations, with torsion playing the role of force. Anyway, classically speaking, it is a matter of convention to describe gravitation with a pseudo-Riemannian with signature +2 (GR) or a Weitzenböck (TG) space-time structure [4]. Moreover, GR and TG can be made equal up to a boundary term (this will be seen later in this work), which means that they produce identical dynamical equations. The relation between the two theories is really important in the sense that in this equivalence the geodesic equation is analog to the Lorentz force equation of electrodynamics.

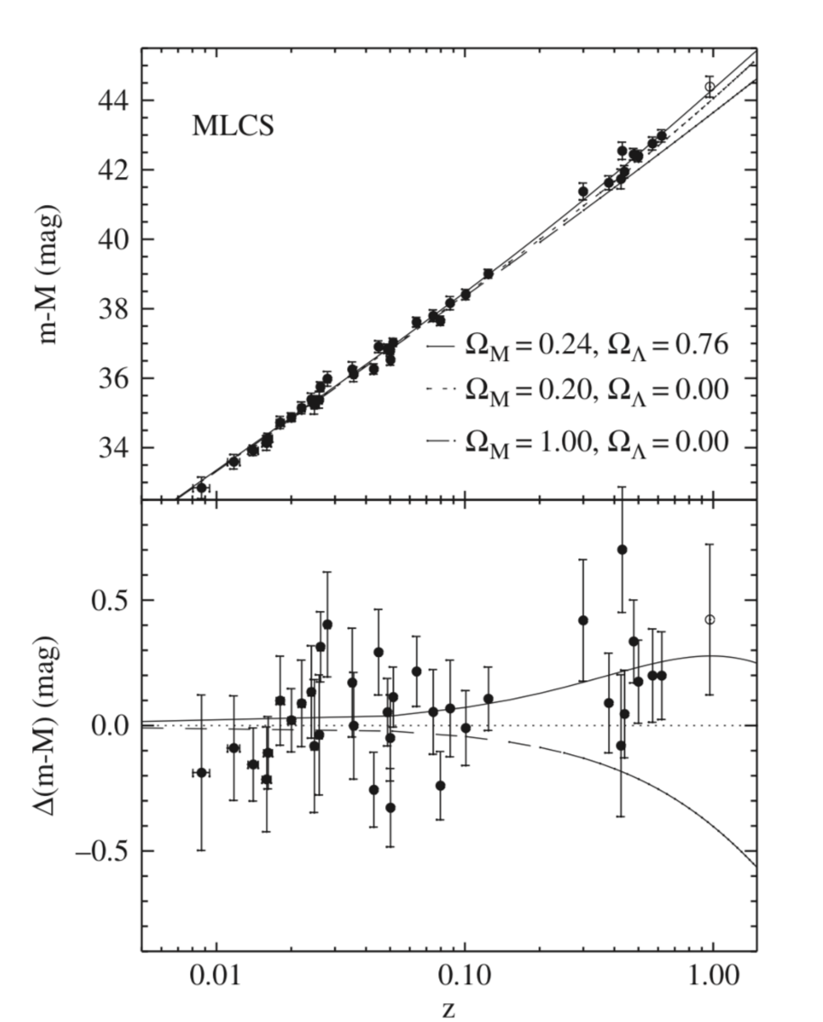

Figure 9: Evolution of the function of luminosity $m - M$ versus redshift z for Type Ia supernovae. The lower part of this curve determines H_0, the upper part demonstrates acceleration. This proclivity of the universe, discovered in 1998 and which motivated the creation of the ΛCDM model, can be reached in TG without introducing any cosmological constant. In a natural way, TG models reproduce this behavior. The figure was taken from [17].

Why are we interested in TG? Why not another modified theory? The reason is that TG features several advantageous properties that should be taken in consideration (see

one reason in Figure 9, which was taken from [17]), such as its similarity to Yang-Mills theory [33], which gives a dimension of particle physics. Another important characteristic of TG is that it allows the definition of an energy-momentum gauge current for the gravitational field, which is covariant under a spacetime general coordinate transformation, and transforms covariantly under a global tangent-space Lorentz transformation; essentially that gauge current is a true spacetime tensor. Therefore, TG seems to provide a more appropriate environment to deal with the energy problem since in GR the energy-momentum density for the gravitational field will always be represented by a pseudotensor [34]. Also, TG does not require the introduction of a Gibbons–Hawking–York boundary term in order to produce a well-defined formulation of the Hamiltonian [35] rendering a more regular theory than GR. TG can also be constructed without the weak equivalence principle [36] (local) while GR cannot; this can be considered as an issue since the equivalence principle has presently passed all experimental tests. Nevertheless, there are many controversies related with its validity, mainly at the quantum level and this might be contemplated as a good point if we want to conciliate gravity with quantum mechanics [37], because the last one is not constructed over this principle but over the uncertainty principle (non-local). There is also an important interest in coupling a fundamental spin-2 field to gravitation; this can be more easily achieved since the description of a gravitationally-coupled spinor field requires the use of the tetrad formalism, which is essentially and naturally used in TG. Making, then, use of the teleparallel paradigm to describe the dynamics of the spin-2 field interpreted as a translational-valued 1-form, a sound spin-2 field theory emerges, and it is both gauge and local-Lorentz invariant, and it preserves the duality symmetry of the free theory [33].
In [38] is given an outlook of generalizations of TG: for example, it can be generalized the lagrangian dependency into a sum of quadratic terms of the torsion tensor; these terms are called vectorial, axial and purely tensorial torsions. Due to all these possibilities of generalization, TG is a theory with many advantages and great transcendence. Something important to be said is that one often looks for simple models, that is, models with the less number of free parameters as possible; this is not only for simplicity purposes (see [39] for a better understanding on this subject). This consideration is taken into account in the proposition of the models that are used in this work, among other reasons that will be explained later.

The theory that is studied and used in this work reflects a different way to look at gravity and to describe this fundamental interaction. A deeper research in TG must be made to probe its power of explaining different phenomena. In this work the H_0 tension is theoretically addressed from a different perspective using, precisely, TG. And to find answers, primordial GWs are suggested to be investigated and studied by using TG extensions, specifically $f(T)$ and $f(T,B)$ models. The deviations from GR at high redshifts will shed light into this problem and perhaps the data found could indicate if TG is, actually, a better theory for describing the gravitational interaction.

In this thesis we use the following convention: $(-,+,+,+)$, and the quantities calculated with the Levi-Civita connection are denoted with a white circle on them; for example, something like $\mathring{A}$. Quantities obtained with the Weitzenböck connection do not have other marks. Throughout the work, we use Latin indices to refer to tangent space coordinates while Greek indices refer to the general manifold. See Table 1.

Table 1	
Symbol	Description
$\mu, \nu, \ldots$	coordinate indices
$a, b, \ldots$	tangent space indices
x^{μ}	space-time coordinates
$f^{c}{}_{ab}$	coefficients of anholonomy
$\eta_{\mu\nu}$	Minkowski spacetime metric
$g_{\mu\nu}$	arbitrary spacetime metric
η_{ab}	Minkowski tangent space metric
$\Lambda^{a}{}_{b}(x)$	local Lorentz transformation
ϵ_{ab}	infinitesimal Lorentz transformation
$\omega^{a}{}_{b\mu}$	teleparallel spin connection
$\mathcal{D}_{\mu}$	covariant derivative of Fock-Ivanenko
$R^{\sigma}{}_{\mu\alpha\nu}$	Riemann curvature tensor of TG
$T^{a}{}_{\mu\nu}$	torsion tensor of $\omega^{a}{}_{b\mu}$
T^{μ}	torsion vector defined by $T^{\nu\mu}{}_{\nu}$
u^{a}	anholonomic 4-velocity
u^{μ}	holonomic 4-velocity
$d\sigma$	Minkowski interval
ds	arbitrary interval
$B^{a}{}_{\mu}$	gauge potential one-form components
e^{a}	non-trivial frame field
e_{a}	non-trivial coframe field
$e^{a}{}_{\mu}$	non-trivial frame field components
$e_{a}{}^{\mu}$	non-trivial coframe field components
e	determinant of the tetrad
$\Gamma^{\lambda}{}_{\mu\nu}$	teleparallel linear (Weitzenböck) connection
∇_{μ}	covariant derivative associated with $\Gamma^{\lambda}{}_{\mu\nu}$
$K^{c}{}_{ba}$	contortion tensor
T	torsion scalar
B	boundary term
R	Ricci scalar of TG
$\mathcal{L}$	Lagrangian density of TG and GR
$\mathcal{S}$	action of TG and GR
$\kappa^2 = 8\pi G$	gravitational coupling constant ($c = 1$)
$S_{a}{}^{\rho\sigma}$	superpotential
$j_{a}{}^{\rho}$	gauge current or energy-momentum pseudo-current
$\mathcal{T}_{a}{}^{\rho}$	matter energy-momentum tensor
ω^{μ}	quantity defined by $\omega^{a}{}_{b\nu}\, h_{a}{}^{\nu}\, h_{s}{}^{\mu}\, \eta^{bs}$
$\mathring{\omega}^{c}{}_{b\mu}$	spin connection of GR
$\mathring{\Gamma}^{\lambda}{}_{\mu\nu}$	Christoffel symbol or general relativity connection

Continuation of Table 1	
Symbol	Description
$\mathring{\nabla}_\mu$	covariant derivative associated with $\mathring{\Gamma}^\lambda{}_{\mu\nu}$
$\mathring{R}^\sigma{}_{\mu\alpha\nu}$	Riemann curvature tensor of GR
$\mathring{R}_{\mu\nu}$	Ricci tensor of GR
$\mathring{R}$	Ricci scalar of GR

Table 1: Notation employed in the description of TG and GR.

Part I

The Background

Chapter 1

General Relativity

Isaac Newton built the laws of motion over the idea that space and time were absolute. This idea was clarified by the philosopher and Anglican cleric Samuel Clarke in his letters to the mathematician and philosopher Gottfried Leibniz [40], and can be stated in the following theses: First, space and time are logically and metaphysically prior to physical bodies and events. Second, physical bodies and events exist within space and time. Third, although we may distinguish regions of space and time, neither space nor time strictly speaking are divisible since no region of space or time could be separated from any other region. Fourth, ontologically speaking, space and time may be identified with attributes of God; infinite space just is the attribute of God's immensity, while infinite time just is the attribute of God's eternity [41]. Leibniz did not share those ideas, and indeed he wrote back to Clarke arguing these points and defending an opposite vision; that space and time were not so much things in which bodies are embedded, can be located and move, but systems of relations holding between material things. Consequently, for Leibniz, space and time were relative. Nevertheless, this relationalist point of view was not really taken into account in the scientific context, perhaps because of the practical success of Newton's substantivalism. In fact, Newtonian mechanics and gravitation were actually successful for two centuries at explaining the solar system and astronomy in general. However, there was a particular phenomenon that was not in agreement with Newton's laws; the precession of Mercury's perihelion. Like the other planets that constitute our solar system, the axes of the elliptical trajectory of Mercury slowly rotate with time. When all the known effects were taken into account using Newtonian physics, the results were not in agreement with observations. This problem could not be solved until the arrival of GR, whose results were correct with high precision. In addition, other facts (which were not known before the 20th century) cannot be explained by Newtonian physics. These are: First, that motion and interaction effectively change the mass of objects; second, that light falls and attracts other objects, even though is has no mass; and third, that time and distances are observer-dependent notions [42]. We cannot blame Newton for not been aware of this since these three facts are neither empirical nor logical at first sight, but Einstein's theory of gravitation changed the concept of space-time and matter in such a way that could explain all these phenomena with great success. It is worth to mention that even though Einstein was inspired by the ideas of the physicist and philosopher

Ernst Mach and by Leibniz vision, GR is not a relationalist theory as it will be seen clearly down below.
The theory of GR, which breaks with the absolute space and absolute time notions, says that the gravitational interaction can be understood with the help of Riemannian geometry; gravity is due to the curvature of space-time, and this is achieved with the Levi-Civita connection, which holds the metricity condition[1] and leaves curvature as the unique basic element. The Levi-Civita connection in spatial coordinates (Christoffel symbols) is given by

$$\mathring{\Gamma}^{\sigma}{}_{\mu\nu} := \frac{1}{2} g^{\sigma\alpha}\Big(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\nu\mu}\Big) \tag{1.1}$$

where repeated indexes indicate a sum, $g_{\mu\nu}$ is the metric tensor and $g^{\mu\nu}$ is the inverse. Considering this connection, the curvature of the manifold is expressed through the Riemann tensor (see Figure 1.1 taken from [43]),

$$\mathring{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \mathring{\Gamma}^{\rho}{}_{\nu\sigma} - \partial_\nu \mathring{\Gamma}^{\rho}{}_{\mu\sigma} + \mathring{\Gamma}^{\rho}{}_{\mu\lambda}\mathring{\Gamma}^{\lambda}{}_{\nu\sigma} - \mathring{\Gamma}^{\rho}{}_{\nu\lambda}\mathring{\Gamma}^{\lambda}{}_{\mu\sigma}, \tag{1.2}$$

which satisfies the following identities:

$$\mathring{R}_{\rho\sigma\mu\nu} = -\mathring{R}_{\sigma\rho\mu\nu} = -\mathring{R}_{\rho\sigma\nu\mu}, \tag{1.3}$$

$$\mathring{R}_{\rho\sigma\mu\nu} = \mathring{R}_{\mu\nu\rho\sigma}, \tag{1.4}$$

where $\mathring{R}_{\rho\sigma\mu\nu} = g_{\rho\alpha}\mathring{R}^{\alpha}{}_{\sigma\mu\nu}$. Also, the Riemann tensor satisfies the first Bianchi identity,

$$\mathring{R}_{\rho\sigma\mu\nu} + \mathring{R}_{\rho\mu\nu\sigma} + \mathring{R}_{\rho\nu\sigma\mu} = 0, \tag{1.5}$$

and the second Bianchi identity,

$$\mathring{\nabla}_\alpha \mathring{R}_{\rho\sigma\mu\nu} + \mathring{\nabla}_\mu \mathring{R}_{\rho\sigma\nu\alpha} + \mathring{\nabla}_\nu \mathring{R}_{\rho\sigma\alpha\mu} = 0, \tag{1.6}$$

in which $\mathring{\nabla}_\alpha$ denotes covariant derivative. Another important object in GR is the Ricci curvature tensor, defined as

$$\mathring{R}_{\mu\nu} = \mathring{R}^{\rho}{}_{\mu\rho\nu}, \tag{1.7}$$

and the scalar curvature (or also called the Ricci scalar) is

$$\mathring{R} = g^{\mu\nu}\mathring{R}_{\mu\nu}. \tag{1.8}$$

[1]The non-metricity tensor is defined as $Q_{\alpha\mu\nu} = \mathring{\nabla}_\alpha g_{\mu\nu}$. The particular case $Q_{\alpha\mu\nu} = 0$ is known as the metricity condition.

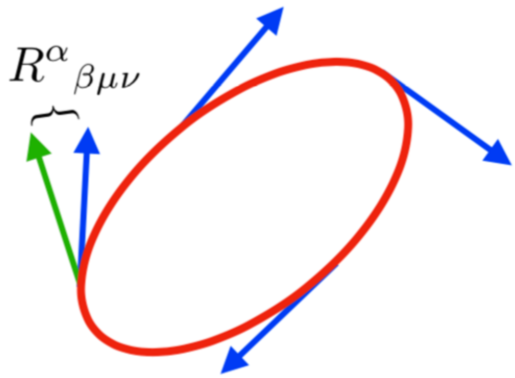

Figure 1.1: The Riemann tensor moves the vectors when they are transported along a closed curve. Figure taken from [43].

The dynamical variable in this theory is the metric, and the action to be considered, known as the Einstein–Hilbert action, is varied with respect to that. Explicitly, the Einstein-Hilbert action is

$$\mathcal{S} = \int d^4x \left[\frac{1}{2\kappa^2}\mathring{R} + \mathcal{L}_{\mathrm{m}}\right]\sqrt{-g}, \tag{1.9}$$

where $\kappa^2 = 8\pi G$, $g = det(g_{\mu\nu})$ and $\mathcal{L}_{\mathrm{m}}$ represents the Lagrangian for matter. Taking the variation we have

$$\delta\mathcal{S} = \int \left[\frac{1}{2\kappa^2}\left(\sqrt{-g}\delta\mathring{R} + \mathring{R}\delta\sqrt{-g}\right) + \delta(\sqrt{-g}\mathcal{L}_{\mathrm{m}})\right] d^4x = 0$$

The variation of the last two terms are

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu},$$

and

$$\delta(\sqrt{-g}\mathcal{L}_{\mathrm{m}}) := -\frac{1}{2}\sqrt{-g}\mathcal{T}_{\mu\nu}\delta g^{\mu\nu}.$$

Now we will analyze the first term, which can be rewritten with the use of Eq.(1.8);

$$\delta\mathring{R} = \delta(\mathring{R}_{\mu\nu}g^{\mu\nu}) = \mathring{R}_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta\mathring{R}_{\mu\nu}.$$

The variation of the Ricci tensor can be expressed as

$$\delta\mathring{R}_{\mu\nu} = \delta\mathring{R}^\rho{}_{\mu\rho\nu} = \mathring{\nabla}_\rho(\delta\mathring{\Gamma}^\rho{}_{\nu\mu}) - \mathring{\nabla}_\nu(\delta\mathring{\Gamma}^\rho{}_{\rho\mu}).$$

where has been used Eq.(1.2). Hence,

$$\delta\mathring{R} = \mathring{R}_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left[\mathring{\nabla}_\rho(\delta\mathring{\Gamma}^\rho{}_{\nu\mu}) - \mathring{\nabla}_\nu(\delta\mathring{\Gamma}^\rho{}_{\rho\mu})\right].$$

Knowing that $\mathring{\nabla}_\alpha g^{\mu\nu} = 0$, which is the metricity condition satisfied by the Levi-Civita connection, the above expression may be expressed as

$$\delta \mathring{R} = \mathring{R}_{\mu\nu}\delta g^{\mu\nu} + \mathring{\nabla}_\rho \Big(g^{\mu\nu}\delta\mathring{\Gamma}^\rho{}_{\nu\mu} - g^{\mu\rho}\delta\mathring{\Gamma}^\nu{}_{\nu\mu} \Big).$$

However, the last term when multiplied by $\sqrt{-g}$ changes to an ordinary derivative,

$$\sqrt{-g}\mathring{\nabla}_\mu A^\mu = \partial_\mu(\sqrt{-g}A^\mu),$$

so this term will not contribute to the field equations and can be neglected. This is because in the extreme values of the integration path there is no variation [44].
The result is, then,

$$\int \left[\frac{1}{2\kappa^2}\Big(\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} \Big) - \frac{1}{2}\mathcal{T}_{\mu\nu} \right] \sqrt{-g}\delta g^{\mu\nu} d^4x = 0.$$

This is only the metric variation. The previous equation can only be satisfied if the integrand is zero identically. The field equations, named the Einstein field equations, are found to be

$$\mathring{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathring{R} = \kappa^2\mathcal{T}_{\mu\nu}, \tag{1.10}$$

in which $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor of matter. At this point non cosmological constant has been taken into account, although, the corresponding term should be introduced in the Einstein-Hilbert action in order to consider a dark energy contribution in the field equations. Something relevant to be said with regard to the physical interpretation of GR is that equations (1.10) illustrate matter as the one in charge to deform space-time and that space-time shows the trajectories that material bodies have to follow (see Figure 1.2). However, according to GR, it is true that, in the absence of matter, a curved universe can exist, even an expansive universe (de Sitter universes). These features confirm that GR is not a relationalist theory since space and time can exist without matter.

Given the Bianchi identities and the metricity condition, the equation

$$\mathring{\nabla}_\nu \mathcal{T}^{\mu\nu} = 0, \tag{1.11}$$

must hold. This is, in fact, the conservation of energy-momentum.

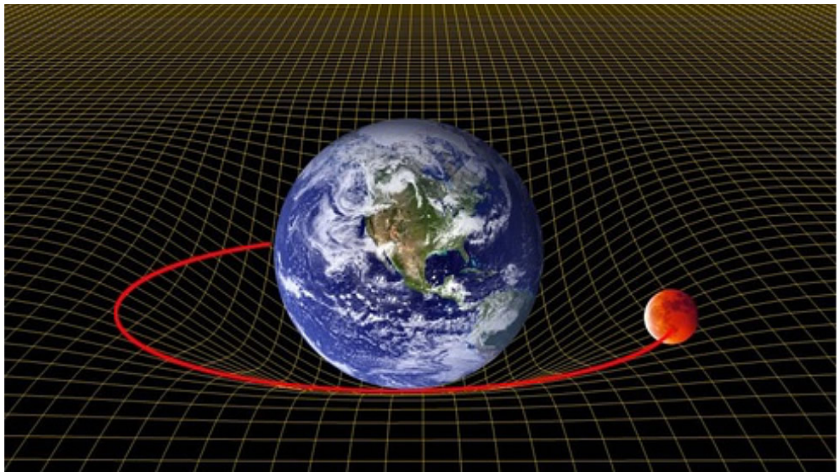

Figure 1.2: The phenomenology of gravity in the context of GR is frequently presented as in this figure, which depicts the famous phrase of John A. Wheeler: *Space-time tells matter how to move; matter tells space-time how to curve.*

The FLRW metric written in coordinates adapted to the symmetries, that is,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2sin^2\theta \end{pmatrix}, \tag{1.12}$$

is a solution of (1.10) taking into account a cosmological constant Λ. In the matrix, $a = a(t)$ is the scale factor and K represents the form of the large scale spatial curvature ($K = -1$, $K = 0$ or $K = 1$ refers to a open, flat or closed universe, respectively). FLRW describes an homogeneous and isotropic universe, in expansion or contraction, which contains an ideal fluid. It is used in the ΛCDM model due to these properties, and it provides the following equations of motion (taking into account that the energy momentum tensor is equally homogeneous and isotropic):

$$H^2 = \frac{8\pi G}{3}\rho_{physical}, \tag{1.13}$$

and

$$\dot{H} = -\frac{8\pi G}{2}(\rho_{physical} + p_{physical}), \tag{1.14}$$

where *physical* denotes the matter content related to baryonic matter and radiation. The perfect fluid is described by a density $\rho_{physical}$ and a pressure $p_{physical}$. Equations (1.13) and (1.14) are called the Friedmann equations, and in this case $\Lambda = 0$ and $K = 0$. In consequence the large scale structure of the universe is flat, and it does not expand or contract.

In the case of Eq.(1.11) we find that

$$\dot{\rho}_{physical} + 3H(\rho_{physical} + p_{physical}) = 0, \tag{1.15}$$

which is basically a continuity equation.

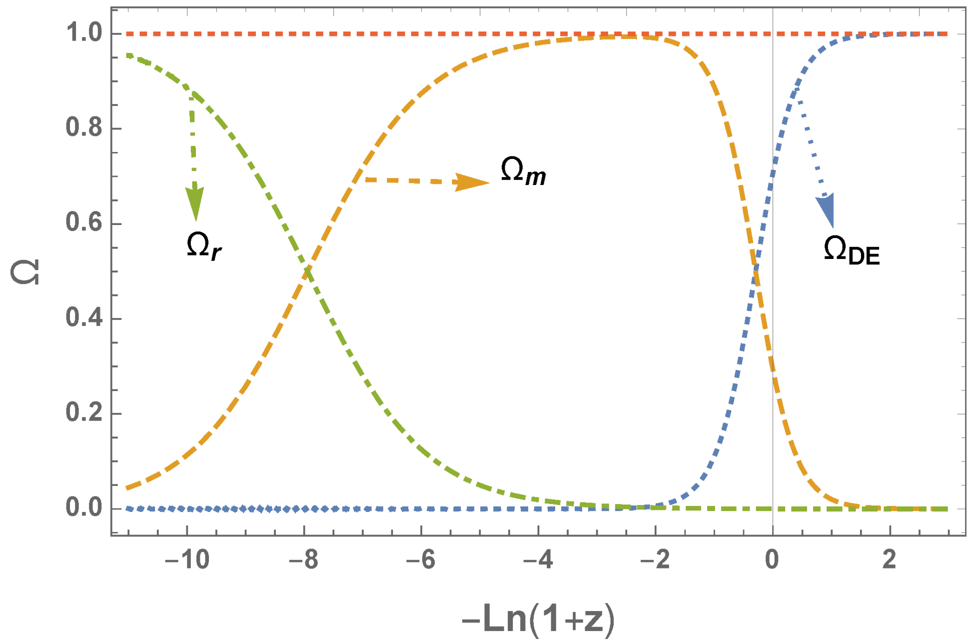

Figure 1.3: This is the evolution of the density parameters; clearly, the sum of all contributions must be equal to unity on a fixed redshift. The figure depicts that, in the history of the universe, there have been three principal epochs; the radiation (r), matter (m) and dark energy (DE) domination era, denoted by green, orange and blue colors, respectively. Notice that this plot neglects the 'curvature density' Ω_k, since we consider a flat universe.

To solve Friedmann equations, (1.13) and (1.14), a relation must be assumed for the physical density. Usually, the behavior is defined with

$$p_{physical} = \omega \rho_{physical}, \tag{1.16}$$

which is known as the equation of state, and ω is the state parameter (could be a constant or maybe a function of something else). The solution of Eq.(1.15) is found when considering

$$\rho_{physical} \propto a^{-3(1+\omega)}.$$

In principle, the state parameter can adopt any value, but it has been found that some particular numbers are special; $\omega = 0$ refers to non-relativistic matter (dust), $\omega = 1/3$ is radiation or relativistic matter and $\omega = -1$ is related to a cosmological constant, which mimics the behaviour of dark energy.

The density parameter is defined as

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_{crit}(t)},$$

where i is a designation for the type of content we are considering (matter, for example). The critical density $\rho_{crit}(t)$, for a flat universe, takes the following form

$$\rho_{crit}(t) = \frac{3H^2}{8\pi G}.$$

The total density parameter is, in this case,

$$\Omega = \Omega_m + \Omega_r + \Omega_K + \Omega_{DE},$$

in which Ω_K (that is zero when we consider a flat universe) stands for the contribution of curvature to the total energy density, and Ω_{DE} refers to the contribution of dark energy. The evolution of the different density parameters is shown in Figure 1.3, and in Figure 1.4 is depicted the relative size of the universe according to the values of those parameters.

For an arbitrary cosmological constant Λ and large scale spatial curvature K, Eq.(1.13) takes the form

$$\dot{a}^2 = \frac{C}{a} + \frac{1}{3}\Lambda a^3 - K, \tag{1.17}$$

where

$$C = \frac{8\pi G}{3} a^3 \rho_{physical}.$$

Taking equation (1.17) into account, the scale factor can be described differently depending on the values of the parameters.

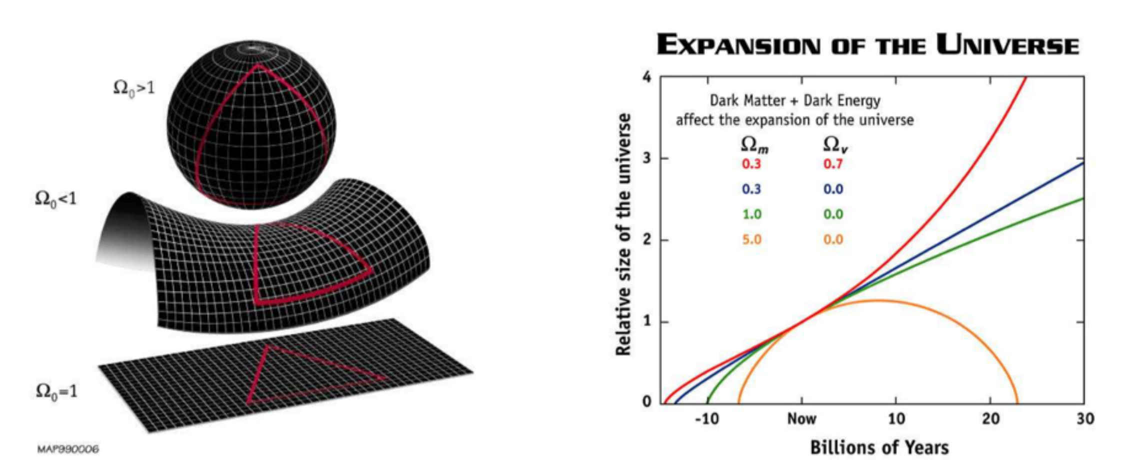

Figure 1.4: In this figure is shown the relative size of the universe depending on the values of the density parameters.

As it was mentioned before, ΛCDM model presents some issues that GR cannot solve. The question, then, is the following: Is there a way to modify GR to adjust predictions? Or, in other words, could one describe the gravitational interaction, in a precise manner, without GR? The answer to both questions is yes. In fact, Lovelock's theorem [12] indicates different forms in which one can modify the field equations of Einstein:

1. Add other field in addition or instead of the metric tensor.
2. Add more than second-order derivatives of the metric.
3. Use more/less than four dimensions of space-time.
4. Consider non-local terms (or what is the same, violate the principle of locality).
5. Derive the field equations from something different of a variational principle (taking an emergent form of gravity).
6. Changing the gravitational connection.

Depending on the changes made, one would be constructing an extended or an alternative theory, but in any case it will be a modified theory of gravity. In simple words: *A modified GR is whatever proposal that add something new or change the elements in the Einstein-Hilbert action* [45]. For example, $f(\mathring{R})$ theories are extended theories of GR since they consider an arbitrary function of the scalar curvature instead of using a linear term as shown in equation (1.9), which is more general. Meanwhile, an alternative theory of GR would be, for example, MOND (Modified Newtonian Dynamics) in which the background is completely different in the elements to those of GR. Diverse modified theories are shown in Figure 1.5, which was taken from [46].

We have seen the phenomenological vision of GR as well as the principal equations and elements of it. The mathematical elegance, the accuracy in describing different astrophysical and cosmological systems and the brilliant explanation of space-time make GR a worldwide accepted theory. However, as it was mentioned in the introduction, there are some tensions which cannot be solved using GR; consequently, modified theories of gravity have appeared in order to give solutions. In the next chapter we study a theory considered as an alternative theory to GR, because it contemplates another dynamical variable instead of the metric, and at the same time considered as an extended theory by reason of its capability to generalize itself with the elements that are used in it.

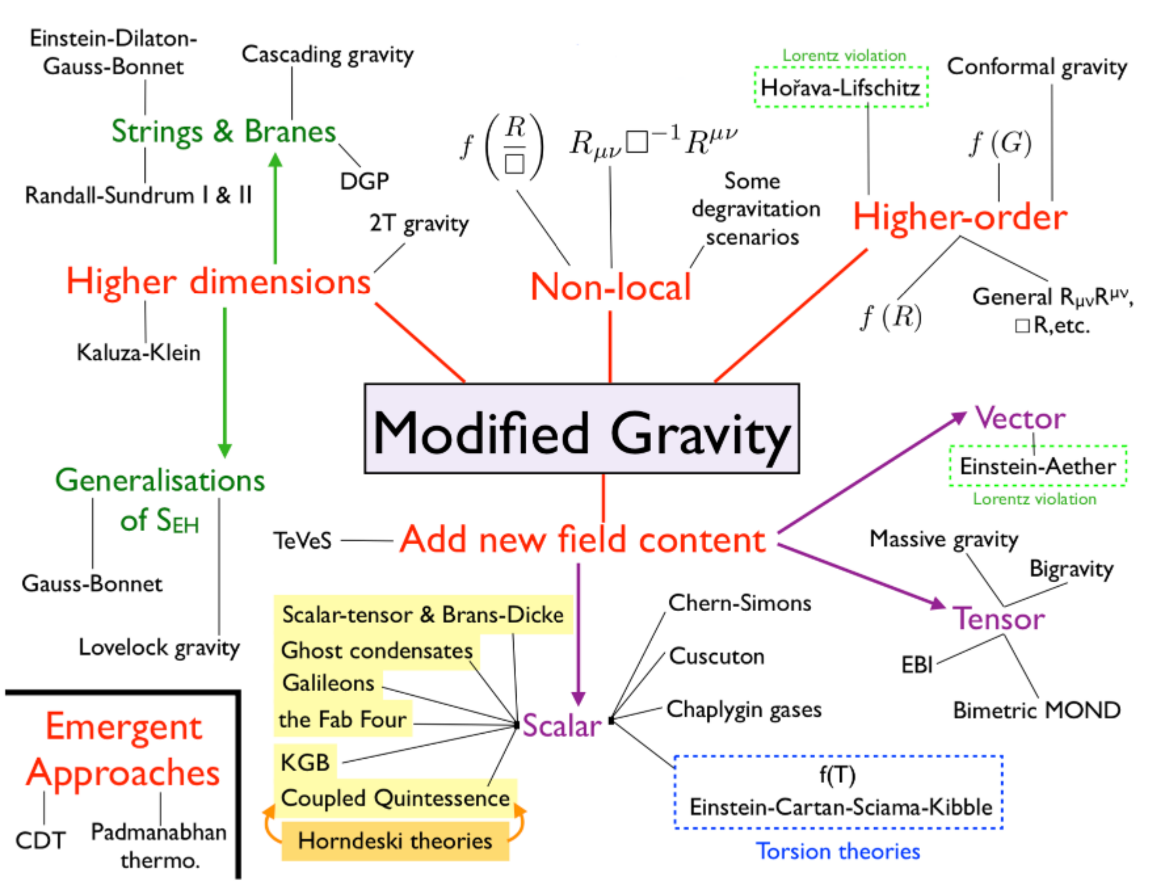

Figure 1.5: This scheme, obtained from [46], depicts some of the most important modified theories, ordered in categories. In red color the (main) first 5 points are shown, through which one can modify the field equations of Einstein. The theory of our interest lies, marked in blue, in the first category; add other field instead of the metric tensor.

Chapter 2

Teleparallel Gravity

An alternative theory to GR that has gained strength is Teleparallel Gravity, a theory that describes the gravitational interaction through torsion only [33]. The Weitzenböck connection, which is the most general linear affine connection that is both curvatureless and satisfies the metricity condition (non-metricity is shown in Figure 2.3, taken from [43]), leaves torsion as the base element to model gravity. It is defined by

$$\Gamma^{\sigma}{}_{\mu\nu} := e_a{}^{\sigma}\partial_{\mu}e^a{}_{\nu} + e_a{}^{\sigma}\omega^a{}_{b\mu}e^b{}_{\nu}, \tag{2.1}$$

where $e^a{}_{\rho}$ is the tetrad field ($e_a{}^{\rho}$ being the transpose), and $\omega^a{}_{b\mu}$ the spin connection. Tetrads relate quantities on the tangent (inertial) space and the general manifold which are represented by Latin and Greek indices, respectively. The role of the spin connection is to retain the invariance of the resulting field equations under local Lorentz transformations (LLTs). Thus, for any choice of tetrad, the spin connection balances this freedom with different inertial contributions. This is thus a flat connection, and can vanish for particular frame (tetrad) choices. These will be related to other frames by appropriate Lorentz matrices. In GR, the associated spin connections are not inertial and are mainly hidden in the inertial structure of the theory. Both, the tetrad and spin connection are the fundamental dynamical objects on which TG is based, and this is invariant under LLTs as well as being generally covariant.

Taking the full breadth of LLTs (Lorentz boosts and rotations), tetrads for a particular system can be related together through

$$e'^a{}_{\mu} = \Lambda^a_b e^b{}_{\mu}, \tag{2.2}$$

where Λ^a_b is the Lorentz matrix which satisfies

$$\eta_{ab} = \eta_{cd}\Lambda^c_a\Lambda^d_b.$$

The spin connection is transformed according to

$$\omega'^a{}_{\mu b} = \Lambda^a_c\omega^c{}_{\mu d}(\Lambda^{-1})^d_b - (\Lambda^{-1})^a_c\partial_{\mu}\Lambda^c_b, \tag{2.3}$$

where (Λ^{-1}) represents the inverse of the Lorentz matrix. Considering the definition of the torsion tensor given later and applying (2.2) and (2.3), we arrive to

$$T'^{a}{}_{\mu\nu} = \Lambda^a_b T^b{}_{\mu\nu}. \tag{2.4}$$

Consequently, the torsion scalar is also covariant under LLTs, and since the field equations are based on T, then they are invariant under LLTs [47]. Clearly, when the spin connection is zero then this is not longer true, in general.

As discussed above, there also exists the so-called good tetrads which allow vanishing spin connection components [36]. This does not necessarily propagate at perturbative level due to the gauge freedom. Tetrad and spin connection pairs produce generally covariant theories which means dynamically equivalent field equations.
In TG, the metric tensor, $g_{\mu\nu}$, is replaced as the fundamental dynamical variable in the theory and subplanted by the tetrad that observes the following consistency relations

$$e^a{}_\mu e_b{}^\mu = \delta^a_b, \quad e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu, \tag{2.5}$$

that form the conditions on which to create inverses of the tetrad fields. Here

$$\delta^\nu_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.6}$$

One direct application of the transformation rules is that between the Minkowski and general spaces which takes the form

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}, \quad \eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}, \tag{2.7}$$

where the tetrad is the principle variable of these relations. The Minkowski metric is, according to our signature,

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.8}$$

It is important to point out that the curvature measured by the Riemann tensor will always vanish in TG due to the curvature-less property of the Weitzenböck connection. Thus, we necessitate a different measure of geometric deformation which in TG takes the form of the torsion tension (see Figure 2.2, taken from [43]) which is defined as an anti-symmetric property on the connection indices

$$T^\sigma{}_{\mu\nu} := 2\Gamma^\sigma{}_{[\nu\mu]}. \tag{2.9}$$

This is a measure of the field strength of gravitation (square brackets represent the anti-symmetric operator $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$) because the non-vanishing torsion is directly related with the gauge field strength. This transforms covariantly under both diffeomorphisms and LLTs (as it was already mentioned). As the Riemann tensor

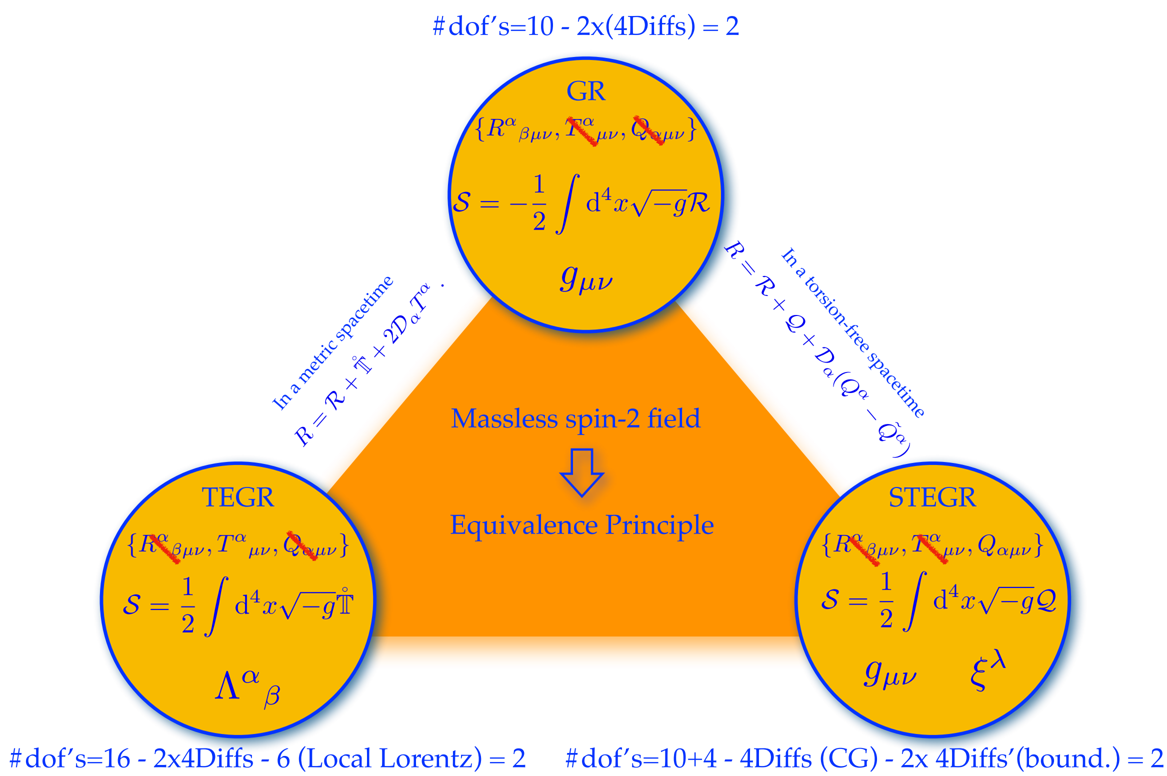

Figure 2.1: This figure, obtained from [43], shows the three alternatives to describe gravity, including the most important elements of each one. In the highest circle there is GR, in which the torsion tensor and the non-metricity vanish, surviving the curvature tensor only, and the fundamental element is the metric tensor. The lower left circle contains TEGR where the curvature tensor and non-metricity vanish, and the fundamental object is the tetrad (in this case, $\Lambda^{\alpha}{}_{\beta}$ represents a tetrad in the covariant formulation). In the lower right circle it can be seen the Symmetric Teleparallel Equivalent of General Relativity (STEGR) in which both the curvature tensor and the torsion tensor vanish, and the basic elements are the metric tensor and a kind of fields that parametrize the connection. At the center of the figure it is shown the fact that gravity, in all these alternatives, is described by a massless spin-2 particle that, overall, implies the equivalence principle.

measure curvature, the torsion tensor is a measure of the deformation of space-time (derivation of equation (2.9) is shown in appendix A, among other details).

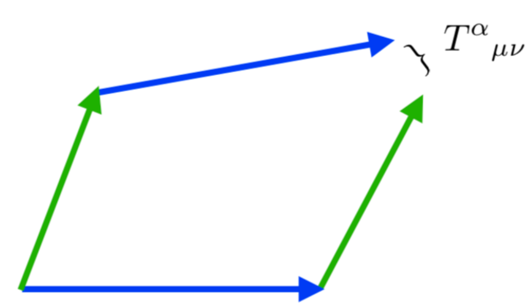

Figure 2.2: The name of 'teleparallelism' or 'distant parallelism' comes from the fact that, in this theory, you can compare the direction of tangent vectors on the manifold in different points due to the geometrical description of the affine connection. The torsion tensor produces a non-closure of infinitesimal parallelograms when two vectors are transported along each other. Figure taken from [43].

Other critical tensors can also be defined too, one of which is the contorsion tensor which measures the difference between the Weitzenböck and Levi-Civita connections

$$K^{\sigma}{}_{\mu\nu} := \Gamma^{\sigma}{}_{\mu\nu} - \mathring{\Gamma}^{\sigma}{}_{\mu\nu} = \frac{1}{2}\Big(T_{\mu}{}^{\sigma}{}_{\nu} + T_{\nu}{}^{\sigma}{}_{\mu} - T^{\sigma}{}_{\mu\nu}\Big), \tag{2.10}$$

where $\mathring{\Gamma}^{\sigma}{}_{\mu\nu}$ is the Levi-Civita connection. This quantity plays an important role in relating TG with Levi-Civita based theories.

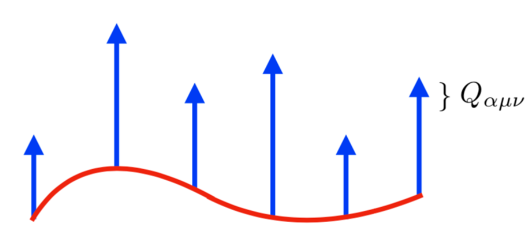

Figure 2.3: The covariant derivative of the metric, that is, the non-metricity tensor, causes variation of the length of vectors when they are transported (parallelly) along a curve. Figure taken from [43].

Another important ingredient in TG is that of the superpotential which is defined as

$$S_{a}{}^{\mu\nu} := K^{\mu\nu}{}_{a} - e_{a}{}^{\nu} T^{\alpha\mu}{}_{\alpha} + e_{a}{}^{\mu} T^{\alpha\nu}{}_{\alpha}. \tag{2.11}$$

This is a representation with a latin index, however it can be rewritten with greek indexes only. The superpotential may play a central role in reformulating TG as a gauge theory of gravity with an associated gravitational energy-momentum tensor.

The contraction of the superpotential and torsion tensors renders the torsion scalar, namely

$$T := S_a{}^{\mu\nu} T^a{}_{\mu\nu}, \tag{2.12}$$

which is solely determined by the Weitzenböck connection (and consequently by the tetrad and the spin connection) in the same way that the Ricci scalar is dependent only on the Levi-Civita connection. Once again, we have here a representation with a latin index in the superpotential and in the torsion tensor. It turns out that (see appendix B) the torsion and Ricci scalars are equal up to a boundary term, i.e. [47]

$$\mathring{R} = -T + \frac{2}{e}\partial_\mu\Big(eT_\sigma{}^{\sigma\mu}\Big) + R := -T + B, \tag{2.13}$$

where $\mathring{R}$ is the Ricci scalar as determined using the Levi-Civita connection (non-zero), R is the Ricci scalar as calculated with the Weitzenböck connection which vanishes, e is the determinant of the tetrad field, $e = \det(e^a{}_\mu) = \sqrt{-g}$, and the boundary term, that is basically a divergence of the torsion vector, is

$$B = \frac{2}{e}\partial_\mu\Big(eT^\mu\Big). \tag{2.14}$$

The torsion vector is the result of a contraction of two indices in the torsion tensor;

$$T^\mu = T_\sigma{}^{\sigma\mu}. \tag{2.15}$$

The action

$$\mathcal{S} = \int d^4x \bigg[-\frac{1}{2\kappa^2}T + \mathcal{L}_{\mathrm{m}}\bigg]e, \tag{2.16}$$

whose dynamical fields are the tetrad field and the spin connection, and where $\kappa^2 = 8\pi G$ and $\mathcal{L}_{\mathrm{m}}$ represents the Lagrangian for matter, turns out to be a very important case in TG called Teleparallel Equivalent of General Relativity (TEGR), this is because the variation of such action with respect to the tetrad field results to completely equivalent dynamical equations to GR. This can be seen from (2.16): Substituting the torsion scalar and the relation $e = \sqrt{-g}$, we have

$$\mathcal{S} = \int d^4x \bigg[\frac{1}{2\kappa^2}(\mathring{R} - B) + \mathcal{L}_{\mathrm{m}}\bigg]\sqrt{-g}. \tag{2.17}$$

Nevertheless, linear boundary terms do not contribute to the field equations. Hence, without loss of generality, we can write

$$\mathcal{S} = \int d^4x \bigg[\frac{1}{2\kappa^2}\mathring{R} + \mathcal{L}_{\mathrm{m}}\bigg]\sqrt{-g}, \tag{2.18}$$

which agrees with (1.9). In consequence, describing the gravitational interaction through curvature or torsion is simply a convention (Einstein could have constructed his theory from torsion instead of curvature without any restriction). In Figure 2.1, taken from

[43], it is shown that, in fact, there are three ways to model the bending of space-time, or in other words, three ways to 'geometrize' gravity: Through curvature, torsion and non-metricity [43]. In this figure, specifically, a limit is depicted in which the three theories are equivalent in the equations of motion. Each theory can be generalized in its elements, losing such equivalence between them. This work focus on TG, so it will not be studied the non-metricity theories, but it is important to notice that curvature is neither the only nor the best element to model gravity; these three possibilities are equally valid. It has to be said, also, that classically it is impossible to make an observation in order to distinguish GR from TEGR, hence all classical experiments already done that confirm GR can also be understood as a confirmation of TERG (and the same with the equivalent limit of non-metricity theories).

The variation of (2.16) reads

$$\delta\mathcal{S} = \int \Big[-\frac{1}{2\kappa^2}(e\delta T + T\delta e) + \delta(e\mathcal{L}_{\mathrm{m}}) \Big] d^4x = 0.$$

Considering the following relations

$$\delta e = e e_a{}^{\lambda}\delta e^a{}_{\lambda},$$

$$\delta T = \frac{1}{4}\delta\Big(T^{\mu\nu\lambda}T_{\mu\nu\lambda}\Big) + \frac{1}{2}\delta\Big(T^{\mu\nu\lambda}T_{\nu\mu\lambda}\Big) - \delta\Big(T^{\mu}T_{\mu}\Big),$$

$$\delta(e\mathcal{L}_{\mathrm{m}}) := e\mathcal{T}_a{}^{\lambda}\delta e^a{}_{\lambda},$$

where $\mathcal{T}_a{}^{\lambda}$ is the energy momentum tensor of matter, the variation can be expressed as

$$\int \Bigg[-\frac{1}{2\kappa^2}\Bigg[e\Bigg(\frac{1}{4}\delta\Big(T^{\mu\nu\lambda}T_{\mu\nu\lambda}\Big) + \frac{1}{2}\delta\Big(T^{\mu\nu\lambda}T_{\nu\mu\lambda}\Big) - \delta\Big(T^{\mu}T_{\mu}\Big)\Bigg) + eTe_a{}^{\lambda}\delta e^a{}_{\lambda}\Bigg] + e\mathcal{T}_a{}^{\lambda}\delta e^a{}_{\lambda} \Bigg] d^4x = 0.$$

Taking into account that

$$\delta\Big(T_{\mu\nu\lambda}T^{\mu\nu\lambda}\Big) = -4T^{\mu\nu\lambda}T_{\mu\nu\beta}e_a{}^{\beta}\delta e^a{}_{\lambda} + 4T_{\mu}{}^{\nu\lambda}e_a{}^{\mu}\partial_{\nu}\delta e^a{}_{\lambda},$$

$$\delta\Big(T_{\mu\nu\lambda}T^{\nu\mu\lambda}\Big) = 2\Big(T^{\beta\nu\mu} - T^{\mu\nu\beta}\Big)T_{\nu\mu\lambda}e_a{}^{\lambda}\delta e^a{}_{\beta} + \Big(T^{\mu}{}_{\nu}{}^{\beta} - T^{\beta}{}_{\nu}{}^{\mu}\Big)e_a{}^{\nu}\partial_{\mu}\delta e^a{}_{\beta},$$

$$\delta\Big(T_{\mu}T^{\mu}\Big) = -2\Big(T^{\beta}T^{\alpha}{}_{\beta\mu} + T^{\alpha}T_{\mu}\Big)e_a{}^{\mu}\delta e^a{}_{\alpha} + 2\Big(T^{\alpha}e_a{}^{\mu} - T^{\mu}e_a{}^{\alpha}\Big)\partial_{\alpha}\delta e^a{}_{\mu},$$

one obtains, after some work and considering that the integrand of the action after being perturbed must be zero, the following field equations

$$-\frac{2}{e}\partial_{\mu}(ee_a{}^{\rho}S_{\rho}{}^{\mu\nu}) + 2e_a{}^{\lambda}T^{\rho}{}_{\mu\lambda}S_{\rho}{}^{\nu\mu} + \frac{1}{2}Te_a{}^{\nu} = \kappa^2 e_a{}^{\rho}\mathcal{T}_{\rho}{}^{\nu}, \tag{2.19}$$

where, again, $\mathcal{T}_\rho{}^\nu$ is the energy momentum tensor of matter.
Now, this variation was done with respect to the tetrad field and not with respect to the spin connection because we are considering the Weitzenböck gauge, that is $\omega^a{}_{b\mu} = 0$, which simplifies the Weitzenböck conection and, hence, all the rest of the elements. Consequently, the field equations are more simple, but the cost is that they are no more invariant under LLTs.

Three things can be said with regard to TG physical interpretation: First, that space-time suffers a natural deformation in this theory [4], something similar as the case of GR. Second, that torsion is the only responsible for the gravitational attraction between objects in nature. Third, that depending on the amount of matter is the magnitude of the force because the bigger is the amount of matter, the bigger is the deformation of space-time. So, the energy-momentum tensor is the source of torsion. There have been, in fact, several researches about torsion and its possible relevance in quantum mechanics and electrodynamics contexts [48], however it is nor clear what torsion really is, physically speaking. In [49] the meaning of torsion in TG theories is investigated, but there are still issues that are opened to debate.

2.1 $f(T)$ gravity

In the same way that GR can be generalized, TG can also be extended in different forms [38]. The action (2.16) can be generalized to arbitrary functions of the torsion scalar to produce $f(T)$ gravity [50, 51, 52, 53, 54] which follows the same reasoning as $f(\mathring{R})$ gravity. $f(T)$ is the first and more direct extension of TG. However, unlike $f(\mathring{R})$ gravity [32, 55, 56] that is a fourth order theory under the metric formalism, $f(T)$ gravity produces generally second-order field equations in the tetrad fields. This is interesting because it means that Lovelock's theorem is weakened in TG [57, 58, 59] which has had interesting consequences for constructing scalar-tensor theories of gravity [59, 60, 61, 62, 63, 64]. There are some works that analyze different $f(T)$ models with some good results; in [65] some $f(T)$ models are studied in the bayesian framework, considering the background and the perturbative behavior simultaneously. It was reported that those specific models, and one of them in particular, had an good effectivity to predict cosmological data. Another example of the relevance of $f(T)$ models is the one reported in [66]; with regard to the dark energy topic an $f(T)$ model (specifically, a combined $f(T)$ theory with both logarithmic and exponential terms) was able to allow the crossing of the line that divides the phantom and non-phantom region in the cosmological evolution. This is an important result in the cosmic expansion context. So, as we can see, $f(T)$ gravity is one of the most studied and used extensions of TG to describe the gravitational interaction.

2.1.1 Field equations

In $f(T)$ gravity the action to be considered is

$$\mathcal{S} = \int d^4x \left[\frac{1}{2\kappa^2} f(T) + \mathcal{L}_\mathrm{m}\right] e, \tag{2.20}$$

and its variation with respect to the tetrad field leads to

$$\delta\mathcal{S} = \int \left[\frac{1}{2\kappa^2}\Big(e f_T \delta T + f\delta e\Big) + \delta(e\mathcal{L}_\mathrm{m})\right] d^4x = 0,$$

where $f_T = df/dT$. Considering previous relations, the variation can be expressed as

$$\int \left[\frac{1}{2\kappa^2}\left[e f_T\Big(\frac{1}{4}\delta\Big(T^{\mu\nu\lambda}T_{\mu\nu\lambda}\Big)+\frac{1}{2}\delta\Big(T^{\mu\nu\lambda}T_{\nu\mu\lambda}\Big)-\delta\Big(T^{\mu}T_{\mu}\Big)\Big)+e f e_a{}^{\lambda}\delta e^a{}_{\lambda}\Big)\right]+e\mathcal{T}_a{}^{\lambda}\delta e^a{}_{\lambda}\right]d^4x = 0.$$

We get, after straightforward calculations,

$$4\partial_\mu\Big(e f_T S_a{}^{\mu\lambda}\Big) - 4 e f_T T^{\sigma}{}_{\mu a} S_\sigma{}^{\lambda\mu} - e f e_a{}^{\lambda} = 2\kappa^2 e \mathcal{T}_a{}^{\lambda},$$

which can be written as

$$4 e f_{TT}\partial_\mu(T) e_a{}^{\nu} S_\nu{}^{\mu\lambda} + 4 f_T \partial_\mu\Big(e e_a{}^{\nu} S_\nu{}^{\mu\lambda}\Big) - 4 e f_T e_a{}^{\nu} T^{\sigma}{}_{\mu\nu} S_\sigma{}^{\lambda\mu} - e f e_a{}^{\lambda} = 2\kappa^2 e e_a{}^{\nu}\mathcal{T}_\nu{}^{\lambda},$$

and where $f_{TT} = d^2f/dT^2$. Final arrangements give us the field equations in $f(T)$ gravity;

$$\frac{2}{e}\partial_\mu(e e_a{}^{\rho} S_\rho{}^{\mu\nu}) f_T + 2 e_a{}^{\rho} S_\rho{}^{\mu\nu}\partial_\mu(T) f_{TT} - 2 f_T e_a{}^{\lambda} T^{\rho}{}_{\mu\lambda} S_\rho{}^{\nu\mu} - \frac{1}{2} e_a{}^{\nu} f = \kappa^2 e_a{}^{\rho}\mathcal{T}_\rho{}^{\nu}. \tag{2.21}$$

These are in general second-order field equations in the tetrad fields pointing to a more generalized Lovelock theorem in TG. This extension has a number of other similarities with GR such as exhibiting the same GW polarizations [8, 67].
Notice that when $f(T) = -T$, which is the TEGR case, Eq.(2.21) is reduced to Eq.(2.19) as it is expected.

2.1.2 Friedmann equations and equation of state

Considering a FLRW metric and an energy momentum tensor associated with a perfect fluid, from (2.21) we obtain the modified Friedmann equations in this case, which are

$$H^2 = \frac{8\pi G}{3}\rho_{physical} + \frac{1}{6}\Big(T - f + 2T f_T\Big) \tag{2.22}$$

and

$$\dot{H} = -\frac{8\pi G}{2}(\rho_{physical} + p_{physical}) + \dot{H}\Big(1 + f_T + 2Tf_{TT}\Big), \tag{2.23}$$

where H is the Hubble parameter, and $\rho_{physical}$ and $p_{physical}$ are the density and pressure, respectively, associated with the perfect fluid, which have a contribution of matter and radiation; that is, $\rho_{physical} = \rho_m + \rho_r$, where ρ_m is the baryonic matter (including dark matter) and ρ_r is radiation (including neutrinos). In the TEGR limit we recover the basic Friedmann equations of GR, as it should be, so the extra terms can be joined together whether we define an effective contribution as follows:

$$\rho_{eff} := \frac{1}{2\kappa^2}\Big(T - f + 2Tf_T\Big), \tag{2.24}$$

$$p_{eff} := -2\frac{\dot{H}}{\kappa^2}\Big(1 + f_T + 2Tf_{TT}\Big) - \rho_{eff}. \tag{2.25}$$

Consequently, Eq.(2.22) and Eq.(2.23) can be rewritten as

$$H^2 = \frac{\kappa^2}{3}\Big(\rho_{physical} + \rho_{eff}\Big), \tag{2.26}$$

$$\dot{H} = -\frac{\kappa^2}{2}\Big(\rho_{physical} + \rho_{eff} + p_{physical} + p_{eff}\Big), \tag{2.27}$$

where the effective density and pressure (ρ_{eff} and p_{eff}, respectively) are related with the contribution of dark energy to the total material content. The effective fluid also satisfies a continuity equation;

$$\dot{\rho}_{eff} + 3H(\rho_{eff} + p_{eff}) = 0. \tag{2.28}$$

An effective equation of state parameter can be defined through $\omega_{eff} := p_{eff}/\rho_{eff}$, which for a general $f(T)$ model has the form

$$\omega_{eff} = -1 - (1+\omega)\frac{\Big(f - 2Tf_T\Big)\Big(1 + f_T + 2Tf_{TT}\Big)}{\Big(f_T + 2Tf_{TT}\Big)\Big(T - f + 2Tf_T\Big)}, \tag{2.29}$$

where ω is given by Eq.(1.16).

2.1.3 $f(T)$ cosmology

In this section some cosmological results that have been obtained in some references for some $f(T)$ models will be reviewed. What one should have in mind is that, ΛCDM results to be a cosmological solution to GR theory with a FLRW metric, however, given the inherent problems of this model, some solutions are searched through modifications of GR, which implies to approve observational tests at a local (solar system) level and a cosmological (large scales) level.

The simplest and more direct $f(T)$ model to be considered is a Power Law Model, that is, a function that depends on a term (apart from the basic term that relates TG with

GR) that is weighted by a coefficient and powered by a free parameter. As it was mentioned in the introduction, one should always look for models with a minimum of free parameters for simplicity not only in the development of the equations but also in the mathematical analysis. There is another reason; the number of restrictions is directly related to the number of free parameters. The constrictions and the phenomenology tend to be more complex and restrictive when one deals with many parameters. In consequence, and following the law of parsimony, it is better and more suitable to work with the minimum number of free parameters as possible; the Power Law Model is one of the simplest models in that aspect [8, 68]. Mathematically, it takes the form

$$f = -T + mT^b, \tag{2.30}$$

with the following derivatives with respect to torsion:

$$f_T = -1 + mbT^{b-1}, \tag{2.31}$$

and

$$f_{TT} = mb(b-1)T^{b-2}. \tag{2.32}$$

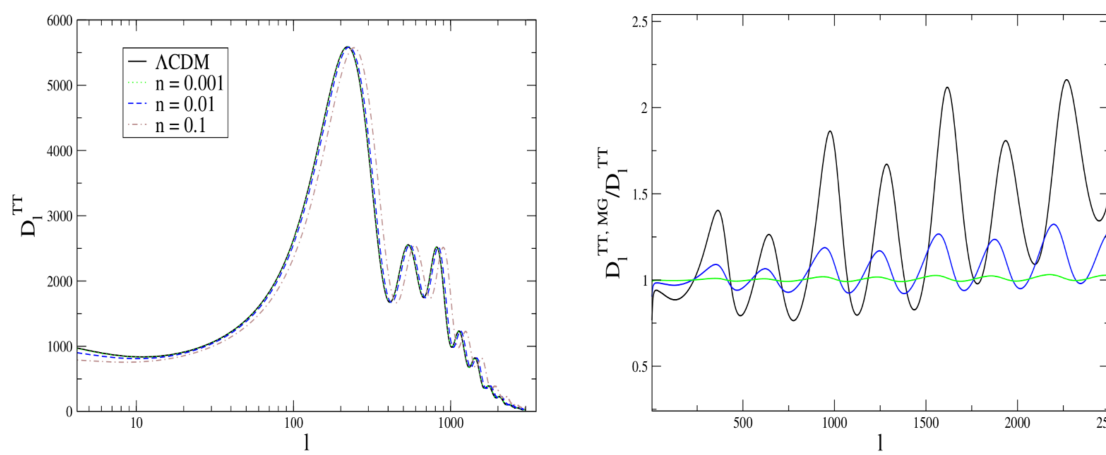

Figure 2.4: In the left panel it is a plot of the CMB TT power spectrum, where $D_l^{TT} = l(l+1)C_l^{TT}/2\pi\mu K^2$, for ΛCDM and for some particular cases of the Power Law Model in $f(T)$, that is Eq.(2.30). The parameter n is just our parameter b. Now, in the right panel it is shown the relative deviation of CMB TT power spectrum from the base line Planck 2015 ΛCDM in comparison with the Power Law Model in $f(T)$; the black, blue and green curve refers to $b = 0.1$, $b = 0.01$ and $b = 0.001$ respectively. Figure taken from [68].

From (2.30) it is observed that $f = -T + constant$ corresponds to the recovery of GR with a cosmological constant. This is achieved by considering $b = 0$ because the additive constant in that case is related with, precisely, the cosmological constant. In this model apparently one has two free parameters; b and m. However, working with Eq.(2.22) at present time we obtain

$$m = (6H_0^2)^{1-b}\Big(\frac{1-\Omega_{m0}-\Omega_{r0}}{2b-1}\Big),$$

so m is just a parameter defined by the initial conditions, and depends on b which turns out to be the only free parameter in this model. Whether m is small, we recover GR solutions in astrophysical scales. Following the restriction (3.18) that will be seen in the next chapter, it is convenient to consider $b > 0$.

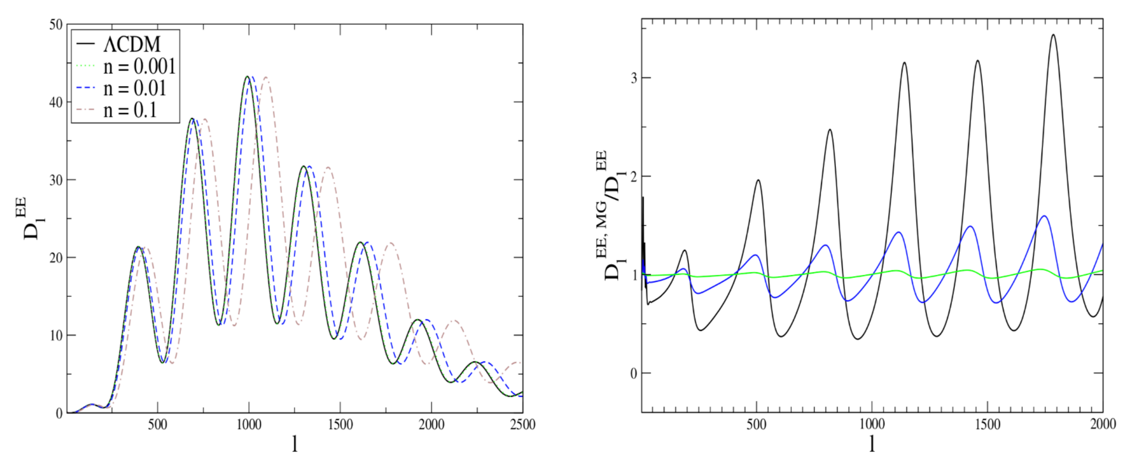

Figure 2.5: These plots are the same as in figure 2.4, but for the CMB EE power spectrum, $D_l^{EE} = l(l+1)C_l^{EE}/2\pi\mu K^2$. Again, the parameter n is just our parameter b, and in the right panel the black, blue and green curve refers to $b = 0.1$, $b = 0.01$ and $b = 0.001$ respectively. Figure taken from [68].

It is expected that an arbitrary $f(T)$ function will cause significant changes on the CMB anisotropy, in all angular scales, since the scalar perturbations in $f(T)$ gravity lead to a non-trivial modification on the dynamics of the scalar modes. Following this, figures 2.4, 2.5 and 2.6, that were taken from [68], present some results for the Power Law Model in the context of power spectrum, but also of a parametric space. Another result of the CMB power spectrum of this model is shown in Figure 2.7 which was taken from [8].

From Figure 2.6 can be seen that the H_0 tension is solved with this model in TG considering $b \in [0.0, 0.1]$. However, the σ_8 tension[1] is not solved with the Power Law Model since from Planck CMB the value of amplitude of matter density fluctuations is $\sigma_8 = 0.831 \pm 0.013$, which is about 2σ higher than $\sigma_8 = 0.75 \pm 0.03$ as given by the Sunyaev-Zeldovich cluster abundances measurements [69], for example. Further investigation on this model should be done in order to find the solution for both tensions.

In Figure 2.7 is shown that for the angular scale $l > 20$ the theoretical predictions for the Power Law Model and ΛCDM are practically the same. In consequence, for small

[1]σ_8 is a cosmology parameter which measures the amplitude of the linear power spectrum on the scale of 8 h^{-1} Mpc. It is a central parameter due to it has a big influence over the growth of fluctuations in the early epoch of the universe. The σ_8 tension is not other thing than the disparity between current observations and theoretical results, just as it happens in the H_0 tension.

angular scales no deviations are expected compared to ΛCDM cosmology. Instead of that, considerable deviations can be observed at large angular scales. In fact, due to the theoretical CMB BB spectrum should present a peak at $l \simeq 5$ (because of the effects of tensor modes on the scattering during the reionization epoch) still to be detected by future experiments, the influence of the Power Law Model can be quantified on the reionization peak, where we can note different predictions for a range of values b compared to the reference ΛCDM scenario.

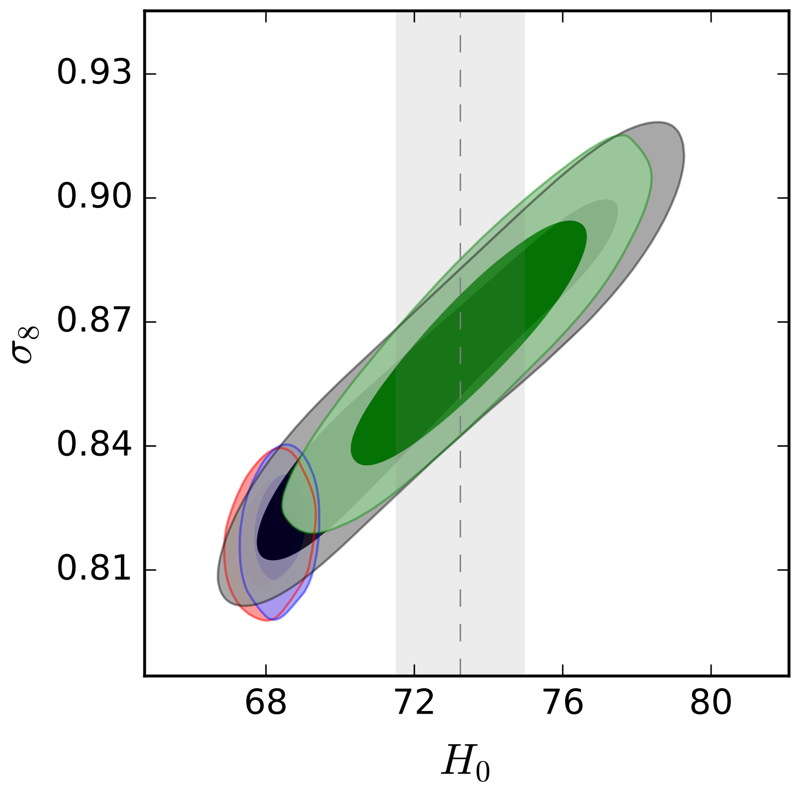

Figure 2.6: In this figure it is presented a parametric space in the plane H_0 - σ_8, where the regions in red (blue) show the constraints for ΛCDM model from CMB + BAO (CMB + BAO + H_0), respectively. The regions in black (green) show the constraints for the Power Law Model in $f(T)$, with $b \in [0.0, 0.1]$, from CMB + BAO (CMB + BAO + H_0), respectively. The vertical gray band corresponds to $H_0 = 73.24 \pm 1.74$ $km\ s^{-1} Mpc^{-1}$. Figure taken from [68].

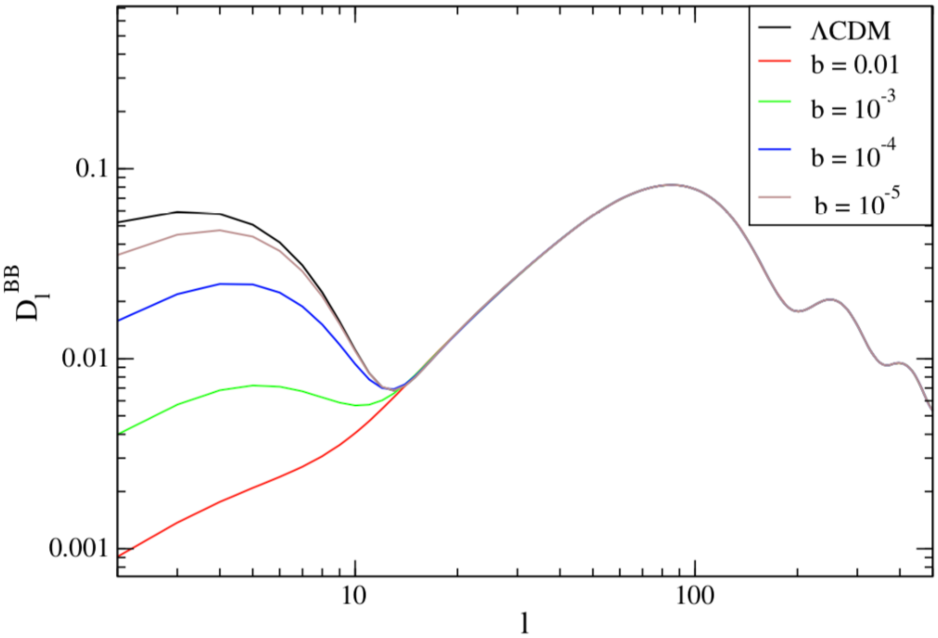

Figure 2.7: This is the CMB BB power spectrum for ΛCDM, and for the Power Law Model for various values of the free parameter b. We can see important deviations at large angular scales; notice that those deviations are more significant when the value of b is higher, close to 1. On the other hand, when the free parameter is practically zero the second term of (2.30) acts like the standard cosmological constant and we recover the ΛCDM cosmology. Figure taken from [8].

2.2 $f(T, B)$ gravity

In order to compare TG with $f(\mathring{R})$ gravity one must consider arbitrary functions of not only the torsion scalar but also the boundary term through $f(T, B)$ gravity (see Figure 2.8, taken from [47]) which provides a much richer class of models at the level of the field equations because of the following: $f(T, B)$ gravity has also been well studied [36, 70, 71, 72, 73, 67, 74, 75] as a possible extension to TEGR; since the boundary term embodies the fourth-order element of the Ricci scalar, $f(T, B)$ gravity is an interesting model in which the second- and fourth-order contributions to the field equations are decoupled. This may be a more natural generalization of gravity in fourth-order theories.

2.2.1 Field equations

Raising the TEGR action to its $f(T, B)$ gravity extension results in the action

$$\mathcal{S} = \int d^4x \left[\frac{1}{2\kappa^2} f(T, B) + \mathcal{L}_{\mathrm{m}}\right] e, \tag{2.33}$$

$$\begin{tikzcd}
f(T,B) \arrow[rr, "f=f(T)"] \arrow[d, "f=f(-T+B)"'] & & f(T) \arrow[d, "f=T"] \\
f(\overset{\circ}{R}) \arrow[rr, "f=\overset{\circ}{R}"'] & & \text{GR or TEGR}
\end{tikzcd}$$

Figure 2.8: Relation between $f(T,B)$ models and its particular cases. This diagram, taken from [47], considers that the TEGR case is given by $f(T)=T$ because of the signature used. In our case, $f(T)=-T$ is the TEGR limit.

where the tetrad field and the spin connection are the variables, but we will only focus on the tetrad for the same reason given below Eq.(2.19). Taking a variation one gets

$$\delta\mathcal{S}=\int\left[\frac{1}{2\kappa^2}\Big(f\delta e+ef_B\delta B+ef_T\delta T\Big)+\delta(e\mathcal{L}_{\mathrm{m}})\right]d^4x=0.$$

We know three terms, those with δe, δT and $\delta(e\mathcal{L}_{\mathrm{m}})$. Basically we need to focus in the variation of the boundary term with respect to the tetrad field. It is possible to find that

$$ef_B\delta B=-\Big(f_BB+2(\partial_\mu f_B)T^\mu\Big)\delta e-2e(\partial_\mu f_B)\delta T^\mu. \tag{2.34}$$

Now, considering that

$$\delta T^\mu=-({e_a}^\mu T^\lambda+g^{\mu\lambda}T_a+{T^\lambda}_a{}^\mu)\delta {e^a}_\lambda+g^{\mu\nu}{e_a}^\lambda(\partial_\lambda\delta {e^a}_\nu-\partial_\nu\delta {e^a}_\lambda),$$

the last term of (2.34) can be written as

$$\begin{aligned}e(\partial_\mu f_B)\delta T^\mu=\Bigg[&\partial_\nu\Big({e_a}^\lambda eg^{\mu\nu}(\partial_\mu f_B)\Big)-\partial_\nu\Big({e_a}^\nu eg^{\mu\lambda}(\partial_\mu f_B)\Big)\\ &-e(\partial_\mu f_B)({e_a}^\mu T^\lambda+g^{\mu\lambda}T_a+{T^\lambda}_a{}^\mu)\Bigg]\delta {e^a}_\lambda.\end{aligned} \tag{2.35}$$

Since the Weitzenböck connection satisfies the metricity condition, we can find that

$$\partial_\lambda g^{\mu\nu}=-({\Gamma_\lambda}^{\nu\mu}+{\Gamma_\lambda}^{\mu\nu}).$$

Knowing this and also that

$$\partial_\lambda e=eg^{\mu\nu}\partial_\lambda g_{\mu\nu},$$

one is able to find that

$$\partial_\lambda e = e\Gamma_\lambda{}^\sigma{}_\sigma.$$

With the relation above and Eq.(2.10) at hand, we arrive to the following expressions:

$$\partial_\nu\Big(e_a{}^\lambda e g^{\mu\nu}(\partial_\mu f_B)\Big) = ee_a{}^\lambda \mathring{\Box} f_B - e(\partial_\mu f_B)(e_a{}^\lambda \Gamma_\nu{}^{\mu\nu} - e_a{}^\lambda \Gamma^{\nu\mu}{}_\nu + \Gamma^{\mu\lambda}{}_a), \tag{2.36}$$

and

$$\partial_\nu\Big(e_a{}^\nu e g^{\mu\lambda}(\partial_\mu f_B)\Big) = ee_a{}^\nu \mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B + e(\partial_\mu f_B)\Big(g^{\mu\lambda}(\Gamma_a{}^\nu{}_\nu - \Gamma_\nu{}^\nu{}_a) - \Gamma_a{}^{\lambda\mu}$$
$$-\Gamma_a{}^{\mu\lambda} + \Gamma^{\lambda\mu}{}_a - K^{\lambda\mu}{}_a\Big). \tag{2.37}$$

Substituting (2.36) and (2.37) in (2.35) we obtain

$$e(\partial_\mu f_B)\delta T^\mu = -\Big[e(\partial_\mu f_B)\Big(e_a{}^\mu T^\lambda + g^{\mu\lambda}T_a + T^\lambda{}_a{}^\mu + g^{\mu\lambda}(\Gamma_a{}^\nu{}_\nu - \Gamma_\nu{}^\nu{}_a) - \Gamma_a{}^{\lambda\mu} - \Gamma_a{}^{\mu\lambda} + \Gamma^{\lambda\mu}{}_a - K^{\lambda\mu}{}_a$$
$$+\Gamma^{\mu\lambda}{}_a + e_a{}^\lambda \Gamma_\nu{}^{\mu\nu} - e_a{}^\lambda \Gamma^{\nu\mu}{}_\nu\Big) - ee_a{}^\lambda \mathring{\Box} f_B + ee_a{}^\nu \mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B\Big]\delta e^a{}_\lambda,$$

which can be simplified as

$$e(\partial_\mu f_B)\delta T^\mu = -\Big[e(\partial_\mu f_B)\Big(e_a{}^\mu T^\lambda + \Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a\Big)$$
$$-ee_a{}^\lambda \mathring{\Box} f_B + ee_a{}^\nu \mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B\Big]\delta e^a{}_\lambda. \tag{2.38}$$

Substituting (2.38) into Eq.(2.34) we have

$$ef_B\delta B = \Big[-f_B B e e_a{}^\lambda - 2ee_a{}^\lambda(\partial_\mu f_B)T^\mu + 2ee_a{}^\mu(\partial_\mu f_B)T^\lambda + 2(\partial_\mu f_B)e(\Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a)$$
$$-2ee_a{}^\lambda \mathring{\Box} f_B + 2ee_a{}^\nu \mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B\Big]\delta e^a{}_\lambda.$$

Introducing the superpotential and rearranging we get

$$ef_B\delta B = \Big[2ee_a{}^\nu \mathring{\nabla}^\lambda \mathring{\nabla}_\nu f_B - 2ee_a{}^\lambda \mathring{\Box} f_B - Bef_B e_a{}^\lambda + 2e(\partial_\mu f_B)(2S_a{}^{\lambda\mu} - K_a{}^{\lambda\mu}$$
$$+\Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a)\Big]\delta e^a{}_\lambda.$$

Notice, however, that $-K_a{}^{\lambda\mu} + \Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a = 0$. Hence, the variation of the boundary term with respect to the tetrad field is

$$ef_B\delta B = \left[2ee_a{}^{\nu}\mathring{\nabla}^{\lambda}\mathring{\nabla}_{\nu}f_B - 2ee_a{}^{\lambda}\mathring{\Box}f_B - Bef_Be_a{}^{\lambda} - 4e(\partial_\mu f_B)S_a{}^{\mu\lambda}\right]\delta e^a{}_{\lambda}.$$

Taking into account the previous equation and the variations seen before, Hamilton's principle applied to Eq.(2.33) lead us to

$$2ee_a{}^{\nu}\mathring{\nabla}^{\lambda}\mathring{\nabla}_{\nu}f_B - 2ee_a{}^{\lambda}\mathring{\Box}f_B - Bef_Be_a{}^{\lambda} - 4e(\partial_\mu f_B)S_a{}^{\mu\lambda} - 4\partial_\mu(ef_TS_a{}^{\mu\lambda})$$
$$+4ef_TT^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu} + efe_a{}^{\lambda} = -2\kappa^2e\mathcal{T}_a{}^{\lambda}.$$

Once again, $\mathcal{T}_a{}^{\lambda}$ is the energy momentum tensor of matter. Modifying a little bit the previous expression, finally we get the field equations in $f(T,B)$ gravity

$$e_a{}^{\mu}\mathring{\Box}f_B - e_a{}^{\nu}\mathring{\nabla}^{\mu}\mathring{\nabla}_{\nu}f_B + \frac{1}{2}Be_a{}^{\mu}f_B + 2\Big[\partial_\nu f_B + \partial_\nu f_T\Big]S_a{}^{\nu\mu} + \frac{2}{e}\partial_\nu\Big(eS_a{}^{\nu\mu}\Big)f_T$$
$$-2f_TT^{\nu}{}_{\sigma a}S_{\nu}{}^{\mu\sigma} - \frac{1}{2}e_a{}^{\mu}f = \kappa^2\mathcal{T}_a{}^{\mu}, \tag{2.39}$$

where the indices of $\mathcal{T}_a{}^{\mu}$ can be acted upon by the tetrad, or contracted with the metric (on general manifold indices) as in standard gravity.

2.2.2 Friedmann equations and equation of state

The modified Friedmann equations in this case are

$$-3H^2\Big(3f_B + 2f_T\Big) + 3H\dot{f}_B - 3\dot{H}f_B + \frac{1}{2}f = \kappa^2\rho_{physical} \tag{2.40}$$

and

$$-\Big(3H^2 + \dot{H}\Big)\Big(3f_B + 2f_T\Big) - 2H\dot{f}_T + \ddot{f}_B + \frac{1}{2}f = -\kappa^2p_{physical}, \tag{2.41}$$

where $\rho_{physical}$ and $p_{physical}$ are the density and pressure, respectively, associated with the perfect fluid, which have a contribution of matter and radiation; that is, $\rho_{physical} = \rho_m + \rho_r$, where ρ_m is the baryonic matter (including dark matter) and ρ_r is radiation (including neutrinos). Compare (2.40) and (2.41) with (2.22) and (2.23); in $f(T,B)$ there are no second derivatives with respect to torsion, but the change of the function f with respect to the boundary term B appears several times. With regard to the effective state parameter in this case, and following the same procedure as in 2.1.2, we can see that for a general $f(T,B)$ model we have

$$\omega_{eff} = -1 + \frac{\ddot{f}_B - 3H\dot{f}_B - 2\dot{H}f_T - 2H\dot{f}_T}{3H^2(3f_B + 2f_T) - 3H\dot{f}_B + 3\dot{H}f_B - \frac{1}{2}f}, \tag{2.42}$$

which is more complex than (2.29).

2.2.3 $f(T, B)$ cosmology

One of the many possibilities for an $f(T, B)$ model is to consider a certain kind of function which is based on the idea that the scalars T and B have to be coupled somehow since as it is noticed from Eqs.(3.28)-(3.29), T and B require to be coupled in order to study any possible deviation from GR. The simplest function of this type is called a Mixed Power Law Model [36] and, without loss of generality, can be described by

$$f = f_0 B^n T^m, \tag{2.43}$$

with the following derivatives with respect to torsion and boundary term

$$f_T = f_0 m B^n T^{m-1}, \tag{2.44}$$

$$f_B = f_0 n B^{n-1} T^m, \tag{2.45}$$

$$\dot{f}_T = 6 f_0 m B^{n-1} T^{m-2} \Big[nT(6H\dot{H} + \ddot{H}) + 2H\dot{H}(m-1)B \Big], \tag{2.46}$$

$$\dot{f}_B = 6 f_0 n B^{n-2} T^{m-1} \Big[(n-1)(6H\dot{H} + \ddot{H})T + 2mBH\dot{H} \Big], \tag{2.47}$$

$$\begin{aligned} \ddot{f}_B = 6 f_0 n \Bigg[\Big[(m-1) B^{n-2} T^{m-2} + 6(n-2) B^{n-3} T^{m-1} \Big(6H\dot{H} + \ddot{H} \Big) \Big] \Big[2mBH\dot{H} \\ + (n-1)\Big(6H\dot{H} + \ddot{H} \Big) \Big] + B^{n-2} T^{m-1} \Big[12H\dot{H}(n-1)\Big(6H\dot{H} + \ddot{H} \Big) \\ + (n-1)T\Big(6\dot{H}^2 + 6H\ddot{H} + \dddot{H} \Big) + 2m\Big(6H\dot{H}(6H\dot{H} + \ddot{H}) + B\dot{H}^2 + BH\ddot{H} \Big) \Big] \Bigg]. \end{aligned} \tag{2.48}$$

In this model, f_0, n and m are free parameters. Actually, it is going to be been seen that at perturbative level f_0 is not of importance because it is naturally eliminated when this model is introduced in (3.29). Evidently, at background level this factor makes a difference, but it is not the case in the perturbative context and can be neglected.
Following the restriction (3.30), it is convenient to consider $m > 0$ and preferably $n \geq 0$.
Figures 2.9, 2.10 and 2.11, taken from [36], depict some cosmological results obtained from the Mixed Power Law Model.

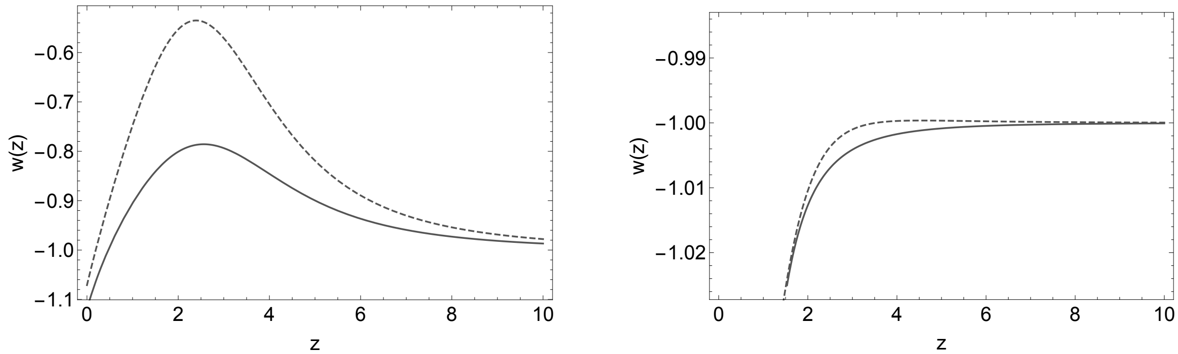

Figure 2.9: This is the evolution of the equation of state (EoS) for the Mixed Power Law. In the left panel we have, resolving for T and B, the case $T < B$ (solid line) and $T > B$ (dashed line). In the right panel it is shown the variation of m and n, with $m < n$ (solid line) and $m > n$ (dashed line). Figure taken from [36].

Parameter	best-fit	mean$\pm\sigma$	95% lower	95% upper
H_0	67.92	$67.86^{+1.2}_{-1.1}$	65.63	70.1
m	36.59	$38.02^{+2.1}_{-2.6}$	33.71	42.45
k	2.55	$2.626^{+0.11}_{-0.13}$	2.396	2.861
c_0	$1.653e+11$	$4.97e+11^{+-2.6e+11}_{-5e+11}$	$2.353e+09$	$4.847e+09$

Figure 2.10: Parameters and mean values for the Mixed Power Law Model model. The parameters k and c_0 are just our parameters n and f_0, respectively. The cosmological tests for the free parameters of this model considered the constraints solutions over T and B imposed in this case, and the specific cosmological parameters were determined, in addition to Ω_m and H_0 late Universe data. The codes CLASS and Monte Python (see [76]) were used to constrain the model using a total sampler of CC+SNeIa+BAO. Table taken from [36].

In the left panel of Figure 2.9, the Mixed Power Law Model model cross the phantom divided-line but preserve its quintessence behaviour until at high redshift both scenarios tend to ΛCDM model. With regard to the right panel of the same figure, at $z < 4$ both scenarios mimic a phantom energy.

Figure 2.11 shows the space parameter for ω_{model} and Ω_m with their probability density function (PDF) versus Ω_m up to $3-\sigma$ confidences levels (CL) using the joint sampler CC+Pantheon+BAO; where CC indicates Cosmic Chronometers which are measurements of $H(z)$ [77], Pantheon refers to an amount of data related with luminosity distances of Type Ia supernovae and BAO are the Baryon Acoustic Oscillations which are frozen relics that provide distance estimates [78]. There are numerical codes as Monte Carlo process for Cosmological Parameter extraction. This kind of process contains likelihood codes of most recent experiments, and interfaces with the Boltzmann code CLASS or computing the cosmological observables.

In summary, the results here discussed, which were obtained from some references [68, 8, 36], show that both the Power Law Model and the Mixed Power Law Model give good results in cosmology and agree with GR in a particular limit, respectively. Such results make us think that both extensions $f(T)$ and $f(T, B)$ build good scenarios, where torsion and the boundary terms, fitted with astrophysical data, can solve doubts in the study of the late-time accelerating universe. Hence, given this reliability, we think they should be tested in other situations, for example in the early universe and in the GWs context. With regard to the GWs subject from the point of view of TG, a theoretical study is done in chapter 3.

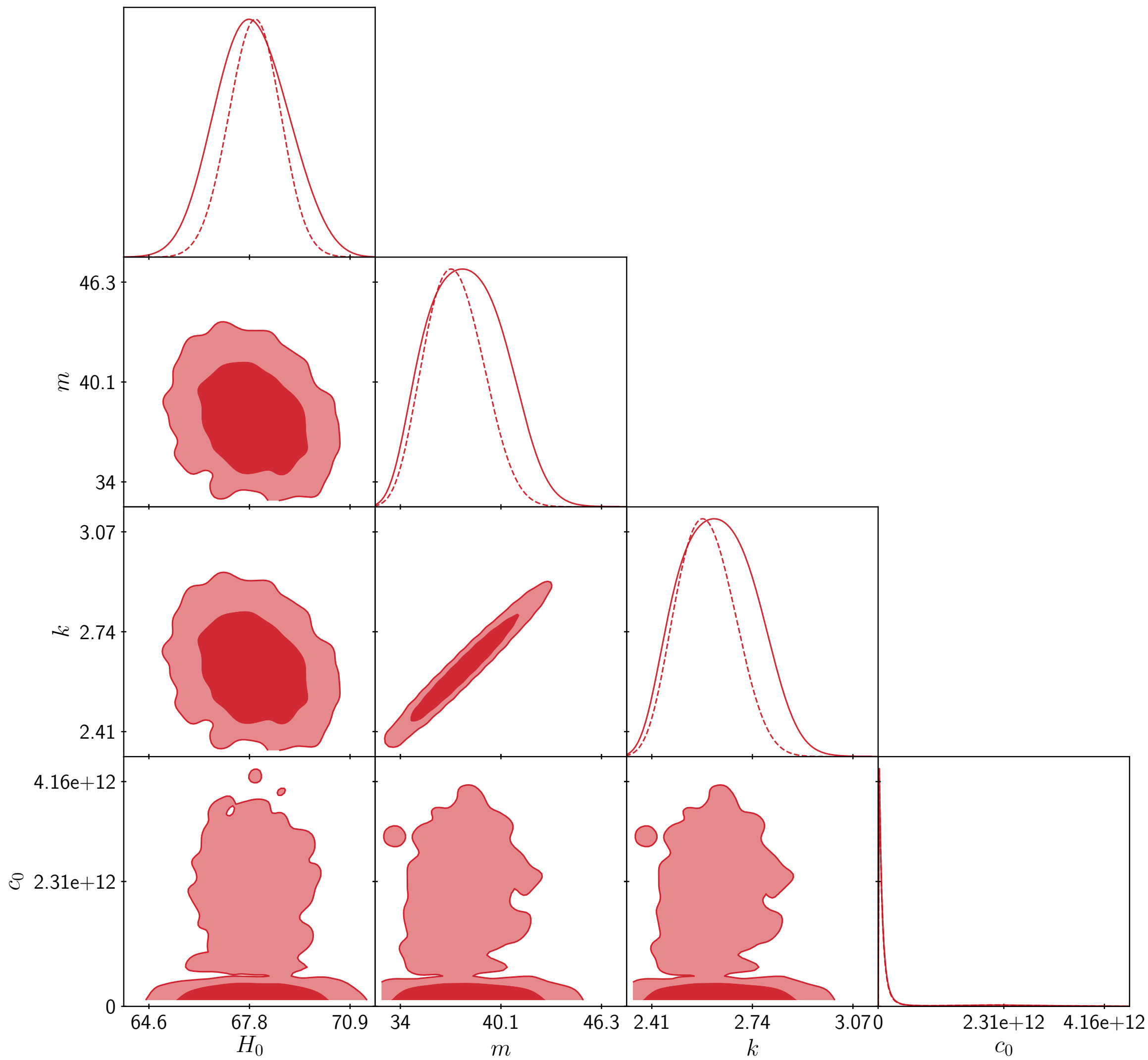

Figure 2.11: One-dimensional marginalised distribution, and two-dimensional contours with 68% and 95% confidence level for the free parameters (H_0, m, k, c_0), which in our case are (H_0, m, n, f_0), for the Mixed Power Law model (2.43) using the constrained solutions for T and B scalars and CC+Pantheon+BAO total sampler. Figure taken from [36].

Part II

The Perturbation

Chapter 3

Perturbations in Teleparallel Gravity

In GR, gravitational waves are 'ripples' in space-time caused by some of the most violent and energetic processes in the Universe, that would travel at the speed of light, carrying with them information about their origins, as well as clues of the nature of gravity itself [79]. As it was mentioned, GR predicted the existence of GWs. The source objects involved must be massive and subjected to gravitational acceleration; this would disrupt space-time so that 'waves' of space-time propagate in all directions away from the event/source. The most energetic GWs are produced by cataclysmic or violent events such as colliding black holes, supernovae (which are massive stars exploding at the end of their lifetimes), and colliding neutron stars. It has also been predicted that the rotation of neutron stars that are not perfect spheres can also cause GWs [80], and even the remnants of gravitational radiation created by the Big Bang may cause them [81]. These last ones are the so known Primordial Gravitational Waves, and its study would bring important data of the beginning of the universe. Information can be obtained, particularly, from the BB-mode correlation angular power spectrum of CMB (see Figure 3.1, taken from [82]).

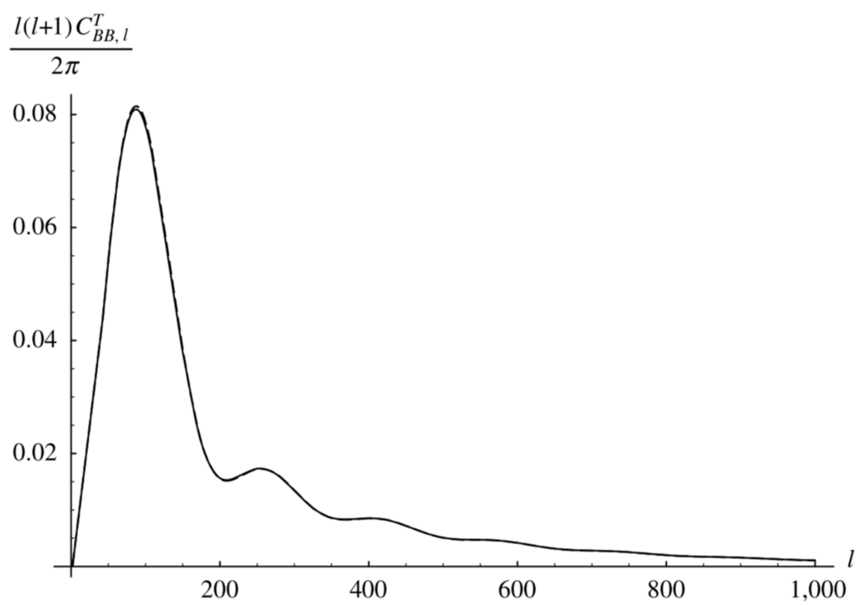

Figure 3.1: One of the multipole coefficients (the one related with gravitational waves) in square microKelvin against the multipole moment. We can see that large scales (low multipole moments) correspond to higher amplitudes of the multipole coefficients, that is, higher temperature fluctuations. GWs are more easily located in this range of observation. This figure was taken from [82].

3.1 Gravitational waves in GR

Mathematically speaking, what has to be done in order to study GWs in GR is to perturb the metric, because a tensorial perturbation on the metric reflects as a perturbation in space-time itself. Usually, the universe is considered to be basically homogeneous and isotropic, with a gravitational field described by the FLRW metric (see Eq.(1.12)). Assuming that all departures from that homogeneity and isotropy in most of the history of the universe are small, then these deviations can be treated as first-order perturbations. The total perturbed metric is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.1}$$

where $\bar{g}_{\mu\nu}$ is the unperturbed $K = 0$ (flat) FLRW metric

$$\bar{g}_{00} = -1, \quad \bar{g}_{i0} = \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = a^2\delta_{ij}, \tag{3.2}$$

and $h_{\mu\nu} = h_{\nu\mu}$ is a small perturbation. The metric perturbation produces a perturbation in the affine connection and, consequently, in all the rest of quantities reviewed in chapter 1. After doing this and remain at first order in the Einstein field equations, one obtains a quite complicated expression. Nevertheless, the spatial isotropy and homogeneity of the unperturbed metric and energy momentum tensor allow to simplify the result by decomposing the perturbations into scalars, vectors and tensors, which are not coupled to each other by the field equations or conservation equations. So we have [82]

$$h_{00} = -E, \tag{3.3}$$

$$h_{i0} = a\Big(\partial_i F + G_i\Big), \tag{3.4}$$

$$h_{ij} = a^2\Big(A\delta_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j + D_{ij}\Big), \tag{3.5}$$

where the perturbations A, B, C_i, D_{ij}, E, F and G_i are functions of the coordinates, satisfying the conditions

$$\partial_i C^i = \partial_i G^i = 0, \tag{3.6}$$

$$\partial_i D^{ij} = 0, \quad D_{ii} = 0. \tag{3.7}$$

Given that the scalar, vector and tensor perturbations are decoupled, one can analyze each one separately in the perturbed Einstein field equations. We want to study the tensor perturbations since from them we can extract information about the GWs. Hence, we arrive to a wave equation called the gravitational wave propagation equation (GWPE), that can be written in terms of h because D_{ij} is directly related with h_{ij}. In vacuum, the GWPE takes the form

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\mathring{\nabla}^2}{a^2} h_{ij} = 0, \tag{3.8}$$

where h_{ij} is the traceless divergenless symmetric [82] perturbation tensor, $\mathring{\nabla}^2$ is the Laplacian operator and a is the scale factor. The dots mean derivative with respect to the physical time. By means of (3.7) we can see from (3.5), when considering tensor perturbations only, that [83]

$$\mathring{\nabla}_i h^{ij} = 0 = g_{ij} h^{ij},$$

which is true in GR but also in TG, as we will see later.
The perturbation tensor is a matrix whose components give information about the polarization of the GW. The unit linear polarization tensors for GWs that travel in the z direction are

$$\mathbf{e}_+ = \mathbf{e}_x \otimes \mathbf{e}_x - \mathbf{e}_y \otimes \mathbf{e}_y \tag{3.9}$$

and

$$\mathbf{e}_\times = \mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x. \tag{3.10}$$

The vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ indicate oscillation in the x-direction and y-direction, respectively, relative to an inertial frame[1]. In addition, the unit circular polarization tensors for GWs are

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_+ + i\mathbf{e}_\times) \tag{3.11}$$

and

$$\mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_+ - i\mathbf{e}_\times), \tag{3.12}$$

where i refers to the imaginary unit. The metric perturbation acquires the following form [11]:

$$h_{jk} = Re\Big[A_0 e^{-i\omega(t-z)} e_{Pjk}\Big], \tag{3.13}$$

in which e_{Pjk} is the polarization tensor. Figures 3.2 and 3.3, taken from [85], show the behavior, in different cases, of a ring of freely-falling particles when it is perturbed by a GW.

The second term of (3.8) is dissipative, so this is a damped wave equation. The factor multiplying $\dot{h}_{ij}$ is called 'coefficient of friction', but in this context this coefficient is variable as the Hubble function depends on the redshift. It is desirable to plot a component of h_{ij} (let's say, h for simplicity) against the scale factor (or the redshift), so one is able to see the evolution of the GW across the history of the universe. Whether one is interested in primordial GWs it is important to pay attention to the lowest range of values of a. TG may provide deviations of h from GR in this range. Tensorial perturbations in TG are studied in the following sections, considering $f(T)$ and $f(T,B)$ models.

[1]General covariance suggests that the free fall trajectories ought to be identified as the inertial trajectories and, hence, the geodesics of space-time. But whether that is so, then space-time is curved. There are inertial trajectories and we can define local inertial frames in curved space-times, but there are no extended inertial systems [84].

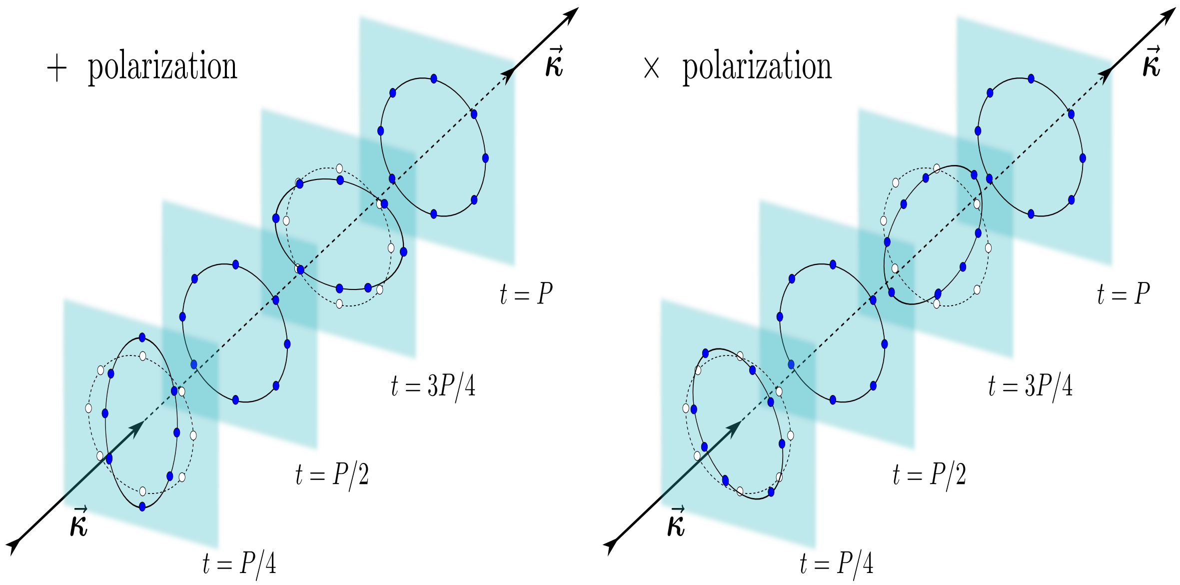

Figure 3.2: These figures, taken from [85], give the schematic deformations produced on a ring of freely-falling particles by GWs that are linear polarized in the + ("plus") and × ("cross") modes. The continuous lines and the dark filled dots show the positions of the particles at different times, while the dashed lines and the open dots show the unperturbed positions.

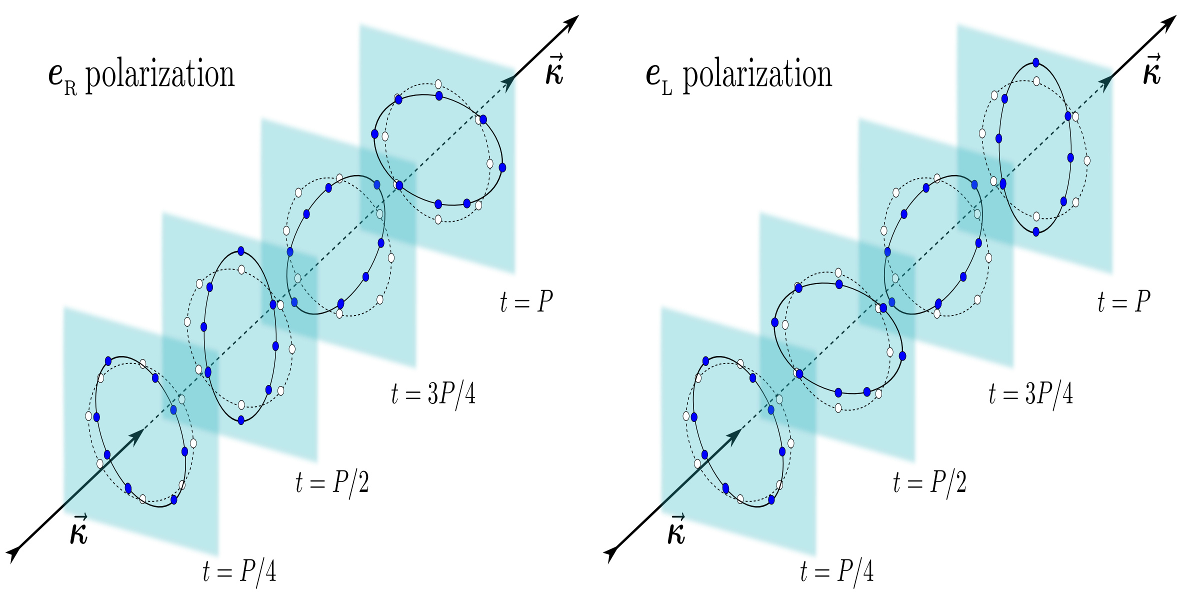

Figure 3.3: These figures, taken from [85], give the schematic deformations produced on a ring of freely-falling particles by GWs that are circularly polarized in the R (clockwise) and L (counter-clockwise) modes. The continuous lines and the dark filled dots show the positions of the particles at different times, while the dashed lines and the open dots show the unperturbed positions.

3.2 Gravitational waves in $f(T)$ gravity

The calculations presented in this section can be found in [8, 86]; we follow these references and discuss some important details.
In order to find a gravitational wave equation we need to make a perturbation in the main quantities of Eq.(2.21). The tetrad can be decomposed as

$$e^a{}_\mu(x) = \bar{e}^a{}_\mu(x) + \xi^a{}_\mu(x),$$

where $\bar{e}^a{}_\mu(x)$ represents the part of the tetrad corresponding to metric components, while $\xi^a{}_\mu(x)$ involves the degrees of freedom released from the local Lorentz transformation (whose number is thus six) [86]. Given that we are interested in gravitational waves we only need to focus on the components of the tetrad corresponding to the components of the metric. Perturbing the tetrad fields $\bar{e}^a{}_\mu(x)$ around a flat FLRW background we obtain:

$$\bar{e}^0{}_\mu = \delta^0_\mu(1+\psi) + a\delta^i_\mu(G_i + \partial_i F),$$

$$\bar{e}^a{}_\mu = a\left[\delta^a_\mu(1-\phi) + \frac{1}{2}\delta^i_\mu\delta^{aj}\Big(h_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j\Big) + \delta^a_0\delta^j_\mu\partial_j\bar{F}\right],$$

$$\bar{e}^\mu{}_0 = \delta^\mu_0(1-\psi) - \frac{1}{a}\delta^{\mu i}(G_i + \partial_i F),$$

$$\bar{e}^\mu{}_a = \frac{1}{a}\left[\delta^\mu_a(1+\phi) - \frac{1}{2}\delta^i_a\delta^{\mu j}\Big(h_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j\Big) - \delta^\mu_0\delta^j_a\partial_j\bar{F}\right],$$

where the Latin indexes from the beginning of the alphabet span the spatial part of the tangent space (the rest of indexes span all coordinates), $a(t)$ is the scale factor, ϕ, ψ, B and F are the scalar modes, C_i and G_i the vector modes, and h_{ij} the tensor mode. Again, $e^a{}_\mu(x)$ is the general tetrad that can be decomposed in two parts, one of them is $\bar{e}^a{}_\mu(x)$ which is the part of the tetrad related with the metric. Given the perturbation over this last tetrad, the standard perturbed metric results to be

$$g_{00} = -1 - 2\psi,$$

$$g_{i0} = -a[G_i + \partial_i F],$$

$$g_{ij} = a^2\left[(1-2\phi)\delta_{ij} + h_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j + \frac{1}{4}h_{ik}h_{j\bar{k}}\delta^{k\bar{k}}\right].$$

Observe that the metric tensor and the tetrad field share the same scalar, vector and tensor perturbation modes, which is not strange because they relate with each other through (2.7). Hence the properties of this modes will remain the same in TG; the vector modes will be transverse and, given that the tensor modes in GR are transverse and traceless, it follows that in TG we will also have [86]

$$\mathring{\nabla}_i h^{ij} = 0 = g_{ij} h^{ij}.$$

The scalar perturbations, related with matter perturbations, and the vector perturbations, which fade away faster, exist under metric perturbations (see appendix C). However, since we are interested in gravitational waves we will focus only and specifically on tensor perturbations. So the perturbed tetrad and metric read as follows

$$\bar{e}^0{}_\mu = \delta^0_\mu, \qquad \bar{e}^a{}_\mu = a\Big[\delta^a_\mu + \frac{1}{2}\delta^i_\mu \delta^{aj} h_{ij}\Big],$$

$$\bar{e}^\mu{}_0 = \delta^\mu_0, \qquad \bar{e}^\mu{}_a = \frac{1}{a}\Big[\delta^\mu_a - \frac{1}{2}\delta^{\mu i}\delta^j_a h_{ij}\Big],$$

$$g_{00} = -1, \qquad g_{i0} = 0, \qquad g_{ij} = \frac{a^2}{4}\Big[4\delta_{ij} + 4h_{ij} + h_{ik}h_{j\bar{k}}\delta^{k\bar{k}}\Big].$$

Using this, one finds the following quantities as the perturbed tetrads are substituted into the respective formulas:

$$T_{i0j} = H\delta_{ij} + \frac{1}{2}\dot{h}_{ij}, \qquad T_{ijk} = \frac{1}{2}(\partial_j h_{ik} - \partial_k h_{ij}), \qquad T = 6H^2 + T^{(2)}, \tag{3.14}$$

$$S_{i0j} = H\delta_{ij} - \frac{1}{4}\dot{h}_{ij}, \qquad S_{ijk} = \frac{1}{4a^2}(\partial_j h_{ik} - \partial_k h_{ij}), \tag{3.15}$$

where H is the Hubble function, $T^{(2)}$ is the second-order part of the torsion scalar expansion and $T^{(1)}$ is the first-order part, but notice that $T^{(1)}$ vanishes identically.
The tetrad determinant turns out to be

$$e = a^3\Bigg[1 + \frac{1}{4}\Big(h^2_{xy} + h^2_{xz} + h^2_{yz} - h_{xx}h_{yy} - h_{xx}h_{zz} - h_{yy}h_{zz}\Big)\Bigg].$$

Now, one introduces this all perturbed quantities in Eq.(2.21) and neglect second and higher order terms in the perturbation. After straightforward calculations one can find the GWPE in vacuum [86];

$$\ddot{h}_{ij} + 3H\Big(1 + \Upsilon\Big)\dot{h}_{ij} - \frac{\mathring{\nabla}^2}{a^2}h_{ij} = 0, \tag{3.16}$$

which is considered the general expression before analyzing a particular case. Here,

$$\Upsilon = \frac{\dot{f}_T}{3Hf_T}. \tag{3.17}$$

Compare (3.16) with (3.8); the coefficient of friction has an extra contribution due to TG. Taking into account $f = -T$ we recover the GWPE of GR.
From Eq.(3.17) one can see that $\Upsilon \to \infty$ when $f_T \to 0$, which means that we need to be careful with the models chosen in order to avoid situations of divergencies. Another thing to be said is that one should consider models such that

$$1+\Upsilon > 0. \tag{3.18}$$

This is because when the coefficient of friction is positive, the wave is damped and, consequently, the amplitude decays. Whether (3.18) is not accomplished in the sense that $1+\Upsilon < 0$, the amplitude grows, which physically speaking has no sense as we expect the GW to decay in time. And, in addition, the case $1+\Upsilon = 0$ does not represent a damped wave. Hence, this restriction is really important, and must be considered always since it eliminates several model possibilities and reduce the values of the parameters considered in each model. It is worth saying that we are avoiding the scalar and vector perturbations, but they exist; the thing is that we are only focusing on tensor perturbations because of the purposes of this work.

Notice the presence of the scale factor in the GWPE. It is true that a has an specific form in TG; the scale factor will depend on the form of f which in this case depends on torsion only (in section 3.3 the boundary term will be included), so we would have to define or specify the Lagrangian f. However, what is usually done is to consider an arbitrary a and plot the perturbation against the scale factor, as it will be shown in Figure 3.4. We use, in section 3.3.1, specific expressions of a because we decided to plot the perturbation against the cosmic time. Nevertheless, as we comment in that section, these plots denote some "toy models" since consider specific era domination. In order to study the GWPE we can evolve the history of the universe using the Cosmic Linear Anisotropy Solving System (CLASS) software [87, 88], which is an excellent tool for cosmologists who want to analyze their theoretical models. One possibility is to work in the code with the effective state parameter; for example, inserting Eq.(2.30), (2.31) and (2.32) in Eq.(2.29) we get

$$\omega_{eff} = -1 + (1+\omega)\frac{H^{2(b-1)} - A}{H^{2(b-1)} - (A/b)}, \tag{3.19}$$

where

$$A = \frac{H_0^{2(b-1)}}{1-\Omega_{m,0}-\Omega_{r,0}}, \tag{3.20}$$

and

$$H = \frac{a'}{a^2}. \tag{3.21}$$

We could proceed to manipulate (3.19) on CLASS and treat TG as an effective fluid (let's say, indirectly), or we can use another method such as changing the correct equations on CLASS and treat TG as a modified theory (let's say, directly). In any case, the result should be the same. There is already a study of GWs through CLASS; in Figure 3.4, taken from [8], can be seen some graphs that show the behavior of the GWs in the $f(T)$ Power Law Model in comparison with ΛCDM. Clearly, when $b \to 0$ the curves tend to the ΛCDM behaviour. Deviations from GR can be seen when $b \to 1$, and apparently these GWs dissipate energy faster. This could serve to establish restrictions on binary pulsars [89], for example. When the universe enters into the accelerated era, only for the cases $b = 10^{-6}; 10^{-7}$ the tensor modes oscillations are non-null. Nonetheless, in the case $b = 10^{-6}$ the modes rapidly decay and only in the case $b = 10^{-7}$ they survive up to present time. This tell us that in this model the 'effective teleparallel fluid', characterized by (3.19), is a powerful dissipative medium. This is because for larger values of b the tensorial modes enter in the cosmological horizon earlier [8], in comparison with the prediction of ΛCDM cosmology, and hence the GW amplitude goes quickly to zero already in radiation and matter epochs; we can see it from the math since

$$\lim_{b\to 1} 3H\Big(1+\Upsilon\Big) \to \infty,$$

so for b values close to unity the damping term in (3.16) becomes huge and the GW decays faster with the running of time. With regard to the phase, it is not observed any difference to GR because there is no phase modification in $f(T)$ gravity.

We tried to reproduce these plots, but for several reasons (different kind of problems in the code) we have not reached that goal. However, we found while working on CLASS, specifically on the Friedmann equation, that a clearer and better behavior of the Hubble parameter requires $-1 \leq b \leq 1$. When the previous relation breaks, a big amount of roots for the Friedmann equation appears, making the Hubble function more complex. Whether $-1 \leq b \leq 1$ is satisfied, only one real root for the Hubble function emerge. Consequently, the Power Law Model is physically viable when $0 < b \leq 1$, which means that the contribution of the 'extra' term in the $f(T)$ function, that is mT^b, should not contribute too much, and of course this helps the GW to keep its energy through time. If one is interested in primordial GWs, the last point is of vital importance since it is expected to measure these particular GWs even though they arrive to us really weak. If the extra term grows, it will be the case that at present time we would not, in theory, be able to measure anything since, as it was mentioned before, the effective teleparallel fluid is significantly dissipative.

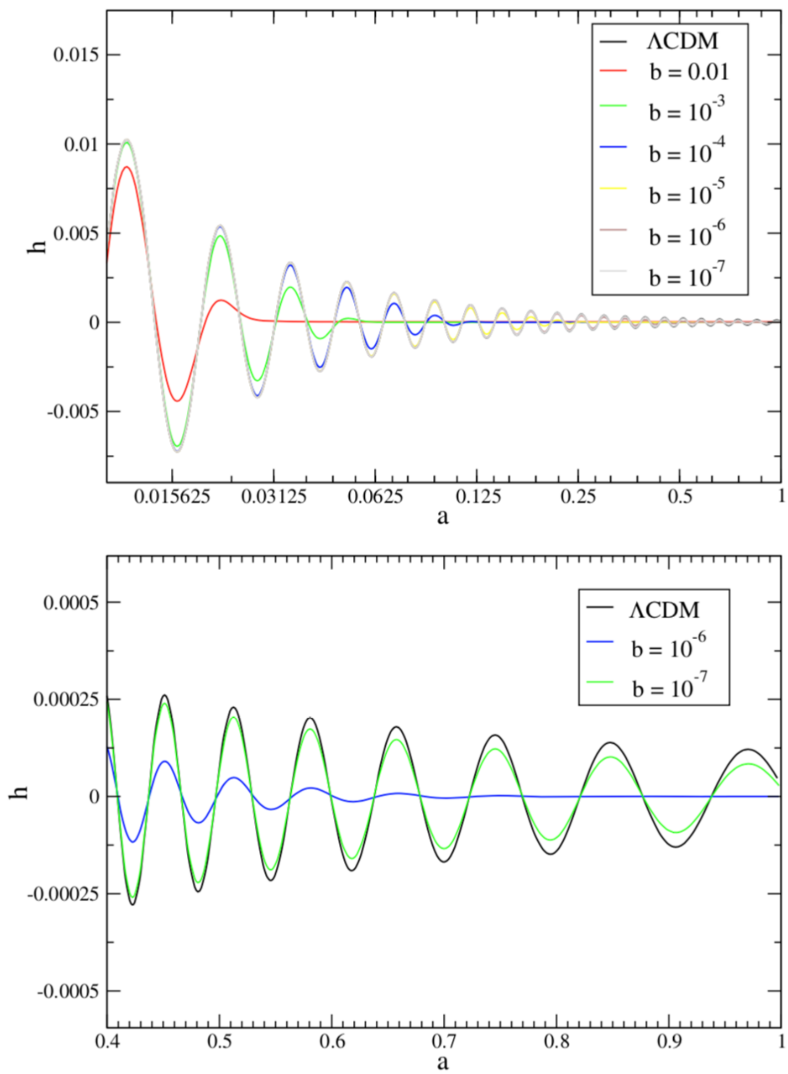

Figure 3.4: These plots were the results of [8] for the Power Law Model we have been discussing. With a wave number $k = 0.01\ Mpc^{-1}$, the upper plot depicts the propagation of the GWs for the ΛCDM model and for the $f(T)$ Power Law Model with some values of the free parameter b over the full cosmological history. The lower plot just shows some of these results in the late-time cosmological history.

3.3 Gravitational waves in $f(T, B)$ gravity

The computation presented in this section was done by us, following the steps of the previous section.
We will consider perturbations on a spatially flat cosmological background

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

which can be straightforwardly described by the tetrad choice

$$e^a{}_\mu = \mathrm{diag}(1, a(t), a(t), a(t)).$$

An interesting feature of this tetrad choice is that it allows vanishing spin connection components and so acts as a good tetrad. Linear tensor perturbations of the cosmological background metric are represented by

$$g_{\mu\nu} = -\delta^\mu_0\delta^\nu_0 + a^2\delta^\mu_i\delta^\nu_j\left(1 + h_{ij}\right)\,,$$

at the level of the metric tensor, and where indices i, j are both spatial. In our setting, these perturbations are transverse, traceless and symmetric, which eliminates superfluous degrees of freedom from the perturbation equations. Following the same procedure as in the background scenario, we can choose components for the linear tetrad perturbation that reproduce the metric perturbations while also having vanishing spin connection components. This is readily achieved for the choice

$$e^a{}_\mu = \delta^0_\mu\delta^a_{(0)} + \frac{1}{2}a\delta^i_\mu\delta^{a(j)}h_{ij}, \tag{3.22}$$

where parenthesis denotes tangent space indices. It is through this perturbation strategy that the GWPE for $f(T, B)$ gravity is obtained in this work.
Regularly, in $f(T, B)$ gravity one can write

$$h_{\mu\nu} = \eta_{\mu a}E^a{}_\nu + \eta_{\nu a}E^a{}_\mu, \tag{3.23}$$

where $|E^a{}_\mu| << 1$. Now, the first order perturbative tetrad $E^a{}_\mu$ is not symmetric due to the $f(T, B)$ gravity is not invariant under a local Lorentz transformation. We can write this perturbation tetrad into symmetric and antisymmetric parts

$$E_{\mu\nu} = E_{(\mu\nu)} + E_{[\mu\nu]},$$

but in fact the antisymmetric part has no physical meaning because it is not involved into the Lagrangian and in the field equations. In consequence we can set to zero the antisymmetric component and Eq.(3.23) is rewritten as

$$h_{\mu\nu} = 2\eta_{\mu a}E^a{}_\nu. \tag{3.24}$$

Hence, it is a matter of convention to use the tetrad or the metric perturbation. Using these tensor perturbations of the flat cosmological tetrad, we can deduce the perturbed TG scalars, which are given by [36]

$$T = 6H^2\,, \tag{3.25}$$

$$B = 6\left(3H^2 + \dot{H}\right)\,, \tag{3.26}$$

and where the tetrad determinant turns out to be

$$e = a^3\left[1 + \frac{1}{4}\Big(h^2_{xy} + h^2_{xz} + h^2_{yz} - h_{xx}h_{yy} - h_{xx}h_{zz} - h_{yy}h_{zz}\Big)\right].$$

Inserting the perturbed quantities in the field equations, including those of (3.14) and (3.15), we obtain the GWPE;

$$\left[\delta^{\mu}_{a}-\frac{1}{2}\delta^{i}_{a}\delta^{\mu j}h_{ij}\right]\ddot{f}_{B}-\left[\delta^{0}_{a}-\frac{1}{2}\delta^{i}_{a}\delta^{0j}h_{ij}\right]\ddot{f}_{B}+\frac{Bf_{B}}{2}\left[\delta^{\mu}_{a}-\frac{1}{2}\delta^{i}_{a}\delta^{\mu j}h_{ij}\right]+\frac{1}{2a}\left(\dot{h}_{a\mu}-\partial_{\mu}h_{a0}\right)\left(\dot{f}_{B}+\dot{f}_{T}\right)$$
$$-\frac{f_{T}H}{a}\left(\dot{h}_{a\mu}-\partial_{\mu}h_{a0}\right)+\frac{f_{T}}{2a}\left(\partial^{\nu}\partial_{\nu}h_{a\mu}-\partial^{\nu}\partial_{\mu}h_{a\nu}\right)$$
$$+\frac{3f_{T}H}{2a}\left(\partial_{\nu}h_{a\mu}-\partial_{\mu}h_{a\nu}\right)-\frac{f_{T}}{4a}A_{a\mu}-\frac{f}{2}\left[\delta^{\mu}_{a}-\frac{1}{2}\delta^{i}_{a}\delta^{\mu j}h_{ij}\right]=0, \tag{3.27}$$

where we have neglected the matter sector and considered that f depends on the time coordinate only. Here,

$$A_{a\mu}=\left(\partial_{\nu}h_{\alpha a}-\partial_{a}h_{\alpha\nu}\right)\left(\partial_{\mu}h_{\alpha\nu}-\partial_{\nu}h_{\alpha\mu}\right),$$

which we will neglect since the contribution of this term is of second order in the perturbation.
After straightforward calculations, our resulting wave equation is

$$\ddot{h}_{ij}+3H\left(1+\beta\right)\dot{h}_{ij}-\frac{\mathring{\nabla}^{2}}{a^{2}}h_{ij}=0, \tag{3.28}$$

where

$$\beta=\frac{2}{Hf_{T}}\left[f_{TB}\left(6H\dot{H}+\ddot{H}\right)+2f_{TT}H\dot{H}\right], \tag{3.29}$$

which agrees with both its $f(T)$ gravity limit and its GR (or TEGR) limit, as one would expect. The GWPE (3.28) is our principal result. About the emission of these GWs, see appendix D.
From Eq.(3.29) we see that torsion needs to be present, that f_0 cancels out at this level and that, to be able to see all the contributions, T and B must be coupled. For this reason, only mixed models should be considered. Similarly to Eq.(3.18) it is said that

$$1+\beta>0. \tag{3.30}$$

Following the restriction (3.30), it is convenient to consider $m>0$ and preferably $n\geq 0$.
The most general source-free GWPE on a flat cosmological background in modified gravity has the generic form [90]

$$\ddot{h}_{ij}+\left(3+\alpha_{M}\right)H\dot{h}_{ij}-\left(1+\alpha_{T}\right)\frac{k^{2}}{a^{2}}h_{ij}=0, \tag{3.31}$$

where α_M is 'Planck mass running rate', and $\alpha_T=c_T^2-1$ is the tensor excess speed. On comparison with (3.28), we can see that, in $f(T,B)$ gravity

- $\alpha_M = 3\beta$, which means that the Planck mass running rate is determined by the torsion scalar and the boundary term.

- $\alpha_T = 0$, which implies, as in the $f(T)$ gravity case, that the GWs travel at the speed of light (the speed is not modified by B). Consequently, $f(T,B)$ gravity exhibits a frictional term which would produce a modification of GW amplitudes but no mass terms are produced meaning that the propagation speed remains the speed of light in agreement with current observations.

Notice also that (3.31) is using the Fourier space since $\mathring{\nabla}^2 \to k^2$.

For the general GWPE that appears in Eq.(3.31), the meaning of the two modifying parameters alter the phase and amplitude of the propagation of a GW through

$$h_{\mathrm{GW}} \sim h_{\mathrm{GR}} \underbrace{e^{-\frac{1}{2}\int \alpha_M \mathcal{H} d\eta}}_{\text{Amplitude}} \underbrace{e^{ik\int \sqrt{\alpha_T + \frac{a^2 m_g^2}{k^2}} d\eta}}_{\text{Phase}},$$

where m_g would be the mass of the graviton, and which is represented in conformal time. In the case of $f(T,B)$ gravity, it turns out that the graviton has no mass resulting in

$$h_{\mathrm{GW}} \sim h_{\mathrm{GR}} e^{-\frac{3}{2}\int \beta \mathcal{H} d\eta},$$

in which is clear that the dampening factor affects the amplitude only. The impact of these alterations to the propagation of GWs also permeates into the GW luminosity distance which in this setting turns out to be related to the electromagnetic luminosity distance through [91]

$$\frac{d_L^g(z)}{d_L^{EM}(z)} = \exp\left[\frac{3}{2}\int_0^z \frac{\beta}{1+z'}\, dz'\right],$$

which will play a crucial role in using GWs as standard sirens in future GW detectors (see [92] for further discussion on this topic).
The approach that we noticed should be used in CLASS code is to add the effective state parameter for each TG model and run the code, so one will assume a different kind of fluid that will affect the dynamics of the GWs. Another approach could be to plot directly the GWPE found, however, solving the equation is quite difficult and complicated, speaking in computational terms. That is the reason the first approach is reasonably more convenient, and that is why we proceeded to work on an expression for the effective state parameter; inserting Eqs.(2.43), (2.44), (2.45), (2.46), (2.47) and (2.48) in Eq.(2.42), we obtain

$$\omega_{eff} = -1 + \Bigg[12mn(m-1)\left(\frac{a''a'}{a^5} - 2\frac{a'^3}{a^6}\right)(6)^{n+m-3}\left(\frac{a'^2}{a^4} + \frac{a''}{a^3}\right)^{n-1}\left(\frac{a'^2}{a^4}\right)^{m-2}$$

$$+6n(n-1)(m-1)\left(\frac{a'''}{a^4} - 6\frac{a'^3}{a^6}\right)(6)^{n+m-4}\left(\frac{a'^2}{a^4} + \frac{a''}{a^3}\right)^{n-2}\left(\frac{a'^2}{a^4}\right)^{m-2}$$

$$+72n\Big(\frac{a'''a''a'}{a^9}-2\frac{a'''a'^3}{a^{10}}-6\frac{a''a'^4}{a^{11}}+12\frac{a'^6}{a^{12}}\Big)(mn+n-m-1)(6)^{n+m-3}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n-2}\Big(\frac{a'^2}{a^4}\Big)^{m-1}$$
$$+6n(n-1)\Big(\frac{a''''}{a^5}+\frac{a'''a'}{a^6}-12\frac{a''a'^2}{a^7}+42\frac{a'^4}{a^8}\Big)(6)^{n+m-2}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n-2}\Big(\frac{a'^2}{a^4}\Big)^{m}$$
$$12mn\Big(\frac{a''^2}{a^6}-13\frac{a''a'^2}{a^7}+22\frac{a'^4}{a^8}\Big)(6)^{n+m-2}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n-1}\Big(\frac{a'^2}{a^4}\Big)^{m-1}$$
$$-2m(2m-1)\Big(\frac{a''}{a^3}-2\frac{a'^2}{a^4}\Big)(6)^{n+m-1}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n}\Big(\frac{a'^2}{a^4}\Big)^{m-1}\Bigg]\Bigg/$$
$$\Bigg[3n\Big(3\frac{a'}{a^2}+1\Big)\frac{a'}{a^2}(6)^{n+m-1}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n-1}\Big(\frac{a'^2}{a^4}\Big)^{m}+(m-\frac{1}{2})(6)^{n+m}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n}\Big(\frac{a'^2}{a^4}\Big)^{m}$$
$$-18n(n-1)\Big(\frac{a'''a'}{a^6}-6\frac{a'^4}{a^8}\Big)(6)^{n+m-2}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n-2}\Big(\frac{a'^2}{a^4}\Big)^{m}$$
$$-6mn\Big(\frac{a''}{a^3}-2\frac{a'^2}{a^4}\Big)(6)^{n+m-1}\Big(\frac{a'^2}{a^4}+\frac{a''}{a^3}\Big)^{n-1}\Big(\frac{a'^2}{a^4}\Big)^{m}\Bigg]. \tag{3.32}$$

which we should use in the code. Nevertheless, as I already mentioned before, we have had some problems in reaching the goal of plotting the GWs. We are still working on that.
Now, it was mentioned that $m > 0$ and $n \geq 0$, but also in consistency with the points discussed in the previous section, it is precise that $0 < m \leq 1$ and $0 \leq n \leq 1$ with one exception; whether $m = 1/2$ and $n = 0$ the effective state parameter (3.32) becomes undetermined. All the rest of possibilities are viable.

3.3.1 Specific cases

Dealing with Eq.(3.28) directly on Mathematica software and assuming a Power Law Model on $f(T, B)$ gravity, that is,

$$f(T,B) = b_0B^k + t_0T^m, \tag{3.33}$$

in which b_0, k, t_0 and m are free parameters, we could plot some cases of perturbations $E(t)$, where t is the cosmic time in gigayears ($1\ Gyr = 10^9\ yrs$) . The quantity $E(t)$, which is the tetrad perturbation, is related to $h(t)$, the metric perturbation, through the equation (3.24); so it is a matter of convention to use one or another quantity. In this case, we have used $E(t)$.
The GW, modeled through Eq.(3.28) using (3.33), is embedded in radiation, matter and dark energy, in an independent way, that is: We plot a GW with (3.28) using a set of parameters in the Power Law Model on $f(T, B)$ gravity considering an specific form of the scale factor; first we use dark energy, then matter and later radiation. Finally, we joined the three curves together in one figure. We repeated the same procedure for different values of the free parameters. The plots are shown in figures 3.5, 3.6 and 3.7.

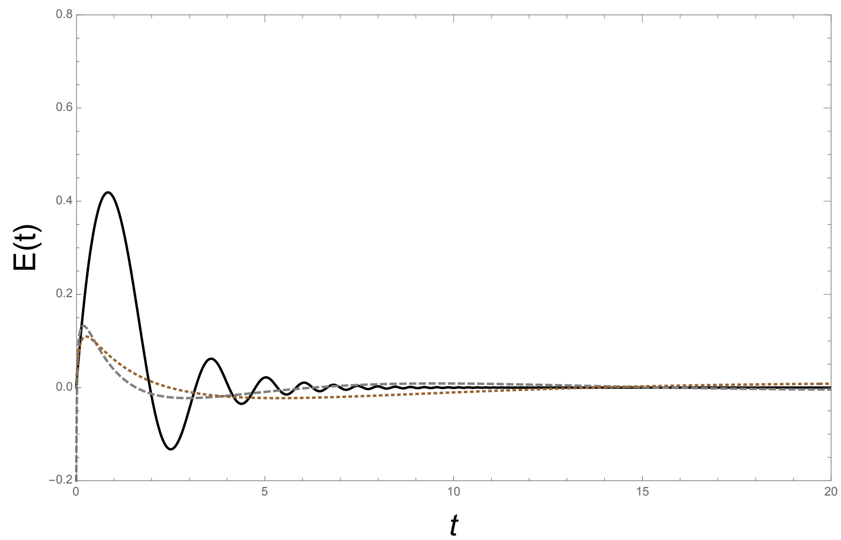

Figure 3.5: $E(t)$, which is the tetrad perturbation (3.24), is plotted against the cosmic time. For this plot we have $b_0 = 0$, $k = 1$, $t_0 = 7$ and $m = 0.5$. The GW embedded in dark energy, matter and radiation is represented by the black, grey and brown curve, respectively. In this case, t_0 is in L^{-1} units, where L is length (according to our conventions, it might be in Planck lengths).

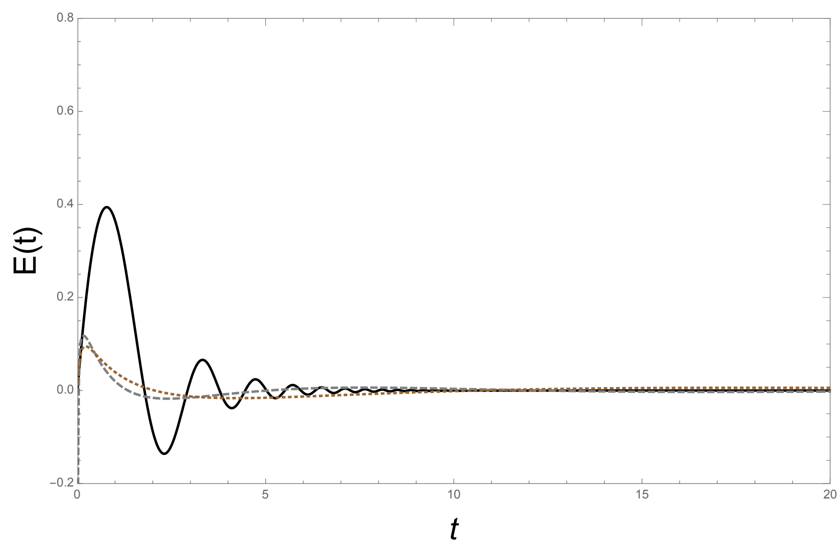

Figure 3.6: $E(t)$, which is the tetrad perturbation (3.24), is plotted against the cosmic time. For this plot we have $b_0 = 0$, $k = 1$, $t_0 = 13$ and $m = 0.4$. The GW embedded in dark energy, matter and radiation is represented by the black, grey and brown curve, respectively. In this case, t_0 is in $L^{-6/5}$ units, where L is length (according to our conventions, it might be in Planck lengths).

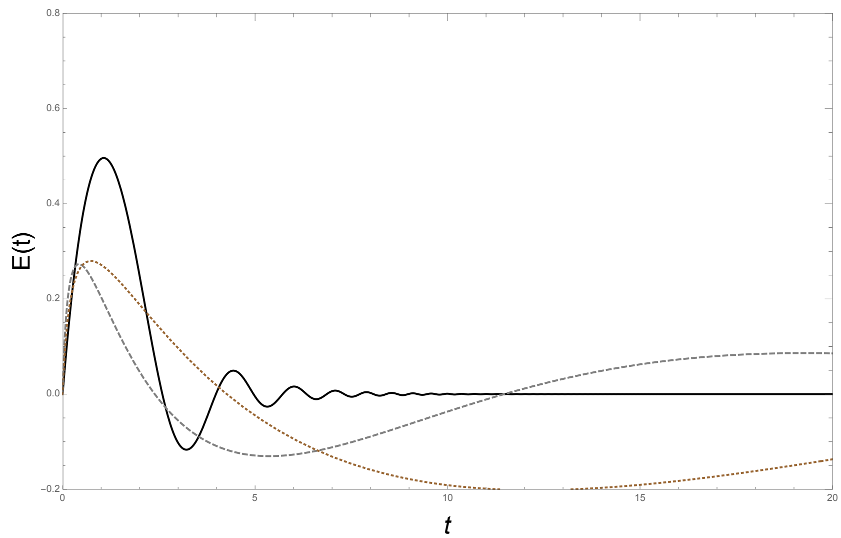

Figure 3.7: $E(t)$, which is the tetrad perturbation (3.24), is plotted against the cosmic time. For this plot we have $b_0 = k = t_0 = m = 1$. The GW embedded in dark energy, matter and radiation is represented by the black, grey and brown curve, respectively. In this case, b_0 and t_0 are dimensionless.

According to our computation, $t \approx 14\ Gyrs$ would be the present time. Now, in all of the figures, we can see that the GW decays faster in dark energy rather than in other substances; that could explain why the amplitude of GWs at present time (dark energy epoch) is almost null. About Figure 3.7, we can see that the amplitude of the GWs embedded in matter and radiation at present time is still huge, which may suggest this is not a good model (we have discussed that, we cannot see all the contribution of terms contained in the damping term when T and B are decoupled, as they are in this case; for that reason, a better behaved GW might be seen in $f(T, B)$ models where both scalars are coupled). With regard to Figure 3.5 and Figure 3.6, the behavior is much better since at present time the amplitude of the waves is small, but notice that in those cases we have a kind of Power Law $f(T)$ model since the boundary term is not present.

Clearly, these plots are not considering different epochs in the history of the universe, but just a wave immerse in an specific medium. Consequently, they are not quite reliable, and, for that reason, we stopped working on this kind of plots. We believe we should work on CLASS (or a similar code) in order to obtain realistic results.

3.3.2 Perspectives

According to the issues discussion along this chapter, we consider the tensor perturbations for specific models, for example a universe dominated solely by dark energy. Up to this point, our equation (3.28) denotes a gravitational wave in a $f(T, B)$ theory. Once the form of this $f(T, B)$ is set, we can proceed with the cosmological solutions at perturbative level. For this, the next stage of this work will be working on CLASS and rewrite the background module, which is initially set in the GR formalism. But there still remain some issues we are trying to solve. The effective state parameter is written in terms of the Hubble parameter (both of them, the one for the Power Law Model and the one for the Mixed Power Law Model), but this needs to be found (as it is not defined in that part of the code yet) with a numerical root finding method using the 'teleparallel' Friedmann equations; we are using Ridders' method, together with some bracketing methods. After that, the effective state parameter needs to be integrated (we are using Simpson's 1/3 rule), but we are having divergences problems. The code seems to be working fine (the numerical part) and the equations are consistent (the theoretical part, which we discuss in this work), so we are actually dealing with this and trying to find where the problem is. The future perspective is to continue working and to solve this issue, we would like to be able to plot different cases for both models and compare our results with those from the references given in this thesis.

Chapter 4

Conclusions

The main goal of this work was to understand and study the gravitational waves derived from Teleparallel theories.

- In chapter 1 a review of GR was made. The principal reasons that motivated Einstein to change the current theory of gravity in that time, that is, Newton's conception of gravity, were studied. The main quantities in GR were also presented, such as the Levi-Civita connection, the Riemann tensor, the Ricci tensor and the Ricci scalar. In addition, the most important equations were reviewed, including the Einstein field equation and the Friedmann equations, obtained considering the FLRW metric. Finally, it was mentioned the different ways in which one can modify GR field equations, following the Lovelock's theorem.

- In chapter 2 we reviewed the main quantities and equations in TG, showing two possible extensions in that theory which involve the torsion scalar and the boundary term. The Friedmann equations and the cosmology behind $f(T)$ and $f(T, B)$ was presented. It can be concluded that TG offers a compelling background scenario, which can be reduced to GR in the TEGR limit. Nonetheless, the features of teleparallel models, in particular of $f(T, B)$ models, over Einstein's theory of gravity make TG a quite recommendable modified theory.

- In chapter 3 we studied the gravitational wave perturbation equations in GR and in $f(T)$ gravity. After that we followed the same procedure and calculated the GWPE for $f(T, B)$ gravity. We discussed some specific cases, assuming a Power Law Model for $f(T, B)$ gravity, and finally we gave the perspectives of this work.

According to the studied in chapter 3 with regard to the GWPEs, two things can be said about the model construction process:

1. In both $f(T)$ and $f(T, B)$ models, it is completely necessary to have terms involving torsion in order to avoid divergences. It seems that in these TG extensions the torsion cannot be neglected.

2. To be able to see the contribution of all terms in $f(T, B)$ models, T and B must be coupled. Simpler GWPEs can be reached by considering both scalars decoupled, or even a model that does not depend on the boundary term.

In relation to the models proposed, a couple of things should be mentioned:

1. The $f(T)$ Power Law Model is quite manageable due to it can be derived easily with respect to the torsion scalar. The case $b = 0$ is an special limit, because $f = -T + constant$ gives GR with cosmological constant. As it was mentioned, $b > 0$ because of the restriction (3.18), but also it was found while working on CLASS that a clearer and better behavior of the Hubble parameter in the Friedmann equation requires $-1 \leq b \leq 1$. Consequently, the Power Law Model is physically viable when $0 < b \leq 1$. This suggests that the 'extra' term, that is mT^b, should not contribute excessively but enough to see deviations from GR at high redshifts.

2. The $f(T, B)$ Mixed Power Law Model is more complex than a simple sum of independent terms of T and B. The necessity of the coupling between both scalars in order to see all the contributions in (3.29) reveal that the torsion and the boundary term are, in a sense, a single contribution to gravity; the gravitational interaction is caused by a unique quantity which is not other thing that the coupling of both scalars. Even if T and B are independent scalars and produce gravity, the boundary term is derived from torsion as it can be seen from (2.14), hence, it is not strange that the coupling of both scalars results to be a good $f(T, B)$ model. Now, the f_0 parameter is not of importance at perturbative level, although, at background level is quite important; f_0 cannot be so large as the matter content term should contribute significantly. Also, it was commented that $m > 0$ and $n > 0$. However, in consistency with point 1, the physically viable Mixed Power Law Model should consider $0 < m \leq 1$ and $0 \leq n \leq 1$, except for the case $m = 1/2$ and $n = 0$. A deeper investigation on the phenomenology of both scalars T and B could tell us more about the cases of torsion domination ($m > n$), and boundary term domination ($m < n$).

Finally, Weitzenböck's vision offers an interesting formalism and explanation about the nature of space-time. One of the most interesting results was the necessity of coupling the torsion scalar and the boundary term in order to see all the contributions of the terms in the GWPE; as we already mentioned, it is not a surprise since the boundary term is obtained directly from torsion, but this kind of details make us thing that, maybe, the best models in TG are those which consider certain kind of coupling between both scalars. $f(T)$ and $f(T, B)$ models have been well studied by most of people dedicated to TG. We would like to focus, in future works, on $f(B)$ models and try to figure out whether they give good results and if this coupling in $f(T, B)$ turns out to be more convenient.

We plan to continue modifying a Boltzmann code and try to find some restrictions over the Power Law and Mixed Power Law Models treated in this thesis. We are also interested in comparing the possible results with LIGO's data bases of GWs [93] and perhaps, in a future, the data produced by the simulations of LISA (Laser Interferometer Space Antenna) [94]. This study will be reported elsewhere.

Appendix A

A brief formalism of Teleparallel Gravity

A.1 The torsion tensor

The following development can be found in [33, 37]. A Lorentz connection or spin connection, which is given by

$$\omega_\mu = \frac{1}{2}\omega^{ab}{}_\mu S_{ab},$$

is a 1-form assuming values in the lie algebra of the Lorentz group. S_{ab} is a given representation of the Lorentz generators which is antisymmetric in ab, and $\omega^{ab}{}_\mu$ must be equally antisymmetric so it can be lorentzian.
The spin connection is of great relevance to understand TG. This connection defines the covariant derivative of Fock-Ivanenko;

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{2}\omega^{ab}{}_\mu S_{ab},$$

whose second part acts only on the algebraic indices (or tangent space indices). Lorentz generators take different values depending on the entity to which is applied the covariant derivative of Fock-Ivanenko. For example, for an scalar field ϕ the generators are:

$$S_{ab} = 0.$$

For a Lorentz vectorial field ϕ^c the generators adopt a form

$$(S_{ab})^c{}_d = i(\eta_{bd}\delta^c_a - \eta_{ad}\delta^c_b),$$

so

$$\mathcal{D}_\mu\phi^c = \partial_\mu\phi^c - \frac{i}{2}\omega^{ab}{}_\mu i(\eta_{bd}\delta^c_a - \eta_{ad}\delta^c_b)\phi^d = \partial_\mu\phi^c + \frac{1}{2}(\omega^a{}_{d\mu}\delta^c_a - \omega^b{}_{\mu d}\delta^c_b)\phi^d$$

$$= \partial_\mu \phi^c + \frac{1}{2}(\omega^c{}_{d\mu} - \omega^c{}_{\mu d})\phi^d.$$

Consequently,

$$\mathcal{D}_\mu \phi^c = \partial_\mu \phi^c + \omega^c{}_{d\mu}\phi^d, \tag{A.1}$$

where the square brackets in the $d\mu$ indices of the spin connection have been omitted. Tetrad fields relate tensors of the tangent (or internal) space tensors with spacetime (or external) tensors. Lets say that ϕ^a is an internal, or Lorentz vector, so

$$\phi^\rho = e_a{}^\rho \phi^a$$

will be a vector in spacetime. Conversely,

$$\phi^a = e^a{}_\rho \phi^\rho.$$

Under these operations, and due to the non-tensorial character, a connection is going to adquire a vacuum (or non-homogeneous) term. For example, for each spin connection $\omega^a{}_{b\mu}$ there is a corresponding general linear connection $\Gamma^\rho{}_{\nu\mu}$ given by

$$\Gamma^\rho{}_{\nu\mu} = e_a{}^\rho \partial_\mu e^a{}_\nu + e_a{}^\rho \omega^a{}_{b\mu} e^b{}_\nu = e_a{}^\rho \Big(\partial_\mu e^a{}_\nu + \omega^a{}_{b\mu} e^b{}_\nu\Big), \tag{A.2}$$

and following (A.1) we get

$$\Gamma^\rho{}_{\nu\mu} = e_a{}^\rho \mathcal{D}_\mu e^a{}_\nu.$$

From (A.2) we see that the inverse expression is

$$\omega^a{}_{b\mu} = e^a{}_\rho \partial_\mu e_b{}^\rho + e^a{}_\rho \Gamma^\rho{}_{\nu\mu} e_b{}^\nu \tag{A.3}$$

or

$$\omega^a{}_{b\mu} = e^a{}_\rho \nabla_\mu e_b{}^\rho. \tag{A.4}$$

In equation (A.4), ∇_μ is the standard covariant derivative in the connection $\Gamma^\rho{}_{\nu\mu}$, which only acts on external indices. And, as we know,

$$\nabla_\mu \phi^\nu = \partial_\mu \phi^\nu + \Gamma^\nu{}_{\rho\mu}\phi^\rho. \tag{A.5}$$

It can be proved that the relation between both covariant derivatives is

$$\mathcal{D}_\mu \phi^d = e^d{}_\rho \nabla_\mu \phi^\rho, \tag{A.6}$$

but is worth to mention that the Fock-Ivanenko covariant derivative can be defined for all fields (tensorial and spinorial), whereas the standard covariant derivative can only be defined for tensorial fields.

Rewriting equation (A.2) as

$$\Gamma^\rho{}_{\nu\mu} e^a{}_\rho = \partial_\mu e^a{}_\nu + \omega^a{}_{b\mu} e^b{}_\nu$$

we notice that

$$\partial_\mu e^a{}_\nu - \Gamma^\rho{}_{\nu\mu} e^a{}_\rho + \omega^a{}_{b\mu} e^b{}_\nu = 0,$$

which express an interesting property; that the total covariant derivative[1] of the tetrad is zero.
Now, when

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} = 0$$

it is said that the metricity condition is satisfied, which means that the connection $\Gamma^\rho{}_{\lambda\mu}$ is compatible with the metric. Rewriting equation (A.3) as

$$\omega^a{}_{b\mu} e_a{}^\nu = \partial_\mu e_b{}^\nu + \Gamma^\nu{}_{\rho\mu} e_b{}^\rho$$

it is possible to see that

$$\partial_\mu \eta_{ab} - \Gamma^\rho{}_{a\mu} \eta_{\rho b} - \Gamma^\rho{}_{b\mu} \eta_{a\rho} = -\omega^c{}_{a\mu} \eta_{cb} - \omega^c{}_{b\mu} \eta_{ac},$$

but the left hand side is zero since

$$\nabla_\mu \eta_{ab} = 0,$$

so

$$-\omega^c{}_{a\mu} \eta_{cb} - \omega^c{}_{b\mu} \eta_{ac} = -\omega_{ba\mu} - \omega_{ab\mu} = 0.$$

Consequently,

$$\omega_{ba\mu} = -\omega_{ab\mu}.$$

Therefore, the fact that the metricity condition is fulfilled (or, said in other words, that the metric is preserved) implies that the spin connection is lorentzian, that is, antysimmetric in the algebraic indices. On the other hand, when $\nabla_\lambda g_{\mu\nu} \neq 0$, the corresponding spin connection does not assume values in the Lie algebra of the Lorentz group (it is not a Lorentz connection).
While the curvature of a Lorentz connection $\omega^a{}_{b\mu}$ is a 2-form assuming values in the Lie algebra of the Lorentz group;

$$R = \frac{1}{4} R^a{}_{b\nu\mu} S_a{}^b dx^\nu \wedge dx^\mu,$$

the torsion is also a 2-form but assumes values in the Lie algebra of the translation group;

$$T = \frac{1}{2} T^a{}_{\nu\mu} P_a dx^\nu \wedge dx^\mu,$$

[1] The total covariant derivative is that which contains both connections; the general linear connection (for external indices) and the spin connection (for internal indices).

where $P_a = \partial_a$ are the translation generators. The components of curvature and torsion are defined, respectively, as

$$R^a{}_{b\nu\mu} = \partial_\nu \omega^a{}_{b\mu} - \partial_\mu \omega^a{}_{b\nu} + \omega^a{}_{e\nu}\omega^e{}_{b\mu} - \omega^a{}_{e\mu}\omega^e{}_{b\nu},$$

$$T^a{}_{\nu\mu} = \partial_\nu e^a{}_\mu - \partial_\mu e^a{}_\nu + \omega^a{}_{e\nu} e^e{}_\mu - \omega^a{}_{e\mu} e^e{}_\nu.$$

It is clear that through the contraction with tetrads, these tensors can be written in spacetime-indexed forms; curvature as follows

$$R^\rho{}_{\lambda\nu\mu} = e_a{}^\rho e^b{}_\lambda R^a{}_{b\nu\mu} = \partial_\nu e^\rho{}_\rho \partial_\mu e_\lambda{}^\rho - \partial_\mu e^\rho{}_\rho \partial_\nu e_\lambda{}^\rho + \partial_\nu \Gamma^\rho{}_{\lambda\mu} - \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \delta^a_a e^b{}_\lambda e^e{}_\rho \partial_\nu e_e{}^\rho \partial_\mu e_b{}^\rho$$
$$+\Gamma^\rho{}_{b\mu}\delta^a_a e^b{}_\lambda \partial_\nu e_e{}^\rho + e^b{}_\lambda e^e{}_\rho \Gamma^\rho{}_{e\nu}\partial_\mu e_b{}^\rho + \Gamma^\rho{}_{e\nu}\Gamma^e{}_{\lambda\mu} - \delta^a_a e^b{}_\lambda e^e{}_\rho \partial_\mu e_e{}^\rho \partial_\nu e_b{}^\rho$$
$$-\Gamma^e{}_{b\nu}\delta^a_a e^b{}_\lambda \partial_\mu e_e{}^\rho - \Gamma^\rho{}_{e\mu} e^b{}_\lambda e^e{}_\rho \partial_\nu e_b{}^\rho - \Gamma^\rho{}_{e\mu}\Gamma^e{}_{\lambda\nu},$$

which is reduced to

$$R^\rho{}_{\lambda\nu\mu} = \partial_\nu \Gamma^\rho{}_{\lambda\mu} - \partial_\mu \Gamma^\rho{}_{\lambda\nu} + \Gamma^\rho{}_{\eta\nu}\Gamma^\eta{}_{\lambda\mu} - \Gamma^\rho{}_{\eta\mu}\Gamma^\eta{}_{\lambda\nu}; \tag{A.7}$$

and torsion as

$$T^\rho{}_{\nu\mu} = e_a{}^\rho T^a{}_{\nu\mu} = e_a{}^\rho \partial_\nu e^a{}_\mu - e_a{}^\rho \partial_\mu e^a{}_\nu + e^\rho{}_\eta \partial_\nu e_\mu{}^\eta + e^\rho{}_\eta \Gamma^\eta{}_{\lambda\nu} e_\mu{}^\lambda - e^\rho{}_\eta \partial_\mu e_\nu{}^\eta - e^\rho{}_\eta \Gamma^\eta{}_{\lambda\mu} e_\nu{}^\lambda$$

that is reduced to

$$T^\rho{}_{\nu\mu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}, \tag{A.8}$$

which is basically equation (2.9).

Being TG a gauge theory it is characterized by a gauge potential with values in the Lie algebra of the translation group,

$$B_\mu = B^a{}_\mu P_a.$$

The 1-form potential is constructed with the generators of infinitesimal translations. The fundamental field strength in TG is due to the gauge potential,

$$F^a{}_{\mu\nu} = \partial_\mu B^a{}_\nu - \partial_\nu B^a{}_\mu. \tag{A.9}$$

A local translation of the tangent space coordinates

$$x'^a = x^a + \alpha^a \tag{A.10}$$

with $\alpha = \alpha(x^\mu)$, defines a gauge transformation. Under such transformation, the gauge potential behaves as

$$B'^a{}_\mu = B^a{}_\mu - \partial_\mu \alpha^a. \tag{A.11}$$

With the above relation we can see that

$$\begin{aligned} F'^{a}{}_{\mu\nu} &= \partial_\mu B'^{a}{}_{\nu} - \partial_\nu B'^{a}{}_{\mu} = \partial_\mu (B^{a}{}_{\nu} - \partial_\nu \alpha^a) - \partial_\nu (B^{a}{}_{\mu} - \partial_\mu \alpha^a) \\ &= \partial_\mu B^{a}{}_{\nu} - \partial_\nu B^{a}{}_{\mu} - \partial_\mu \partial_\nu \alpha^a + \partial_\nu \partial_\mu \alpha^a = \partial_\mu B^{a}{}_{\nu} - \partial_\nu B^{a}{}_{\mu} = F^{a}{}_{\mu\nu}, \end{aligned}$$

so the field strength is invariant.

The gauge potential appears as the non-trivial part of the tetrad field

$$e^{a}{}_{\mu} = \partial_\mu x^a + B^{a}{}_{\mu}. \tag{A.12}$$

Substituting (A.10) and (A.11) in (A.12) prime, we have

$$\begin{aligned} e'^{a}{}_{\mu} = \partial_\mu x'^a + B'^{a}{}_{\mu} = \partial_\mu (x^a + \alpha^a) + (B^{a}{}_{\mu} - \partial_\mu \alpha^a) &= \partial_\mu x^a + B^{a}{}_{\mu} \\ &= e^{a}{}_{\mu}, \end{aligned}$$

so the tetrad field is also invariant.

The non-vanishing torsion in Eq.(A.8) relates directly with the gauge field strength of Eq.(A.9) through the relation

$$T^{\rho}{}_{\mu\nu} = e_{a}{}^{\rho} F^{a}{}_{\mu\nu}.$$

This is the reason why in chapter 2 it was mentioned that the torsion tensor was a measure of the field strength of gravitation.

A.2 Is the weak equivalence principle necessary in TG?

The universality of free fall, a claim that comes directly from Galileo Galilei, asserts that: *All test bodies fall in a gravitational field with the same acceleration regardless of their mass or internal composition.* This is in agreement with Newtonian mechanics and is equivalent the requirement of equality between inertial and gravitational masses [95]. This claim is also known as the weak equivalence principle (see Figure A.1, taken from [47]), and in GR its validity is a necessity. We will see, following [37], that in TG this is not the case.

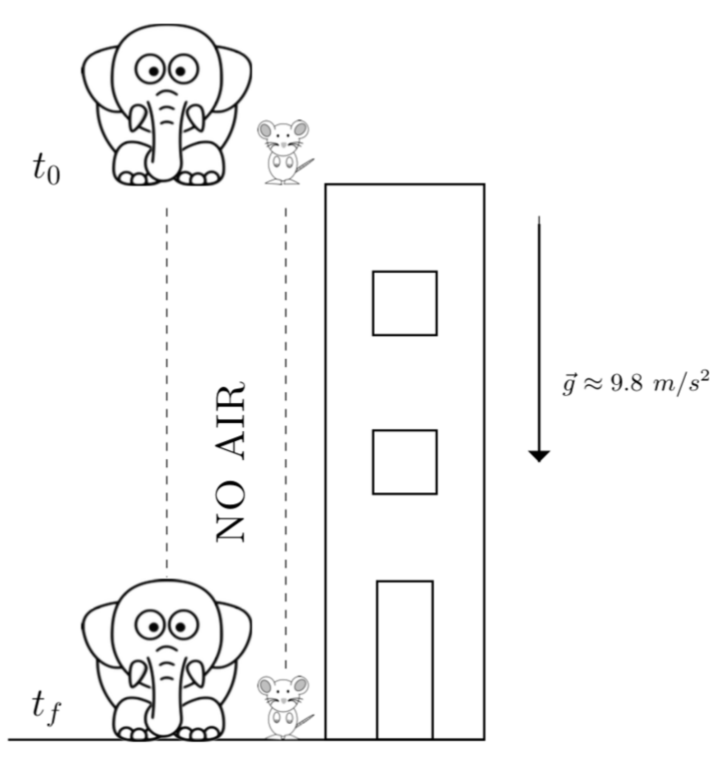

Figure A.1: It does not matter the weight of the test body, it will always fall with the same acceleration. Hence, two different bodies with different weights will reach the floor at the same time. This is the weak equivalence principle. Image taken from [47].

The action of a spinless particle in a gravitational field ${B^a}_\mu$ is, equivalently to electromagnetism,

$$S = \int_1^2 \Big[- m_i d\sigma - m_g {B^a}_\mu u_a dx^\mu \Big],$$

where u^a is the particle four-velocity seen from the tetrad frame and

$$d\sigma = \sqrt{\eta_{ab} dx^a dx^b}$$

is the Minkowski tangent-space invariant interval. In gauge theories is possible to represent, separately, the inertial and the gravitational parts of the particle. The first term in the action contains the inertial mass m_i and the second term refers to the coupling of such particle to the gravitational field through its gravitational mass m_g. This cannot be done in GR as this is not a gauge theory.
The action can be expressed as

$$S = \int_1^2 m_i \Big[- \frac{d\sigma}{ds} - \frac{m_g}{m_i} {B^a}_\mu u_a \frac{dx^\mu}{ds} \Big] ds,$$

where

$$ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$$

is the space-time invariant. The four-velocity results to be holonomic in terms of the tangent-space line element;

$$u^a = \frac{dx^a}{d\sigma},$$

and on the other side

$$u^\mu = \frac{dx^\mu}{ds} = e_a{}^\mu u^a.$$

In consequence,

$$S = \int_1^2 m_i \Big[-\frac{d\sigma}{ds} - \frac{m_g}{m_i} B^a{}_\rho u_a u^\rho \Big] ds.$$

Taking the variation of this action (with respect to the space-time coordinates) we obtain

$$\delta S = \int_1^2 m_i \Bigg[\Big(\partial_\mu x^a + \frac{m_g}{m_i} B^a{}_\mu \Big) \frac{du_a}{ds} - \frac{m_g}{m_i} \Big(\partial_\mu B^a{}_\rho - \partial_\rho B^a{}_\mu \Big) u_a u^\rho \Bigg] \delta x^\mu ds = 0$$

Using Eq.(A.9), the action takes the form

$$\int_1^2 m_i \Bigg[\Big(\partial_\mu x^a + \frac{m_g}{m_i} B^a{}_\mu \Big) \frac{du_a}{ds} - \frac{m_g}{m_i} F^a{}_{\mu\rho} u_a u^\rho \Bigg] \delta x^\mu ds = 0$$

Since δx^μ is arbitrary, the following relation must be true:

$$\Big(\partial_\mu x^a + \frac{m_g}{m_i} B^a{}_\mu \Big) \frac{du_a}{ds} = \frac{m_g}{m_i} F^a{}_{\mu\rho} u_a u^\rho.$$

This is the gravitational force equation. Evidently, the particular case $m_i = m_g$ means that the weak equivalence principle holds. However, at this point nothing with regard to the masses have been assumed, and, a priori, nothing can be said. What must be observed is that, even though the equation of motion depends explicitly on the ratio m_g/m_i, neither the gauge potential nor the field strength depends on this relation. Consequently, the teleparallel field equation and the equation of motion can be obtained with the gauge potential independently of the validity of the weak equivalence principle. It is concluded that the weak equivalence principle is not a necessity in TG.

Appendix B

Relation between scalars

The following can be found in [47]. The Riemann tensor is written in terms of the Weitzenböck connection as

$$R^{\mu}{}_{\alpha\gamma\lambda} = \partial_{\gamma}\Gamma^{\mu}{}_{\alpha\lambda} - \partial_{\lambda}\Gamma^{\mu}{}_{\alpha\gamma} + \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\beta}{}_{\alpha\lambda} - \Gamma^{\mu}{}_{\beta\lambda}\Gamma^{\beta}{}_{\alpha\gamma}, \tag{B.1}$$

as it was seen on Eq.(A.7). Nevertheless, the Weitzenböck connection can be expressed as

$$\Gamma^{\lambda}{}_{\mu\nu} = \mathring{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} \tag{B.2}$$

With this expression at hand, the Riemann tensor takes the form

$$R^{\mu}{}_{\alpha\gamma\lambda} = \mathring{R}^{\mu}{}_{\alpha\gamma\lambda} + \mathring{\nabla}_{\gamma}K_{\alpha}{}^{\mu}{}_{\lambda} - \mathring{\nabla}_{\lambda}K_{\alpha}{}^{\mu}{}_{\gamma} + K_{\beta}{}^{\mu}{}_{\gamma}K_{\alpha}{}^{\beta}{}_{\lambda} - K_{\beta}{}^{\mu}{}_{\lambda}K_{\alpha}{}^{\beta}{}_{\gamma}. \tag{B.3}$$

The Ricci tensor is, then,

$$R_{\alpha\lambda} = \mathring{R}_{\alpha\lambda} + \mathring{\nabla}_{\mu}K_{\alpha}{}^{\mu}{}_{\lambda} - \mathring{\nabla}_{\lambda}K_{\alpha}{}^{\mu}{}_{\mu} + K_{\beta}{}^{\mu}{}_{\mu}K_{\alpha}{}^{\beta}{}_{\lambda} - K_{\beta}{}^{\mu}{}_{\lambda}K_{\alpha}{}^{\beta}{}_{\mu}, \tag{B.4}$$

and the Ricci scalar results

$$R = \mathring{R} + \mathring{\nabla}_{\mu}K^{\lambda\mu}{}_{\lambda} - \mathring{\nabla}_{\lambda}K^{\lambda\mu}{}_{\mu} + K_{\beta}{}^{\mu}{}_{\mu}K^{\lambda\beta}{}_{\lambda} - K_{\beta}{}^{\mu}{}_{\lambda}K^{\lambda\beta}{}_{\mu}. \tag{B.5}$$

Given the contorsion tensor

$$K_{\mu}{}^{\lambda}{}_{\nu} = \frac{1}{2}\Big(T^{\lambda}{}_{\mu\nu} - T_{\nu\mu}{}^{\lambda} + T_{\mu}{}^{\lambda}{}_{\nu}\Big), \tag{B.6}$$

the Ricci scalar can be rewritten, after straightforward calculations, as

$$R = \mathring{R} - 2\mathring{\nabla}_{\mu}T_{\lambda}{}^{[\lambda\mu]} + T = \mathring{R} - \frac{2}{e}\partial_{\mu}(eT^{\mu}) + T, \tag{B.7}$$

which means that

$$\mathring{R} = -T + B + R, \tag{B.8}$$

where the definition of the boundary term was applied.

Appendix C

Scalar and vector perturbations in $f(T)$ and $f(T,B)$

The information presented here (taken from [96]) refers to the non-tensor perturbed quantities in TG, considering the signature $(+,-,-,-)$.

C.1 Vector and pseudovector perturbations

Usually, the vector modes are ignored since there are few known physical processes in which they can be generated, and also these kind of modes are diluted more quickly. The non-zero components of the vectorial and pseudo vectorial perturbations for the torsion tensor and the superpotential are

$$\begin{aligned}
\delta T^{0}{}_{0i} &= a\dot{\beta}_i\,, && \text{(C.1)}\\
\delta T_{i0j} &= 2\partial_i\dot{h}_j - \frac{1}{a}\partial_j b_i - \epsilon_{kij}\dot{\sigma}_k\,, && \text{(C.2)}\\
\delta T^{0}{}_{ij} &= a(\partial_i\beta_j - \partial_j\beta_i)\,, && \text{(C.3)}\\
\delta T_{ijk} &= 2(\partial_i\partial_j h_k - \partial_i\partial_k h_j) + \epsilon_{ijl}\partial_k\sigma_l - \epsilon_{ikl}\partial_j\sigma_l\,, && \text{(C.4)}\\
\delta S_{00i} &= -\frac{1}{2a^2}\Big[2aH(b_i-\beta_i) + \epsilon_{ilk}\partial_k\sigma_l\Big]\,, && \text{(C.5)}\\
\delta S_{i0j} &= -\frac{1}{2a}\left[\frac{1}{2}\Big(\partial_i(b_j+\beta_j-a\dot{h}_j) + \partial_j(b_i-\beta_i-a\dot{h}_i)\Big)\right]\,, && \text{(C.6)}\\
\delta S_{0ij} &= -\frac{1}{4a^3}\Big[\partial_i(b_j-\beta_j+2a\dot{h}_j) - \partial_j(b_i-\beta_i+2a\dot{h}_i) - 2a\epsilon_{lij}\dot{\sigma}_l\Big]\,, && \text{(C.7)}\\
\delta S_{ijk} &= -\frac{1}{2a^2}\Big[\delta_{im}\epsilon_{kjl}\partial_l\sigma_m + \delta_{ij}\Big(2aH(b_k-\beta_k) - a\dot{\beta}_k - 2\partial^2 h_k\Big)\\
&\quad -\delta_{ik}\Big(2aH(b_j-\beta_j) - a\dot{\beta}_j - 2\partial^2 h_j\Big) - 2\delta_{il}\partial_k\partial_l h_j + 2\delta_{kl}\partial_i\partial_j h_l\Big]\,, && \text{(C.8)}
\end{aligned}$$

and the perturbations related to the torsion and boundary term scalars are

$$\begin{aligned}
\delta T &= 0\,, && \text{(C.9)}\\
\delta B &= 0\,. && \text{(C.10)}
\end{aligned}$$

C.2 Scalar and pseudo scalar perturbations

These kind of modes represent perturbations in the energy density of the cosmological fluid that is been considered at last scattering, and they also are the only fluctuations which can form structure though gravitational instability. The components of the torsion tensor and the superpotential for scalar and pseudo scalar perturbations up to first order are

$$
\begin{aligned}
\delta T^{0}{}_{0i} &= \partial_i(a\dot{\beta}-\phi)\,, && \text{(C.11)}\\
\delta T_{i0j} &= \partial_i\partial_j(\dot{h}-a^{-1}b)-\epsilon_{lij}\partial_l\dot{\sigma}-\dot{\psi}\delta_{ij}\,, && \text{(C.12)}\\
\delta T^{0}{}_{ij} &= 0\,, && \text{(C.13)}\\
\delta T_{ijk} &= \delta_{ij}\partial_k\psi-\delta_{ik}\partial_j\psi+\delta_{il}(\epsilon_{klm}\partial_j\partial_m\sigma-\epsilon_{jlm}\partial_k\partial_m\sigma)\,, && \text{(C.14)}\\
\delta S_{00i} &= -\frac{H}{a}\partial_i\Big(b-\beta-(aH)^{-1}\psi\Big)\,, && \text{(C.15)}\\
\delta S_{i0j} &= \Big[(2H\phi+\dot{\psi})\delta_{ij}+\frac{1}{2}\partial_i\partial_j(\dot{h}-a^{-1}b)-\frac{1}{2}\partial^2(\dot{h}-a^{-1}b)\delta_{ij}\Big]\,, && \text{(C.16)}\\
\delta S_{0ij} &= \frac{1}{2a^2}\epsilon_{ijk}\partial_k\dot{\sigma}\,, && \text{(C.17)}\\
\delta S_{ijk} &= \frac{1}{2a^2}\Big[\delta_{ik}\partial_j\Big(2aH(b-\beta)+\phi-\psi-a\dot{\beta}\Big)-\delta_{ij}\partial_k\Big(2aH(b-\beta)\\
&\quad+\phi-\psi-a\dot{\beta}\Big)\Big]\,, && \text{(C.18)}
\end{aligned}
$$

and the perturbations up to first order to the scalar torsion and boundary term become

$$
\begin{aligned}
\delta T &= 4H\Big(3H\phi+3\dot{\psi}+\frac{1}{a}\partial^2 b-\partial^2\dot{h}\Big)\,, && \text{(C.19)}\\
\delta B &= -\Big[H\Big(\frac{1}{a}\partial^2(6\beta-10b)-6(6\dot{\psi}+\dot{\phi}-2\partial^2\dot{h}+6H\phi)\Big)+\frac{2}{a}\partial^2(\dot{\beta}-\dot{b})+\frac{2}{a^2}\partial^2(2\psi-\phi)\\
&\quad+2(\partial^2\ddot{h}-6\dot{H}\phi-3\ddot{\psi})\Big]\,. && \text{(C.20)}
\end{aligned}
$$

Then, the perturbation conservation equations become

$$
\begin{aligned}
\mathring{\nabla}_\mu\mathcal{T}_0{}^\mu &= \delta\dot{\rho}+3H(\delta P+\delta\rho)+\frac{\partial^2 v(P+\rho)}{a}-3\dot{\psi}(P+\rho)\\
&\quad+\partial^2\dot{h}(P+\rho)=0\,, && \text{(C.21)}\\
\mathring{\nabla}_\mu\mathcal{T}_i{}^\mu &= \partial_i\Big[\delta P+(\rho+P)\Big(4aH(b+v-\beta)+\phi+a(\dot{b}-\dot{\beta}+\dot{v})\Big)\\
&\quad+a(\dot{\rho}+\dot{P})(v+b-\beta)\Big]=0\,. && \text{(C.22)}
\end{aligned}
$$

Appendix D

Emission of Gravitational Waves in TG

Einstein field equations can be modified in order to have fourth order terms in the derivative of the metric tensor. Such modification, which has been proposed in order to explain different phenomena like the cosmic acceleration, consist in adding a term containing the fourth derivative of the metric tensor $g_{\mu\nu}$ as follows

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} = \kappa^2 \mathcal{T}_{\mu\nu} + L^2 \mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta}, \tag{D.1}$$

where $\kappa^2 = 8\pi G$ (in units of $c = 1$). L is the length scale at which the modification to the field equations becomes important. We pinpoint that these field equations do not arise from an action principle, and hence they should be treated as an effective theory in which the scale L arises from an effective treatment of an underlying theory which does derive from an action. The left hand side can be reached in the TEGR limit, while the extra term (the second term of the right hand side) represent the fourth order modification of standard GR, which can be achieved in $f(T, B)$ gravity. Precisely, the underlying theory which does derive from an action can be TG, moreover since the decomposition of second and fourth order terms in the derivative of the tetrad field is a natural feature of this theory.
In order to obtain the vacuum solutions, we set $\mathcal{T}_{\mu\nu} = 0$ and arrive to

$$\mathring{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathring{R} - L^2 \mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta} = 0. \tag{D.2}$$

The wave equation satisfied by the gravitational wave can be found when we perturb the metric up to $\mathcal{O}(1)$, that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (\text{where } h_{\mu\nu} \ll 1). \tag{D.3}$$

Here $\eta_{\mu\nu}$ is the Minkowski metric $(-, +, +, +)$ and $h_{\mu\nu}$ is the small perturbation to the flat Minkowski space-time. Perturbing Eq.(D.2) we obtain

$$\delta \mathring{R}_{\mu\nu} + \frac{L^2}{2} \delta[g_{\mu\nu} g^{\rho\sigma} \mathring{R}^{;\gamma\delta}_{\rho\gamma\sigma\delta}] = L^2 \delta[\mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta}]. \tag{D.4}$$

The second term on the left hand side is the Ricci scalar term which is obtained by taking the trace of the field equation (D.2). In order to simplify the mathematical formulation, we can choose the Lorentz gauge, which is

$$\partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h = 0. \tag{D.5}$$

Using this gauge, the perturbed field equations become

$$\begin{aligned} &\Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h + \frac{L^2}{2}\eta_{\mu\nu}\partial^\alpha\partial^\beta\Box h_{\alpha\beta} \\ &= L^2\left(\Box^2 h_{\mu\nu} - \partial^\beta\partial_\nu\Box h_{\mu\beta} - \partial^\alpha\partial_\mu\Box h_{\alpha\nu} + \partial^\alpha\partial^\beta\partial_\mu\partial_\nu h_{\alpha\beta}\right) \end{aligned} \tag{D.6}$$

We can simplify even more whether we apply the TT (traceless and transverse) gauge which is the general convention used to study GWs. The TT gauge is given by

$$h^\mu_\mu = 0 \quad ; \; \partial^\nu h_{\mu\nu} = 0. \tag{D.7}$$

This gauge reflects the transverse nature of the GWs. Applying TT gauge, we obtain the modified wave equations (free space) as

$$(L^2\Box^2 - \Box)h_{\mu\nu} = 0 \tag{D.8}$$

where $\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu$ is the D'Alembertian operator. The first term inside the bracket gives the fourth order modification to the free space wave equation. Clearly, we can recover the wave equation for standard GR when $L = 0$. Now, following the same procedure as for the vacuum case, one can obtain the modified linearized Einstein's equation in TT gauge as

$$\Box\left(\Box - \frac{1}{L^2}\right)h_{\mu\nu} = \frac{16\pi G}{L^2}\mathcal{T}_{\mu\nu}. \tag{D.9}$$

The solution of the above equation can be obtained using a Green's function. Let's consider that the source is compact, and located in a region $\mathbf{x}'$ which includes the origin $\mathbf{x}' = 0$, and the observer is far away, at $\mathbf{r} = \mathbf{x}$. Let's also restrict ourselves to slowly moving compact sources, and because the system is compact it follows that $|\mathbf{x}'| \ll |\mathbf{x}|$, which implies that most radiation is emitted at frequencies such that $r \gg 1/\omega$. Hence, we can take $\mathbf{r} \approx \mathbf{x} - \mathbf{x}'$. The equation satisfied by the Green's function $G(r, t-t')$ is

$$\Box\left(\Box - \frac{1}{L^2}\right)G(r, t-t') = 4\pi\delta(t-t')\delta^3(\mathbf{r}), \tag{D.10}$$

where $r = |\mathbf{x} - \mathbf{x}'|$ and $\delta(\mathbf{r})$ denotes the Dirac delta function. The corresponding retarded Green's function in this particular case can be shown to be

$$G(r, t-t') = \frac{L\mathcal{J}_1(\sqrt{(t-t')^2 - r^2}/L)}{\sqrt{(t-t')^2 - r^2}}\Theta(t-t'-r), \tag{D.11}$$

where $\mathcal{J}_n$ is the Bessel function of the first kind and Θ is the Heaviside step function. The general solution for the GW can be obtained by integrating the Green's function over all the sources

$$h_{\mu\nu}(\mathbf{x},t) = \frac{4G}{L^2}\int d^3x'dt'G(r,t-t')\mathcal{T}_{\mu\nu}(\mathbf{x}',t'). \tag{D.12}$$

Using Eq.(D.11), we integrate Eq.(D.12) over t' and $\mathbf{x}'$, so we have

$$h_{\mu\nu}(\mathbf{x},t) = \frac{4G}{L}\int\int d\mathbf{x}'\ dt'\ \frac{\mathcal{J}_1(\sqrt{(t-t')^2-r^2}/L)}{\sqrt{(t-t')^2-r^2}}\Theta(t-t'-r)\mathcal{T}_{\mu\nu}(t',x'). \tag{D.13}$$

Now, as a consequence of the Bianchi identities, the energy conservation equations for the modified field equations are given by

$$(\kappa^2\mathcal{T}_{\mu\nu} + L^2\mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta})^{;\nu} = 0. \tag{D.14}$$

We now perturb the above equations in the weak field limit (up to first order) and use the form of $\delta\mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta}$ from the right hand side of (D.6). Writing the covariant derivative in terms of the partial derivative and Christoffel symbols, we can neglect the terms involving the Christoffel symbols since they produce higher order corrections. Hence retaining only the partial derivative, we get the energy conservation equations as

$$(\kappa^2\delta\mathcal{T}_{\mu\nu} + L^2\delta\mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta})^{,\nu} = 0. \tag{D.15}$$

Since we are considering weak field limit, the magnitude of energy momentum tensor must be small. Hence we are ignoring the higher order corrections to $\mathcal{T}_{\mu\nu}$ such that the lowest non-vanishing term is the same as the order of magnitude as the perturbation. In consequence, we will denote $\delta\mathcal{T}_{\mu\nu}$ as $\mathcal{T}_{\mu\nu}$.
Taking the partial derivative of the right hand side of (D.6) (which denotes $\delta\mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta}$) and applying Lorentz and TT gauge, it can be easily seen that $(\delta\mathring{R}^{;\alpha\beta}_{\mu\alpha\nu\beta})^{,\nu} = 0$. Using this result and differentiating Eq.(D.15) with respect to time, we get

$$\frac{\partial^2}{\partial t_r^2}\mathcal{T}_{00} = -\frac{\partial^2}{\partial x_i'\partial t_r}\mathcal{T}_{0i} = \frac{\partial^2}{\partial x_i'\partial x_j'}\mathcal{T}_{ij}. \tag{D.16}$$

Multiplying the above equation with $x_i'x_j'$ and integrating, we have

$$\frac{d^2}{dt_r^2}\int d^3x'x_i'x_j'\mathcal{T}_{00} = \int d^3x'x_i'x_j'\frac{\partial^2}{\partial x_k'x_l'}\mathcal{T}_{kl} = 2\int d^3x'\mathcal{T}_{ij}. \tag{D.17}$$

The quadrupole moment of the source is defined in terms of the energy momentum tensor as follows

$$I_{ij}(t') = \int d^3x'x_i'x_j'\mathcal{T}_{00}(t',\mathbf{x}'). \tag{D.18}$$

Given that in the chosen gauge, h_{ij} is traceless and transverse, it is more convenient to replace I_{ij} by its traceless (reduced) quadrupole moment, which is defined as

$$Q_{ij}(t') = \int d^3x'(x'_i x'_j - \frac{1}{3}\delta_{ij}r'^2)\mathcal{T}_{00}(t', \mathbf{x}'), \tag{D.19}$$

where $\mathcal{T}_{00}$ is related with the physical density. In $f(T, B)$ gravity we would have, for a FLRW metric,

$$Q_{ij}(t') = \int d^3x' \Big(x'_i x'_j - \frac{1}{3}\delta_{ij}r'^2\Big)\frac{1}{\kappa^2}\Big[-3H^2\Big(3f_B + 2f_T\Big) + 3H\dot{f}_B - 3\dot{H}f_B + \frac{1}{2}f\Big]. \tag{D.20}$$

With the intention to make Q_{ij} transverse, we project its components on a transversal plane using the projection operator $P^a_b(x') = \delta^a_b - \frac{x'^a x'_b}{r'^2}$, where $\mathbf{x}' = (x', y', z')$ and $r' = |\mathbf{x}'|$. Using the basic properties of the projection operator, that is, $P^2 = P$ and $P^b_a P^a_c = P^b_c$, we get the traceless-transverse quadrupole moment:

$$\bar{Q}_{ij} = P^a_i Q^{ab} P^b_j - \frac{1}{2}P^{ab}Q_{ab}P_{ij}. \tag{D.21}$$

Using the previous relations, we obtain the quadrupole formula for the emission of GWs as

$$\begin{aligned} h_{ij} &= -\frac{2G}{L}\int_{-\infty}^{t-r/c} \frac{\mathcal{J}_1\left(\frac{\sqrt{\tau^2 - r^2}}{L}\right)}{\sqrt{\tau^2 - r^2}}\ddot{\bar{Q}}_{ij}(t')dt' \\ &= \frac{2G}{L}\int_0^\infty \frac{\mathcal{J}_1(s)}{\sqrt{s^2+\chi^2}}\ddot{\bar{Q}}_{ij}(t'_r)ds. \end{aligned} \tag{D.22}$$

Here $\tau = t - t'$, $\chi = r/L$, $s = c\sqrt{\tau^2 - r^2/c^2}/L$ and $t'_r = \left(t - \frac{L\sqrt{s^2+\chi^2}}{c}\right)$. Eq.(D.22) is valid in TG because, as we have mentioned, the fourth order modification of Einstein field equations is equivalent to $f(T, B)$ gravity. Consequently, the emission of GWs in TG comes from a quadrupole moment; the behavior of matter is the same as in GR. This is interesting because neither the dominant moment of GR, nor the speed, nor even the polarization modes, is different from TG. The main difference, as we showed in chapter 3, is the amplitude of the GW.

Bibliography

[1] S. M. Bilenky. Basics of General Theory of Relativity for Beginners. 10 2020.

[2] J. O. Stenflo. Origin of the cosmological constant. *Astrophys. Space Sci.*, 364(9):143, 2019.

[3] Andrzej Trautman. Einstein-Cartan theory. 6 2006.

[4] H. I. Arcos and J. G. Pereira. Torsion gravity: A Reappraisal. *Int. J. Mod. Phys. D*, 13:2193–2240, 2004.

[5] A. Unzicker and Timothy J. Case. Translation of einstein's attempt of a unified field theory with teleparallelism. *arXiv: History and Philosophy of Physics*, 2005.

[6] Geovanny A. Rave-Franco, Celia Escamilla-Rivera, and Jackson Levi Said. Dynamical complexity of the Teleparallel gravity cosmology. 1 2021.

[7] Simon A. Mitton. Georges Lemaître and the foundations of Big Bang cosmology. June 2020.

[8] Rafael C. Nunes, Supriya Pan, and Emmanuel N. Saridakis. New observational constraints on $f(T)$ gravity through gravitational-wave astronomy. *Phys. Rev. D*, 98(10):104055, 2018.

[9] LIGO. Facts. `https://www.ligo.caltech.edu/page/facts`, 2021.

[10] Joel R. Primack. Dark Matter and Galaxy Formation. *AIP Conf. Proc.*, 1192(1):101–137, 2009.

[11] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler. *Gravitation.* W. H. Freeman and Company, 1973.

[12] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified Gravity and Cosmology. *Phys. Rept.*, 513:1–189, 2012.

[13] Laura Baudis. Dark matter searches. *Annalen Phys.*, 528:74–83, 2016.

[14] Gianfranco Bertone, Dan Hooper, and Joseph Silk. Particle dark matter: Evidence, candidates and constraints. *Phys. Rept.*, 405:279–390, 2005.

[15] Adam G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *Astron. J.*, 116:1009–1038, 1998.

[16] S. Perlmutter et al. Measurements of Ω and Λ from 42 high redshift supernovae. *Astrophys. J.*, 517:565–586, 1999.

[17] Bernard Schutz. *A First Course in General Relativity.* Cambridge University Press, 2009.

[18] J. Alberto Vázquez, Luis E. Padilla, and Tonatiuh Matos. Inflationary Cosmology: From Theory to Observations. 10 2018.

[19] Robert J. Scherrer. Mapping the Chevallier-Polarski-Linder parametrization onto Physical Dark Energy Models. *Phys. Rev. D*, 92(4):043001, 2015.

[20] David H. Lyth and Andrew R. Liddle. *The Primordial Density Perturbation.* Cambridge University Press, 2009.

[21] Kayll Lake. The Flatness problem and lambda. *Phys. Rev. Lett.*, 94:201102, 2005.

[22] George Efstathiou and Steven Gratton. The evidence for a spatially flat Universe. *Mon. Not. Roy. Astron. Soc.*, 496(1):L91–L95, 2020.

[23] Lucas Lolli Savi. Reconsidering the horizon problem. 3 2017.

[24] G. R. Dvali, Alejandra Melfo, and Goran Senjanovic. Is There a monopole problem? *Phys. Rev. Lett.*, 75:4559–4562, 1995.

[25] Ray d'Inverno. *Introducing Einstein's Relativity.* Oxford University Press, 1992.

[26] N. Aghanim et al. Planck 2018 results. VI. Cosmological parameters. *Astron. Astrophys.*, 641:A6, 2020.

[27] P. A. R. Ade et al. Planck 2015 results. XIII. Cosmological parameters. *Astron. Astrophys.*, 594:A13, 2016.

[28] Kenneth C. Wong et al. H0LiCOW – XIII. A 2.4 per cent measurement of H0 from lensed quasars: 5.3σ tension between early- and late-Universe probes. *Mon. Not. Roy. Astron. Soc.*, 498(1):1420–1439, 2020.

[29] Martin Bucher. Physics of the cosmic microwave background anisotropy. *Int. J. Mod. Phys. D*, 24(02):1530004, 2015.

[30] Leor Barack et al. Black holes, gravitational waves and fundamental physics: a roadmap. *Class. Quant. Grav.*, 36(14):143001, 2019.

[31] Antony Lewis and Anthony Challinor. Weak gravitational lensing of the CMB. *Phys. Rept.*, 429:1–65, 2006.

[32] Salvatore Capozziello and Mariafelicia De Laurentis. Extended Theories of Gravity. *Phys. Rept.*, 509:167–321, 2011.

[33] R. Aldrovandi and J. G. Pereira. *Teleparallel Gravity.* Springer, 2013.

[34] V. C. De Andrade, L. C. T. Guillen, and J. G. Pereira. Teleparallel gravity: An Overview. In *9th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG 9)*, 11 2000.

[35] Yi-Fu Cai, Salvatore Capozziello, Mariafelicia De Laurentis, and Emmanuel N. Saridakis. f(T) teleparallel gravity and cosmology. *Rept. Prog. Phys.*, 79(10):106901, 2016.

[36] Celia Escamilla-Rivera and Jackson Levi Said. Cosmological viable models in $f(T, B)$ theory as solutions to the H_0 tension. *Class. Quant. Grav.*, 37(16):165002, 2020.

[37] R. Aldrovandi, J. G. Pereira, and K. H. Vu. Doing without the equivalence principle. In *10th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG X MMIII)*, pages 1505–1512, 10 2004.

[38] M. Krssak, R. J. van den Hoogen, J. G. Pereira, C. G. Böhmer, and A. A. Coley. Teleparallel theories of gravity: illuminating a fully invariant approach. *Class. Quant. Grav.*, 36(18):183001, 2019.

[39] Andrew R. Liddle. How many cosmological parameters? *Mon. Not. Roy. Astron. Soc.*, 351:L49–L53, 2004.

[40] Timothy Yenter and Ezio Vailati. Samuel Clarke. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2020 edition, 2020.

[41] Jeffrey K. McDonough. Leibniz's Philosophy of Physics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2019 edition, 2019.

[42] Pierre Fleury. Gravitation: from Newton to Einstein. SpringerBriefs in Physics, page pp., Berlin, 2019. Springer.

[43] Jose Beltrán Jiménez, Lavinia Heisenberg, and Tomi S. Koivisto. The Geometrical Trinity of Gravity. *Universe*, 5(7):173, 2019.

[44] L. D. Landau and E. M. Lifshitz. *Mechanics*. Butterworth-Heinemann, 1993.

[45] Celia Escamilla-Rivera and Júlio C. Fabris. A non-Lagrangian Theory of Gravity. 5 2020.

[46] Philip Bull et al. Beyond ΛCDM: Problems, solutions, and the road ahead. *Phys. Dark Univ.*, 12:56–99, 2016.

[47] S. Bahamonde. *Modified Teleparallel theories of gravity*. PhD thesis, University College London, London, England, 2018.

[48] V. de Sabbata and C. Sivaram. *Spin and torsion in gravitation.* World Scientific, 1994.

[49] J. B. Formiga. The meaning of torsion in teleparallel theories. 4 2020.

[50] Rafael Ferraro and Franco Fiorini. Modified teleparallel gravity: Inflation without inflaton. *Phys. Rev. D*, 75:084031, 2007.

[51] Rafael Ferraro and Franco Fiorini. On Born-Infeld Gravity in Weitzenbock spacetime. *Phys. Rev. D*, 78:124019, 2008.

[52] Gabriel R. Bengochea and Rafael Ferraro. Dark torsion as the cosmic speed-up. *Phys. Rev. D*, 79:124019, 2009.

[53] Eric V. Linder. Einstein's Other Gravity and the Acceleration of the Universe. *Phys. Rev. D*, 81:127301, 2010. [Erratum: Phys.Rev.D 82, 109902 (2010)].

[54] Shih-Hung Chen, James B. Dent, Sourish Dutta, and Emmanuel N. Saridakis. Cosmological perturbations in f(T) gravity. *Phys. Rev. D*, 83:023508, 2011.

[55] Thomas P. Sotiriou and Valerio Faraoni. f(R) Theories Of Gravity. *Rev. Mod. Phys.*, 82:451–497, 2010.

[56] Valerio Faraoni. f(R) gravity: Successes and challenges. In *18th SIGRAV Conference*, 10 2008.

[57] D. Lovelock. The Einstein tensor and its generalizations. *J. Math. Phys.*, 12:498–501, 1971.

[58] P. A. Gonzalez and Yerko Vasquez. Teleparallel Equivalent of Lovelock Gravity. *Phys. Rev. D*, 92(12):124023, 2015.

[59] Sebastian Bahamonde, Konstantinos F. Dialektopoulos, and Jackson Levi Said. Can Horndeski Theory be recast using Teleparallel Gravity? *Phys. Rev. D*, 100(6):064018, 2019.

[60] Sebastian Bahamonde, Ugur Camci, Salvatore Capozziello, and Mubasher Jamil. Scalar-Tensor Teleparallel Wormholes by Noether Symmetries. *Phys. Rev. D*, 94(8):084042, 2016.

[61] Manuel Hohmann, Laur Järv, and Ulbossyn Ualikhanova. Covariant formulation of scalar-torsion gravity. *Phys. Rev. D*, 97(10):104011, 2018.

[62] Manuel Hohmann. Scalar-torsion theories of gravity I: general formalism and conformal transformations. *Phys. Rev. D*, 98(6):064002, 2018.

[63] Maria A. Skugoreva, Emmanuel N. Saridakis, and Alexey V. Toporensky. Dynamical features of scalar-torsion theories. *Phys. Rev. D*, 91:044023, 2015.

[64] Manuel Hohmann. Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants. *Phys. Rev. D*, 98(6):064004, 2018.

[65] Fotios K. Anagnostopoulos, Spyros Basilakos, and Emmanuel N. Saridakis. Bayesian analysis of $f(T)$ gravity using $f\sigma_8$ data. *Phys. Rev. D*, 100(8):083517, 2019.

[66] Kazuharu Bamba, Chao-Qiang Geng, Chung-Chi Lee, and Ling-Wei Luo. Equation of state for dark energy in $f(T)$ gravity. *JCAP*, 01:021, 2011.

[67] Gabriel Farrugia, Jackson Levi Said, Viktor Gakis, and Emmanuel N. Saridakis. Gravitational Waves in Modified Teleparallel Theories. *Phys. Rev. D*, 97(12):124064, 2018.

[68] Rafael C. Nunes. Structure formation in $f(T)$ gravity and a solution for H_0 tension. *JCAP*, 05:052, 2018.

[69] P. A. R. Ade et al. Planck 2013 results. XX. Cosmology from Sunyaev–Zeldovich cluster counts. *Astron. Astrophys.*, 571:A20, 2014.

[70] Sebastian Bahamonde, Christian G. Böhmer, and Matthew Wright. Modified teleparallel theories of gravity. *Phys. Rev. D*, 92(10):104042, 2015.

[71] Salvatore Capozziello, Maurizio Capriolo, and Maria Transirico. The gravitational energy-momentum pseudotensor: the cases of $f(R)$ and $f(T)$ gravity. *Int. J. Geom. Meth. Mod. Phys.*, 15:1850164, 2018.

[72] Sebastian Bahamonde and Salvatore Capozziello. Noether Symmetry Approach in $f(T, B)$ teleparallel cosmology. *Eur. Phys. J. C*, 77(2):107, 2017.

[73] Andronikos Paliathanasis. de Sitter and Scaling solutions in a higher-order modified teleparallel theory. *JCAP*, 08:027, 2017.

[74] Sebastian Bahamonde, M. Zubair, and G. Abbas. Thermodynamics and cosmological reconstruction in $f(T, B)$ gravity. *Phys. Dark Univ.*, 19:78–90, 2018.

[75] Matthew Wright. Conformal transformations in modified teleparallel theories of gravity revisited. *Phys. Rev. D*, 93(10):103002, 2016.

[76] Benjamin Audren. Monte Python. `https://baudren.github.io/montepython.html`, 2021.

[77] Michele Moresco, Lucia Pozzetti, Andrea Cimatti, Raul Jimenez, Claudia Maraston, Licia Verde, Daniel Thomas, Annalisa Citro, Rita Tojeiro, and David Wilkinson. A 6% measurement of the Hubble parameter at $z \sim 0.45$: direct evidence of the epoch of cosmic re-acceleration. *JCAP*, 05:014, 2016.

[78] D. M. Scolnic et al. The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample. *Astrophys. J.*, 859(2):101, 2018.

[79] LIGO. What are Gravitational Waves? `https://www.ligo.caltech.edu/page/what-are-gw`, 2021.

[80] D. I. Jones. Gravitational waves from rotating neutron stars. *Class. Quant. Grav.*, 19:1255–1266, 2002.

[81] Nicolás Bernal, Anish Ghoshal, Fazlollah Hajkarim, and Gaetano Lambiase. Primordial Gravitational Wave Signals in Modified Cosmologies. *JCAP*, 11:051, 2020.

[82] Steven Weinberg. *Cosmology.* Oxford University Press, 2008.

[83] J. Tafel. All transverse and TT tensors in flat spaces of any dimension. *Gen. Rel. Grav.*, 50(3):31, 2018.

[84] Robert DiSalle. Space and Time: Inertial Frames. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Metaphysics Research Lab, Stanford University, winter 2020 edition, 2020.

[85] Nigel T. Bishop and Luciano Rezzolla. Extraction of Gravitational Waves in Numerical Relativity. *Living Rev. Rel.*, 19:2, 2016.

[86] Yi-Fu Cai, Chunlong Li, Emmanuel N. Saridakis, and Lingqin Xue. $f(T)$ gravity after GW170817 and GRB170817A. *Phys. Rev. D*, 97(10):103513, 2018.

[87] Julien Lesgourgues. The Cosmic Linear Anisotropy Solving System (CLASS) I: Overview. 4 2011.

[88] Diego Blas, Julien Lesgourgues, and Thomas Tram. The Cosmic Linear Anisotropy Solving System (CLASS) II: Approximation schemes. *JCAP*, 07:034, 2011.

[89] John Antoniadis. Gravitational Radiation from Compact Binary Pulsars. *Astrophys. Space Sci. Proc.*, 40:1–22, 2015.

[90] Alexander Bonilla, Rocco D'Agostino, Rafael C. Nunes, and José C. N. de Araujo. Forecasts on the speed of gravitational waves at high z. *JCAP*, 03:015, 2020.

[91] Jose María Ezquiaga and Miguel Zumalacárregui. Dark Energy in light of Multi-Messenger Gravitational-Wave astronomy. *Front. Astron. Space Sci.*, 5:44, 2018.

[92] Maxence Corman, Celia Escamilla-Rivera, and M. A. Hendry. Constraining extra dimensions on cosmological scales with LISA future gravitational wave siren data. 4 2020.

[93] LIGO. Use LIGO data. `https://www.ligo.caltech.edu/page/ligo-data`, 2021.

[94] James Ira Thorpe. LISA. `https://lisa.nasa.gov`, 2021.

[95] Elias Okon and Craig Callender. Does quantum mechanics clash with the equivalence principle - and does it matter? *European Journal for Philosophy of Science*, 1, 08 2010.

[96] Sebastian Bahamonde, Viktor Gakis, Stella Kiorpelidi, Tomi Koivisto, Jackson Levi Said, and Emmanuel N. Saridakis. Cosmological perturbations in modified teleparallel gravity models: Boundary term extension. *Eur. Phys. J. C*, 81(1):53, 2021.