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PRESENTA:
JORGE RENÉ GONZÁLEZ MARTÍNEZ

DIRECTOR DE LA TESIS: LUIS NÚÑEZ BETANCOURT
CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS.

MIEMBROS DEL COMITÉ TUTOR:
MARTHA TAKANE IMAY
INSTITUTO DE MATEMÁTICAS
RAFAEL VILLARREAL RODRÍGUEZ
DEPARTAMENTO DE MATEMÁTICAS CINVESTAV

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A mis hijas Sarahí y Tania.

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Contents

Introducción

Sea K un campo, $\mathbf{X}_{mn} = (x_{ij})_{ij}$ un matriz de variables, y $R := K[\mathbf{X}]$ un anillo de polinomios en estas variables. Sea $I_k(\mathbf{X})$ el ideal de $k \times k$-menores de $\mathbf{X}$. Notamos que el conjunto de ceros de $I_k(\mathbf{X})$ consiste en las matrices con coeficientes en K de rango menor a k.

El anillos cociente $R/I_k(\mathbf{X})$ es llamado un *anillo determinantal*. Estos anillos son dominios Cohen-Macaulay [HE71]. Estos son Gorenstein si y solo si $m = n$ [Sva74]. Diversas variaciones de anillos determinantales han sido estudiadas, tomando, por ejemplo, ideales generados por menores de diversos tamaños [dCEP80]. Hay otros tipos de variantes, por ejemplo tomando los *determinantales de escalera* y los *determinantales de escalera mezclados*. En ambos casos estos ideales son primos y tienen la propiedad de ser Cohen-Macaulay. Adicionalmente, se conoce una caracterización para la propiedad de ser Gorenstein [Con95, GM00].

Por otro lado, un $k \times k$ *menor adyacente* de $\mathbf{X}_{mn}$ es el determinante de una submatriz que tiene indices de renglones $r_1, \ldots, r_k$ e indices de columnas $c_1, \ldots, c_k$ donde estos índices son enteros consecutivos. Denotamos por $I_{mn}(k)$ al ideal generado por todos los $k \times k$ menores adyacentes de $\mathbf{X}_{mn}$. El ideal ideal $I_{mn}(k)$ dista mucho de ser ideal primo. Este tipo de ideales aparecieron por primera vez para el caso $k = 2$ donde se da la descomposición primaria de $I_{2n}(2)$ y de $I_{44}(2)$. La motivación para estudiar $I_{mn}(2)$ viene del área de *estadística algebráica* [PRW01, Stu02].

Nuestro objetivo principal es estudiar propiedades homológicas de los ideales binomiales de aristas, este tipo de ideales es una generalización de los ideales determinantales y de los ideales generados por los 2-menores adyacentes de una $2 \times n$ matriz genérica. Los ideales binomiales de aristas fueron introducidos independientemente por Herzog, Hibi, Hreindóttir, Kahle y Rauh [HHH$^+$10] y por Ohtani [Oht11]. Los ideales binomiales de aristas son ideales generados por una colección arbitraria de 2-menores de una $2 \times n$ matriz cuyas entradas son todas indeterminadas. Los generadores de este tipo de ideales son de la forma $f_{ij} = x_i y_j - x_j y_i$ con $i < j$. Es natural asociar a cada ideal de esta forma una gráfica G en el conjunto de vértices $[n] := \{1, 2, \ldots n\}$ para la cual $\{i, j\}$ es una arista si y solo si f_{ij} es un generador de el ideal. Es por este motivo que este tipo de ideales llevan ese nombre. Denotamos el ideal binomial de aristas de G por $\mathcal{J}_G$.

Sea G una gráfica simple, esto es, G no tiene lazos ni aristas multiples, en el conjunto de vértices $V(G) = [n] = \{1, \ldots, n\}$ con aristas $E(G)$. Sea $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ el anillo de polinomios en $2n$ variables sobre un campo K, y sea $f_{ij} = x_i y_j - x_j y_i$ donde $i < j$. el *ideal binomial de aristas* $\mathcal{J}_G$ de G es:

$$\mathcal{J}_G := \langle f_{ij} | \{i, j\} \in E(G) \rangle. \tag{1}$$

Clasicamente se estudian los ideales monomiales de aristas asociados a una gráfica G. Este tipo de ideales son generados por los monomios de la forma $x_i x_j$ donde $\{i, j\}$ es una arista de G. El ideal de aristas de una gráfica fue introducido por Villarreal [Vil90], donde se estudia la propiedad de ser Cohen-Macaulay.

Las propiedades de los ideales binomiales de aristas han sido ampliamente estudiadas por muchos investigadores, como por ejemplo:

- La propiedad de ser Cohen-Macaulay [BNnB17, BMS18, EHH11, HHH$^+$10, KSM15, MR18, RR14, Rin13, Rin19, Zaf12],
- Números de Betti y regularidad [Bas16, CDI16, dAH18, EZ15, KSM14, KSM16, MM13, SMK12, SMK18, SZ14],
- La propiedad de ser álgebra de Koszul [BEI18, EHH14, EHH15, Kiv14],
- Bases de Gröbner [BBS17, CR11, HHH$^+$10, Oht11],
- Generalizaciones de estos ideales [CDNG18, EHHQ14, Rau13, SMK13].

Una linea de investigación reciente se ha enfocado en la relación entre las propiedades combinatorias de la gráfica G, y las propiedades algebráicas del anillo dado por el ideal binomial de aristas $S/\mathcal{J}_G$ [BNnB17, EHH14].

Herzog, Hibi, Hreinsdóttir, Kahle y Rauh caracterizaron las gráficas tales que su ideal binomial de aristas tiene una base de Gröbner cuadrática. Para una gráfica G, los generadores f_{ij} de $\mathcal{J}_G$ forman una base de Gröbner cuadrática si y solo si para todas las aristas $\{i, j\}$ y $\{k, l\}$ con $i < j$ y

$k < l$, se tiene que $\{j, l\} \in E(G)$ si $i = k$, y $\{i, k\} \in E(G)$ si $j = l$ [HHH+10]. Una gráfica se denomina *cerrada con respecto al etiquetado de sus vértices* si cumple con esta condición, y se denomina *cerrada* si existe un etiquetado de sus vértices bajo la cual es cerrada con respecto a dicho etiquetado.

Ene, Herzog e Hibi probaron que si G es una gráfica cerrada, entonces $S/\mathcal{J}_G$ es Gorenstein si y solo si G es un camino [EHH11]. Esto motivó el principal resultado de esta tesis.

Teoremas 7.1.7 y 7.1.8. Sea $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Sea G una gráfica conexa tal que $S/\mathcal{J}_G$ es Gorenstein. Entonces G es un camino.

Esto se logra usando aplicaciones de métodos en característica prima. Un resultado clave para obtener el teorema 7.1.7 es el hecho de que la acción de Frobenius en la cohomología local de $S/\mathcal{J}_G$ es siempre inyectiva. Esta es una propiedad deseable que se relaciona con el tipo de singularidad de la variedad asociada al anillo [Fed83, Sch09].

Teorema 7.1.6. Sea $S = K[x_1, \ldots, x_n]$ un anillo de polinomios sobre un campo, K, de característica prima. Sea I un ideal y $<$ un orden monomial tal que $\mathrm{In}_<(I)$ es libre de cuadrados. Entonces S/I es F-inyectivo.

El resultado anterior es de interés por sí mismo para singularidades en característica prima. Hacemos mención de que este teorema fue obtenido de manera independiente y simultanea por Varbaro y Koley [VK].

Otro ingrediente clave son los umbrales F-puros [TW04]. Vagamente hablando, este invariante nos da el orden asintótico de la escisión de un anillo. En este trabajo calculamos el umbral F-puro de los ideales binomiales de aristas asociados a gráficas cerradas. En particular mostramos que el umbral F-puro de un ideal binomial de aristas coincide con el umbral F-puro del ideal inicial del ideal binomial de aristas para gráficas cerradas.

Corollary 6.2.3. Sean G una gráfica cerrada y $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Entonces, $\mathrm{fpt}(S/\mathcal{J}_G) = \mathrm{fpt}(S/\mathrm{In}_<(\mathcal{J}_G)) = 2$.

Este último resultado sigue la linea de investigación que establece que los ideales binomiales de aristas y us respectivos ideales iniciales, tienen propiedades similares para gráficas cerradas [dAH18, EHH11].

Introduction

Let K be a field, $\mathbf{X}_{mn} = (x_{ij})_{ij}$ be a matrix of indeterminates, and $R = K[\mathbf{X}]$ be a polynomial ring in these variables. Let $I_k(\mathbf{X})$ denote the ideal of $k \times k$-minors of $\mathbf{X}$. We note that the vanishing set of $I_k(\mathbf{X})$ is the set of matrices with coefficients in K of rank less than k.

The quotient $R/I_k(\mathbf{X})$ is called a *determinantal ring*. This rings are Cohen-Macaulay domains [HE71]. They are Gorenstein if and only if $m = n$ [Sva74]. Variations of determinantal rings have been studied taking mixed minors of different sizes [dCEP80]. There are also many variations such as *ladder determinantal ideals* and *mixed ladder determinantal ideals*. In both cases these rings are prime and Cohen-Macaulay, and criteria for when they are Gorenstein is characterized [Con95, GM00].

A $k \times k$ *adjacent minor* of $\mathbf{X}_{mn}$ is the determinant of a submatrix with row indices $r_1, \ldots, r_k$ and column indices $c_1, \ldots, c_k$, where this indices are consecutive integers. Let $I_{mn}(k)$ be the ideal generated by all the $k \times k$ adjacent minors of $\mathbf{X}_{mn}$. This ideal is far from being prime. It first appeared for the case $k = 2$ where primary decomposition of $I_{2n}(2)$ and $I_{44}(2)$ were given. The motivation for studying $I_{mn}(2)$ comes from the field of *algebraic statistics* [PRW01, Stu02].

The main goal of this work is to study homological properties of binomial edge ideals. These kind of ideals are a generalization of determinantal ideals and ideals generated by adjacent 2-minors in a $2 \times n$ generic matrix. These ideals were introduced by Herzog, Hibi, Hreindóttir, Kahle, and Rauh [HHH$^+$10], and by Ohtani [Oht11] independently and about the same time. Binomial edge ideals are, in simple terms, ideals generated by an arbitrary collection of 2-minors of a $2 \times n$ matrix whose entries are all indeterminates. The generators of this ideals are of the form $f_{ij} = x_i y_j - x_j y_i$ with $i < j$. Then, it is natural to associate to every ideal of this form a graph G on the vertex set $[n] = \{1, 2, \ldots n\}$ for which $\{i, j\}$ is an edge if and only if f_{ij} belongs to our ideal. Explaining the naming of this type of ideals. We denote the binomial edge ideal of G by $\mathcal{J}_G$.

By a simple graph we mean an undirected graph with no loops, no weights, and no multiple edges. Let G a simple graph on the vertex set $V(G) = [n] = \{1, \ldots, n\}$ with edge set $E(G)$. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the polynomial ring on $2n$ variables over a field K, and let $f_{ij} = x_i y_j - x_j y_i$ for $i < j$. The *binomial edge ideal* $\mathcal{J}_G$ *of* G is

$$\mathcal{J}_G = (f_{ij} | \{i, j\} \in E(G)). \tag{2}$$

Clasically, monomial edge ideals of a graph G have been studied. This kind of ideals are generated by the monomials $x_i x_j$ where $\{i, j\}$ is an edge of G. The edge ideal of a graph was introduced by Villarreal [Vil90] where he studied the Cohen-Macaulay property of such ideals.

Properties of binomial edge ideals that have been studied vastly by many researchers include

- Cohen-Macaulayness [BNnB17, BMS18, EHH11, HHH+10, KSM15, MR18, RR14, Rin13, Rin19, Zaf12],
- Betti numbers and regularity [Bas16, CDI16, dAH18, EZ15, KSM14, KSM16, MM13, SMK12, SMK18, SZ14],
- Koszulness [BEI18, EHH14, EHH15, Kiv14],
- Gröbner basis [BBS17, CR11, HHH+10, Oht11],
- generalizations of the binomial edge ideals [CDNG18, EHHQ14, Rau13, SMK13].

A line of research has focused on the relation between the the combinatorial properties of the graph G and the algebraic properties of the ring given by the binomial edge ideal $S/\mathcal{J}_G$ [BNnB17, EHH14].

Herzog, Hibi, Hreinsdóttir, Kahle and Rauh characterized the graphs whose binomial edge ideal has quadratic Gröbner base. For a graph G, the generators f_{ij} of $\mathcal{J}_G$ form a quadratic Gröbner basis if and only if for all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$ [HHH+10, Theorem 1.1]. A graph G that satisfies the aforementioned condition is called *closed with respect to the given labelling of the vertices*. We say that a graph G is *closed* if there exists a labeling of its vertices such that G is closed with respect to that labeling.

Ene, Herzog and Hibi proved that if G is a closed graph, then $S/\mathcal{J}_G$ is Gorenstein if and only if G is a path [EHH11, Corollary 3.4]. This motivated the main result of this thesis.

Theorems 7.1.7 y 7.1.8. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$. Let G be a connected graph such that $S/\mathcal{J}_G$ is Gorenstein. Then, G is a path.

This is achieved through the applications of methods in prime characteristic. A key result to obtain Theorem 7.1.7 is the fact that the action of Frobenius in the local cohomology of $S/\mathcal{J}_G$ is always injective. This a desirable property that relates to the singularity of the variety associated to a ring [Fed83, Sch09].

Theorem 7.1.6. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field, K, of prime characteristic. Let I be an ideal and $<$ a monomial order such that $\operatorname{In}_<(I)$ is square-free. Then, S/I is F-injective.

The previous result is of independent interest for singularities in prime characteristic. We point out that this theorem was obtained independently and simultaneously by Varbaro and Koley [VK].

Another key ingredient are the F-pure thresholds [TW04]. Roughly speaking, this invariant gives the asymptotic splitting order of a ring. In this work compute the F-pure threshold of binomial edge ideals associated to closed graphs. In particular, we show that the F-pure threshold of the binomial edge ideal coincide with the F-pure threshold of the initial ideal of the binomial edge ideal for closed graphs.

Corollary 6.2.3. Let G be a closed graph, and $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$. Then, $\operatorname{fpt}(S/\mathcal{J}_G) = \operatorname{fpt}(S/\operatorname{In}_<(\mathcal{J}_G)) = 2$.

The previous result follows the line of research which establish that the binomial edge ideal and its initial ideal have similar properties for closed graphs [dAH18, EHH11].

Chapter 1

Preliminaries

1.1 Graded rings

Through this document all the rings are commutative with 1 and all homomorphisms take 1 to 1.

Let R be a ring. A *gradation* on R is a decomposition $R = \oplus_{n\geq 0} R_n$, of R as a direct sum of subgroups R_n of R, where n runs over the set of all nonnegative integers, such that $R_n R_m \subseteq R_{n+m}$ for all n, $m \in \mathbb{N}$. A ring with such gradation is called a *graded ring*.

The group R_n is called the *homogeneous component* of R of degree n, and the elements of R_n are called *homogeneous elements* of degree n. Thus 0 is homogeneous of every degree. Every $r \in R$ has a unique expression

$$r = \sum_{n\geq 0} r_n$$

with $r_n \in R_n$ for every n and $r_n = 0$ for almost all n. This expression is called a *homogeneous decomposition* of r, and r_n is called the *homogeneous component* of r of degree n. We denote the R-submodule $\oplus_{n\geq 1} R_n$ of R by R_+

Let K be an arbitrary field. By a *standard graded K-algebra* $(R, \mathfrak{m}, K)$ we mean a graded ring such that $R_0 = K$, $\mathfrak{m} = \oplus_{n\geq 1} R_n$ and R is finitely generated as K-algebra over $R_0 = K$ by R_1. Since $K \cong R/\mathfrak{m}$, $\mathfrak{m}$ is a maximal ideal, and it is called the maximal graded ideal of R.

Proposition 1.1.1. Let R be a graded ring. Then

i) R_0 is a subring of R, and each R_n is an R_0-submodule of R.

ii) R_+ is an ideal of R, and $R_0 \cong R/R_+$.

Let R and S graded rings. We say that a ring homomorphism $f : R \to S$ is *graded* if $f(R_n) \subseteq S_n$ for all n. Some ideals of a graded ring preserve the gradation as the next lemma shows.

Lemma 1.1.2. For an ideal $\mathfrak{a}$ of a graded ring R, the following conditions are equivalent:

i) For every $a \in \mathfrak{a}$, all homogeneous components of a belong to $\mathfrak{a}$.

ii) $\mathfrak{a} = \oplus_{n\geq 0}(\mathfrak{a} \cap R_n)$.

iii) $\mathfrak{a}$ is generated as an ideal by homogeneous elements.

By a *homogeneous ideal* of R we mean an ideal that satisfies any of the equivalent contditions of Lemma 1.1.2.

Graded rings share some properties with local rings, as the next two lemmas shows.

Lemma 1.1.3 (Nakayama's Lemma)**.** Let be $(R, \mathfrak{m}, K)$ a local ring and $\mathfrak{a}$ an ideal of R, and let M be a finitely generated R-module. If $\mathfrak{a}M = M$ then $M = 0$

Lemma 1.1.4 (Graded version of Nakayama's Lemma)**.** Let be $(R, \mathfrak{m}, K)$ a standard graded ring and $\mathfrak{a}$ an ideal contained in the maximal graded ideal $\mathfrak{m}$ of R, and let M be either a finitely generated graded R-module or a graded R-module such that $M_{-n} = 0$ for $n \gg 0$. If $\mathfrak{a}M = M$ then $M = 0$

If $\mathfrak{a}$ is a graded ideal of R then the quotient ring $R/\mathfrak{a}$ acquires a gradation given by $(R/\mathfrak{a})_n = R_n/\mathfrak{a}_n$, called the *quotient gradation*. This is the unique gradation on $R/\mathfrak{a}$ for wich the natural surjection $R \to R/\mathfrak{a}$ is a graded homomorphism.

For a graded ring R there are modules that behave well with the action of graded elements of R. Let R be a graded ring, an M an R-module. A *gradation* on M is a decomposition $M = \oplus_{n\in\mathbb{Z}} M_n$ of M as a direct sum of subgroups M_n such that $R_n M_m \subseteq M_{m+n}$ for all m, n. Such a module is called *graded module*. Since $R_0 M_n \subseteq M_n$, each M_n can be seen as an R_0-module, and M_n is called the *homogeneous component* of M of degree n, and its elements are called *homogeneous elements* of degree n. Thus 0 is homogeneous of every degree. Each $x \in M$ has a unique expression

$x = \sum_{n \geq 0} x_n$ with $x_n \in M_n$ for every n and $x_n = 0$ for almost every n. This expression is called the *homogeneous decomposition* of x, and x_n is called the *homogeneous component* of x of degree n.

Let M and N graded R-modules, and let d be an integer. A R-homomorphism $f : M \to N$ is said to be *a graded homomorphism of degree d* if $f(M_n) \subseteq N_{n+d}$ for every n, or just a *graded homomorphism* if $d = 0$.

As for ideals of graded rings, for a graded module M there are submodules that preserves the gradation.

Lemma 1.1.5. For a submodule N of a graded submodule M, the following are equivalent:

i) For every $x \in N$ the homogeneous components of x belong to N.

ii) $N = \oplus_{n \in \mathbb{Z}}(N \cap M_n)$.

iii) N is generated as a R-submodule by homogeneous elements

A submodule of a graded R-module is called a *graded submodule* if it satisfies any of the equivalent conditions of Lemma 1.1.5.

If N is a graded submodule of a graded R-module M then the quotient module M/N acquires a gradition given by $(M/N)_n = M_n/N_n$ for every n, called the *quotient gradation*, and it is the unique gradation on M/N such that the natural surjection $M \to M/N$ is a graded homomorphism.

In the definition of graded rings and modules the index set is the non negative integers and the integers respectively but we can have gradations by any monoid.

Definition 1.1.6. Let G be a monoid. A G-graded ring R is a ring R with a decomposition $R = \oplus_{g \in G} R_g$, of R as a direct sum of subgroups R_g of R, such that $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$.

We have similar results for G-grades rings for some of the statements in this section. But we will not proceed further in that direction.

Let R be an arbitrary ring and $\mathfrak{a}$ a proper ideal of R. Let $S_n = \mathfrak{a}^n/\mathfrak{a}^{n+1}$ where we put $R_0 = R$, and let $S = \oplus_{n \geq 0} S_n$. Each S_n is an R-module, so S is an R-module. We define multiplication in S as follows: Let $\rho_n \in S_n$ and $\rho_m \in S_m$. Choose r_n and r_m such that ρ_n and ρ_m are the images of r_n and r_m under the natural projection modulo $\mathfrak{a}^{n+1}$ and $\mathfrak{a}^{m+1}$, respectively. Note that $r_n r_m \in S_{n+m}$. Then we define $\rho_n \rho_m$ to be the natural image of $r_n r_m$ in S_{m+n}. It is checked that this independent of the representatives r_n and r_m of ρ_n and ρ_m.

With this definition we say that S is the *associated graded ring* of R with respect to $\mathfrak{a}$, and it is denoted by $\mathrm{Gr}_{\mathfrak{a}}(R)$.

Now, let M be a n R-module. The *associated graded module* of M with respect to $\mathfrak{a}$, denoted by $\mathrm{Gr}_{\mathfrak{a}}(M)$ is defined in a similar way. Namely, is the $\mathrm{Gr}_{\mathfrak{a}}(R)$-module constructed as follows: $\mathrm{Gr}_{\mathfrak{a}}(M) = \oplus_{n\in\mathbb{Z}} N_n$, where $N_n = \mathfrak{a}^n M/\mathfrak{a}^{n+1}M$. For $\rho \in S_n$ and $y \in N_m$ we choose r and x such that $\rho \in S_n$ and $y \in N_m$ are the natural projections of r and x modulo $\mathfrak{a}^{n+1}$ and $\mathfrak{a}^{m+1}M$ respectively. Note that $rx \in \mathfrak{a}^{n+m}M$ and define ρy to be the natural image of rx modulo $\mathfrak{a}^{n+m+1}M$. Again it is verified that this choice is independent of the representatives so it defines a well defined escalar product on homogeneous elements wich extends by distributivity to all $\mathrm{Gr}_{\mathfrak{a}}(R)$.

Chapter 2

Cohen-Macaulay rings and modules

2.1 Dimension

The most fundamental numerical invariant of a Noetherian ring R is the *dimension* of the ring R, and *depth*, previously known as *homological dimension*, is the second most fundamental invariant of R. For a *chain of prime ideals* of R we mean a sequence of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

Such a chain is said to have *length* n (the number of links).

Definition 2.1.1. The *dimension* (also called *Krull dimension*) of a ring R, written $\dim(R)$, is the supremum of the lengths of chains of prime ideals of R.

2.2 Regular sequences

While depth is defined in terms of regular sequences, is measured by certain Ext modules. With this connection we can use homological methods to investigate commutative rings.

Definition 2.2.1. Let M be a module over a ring R. We say that $x \in R$ is an *M-regular element* (or a regular element if $M = R$) if $xz = 0$ for $z \in R$

implies $z = 0$, in other words, x is not a zero-divisor of M. We can compose successively regular elements.

Definition 2.2.2. Let R be a ring. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an *M-regular sequence* or *M-sequence* for short if the following contidions are satisfied:

i) x_1 is an M-regular element

ii) x_i is a $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 2, \ldots, n$, and

iii) $M/\mathbf{x}M \neq 0$.

We shall also say that M is an *$\boldsymbol{x}$-regular module*. A *regular sequence* is an R-regular sequence.

Remark 2.2.3. Suppose that $R = (R, \mathfrak{m}, K)$ is a local ring or a standard graded K-algebra, suppose that $M \neq 0$ is a finite R-module and finally suppose that $\mathbf{x} \subset \mathfrak{m}$. Then Condition iii is satisfied automatically because of the local and graded versions of Nakayama's lemma.

Remark 2.2.4. A permutation of a regular sequence need not to be a regular sequence. As an example of this fact we take $R = K[X, Y, Z]$, K a field. The sequence $X, Y(1-X), Z(1-X)$ is an R-sequence, but $Y(1-X), Z(1-X), X$ is not. There are conditions under which regular sequences can be permuted.

Proposition 2.2.5. Let $x_1, x_2, \ldots, x_n$ be an M-regular sequence. If one of the next two conditions hold

- R is a Noetherian local ring.
- R is a graded ring, M is a graded R-module and $x_1, x_2, \ldots, x_n \in R_+$.

Then every permutation of $x_1, x_2, \ldots, x_n$ is again M-regular.

Let R be a ring, M an finite R-module, and $\mathbf{X} = X_1, X_2, \ldots, X_n$ indeterminates over R. Then we write $M[\mathbf{X}]$ for $M \otimes R[\mathbf{X}]$ and call its elements *polynomials with coefficients in M*. If $\mathbf{x} = x_1, x_1, \ldots, x_n$ is a sequence of elements of R, then there is an R-algebra homomorphism $\varphi : R[\mathbf{X}] \to R$ and also an R-module homomorphism $\psi : M[\mathbf{X}] \to M$ induced by the sustitution $X_i \mapsto x_i$. We denote by $F(\mathbf{x})$ the image of $F \in R[\mathbf{X}]$ (or $F \in M[\mathbf{X}]$) under this map.

The classical example of a regular sequence is the sequence $\mathbf{X} = X_1, \ldots, X_n$ of indeterminates in a polynomial ring $R = S[X_1, \ldots, X_n]$. Moreover, every M-sequence behaves in some sense like a sequence of inderteminates, as shown in the next theorem.

Theorem 2.2.6. Let R be a ring, M and R-module, $\mathbf{x} = x_1, \ldots, x_n$ an M-sequence, and $I = (\mathbf{x})$. Then the map

$$\begin{aligned} (M/IM)[X_1, \ldots, X_n] &\to \mathrm{Gr}_I(M) \\ X_i &\mapsto \bar{x}_i \in I/I^2 \end{aligned}$$

is an isomorphism.

This theorem says precisely how a regular sequence behaves like a sequence of indeterminates: the residue classes $\bar{x}_i \in I/I^2$ operate on $\mathrm{Gr}_I(M)$ exactly like indeterminates. A sequence $\mathbf{x}$ that satisfies the condition of Theorem 2.2.6 is called *M-quasi-regular sequence*. Not every M-quasi-regular sequence is a regular sequence, since a regular sequence may lose regularity after a permutation as shown in Remark 2.2.4, but permutations does not affect the conditions of the theorem.

2.3 Depth

Let R be a Noetherian ring, and M be a finitely generated R-module. If $\mathbf{x} = x_1, \ldots, x_n$ is an M-sequence, then the sequence

$$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \cdots, x_n)$$

ascends strictly. Therefore an M-sequence can be extended to a maximal sequence. An M-sequence $\mathbf{x} = x_1, \ldots, x_n$ contained in an ideal I is said to be *maximal in* I, if $x_1, x_2, \ldots, x_n, x_{n+1}$ is not an M-sequence for any $x_{n+1} \in I$.

Lemma 2.3.1. For an ideal I such that $IM \neq M$, the following two conditions are equivalent:

i) $\hom_R(R/I, M) = 0$.

ii) I contains an M-regular element.

Proposition 2.3.2. For an ideal I of R such that $IM \neq M$, and for an integer $r > 0$, the following two conditions are equivalent:

i) $\operatorname{Ext}^i_R(R/I, M) = 0$ for every i, $0 \leq \ i \leq r-1$.

ii) I contains an M-regular sequence of length r.

Proposition 2.3.2, tell us precisely that every maximal M-sequence has the same length. Which motivates the next definition.

Definition 2.3.3. Let I be an ideal of R such that $IM \neq M$. The length of any maximal M-sequence in I is called the *I-depth* of M and is denoted by $\operatorname{depth}_I M$. This is well defined in view of the above proposition.

If $(R, \mathfrak{m})$ is a Noetherian local ring and M is a nonzero finitely generated R-module (so that $\mathfrak{m}M \neq M$ by Nakayama 1.1.3) then the $\mathfrak{m}$-depth of M is called the *depth* of M, and in this case is denoted by $\operatorname{depth} M$.

2.4 Depth and dimension

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finite R-module. All the minimal elements of $\operatorname{Supp} M$ belong to $\operatorname{Ass} M$. Therefore, if $x \in \mathfrak{m}$ is an M-regular element, then $x \notin \mathfrak{p}$ for all minimal elements of $\operatorname{Supp} M$, and inductions yields $\dim M/\mathbf{x}M = \dim M - n$ if $\mathbf{x} = x_1, \ldots, x_n$ is an M-sequence. We just proved:

Proposition 2.4.1. Let $(R, \mathfrak{m})$ a Noetherian local ring and $M \neq 0$ a finite R-module. Then every M-sequence is part of a system of parameters of M. In particular $\operatorname{depth} M \leq \dim M$

The inequality of Proposition 2.4.1, can be refined:

Proposition 2.4.2. Let $(R, \mathfrak{m})$ a Noetherian local ring and $M \neq 0$ a finite R-module. Then $\operatorname{depth} M \leq \dim R/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$.

A variant of Proposition 2.4.1 says that depth id bounded by height, more precisely:

Proposition 2.4.3. Let R be a Noetherian ring and $I \subseteq R$ and ideal. Then $\operatorname{depth}_I M \leq \operatorname{height} I$.

We are now ready to present an invariant that refines the information given by depth:

Definition 2.4.4. Let M be a module over a local ring $(R, \mathfrak{m}, K)$, and $\mathbf{x}$ a maximal M-sequence. Then

$$\operatorname{Soc} M = (0 : \mathfrak{m})_M \cong \hom_R(K, M)$$

is called the *socle of* M, and the number

$$r(M) = \dim_K \operatorname{Soc}(M/\mathbf{x}M)$$

is called the *type of* M.

2.5 Depth and projective dimension

Let R be a ring, and M an R-module; M has a projective resolution

$$P_\bullet : \cdots \to P_n \xrightarrow{\varphi_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\varphi_1} P_0 \to 0.$$

Set $M_0 = M$ and $M_i = \operatorname{Im} \varphi_i$ for $i \geq 1$. We note that the modules M_i depend on $P_\bullet$. However, M determines M_i up to projective equivalence, therefore is justified to call M_i the *i-th syzygy of M*. The *projective dimension of M*, denoted by $\operatorname{Proj.Dim} M$, is infinity if none of the modules M_i is projective. Otherwise $\operatorname{Proj.Dim} M$ is the least integer n for which M_n is projective; replacing P_n by M_n one gets a projective resolution of M of lenght n:

$$0 \to M_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

For a finite module M over a Noetherian local ring $(R, \mathfrak{m}, K)$ there is a very natural condition which, if satisfied by $P_\bullet$, determines $P_\bullet$ uniquely. It is a consequence of Nakayama's lemma that $x_1, \ldots, x_m$ form a minimal system of generators of M if and only if the residue classes $\bar{x_1}, \ldots, \bar{x_m} \in M/\mathfrak{m}M \cong M \otimes K$ are a K-basis of the K-vector space $M \otimes K$, and

$$\mu(M) = \dim_K M \otimes K$$

is the *minimal number of generators of M*. Set $\beta_0 = \mu(M)$. We choose a minimal system $x_1, \ldots, x_{\beta_0}$ of generators of M and specify an epimorphism

$$\begin{aligned} \varphi_0 : R^{\beta_0} &\to M \\ e_i &\to x_i \end{aligned}$$

where $e_1, \dots, e_{\beta_0}$ is the canonical basis of R^{β_0}. now we set $\beta_1 = \mu(\ker \varphi_0)$ and define

$$\varphi_1 : R^{\beta_1} \to \ker \varphi_0$$
$$e_i \to x_i$$

where now $e_1, \dots, e_{\beta_1}$ is the canonical basis of R^{β_1}, and $x_1, \dots, x_{\beta_1}$ are a minimal system of generators of $\ker \varphi_0$. Proceeding in this manner, we construct a *minimal free resolution*

$$F_\bullet : \cdots \to R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \to \cdots \to R^{\beta_1} \xrightarrow{\varphi_1} \beta_0 \to 0.$$

Remark 2.5.1. $F_\bullet$ is determined by M up to an isomorphism of complexes.

Definition 2.5.2. The number $\beta_i(M) = \beta_i$ is called the *i-th betti number of* M.

The next proposition tell us exactly when a free resolution is minimal.

Proposition 2.5.3. Let $(R, \mathfrak{m}, K)$ be a Noetherian loca ring, M a finite R-module, and

$$F_\bullet : \cdots \to F_n \xrightarrow{\varphi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \to 0$$

a free resolution of M. Then the following are equivalent:

i) $F_\bullet$ is minimal;

ii) $\varphi_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all $i \geq 1$;

iii) $\operatorname{rank} F_i = \dim_K \operatorname{Tor}_i^R(M, K)$ for all $i \geq 0$;

iv) $\operatorname{rank} F_i = \dim_K \operatorname{Ext}_R^i(M, K)$ for all $i \geq 0$.

Now we can characterize the projective dimension of a module.

Corollary 2.5.4. Let $(R, \mathfrak{m}, K)$ be a Noetherian loca ring, M a finite R-module. Then $\beta_i(M) = \dim_K \operatorname{Tor}_i^R(M, K)$ for all i and

$$\operatorname{Proj.Dim} M = \sup\{i | \operatorname{Tor}_i^R(M, K) \neq 0\}.$$

The following theorem known as the *Auslander-Buchsbaum formula* is an effective instrument to compute the depth of a module.

Theorem 2.5.5 (Auslander-Bauchsbaum)**.** Let $(R, \mathfrak{m}, K)$ be a Noetherian loca ring, $M \neq 0$ a finite R-module. If $\operatorname{Proj.Dim} M < \infty$, then

$$\operatorname{Proj.Dim} M + \operatorname{depth} M = \operatorname{depth} R$$

the proof of this theorem is by induction on $\operatorname{depth} R$.

2.6 Cohen-Maculay rings and modules

Let R be a Noetherian local ring, and M a finite R-module. We have presented two invariants of M. A "geometric" invariant and an "homological" invariant. We say that M is Cohen-Macaulay if both invariants coincide.

Definition 2.6.1. Let R be a Noetherian local ring. A finite R-module $M \neq 0$ is a *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. If R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulay ring.* A *maximal* Cohen-Macaulay module is a Cohen-Macaulay module M such that $\dim M = \dim R$.

In general, if R is a Noetherian ring and M is a finite R-module, Then M is a *Cohen-Macaulay module* if $M_{\mathfrak{m}}$ is a Cohen-Macaulay $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \in \operatorname{Supp} M$. For a module M to be a *maximal* Cohen-Macaulay R-module we require that $M_{\mathfrak{m}}$ is a maximal $R_{\mathfrak{m}}$ module for each maximal ideal $\mathfrak{m}$ of R. As in the local case, R is a *Cohen-Macaulay ring* if it is a Cohen-Macaulay R-module.

Definition 2.6.2 (Chain conditions in Cohen-Macaulay rings)**.** A Noetherian ring R is *catenary* if every saturated chain joining prime ideals $\mathfrak{p}$ and $\mathfrak{q}$, such that $\mathfrak{p} \subseteq \mathfrak{q}$, has maximal length and it is equal to $\operatorname{height} \mathfrak{q}/\mathfrak{p}$. We say that R is *universally catenary* if all the polynomial rings $R[X_1, \ldots, X_n]$ are catenary.

We show some properties of Cohen-Macaulay rings and Cohen-Macaulay modules.

Theorem 2.6.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $M \neq 0$ a Cohen-Macaulay R-module. Then

i) $\dim R/\mathfrak{p} = \operatorname{depth} M$ for all $\mathfrak{p} \in \operatorname{Ass} M$,

ii) $\operatorname{depth}_I M = \dim M - \dim M/IM$ for all ideals $I \subseteq \mathfrak{m}$,

iii) $\mathbf{x} = x_1, x_2, \ldots, x_n$ is an M-sequence if and only if

$$\dim M/\mathbf{x}M = \dim M - r, \tag{2.1}$$

iv) $\mathbf{x}$ is an M-sequence if and only if it is part of a system of parameters of M.

Theorem 2.6.4. If R is Cohen-Macaulay then the polynomial ring

$$R[X_1, \ldots, X_n]$$

is also Cohen-Macaulay.

Corollary 2.6.5. A Cohen-Macaulay ring is universally catenarian.

An ideal $\mathfrak{a}$ of R is said to be unmixed if $\operatorname{height} \mathfrak{p} = \operatorname{height} a$ for every $\mathfrak{p} \in \operatorname{Ass} A/a$. We say unmixedness holds for R if, for every $r \geq 0$, every ideal of R of height r and generated by r elements is unmixed.

Theorem 2.6.6. A ring R is Cohen-Macaulay if and only if unmixedness holds for R.

2.7 Homogeneous case

Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra with maximal homogeneous ideal $\mathfrak{m}$, and let M be a finitely generated graded R-module not equal to the zero module.

Definition 2.7.1. An homogeneous element x of R is said to be *M-regular* if $x \in \mathfrak{m}$ and x is a nonzero divisor on M, that is, the map

$$M \xrightarrow{x} M$$
$$m \mapsto x \cdot m$$

is an injection but not a surjection, the second condition is a consequence of the graded version of Nakayama's lemma, wich states that if $\mathfrak{m}M = M$, then $M = 0$. The definition of regular element can be extended to a sequence as follows: A sequence $x_1, x_2, \ldots, x_r$ of homogeneous elements of R is said to be M-regular if x_i is $M/(x_1, \ldots, x_{i-1})M$-regular for every i, $1 \leq i \leq r$. For $i = 1$, the condition means that x_1 is M-regular.

If $x_1, \ldots, x_r$ is an M-regular sequence in $\mathfrak{a}$ then the integer r is called the *lenght* of the sequence.

If I is an homogeneous ideal of R we have that $IM \neq M$. It turns out that there are maximal regular sequences on M contained on I, and every maximal regular sequence have the same length, called the *graded depth* of M on I.

The *graded Krull dimension* of a graded ring R is defined as the supremum of lengths of chains of homogeneous prime ideals of R, where the length of the chain

$$\mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \cdots \supsetneqq \mathfrak{p}_n$$

is n.

By a *homogenous system of parameters* for R, we mean a sequence of homogeneous elements $F_1, \ldots, F_n$ of positive degree, in R such that $n = \dim(R)$ and $R/(F_1, \ldots, F_n)R$ has graded Krull dimension 0. A homogeneous system of parameters always exist. Moreover, if $F_1, \ldots, F_n$ is a sequence of homogeneous elements of positive degree, then the following statements are equivalent.

1. $F_1, \ldots, F_n$ is a homogeneous system of parameters.

2. $\mathfrak{m}$ is nilpontent modulo $(F_1, \ldots, F_n)R$.

3. $R/(F_1, \ldots, F_n)R$ is finite-dimensional as a K-vector space.

4. R is a finite module over the subring $K[F_1, \ldots, F_n]$.

When these conditions hold. $F_1, \ldots, F_n$ are algebraically independent over K, so that $K[F_1, \ldots, F_n]$ is a polynomial ring.

A graded R ring is called *Cohen-Macaulay* if some homogeneous system of parameters is a regular sequence on R. In our case, when R is a standard graded algebra, the following conditions are equivalent.

i) Some homogeneous system of parameters is a regular sequence.

ii) Every homogeneous system of parameters is a regular sequence.

iii) Form every homogeneous system of parameters $F_1, \ldots, F_n$, R is a free-module over $K[F_1, \ldots, F_n]$.

iv) Form some homogeneous system of parameters $F_1, \ldots, F_n$, R is a free-module over $K[F_1, \ldots, F_n]$.

v) $R_\mathfrak{m}$ is a local Cohen-Macaulay ring (in the usual sense).

vi) R is a graded Cohen-Macaulay ring.

vii) R is a Cohen-Macaulay ring.

Chapter 3

Gorenstein modules and rings

3.1 Injective modules

Definition 3.1.1. Let R be a ring. An R module M is *injective* if the functor $\hom_R(\_, R)$ is exact.

There are some useful characterizations of injective modules. Here we list some of them.

Proposition 3.1.2 ([BH93, Proposition 3.1.2])**.** Let R be a ring and I, M and N R-modules. The following conditions are equivalent:

(a) I is injective;

(b) for every $\varphi : M \to N$ monomorphism, and every $\alpha : M \to I$, there exists a homomorphism $\beta : N \to I$ such that $\alpha = \beta \circ \varphi$;

(c) if $N \subseteq M$, and $\alpha : N \to I$, there exists $\beta : M \to I$ such that $\beta|_N = \alpha$;

(d) for all $J \subseteq R$ ideal, every R-homomorphism $\varphi : J \to I$ can be extended to R;

(e) for all $J \subseteq R$ ideal, $\mathrm{Ext}^1_R(R/J, I) = 0$;

(f) $\mathrm{Ext}^1_R(M, I) = 0$ for all M;

(g) $\mathrm{Ext}^i_R(M, I) = 0$ for all M and $i > 0$.

Definition 3.1.3. Let R be a ring and M an R-module. A complex

$$I^\bullet : 0 \to I^0 \to I^1 \to I^2 \to \cdots$$

with injective modules I^i is an *injective resolution* of M if $H^0(I^\bullet) \cong M$ and $H^i(I^\bullet) = 0$ for $i > 0$.

It's not clear from this definition that injective resolutions do exist. The next theorem indicates that injective resolutions always exist.

Theorem 3.1.4 ([BH93, Theorem 3.1.8])**.** Let R be a ring. Every R-module can be embedded into an injective R-module. In fact, this embedding can be achieved canonically, that is, without making arbitrary choices.

Definition 3.1.5. Let R be a ring and M an R-module. The *injective dimension* of M, denoted $\operatorname{Inj.Dim}_R(M)$ or $\operatorname{Inj.Dim}(M)$ when R is clear in the context, is the smallest integer n for wich there exists an injective resolution $I^\bullet$ of M with $I^m = 0$ for $m > n$. If there is no such n, we define the injective dimension of M to be infinite.

Proposition 3.1.6 ([BH93, Proposition 3.1.10])**.** Let R be a ring and M an R-module. The following are equivalent:

(a) $\operatorname{Inj.Dim}(M) \leq n$;

(b) $\operatorname{Ext}_R^{n+1}(N, M) = 0$ for all R-modules N;

(c) $\operatorname{Ext}_R^{n+1}(R/J, M) = 0$ for all ideals $j \subseteq R$.

Definition 3.1.7. A Noetherian local ring R is a *Gorenstein ring* if $\operatorname{Inj.Dim}_R R < \infty$. A Noetherian ring is a *Gorenstein ring* if its localization al every maximal ideal is a Gorenstein local ring.

Definition 3.1.8. The *type* of a (finitely generated) Cohen-Macaulay module M over a local ring $(R, \mathfrak{m}, K)$ is the dimension as K-vector space of $\operatorname{Ext}_r^d(K, M)$, where $d = \dim(M)$.

Theorem 3.1.9 ([BH93, Theorem 3.2.10])**.** A local ring R is Gorenstein if and only if R is a Cohen-Macaulay ring of type one.

The Gorenstein property is stable under standard ring operations. By this we mean the following proposition.

Proposition 3.1.10 ([BH93, Proposition 3.1.9.]). Let R be a Noetherian ring:

1. Suppose R is Gorenstein. Then for every multiplicatively closed set S in R the localized ring R_S is also Gorenstein. In particular $R_{\mathfrak{p}}$ is Gorenstein for every $\mathfrak{p} \in \operatorname{Spec} R$.

2. Suppose $x_1, \dots, x_n$ is an R-regular sequence. If R is Gorestein, then so is $R/(x_1, \dots, x_n)R$. The converse holds when R is local.

3. Suppose R is local. Then R is Gorenstein if and only if its completion $\hat{R}$ is Gorenstein.

For a Noetherian local ring we have the following implications:

$$\begin{aligned} R \text{ is regular} &\Rightarrow R \text{ is a complete intersection} \\ &\Rightarrow R \text{ is Gorenstein} \\ &\Rightarrow R \text{ is Cohen-Macaulay}, \end{aligned}$$

where all the implications are strict.

Chapter 4

The local cohomology functors

4.1 The torsion functor

Definition 4.1.1. Let M be an R module, set

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^t m = 0 \text{ for some } t \in \mathbb{N}\}.$$

We note that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M. When $\Gamma_{\mathfrak{a}}(M) = M$, M is said to be $\mathfrak{a}$-*torsion*, and when $\Gamma_{\mathfrak{a}}(M) = 0$, M is said to be $\mathfrak{a}$-*torsion-free*. For a homomorphism of R-modules $\varphi : M \to N$ we have that $\varphi(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$, and so there is an induced homomorphism of R-modules

$$\Gamma_{\mathfrak{a}}(\varphi) : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N)$$

wich agrees with φ on every element of $\Gamma_{\mathfrak{a}}(M)$.

If $\psi : M \to N$ and $\vartheta : N \to L$ are further homomorphisms of R-modules and $r \in R$, then $\Gamma_{\mathfrak{a}}(\vartheta \circ \varphi) = \Gamma_{\mathfrak{a}}(\vartheta) \circ \Gamma_{\mathfrak{a}}(\varphi)$, $\Gamma_{\mathfrak{a}}(\varphi + \psi) = \Gamma_{\mathfrak{a}}(\varphi) + \Gamma_{\mathfrak{a}}(\psi)$, $\Gamma_{\mathfrak{a}}(r\varphi) = r\Gamma_{\mathfrak{a}}(\varphi)$ and $\Gamma_{\mathfrak{a}}(\mathrm{Id}_M) = \mathrm{Id}_{\Gamma_{\mathfrak{a}}(M)}$. Thus, $\Gamma_{\mathfrak{a}}(\_)$ becomes a covariant, R-linear functor from the category of R-modulos $\mathcal{C}(R)$ to itself, and it is called the $\mathfrak{a}$-*torsion functor* (A functor T is called R-linear if it is additive and $T(r\varphi) = rT(\varphi)$ for all $r \in R$ and all homomorphisms φ of R-modules). It extends to a functor in the category of complexes of R-modules.

Lemma 4.1.2 ([BS13, Lemma 1.1.6])**.** The functor $\Gamma_{\mathfrak{a}}(\_)$ is left exact.

4.2 Local cohomology modules

Definition 4.2.1. For $i \in \mathbb{N}_0$, the i-right derived functor of $\Gamma_{\mathfrak{a}}(\_)$ is denoted by $H^i_{\mathfrak{a}}(\_)$ and is referred to as the *i-th local cohomology funtor with respect to $\mathfrak{a}$*.

For an R-module M, we shall refer to $H^i_{\mathfrak{a}}(M)$, the result of applying the functor $H^i_{\mathfrak{a}}(\_)$ to M, as the *i-th local cohomology module of M with respect to $\mathfrak{a}$*.

Properties of local cohomology modules 4.2.2. Let M be an arbitrary R-module.

(i) To calculate $H^i_{\mathfrak{a}}(M)$, one proceeds as follows. Take an injective resolution
$$I^\bullet : 0 \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \to \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \cdots$$
of M, so that there si an R-homomorphism $\varphi : M \to I^0$ such that the sequence
$$0 \to M \xrightarrow{\varphi} I^0 \xrightarrow{d^0} I^1 \to \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \cdots$$
is exact. Apply the functor $\Gamma_{\mathfrak{a}}(\_)$ to the complex $I^\bullet$ to obtain
$$0 \to \Gamma_{\mathfrak{a}}(I^0) \xrightarrow{\Gamma_{\mathfrak{a}}(d^0)} \Gamma_{\mathfrak{a}}(I^1) \to \cdots \to \Gamma_{\mathfrak{a}}(I^i) \xrightarrow{\Gamma_{\mathfrak{a}}(d^i)} \Gamma_{\mathfrak{a}}(I^{i+1}) \to \cdots$$
and take the i-th cohomology module of this complex; the result
$$\ker(\Gamma(d^i))/\operatorname{Im}(\Gamma(d^{i-1})),$$
wich, by a standar fact of homological algebra, is independent of the choice of injective resolution $I^\bullet$ of M, is $H^i_{\mathfrak{a}}(M)$.

(ii) Since $\Gamma_{\mathfrak{a}}(\_)$ is covariant and R-linear, it is a consequence that each $H^i_{\mathfrak{a}}(\_)$ is again covariant and R-linear.

(iii) Since $\Gamma_{\mathfrak{a}}(\_)$ is left exact, there is a natural equivalence between $H^0_{\mathfrak{a}}(\_)$ and $\Gamma_{\mathfrak{a}}(\_)$. We use this equivalence to identify these two functors.

(iv) Let $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ be an exact sequence of R-modules and R-homomorphisms. Then, for each $i \in \mathbb{N}_0$, there is a connecting homomorphism $H^i_{\mathfrak{a}}(N) \xrightarrow{\delta_i} H^{i+1}_{\mathfrak{a}}(L)$, and these homomorphisms make the resulting long sequence

$$\begin{aligned}
0 \to H^0_{\mathfrak{a}}(L) &\xrightarrow{H^0_{\mathfrak{a}}(\varphi)} H^0_{\mathfrak{a}}(M) \xrightarrow{H^0_{\mathfrak{a}}(\psi)} H^0_{\mathfrak{a}}(N)\\
\xrightarrow{\delta_0} H^1_{\mathfrak{a}}(L) &\xrightarrow{H^1_{\mathfrak{a}}(\varphi)} H^1_{\mathfrak{a}}(M) \xrightarrow{H^1_{\mathfrak{a}}(\psi)} H^1_{\mathfrak{a}}(N)\\
\xrightarrow{\delta_1} \cdots &\\
\xrightarrow{\delta_{i-1}} H^i_{\mathfrak{a}}(L) &\xrightarrow{H^i_{\mathfrak{a}}(\varphi)} H^i_{\mathfrak{a}}(M) \xrightarrow{H^i_{\mathfrak{a}}(\psi)} H^i_{\mathfrak{a}}(N)\\
\xrightarrow{\delta_i} H^{i+1}_{\mathfrak{a}}(L) &\xrightarrow{H^{i+1}_{\mathfrak{a}}(\varphi)} \cdots
\end{aligned}$$

exact. These long exact sequences have 'natural' properties. By natural properties we mean the following: If

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\varphi"] \arrow[d, "\lambda"] & M \arrow[r, "\psi"] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & 0\\
0 \arrow[r] & L' \arrow[r, "\varphi'"] & M' \arrow[r, "\psi'"] & N' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram of R-modules and R-homomorphisms with exact rows, then, for each $i \in \mathbb{N}_0$, we have the following commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{a}}(L) \arrow[r, "H^i_{\mathfrak{a}}(\varphi)"] \arrow[d, "H^i_{\mathfrak{a}}(\lambda)"] & H^i_{\mathfrak{a}}(M) \arrow[r, "H^i_{\mathfrak{a}}(\psi)"] \arrow[d, "H^i_{\mathfrak{a}}(\mu)"] & H^i_{\mathfrak{a}}(N) \arrow[d, "H^i_{\mathfrak{a}}(\nu)"]\\
H^i_{\mathfrak{a}}(L') \arrow[r, "H^i_{\mathfrak{a}}(\varphi')"] & H^i_{\mathfrak{a}}(M') \arrow[r, "H^i_{\mathfrak{a}}(\psi')"] & H^i_{\mathfrak{a}}(N')
\end{tikzcd}$$

simply because $H^i_{\mathfrak{a}}(\_)$ is a functor, but we also have a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{a}}(N) \arrow[r, "\delta_i"] \arrow[d, "H^i_{\mathfrak{a}}(\nu)"] & H^{i+1}_{\mathfrak{a}}(L) \arrow[d, "H^i_{\mathfrak{a}}(\lambda)"]\\
H^i_{\mathfrak{a}}(N') \arrow[r, "\delta'_i"] & H^{i+1}_{\mathfrak{a}}(L')
\end{tikzcd}$$

in which the horizontal homomorphisms are the appropiate connecting homomorphisms.

(v) Let $\mathfrak{b}$ be a second ideal of R such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then $H^i_{\mathfrak{a}}(\_) = H^i_{\mathfrak{b}}(\_)$ for all $i \in \mathbb{N}_0$.

Here we present a method for computing local cohomology as a direct limit of some suitable Ext functors.

Theorem 4.2.3 ([ILL+07, Theorem 7.8])**.** For each R-module M, there are natural isomorphisms

$$\varinjlim_{t} \operatorname{Ext}^i_R(R/\mathfrak{a}^t, M) \cong H^i_{\mathfrak{a}}(M) \quad \text{for each } j \geq 0.$$

Chapter 5

Binomial Edge Ideals

5.1 Basic properties of binomial edge ideals

Definition 5.1.1 ([HHH+10]). Let G be a simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$, This means that G has no loops and no multiple edges. Let K be a field and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ the ring of polynomials in $2n$ variables. For $i < j$ we set $f_{ij} = x_i y_j - x_j y_i$. We define the *binomial edge ideal* $\mathcal{J}_G \subseteq S$ of G as the ideal generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ such that $i < j$ and $\{i, j\} \in E(G)$

The binomials f_{ij} satisfy a relation that is useful in our work.

Proposition 5.1.2 (Plücker relation). let G be a simple graph, for $i < j < k < l$ vertices of G the following equation hold:

$$f_{ij} f_{kl} - f_{ik} f_{jl} + f_{il} f_{jk} = 0. \tag{5.1}$$

We present a characterization of when $\mathcal{J}_G$ has a quadratic Gröbner basis.

Theorem 5.1.3 ([HHH+10, Theorem 1.1]). Let G be a simple graph on the vertex set $[n]$, and let $<$ be the lexicographic order on $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. Then the following conditions are equivalent:

1. The generators f_{ij} of $\mathcal{J}_G$ form a quadratic Gröbner basis;
2. For all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$ (Figure 5.1).

Figure 5.1: Graphically, the Condition 2 of Theorem 5.1.3 says that if the solid lines are edges of G then the dashed lines are also edges of G.

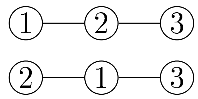

Figure 5.2: Two isomorphic graphs with different labelings. The first one satisfies the Condition 2 in Theorem 5.1.3 but the second not.

We note here that Condition 2 in Theorem 5.1.3 does not only depend on the isomorphism type of the graph, but also on the labeling of its vertices. In Figure 5.2 we show two isomorphic graphs. The first one satisfies the Condition 2 of Theorem 5.1.3, but the second not.

We can associate to every graph G on the vertex set $[n]$, a digraph G^* on the vertex set $[n]$ defined as follows: the ordered pair (i, j) is an arrow of G^* if $\{i, j\}$ is an edge of G and $i < j$. The digraph G^* is *acyclic*, that is, it has no directed cycles. We call G^* the *associated acyclic directed graph* G^* of G. In this case, Condition 2 of Theorem 5.1.3 is a condition of the associated directed graph G^* of G.

We say that a graph G on $[n]$ is *closed with respect to the given labeling of the vertices*, if G satisfies Condition 2 of Theorem 5.1.3, and we say that a graph G on $[n]$ is *closed*, if G admits a labeling of its vertices such that G is closed with respect to this labeling. When we say that a graph G is closed, we assume that G is closed with respect to the given labeling of its vertices.

A graph G is *chordal*, if all cycles of more than four vertices has a chord.

A graph with three diferent edges e_1, e_2, e_3 such that $e_1 \cap e_2 \cap e_3 \neq \emptyset$ is called a *claw*.

Proposition 5.1.4 ([HHH+10, Proposition 1.2])**.** If a graph G is closed, then G is chordal and claw-free.

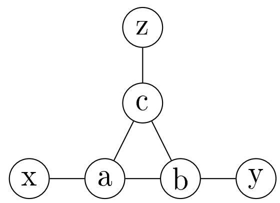

Figure 5.3: An example of a graph that is chordal without a claw, but is not closed.

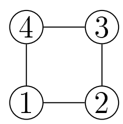

Figure 5.4: A graph that is not closed because is not chordal. The path $2, 3, 4$ is directed, while $2, 1, 4$ is not directed. But both paths are shortest paths between 2 and 4.

Corollary 5.1.5 ([HHH+10, Corollary 1.3])**.** A bipartite graph is closed if and only if it is a line.

The conditions for being a closed graph stated in Proposition 5.1.4 are necessary but not sufficient as shown in Figure 5.3.

Now we present a caracterization of graphs wich are closed with respect to a given labeling. Let G be a graph, and let u and v be vertices of G. A *path* π from u to v is a sequence of vertices $u = v_0, v_1, \ldots, v_\ell = v$ such that each $\{v_i, v_{i+1}\}$ is an edge of the underlying graph. If G is a digraph, then the path π is called *directed*, if either (v_i, v_{i+1}) is an arrow for all i, or (v_{i+1}, v_i) is an arrow for all i.

Proposition 5.1.6 ([HHH+10, Proposition 1.4])**.** A graph G on $[n]$ is closed with respect to the given labeling, if and only if for any two vertices $i \neq j$ of the associated directed graph G^*, all paths of shortest length from i to j are directed.

In Proposition 5.1.6 is required that *all* paths of shortest length from i to j are directed in order to conclude that G is closed as shown in Figure 5.4.

Let G be a graph. Since the set of graphs on $[n]$ containing G and that are closed with respect the given labeling is not empty. And since the intersection of two closed graphs containing G is again a closed graph containing G. This yields the next proposition.

Proposition 5.1.7. Let G be a simple graph on $[n]$. Then there exists a unique minimal graph $\bar{G}$ on $[n]$ with respect to inclusion of edges, that is closed with respect to the given labeling and such that G is a subgraph of $\bar{G}$.

5.2 The reduced Gröbner basis of a binomial edge ideal

Let G be a simple graph on $[n]$, and let i and j be two vertices of G with $i < j$. A path $i = i_0, i_1, \ldots, i_r = j$ is called *admissible*, if

i) $i_k \neq i_\ell$ for $k \neq \ell$;

ii) for each $k = 1, \ldots, r-1$ one has either $i_k < i$ or $i_k > j$;

iii) for any proper subset

$$\{j_1, \ldots, j_s\} \subsetneq \{i_1, \ldots, i_{r-1}\},$$

the sequence $i, j_1, \ldots, j_s, j$ is not a path.

Given an admissible path

$$\pi : i = i_0, i_1, \ldots, i_r = j$$

from i to j, where $i < j$, we associate the monomial

$$u_\pi = \left(\prod_{i_k > j} x_{i_k} \right) \left(\prod_{i_\ell < i} y_{i_\ell} \right)$$

Theorem 5.2.1 ([HHH+10, Theorem 2.1])**.** Let G a simple graph on $[n]$. Let $<$ the monomial order introduced in Theorem 5.1.3. Then, the set of binomials

$$\mathcal{G} = \bigcup_{i<j} \{u_\pi f_{ij} \,:\, \pi \text{ is an admissible path from } i \text{ to } j\}$$

is a reduced Gröbner basis of $\mathcal{J}_G$.

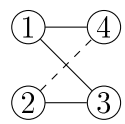

Figure 5.5: $\{2,4\}$ belongs to $\bar{G}$, but there is not admissible path from 2 to 4

Corollary 5.2.2 ([HHH$^+$10, Corollary 2.2])**.** $\mathcal{J}_G$ is a radical ideal.

As a consequence of Theorem 5.2.1 we see that all admissible paths of a graph G can be determined computing the reduced Gröbner basis of $\mathcal{J}_G$.

It is not the case that if the edge $\{i,j\}$ belongs to $\bar{G}$, there must exist an admissible path from i to j, as shown in Figure 5.5.

5.3 The minimal prime ideals of a binomial edge ideal

Let G be a simple graph on $[n]$. For each subset $\mathcal{S} \subseteq [n]$ we define a prime ideal $P_{\mathcal{S}}$. Let $T = [n]\backslash\mathcal{S}$, and let $G_1, \ldots, G_{c(\mathcal{S})}$ be the connected components of G_T. By G_T we mean the *restriction* of G to T, whose edges are exactly those edges $\{i,j\}$ of G for which $i,j \in T$. We denote by $\tilde{G}_i$ the complete graph on the vertex set $V(G_i)$. We set

$$P_{\mathcal{S}}(G) = \left(\bigcup_{i \in \mathcal{S}} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(\mathcal{S})}} \right).$$

Each $J_{\tilde{G}_i}$ is the ideal of 2-minors of a generic $2 \times n_j$-matrix with $n_j = |V(G_j)|$. Since all the ideals $J_{\tilde{G}_j}$, as well the ideal $\bigcup_{i \in S}\{x_i, y_i\}$ are prime in pairwise different set of variables, $P_{\mathcal{S}}(G)$ is a prime ideal too.

Lemma 5.3.1 ([HHH$^+$10, Lemma 3.1])**.** Let G be a simple graph on $[n]$, and let be $S \subseteq [n]$. Then height $P_{\mathcal{S}}(G) = |S| + (n - c(\mathcal{S}))$.

Theorem 5.3.2 ([HHH$^+$10, Theorem 3.2])**.** Let G be a simple graph on the vertex set $[n]$. Then $\mathcal{J}_G = \bigcap_{\mathcal{S} \subseteq [n]} P_{\mathcal{S}}(G)$.

Lemma 5.3.1 and Theorem 5.3.2 yield the following result.

Corollary 5.3.3 ([HHH+10, Corollary 3.3])**.** Let G be a simple graph on $[n]$. Then

$$\dim S/\mathcal{J}_G = \max\{(n - |\mathcal{S}|) + c(\mathcal{S}) : \mathcal{S} \in [n]\}$$

Since $P_\emptyset(G)$ does not contain monomials, it follows that $P_\mathcal{S}(G) \nsubseteq P_\emptyset(G)$. Theorem 5.3.2 implies that $P_\emptyset(G)$ is a minimal prime ideal of $\mathcal{J}_G$. From this observation follows:

Corollary 5.3.4 ([HHH+10, Corollary 3.4])**.** Let G be a simple graph on $[n]$ with c connected components, If $S/\mathcal{J}_G$ is Cohen-Macaulay, then $\dim(S/\mathcal{J}_G) = n + c$.

We want to know which of the ideals $P_S(G)$ are minimal prime ideals of $\mathcal{J}_G$. The next result is helpful to find them.

Proposition 5.3.5 ([HHH+10, Proposition 3.8])**.** Let G be a simple graph on $[n]$, and let S and T be subsets of $[n]$. Let $G_1, \ldots, G_r$ be the connected components of $G_{[n]\setminus S}$, and $H_1, \ldots, H_t$ the connected components of $G_{[n]\setminus T}$. Then $P_T(G) \subseteq P_S(G)$ if and only if $T \subseteq S$ and for all $i = 1, \ldots, t$ one has $V(H_i)\setminus S \subseteq V(G_j)$ for some j.

Let $G_1, \ldots, G_r$ be the connected components of G. If we know the minimal prime ideals of $\mathcal{J}_{G_i}$, then we can know the minimal prime ideals of $\mathcal{J}_G$. Since the ideals $\mathcal{J}_{G_i}$ are ideals in different sets of variables, it follows that the minimal prime ideals of $\mathcal{J}_G$ are exactly the ideals $\sum_{i=1}^r P_i$, where P_i is a minimal prime ideal of $\mathcal{J}_{G_i}$.

The next result detects the minimal prime ideal of connected graphs.

Corollary 5.3.6 ([HHH+10, Corollary 3.9])**.** Let G be a connected simple graph on the vertex set $[n]$, and $S \subseteq [n]$. Then $P_S(G)$ is a minimal prime ideal of $\mathcal{J}_G$ if and only if $S = \emptyset$, or $S \neq \emptyset$ and for each $i \in S$ one has $c(S\setminus\{i\}) < c(S)$.

In graph theory terminology, Corollary 5.3.6 says that if G is a connected graph, then $P_S(G)$ is a minimal prime of $\mathcal{J}_G$ if and only if each $i \in S$ is a cut-point of the graph $G_{([n]\setminus S)\cup\{i\}}$.

Lemma 5.3.7 ([HHH+10, Lemma 3.1])**.** Let G be a simple graph on the set $[n]$, and $S \subseteq [n]$. Then

$$\text{height } P_S(G)) = n + |S| - c(S).$$

As a consequence

$$\dim S/\mathcal{J}_G = \max\{(n - |S|) + c(S) \,:\, S \subseteq [n]\}.$$

5.4 Classes of chordal graphs with Cohen-Macaulay binomial edge ideal

Now we want investigate when the binomial edge ideal of a graph is Cohen-Macaulay. We say that a graph G is Cohen-Macaulay if $S/\mathcal{J}_G$ is a Cohen-Macaulay ring. For a graph G, this is the case if and only if the binomial edge ideal of each of its components is Cohen-Macaulay. Thus, we consider only connected graphs.

We denote by $N_G(j)$ the set of *neighbours* of j on G. Recall that, by a result of Dirac [Dir61], a graph is chordal if and only if it admits a *perfect elimination order*. That is, its vertices can be labeled $1, 2, \dots, n$ such that for all $j \in [n]$, $C_j = \{i| \, i \leq j\} \cap N_G(j)$ is a clique of G. A clique is a set of vertices such that the subgraph induced by this set is a complete subgraph of G.

A simplicial complex Δ on the set $[n]$ is a collection of non-empty subsets of $[n]$ such that $F \in \Delta$ whenever $F \subseteq G$ for some $G \in \Delta$. The elements of Δ are called *faces*. Faces containing only one element are called *vertices* and maximal faces are called *facets*. For each face $F \in \Delta$, we define $\dim(F) = |F| - 1$ to be the *dimension of the face* F. We define $\dim(\Delta) = \max\{\dim(F) : F \in \Delta\}$ to be the *dimension of the simplicial complex* Δ. If Δ is a simplicial complex with only one facet and n vertices we call Δ an n-simplex.

From the definition follows that a simplicial complex can be described completely by its facets, since every face is a subset of a facet and every subset of every facet is in a simplicial complex. So, if Δ has facets $F_0, \dots, Fq$, we use the notation $\langle F_0, \dots, F_q \rangle$ to describe Δ.

There is a characterization of chordal graphs in terms of its maximal cliques. This characterization is used to prove Theorem 5.4.1. Let Δ be a simplicial complex. A facet F of Δ is called a *leaf*, if either F is the only facet, or else there exist a facet G, called a *branch* of F, wich intersects F maximally. In other words, for each facet H of Δ with $H \neq F$ one has $H \cap F \subseteq G \cap F$. Each leaf has at least one *free vertex*, that is, a vertex wich belongs only to F.

The simplicial complex Δ is called a *quasi-forest* if its facets can be ordered $F_1, \dots, F_r$ sucha that for all $i > 1$ F_i is a leaf of the simplicial complex $\langle F_1, \dots, F_i \rangle$. Such an order of the facets is called a *leaf order*. A connected quasi-forest is called a *quasi-tree*.

Now we can associate to every graph G a simplicial complex wich has faces the cliques of G. This complex is denoted by $\Delta(G)$, and its called the *clique complex* of G. The equivalent statement to Dirac's Theorem now says that G is chordal if and only if $\Delta(G)$ is a quasi-forest.

Ene, Herzog and Hibi [EHH11] computed the depth of $S/\mathcal{J}_G$ for a class of chordal graphs, all forests included. As a consequence, we show that a forest has Cohen-Macaulay binomial edge ideal if and only if all of its components are path graphs.

The next theorem shows a bridge between the homological properties of $\mathcal{J}_G$ and the combinatorics of chordal graphs.

Theorem 5.4.1 ([EHH11, Theorem 1.1])**.** Let G be a chordal graph on $[n]$ such that any two distinct maximal cliques intersect in at most one vertex. Then $\operatorname{depth} S/\mathcal{J}_G = n + c$, where c is the number of connected components of G.

Moreover, the following conditions are equivalent:

i) $\mathcal{J}_G$ is unmixed.

ii) $\mathcal{J}_G$ is Cohen-Macaulay.

iii) Each vertex of G is the intersection of at most two facets of $\Delta(G)$.

Corollary 5.4.2 ([EHH11, Corollary 1.2])**.** Let G be a forest on the vertex set $[n]$. Then

$$\operatorname{depth} S/\mathcal{J}_G = n + c,$$

where c is the number of connected components of G. Moreover, the following conditions are equivalent:

i) $\mathcal{J}_G$ is unmixed.

ii) $\mathcal{J}_G$ is Cohen-Macaulay.

iii) $\mathcal{J}_G$ is a complete intersection.

iv) Each component of G is a path graph.

The formula for depth in Theorem 5.4.1 is not valid for arbitrary chordal graphs. For G as in Figure 5.6 we have that $\operatorname{depth} S/\mathcal{J}_G = 5$ and not 6 as one would expect from Theorem 5.4.1. It is also an example of a graph such that $\mathcal{J}_G$ is unmixed but not Cohen-Macaulay.

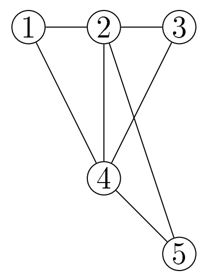

Figure 5.6: The maximal cliques $\{1, 2, 4\}$ and $\{2, 3, 4\}$ intersect at $\{2, 4\}$, so the hypothesis of Theorem 5.4.1 is not satisfied.

5.5 Closed graphs with Cohen-Macaulay binomial edge ideal

Theorem 5.5.1 ([EHH11, Theorem 2.2])**.** Let be G a simple graph on $[n]$. The following conditions are equivalent:

i) G is closed;

ii) there exists a labeling of G such that all facets of $\Delta(G)$ are intervals $[a, b] \subseteq [n]$.

Moreover, if the equivalent conditions hold and the facets $F_1, \ldots, F_r$ of $\Delta(G)$ are labeled such that $\min\{F_1\} < \min\{F_2\} < \ldots < \min\{F_r\}$, then $F_1, \ldots, F_r$ is a leaf order of $\Delta(G)$.

With the description given in Theorem 5.5.1 we can now classify all closed graphs with Cohen-Macaulay binomial edge ideal.

Theorem 5.5.2 ([EHH11, Theorem 3.1])**.** Let G be a connected graph on $[n]$ wich is closed with respect to the given labeling. Then the following conditions are equivalent.

i) $\mathcal{J}_G$ is unmixed;

ii) $\mathcal{J}_G$ is Cohen-Macaulay;

iii) $\mathrm{In}(\mathcal{J}_G)$ is Cohen-Macaulay;

iv) G satisfies the condition that whenever $\{i, j+1\}$ with $i < j$ and $\{j, k+1\}$ with $j < k$ are edges of G, then $\{i, k+1\}$ is an edge of G;

v) there exist integers $1 = a_1 < a_2 < \ldots < a_r < a_{r+1} = n$ and a leaf order of the facets $F_1, F_2, \ldots, F_r$ of $\Delta(G)$ such that $F_i = [a_i, a_{i+1}]$ are intervals for every $i = 1, \ldots, r$.

For a closed graph G, $\mathcal{J}_G$ share a lot of properties with his initial ideal $\mathrm{In}(\mathcal{J}_G)$, as Theorem 5.5.2 shows one of them. We also have the following nice property.

Proposition 5.5.3 ([EHH11, Proposition 3.2])**.** Let G be a closed graph with Cohen-Macaulay binomial edge ideal. Then $\beta_{i,j}(\mathcal{J}_G) = \beta_{i,j}(\mathrm{In}\,\mathcal{J}_G)$.

Corollary 5.5.4 ([EHH11, Corollary 3.4])**.** Let G be a closed graph with Cohen-Macaulay binomial edge ideal, and assume that $F_1, \ldots, F_r$ are the facets of $\Delta(G)$ with $k_i = |F_i|$ for $i = 1, \ldots, r$. Then the Cohen-Macaulay type of $S/\mathcal{J}_G$ is equal to $\prod_{i=1}^{r}(k_i - 1)$. In particular , $S/\mathcal{J}_G$ is Gorenstein if and only if G is a path graph.

Based on Corollary 5.5.4 Matsuda and Murai [MM13] conjectured that $S/\mathcal{J}_G$ is Gorenstein if and only if G is a path graph, wich is proved later in this work.

Let G de a closed graph with Cohen-Macaulay binomial edge ideal, and assume that $F_1 = [a_1, a_2], \ldots, F_r = [a_r, a_{r+1}]$, where $1 = a_1 < a_2 < \ldots < a_r < a_{r+1} = n$, are the facets of $\Delta(G)$ and $k_i = |F_i|$ for $i = 1, \ldots, r$. It is a well known fact that $S/\mathcal{J}_G$ and $S/\,\mathrm{In}(\mathcal{J}_G)$ have the same hilbert series, one easily gets the Hilbert series of $S/\mathcal{J}_G$,

$$H_{S/\mathcal{J}_G}(t) = \frac{\prod_{i=1}^{r}[(k_i - 1)t + 1]}{(1-t)^{n+1}}.$$

In particular, the multiplicity of $S/\mathcal{J}_G$ is $e(S/\mathcal{J}_G) = k_1 \cdots k_r$ and the a-invariant is $a(S/\mathcal{J}_G) = r - n - 1$.

By using the associativity formula for multiplicities we obtain a different expression for the multiplicity as the one given above.

Proposition 5.5.5. $P_S(G)$ is minimal prime of $\mathcal{J}_G$ if and only if S is empty or of the form $S = \{a_{j_1}, \ldots, a_{j_s}\}$ for some $2 \leq j_1 < j_2 < \cdots < j_s \leq r$ such that $a_{j_{q+1}} - a_{j_q} \geq 2$ for all $1 \leq q \leq s-1$.

In this case, the multiplicity of $S/P_S(G)$ is

$$e(S/P_S(G)) = (a_{j_1} - 1)(a_{j_2} - a_{j_1} - 1)\cdots(a_{j_s} - a_{j_{s-1}} - 1)(n - a_{j_s}).$$

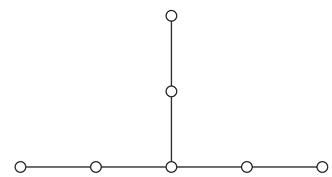

Figure 5.7: A graph with 7 vertices and with length of its longest induced path 4. However, the regularity of the polynomial ring modulo the binomial edge ideal of this graph is 5.

5.6 Regularity of binomial edge ideals

There are sharp bounds for the regularity of binomial edge ideals. The binomial edge ideal of a path P_n with n vertices is a complete intersection having $n-1$ generators of degree 2 and $\operatorname{reg}(S/\mathcal{J}_{P_n}) = n-1$ [EZ15]. The following results state that regularity $n-1$ implies that the graph is a path.

Theorem 5.6.1 ([MM13, Theorem 1.1])**.** Let G be a simple graph on $[n]$ and let ℓ be the length of the longest induced path of G. Then

$$\ell \leq \operatorname{reg}(S/\mathcal{J}_G) \leq n-1.$$

Both inequalities in Theorem 5.6.1 could be strict. Figure 5.7 is an example of this affirmation.

Theorem 5.6.2 ([KSM16, Theorem 3.4])**.** Let G be a graph which is not a path. Then $\operatorname{reg} S/\mathcal{J}_G \leq n-2$.

The formula in Theorem 5.6.2 is used in our main result. In Chapter 7 we prove that if $R/\mathcal{J}_G$ is Gorenstein then the regularity of $R/\mathcal{J}_G$ must be $n-1$.

Chapter 6

F-pure thresholds of graded rings

6.1 Frobenius endomorphism

In this section we introduce the basic definitions and properties for the F-pure threshold of standard graded rings.

Definition 6.1.1. Let R be a Noetherian ring of positive characteristic p. We say that R is *F-finite* if it is finitely generated R-module via the action induced by the Frobenius endomorphism

$$\begin{aligned} F : R &\to R \\ r &\mapsto r^p. \end{aligned}$$

For $e \in \mathbb{N}$, let $F^e : R \to R$ the e-th iteration of the Frobenius endomorphism on R. If R is reduced, R^{1/p^e} denotes the ring of p^e-th roots of R. We can identify F^e with the inclusion $R \subseteq R^{1/p^e}$. In this case R is F-finite is equivalent to $R^{1/p}$ is a finitely generated R-module. For a standard graded K-algebra $(R, \mathfrak{m}, K)$, R is F-finite if and only if K is F-finite. That is, if and only if $[K : K^p] < \infty$. We say that R is *F-pure* if F is a pure homomorphism of R-modules. That is R is F-pure if $F \otimes 1 : R \otimes M \to R \otimes M$ is injective for all R-modulles M. A ring R is called *F-split* if F is a split monomorphism. Let I be and ideal of R, we denote by $J^{[p]} = (x^p | \, x \in J)$.

Lemma 6.1.2 (Fedder's criterion for graded rings [Fed83, Theorem 1.12])**.** Let K be a perfect field of characteristic $p > 0$ and $R = K[x_1, \ldots, x_n]$ the polynomial ring over K. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the irrelevant maximal ideal of R and let $I \subseteq \mathfrak{m}$ be a homogeneous ideal of R. Then R/I is F-pure if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$.

Definition 6.1.3. Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra wich is F-finite and F-pure, and let $I \subseteq R$ be a homogeneous ideal. For a real number $\lambda \geq 0$, we say that (R, I^{λ}) is *F-pure* if for every $e \gg 0$, there exists an element $f \in I^{\lfloor (p^e-1)\lambda \rfloor}$ such that the inclusion of R-modules $f^{1/p^e}R \subseteq R^{1/p^e}$ splits.

Definition 6.1.4 ([TW04, Definition 2.1])**.** Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra wich is F-finite and F-pure. Let $I \subseteq R$ be a homogeneous ideal. The *F-pure threshold* of I is defined by

$$\operatorname{fpt}(I) = \sup\{\lambda \in \mathbb{R}_{\geq 0} | (R, I\lambda) \text{ is } F \text{ pure}\}.$$

If $I = \mathfrak{m}$, we denote the F-pure threshold by $\operatorname{fpt}(R)$.

Definition 6.1.5. Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra wich is F-finite and F-pure. We define

$$I_e(R) = \{r \in R | \varphi(r^{1/p^e}) \in \mathfrak{m} \text{ fore every } \varphi \in \operatorname{Hom}(R^{1/p^e}, R)\}. \tag{6.1}$$

Definition 6.1.6. Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra. Let $J \subseteq R$ be a homogeneous ideal. Then we define

$$b_J(p^e) = \max\{r | J^r \nsubseteq I_e(R)\}. \tag{6.2}$$

Lemma 6.1.7 ([DSNnB18, Lemma 3.9])**.** Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra wich is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous ideal. Then $p \cdot b_J(p^e) \leq b_J(p^{e+1})$.

Proposition 6.1.8 ([DSNnB18, Proposition 3.10])**.** Let $(R, \mathfrak{m}, K)$ be a standard graded K-algebra wich is F-finite and F-pure. Let $J \subseteq R$ be a homogeneous ideal. Then

$$\operatorname{fpt}(J) = \lim_{e \to \infty} \frac{b_J(p^e)}{p^e}. \tag{6.3}$$

Lemma 6.1.9 ([DSNnB18, Lemma 4.2])**.** Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over an F-finite field K. Let $\mathfrak{n} = (x_1, \ldots, x_n)$ denote the maximal homogeneus ideal. Let $I \subseteq S$ be an homogeneous ideal such that $R = S/I$ is an F-pure ring, and let $\mathfrak{m} = \mathfrak{n}R$. Then,

$$\min\left\{ s \in \mathbb{N} \middle| \left[\frac{I^{[p^e]} : I + \mathfrak{n}^{[p^e]}}{\mathfrak{n}^{[p^e]}}\right]_s \neq 0 \right\} = n(p^e - 1) - b_{\mathfrak{m}}(p^e). \tag{6.4}$$

Proposition 6.1.10 ([DSNnB18, Theorem 7.3])**.** Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over an F-finite field K of positive characteristic p. Let I be a homogeneous ideal such that $R = S/I$ is F-pure and Gorenstein. Then,

$$\mathrm{reg}_S(R) = \dim(R) - \mathrm{fpt}(R). \tag{6.5}$$

Theorem 6.1.11. Let $R = K[x_1, \dots, x_n, y_1, \dots, y_m]$ the ring of polynomials in $n+m$ variables over a field K, and let $J = \mathfrak{a} + \mathfrak{b} \subseteq R$ an ideal such that $\mathfrak{a}$ is an ideal in the variables $\{x_1, \dots, x_n\}$ and $\mathfrak{b}$ is an ideal in the variables $\{y_1, \dots, y_m\}$. Then

$$R/J \cong K[x_1, \dots, x_n]/\mathfrak{a} \otimes K[y_1, \dots, y_m]/\mathfrak{b}, \tag{6.6}$$

and

$$\mathrm{fpt}(R/J) = \mathrm{fpt}(K[x_1, \dots, x_n]/\mathfrak{a}) + \mathrm{fpt}(K[y_1, \dots, y_m]/\mathfrak{b}). \tag{6.7}$$

Proof. The first part follows from the properties of tensor product. For the second part it suffices to prove that $b_J(p^e) = b_{\mathfrak{a}}(p^e) + b_{\mathfrak{b}}(p^e)$.

Set $b = b_J(p^e)$. By the definition of $b_J(p^e)$ we have that $J^b \nsubseteq I_e(R)$. This means that there exists $r \in J^b$ a generator of this ideal that can be written as $r = a_{i_1}a_{i_2}, \dots, a_{i_s}b_{j_1}, \dots, b_{j_t}$ for some generators $\{a_{i_1}a_{i_2}, \dots, a_{i_s}\}$ of $\mathfrak{a}$ and some generators $\{b_{j_1}, \dots, b_{j_t}\}$ of $\mathfrak{b}$ with $b = s+j$ and also a morphism $\varphi \in \mathrm{Hom}(R^{1/p^e}, R)$ such that $\varphi(r^{1/p^e}) \notin \mathfrak{m}$. This element correspond by

$$R \cong K[x_1, \dots, x_n] \otimes K[y_1, \dots, y_m] \tag{6.8}$$

to a element $\alpha \otimes \beta$ in the tensor product with $\alpha \in \mathfrak{a}^s$ and $\beta \in \mathfrak{b}^t$. Then we have the next composition of morphisms:

$$\begin{array}{ccccccc} K[x_1, \dots, x_n]^{1/p^e} & \longrightarrow & R^{1/p^e} & \xrightarrow{\varphi} & R & \twoheadrightarrow & K[x_1, \dots, x_n] \\ \alpha^{1/p^e} & \longmapsto & r^{1/p^e} & \longmapsto & \varphi(r^{1/p^e}) & \mapsto & \overline{\varphi(r^{1/p^e})} \end{array}$$

where the leftmost morphism send $x^{1/p^e} \mapsto x^{1/p^e} \otimes \beta^{1/p^e}$ and the rightmost is the natural projection. We have an element $\alpha \in \mathfrak{a}^s$ and a morphism $\varphi* \in \mathrm{Hom}(K[x_1, \dots, x_n]^{1/p^e}, K[x_1, \dots, x_n])$ (the composition of the morphisms above), which sends α^{1/p^e} to $\overline{\varphi(r^{1/p^e})}$ which is not in the maximal ideal of $K[x_1, \dots, x_n]$, because $\varphi(r^{1/p^e}) \notin \mathfrak{m}$. Then

$$b_{\mathfrak{a}}(p^e) = \max(k | \mathfrak{a}^k \nsubseteq I_e(K[x_1, \dots, x_n])) \geq s.$$

By symmetric argument, we obtain

$$b_{\mathfrak{b}}(p^e) = \max(k|\mathfrak{b}^k \nsubseteq I_e(K[y_1, \dots, y_m])) \geq t.$$

Proving that $b_J(p^e) \leq b_{\mathfrak{a}}(p^e) + b_{\mathfrak{b}}(p^e)$.

For the reverse inequality let $s = b_{\mathfrak{a}}(p^e)$, $t = b_{\mathfrak{b}}(p^e)$, $\alpha \in \mathfrak{a}^s$, $\beta \in \mathfrak{b}^t$

$$\varphi \in \operatorname{Hom}(K[x_1, \dots, x_n]^{1/p^e}, K[x_1, \dots, x_n])$$

and

$$\psi \in \operatorname{Hom}(K[y_1, \dots, y_m]^{1/p^e}, K[y_1, \dots, y_m]),$$

such that $\varphi(\alpha^{1/p^e})$ and $\psi(\beta^{1/p^e})$ are not in their respective maximal ideals. Then by the isomorphism 6.8 $\alpha \otimes \beta$ corresponds to an element $\alpha\beta \in J^{t+s}$ but

$$(\varphi \otimes \psi)(\alpha^{1/p^e} \otimes \beta^{1/p^e}) = \varphi(\alpha^{1/p^e})\psi(\beta^{1/p^e}) \notin \mathfrak{m}.$$

Then $b_J(p^e) \geq t + s$. □

Theorem 6.1.12 ([DSNnB18, Theorem 4.7])**.** Let $(R, \mathfrak{m}, K)$ a standard graded K-algebra wich is F-finite and F-pure, and let $J \subseteq R$ a compatible ideal. Then

$$\operatorname{fpt}(R) \leq \operatorname{fpt}(R/J).$$

In particular

$$\operatorname{fpt}(R) \leq \operatorname{fpt}(R/\mathfrak{p})$$

for every minimal prime ideal $\mathfrak{p}$ of R.

6.2 F-pure thresholds of binomial edge ideals

For a sequence $v_1, \dots, v_s$ of natural numbers, we set

$$f_{v_1,\dots,v_s} = f_{v_1 v_2}^{p-1} \cdots f_{v_{n-1} v_n}^{p-1}.$$

Defining $f_{ji} = -f_{ij}$ for $j >$i.

Proposition 6.2.1. If $\{a, b\} \in E(G)$, then

$$f_{v_1,\dots,c,a,b,d,\dots,v_s} \equiv f_{v_1,\dots,c,b,a,d,\dots,v_s} \mod \mathcal{J}_G.$$

Proof. By the Plücker relation stated above

$$\begin{aligned}
f_{ca}^{p-1} f_{ab}^{p-1} f_{bd}^{p-1} &= f_{ab}^{p-1}(f_{cb}f_{ad} - f_{cd}f_{ab})^{p-1} \\
&= f_{ab}^{p-1} \sum_{i=0}^{p-1} \binom{p-1}{i} (-1)^i f_{cb}^{p-i-1} f_{ad}^{p-i-1} f_{cd}^{i} f_{ab}^{i} \\
&= \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} f_{cb}^{p-i-1} f_{ad}^{p-i-1} f_{cd}^{i} f_{ab}^{p+i-1}.
\end{aligned}$$

By assumption $f_{ab} \in \mathcal{J}_G$, so all the terms of the sum with $i > 0$ are contained in $\mathcal{J}_G^{[p]}$. This gives

$$\begin{aligned}
f_{v_1,\dots,c,a,b,d,\dots,v_s} &= f_{v_1v_2}^{p-1} \cdots f_{ca}^{p-1} f_{ab}^{p-1} f_{bd}^{p-1} \cdots f_{v_{n-1}v_n}^{p-1} \\
&\equiv f_{v_1v_2}^{p-1} \cdots f_{cb}^{p-1} f_{ad}^{p-1} f_{ab}^{p-1} \cdots f_{v_{n-1}v_n}^{p-1} \mod \mathcal{J}_G \\
&= f_{v_1,\dots,c,b,a,d,\dots,v_s}.
\end{aligned}$$

□

Theorem 6.2.2 ([GM20, Theorem 3.2])**.** Let G a simple connected closed graph which is not the complete graph and let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ and $\mathfrak{m}$ the maximal homogeneous ideal. Then, $S/\mathcal{J}_G$ is F-pure and

$$\min\left\{ s \in \mathbb{N} \,\middle|\, \left[\frac{\mathcal{J}_G^{[p]} : \mathcal{J}_G + \mathfrak{m}^{[p]}}{\mathfrak{m}^{[p]}}\right]_s \neq 0 \right\} \leq 2(n-1)(p-1).$$

Proof. By the Fedder's criterion 6.1.2, it suffices to show that

$$f_{1,2,\dots,n} \in (\mathcal{J}_G^{[p]} : \mathcal{J}_G) \backslash \mathfrak{m}^{[p]}.$$

First we prove that $f_{1,\dots,n} \notin \mathfrak{m}$. Let $<$ the lexicographic order on S induced by $x_1 > \dots > x_n > y_1 > \dots > y_n$. Then

$$\operatorname{In} f_{1,\dots,n} = x_1^{p-1} \cdots x_{n-1}^{p-1} y_2^{p-1} \cdots y_n^{p-1} \notin \mathfrak{m}^{[p]}.$$

Next we show that $f_{1,\dots,n} \in \mathcal{J}_G^{[p]} : \mathcal{J}_G$, it is enough to show that $f_{1,\dots,n} f_{ij} \in \mathcal{J}_G^{[p]}$ for all $\{i,j\} \in E(G)$. Assume that $\{i,j\} \in E(G)$, if $j = i+1$. Then, $f_{1,\dots,n} f_{ij} \in \mathcal{J}_G^{[p]}$. We can assume that $j > i+1$. If $j \neq n$. Then, for all

$k \in \{i+1, \ldots, j-1\}$, we have that $\{k, j\} \in E(G)$ [Mat18]. Hence, by using repeatedly Proposition 6.2.1, we obtain

$$\begin{aligned} & f_{1,2,\ldots,i,i+1,\ldots,j-1,j,j+1,\ldots,n} \\ \equiv\ & f_{1,2,\ldots,i,j,i+1,\ldots,j-1,j+1,\ldots,n} \mod \mathcal{J}_G^{[p]}. \end{aligned}$$

Since f_{ij}^{p-1} is a factor of the last expression, we have that $f_{1,\ldots,n} \in \mathcal{J}_G^{[p]} : \mathcal{J}_G$.

If $j = n$, then $i \neq 1$ because if $\{1, n\} \in E(G)$, then G is complete [Mat18]. By iterating Proposition 6.2.1,

$$\begin{gathered} f_{1,2,\ldots,i,i+1,\ldots,j-1,j,j+1,\ldots,n} \\ \equiv f_{1,2,\ldots,i-1,i+1,\ldots,i,n} \mod \mathcal{J}_G^{[p]}. \end{gathered}$$

Then, f_{in}^{p-1} is a factor of the last expression and $f_{1,\ldots,n} \in \mathcal{J}_G^{[p]} : \mathcal{J}_G$. □

Corollary 6.2.3 ([GM20, Corollary 3.3])**.** Let G be a closed graph, and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Then, $\operatorname{fpt}(S/\mathcal{J}_G) = \operatorname{fpt}(S/\operatorname{In}_<(\mathcal{J}_G)) = 2$.

Proof. If G is complete, $S/\mathcal{J}_G$ is determinantal, and $\operatorname{fpt}(S/\mathcal{J}_G) = 2$ [MSV14, Proposition 4.3]. First we prove by induction on e that if $f^{p-1} \in (I^{[p]} : I) \backslash \mathfrak{m}^{[p]}$, then $f^{p^e-1} \in (I^{[p^e]} : I) \backslash \mathfrak{m}^{[p^e]}$. The base step follows from our initial assumption.

For $f^{p^e-1} \notin \mathfrak{m}^{[p^e]}$ we have that

$$\begin{aligned} & (\mathfrak{m}^{[p^e]} : f^{p^e-1}) \subseteq \mathfrak{m} \\ \Rightarrow & (\mathfrak{m}^{[p^{e+1}]} : f^{p^{e+1}-p}) \subseteq \mathfrak{m}^{[p]} \\ \Rightarrow & ((\mathfrak{m}^{[p^{e+1}]} : f^{p^{e+1}-p}) : f^{p-1}) \subseteq (\mathfrak{m}^{[p]} : f^{p-1}) \subseteq \mathfrak{m} \\ \Rightarrow & (\mathfrak{m}^{[p^{e+1}]} : f^{p^{e+1}-1}) \subseteq \mathfrak{m}. \end{aligned}$$

This means that $f^{p^{e+1}-1} \notin \mathfrak{m}^{[p^{e+1}]}$. If $f^{p^e-1} \in I^{[p^e]} : I$, then

$$\begin{aligned} & f^{p-1} I \subseteq I^{[p]} \\ \Rightarrow & (f^{p-1} I)^{[p^e]} \subseteq I^{[p^{e+1}]} \\ \Rightarrow & f^{p^{e+1}-p^e} I^{[p^e]} \subseteq I^{[p^{e+1}]} \\ \Rightarrow & f^{p^{e+1}-p^e} (f^{p^e-1} I) \subseteq f^{p^{e+1}-p^e} (I^{[p^e]}) \subseteq I^{[p^{e+1}]} \\ \Rightarrow & f^{p^{e+1}-1} I \subseteq I^{[p^{e+1}]}. \end{aligned}$$

Then $f^{p^{e+1}-1} \in I^{[p^{e+1}]} : I$. This means in that

$$(x_1 x_2 \cdots x_{n-1} y_2 \cdots y_n)^{p^e-1} \in (\mathcal{J}_G^{[p^e]} : \mathcal{J}_G) \backslash \mathfrak{m}^{[p^e]}.$$

Using Lemma 6.1.9 we deduce that

$$\begin{aligned}
2n(p^e-1) - b_{\mathfrak{m}}(p^e) &\leq 2(n-1)(p^e-1) \\
-b_{\mathfrak{m}}(p^e) &\leq -2(p^e-1) \\
\frac{b_{\mathfrak{m}}(p^e)}{p^e} &\geq \frac{2(p^e-1)}{p^e} \\
\operatorname{fpt}(S/\mathcal{J}_G) = \lim_{e\to\infty} \frac{b_{\mathfrak{m}}(p^e)}{p^e} &\geq \lim_{e\to\infty} \frac{2(p^e-1)}{p^e} = 2.
\end{aligned}$$

Since $\mathcal{J}_{K_n}$ is a minimal prime over $\mathcal{J}_G$, the reverse inequality is a consequence of Theorem 6.1.12 and the fact that $\operatorname{fpt} \mathcal{J}_{k_n} = 2$. $\square$

Chapter 7

Main result

7.1 Gorenstein binomial edge ideals

Theorem 7.1.1. Let I be a monomial square-free ideal of $K[x_1, \ldots, x_n]$. Then $\mathrm{fpt}(R/I)$ is equal to the number of variables that do not appear in its minimal set of generators.

Proof. Let $\{x_{i_1}, \ldots, x_{i_t}\}$ be the set of variables that appear in I. Then,

$$x_{i_1}^{p^e-1} \cdots x_{i_t}^{p^e-1} \in (I^{[p^e]} : I) \backslash \mathfrak{m}^{[p^e]}.$$

Set

$$d = \deg x_{i_1}^{p^e-1} \cdots x_{i_t}^{p^e-1}.$$

For every monomial m of degree less than d, we have that $m \notin (I^{[p^e]} : I) \backslash \mathfrak{m}^{[p^e]}$. By Lemma 6.1.9 $b_{\mathfrak{m}}(p^e) = (n-t)(p^e-1)$. Dividing both sides by p^e and taking the limit as e go to infinity yields the desired result. $\square$

Theorem 7.1.2. Let G be a connected graph on $[n]$. Then x_n and y_1 are the only variables that do not appear in the minimal generating set of $\mathrm{In}(\mathcal{J}_G)$.

Proof. The set of binomials

$$\mathcal{G} = \bigcup_{i<j} \{u_\pi f_{ij} \,:\, \pi \textit{ is an admisible path from } i \textit{ to } j \,\}$$

is a reduced Gröbner basis of $\mathcal{J}_G$ (Theorem 5.2.1). Hence, the set

$$\mathcal{H} = \bigcup_{i<j} \{u_\pi x_i y_j \,:\, \pi \textit{ is an admisible path from } i \textit{ to } j \,\}$$

generates $\operatorname{In}(\mathcal{J}_G)$. We claim that this set is a minimal generating set of $\operatorname{In}(\mathcal{J}_G)$. If $x_iy_ju_\pi$ and $x_iy_ju_{\pi'}$ are distinct elements of $\mathcal{H}$ then by the definition of admisible paths, the vertices of the path π can't be a subset of the vertices of the path π' In this case $x_iy_ju_\pi$ do not divide $x_iy_ju_{\pi'}$. Now, if $x_iy_ju_\pi$ and $x_ky_\ell u_{\pi'}$ are distinct elements of $\mathcal{H}$ with $\{i,j\} \neq \{k,\ell\}$. Then, by the definition of admisible path, x_iy_j can't be a factor of $x_ky_\ell u_{\pi'}$. In this case $x_iy_ju_\pi$ do not divide $x_iy_ju_{\pi'}$ neither. In every of these monomials, the variables x_n and y_1 do not appear. We show now that they are the only variables with this property. Let $i \in \{1,\ldots,n-1\}$. Then, there exists a path from i to $i+1$, a path of minimal lenght, say π, is admisible. Then x_i and $y_i + 1$ appear in $\mathcal{H}$ for $i \in \{1,\ldots,n-1\}$. □

Corollary 7.1.3. Let G be a connected graph on $[n]$. Then,

$$\operatorname{fpt}(R/\operatorname{In}(\mathcal{J}_G)) = 2.$$

Proof. This follows form Theorem 7.1.1 and Theorem 7.1.2. □

In this section we prove our main result. First we start with a preparation theorem regarding F-injectivity of square Gröbner deformations. We first need to introduce notation.

Notation 7.1.4. Let $S = K[x_1,\ldots,x_n]$ be a polynomial ring over a field with maximal homogeneous ideal $\mathfrak{m}$. Let I be an ideal and $<$ a monomial order such that $\operatorname{In}(I)$ is square-free. There exists a vector $w \in \mathbb{N}^n$ such that $\operatorname{In}_<(I) = \operatorname{In}_w(I)$ [Stu96, Proposition 1.11]. Let $A = K[t]$ be a polynomial ring, $L = \operatorname{frac}(A)$, and $T = A \otimes_K S$. We set $J = \operatorname{Hom}_w(I) \subseteq T$ the homogenization of I, and $R = T/J$.

Remark 7.1.5. Under Notation 7.1.4, it is well known that

1. $A \to R$ is flat;
2. $R/tR = S/\operatorname{In}_<(I)$;
3. $R/(t-a)R = S/I$ for every $a \in K \setminus \{0\}$;
4. $R \otimes_A L = S/I \otimes_K L$;

The following result was obtained independently and simultaneously by Varbaro and Koley [VK].

Theorem 7.1.6. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field, K, of prime characteristic. Let I be an ideal and $<$ a monomial order such that $\mathrm{In}_<(I)$ is square-free. Then S/I is F-injective.

Proof. We use Notation 7.1.4 and the facts in Remark 7.1.5 in this proof. We have that R/tR is an Stanley-Reisner ring, and so, F-pure. Since F-pure rings are F-full and F-injective, R/tR satisfies these properties. Since t is a nonzero divisor, we have that R is also F-full and F-injective [MQ18, Theorem 1.1]. Then, $R \otimes_A L = S/I \otimes_K L$ are both F-injective and F-full, because these properties are preserved under localization. We note that S/I is a direct summand of $S/I \otimes_K L$. We note that $\mathfrak{m}$ expands the maximal homogeneous ideal in $S/I \otimes_K L$. Then, we have a commutative diagram

$$\begin{tikzcd}
H^i_{\mathfrak{m}}(S/I) \arrow[r, "\alpha"] \arrow[d, "F_{S/I}"] & H^i_{\mathfrak{m}}(S/I \otimes_K L) \arrow[d, "F_{S/I\otimes_K L}"] \\
H^i_{\mathfrak{m}}(S/I) \arrow[r, "\alpha"] & H^i_{\mathfrak{m}}(S/I \otimes_K L)
\end{tikzcd},$$

where α denotes the maps induced by the inclusion $S/I \to S/I \otimes_K L$. Since the horizontal maps split, they are injective. Since $S/I \otimes_K L$ is F-injective, we have that $F_{S/I\otimes_K L} \circ \alpha = \alpha \circ F_{S/I}$ is injective. Hence, $F_{S/I}$ is injective, and the result follows. $\square$

We are now ready to show our main result in prime characteristic.

Theorem 7.1.7 ([GM20, Theorem 4.4])**.** Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Suppose that $\mathrm{char}(K) = p > 0$. Let G be a connected graph such that $S/\mathcal{J}_G$ is Gorenstein. Then G is a path.

Proof. By Theorem 7.1.6, we have that $S/\mathcal{J}_G$ is F-injective. Since $S/\mathcal{J}_G$ is Gorenstein, we have that $S/\mathcal{J}_G$ is F-pure [Fed83, Lemma 3.3].

Since G is connected, $\mathcal{J}_{k_n}$ is a minimal prime over $\mathcal{J}_G$ [HHH+10] and its dimension is $n + 1$. Then

$$\mathrm{reg}(R/\mathcal{J}_G) = \dim(R/\mathcal{J}_G) - \mathrm{fpt}(R/\mathcal{J}_G) \geq (n + 1) - 2 = n - 1 \qquad (7.1)$$

where the inequality comes from the fact that if $I \subseteq J$ then $\mathrm{fpt}(I) \leq \mathrm{fpt}(J)$. In this case $\mathrm{fpt}(\mathcal{J}_G) \leq \mathrm{fpt}(\mathcal{J}_{K_n}) = 2$. Hence, G is a path by Theorem 5.6.2. $\square$

In the previous result we estimate the regularity of $R/\mathcal{J}_G$ using F-pure thresholds. We point out that the extremal Betti numbers of $R/\mathcal{J}_G$

and $R/\operatorname{In}(\mathcal{J}_G)$ coincide, in particular, $\operatorname{reg}(R/\mathcal{J}_G) = \operatorname{reg}(R/\operatorname{In}(\mathcal{J}_G))$ [CV18, Corollary 2.7].

We are now ready to prove the main result in this thesis in characteristic zero.

Theorem 7.1.8 ([GM20, Theorem 4.5])**.** Let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$. Suppose that $\operatorname{char}(K) = 0$. Let G be a connected graph such that $S/\mathcal{J}_G$ is Gorenstein. Then G is a path.

Proof. Since field extensions do not affect whether a ring is Gorenstein, without loss of generality we can assume that $K = \mathbb{Q}$.

Let $A = \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$ and

$$J = (x_i y_j - x_j y_i : \{i, j\} \in G \text{ and } i < j)A.$$

Then,

$$\operatorname{reg}_S(S/\mathcal{J}_G) = \operatorname{reg}_{A \otimes_{\mathbb{Z}} \mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}/J \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{reg}_{A \otimes_{\mathbb{Z}} \mathbb{F}_p}(A/J \otimes_{\mathbb{Z}} \mathbb{F}_p)$$

and $A/J \otimes_{\mathbb{Z}} \mathbb{F}_p$ is Gorenstein for $p \gg 0$ [HH, Theorem 2.3.5]. Then, $\operatorname{reg}_S(S/\mathcal{J}_G) \geq n - 1$ from the proof of Theorem 7.1.7. Hence, G is a path by Theorem 5.6.2. $\square$

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