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Resumen

El presente documento es una recopilación del trabajo realizado por el autor en su programa doctoral.

Dentro de la teoría de los Espacios Discretamente Generados se resuelven los Problemas 3.19 y 3.3 en [41]. Problema 3.19: ¿El espacio $\{\xi\} \cup \omega$ se encaja en un producto caja de rectas reales, para algún $\xi \in \beta\omega \setminus \omega$? Respondemos negativamente. Problema 3.3: ¿Es cualquier producto caja de espacios primero numberables discretamente generado? Respondemos positivamente suponiendo que los factores son regulares.

Posteriormente, probamos un resultado sobre uno de los objetos importantes de la Teoría de Conjuntos, los llamados $\Psi(\mathcal{A})$ definidos a través de una colección $\mathcal{A}$ de subconjuntos de números naturales. Se demuestra que existen $\mathfrak{c}$ espacios $\Psi(\mathcal{A})$ no homeomorfos por pares, pero cuyas colecciones subyascentes sí son homeomorfas por pares vistas como subespacios del conjunto de Cantor.

Finalmente, el trabajo principal de este documento trata sobre el problema de la normalidad en los Productos Caja numerables: no se sabe si el producto caja numerable de la sucesión convergente es normal en ZFC o no normal en algún modelo de ZFC. Realizamos un estudio amplio de este tema abriendo una nueva dirección de investigación que involucra la propiedad topológica de normalidad monótona.

Palabras Clave: Normalidad monótona, Principio Δ, Ψ-espacios, Discretamente Generado, Paracompacidad

Abstract

This document collect the work done by the author in his PhD program.

In the theory of Discretely Generated Spaces are solved the Problems 3.19 and 3.3 in [41]. Problem 3.19: Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines when $\xi \in \beta\omega \setminus \omega$? We answer negatively. Problem 3.3: Is any box product of first countable spaces discretely generated? We answer positively by assuming the factors are regular.

Later, we proved a result about one of the important objects in Set Theory, the so called $\Psi(\mathcal{A})$-spaces defined from a collection $\mathcal{A}$ of subsets of natural numbers. It is proved that there are $\mathfrak{c}$-many $\Psi(\mathcal{A})$-spaces pairwise non-homeomorphic whose underlying collections are pairwise homeomorphic view as subspaces of the Cantor set.

Finally, the main work in this document is about the problem of normality in countable Box Products: it is unknown if the countable box product of the convergent sequence is normal in ZFC or if it is non-normal in some model of ZFC. We give a wide study of the topic and open a new line of research involving the topological property of monotone normality.

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Introduction

Topology is one of the fields that compound Mathematics. It studies properties of spaces that are invariant under continuous deformations. Although topology grew naturally out of calculus and analysis, first topological works *per se* date back to the early 1900's, making Topology a relatively new branch of Mathematics. The following are some subfields of Topology: General Topology (or Point-Set Topology), Combinatorial Topology, Algebraic Topology and Differential Topology. General Topology commonly considers local properties of spaces, and it is closely related to Analysis. The initial concepts of General Topology are the concepts of a *topological space* and *continuous mapping*, introduced by F. Hausdorff in 1914. Although, other related concepts, such as *limit point* and notions of infinity, were introduced by George Cantor (-the father of Set Theory-) in the 1870's. Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used to define nearly all mathematical objects. The reader will find that the content in this document moves from General Topology to Set theory and vice versa; but this thesis falls mostly into the scope of General Topology.

With these concepts in hand, Hausdorff proceeded to develop the elementary theory of topological spaces (see [17]) in a form that has remained almost unchanged up to the present. He presented the concepts of open and closed sets, interior and boundary points, accumulation points, relative topologies, denseness and nowhere-denseness, and connectedness just as we understand them today.

The fast growth of General Topology led to the challenge of defining objects in optimal ways. Sometimes it would seem easy to define an object from intuition, but intuitive definitions are not satisfactory all the time. For example, in terms of topological spaces, it is more intuitive to define the box product topology rather than the Tychonoff product topology (see Chapter 1). However, the Tychonoff product topology is preferred by topologists given that it preserves important topological properties such as compactness and connectedness, and this is not the case of the infinite box product topology (see Theorem 16). Much of this is due to the fact that the Tychonoff product is the correct definition in the sense of category theory as a limit of a discrete diagram. Given a family of topological spaces $\{X_i : i \in I\}$, the *box product* $\square_i X_i$, is the space with underlying set $\prod_i X_i$ and basis all sets of the form $\prod_i U_i$, where each U_i is open in X_i.

Another challenge of a growing field is the process of formulation and solution of problems. Easy problems usually carry quick solutions. However, not every simple formulation of a question has a trivial answer. One of the easiest formulations within Mathematics is: given an object with property $\mathcal{P}$, do “products" of objects with property $\mathcal{P}$ have property $\mathcal{P}$? That is, if property $\mathcal{P}$ is preserved under (finite or infinite) products. In this case, “products" means Cartesian-like products inherited with a particular structure. Thus, the object could be a topological space, a graph, a group, a manifold, etc.; and the property $\mathcal{P}$ could be normality, conectedness, commutativity, dimension, respectively.

Large portion of this thesis is devoted to the study of preservation of normality by box-products. It is said that a topological space is *normal* if every pair of disjoint closed subsets can be separated by open sets. Even though the notion of normality is on surface quite simple, problems involving normality are usually quite complex. It is known that compact Hausdorff spaces and metrizable spaces are normal. Hence, Tychonoff products of compact Hausdorff spaces and countable Tychonoff products of metrizable spaces are normal. Also, it is known that there are non-normal uncountable Tychonoff products of metrizable spaces (e.g. ω^{ω_1} is not normal). On the other hand, a result Lawrence, Theorem 63, shows that uncountable box products of infinite compact metrizable spaces are not normal. Nonetheless, it is unknown if every countable box product of compact metrizable spaces is normal. This is *the box products' problem* and is treated in Chapter 4.

Interest in box products initially came from the fact that they are a natural construction and a useful source of counterexamples. Perhaps the most important application of box products is the famous first example in ZFC of a Dowker space constructed by M. E. Rudin (see [39]). Such space is a subspace of a countable box product. As the work on box products proceeded, a third source of interest appeared: questions about box products with countably many factors are closely related to the combinatorics of one of the most basic and natural set theoretic objects: The Baire space ω^ω.

The first version of the problem of box products was attributed to Tietze in the 1940's: Is $\square\mathbb{R}^\omega$ normal? A second version was Stone's question from the 1950's: Is $\square_{n\in\omega} X_n$ normal if each X_n is separable metric? The fact that these questions were initially asked for a box product with countably many factors is probably attributable to the reasonableness of a conjecture of "yes", but half a century later it turned out that this restriction was necessary. The following is a brief summary of results concerning countable box products, but more details can be found in [38], [10], [24], [26], [33]:

- If even one of the factors is too far away from such variations of compactness as local compactness or σ-compactness, then the countable box product is not normal.
- Many countable box products, in which the factors are too big (in the sense of cardinality, weight or character), are not normal.
- In fact, it's consistent that even one factor which is too big gives a non-normal countable box product.
- On the other hand, if all factors are compact first countable, then it's consistent that the countable box product is not only normal but also paracompact.
- As the set-theoretic hypotheses are weaken, the requires properties (e.g., compact metrizable instead of compact first countable) of the factors are strengthened.

In Chapter 1 we introduced basic definitions, state basic facts on Topology and define the box topology. Besides *the box products' problem*, in

Chapter 2 we study the theory of discretely generated spaces introduced by Dow, Tkachuk, Tkachenko and Wilson [12], and we solve two problems from [41] related to box products. Problem 3.19: Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines when $\xi \in \beta\omega \setminus \omega$? We answer it in the negative. Problem 3.3: Is any box product of first countable spaces discretely generated? We answer this positively assuming the factors are regular.

Another particular object in General Topology is the topological space $\Psi(\mathcal{A})$ associated to an almost disjoint family $\mathcal{A}$ on ω (a family of subsets of ω in which every pair of elements have finite intersection). These spaces are useful objects since they contain combinatorial properties and are commonly used as counterexamples. Given an almost disjoint family $\mathcal{A}$, the space $\Psi(\mathcal{A})$ has as underlying set $\mathcal{A} \cup \omega$ and the topology is defined as: ω is a discrete set, and neighborhoods around a point $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$, where $F \subseteq \omega$ is finite. In Chapter 3 we study the spaces $\Psi(\mathcal{A})$ and prove that there can be $\mathfrak{c}$-many distinct pairwise homeomorphic almost disjoint families $\mathcal{A}$ on ω (viewed as subsets of the Cantor set) whose associated spaces $\Psi(\mathcal{A})$ are pairwise non-homeomorphic.

Finally, in Chapter 4 we study the 'nabla products', which quotients of box products, and we focus on the work of Roitman in [31], [32], [33] and [34]. Nabla products are important in this topic due to a result of Kunen [22]: if $\{X_n : n \in \omega\}$ be a family of compact spaces, $\square_n X_n$ is paracompact if and only if $\nabla_n X_n$ is paracompact. Althouth in ZFC it is unknown whether $\square(\omega+1)^\omega$ (or $\nabla(\omega+1)^\omega$) is normal or not, it is known that it is consistently paracompact (e.g. assuming CH, $\mathfrak{b} = \mathfrak{d}$, or $\mathfrak{d} = \mathfrak{c}$), hence normal. Roitman extracted a beautiful combinatorial principle called Δ, which is implied by the set-theoretic axioms mentioned before. The principle Δ implies that $\nabla(\omega+1)^\omega$ is paracompact. We develop a new direction on nabla spaces by showing that Δ holds if and only if the space $\nabla(\omega+1)^\omega$ is monotonically normal (see Theorem 79). Monotone normality and paracompactness are not related. We list a brief summary of results obtained in this line of research:

- We uncover Δ in terms of halvability, (see Lemma 78).
- Example 80 points out a gap in Roitman's argument '∇^* paracompact implies $\nabla(\omega+1)^\omega$ paracompact'.

- In the class of nabla products of metrizable factors, monotone normality implies paracompactness (see Theorem 82). We do this by showing that stationary subsets of regular uncountable cardinals can not be embedded as closed copies into any nabla product in this class,

- Proposition 86 and Lemma 87 show that some nabla products of non-metrizable factors accept copies of stationary subsets of regular uncountable cardinals.

- We define Δ-like principles for nabla products of metrizable, ordinal and one-point compactification factors and state corresponding versions of Theorem 79 for these classes.

- We give a counterexample of a result of Roitman, Theorem 105.2. This result is derived from Theorem 105.3. We mention a gap in the proof of Theorem 105.3.

- Corollary 107 shows that the countable nabla product of the one-point compactification of a discrete set of size ω_1 and the countable nabla product of $\omega_1 + 1$ are consistently non-hereditarily normal, answering a Roitman's question.

- We show that an instance of Δ is true in ZFC and it is related to the notion of tangle-free filters introduced by Gartside (see Proposition 116).

- We state a Δ-like principle for basic paracompactness of a subspace of $\nabla(\omega + 1)^\omega$ (see Proposition 119).

- Variations of monotone normality such as halvability and utterly ultra normality are related in countable nabla products of metrizable, ordinal and one-point compactification factors (see Corollary 126).

- Section 4.9 lists some open questions for the author's interest.

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Chapter 1

Infinitary Combinatorics and Topology

In this chapter we introduce fundamental tools in Topology and Set Theory such as basic notions, some notation and well known results. Most of the results stated in this chapter do not include a proof, instead we provide references so the reader can verify any stated result. Most of our notation is standard in the area and we will follow Kunen [23] and Engelking [13]. The set of natural numbers is denoted by ω and we use the symbol $\mathbb{R}$ for the real line. The binary Cantor set is denoted by 2^ω and the irrational numbers by ω^ω. Denote by $[\omega]^\omega$ the set of infinite subsets of ω.

1.1 Topology

Most of the results presented in this section can be found in [13] and [24].

Definition 1. A *topological space* is a pair (X, τ), where X is a set and τ is a collection of subsets of X (called *topology* for X) satisfying the following properties:

- $\emptyset, X \in \tau$,
- if $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$, and
- if $\mathcal{U} \subseteq \tau$ is finite, then $\bigcap \mathcal{U} \in \tau$.

Elements of τ are called *open sets* and the complement of an open set is a *closed set*. A subset A of a topological space X is itself a topological space, where a set $U \subseteq A$ is open if and only if there is an open set V of X such that $U = A \cap V$.

The examples that we are going to use in this document are mostly subsets of the *real numbers* $\mathbb{R}$: the topology of $\mathbb{R}$ is the generated by the intervals, 2^ω and ω^ω have the product topology, that coincides with the topology generated by sets of the form $\langle t \rangle = \{f : t \subseteq f\}$ (the cone of t), where t is a finite partial function. These topological spaces are also examples of metric spaces.

Definition 2. A *metric space* is an ordered pair (X, ρ), where X is a set and $\rho : X^2 \to [0, \infty)$ is a function with the following properties:

- (reflexivity) for all $x, y \in X$, $\rho(x, y) = 0$ if and only if $x = y$,
- (symmetry) for all $x, y \in X$, $\rho(x, y) = \rho(y, x)$, and
- (triangle inequality) for all $x, y, z \in X$, $\rho(x, y) + \rho(y, z) \geqslant \rho(x, z)$.

Given $x \in X$ and $\epsilon > 0$, the set $B(x, \epsilon) = \{y \in X : \rho(x, y) < \epsilon\}$ is called the *ball* with center x and radius ϵ. In a metric space, a set U is open if for all $x \in U$ there is $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. It follows that the collection of open sets of a metric space X forms a topology for X. The euclidean metric in $\mathbb{R}$, defined as $\rho(x, y) = |x - y|$, generates the topology of the intervals, and the product topology is generated by the metric $\rho(f, g) = 2^{-\Delta(f,g)}$, where $\Delta(f, g) = \min\{k \in \omega : f(n) \neq g(n)\}$, if $f \neq g$ and $\Delta(f, g) = \infty$ if $f = g$. Main concepts we will study are the different notions of compactness.

Definition 3. A space X is *(countably) compact* if every (countable) open cover of X, i.e. a collection of open sets whose union is X, has a finite subcover.

There are many weaker versions of compactness, we introduce the ones relevant for us. Let $\mathcal{C}$ be a collection of subsets of a topological space X. We define the *star of* x with respect to $\mathcal{C}$ as $St_{\mathcal{C}}(x) = \{C \in \mathcal{C} : x \in C\}$. We say that $\mathcal{C}$ is *locally finite* if for any point $x \in X$, there is an open set U containing x such that $\{C \in \mathcal{C} : C \cap U \neq \emptyset\}$ is finite, and $\mathcal{C}$ is *point-finite* if for any point $x \in X$, $St_{\mathcal{C}}(x)$ is finite. For two collections $\mathcal{C}$ and $\mathcal{C}'$ of subsets of X, we say that $\mathcal{C}'$ *refines* $\mathcal{C}$ if any element in $\mathcal{C}'$ is contained in some element of $\mathcal{C}$ (in this case, we say that $\mathcal{C}'$ is a *refinement* of $\mathcal{C}$).

Definition 4. A space X is (*countably*) *para*[*meta*]{*ortho*}*compact* if every (countable) open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ covering X which is locally finite [point-finite] {for any $x \in X$, $\bigcap St_{\mathcal{V}}(x)$ is open}.

Definition 5. A space X is said to be *Lindelöf* if every open cover of X has a countable subcover.

Besides compactness, other interesting topological properties are the so called Countability Axioms and Separations Axioms T_i. A space X is *first countable* if every point has a countable local basis, and X is *second countable* if it has a countable base. A space X is T_0 if for any $x, y \in X$ there is an open set U such that $x \in U$ and $y \notin U$ or vice versa; X is T_1 if every singleton is closed in X; X is T_2 (*Hausdorff*) if for any $x, y \in X$ there there are open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$; X is T_3 (*regular*) if for any $x \in X$ and a closed set $F \subseteq X$ with $x \notin F$ there are open sets U, V such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$; X is $T_{3\frac{1}{2}}$ (*Tychonoff or completely regular*) if it is T_1 and for $x \in X$ and closed set $F \subseteq X$ with $x \notin F$, they are separated by a continuous function, that is, there is $f : X \to [0, 1]$ continuous such that $f(x) = 0$ and $f[F] \subseteq \{1\}$; X is T_4 (*normal*) if it is T_1 and for any disjoint closed sets $F, G \subseteq X$ there are open sets U, V such that $F \subseteq U, G \subseteq V$ and $U \cap V = \emptyset$; X is T_5 if it is hereditarily normal; X is T_6 (*perfectly normal*) if it is T_4 and every closed set is a G_δ-set (see the Borel Hierarchy in the next section).

(Countable) compactness implies (countable) paracompactness implies (countable) metacompactness implies (countable) orthocompactness. Any of these properties imply its 'countable' version. Also, every property T_j implies T_i, for $i < j$. Moreover (see [13]), a paracompact Hausdorff space is normal; a Lindelöf space is paracompact; (Nagata-Smirnov Theorem) a space is metrizable if and only if it is regular and has a σ-locally finite base.

1.2 Descriptive Topology

We will state one of the most important results in the theory of complete metric spaces, the *Baire Category Theorem*. For this, we introduce some important notions.

A topological space is *Polish* if it is separable and completely metrizable. A space is *separable* if it has a countable dense subset (a set intersecting every non-empty open set). A space is *completely metrizable* if it admits a metric inducing its topology such that every Cauchy sequence has a limit point in the space.

Definition 6. Let X be a topological space. A subset $N \subseteq X$ is *nowhere dense* if for every non-empty open set U of X, there is a non-empty open set V in X such that $V \subseteq U$ and $V \cap N = \emptyset$. A subset $M \subseteq X$ is *meager* if there is a countable collection $\{N_n : n \in \omega\}$ of nowhere dense sets such that $M \subseteq \bigcup_{n\in\omega} N_n$.

Theorem 7 (Baire Category Theorem)**.** *If X is a complete metric space or a compact topological space, then X is not meager.*

We will be interested in studying the class of definable sets of the reals. In particular, we are interested in the structure of the Borel sets.

Definition 8. Given a topological space X, the class of the *Borel sets* of X is the minimal σ-algebra (a collection of sets closed under complements and countable unions) containing the open sets of X. We will denote this class by $Borel(X)$.

The class $Borel(X)$ can be analyzed in a transfinite hierarchy of length ω_1, this transfinite hierarchy is called the *Borel Hierarchy*: In the lowest level we have the open sets and the closed sets, then we have the G_δ-sets (countable intersections of open sets) and the F_σ-sets (countable unions of closed sets), then the $G_{\delta\sigma}$-sets (countable unions of G_δ-sets) and the $F_{\sigma\delta}$-sets (countable intersection of F_σ-sets), and so on. In general, these classes are denoted by $\sum_\alpha^0$, $\prod_\alpha^0$, where $\sum_1^0$ is the class of open sets, $\prod_1^0$ is the class of closed sets, and if α is such that $1 < \alpha < \omega_1$, $\sum_\alpha^0$ is the collection of countable unions of elements in $\bigcup_{\beta<\alpha} \prod_\beta^0$ and $\prod_\alpha^0$ is the collection of complements of $\sum_\alpha^0$. Therefore, $\sum_2^0 = \{F_\sigma\text{-sets}\}$, $\prod_2^0 = \{G_\delta\text{-sets}\}$, $\sum_3^0 = \{G_{\delta\sigma}\text{-sets}\}$ and $\prod_3^0 = \{F_{\sigma\delta}\text{-sets}\}$. It is easy to see that $Borel(X) = \bigcup_{\alpha<\omega_1} \sum_\alpha^0 = \bigcup_{\alpha<\omega_1} \prod_\alpha^0$. Also, it is possible to show that these hierarchy classes are different from each other (see [20]).

Another important combinatorial tool is the concept of a tree. There are different versions for the definition of a tree. We will use the following.

Definition 9. A *tree* on a set X is a subset $T \subseteq X^{<\omega}$ closed under initial segments (i.e. if $t \in T$ and $s \subseteq t$ then $s \in T$).

The elements of a tree are called *nodes*. The *stem* of a tree T, denoted by $stem(T)$ is the $\subseteq$-maximal node that is compatible with every node of T. A node $t \in T$ is a *splitting node* if there are different i, j such that both $t^\frown i$ and $t^\frown j$ are elements of T. A *pruned tree* is a tree without $\subseteq$-minimal nodes. A *branch of T* is an element $x \in X^\omega$ such that, for every $n \in \omega$, $x \restriction n \in T$. We will denote the set of all branches of T by $[T]$. For any set X, there is a natural metric on X^ω: if $f, g \in X^\omega$ with $f \neq g$, then $d(f, g) = \frac{1}{2^{\Delta(f,g)}}$, where $\Delta(f, g) = \min\{n \in \omega : f(n) \neq g(n)\}$. The following proposition is straightforward from the definition:

Proposition 10. *For every pruned tree T on X, $[T]$ is a non empty closed set in X^ω. Moreover, if $C \subseteq X^\omega$ is a non-empty closed set, there is a pruned tree T such that $C = [T]$.*

Definition 11. A collection $\mathcal{F}$ of subsets of a set S is called *(ultra) filter* if:

- $\emptyset \notin \mathcal{F}$,
- $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$,
- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and
- $(A \subseteq S$ implies $A \in \mathcal{F}$ or $S \setminus A \in \mathcal{F})$.

1.3 The Cohen Model, $\mathfrak{b}$ and $\mathfrak{d}$

In 1963, Paul Cohen (see [6] and [7]) proved the independence of the continuum hypothesis CH and the axiom of choice AC. He did that using a technique called forcing, which consist of adding a generic filter to a model of set theory. The reader can consult [23] and [16] for an introduction to this topic.

A *forcing* (which is also known as a separative partial order) is a partial order $(P, \leqslant)$ such that for every $p \in P$ there are $r, s \leqslant p$ such that there is no $q \in P$ such that q is smaller than both r and s. The *Cohen forcing* is the set of finite functions from ω to ω, denoted by $\omega^{<\omega}$, ordered by inclusion, 'larger functions are stronger'. Any countable forcing is equivalent to Cohen's forcing. The *Cohen model* is obtained by forcing with the finite support iteration of length ω_2 of the Cohen forcing.

Definition 12. Let M, N be models of set theory such that $M \subseteq N$. Then $x \in N \cap \omega^\omega$ is a *Cohen real* over M if x is not in any meager set coded in M.

Denote by $\mathfrak{c}$ the cardinality $|\mathbb{R}|$. Recall the *bounding number* $\mathfrak{b}$ and *dominating number* $\mathfrak{d}$ satisfying $\mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$: For $f, g \in \omega^\omega$, define $f \leqslant^* g$ iff $\exists n \in \omega\ \forall m \geqslant n\ (f(m) \leqslant g(m))$. A family $\mathcal{F} \subseteq \omega^\omega$ is $\leqslant^*$*-bounded* if $\exists g \in \omega^\omega\ \forall f \in \mathcal{F}\ (f \leqslant^* g)$. A family $\mathcal{F} \subseteq \omega^\omega$ is $\leqslant^*$*-dominant* if $\forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ (g \leqslant^* f)$.

- $\mathfrak{b} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ is not $\leqslant^*$-bounded$\}$
- $\mathfrak{d} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega$ is $\leqslant^*$-dominant$\}$

Proposition 13. *Let M be a model of set theory and N be the model of set theory obtained by forcing with the Cohen forcing over M. If $x \in N \cap \omega^\omega$ is a Cohen real over M, then x is $\leqslant^*$-unbounded on $M \cap \omega^\omega$.*

If we iterated ω_2 times the Cohen forcing over a model of ZFC in which CH holds, we will have the following in the forcing extension: $\mathfrak{b} = \omega_1$, $\mathfrak{d} = \mathfrak{c} = \omega_2$.

1.4 The Box Topology

In Chapter 2 and Chapter 4 we will be studying box products. In the following, we will assume that any box product consist of infinite factors. We state some facts about this class of spaces.

Definition 14. Let $\{X_i : i \in I\}$ be a family of topological spaces. A *box* is a set $\prod_i U_i$, where each U_i is open in X_i. The *box product*, $\Box_i X_i$, is the space with underlying set $\prod_i X_i$ and basis all boxes. For every $j \in I$, denote by $\pi_i : \Box_i X_i \to X_j$ the canonical projection defined as $\pi_j(x) = x(j)$, $x \in \Box_i X_i$.

Proposition 15. *Suppose $\{X_i : i \in I\}$ is a family of topological spaces.*

- *For every $j \in I$, π_j is continuous and open.*
- *$\prod_i F_i$ is closed (open) in $\Box_i X_i$ if and only if for every $j \in I$, F_j is closed (open) in X_j.*
- *$\Box_i X_i$ is Hausdorff (regular, completely regular) if and only if for every $j \in I$, X_j is Hausdorff (respectively, regular, completely regular)*

Theorem 16. *Infinite Box products are not any of the following: (i) locally compact, (ii) separable, (iii) connected or locally connected, (iv) first countable, (v) perfect (closed subspaces are G_δ).*

Definition 17. A space X is a *P-space* if every G_δ-set is open.

Theorem 18 (Katětov)**.** *If $X \times Y$ is hereditarily normal, then either X is a P-space or Y is perfect.*

Theorem 19. *If a box product $\Box_i X_i$ is hereditary normal, then it is a P-space.*

Proof. Fix $j \in I$ and set $\Box' = \Box_{i \neq j} X_i$. Then associativity of box products allow $\Box_i X_i = X_j \times \Box'$. According to Theorem 16 (v), $\Box'$ has a closed set which is not G_δ. Now, apply Katětov's Theorem and conclude that X_j is a P-space, for every $j \in I$. It is clear that box products of P-spaces are P-spaces. ■

Corollary 20. *If X_i is compact or first countable, for $i \in I$ where I is infinite, then $\Box_i X_i$ has a non-normal subspace.*

Chapter 2

Discretely Generated Spaces

2.1 Introduction

In this chapter we study the class of the discretely generated spaces. The results are contained in [3]. We say that D is a *discrete* subset of a space X if for every $d \in D$ there is a set U open in X such that $U \cap D = \{d\}$. A topological space X is *discretely generated* if for any $A \subseteq X$ and $x \in \overline{A}$ there exists a discrete set $D \subseteq A$ such that $x \in \overline{D}$. Many results concerning discretely generated spaces appear in [12] and [41]. We solved the Problems 3.19 and 3.3 in [41]. Problem 3.19: Does the space $\{\xi\} \cup \omega$ embed into a box product of real lines when $\xi \in \beta\omega \setminus \omega$? Corollary 29 answers negatively. Problem 3.3: Is any box product of first countable spaces discretely generated? Theorem 32 answers positively by assuming the factors are regular.

For this chapter, denote by $\mathcal{V}$ the countable regular maximal space due to Eric van Douwen [11]. It was shown in [12] that $\mathcal{V}$ is not discretely generated. Since $\square\mathbb{R}^\kappa$ is discretely generated and this property is hereditary, there is no embedding of $\mathcal{V}$ to $\square\mathbb{R}^\kappa$. The authors of [41] asked whether if there were more countable regular spaces (for example, $\{\xi\} \cup \omega$ which is, in fact, discretely generated) that do not embed into a box product of real lines, this is the motivation of Problem 3.19 in [41]. Compare with their Example 2.10, part b).

Tkachuk and Wilson showed in [41] that if X_t is a monotonically normal space (see Chapter 4 for this concept), then the box product $\square_{t\in T}X_t$ is discretely generated. Hence, the spaces $\square\mathbb{R}^\kappa$, $\square(\omega+1)^\kappa$ and $\square(\{\xi\}\cup\omega)^\kappa$ are discretely generated, for any cardinal κ. There is no relation between first countability and monotone normality. For example, $\{\xi\}\cup\omega$ is a monotonically normal non-first countable space. On the other hand, if $\mathcal{A}$ is an AD family whose $\Psi(\mathcal{A})$ is not normal, then $\Psi(\mathcal{A})$ is a regular first countable non-monotonically normal space. However, the space $\square\Psi(\mathcal{A})^\kappa$ is discretely generated, for any κ, by our result.

We prove Corollary 29 in Section 2.3 and Theorem 32 in Section 2.4. In Section 2.2 we state some notations and useful tools.

2.2 Notation and Basic Facts

We use standard terminology and follow Engelking [13]. All spaces we consider are asummed to be Hausdorff. Let X be a set, $A\subseteq X^\kappa$, κ a cardinal, $S\subseteq\kappa$ and $a,b\in X^\kappa$. We denote the *domain of* $a\in X^\kappa$ with respect to b by $dom_b(a)=\{\alpha\in\kappa : a(\alpha)\neq b(\alpha)\}$. The *restriction of* a *to* S is the element $a\restriction S\in X^S$ defined as $(a\restriction S)(s)=a(s)$, as well as $A_{S,b}=\{a\in A : dom_b(a)=S\}$ and $A\restriction S=\{a\restriction S\in X^S : a\in A\}$. For every $x\in\square_{t\in T}X_t$ and $A\subseteq\square_{t\in T}X_t$, define $A_{\infty,x}=\{a\in A : |dom_x(a)|=\omega\}$ and $A_{<\infty,x}=\{a\in A : |dom_x(a)|<\omega\}=\bigcup_{F\in[T]^{<\omega}}A_{F,x}$.

We will focus mainly on countable box products. We denote by c_ω the constant ω function in $\square(\omega+1)^\omega$. When we talk about the domain in $\square(\omega+1)^\omega$ with respect to c_ω, we use $dom(a)$ instead of $dom_{c_\omega}(a)$ and A_S instead of A_{S,c_ω}.

Let $\beta\omega$ denote the Stone-Čech compactification of ω. If $\xi\in\beta\omega\setminus\omega$ is an ultrafilter on ω, then $\{\xi\}\cup\omega$ inherits the subspace topology from $\beta\omega$. Explicitly, every element in ω is isolated and neighborhoods around ξ are $\{\xi\}\cup U$, where $U\in\xi$.

Lemma 21. *Suppose that X_n is first countable, $n \in \omega$, $x \in \Box_n X_n$ and $A \subseteq \Box_n X_n$ is such that $|A_{\infty,x}| < \mathfrak{b}$. Then $x \notin \overline{A_{\infty,x}}$.*

Proof. Since x has a local basis $\{\Box_n B^n_k : k \in \omega\}$ indexed by ω^ω, for every $a \in A_{\infty,x}$ there is $f_a \in \omega^\omega$ such that $a \notin \Box_n B^n_{f(n)}$. Hence, there exists $h \in \omega^\omega$ that $\leqslant^*$-dominates every f_a. Now is immediate that $\Box_n B^n_{h(n)} \cap A_{\infty,x} = \emptyset$. ∎

Denote the neighborhoods $\Box_n B^n_{f(n)}$ of x by $N(x, f)$. Lemma 21 will help us to isolate a set of those elements with infinity domain as long as the size of the set is less than $\mathfrak{b}$. We can also isolate sets consisting of elements with finite domain under certain conditions which we will establish in the next lemma.

Lemma 22. *Suppose that X_n is first countable, $n \in \omega$, $x \in \Box_n X_n$ and $A \subseteq \Box_n X_n$ is such that $|A_{F,x}| < \omega$, for every $F \in [\omega]^{<\omega}$. Then $x \notin \overline{A_{<\infty,x}}$.*

Proof. Recall that $A_{<\infty,x} = \bigcup_{F \in [\omega]^{<\omega}} A_{F,x}$. For every $n \in \omega$, $\{F \in [\omega]^{<\omega} : F \subseteq n\}$ is finite and so is $\bigcup_{F \subseteq n} A_F$. Hence, for every $n \in \omega$ choose $h(n) \in \omega$ so $B^n_{h(n)} \cap \pi_n[\bigcup_{F \subseteq n} A_F] = \emptyset$, where $\pi_k : \Box_n X_n \to X_k$ is the canonical projection. Then $h \in \omega^\omega$ satisfies $\Box_n B^n_{h(n)} \cap (\bigcup_{F \in [\omega]^{<\omega}} A_{F,x}) = \emptyset$. ∎

Lemma 21 and Lemma 22 try to "ward off" from $x \in \Box_n X_n$ certain type of elements in a set of small size. However, the role of the domain of the elements is important even for small sets. The next example provides a non-trivial countable set whose elements have finite domain and it has a limit point in its closure. We mean "trivial" if a limit point x belongs to $\overline{A_{F,x}}$, for some $F \in [\omega]^{<\omega}$.

Example 23. There exists a countable set $A \subseteq \Box(\omega+1)^\omega$ of elements with finite support such that $c_\omega \notin \overline{A_F}$, for any $F \in [\omega]^{<\omega}$, but $c_\omega \in \overline{A}$.

Proof. For every $n \in \omega$, consider the set $A_n = \{a \in \Box(\omega+1)^\omega : a(0) = n \text{ and } dom(a) = n\}$. It is clear that every element in $A = \bigcup_{n \in \omega} A_n$ has domain equal n, for some $n \in \omega$. Moreover, $c_\omega \notin \overline{A_n}$ because $a(0) = n$, for

$a \in A_n$. Now, to see that $c_\omega \in \overline{A}$ consider a function $h \in \omega^\omega$. Let $k \in \omega$ such that $k > h(0)$. So, there is $a \in A_k$ such that $\forall i \leqslant k$, $a(i) > h(i)$ and $\forall i > k$, $a(i) = \omega$. That is, $a \in N(c_\omega, h)$. ■

Let X be a space. Given $A \subseteq X$, define A^* as the set of accumulation points of A. Observe that $A^* \subseteq \overline{A}$. We say that X has the *property* $\mathcal{P}$ at (x, A) if there are disjoint sets $B, C \subseteq A$ such that $x \in B^* \cap C^*$. Also, X has the *property* $\mathcal{P}^+$ if for every set $A \subseteq X$ there are disjoint sets $B, C \subseteq A$ such that $A^* = B^* = C^*$. It is clear that property $\mathcal{P}^+$ is stronger than property $\mathcal{P}$ and both of them are topological properties. It is worth it to mention that the space $\{\xi\} \cup \omega$ does not have the property $\mathcal{P}$ at any (ξ, U), with $U \in \xi$, otherwise it contradicts ξ being 'ultra'. This implies that if the space $\{\xi\} \cup \omega$ embeds into X via φ, then X does not have the property $\mathcal{P}$ at $(\varphi(\xi), \varphi[\omega])$. In the next lemma we will use this properties.

Lemma 24. *Any Polish space X has the property $\mathcal{P}^+$.*

Proof. Let $A \subseteq X$. Since $\overline{A}$ is a Polish space, let $D \subseteq A$ be countable dense in A, so in $\overline{A}$. Also, let $\{U_n : n \in \omega\}$ be a countable base for $\overline{A}$. Note that $\overline{A} = P \cup C$, where P is a perfect set (possibly empty) and C is countable. Consider $N = \{n \in \omega : U_n \text{ contains an accumulating point}\} = \{k_n : n \in \omega\}$. For $k_n \in N$, there are $d_n^0, d_n^1 \in D \cap (U_{k_n} \setminus \{d_j^i : j < n,\ i \in 2\})$. Let $D_i = \{d_n^i : n \in \omega\} \subseteq A$, $i \in 2$.

We claim that $A^* = D_0^* = D_1^*$: let $x \in A^*$ be an accumulating point and U any neighborhood at x. Then there is $n \in \omega$ such that $x \in U_{k_n}$. Hence, $d_n^0, d_n^1 \in U$, as claimed. ■

Proposition 25. *Suppose $A \subseteq \square(\omega+1)^\omega$ and $c_\omega \in A^*_{<\infty, c_\omega}$. Then, $\square(\omega+1)^\omega$ has property $\mathcal{P}$ at (c_ω, A).*

Proof. For every $F \in [\omega]^{<\omega}$, $A_{F,x}$ is homemorphic to a subspace of $(\omega+1)^n$. The space $(\omega+1)^n$ is Polish and has property $\mathcal{P}^+$. Hence, there are disjoint sets $B_F, C_F \subseteq A_{F,c_\omega}$ so that $A^*_{F,c_\omega} = B^*_F = C^*_F$. The sets $B = \bigcup_{F \in [\omega]^{<\omega}} B_F$ and $C = \bigcup_{F \in [\omega]^{<\omega}} C_F$ are disjoint since for $F \neq F'$, we have $A_{F,c_\omega} \cap A_{F',c_\omega} = \emptyset$.

We claim that $c_\omega \in B^* \cap C^*$: let $N(c_\omega, h)$ be a neighborhood of c_ω, for some $h \in \omega^\omega$. Since we are assuming $c_\omega \in A^*_{<\infty, c_\omega}$, there is $F \in [\omega]^{<\omega}$ such that $N(c_\omega, h) \cap A_{F,c_\omega}$ is infinite, otherwise we would contradict Lemma 22. The set $N(c_\omega, h) \cap A_{F,c_\omega}$ being infinite and $(\omega+1)^F$ compact and 0-dimensional ($N(c_\omega, h)$ is clopen), $N(c_\omega, h) \cap A_{F,c_\omega}$ has an accumulating point x in $(\omega + 1)^F \cap N(c_\omega, h)$. Necessarily there are $b \in B_F \subseteq B$ and $c \in C_F \subseteq C$ such that $b, c \in N(c_\omega, h)$. ■

2.3 No Embeddings from $\{\xi\} \cup \omega$ into Box Products

We now prove the main result of this section, Theorem 27, which solves Problem 3.19 in [41]. In what follows, we relate in a "natural way" both spaces $\square(\omega + 1)^\omega$ and $\square_n X_n$, where X_n is first countable and regular.

Suppose $\{X_n : n \in \omega\}$ is a family of regular first countable spaces and fix $p \in \square_n X_n$. For each $n \in \omega$, let $\{W^n_k : k \in \omega\}$ be a countable base for $p(n)$ such that $W^n_0 = X_n$ and $\overline{W^n_{k+1}} \subseteq W^n_k$, $k \in \omega$. For every $n \in \omega$, define $f_n : X_n \to \omega + 1$ as

$$f_n(x) = \begin{cases} k, & if\ x \in W^n_k \setminus W^n_{k+1}; \\ \omega, & if\ x = p(n). \end{cases}$$

Let $\Phi_p : \square_n X_n \to \square(\omega + 1)^\omega$ defined as $\Phi_p(x) = \langle f_n(x(n)) : n \in \omega \rangle$. Observe that Φ_p is well defined, $\Phi_p(p) = c_\omega$ and Φ_p defines the equivalence relation on $\square_n X_n$ as $a \sim_p b$ iff $\Phi_p(a) = \Phi_p(b)$. The next observations about Φ are immediate, but very useful.

Remark 26. Given $A \subseteq \square_n X_n$, let $R_A \subseteq A$ be a maximal set of elements which are $\sim_p$-representative. Then we have the following properties:

1. $p \in \overline{A}$ if and only if $p \in \overline{R_A}$,
2. $p \in \overline{R_A}$ if and only if $c_\omega \in \overline{\Phi[R_A]}$,
3. the restriction $\Phi_p \upharpoonright R_A$ is a one-to-one mapping, and

4. for every $h \in \omega^\omega$, $\Phi_p[\square_{n\in\omega} W^n_{h(n)}] = N(c_\omega, h)$ and $\square_{n\in\omega} W^n_{h(n)} = \Phi_p^{-1}[N(c_\omega, h)]$.

Theorem 27. *Suppose X_n is regular first countable, $n \in \omega$. Then, there is no bijective continuous function $\varphi : \{\xi\} \cup \omega \to \square_n X_n$,*

Proof. Suppose there is such φ. Let $A = \varphi[\omega] \subseteq \square_n X_n$, $p = \varphi(\xi)$ and consider the partition induced by p on $\square_n X_n$. Since A is countable, by Lemma 21 and Lemma 22, we may assume that for every $a \in A$ the $dom_p(a)$ is finite. Let $R_A \subseteq A$ be a maximal set of $\sim_p$-representative elements. Observe that $\xi \in \overline{\varphi^{-1}[R_A]}$ and by Remark 26, $c_\omega \in \overline{\Phi_p[R_A]}$.

Now, apply Proposition 25 to find $B_0, C_0 \subseteq \Phi_p[R_A]$ such that $B_0 \cap C_0 = \emptyset$ and $c_\omega \in \overline{B_0} \cap \overline{C_0}$. Let $B = \Phi_p^{-1}[B_0] \cap R_A$ and $C = \Phi_p^{-1}[C_0] \cap R_A$. It is clear that $B \cap C = \emptyset$. Remark 26 applies again, so $p \in \overline{B} \cap \overline{C}$.

Hence, property $\mathcal{P}$ holds at (p, A). This contradicts the fact that $\{\xi\} \cup \omega$ does not have $\mathcal{P}$ at any pair. That is, the subsets $U = \varphi^{-1}[B]$ and $V = \varphi^{-1}[C]$ of $\{\xi\} \cup \omega$ satisfy $U \cap V = \emptyset$ and $\xi \in \overline{U} \cap \overline{V}$, which contradicts the property of being 'ultra' in ultrafilter. ∎

Corollary 28. *Suppose κ is a cardinal and X_t is first countable, $t \in \kappa$. Then, there is no embedding from $\{\xi\} \cup \omega$ to $\square_t X_t$.*

Proof. Suppose again that φ is an embedding. Let $p = \varphi(\xi)$ and $A = \varphi[\omega]$. Split $A = A_{<\infty} \cup A_\infty$, where $A_{<\infty} = \{a \in A : |dom_p(a)| < \omega\}$ and $A_\infty = \{a \in A : |dom_p(a)| \geqslant \omega\}$.

For every $a \in A_\infty$, consider a fixed countable set $S_a \subseteq dom_p(a)$. Since A_∞ is countable, so is $S = \bigcup_{a\in A_\infty} S_a$. By Lemma 21, there is a function $h \in \omega^S$ such that the function on ω^κ defined as $\tilde{h}(t) = h(t)$, if $t \in S$, $\tilde{h}(t) = 0$, if $t \notin S$, satisfies $N(p, h) \cap A_\infty = \emptyset$.

Thus, $p \in \overline{A_{<\infty}}$. Observe that, $M = \bigcup_{a\in A_{<\infty}} dom_p(a)$ is countable, and hence $\{p\} \cup A_{<\infty}$ is homeomorphic to a subspace of $\square_{t\in S} X_t$. Now, apply Theorem 27 to contradict φ is an embedding. ∎

Corollary 29. *For any ultrafilter $\xi \in \beta\omega \setminus \omega$ and any cardinal κ, there is no embedding from $\{\xi\} \cup \omega$ into $\square\mathbb{R}^\kappa$.*

2.4 Box Products of Regular First Countable Spaces are Discretely Generated

Our main result in this section is Theorem 32 which solves Problem 3.3 in [41].

In a similar fashion as in Section 2.3 we can define the function $\Phi_p : \square_{t\in T} X_t \to \square(\omega+1)^T$ defined as $\Phi_p(x) = \langle f_t(x(t)) : t \in T\rangle$. Denote again by c_ω the constant function equal ω in every coordinate. We have $\Phi_p(p) = c_\omega$, and $a \sim_p b$ if and only if $\Phi_p(a) = \Phi_p(b)$ is an equivalence relation. We will use the following important result of Tkachuk and Wilson.

Theorem 30 ([41]). *If X_t is a monotonically normal space, for $t \in T$, then the box product $\square_{t\in T} X_t$ is discretely generated.*

Definition 31. If κ is a cardinal, we define $\leqslant^*$ on ω^κ such that for $f, g \in \omega^\kappa$, $f \leqslant^* g$ if and only if $\{\alpha \in \kappa : f(\alpha) > g(\alpha)\}$ is finite. We say that $\mathcal{D} \subseteq \omega^\kappa$ is a $\leqslant^*$-dominant family if $\forall f \in \omega^\kappa\ \exists g \in \mathcal{D}\ (f \leqslant^* g)$. Recall that

$$\mathfrak{d}(\kappa) = \min\{|\mathcal{D}| : \mathcal{D} \subseteq \omega^\kappa \text{ is a } \leqslant^* \text{-dominant family}\}.$$

If $\mathcal{D} = \{g_\alpha \in \omega^\kappa : \alpha < \mathfrak{d}(\kappa)\}$ is a $\leqslant^*$-dominant family, then $\mathcal{D}^* = \{h \in \omega^\kappa : |\{\beta \in \kappa : h(\beta) \neq g_\alpha(\beta)\}| < \omega\ \wedge\ g_\alpha \in \mathcal{D}\}$ is a $\leqslant$-dominant family, with $|\mathcal{D}^*| = \kappa \cdot \mathfrak{d}(\kappa)$ and $\{N(c_\omega, h) : h \in \mathcal{D}^*\}$ is a local basis of c_ω. By Lemma 2.1 in [8], $\mathfrak{d}(\kappa) = \kappa \cdot \mathfrak{d}(\kappa)$. Thus, enumerating $\mathcal{D}^* = \{h_\alpha \in \omega^\kappa : \alpha \in \mathfrak{d}(\kappa)\}$ we conclude that $\{N(c_\omega, h_\alpha) : \alpha < \mathfrak{d}(\kappa)\}$ is a local basis of c_ω.

Theorem 32. *Suppose $\{X_t : t \in T\}$ is a family of regular first countable spaces. Then $\square_{t\in T} X_t$ is discretely generated.*

Proof. Let $A \subseteq \square_{t\in T} X_t$ and $p \in \overline{A} \setminus A$. We may assume that A consists only of $\sim_p$-representative elements. Let $E = \Phi_p[A] \subseteq \square(\omega+1)^T$ and note that $\Phi_p(p) = c_\omega \in \overline{E}$. By Theorem 30, $\square(\omega+1)^T$ is discretely generated, and thus, there is a discrete $D \subseteq E$ such that $c_\omega \in \overline{D}$. Consider a local basis $\{N_{h_\alpha} : \alpha < \mathfrak{d}(\kappa)\}$ of c_ω, write N_g for $N(c_\omega, g)$.

Recursively construct a set $D_{\mathfrak{d}(\kappa)} = \{d_\alpha : \alpha < \mathfrak{d}(\kappa)\} \subseteq D$ so the preimage under Φ_p is discrete and generates p as follows: take $d_0 \in N_{h_0} \cap D$.

Now, suppose constructed $D_{<\alpha} = \{d_\beta : \beta < \alpha\}$. If $c_\omega \in \overline{D_{<\alpha}}$, we are done. If not, there is $e_\alpha \in \omega^\kappa$ such that $N_{e_\alpha} \cap D_{<\alpha} = \emptyset$. Since $c_\omega \in \overline{D}$, there is $d_\alpha \in N_{e_\alpha+2} \cap N_{h_\alpha} \cap D$.

The reason to add 2 to the functions e_α is to choose the elements d_α sufficiently far away from each other to assure their preimages under Φ are still enough apart from each other, that will allow us to find a discrete subset of A.

We may assume that the process ends until $c_\omega \in \overline{D_{\mathfrak{d}(\kappa)}}$. Note that $D_{\mathfrak{d}(\kappa)}$ is discrete because it is contained in D. Now, let $G = \Phi_p^{-1}[D_{\mathfrak{d}(\kappa)}] \cap A$. From the construction of $D_{\mathfrak{d}(\kappa)}$, we have the following: for every $\beta < \alpha < \mathfrak{d}(\kappa)$ there is $t \in dom_{c_\omega}(d_\beta)$ with $d_\beta(t) + 2 < d_\alpha(t)$.

We check that $G \subseteq A$ is discrete and $p \in \overline{G}$. It is clear that $p \in G$, since $\Phi_p(p) = c_\omega \in \overline{D_{\mathfrak{d}}(\kappa)}$. To see that G is discrete, fix $g \in G$. Since G is a set of $\sim_p$-representative elements, there is $\alpha < \mathfrak{d}(\kappa)$ such that $d_\alpha = \Phi_p(g)$. For every $t \in dom_p(g)$, there is a unique $k_t \in \omega$ such that $g(t) \in W^t_{k_t} \setminus W^t_{k_t+1}$. If $t \in dom_p(g)$, let $W_t = W^t_{k_t} \setminus \overline{W^t_{k_t+2}}$; otherwise let $W_t = X_t$. By regularity, for every $t \in T$, we have that $g(t) \in W^t_{k_t} \setminus W^t_{k_t+1} \subseteq W_t$ and W_t is open. Then, $W = (\square_{t\in T} W_t) \cap (\square_{t\in T} W^t_{e_\alpha(t)+2})$ is a neighborhood of g.

Now, suppose for a contradiction that $f \in W \cap G$ for some $f \neq g$. There is $\beta < \mathfrak{d}(\kappa)$ such that $d_\beta = \Phi_p(f)$. Observe that $dom_p(g) = dom_{c_\omega}(d_\alpha)$, $dom_p(f) = dom_{c_\omega}(d_\beta)$ and for every $t \in dom_p(g)$, $d_\alpha(t) = k_t$. We have two cases:

- if $\alpha < \beta$, then by the property above there is $t \in dom_{c_\omega}(d_\alpha)$ such that $d_\alpha(t) + 2 < d_\beta(t)$. That is, $f(t) \notin W^t_{k_t} \setminus \overline{W^t_{k_t+2}} = W_t$, and this contradicts $f \in W$.
- If $\beta < \alpha$, by the construction we have $N_{e_\alpha} \cap D_\alpha = \emptyset$. Since d_β is an element of D_α, there exists $t \in dom_{c_\omega}(d_\beta)$ such that $e_\alpha(t) > d_\beta(t)$. Thus, $d_\beta(t) < e_\alpha(t) + 2$. It follows that $f(t) \notin W^t_{e_\alpha(t)+2}$ contradicting again the assumption $f \in W$.

This concludes the proof. ■

Question 33. Is the box product of regular Fréchet-Urysohn spaces discretely generated?

Chapter 3

$\mathfrak{c}$-*Many Types of a* Ψ-*Space*

3.1 Introduction

In this chapter we study the well known Ψ-spaces defined from almost disjoint families. The results are contained in [4]. We show that there are $\mathfrak{c}$-many almost disjoint families of the same (uncountable) size whose Ψ-spaces are pairwise non-homeomorphic and they can be Luzin families or branch families of 2^ω. An *almost disjoint family* (AD family, for short) of subsets of the natural numbers ω (or any other countable set) is a family of infinite subsets of ω so that any two different elements of the family have finite intersection. If $\mathcal{A}$ is an AD family on ω, define the topological space $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$ as follows: ω is a discrete subset of $\Psi(\mathcal{A})$; basic neighborhoods of a point $x \in \mathcal{A}$ are of the form $\{x\} \cup (x \setminus F)$, where $F \subseteq \omega$ is finite. The Ψ-spaces have been well studied through the years since they are candidates to give examples or counterexamples of many topological concepts. There are nice properties Ψ-spaces satisfy: they are Hausdorff, separable, first countable, locally compact and zero dimensional. For topological and combinatorial aspects of Ψ-spaces see [18] and [19], respectively.

Daniel Bernal-Santos and Salvador García-Ferreira asked what is the relation between $C_p(\Psi(\mathcal{A}))$ and $\Psi(\mathcal{A})$. Specifically, they asked if $C_p(\Psi(\mathcal{A}))$ and $C_p(\Psi(\mathcal{B}))$ were homeomorphic whenever $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic as subspaces (considered as sets of characteristic functions) of the Cantor set 2^ω with the usual topology. To understand better the space $C_p(\Psi(\mathcal{A}))$, in personal communications, they asked us for a more elementary question:

Question 34 (Bernal-Santos, García-Ferreira)**.** If $X, Y \subseteq 2^\omega$ are homeomorphic, are $\Psi(\mathcal{A}_X)$ and $\Psi(\mathcal{A}_Y)$ homeomorphic?

Here, $\mathcal{A}_X := \{\{x \restriction n : n \in \omega\} : x \in X\}$ is the almost disjoint family of branches determined by the elements of $X \subseteq 2^\omega$. It is well known that under $\mathsf{MA} + \neg\mathsf{CH}$, every set $X \subseteq 2^\omega$ of size less than the continuum is a *Q-set* (recall that a separable metrizable space X is a Q-set if every subset of X is G_δ in X), and thus, $\Psi(\mathcal{A}_X)$ is normal. There are many other topological properties of $X \subseteq 2^\omega$ having effect on the space $\Psi(\mathcal{A}_X)$. One might think that $\mathsf{MA} + \neg\mathsf{CH}$ is a good ingredient to conjecture that the answer is affirmative. However, we answer negatively Question 34 since Theorem 46 shows that in ZFC there are different types of spaces $\Psi(\mathcal{A}_X)$, $\Psi(\mathcal{A}_Y)$ even when X and Y are homeomorphic.

Recall that an AD family $\mathcal{A}$ is *Luzin* if it can be enumerated as $\mathcal{A} = \langle A_\alpha : \alpha < \omega_1\rangle$ in such way that $\forall \alpha < \omega_1\ \forall n \in \omega\ \ (|\{\beta < \alpha : A_\alpha \cap A_\beta \subseteq n\}| < \omega)$. Branch and Luzin families are in some sense "orthogonal", precisely because the normality of their Ψ-spaces might hold in the former and breaks down badly in the latter. We show in Theorem 45 that in ZFC there are different types of Ψ-spaces for Luzin families.

Focusing on AD families of size ω_1, Michael Hrušák formulated the following question in a local seminar.

Question 35 (Hrušák)**.** Is it consistent that there is an uncountable almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A}) \simeq \Psi(\mathcal{B})$, whenever $\mathcal{B} \subseteq \mathcal{A}$ and $|\mathcal{A}| = |\mathcal{B}|$?

Observe that $2^\omega < 2^{\omega_1}$ (in particular CH) implies that the answer to Question 35 is negative by the simple fact that given an AD family $\mathcal{A}$ of size ω_1, there are only $\mathfrak{c}$-many subspaces $\Psi(\mathcal{B})$ for which $\Psi(\mathcal{A}) \simeq \Psi(\mathcal{B})$ (there are only $\mathfrak{c}$-many permutations of ω), and there are 2^{ω_1}-many subsets of $\mathcal{A}$ of size ω_1. We believe that it is a very interesting question; we conjecture that the answer is negative but our methods do not work to solve it.

3.2 Basic Facts

Our notation is standard and follows closely [18] and [19]. We use, like them, $f(A)$ to denote the evaluation of the function f at the point A in its domain while $f[A]$ denotes the image of the set A under the function f. For sets A and B, we say that $A \subseteq^* B$, in words that A is almost contained in B, if $A \setminus B$ is a finite set. Likewise, $A =^* B$ if and only if $A \subseteq^* B$ and $B \subseteq^* A$. For a set Z and a cardinal κ, denote by $[Z]^\kappa$, $[Z]^{<\kappa}$ and $[Z]^{\leqslant\kappa}$ the families of all subsets of Z of size κ, less than κ and less or equal to κ, respectively. If $x \in 2^\omega$, we denote

$$\widehat{x \downarrow n} = \{x \upharpoonright k \in 2^{<\omega} : n \leqslant k\} \quad \text{and} \quad \widehat{x} := \widehat{x \downarrow 0}.$$

The families $\mathcal{A}_X$ defined above, where $X \subseteq 2^\omega$, are canonical AD families on $2^{<\omega}$, and they can reach any size below the continuum. Under a bijection between ω and $2^{<\omega}$ we may consider the Ψ-space associated to $\mathcal{A}_X$. Perhaps the families $\mathcal{A}_X$ were first studied by Frank Tall [40] when he showed that if $X \subseteq 2^\omega$, then X is a Q-set if and only if $\Psi(\mathcal{A}_X)$ is normal.

The following lemma shows how a homeomorphism between Ψ-spaces looks like.

Lemma 36. *Let $\mathcal{A}, \mathcal{B}$ be AD families on ω and let $H : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ be a bijection. Then, H is a homeomorphism if and only if $H[\omega] = \omega$ and for every $x \in \mathcal{A}$, $H[x]$ and $H(x)$, as subsets of ω, are almost equal.*

Proof. $\Rightarrow$) Since H is bijective and must send isolated points to isolated points, it is clear that $H[\omega] = \omega$, that is, H is a permutation on ω. Now, let $x \in \mathcal{A}$ and $\{H(x)\} \cup (H(x) \setminus F)$ be a neighborhood of $H(x)$, where $F \in [\omega]^{<\omega}$. By continuity, there is $F' \in [\omega]^{<\omega}$ such that $H[\{x\} \cup (x \setminus F')] \subseteq \{H(x)\} \cup (H(x) \setminus F)$. Notice that the former set $H[\{x\} \cup (x \setminus F')]$ is the set $\{H(x)\} \cup H[x \setminus F']$. Then $H[x \setminus F'] \subseteq H(x) \setminus F$, and thus, $H[x] \subseteq^* H(x)$. Use the fact that H is open and similar arguments to get $H[x] \supseteq^* H(x)$.

$\Leftarrow$) We will see that H is continuous; to see that H is open use similar arguments as above. Let $x \in \mathcal{A}$ and $\{H(x)\} \cup (H(x) \setminus F)$ be a neighborhood of $H(x)$, where $F \in [\omega]^{<\omega}$. Since $H[x] =^* H(x)$, there is $F' \in [\omega]^{<\omega}$ such that $H[x] \setminus H[F'] \subseteq H(x) \setminus F$. Since H is a permutation on ω, we have $H[x \setminus F'] = H[x] \setminus H[F']$, and then, $H[\{x\} \cup (x \setminus F')] \subseteq \{H(x)\} \cup (H(x) \setminus F)$. ■

Recall that for $s \in 2^{<\omega}$, we let $\langle s \rangle = \{t \in 2^{<\omega} : s \subseteq t\}$ and $[\langle s \rangle] = \{x \in 2^{\omega} : s \subseteq x\}$.

Lemma 37. *Let $X \subseteq 2^{\omega}$ be a set of size κ, $cf(\kappa) > \omega$. Then there are infinitely many $n \in \omega$ for which there are different elements $s, t \in 2^n$ such that $|[\langle s \rangle] \cap X| = \kappa = |[\langle t \rangle] \cap X|$.*

Proof. Suppose for a contradiction that for every $n \in \omega$ there is a unique $s_n \in 2^n$ such that $X_n := [\langle s_n \rangle] \cap X$ has size κ. Let $Y_n = X \setminus X_n$. Notice that $|Y_n| < \kappa$, and since $cf(\kappa) > \omega$, $Y = \bigcup_{n \in \omega} Y_n$ has size less than κ. This is a contradiction because $X \setminus Y = \bigcap_{n \in \omega} X_n$ has size κ and it is contained in the set $\bigcap_{n \in \omega} [\langle s_n \rangle]$ that has at most one element. ■

Notice that by Lemma 37, one can actually get infinitely many $n \in \omega$ for which there is $s \in 2^n$ such that $|[\langle s^\frown 0 \rangle] \cap X| = \kappa = |[\langle s^\frown 1 \rangle] \cap X|$. For an AD family $\mathcal{A}$ on ω, we obtain the next observation by considering $\{\chi_A : A \in \mathcal{A}\} \subseteq 2^{\omega}$, where χ_A is the characteristic function of A.

Remark 38. Let $\mathcal{A}$ be an *AD* family of size κ with $cf(\kappa) > \omega$. Then

$$\forall n \in \omega\ \exists m > n\ (|\{x \in \mathcal{A} : m \in x\}| = |\{x \in \mathcal{A} : m \notin x\}| = \kappa).$$

Lemma 39. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ with $cf(\kappa) > \omega$ and $h : \mathcal{A} \to \mathcal{B}$ be a bijection. Then for all $n \in \omega$ there are $x, y, z \in \mathcal{A}$ such that*

1. $\max\{x \cap y\} > n \ \wedge\ x \cap y \subsetneq x \cap z$; *and*
2. $\max\{h(x) \cap h(y)\} > n \ \wedge\ h(x) \cap h(y) \subsetneq h(x) \cap h(z)$.

Proof. Fix $n \in \omega$. By Remark 38, choose $m_0, m_1 > n$ and $\mathcal{A}_0 \in [\mathcal{A}]^\kappa$ such that for every $x \in \mathcal{A}_0$, $m_0 \in x$ and $m_1 \in h(x)$. Now, fix $y \in \mathcal{A}_0$ and apply Pigeonhole Principle to the set $\{x \cap y : x \in \mathcal{A}_0 \setminus \{y\}\}$. There are $F_0 \in [\omega]^{<\omega}$ and $\mathcal{A}_1 \in [\mathcal{A}_0]^\kappa$ such that for all $x \in \mathcal{A}_1$, $x \cap y = F_0$. There are also $G_0 \in [\omega]^{<\omega}$ and $\mathcal{B}_1 \in [h[\mathcal{A}_1]]^\kappa$ such that for all $w \in \mathcal{B}_1$, $w \cap h(y) = G_0$. Let $\mathcal{A}_2 = h^{-1}[\mathcal{B}_1]$.

At this point we have that for any $\{x, z\} \in [\mathcal{A}_2]^2$, $F_0 = x \cap y = z \cap y$ and $G_0 = h(x) \cap h(y) = h(z) \cap h(y)$, simultaneously. This already implies that $x \cap y \subseteq x \cap z$ and $h(x) \cap h(y) \subseteq h(x) \cap h(z)$. To find elements so that the inclusions are strict, since $|\mathcal{A}_2| = \kappa$, use again Remark 38 to get $m'_0, m'_1 > \max(F_0 \cup G_0 \cup \{m_0, m_1\})$ and $\mathcal{A}_3 \in [\mathcal{A}_2]^\kappa$ such that for any $x \in \mathcal{A}_3$, $m'_0 \in x$ and $m'_1 \in h(x)$. Now, if $\{x, z\} \in [\mathcal{A}_3]^2$, then $x \cap y = F_0 \subsetneq F_0 \cup \{m'_0\} \subseteq x \cap z$ and $h(x) \cap h(y) = G_0 \subsetneq G_0 \cup \{m'_1\} \subseteq h(x) \cap h(z)$.

It is clear that for any $x \in \mathcal{A}_3$, $\max\{x \cap y\} \geqslant m'_0 > n$ and $\max\{h(x) \cap h(y)\} \geqslant m'_1 > n$. ∎

Definition 40. Let $\mathcal{A}, \mathcal{B}$ be AD families on ω of size κ and $h : \mathcal{A} \to \mathcal{B}$ be bijective. We say that h is of *dense oscillation* if for each $\mathcal{A}' \in [\mathcal{A}]^\kappa$ there are $x, y, z \in \mathcal{A}'$ such that $|x \cap z \setminus x \cap y| \neq |h(x) \cap h(z) \setminus h(x) \cap h(y)|$.

Proposition 41. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ with $cf(\kappa) > \omega$ and $h : \mathcal{A} \to \mathcal{B}$ be of dense oscillation. Then, there is no homeomorphism from $\Psi(\mathcal{A})$ to $\Psi(\mathcal{B})$ that extends h.*

Proof. Suppose for a contradiction that $H : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ is a homeomorphism extending h. By Lemma 36, for every $A \in \mathcal{A}$, $H[A] =^* H(A)$. So, for $A \in \mathcal{A}$, consider the finite sets $F_A = \{n \in A : H(n) \notin H(A)\}$ and $G_A = \{n \in H(A) : H^{-1}(n) \notin A\}$.

There are $\mathcal{A}' \in [\mathcal{A}]^\kappa$ and $F, G \in [\omega]^{<\omega}$ such that for all $A \in \mathcal{A}'$, $F = F_A$ and $G = G_A$. If $x, y, z \in \mathcal{A}'$ are different, then $(x \cap z \setminus x \cap y) \cap F = \emptyset$ and $(H(x) \cap H(z) \setminus H(x) \cap H(y)) \cap G = \emptyset$. Moreover, $m \in x \setminus F$ implies $H(m) \in H(x)$, and $H(m) \in H(x) \setminus G$ implies $m \in x$. From this, one can deduce that $|x \cap z \setminus x \cap y| = |H(x) \cap H(z) \setminus H(x) \cap H(y)|$, contradicting the dense oscillation property of $H \restriction \mathcal{A} = h$. ∎

Given two sets $A, B \subseteq \omega$, we say that A and B are *oscillating* if for $x, y \in A,\ w, z \in B$, we have $|y-x| \neq |z-w|$. Similarly, A and B are *almost oscillating* if there is $n \in \omega$ such that $A \setminus n$ and $B \setminus n$ are oscillating.

Proposition 42. *There are $\mathfrak{c}$-many infinite subsets of ω pairwise almost oscillating.*

Proof. From ω, we first construct two oscillating sets $A = \bigcup_{n\in\omega} A_n$, $B = \bigcup_{n\in\omega} B_n$. Fix $A_0 = \{0\}, B_0 = \{1\}$. Suppose that we constructed oscillating sets $A_n = \{a_0, \ldots, a_n\}$ and $B_n = \{b_0, \ldots, b_n\}$. Let $a_{n+1} \in \omega$ such that $a_{n+1} - a_n > b_n - b_0$ and $b_{n+1} \in \omega$ such that $b_{n+1} - b_n > a_{n+1} - a_0$. Observe that $A_{n+1} = A_n \cup \{a_{n+1}\}$ and $B_{n+1} = B_n \cup \{b_{n+1}\}$ are oscillating, and so are A and B.

Notice that the construction is hereditary. That is, for any $X \in [\omega]^\omega$, there are oscillating sets $A, B \in [X]^\omega$. This allows to define a Cantor tree induced by this partitions. Each branch of the Cantor set, $f \in 2^\omega$, represents a decreasing sequence $\langle A_{f\restriction n} : n \in \omega\rangle$ of infinite sets of ω such that for any other branch $g \in 2^\omega$, we have that $A_{f\restriction k}, A_{g\restriction l}$ are oscillating whenever $k, l > \Delta(f, g)$. Now, for every sequence $\langle A_{f\restriction n} : n \in \omega\rangle$, consider a pseudointersection P_f of $\{A_{f\restriction n} : n \in \omega\}$. Observe that for any two sequences $\langle A_{f\restriction n} : n \in \omega\rangle$, $\langle A_{g\restriction n} : n \in \omega\rangle$, their pseudointersections P_f, P_g are almost oscillating. ■

Corollary 43. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ, with $cf(\kappa) > \omega$, and $h : \mathcal{A} \to \mathcal{B}$ be a bijection. If $A = \{|x \cap y| : x, y \in \mathcal{A}\}$ and $B = \{|x \cap y| : x, y \in \mathcal{B}\}$ are almost oscillating, then there is $\mathcal{A}' \in [\mathcal{A}]^\kappa$ such that $h \restriction \mathcal{A}'$ is of dense oscillation.*

Proof. Let $n \in \omega$ such that $A \setminus n$ and $B \setminus n$ are oscillating. Iterating $n + 1$-many steps Remark 38, we can find a subfamily $\mathcal{A}_0 \in [\mathcal{A}]^\kappa$ such that for any $x, y \in \mathcal{A}_0$, $|x \cap y| \geqslant n + 1$. In the same fashion, we can find a subfamily $\mathcal{B}_1 \in [h[\mathcal{A}_0]]^\kappa$ such that for any $w, z \in \mathcal{B}_1$, $|w \cap z| \geqslant n + 1$. Define $\mathcal{A}_1 := h^{-1}[\mathcal{B}_1]$. Note that $\mathcal{A}_1 \in [\mathcal{A}_0]^\kappa$ and for any $x, y \in \mathcal{A}_1$, $n + 1 \leqslant \min\{|x \cap y|, |h(x) \cap h(y)|\}$. We are looking for "an oscillation"; the last inequality avoids the possibility to obtain small intersections (of size less than n).

To see that $\mathcal{A}' := \mathcal{A}_1$ is the desired family, choose $\mathcal{A}'' \in [\mathcal{A}']^\kappa$. Apply Lemma 39 to n and $h \restriction \mathcal{A}'' : \mathcal{A}'' \to h[\mathcal{A}'']$, and get $x, y, z \in \mathcal{A}''$ such that $x \cap y \subsetneq x \cap z$, $h(x) \cap h(y) \subsetneq h(x) \cap h(z)$ and $n < \min\{\max\{x \cap y\}, \max\{h(x) \cap h(y)\}\}$. Thus, there are $a_0, a_1 \in A \setminus n$ and $b_0, b_1 \in B \setminus n$ such that $|x \cap z \setminus x \cap y| = a_0 - a_1 \neq b_0 - b_1 = |h(x) \cap h(z) \setminus h(x) \cap h(y)|$. As desired. ■

Corollary 44. *Let $\mathcal{A}, \mathcal{B}$ be AD families of size κ, with $cf(\kappa) > \omega$, and $h : \mathcal{A} \to \mathcal{B}$ be a bijection. If $\{|x \cap y| : x, y \in \mathcal{A}\}$ and $\{|x \cap y| : x, y \in \mathcal{B}\}$ are almost oscillating, there is no homeomorphism from $\Psi(\mathcal{A})$ to $\Psi(\mathcal{B})$ that extends h.*

Proof. If $H : \Psi(\mathcal{A}) \to \Psi(\mathcal{B})$ is such homeomorphism, by Corollary 43 there is $\mathcal{A}' \in [\mathcal{A}]^\kappa$ such that $H \restriction \mathcal{A}' : \mathcal{A}' \to H[\mathcal{A}']$ is of dense oscillation. If $A = \bigcup \mathcal{A}'$, then $Z = \mathcal{A}' \cup A$ is a subspace of $\Psi(\mathcal{A})$ and $H \restriction Z$ is a homeomorphism, contradicting Proposition 41. ■

3.3 $\mathfrak{c}$-Many Pairwise Non-Homeomorphic Ψ-Spaces

Next we construct $\mathfrak{c}$-many AD families of the same size whose Ψ-spaces are pairwise non-homeomorphic for each of the classes of Luzin families and branch families of 2^ω.

Theorem 45. *There are $\mathfrak{c}$-many distinct Luzin families (of size ω_1) with non-homeomorphic Ψ-spaces.*

Proof. Given $L = \{k_n : n \in \omega\} \subseteq \omega$ such that $k_n > \sum_{i<n} k_i$, construct a Luzin family $\mathcal{A}_L$ as follow: choose a partition $\{A_n : n \in \omega\}$ of ω into infinite sets. Suppose that we constructed sets A_β, for $\beta < \alpha < \omega_1$. Let $\{B_n : n \in \omega\}$ be an enumeration with no repetitions of $\{A_\beta : \beta < \alpha\}$ and for each $n \in \omega$, pick $a_n \subseteq B_n \setminus \bigcup_{i<n} B_i$ such that $|(\bigcup_{i\leqslant n} a_i) \cap B_n| = k_n$. Let $A_\alpha = \bigcup_{n\in\omega} a_n$ and $\mathcal{A}_L = \{A_\alpha : \omega < \alpha < \omega_1\}$. It is easy to see that $\mathcal{A}_L$ is a Luzin family. Observe that

$$\forall \omega < \alpha, \beta < \omega_1 \ \exists n \in \omega \ (|A_\alpha \cap A_\beta| = k_n). \tag{3.1}$$

This is how we construct a Luzin family $\mathcal{A}_L$ from a given set of natural numbers L. All Luzin families considered in the following are constructed from a fixed partition $\{A_n : n \in \omega\}$ of ω.

By Proposition 42, let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a pairwise almost oscillating family of sets of ω. For every $\alpha < \mathfrak{c}$, let $Q_\alpha = \{q_n^\alpha : n \in \omega\} \subseteq P_\alpha$ such that for every $n \in \omega$, $q_n^\alpha > \sum_{i<n} q_i^\alpha$. Notice that $\{Q_\alpha : \alpha < \mathfrak{c}\}$ is also a pairwise almost oscillating family of sets of ω. It follows from Equation (3.1) that for any $\alpha < \mathfrak{c}$, $\{|x \cap y| : x, y \in \mathcal{A}_{Q_\alpha}\} \subseteq Q_\alpha$. Since "almost oscillating" is a hereditary property, for any $\omega < \beta, \alpha < \mathfrak{c}$, the sets $\{|x \cap y| : x, y \in \mathcal{A}_{Q_\alpha}\}$, $\{|x \cap y| : x, y \in \mathcal{A}_{Q_\beta}\}$ are almost oscillating. By Corollary 44, $\{\mathcal{A}_{Q_\alpha} : \alpha < \mathfrak{c}\}$ is the desired collection of Luzin families. ■

Theorem 46. *Given a cardinal $\kappa \leqslant \mathfrak{c}$ of uncountable cofinality, there are $\mathfrak{c}$-many distinct homeomorphic subsets of 2^ω of size κ with non-homeomorphic Ψ-spaces.*

Proof. Given $A \in [\omega]^\omega$, consider the tree $S_A \subseteq 2^{<\omega}$ defined by $\emptyset \in S_A$ and $s \in Lev_n(S_A) \implies (s^\frown 1 \in S_A) \wedge (s^\frown 0 \in S_A \longleftrightarrow n \in A)$. Observe that $\forall x, y \in [S_A]$, $\Delta(x, y) = |\widehat{x} \cap \widehat{y}| \in A$.

Again, by Proposition 42, let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be a pairwise almost oscillating family of sets of ω. Note that if $A, B \in [\omega]^\omega$, then $[S_A] \simeq [S_B] \simeq 2^\omega$, and $A \cap B =^* \emptyset$ implies that $|[S_A] \cap [S_B]| < \omega$. Hence, we can choose $X_\alpha \subseteq [S_{P_\alpha}]$ of size κ such that the X_α's are all different, but $X_\alpha \simeq X_\beta$, whenever $\alpha, \beta < \mathfrak{c}$. Recall that $\{|\widehat{x} \cap \widehat{y}| : x, y \in X_\alpha\} \subseteq P_\alpha, \alpha < \mathfrak{c}$. Also, $\{|\widehat{x} \cap \widehat{y}| : x, y \in X_\alpha\}$, $\{|\widehat{x} \cap \widehat{y}| : x, y \in X_\beta\}$ are almost oscillating, $\beta, \alpha < \mathfrak{c}$. By Corollary 44, $\{X_\alpha : \alpha < \mathfrak{c}\}$ is the desired collection of subsets of 2^ω. ■

Corollary 47. *Let $\mathcal{A}$ be an AD family of size κ. If there are $\mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{A}]^\kappa$ such that $\{|x \cap y| : x, y \in \mathcal{A}_0\}$ and $\{|x \cap y| : x, y \in \mathcal{A}_1\}$ are almost oscillating, then $\Psi(\mathcal{A}) \not\simeq \Psi(\mathcal{A}_0)$.*

Proof. If $h : \mathcal{A}_0 \to \mathcal{A}$ is a bijection, use Corollary 43 to find $\mathcal{A}_0' \in [h^{-1}[\mathcal{A}_1]]^\kappa$ such that $h \restriction_{\mathcal{A}_0'} : \mathcal{A}_0' \to h[\mathcal{A}_0']$ is of dense oscillation. Now, it follows from Proposition 41 that there cannot exist a homeomorphism between $\Psi(\mathcal{A}_0')$ and $\Psi(h[\mathcal{A}_0'])$ extending $h \restriction_{\mathcal{A}_0'}$. This implies that it cannot be a homeomorphism between $\Psi(\mathcal{A}_0)$ and $\Psi(\mathcal{A})$ that extends h. ■

Motivated by Corollary 47, a positive answer to the following question gives raise a negative answer to Question 35. However, we do not even know if CH answers:

Question 48. Let $\mathcal{A}$ be an AD family on ω of size ω_1. Are there $\mathcal{A}_0, \mathcal{A}_1 \in [\mathcal{A}]^{\omega_1}$ such that $\{|x \cap y| : x, y \in \mathcal{A}_0\}$ and $\{|x \cap y| : x, y \in \mathcal{A}_1\}$ are almost oscillating?

The arguments under CH below Question 35 say that if $\mathcal{A}$ is an AD family of size ω_1, then there is $\mathcal{A}_0 \in [\mathcal{A}]^{\omega_1}$ such that $\Psi(\mathcal{A}) \not\approx \Psi(\mathcal{A}_0)$. In that case however, the sets $\{|x \cap y| : x, y \in \mathcal{A}_0\}$ and $\{|x \cap y| : x, y \in \mathcal{A}\}$ are far from being almost oscillating (the former is contained in the latter).

Chapter 4

Nabla Products

4.1 Introduction

This is the main chapter of the document. The work developed here in collaboration with Professor Paul Gartside [2] is in progress to be published. We focus on countable box and nabla products. Let $\{X_i : i \in I\}$ be a family of topological spaces. Recall that a *box* is a set $\prod_i U_i$, where each U_i is open in X_i. The *box product*, $\Box_i X_i$, is the space with underlying set $\prod_i X_i$ and basis all boxes. Two elements $\overline{x}$ and $\overline{y}$ of $\Box_i X_i$ are *mod-finite equivalent*, denoted $\overline{x} \sim \overline{y}$, if the set $\{i \in I : \overline{x}(i) \neq \overline{y}(i)\}$ is finite. The *nabla product*, $\nabla_i X_i$, is the quotient space, $\Box_i X_i/\sim$.

It is unknown, in ZFC, whether the countable box product $\Box[0,1]^\omega$, or even its closed subspace, $\Box(\omega+1)^\omega$, is normal. With high probability, this is the oldest problem in General Topology. This question was asked (orally) for the first time by Tietze sometime in the 1940's. A second version is attributed to Arthur Stone in the 1950's: Is the box product of countably many separable metrizable spaces normal? A positive answer to Stone's question under CH for compact metrizable spaces was found by Mary Ellen Rudin [37] in 1972; she actually found that the box product was paracompact. Thus paracompactness entered to the picture. See [36] for a survey of the box product problem.

Central to almost all positive results on paracompactness, and hence normality, of box products, is a connexion to the nabla product due to Kenneth Kunen [22]: let $\{X_n : n \in \omega\}$ be a family of compact spaces, then, $\square_n X_n$ is paracompact if and only if $\nabla_n X_n$ is paracompact. In particular, it is now known that under certain small cardinal conditions, namely, $\mathfrak{b} = \mathfrak{d}$ ([10]), or $\mathfrak{d} = \mathfrak{c}$ ([31], [25], [43]), the nabla product $\nabla(\omega + 1)^\omega$ is paracompact and so the box product $\square(\omega + 1)^\omega$ is paracompact. Another statement implying the paracompactness of $\nabla(\omega+1)^\omega$ that differs from any cardinal arithmetic is the so called *Model Hypothesis* [33], that follows from $\mathfrak{d} = \mathfrak{c}$, and holds in any forcing extension by uncountably many Cohen reals. At the end of the introduction we state several important known results up to date.

In an insightful analysis of these results, Judy Roitman [33] extracted a combinatorial principle that she called Δ which is a consequence of each of the set-theoretic axioms mentioned above. She further claimed that Δ implies the paracompactnes of $\nabla(\omega + 1)^\omega$. The principle Δ is the most recent progress concerning to the paracompactness of $\nabla(\omega + 1)^\omega$ and it is unknown if $\neg\Delta$ is consistent with ZFC or if Δ true in ZFC.

This work focuses in finding topological aspects of Δ. These aspects can be stronger or weaker versions of paracompactness and normality, such as being pseudonormal, coleccionwise Hausdorff, colectionwise normal, monotonically normal, paranormal, metanormal, metacompact, orthocompact, etc. And 'hereditarily' and 'countably' versions. See Chapter 1 for the definitions of this concepts. Everything started here, the moment we found a topological characterization of Δ (see Theorem 79): Δ holds if and only if $\nabla(\omega + 1)^\omega$ is monotonically normal.

Recall that a space X with topology τ is *monotonically normal* if there exists an operator $G : X \times \tau \to \tau$ such that $x \in G(x, U) \subseteq U$; and if $G(x, U) \cap G(y, V) \neq \emptyset$, then $x \in V$ or $y \in U$. We record three facts about monotone normality that are all easy to check: (a) if $\mathcal{B}$ is a base for τ, then in both domain and range of the operator G we can replace τ with $\mathcal{B}$; (b) the restriction of a monotone normality operator to a subspace yields a monotone normality operator for the subspace, and so monotone normality is hereditary; and (c) monotone normality implies (collectionwise) normality.

We note that monotone normality does not transfer from $\nabla(\omega+1)^\omega$ to $\Box(\omega+1)^\omega$. Indeed, see Corollary 20 [24], if $\{X_i : i \in I\}$ is a family of compact or first countable spaces, then $\Box_i X_i$ is not hereditarily normal. On the other hand, Gartside proved in [15] that if a space X has monotonically normal square, then X is hereditarily paracompact. Hence, if a nabla space of the form ∇X^ω is monotonically normal, then it is hereditarily paracompact because it is homeomorphic to its square. What if infinitely many factors are different? We have showed, Theorem 82, that if the factors X_n are metrizable, then if $\nabla_n X_n$ is monotonically normal, it must be hereditarily paracompact.

In Section 4.2 we introduce technical notation and prove basic facts about nabla products. Theorem 79 and Roitman's Theorem 73 'Δ implies $\nabla(\omega+1)^\omega$ is paracompact' are proved in Section 4.3. Theorem 82 is proved in Section 4.4 where we study embeddings into nabla products of metrizable and non-metrizable factors. In Section 4.5 we introduce Δ-like statements concerning to nabla products of metrizable, ordinal and one point compactification spaces. These statements play the roll of Δ by showing that the corresponding nabla products are monotonically normal (see Proposition 89, Proposition 94 and Corollary 96). In Section 4.6 we prove that some nabla products are (consistently) non hereditarily normal solving the Roitman's open Question 106. In Section 4.7 we study subspaces of $\nabla(\omega+1)^\omega$ and restrictions of Δ. In particular, we mention that an instance of Δ is true in ZFC (this was showed by Roitman in [33]). We relate such instance with the so called 'tangled-free' filters introduced by Gartside in [15]. In Section 4.8 we study variants of monotone normality and showed these variants are equivalent for certain nabla products. Finally, in Section 4.9 we talk about many open questions.

§ 4.1.1 Positive Results

None of the following are ZFC results.

Theorem 49 (Roitman, 1979 [31])**.** *The box products of countably many compact first countable spaces is paracompact if* $\mathfrak{d} = \mathfrak{c}$ *or if* MH *holds.*

Theorem 50 (Williams, 1984 [24]). *The box products of countably many compact spaces of weight $\leqslant \omega_1$ is paracompact if $\mathfrak{d} = \omega_1$.*

Theorem 51 (van Douwen, 1980 [10]; [42]). *The box product of countably many compact metrizable spaces is paracompact if $\mathfrak{b} = \mathfrak{d}$.*

Theorem 52 (Kunen, 1978 [22]). *The box product of countably many compact scattered spaces is paracompact if* CH *holds.*

Theorem 53 (Lawrence, 1988 [25]). *The box product of countably many countable metrizable spaces is paracompact if $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$.*

Theorem 54 (Wingers, 1994 [43]). *The box product of countably many σ-compact 0-dimensional first countable spaces of cardinality $\leqslant \mathfrak{c}$ is paracompact if $\mathfrak{d} = \mathfrak{c}$.*

Theorem 55 (Roitman, 2015 [36]). *If Δ holds, then $\square(\omega + 1)^\omega$ is paracompact.*

§ 4.1.2 Not Normal

Almost all of these are ZFC results. For an ordinal α, 2^α_{LEX} is the space 2^α with the following order $\prec$: for $f, g \in 2^\alpha$, $f \prec g$ if there is $\beta \in \alpha$ so that $f(\beta) < g(\beta)$ and for $\gamma \in \beta$, $f(\gamma) = g(\gamma)$.

Theorem 56 (van Douwen, 1980 [10]). *$\omega^\omega \times \square(\omega + 1)^\omega$ is not normal; hence $\square(\omega + 1)^\omega$ is not hereditarily normal.*

Theorem 57 (Kunen, 1978 [22]; [14]). *$\square(2^{\mathfrak{c}^+})^\omega$ is not normal.*

Theorem 58 (Kunen, 1973 [21]; [14]). *If $\mathfrak{b} = \mathfrak{d}$, then $\mathfrak{d} \times \square(\omega + 1)^\omega$ is not normal.*

Theorem 59 (Kunen, Erdös, Rudin, 1973 [14]). *$\square(2^{\omega_1+1}_{LEX})^\omega$ is not normal.*

Theorem 60 (van Douwen, 1977 [9]). *$\square(2^{\omega_2})^\omega$ is not normal.*

Theorem 61 (Wingers, 1995 [44]). *If X is Lindelöf, not Hurewicz, with a dense Hurewicz subspace, then $X \times \square(\omega + 1)^\omega$ is not normal.*

Theorem 62 (Wingers, 1995 [44]). *Assume* MA. *Then, there is $X \subseteq \mathbb{R}$, X Hurewicz, $X \times \square(\omega + 1)^\omega$ paracompact but $X^2 \times \square(\omega + 1)^\omega$ is not normal.*

Theorem 63 (Lawrence, 1996 [27]). *$\square(\omega + 1)^{\omega_1}$ is not normal.*

4.2 Notation

Here we state the notation we will use throughout the chapter. Due to Lawrence's Theorem 63, we will focus only on countable box and nabla products.

If $x \in \Box_n X_n$ or $\overline{x} \in \nabla_n X_n$, and $U = \langle U_n \rangle_{n\in\omega}$ is a sequence of open sets, where $x(n) \in U_n \subseteq X_n$, define the basic neighborhood around x as $N(x, U) = \Box_n U_n \subseteq \Box_n X_n$ and $N(\overline{x}, U) = \nabla_n U_n \subseteq \nabla_n X_n$. If the X_n's are first countable, a basis of x or $\overline{x}$ is codified by ω^ω: if $\{U_n^k : k \in \omega\}$ is a base at $x(n)$, we will write $N(x, f) = \Box_n U_n^{f(n)}$ and $N(\overline{x}, f) = \nabla_n U_n^{f(n)}$, where $f \in \omega^\omega$. We will make no difference denoting the elements of $\Box_n X_n$ and $\nabla_n X_n$ if there is no chance to confusion. A function $F : X \to \tau$, where τ is the topology of X, is a neighborhood assignment if $x \in F(x)$, for $x \in X$. We will write "*neighbornet*" for "neighborhood assignment".

In order to state Δ we introduce some specific notation for $\Box(\omega + 1)^\omega$ and $\nabla(\omega + 1)^\omega$. Let $\omega^{\subset\omega} = \{x : N \to \omega : N \subseteq \omega \text{ is infinite-coinfinite}\}$. For $k \leqslant \omega$, denote c_k be the constant k function. If $x \in \omega^{\subset\omega}$, $\tilde{x} \in (\omega+1)^\omega$ is defined as $\tilde{x}(n) = x(n)$ if $n \in dom(x)$, and $\tilde{x}(n) = \omega$, otherwise. If $x \in \omega^{\subset\omega}$ and $f \in \omega^\omega$, we will use $\overline{x}, \overline{f}, \overline{c}_\omega$ as the equivalent classes $[\tilde{x}]_\sim, [f]_\sim, [c_\omega]_\sim$, respectively, in $\nabla(\omega + 1)^\omega$. Let $\nabla^* = \{\overline{x} \in \nabla(\omega + 1)^\omega : x \in \omega^{\subset\omega}\}$. For partial functions $x, y \in \omega^{\subset\omega}$, let $x \setminus y$ be the partial function defined as $x \restriction (dom(x) \setminus dom(y))$ on ω (possibly finite); also, we say that $x \subseteq^* y$ if for all but finitely many $n \in dom(x)$, $x(n) = y(n)$. If x, y are *almost compatible* partial functions ($x \restriction dom(x) \cap dom(y) =^* y \restriction dom(x) \cap dom(y)$), then x and y have a $\subseteq^*$-greatest lower bound $x \wedge y = x \restriction dom(x) \cap dom(y) =^* y \restriction dom(x) \cap dom(y)$ and a $\subseteq^*$-lowest greater bound $x \vee y =^* x \cup y$, in $\nabla(\omega+1)^\omega$. For $x \in \omega^{\subset\omega} \cup \omega^\omega$ and $h \in \omega^\omega$ we say that $x \geqslant^* h$ ($x >^* h$) if for all but finitely many $n \in dom(x)$, $x(n) \geqslant h(n)$ ($x(n) > h(n)$). Basic neighborhoods around $\overline{c}_\omega$ are of the form $N(\overline{c}_\omega, h) = \{\overline{y} \in \nabla(\omega + 1)^\omega : y \geqslant^* h\}$ and basic neighborhoods around $\overline{x}$ are of the form $N(\overline{x}, h) = \{\overline{y} \in \nabla(\omega + 1)^\omega : x \subseteq^* y \text{ and } y \setminus x \geqslant^* h\}$, where $h \in \omega^\omega$. Notice that elements $\overline{f}$, where $f \in \omega^\omega$, are isolated points in $\nabla(\omega + 1)^\omega$.

If $A \subseteq \nabla(\omega+1)^\omega$, we denote $N_A(\overline{x}, f) = N(\overline{x}, f) \cap A$ the neighborhoods in the subspace. If A is any subset of $\omega^{\subset\omega}$, write $\nabla^*(A)$ for the subspace

$\{\overline{x} : x \in A\}$ of $\nabla(\omega+1)^\omega$ and set define $\nabla(A)$ to be $\nabla^*(A) \cup \omega^\omega \cup \{\overline{c}_\omega\}$. Abbreviate $\nabla^*(\omega^{\subset\omega})$ by ∇^*, and $\nabla(\omega^{\subset\omega}) = \nabla(\omega+1)^\omega$ by ∇.

§ 4.2.1 Basic Facts on $\nabla_n X_n$

Lemma 64 ([33]). *Let $x, y \in \omega^{\subset\omega}$, $f \in \omega^\omega$, and suppose that $x \subseteq^* y$. The following are equivalent: (1) $\overline{y} \notin N(\overline{x}, f)$, (2) $f \nleq^* y \setminus x$, (3) $\forall g \in \omega^\omega$, $N(\overline{x}, f) \cap N(\overline{y}, g) = \emptyset$.*

Proof. (1) implies (2): this is definition of $\overline{y}$ not being in $N(\overline{x}, f)$. (2) implies (3): let $N = \{n \in \omega : f(n) > y(n)\}$. Since every element $\overline{z} \in N(\overline{y}, g)$ satisfies $z(n) = y(n)$, $n \in dom(y)$, then $f(n) > z(n)$ for $n \in N$. Hence, $\overline{z} \notin N(\overline{x}, f)$. (3) implies (1): this is trivial. ■

The following remark will be used in repeated occasions. It explains how any two elements of ∇^* look like. The proof is easily checked.

Remark 65. Given $x, y \in \omega^{\subset\omega}$ we have three cases: (i) $x \subseteq^* y$ or $y \subseteq^* x$, (ii) there are infinitely many $n \in \omega$, $x(n) \neq y(n)$, and (iii) $|x \setminus y| = |y \setminus x| = \omega$ and $|\{n \in dom(x) \cap dom(y) : x(n) \neq y(n)\}| < \omega$.

Remark 66. Let $f, g \in \omega^\omega$. Suppose $x, y \in \omega^{\subset\omega}$ satisfy $|x \setminus y| = |y \setminus x| = \omega$ and $|\{n \in dom(x) \cap dom(y) : x(n) \neq y(n)\}| < \omega$. Then, $x \setminus y \not>^* g$ or $y \setminus x \not>^* f$ if and only if $N(\overline{x}, f) \cap N(\overline{y}, g) = \emptyset$.

Proof. $\Longrightarrow$) Suppose $x \setminus y \not>^* g$. Hence, there are infinitely many $n \in dom(x) \setminus dom(y)$ such that $x(n) < g(n)$. Since every $\overline{z} \in N(\overline{x}, f)$ satisfies $dom(x) \subseteq^* dom(z)$, then there are infinitely many $n \in dom(z) \setminus dom(y)$ such that $z(n) < g(n)$, hence $\overline{z} \notin N(\overline{y}, g)$.

$\Longleftarrow$) Suppose $x \setminus y >^* g$ and $y \setminus x <^* f$. Modify x finitely many coordinates so $z = x \cup y$ is well defined. Hence $x, y \subseteq^* z$ and $z \setminus x >^* f$, $z \setminus y >^* g$. This is the definition of $\overline{z}$ being in both $N(\overline{x}, f)$ and $N(\overline{y}, g)$, contradiction. ■

Proposition 67 ([31]). *1. If $F \subseteq \omega^\omega$ and $|F| < \mathfrak{b}$ then F is $\leqslant^*$-bounded.*

2. $\mathfrak{b} = \mathfrak{d}$ *iff there is a dominating family* $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ *so that if* $\alpha < \beta$ *then* $f_\alpha \leqslant^* f_\beta$. *Such a family is called a* scale.

3. *If* $\mathcal{G} \subseteq \omega^\omega, \mathcal{A} \subseteq \mathcal{P}(\omega)$ *and* $|\mathcal{G}|, |\mathcal{A}| < \mathfrak{d}$, *then there is a function* $f \in \omega^\omega$ *so that for any* $g \in \mathcal{G}$ *and* $a \in \mathcal{A}$, $|\{n \in a : f(n) > g(n)\}| = \omega$.

Definition 68. A space is a P_κ*-space* if the intersection of fewer than κ-many open sets is open- The P_{ω_1}-spaces are called P-spaces.

Definition 69. A space is κ*-Lindelöf* if every open cover has a subcover of size $< \kappa$.

Proposition 70 ([33]). 1. *If each* X_n *is a space,* $\nabla_n X_n$ *is a* P*-space.*

2. *If each* X_n *is first countable, then* $\nabla_n X_n$ *is a* $P_\mathfrak{b}$*-space.*

3. *If each* X_n *is first countable,* $\beta < \mathfrak{d}$ *and* $\{H_\alpha : \alpha < \beta\}$ *is a collection of closed boxes in the nabla product, then* $\bigcup_{\alpha<\beta} H_\alpha$ *is closed.*

Proof. For (1), use the fact that every countable collection of functions in ω^ω is $\leqslant^*$-bounded. For (2), use the fact that every set of fewer than $\mathfrak{b}$ functions in ω^ω is $\leqslant^*$-bounded. Observe that part (3) of Proposition 67 implies (3). ■

The *character* of a topological space X at a point x is the cardinality $\chi(x, X)$ of the smallest local base for x. The character of X is $\chi(X) = \sup\{\chi(x, X) : x \in X\}$. The *tightness* $t(x, X)$ is the smallest cardinal number κ such that whenever $x \in \overline{Y}$ for some $Y \subseteq X$, there exists a subset $Z \subseteq Y$, with $|Z| \leqslant \kappa$. The tightness of a space X is $t(X) = \sup\{t(x, X) : x \in X\}$.

Lemma 71. *Let* $\{X_n : n \in \omega\}$ *be a family of first countable spaces. For any point* $x \in \overline{A} \setminus A$, *where* $A \subseteq \nabla_n X_n$, *we have the equalities* $t(x, A \cup \{x\}) = \mathfrak{d} = \chi(x, A \cup \{x\})$.

Proof. Since x is a limit point of A, then for infinitely many $n \in \omega$, $x(n)$ is non-isolated in X_n. Thus, without loss of generality we may assume that for all $n \in \omega$, $x(n)$ is non-isolated in X_n. Let $\{B_m(n) : m \in \omega\}$ be a decreasing countable local basis for $x(n)$, $n \in \omega$.

Since $t(x, A \cup \{x\}) \leqslant \chi(x, A \cup \{x\}) \leqslant \chi(x, \nabla_n X_n) = \mathfrak{d}$ (the equality holds because a local basis of x can be represented by a dominating family of ω^ω), we only have to prove that $t(x, A \cup \{x\}) \geq \mathfrak{d}$.

Suppose for a contradiction that there is $Z \subseteq A$ with $|Z| < \mathfrak{d}$ and $x \in cl_{A\cup\{x\}}(Z)$. For every $z \in Z$, since $z \neq x$, there is an infinite set $a_z \subseteq \omega$ such that $z(n) \neq x(n)$, $n \in \omega$. Also, for every $z \in Z$ there is a function $f_z \in \omega^\omega$ such that for $n \in a_z$, $z(n) \notin B_{f_z(n)}(n)$, and thus, $z \notin N(x, f_z)$. Let $\mathcal{G} = \{f_z : z \in Z\}$ and $\mathcal{A} = \{a_z : z \in Z\}$. By Proposition 67, there is $f \in \omega^\omega$ diagonalizing the families $\mathcal{G}$ and $\mathcal{A}$. Then, for any $z \in Z$, $z \notin N(x, f)$, contradicting that $x \in cl_{Z\cup\{x\}}(Z)$. ■

4.3 Parametrizing Roitman's Principle Δ

Here we prove the main result of this section, Theorem 79. Monotone normality entered the picture, opening a new line of results in the following sections.

The way Roitman proved Theorem 55 in [33] follow these steps: the principle Δ implies ∇^* is paracompact. Also, ∇^* is obtained from $\nabla(\omega+1)^\omega$ by removing all isolated points and the limit point c_ω. Then, she thought that '∇^* is paracompact if and only if $\nabla(\omega + 1)^\omega$ is paracompact', but there is a gap. We talk about this gap in Section 4.9. However, it can be proved that Δ implies $\nabla(\omega + 1)^\omega$ is hereditarily basic ultraparacompact.

At the end of this section, Example 80 shows a subspace A of a space X obtained by removing all isolated points and one limit point of X, for which A is paracompact but X is not.

Definition 72. Let A be any subset of $\omega^{\subset\omega}$. Then $\Delta(A)$ is the statement: there exists $F : A \longrightarrow \omega^\omega$ such that if $x, y \in A$ with $|x \setminus y| = |y \setminus x| = \omega$ and $|\{n \in dom(x) \cap dom(y) : x(n) \neq y(n)\}| < \omega$, then $x \setminus y \not>^* F(y)$ or $y \setminus x \not>^* F(x)$.

Abbreviate $\Delta(\omega^{\subset\omega})$ to Δ, this is Roitman's combinatorial principle in [33] and [36]. It is known to be consistently true, but it is unknown if it can be consistently false, or it is true in ZFC. For the sake of completeness, we prove the following Roitman's result in [33].

Theorem 73 (Roitman)**.** *If Δ holds, then ∇^* is ultraparacompact (every open cover has a pairwise disjoint open refinement).*

Proof. Let $\{x_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^{\subset\omega}$ be such that $\forall x \in \omega^{\subset\omega} \exists \alpha\ (x \subseteq^* x_\alpha)$. Define $\nabla_\alpha = \{\overline{x} \in \nabla^* : \alpha \text{ is the least with } x \subseteq^* x_\alpha\}$. Let $\mathcal{U}$ be an open cover of ∇^*. If $\overline{x} \in \nabla_\alpha$ choose $f_x \in \omega^\omega$ such that $f_x > \max\{F(x), x_\alpha\}$ and $N(\overline{x}, f_x)$ refines some element $U \in \mathcal{U}$. Note that $\mathcal{W}_\alpha = \{N(\overline{x}, f_x) : \overline{x} \in \nabla_\alpha\}$ is pairwise disjoint.

For $\alpha < \mathfrak{c}$, we will define $D_\alpha \subseteq \nabla_\alpha$ and a clopen family $\mathcal{V}_\alpha$ covering D_α so that $\mathcal{V} = \bigcup\bigcup_{\alpha<\mathfrak{c}} \mathcal{V}_\alpha$ covers ∇^* and $\mathcal{V}$ is a pairwise disjoint refinement of $\mathcal{U}$.

Let $D_0 = \nabla_0$, $\mathcal{V}_0 = \mathcal{W}_0$. Given $D_\beta, \mathcal{V}_\beta$, for $\beta < \alpha$, define $D_\alpha = \nabla_\alpha \setminus \bigcup\bigcup_{\beta<\alpha} \mathcal{V}_\beta$, and $\mathcal{V}_\alpha = \{N(\overline{x}, f_x) : \overline{x} \in D_\alpha\}$. To show $\mathcal{V}$ is pairwise disjoint, pick $N(\overline{x}, f_x), N(\overline{y}, f_y) \in \mathcal{V}$, $\overline{x} \neq \overline{y}$. If there is α with $\overline{x}, \overline{y} \in D_\alpha$, we are done, since $\mathcal{W}_\alpha$ is pairwise disjoint. The case (ii) in Remark 65 is trivial. Suppose $\overline{y} \in D_\beta, \overline{x} \in D_\alpha$ and $\alpha < \beta$. It is impossible $y \subseteq^* x$ by the minimality of β. So if $x \subseteq^* y$, then since $y \notin N(\overline{x}, f_x)$, Lemma 64 applies, hence $N(\overline{x}, f_x) \cap N(\overline{y}, f_y) = \emptyset$. The remaining case (case (iii) in Remark 65) is 'x and y are as in Δ', in which case the property of F implies $N(\overline{x}, F(x)) \cap N(\overline{y}, F(y)) = \emptyset$. ■

We will reuse this method again in next sections. So, we restate the previous construction.

For an open cover $\mathcal{U}$ of ∇^*, $A \subseteq \omega^{\subset\omega}$ and $F : A \to \omega^\omega$ a neighbornet refining $\mathcal{U}$, we denote by $\mathcal{V}(\mathcal{U}, A, F)$ the open refinement of $\mathcal{U}$ constructed as follows: let $\{x_\alpha : \alpha < \mathfrak{c}\} \subseteq \omega^{\subset\omega}$ such that $\forall x \in \omega^{\subset\omega} \exists \alpha\ (x \subseteq^* x_\alpha)$. Define $\nabla_\alpha = \{\overline{x} \in \nabla^* : \alpha \text{ is least with } x \subseteq^* x_\alpha\}$. For $\overline{x} \in \nabla_\alpha$, let $f_x = \max\{x_\alpha, F(x)\}$ be a second neighbornet. Let $A_\alpha = \nabla_\alpha \cap A$ and note that $\mathcal{W}_\alpha = \{N(\overline{x}, f_x) : \overline{x} \in A_\alpha\}$ is a discrete family. For $\alpha < \mathfrak{c}$ we define $D_\alpha \subseteq A_\alpha$ and a clopen family $\mathcal{V}_\alpha$ covering D_α: let $D_0 = A_0$, $\mathcal{V}_0 = \{N(\overline{x}, f_x) : \overline{x} \in D_0\}$. Given D_β, $\mathcal{V}_\beta$ for all $\beta < \alpha$ we define $D_\alpha = A_\alpha \setminus \bigcup\bigcup_{\beta<\alpha} \mathcal{V}_\beta$ and $\mathcal{V}_\alpha = \{N(\overline{x}, f_x) : \overline{x} \in D_\alpha\}$. Finally, let $\mathcal{V}(\mathcal{U}, A, F) = \bigcup_{\alpha<\mathfrak{c}} \mathcal{V}_\alpha$.

Remark 74. If $\mathcal{U}$ is an open cover of ∇^*, $A \subseteq \omega^{\subset\omega}$ and F is a neighbornet refining $\mathcal{U}$, then $\mathcal{V}(\mathcal{U}, A, F)$ and $\mathcal{V}_\alpha$ defined above have the following properties.

1. For any $N(\overline{x}, f_x) \in \mathcal{V}$, $N(\overline{x}, f_x) \cap \bigcup\{N(\overline{y}, f_y) \in \mathcal{V} : y \subseteq^* x\} = \emptyset$.
2. For every $\alpha < \mathfrak{c}$, $\mathcal{V}_\alpha$ is pairwise disjoint.
3. $\mathcal{V}(\mathcal{U}, A, F)$ covers A.

Proof. (1) follows from the minimality of ∇_α and the constructions of the $\mathcal{V}_\alpha$'s. For (2), observe that $\mathcal{V}_\alpha \subseteq \mathcal{W}_\alpha$. Part (3) is immediate. ■

Definition 75 (Roitman)**.** The *Model Hypothesis* MH is the following statement: For some κ, $H(\omega_1) = \bigcup_{\alpha<\kappa} H_\alpha$, where each H_α is an elementary submodel of $(H(\omega_1), \in)$ and each $H_\alpha \cap \omega^\omega$ is not dominating.

Here $H(\kappa)$ is the collection of all sets whose transitive closures have size less than κ. In particular, both $\omega^\omega, \mathcal{P}(\omega)$ are contained in $H(\omega_1)$, and a space of countable weight can be coded as (hence is isomorphic to) a subset of $H(\omega_1)$. See [23] for more on $H(\kappa)$.

Theorem 76 (Roitman [33])**.** *If any of $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} = \mathfrak{c}$ or* MH *hold, then Δ holds. Also, Δ holds in any forcing extension by adding cofinally many Cohen reals.*

We only know about consistent proofs for Δ. However, taking the family $INC \subseteq \omega^{\subset\omega}$ of all increasing partial functions we get a positive result in ZFC. That is, $\Delta(INC)$ is true in ZFC as we will see in Section 4.7. In the following we uncover the combinatorics of Δ and we extract topological aspects of it.

Definition 77. Let X be a space and $A \subseteq X$. A neighbornet T is *halvable* if there is a neighbornet S for A such that: if $S(x) \cap S(y) \neq \emptyset$ then $x \in T(y)$ or $y \in T(x)$. Thus, we say that X is *halvable* if every neighbornet for X is halvable.

Observe that if a space X is monotonically normal, then it is halvable.

Lemma 78. *Let X be a space with partial order $\preceq$ and neighborhood bases, $\mathcal{B}_x$, $x \in X$, such that: (1) $\downarrow x = \{y : y \preceq x\}$ is open for all x, and (2) if $y \in B \subseteq \downarrow x$, where $B \in \mathcal{B}(x)$, then the interval $[y, x] = \{z : y \preceq z \preceq x\}$ is contained in B. Then, X is monotonically normal if and only if the neighbornet $T(x) = \downarrow x$ is halvable.*

Proof. We need to show if S halves $T(x) = \downarrow x$, then X is monotonically normal. For any x in $B \in \mathcal{B}_x$, where $B \subseteq \downarrow x$, define $G(x, B) = S(x) \cap B$. This is an operator of monotone normality for X: suppose $z \in G(x, B) \cap G(x', B')$. Then $S(x)$ meets $S(x')$, and suppose without loss of generality that $x' \in \downarrow x$, that is, $x' \preceq x$. As $z \in B \subseteq \downarrow x$, we have $z \preceq x$. Hence, $[z, x]$ is contained in B. But as $z \in B' \subseteq \downarrow x'$, we have $z \preceq x'$. Hence x' is in $[z, x]$, and so in B. ■

Observe that any subspace $A \subseteq \nabla(\omega + 1)^\omega$ with the partial order $\subseteq^*$ and basic neighborhoods $N_A(\overline{x}, f)$ satisfies (1) and (2) of the above lemma. Hence, we deduce that $\nabla(\omega + 1)^\omega$ is monotonically normal if and only if we can halve the neighbornet $T(x) = N(x, c_0)$.

Theorem 79. *Let A be any subset of $\omega^{\subset\omega}$ with the property that for almost compatible functions $x, y \in A$, $x \vee y \in A$. Then the following are equivalent: (1) $\Delta(A)$ holds, (2) $\nabla^*(A)$ is monotonically normal, and (3) $\nabla(A)$ is monotonically normal.*

Proof. Since monotone normality is hereditary, (3) implies (2). We prove, (2) implies (1) and then (1) implies (3).

Assume that $\nabla^*(A)$ is monotonically normal. By Lemma 78, the neighbornet $T(x) = N_A(\overline{x}, c_0)$ is halvable with witness S. For every $x \in A$, let $f_x \in \omega^\omega$ so $N_A(\overline{x}, f_x) \subseteq S(x)$. We check that $F(x) = f_x$ satisfies $\Delta(A)$: if $x, y \in A$ are as in Definition 72, then $x \notin N_A(y, c_0)$ and $y \notin N_A(x, c_0)$, though $x \vee y \in N_A(x, c_0) \cap N_A(y, c_0)$. By halvability, $S(x) \cap S(y) = \emptyset$, hence $N_A(\overline{x}, f_x) \cap N_A(\overline{y}, f_y) = \emptyset$. Hence $x \vee y$ is not in $N_A(\overline{x}, f_x)$ or $N_A(\overline{y}, f_y)$, suppose not in $N_A(\overline{x}, f_x)$. Since $x \subseteq^* x \vee y$, Lemma 64 applie so $x \vee y \notin N_A(\overline{x}, f_x)$ if and only if $y \setminus x =^* (x \vee y) \setminus x \not\geq^* f_x$, as desired.

Now assume $\Delta(A)$ holds with witness $F : A \to \omega^\omega$. Let $S(x) = N_{\nabla(A)}(\overline{x}, F(x))$, we check that S halves $T(x) = N_{\nabla(A)}(\overline{x}, c_0)$: pick $x, y \in$

$\nabla(A)$. If one of x or y is c_ω or belongs to ω^ω, halvability is easily checked. Suppose $x, y \in \omega^{\subset\omega}$. If $x \subseteq^* y$ (Remark 65 (i)), by Lemma 64 $y \notin S(x)$ if and only if $S(x) \cap S(y) = \emptyset$. If $\exists^\infty n \in dom(x) \cap dom(y)$ so $x(n) \neq y(n)$ (Remark 65 (ii)), clearly $S(x) \cap S(y) = \emptyset$. Finally, if x, y are as in $\Delta(A)$ (Remark 65 (iii)), then $x \setminus y \not\geq^* F(y)$ or $y \setminus x \not\geq^* F(x)$. In either case, Remark 66 indicates $N(x, F(x)) \cap N(y, F(y)) = \emptyset$, and we are done. ■

Example 80. A space X and a subspace $A \subseteq X$ obtained from X by removing all of its isolated points plus one limit point such that A is paracompact, but X is not.

Proof. Consider an almost disjoint family $\mathcal{A}$ on ω of size $\mathfrak{c}$. Let X be the space $\Psi(\mathcal{A}) = \mathcal{A} \cap \omega$ (see Chapter 3). The set of isolated points in X is ω and every point in $\mathcal{A}$ is an accumulating point in X. Fix $x \in \mathcal{A}$ and let A be $X \setminus (\{x\} \cup \omega) = \mathcal{A} \setminus \{x\}$. Then, A is the desired subspace.

Observe that A is a closed discrete space, hence paracompact. However, since $|\mathcal{A}| = \mathfrak{c}$, X is not normal (hence, not paracompact) by the next result known as the Jones' Lemma. ■

Lemma 81 (Jones, [13])**.** *Let X be a separable and normal space. Then for any set D that is a closed and discrete set in X, we have $2^{|D|} \leq 2^\omega$.*

4.4 Embeddings into Nabla Products

§ 4.4.1 Metrizable Factors

This section strongly involves the property of monotone normality. The key result of this section is the following.

Theorem 82. *Let $\{X_n : n \in \omega\}$ be a family of metrizable spaces. If $\nabla_n X_n$ is monotonically normal then it is hereditarily paracompact.*

This follows immediately from the next proposition and Balogh and Rudin's characterization of paracompactness in monotonically normal spaces:

Theorem 83 ([1]). *Suppose X is a monotonically normal space. Then, X is paracompact if and only if X does not contain closed copies of stationary subsets of regular uncountable cardinals.*

Proposition 84. *Let $\{X_n : n \in \omega\}$ be a family of metrizable spaces and S a stationary subset of a regular uncountable cardinal κ. Then S doesn't embed into $\nabla_n X_n$.*

Proof. Suppose, for a contradiction, $\varphi : S \to \nabla_n X_n$ is an embedding. We split the proof in two cases. If $\kappa \leqslant \mathfrak{d}$, then any α in $Lim(S)$ has $cf(\alpha) \neq \mathfrak{d}$, so S, and $A = \varphi(S)$, have limit points but no points of character $\mathfrak{d}$, contradicting Lemma 71. If $\kappa > \mathfrak{d}$, by Lemma 85 the map φ is eventually constant, thus it cannot be an embedding. ■

Lemma 85. *Let $\{(X_n, d_n) : n \in \omega\}$ be a family of metric spaces and S a stationary subset of a regular uncountable cardinal $\kappa > \mathfrak{d}$. Then, every continuous function $\varphi : S \to \nabla_n X_n$ is eventually constant.*

Proof. For each $n \in \omega$, write $B_n(a, \varepsilon)$ for the ε-ball around a in X_n with respect to the metric d_n. Let $\{f_\mu : \mu < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family. Then, for every $x \in \nabla_n X_n$ and $\mu < \mathfrak{d}$, define $N(x, f_\mu) = \nabla_{n\in\omega} B_n(x(n), {}^1/_{f_\mu(n)})$; $\{N(x, f_\mu) : \mu < \mathfrak{d}\}$ is a local basis at x.

Fix, for the moment, $\mu < \mathfrak{d}$. For every $\alpha \in Lim(S)$, pick $g_\mu(\alpha) < \alpha$, $g_\mu(\alpha) \in S$, such that $\varphi[(g_\mu(\alpha), \alpha]] \subseteq N(\varphi(\alpha), 2f_\mu)$. Then, g_μ is a regressive function and by the Pressing Down Lemma, there is an α_μ in S and stationary $S' = S'_\mu \subseteq S$ such that $\forall \beta \in S'$ $(g_\mu(\beta) = \alpha_\mu)$.

We claim that for all $\delta, \gamma > \alpha_\mu$, $\delta, \gamma \in S$, we have $\varphi(\gamma) \in N(\varphi(\delta), f_\mu)$: to see this, take any δ and γ strictly larger than α_μ in S. Since S' is stationary, there is a β in $Lim(S')$ with $\beta > \max\{\gamma, \delta\}$. Then, $\{\varphi(\gamma), \varphi(\delta)\} \subseteq \varphi[(g_\mu(\beta), \beta]] \subseteq N(\varphi(\beta), 2f_\mu)$. So by definition of $N(\varphi(\beta), 2f_\mu)$, for all $n \in \omega$ we have $\varphi(\gamma)(n), \varphi(\delta)(n) \in B_n(\varphi(\beta)(n), {}^1/_{2f_\mu(n)})$. Then by symmetry and the triangle inequality, for all n, we see $d_n(\varphi(\gamma)(n), \varphi(\delta)(n)) < {}^1/_{f_\mu(n)}$. Now by definition of $N(\varphi(\delta), f_\mu)$ we have $\varphi(\gamma) \in N(\varphi(\delta), f_\mu)$, as claimed.

Now, as we let μ run over all values below $\mathfrak{d}$, since $\kappa > \mathfrak{d}$, there is a least upper bound α_∞ of $\{\alpha_\mu : \mu < \mathfrak{d}\}$ in S. Notice that by the claim above, for any $\mu < \mathfrak{d}$ and $\gamma, \delta \in S \setminus \alpha_\infty$, we have $\varphi(\gamma) \in N(\varphi(\delta), f_\mu)$, and so $\varphi(\gamma) = \varphi(\delta)$. Hence φ is constant from α_∞ on, as desired. ∎

§ 4.4.2 Non-Metrizable Factors

In this section we check some embeddings into nabla products of non-metrizable factors.

Proposition 86. *There is an embedding φ from the stationary set $E^{\omega_2}_{\omega_1} = \{\alpha \in \omega_2 : cf(\alpha) = \omega_1\}$ to $\nabla(\omega_2+1)^\omega$. Moreover, $\overline{\varphi[E^{\omega_2}_{\omega_1}]} = \varphi[E^{\omega_2}_{\omega_1}] \cup \{c_{\omega_2}\}$.*

Proof. Define $\varphi : E^{\omega_2}_{\omega_1} \to \nabla(\omega_2 + 1)^\omega$ as $\varphi(\alpha) = c_\alpha$, where c_α is the constant α function. Clearly, φ is injective. We will show that it is open and continuous: for any $\alpha \in E^{\omega_2}_{\omega_1}$ and open $U = (\beta, \alpha] \cap E^{\omega_2}_{\omega_1}$, we see that $\varphi[U]$ is open in $\varphi[E^{\omega_2}_{\omega_1}]$. Take $c_\gamma \in \varphi[U]$, then the $V = \varphi[E^{\omega_2}_{\omega_1}] \cap \nabla(\beta, \gamma]^\omega$ is an open set in $\varphi[E^{\omega_2}_{\omega_1}]$ contained in $\varphi[U]$. Now, to see that φ is continuous, let $V = N(c_\alpha, f)$ be a neighborhood of $\varphi(\alpha)$, where $f \in \alpha^\omega$. Since $cf(\alpha) = \omega_1$, there is a supremum $\alpha' = \sup\{f(n) : n \in \omega\} < \alpha$. Hence, $\varphi[(\alpha', \alpha] \cap E^{\omega_2}_{\omega_1}] \subseteq V$.

It remains to prove $\overline{\varphi[E^{\omega_2}_{\omega_1}]} = \varphi[E^{\omega_2}_{\omega_1}] \cup \{c_{\omega_2}\}$. To see $c_{\omega_2} \in \overline{\varphi[E^{\omega_2}_{\omega_1}]}$ take $f \in \omega_2^\omega$. If $\alpha' = \sup\{f(n) : n \in \omega\}$, then $\{c_\alpha : \alpha > \alpha'\} \subseteq N(c_{\omega_2}, f)$. So, $\varphi[E^{\omega_2}_{\omega_1}] \cap N(c_{\omega_2}, f) \neq \emptyset$. Now, fix $x \notin \varphi[E^{\omega_2}_{\omega_1}] \cup \{c_{\omega_2}\}$. We will prove $x \notin \overline{\varphi[E^{\omega_2}_{\omega_1}]}$. There are two options for x, either x is a constant function or not. If x is constant, then $x = c_\beta$, for some β with $cf(\beta) < \omega_1$. If x is not constant, then there are disjoint infinite sets $N = \{n_k : k \in \omega\}, M = \{m_k : k \in \omega\} \subseteq \omega$ such that for $n \in N, m \in M$, $x(n) \neq x(m)$. For the first case, if $cf(\beta) = 1$, then x is isolated and we're done. If $cf(\beta) = \omega$, let $\{\gamma_n : n \in \omega\}$ be cofinal in β. Define $f(n) = \gamma_n$ and note that $c_\alpha \notin N(x, f)$, for any $\alpha \in E^{\omega_2}_{\omega_1}$. Assume now the second case holds. Then, for every $k \in \omega$ there are $\alpha_k \leqslant x(n_k)$ and $\beta_k \leqslant x(m_k)$ such that $(\alpha_k, x(n_k)] \cap (\beta_k, x(n_k)] = \emptyset$. Consider $f(n) = \alpha_k$, if $n = n_k \in N$, $f(n) = \beta_k$, if $n = m_k \in M$ and $f(n) = 0$, otherwise. It is easy to check that $\varphi[E^{\omega_2}_{\omega_1}] \cap N(x, f) = \emptyset$. This concludes the proof. ∎

If λ, κ are cardinals, denote by $D(\kappa)$ the discrete space of size κ and $L_\lambda(\kappa) = D(\kappa) \cup \{\kappa\}$, where neighborhoods around κ have the form $\{\kappa\} \cup (D(\kappa) \setminus C)$ and $C \subseteq D(\kappa)$ has size less $< \lambda$. Write $A(\kappa)$ for $L_\omega(\kappa)$, the one-point compactification of $D(\kappa)$, and $L(\kappa) = L_{\omega_1}(\kappa)$ for the one-point Lindelofication of $D(\kappa)$.

Lemma 87. *For any $n \in \omega$,*

1. *$L_{\omega_n}(\omega_n)$ embeds into $\nabla(\omega_n + 1)^\omega$, and*
2. *$L(\omega_n)$ embeds into $\nabla A(\omega_n)^\omega$.*

Proof. (1) for every isolated point $\alpha \in \omega_n + 1$, consider $c_\alpha \in \nabla(\omega_n + 1)^\omega$ the constant α function. Note that c_α is isolated. Observe that every neighborhood $N(c_{\omega_n}, f)$, where $f \in \omega_n^\omega$, contains all but $< \omega_n$ many elements c_α. Hence, $\{c_\alpha : \alpha \text{ is isolated in } \omega_n\} \cup \{c_{\omega_n}\}$ is homeomorphic to $L_{\omega_n}(\omega_n)$.

For (2), we check that $\{c_\alpha : \alpha \in D(\omega_n)\} \cup \{c_{\omega_n}\} \subseteq \nabla A(\omega_n)^\omega$ is homeomorphic to $L(\omega_n)$. Every c_α is isolated. Observe that a neighborhood around c_{ω_n} has the form $N(c_{\omega_n}, \langle G_k \rangle_k) = \nabla_k(A(\omega_n) \setminus G_k)$, where $G_k \subseteq D(\omega_n)$ is finite. Hence, for every α outside the countable set $\bigcup_{k \in \omega} G_k$, it happens $c_\alpha \in N(c_{\omega_n}, \langle G_k \rangle_k)$. ■

4.5 Δ-like Principles

In this section we define other combinatorial statements similar to Δ for countable nabla products when the factors are metrizable, ordinals and one-point compactifications of a discrete set. Hence, Δ is the core principle among the statements we will present. The motivation is proving that these statements are equivalent to the monotone normality of the corresponding nabla products. This establish a hierarchy of monotone normality on the class of countable nabla products for certain type of factors.

§ 4.5.1 Metrizable Factors

For this section, $\{(X_n, d_n) : n \in \omega\}$ will be a family of metric spaces. For $x, y \in \nabla_n X_n$ and $f \in \omega^\omega$, define $N(x, f) = \nabla_n B(x(n), {}^1/_{f(n)})$ and $M(x, f; y) = \{n \in \omega : y(n) \notin B(x(n), {}^1/_{f(n)})\}$. If $x(n)$ is isolated, declare ${}^1/_{f(n)}= 0$, that is $B(x(n), {}^1/_{f(n)}) = \{x(n)\}$.

Definition 88. Let $\{(X_n, d_n) : n \in \omega\}$ be a family of metric spaces and $R \subseteq \prod_n X_n$ be a maximal family of representative elements of $\nabla_n X_n$. Then $\Delta_{(X_n,d_n)_n}$ is the statement: there is $F : R \times \omega^\omega \to \omega^\omega$, write $f_x := F(x, f)$, such that for any $(x, f), (y, g) \in R\times\omega^\omega$ for which $M(x, f; y)$ and $M(y, g; f)$ are almost disjoint infinite sets, then ${}^1/_{f_x(n)}+{}^1/_{g_y(n)}< d_n(x(n), y(n))$ holds for infinitely many n's in ω.

Proposition 89. *$\Delta_{(X_n,d_n)_n}$ holds if and only if $\nabla_n X_n$ is monotonically normal.*

Proof. $\Longrightarrow$) Let F witness of $\Delta_{(X_n,d_n)_n}$ and define $G : \nabla_n X_n \times \mathcal{B} \to \tau$ as $G(x, N(x, f)) = N(x, \max\{2f, f_x\})$, where $\mathcal{B}$ is a base for the topology τ of $\nabla_n X_n$. Here, we are assuming that for any elements a, b in the same equivalent class of $\nabla_n X_n$, $f_a = f_b$.

We prove that G is a monotone normality operator. First observe that $x \in N(x, \max\{2f, f_x\}) \subseteq N(x, f)$. Now, to prove the second property of monotone normality, let $x, y \in \nabla_n X_n$, $f, g \in \omega^\omega$ and assume that $y \notin N(x, f)$ and $x \notin N(y, g)$. We have to prove that $N(x, \max\{2f, f_x\}) \cap N(y, \max\{2g, g_y\}) = \emptyset$. There are three cases for the sets $M(x, f; y)$ and $M(y, g; x)$:

- $M(x, f; y)$ or $M(y, g; x)$ is finite: if $M(x, f; y)$ is finite, then by its definition, $y(n) \in B(x(n), f(n))$ for all but finitely many $n \in \omega$. Hence, $y \in N(x, f)$ which is not possible by our assumption.
- $M(x, f; y) \cap M(y, g; x)$ is infinite: let $Z = M(x, f; y) \cap M(y, g; x)$. Then $y(n) \notin B(x(n), f(n))$ and $x(n) \notin B(y(n), g(n))$, for $n \in Z$. By triangle inequality, $B(x(n), 2f(n)) \cap B(y(n), 2g(n)) = \emptyset, n \in Z$. Thus, the sets $N(x, \max\{2f, f_x\})$ and $N(y, \max\{2g, g_y\})$ are disjoint.
- $M(x, f; y)$, $M(y, g; x)$ are infinite almost disjoint sets: by $\Delta_{(X_n,d_n)_n}$, for infinitely many $n \in \omega$, ${}^1/_{f_x(n)}+{}^1/_{g_y(n)}< d_n(x(n), y(n))$. This implies

that $B(x(n), f_x(n)) \cap B(y(n), g_y(n)) = \emptyset$ for infinite n's, and implies $N(x, \max\{2f, f_x\}) \cap N(y, \max\{2g, g_y\}) = \emptyset$.

$\Longleftarrow$) Now, assume that $\nabla_n X_n$ is monotonically normal with operator G. Define $F : R\times\omega^\omega \to \omega^\omega$ as $F(x, f) = f_x$, where $N(x, f_x) \subseteq G(x, N(x, f))$.

We prove that F witnesses $\Delta_{(X_n,d_n)_n}$. Let $(x, f), (y, g)$ such that the sets $M(x, f; y)$ and $M(y, g; f)$ are infinite almost disjoint, thus, $y \notin N(x, f)$ and $x \notin N(y, g)$. Since G is an operator of monotone normality, $G(x, N(x, f))\cap G(y, N(y, g)) = \emptyset$, and $N(x, f_x) \cap N(y, g_y) = \emptyset$. Hence, $B(x(n), f_x(n)) \cap B(y(n), g_y(n)) = \emptyset$ for infinitely many $n \in \omega$, and for these n's we have the inequality ${}^1/_{f_x(n)}+{}^1/_{g_y(n)}< d_n(x(n), y(n))$, as desired. ■

For a sequence of metrizable spaces $\{X_n : n \in \omega\}$, $\Delta_{(X_n)_n}$ means '$\Delta_{(X_n,d_n)_n}$ holds for some choice of compatible metrics d_n'. Abbreviate $\Delta_{(X)_n}$ to Δ_X. Also, for a class C of spaces, Δ_C means '$\Delta_{(X_n)_n}$ holds for any sequence $(X_n)_n$ of spaces from C'.

Write $\mathcal{M}$ for the class of all metrizable spaces, $\mathcal{M}(\kappa)$ for the class of all metrizable spaces of cardinality $\leq \kappa$, and $\mathcal{SM}$ for the class of separable metrizable spaces. The paragraph after Definition 2.3 in [33] mention that any metrizable separable space (second countable) is isomorphic to a subset of $H(\omega_1)$. Thus, if $\{X_n : n \in \omega\}$ is a family of separable metrizable spaces, then $\omega^\omega, \nabla_n X_n \subseteq H(\omega_1)$.

Proposition 90.

- $\mathfrak{d} = \mathfrak{b}$ *implies* $\Delta_{\mathcal{M}}$,
- $\mathfrak{d} = \mathfrak{c}$ *implies* $\Delta_{\mathcal{M}(\mathfrak{c})}$, *and*
- MH *implies* $\Delta_{\mathcal{SM}}$.

Proof. We will prove each fact separately.

For $\mathfrak{d} = \mathfrak{b}$: Let $\{f_\alpha : \alpha < \mathfrak{d}\}$ be an scale on ω such that f_α dominates $\{2f_\beta : \beta < \alpha\}$. Let $\downarrow f_\alpha = \{f \in \omega^\omega : f \leqslant^* f_\alpha\}$ and define the function

$F : \nabla_n X_n \times \omega^\omega \to \omega^\omega$ as $F(x, f) = 2f_\alpha$ if and only if α is the minimum such that $f \in\downarrow f_\alpha$.

Pick $(x, f), (y, g) \in \nabla_n X_n \times \omega^\omega$, such that $M(x, f; y)$ and $M(y, g; f)$ are infinite almost disjoint. We may assume that $f \in\downarrow f_\alpha$, $g \in\downarrow f_\beta$, for minimum β, α and $\beta \leqslant \alpha$. Then, $g \leqslant^* f_\beta \leqslant^* 2f_\beta \leqslant^* f_\alpha$. Now, $n \in M(y, g; x)$ implies $x(n) \notin B(y(n), g(n))$ implies $1/_{g(n)} < d_n(y(n), x(n))$ implies $1/_{2g(n)} + 1/_{2f_\alpha(n)} < d_n(y(n), x(n))$ implies $1/_{2f_\beta(n)} + 1/_{2f_\alpha(n)} < d_n(y(n), x(n))$. Since $M(y, g; x)$ is infinite, we are done.

For $\mathfrak{d} = \mathfrak{c}$: Since X_n is metrizable separable, then $|X_n| = |\nabla_n X_n| = \mathfrak{d}$. Enumerate $\nabla_n X_n \times \omega^\omega = \{(x_\alpha, f_\alpha) : \alpha < \mathfrak{d}\}$. Fix $\alpha < \mathfrak{d}$ and suppose F is defined on $\{(x_\beta, f_\beta) : \beta < \alpha\}$ with $F(x_\beta, f_\beta) \geqslant^* 2f_\beta$ and satisfies $\Delta_{(X_n)_n}$. The sets $\mathcal{F} = \{2f_\beta : \beta < \alpha\}$ and $\mathcal{A} = \{M(x_\beta, f_\beta; x_\alpha) : \beta < \alpha\}$ have size less than $\mathfrak{d}$. Proposition 67 applies, so there is $f'_\alpha \in \omega^\omega$ such that $2f_\beta \restriction M(x_\beta, f_\beta; x_\alpha) \ngeqslant^* f'_\alpha$, $\beta < \alpha$. Define $F(x_\alpha, f_\alpha) = 2\max\{f_\alpha, f'_\alpha\}$. This construction defines F on $\nabla_n X_n \times \omega^\omega$. To see that F witness $\Delta_{(X_n)_n}$ pick $(x_\beta, f_\beta), (x_\alpha, f_\alpha)$ so $M(x_\beta, f_\beta; x_\alpha)$ and $M(x_\alpha, f_\alpha; x_\beta)$ are infinite almost disjoint, and assume $\beta < \alpha$. Let $M = \{n \in M(x_\beta, f_\beta; x_\alpha) : f'_\alpha(n) > 2f_\beta\}$. Hence, for $n \in M$, $x_\alpha(n) \notin B(x_\beta(n), 1/_{f_\beta(n)})$ and $f'_\alpha(n) > 2f_\beta$. It is clear that for $n \in M$, $1/_{F(x_\beta, f_\beta)(n)} + 1/_{F(x_\alpha, f_\alpha)(n)} < d_n(x_\beta(n), x_\alpha(n))$.

For MH: Let H_α be as in MH and f_α be a witness that $H_\alpha \cap \omega^\omega$ is not dominant. We may assume that $H_\alpha \subseteq H_{\alpha+1}$ and that $f_\alpha \in H_{\alpha+1}$. Define $F : \nabla_n X_n \times \omega^\omega \to \omega^\omega$ as $F(x, f) = 2f_\alpha$ if and only if α is the least such that $(x, f) \in H_\alpha$, this is possible since $\nabla_n X_n \times \omega^\omega \subseteq H(\omega_1)$. Now, if $(x, f), (y, g) \in \nabla_n X_n \times \omega^\omega$, then $(x, f) \in H_\beta, (y, g) \in H_\alpha$, form some $\beta \leqslant \alpha$. Notice that $2g, 2f, M(x, f; y)$ and $M(y, g; f)$ are in H_α, and for any $h \in H_\alpha \cap \omega^\omega$ and $a \in H_\alpha \cap [\omega]^\omega$, $h \restriction a \ngeqslant^* f_\alpha$. Use similar arguments from previous cases to conclude that for infinitely many $n \in \omega$, $1/_{F(x,f)(n)} + 1/_{F(y,g)(n)} < d_n(x(n), y(n))$. ■

§ 4.5.2 Ordinal Factors

Fix a limit ordinal α. For any $x \in \alpha$ we have two cases: (i) there is $\beta_x \in \alpha$ such that $x \in (\beta_x, \beta_x + \omega]$, in which case x is isolated or $x = \beta_x + \omega$, or (ii) x is a limit of limits. If (i) holds, write $S_x = (\beta_x, \beta_x + \omega]$.

Definition 91. Let α be any ordinal and $x, y \in \nabla(\alpha+1)^\omega$. We say that x, y *switch* if there are infinite sets $M_x, M_y \subseteq \omega$ such that for $n \in M_y$, $x(n) < y(n) \in Lim(\alpha+1)$, for $n \in M_x$, $y(n) < x(n) \in Lim(\alpha+1)$, and $|\{n \in \omega : x(n), y(n) \text{ are isolated and } x(n) \neq y(n)\}| < \omega$.

Remark 92. If $x, y \in \nabla(\alpha+1)^\omega$ switch, then $x \notin N(y, c_0)$ and $y \notin N(x, c_0)$, where c_0 is the constant 0 function.

Proof. For any $n \in M_y$, $x(n) < y(n) \in Lim(\alpha+1)$, Then, $y(n) \notin [0, x(n)]$, $n \in M_y$. Hence $y \notin N(x, c_0)$. The other case is similar. ■

Definition 93. $\Delta[\alpha]$ is the statement: there is $F : \nabla(\alpha+1)^\omega \to \alpha^\omega$ such that if $x, y \in \nabla(\alpha+1)^\omega$ switch, then $y(n) < F(x)(n) < x(n)$ for infinitely many $n \in M_x$ or $x(n) < F(y)(n) < y(n)$ for infinitely many $n \in M_y$ (the conclusion implies that $N(x, F(x)) \cap N(y, F(y)) = \emptyset$).

Proposition 94. *The following are equivalent: (i) $\Delta[\alpha]$ holds, (2) $\nabla(\alpha+1)^\omega$ is halvable, and (3) $\nabla(\alpha+1)^\omega$ is monotonically normal.*

Proof. (3) implies (2) is clear. For (1) implies (3), define $H : \nabla(\alpha+1)^\omega \to \alpha^\omega$ as $H(x)(n) = x(n)$ if $x(n)$ is isolated, $H(x)(n) = \min\ S_x$, if $x = \beta_x + \omega$, and $H(x)(n) = 0$, if x is a limit of limits; recall $S_x = (\beta_x, \beta_x + \omega]$. Now define $G : \nabla(\alpha+1)^\omega \times \alpha^\omega$ as $G(x, f) = N(x, f_x)$, where $f_x = \max\{f, H_0(x), F(x)\}$.

We prove that the operator G is monotonically normal: pick elements $(x, f), (y, g) \in \nabla(\alpha+1)^\omega \times \alpha^\omega$. Let $N_0 = \{n \in \omega : x(n) = y(n)\}$, $N_1 = \{n \in \omega : x(n) < y(n)\}$ and $N_2 = \{n \in \omega : y(n) < x(n)\}$. There are three cases for x and y, (i) there are infinitely many $n \in N_1$ where $y(n) \in Lim(\alpha)$ and infinitely many $n \in N_2$ where $x(n) \in Lim(\alpha)$, (ii) there is $N \subseteq N_1$ infinite such that $x(n)$ is isolated, or similarly for y, and (iii) N_2 is empty and for every $n \in N_1$, $y(n) \in Lim(\alpha)$ or vice versa for N_1, N_1 and x. If (i) holds, then x and y switch, hence $\Delta[\alpha]$ applies and $G(x, f) \cap G(y, g) = \emptyset$. If (ii) holds, then for $n \in N$, $[0, y(n)] \cap \{x(n)\} = \emptyset$, hence $G(x, f) \cap G(y, g) = \emptyset$. Finally, assume (iii). Hence, $G(x, f) \cap G(y, g) \neq \emptyset$ implies $x \in G(y, g) \subseteq N(y, g)$.

For (2) implies (1), consider the neighbornet $T(x) = N(x, c_0)$. Then, there is a neighbornet S that halves T. For $x \in \nabla(\alpha+1)^\omega$, let $F(x)$ such that $N(x, F(x)) \subseteq S(x)$. To see that F satisfies $\Delta[\alpha]$, pick x, y that switch. By Remark 92, $x \notin N(y, c_0)$ and $y \notin N(x, c_0)$, hence by halvability,

$N(x, F(x)) \cap N(y, F(y)) = \emptyset$. It is clear that $y(n) < F(x)(n) < x(n)$ for infinitely many $n \in M_x$ or $x(n) < F(y)(n) < y(n)$ for infinitely many $n \in M_y$. ■

§ 4.5.3 $A(\kappa)$ Factors

We may redefine that two elements switch, now for elements $x, y \in \nabla A(\kappa)^\omega$, if $x(n) \in D(\kappa), y(n) = \kappa$ and $y(n) \in D(\kappa), x(n) = \kappa$ for infinitely many $n \in \omega$, and $|\{n \in \omega : x(n), y(n) \in D(\kappa) \text{ and } x(n) \neq y(n)\}| < \omega$. Denote by $D(\kappa)^{\subset\omega}$ the set of partial functions from ω to $D(\kappa)$ with infinite-coinfinite domain.

Definition 95. $\Delta[A(\kappa)]$ is the statement: there is $F : D(\kappa)^{\subset\omega} \longrightarrow ([\kappa]^{<\omega})^\omega$ such that for $x, y \in D(\kappa)^{\subseteq\omega}$ with (i) $|x \setminus y| = |y \setminus x| = \omega$ and (ii) $\{n \in dom(x) \cap dom(y) : x(n) \neq (n)\}$ is finite, then $(x \setminus y)(n) \in F(y)(n)$ or $(y \setminus x)(n) \in F(x)(n)$ for infinitely many $n \in \omega$.

Let $\nabla^* A(\kappa) = \{\overline{x} \in \nabla A(\kappa)^\omega : x \in D(\kappa)^{\subseteq\omega}\}$.

Corollary 96. *$\Delta(A(\kappa))$ holds if and only if $\nabla A(\kappa)^\omega$ is monotonically normal if and only if $\nabla^* A(\kappa)$ is monotonically normal.*

Proof. The proof is similar to Theorem 79 and uses Lemma 78. ■

Corollary 97. *The following are equivalent: $\nabla(\omega + 1)^\omega$ is monotonically normal, Δ holds, $\Delta_{\omega+1}$ holds, $\Delta[\omega]$ holds and $\Delta[A(\omega)]$ holds.*

4.6 Non-Hereditarily Normal Nabla Products

In this section we check some nabla products which are (consistently) non-hereditarily normal. We answer Roitman's Question 106: Is $\nabla A(\omega_1)^\omega$ consistently non-hereditarily normal? Also, we discuss a gap in the proof of a result in [33], Theorem 105.3, and give a counterexample. The following result can be found in [34], and show that $\Delta[A(\omega_2)]$ is false. However, we give another proof that involves the spaces $L(\kappa)$ from Section 4.4.2.

Theorem 98 (Roitman)**.** *$\nabla A(\omega_2)^\omega$ is not hereditarily normal.*

Proof. Note that $L(\omega_2)$ is embedded into $\nabla A(\omega_2)^\omega$ and $\nabla A(\omega_2)^\omega$ is homeomorphic to its square. Then, Lemma 100 applies. ■

Lemma 99. *For any $\alpha < \omega_2$ choose $A_\alpha \in [\omega_2]^\omega$. Then there is $\delta \geqslant \omega_1$ such that for any $\alpha < \delta$, $A_\alpha \subseteq \delta$.*

Proof. Let $M \prec H(\omega_3)$ be an elementary submodel of size ω_1 such that $\{A_\alpha : \alpha < \omega_2\} \in M$. Let $\delta = M \cap \omega_2 \in \omega_2$. Now, for $\alpha \in M$, M thinks 'A_α is contained in M', and so does $H(\omega_3)$. ■

Lemma 100. *$L(\omega_2)^2$ is not hereditarily normal.*

Proof. We check that $Y = L(\omega_2) \times L(\omega_2) \setminus \{(\omega_2, \omega_2)\}$ is not normal. Note that the sets $H = (L(\omega_2) \setminus \{\omega_2\}) \times \{\omega_2\}$ and $K = (\{\omega_2\} \times L(\omega_2)) \setminus \{\omega_2\}$ are disjoint and closed in Y. By hypothesis, there are U, V separating H and K. So, for every $(\alpha, \omega_2) \in H$ choose $A_\alpha \in [D(\omega_2)]^\omega$ so $\{\alpha\} \times (L(\omega_2) \setminus A_\alpha) \subseteq U$, and similarly elements $(\omega_2, \beta) \in K$ are choosing sets B_β so $(L(\omega_2) \setminus B_\beta) \times \{\beta\} \subseteq V$. By Lemma 99, there is $\delta \geqslant \omega_1$ so for every $\alpha < \delta$, if $(\alpha, \omega_2) \in H$, then $A_\alpha \subseteq \delta$. Pick $\beta > \delta$. Since δ is uncountable and B_β countable, there is $\alpha \in \delta \setminus B_\beta$. Hence $\{\alpha\} \times (L(\omega_2) \setminus A_\alpha) \cap (L(\omega_2) \setminus B_\beta) \times \{\beta\} \neq \emptyset$. ■

The following result shows that $\Delta[\omega_2]$ is false.

Theorem 101. *The space $\nabla(\omega_2 + 1)^\omega$ is not hereditarily normal.*

Proof. From Proposition 86, let $S = E^{\omega_2}_{\omega_1}$. Then, S and $\overline{S} = S \cup \{c_{\omega_2}\}$ embed into $\nabla(\omega_2 + 1)^\omega$. Now, Lemma 102 applies. ■

Lemma 102. *Let S be a stationary subset of a regular uncountable cardinal κ. Then, $S \times \overline{S}$ (as subspace of κ^2) is not normal, where $\overline{S} = S \cup \{\kappa\}$.*

Proof. Consider the diagonal $H = \{(\alpha, \alpha) : \alpha \in S\}$ and the top edge $K = \{(\alpha, \kappa) : \alpha \in S\}$. Note that H and K are closed disjoint sets. Now, if U, V are neighborhoods of H, K, respectively, let $g(\alpha) \in \alpha \cap S$ so that $[g(\alpha), \alpha] \times [g(\alpha), \alpha] \subseteq U$ (whenever we write an interval $[\theta, \lambda]$ we mean $S \cap [\theta, \lambda]$). Since g is a pressing down function, there is some $\delta \in S$ and there is a stationary set $S_0 \subseteq S$ such that all points in S_0 are mapped to δ

by g. Specifically, for each $\alpha \in S_0$, we have $[\delta,\alpha]\times[\delta,\alpha] \subset U$. Choose $\beta \in S_0 \setminus \delta$ an isolated point (every stationary set has isolated points). We have $(\beta,\omega_1) \in K \subset V$. Choose $\gamma > \delta$ such that $\{\beta\}\times[\gamma,\omega_1] \subset V$. Choose $\alpha \in S_0$ such that $\alpha > \max\{\gamma,\beta\}$. We have the following set inclusions: $\{\beta\}\times[\gamma,\alpha] \subset [\delta,\alpha]\times[\delta,\alpha] \subset U$ and $\{\beta\}\times[\gamma,\alpha] \subset \{\beta\}\times[\gamma,\omega_1] \subset V$. Thus $U \cap V \neq \emptyset$. Hence, H and K cannot be separated by open sets. ■

Definition 103. A space X is *κ-metrizable* if it has an open base $\mathcal{B} = \{U_{x,\alpha} : \alpha < \kappa,\ x \in X\}$ so that $\{U_{x,\alpha} : \alpha < \kappa\}$ is a neighborhood base at x, and given two points x, y and two ordinals $\alpha \leqslant \beta < \kappa$ then (i) if $y \in U_{x,\alpha}$ then $U_{y,\beta} \subseteq U_{x,\alpha}$; and (ii) if $y \notin U_{x,\alpha}$ then $U_{y,\beta} \cap U_{x,\alpha} = \emptyset$.

Remark 104. Suppose X is κ-metrizable with basis $\{U_{x,\alpha} : \alpha < \kappa\}$ at x, $x \in X$. Then for every $\alpha < \kappa$, $\{U_{x,\alpha} : x \in X\}$ is pairwise disjoint.

A κ-metrizable space is paracompact and monotonically normal, hence hereditarily normal. The following results are Proposition 6.3, Theorem 6.1 and Proposition 6.4 in [33].

Theorem 105 (Roitman)**.**

1. *If $\mathfrak{b} = \mathfrak{d}$ then the nabla product of countably many compact metrizable spaces is $\mathfrak{b}$-metrizable.*
2. *If $\mathfrak{b} = \mathfrak{d} < \aleph_\omega$ and each X_n has weight $\leqslant \mathfrak{d}$ then $\nabla_n X_n$ is $\mathfrak{b}$-metrizable.*
3. *Let $\kappa < \mathfrak{b} = \mathfrak{d} < \aleph_\omega$. If the nabla product of countably many spaces of weight κ is $\mathfrak{b}$-metrizable, then the nabla product of countably many spaces of weight κ^+ is $\mathfrak{b}$-metrizable.*

Part 3 implies part 2 of previous theorem by finite induction. However, Roitman's proof of part 2 contains a gap. She used part 1 to upgrade a construction of a basis witnessing κ-metrizability, and the construction required first countability, such thing cannot be assured in the process.

Here is a counterexample to previous theorem. The compact spaces $A(\omega_2)$ and (ω_2+1) have weight ω_2, but $\nabla A(\omega_2)^\omega$ and $\nabla(\omega_2+1)^\omega$ can not be κ-metrizable as shown in Theorem 98 and Theorem 101. This shows that Theorem 105.2 fails (Theorem 105.3 fails as well, but Theorem 105.1 is true).

§ 4.6.1 $\nabla A(\omega_1)^\omega$ and $\nabla(\omega_1+1)^\omega$

In this section we prove Corollary 107, which shows that $\Delta[A(\omega_1)]$ and $\Delta[\omega_1]$ are independent from ZFC. This solves the following:

Question 106 (Roitman, [34])**.** Is $\nabla A(\omega_1)^\omega$ consistently non-hereditarily normal?

Corollary 107. *If $\mathfrak{b} > \omega_1$, then $\nabla A(\omega_1)^\omega$ and $\nabla(\omega_1+1)^\omega$ are non hereditarily normal.*

Proof. The spaces $L(\omega_1)$ and $\nabla(\omega+1)^\omega$ embed into both $\nabla A(\omega_1)^\omega$ and $\nabla(\omega_1+1)^\omega$. Hence, Theorem 108 applies. ■

We weill prove Theorem 108 with help of Lemma 109. For this, we are going to use a well known object in Descriptive Set Theory, the so called K-Luzin sets. A subset L of ω^ω is a *K-Luzin* set if it is uncountable and meets every infinite compact of ω^ω (ω^ω with the product topology) in a countable set, or equivalently, for every $g \in \omega^\omega$, the set $\{f \in L : f \leqslant^* g\}$ is countable. Observe that any subspace of K-Luzin is K-Luzin, hence the existence of a K-Luzin set is equivalent to $\mathfrak{b} = \omega_1$. Also, $|L| \leqslant \mathfrak{d}$.

Denote by $X(\omega^\omega, \leqslant^*)$ the subspace $\omega^\omega \cup \{c_\omega\}$ of $\nabla(\omega+1)^\omega$ and write $N(c_\omega, f) = \{\overline{g} \in \omega^\omega : g \geqslant^* f\} \cup \{c_\omega\}$, $f \in \omega^\omega$, the neighborhoods around c_ω in $X(\omega^\omega, \leqslant^*)$.

Theorem 108. *The space $L(\omega_1) \times X(\omega^\omega, \leqslant^*)$ is hereditarily normal if and only if $\mathfrak{b} = \omega_1$.*

Proof. We prove the equivalence '$L(\omega_1) \times X(\omega^\omega, \leqslant^*)$ is hereditarily normal if and only if there is a K-Luzin set'.

For the sufficiency, let $p = (\omega_1, c_\omega)$ be the top-right corner of the given product. In $X' = L(\omega_1) \times X(\omega^\omega, \leq^*) \setminus \{p\}$ the top edge, $T = L(\omega_1) \times \{c_\omega\} \setminus \{p\}$, and right edge, $R = \{\omega_1\} \times X(\omega^\omega, \leq^*) \setminus \{p\}$, are disjoint closed sets. Hence there are disjoint open U and V such that $T \subseteq U$ and $R \subseteq V$. For each $\alpha < \omega_1$, pick f_α such that $\{\alpha\} \times N(c_\omega, f_\alpha) \subseteq U$. For each g in ω^ω pick countable $C_g \subseteq \omega_1$ such that $(L(\omega_1) \setminus C_g) \times \{g\} \subseteq V$.

Let $A = \{f_\alpha : \alpha < \omega_1\}$. The choice of the f_α's can be in such way so they are all different, so the enumeration of A is injective. We check that A is K-Luzin. Take any g in ω^ω, then for any α not in C_g, as U and V are disjoint, (α, g) is not in $\{\alpha\} \times N(c_\omega, f_\alpha)$, so $\neg(f_\alpha <^* g)$. Hence, $\{\alpha \in \omega_1 : f_\alpha \leq^* g\}$ is contained in C_g, and so is countable.

Now, the necessity follows from Lemma 109 and the fact that $L(\omega_1) \times X(\omega^\omega, \leq^*)$ is regular and points in $(L(\omega_1) \setminus \{\omega_1\}) \times \omega^\omega$ are isolated. ■

Lemma 109. *Suppose $A \subseteq T, B \subseteq R$ (T, R as in the proof of previous theorem). If there is a K-Luzin set, then there are sets U, V open in $L(\omega_1) \times X(\omega^\omega, \leq^*)$ separating A and B.*

Proof. If $A = \{(\alpha, c_\omega) : \alpha \in S\}$, for some $S \subseteq \omega_1$, is countable, the result is clear. Hence, suppose S is uncountable. Let $L = \{f_\alpha : \alpha \in S\} \subseteq \omega^\omega$ be a K-Luzin set such that the enumeration is bijective. For every $g \in \omega^\omega$, $C_g = \{\alpha \in S : f_\alpha \leqslant^* g\}$ is countable. Hence $U = \bigcup_{\alpha \in S}\{\alpha\} \times N(c_\omega, f_\alpha)$ and $V = \bigcup_{(g,\omega_1) \in B}(L(\omega_1) \setminus C_g) \times \{g\}$ work. ■

4.7 Subspaces of $\nabla(\omega + 1)^\omega$ and Restrictions on Δ

In this section we mention some results in [32] and [33] concerning to subspaces of $\nabla(\omega + 1)^\omega$ and their paracompactness. We also study the notion of tangle-free filter due to Gartside (see [15]), and we connect it to a restriction on Δ which is true in ZFC. Finally, we introduce the principle Λ which is a weaker variant of Δ and characterize basic ultraparacompactness of ∇^*. We begin stating the machinery.

Definition 110. A subspace Y of X is *strongly separated* if there is a discrete open collection $U = \{U_y : y \in Y\}$ with $Y \cap U_y = \{y\}$. We say that U strongly separates Y.

If $x \in \nabla^*$, define $x^\perp(n) = x(n)$, if $x(n) \leqslant x(m)$ for $m \in dom(x) \setminus n$, and $x^\perp(n) = \omega$, otherwise. Let $x_0 = x^\perp$ and $x_{n+1} = (x \setminus x_n)^\perp$. For $n \in \omega$, let $\nabla_n = \{x \in \nabla^* : x_{n+1} = c_\omega\}$ and $\nabla_\omega = \nabla^* \setminus \bigcup_n \nabla_n$. Note that ∇_0 is the set of non-decreasing partial functions in ∇^* ($x \in \nabla_0$ if and only if $x_0 =^* x$) and $x \in \nabla_n$ if and only if $x =^* \bigcup_{i \leqslant n} x_i$.

Theorem 111 (Roitman, [32])**.** *Each* $\nabla_n \subseteq \nabla^*$ *is discrete. The set* $INC = \nabla_0$ *is strongly separated.*

Observe that the set INC being strongly separated implies the existence of a function $F : INC \to \omega^\omega$ so $N(x, F(x)) \cap N(y, F(y)) = \emptyset$, for $x \neq y$ in INC. In particular, x and y may satisfy the conditions of Δ. In other words, $\Delta(INC)$ is true in ZFC. We show now that $\Delta(INC)$ implies another condition on Δ.

Definition 112. The statement $\Delta[AD]$ is: there is $F : \omega^{\subset\omega} \to \omega^\omega$ such that for $x, y \in \omega^{\subset\omega}$ with $|dom(x) \cap dom(y)| < \omega$ (x, y almost disjoint), then $x \not\geqslant^* F(y)$ or $y \not\geqslant^* F(x)$.

Lemma 113. $\Delta(INC)$ *implies* $\Delta[AD]$.

Proof. For $x \in \omega^{\subset\omega}$ consider any operation converting x to a non-decreasing partial function x' with $dom(x) = dom(x')$ (for example, $x'(n) = (\sum_{i\leqslant n} x(i))$ iff $n \in dom(x)$). Let F' be a witness of $\Delta(INC)$.

Let $F : \omega^{\subset\omega} \to \omega^\omega$ given by $F(x) = F'(x')$. Now, pick almost disjoint $x, y \in \omega^{\subset\omega}$. We claim that $x \not\geqslant^* F(y)$ or $y \not\geqslant^* F(x)$: since $dom(x) \cap dom(y)$ is finite, $dom(x') \cap dom(y')$ is finite as well. Since $\Delta(INC)$ applies, $x' =^* x' \setminus y' \not\geqslant^* F'(y') = F(y)$ or $y' =^* y' \setminus x' \not\geqslant^* F'(x') = F(x)$. Also, $x \leqslant^* x'$ and $y \leqslant^* y'$. Thus, $x =^* x \setminus y \not\geqslant^* F(y)$ or $y =^* y \setminus x \not\geqslant^* F(x)$, as desired. ■

§ 4.7.1 Tangle Free Filters

Definition 114. We say that a filter $\mathcal{F}$ on a set S is *tangle-free* if there is $T : S \to \mathcal{F}$ such that for $s, t \in S$, $s \notin T(t)$ or $t \notin T(s)$.

In [15], Gartside related this notion about filters to the monotone normality of the square of a topological space. He also showed an interesting example of a topological group, constructed based on a tangle-free filter, all of whose finite powers are monotonically normal, but which is not linearly stratifiable.

Theorem 115 (Gartside)**.** *If a space X has monotonically normal square then every neighborhood filter of a point x in X, considered as a free filter on $X \setminus \{x\}$, is tangle free.*

Gartside believed tangle-freeness has something to do with some instance of Δ. Indeed, we show the following result for the neighborhood filter $\mathcal{F}_{c_\omega} = \{N(c_\omega, f) : f \in \omega^\omega\}$ of c_ω in $\nabla(\omega+1)^\omega$.

Proposition 116. *The filter $\mathcal{F}_{c_\omega}$ is tangle-free if and only if $\Delta[AD]$ holds.*

Proof. Assume that $\mathcal{F}_{c_\omega}$ is tangle-free with witness $T : S \to \mathcal{F}_{c_\omega}$, where $S = \nabla(\omega+1)^\omega \setminus \{c_\omega\}$. Define $F : \omega^{\subset\omega} \to \omega^\omega$ such that $T(\overline{x}) = N(c_\omega, F(x))$. We show F witnesses $\Delta[AD]$: suppose $x, y \in \omega^{\subset\omega}$ are almost disjoint. It is easy to see that '$\overline{x} \notin T(\overline{y})$ or $\overline{y} \notin T(\overline{x})$' if and only if '$\overline{x} \notin N(c_\omega, F(y))$ or $\overline{y} \notin N(c_\omega, F(x))$' if and only if '$x =^* x \setminus y \not\geqslant^* F(y)$ or $y =^* y \setminus x \not\geqslant^* F(x)$'. This prove the first implication.

Now, suppose $\Delta[AD]$ holds via F. Replace F for $F' : \omega^{\subset\omega} \to \omega^\omega$ defined as $F'(x)(n) = x(n) + 1$, if $n \in dom(x)$, and $F'(x)(n) = F(x)(n)$, otherwise. Define $T : S \to \mathcal{F}_{c_\omega}$ given by $T(\overline{x}) = N(c_\omega, F'(x))$, if $x \in \omega^{\subset\omega}$, and $T(\overline{f}) = N(c_\omega, f + 1)$, if $f \in \omega^\omega$. We show that $\mathcal{F}_{c_\omega}$ is tangle-free via T: take $\overline{a}, \overline{b} \in S$. Suppose $\overline{a} \in T(\overline{b})$, then we have to prove that $\overline{b} \notin T(\overline{a})$. We have two cases.

- $dom(a) \cap dom(b)$ is finite: by $\Delta[AD]$, since $a \geqslant^* F'(b)$, then $b =^* b \setminus a \not\geqslant^* F'(a)$. This implies $\overline{b} \notin N(c_\omega, F'(a))$ and $\overline{b} \notin T(\overline{a})$
- $dom(a) \cap dom(b)$ is infinite: $\overline{a} \in T(\overline{b}) = N(c_\omega, F'(b))$ implies that for all but finitely many $n \in dom(a) \cap dom(b)$, $F'(b)(n) = b(n) + 1 \leqslant a(n) < a(n) + 1 = F'(a)(n)$. This implies $\overline{b} \not\geqslant^* F'(a)$, hence $\overline{b} \notin T(\overline{a})$.

This finishes the proof. ∎

§ 4.7.2 The Principle Λ

Here we present the statement Λ. It is a weaker version of Δ motivated by the possibility to be proved in ZFC. We know that Δ implies that ∇^*

is basic ultraparacompact. We characterize basic ultraparacompactness by this Λ principle. Fix a space X and a base $\mathcal{B}$ for X. We say that X is *($\mathcal{B}$) ultraparacompact* if every open cover of X has a pairwise disjoint open refinement (consisting of elements in $\mathcal{B}$). For a nabla product ∇X_n, we just say that ∇X_n is basic ultraparacompact if it is $\mathcal{B}$ ultraparacompact, for the canonical base $\mathcal{B} = \{N(x, B) : x \in \nabla X_n,\ B \in \mathcal{B}\}$. Also, if $A \subseteq X$ and τ is the topology of X, we say that a neighborhnet $F : A \to \tau$ is *nested* over A, if for $x, y \in A$ such that $y \in F(x)$, then $F(y) \subseteq F(x)$. Recall the refinement $\mathcal{V}(\mathcal{U}, A, F)$ constructed in Section 4.3, we will use this process in the following results.

Lemma 117 (Footnote 14 in [10])**.** *If X is paracompact regular and P-space, then it is ultraparacompact.*

Proof. A regular P-space is 0-dimensional. So, every open cover $\mathcal{U}$ of X has a locally finite refinement $\mathcal{V}$ by clopen sets. Enumerate $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$. Then $\mathcal{W} = \{V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta : \alpha < \kappa\}$ is a disjoint refinement of $\mathcal{V}$ covering X. Since $\mathcal{V}$ consist of clopen sets and it is locally finite, the initial unions $\bigcup_{\beta<\alpha} V_\beta$ are closed. Hence, elements in $\mathcal{W}$ are open and $\mathcal{W}$ is the desired refinement. ■

Definition 118. Λ is the statement: for every neighbornet $F' : \omega^{\subset\omega} \to \omega^\omega$ there is $A \subseteq \omega^{\subset\omega}$ and a nested neighbornet $F : A \to \omega^\omega$ refining F' such that F witnesses $\Delta(A)$ and $\{N(x, F(x)) : x \in A\}$ covers ∇^*.

See above Remark 74 for the construction of $\mathcal{V}(\mathcal{U}, A, F)$ from a given open cover $\mathcal{U}$ of ∇^*, $A \subseteq \omega^{\subset\omega}$ and a neighbornet F refining $\mathcal{U}$. We will use this construction and related elements in the following proposition.

Proposition 119. *Λ holds if and only if ∇^* is basic ultraparacompact.*

Proof. $\Longrightarrow$) Suppose first that Λ holds. Let $\mathcal{U}$ be an open cover of ∇^* and $F' : \omega^{\subset\omega} \to \omega^\omega$ be a neighbornet refining $\mathcal{U}$. There are $A \subseteq \omega^{\subset\omega}$ and a nested neighbornet $F : A \to \omega^\omega$ refining F' with the properties of Λ. Let $\mathcal{V} = \mathcal{V}(\mathcal{U}, A, F)$. To see that $\mathcal{V}$ covers ∇^*, fix $\overline{z} \in \nabla^*$. By Λ, there is $y \in A$ such that $\overline{z} \in N(\overline{y}, F(y))$. Now, $\overline{y}$ was either in D_α or in $\bigcup\bigcup_{\beta<\alpha} \mathcal{V}_\beta$, for some $\alpha < \mathfrak{c}$. If $\overline{y} \in D_\alpha$, then $N(\overline{y}, F(y)) \in \mathcal{V}_\alpha$ and $\overline{z} \in \bigcup \mathcal{V}$. If $\overline{y} \in \bigcup\bigcup_{\beta<\alpha} \mathcal{V}_\beta$, then $\overline{y} \in N(\overline{w}, F(w))$, for some $N(\overline{w}, F(w)) \in \mathcal{V}_\beta$ and

$\overline{w} \in A$. Since F is nested, $\overline{z} \in N(\overline{y}, F(y)) \subseteq N(\overline{w}, F(w))$ and thus, $\overline{z} \in \bigcup \mathcal{V}$, as claimed.

To prove $\mathcal{V}$ is pairwise disjoint, suppose $N(\overline{x}, F(x)), N(\overline{y}, F(y)) \in \mathcal{V}$, $\overline{x}, \overline{y} \in A$, then $\overline{x} \in D_\beta, \overline{y} \in D_\alpha$, $\beta \leqslant \alpha$. If $\beta = \alpha$, we are done, since $\mathcal{V}_\alpha$ is pairwise disjoint. Suppose $\beta < \alpha$. Note that $\overline{y} \notin N(\overline{x}, F(x))$ by the construction of $\mathcal{V}_\alpha$, and also, it can not happen $y \subseteq^* x$ by the minimality of α. If $x \subseteq^* y$, Lemma 64 applies, hence $N(\overline{x}, F(x)) \cap N(\overline{y}, F(y)) = \emptyset$. The remaining case is x, y as in Δ, in which case $\Delta(A)$ implies $N(\overline{x}, F(x)) \cap N(\overline{y}, F(y)) = \emptyset$.

$\Longleftarrow$) Now suppose ∇^* is basic ultraparacompact. If $F' : \omega^{\subset\omega} \to \omega^\omega$ is a neighbornet, consider the open cover $\mathcal{U} = \{F'(x) : \overline{x} \in \nabla^*\}$. Then, there is a pairwise disjoint open refinement $\mathcal{W} = \{N(x_\alpha, f_\alpha) : \alpha < \kappa\}$ of $\mathcal{V}(\mathcal{U}, \omega^{\subset\omega}, F')$ by basic sets, for some κ. Let $A = \{x_\alpha : \alpha < \kappa\} \subseteq \omega^{\subset\omega}$ and define $F : A \to \omega^\omega$ as $F(x_\alpha) = f_\alpha$.

If $x, y \in A$ are as in Δ, then $N(x, F(x)) \cap N(y, F(y)) = \emptyset$ implies $x \setminus y \not\geqslant^* F(y)$ or $y \setminus x \not\geqslant^* F(x)$, that is, F witnesses $\Delta(A)$. Finally, F is trivially nested over A because the $N(x, F(x))$'s are disjoint. ∎

Remark 120. If X is κ-metrizable, then there is a nested neighbornet over X.

Proof. Let $\mathcal{B} = \{B_{x,\alpha} : x \in X, \alpha < \kappa\}$ be a basis for X witnessing that X is κ-metrizable. For $\alpha < \kappa$, define $F_\alpha : X \to \tau$ as $F_\alpha(x) = B_{x,\alpha}$. Observe that $\{F_\alpha(x) : x \in X\}$ is pairwise disjoint, hence, F_α is nested, for every $\alpha < \kappa$. ∎

Lemma 121. *∇^* does not admit nested neighbornets by sets of the form $N(x, f)$ over $\omega^{\subset\omega}$.*

Proof. Split $\omega = P_0 \cup P_1$ into disjoint infinite sets. Let $\{C_\alpha : \alpha < \mathfrak{b}\}$ be a linearly ordered chain of infinite subsets of P_0, that is, $C_\alpha \subseteq C_\beta$ and $|C_\beta \setminus C_\alpha| = \omega$ iff $\alpha < \beta$. Also, consider $\{F_\alpha : \alpha < \mathfrak{b}\}$ an unbounded family of increasing functions well ordered by $\leqslant^*$.

Now, suppose for a contradiction that $F : \omega^{\subset\omega} \to \omega^\omega$ is nested. We will construct a compatible $\subseteq$-chain of partial functions whose union is well defined: let x_0 be any partial function in $\omega^{C_0} \subseteq \omega^{\subset\omega}$. Supposed constructed $\{x_\beta : \beta < \alpha\}$, $\alpha < \mathfrak{b}$, with the following properties: if $\gamma < \beta$, then $dom(x_\beta) = C_\beta$ and $x_\beta \in N(x_\gamma, \max\{f_\gamma, F(x_\gamma)\})$ (we can even have the inclusion $x_\gamma \subseteq x_\beta$). In particular, nestedness implies $N(x_\beta, \max\{f_\beta, F(x_\beta)\}) \subseteq N(x_\gamma, \max\{f_\gamma, F(x_\gamma)\})$. Since ∇^* is a $P_\mathfrak{b}$-space, consider any element $x_\alpha \in \bigcap_{\beta<\alpha} N(x_\beta, \max\{f_\beta, F(x_\beta)\})$ with $dom(x_\alpha) = C_\alpha$ and if $E_\alpha = C_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta$ is infinite, then $x_\alpha \restriction E_\alpha \geqslant^* f_\alpha$. Again, nestedness implies $N(x_\alpha, \max\{f_\alpha, F(x_\alpha)\}) \subseteq N(x_\beta, \max\{f_\beta, F(x_\beta)\})$, $\beta < \alpha$.

We have constructed a $\subseteq$-chain $\{x_\alpha : \alpha < \mathfrak{b}\}$ of compatible partial functions. Hence, the element $x = \bigcup_{\alpha<\mathfrak{b}} x_\alpha$ is a partial function with $dom(x) = \bigcup_\alpha C_\alpha \subseteq P_0$. Now, observe that there is no function $g \in \omega^\omega$ so $N(x, g)$ is contained in $\bigcap_{\alpha<\mathfrak{b}} N(x_\alpha, \max\{f_\alpha, F(x_\alpha)\})$ since the family of the f_α's is unbounded. This contradicts nestedness. ∎

The two previous observations show that under $\mathfrak{b} = \mathfrak{d}$, $\nabla(\omega + 1)^\omega$ is $\mathfrak{b}$-metrizable, but it cannot be $\mathfrak{b}$-metrizable by basic open sets of the form $N(x, f)$. Moreover, if F is a witness of Δ, then F cannot be nested. However, there could be subsets $A \subseteq \omega^{\subset\omega}$ that admit nested neighbornets.

4.8 Miscellaneous on Δ

§ 4.8.1 Utterly Ultra Normal Spaces

In [30], Peter Nyikos used the concept of utterly ultranormal spaces. He showed that different variants of this notion are equivalent on trees. We define and show that this variants are equivalent also for some countable nabla products. For more on utterly ultranormal spaces we refer to the reader to [5].

Definition 122. A space X is *halvable* (*utterly halvable*)[*utterly ultra halvable*] if for any neighborhood assignment $U : X \to \tau$, there is a neighborhood assignment $V : X \to \tau$ refining it such that if $V(x) \cap V(y) \neq \emptyset$, then $x \in U(y)$ or $y \in U(x)$ ($x \in \overline{V(y)}$ or $y \in \overline{V(x)}$)[$x \in V(y)$ or $y \in V(x)$].

Definition 123. A space X with topology τ is *utterly monotonically normal* (*utterly ultra monotonically normal*) if there exists an operator $G : X \times \tau \to \tau$ such that $x \in G(x, U) \subseteq U$, and if $G(x, U) \cap G(y, V) \neq \emptyset$, then $x \in \overline{G(y, V)}$ or $y \in \overline{G(x, U)}$ ($x \in G(y, V)$ or $y \in G(x, U)$).

Definition 124. A space X is *utterly normal* (*UNO*)[*utterly ultra normal*] if X is regular and there is a base system $\{\mathcal{U}_x : x \in X\}$, where $\mathcal{U}_x \subseteq \mathcal{N}_x$, and if $U_x \cap U_y \neq \emptyset$, then $x \in \overline{U_y}$ or $y \in \overline{U_x}$ (plus all members of $\mathcal{U}_x$ are open)[plus all members of $\mathcal{U}_x$ are clopen].

Uterlly normal (UNO)[utterly ultra normal] $\Longrightarrow$ monotonically normal (utterly monotonically normal)[utterly ultra monotonically normal] $\Longrightarrow$ halvable (utterly halvable)[utterly ultra halvable]. Also, 'utterly ultra $\mathcal{P}$' $\Longrightarrow$ 'utterly $\mathcal{P}$' $\Longrightarrow$ '$\mathcal{P}$', where $\mathcal{P}$ can be halvability, normality or monotone normality.

Proposition 125. *Let X be a monotonically normal space. If X is a P-space, then it is utterly ultra normal.*

Proof. Fix a monotonically normal operator $G : X \times \tau \to \mathcal{B}$, where $\mathcal{B}$ is a basis for X. For point and open set (x, U), define $G^0(x, U) = U$ and inductively $G^{n+1}(x, U) = G(x, G^n(x, U))$. Note that for every n we have $G^{n+1}(x, U) \subseteq \overline{G^{n+1}(x, U)} \subseteq G^n(x, U)$. Since X is a P-space, $G^\omega(x, U) = \bigcap_{n\in\omega} G^n(x, U)$ is a clopen neighborhood of x. Observe that if A is any infinite subset of ω then $G^\omega(x, U) = \bigcap_{n\in\omega} G^n(x, U) = \bigcap_{n\in A} G^n(x, U)$.

Declare $\mathcal{U}_x = \{U_x := G^\omega(x, U) : x \in U \text{ and } U \text{ is open}\}$. Notice that $\{\mathcal{U}_x : x \in X\}$ is a base system. We check that it satisfies the conditions of 'utterly ultra normal': suppose $G^\omega(x, U) \cap G^\omega(x', U') \neq \emptyset$. Then for all n, $G^{n+1}(x, U) \cap G^{n+1}(x', U') \neq \emptyset$, and so $x \in G^n(x', U')$ or $x' \in G^n(x, U)$. Let $A = \{n \in \omega : x \in G^n(x', U')\}$ and $A' = \{n \in \omega : x' \in G^n(x, U)\}$. At least one of A and A' is infinite. Say, with no loss of generality, A is infinite. Then, $x \in \bigcap_{n\in A} G^n(x', U') = G^\omega(x, U')$, as claimed. ∎

Corollary 126. *Let κ be any cardinal* (*ordinal*)*, and let X_n be a metrizable space, $n \in \omega$. The following are equivalent:*

1. $\Delta[A(\kappa)]$ $(\Delta[\kappa])$ $[\Delta_{(X_n)_n}]$ *holds,*

2. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) is halvable*
3. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) is utterly halvable*
4. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) is utterly ultra halvable,*
5. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) [$\nabla_n X_n$] is monotonically normal,*
6. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) [$\nabla_n X_n$] is uterly monotonically normal,*
7. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) [$\nabla_n X_n$] is utterly ultra monotonically normal,*
8. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) [$\nabla_n X_n$] is utterly normal,*
9. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) [$\nabla_n X_n$] is UNO,*
10. *$\nabla A(\kappa)^\omega$ ($\nabla(\kappa+1)^\omega$) [$\nabla_n X_n$] is utterly ultra normal.*

Proof. Use Proposition 125 and Corollary 96, Proposition 94, Proposition 89 for $\Delta[A(\kappa)]$, $\Delta[\kappa]$, $\Delta_{(X_n)_n}$, respectively. ∎

§ 4.8.2 The Scope of Δ

In [34], Roitman proved another interesting result showing that Δ implies the nabla products of certain one point compactification factors are basic ultraparacompact. Here we show that Δ implies the nabla product of infinite compact subspaces of $\mathbb{R}$ with finitely many limit points is monotonically normal.

Theorem 127 (Roitman)**.** *Assume $\nabla(\omega+1)^\omega$ is basic ultraparacompact. If $\kappa < \aleph_\omega$, then $\nabla A(\kappa)^\omega$ is basic ultraparacompact.*

Corollary 128. *If Δ holds, then $\nabla A(\kappa)^\omega$ is normal, $\kappa < \aleph_\omega$.*

Proposition 129. *$\nabla(\omega+1)^\omega$ is monotonically normal if and only if $\nabla(\omega\cdot n+1)^\omega$ is monotonically normal.*

Proof. One direction is immediate since monotone normality is hereditary. For the other direction, fix $n \in \omega$ and define $I_k = (\omega \cdot k, \omega \cdot (k+1)]$, for $k < n$. Also, for $f \in \nabla n^\omega$, define $\nabla_f = \nabla_k I_{f(k)}$ (which is homeomorphic to $\nabla(\omega+1)^\omega$). We show that ∇_f is a clopen set: pick $x \in \nabla_f$. For $k \in \omega$, let $U_k = \{x(k)\}$, if $x(k)$ is isolated, and $U_k = I_{f(k)}$, otherwise. Thus, $\nabla_k U_k$ is a neighborhood of x contained in ∇_f (the same argument shows that ∇_f is closed).

Now we prove that $\nabla(\omega \cdot n + 1)^\omega$ is the disjoint union of ∇_f, $f \in \nabla n^\omega$, in which case we will be done since the disjoint union of open monotonically normal spaces is a monotonically normal space. To see $\bigcup_{f \in \nabla n^\omega} \nabla_f$ is a covering, pick $x \in \nabla(\omega \cdot n + 1)^\omega$. Hence, for every $k \in \omega$, $x(k) \in I_{f(k)}$, for some $f(k) \in \omega$. Then, $x \in \nabla_f$. Finally, if $f, g \in \nabla n^\omega$, then there are infinitely many $k \in \omega$, $f(k) \neq g(k)$. Hence for infinitely many $k \in \omega$, $I_{f(k)} \cap I_{g(k)} = \emptyset$, thus, $\nabla_f \cap \nabla_g = \emptyset$. ■

4.9 Open Questions

The focus is on box products of *compact metrizable spaces*, and nabla products of *separable metrizable spaces*.

§ 4.9.1 The Gap

In the second paragraph of Section 4.3, we mentioned a gap in Roitman's argument '∇^* paracompact implies $\nabla(\omega+1)^\omega$ paracompact'. We discovered this in personal comunications with Judy Roitman [35]. Then, she claimed the following is still an open question.

Question 130. Is it true that ∇^* is paracompact if and only if $\nabla(\omega+1)^\omega$ is paracompact?

§ 4.9.2 Hereditary Properties of $\square = \square(\omega+1)^\omega$

We know that infinite box products $\square_n X_n$ are never hereditarily normal (Corollary 20). Basically because if $(\omega+1) \times X$ is hereditarily normal then

X is perfect (Theorem 16).

E. van Douwen showed in [10] that $\omega^\omega \times \square$ is not paranormal. Hence $\square$ is not hereditarily paranormal (Theorem 7.2). Paranormality is implied by normality and by countably paracompactness. So $\square$ is never hereditarily countably paracompact. He also showed that $\omega^\omega \times \square$ is not $\mathfrak{d}$-collectionwise Hausdorff (Remark 12.7).

Question 131. Is $\square$ non-hereditarily collectionwise Hausdorff?

However, van Douwen asked (Question 13.13) whether $\omega^\omega \times \square$ is countably orthocompact, or $\square$ is hereditarily orthocompact. Recall that metacompactness implies orthocompactness. So even, '$\square$ is not hereditarily metacompact' would be a new result.

Question 132. Can $\square_n X_n$ be hereditarily metacompact or hereditarily collectionwise Hausdorff?

Question 133. For which topological property $\mathcal{P}$ can we show that $\square_n X_n$ is never hereditarily $\mathcal{P}$?

Question 134. If $(\omega + 1) \times X$ is hereditarily metacompact, hereditarily countably paracompact or hereditarily collectionwise Hausdorff, what can we say about X?

§ 4.9.3 Relating Properties of $\square$ and ∇

How are separation and covering properties of $\square_n X_n$ and $\nabla_n X_n$ related? When the factors X_n are compact, we know the quotient map $\sigma : \square_n X_n \to \nabla_n X_n$ is *closed*, *open* and its fibres, $\sigma^{-1}\{y\}$, are *σ-(compact metric)*. Many covering and separation properties are preserved by closed images. Many covering and separation properties are preserved by inverse images of perfect maps. We recall Kunen's Theorem:

Theorem 135. *If X_n is compact, then $\square_n X_n$ is paracompact if and only if $\nabla_n X_n$ is paracompact.*

Paracompactness is preserved by inverse images of maps which are closed and have Lindelöf fibres, see [22], [28] and [29].

Question 136. If $\square_n X_n$ has covering or separation property $\mathcal{P}$ then what properties does $\nabla_n X_n$ have?

Question 137. If $\nabla_n X_n$ has covering or separation property $\mathcal{P}$ then what properties does $\square_n X_n$ have?

In particular, we are interested in formulating and proving general theorems of the type: 'if $f : X \to Y$ is closed (onto) with *nice* fibres, and Y has property $\mathcal{P}$ then X has $\mathcal{P}$'.

'Nice' fibres might be: Lindelof, σ-compact or σ-(compact metric). Possible properties $\mathcal{P}$: metacompactness, countably paracompactness, collectionwise normality, normality, collectionwise Hausdorffness; hereditary versions.

§ 4.9.4 'Basic' Properties

Some covering and separation properties only require knowledge of *basic* open sets. Most do not. For example paracompact says: 'every cover by open sets has a refinement by open sets'. We can replace the first 'open' by 'basic open'; but not the second (that would give 'basic paracompactness'). When we try to translate a topological property of $\nabla(\omega + 1)^\omega$ to a combinatorial one we really need 'basic' properties.

So, paracompactness, metacompactness, countable paracompactness, normality and collectionwise normality are all not basic. But monotone normality, basic paracompactness and collectionwise Hausdorff are basic.

In addition to show that $\nabla(\omega + 1)^\omega$ is normal (or not normal) we need to understand the (disjoint pairs of) closed subsets of $\nabla(\omega + 1)^\omega$. Closed discrete subsets are examples of closed subsets. (And any partition of a closed discrete set gives a disjoint pair of closed sets.)

Question 138. What are the closed discrete subsets of ∇? Can they all be separated by open sets? (So $\nabla(\omega + 1)^\omega$ collectionwise Hausdorff.)

Question 139. Is there a combinatorial statement equivalent to '$\nabla(\omega + 1)^\omega$ is collectionwise Hausdorff'?

Question 140. Does ultraparacompactness of $\nabla(\omega + 1)^\omega$ implies its basic ultraparacompactness?

§ 4.9.5 Monotone Normality of Nabla Products

There is a family of combinatorial properties $\Delta_{(X_n)_n}$ equivalent to $\nabla_n X_n$ being monotonically normal.

Write Δ_X for $\Delta_{(X)_n}$. If $A \subseteq X$, then Δ_X implies Δ_A. We know that if X is not discrete then Δ_X implies $\Delta_{\omega+1}$. Hence $\Delta_{\omega+1}$ is the weakest Δ_X property, and Δ_H, where $H = [0, 1]^\omega$ is the Hilbert cube, is the strongest among the separable metrizable spaces. We know that $\Delta_{\omega+1}$ is equivalent to the original Δ. Also, we have shown that Δ implies $\Delta_{(\omega \cdot n+1)}$, for any $n \in \omega$.

Question 141. We have a 'formal' hierarchy of Δ_X properties. How can we distinguish them?

Question 142. For which X does $\Delta_{\omega+1}$ imply Δ_X? Does Δ imply $\Delta_{\omega\cdot\omega+1}$?

Question 143. If $\nabla(\omega + 1)^\omega$ is monotonically normal, is $\nabla(\omega \cdot \omega + 1)^\omega$ normal?

Question 144. If for every $\alpha < \omega_1$, $\nabla(\alpha + 1)^\omega$ is monotonically normal, is $\nabla(\omega_1 + 1)^\omega$ normal?

For $\nabla(\omega + 1)^\omega$ we know that being monotonically normal is equivalent to being halvable, Lemma 78. What about other ∇-products?

Question 145. If $\nabla_n X_n$ is halvable then is $\nabla_n X_n$ monotonically normal?

Orderable spaces (also called, LOTS) are monotonically normal. In fact they are nested monotonically normal (more usually called, strongly monotonically normal). As monotone normality and nested monotone normality are hereditary, sub-orderable spaces (also called GO spaces) are nested monotonically normal.

Question 146. If $\nabla(\omega + 1)^\omega$ is monotonically normal then is it orderable or sub-orderable?

§ 4.9.6 Further Questions

Question 147. Does (basic) metacompactness of ∇^* implies its (basic) paracompactness?

Question 148. Does (basic) orthocompactness of ∇^* implies its (basic) metacompactness?

Question 149. If $\nabla(\omega + 1)^\omega$ is paracompact, must it be halvable?

Question 150. Is ∇^* countably metacompact?

Finally, the central motivation of this work.

Question 151. Is Δ or MH true in ZFC?

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