UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y
DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

“TWO PROBLEMS IN DISCRETE GEOMETRY RELATED WITH METRIC POLYTOPES”

TESIS
QUE PARA OPTAR POR EL GRADO DE:
MAESTRO EN CIENCIAS

PRESENTA:
GYIVAN ERICK LÓPEZ CAMPOS

DIRECTORA
DRA. DEBORAH OLIVEROS BRANIFF,
INSTITUTO DE MATEMÁTICAS (UNAM)

CIUDAD UNIVERSITARIA, CIUDAD DE MÉXICO 20 MARZO DE 2020

Agradezco profundamente:

A mis padres y a mi hermana, que como muchos, me han acompañado en cada una de las etapas de mi vida, pero que, como pocos, entienden el trabajo y esfuerzo que me costó triunfar en cada una de ellas.

A mis abuelos, que me han dado la sabiduría y el camino para andar en la vida.

A mi amada novia, que me ha brindado su amor y apoyo incondicional, aún en los tiempos más difíciles.

A mis tíos y primos, que han hecho de mi una persona alegre y productiva, particularmente a mi tía Eugenia que me ha apoyado incondicionalmente como si fuera su propio hijo.

A mis profesores que me han enseñado, instruido y orientado en mi vida académica y profesional, particularmente al Dr. Luis Montejano y la Dra. Adriana Hansberg que tanto en la licenciatura como en la maestría me han procurado, guiado y aconsejado inumerables veces

Al Dr. Luis Montejano, Dr. Edgardo Roldán, Dr. Leonardo Martínez, Dr. Isaac Arelio y Dra. Adriana Hansberg que han dedicado su invaluable tiempo a revisar, comentar y contribuir con sus observaciones a este trabajo.

A mis amigos, con los que he vivido momentos memorables y han sido una parte fundamental en mi vida.

Al Posgrado UNAM y al Instituto de Matemáticas de la UNAM por recibirme entre sus paredes tanto tiempo y permitirme desarrollarme como profesionista.

Y de manera muy especial a la Dra. Déborah Oliveros, que ha sido una excelente profesora y una gran amiga, que con mucho trabajo y paciencia me ha guiado e instruido a la investigación matemática, y producto de ello es este trabajo que solo ella y yo conocemos el gran esfuerzo que está grabado en cada palabra de éste.

Gracias a todos por siempre estar conmigo cuando los he necesitado, y cuando he creído no necesitarlos también.

Gyivan Erick López Campos

Contents

Introduction

After some time of doing mathematics, you may get some strategies, techniques, and even some magical intuition about how to solve problems or at least on how to try to solve them. Sometimes it works, sometimes it does not, but does it really matter? I think mathematics is beautiful by itself, even when you fall in proving something, or when a problem you thought was easy gets more and more complicated. At the end of the day, after very hard work, you may connect the dots, and perhaps you will have done a piece of art. I had not realized this until this work.

I have to be honest, reviewing papers [9, 2, 10] and books [11, 19], was not an easy task for me at the beginning, considering that it was my first time doing mathematical research. So I was totally lost, since all of them are very technical. Without a proper introduction everything seemed very confusing.

Finally, when Deborah told me her ideas about how to generalize a problem in discrete geometry for higher dimensions using strategies and tools that were in the papers and books that I was reading, each idea started to make sense in my mind and I finally connected the dots. There were constantly repeating structures in all the problems we were working on, and those structures helped us understand the behavior in higher dimensions. Those structures are the *metric polytopes*.

In this thesis, we will present two discrete geometrical problems we worked on during my masters, and all the properties that metric polytopes have.

The first one is related to bodies of constant width. The main idea was to generalize the algorithm proposed by Luis Montejano and Edgardo Roldán in [13] to create families of bodies of constant width in $\mathbb{R}^4$, using metric polytopes.

The second one is related with the problem of the maximum number of diameters of a set of points in Euclidean space. For dimensions one, two and three the vertices of metric polytopes provide as examples of maximum frequent diameters configurations [9, 14, 11] that we call *extremal configurations*. In 2009, K. Swanepoel proved that for higher dimensions [16], the set of points has to be in a strict configuration called *Lenz configuration*. The main goal was to study this paper in order to verify if the vertices of metric polytopes still worked for higher dimensions as extremal configuration families.

Therefore, the main question of this thesis is: for problems in discrete geometry, where metric polytopes are solutions in lower dimensions, could metric polytopes help to construct families and examples in higher dimensions as well?

This thesis is distributed in four chapters. In Chapter 1, we give all the basic tools, characterizations, properties and main definitions to understand the metric polytopes.

In Chapter 2, we will define *bodies of constant width*, an give little about their history and some algorithms to create them. The main result of this thesis (Theorem 2.2.2) is in this chapter, and it is about how algorithms that create bodies of constant width in $\mathbb{R}^2$ and $\mathbb{R}^3$ no longer work in higher dimensions.

The third chapter is about the problem of the maximum number of diameters of a set of points in Euclidean spaces and the fourth one contains the conclusions and possible direction of future work.

Chapter 1
Definitions and Important Results

In this first chapter we will introduce general facts that will be needed in the rest of this thesis, such as notation, basic definitions and results related with polytopes and convexity.

1.1 Convexity

Convexity is a mathematical notion in geometry which is strongly related with discrete geometry, graph theory, combinatorics, analysis, etc.

Convexity has been studied for a long time. The first notions in this area appeared in Archimides' book "On the Sphere and Cylinder", maybe not as convexity itself, but he uses the notions to define curves and surfaces. [11]

Nowadays, the most common definition of a convexity set is the following:

Definition 1.1.1: A set $K \subseteq \mathbb{R}^d$ is called a *convex set* if for every $x,\, y \in K$ and $t \in [0,\, 1]$, the point $tx + (1-t)\, y \in K$. The empty set is convex by vacuity.

Definition 1.1.2: Let $A \subseteq \mathbb{R}^d$. The *convex hull* of A denoted by $cc\,(A)$ is the intersection of all the convex sets K such that $A \subseteq K$, i.e.

$$cc\,(A) \quad = \quad \bigcap_{K \subseteq A} K.$$

Definition 1.1.3: The vector $x \in \mathbb{R}^d$ is a *convex combination* of the points $x_1, ..., \, x_n$ if there are real numbers $\lambda_1, ..., \, \lambda_n$ satisfying $\sum_{i=1}^n \lambda_i = 1$, $\lambda_i \geq 0$ and $x = \sum_{i=1}^n \lambda_i x_i$.

With these three main definitions, some basic properties of convexity can be proved (see [11]) .

Proposition 1.1.1: *The intersection of convex sets is convex.*

Proof: Let $\{K_i\}_{i \in I}$ be a family of convex sets. If $\cap K_i = \emptyset$ or $|\cap K_i| = 1$ by vacuity the intersection is convex. Let us say there are at least two points in $\cap_{i\in I} K_i$, then $x,\, y \in \cap K_i$. Because $x,\, y \in K_i$ for every $i \in I$, then $tx + (1-t)\, y \in K_i$ for every $t \in [0,\, 1]$ because each K_i is convex. Therefore $tx + (1-t)\, y \in \cap K_i$ which implies that $\cap_{i \in I} K_i$ is convex. ∎

Proposition 1.1.2: *Let $K \subset \mathbb{R}^d$ be a convex set, $x \in \mathbb{R}^d$ is a convex combination of points in K if and only if $x \in K$.*

Proof: If $x \in K$, then x is a convex combination by itself. If x is convex combination we proceed by induction on the number of vectors. If x is convex combination of just two points, by definition of convexity x is in K. Suppose that if x is convex combination of $n-1 > 2$ points then $x \in K$. If x is convex combination of n points $x_1, ..., \, x_n$, then $\lambda_n \neq 1$ otherwise $x = x_n$. Thus:

$$\begin{aligned} x &= \sum_{i=1}^{n} \lambda_i x_i \\ &= (1-\lambda_n) \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i + \lambda_n x_n \end{aligned}$$

Because $\frac{\lambda_i}{1-\lambda_n} \geq 0$ and

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} &= \frac{1}{1-\lambda_n} \sum_{i=1}^{n-1} \lambda_i \\ &= \frac{1}{1-\lambda_n} (1-\lambda_n) = 1 \end{aligned}$$

$y = \sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i$ is convex combination of $n-1$ points of K so by the inductive hypothesis $y \in K$. Since $x = (1-\lambda_n) y + \lambda_n x_n$, then x is in K by definition of convexity. ■

Proposition 1.1.3: *Let A and B be sets in $\mathbb{R}^d$ such that $A \subset B$, then $cc(A) \subset cc(B)$.*

Proof: Let K any convex set such that $B \subset K$, then $A \subset K$. Therefore:

$$cc(A) = \bigcap_{A \subset K} K \quad \subset \bigcap_{B \subset K} K = cc(B) . \blacksquare$$

Proposition 1.1.4: *If K is a convex set, then $cc(K) = K$.*

Proof: Let $x \in K$, so $x \in A$ for all convex such that $K \subset A$, so $x \in \cap_{K \subset A} A = cc(K)$. Then $K \subset cc(K)$.

Now suppose that $x \in cc(K)$. Since K is convex, then $cc(K) \subset K$ by definition, so $x \in K$. Therefore $K = cc(K)$

Corollary 1.1.1: *Let A, K be subsets of $\mathbb{R}^d$, K convex and $A \subset K$, then $cc(A) \subset K$.*

Proof: Since $A \subset K$ then $cc(A) \subset cc(K)$ by proposition 1.1.3 and since K is convex, $cc(K) = K$ by proposition 1.1.4. Therefore $cc(A) \subset K$. ■

1.2 Geometric Polytopes

In order to start defining what is a metric polytope, we need to understand first what is a *geometric polytope*. There are two ways to define a geometric polytope:

Definition 1.2.1: $\mathcal{P}$ is a $\mathcal{V}-$*polytope* in $\mathbb{R}^d$ if it is the convex hull of a finite set of points in $\mathbb{R}^d$.

A $\mathcal{V}-$*polytope* is illustrated in $\mathbb{R}^2$ as the convex hull of the points A, B, C, D and E in Figure 1.2.1

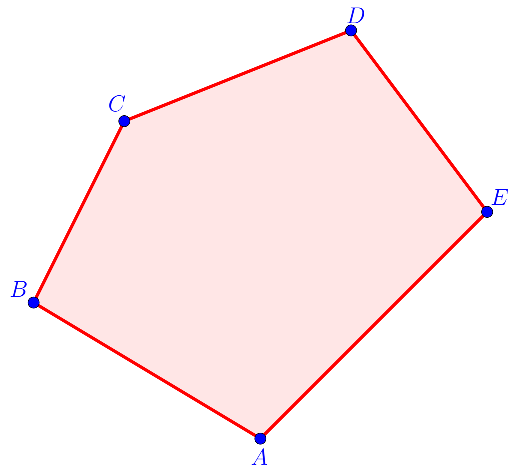

Figure 1.2.1

Definition 1.2.2: $\mathcal{P}$ is an $\mathcal{H}-polytope$ in $\mathbb{R}^d$ if it is a bounded intersection of finite set of half-spaces in $\mathbb{R}^d$.

A $\mathcal{H}-polytope$ it is illustrated in $\mathbb{R}^2$ as the intersection of the half-planes a, b, c, d and e in Figure 1.2.2.

Notice in Figure 1.2.1 and Figure 1.2.2 that the $\mathcal{V}-polytope$ and the $\mathcal{H}-polytope$ have the same geometric representation. This is not a coincidence, actually, even in higher dimensions a $\mathcal{V}-polytope$ can be represented as an $\mathcal{H}-polytope$ and vice versa. So this leads us to Theorem 1.2.1, whose proof can be found in [19].

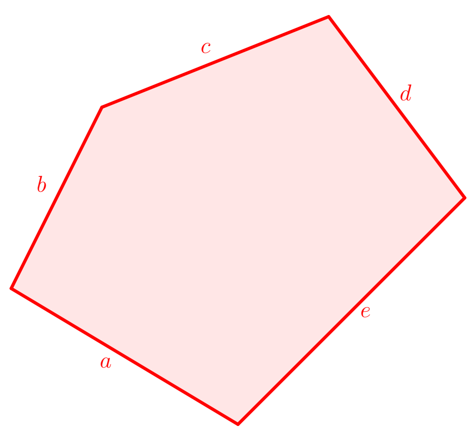

Figure 1.2.1

Theorem 1.2.1 (Main theorem for geometric polytopes): *A subset $\mathcal{P} \subset \mathbb{R}^d$ is the convex hull of a finite point set (a $\mathcal{V}-polytope$) if only if it is a bounded intersection of half-spaces (an $\mathcal{H}-polytope$).*

So, a geometric polytope can be understood either Definition 1.2.1 or Definition 1.2.2. In other words:

Definition 1.2.3: $\mathcal{P}$ is a geometric polytope of dimension d or $d-polytope$, if it is a $\mathcal{V}-polytope$ or $\mathcal{H}-polytope$

in $\mathbb{R}^d$ that can not be embed in $\mathbb{R}^{d-1}$.

Definition 2.1.4: H is a supporting hyperplane of a body K if K is totally contained in only one of the closed half-spaces induced by H and $H \cap \mathrm{bd}\,(K) \neq \emptyset$

Definition 1.2.5: f is a *face* of a $d-$polytope $\mathcal{P}$ if f is the intersection of $\mathcal{P}$ and a supporting hyperplane of $\mathcal{P}$. The *dimension of a face* is given by the dimension of the minimal affine space containing the intersection.

We use $F\,(\mathcal{P})$ to denote the set of faces of a $\mathcal{P}$ polytope, and F_d to denote the set of $d-$ *dimensional* faces. Commonly if $\mathcal{P}$ is a $d-$polytope, the $(d-1)-$dimensional faces of $\mathcal{P}$ are called a *facets* of $\mathcal{P}$, the $d-$dimensional face is called the *complete face*, and F_0 are the *vertices* of $\mathcal{P}$. This last definition is based in the fact that F_0 and F_1 induce a graph in the common sense and it is going to be defined in Subsection 1.6.

One topological observation is that every geometric $d-$polytope $\mathcal{P}$ is isomorphic to the $d-$ *unit ball* and $\mathrm{bd}(\mathcal{P})$ is homomorphic to S^{d-1}. So $\mathrm{bd}(\mathcal{P})$ can be embedded on S^{d-1}.

Another important observation is that since $\mathcal{P}$ is an intersection of finite hyperplane, $\mathcal{P}$ can be expressed also as:

$$\mathcal{P} \quad = \quad \left\{x \in \mathbb{R}^d \mid A \cdot x \leq \mathbf{1}\right\}, \text{ with } A \text{ an square matrix and } \mathbf{1} \text{ a vector of ones.}$$

where each line of A corresponds to the equation of each hyperplane.

1.3 Abstract Polytopes

In a more general way, we may define an abstract polytope as a set which satisfies the following four properties . It is expected that every geometric polytope is an abstract polytope.

For more details see [1, 12].

Definition 1.3.1: An abstract polytope of dimension d, or simply a $d-$*polytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function with range $\{-1, 0, ..., d\}$ satisfying the following conditions:

1. The elements of rank j are called are called the $j-faces$ of $\mathcal{P}$, in particular we have vertices, edges, and facets of $\mathcal{P}$ for $j = 0$, $j = 1$ or $d-1$ respectively.
2. Each $flag$ (maximal totally ordered subset) of $\mathcal{P}$ contains exactly $d+2$ faces, including a unique minimal face F_{-1} (of rank -1) and a unique maximal face F_d (of rank d).
3. $\mathcal{P}$ satisfies the *diamond property*, namely if F is a $(j-1)$-face and G a $(j+1)-$face with $F < G$, then there are exactly two $j-$faces H such that $F < H < G$.
4. $\mathcal{P}$ is *strongly flag-connected* meaning that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, ..., \Phi_{l-1}, \Phi_l = \Psi$, all containing $\Phi \cap \Psi$, such that Φ_{i-1}, Φ_i are adjacent (they differ by exactly one face) for each $i = 1, ..., l$.

Notice that every geometric polytope is an abstract polytope, where the faces of the geometric polytope are the elements ordered by the inclusion. It is easy to see that this partially order set satisfies the definition of abstract polytope definition. Let this partially order be called as the *face-lattice* of the polytope $\mathcal{P}$ and denoted it by $\mathcal{L}(\mathcal{P})$.

Although this new way of defining a polytope preserves very important combinatorial information about the geometric polytopes, we are going to lose some other important properties. For example, the face-lattice loses the geometric symmetries, the distances, etc. of a geometric polytope.

In order to visualize of the face-lattice, we can represent it with the *Hasse diagram* in which by convention, faces of equal rank are placed on the same vertical level. Each edge between faces, say F,G, indicates an order relation $<$ such that $F < G$ where F is below G in the diagram. The Hasse diagram defines the poset uniquely and therefore fully captures the structure of the polytope.[12]

As an example, Figure 1.3.1 is the face-lattice of the tetrahedra represented by the Hasse diagram, where the red letters represent the faces.

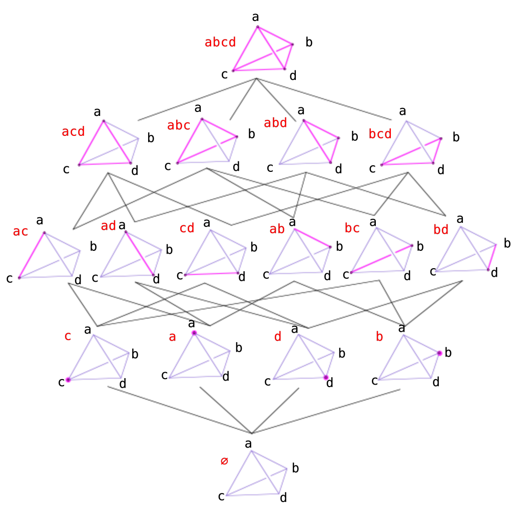

Figure 1.3.1

Definition 1.3.2: Two polytopes $\mathcal{P}$ y $\mathcal{Q}$ are said to be *combinatorially isomorphic* to each other, denoted by $\mathcal{P} \cong \mathcal{Q}$, if $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}^{*})$.

Combinatorially isomorphic polytopes give rise to isomorphic Hasse diagrams, and vice versa.

1.4 Duality

Duality of "something" is a very common definition in several fields in mathematics. There are a lot of types of duality, even for polytopes, but the general definition is given in terms of the face lattice and is as follows:

Definition 1.4.1: $\mathcal{P}^{*}$ is the *dual* of a polytope $\mathcal{P}$ if their face lattice are anti-isomorphic; that is, there is a one-to-one inclusion-reversing map τ from $\mathcal{L}(\mathcal{P})$ to $\mathcal{L}(\mathcal{P}^{*})$.

In geometric polytopes, the most common duality is given by the *polar dual* described in Definition 1.4.2. Without loss of generality assume for the rest of this thesis that every geometric polytope $\mathcal{P}$ in $\mathbb{R}^d$ contains the origin in its interior.

Definition 1.4.2: For a polytope $\mathcal{P} = \left\{x \in \mathbb{R}^d \mid A \cdot x \leq \mathbf{1}\right\}$, where $A \in M_{k\times d}$ is a real matrix, then the *polar dual* of polytope $\mathcal{P}$ is given as follows:

$$\mathcal{P}^* \quad = \quad \left\{x \in \mathbb{R}^d \mid v \cdot x \leq 1, \text{ for every } v \in \mathcal{V}(\mathcal{P})\right\}.$$

In [p59-p64, [19]], there are a lot of interesting results related polar dual polytopes. It is proved that if $\mathcal{P}^*$ is a polytope with the origin in its interior, the dual of the dual is the same polytope, i.e $\mathcal{P} = \mathcal{P}^{**}$ and most importantly for us, the face lattice of $\mathcal{P}^*$ is the opposite of the face lattice of $\mathcal{P}$, i.e. $\mathcal{L}(\mathcal{P}) \cong \mathcal{L}(\mathcal{P}^*)^{op}$. This exactly Definition 1.4.1.

An example of this duality in $\mathbb{R}^2$ is in Figure 1.4.1, where $\mathcal{P}$ is the red polytope and $\mathcal{P}^*$ is the green one and the inclusion reversing map is following:

$\mathcal{P}$ face-lattice		$\mathcal{P}^*$ face-lattice	$\mathcal{P}$ face-lattice		$\mathcal{P}^*$ face-lattice
$ABCDE$	$\to$	$\emptyset$	A	$\to$	GH
AB	$\to$	H	B	$\to$	HI
BC	$\to$	I	C	$\to$	IJ
CD	$\to$	J	D	$\to$	JF
DE	$\to$	F	E	$\to$	FG
EA	$\to$	G	$\emptyset$	$\to$	$FGHIJ$

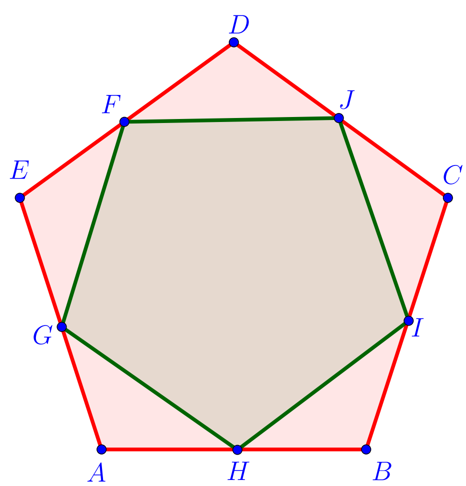

Figure 1.4.1

Definition 1.4.4: If $\mathcal{P} \cong \mathcal{P}^*$, $\mathcal{P}$ is said to be *self-dual*.

Figure 1.4.1 also shows an example of a self-dual polytope, because the red polytope lattice is isomorphic to the green one (both are pentagons).

$\mathcal{P}$ face-lattice		$\mathcal{P}^*$ face-lattice	$\mathcal{P}$ face-lattice		$\mathcal{P}^*$ face-lattice
$ABCDE$	$\leftrightarrow$	$FGHIJ$	A	$\leftrightarrow$	J
AB	$\leftrightarrow$	FJ	B	$\leftrightarrow$	F
BC	$\leftrightarrow$	FG	C	$\leftrightarrow$	G
CD	$\leftrightarrow$	GH	D	$\leftrightarrow$	H
DE	$\leftrightarrow$	HI	E	$\leftrightarrow$	I
EA	$\leftrightarrow$	JI	$\emptyset$	$\leftrightarrow$	$\emptyset$

The Hasse diagram of a self-dual polytope must be symmetrical about the horizontal axis half-way between the top and bottom [17], hence, all the polytopes in $\mathbb{R}^2$ are self-dual because their Hasse diagrams are symmetric. In a geometric way, $\mathcal{P}$ could be said to be self-dual if $\mathcal{P}^*$ has the same geometric "shape" than $\mathcal{P}$, not necessarily with the same lengths.

Commonly, if $\mathcal{P}$ is a self-dual polytope, $\mathcal{P}$ and $\mathcal{P}^*$ are represented with the same picture, not with two. For example, in Figure 1.4.1 we can delete the green pentagon and just rename all the edges an vertices with "$*$" as in Figure 1.4.2. This is an easier representation to understand self-duality, because we omit one polytope and the correspondence is clearer.

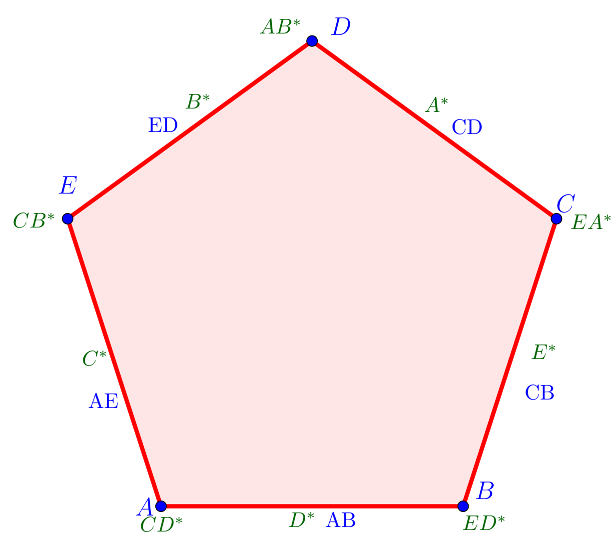

Figure 1.4.2

More examples of self-dual polytopes families are in [2], for example Figure 1.4.3, which is like a pyramid with n levels in $\mathbb{R}^3$ seen from above. This polytope is called the 3-dimensional $n-\textit{prismoidal}$ with triangular base.

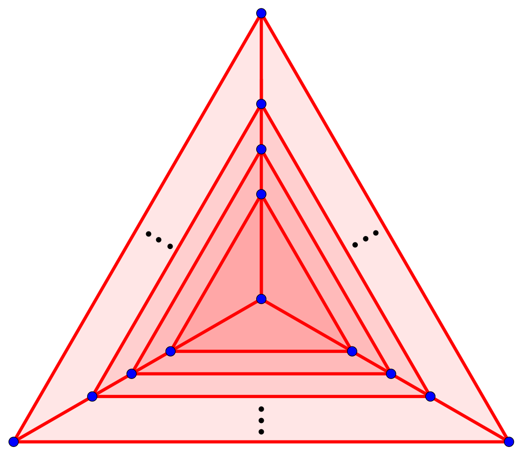

Figure 1.4.3

An stronger property related with duality that we will need is the following:

Definition 1.4.5: A $d-$polytope $\mathcal{P}$ is *involutory* if there is a dual polar function τ such that $\tau^2 = I$, that is $\mathcal{P}^{**} = \mathcal{P}$

Even thought we already know that polar duality induces an involutory mapping if the polytope contains the origin, if we take any τ dual polar function this is not necessarily true. In 1962 B. Grünbaum and G.C. Shepard [4](p729-p733) asked whether every self-dual convex polyhedron P has rank 2 or, equivalent. whether every self-dual P admits an involutory self-duality map. In [7] (p325-p733) S. Jendrol gave a negative answer to this question by showing an example of a polytope that is self-dual but not involutory as the one in Figure 1.4.4. A self-dual map of P is given by the permutation $(AaBb)\,(CcDd)\,(EeFf)\,(JjKk)\,(LlMm)\,(NnOo)$. Clearly the permutation function $p$satisfy $p^4 = p$ (see Figure 1.4.4).

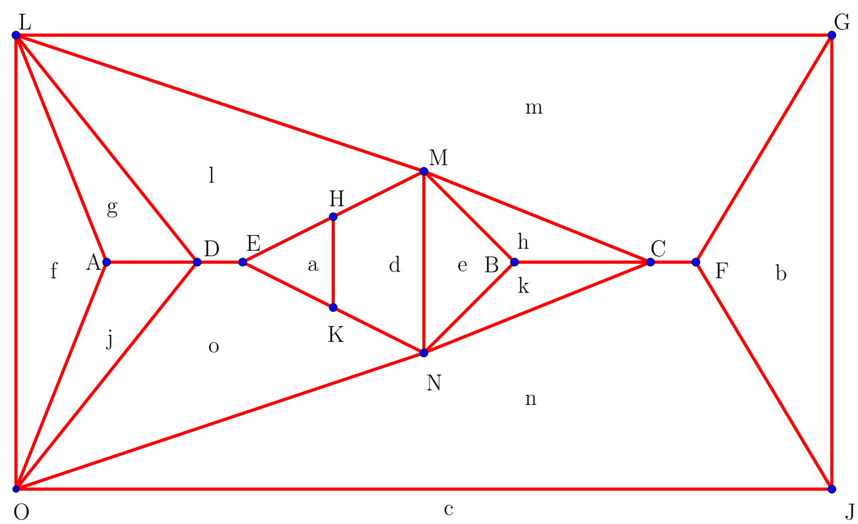

Figure 1.4.4

Definition 1.4.6: Let $\mathcal{P}$ be a $d-$polytope in $\mathbb{R}^d$. The *dual face* of a face $f \subset \mathcal{F}(\mathcal{P})$ is $f^* = \tau(f)$.

1.5 Metric Polytopes

We are ready to define the polytopes we are really interested in, the metric ones!

Definition 1.5.3: The *diameter* of a convex set $K \in \mathbb{R}^d$ is the maximum distance between any two points in $\mathcal{P}$ with the Euclidean metric. It is denoted by $diam\,(\mathcal{P})$.

Definition 1.5.1: Let $\mathcal{P}$ be a self-dual polytope in $\mathbb{R}^d$. The *dual-diagonal* of $\mathcal{P}$ are the segment lines that go from any vertex $v \in F_0\,(\mathcal{P})$ to any vertex $w \in F_0\,(\mathcal{P})$ such that $w \in v^*$.

Definition 1.5.2: Let $\mathcal{P}$ be a convex self-dual d-polytope, we call $\mathcal{P}$ a *metric polytope* with length h if:

1. Every dual diagonal has length h.
2. $diam\,(\mathcal{P}) = h$.

In other words, $\mathcal{P}$ is a *metric polytope* if all its dual-diagonals are diameters. We will denote a $\mathcal{P}$ *metric polytope* with length h as $\mathcal{P}_h$.

Examples of metric polytopes in $\mathbb{R}^2$ are the regular *n-agons*. There is a pentagon in Figure 1.5.1 to illustrate this fact.

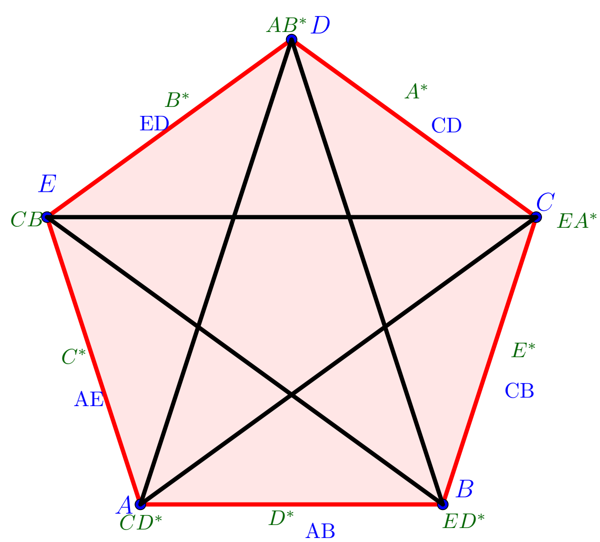

Figure 1.5.1

The examples in [2] can also be realized as metric polytopes. For example, in Figure 1.5.2 we have the 2-prismoidal with triangular base and some of its diagonals which have the same length.

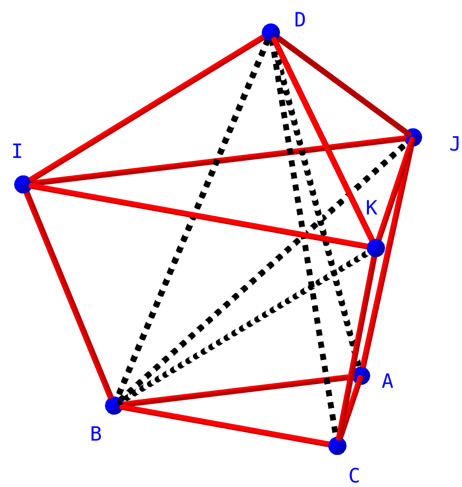

Figure 1.5.2

We believe that condition number two of Definition 1.5.2 could be removed. However, we could not do it. We studied a lot of examples but we failed. So we decided to leave this claim as a conjecture.

Conjecture 1.5.1: *If a d-polytope $\mathcal{P}$ has the property that every dual-diagonal has length h, then* $diam(\mathcal{P}) = h$.

Other way to define metric polytopes is by keeping condition 2, and requiring all the vertices to be on a sphere. This definition has been used several times, for example in [2] where some of the vertices are required to be on a sphere, or as in [10] where Lovász requires all the vertices to be on a sphere. For us it is enough to keep definition we gave, because we only want to require $diam(\mathcal{P}) = h$ and this is less restrictive. Let us prove it

Notice the following example of a metric polytope that is not embedded on a sphere (see Figure 1.5.3). However, as we will see next, if a $d-$polytope has all its diagonals of length h and all its vertices are on a sphere then it has diameter h.

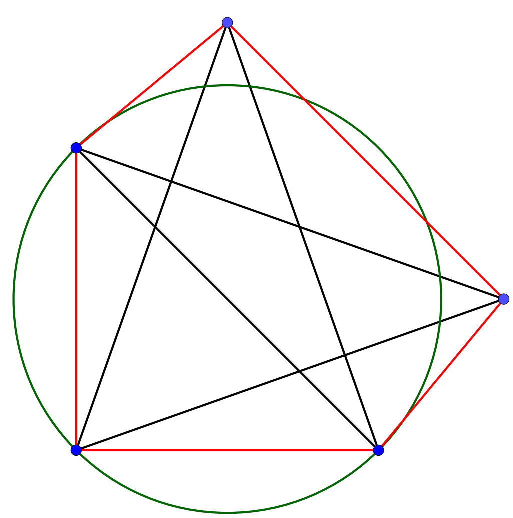

Figure 1.5.3

Lemma 1.5.1: *Let $v_1 v_2 w_1 w_2$ be vertices in ciclic order on a circle and $d(v_1, w_1) = d(v_1, w_2) = h$. Then $d(v_1, v_2) \leq h$.* (see Figure 1.5.3)
Proof: By the triangle inequality we have that:

$$\begin{aligned} d(w_1, w_2) + d(v_2, w_1) &\geq d(v_2, w_2) \\ \Rightarrow 1 &\geq \frac{d(v_2, w_2) - d(v_2, w_1)}{d(w_1, w_2)} \end{aligned} \tag{1}$$

By Ptolomeo's theorem, we know that

$$\begin{aligned} d(v_2, w_1) \cdot d(v_1, w_2) + d(v_1, v_2) \cdot d(w_1, w_2) &= d(v_1, w_1) \cdot d(v_2, w_2) \\ \Rightarrow d(v_2, w_1) \cdot h + d(v_1, v_2) \cdot d(w_1, w_2) &= h \cdot d(v_2, w_2) \\ \Rightarrow d(v_1, v_2) &= h \cdot \left(\frac{d(v_2, w_2) - d(v_2, w_1)}{d(w_1, w_2)} \right) \\ \Rightarrow d(v_1, v_2) &\leq h \text{ by } (1) \blacksquare \end{aligned}$$

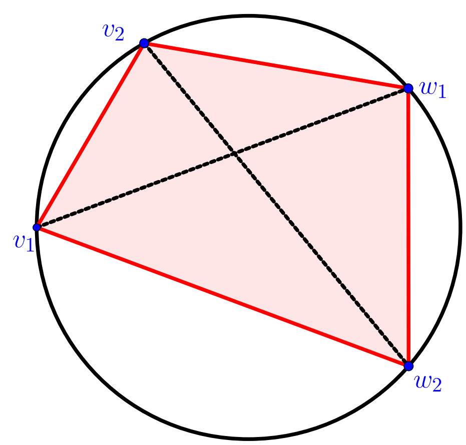

Figure 1.5.4

As notation, $S_t^{d-1}(v)$ is the $(d-1)$-sphere of radius t and center in v and S_t^{d-1} the $(d-1)$-sphere of radius t and center in the origin.

Lemma 1.5.2: *Let $C \subset S_t^{d-1}$ be an hyper-spherical cap induced by the hyperplane H, i.e one of the subsets of S_t^{d-1} left in one side by H. If $v_1, v_2 \in C$ and the distance between v_1 and every point of $S_t^{d-1} \cap H$ is h (v_1 is the pole of the cap), then $d(v_1, v_2) \leq h$.*
Proof: Let $w_1, w_2 \in S_t^{d-1} \cap H$ be points such that w_1 and w_2 are in the geodesic generated by v_1 and v_2, then there is a plane P that contains v_1, v_2, w_1, and w_2 i.e, $v_1, v_2, w_1, w_2 \in \left(P \cap S_t^d\right) \cong S_t^1$. Therefor $d(v_1, v_2) \leq h$ by Lemma 1.5.1 . $\blacksquare$

Lemma 1.5.3: *If $\mathcal{P}$ is a self-dual d-polytope with all of its dual-diagonals having length h and all its vertices in a S_t^{d-1} sphere in $\mathbb{R}^d$ then the distance between any two vertices is less than or equal to h.*
Proof: Notice that for each vertex v of $\mathcal{P}$, there is a supporting hyperplane H_v such that $v^* \subset H_v$. This hyperplane induces two spherical caps, let C_v be the one that contains v and C_v^* the other one. Furthermore, since v^* has at

least d points in order to have dimension $d-1$, there is only one $(d-2)$-sphere S containing all the vertices of v^*, so $S = S_t^{d-1} \cap H_v$.

Since all the dual-diagonals have length h, the sphere $S_h^{d-1}(v)$ with radius h and center in v contains also all the vertices of v^*, so the distance between v and $H_v \cap S_h^{d-1}(v) = S_t^{d-1} \cap H_v$ is h.

If we suppose there are v_1 and v_2 vertices of $\mathcal{P}$ such that $d(v_1, v_2) > h$, then v_2 has to be in $C_{v_1}^*$ by Lemma 1.5.2, then H_{v_1} is not a supporting hyperplane because leaves v_1 in one half-space and v_2 in the other one, a contradiction. ■

Lemma 1.5.4: *If $\mathcal{P}$ is a d-polytope with diameter h, then the length between $x, y \in \mathcal{P}$ is h only if both are vertices of $\mathcal{P}$.*

Proof: Let x and y points in $\mathcal{P}$ such that x, y are the vertices of a diameter. Without loss of generality, if x is not a vertex, then either x is $int\,\mathcal{P}$ or x is in a facet. If $x \in int\,\mathcal{P}$, then we get a contradiction because there is a $x_1 \notin int\,\mathcal{P}$ such that y, x and x_1 are colinear in that order, so the distance between x_1 and y is greater than the distance between x and y.

If x is in a facet, there is a segment line $x_1y_1 := l$ contained in such facet, such that $x \in int\, l$ for some x_1 and $y_1 \in \mathcal{P}$. Notice that l and xy has one angle greater than or equal to 90º, without loss of generality let x_1xy be such angle (see Figure 1.5.5). Then distance between x_1 and y is greater the diameter. A contradiction. ■

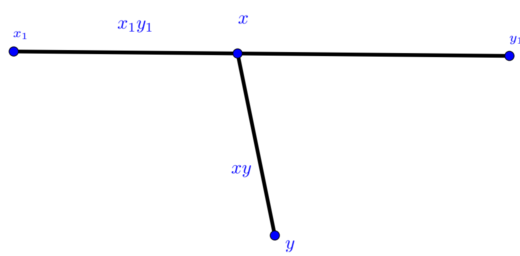

Figure 1.5.5

Theorem 1.5.1: *Every self-dual d-polytope $\mathcal{P}$ with all its dual-diagonals having length h and all its vertices in a sphere in $\mathbb{R}^d$ has diameter h.*

Proof: By Lemma 1.5.4 we have that all the diameters have to be realized by pairs of vertices of $\mathcal{P}$ and by Lemma 1.5.3 the distance between any two vertices of $\mathcal{P}$ is less than or equal to h, so $diam(\mathcal{P}) = h$. ■

Corollary 1.5.1: *Let $\mathcal{P}$ be a self dual d-polytope with all its dual-diagonals having length h and all its vertices in a sphere. Any two vertices v_1 and v_2 of are at distance h if and only if $v_1 \in v_2^*$.*

Proof: If $v_1 \in v_2^*$ by definition we know they are at distance h. Now suppose that v_1 and v_2 are at distance h but $v_1 \notin v_2^*$, since v_2^* is a $d-1$ dimensional face, then there is just one $d-1$ sphere containing the vertices of v_2^* and v_1, but the center of this sphere is in v_2 which is a contradiction since v_2 has to be in this sphere too by the hypothesis that are the vertices of $\mathcal{P}$ are in a sphere . ■

Chapter 2
Meissner Polytopes and Bodies of constant width

An important reason to study metric polytopes is that this kind of polytopes, could work as the "structure" for bodies of constant width, and such bodies have several interesting applications, see for instance [11] for concrete applications. Furthermore, the study of them is very relevant in several branches of mathematics. What is a body of constant width?

2.1 Bodies of constant width

Definition 2.1.1: Let K be a convex set in $\mathbb{R}^d$. The *width* $w(u)$ of K in the direction of the unitary vector u is the length of the largest diameter of K in direction u.

Definition 2.1.2: K is said to be a *body of constant width* (BCW) if in every direction u, $w(u)$ is constant.

To simplify notation, BCW means body or bodies of constant width.

It is often thought that the only BCW are the d-dimensional spheres, whose diameters are two times their radii in any direction, but this is totally false. There are many BCW.

As an example, the triangle in Figure 2.1.1 is not a BCW because the width between the black lines is shorter than the width between the orange lines. But in the second figure, it can be proved that in any direction the width is constant. This figure is well known as the Reuleaux triangle, named after its creator, the scientific and engineer Franz Reuleaux.

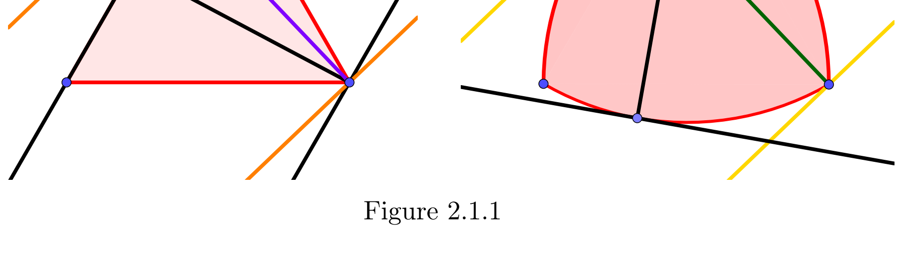

Figure 2.1.1

An interesting question is, how is the Reuleaux triangle created in order to have constant width? Figure 2.1.2 shows an equilateral triangle and arcs of circumferences though exactly two vertices, with center in the opposite vertex and radius l (the length of the triangle side). More precisely, this construction is called the *ball-polytope* of the equilateral triangle.

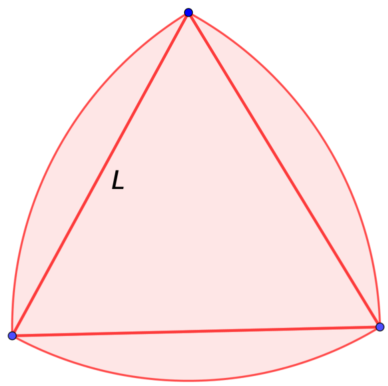

Figure 2.1.2

In order to prove the fact that the Reuleaux triangle is a BCW, it is necessary to mention some basic BCW's results. Some of them are easy to understand geometrically, but the proofs are very technical, so they will be skipped. The proofs, details can be found in [11].

Lemma 2.1.1: *Every supporting hyperplane H of a BCW Φ touches Φ at exactly one point.*

Theorem 2.1.1: *The line segment joining the point of contact between a BCW and two of its parallel supporting hyperplanes is perpendicular to them.*

Definition 2.1.3: A chord ϕ in a body C is called *normal* if at least one of the supporting hyperplane at the end of the chord ϕ is perpendicular to ϕ. When both hyperplanes are perpendicular then ϕ is called *binormal*.

Theorem 2.1.2: *A body Φ has constant width if and only if it has a binomial in every direction.*

Theorem 2.1.3: *A body Φ is of constant width if and only if each normal is a binormal.*

The last two theorems are BCW characterizations that often simplify the proofs that a body is a BCW.

Notice that the Reuleaux triangle has a binormal at each direction and that each supporting line touches it in just one point (see Figure 2.1.3), so the Reuleaux triangle is a BCW.

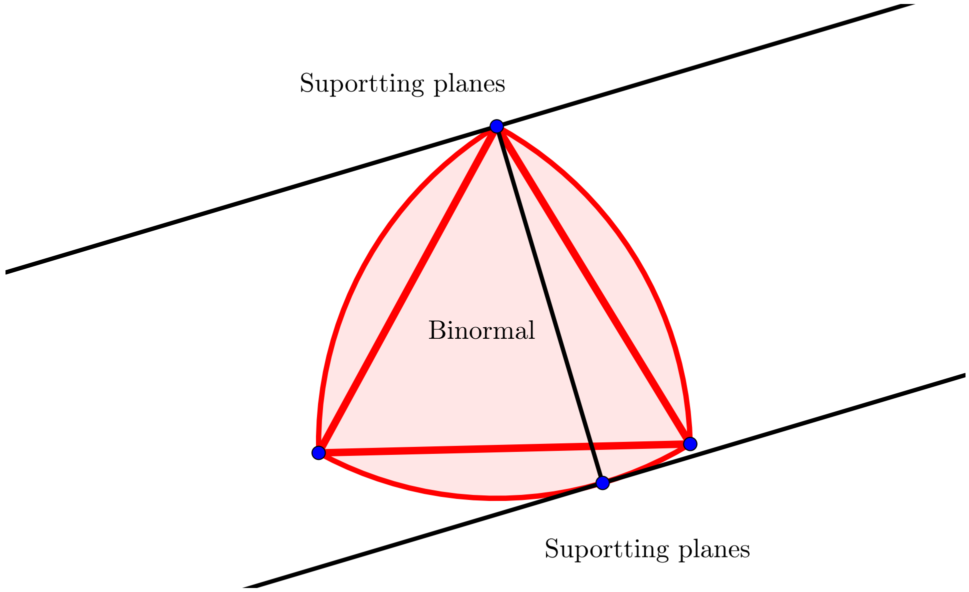

Figure 2.1.3

2.2 Constructing BCW

In some cases, creating algorithms to make families or sets with some properties is hard. This is why the main goal of this chapter is to establish an algorithm or a parametrization to create families of BCW in higher dimensions, at least in $\mathbb{R}^4$.

Are there algorithms to create families of BCW? There are different ways to construct a BCW, for example, the solutions to a particular differential equation can be BCW [11]. Since this thesis focuses on metric polytopes properties, algorithms that do not include these polytopes will be omitted. For other kind of constructions see [11].

So, is there an algorithm to create families of BCW using metric polytopes? The answer is yes, and in order to describe this algorithm we need the following definitions.

Definition 2.2.1: Let $T \subset \mathbb{R}^d$ be a finite set of at least d points. The ball-polytope $\mathcal{B}_r(T)$ associated to T with radius r is the intersection of all the balls with center at the vertices of T and radius r, i.e.

$$\mathcal{B}_r(T) \quad = \quad \bigcap_{p \in T} B_r(p)$$

Definition 2.2.2: Let $T = \{p_1, ..., p_k\} \subset \mathbb{R}^d$ be a finite set with at least $d+1$ affinely independent points. Denote by $S_r(p_i)$ the boundary of each $B_r(p_i)$, then the *ball-polytope* $\Phi = \mathcal{B}_r(T)$ has on its border $bd(\Phi)$ d types of points:

- 0-singular points, points $p \in$bdΦ such that $p \in \bigcap_{i \in I} S_r(p_i)$, for some $|I| = d$
- $k-$singular points, which are the points $p \in$ bdΦ such that $p \in \bigcap_{i \in I} S_r(p_i)$ for some $|I|$ $=d-k$, with $k = 1, ..., d-2$.
- Regular points, which are the points $p \in$ bdΦ such that p is in only one boundary sphere of a ball.

Basically, Definition 2.2.2 says that the intersection of at least d, $(d-1)$-spheres is a point which is a $0-$sphere, the intersection of k, $(d-1)$-spheres is a $(d-k-1)$-dimensional surface on a $(d-k-1)$-sphere, because there are tangent spheres and the regular points are in a surface of a $(d-1)$-sphere by definition. So this induces a face lattice $\mathcal{L}(\Phi)$ by inclusion in the following way:

Definition 2.2.3: Let Φ be as in Definition 2.2.1 and let G_Φ be the induced graph of Φ defined by vertices $V(G_\Phi)$ that are the $0-$singular points of Φ and whose edges correspond to arcs of circles that contain only 1-singular points and connect pairs of points of $V(G_\Phi)$.

Definition 2.2.4: Let $\Phi = \mathcal{B}_h(T)$ be such that for every subset $T' \subsetneq T$, we have $\Phi \neq \mathcal{B}_h(T')$. A *supporting sphere* S^l is a sphere of dimension l, where $0 \leq l \leq n-1$, which can be obtained as the intersection of some of the spheres in $\{S_h(x)\}_{x\in T}$. We will say that Φ is *standard* if for any supporting sphere S^l the intersection $\Phi \cap S^l$ is *spherical convex*, this is for every two points x, $y \in \Phi \cap S^l$ there is a geodesic joining x and y totally contained in $\Phi \cap S^l$.

Definition 2.2.5: A *Reuleaux polytope* is a convex set $\Phi \subset \mathbb{R}^n$ satisfying the following properties:

1. There is a set of points $T \subset \mathbb{R}^n$ with $\Phi = \bigcap_{x\in T} B_h(x) = \mathcal{B}_h(T)$,
2. Φ is a standard ball polytope, and
3. the set of 0-singular points of bdΦ is T.

A Reuleaux polytope is an spherical convex polytope (satisfying the conditions of an abstract polytope), the faces of this convex polytope are as follows:

- The empty face is the empty set.
- Each 0-singular point is a 0-face.
- The closed set (this includes the 0-faces) of 1-singular points joining two 0-faces is a 1-face.
- The closed set (this includes the $(k-1)$-faces) of k-singular points joining $(k-1)$-faces in a spherical convex $(d-k-1)$-dimensional surface on a $(d-k-1)$-sphere is a $k-$face.
- The closed set of regular points joining $(d-2)$-faces in a spherical convex $(d-1)$- dimensional surface on a $(d-1)$-sphere is a facet.
- Φ is the total face.

In the same way a start analyzing the relation between Reuleaux polytopes, BCW and metric polytopes in the plane. Reuleaux triangle was created, we may construct others by using different 2-polytopes, or more commonly called *polygons*. For example, we can create the regular Reuleaux Pentagon created by the vertices of a regular pentagon (Figure 2.2.1).

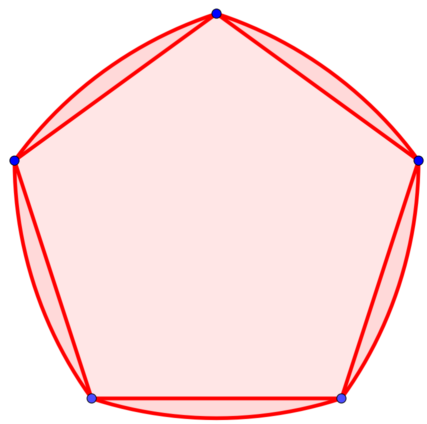

Figura 2.2.1

Reuleaux polytopes in $\mathbb{R}^2$ are called Reuleaux polygons and all Reuleaux polygons are BCW [11], furthermore, the vertices of a Reuleaux polygon are vertices of a regular polygon with an odd number of vertices. Now the question is if metric polygons are related in some way to Reuleaux polygons. What we found was a very strong relation between metric polygons and Reuleaux polygons.

Lemma 2.2.1: *A set of points V induces a Reuleaux polygon if and only if V is the set of vertices of a metric polygon in $\mathbb{R}^2$.*
PROOF: It is enough to prove that every metric polygon in $\mathbb{R}^2$ is a polygon (can be not regular) with an odd number of vertices, because in [11] they prove that a body is a Reuleaux polygon if and only if it is induced by the vertices of a polygon with an odd number of vertices polygon in such a way that each dual-diagonal is a diameter, which we have for free by definition of metric polytopes.

Let $V = \{v_1, ..., v_n\}$ be ordered by the adjacency in the metric polygon, i.e. $v_i v_{i+1}$ is an edge for every $i \mod n$. Since $\mathcal{P}$ is metric, the line l induced by the midpoint of the edge $v_k v_{k+1} = v_1^*$ for some k and v_1, is orthogonal to v_1^*. Note that l leaves $v_2, ..., v_k$ and $v_{k+1}, ..., v_n$ in different open half-spaces A_k and A_{k+1} respectively. Let us notice that v_k^* has to contain v_1 by the inverse inclusion duality function.

If v_k^* is in A_k then $v_k^* = v_1 v_2$, $v_2^* = v_k v_{k-1}$, and so on, then at the end, the edge $v_{\lfloor \frac{k}{2} \rfloor} v_{\lceil \frac{k}{2} \rceil}$ is going to be a polar edge of either $v_{\lfloor \frac{k-1}{2} \rfloor}$ or $v_{\lceil \frac{k-1}{2} \rceil}$ but this is a contradiction by the metric condition. Then v_k^* has to be in A_{k+1}.

Since v_k^* is in A_{k+1}, by the inverse inclusion duality function again $v_{k-1}^*, ..., v_2^*$ have to be in A_{k+1}, so A_{k+1} has $k-1$ edges, and the only way this could happen is when A_{k+1} has $k-1$ vertices.

Therefore $\mathcal{P}$ has to have an odd number of vertices. ■

Theorem 2.2.1: *Let T be the set of vertices of a metric polygon $\mathcal{P}_h$, then $\mathcal{B}_h(T)$ is a BCW.*
PROOF: The result follows since we know that the vertices of $\mathcal{P}_h$ induce a Reuleaux polygon by Lemma 2.2.1 and we know that every Reuleaux polygon is a BCW. ■

In $\mathbb{R}^3$ the things are quite different. In [13] it is proved that if $\Phi \subset \mathbb{R}^3$ is a Reuleaux polyhedron, then G_Φ is a self-dual graph, where the dual function τ is given by $\tau(x) = S(x, h) \cap \Phi$ and τ is also an involution in the sense that a vertex x belongs to the cell $\tau(y)$ if and only if the vertex y belongs to the cell $\tau(x)$. So in $\mathbb{R}^3$ not all Reuleaux polyhedron come from a metric polytope, even worse, they do not necessarily comes from a polytope. In the example in Figure 2.2.2 we can see that G_Φ is not the 1-skeleton of a polytope but Φ induces a Reuleaux polyhedron:

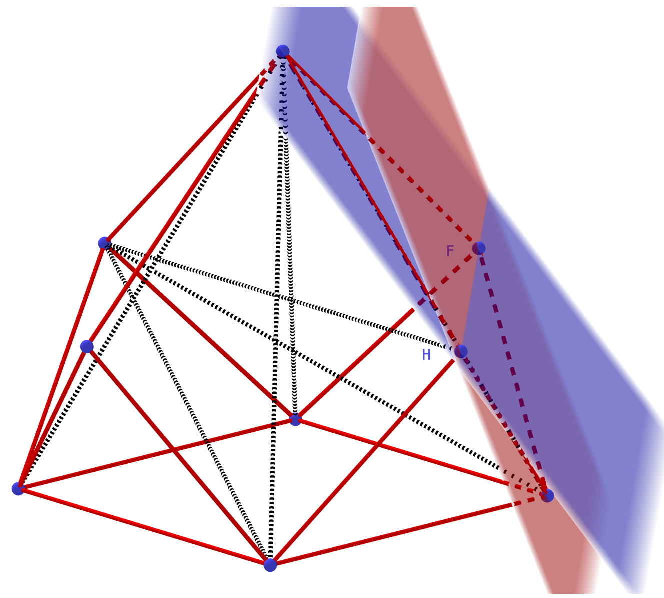

Figure 2.2.2, example from [13]

Figure 2.2.2 it is called the 4-hyper-wheel because its base is a quadrilateral and the parametrization of this figure could be given in general for a $2n$-hyper-wheel such that it is the 1-skeleton of a Reuleaux polyhedron. One parametrization is: the vertices are in the n-gon with vertices $v_{0,\,k}\left(\cos\frac{k\pi}{n},\, \sin\frac{k\pi}{n},\, 0\right)$, with $k = 1, \ldots, 2n$, the other layer of vertices are $v_{1,\,k} = \left(\cos\left(\frac{k\pi}{n} + \frac{\pi}{2}\right),\, \sin\left(\frac{k\pi}{n} + \frac{\pi}{2}\right),\, h\sqrt{\left(1 - \frac{(1+cos(\pi/n))^2}{4}\right)}\right)$, with $k = 1, \ldots, 2n$ and h is the radius of the ball polytope, and the apex is going to be in $v = \left(0, 0, \frac{\sqrt{3}h}{2}\right)$. The edges are going to be formed by

1. $v_{1,\,j}v_{0,\,j}$, with j mod $2n$.
2. $v_{1,\,k}v_{0,\,j+1}$ with j mod $2n$ and $k = 1, \ldots, 2n$.
3. $vv_{1,\,k}$, with $k = 1, \ldots, 2n$.

With this parametrization we can see that G_Φ is not the 1-skeleton of a polytope because in the case of Figure 2.2.2, the blue plane and the red are support planes but the edge FH is missing in the graph.

In other words what we are saying is that metric polytopes induce concrete examples of Reuleaux polyhedrons but they do not say a lot about the whole family of Reuleaux polyhedrons.

Now, is every Reuleaux polyhedron is a BCW? It is natural to think so, since all Reuleaux polygons are BCW, but this is not true, because if we take a Reuleaux polyhedron created by a ball-polytope of radius h in $\mathbb{R}^3$, its diameter is a little bit larger than h as the following example.

We may note that the ball-polytope of the tetrahedra with vertices a, b, c and d and radius 1 is shown in Figure 2.2.3 has arcs ab, ac, ad, bc, bd and cd. According to [11], the length of every chord from the midpoint of every arc to the midpoint of its dual edge is $\left(\sqrt{3}-\frac{\sqrt{2}}{2}\right) \approx 1.02$, meanwhile the length of every chord containing a vertex is 1. See chords d_1 and d_2 in Figure 2.2.3 which represent these two types of chords. Consider the chord d_1 from the midpoint of arc ab to the midpoint of the arc cd, and the chord d_2 which goes from point b to the barycenter of the face adc of the tetrahedra ball polytope. Then d_1 is longer than d_2. So Figure 2.2.3 is not a BCW.

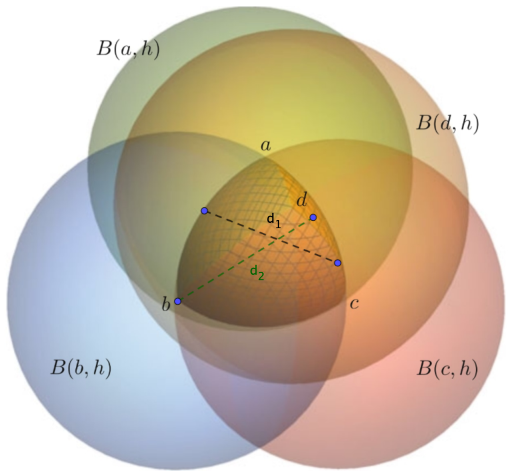

Figure 2.2.3. Image extracted from [11] with the author's consent

Since the length difference between these two types of chords is pretty small and the tetrahedron ball polytope is "almost" a BCW, the mathematician E. Meissner proposed the very clever idea of "fixing" this figure in order to make it with constant width, "shaving" a little bit the figure where the chords are larger than one. Loosely speaking, this shaving consist on leaving intact one of the edges of the Reuleaux tetrahedron and shaving the corresponding dual edge, for example, if the tetrahedron has vertices a, b, c and d, if the arc ab remain intact, then the dual arc cd must be shaved until all chords connecting both edges (chords as d_1 in Figure 2.2.3) become of length one. After this shave operation we have what is called the *Meissner tetrahedron*.

As in [13], we call this "shave" a *surgery*. This procedure is described in Boltianski and Yaglom's book [18] using the tetrahedra ball polytope. Later, Luis Montejano and Edgardo Roldán generalized this procedure to any Reuleaux polyhedron in $\mathbb{R}^3$ [13]. These bodies are called *Meissner polyhedron* (see Figure 2.2.6)

Definition 2.2.5: A *Meissner polyhedron* is any BCW that can be obtained from a Reuleaux polyhedron in $\mathbb{R}^3$ by performing surgery over it.

Figure 2.2.6

In other words, not all the Reuleaux polyhedron are induced by a metric polytope, but all metric polytopes induce a Reuleaux polyhedron. With a similar procedure to that used by Meissner, it is possible to transform any Reuleaux polyhedron into a Meissner polyhedron which is a BCW, so if we take any metric polytope, this induce a Reuleaux polyhedron and then this induce a Meissner polyhedron. So metric polytopes, Reuleaux polyhedron and BCW are related to each other.

One of the main goals of this work was to try to make a similar algorithm to the one defined by Montejano and Roldán in $\mathbb{R}^3$ (see [13]), to create families of BCW in spaces of dimension larger than 3, because it is no such thing in the literature. Actually, in general, we have not found examples of BCW even in $\mathbb{R}^4$ besides the ones that are in [2].

To understand what we wish to generalize, we now describe some ideas in the steps required to perform a surgery in $\mathbb{R}^3$.

First, we have to choose a Reuleaux polyhedron. Once we have chosen a Reuleaux polyhedron Φ in $\mathbb{R}^3$, we have to find a function $g: F_1(\Phi) \to \{0, 1\}$ such that $g(A) \neq g(A^*)$ for every edge A. This is important because the distance between dual edges are the ones greater than h, so the surgery is applied in one edge of each pair of dual edges, i.e. either all the edges with number 1 or all with number 0 will be modified.

In Figure 2.2.6 we can see all the pairs of dual edges painted by different colors and an example of how g could be defined. There are two different types of Meissner polyhedron induced by this example, the Meissner polyhedron would have a 2-face with all its edges shaved, remaining intact all edges with label 0, and if we perform surgery over 0, the Meissner polyhedra would have a vertex with all its edges shaved and one 2-face unchanged.

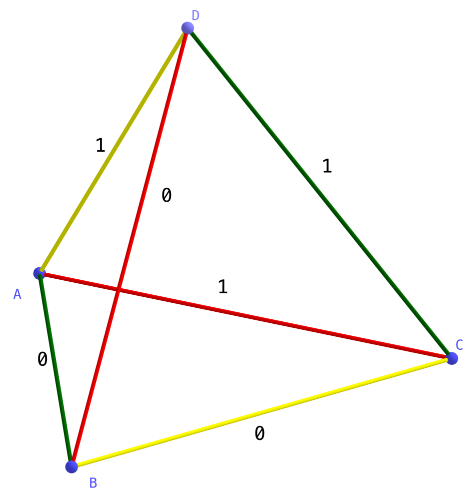

Figure 2.2.6

The next step is the surgery itself to make the width constant in every direction, for more details on how do the surgery see [18, 13]. After the surgery we obtain a Meissner polyhedron.

Once we analyze step by step the surgery in $\mathbb{R}^3$, we realize that for any $\Phi = \bigcap_{x\in X} B_h(x)$ Reuleaux polyhedron in $\mathbb{R}^4$, we would want to perform surgery either one edge and leave unchanged the complete dual 2-face (this is the 2-face together with edges) or shave a complete 2-face and leave unchanged the dual edge. This is because the distance between the center of mass of a 2-face to its dual edge is a little bit larger than h, furthermore, the distance between the midpoint of every edge and the midpoint of every edge of its dual 2-face is larger than h [2]. So in order to perform surgery as in [13] the function g defined above has to be now as follows: $g:\, F\setminus\{F_4,\, F_3\, F_0,\, \emptyset\}\to\{0,\,1\}$ such that:

1. If $C\leq A$, then $g(C)=g(A)$.
2. If $f(A)=A^*$, $g(A)\neq g(A^*)$.

The existence of this function g is strictly necessary because either all the faces in $g^{-1}(1)$ get surgery or all the faces in $g^{-1}(0)$ do, otherwise the surgery can not be performed as in [13].

We decided to work first with Reuleaux polytopes induced by metric polytopes, because they are easier to understand and we have concrete examples of metric polytopes as we saw in Section 1.5. Unfortunately, after several attempts to define a function g over these Reuleaux polyhedrons in $\mathbb{R}^4$, we proved that such function does not exist, even worse, for any n-dimensional Reuleaux polyhedron such function does not exist for $n>3$. Therefore, surgery in the sense of [13] can not be performed in any dimension larger than 3 to any Reuleaux polyhedron as we proved in the following theorem:

Theorem 2.2.2: *Let Φ be a Reuleaux polyhedron in $\mathbb{R}^n$, with $n>3$. There is not a function*

$$g:\, F(\Phi)\setminus\{F_n,\, F_{n-1},\, F_0,\, \emptyset\}\quad\to\quad\{0,\,1\}$$

such that for every pair of faces C,$A\in F(\Phi)$:

1. *If $C\leq A$, then $g(C)=g(A)$.*

2. $g(A) \neq g(A^*)$.

Proof: Suppose there is a function g with the required properties. If e is an edge of Φ, then $e^* \in F_{n-2}$, so $g|_{F_2}$ is onto. There is at least one vertex v in Φ with edges e_1 and h_1 such that $g(e_1) = 0$ and $g(h_1) = 1$ because Φ is strongly connected.

Let us take two flags Ψ_0, Ψ_1 with the vertex v and e_1 as an edge of Ψ_0 and h_1 as an edge of Ψ_1, i.e.

$$\begin{aligned} \Psi_0 &= \{\emptyset, v, e_1, e_2, ..., e_{n-1}, \Phi\} \\ \Psi_1 &= \{\emptyset, v, h_1, h_2, ..., h_{n-1}, \Phi\} \end{aligned}$$

By the first property of g, $g(e_i) = 0$ and $g(h_i) = 1$ for every $i = 1, ..., n-2$.

The fourth axiom says there has to be a sequence from Ψ_0 to Ψ_1, changing just one face at the time. Let us say that $e_1 = c_1 \to c_2 \to ... \to c_m = h_1$ is the path of vertices to go from e_1 into h_1, then there has to be an i such that $g(c_i) = 0$ but $g(c_{i+1}) = 1$. Let Ψ_0' be the last flag with c_i as an edge and Ψ_1' the first flag with c_{i+1} as an edge in the sequence from Ψ_0 to Ψ_1, so Ψ_0' is followed by Ψ_1' in this sequence.

Let s be the 2-*face* of Ψ_0', therefore s is also the $2-face$ of Ψ_1' because there has to be just one different face between Ψ_0' and Ψ_1', which are already the edges. So c_i and c_{i+1} are edges of s. But this yields a contradiction because neither $g(s) = 0$ is possible because $g(c_{i+1}) = 1$ nor $g(s) = 1$ is possible because $g(c_i) = 0$. ■

Corollary 2.2.1: *There are no Meissner polyhedra in $\mathbb{R}^d$ for $d \geq 4$ in the sense of [13].*
Proof: Let Φ be a Reuleaux polyhedron. For every face $f_1 \in F(\Phi) \setminus \{F_n, F_{n-1}, F_0, \emptyset\}$, the chord from any center for mass of any sub-faces of f_1 to any center for mass of any sub-faces of f_1^* is larger than one. So we have to perform surgery either all the faces of f_1 or all the faces of f_1^*.

Then we need a function g as in Theorem 2.2.2 in order to perform surgery. But this function does not exist, so there are no Meissner polyhedra in $\mathbb{R}^d$ for $d \geq 4$. ■

An interesting question would be whether there is another construction of BCW using Reuleaux polytopes. There is a new idea that could work to create families of BCW using surgery, but this time shaving the ball polytope equally and avoiding the function g, or to define a symmetric transformation over Reuleaux polytopes to get BCW. Deborah Oliveros and Isaac Arelio are currently working on this idea.

In Figure 2.2.8 [8] we can find this idea applied to Reuleaux tetrahedra by performing a Minkowski sum (middle) of the two types of Meissner's tetrahedra described before, the M_V which is the one with rounded edges meeting in a vertex (left) and the other one with M_F with rounded edges surrounding a face (right).

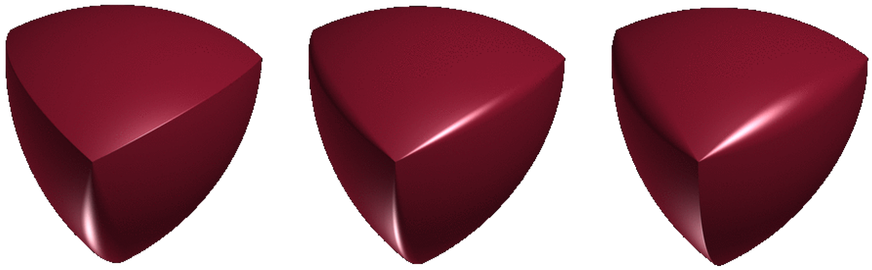

Fig. 2.2.8. Extracted from [8]

In order to prove that the new figure in [8] is a BCW, let us remember some properties of the Minkowski sum.

Definition 2.2.6: The *Minkowski* sum of S and T is the set

$$S + T \quad = \quad \{a + b \mid a \in S,\, b \in T\}$$

Some times it is useful to think of the Minkowski sum $S + T$ as the union of all translated copies of T by vectors of S. That is:

$$S + T \quad = \quad \bigcup_{a \in S} (a + T)$$

The proves of the following statements can be found in ([11], p.20, p70)

Lemma 2.2.2: *For given sets R, S, $T \subset \mathbb{R}^n$ and a real number $\lambda \in \mathbb{R}$ we have*

- $S + T = T + S$,
- $(S + T) + R = S + (T + R)$,
- $\lambda (S + T) = \lambda S + \lambda T$.

Theorem 2.2.3: *A convex set Φ has constant width if and only if $\Phi + (-\Phi)$ is a ball.*

Corollary 2.2.2: *The Minkowski sum of two BCW Φ and Ψ is a BCW*

Proof: It is enough to prove that $\Psi + \Phi - (\Psi + \Phi)$ is a sphere by Theorem 2.2.3. But this is true just applying the properties of Minkowski sum in Lemma 2.2.3.

$$\begin{aligned} \Psi + \Phi - (\Psi + \Phi) &= \Psi + \Phi - \Psi - \Phi \\ &= \Psi - \Psi + \Phi - \Phi \\ &= B_1 + B_2 \text{ applying Theorem 2.2.3 to } \Phi \text{ and } \Psi \end{aligned}$$

Since B_1 and B_2 are balls, and the sum of two balls is a ball, the result is proved. ∎

Although this new body is a BCW, there are several questions around this construction. How do we know that the Minkowski sum of $M_V + M_F$ is actually a figure with symmetrical the symmetrical properties that we want? This is not proved mathematically in [8], they just present a picture that looks similar because they were more interested in a create a computational construction to solve some visualizations problems than to prove geometrical properties. However, we think this fact is not very hard to prove it.

An other question would be, what do we have to sum in order to have a similar construction in higher dimensions? In $\mathbb{R}^4$ there are not Meissner polyhedra as we said in Corollary 2.2.1.

Chapter 3
The Frequent Large Distance Problem

Usually, one of the hardest and most interesting type of problems in mathematics involves finding maximums and minimums, for example, in graph theory, there is a huge number of people working on extremal graph theory, where the main goal is to find the largest graph that avoids some subgraph as a substructure. In discrete geometry, finding Helly's numbers, which are the minimum numbers required to have a Helly type theorem, is sometimes hard but very interesting. Another example is to find algorithms that minimize the running time and maximize the information about a problem. In numeric analysis, minimizing the error of the numeric methods is very important to approximate solutions to problems (that do not have analytic solutions).

Another problem of this type in discrete geometry consists on finding the maximum number of distances in the Euclidean space: Let V be a set of $n \geq 2$ points in Euclidean d-space. The *diameter* of V ($diam\, V$) is the largest distance between points of V (note that this definition has been given in Definition 1.5.3 for polytopes). We denote by $e\,(V)$ the number of pairs $\{x,\, y\} \subset V$ such that $\|x - y\| = \operatorname{diam} V$. Now, what arrangement of points V attains the maximum number $e\,(V)$? We denote $e\,(d,\, n) = \max e\,(V)$, with $|V_n| = n$, and we call V an *extremal configuration* if $e\,(V) = e\,(d,\, n)$ for $n > d$. This problem was formulated originally by Erdős, and solved by Hopf and Pannwitz [6] in dimension 2. In dimension 3 a similar problem was proposed by Vázsonyi and solved by Grünbaum [3], Heppes [5] and Straszewicz [15], see Section 3.2.

The main goal of this chapter is to verify if the set of vertices of a metric polytope can be an extremal configuration in any dimension, because if not, a some interesting questions would be, what is the maximum number of diameters that a metric polytope with n vertices can have? Does every metric polytope with n vertices reach this number?

3.1 Extremal configuration in $\mathbb{R}^2$

You might be wondering, how is this problem related with polytopes? In $\mathbb{R}$ the problem is trivial, because $e\,(1,\, n) = 1$ for all $n \geq 2$, but in $\mathbb{R}^2$ something very interesting happens, $e\,(2,\, n) = n$ [14]. Furthermore, there is a characterization of extremal configurations in [9] that states the following: $V \subset \mathbb{R}^2$ is an extremal configuration if and only if *vertices* $(\Phi) \subseteq V \subseteq \operatorname{bd} \Phi$ for some Reuleaux polygon Φ. This is amazing! We can check if V is an extremal configuration (at least in $\mathbb{R}^2$) with all the characterizations we have talked about in the previous chapter about Reuleaux polytopes.

Particularly, the vertices of a metric polygon yield an extremal configuration because they induce a Reuleaux polyhedron. So we have a family of extremal configurations with an odd number of points in $\mathbb{R}^2$.

In Figure 3.1.1 we have a heptagon as an example of a metric polygon together with its diagonals.

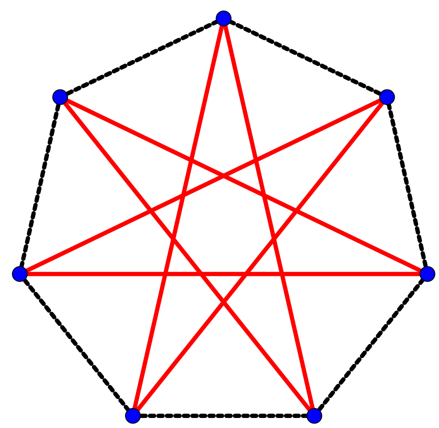

Fig 3.1.1

In general, since every metric polygon with n vertices has n diagonals, the following interesting result holds and it is equivalent to Corollary 1.5.1 for dimension 2.

Theorem 3.1.1: *Two points v_1 and v_2 define a diameter of a metric polygon $\mathcal{P}$ in $\mathbb{R}^2$ if and only if both are vertices of $\mathcal{P}$ and $v_1 \in v_2^*$.*

Proof: By Lemma 1.5.4 we know that the diameters are only induced by vertices of $\mathcal{P}$, if there is a diameter v_1v_2 such that $v_1 \notin v_2^*$ then we would have at least $n+1$ diameters because there are n dual-diagonals in $\mathcal{P}$, which are diameters by definition plus v_1v_2 which is not a dual diagonal. But this is not possible since n is the maximum number of diameters in $\mathbb{R}^2$ for a set of n points. ■

Corollary 3.1.1 *Every set of vertices V of a metric polygon $\mathcal{P}$ in $\mathbb{R}^2$ is an extremal configuration.*

Proof: This follows from Lemma 2.1.1 and the fact that a set V is an extremal configuration if and only if there is Reuleaux polygon such that $V(\Phi) \subseteq V \subseteq \operatorname{bd}\Phi$.

In summary, the vertices of every metric polygon from an extremal configuration. In other words, metric polygons are solutions in $\mathbb{R}^2$. Let us jump to the next dimension!

3.2 Extremal configuration in $\mathbb{R}^3$

As we mentioned before, in $\mathbb{R}^3$ B. Grünbaum [3], A. Heppes [5] and S. Straszewicz [15] proved independently the Vázsonyi conjecture that claimed that $e(3, n) = 2n-2$ (see [14] and the references therein). This equality is known as the GHS-theorem.

Even though we know the exact number of diameters that a set of points has to have to be an extremal configuration in $\mathbb{R}^3$, the extremal configurations are mostly uncharted territory. Very few families of extremal configurations have been described in the literature. In [9] some constructions of extremal configurations using ball-polytopes are given.

A very interesting example of extremal configurations in $\mathbb{R}^3$ is the set of vertices of a 3-dimensional k-prismoidal with n-*gon* base (described in Chapter 1, see Figure 3.1.2, is described in [2] by T. Bisztrizcky and D. Oliveros), for any k and any odd n. Because of this, a metric polytope reaches the bound and this creates an infinite extremal configurations' family.

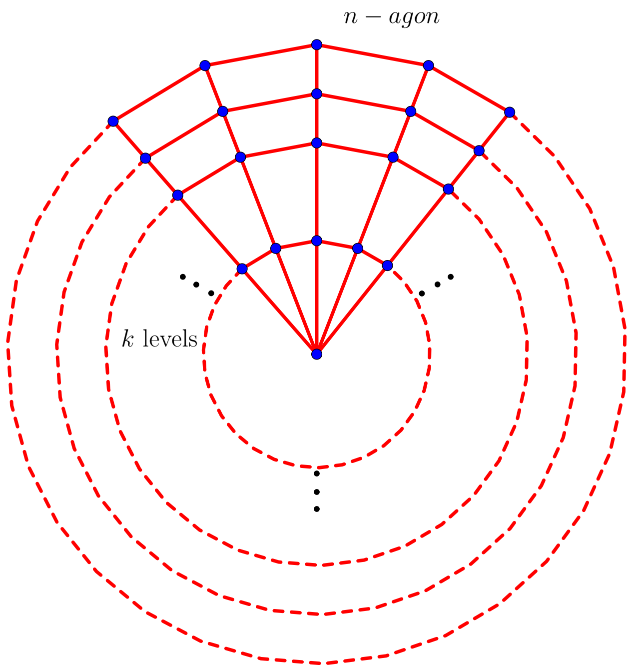

Figure 3.1.2

Lemma 3.2.1: *In any metric polyhedron $\mathcal{P}_h$, the number of edges is equal to the number of dual-diagonals.*

Proof: Let v_1 and v_2 vertices of $\mathcal{P}_h$ and m the number of dual-diagonals in $\mathcal{P}_h$. Observe that $v_1 \in v_2^*$ if and only if $v_2 \in v_1^*$, then v_1 has to be exactly in the same number of facets than the number of vertices that are in v_2^*. So if v_2^* has k vertices, then v_1 has to be in exactly in k facets.

Since $G_{\mathcal{P}_h}$ is a planar graph, the only way that v_1 could be in k facets is when $\delta(v_1) = k$, each of the edges splitting a facet, then the degree of v_1 is k. But this exactly the number of dual-diagonals coming from v_1 to v_2^*. Let us denote $diag(v)$ be the number of dual-diagonals coming from v, then

$$\begin{aligned} \sum_{v \in V(G_{\mathcal{P}_h})} diag(v) &= \sum_{v \in V(G_{\mathcal{P}_h})} \delta(v) \\ \Rightarrow 2m &= 2\left|E(G_{\mathcal{P}_h})\right| \end{aligned}$$

Therefore $\mathcal{P}_h$ has the same number of edges and dual-diagonals. ∎

Theorem 3.2.1: *The set of vertices of any metric polyhedron $\mathcal{P}_h$ is an extremal configuration.*

Proof: Since $\mathcal{P}_h$ is self dual, then $|V(G_{\mathcal{P}_h})| = |F_0(\mathcal{P}_h)| = |F_2(\mathcal{P}_h)|$, so by Euler's formula we have that:

$$\begin{aligned} |F_0(\mathcal{P}_h)| + |F_2(\mathcal{P}_h)| &= 2 + |F_1(\mathcal{P}_h)| \\ 2\,|F_0(\mathcal{P}_h)| &= 2 + |F(\mathcal{P}_h)| \\ |F_1(\mathcal{P}_h)| &= 2\,|F_0(\mathcal{P}_h)| - 2 \\ |E(G_{\mathcal{P}_h})| &= 2\,|V(G_{\mathcal{P}_h})| - 2 \end{aligned}$$

by Lemma 3.2.1 there are $2\left|V\left(G_{\mathcal{P}_h}\right)\right|-2$ dual-diagonals and since all of them are diameter then $\mathcal{P}_h$ is an extremal configuration. ■

Corollary 3.2.1: *Two points v_1 and v_2 are diameters of a metric polyhedron $\mathcal{P}$ if and only if both are vertices of $\mathcal{P}$ and $v_1 \in v_2^*$.*

PROOF: Since the number of dual-diagonals is $2\left|V\left(G_{\mathcal{P}_h}\right)\right|-2$ for every metric polyhedron, then all the diameters are dual-diagonals. ■

Theorem 3.2.1 is amazing because it give us hope to find extremal configurations via metric polytopes in higher dimensions. If metric polytopes worked fine in $\mathbb{R}^2$ and $\mathbb{R}^3$ for this problem, we can think that they might help us in $\mathbb{R}^4$ at least.

3.3 Extremal configurations in higher dimensions

So, what happens in $\mathbb{R}^4$?, Do the vertices of metric polytopes still work as extremal configurations?

In [16], Swanepoel defined an structured called *Lenz configurations,* which is the solution to this problem for any dimension $d \geq 4$ if the set of points is sufficiently large.

Definition 3.3.1: We define a *Lenz configuration* in dimension d to be any translate of a finite subset of $\bigcup_{i=1}^{p} C_i$ or $\Sigma \cup \bigcup_{i=1}^{p} C_i$, where:

- For $d \geq 4$, d even, let $p = d/2$ and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus ... \oplus V_p$, where each V_i is 2-dimensional. In each V_i, let C_i be the circle with centre the origin o and radius r_i such that $r_i^2 + r_j^2 = 1$ for all distinct i and j.
- For $d \geq 5$, d odd, let $p = \lfloor d/2 \rfloor$ and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus ... \oplus V_p$, where V_1 is a 3-dimensional space and each V_i is 2-dimensional, for $i \geq 2$. Let Σ be the sphere in V_1 with centre o and radius r_1 and for each $i = 1, ..., p$, let C_i be the circle with centre o and radius r_i, such that $r_i^2 + r_j^2 = 1$ for all distinct i,j.

Theorem 3.3.1: *For each $d \geq 4$ there exists $N(d)$ such that all extremal sets of $n \geq N(d)$ points are Lenz configurations.*

The proof of this theorem is based on typical techniques in extremal graphs, hyper-graphs theory and some stability results for sets that are close to extremal.

For now, everything looks fine. Vertices of metric polytopes can be in a Lenz configuration without any problem, but how do the faces have to look like? In order to partially answer this question, Swanepoel also gives the exact number of diameters for higher dimensions for sufficiently large n.

Corollary 3.3.1: *For sufficiently large n:*

- $e(4, n) = \begin{cases} t_2(n) + \left\lceil \frac{n}{2} \right\rceil + 1 & \textit{if } n \not\equiv 3 \pmod 4; \\ t_2(n) + \left\lceil \frac{n}{2} \right\rceil & \textit{if } n \equiv 3 \pmod 4, \end{cases}$

- $e(5,\, n) = t_2(n) + n$,
- $e(d,\, n) = t_p(n) + p$ *for even* $d \geq 6$, *where* $p = d/2$,
- $e(d,\, n) = t_p(n) + \lceil n/p \rceil + p - 1$ *for odd* $d \geq 7$, *where* $p = \lfloor d/2 \rfloor$.

Where $t_p(n) = \frac{p-1}{2p}n^2 - O(1)$ is the Turán's number.

After working with Lenz configurations for a while we concluded that for a sufficiently large n, there are no metric d-polytopes in $\mathbb{R}^d$ whose n vertices yield Lenz extremal configurations, for all $d \geq 4$. Let us suppose again without loss of generality that the diameter is one. The details about Lenz configuration in each case are in [16].

Definition 3.3.2: Let V be a set in Lenz configuration, we call a C_i-*diameter* any diameter among vertices in C_i.

Theorem 3.3.2: *Besides the 4-tetrahedron, there are no 4-polytopes in* $\mathbb{R}^4$ *whose vertices conform a Lenz extremal configuration.*

Proof: We begin by noticing that the 4-tetrahedron allows a metric embedding in a Lenz configuration with the following parametrization:

- Let $r_1 = \frac{1}{\sqrt{3}}$ and $C_1 = \left\{(x,\, y,\, 0,\, 0) \in \mathbb{R}^4 : x^2 + y^2 = \frac{1}{3}\right\}$, then we can embed and equilateral triangle in C_1 with coordinates $\left(\frac{1}{\sqrt{3}}\cos\left(\frac{\pi}{3}\right), \frac{1}{\sqrt{3}}\sin\left(\frac{\pi}{3}\right),\, 0,\, 0\right)$, $\left(\frac{1}{\sqrt{3}}\cos\left(\frac{2\pi}{3}\right), \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi}{3}\right),\, 0,\, 0\right)$ and $\left(\frac{1}{\sqrt{3}}, 0, 0, 0\right)$. Since the edges of the equilateral triangle are length one, these edges are diameters.
- Let $r_2 = \sqrt{\frac{2}{3}}$ and $C_2 = \left\{(0,\, 0,\, x,\, y) \in \mathbb{R}^4 : x^2 + y^2 = \frac{2}{3}\right\}$. Notice that the distance between the points $\left(0,\, 0,\, \frac{\sqrt{6}}{12},\, \frac{\sqrt{10}}{4}\right)$ and $\left(0,\, 0,\, \sqrt{\frac{2}{3}},\, 0\right)$ is one and both vertices are in C_2.

Then the points $\left(\frac{1}{\sqrt{3}}\cos\left(\frac{\pi}{3}\right), \frac{1}{\sqrt{3}}\sin\left(\frac{\pi}{3}\right),\, 0,\, 0\right)$, $\left(\frac{1}{\sqrt{3}}\cos\left(\frac{2\pi}{3}\right), \frac{1}{\sqrt{3}}\sin\left(\frac{2\pi}{3}\right),\, 0,\, 0\right)$, $\left(\frac{1}{\sqrt{3}}, 0, 0, 0\right)$, $\left(0,\, 0,\, \frac{\sqrt{6}}{12},\, \frac{\sqrt{10}}{4}\right)$ and $\left(0,\, 0,\, \sqrt{\frac{2}{3}},\, 0\right)$ are in Lenz extremal configuration. Furthermore the convex hull of these five points yields a metrical embedding of the tetrahedron ■

Now assume $n \geq 5$, by Theorem 3.3.1 and Corollary 3.3.1 we know that $N(4) = 6$, and $e(4,\, n) = t_2(n) + \left\lceil \frac{n}{2} \right\rceil + 1$. We proceed by contradiction. Suppose there is a metric polytope $\mathcal{P}$ with $n \geq 6$ vertices in a Lenz extremal configuration. Let C_1 and C_2 be the orthogonal circles of Lenz configuration where all the vertices lie. Again, according to Corollary 1, the vertices have to be split as $\left\lceil \frac{n}{2} \right\rceil$ vertices in say C_1 and $\left\lfloor \frac{n}{2} \right\rfloor$ in C_2.

Since C_1 and C_2 are orthogonal and $r_1^2 + r_2^2 = 1$, all the edges with one end in C_1 and other end in C_2 are at distance one, therefore are diameters, furthermore we have t_2 of them, some of which may be dual diameters or not .Thus the $\left\lceil \frac{n}{2} \right\rceil + 1$ missing diameters must be C_i-diameters for $i = \{1,\, 2\}$.

1. Given a vertex $v \in C_i$, for some $i = 1,\, 2$, v^* must contain vertices in both circles C_1 and C_2. We observe that if every vertex of v^* lies only in C_j for some $j \in \{1,\, 2\}$ since are coplanar, v^* is not a facet (3-dimensional).
2. At most two vertices v_i and w_i in each C_i, i =1, 2 satisfy that v_i^* and w_i^* has three or more vertices in $C_{i+1 \pmod 2}$. Assume v_i^* contains three or more vertices in $C_{i+1 \pmod 2}$ for some $i = 1,\, 2$. Then v^* contains (since all are coplanar), all the vertices in $C_{i+1 \pmod 2}$. Since every 2-face in $\mathcal{P}$ is contained in exactly two

facets then only one w_i^* contains such a 2-face. Note that v_i and w_i may or may not exist, but if one of them does, the corresponding w_i as well.

Let $\left\{u_{1,1},\, u_{1,2}, ..., u_{1,\lceil \frac{n}{2} \rceil - s}, v_2, w_2\right\}$ vertices of C_2, where $r,\, s \in \{0,\, 2\}$ depending upon v_i existing $(s = r = 2)$ or not $(s = r = 0)$ for $i \in \{1,\, 2\}$.

3. Every $u_{i,j}^*$ have one edge in C_i (that is contained in two dual-diagonals in C_i). Sin $u_{i,j}^*$ contains at most two vertices in $C_{i+1(\mod 2)}$, then by 1) and by the fact that $u_{i,j}^*$ is a 3-facet, then $u_{i,j}^*$ have at least two vertices in C_i.

4. By 1) and 2), if v_i and w_i exist then then v_i^* and w_i^* have at least one edge in C_i.

Then, by 3) and 4) the graph consisting of dual-diagonals in C_i (C_i-diagonals) contains at least a path of length $\lceil \frac{n}{2} \rceil$ in C_1 and a path of length $\lfloor \frac{n}{2} \rfloor$ in C_2 thus the total number of C_i-diagonals will be $n-2$. Then this Lenz configuration have more diagonals than the maximum number of possible diagonals in any configuration of points $n \geq 7$, yielding a contradiction. ■

Theorem 3.3.3: *For a sufficiently large n. There are no metric $5-$polytopes in $\mathbb{R}^5$ whose n vertices are in a Lenz extremal configuration.*

Proof: We proceed by contradiction. Suppose there is a metric 5-polytope $\mathcal{P}$ for a sufficiently large n. Recall that all Lenz extremal configurations is a translation of $\Sigma \cup C_2$. Furthermore, it is known that there are two types of Lenz extremal configuration in $\mathbb{R}^5$, see ([16], p16).

The first type of Lenz extremal configuration is obtained by taking $n_1 := \lfloor n/2 \rfloor + 1$ or $\lceil n/2 \rceil + 1$ vertices in Σ, with $n_1 \geq 4$, $n_1 \neq 5$, the remainder of the vertices in C_2, and only one C_2-diameter $e = \{x, y\}$ with end points in C_2.

Then we will show that $\mathcal{P}$ is not a self-dual polytope; in particular we will show that every vertex v in C_2 different from x or y its dual facet v^* is not fully dimensional. Let $v \in C_2$, $v \notin \{x,\, y\}$ observe that if w is a vertex v^* such that $w \in C_2$ then $e_1 = \{v,\, w\}$ is a dual-diagonal, with $e_1 \neq e$ which is a contradiction to the unicity of e. Then there are no C_2-diagonal containing v and then all vertices of v^* lie in Σ yielding a contradiction since every polytope with vertices in Σ is at most 3-dimensional, instead of 4-dimensional.

Then, we will show the following claim:

Claim 1: For every $v \in \Sigma$, v^* has one edge in Σ and the 2-face generated by all the vertices in C_2.

In order to prove such a claim we will first show that v^* has one edge in Σ and the 2-face generated by al the vertices in C_2.

As before, for each vertex $v \in \Sigma$, v^* most have vertices in Σ and in C_2 in order to be 4-dimensional. We will observe that there are exactly two vertices in $v^* \cap \Sigma$, and at least three vertices in $v^* \cap C_2$. Furthermore, v has two Σ-diameters because:

- If v does not have any Σ-diameter, then v^* does not have vertices in Σ, yielding a contradiction,
- if v has one Σ-diameter, then v^* is at most a 3-dimensional face, yielding a contradiction, and
- if v has more than two Σ-diameter, there is a vertex $v_1 \in \Sigma$ with at least one Σ-diameters because there are only n_1 Σ-diameters. This yields a contradiction because v_1^* would be at most a 3-dimensional face.

Notice that if v^* contains three or more vertices in C_2 (since all are coplanar), then v^* contains all the vertices in C_2.

Since each vertex $v \in \Sigma$ has two Σ-diameters, v^* has at least three vertices in C_2 other wise v^* would be at most a 3-dimensional face. Then all the vertices in C_2 are in the dual faces of all vertices in Σ. Let us call f_{C_2} the 2-face with vertices in C_2.

Therefore $f^*_{C_2}$, that is a 2-face, has to have all the vertices in Σ by the inverse inclusion property, which implies that all the vertices in Σ are in a 2-face, i.e. all the vertices in Σ are in a circumference. Then all the vertices of $\mathcal{P}$ are in two orthogonal circumferences, yielding a contradiction because $\mathcal{P}$ is at most a metric 4-polytope ■

Theorem 3.3.4: *There are not metric d-polytopes in $\mathbb{R}^d$ for even $d \geq 6$ with their vertices in extremal configuration.*

Proof: In [16] is proved that $N(d) = d+1$, so $e(d, n)$ is exactly as in Corollary 3.1.3 for all $n \geq d+1$. We proceed by contradiction.

Suppose there is a metric polytope $\mathcal{P}$ with its n vertices in extremal configuration, so $n \geq d+1$, and let C_1, $C_2, \ldots, C_{d/2}$ be the orthogonal circumferences of the Lenz configuration where all the vertices are.

Since $e(d, n)$ can only be reached by dividing the n points as equally as possible between the $d/2$ circles, and ensuring that there is going to be a C_i-diameter for each i (see [16]), there is going to be at least one vertex v in some C_i without a C_i-diameter, so v^* is going to be at most a $d-2$ dimensional face, yielding a contradiction. ■

Theorem 3.3.5: *For a sufficiently large n, there are no metric d-polytopes in $\mathbb{R}^d$ for odd $d \geq 7$ with n vertices in extremal configuration.*

Proof: Suppose there is a metric polytope $\mathcal{P}$ with n vertices in extremal configuration, $p := \left\lfloor \frac{d}{2} \right\rfloor$, and let C_1, $C_2, \ldots, C_{p-1}$ be the orthogonal circumferences and Σ the 2-sphere of Lenz configuration where all the vertices lay one .

The bound $e(d, n)$ can only be reached satisfying all of the following conditions:

- Dividing the $n \geq d+1$ points as equally as possible between the $p-1$ circles and Σ.
- Ensuring that there is going to be a C_i-diameter for each i (see [16]).
- The number of vertices in Σ is $\left\lceil \frac{n}{p} \right\rceil$.
- There is going to be $\left\lceil \frac{n}{p} \right\rceil - 1$ $\Sigma-$diagonals.

Then, there are going to be at least three vertices in at least one C_i, so there is a vertex v in C_i without a C_i-diameter, then v^* is going to be at most a $d-2$ dimensional face, contradiction. ■

Finally, as a corollary, Theorem 3.3.2, Theorem 3.3.3, Theorem 3.3.4 and Theorem 3.3.5 we have the following theorem:

Theorem 3.3.2: *For a sufficiently large n, there are not metric d-polytopes in $\mathbb{R}^d$ for $d \geq 4$ with their n vertices in extremal configuration.*

Chapter 4
Conclusions and future work

As a general conclusion I can say that metric polytopes work fine in lower dimensions for several problems in discrete geometry, but for higher dimensions they do not solve the problems studied in this thesis. Nevertheless, a lot of questions and a conjecture can be stated about them.

For example, is the second condition about the metric polytopes definition (Definition 1.5.2) really necessary? For several examples I did in $\mathbb{R}^2$ it seems that the first condition implies the second one. So at least in $\mathbb{R}^2$, I am pretty sure that:

Conjecture 4.1: *Every geometric self-dual polytope in* $\mathbb{R}^2$ *where all its dual-diagonals have the same length is a metric polytope.*

In Chapter 2, metric polytopes helped us a lot to understand the structure of the BCW, even when there is a better structure to work with (Reuleaux Polytopes). Despite of that, metric polytopes gave us the main idea on how the surgery in higher dimensions was not going to work, so they were very useful in that sense. This taught me a lesson, always try to do specific (the easiest) cases of a huge problem, they could give you a clue about how to solve it.

Of course there is a lot of work to do in order to perform a proper surgery in a Reuleaux polytope to create BCW. The idea exposed in Chapter 2 about performing symmetrical surgery could be a good approach in order to solve the problem in higher dimensions.

In Chapter 3, we proved that in $\mathbb{R}^d$ there are no metric d-polytopes that attain the maximum number of diameters. However, an interesting question could be, what is the maximum number of diameters of n points such that these points induce a metric d-polytope?

Another interesting question arises from points in an extremal configurations. Can these points induce a Reuleaux polytope? i.e. if we take the ball-polytope of a set of points in extremal configuration, is this a Reuleaux polytope? Remember that no all the Reuleaux polytopes are induced by a metric polytope necessarily (see Figure 2.2.2), so we think that extremal configuration in higher dimensions might induce Reuleaux polytopes, because each point would be on the boundary. What we do not know is if all the points are 0-singular.

In a few words I could say that this thesis taught me a lot about discrete geometry. It showed me a lot of techniques and strategies that could work in order to solve these kind of problems. Although the original ideas we had about metric polytopes did not work, they gave me a panoramic view of the problems in the area.

References

[1] Gabriela Araujo-Pardo, Isabel Hubard, Deborah Oliveros, and Egon Schulte. Colorful associahedra and cyclohedra. *Journal of Combinatorial Theory, Series A*, 129:122 – 141, 2015.

[2] T. Bisztriczky and D. Oliveros. d-dimensional self-dual polytopes and Meissner polytopes. *Mathematics subject Classification*, 2010.

[3] B. Grümbaum. A proof of vázsonyi's conjecture. *Bull. Res.*, Council Israel Sect. A 6:77–78, 1956.

[4] B. Grünbaum and G. C. Shephard. Is selfduality involutory? *The American Mathematical Monthly*, 95(8):729–733, 1988.

[5] A. Heppes. Beweis einer vermutung von a. vázsonyi. *Acta Mathematica Academiae Scientiarum Hungarica*, 7:463–466, 1956.

[6] H. Hopf and E. Pannwitz. Aufgabe. *Jahresbericht Deutsch. Math.- Verein*, Nr 167:114.

[7] S. Jendrol'. A non-involutory selfduality. *Discret. Math.*, 74:325–326, 1989.

[8] B. Kawohl and C. Weber. Meissner's mysterious bodies. *The Mathematical Intelligencer*, 33:94–101, 2011.

[9] Y. Kupitz, H. Martini, and M. Perles. Ball polytopes and the vázsonyi problem. *Acta Mathematica Hungarica*, 126:99–163, 2009.

[10] L. Lovász. Self-dual polytopes and the chromatic number of distance graphs on the sphere. *Acta Sci. Math, 45*, 1983.

[11] H. Martini, L. Montejano, and D. Oliveros. *Bodies of Constant Width: An Introduction to Convex Geometry with Applications.* Springer International Publishing, 2019.

[12] P. McMullen and E. Schulte. *Abstract Regular Polytopes.* Number v. 92 in Abstract Regular Polytopes. Cambridge University Press, 2002.

[13] L. Montejano and E. Roldán-Pensado. Meissner polyhedra. *Acta Mathematica Hungarica*, 151:482–494, 2017.

[14] J. Pach and P.K. Agarwal. *Combinatorial geometry.* Wiley-Interscience series in discrete mathematics and optimization. Wiley, 1995.

[15] S. Straszewicz. Sur un problème géométrique de P. Erdös. *Bull. Acad. Polon. Sci. Cl.III 5*, pages 39–40, 1957.

[16] Konrad J. Swanepoel. Unit distances and diameters in euclidean spaces. *Discrete & Computational Geometry*, 41:1–27, 2009.

[17] Wikipedia. Abstract polytopes duality, 1999. https://en.wikipedia.org/wiki/Abstract_polytope#Duality.

[18] M. Yaglom and V. Boltyanskii. *Convex Figures.* Holt, Rinehart and Winston, 1961.

[19] G.M. Ziegler. *Lectures on Polytopes.* Graduate Texts in Mathematics. Springer New York, 2012.