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Directed Polymers in a Random Environment

TESIS
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Resumen

En esta tesis discutimos el comportamiento de polímeros dirigidos en un ambiente aleatorio. Para ello utilizamos la medida polimeral, la cual es una generalización del modelo canónico. Demostramos que la medida polimeral tiene la propiedad de Markov y calculamos sus coeficientes de transición. Posteriormente estudiamos la energía libre de Gibbs y encontramos una cota superior para dicha energía. Así mismo analizamos las trayectorias del polímero y mostramos que éstas se encuentran en una vecindad alrededor de la trayectoria favorita. Finalmente, simulamos un polímero libre en presencia de tres diferentes ambientes aleatorios (exponencial, normal y pareto) a diferentes temperaturas. Cabe señalar que calcular la función de partición tanto analíticamente como computacionalmente es sumamente complicado por lo que en vez de graficar la medida polimeral se graficó el orden de magnitud del exponente de la medida polimeral, y encontramos que el polímero se encuentra en una vecindad alrededor de la trayectoria favorita, como se esperaba.

Abstract

In this thesis, we discuss the behavior of directed polymers in a random environment. To do that we use the polymer measure, which is a generalization of the canonical model. We prove that the polymer measure has a Markov property, and we calculate the transitions probabilities. Then we study the free energy of the polymer, and we showed an upper bound. Then we analyze the paths of the polymer, and we prove that they are in a neighborhood of the favorite path. Finally, we simulate a free polymer in the presence of three different environments (exponential, Gaussian, and Pareto) at different temperatures. It should be noted that to calculate the partition function both analytically and computationally is very difficult, so instead of plotting the polymer measure, we plot the order of the polymer's measure, and we find, that they are actually in a neighborhood of the favorite path, as expected.

Contents

Chapter 1

Motivation

Polymers are one of the essential structures in nature; also, polymers have a significant presence in the industry. Some examples of important polymers are cellulose, proteins, DNA, vinyl polychloride, polystyrene, polypropylene, among others. We present a more detailed introduction in Chapter 2. Due to the relevance they possess, numerous studies of their properties have been made. This thesis analyses a particular model for the behavior of the polymer in the presence of a random potential. To motivate this study, and to make this easy to understand, let us take a look at the next example taken from [4].

1.1 Example

Consider a hydrophilic polymer wafting in water. We suppose that the water contains randomly placed hydrophobic molecules as impurities, which repel the monomers of the polymer. Due to the thermal fluctuation, the shape of the polymer could be understood as a random object[1]. The goal is to describe how impurities affect the global shape of the polymer. A model named directed polymers in a random environment was developed to achieve this. Like all models in mathematical physics, it is complicated to face a problem without a simplification picture of the initial problem. In this model, the simplification goes as follows. The entanglement, U-turns, and self-interactions of the polymer are suppressed; then the polymer is represented by a graph $\{(j, x_j)\}_{j=1}^{n}$ in $\mathbb{N} \times \mathbb{Z}^d$, so the polymer is supposed to live in $(1+d)-$dimensional discrete lattice and to stretch in the direction of the first

[1]A rigorous analysis of this system is done in [27].

coordinate. It is assumed that the transversal motion $\mathbf{x} = \{x_j\}_{j=1}^n$ performs a simple random walk in $\mathbb{Z}^d$, if the impurities are absent; this accounts for consecutive monomers in the chain. Now, we define $\{\omega(n, x) : n \geq 1, x \in \mathbb{Z}^d\}$ as a random field of independent and identically distributed random variables. Where $\omega(n, x)$ describes the presence or the strength of an impurity at site (n, x) when $\omega(n, x)$ is negative, and the presence of a water molecule when $\omega(n, x)$ is positive, i.e., each time the path steps on an impurity it gets a penalty, and every step on a water molecule brings a reward. The sum of all penalties and rewards obtained by the random walk is known as the total energy of the path, also known as the Hamiltonian of the system. Thus, the typical shape $\{(j, x_j)\}_{j=1}^n$ of the polymer is given by the one that maximizes the energy. For example, suppose that $\omega(n, x_n)$ takes only two different values $+1$ and -1, the first one to describe the presence of a water molecule at (n, x_n), and the second one to describe the presence of the hydrophobic impurity at (n, x_n). This means that the energy of the polymer is increased (decreased) by 1 each time a monomer is in contact with a water molecule (impurity). Therefore, the typical shape of the polymer for each given configuration of $\omega(j, x_j)$ is given by the one who tries to avoid the impurities as much as possible. See figure 1.1.

As we want to study large systems, two possible scenarios appear. The first scenario is when the dimension is large, and the temperature is high, the impurities should be ignored, so the global shape of the polymer is not affected. In the second scenario, the dimension is small, or the environment is strong, then the polymer will not be able to avoid the impurities. Thus the global shape of the polymer path changes drastically.

To make a mathematical framework of this model, we have to introduce statistical mechanics theory, stochastic process theory, and in particular the Gibbs measure. All of this will be done in the following chapters. We want to emphasize that the main contents of this thesis follow the lines given on Chapters 2 and 6 of Francis Comets [4].

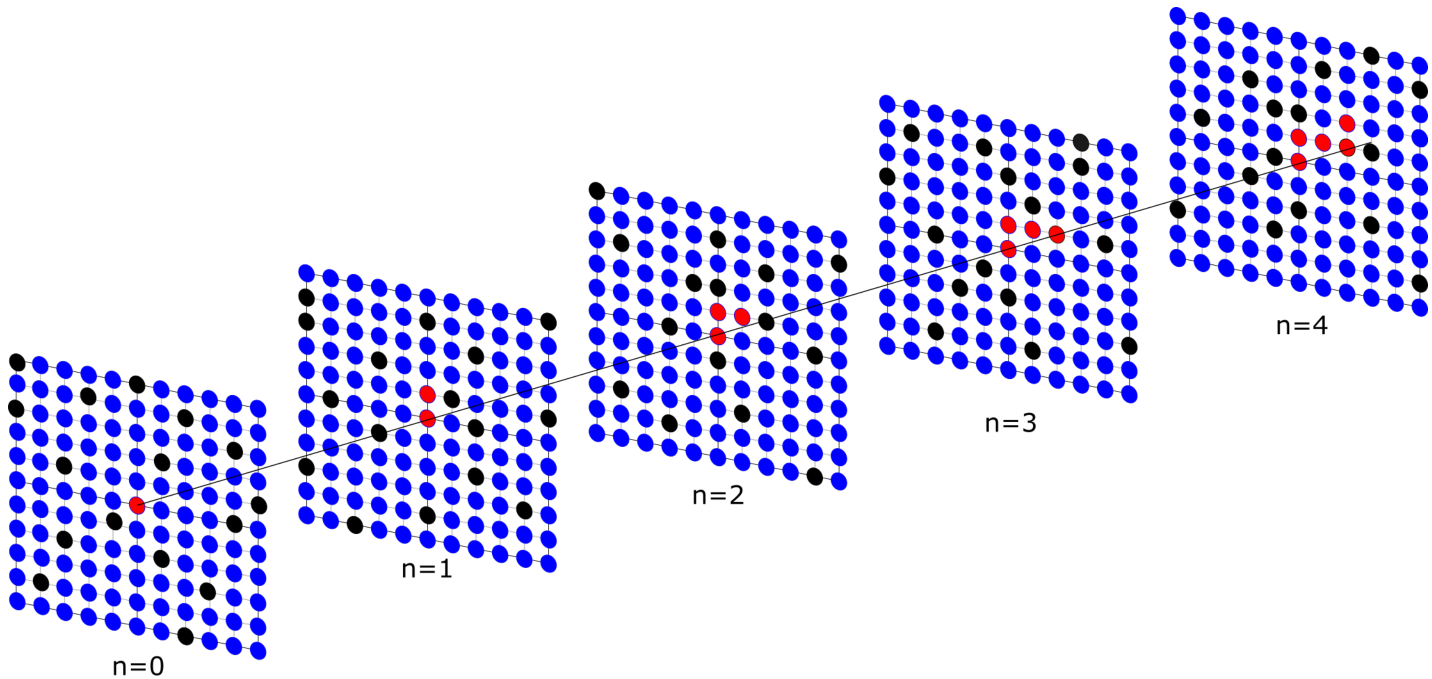

Figure 1.1: Two dimensional hydrophilic polymer wafting in water (blue dots) with hydrophobic impurities (black dots).

Chapter 2

Introduction

2.1 Polymers

A *polymer* is a large molecule built up from smaller molecules, known as *monomers*; that are tied together by chemical bonds. The polymers can be either large units with internal structure (as the adenine-thymine which is a base in the DNA structure) or small units (such as ethylene $[CH_2]_2$). Polymers abound in nature because of the multivalency of atoms like carbon, oxygen, phosphorus, and nitrogen, which are prone to concatenate into large structures.

2.2 Polymer classification.

There are several methods of classifying polymers. One is to adopt the approach of using their response to thermal treatment and divide them into thermosets and thermoplastic. Thermoplastic polymers melt when heated and resolidify when cooled. By contrast, thermosets are those who do not melt when heated and at high temperature, decompose irreversibly.

Another classification system is based on the nature of the chemical reactions during polymerization. The two principal groups are the condensation and the addition polymers. Condensation polymers are those formed by monomers, but the reaction is accompanied by the loss of a small molecule, usually of water, while addition polymers are those formed by the addition reaction of unsaturated monomers without the co-generation of other products.

In general, polymers come in two varieties: 1) Homopolymers, with all its components being identical monomers (such as polyethylene $[C_2H_4]_n$); 2) Copolymers, with two or more different types of monomers (such as RNA). The order of monomers in copolymers can be periodic (as Bacterial polysaccharides) or random (as plant polysaccharides): an example for each type is the agar and the carrageenan.

Synthetic polymers and natural polymers give another classification of polymers important for the industry. Some examples of these both types are in the table below.

Synthetic P.	Natural P.
nylon	proteins
polyethylene	nucleic acids
polystyrene	polysaccharides

Table 2.1: Examples of synthetic and natural polymers.

All the examples of natural polymers in the table 2.1 are organic materials[1]. However, there are also inorganic examples, like minerals. Synthetic polymers typically are homopolymers, while natural polymers are usually copolymers. An example of a copolymer is one whose monomers carry positive and negative charges, randomly arranged along the chain. Another example of copolymers is a polymer consisting of hydrophobic and hydrophilic monomers, such as in the example of section 1.1.

Finally, another classification of polymers is Linear and Branched. In the former case, the monomers have one reactive group (such as CH_2), leading to a linear organization as a result of the polymerization process. In the second group, the monomers have two or more reactive groups (such as hydroxy acid), leading to a network with multiple cross-connection. A lot of natural polymers are linear like DNA and proteins. An example of branched polymers is amylopectin. A more detailed classification can be seen in [6], [7] and [23].

[1]Organic materials are defined as combinations of few lightest elements, mainly hydrogen, carbon, nitrogen, and oxygen.

Now polymerization is the chemical process of building a polymer from monomers is called polymerization. The degree of polymerization is the number of constituent monomers and may vary from 10^3 up to 10^{10}. Human DNA has $10^9 - 10^{10}$ base pairs, while polysaccharides carry $10^3 - 10^4$ glucose units. For more background on the structure, classification, and polymerization of polymers see [9], [20] and [28].

The chemical bonds in a polymer are flexible so the polymer can arrange itself in many different configurations. The longer the chain, the more involved these configurations tend to be. Sometimes, the polymer can wind around itself to form knots, or can collapse to a ball due to attractive van der Waals forces (such as graphene-polymer nanocomposites). It can also interact with a surface on which it may or may not be adsorbed, or it can live in a slit between two confining surfaces.

2.3 Statistical thermodynamics

The model study in this work is based in the Gibbs measure, Thus, it is essential to introduce some quantities related to statistical thermodynamics, in particular: the partition function, the Gibbs measure, the average Gibbs measure, the free energy, and the Gibbs entropy. Before defining each quantity, let us introduce some basic definitions of the statistical mechanics.

2.3.1 Canonical ensemble

The canonical ensemble is a system with a specified volume V, and number of particles N. The particles are in contact with an infinitely large heat reservoir of constant temperature T. Similarly, in the canonical ensemble, we expect that the energy will differ from system to system. The probability density function this ensemble follows is given by

$$P(E_i) = \frac{e^{-\beta E_i}}{Z}, \tag{2.1}$$

where β is the inverse of the product of the temperature with the Boltzmann's constant i.e. $\beta = (Tk_B)^{-1}$, where the Boltzmann's constant k_B is approximately $1.38 \times 10^{-23} JK^{-1}$, and E_i is the energy of the $i-$th state. In this thesis, the Hamiltonian describes the total energy. Also, the constant Z, which is the normalization constant for this model, is known as the partition function. It is usually denoted by "Z" and is defined as

$$Z := \sum_{i=1}^{n} e^{-\beta E_i}, \tag{2.2}$$

where the sum is over all states, the dependence of Z on T is quite apparent. However, there is an essential relation between the partition function and the total average energy, that is For a classical system with a discrete set of microstates, if E_i is the energy of microstate i, and p_i is the probability that it occurs, one can show that:

$$\langle E \rangle := \sum_i E_i p_i = -\frac{\partial \ln Z}{\partial \beta}. \tag{2.3}$$

For a more in-depth description, [11], [13] and [26] can be consulted.

Now we will define a measure very similar to the canonical model, where we will introduce a potential for interaction and, therefore, a Hamiltonian. This measure is known as the Gibbs measure and is given by:

$$P(\mathbf{x}) = \frac{e^{-\beta H_n(\mathbf{x})}}{Z_n}, \tag{2.4}$$

where $\mathbf{x}$ is a path configuration of the system, n is the length of the system, $H_n(\mathbf{x})$ the Hamiltonian of the system, which is the total energy of the path $\mathbf{x}$ and the partition function Z_n is given by:

$$Z_n := \sum_{\mathbf{x} \in \mathbf{X}} e^{-\beta H_n(\mathbf{x})}, \tag{2.5}$$

where X is the set of all possible paths of length n in the system. See [14], [15] and [21].

The Gibbs entropy is the generalization of the Boltzmann entropy[2]. For a classical system with a discrete set of microstates, if E_i is the energy of microstate i, and p_i is the probability that it occurs during the system fluctuations, then the entropy of the system is defined by

[2]Recall that the Boltzmann entropy is defined as $S = k_B \ln W$, where W denotes the number of real microstates of a single isolated a system.

$$S = -k_B \sum_i p_i \ln p_i. \tag{2.6}$$

The principal difference between the Gibbs and the Boltzmann entropy is that the Boltzmann entropy considers that all the energy states have the same probability, while the Gibbs measure gives each energy state a single probability p_i, see [12], [19] and [29] for a general relationship between the Boltzmann entropy and Gibbs entropy. In the following chapters, we will consider the Boltzmann's constant as one i.e., $k_B = 1$.

2.4 The free energy

The *free energy* is defined as

$$p_n = \frac{1}{n} \ln Z_n, \tag{2.7}$$

when H_n assigns an attractive self-interaction to the system. It can be proven that the partition function Z_n satisfies the inequality

$$Z_n \geq Z_m Z_{n-m} \quad \forall\, 0 \leq m \leq n. \tag{2.8}$$

In that case the inequality follows by viewing the $n-$length-system as a concatenation of two systems of length m and $n-m$. Applying logarithm to (2.8) it is obtained

$$\ln Z_n \geq \ln Z_m + \ln Z_{n-m} \quad \forall\, 0 \leq m \leq n \tag{2.9}$$

i.e., $\ln Z_n$ is a superadditive sequence. In terms of the p_n, this last inequality implies

$$p = \lim_{n\to\infty} p_n = \sup_{n\in\mathbb{N}} p_n \quad \text{exist in } (-\infty, \infty] \tag{2.10}$$

The same occurs if H_n assigns a repulsive self-interaction to the system; in which case the inequalities in (2.8) and (2.9) are reversed, and sup in (2.10) is replaced by inf. In the case H_n assigns both repulsive and attractive interactions, then the above argument is generally not available; thus, to determinate the free energy, it has to be established other means. The situation, drastically, change in the presence of a random environment ω, the superadditivity property in (2.9), is replaced by a Markov property.

Chapter 3

Polymer model

We present in this Chapter the model that will be studied all through this thesis. It is defined as a particular example of a Simple Random Walk (SRW) in a random potential. A standard reference about SRW is [25].

3.1 A model for directed polymers

Our polymers will live on the $d-$dimensional Euclidean lattice $\mathbb{Z}^d$, $d \in \{1, 2, ...\}$. They will be modeled as random paths on this lattice, where the monomers are the vertices in the path.

The simple random walk satrting at $x \in \mathbb{Z}^d$, denoted by $(\mathbf{S} = \{S_n\}_{n\geq 0}, P_x)$ is defined on $(\Omega_{\text{traj}}, \mathcal{F}, P_x)$, where $\Omega_{\text{traj}} = (\mathbb{Z}^d)^{\mathbb{N}}$, equipped with the cylindric $\sigma-$field $\mathcal{F}$, and a probability measure P_x such that, under P_x, the jumps $S_1 - S_0, ..., S_n - S_{n-1}$ are independent with transitions

$$P_x(S_0 = x) = 1, \quad P_x(S_n - S_{n-1} = \pm e_j) = (2d)^{-1}, \quad j = 1, 2, ..., d.$$

where $e_j = (\delta_{kj})_{k=1}^d$ is the $j-$th vector of the canonical basis of $\mathbb{Z}^d$. In the sequel, P_0 will be shortened as P, and we will refer to the tern $(\Omega_{\text{traj}}, \mathcal{F}, P_x)$ as the path space. Also, we define $P(\cdot)$ and $E(\cdot)$ as the probability measure and the expectation value relating to the path space. Finally we define $P(S_{[1,n]} = x_{[1,n]})$ as

$$P(S_{[1,n]} = x_{[1,n]}) := P(S_n = x_n, S_{n-1} = x_{n-1}, \cdots, S_1 = x_1).$$

The random environment: $\omega = \{\omega(n,x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$ is a set of random variables defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, which are real valued, non-constant, and independent identically distributed (i.i.d.) such that

$$\mathbb{E}\left[\exp \beta\omega(n,x)\right] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{3.1}$$

We define $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ as the probability measure and the expectation value related to $(\Omega, \mathcal{G}, \mathbb{P})$.

3.2 The polymer measure

For any $n > 0$, define the probability measure $P_n^{\beta\omega}$ on the path space starting at the origin by

$$P_n^{\beta,\omega}(\mathbf{x}) = \frac{e^{\beta H_n(\mathbf{x})}}{Z_n(\omega,\beta)} P(\mathbf{x}), \tag{3.2}$$

where $\beta > 0$ is the inverse of the temperature, and $P(\mathbf{x})$ is a simplified notation of the probability that the random walk takes the path $\mathbf{x} = (0, x_1, ..., x_n)$ i.e., $P(\mathbf{x}) = P(S_0 = 0, S_1 = x_1, ..., S_n = x_n)$, also, $H_n(\mathbf{x})$ is the total energy of the path $\mathbf{x}$ in environment ω, which is called the Hamiltonian [10]. It is defined by:

$$H_n(\mathbf{x}) = \sum_{1 \le j \le n} \omega(j, x_j), \tag{3.3}$$

if the random environment is fixed, then we will simplify the notation as H_n. Finally, the partition function is given by:

$$Z_n(\omega,\beta) = E\left[e^{\beta \sum\limits_{j=1}^{n} \omega(j,S_j)}\right], \tag{3.4}$$

As we say before, it is the normalizing constant to make $P_n^{\beta,\omega}$ a probability measure. Since the partition function is an expected value, we can rewrite (3.4) as

$$Z_n(\omega, \beta) = \sum_{\mathbf{x} \in \mathbf{X}} (2d^{-n}) e^{\beta H_n(\mathbf{x})},$$

where $\mathbf{X}$ is the set of the $(2d)^n$ possible paths of length n for the simple random walk. As long as the environment and the inverse of the temperature are always a parameter to the partition function, sometimes we write Z_n.

Equation (3.2) is known as the polymer measure and it is similar to the Gibbs measure (2.4). The polymer measure is the principal object to be analyzed in this work.

One of the principal properties of the polymer measure is that the paths with low energy have a high probability, while paths with high energy have a low probability, at a fixed β.

In general $\{P_n^{\beta,\omega}\}_{n \in \mathbb{N}}$ is not a consistent family of probability distributions., i.e., $P_{n+1}^{\beta,\omega}$ is not obtained by summing out the energy of the $(n+1)$-th monomer to $P_n^{\beta,\omega}$, this due to the partition function, would have to take into account all the paths of length $(n+1)$, and not all paths would end up in the position of the monomer $(n+1)$. So, we have a different polymer measure, modeling the polymer at the length $(n+1)$.

It is important to point out that as the environment can take either, positive and negative values, we say that the polymer is attracted to sites where the random environment is positive, and repelled by sites where the environment is negative.

The dependence on the environment ω of both $Z_n(\beta, \omega)$ and $P_n^{\beta\omega}$ is evident. Then, these quantities are random variables on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, so we can calculate its probability with the product measures. by:

$$\mathbb{P}_n^{\beta}(\omega, \mathbf{x}) = P_n^{\beta,\omega}(\mathbf{x}) \mathbb{P}(\omega),$$

and expected value by:

$$\mathbb{E}\left(P_n^{\beta\omega}(\mathbf{x})\right) = \sum_{\omega} P_n^{\beta,\omega}(\mathbf{x}) \mathbb{P}(\omega).$$

This quantity is known as the **average Gibbs measure**. This model is used to describe a polymer whose random environment is not frozen but takes part in the equilibrium.

From now onwards, we will mostly be interested in the behavior of the polymer measure in the limits $n \to \infty$, and $\beta \to \infty$.
We will not consider models where the length or the configuration of the polymer variates in time.

Before we conclude the section, it is convenient to say that another property of the polymer measure is that this quantity maximizes the Gibbs entropy (2.6), or, equivalently, it minimizes the free energy (2.7). The following chapter presents a deeper analysis of the free energy in the presence of a random environment.

Chapter 4

Thermodynamics and phase transitions

Throughout this chapter, we will analyze the basic properties of the polymer measure.to do that let us define an important quantity for this model is the cumulant generating function λ of $\omega(n, x)$,

$$\lambda(\beta) = \ln \mathbb{E}\left[\exp\left(\beta\omega(n, x)\right)\right], \tag{4.1}$$

which is finite for all β due to hypothesis (3.1). Along this thesis, we are going to consider β as positive.

4.1 Markov property and the partition function

This chapter introduces the Markov property of the paths and how it affects the polymer measure and the partition function. However, before that, let us introduce the shift operator.

Definition 4.1.1. *The shift operator on the space Ω of environments, is defined as $\theta_{i,x} : \Omega \to \Omega$, with $i \geq 1$, and $x \in \mathbb{Z}^d$, given by $\omega \mapsto \theta_{i,x}\omega$,*

$$(\theta_{i,x}\omega)(t, y) = \omega(i + t, x + y). \tag{4.2}$$

Note: Since the environment, ω is a collection of i.i.d.r.v., the law of ω and $\theta_{i,x}\omega$ is the same.

For $n, m \geq 1$, $x \in \mathbb{Z}^d$, the random variable

$$\begin{aligned} Z_m(\theta_{n,x}(\omega), \beta) &= E\left[\exp\left(\sum_{1\leq t\leq m} \beta\omega(t+n, S_{t+n})\right)\middle| S_n = x\right] \\ &= E\left[\exp\left(\sum_{1\leq t\leq m} \beta\omega(t+n, S_t)\right)\middle| S_0 = x\right] \\ &= E_x\left[\exp\left(\sum_{1\leq t\leq m} \beta\omega(t+n, S_t)\right)\right], \end{aligned} \tag{4.3}$$

is the partition function of the polymer of length m starting at x at time n. Sometimes we write $Z_m \circ \theta_{n,x}$ because the environment and the inverse of the temperature are always a parameter to it.

Proposition 4.1.2. *Since ω and its shift $\theta_{n,x}(\omega)$ have the same law, then $Z_m \circ \theta_{n,x}$ has the same law as Z_m.*

Proof. It is known that $\omega \stackrel{D}{=} \theta_{n,x}\omega$ if and only if for every continuous and bounded function f we have $E(f(\omega)) = E(f(\theta_{n,x}(\omega)))$.
So define

$$f(q) = \exp\left(\beta \sum_{1\leq t\leq m} q_t\right),$$

where $q_t \in \mathbb{R}$ for all t. Clearly $f(\cdot)$ is a continuous function and its bounded by (3.1). Then

$$E(f(\theta_{n,x}\omega)) = E(f(\omega)),$$

i.e.,

$$E\left[\exp\left(\beta \sum_{1\leq t\leq m} \omega(t+n, S_t + x)\right)\right] = E\left[\exp\left(\beta \sum_{1\leq t\leq m} \omega(t, S_t)\right)\right].$$

The right hand side of the previous equality is $Z_m(\omega)$ and the left hand side is

$$\begin{aligned} E\left[\exp\left(\beta \sum_{1\leq t\leq m} \omega(t+n, S_t + x)\right)\right] &= E_x\left[\exp\left(\beta \sum_{1\leq t\leq m} \omega(t+n, S_t)\right)\right] \\ &= Z_m \circ \theta_{n,x}(\omega). \end{aligned}$$

In consequence, $Z_m(\omega) = Z_m \circ \theta_{n,x}(\omega)$ for every $\omega \in \Omega$. Therefore,

$$\mathbb{P}(Z_m \circ \theta_{n,x} \leq x) = \mathbb{P}(Z_m \leq x) \quad \forall x \in \mathbb{R},$$

thus $Z_m \circ \theta_{n,x} \stackrel{D}{=} Z_m$.

□

Note:. *We can also write (4.3) in form of a conditional expectation given* $\mathcal{F}_n = \sigma\{S_t, t \leq n\}$ *as:*

$$Z_m \circ \theta_{n,x}(\omega) = E\left[e^{\beta\{H_{n+m}(S)-H_n(S)\}}|\mathcal{F}_n\right] \quad \textit{on the event } \{S_n = x\}. \tag{4.4}$$

Proof. On the event $\{S_n = x\}$ we have the equalities:

$$\begin{aligned} E\left[e^{\beta\{H_{n+m}(S)-H_n(S)\}}|\mathcal{F}_n\right] =& E\left[e^{\beta\left\{\sum\limits_{n+1\leq t\leq n+m}\omega(t,S_t)\right\}}\middle| S_n = x\right] \\ =& E_x\left[e^{\beta\left\{\sum\limits_{1\leq t\leq m}\omega(t+n,S_t)\right\}}\right] \\ =& Z_m \circ \theta_{n,x}(\omega). \end{aligned}$$

□

For $n, m \geq 1$, we can express the partition function of the polymer of length $n+m$ by conditioning:

$$\begin{aligned} Z_{n+m} &= E\left[e^{\beta H_{n+m}(S)}\right] \\ &= E\left[e^{\beta H_n(S)} e^{\beta(H_{n+m}(S)-H_n(S))}\right] \\ &= E\left[e^{\beta H_n(S)} E\left(e^{\beta(H_{n+m}(S)-H_n(S))}|\mathcal{F}_n\right)\right] \\ &= E\left[e^{\beta H_n(S)} \times Z_m \circ \theta_{n,S_n}\right], \end{aligned}$$

where we use (4.4) in the last equality. This important identity will be referred to as the Markov property. It can be reformulated as

$$
\begin{aligned}
E\left[e^{\beta H_n(S)} \times Z_m \circ \theta_{n,S_n}\right] &= Z_n \times E\left[\frac{1}{Z_n} e^{\beta H_n(S)} Z_m \circ \theta_{n,S_n}\right] \\
&= Z_n \times \sum_{\mathbf{x}\in\mathbf{X}} Z_m \circ \theta_{n,S_n} \frac{1}{Z_n} e^{\beta H_n(\mathbf{x})} P(\mathbf{x}) \\
&= Z_n \times \sum_{\mathbf{x}\in\mathbf{X}} (Z_m \circ \theta_{n,S_n}) P_n^{\beta,\omega}(\mathbf{x}) \\
&= Z_n \times E_n^{\beta,\omega}[Z_m \circ \theta_{n,S_n}],
\end{aligned}
$$

so we have proved:

Proposition 4.1.3. ***Markov property:*** *The partition function of length $n+m$ can be seen as the product of the partition function up to length n and the expected value respect the polymer measure of the shift up to (n, S_n) of the partition function of length m, i.e.*

$$
Z_{n+m} = Z_n \times E_n^{\beta,\omega}[Z_m \circ \theta_{n,S_n}]. \tag{4.5}
$$

Now we introduce an useful version for the conditional expectancy of the shift operator applied on the partition function, given S_n. We will show that $e^{\beta \sum_{t=1}^{m} \omega(t+n,S_t+x)}$ is a version of $Z_m \circ \theta_{n,x}$ i.e.

$$
E\left[Z_m \circ \theta_{n,x} \mathbb{I}_{S_n=x}\right] = E\left(e^{\beta \sum_{t=1}^{m} \omega(t+n,S_t+x)} \mathbb{I}_{S_n=x}\right). \tag{4.6}
$$

Proof. First, let us note that $e^{\beta \sum_{t=1}^{n} \omega(t+n,S_t+x)} \in \mathcal{L}^1$.

By (4.3), we can directly compute:

$$
\begin{aligned}
E\Big[E_x\Big(e^{\beta \sum_{t=1}^{m} \omega(t+n,S_t)}\Big)\mathbb{I}_{S_n=x}\Big] &= E\Big[E\Big(e^{\beta \sum_{t=1}^{m} \omega(t+n,S_t)}\Big| S_0 = x\Big)\mathbb{I}_{S_n=x}\Big] \\
&= E\Big[E\Big(e^{\beta \sum_{t=1}^{m} \omega(t+n,S_{t+n})}\Big| S_n = x\Big)\mathbb{I}_{S_n=x}\Big] \\
&= E\left(e^{\beta \sum_{t=1}^{m} \omega(t+n,S_t+x)} \mathbb{I}_{S_n=x}\right).
\end{aligned}
$$

□

4.2 The polymer measure as a Markov chain

In this section, we will discuss some basic properties of the polymer measure $P_n^{\beta,\omega}$. Let us fix the environment ω.

Proposition 4.2.1. *Under the polymer measure $P_n^{\beta,\omega}$, the path S is a Markov chain, with transition probabilities*[1]

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)=\frac{e^{\beta w(i+1,y)}Z_{n-i-1}\circ\theta_{i+1,y}}{Z_{n-i}\circ\theta_{i,x}}P(S_i=y|S_0=x), \quad (4.7)$$

for $0\le i<n$, and

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)=P(S_{i+1}=y|S_i=x) \quad \text{for } i\ge n.$$

Proof. For a given path $(x_0=0,x_1,....,x_n)$, the following product is telescopic,

$$\prod_{i=0}^{n-1}\frac{e^{\beta w(i+1,x_{i+1})}Z_{n-i-1}\circ\theta_{i+1,x_{i+1}}}{Z_{n-i}\circ\theta_{i,x_i}}P(S_1=x_{i+1}|S_0=x_i).$$

So, let us see what happens with the terms of the form $\frac{e^{\beta w(i+1,x_{i+1})}Z_{n-i-1}\circ\theta_{i+1,x_{i+1}}}{Z_{n-i}\circ\theta_{i,x_i}}$, recall that $Z_0=1$ and $Z_n\circ\theta_{0,x_0}=Z_n$, because we already set all the paths to start in x_0. Then,

$$\begin{aligned}\prod_{i=0}^{n-1}\frac{e^{\beta w(i+1,x_{i+1})}Z_{n-i-1}\circ\theta_{i+1,x_{i+1}}}{Z_{n-i}\circ\theta_{i,x_i}} &=\frac{1}{z_n}e^{\beta w(1,x_1)}e^{\beta w(2,x_2)}\cdots e^{\beta w(n-1,x_{n-1})}e^{\beta w(n,x_n)}\\ &=\frac{1}{z_n}e^{\beta\sum_{i=n}^{n}\omega(i,x_i)}=\frac{1}{z_n}e^{\beta H_n(\mathbf{x})}.\end{aligned}$$

Now, we analyze the product of $P(S_1=x_{i+1}|S_0=x_0)$. Recall that

[1]The step by step proof is in (B.1.1)

$P(S_0 = x_0) = 1$. We can compute

$$\begin{aligned} P(S_{[1,n]} = x_{[1,n]}) =& P(S_n = x_n, S_{n-1} = x_{n-1}, \cdots, S_2 = x_2, S_1 = x_1)P(S_0 = x_0) \\ =& \prod_{i=1}^{n} P(S_i - S_{i-1} = x_i - x_{i-1})P(S_0 = 0) \\ =& \prod_{i=1}^{n} P(S_i = x_i | S_{i-1} = x_{i-1}), \end{aligned}$$

so

$$P(S_{[1,n]} = x_{[1,n]}) = \prod_{i=1}^{n} P(S_i = x_i | S_{i-1} = x_{i-1}).$$

Since

$$\frac{1}{z_n} e^{\beta H_n(\mathbf{x})} P(S_{[1,n]} = x_{[1,n]}) = P_n^{\beta,\omega}(S_{[1,n]} = x_{[1,n]}),$$

We conclude

$$\prod_{i=0}^{n-1} \frac{e^{\beta w(i+1,x_{i+1})} Z_{n-i-1} \circ \theta_{i+1,x_{i+1}}}{Z_{n-i} \circ \theta_{i,x_i}} P(S_1 = x_{i+1} | S_0 = x_i) = P_n^{\beta,\omega}(S_{[1,n]} = x_{[1,n]}). \tag{4.8}$$

Let us define

$$\begin{aligned} A :=& \prod_{k=0}^{i-2} e^{\beta\omega(k+1,x_{k+1})} \frac{Z_{n-k-1} \circ \theta_{k+1,x_{k+1}}}{Z_{n-k} \circ \theta_{k,x_k}} P(S_1 = x_{k+1} | S_0 = x_0), \\ B :=& e^{\beta\omega(i,x)} \frac{Z_{n-i} \circ \theta_{i,x}}{Z_{n-i-1} \circ \theta_{i-1,x_{i-1}}} P(S_1 = x | S_0 = x_{i-1}), \\ C :=& \prod_{k=i+1}^{n} e^{\beta\omega(k+1,x_{k+1})} \frac{Z_{n-k-1} \circ \theta_{k+1,x_{k+1}}}{Z_{n-k} \circ \theta_{k,x_k}} P(S_1 = x_{k+1} | S_0 = x_0). \end{aligned}$$

Now we will calculate

$$P_n^{\beta,\omega}(S_{i+1} = y | S_{[1,i-1]} = x_{[1,i-1]}, S_i = x, S_{[i+2,n]} = x_{[i+2,n]}). \tag{4.9}$$

By expressing (4.9) in its product form we have

$$P_n^{\beta,\omega}(S_{i+1}=y|S_{[1,i-1]}=x_{[1,i-1]},S_i=x,S_{[i+2,n]}=x_{[i+2,n]})$$

$$=\frac{P_n^{\beta,\omega}(S_{i+1}=y,S_{[1,i-1]}=x_{[1,i-1]},S_i=x,S_{[i+2,n]}=x_{[i+2,n]})}{P_n^{\beta,\omega}(S_{[1,i-1]}=x_{[1,i-1]},S_i=x,S_{[i+2,n]}=x_{[i+2,n]})}$$

$$=\frac{A\cdot B\cdot e^{\beta\omega(i+1,y)}\frac{Z_{n-i-1}\circ\theta_{i+1,y}}{Z_{n-i}\circ\theta_{i,x}}P(S_1=y|S_0=x)\cdot C}{A\cdot B\cdot C}$$

$$=e^{\beta\omega(i+1,y)}\frac{Z_{n-i-1}\circ\theta_{i+1,y}}{Z_{n-i}\circ\theta_{i,x}}P(S_1=y|S_0=x).$$

After last computations, it is clear to establish the Markov property, because we start conditioning by $(S_{[1,i-1]}=x_{[1,i-1]},S_i=x,S_{[i+2,n]}=x_{[i+2,n]})$ and we end with only one condition $(S_0=x)$. Finally, after identifying this last equality with (4.8) we get:

$$e^{\beta\omega(i+1,y)}\frac{Z_{n-i-1}\circ\theta_{i+1,y}}{Z_{n-i}\circ\theta_{i,x}}P(S_1=y|S_0=x)=P_n^{\beta,\omega}(S_{i+1}=y|S_i=x).$$

□

Observation. *We can re-write (4.7) as*[2]

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)=P_{n-i}^{\beta,\theta_{i,x}\omega}(S_1=y-x) \tag{4.10}$$

Proof. Applying (4.7) to the right-hand-size of (4.10), we get:

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)=\frac{e^{\beta\omega(i+1,y)}z_{n-i-1}\circ\theta_{i+1,y}}{z_{n-i}\circ\theta_{i,x}}P(S_{i+1}=y|S_i=x),$$

as $e^{\beta\omega(i+1,y)}$ is measurable respect to $S_{i+1}=y$ we get:

[2]The step by step prove is in (B.1.2).

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x) = \frac{E\left(e^{\beta\omega(i+1,y)}e^{\beta\sum\limits_{t=1}^{n-i-1}\omega(t+i+1,S_t+S_{i+1})}|S_{i+1}=y\right)}{z_{n-i}\circ\theta_{i,x}}E(\mathbb{I}_{S_{i+1}=y}|S_i=x),$$

we can simplify the previous equation due to the Independence respect to $S_i = x$, so

$$\begin{aligned} P_n^{\beta,\omega}(S_{i+1}=y|S_i=x) &= \frac{E\left(e^{\beta\sum\limits_{t=0}^{n-i-1}\omega(t+i+1,S_t+y)}\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}} \\ &= \frac{E\left(e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}\mathbb{I}_{S_{i+1}=y}\mathbb{I}_{S_i=x}\right)}{z_{n-i}\circ\theta_{i,x}P(S_i=x)} \\ &= \frac{E\left(e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}}. \end{aligned}$$

Now, as $e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}$ is $S_i = x$ measurable, we obtain:

$$\begin{aligned} P_n^{\beta,\omega}(S_{i+1}=y|S_i=x) &= \frac{e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}E\left(\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}} \\ &= \frac{e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}}{z_{n-i}\circ\theta_{i,x}}P\left(S_{i+1}=y|S_i=x\right) \\ &= \frac{e^{\beta\sum\limits_{t=1}^{n-i}\theta_{i,x}\omega(t,S_t)}}{z_{n-i}\circ\theta_{i,x}}P\left(S_1=y-x\right). \end{aligned}$$

□

Since the transition probabilities depend on the environment at the current time, i.e., the transition probabilities at step i depend on i, the Chain is time inhomogeneous. Since the transitions probabilities also depend on the time horizon n, the family of measures $P_n^{\beta,\omega}$ is not consistent, except for the trivial case $\beta = 0$. A final observation is the following.

Observation. *For* $0 \leq n, m$.

$$P_{m+n}^{\beta,\omega}\left(S_{[1,n]} = \cdot\,|S_n = y\right) = P_n^{\beta,\omega}\left(S_{[1,n]} = \cdot\,|S_n = y\right), \tag{4.11}$$

$$P_{m+n}^{\beta,\omega}\left(S_{[n,n+m]} = y + \cdot\,|S_n = y\right) = P_m^{\beta,\theta_{n,y}\omega}\left(S_{[0,m]} = \cdot\right). \tag{4.12}$$

Proof. To prove (4.11), recall equation (4.8), i.e.

$$\begin{aligned} P_{m+n}^{\beta,\omega}\left(S_{[1,n]} = \cdot\,|S_n = y\right) = \prod_{i=0}^{n-1} \Bigg[& \frac{e^{\beta w(i+1,x_{i+1})} Z_{n-i-1} \circ \theta_{i+1,x_{i+1}}}{Z_{n-i} \circ \theta_{i,x_i}} \\ & P(S_1 = x_{i+1}|S_0 = x_i, S_n = y) \Bigg] \\ = & P_n^{\beta,\omega}(S_{[1,n]} = \cdot|S_n = y). \end{aligned}$$

Now, to prove (4.12) we will apply (4.10):

$$\begin{aligned} P_{m+n}^{\beta,\omega}\left(S_{[n,n+m]} = y + \cdot\,|S_n = y\right) &= P_m^{\beta,\theta_{n,y}\omega}\left(S_{[0,m]} = y + \cdot - y\right) \\ &= P_m^{\beta,\theta_{n,y}\omega}\left(S_{[0,m]} = \cdot\right). \end{aligned}$$

□

4.3 The free energy

In statistical mechanics, there is a quantity that encode plenty of information about the Gibbs measure. We will refer to that quantity as the (finite volume, specific) free energy and its expression for a polymer of length n is:

$$p_n(\omega, \beta) = \frac{1}{n} \ln Z_n(\omega, \beta). \tag{4.13}$$

As the environment and the inverse of the temperature are always a parameter to the free energy, sometimes we write p_n instead of $p_n(\omega, \beta)$. Also we can study its behavior as the polymer length tends to infinity.

Theorem 4.3.1. *As* $n \to \infty$,

$$p_n(\omega;\beta) \to p(\beta) = \sup_n \frac{1}{n}\mathbb{E}\big[\ln Z_n(\omega;\beta)\big]. \tag{4.14}$$

$\mathbb{P}$*-a.s. and in* L^p*-norm, for all* $p \in [1,\infty)$.

This theorem states that the sequence $p_n(\omega,\beta)$ converges almost surely (a.s.) to a limit, and the limit is deterministic. It is given as a supremum over the polymer length. The limit p is called the (infinity volume, specific) free energy.

Proof. First, we will prove that the expectations converge, to do this we will consider expected values and show that

$$\lim_{n\to\infty} \mathbb{E}(p_n) = \sup_{n\in\mathbb{N}} \mathbb{E}(p_n) < \infty. \tag{4.15}$$

For $m, n \geq 1$, recall the Markov property (4.5), and also that Z_m and $Z_m \circ \theta_{n,x}$ have the same law. Using Jensen's inequality, we obtain

$$\begin{aligned}
\ln Z_{n+m} &= \ln\big(Z_n \cdot E_n^{\beta,\omega}\big[Z_m \circ \theta_{n,x}\big]\big) \\
&= \ln Z_n + \ln E_n^{\beta,\omega}\big[Z_m \circ \theta_{n,x}\big] \\
&\geq \ln Z_n + E_n^{\beta,\omega}\big[\ln Z_m \circ \theta_{n,x}\big] \\
&= \ln Z_n + \sum_{\mathbf{x}\in\mathbf{X}} P_n^{\beta,\omega}(S_n = x) \ln Z_m(\theta_{n,x}\omega).
\end{aligned}$$

Taking expectation and using independence of the $\omega(i,y)'s$, it follows that

$$\begin{aligned}
\mathbb{E}\left[\ln Z_{n+m}\right] \geq& \mathbb{E}\left[\ln Z_n\right] + \sum_{\mathbf{x}\in\mathbf{X}} \mathbb{E}\left[P_n^{\beta,\omega}(S_n = x)\ln Z_m\right] \\
=& \mathbb{E}\left[\ln Z_n\right] + \mathbb{E}\left[\ln Z_m\right] \sum_{\mathbf{x}\in\mathbf{X}} \mathbb{E}\left[P_n^{\beta,\omega}(S_n = x)\right] \\
=& \mathbb{E}\left[\ln Z_n\right] + \mathbb{E}\left[\ln Z_m\right] \mathbb{E}\left[\sum_{\mathbf{x}\in\mathbf{X}} P_n^{\beta,\omega}(S_n = x)\right] \\
=& \mathbb{E}\left[\ln Z_n\right] + \mathbb{E}\left[\ln Z_m\right],
\end{aligned} \tag{4.16}$$

i.e., $\mathbb{E}[\ln Z_n]$ is super-additive. By the Kingman's super-additive ergodic theorem (see lemma(A.1.1) in the Appendix), we conclude that

$$\lim_{x\to\infty}\frac{1}{n}\mathbb{E}[\ln Z_n]=\sup_{n\in\mathbb{N}}\mathbb{E}(\ln Z_n).$$

Now, the finiteness of p follows from the annealed bound (4.19) below.

To prove the almost sure convergence, for all $\varepsilon>0$, define $A_n=|p_n-\mathbb{E}[p_n]|>\varepsilon$. then by (A.2.1)

$$\mathbb{P}(A_n)=\mathbb{P}(|p_n-\mathbb{E}[p_n]|>\varepsilon)\leq 2e^{-nc\varepsilon^2}=:\varepsilon(n).$$

Then:

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n)<\infty,$$

so by Borel-Cantelli lemma,

$$\mathbb{P}\left(\limsup_{n\to\infty}|p_n-\mathbb{E}[p_n]|>\varepsilon\right)=0,$$

which implies that $\limsup_n|p_n-\mathbb{E}[p_n]|\leq\varepsilon$, $\mathbb{P}$-a.s.. Hence,

$$\limsup_{n\to\infty}|p_n-\mathbb{E}[p_n]|=0\quad\mathbb{P}\text{-a.s.}.$$

This equality together with (4.15) implies

$$\lim_{n\to\infty}|p_n-\mathbb{E}[p_n]|=0\quad\mathbb{P}\text{-a.s.}.$$

Now to prove L^p-convergence we will use the concentration inequality (A.2) stated in the Appendix. we have that

$$\begin{aligned}
\mathbb{E}\left(|p_n-\mathbb{E}[p_n]|^p\right)&=\int_0^\infty\mathbb{P}\left(|p_n-\mathbb{E}[p_n]|>r^{1/p}\right)dr\\
&\leq 2\int_0^\infty\exp\{-nC(r^{1/p}\wedge r^{2/p})\}dr\quad\text{by }(A.2)\\
&=2\int_0^1\exp\{-nCr^{2/p}\}dr+2\int_1^\infty\exp\{-nCr^{1/p}\}dr\\
&\leq 2\int_0^\infty\exp\{-nCr^{2/p}\}dr+2\int_1^\infty\exp\{-nCr^{1/p}\}dr\\
&=C'n^{-p/2}+\mathcal{O}(\exp\{-Cn\}),
\end{aligned}$$

where $C' = \int_0^\infty \exp\{-Cv^{2/p}\}dv < \infty$. This implies L^p-convergence. $\square$

4.4 Upper bounds

Computing the value of the free energy is difficult in general. Hence it is necessary to estimate it. In this section, we obtain an upper bound for the free energy, and then it will be improved.

4.4.1 The annealed bound

For every path $\mathbf{x}$,

$$\begin{aligned}\mathbb{E}\left[e^{\beta H_n(\mathbf{x})}\right] &= \mathbb{E}\left[e^{\beta \sum_{j=1}^{n} \omega(j,x_j)}\right] \\ &= \left(\mathbb{E}\left[e^{\beta\omega(j,x_j)}\right]\right)^n \\ &= \exp n \ln\left(\mathbb{E}\left[e^{\beta\omega(j,x_j)}\right]\right) \\ &= \exp n\lambda(\beta),\end{aligned}$$

Where we used the independence and identical distribution of the random variables ω's. Then,

$$\mathbb{E}\left[e^{\beta H_n(\mathbf{x})}\right] = \exp\left(n\lambda(\beta)\right), \tag{4.17}$$

and by Fubini's theorem,

$$\begin{aligned}\mathbb{E}\left[Z_n\right] &= \mathbb{E}\left[E\left(e^{\beta H_n(\mathbf{x})\}}\right)\right] \\ &= E\left(\mathbb{E}\left[e^{\beta H_n(\mathbf{x})\}}\right]\right) \\ &= E\left(\exp\{n\lambda(\beta)\}\right) \\ &= \exp\left(n\lambda(\beta)\right).\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\left[p_n(\omega,\beta)\right] &= \mathbb{E}\left[\frac{1}{n}\ln Z_n(\omega,\beta)\right] \\
&\leq \frac{1}{n}\ln \mathbb{E}\left[Z_n(\omega,\beta)\right] \\
&= \frac{1}{n}\ln\exp\left(n\lambda(\beta)\right) \\
&= \lambda(\beta).
\end{aligned}$$

So

$$\mathbb{E}\left[p_n(\omega,\beta)\right] \leq \lambda(\beta). \tag{4.18}$$

Hence, taking supremum over n, we conclude that

$$p(\beta) \leq \lambda(\beta). \tag{4.19}$$

4.4.2 Improving the annealed bound

To improve (4.19) we will use monotonicity properties.

Proposition 4.4.1. *For all fixed ω, we have:*

1. *The function $p_n(\omega,\beta)$ is convex in β, with $p_n(\omega;0)=0$.*
2. *$\beta \mapsto \beta^{-1}p_n(\omega,\beta)$ is increasing.*
3. *$\beta \mapsto \beta^{-1}\left[p_n(\omega,\beta)+\ln 2d\right]$ is decreasing.*

The function $p(\beta)$ also satisfies properties $1-3$.

Proof. 1. Note that $\beta \mapsto p_n(\omega,\beta)$ is C^∞. By differentiation, one gets

$$\begin{aligned}
\frac{d}{d\beta}np_n = \frac{d}{d\beta}\ln Z_n(\omega,\beta) = \frac{\frac{d}{d\beta}Z_n}{Z_n} &= \frac{\frac{d}{d\beta}E\left[e^{\beta H_n(\mathbf{x})}\right]}{Z_n} \\
= \frac{E\left[\frac{d}{d\beta}e^{\beta H_n(\mathbf{x})}\right]}{Z_n} &= \frac{E\left[H_n(\mathbf{x})e^{\beta H_n(\mathbf{x})}\right]}{Z_n} \\
= \frac{\sum_{\mathbf{x}} H_n(\mathbf{x})e^{\beta H_n(\mathbf{x})}P(\mathbf{x})}{Z_n} &= E_n^{\beta,\omega}\left(H_n(\mathbf{x})\right).
\end{aligned}$$

Now we take the second derivative of np_n respect β.

$$\begin{aligned}
\frac{d^2}{d\beta^2} np_n &= \frac{d}{d\beta}\frac{d}{d\beta} np_n = \frac{d}{d\beta} \frac{\sum_{\mathbf{x}} H_n(\mathbf{x}) e^{\beta H_n(\mathbf{x})} P(\mathbf{x})}{Z_n} \\
&= \frac{\sum_{\mathbf{x}} H_n^2(\mathbf{x}) Z_n e^{\beta H_n(\mathbf{x})} P(\mathbf{x}) - \left(\sum_{\mathbf{x}} H_n(\mathbf{x}) e^{\beta H_n(\mathbf{x})} P(\mathbf{x})\right)^2}{Z_n^2} \\
&= \frac{\sum_{\mathbf{x}} H_n^2(\mathbf{x}) e^{\beta H_n(\mathbf{x})} P(\mathbf{x})}{Z_n} - \left(\frac{\sum_{\mathbf{x}} H_n(\mathbf{x}) e^{\beta H_n} P(\mathbf{x})}{Z_n}\right)^2 \\
&= \mathbb{E}_n^{\beta,\omega}\left(H_n^2\right) - \left(\mathbb{E}_n^{\beta,\omega}\left(H_n\right)\right)^2 \\
&= \mathbb{V}ar_n^{\beta,\omega}(H_n) > 0.
\end{aligned}$$

We have that

$$p_n(\omega, 0) = \frac{1}{n} \ln Z_n(\omega, 0) = \frac{1}{n} \ln 1 = 0.$$

Thus p_n is convex in β, proving (1).

2. We have $\beta^{-1} p_n(\omega, \beta) = \beta^{-1}\left[p_n(\omega, \beta) - p_n(\omega, 0)\right]$, thus by convexity, it is non-decreasing in β.

3. We have the identity

$$\begin{aligned}
\frac{d}{d\beta}\left(\frac{1}{\beta}\left[p_n + \ln(2d)\right]\right) &= \frac{1}{n\beta} E_n^{\beta,\omega}(H_n) - \frac{p_n + \ln 2d}{\beta^2} \\
&= \frac{1}{n\beta^2}\Bigg[\sum_{\mathbf{X}} P_n^{\beta,\omega}(x)\beta H_n - \sum_{\mathbf{X}} P_n^{\beta,\omega}(x) \ln Z_n \\
&\qquad + \sum_{\mathbf{X}} P_n^{\beta,\omega}(x) \ln P(x)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n\beta^2} \sum_{\mathbf{x}} P_n^{\beta,\omega}(x) \left[\ln e^{\beta H_n} P(x) - \ln Z_n\right] \\
&= \frac{1}{n\beta^2} \sum_{\mathbf{x}} P_n^{\beta,\omega}(x) \left[\ln \frac{e^{\beta H_n} P(x)}{Z_n}\right] \\
&= \frac{1}{n\beta^2} \sum_{\mathbf{x}} P_n^{\beta,\omega}(x) \ln P_n^{\beta,\omega}(x) \\
&= \frac{1}{n\beta^2} h(P_n^{\beta,\omega}),
\end{aligned}
$$

where $h(\nu)$ is the Boltzmann entropy of a probability measure ν on the n steps path space,

$$h(\nu) := \sum_{\mathbf{x}} \nu(x) \ln \nu(x). \tag{4.20}$$

Finally we have that $h(\nu) \leq 0$ for all ν, which ends the prove[3]. □

Now, we will derive a better upper bound for the annealed bound.

Proposition 4.4.2. *We have*

$$p(\beta) \leq \beta \inf_{b \in (0,\beta]} \frac{\lambda(b) + \ln(2d)}{b} - \ln(2d). \tag{4.21}$$

Hence, under condition (T),

$$(T): \qquad \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d), \tag{4.22}$$

we have

$$p(\beta) < \lambda(\beta). \tag{4.23}$$

More precisely, if there exist a positive root β_1 to the equation $\beta\lambda'(\beta) = \lambda(\beta) + \ln(2d)$, then for all $\beta > \beta_1$ it holds

$$p(\beta) \leq \frac{\beta}{\beta_1} [\lambda(\beta_1) + \ln(2d)] - \ln(2d) < \lambda(\beta). \tag{4.24}$$

[3]The step by step prove of 4.4.1-3 is in (B.1.3)

Proof. To simplify the notation, we introduce

$$g(\beta) = \beta\lambda'(\beta) - \lambda(\beta), \qquad f(\beta) = \frac{\lambda(\beta) + \ln(2d)}{\beta}. \tag{4.25}$$

Since λ is smooth and convex,we have that

$$\begin{aligned} g'(\beta) &= \frac{d}{d\beta} g(\beta) = \frac{d}{d\beta}\left[\beta\lambda'(\beta) - \lambda(\beta)\right] \\ &= \lambda'(\beta) + \beta\lambda''(\beta) - \lambda'(\beta) = \beta\lambda''(\beta). \end{aligned}$$

So $g'(\beta) = \beta\lambda''(\beta)$ and since λ is convex, then its second derivative is non-negative in $\mathbb{R}^+$, so $\beta\lambda''(\beta)$ has the same sing of β, thus $g'(\beta)$ has the same sign of β, and $g(\beta)$ is increasing on $\mathbb{R}^+$.

Now, we introduce the convex conjugate of λ denoted as λ^*, with

$$\lambda^*(u) = \sup_{\beta} \beta u - \lambda(\beta), \qquad \text{for } u \in \mathbb{R}. \tag{4.26}$$

We have $g(\beta) = \lambda^*(u)$, when $u = \lambda'(\beta)$, where β satisfy (4.26). Moreover,

$$\begin{aligned} f'(\beta) &= \frac{d}{d\beta} f(\beta) = \frac{\beta\lambda'(\beta) - \lambda(\beta) - \ln(2d)}{\beta^2} \\ &= \frac{g(\beta) - \ln(2d)}{\beta^2}. \end{aligned}$$

So we can write

$$\begin{aligned}
\mathbb{E}\left[p_n(\omega,\beta)+\ln(2d)\right] &= \beta\frac{1}{\beta}\mathbb{E}\left[p_n(\omega,\beta)+\ln(2d)\right] \\
&= \beta \inf_{b\in(0,\beta]} \frac{1}{b}\mathbb{E}\left[p_n(\omega,b)+\ln(2d)\right] \qquad \text{by Proposition 4.4.1.3} \\
&= \beta \inf_{b\in(0,\beta]} \frac{1}{b}\left(\mathbb{E}\left[p_n\right]+\ln(2d)\right) \\
&\le \beta \inf_{b\in(0,\beta]} \frac{1}{b}\left(\lambda(b)+\ln(2d)\right) \qquad \text{by (4.18)} \\
&= \beta \inf_{b\in(0,\beta]} f(b).
\end{aligned}$$

Letting $n\to\infty$ we obtain

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left[p_n(\omega,\beta)+\ln(2d)\right] &\le \lim_{n\to\infty}\beta \inf_{b\in(0,\beta]} f(b) \\
\mathbb{E}\left[\lim_{n\to\infty}p_n(\omega,\beta)+\ln(2d)\right] &\le \beta \inf_{b\in(0,\beta]} f(b) \\
\mathbb{E}\left[p(\beta)+\ln(2d)\right] &\le \beta \inf_{b\in(0,\beta]} f(b) \\
p(\beta)+\ln(2d) &\le \beta \inf_{b\in(0,\beta]} \frac{\lambda(b)+\ln(2d)}{b}.
\end{aligned}$$

therefore

$$p(\beta)\le \beta \inf_{b\in(0,\beta]} \frac{\lambda(b)+\ln(2d)}{b}-\ln(2d),$$

so we have obtained (4.21). Now let us define $\beta_1\in(0,\infty]$ by

$$\beta_1=\inf\{\beta>0 : g(\beta)\ge\ln(2d)\}$$

allowing it to take an infinite value. Then, f reaches its minimum at β_1, because

$$f'(\beta) = \frac{g(\beta) - \ln(2d)}{\beta^2} = 0 \iff g(\beta) = \ln(2d) \iff \beta = \beta_1$$

and

$$f''(\beta_1) = \frac{\beta_1^3 \lambda''(\beta_1) - 2\beta_1(g(\beta_1) - \ln(2d))}{\beta_1^4},$$

$$f''(\beta_1) = \frac{\lambda''(\beta_1)}{\beta_1} > 0.$$

This last equality is due to $1/\beta_1 > 0$ for $\beta_1 \in (0, \infty]$.

To make this easier to understand, look at Figure 4.1

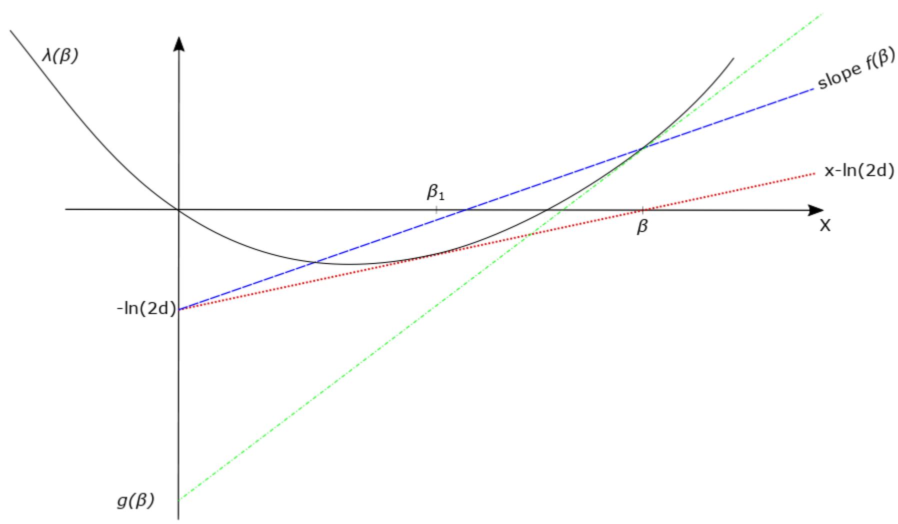

Figure 4.1: Constructing the upper bound of Proposition 2.2.

$$f(\beta) = \frac{\lambda(\beta) + \ln(2d)}{\beta}.$$

So $f(\beta)$ is the line between the points $(0, -\ln(2d))$ and $(\beta, \lambda(\beta))$.

While $g(\beta) = \beta\lambda'(\beta) - \lambda(\beta) = 0$ is the $Y-$axes interception of the tangent of λ at β.

For β as in Figure 4.1, the infimum of $f(\hat{\beta})$ is in $\hat{\beta} = \beta_1$ where the tangent to λ intercepts the vertical axis at $-\ln(2d)$.

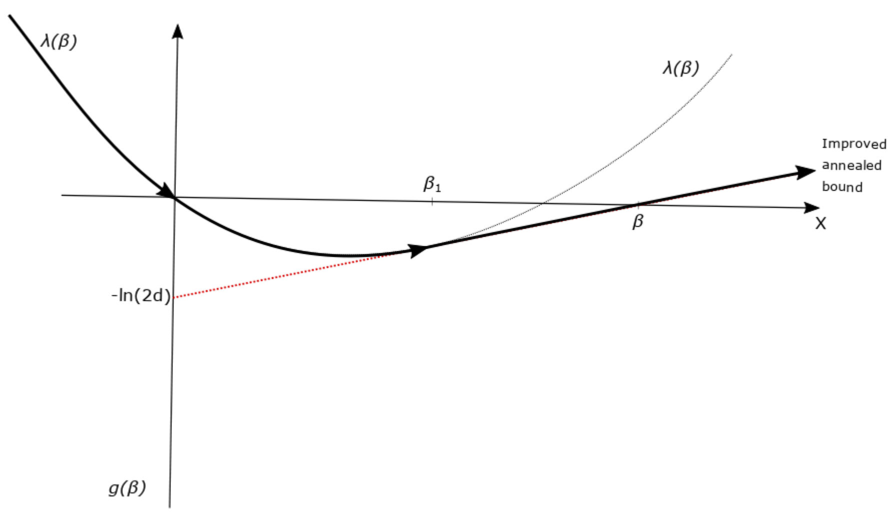

Figure 4.2: Improved annealed bound.

If λ has a tangent at some $\beta_1 > 0$ which intersects the vertical axis at $-\ln(2d)$ the bound for p on $\mathbb{R}^+$ is improved for the annealed bound by that tangent, for values $\beta > \beta_1$.

Summarizing, if f has its minimum at β_1 we have

$$\inf_{\hat\beta\in(0,\beta]} f(\hat\beta) = \begin{cases} f(\beta), & \text{if } \beta \le \beta_1, \\ f(\beta_1), & \text{if } \beta \ge \beta_1. \end{cases}$$

Using last equality and (4.21) we have

$$\begin{aligned} p(\beta) &\le \frac{\beta}{\beta_1}\left[\lambda(\beta_1) + \ln(2d)\right] - \ln(2d) \\ &< \frac{\beta}{\beta}\left[\lambda(\beta) + \ln(2d)\right] - \ln(2d) = \lambda(\beta). \end{aligned}$$

This last inequality proves (4.24). Finally note that by the strict convexity of λ the condition $\beta > \beta_1$ is equivalent to condition (**T**) (4.22). □

Now we are looking for conditions ensuring $p(\beta) < \lambda(\beta)$, for large β, in terms of the marginal distribution of $\omega(n, x)$.

Proposition 4.4.3. *Set $q(dh) = \mathbb{P}(\omega(n,x) \in dh)$ and $s = \sup supp[q]$. If $s = +\infty$ or $q(\{s\}) < \frac{1}{2d}$, then, there exist $\beta_1 \in (0,\infty)$ such that $p(\beta) < \lambda(\beta)$, for $\beta > \beta_1$.*

Proof. Let λ^* the Legendre transform of λ,

$$\lambda^*(u) = \sup_\beta \{u\beta - \lambda(\beta)\}$$

then, $\beta\lambda'(\beta) - \lambda(\beta) = \lambda^*(\lambda'(\beta))$.

On the other hand, let us calculate $\mathbb{E}(\omega(n,x)e^{\beta\omega(n,x)})$:

$$\mathbb{E}(te^{\beta t}) = \int te^{\beta t} q_t(dh) = \int_0^s te^{\beta t} q_t(dh).$$

Integrating by parts

$$\begin{aligned} \mathbb{E}(te^{\beta t}) &= \left[\frac{te^{\beta t}}{\beta}\right]_0^s - \int_0^s \frac{e^{\beta t}}{\beta} dq_t(dh) \\ &= \frac{se^{\beta s}}{\beta} - \frac{1}{\beta}\mathbb{E}(e^{\beta t}). \end{aligned}$$

So $\beta M'_\omega(\beta) = \beta\mathbb{E}(\omega e^{\beta\omega}) = se^{\beta s} - \mathbb{E}(e^{\beta\omega}) = se^{\beta s} - M_\omega(\beta)$,

where $M_\omega(\beta)$ is the moment generating function of $\omega(n, x)$.
In the following we take the limit $\beta \to \infty$ in the derivative of $\lambda(\beta)$.

Recall that

$$\lambda(\beta) = \ln \mathbb{E}(e^{\beta\omega}).$$

Then,

$$\begin{aligned}
\lambda'(\beta) &= \frac{d}{d\beta} \ln \mathbb{E}(e^{\beta\omega}) = \frac{\mathbb{E}(\frac{d}{d\beta} e^{\beta\omega})}{\mathbb{E}(e^{\beta\omega})} \\
&= \frac{\mathbb{E}(\omega e^{\beta\omega})}{\mathbb{E}(e^{\beta\omega})} = \frac{\mathbb{E}(\omega e^{\beta\omega})}{M_\omega(\beta)} \\
&= \frac{se^{\beta s}}{\beta M_\omega(\beta)} - \frac{M_\omega(\beta)}{\beta M_\omega(\beta)} \\
&= \frac{se^{\beta s}}{\beta M_\omega(\beta)} - \frac{1}{\beta}.
\end{aligned}$$

Now, taking limits in both sides of the equation we obtain:

$$\begin{aligned}
\lim_{\beta\to\infty} \lambda'(\beta) &= \lim_{\beta\to\infty} \left[\frac{se^{\beta s}}{\beta M_\omega(\beta)} - \frac{1}{\beta} \right] \\
&= \lim_{\beta\to\infty} \frac{se^{\beta s}}{\beta M_\omega(\beta)}.
\end{aligned}$$

By L'Hopital rule:

$$\begin{aligned}
\lim_{\beta\to\infty} \lambda'(\beta) &= \lim_{\beta\to\infty} \frac{s^2 e^{\beta s}}{M_\omega(\beta) + \beta M'_\omega(\beta)} \\
&= \lim_{\beta\to\infty} \frac{s^2 e^{\beta s}}{se^{\beta s}} \\
&= \lim_{\beta\to\infty} s = s.
\end{aligned}$$

Thus, $\lambda'(\beta) \to sup\, supp\, q$ when $\beta \to \infty$.

Finally, we have two cases.

1. If $s = \infty$, then $\beta\lambda'(\beta) - \lambda(\beta) \to \infty$, because $\lambda(\beta)$ is finite for all β by assumption. And (4.22) holds for large β.

2. If $s < \infty$ and $q(\{s\}) < 1/2d$, then by [8] we know that

$$1 \lim_{\beta\to\infty} \lambda^*(\lambda'(\beta)) = -\ln q(\{s\}) > -\ln\left(\frac{1}{2d}\right) = \ln(2d),$$

   and (4.22) holds for large β.

□

Note: Easily we can find some lower bounds, but they are useless for our analysis. Although we will show some of them using **Theorem 4.3.1**. For example:

$$p(\beta) = \sup_n \frac{1}{n}\mathbb{E}\left[\ln Z_n(\omega;\beta)\right] \geq \mathbb{E}\left[\ln E\left(e^{\beta\omega_1}\right)\right],$$

which we can improve by Jensen inequality and Fubini's theorem

$$\begin{aligned} p(\beta) &\geq \mathbb{E}\left[\ln E\left(e^{\beta\omega(1,S_1)}\right)\right] \\ &\geq \mathbb{E}\left(\ln e^{E[\beta\omega(1,S_1)]}\right) \\ &= \mathbb{E}\left(\beta E\left[\omega(1,S_1)\right]\right) = \beta\mathbb{E}E[\omega(1,S_1)] \\ &= \beta E\mathbb{E}[\omega(1,S_1)] = \beta\mathbb{E}[\omega(t,x)]. \end{aligned} \tag{4.27}$$

But all of this bounds are local optimizations.

4.5 Monotonicity

We have seen in Proposition (4.4.1) that $\beta \mapsto p_n(\omega;\beta)$ is convex and differentiable. So we can compute its derivative:

$$\begin{aligned}\frac{\partial}{\partial\beta}p_n(\omega,\beta) &= \frac{1}{n}\frac{\partial}{\partial\beta}\ln Z_n(\omega,\beta) = \frac{1}{n}\frac{\frac{\partial}{\partial\beta}Z_n(\omega,\beta)}{Z_n(\omega,\beta)}\\ &= E\left[\frac{\frac{\partial}{\partial\beta}e^{\beta H_n(\mathbf{x})}}{nZ_n}\right] = E\left[\frac{H_n(\mathbf{x})e^{\beta H_n(\mathbf{x})}}{nZ_n}\right]\\ &= E_n^{\beta,\omega}\left[\frac{H_n(\mathbf{x})}{n}\right],\end{aligned}$$

which is the specific (internal) energy. What this says is that $\beta p_n'$ is the limiting energy per monomer under the polymer measure [7]. Let $\mathcal{D} = \mathcal{D}(p)$ be the set of $\beta's$ such that the limit p is differentiable in β. Since p is convex we have from [2] that $\mathcal{D}^c$ is at most numerable. In fact, p is $\mathcal{C}^1$ on $\mathcal{D}$, and since p is the limit of $p_n(\omega;\beta)$ a.s. and in L^1-norm, we deal with convex functions that have third derivative. The next result follows from this last fact and general results of convergence of convex functions and its derivatives (The details can be consulted in [24]).

Proposition 4.5.1. *For all $\beta \in \mathcal{D}$ and almost every environment ω,*

$$\lim_{n\to\infty} E_n^{\beta,\omega}\left[H_n(S)/n\right] = \lim_{n\to\infty}\mathbb{E}\left(E_n^{\beta,\omega}\left[H_n(S)/n\right]\right) = p'(\beta).$$

More over, for all $\beta \in \mathbb{R}^+$, we could write bounds involving the left and right derivatives:

$$p'(\beta^-) \le \liminf_{n\to\infty} E_n^{\beta,\omega}\left[H_n(S)/n\right] \le \limsup_{n\to\infty} E_n^{\beta,\omega}\left[H_n(S)/n\right] \le p'(\beta^+).$$

The notation $p'(\beta^{\pm})$ denote the limit of $p'(b)$ when b increase (decrease) to β in $\mathcal{D}$. Finally we have

$$\mathbb{E}[\omega(t,x)] \le p'(\beta) \le \lambda'(\beta). \qquad\qquad \beta \ge 0, \beta \in \mathcal{D}.$$

Proof. First we have

$$\lim_{n\to\infty} E_n^{\beta,\omega}\left[H_n(s)/n\right] = \lim_{n\to\infty} \frac{\partial}{\partial\beta} p_n(\omega,\beta) = \frac{\partial}{\partial\beta} \lim_{n\to\infty} p_n(\omega,\beta)$$
$$= \frac{\partial}{\partial\beta} p(\beta) = p'(\beta).$$

We also have:

$$\lim_{n\to\infty} E_n^{\beta,\omega}\left[H_n(s)/n\right] = \lim_{n\to\infty} \frac{\partial}{\partial\beta} p_n(\omega,\beta) = \frac{\partial}{\partial\beta} \lim_{n\to\infty} \frac{1}{n} \ln Z_n(\omega,\beta).$$

Now, by (4.1) and the Kingman's superadditive ergodic theorem (A.1), we get:

$$\lim_{n\to\infty} E_n^{\beta,\omega}\left[H_n(s)/n\right] = \frac{\partial}{\partial\beta} \lim_{n\to\infty} \frac{1}{n} \ln Z_n(\omega,\beta) = \frac{\partial}{\partial\beta} \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\ln Z_n(\omega,\beta)\right]$$
$$= \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\frac{\partial}{\partial\beta} \ln Z_n(\omega,\beta)\right] = \lim_{n\to\infty} \mathbb{E}\left(E_n^{\beta,\omega}\left[H_n(S)/n\right]\right).$$

The inequalities of the second statement follow immediately from the previous one. Finally, the last inequality comes from (4.27) and (4.29) below. □

The relevance of this analysis is to show that $\lambda - p$ has a remarkable monotonicity property.

Theorem 4.5.2. *The functions $\beta \mapsto \lambda(\beta) - \mathbb{E}[p_n(\omega,\beta)]$ and $\beta \mapsto \lambda(\beta) - p(\beta)$ are non-decreasing in $\mathbb{R}^+$, and non-increasing in $\mathbb{R}^-$.*

Proof. First recall the notation of proposition (4.4.3) for the law of $\omega(t,x)$, and define

$$\zeta_n(S) = e^{\beta H_n(S)}.$$

Let us calculate the derivative respect to β of $\ln Z_n$:

$$\frac{\partial}{\partial\beta} \mathbb{E}\left[\ln Z_n\right] = \mathbb{E}\left[\frac{\partial}{\partial\beta} \ln Z_n\right] = \mathbb{E}\left[\frac{\frac{\partial}{\partial\beta} Z_n}{Z_n}\right]$$
$$= \mathbb{E}\left[\frac{E\left(H_n e^{\beta H_n}\right)}{Z_n}\right] = E\left[\mathbb{E}\left(Z_n^{-1} H_n \zeta_n\right)\right]. \tag{4.28}$$

Now, fix a path $\mathbf{x} \in \mathbf{X}$. Define a probability measure as follows:

$$d\hat{\mathbb{P}}^{\mathbf{x}} := \zeta_n(\mathbf{x}) e^{-n\lambda(\beta)} d\mathbb{P}.$$

It is easy to see that $e^{n\lambda(\beta)}$ is the normalization of $\zeta_n d\mathbb{P}$. Under this measure the r.v. $\omega' s$ are independent (though no identically distributed). We prove this afirmation in the following.

$$\begin{aligned}
\hat{\mathbb{P}}^{\mathbf{x}}(\omega(t,x_t) \in dA, \omega(r,x_r) \in dB) &= e^{\beta(\omega_t+\omega_r)} e^{-2\lambda(\beta)} \mathbb{P}(\omega(t,x_t) \in dA, \omega(r,x_r) \in dB) \\
&= e^{\beta\omega_t} e^{-\lambda(\beta)} e^{\beta\omega_r} e^{-\lambda(\beta)} \mathbb{P}(\omega(t,x_t) \in dA) \mathbb{P}(\omega(r,x_r) \in dB) \\
&= e^{\beta w_t} e^{-\lambda(\beta)} \mathbb{P}(\omega(t,x_t) \in dA) e^{\beta w_r} e^{-\lambda(\beta)} \mathbb{P}(\omega(r,x_r) \in dB) \\
&= \hat{\mathbb{P}}^{\mathbf{x}}(\omega(t,x_t) \in dA) \hat{\mathbb{P}}^{\mathbf{x}}(\omega(r,x_r) \in dB),
\end{aligned}$$

which proves the independence between the random variables. We proceed now to show that the random variables are not identically distributed. Let us take a vertex in the path $x_t \in \mathbf{x}$, then

$$\begin{aligned}
\hat{\mathbb{P}}^{\mathbf{x}}(\omega(t,x_t) \in dh) &= e^{\beta\omega_t} e^{-\lambda(\beta)} \mathbb{P}(\omega(t,x_t) \in dh) \\
&= e^{\beta\omega_t - \lambda} \mathbb{P}(\omega(t,x_t) \in dh) = e^{\beta\omega_t - \lambda} q(dh).
\end{aligned}$$

While if $x_t \notin \mathbf{x}$, then its contribution to the Hamiltonian is equal to zero. Then:

$$\begin{aligned}
\hat{\mathbb{P}}^{\mathbf{x}}(\omega(t,x_t) \in dh) &= e^{\beta\omega_t \mathbb{1}(x_t \in \mathbf{x})} e^{-\ln \mathbb{E}\left(e^{\beta\omega_t \mathbb{1}(x_t \in \mathbf{x})}\right)} \mathbb{P}(\omega(t,x_t) \in dh) \\
&= e^0 e^{-\ln 1} \mathbb{P}(\omega(t,x_t) \in dh) = e^0 \mathbb{P}(\omega(t,x_t) \in dh) = q(dh).
\end{aligned}$$

This shows that the distribution of ω under the measure $\hat{\mathbb{P}}^{\mathbf{x}}$ depends on the path.

Now, we will proof two equalities that we will use below.

$$\begin{aligned}
\lambda'(\beta) = \frac{\partial}{\partial\beta} \ln \mathbb{E}\left(e^{\beta\omega}\right) &= \frac{\mathbb{E}\left(\frac{\partial}{\partial\beta} e^{\beta\omega}\right)}{\mathbb{E}\left(e^{\beta\omega}\right)} \\
&= \frac{\mathbb{E}\left(\omega e^{\beta\omega}\right)}{e^{\ln \mathbb{E}(\beta\omega)}} = \frac{\mathbb{E}\left(\omega e^{\beta\omega}\right)}{e^{\lambda(\beta)}}.
\end{aligned}$$

So

$$e^{\lambda(\beta)} \lambda'(\beta) = \mathbb{E}\left(\omega e^{\beta\omega}\right).$$

The second equality that we need is

$$n\lambda'(\beta)e^{n\lambda(\beta)} = n\frac{\mathbb{E}\left(\omega e^{\beta\omega}\right)}{e^{\lambda(\beta)}}e^{n\lambda(\beta)} = n\mathbb{E}\left(\omega e^{\beta\omega}\right)e^{(n-1)\lambda(\beta)}$$
$$= \mathbb{E}\left(n\omega e^{\beta\omega}\right)\left(e^{\lambda(\beta)}\right)^{(n-1)},$$

as $\omega' s$ are identically distributed,

$$n\lambda'(\beta)e^{n\lambda(\beta)} = \mathbb{E}\left[e^{\beta\omega}\left(\omega_1 + \cdots + \omega_n\right)\right]\left[\mathbb{E}\left(e^{\beta\omega}\right)\right]^{(n-1)},$$

by independence of the $\omega' s$,

$$n\lambda'(\beta)e^{n\lambda(\beta)} = \left[\mathbb{E}\left(e^{\beta\omega_1}\omega_1\right) + \cdots + \mathbb{E}\left[e^{\beta\omega_n}\omega_n\right]\right]\left[\mathbb{E}\left(\prod_{n-1} e^{\beta\omega}\right)\right]$$
$$= \mathbb{E}\left(e^{\beta\omega_1}\omega_1\right)\mathbb{E}\left(e^{\beta\sum\limits_{n-1}\omega}\right) + \cdots + \mathbb{E}\left[e^{\beta\omega_n}\omega_n\right]\mathbb{E}\left(e^{\beta\sum\limits_{n-1}\omega}\right),$$

if in each element of the sum we do not take the element ω_i over the sum in the exponent, we have independence. Thus

$$n\lambda'(\beta)e^{n\lambda(\beta)} = \mathbb{E}\left[e^{\beta\omega_1}\omega_1 e^{\beta\sum\limits_{n-1\neq 1}\omega}\right] + \cdots + \mathbb{E}\left[e^{\beta\omega_n}\omega_n e^{\beta\sum\limits_{n-1\neq n}\omega}\right]$$
$$= \mathbb{E}\left[\omega_1 e^{\beta\sum\limits_{n}\omega}\right] + \cdots + \mathbb{E}\left[\omega_n e^{\beta\sum\limits_{n}\omega}\right] = \mathbb{E}\left[e^{\beta H_n}\sum_n \omega\right]$$
$$= \mathbb{E}\left[H_n e^{\beta H_n}\right] = \mathbb{E}\left[\zeta_n H_n\right].$$

So

$$n\lambda'(\beta)e^{n\lambda(\beta)} = \mathbb{E}\left[\zeta_n H_n\right].$$

But, we also have shown that

$$e^{n\lambda(\beta)} = \mathbb{E}\left[e^{\beta H_n}\right] = \mathbb{E}\left[\zeta_n\right].$$

Returning to the proof, since the r.v. $\omega' s$ are independent under $\hat{\mathbb{P}}^{\mathbf{x}}$, then by the **Harris-FKG Theorem** (A.5) we have that they are positively associated. Note that the function H_n is increasing in ω, while $(Z_n)^{-1}$ is decreasing for $\beta \geq 0$, hence $-(Z_n)^{-1}$ is increasing for $\beta \geq 0$. In such manner, for fixed $\mathbf{x}$ we find

$$\begin{aligned}\mathbb{E}[-(Z_n)^{-1}H_n\zeta_n] &= \mathbb{E}[-(Z_n)^{-1}\zeta_n\zeta_n^{-1}H_n\zeta_n] \\ &\geq \mathbb{E}\left[-(Z_n)^{-1}\zeta_n\zeta_n^{-1}\right]\mathbb{E}\left[H_n\zeta_n\right],\end{aligned}$$

applying again (A.5), we get:

$$\mathbb{E}[-(Z_n)^{-1}H_n\zeta_n] \geq \mathbb{E}\left[-(Z_n)^{-1}\zeta_n\right]\mathbb{E}\left[\zeta_n^{-1}\right]\mathbb{E}\left[H_n\zeta_n\right],$$

by Jensen's inequality

$$\mathbb{E}[-(Z_n)^{-1}H_n\zeta_n] \geq \mathbb{E}\left[-(Z_n)^{-1}\zeta_n\right]\left(\mathbb{E}\left[\zeta_n\right]\right)^{-1}\mathbb{E}\left[H_n\zeta_n\right].$$

Thus

$$\begin{aligned}\mathbb{E}[(Z_n)^{-1}H_n\zeta_n] &\leq \mathbb{E}\left[(Z_n)^{-1}\zeta_n\right]\left(\mathbb{E}\left[\zeta_n\right]\right)^{-1}\mathbb{E}\left[H_n\zeta_n\right] \\ &= \mathbb{E}\left[\zeta_n(Z_n)^{-1}\right]e^{-n\lambda(\beta)}n\lambda'(\beta)e^{n\lambda(\beta)} \\ &= \mathbb{E}\left[\zeta_n(Z_n)^{-1}\right]n\lambda'(\beta).\end{aligned}$$

Then, if we integrate (4.28) respect to P, we obtain:

$$\begin{aligned}\frac{\partial}{\partial\beta}E\mathbb{E}\ln Z_n &\leq n\lambda'(\beta)E\left(\mathbb{E}\left[(Z_n)^{-1}\zeta_n\right]\right) \\ &= n\lambda'(\beta)\mathbb{E}\left(E\left[(Z_n)^{-1}\zeta_n\right]\right),\end{aligned}$$

using Fubini's Theorem

$$\begin{aligned}\frac{\partial}{\partial\beta}E\mathbb{E}\ln Z_n &\leq n\lambda'(\beta)\mathbb{E}\left((Z_n)^{-1}E\left[\zeta_n\right]\right) \\ &= n\lambda'(\beta)\mathbb{E}\left((Z_n)^{-1}Z_n\right) \\ &= n\lambda'(\beta). \qquad (4.29)\end{aligned}$$

Therefore

$$0 \leq \lambda'(\beta) - \frac{\partial}{\partial\beta}\mathbb{E}\left[p_n\right]. \qquad (4.30)$$

Finally, since p_n and λ are both equal to zero when $\beta = 0$ and $\lambda(\beta)-\mathbb{E}\left[p_n\right]$ has non-negative slope over all positive $\beta's$ we conclude that $\lambda(\beta)-\mathbb{E}\left[p_n\right]$ are non-decreasing on $\mathbb{R}^+$.

If we take $\sup_n$ on (4.30), we obtain

$$0 \leq \lambda'(\beta) - p'(\beta).$$

Once again, p and λ are both equal to zero when $\beta = 0$ and the slope of $\lambda(\beta) - p(\beta)$ are non-negative over all positive $\beta's$ we have that $\lambda(\beta) - p(\beta)$ are non decreasing on $\mathbb{R}^+$.
The second part of this Theorem is analogue. □

Another important result is that the r.v.'s $Z_n(\beta,\omega)e^{-n\lambda(\beta)}$ are increasing in β in the convex order.

For two integrable r.v.'s X, Y, we say that X is smaller than Y in convex order if

$$\mathbb{E}\left[\phi(X)\right] \leq \mathbb{E}\left[\phi(Y)\right],$$

for all convex $\phi : \mathbb{R} \mapsto \mathbb{R}$ such that the expectation exist. In that case we write $X \leq_{cx} Y$.

Theorem 4.5.3. *For fixed $n \in \mathbb{N}$, the process $\beta \mapsto Z_n(\omega,\beta)e^{-n\lambda(\beta)}$ is increasing in the convex order.*

Proof. Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ a convex function. We need to prove that the function $\beta \mapsto \mathbb{E}\left[\phi\left(Z_n(\omega;\beta)e^{-n\lambda(\beta)}\right)\right]$ is increasing in β. We have

$$\frac{\partial}{\partial\beta}\mathbb{E}\left[\phi\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right] = \mathbb{E}\left[\frac{\partial}{\partial\beta}\phi\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right], \tag{4.31}$$

by the chain rule, (4.31) is equal to

$$\begin{aligned}
&\mathbb{E}\left[\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\left(E\left[H_n(\mathbf{x})e^{\beta H_n(\mathbf{x})}\right]e^{-n\lambda(\beta)} + Z_n(\omega,\beta)\left(-n\lambda'(\beta)\right)e^{-n\lambda(\beta)}\right)\right]\\
=&\mathbb{E}\left[\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\left(E\left[H_n(\mathbf{x})e^{\beta H_n(\mathbf{x})-n\lambda(\beta)}\right] + E\left[-n\lambda'(\beta)e^{\beta H_n(\mathbf{x})-n\lambda(\beta)}\right]\right)\right]\\
=&\mathbb{E}E\left[\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\left(H_n(\mathbf{x}) - n\lambda'(\beta)\right)e^{\beta H_n(\mathbf{x})-n\lambda(\beta)}\right],
\end{aligned}$$

by Fubini's Theorem:

$$\begin{aligned}
\frac{\partial}{\partial\beta}\mathbb{E}\left[\phi\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right] &= E\mathbb{E}\left[\left(H_n(\mathbf{x}) - n\lambda'(\beta)\right)e^{\beta H_n(\mathbf{x})-n\lambda(\beta)}\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right]\\
&= E\hat{\mathbb{E}}^{\mathbf{x}}\left[\left(H_n(\mathbf{x}) - n\lambda'(\beta)\right)\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right].
\end{aligned}$$

As we proved previously, the r.v. ω's are independent under $\hat{\mathbb{P}}^{\mathbf{x}}$, then by Harris-FKG, they are positively associated and these two following applications are increasing for $\beta \geq 0$

$$\begin{aligned} &\omega \mapsto H_n(\mathbf{x}), \\ &\omega \mapsto \phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right). \end{aligned}$$

Since they are positively associated we have

$$\begin{aligned} \frac{\partial}{\partial\beta}\mathbb{E}\left[\phi\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right] &= E\hat{\mathbb{E}}^{\mathbf{x}}\left[\left(H_n(\mathbf{x}) - n\lambda'(\beta)\right)\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right] \\ &\geq E\hat{\mathbb{E}}^{\mathbf{x}}\left[\left(H_n(\mathbf{x}) - n\lambda'(\beta)\right)\right] E\hat{\mathbb{E}}^{\mathbf{x}}\left[\phi'\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right]. \end{aligned}$$

In particular

$$\begin{aligned} E\hat{\mathbb{E}}^{\mathbf{x}}\left[H_n(\mathbf{x}) - n\lambda'(\beta)\right] &= E\mathbb{E}\left[\left(H_n(\mathbf{x}) - n\lambda'(\beta)\right)e^{\beta H_n(\mathbf{x}) - n\lambda(\beta)}\right] \\ &= E\mathbb{E}\left[\frac{\partial}{\partial\beta}e^{\beta H_n(\mathbf{x}) - n\lambda(\beta)}\right] = \frac{\partial}{\partial\beta}E\left(\mathbb{E}\left[e^{\beta H_n(\mathbf{x})}\right]e^{-n\lambda(\beta)}\right) \\ &= \frac{\partial}{\partial\beta}E\left(e^{n\lambda(\beta)}e^{-n\lambda\beta}\right) = 0. \end{aligned}$$

Thus

$$\frac{\partial}{\partial\beta}\mathbb{E}\left[\phi\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)\right] \geq 0,$$

which yields the result. □

This result can be generalized in classical spin glass models such as

- Sherrington-Kirkpatrick (SK) model,
- Edwards-Anderson (EA) model,
- Random field Ising (RFIM) model.

Returning to the r.v. $Z_n(\omega,\beta)e^{-n\lambda(\beta)}$, as they are increasing in convex order, i.e. $Z_n(\omega,\beta) \leq_{cv} Z_n(\omega,\beta')$, for some β and β' in $\mathbb{R}$, such that $\beta \leq \beta'$, and since $\phi(x) = x$ and $\phi(x) = -x$ are convex functions, then

$$\mathbb{E}\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right) \leq \mathbb{E}\left(Z_n(\omega,\beta')e^{-n\lambda(\beta')}\right),$$

and also

$$\mathbb{E}\left(-Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right) \leq \mathbb{E}\left(-Z_n(\omega,\beta')e^{-n\lambda(\beta')}\right)$$
$$\mathbb{E}\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right) \geq \mathbb{E}\left(Z_n(\omega,\beta')e^{-n\lambda(\beta')}\right).$$

Therefore

$$\mathbb{E}\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right) = \mathbb{E}\left(Z_n(\omega,\beta')e^{-n\lambda(\beta')}\right).$$

Also, since $\phi(x) = x^2$ is convex, we have

$$\mathbb{E}\left(Z_n^2(\omega;\beta)e^{-n2\lambda(\beta)}\right) \leq \mathbb{E}\left(Z_n^2(\omega,\beta')e^{-n2\lambda(\beta')}\right)$$
$$\mathbb{E}\left(Z_n^2(\omega,\beta)e^{-n2\lambda(\beta)}\right) - \mathbb{E}\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right)^2 \leq \mathbb{E}\left(Z_n^2(\omega,\beta')e^{-n2\lambda(\beta')}\right) - \mathbb{E}\left(Z_n(\omega,\beta')e^{-n\lambda(\beta')}\right)^2$$
$$Var\left(Z_n(\omega,\beta)e^{-n\lambda(\beta)}\right) \leq Var\left(Z_n(\omega,\beta')e^{-n\lambda(\beta')}\right).$$

Since they have the same expectation, this is a quantitative way to say that $Z_n(\omega,\beta)e^{-n\lambda(\beta)}$ is less dispersed than $Z_n(\omega,\beta')e^{-n\lambda(\beta')}$.

In our case, the process $\beta \mapsto Z_n(\omega,\beta)e^{-n\lambda(\beta)}$ is increasing in the convex order. This is a mathematical formulation of an intuitive physic property: The fluctuations of $Z_n(\omega,\beta)e^{-n\lambda(\beta)}$ increase as β grows.

4.6 Phase transitions.

Theorem 4.6.1. ***Critical Temperature*** *There exist $\beta_c = \beta_c(\mathbb{P}, d) \in [0,\infty]$ such that*

$$\begin{cases} p(\beta) = \lambda(\beta) & \text{if } 0 \leq \beta \leq \beta_c, \\ p(\beta) < \lambda(\beta) & \text{if } \beta > \beta_c. \end{cases}$$

Proof. We define $\beta_c := \inf\{\beta \geq 0 : p(\beta) < \lambda(\beta)\}$.

As we consider $\beta_c \in [0,\infty]$, by Theorem (4.5.2) β_c exit. Also, by proposition (4.5.1), we have that $\beta \mapsto \lambda(\beta) - p(\beta)$ is non-decreasing, and $\lambda(0) = p(0) = 0$, thus

$$\forall \quad \beta_c > \beta \geq 0 \quad \lambda(\beta) - p(\beta) = 0 \Rightarrow \lambda(\beta) = p(\beta)$$

(otherwise, for some $\beta \in [0, \beta_c)$, we have $\lambda(\beta) - p(\beta) > 0$. Then $\lambda(\beta) > p(\beta)$, which contradicts the definition of β_c).

Also, since $\lambda(\beta) - p(\beta)$ is non-decreasing, if $\beta > \beta_c$).

$$\lambda(\beta) - p(\beta) > \lambda(\beta_c) - p(\beta_c) > 0.$$

□

Definition 4.6.2. *We call the high temperature region (or small β region) to the set of $\beta's$ such that $p = \lambda$, and the low temperature region (or large β region) the set of $\beta's$ such that $p < \lambda$.*

Now we have some observations:

1. $\beta = 0$ is in the high temperature region.
2. Theorem (4.6.1) implies the absence of re-entrant phase transition in the phase diagram of the model. But we can also have just one of the two regimes of the diagram if $\beta_c = 0$ or $\beta_c = \infty$.
3. It is natural that the polymer measure has a different behavior in these two regions. In the high temperature region, the Gibbs measure is a small perturbation of the random walk. In the low temperature region, the polymer strongly feels the environment.

Now, recall that a function is analytic at point β_0 in the interior of its domain. If it is equal in a neighborhood of β_0 to a power series in $\beta - \beta_0$ with a positive radius of convergence. By definition of λ, it is analytic in $\mathbb{R}$. We also have that $p_n(\omega; \beta)$ is C^∞; since is the logarithm of a finite sum of smooth terms. However the limit $p(\cdot)$ may not be analytic.
If β_c is strictly positive and finite, the high and low temperature regions have non-empty interior, so the function $p(\cdot)$ is non-analytic in β_c. Because $p(\beta) = \lambda(\beta)$ when $\beta \in [0, \beta_c]$, and the analytic continuation on $\mathbb{R}$ is $\lambda(\beta)$, but by (4.6.1) we have that $p(\beta) < \lambda(\beta)$ for $\beta > \beta_c$, $p(\cdot)$ is non-analytic in β_c. For this reason β_c is called **critical** (in the mathematical point of view).

The following diagram, depicts phases according to the parameter value β, with $\beta \in [0, \infty)$. The case when β is negative could be obtained by changing the environment ω into $-\omega$.

β	0 β_c ∞	
Phase	Delocalized	Localized
Temperature	High temperature	Low temperature
Free energy	$p = \lambda$	$p < \lambda$

Figure 4.3: Behavior of the Free energy, Phase, and Temperature as a function of β.

Chapter 5

The localized phase

In this chapter, we will focus on the low temperature region. In this region its known by [5], [16], [3] and [1] that the phase is localized for low dimensions $(1+1)$ and $(1+2)$. Therefore we can find a corridor where the polymer is pinned. We can build a "favorite path" (that depends on the environment and the temperature) such that, the time that the polymer spends in the path is positive. Moreover, the limit approximates its maximum value as the temperature vanishes.

5.1 Path localization

Consider the Gibbs measure and a Gaussian environment.

$$\omega(t,x) \sim N(0,1),$$

for $\mathbf{y} = (y_t)_t : \mathbb{N} \to \mathbb{Z}^d$ and S a path, we define

$$N_n(S,\mathbf{y}) = \sum_{t=1}^{n} \mathbb{1}_{S_t = y_t}.$$

This last quantity represents the number of intersections between S and $\mathbf{y}$ up to time n. Now, define the parameter region

$$\mathcal{C} = \{\beta > 0 : \text{p is differentiable at } \beta, p'(\beta) < \lambda'(\beta)\}.$$

By convexity the set where p is non-differentiable is at most countable.

Theorem 5.1.1. *Assume that the environment is Gaussian. Then there exist* $\boldsymbol{y}^{(n)} : [0,n] \to \mathbb{Z}^d$*, such that*

$$\liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left[\frac{N_n(S,\boldsymbol{y}^{(n)})}{n}\right] \geq 1 - \frac{p'(\beta)}{\lambda'(\beta)} > 0, \tag{5.1}$$

for all $\beta \in \mathcal{C}$*. Moreover*

$$\lim_{\beta\to\infty}\liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left[\frac{N_n(S,\boldsymbol{y}^{(n)})}{n}\right] = 1 \tag{5.2}$$

Proof. Since the environment is Gaussian, we can use (A.4), to get

$$\begin{aligned}
\frac{d}{d\beta}\mathbb{E}\left[p_n(\omega,\beta)\right] &= \mathbb{E}E_n^{\beta,\omega}\left[\frac{H_n(S)}{n}\right] \\
&= \frac{1}{n}\sum_{t\leq n}\mathbb{E}E_n^{\beta,\omega}\left[\omega(t,x)\right] \\
&= \frac{1}{n}\sum_{t\leq n}\sum_{x}\mathbb{E}\ \left[\omega(t,x)P_n^{\beta,\omega}(S_t=x)\right] \\
&= \frac{\beta}{n}\sum_{t\leq n}\sum_{x}\mathbb{E}\left[P_n^{\beta,\omega}(S_t=x) - P_n^{\beta,\omega}(S_t=x)^2\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\beta}{n}\left[\sum_{t\leq n}\mathbb{E}\sum_{x}P_n^{\beta,\omega}(S_t=x) - \mathbb{E}\sum_{t\leq n}\sum_{x}\frac{e^{\beta H_n(S)+\beta H_n(\tilde S)}P(S=x)P(\tilde S=x)}{Z_n^2}\mathbb{1}_{S_t=\tilde S_t}\right] \\
&= \frac{\beta}{n}\left[\sum_{t\leq n}\mathbb{E}(1) - \mathbb{E}\sum_{x}\sum_{t\leq n}\mathbb{1}_{S_t=\tilde S_t}\frac{e^{\beta H_n(S)+\beta H_n(\tilde S)}P(S=x)P(\tilde S=x)}{Z_n^2}\right] \\
&= \frac{\beta}{n}\left(\sum_{t\leq n}1 - \mathbb{E}P_n^{\beta,\omega\otimes 2}\left[N_n(S,\tilde S)\right]\right) \\
&= \beta\left(1 - \mathbb{E}P_n^{\beta,\omega\otimes 2}\left[\frac{N_n(S,\tilde S)}{n}\right]\right).
\end{aligned}$$

So

$$1-\frac{1}{\beta}\frac{d}{d\beta}\mathbb{E}\left[p_n(\omega,\beta)\right]=\mathbb{E}P_n^{\beta,\omega^{\otimes 2}}\left[\frac{N_n(S,\tilde{S})}{n}\right],$$

where $P_n^{\beta,\omega^{\otimes 2}}(\cdot)$, denotes the product measure of two polymers measures, also the coupling $(S,\tilde{S})$ are two independent paths over Ω. When $\beta\in\mathcal{C}$, the limit is differentiable at β, and convexity implies the existence of the following limit:

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{E}P_n^{\beta,\omega^{\otimes 2}}\left[\frac{N_n(S,\tilde{S})}{n}\right]&=\liminf_{n\to\infty}\quad 1-\frac{1}{\beta}\frac{d}{d\beta}\mathbb{E}\left[p_n(\omega,\beta)\right]\\&=1-\frac{1}{\beta}\frac{d}{d\beta}\lim_{n\to\infty}\mathbb{E}\left[p_n(\omega,\beta)\right]\\&=1-\frac{p'(\beta)}{\beta}.\end{aligned}\tag{5.3}$$

By the Gaussian environment hypothesis we can compute directly

$$\lambda(\beta)=\ln\mathbb{E}\left[e^{\beta\omega(t,x)}\right]=\ln e^{\beta^2/2}=\frac{\beta^2}{2}.$$

Then $\beta=\lambda'(\beta)$. Replacing this in (5.3) we get

$$\liminf_{n\to\infty}\mathbb{E}P_n^{\beta,\omega^{\otimes 2}}\left[\frac{N_n(S,\tilde{S})}{n}\right]=1-\frac{p'(\beta)}{\lambda'(\beta)}.$$

Now, for fixed n,β,ω we define the the "favorite path" by

$$\mathbf{y}^{(n)}=arg\max_{x\in\mathbb{Z}P_n^{\beta,\omega}}(S_t=x)\qquad\qquad t=1,2,...,n.\tag{5.4}$$

Also, by definition,

$$\begin{aligned}P_n^{\beta,\omega^{\otimes 2}}(S_t=\tilde{S}_t)&=\sum_x\mathbb{1}_{S_t=\tilde{S}_t}\frac{e^{\beta H_n(S)+\beta H_n(\tilde{S})}P(S=x)P(\tilde{S}=x)}{Z_n^2}\\&\leq P_n^{\beta,\omega}(S_t=\mathbf{y}^{(n)}),\end{aligned}$$

where the last inequality is due to the fact that the indicator function reduces to a single path, and also as $P_n^{\beta,\omega^{\otimes 2}}$ is the product measure of $P_n^{\beta,\omega}$, its value is less than or equal to the value of one of its parts; by definition it is less than or equal to $P_n^{\beta,\omega}\left(S_t = \mathbf{y}^{(n)}\right)$. So, we obtain

$$\mathbb{E}E_n^{\beta,\omega^{\otimes 2}}\left[\frac{N_n(S,\tilde{S})}{n}\right] \leq \mathbb{E}E_n^{\beta,\omega}\left[\frac{N_n(S,\mathbf{y}^{(n)})}{n}\right].$$

Thus

$$\liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left[\frac{N_n(S,\mathbf{y}^{(n)})}{n}\right] \geq 1 - \frac{p'(\beta)}{\lambda'(\beta)} > 0,$$

which implies (5.1) (the last inequality follows from $\beta \in \mathcal{C}$ which implies $p'(\beta) < \lambda'(\beta)$ so $1 - p'(\beta)/\lambda'(\beta) > 0$).

Recalling the improved annealed bound (4.21), we see that β grows linearly. Which implies, by convexity, that $p'(\beta) \leq C < \infty$ for some C. Then

$$1 - \frac{p'(\beta)}{\lambda'(\beta)} \geq 1 - \frac{C}{\beta},$$

and taking the limit $\beta \to \infty$ we get

$$\lim_{\beta\to\infty} \quad 1 - \frac{p'(\beta)}{\lambda'(\beta)} \geq 1.$$

Finally, since we have that $N_n(S,\mathbf{y}^{(n)}) \leq n$, we obtain

$$1 \geq \lim_{\beta\to\infty} \liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left[\frac{N_n(S,\mathbf{y}^{(n)})}{n}\right] \geq 1.$$

So we conclude (5.2)

$$\lim_{\beta\to\infty} \liminf_{n\to\infty} \mathbb{E}E_n^{\beta,\omega}\left[\frac{N_n(S,\mathbf{y}^{(n)})}{n}\right] = 1.$$

□

5.2 Simulations of the localized phase

The following results come from the simulation of two polymers of length n. The first one has a standard normal environment, and the second one has an exponential environment with a parameter $\lambda = 10$. There is a plot for each value of the temperature in $\{10, 50, 100, 1000\}$. The simulations are in dimension $D = 1 + 1$. Therefore, by (5.1) the polymer is localized.

The first plots in each section show the localization of the favorite path for the two environments. The subsequent plots show a map of the order of the maximum path to each point.

5.2.1 Standard normal environment.

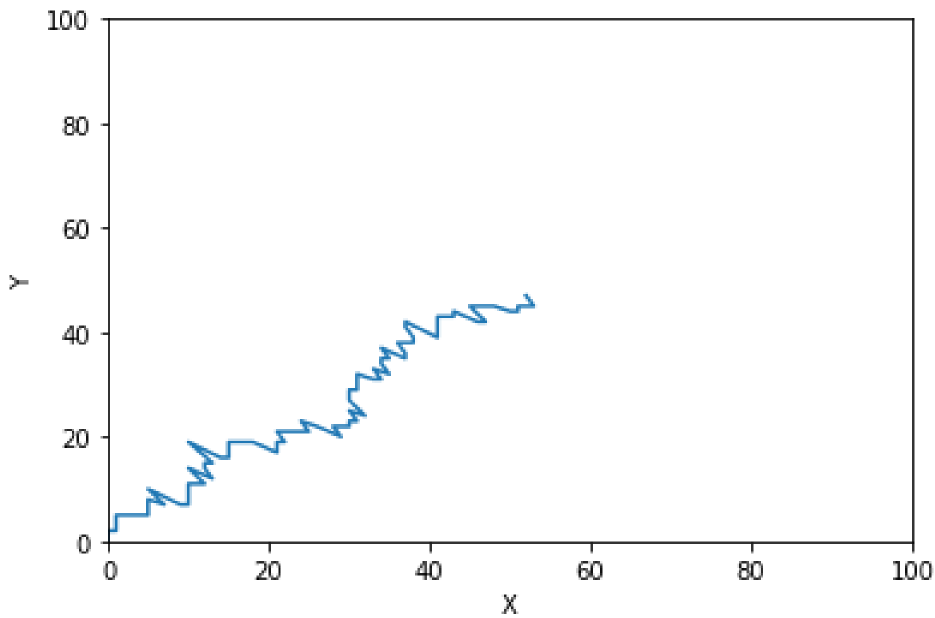

Figure 5.1: Localization of the favorite path for normal environment. The jumps below to suboptimal paths at low temperatures. Jumps tend to disappear as temperature grows.

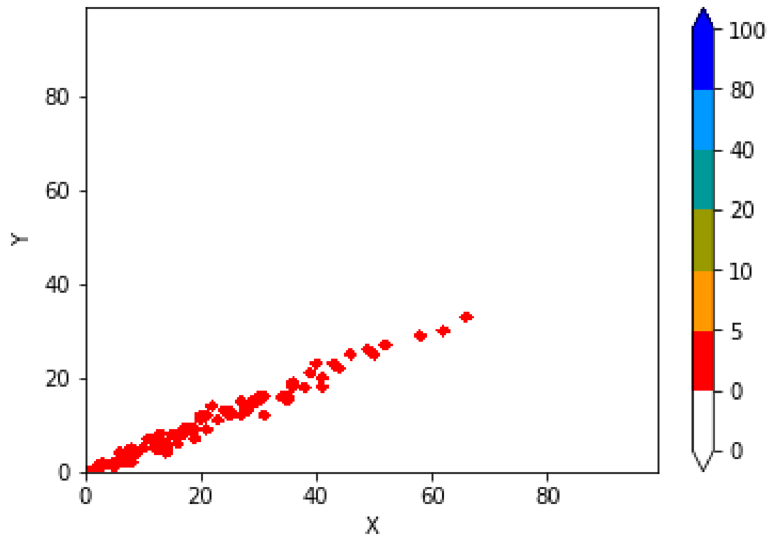

Figure 5.2: Mass of the order of the maximum at each point with temperature $T = 10$. (The mass decreases from blue to withe).

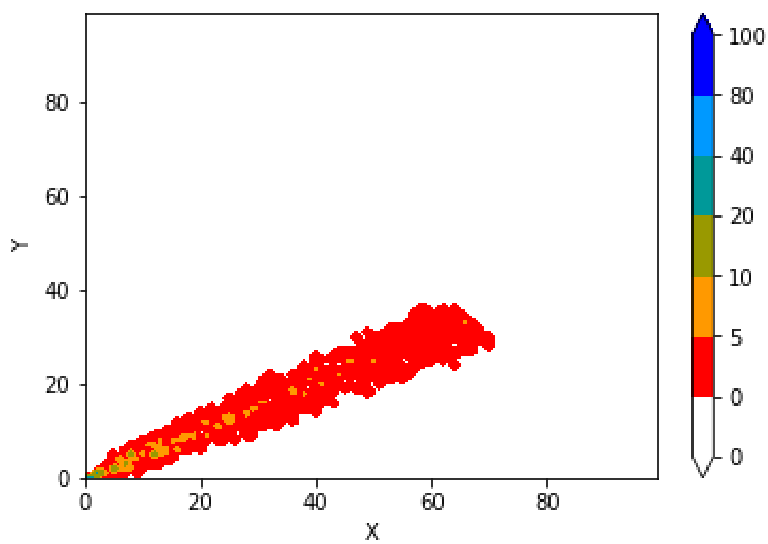

Figure 5.3: Mass of the order of the maximum at each point with temperature $T = 50$.

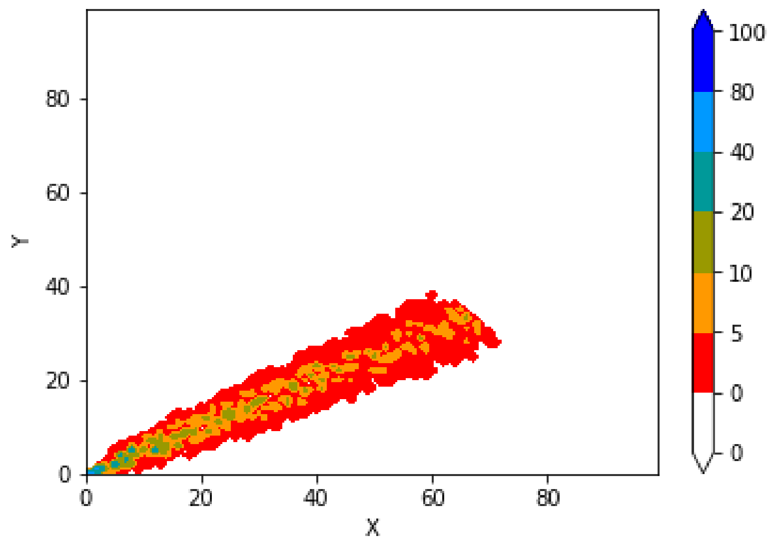

Figure 5.4: Mass of the order of the maximum at each point with temperature $T = 100$.

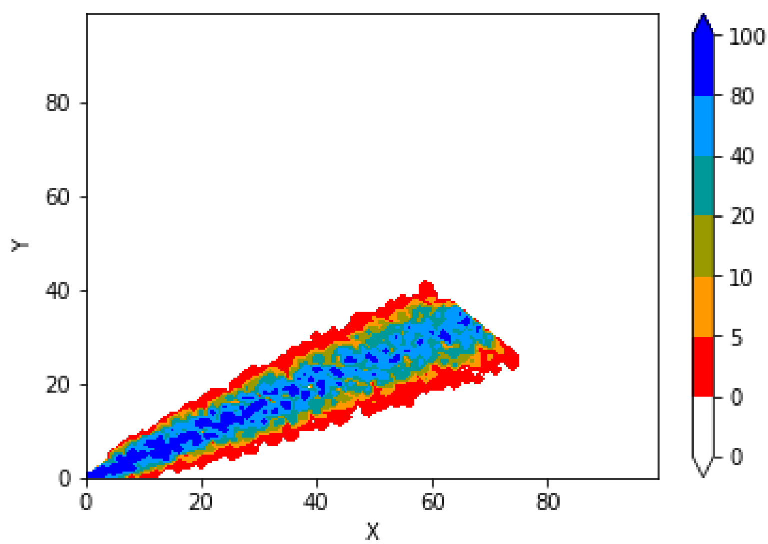

Figure 5.5: Mass of the order of the maximum at each point with temperature $T = 1000$.

- In the case $T = 10, 50$ in the normal environment, the localization is strong, most of the time, the order of mass at each point is lower than 10. Only at the begin of the polymer, there are few locations where the mass is not too small, and there is a sharp corridor where the polymer tends to lie.

- When $T = 100, 1000$ the plots show that the mass decay slowly around the favorite path Fig.(5.1), resulting in a broad corridor.

5.2.2 Exponential environment.

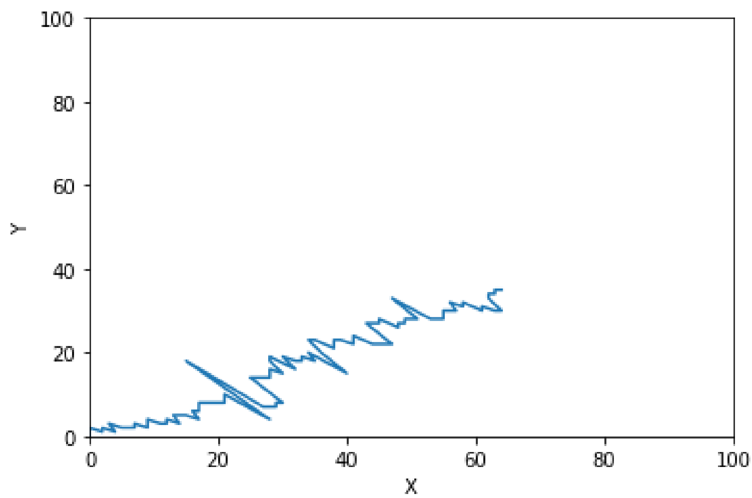

Figure 5.6: Localization of the favorite path for exponential environment with $\lambda = 10$.

- For $T = 1$, points in the plot are around the favorite path, and their mass is lower than 5. Also, jumps are due to the inner jumps in the favorite path.

- For $T = 10$, unlike the normal environment, in the exponential case, we have a corridor around the favorite path, but just in points near to the favorite path, the order of their mass is significant. Similarly, the mass decreases quickly as the points separate from the favorite path.

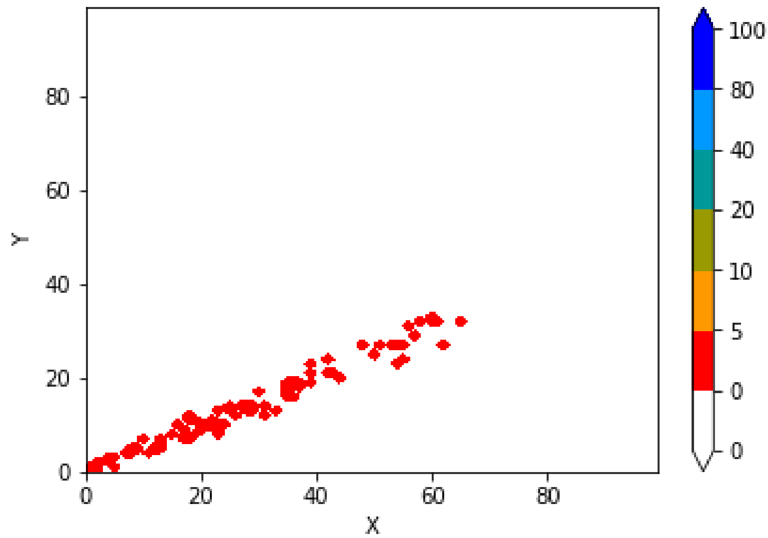

Figure 5.7: Mass of the order of the maximum at each point with temperature $T = 1$. (The mass decreases from blue to white).

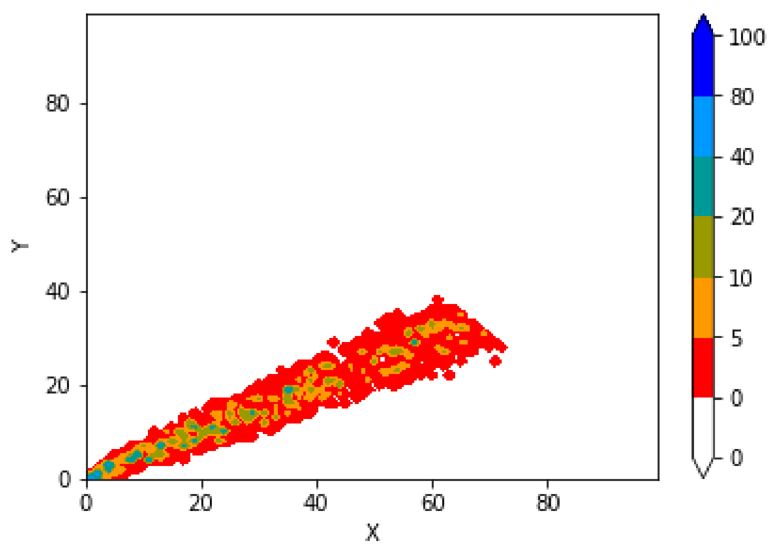

Figure 5.8: Mass of the order of the maximum at each point with temperature $T = 10$. (The mass decreases from blue to white).

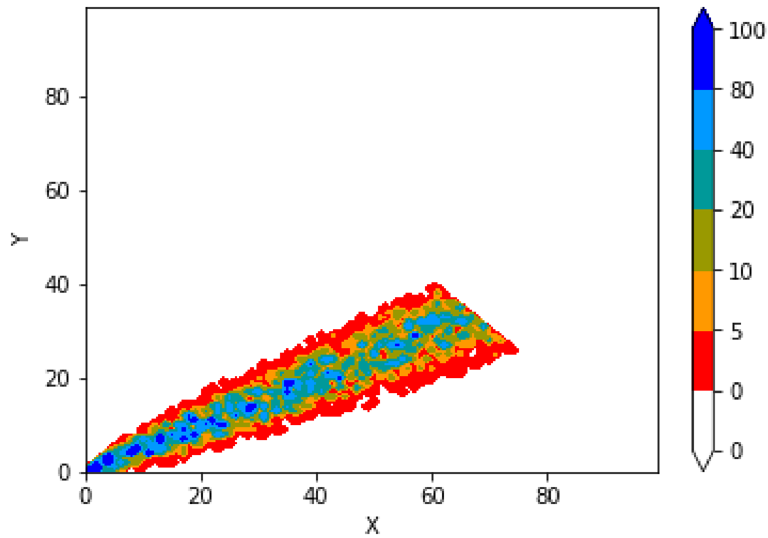

Figure 5.9: Mass of the order of the maximum at each point with temperature $T = 50$. (The mass decreases from blue to white).

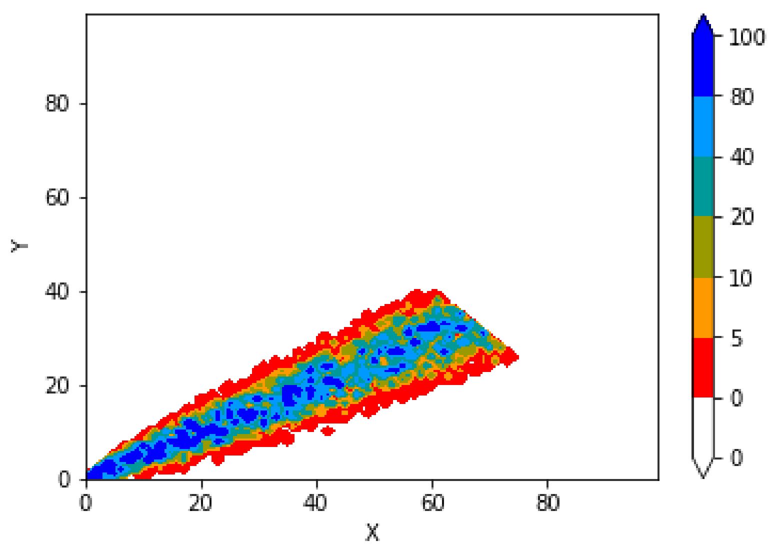

Figure 5.10: Mass of the order of the maximum at each point with temperature $T = 100$. (The mass decreases from blue to white).

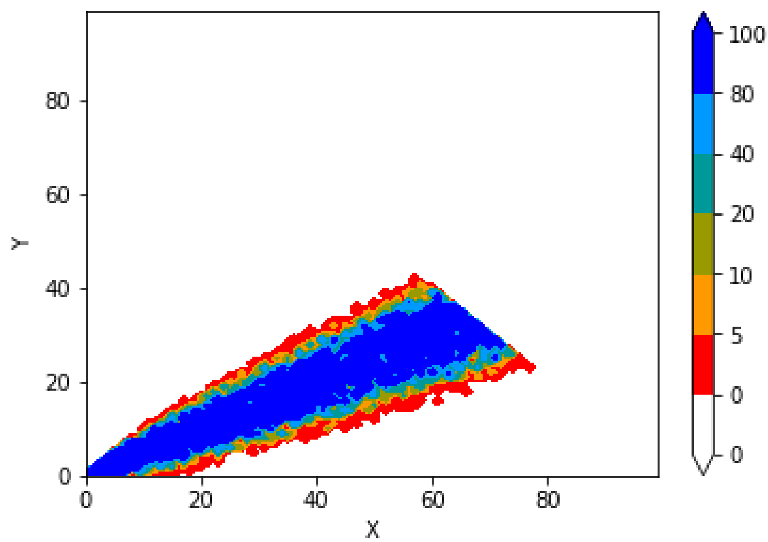

Figure 5.11: Mass of the order of the maximum at each point with temperature $T = 1000$. (The mass decreases from blue to white).

- For $T = 50, 100$ both corridors have the same shape, there are two differences between them, the first one is that the order of the points with $t = 100$ is, obviously, more prominent than the order of the points with $t = 50$. The second difference is that the order of the points decreases slowly when $T = 100$.

- for $T = 1000$, the plot shows a wide corridor, with the order of the points bigger than 100. In this case, the maximum path is no longer relevant. Also, the order decreases rapidly at the edge of the corridor.

5.2.3 Pareto environment.

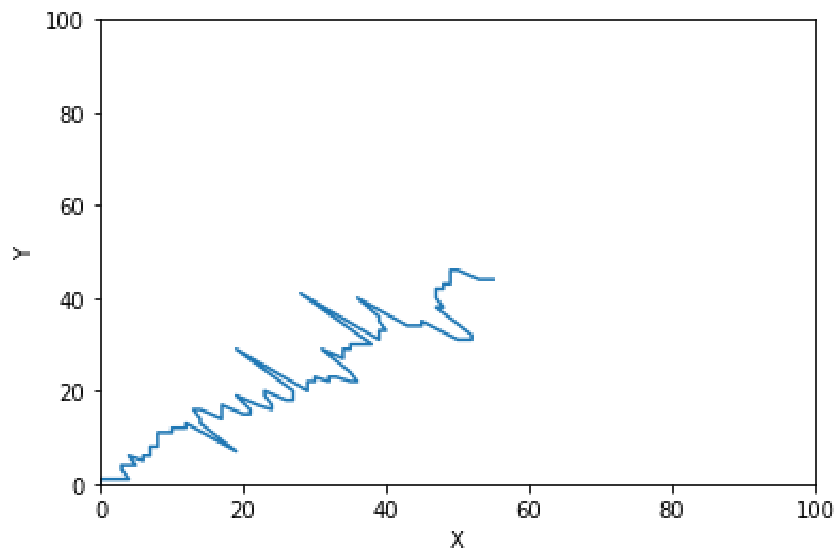

Figure 5.12: Localization of the favorite path for Pareto environment with $\alpha = 10$, and scale $m = 200$.

- For $T = 1$, the plot already shows a smooth corridor, and the point mass decay slowly as they get away from the favorite path.
- For $T = 10$, in this case, there is a smooth corridor where the point mass decay slowly near the favorite path, but also decay quickly in the edge of the corridor.
- For $T = 100$, unlike the previous case, we have a thick central corridor, where almost all points have the same mass (over 100), but at the edge, there are points which masses decay very quickly.
- for $T = 1000$, starting from the previous case, we have almost the same shape; the only difference is that the edge where the mass decay is thinner than the showed in the case $t = 100$.

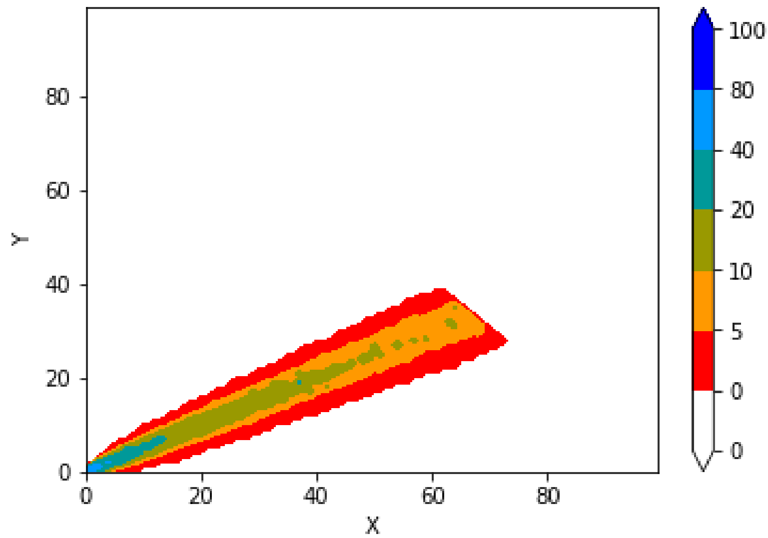

Figure 5.13: Mass of the order of the maximum at each point with temperature $T = 1$. (The mass decreases from blue to white).

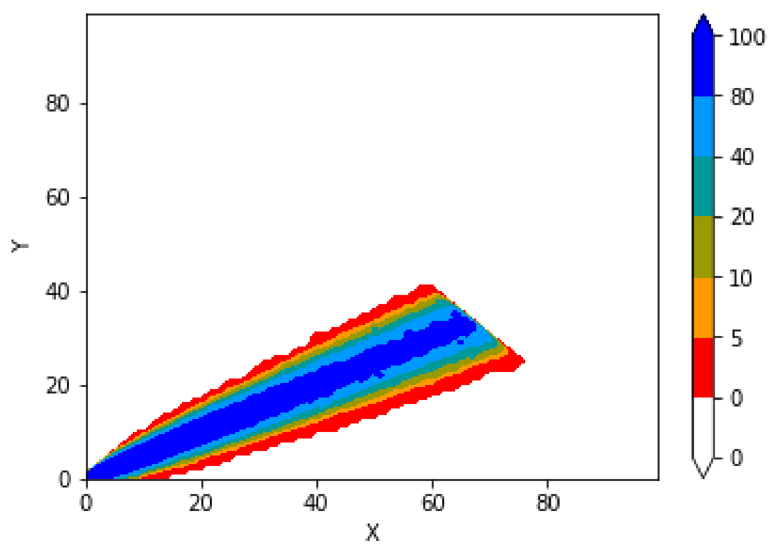

Figure 5.14: Mass of the order of the maximum at each point with temperature $T = 10$. (The mass decreases from blue to white).

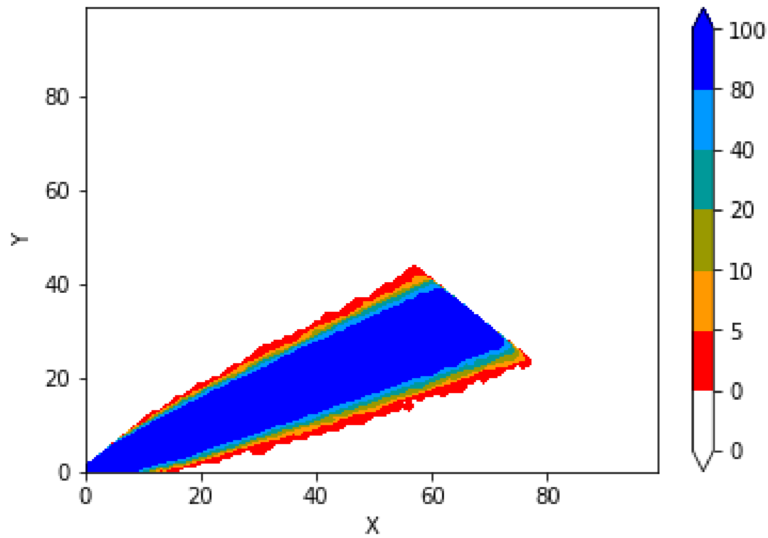

Figure 5.15: Mass of the order of the maximum at each point with temperature $T = 100$. (The mass decreases from blue to white).

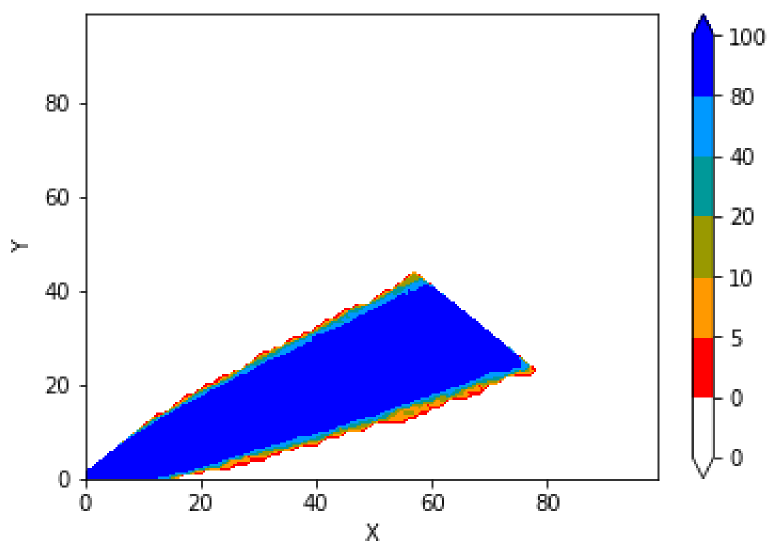

Figure 5.16: Mass of the order of the maximum at each point with temperature $T = 1000$. (The mass decreases from blue to white).

Chapter 6

Conclusions.

In the theory of directed polymers in random environments, much information is encoded in the polymer measure, the partition function, and the free energy. The polymer measure mediates the interaction between the random walk and the environment, when the temperature tends to zero, the polymer measure concentrates on the favorite path. On the other hand, when the temperature tends to infinite, the environment no longer affects the random walk. In general, the partition function has a superadditive (subadditive) properties if the Hamiltonian has an attractive (repulsive) self-interaction. In the case of random environments, the partition function defined from the polymer measure satisfies a Markov property, and the transition probabilities can be calculated explicitly. The principal quantity related to the behavior of the polymer is in the free energy. In this thesis, it was shown that the free energy converges a.s. as the length of the polymer tends to infinite. Furthermore, this limit is bounded by the logarithm of the moment generating function of $\omega(n, x)$. After that, the bound was improved, and it was shown that the free energy is a convex function on β, and it has a monotonicity property. Due to the in-depth analysis of the free energy, it was found a critical temperature, where the limit of the free energy is no longer analytic. This critical temperature indicates a phase transition. Then, it was studied the path localization, where it was shown (for the Gaussian environment) that the polymer tends to live near to a favorite path when the temperature tends to zero. Finally, as the phase is localized for low dimensions ($d = 1, 2$), the simulations made shows the corridor where the polymer tends to live, as predicted by theory. For low temperatures, the corridor is just a low variation of the favorite path, and as the temperature starts to grow, a corridor appears. The mass of the points decreases as they move away from the favorite path. The thickness and the rapidness of the mass decrease depend on the distribution followed by the environment.

Appendix A

Complementary theory

Let us state some results that have been used in this thesis.

A.1 Kingman's subadditive ergodic theorem.

Theorem A.1.1. ***Kingman's subadditive ergodic theorem.*** *Suppose $(X_{m,n})$ a sequence of r.v. such that, for all $0 \leq m \leq n$*

1) $X_{0,n} \leq X_{0,m} + X_{m,n}$,

2) The joint distribution of $\{X_{m+1,m+k+1} : k \geq 1\}$ is the same as $\{X_{m,m+k}, k \geq 1\}$ for each positive m,

3) for each $k \geq 1$, the set $\{X_{nk,(n+1)k}, n \geq 1\}$ is a ergodic process.

Then:

$$X := \lim_{n\to\infty} \frac{X_{0,n}}{n} = \lim_{n\to\infty} \frac{\mathbb{E}(X_{0,n})}{n} = \inf_n \frac{\mathbb{E}(X_{0,n})}{n} \tag{A.1}$$

and the limit X is a constant random variable a.s.

This result is stated by Thomas Liggett in [17].

A.2 Concentration inequality

Theorem A.2.1. ***General concentration inequality for the free energy:*** *Assume that the environment has all exponential moments (3.1). Then,*

$$\mathbb{E}\left[|p_n - \mathbb{E}\left(p_n\right)| \geq r\right] \leq \begin{cases} 2\exp\left\{-nCr^2\right\} & \textit{if } 0 \leq r \leq 1, \\ 2\exp\left\{-nCr\right\} & \textit{if } r \geq 1. \end{cases} \tag{A.2}$$

for some constant $C > 0$.

In [18] Lui and Watbled make a rigorous proof of this theorem.

A.3 Integration by parts.

Lemma A.3.1. ***Integration by Part Formula*** *If X is centered normal random variable, and f is a smooth function which does not grow too fast at infinity, i.e.,*

$$\lim_{|x|\to\infty} f(x)\exp -x^2/(2E(X^2)) = 0,$$

then

$$E(Xf(X)) = E(X^2)E(f'(X)) \tag{A.3}$$

A proof of this lemma is shown in [22] Chapter 5. Using that we can prove the proposition below.

Proposition A.3.2. *If $\omega(t,x)$ is centered normal, using integration by part formula (A.3), we get*

$$\mathbb{E}\left[\omega(t,x)P_n^{\beta,\omega}(S_t = x)\right] = \beta\mathbb{E}\left[P_n^{\beta\omega}(S_t = x) - P_n^{\beta\omega}(S_t = x)^2\right] \tag{A.4}$$

Proof.

$$\begin{aligned} \mathbb{E}\left[\omega(t,x)P_n^{\beta,\omega}(S_t = x)\right] &= \mathbb{E}\left[\omega^2(t,x)\right]\mathbb{E}\left[\frac{\partial}{\partial\omega(t,x)}P_n^{\beta\omega}(S_t = x)\right] \\ &= 1\cdot\mathbb{E}\left[\frac{\partial}{\partial\omega(t,x)}\left(\frac{e^{\beta\sum_{i\leq n}\omega(i,x_i)}P(S_t = x)}{Z_n^{\beta\omega}}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{\beta e^{\beta\sum_{i\le n}\omega(i,x_i)}P(S_t=x)}{Z_n^{\beta\omega}} - \frac{e^{\beta\sum_{i\le n}\omega(i,x_i)}P(S_t=x)\sum_{\hat{\mathbf{x}}\in\mathbb{X}}e^{\beta\sum H_n(\hat{\mathbf{x}})}\beta\mathbb{I}_{\omega(t,\hat{x}_t)=\omega(t,x)}P(\hat{\mathbf{x}})}{\left(Z_n^{\beta\omega}\right)^2}\right]\\
&= \beta\mathbb{E}\left[\frac{e^{\beta\sum_{i\le n}\omega(i,x_i)}P(S_t=x)}{Z_n^{\beta\omega}} - \frac{e^{\beta\sum_{i\le n}\omega(i,x_i)}P(S_t=x)}{Z_n^{\beta\omega}}\frac{\sum_{\hat{\mathbf{x}}\in\mathbb{X}}e^{\beta H_n(\hat{\mathbf{x}})}\mathbb{I}_{\omega(t,\hat{x}_t)=\omega(t,x)}P(\hat{\mathbf{x}})}{Z_n^{\beta\omega}}\right]\\
&= \beta\mathbb{E}\left[P_n^{\beta\omega}(S_t=x) - P_n^{\beta\omega}(S_t=x)E_n^{\beta\omega}\left(\mathbb{I}_{\omega(t,\hat{x}_t)=\omega(t,x)}\right)\right]\\
&= \beta\mathbb{E}\left[P_n^{\beta\omega}(S_t=x) - P_n^{\beta\omega}(S_t=x)E_n^{\beta\omega}\left(\mathbb{I}_{S_t=x}\right)\right]\\
&= \beta\mathbb{E}\left[P_n^{\beta\omega}(S_t=x) - P_n^{\beta\omega}(S_t=x)P_n^{\beta\omega}\left(S_t=x\right)\right]\\
&= \beta\mathbb{E}\left[P_n^{\beta\omega}(S_t=x) - P_n^{\beta\omega}(S_t=x)^2\right]
\end{aligned}
$$

□

A.4 FKG-Harris inequality

Recall that a function $f:\mathbb{R}^k\to\mathbb{R}$ is increasing if $f(x)\le f(y)$ whenever it $x_i\le y_i\,\forall\, i\le k$. This is equivalent to f being coordinatewise increasing, i.e., $f(x)\le f(y)$ for all y such that $y_i=x_i$ for all i,j with $i\ne j$ and $x_j<y_j$.

Definition A.4.1. *A family $X=(X_i;1\le i\le k)$ of real random variables defined on the same probability space are called **positively associated** if for any $f,g:\mathbb{R}^k\to\mathbb{R}$ bounded increasing function,*

$$\mathbb{E}\left[f(X)g(X)\right]\ge\mathbb{E}[f(X)]\mathbb{E}[g(X)]. \tag{A.5}$$

The inequality (A.5) is called the Fortuyn-Kasteleyn-Ginibre (FKG) inequality. The inequality simply means that increasing functions are positively correlated.

Proposition A.4.2. ***FKG-Harris Inequality*** *A family of independent, real random variables is positively associated.*

Appendix B

Complete proves.

B.1

Proposition B.1.1. *Step by step proof of (4.7), Under the polymer measure $P_n^{\beta,\omega}$, the path S is a Markov chain, with transition probabilities*

$$P_n^{\beta,\omega}(S_{i+1} = y|S_i = x) = \frac{e^{\beta w(i+1,y)} Z_{n-i-1} \circ \theta_{i+1,y}}{Z_{n-i} \circ \theta_{i,x}} P(S_i = y|S_0 = x) \quad \text{(B.1)}$$

for $0 \leq i < n$, and

$$P_n^{\beta,\omega}(S_{i+1} = y|S_i = x) = P(S_{i+1} = y|S_i = x) \quad \text{for } i \geq n.$$

Proof. For a given path $(x_0 = 0, x_1, ..., x_n)$, the following product is telescopic,

$$\prod_{i=0}^{n-1} \frac{e^{\beta w(i+1,x_{i+1})} Z_{n-i-1} \circ \theta_{i+1,x_{i+1}}}{Z_{n-i} \circ \theta_{i,x_i}} P(S_1 = x_{i+1}|S_0 = x_i)$$
$$= \frac{e^{\beta w(1,x_1)} Z_{n-1} \circ \theta_{1,x_1}}{Z_n \circ \theta_{0,x_0}} P(S_1 = x_1|S_0 = x_0) \times \frac{e^{\beta w(2,x_2)} Z_{n-2} \circ \theta_{2,x_2}}{Z_{n-1} \circ \theta_{1,x_1}} P(S_1 = x_2|S_0 = x_1)$$
$$\times \cdots \times \frac{e^{\beta w(n-1,x_{n-1})} Z_1 \circ \theta_{n-1,x_{n-1}}}{Z_2 \circ \theta_{n-2,x_{n-2}}} P(S_1 = x_{n-1}|S_0 = x_{n-2})$$
$$\times \frac{e^{\beta w(n,x_n)} Z_0 \circ \theta_{n,x_n}}{Z_1 \circ \theta_{n-1,x_{n-1}}} P(S_1 = x_n|S_0 = x_{n-1}).$$

First, let us see what happens with the terms of the form $\frac{e^{\beta w(i+1,x_{i+1})}Z_{n-i-1}\circ\theta_{i+1,x_{i+1}}}{Z_{n-i}\circ\theta_{i,x_i}}$:

$$\frac{e^{\beta w(1,x_1)}Z_{n-1}\circ\theta_{1,x_1}}{Z_n\circ\theta_{0,x_0}}\times\frac{e^{\beta w(2,x_2)}Z_{n-2}\circ\theta_{2,x_2}}{Z_{n-1}\circ\theta_{1,x_1}}\times\cdots\times\frac{e^{\beta w(n-1,x_{n-1})}Z_1\circ\theta_{n-1,x_{n-1}}}{Z_2\circ\theta_{n-2,x_{n-2}}}$$
$$\times\frac{e^{\beta w(n,x_n)}Z_0\circ\theta_{n,x_n}}{Z_1\circ\theta_{n-1,x_{n-1}}}$$
$$=\frac{e^{\beta w(1,x_1)}\cancel{Z_{n-1}\circ\theta_{1,x_1}}}{Z_n\circ\theta_{0,x_0}}\times\frac{e^{\beta w(2,x_2)}\cancel{Z_{n-2}\circ\theta_{2,x_2}}}{\cancel{Z_{n-1}\circ\theta_{1,x_1}}}\times\cdots\times\frac{e^{\beta w(n-1,x_{n-1})}\cancel{Z_1\circ\theta_{n-1,x_{n-1}}}}{\cancel{Z_2\circ\theta_{n-2,x_{n-2}}}}$$
$$\times\frac{e^{\beta w(n,x_n)}Z_0\circ\theta_{n,x_n}}{\cancel{Z_1\circ\theta_{n-1,x_{n-1}}}}.$$

We also have that $Z_0=1$ and $Z_n\circ\theta_{0,x_0}=Z_n$, because we already set all the paths to start in x_0. Then

$$\prod_{i=0}^{n-1}\frac{e^{\beta w(i+1,x_{i+1})}Z_{n-i-1}\circ\theta_{i+1,x_{i+1}}}{Z_{n-i}\circ\theta_{i,x_i}}=\frac{1}{z_n}e^{\beta w(1,x_1)}e^{\beta w(2,x_2)}\cdots e^{\beta w(n-1,x_{n-1})}e^{\beta w(n,x_n)}$$
$$=\frac{1}{z_n}e^{\beta\sum_{i=n}^{n}\omega(i,x_i)}=\frac{1}{z_n}e^{\beta H_n(\mathbf{x})}.$$

Now, we analyze the product of $P(S_1=x_{i+1}|S_0=x_0)$. Recall that $P(S_0=x_0)=1$. Now we can compute

$$\begin{aligned}P(S_{[1,n]}=x_{[1,n]}) =&P(S_n=x_n,S_{n-1}=x_{n-1},\cdots,S_2=x_2,S_1=x_1)\\
=&P(S_n=x_n,S_{n-1}=x_{n-1},\cdots,S_2=x_2,S_1=x_1)\\
=&P(S_n=x_n,S_{n-1}=x_{n-1},\cdots,S_2=x_2,S_1=x_1,S_0=x_0)\\
=&P(S_n-S_{n-1}=x_n-x_{n-1},S_{n-1}-S_{n-2}=x_{n-1}-x_{n-2},\cdots\\
&,S_2-S_1=x_2-x_1,S_1-S_0=x_1-x_0,S_0=x_0)\\
=&P(S_n-S_{n-1}=x_n-x_{n-1})P(S_{n-1}-S_{n-2}=x_{n-1}-x_{n-2})\cdots\\
&P(S_2-S_1=x_2-x_1)P(S_1-S_0=x_1-x_0)P(S_0=x_0)\\
=&P(S_1-S_0=x_n-x_{n-1})P(S_1-S_0=x_{n-1}-x_{n-2})\cdots\\
&P(S_1-S_0=x_2-x_1)P(S_1-S_0=x_1-x_0)\cdot 1\\
=&P(S_1=x_n|S_0=x_{n-1})P(S_1=x_{n-1}|S_0=x_{n-2})\cdots\\
&\cdots P(S_1=x_2|S_0=x_1)P(S_1=x_1|S_0=x_0),\end{aligned}$$

so

$$P(S_{[1,n]} = x_{[1,n]}) = P(S_1 = x_n | S_0 = x_{n-1}) \cdots P(S_1 = x_1 | S_0 = x_0).$$

Since

$$\frac{1}{z_n} e^{\beta H_n(\mathtt{x})} P(S_{[1,n]} = x_{[1,n]}) = P_n^{\beta,\omega}(S_{[1,n]} = x_{[1,n]}),$$

We conclude

$$\prod_{i=0}^{n-1} \frac{e^{\beta w(i+1,x_{i+1})} Z_{n-i-1} \circ \theta_{i+1,x_{i+1}}}{Z_{n-i} \circ \theta_{i,x_i}} P(S_1 = x_{i+1} | S_0 = x_i) = P_n^{\beta,\omega}(S_{[1,n]} = x_{[1,n]}). \tag{B.2}$$

Let us define

$$\begin{aligned} A :=& \prod_{k=0}^{i-2} e^{\beta\omega(k+1,x_{k+1})} \frac{Z_{n-k-1} \circ \theta_{k+1,x_{k+1}}}{Z_{n-k} \circ \theta_{k,x_k}} P(S_1 = x_{k+1} | S_0 = x_0), \\ B :=& e^{\beta\omega(i,x)} \frac{Z_{n-i} \circ \theta_{i,x}}{Z_{n-i-1} \circ \theta_{i-1,x_{i-1}}} P(S_1 = x | S_0 = x_{i-1}), \\ C :=& \prod_{k=i+1}^{n} e^{\beta\omega(k+1,x_{k+1})} \frac{Z_{n-k-1} \circ \theta_{k+1,x_{k+1}}}{Z_{n-k} \circ \theta_{k,x_k}} P(S_1 = x_{k+1} | S_0 = x_0). \end{aligned}$$

Now we will calculate

$$P_n^{\beta,\omega}(S_{i+1} = y | S_{[1,i-1]} = x_{[1,i-1]}, S_i = x, S_{[i+2,n]} = x_{[i+2,n]}). \tag{B.3}$$

By expressing (4.9) in its product form we have

$$P_n^{\beta,\omega}(S_{i+1} = y | S_{[1,i-1]} = x_{[1,i-1]}, S_i = x, S_{[i+2,n]} = x_{[i+2,n]})$$

$$= \frac{P_n^{\beta,\omega}(S_{i+1} = y, S_{[1,i-1]} = x_{[1,i-1]}, S_i = x, S_{[i+2,n]} = x_{[i+2,n]})}{P_n^{\beta,\omega}(S_{[1,i-1]} = x_{[1,i-1]}, S_i = x, S_{[i+2,n]} = x_{[i+2,n]})}$$

$$= \frac{A \cdot B \cdot e^{\beta\omega(i+1,y)} \frac{Z_{n-i-1} \circ \theta_{i+1,y}}{Z_{n-i} \circ \theta_{i,x}} P(S_1 = y | S_0 = x) \cdot C}{A \cdot B \cdot C}$$

$$= e^{\beta\omega(i+1,y)} \frac{Z_{n-i-1} \circ \theta_{i+1,y}}{Z_{n-i} \circ \theta_{i,x}} P(S_1 = y | S_0 = x).$$

□

Proposition B.1.2. *Step by step proof of (4.10). The following equality holds*

$$P_n^{\beta,\omega}(S_{i+1} = y | S_i = x) = P_{n-i}^{\beta,\theta_{i,x}\omega}(S_1 = y - x). \tag{B.4}$$

Proof. By(4.7)

$$P_n^{\beta,\omega}(S_{i+1} = y | S_i = x) = \frac{e^{\beta\omega(i+1,y)} z_{n-i-1} \circ \theta_{i+1,y}}{z_{n-i} \circ \theta_{i,x}} P(S_{i+1} = y | S_i = x),$$

recall that $e^{\beta\omega(t+i,y)}$ is measurable respect to $S_{i+1} = y$:

$$\begin{aligned} P_n^{\beta,\omega}(S_{i+1} = y | S_i = x) &= \frac{e^{\beta\omega(i+1,y)} E\left(e^{\beta \sum_{t=1}^{n-i-1} \omega(t+i+1, S_{t+i+1})} | S_{i+1} = y \right)}{z_{n-i} \circ \theta_{i,x}} E(\mathbb{I}_{S_{i+1}=y} | S_i = x) \\ &= \frac{E\left(e^{\beta\omega(i+1,y)} e^{\beta \sum_{t=1}^{n-i-1} \omega(t+i+1, S_{t+i+1})} | S_{i+1} = y \right)}{z_{n-i} \circ \theta_{i,x}} E(\mathbb{I}_{S_{i+1}=y} | S_i = x), \end{aligned}$$

we can re-write the previous equation as

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)=\frac{E\left(e^{\beta\omega(i+1,y)}e^{\beta\sum\limits_{t=1}^{n-i-1}\omega(t+i+1,S_t+y)}\right)}{z_{n-i}\circ\theta_{i,x}}E(\mathbb{I}_{S_{i+1}=y}|S_i=x),$$

now, by independence respect to $S_i=x$, we obtain

$$P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)=\frac{E\left(e^{\beta\omega(i+1,y)}e^{\beta\sum\limits_{t=1}^{n-i-1}\omega(t+i+1,S_t+y)}\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}},$$

finally

$$\begin{aligned}
P_n^{\beta,\omega}(S_{i+1}=y|S_i=x)&=\frac{E\left(e^{\beta\omega(i+1,y)}e^{\beta\sum\limits_{t=1}^{n-i-1}\omega(t+i+1,S_t+y)}\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}}\\
&=\frac{E\left(e^{\beta\omega(i+1,y)}e^{\beta\sum\limits_{t=1}^{n-i-1}\omega(t+i+1,S_t+y)}\mathbb{I}_{S_{i+1}=y}\mathbb{I}_{S_i=x}\right)}{z_{n-i}\circ\theta_{i,x}P(S_i=x)}\\
&=\frac{E\left(e^{\beta\sum\limits_{t=0}^{n-i-1}\omega(t+i+1,S_t+y)}\mathbb{I}_{S_{i+1}=y}\mathbb{I}_{S_i=x}\right)}{z_{n-i}\circ\theta_{i,x}P(S_i=x)}\\
&=\frac{E\left(e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}\mathbb{I}_{S_{i+1}=y}\mathbb{I}_{S_i=x}\right)}{z_{n-i}\circ\theta_{i,x}P(S_i=x)}\\
&=\frac{E\left(e^{\beta\sum\limits_{t=1}^{n-i}\omega(t+i,S_t+x)}\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}P(S_i=x)}.
\end{aligned}$$

Thus

$$
\begin{aligned}
P_n^{\beta,\omega}(S_{i+1}=y|S_i=x) &= \frac{e^{\beta\sum_{t=1}^{n-i}\omega(t+i,S_t+x)}E\left(\mathbb{I}_{S_{i+1}=y}|S_i=x\right)}{z_{n-i}\circ\theta_{i,x}P(S_i=x)} \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\omega(t+i,S_t+x)}}{z_{n-i}\circ\theta_{i,x}}P\left(S_{i+1}=y|S_i=x\right) \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\omega(t+i,S_t+x)}}{z_{n-i}\circ\theta_{i,x}}\frac{P\left(S_{i+1}=y,S_i=x\right)}{P(S_i=x)} \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\omega(t+i,S_t+x)}}{z_{n-i}\circ\theta_{i,x}}\frac{P\left(S_{i+1}=y,S_i=x\right)}{P(S_i=x)} \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\omega(t+i,S_t+x)}}{z_{n-i}\circ\theta_{i,x}}\frac{P\left(S_{i+1}-S_i=y-x,S_i=x\right)}{P(S_i=x)} \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\omega(t+i,S_t+x)}}{z_{n-i}\circ\theta_{i,x}}\frac{P\left(S_{i+1}-S_i=y-x\right)P(S_i=x}{P(S_i=x)} \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\theta_{i,x}\omega(t,S_t)}}{z_{n-i}\circ\theta_{i,x}}P\left(S_1-S_0=y-x\right) \\
&= \frac{e^{\beta\sum_{t=1}^{n-i}\theta_{i,x}\omega(t,S_t)}}{z_{n-i}\circ\theta_{i,x}}P\left(S_1=y-x\right) \\
&= P_{n-i}^{\beta,\theta_{i,x}\omega}(S_1=y-x).
\end{aligned}
$$

□

Proposition B.1.3. *Step by step proof of (4.4.1-3), the following function is decreasing*

$$\beta \mapsto \beta^{-1}[p_n(\beta,\omega)+\ln 2d].$$

Proof. We have the identity

$$
\begin{aligned}
\frac{d}{d\beta}\left(\frac{1}{\beta}\left[p_n+\ln(2d)\right]\right) &= \frac{1}{n\beta}E_n^{\beta,\omega}(H_n)-\frac{p_n+\ln 2d}{\beta^2} \\
&= \frac{1}{n\beta^2}\left[\beta E_n^{\beta,\omega}(H_n)-\ln Z_n - n\ln 2d\right] \\
&= \frac{1}{n\beta^2}\left[\beta E_n^{\beta,\omega}(H_n)-1\cdot\ln Z_n + 1\cdot\ln(2d)^{-n}\right] \\
&= \frac{1}{n\beta^2}\left[\beta\sum_{\mathbf{x}}\frac{e^{\beta H_n}}{Z_n}H_n P(x)-\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\ln Z_n\right. \\
&\qquad\left.+\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\ln(2d)^{-n}\right] \\
&= \frac{1}{n\beta^2}\left[\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\beta H_n-\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\ln Z_n\right. \\
&\qquad\left.+\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\ln P(x)\right] \\
&= \frac{1}{n\beta^2}\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\left[\beta H_n+\ln P(x)-\ln Z_n\right] \\
&= \frac{1}{n\beta^2}\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\left[\ln e^{\beta H_n}P(x)-\ln Z_n\right] \\
&= \frac{1}{n\beta^2}\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\left[\ln\frac{e^{\beta H_n}P(x)}{Z_n}\right] \\
&= \frac{1}{n\beta^2}\sum_{\mathbf{x}}P_n^{\beta,\omega}(x)\ln P_n^{\beta,\omega}(x) \\
&= \frac{1}{n\beta^2}\sum_{\mathbf{x}}P_n^{\beta,\omega}\ln P_n^{\beta,\omega} := \frac{1}{n\beta^2}h(P_n^{\beta,\omega}),
\end{aligned}
$$

where $h(\nu)$ is the Boltzmann entropy of a probability measure ν on the n steps path space,

$$h(\nu) := \sum_{\mathbf{x}}\nu(x)\ln\nu(x). \tag{B.5}$$

Finally we have that $h(\nu)\leq 0$ for all ν, which ends the prove. □

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