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Extended gravity in different formalisms with curvature-matter couplings motivated by astrophysical phenomenology

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PRESENTA

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To my family and friends, who rejoice for each step and achievement I make. You can be sure that I will always give my best.

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Resumen

Bernal *et al.* (2011) propusieron una acción relativista, la cual, en su límite de campo débil y para el caso límite donde el radio local de curvatura es mayor que la longitud característica del sistema r, lleva a la expresión MONDiana de la aceleración para una partícula puntual. Esta propuesta parte de una acción adimensional $f(\chi)$ que introduce una constante de acoplamiento L_M la cual después de ajustarse presenta una dependencia explícita en la masa del sistema. El análisis de dicha propuesta se hizo en el formalismo puramente métrico, es decir, se consideró a la métrica $g_{\mu\nu}$ como el único campo fundamental del problema.

El trabajo de partida de esta tesis son los resultados obtenidos por Bernal *et al.* (2011). Retomando la acción $f(\chi)$ se estudiaron las consecuencias de considerar a la conexión como un campo independiente a la métrica. En un principio se mantuvo la condición de simetría de la conexión, pero se relajó la condición de compatibilidad métrica (formalismo Palatini). Como resultado de este formalismo se obtiene que la conexión general es la conexión de Levi-Civita para una métrica $h_{\mu\nu}$ que está conformemente relacionada a $g_{\mu\nu}$. Para recuperar la aceleración MONDiana en el campo débil de esta teoría en este formalismo, se encontró que la forma funcional $f(\chi) = \chi^{3/2}$ es necesaria. Este resultado concuerda con el encontrado en el formalismo métrico puro. Desafortunadamente, este formalismo no representa ninguna ventaja respecto al formalismo métrico. En primer lugar porque la conexión resulta ser la conexin de Levi-Civita de una mtrica relacionada conformemente con $g_{\mu\nu}$ por lo que un segundo marco entra en escena, y la relación de los tensores entre ambos marcos no resulta fácil de manipular, sino que se debió hacer una suposición extra a fin de poder simplificar los cálculos. Además de lo anterior, la constante de acoplamiento sigue requiriendo la presencia explícita de la masa, lo que implica que la teoría sigue

siendo no-local/no-covariante. Estos resultados fueron publicados (Barrientos and Mendoza, 2016) y constituyen el contenido del capítulo 2 de esta tesis.

De nuestros estudios concluimos que si deseamos obtener MOND como el límite de campo débil de una teoría relativista, modificando únicamente el sector de la curvatura en la acción de Hilbert-Einstein a través de una generalización que involucre sólo términos geométricos, la constante de acoplamiento siempre tendrá una dependencia en la masa. De esta forma, en el capítulo 3 mostramos una propuesta donde modificamos no sólo el sector geométrico de nuestra acción, sino que también el sector que envuelve a los términos de materia mediante derivadas del lagrangiano de materia. El hecho de introducir derivadas y no otro tipo de modificación vino del análisis dimensional que habíamos estado efectuando a nuestros modelos previos. Además de las modificaciones ya mencionadas, de igual manera se manejó a la conexión como un campo independiente, pero en este caso se relajó la condición de simetría, de tal forma que se permitió la existencia de torsión. Los resultados obtenidos de esta propuesta pueden ser revisados con más detalle en Barrientos and Mendoza (2017) pero pueden ser resumidos por lo siguiente: La inclusión de la torsión tampoco representa ninguna ventaja desde el punto de vista operacional, pues para facilitar el manejo de las ecuaciones resultantes, de igual manera se debe hacer una suposición adicional respecto a la naturaleza de la torsión. El significado físico de la inclusión de las derivadas del lagrangiano de materia no es muy claro, su inclusión en un principio se debe únicamente a un requerimento matemático que permite obtener la aceleración MONDiana. Lo que sí representa un avance respecto a las propuestas previas es que la constante de acoplamiento de esta teoría depende exclusivamente de las constantes fundamentales c, G y a_0, convirtiendo así a la teoría en un caracter covariante/local. De esta forma hemos dado un paso hacia adelante respecto a uno de los puntos que ha sido ampliamente debatido en los modelos previos.

Hasta este punto, nuestras modificaciones se han restringido a modificaciones geométricas e introducir derivadas del lagrangiano de materia para el sector de materia en la acción, es decir, se han mantenido separadas a la curvatura de la materia a nivel de la acción. Otro tipo de modificación que hemos estudiado es el acoplamiento entre materia y curvatura. El capítulo 4 presenta los resultados del análisis en el marco de Palatini para una teoría $f(R,T)$ donde T representa a la traza del tensor de energía-momento. Además de presentar las ecuaciones de campo se introducen algunas aplicaciones astrofísicas. Una de las características más resaltantes de estas teorías con acoplamiento es la aparición de una fuerza extra en la ecuación geodésica. Éste termino se da debido a que la divergencia del tensor de energía-momento no es nula. Los resultados presentados en dicho capítulo son encontrados en Barrientos *et al.* (2018).

El capítulo 5 que presenta los resultados de (Barrientos and Mendoza, 2018) introduce una teoría con acoplamiento curvatura-materia en el formalismo métrico puro $f(\chi, \xi)$. Esta acción es de nueva cuenta adimensional, donde las constantes de acoplamiento se construyen a priori a partir de las constantes fundamentales G, c y a_0. Esta teoría representa un avance en nuestro objetivo de construir una teoría relativista de MOND no solo por el hecho de que las constantes de acoplamiento no dependan de la masa, sino que al analizar las ecuaciones de campo de manera perturbativa respecto a ordenes de $1/c$. Y por lo tanto podemos escoger los coeficientes de $f(\chi, \xi) = \chi^\gamma \xi^\beta$ de tal forma que los términos con orden dominate reproduzcan la acelaración MONDiana ($\gamma = -\beta = 3$) o la aceleración Newtoniana ($\gamma + \beta = 1$) sin ninguna suposición adicional . Esto tal vez podría darnos un poco de luz respecto a la forma de empalmar ambas teorías en algún límite, característica que hasta el momento no ha podido resolverse.

Artículos

Como resultado del trabajo de tesis, los siguientes artículos fueron publicados:

- *A relativistic description of MOND using the Palatini formalism in an extended metric theory of gravity*
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Chapter 1

Introduction

Kepler's laws succeed in explaining the movement of planets about the sun. Kepler's greatest achievement consisted in expressing these motions in a mathematical language, but from a physical point of view they have a flaw: they do not come from a first principle theory. It was Newton who developed a satisfactory non-relativistic gravitational theory. The main success of Newton's theory comes from the fact that Kepler's laws can be straight forwardly obtained from it.

In particular, for circular orbits Kepler's third law of motion describes the orbital velocity v of a planet about the sun as a function of the sun's mass M and the separation r between them as:

$$v \propto \frac{M^{1/2}}{r^{1/2}}, \tag{1.1}$$

Since the planet orbits about the sun due to an equilibrium between its centrifuge acceleration $a = v^2/r$ and its gravitational attraction towards the sun, then:

$$a = -GM/r^2, \tag{1.2}$$

where G represents Newton's constant of gravitation and is introduced as a proportionality constant in the previous relation. The minus sign corresponds to the correct attractive nature of gravity. This relation is the simplest way to represent the Newtonian universal law of gravity.

Newton's gravity law was the prevailing idea to describe any gravitational phenomenon for more than two hundred years. In fact, it still widely used nowadays to explain non-relativistic gravitational phenomena. The discovery of an anomaly in the precession of Mercury's perihelion yielded the scientific community to the search of an improvement in the gravitational theory[1]. This comes with Einstein's general relativity (GR). Such is a theory expressed in tensor language, where gravitation is expressed as a consequence of the curvature of space-time; giving an idea of a mathematical relation between matter and curvature. Furthermore standard Newtonian gravity can be recovered from GR in its weak field limit of approximation. GR has proven extremely well at mass to length ratio scales similar to the ones associated to our solar system (Will, 1971a,b,c,d; Hulse and Taylor, 1975; Kramer *et al.*, 2006; Kramer and Champion, 2013; Kramer *et al.*, 2005a,b; Kramer, 2013). For these reasons, GR has been taken as the correct theory to describe gravitation at such scales.

Observational data of astrophysical systems, including individual, groups and clusters of galaxies, and the universe in a cosmological context, show that in order to maintain the standard gravitational field equations of general relativity, including their Newtonian non-relativistic weak field limit, it is necessary to postulate the existence a new kind of non-baryonic dark matter (Oort, 1932; Zwicky, 1933; Ostriker and Peebles, 1973; Bosma, 1981; Rubin *et al.*, 1982; Gunn *et al.*, 1978; Vittorio and Silk, 1984; Tremaine and Gunn, 1979; Bond and Efstathiou, 1984; Blumenthal *et al.*, 1984; White *et al.*, 1983; Frenk *et al.*, 1988). Although, current research is usually done assuming the existence of this undetected dark matter, there exists an alternative scenario, consinsting on changing the field equations of gravitation at those astrophysical and cosmological scales.

It was under this point of view that in 1983 Milgrom proposed MOND (MOdified Newtonian Dynamics), a theory that introduced a modification to Newton's second law in order to explain the flattening of rotation curves in disc galaxies (Milgrom, 1983a,b). From this empirical proposal it is possible to recover the baryonic Tully-Fisher relation, for which the rotation velocity V scales as a power of the baryonic mass (composed of stars and dust) M, i.e.

[1]The scientist searched almost any explanation for this discrepancy, a well-known one was that that propose a flattening in the sun's poles, which results in a non-spherical potential

$$V \propto M^{1/4}. \tag{1.3}$$

Although in principle, the Tully-Fisher relation was found for disc galaxies, in recent years, through dynamical observations has been shown to be true for spiral, elliptical and dwarf spheroidal galaxies (Famaey and McGaugh, 2012; Hernandez *et al.*, 2010) and globular clusters (Hernandez and Jiménez, 2012; Hernandez *et al.*, 2013). Also there are strong evidence that wide open binaries do not follow the Newtonian gravity (Hernandez *et al.*, 2012; Banik and Zhao, 2018). Moreover, the astrophysical observations strongly suggest that MONDian gravity accurately describes pressure supported systems across 12 order of magnitude in mass (Hernandez *et al.*, 2015). It has became clear that at certain scales of mass and length, where the induced gravitational accelerations on test particles fall below of a certain value $a_0 = 1{,}2 \times 10^{-10}\, m/s^2$, Kepler's third law appears not to hold in its classical form on these systems, but rather obey the Tully-Fisher law[2]:

Following Mendoza (2012); Mendoza and Olmo (2014), we assume that at some regime, gravity follows Kepler's third law (1.1), and at some other it follows the Tully-Fisher law. As such, when the Tully-Fisher regime is reached, the acceleration exerted by a test particle at a distance r from a point mass source M generating a gravitational field is given by:

$$a = \frac{v^2}{r} \propto -\frac{M^{1/2}}{r}. \tag{1.4}$$

As noted by Mendoza (2012); Mendoza and Olmo (2014), the proportionality constant can be written as $\sqrt{Ga_0}$, where $a_0 \approx 10^{-10}\,\mathrm{m\,s^{-2}}$ is Milgrom's acceleration constant. Using this, the previous relation can be written as:

$$a = \frac{\sqrt{a_0 GM}}{r}. \tag{1.5}$$

All current observations (Sanders, 1990; McGaugh, 2004, 1999; Scarpa, 2003;

[2]To be more precise, the baryonic Tully-Fisher relation is only observed in spiral galaxies. For elliptical galaxies an analogous relation is also observed and is known as the Faber-Jackson relation (Famaey and McGaugh, 2012). We are using both relations as to mean the same physical idea, that the scaling with velocity -or velocity dispersion for pressure supported systems- with the mass is the one shown in eq. (1.3).

McGaugh, 2005; Sanders and McGaugh, 2002; Milgrom, 1989; Brada and Milgrom, 1999) show that Newtonian gravity is reached when test particles acquire an acceleration greater than a_0 and a full MONDian regime is obtained when those accelerations are smaller than a_0. View in this way, all systems with accelerations $a \lesssim a_0$ are the ones that are commonly viewed as systems where non-baryonic dark matter is required to explain the observed dynamics.

Despite the success that MOND has at the phenomenological level, it is an incomplete theory, in the sense that is an non-relativistic description and also since at the non-relativist limit, it does not come from a Poisson field equation. From a mathematical point of view, MOND should be conceived as the weak field limit of a relativistic proposal. Several attempts have been done towards building a relativistic version of MOND, amongst the many proposals in this direction we can name the Tensor-Vector-Scalar theories (Bekenstein and Sanders, 2012; Bekenstein, 2004; Zlosnik *et al.*, 2006; Sanders, 1997; Zlosnik *et al.*, 2007; Sanders, 2005; Skordis, 2009), galileons (Babichev *et al.*, 2011), AQUAL (Bekenstein and Milgrom, 1984) bimetric theories (Milgrom, 2009; Clifton and Zlosnik, 2010), non-local theories (Deffayet *et al.*, 2011a), modified energies (Demir and Karahan, 2014) and field theories (Bruneton and Esposito-Farèse, 2007), $f(R)$ theories (Sotiriou and Faraoni, 2010; Sotiriou and Liberati, 2007; Capozziello and Faraoni, 2011; Capozziello and de Laurentis, 2011; Nojiri and Odintsov, 2011), dipolar dark matter (Blanchet, 2007a,b; Blanchet and Le Tiec, 2008) and nonlocal theories(Soussa and Woodard, 2003; Deffayet *et al.*, 2011b) to name a few.

Bernal *et al.* (2011) built a relativistic proposal for MOND in the pure metric formalism. This theory is based on a dimensionally correct action for a $f(\chi)$ function, where χ is a dimensionless Ricci scalar defined as: $\chi = L_M^2 R$, and L_M is a free coupling parameter of the theory with length dimensions, which is fixed by recovering MOND in the weak field limit. The value $\chi^{3/2}$ was found as the function that turns into MOND on its weak field limit. This value is coincident with the results obtained in some cosmological applications (Capozziello, 2002; Capozziello and DeFelice, 2008).

The $f(\chi)$ theory explains not only the flattening of rotation curves, but the correct bending angle of light for gravitational lensing in individual, groups and clusters of galaxies (Mendoza *et al.*, 2013). However, this proposal possess a mathematical inconvenient since the coupling constant L_M has an explicit mass dependence and so, makes this proposal formally not covariant and non-local (there is however a mathematical way to deal with this caveat as explained by Carranza *et al.* (2013a); Mendoza (2015)). As such, the $f(\chi)$ action must not to be seen as a complete theory but as a particular case of a more general idea.

Barrientos and Mendoza (2016) built a relativistic metric description of MOND using the Palatini formalism following the $f(\chi) = \chi^b$ description of (Bernal *et al.*, 2011). It was shown that in order to recover the non-relativistic MOND regime where, for circular orbits the Tully-Fisher law replaces Kepler's third law, the value of the parameter $b = 3/2$, which is coincident with the value found using a pure metric formalism (Bernal *et al.*, 2011). Unlike this pure metric formalism, which yields fourth order field equations, the Palatini approach yields second order field equations, which is a desirable requirement from a theoretical perspective. Thus, the phenomenology associated to astrophysical phenomena with Tully-Fisher scaling can be accounted for using this proposal, without the need to introduce any non-baryonic dark matter particles.

Barrientos and Mendoza (2017) introduced a relativistic action, which in its weak field limit reduces to MOND, but unlike the $f(\chi)$ theory, the coupling constants have exclusive dependence in pure physical constants: Newton's gravitational constant G, the speed of light c and Milgrom's acceleration constant a_0, making the action entirely covariant. This theory has two departures with respect general relativity. On the one hand, in the geometrical sector, we use a $f(R)$ theory with torsion. From the cosmological point of view, it has been proven that the torsion has interesting implications in order to explain the accelerated expansion of the universe (Capozziello *et al.*, 2008, 2007). Our approach in this work was to find a MONDian behaviour in extended metric theories of gravity with torsion. On the other hand, based on the $f(\Sigma)$ and $f(\mathcal{L}_\mathrm{m})$ theories(Harko *et al.*, 2011a; Haghani *et al.*, 2013; Lobo and Harko, 2012), where Σ is the trace of the energy momentum tensor $\Sigma_{\mu\nu}$ and $\mathcal{L}_\mathrm{m}$ is the matter Lagrangian, we also modify the matter sector with an action which for this particular case is only dependent on derivatives of the matter Lagrangian.

Barrientos *et al.* (2018) studied $f(R, T)$ theories of gravity, where T is the trace of the energy-momentum tensor $T_{\mu\nu}$, with independent metric and affine connection (metric-affine theories). We find that the resulting field equations share a close resemblance with their metric-affine $f(R)$ relatives once an effective energy-momentum tensor is introduced. As a result, the metric field equations are second-order and no new propagating degrees of freedom arise as compared to GR, which contrasts with the metric formulation of these theories, where a dynamical scalar degree of freedom is present. Analogously to its metric counterpart, the field equations impose the non-conservation of the energy-momentum tensor, which implies non-geodesic motion and consequently leads to the appearance of an extra force. The weak field limit leads to a modified Poisson equation formally identical to that found in Eddington-inspired Born-Infeld gravity. Furthermore, the coupling of these gravity theories to perfect fluids, electromagnetic, and scalar fields, and their potential applications are discussed.

Barrientos and Mendoza (2018) built an extended relativistic $f(R)$ theory of gravity with matter-curvature couplings $F(R, \mathcal{L}_{\mathrm{matt}})$ for which its weak field limit of approximation recovers the simplest version of MOND. This was made by (a) performing an order of magnitude approach and (b) by perturbing the resulting field equations of the theory to the weakest field limit of approximation. Also a computation of the geodesic equation of the resulting theory was performed and showed that it has an extra force, a fact that commonly appears in general curvature-matter couplings.

Chapter 2

A relativistic description of MOND using the Palatini formalism in an extended metric theory of gravity

2.1. Introduction

In this article, we search for a possible extended metric theory of gravity using the Palatini formalism, which recovers MOND on its weak field limit of approximation. In sect. 2.2 we briefly introduce the relevant equations for the Palatini metric formalism useful for our further developments. In sect. 2.3 we propose a power of the Ricci scalar for the gravitational action and we find an expression for the Ricci scalar curvature as function of the trace of the energy-momentum tensor. In sect. 2.4 we explore the non-relativistic weak-field limit of the theory and expand the metric as Minkowskian plus a second order perturbation to arrive at a non-relativistic equation for the acceleration as function of the energy-momentum tensor. In sect. 5.4 we fix the free parameters of our theory such that in the weak-field limit of approximation the acceleration converges to the simplest MONDian description of eq. (1.5). In sect. 3.4.3 we perform a Parametrised Post Newtonian (PPN) analysis to second order of our field equations in order to complement the results of sect. 2.4. In sect. 2.7 we conclude and discuss our results.

2.2. $f(\chi)$ in Palatini formalism

Many of the results mentioned in this section are well known on the studies of the Palatini formalism for metric $F(R)$ theories of gravity. For further information, the reader is referred to the excellent introductory texts by (Sotiriou and Faraoni, 2010; Sotiriou and Liberati, 2007; Capozziello and Faraoni, 2011; Capozziello and de Laurentis, 2011; Nojiri and Odintsov, 2011).

Let us start with the action for the gravitational field motivated by the one built by (Bernal *et al.*, 2011):

$$\mathcal{S} = -\frac{c^3}{16\pi G L_M^2} \int f(\chi)\sqrt{-g}\,\mathrm{d}^4x - \frac{1}{c}\int \mathcal{L}_{\mathrm{matt}}\sqrt{-g}\,\mathrm{d}^4x, \tag{2.1}$$

where L_M is a coupling constant with dimensions of length and the dimensionless Ricci scalar χ is given by:

$$\chi := L_M^2 R, \tag{2.2}$$

where R is a non-traditional Ricci scalar, (not to be confused with the standard Levi-Civita Ricci's one $\tilde{R}$) defined by:

$$R := g^{\mu\nu} R_{\mu\nu}. \tag{2.3}$$

In the previous equation, and in what follows, we use Einstein's summation convention, greek and latin indices take values from 0 to 4 and from 0 to 3 respectively. The tensor $g_{\mu\nu}$ represents the metric tensor and $R_{\mu\nu}$ is a non-traditional Ricci tensor defined exclusively in terms of the affine connection $\Gamma^{\alpha}{}_{\mu\nu}$ through the following equation:

$$R_{\mu\nu} := \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\rho}_{\mu\nu}\,\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\rho}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\rho}. \tag{2.4}$$

In the Palatini formalism, the connection $\Gamma^{\alpha}{}_{\mu\nu}$ has no relation with the standard Levi-Civita connection $\tilde{\Gamma}^{\alpha}{}_{\mu\nu}$.

The null variations of the action (2.1) with respect to the metric $g_{\mu\nu}$ yield the following field equations:

$$f'(\chi)\chi_{\mu\nu} - \frac{1}{2}f(\chi)g_{\mu\nu} = \frac{8\pi G L_M^2}{c^4}T_{\mu\nu}, \tag{2.5}$$

where the dimensionless tensor

$$\chi_{\mu\nu} := L_M^2 R_{\mu\nu} \tag{2.6}$$

and $f'(\chi) := \mathrm{d}f(\chi)/\mathrm{d}\chi$. The energy-momentum tensor $T_{\mu\nu}$ is given by (Misner *et al.*, 1973):

$$T_{\mu\nu} := -\frac{2}{\sqrt{-g}}\frac{\delta(\mathcal{L}_{\text{matt}}\sqrt{-g})}{\delta g^{\mu\nu}}. \tag{2.7}$$

The contraction of eq. (2.5) with $g^{\mu\nu}$ yields:

$$L_M^2 f'(\chi)R - 2f(\chi) = \frac{8\pi G L_M^2}{c^4}T, \tag{2.8}$$

for all $f(\chi) \neq \chi^2$. Under the assumption of a torsion free connection, i.e. imposing a symmetric connection $\Gamma^{\alpha}{}_{\mu\nu}$, the null variations of the action (2.1) with respect to this affine connection yield:

$$\nabla_\lambda\left(\sqrt{-g}\, f'(\chi)\, g^{\mu\nu}\right) = 0. \tag{2.9}$$

The usual approach to solve this equation, consists on performing the following conformal transformation to the metric tensor:

$$h_{\mu\nu} = f'(\chi)g_{\mu\nu}. \tag{2.10}$$

Substitution of this last equation into relation (2.9) gives:

$$\nabla_\lambda\left(\sqrt{-h}h^{\mu\nu}\right) = 0, \tag{2.11}$$

where $h := h^{\nu}{}_{\nu}$. Equation (2.11) is known as the metricity condition and states that $\Gamma^{\alpha}{}_{\mu\nu}$ is the Levi-Civita connection of the $h_{\mu\nu}$ metric, i.e.:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} h^{\lambda\rho} \left(h_{\rho\mu,\nu} + h_{\rho\nu,\mu} - h_{\mu\nu,\rho} \right). \tag{2.12}$$

For the conformal transformation (2.10), the tensor $R_{\mu\nu}(\Gamma)$ is related to the usual Ricci tensor $\tilde{R}_{\mu\nu}(\tilde{\Gamma})$ defined in terms of the Levi-Civita connection of the metric $g_{\mu\nu}$ by (Capozziello and Faraoni, 2011; Carroll, 2004):

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{f'} \nabla_\mu \nabla_\nu f' - \frac{1}{2f'} g_{\mu\nu} \Delta f' + \frac{3}{2f'^2} \nabla_\mu f' \nabla_\nu f', \tag{2.13}$$

The contraction of this last result with the metric $g^{\mu\nu}$ yields:

$$R = \tilde{R} - \frac{3}{f'} \Delta f' + \frac{3}{2f'^2} \nabla_\mu f' \nabla^\mu f'. \tag{2.14}$$

Note that $\mathcal{R}$ is not the Ricci scalar for the $h_{\mu\nu}$ metric, since it is built by its contraction with the conformal metric $h^{\mu\nu}$.

In what follows we are going to work extensively with eqs. (2.8) and (2.14), since using the former it is possible to find R as a function of the trace of the energy-momentum tensor, i.e.: $R = R(T)$. Substitution of this result on the latter, and bearing in mind the fact that f' is a function of R and hence of T, the solution $\tilde{R} = \tilde{R}(T)$ can be found.

2.3. $f(\chi)$ as a power law

Let us now assume that:

$$f(\chi) = \chi^b. \tag{2.15}$$

and substitute it into relations (2.14) and (2.8) to obtain:

$$R = \tilde{R} + \frac{3}{2} R^{2-2b} \nabla_\mu R^{b-1} \nabla^\mu R^{b-1} - 3R^{1-b} \Delta R^{b-1}, \tag{2.16}$$

and

$$R^b = \frac{\alpha T}{b-2}, \tag{2.17}$$

where

$$\alpha := \frac{8\pi G L_M^{2(1-b)}}{c^4}. \tag{2.18}$$

In order to obtain an equation that relates the curvature $\tilde{R}$ with the trace T of the energy-momentum tensor, eq. (2.17) must be substituted into (2.16). Since this procedure yields a complex equation, we will tackle the problem in a different manner.

Let us then proceed by expressing relation (2.13) as:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + H_{\mu\nu}(R), \tag{2.19}$$

where we have used the fact that $f' = f'(R)$. The tensor $H_{\mu\nu}(R)$ has a complicated algebraic form which will be determined in sect. 3.4.3, and we will show an explicit functional form which allows our proposal to have full consistency at the lowest second perturbation order. The trace of eq. (2.19) is:

$$R = \tilde{R} + H(R), \tag{2.20}$$

which is another way to express eq. (2.14). A Taylor expansion of the function $H(R)$ yields the following linear relation:

$$H(R) = \kappa R + \mathcal{O}(R^2), \tag{2.21}$$

since $H(R=0) = 0$ according to eq. (2.20). Substitution of eq. (2.21) into eq. (2.20) yields:

$$\tilde{R} = \kappa' R, \qquad \text{where:} \qquad \kappa' := 1 - \kappa. \tag{2.22}$$

Using this result in eq. (2.17), we obtain:

$$\tilde{R} = \kappa' \left[\frac{\alpha T}{b-2} \right]^{1/b} . \tag{2.23}$$

2.4. Weak field limit

Our main target is to find b such that in the weakest (non-relativistic) limit of the theory, the acceleration of a test particle in a gravitational field produced by a point mass source M is reduced to the MONDian one (1.5).

For this purpose we take the background metric as the Minkowsky space-time plus a small perturbation expanded in powers of $1/c$, which we call perturbation orders. As an example, a second perturbation order is proportional to $1/c^2$ and zeroth order terms have no dependence on the speed of light. The next perturbation expansion to the Minkowsky background metric is of second order (Will, 1993), and since we are interested in the weakest limit of the theory describing the motion of non-relativistic massive test particles, this correction is enough for our study.

At this point we stress that in eq. (3.45) we have returned to the original metric $g_{\mu\nu}$. The conformal transformation was just a mathematical tool in order to manipulate more easily the resulting equations. Therefore, the expansion used below is justified.

For the second perturbation order, we take as base the work of (Mendoza and Olmo, 2014), in which they proved that, to be in accordance with astronomical observations of the deflection of light of individual, groups and clusters of galaxies together with the Tully-Fisher law (1.3) for material particles, the metric coefficients at second perturbation order are given by:

$$\begin{aligned} g_{00} &= {}^{(0)}g_{00} + {}^{(2)}g_{00} = 1 + \frac{2\phi}{c^2}, \\ g_{ij} &= {}^{(0)}g_{ij} + {}^{(2)}g_{ij} = \delta_{ij} \left(-1 + \frac{2\phi}{c^2} \right), \\ g_{0i} &= 0. \end{aligned} \tag{2.24}$$

which implies that the PPN parameter $\gamma = 1$ according to (Mendoza and Olmo, 2014). In the previous equation and in what follows the left superscript in parenthesis on a given quantity denotes its perturbation order. Ricci's scalar of the previous metric at the same perturbation order is given by:

$$^{(2)}\tilde{R} = -\frac{2\nabla^2\phi}{c^2}. \tag{2.25}$$

Since the Tully-Fisher law describes the motion of non-relativistic dust particles, then the energy-momentum tensor trace is

$$T = \rho c^2, \tag{2.26}$$

where ρ is the mass density. Thus, eq. (3.45) turns into:

$$-\frac{2\nabla^2\phi}{c^2} = \kappa' \left[\frac{8\pi G L_M^{2(1-b)}\rho}{c^2(b-2)}\right]^{1/b}. \tag{2.27}$$

Since the acceleration is defined by: $|\boldsymbol{a}| := |\nabla\phi|$, then

$$-\frac{2}{c^2}\nabla\cdot\boldsymbol{a} = \kappa' \left[\frac{8\pi G L_M^{2(1-b)}\rho}{c^2(b-2)}\right]^{1/b}. \tag{2.28}$$

This last equation will allow us to fix the parameter b such that it is possible to recover a MONDian acceleration (1.5).

2.5. Recovering MOND

At order of magnitude, eq. (2.28) turns into:

$$\frac{a}{c^2 r} \approx \left[\frac{G L_M^{2-2b}\rho}{c^2}\right]^{1/b}. \tag{2.29}$$

For a point mass source located at the origin, the density ρ is given by:

$$\rho = M\delta(\boldsymbol{r}), \tag{2.30}$$

where $\delta(\boldsymbol{r})$ is the three-dimensional Dirac's delta distribution in spherical coordinates. Approximating the previous equation to the same order of magnitude yields

$\rho \approx M/r^3$, and so expression (2.29) reduces to

$$a \approx (GM)^{1/b} L_M^{2(1-b)/b} c^{2(b-1)/b} r^{(b-3)/b}. \tag{2.31}$$

On the one hand, the flattening of rotation curves requires $a \propto r^{-1}$, and so:

$$b = 3/2. \tag{2.32}$$

On the other hand, the weakest field limit of approximation yields a non-relativistic description of gravity and as such, the velocity of light should not appear on eq. (3.78). In other words,

$$L_M \propto c. \tag{2.33}$$

As noted by (Bernal *et al.*, 2011), using dimensional analysis in the description of a point mass source for a relativistic version of MOND, it is possible to construct two independent fundamental lengths and it is expected that the length L_M should be a function of those two lengths, in other words:

$$L_M := \zeta r_g^{\alpha} l_M^{\beta}, \tag{2.34}$$

where

$$r_g := \frac{GM}{c^2}, \qquad l_M := \left(\frac{GM}{a_0} \right)^{1/2}, \tag{2.35}$$

represent the gravitational radius and a MONDian "mass-length" scale respectively. The constant ζ is a proportionality factor and the exponents α and β must satisfy the condition $\alpha + \beta = 1$ so that eq. (2.34) is dimensionally correct. With the aid of eq. (2.33) it follows that $\alpha = -1/2$, and so, $\beta = 3/2$. In other words:

$$L_M = \zeta \left(\frac{GM}{a_0^3} \right)^{1/4} c. \tag{2.36}$$

Using this expression for L_M and the value for b previously found, at order of magnitude the acceleration (3.78) reaches a MONDian value: $a \approx (GMa_0)^{1/2}/r$.

In order to fully show that a MONDian non-relativistic limit is obtained in the weak-field limit of the theory, we proceed as follows. Direct substitution of the values obtained for b and L_M, into eq. (2.28) yields:

$$-2\nabla \cdot \boldsymbol{a} = \kappa' \left(a_0 GM\right)^{1/2} \left[\frac{4\delta(r)}{\zeta r^2}\right]^{2/3}. \tag{2.37}$$

where we have used the fact that the three-dimensional Dirac's delta function is given by $\delta(\boldsymbol{r}) = \delta(r)/4\pi r^2$.

For Schwartz distributions it is impossible in general terms, to define a product in such a way that the resulting distribution forms an algebra with acceptable topological properties (Schwartz, 1954). Schwartz's impossibility result states that it is not possible to have a differential algebra that contains the space of distributions and preserves the product of continuous functions. To overcome these disadvantages, (Colombeau, 1990, 1985) has developed a theory of generalised functions, which allows to define a fully consistent product of distributions. As such, we can consider Dirac's delta distribution as a standard function so that we can write the following identity

$$[\delta(r)]^{2/3} = [\delta(r)]^{-1/3}\ \delta(r). \tag{2.38}$$

With this relation, eq. (3.32) turns into:

$$-2\nabla \cdot \boldsymbol{a} = \kappa' \left(a_0 GM\right)^{1/2} \left(\frac{4}{\zeta}\right)^{2/3} \left[\frac{1}{r^4 \delta(r)}\right]^{1/3} \delta(r). \tag{2.39}$$

Since we are searching for a MONDian value for the acceleration, let us assume it obeys the following general power law:

$$\boldsymbol{a} = \lambda r^{\sigma} \boldsymbol{e_r}, \tag{2.40}$$

where $\boldsymbol{e_r}$ is a unitary vector in the radial direction, λ and σ are constants so that:

$$\nabla \cdot \boldsymbol{a} = \lambda(\sigma + 2) r^{\sigma - 1}. \tag{2.41}$$

Substitution of this last equation into (2.39) and performing an integration over

r yields:

$$-\left.\frac{2\lambda(\sigma+2)}{\sigma}r^{\sigma}\right|_{r=0}^{r=\infty}=\kappa'\left(a_0GM\right)^{1/2}\left(\frac{4}{\zeta}\right)^{2/3}\left.\left[\frac{1}{r^4\delta(r)}\right]^{1/3}\right|_{r=0}. \tag{2.42}$$

Let us now use the fact that $\delta(0)$ can be obtained from the following relation (Gsponer, 2008):

$$\delta(r=0)=\lim_{r\to 0}\frac{1}{2\pi r}, \tag{2.43}$$

and substitute it into eq. (2.42) in order to obtain:

$$-\left.\frac{\lambda(\sigma+2)}{\gamma}r^{\sigma}\right|_{r=0}^{r=\infty}=\kappa'\left(a_0GM\right)^{1/2}\left(\frac{4\pi}{\zeta^2}\right)^{1/3}\left.\frac{1}{r}\right|_{r=0}. \tag{2.44}$$

Since ζ and λ are constants, the following relation is necessarily satisfied:

$$\sigma=-1, \tag{2.45}$$

which is an expected result from the order of magnitude analysis developed above in order to obtain flat rotation curves. Equation (2.44) is then reduced to:

$$-\lambda=\kappa'\left(a_0GM\right)^{1/2}\left(\frac{4\pi}{\zeta^2}\right)^{1/3}. \tag{2.46}$$

In order to recover a MONDian acceleration (1.5) limit, it is necessary that $\lambda = -\left(a_0GM\right)^{1/2}$ and so:

$$\zeta=2\left(\kappa'^3\pi\right)^{1/2}. \tag{2.47}$$

2.6. Second order perturbation analysis

In order to show that a MONDian solution is directly obtained from the field equations of the previous analysis, let us proceed as follows. Substituting the value $b=3/2$ in eq. (2.5) and (2.17), the field equations and the trace take the following form:

$$3R^{1/2}R_{\mu\nu} - g_{\mu\nu}R^{3/2} = \frac{16\pi G}{c^4 L_M} T_{\mu\nu}, \tag{2.48}$$

and

$$-R^{3/2} = \frac{16\pi G}{c^4 L_M} T. \tag{2.49}$$

This last equation is meaningless unless the energy-momentum tensor is defined with a minus sign on the right-hand side of eq. (2.7). This fact is closely related to the multiple branches that the solution space of any $F(\tilde{R})$ theory of gravity has, which is usually ascribed to the choice of the Riemann tensor (see e.g. the discussion on the appendix of (Mendoza *et al.*, 2013)). Quite curiously for the previous and following discussions it is not necessary at all to enter into further discussions about this, since the obtained results require only the square of eq. (2.49). Substituting eqs. (2.49), (2.36) and (2.19) into (2.48), we obtain the following field equations:

$$3(\tilde{R}_{\mu\nu} + H_{\mu\nu}) = \left(\frac{16\pi}{c^5\zeta}\right)^{2/3} \frac{(a_0 G)^{1/2}}{M^{1/6}} \frac{(T g_{\mu\nu} - T_{\mu\nu})}{T^{1/3}}. \tag{2.50}$$

If we now perturb the metric $g_{\mu\nu}$ about a flat Minkowsky space-time $\eta_{\mu\nu}$ we obtain:

$$g_{\mu\nu} = \eta_{\mu\nu} + \xi_{\mu\nu}. \tag{2.51}$$

The quantity $\xi_{\mu\nu}$ is the perturbation expanded in powers of $1/c$. To first order in ξ (second order in $1/c$), the time and space components of $\tilde{R}_{\mu\nu}$ are (Will, 1993):

$$^{(2)}\tilde{R}_{00} = \frac{1}{2}\nabla^2 \xi_{00}, \tag{2.52}$$

and

$$^{(2)}\tilde{R}_{ij} = \frac{1}{2}\nabla^2 \xi_{ij}, \tag{2.53}$$

where we have suppressed the upper index in ξ in the understanding that only the second perturbation order is relevant in our analysis. We have also chosen the PPN gauge for which: $\xi^{\mu}{}_{i,\mu} - 1/2\xi^{\mu}{}_{\mu,i} = 0$.

The constraint eq. (2.21) implies that $H_{\mu\nu}$ is a linear function in R and so by eq. (2.22) it is also linear $\tilde{R}$, but now the proportionality constant is a second rank tensor $\kappa_{\mu\nu}$, i.e.:

$$H_{\mu\nu} = \kappa_{\mu\nu} R. \tag{2.54}$$

Thus, if the first perturbative term of R is a second order term, $H_{\mu\nu}$ would also be as such.

The spatial components of eq. (2.50) are:

$$3\left(\frac{1}{2}\nabla^2\xi_{ij} + {}^{(2)}H_{ij}\right) = \left(\frac{16\pi}{c^5\zeta}\right)^{2/3} \frac{(a_0 G)^{1/2}}{M^{1/6}} \frac{(Tg_{ij} - T_{ij})}{T^{1/3}}. \tag{2.55}$$

The left-hand side of this relation is of second order and so, to obtain a second order term on the right-hand side, the last factor involving only T must be of order $\mathcal{O}(c^{4/3})$. For dust, the lowest perturbation order on T implies that: $T = \rho c^2$ and $T_{ij} = 0$, satisfying the previous requirement. This is a consistency check that our proposal is coherent at the lowest perturbation order. Thus, for dust and a point mass source, eq. (2.55) turns into:

$$3\left(\frac{1}{2}\nabla^2\xi_{ij} + {}^{(2)}H_{ij}\right) = -\left(\frac{4\delta(r)}{r^2\zeta}\right)^{2/3} \frac{(a_0 GM)^{1/2}}{c^2}\delta_{ij}. \tag{2.56}$$

Comparison of this expression with (3.32), yields:

$$3\left(\frac{1}{2}\nabla^2\xi_{ij} + {}^{(2)}H_{ij}\right) = \frac{2\nabla^2\phi}{\kappa' c^2}\delta_{ij}. \tag{2.57}$$

In order to recover the value of ξ_{ij} consistent with an isotropic metric, i.e. $\xi_{ij} = 2\phi/c^2\delta_{ij}$, the following value of H_{ij} is obtained:

$$ {}^{(2)}H_{ij} = \frac{\nabla^2\phi}{c^2}\delta_{ij}\left(\frac{2}{3\kappa'} - 1\right). \tag{2.58}$$

An analogous procedure for the time component yields:

$$^{(2)}H_{00} = -\frac{\nabla^2 \phi}{c^2}. \qquad (2.59)$$

Physically in a weak field limit it is expected that the Jordan and the Einstein frames, with metrics $g_{\mu\nu}$ and $h_{\mu\nu}$ respectively, lead to the same physical results. This means that the contributions of the tensor $H_{\mu\nu}$ must be sufficiently small. Bearing this in mind and the arbitrariness of the constant κ', let us choose:

$$\kappa' = \frac{2}{3}, \qquad \text{for which:} \qquad \kappa = \frac{1}{3}. \qquad (2.60)$$

Using these results together with eqs. (2.54), (2.58), (2.22) and (3.69), we obtain that the only non vanish component of the tensor $\kappa_{\mu\nu}$ is $\kappa_{00} = 1/3$.

Finally, from eq. (2.47), we find:

$$\zeta = \left(\frac{32\pi}{27} \right)^{1/2}. \qquad (2.61)$$

2.7. Discussion

It has become quite challenging to find a general expression that could potentially yield MOND on the weak-field limit of approximation (Bekenstein, 2004; Sanders, 1997; Zlosnik *et al.*, 2007; Sanders, 2005; Babichev *et al.*, 2011; Bekenstein and Milgrom, 1984; Milgrom, 2009; Deffayet *et al.*, 2011a; Bruneton and Esposito-Farèse, 2007; Skordis, 2008; Zhao, 2007). Many of the proposal fail since the metric coefficients (2.24) at second perturbation order are in no agreement with the mathematical particularities of the theories involved. Most importantly, it has always been desired that the field equations of a relativistic version of MOND are of the second order and involve only a power law function of the Ricci scalar. In this article, we have shown how to build such a second order field equations theory based on the metric coefficients (2.24) that converges to the simplest form of MOND (1.5) on its weakest limit of approximation

It is worth noticing at this point that the developed formalism in this article is such that the "coupling constant length" L_M of the gravitational action (2.1) is a proportional to $M^{1/4}$. (Bernal *et al.*, 2011; Sobouti, 2007; Mendoza and Rosas-Guevara,

2007) have all encountered this particularity when trying to build relativistic versions of MOND for metric formulations of gravity. Since it is customary that the gravitational action does not depend on the mass (or the energy-momentum tensor) then these authors have noticed that "one should not be surprised if some of the commonly accepted notions, even at the fundamental level of the action, require generalisations and re-thinking". An extended metric theory of gravity goes beyond the traditional general relativity ideas and in this way, we should change some of our standard views regarding its fundamental principles. Accepting this we can formally write the gravitational action S_g -first term on the right-hand side of eq. (2.1)- inspired by the generalisations made by (Harko *et al.*, 2011a; Haghani *et al.*, 2013; Lobo and Harko, 2012; Harko *et al.*, 2013, 2014a) and following a similar approach as that of (Carranza *et al.*, 2013b):

$$S_g = -\frac{c^3}{16\pi G} \int \frac{f(\chi)}{L_M^2} \sqrt{-g}\, \mathrm{d}x^4, \tag{2.62}$$

where following the results of eq. (2.36):

$$\begin{aligned} L_M &= c \left(\frac{G}{a_0^3} \right)^{1/4} \int \rho \mathrm{d}^3 x, \\ &= \frac{1}{c} \left(\frac{G}{a_0^3} \right)^{1/4} \int \mathcal{L}_{\text{matt}} \mathrm{d}^3 x, \end{aligned} \tag{2.63}$$

and we have used the fact that the matter Lagrangian $\mathcal{L}_{\text{matt}} = \rho c^2$ for dust, and for systems with sufficient degree of symmetry, e.g. isotropic or spherically symmetric space-times, the integral is taken over all the causally connected masses related to a particular problem. For the single point mass source discussed in this article, $\rho = M\delta(\boldsymbol{r})$ and in this case, eq. (2.62) converges to the gravitational action (2.1).

At this point, it is important to note that usually in the analysis of $F(R)$ theories on a Post-Newtonian frame, the comparison with a Brans-Dicke-like scalar-tensor theory can be achieved (Olmo, 2005). In our work, we do not appeal to this analogy and we keep the original equations throughout our analysis.

We choose to work in the frame of the Palatini formalism since it provides a deeper understanding of our proposal than the metric formalism because we do not restrict to a special kind of connection. While it is true that in standard general relativity, the Palatini formalism does not seem to bring something new, its use in areas where general relativity is not tested has been extended (Olmo *et al.*, 2015; Bazeia *et al.*, 2014; Lobo *et al.*, 2014).

The value of the parameter $b = 3/2$ required for an extended metric theory of gravity $f(\chi) = \chi^b$ in the Palatini formalism to yield a MONDian behaviour has appeared on many other works related to the cosmology (Capozziello, 2002; Capozziello and DeFelice, 2008) and to MOND using a pure metric approach to the problem (Bernal *et al.*, 2011) using Noether's symmetry. It is quite interesting that this value also appears in the Palatini formalism presented in this article and together with the previous findings sheds some light into a deepest understanding of gravitational phenomena beyond Einstein's general relativity.

The analysis performed in this article shows that it is possible to explain the flattening of rotation curves and the Tully-fisher law from our $f(\chi) = \chi^{3/2}$ theory using the Palatini formalism. By construction it not only reproduces the dynamics of material particles required to flatten rotation curves which show a Tully-Fisher scaling, but also reproduces bending of light associated to individual, groups and clusters of galaxies. This approach can be tested in cosmological models dealing with the accelerated expansion of the universe and in complex gravitational lensing, such as for example the ones produced by collisional clusters of galaxies. The fourth perturbation order of the theory can be also used to model the dynamics of clusters of galaxies in a completely analogous way as it was done in (Bernal *et al.*, 2015). These analyses are beyond the scope of this article and will be studied elsewhere.

Chapter 3

MOND as the weak field limit of an extended metric theory of gravity with torsion

3.1. Introduction

In this article, we introduce a relativistic action, which in its weak field limit reduces to MOND, but unlike the $f(\chi)$ theory, the coupling constants has exclusive dependence in pure physical constants: Newton's gravitational constant G, the speed of light c and Milgrom's acceleration constant a_0, making the action entirely covariant and local. This theory has two departures with respect general relativity. On the one hand, in the geometrical sector, we work with a $f(R)$ theory with torsion. From the cosmological point of view, it has been proven that the torsion has interesting implications in order to explain the accelerated expansion of the universe (Capozziello *et al.*, 2008, 2007). Our approach in this work is to find a MONDian behaviour in extended metric theories of gravity with torsion. On the other hand, based on the $f(\Sigma)$ and $f(\mathcal{L}_\mathrm{m})$ theories(Harko *et al.*, 2011a; Haghani *et al.*, 2013; Lobo and Harko, 2012), where Σ is the trace of the energy momentum tensor $\Sigma_{\mu\nu}$ and $\mathcal{L}_\mathrm{m}$ is the matter Lagrangian, we also modify the matter sector with an action which for this particular case is only dependent on derivatives of the matter Lagrangian.

The article is organised as follows. Section 3.2 introduces some of the theoretical background needed for torsion and for the weak field limit of a general metric theory of gravity. In section 3.3 we present our preliminary attempts which yield the correct MONDian proposal described in section 3.4. Finally, in section 3.5 we discuss our results.

3.2. Background information

Before dealing with our action proposals, we first introduce some of the mathematical concepts which we will use throughout our work. The reader is referred to the extensive reviews of Hehl (1973); Hehl *et al.* (1976) and the summaries of Capozziello *et al.* (2008, 2007) for further information. As we are interested in a general scenario where there exists two fundamental variables, the metric $g_{\mu\nu}$ and a priori non-symmetric connection $\Gamma^{\lambda}{}_{\mu\nu}$, let us start defining the torsion tensor $S^{\lambda}{}_{\mu\nu}$ as:

$$S^{\lambda}{}_{\mu\nu} := \frac{1}{2}\left(\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}\right). \tag{3.1}$$

If we demand that this connection holds the metric compatibility $\nabla_{\lambda} g_{\mu\nu} = 0$, then it is possible to relate it with the Levi-Civita connection $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$ of the metric $g_{\mu\nu}$, through the following expression:

$$\Gamma^{\lambda}{}_{\mu\nu} = \{\} - K^{\lambda}{}_{\mu\nu}, \tag{3.2}$$

where the contorsion tensor $K^{\lambda}{}_{\mu\nu}$ is given by (Capozziello *et al.*, 2008):

$$K^{\lambda}{}_{\mu\nu} := -S^{\lambda}{}_{\mu\nu} + S_{\mu\nu}{}^{\lambda} - S_{\nu}{}^{\lambda}{}_{\mu}. \tag{3.3}$$

The Riemann tensor is a geometric quantity defined entirely in terms of a general connection by:

$$R^{\alpha}{}_{\epsilon\mu\nu} := \partial_{\mu}\Gamma^{\alpha}{}_{\nu\epsilon} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\epsilon} + \Gamma^{\sigma}{}_{\nu\epsilon}\Gamma^{\alpha}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\mu\epsilon}\Gamma^{\alpha}{}_{\nu\sigma}. \tag{3.4}$$

Substitution of eq.(3.2) into the previous equation yields a relation between the general Riemann tensor $R^{\alpha}{}_{\epsilon\mu\nu}$ and the standard Riemann tensor built exclusively in terms of the Levi-Civita connection $\tilde{R}^{\alpha}{}_{\epsilon\mu\nu}$:

$$R^{\alpha}{}_{\epsilon\mu\nu} = \tilde{R}^{\alpha}{}_{\epsilon\mu\nu} + \tilde{\nabla}_{\nu} K^{\alpha}{}_{\mu\epsilon} - \tilde{\nabla}_{\mu} K^{\alpha}{}_{\nu\epsilon} + K^{\sigma}{}_{\nu\epsilon} K^{\alpha}{}_{\mu\sigma} - K^{\sigma}{}_{\mu\epsilon} K^{\alpha}{}_{\nu\sigma}. \tag{3.5}$$

The Ricci tensor is defined by the contraction of the first and third index: $R_{\mu\nu} := R^{\alpha}{}_{\mu\alpha\nu}$. Performing this contraction in eq.(3.5), yields to:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \tilde{\nabla}_{\nu} K^{\alpha}{}_{\alpha\mu} - \tilde{\nabla}_{\alpha} K^{\alpha}{}_{\nu\mu} + K^{\sigma}{}_{\nu\mu} K^{\alpha}{}_{\alpha\sigma} - K^{\sigma}{}_{\alpha\mu} K^{\alpha}{}_{\nu\sigma}, \tag{3.6}$$

where $\tilde{\nabla}$ is the covariant derivative defined in terms of Levi-Civita connection only.

Sometimes, instead of working with the torsion tensor, it is useful to express the results in terms of the torsion's contraction. In order to simplify the notation we define the following tensor:

$$T^{\lambda}{}_{\mu\nu} := S^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{\mu} S_{\nu} - \delta^{\lambda}_{\nu} S_{\mu}, \tag{3.7}$$

called the modified torsion tensor, and where $S_{\mu} := S^{\lambda}{}_{\mu\lambda}$.

In this work, we use a simplified torsion term which is only vectorial as described in the work of Capozziello *et al.* (2001). For this particular kind of torsion, the Ricci tensor is given by:

$$R = \tilde{R} - 2\tilde{\nabla}_{\alpha} T^{\alpha} - \frac{2}{3} T^{\alpha} T_{\alpha}. \tag{3.8}$$

Since we are assuming the existence of torsion, there are two main differences when performing the variations and the use of Gauss'theorem when integrating by parts, as compared to the purely metric formalism. Such differences are expressed in the following equations:

$$\delta R_{\mu\nu} = \nabla_{\alpha} \delta\Gamma^{\alpha}_{\nu\mu} - \nabla_{\nu} \delta\Gamma^{\alpha}_{\alpha\mu} + 2 S^{\alpha}{}_{\lambda\nu} \delta\Gamma^{\lambda}_{\alpha\mu}, \tag{3.9}$$

and

$$\nabla_{\mu} \mathcal{W}^{\mu} = \partial_{\mu} \mathcal{W}^{\mu} + 2 S^{\mu}{}_{\mu\nu} \mathcal{W}^{\nu}, \tag{3.10}$$

where $\mathcal{W}^\mu$ is a density tensor of weight $+1$.

Also, since we are interested in the non-relativistic weak field limit for our proposals, the metric is expanded as a Minkowskian background plus a small perturbation. The perturbations are given in factor terms of order $1/c$. For the purposes of this work, a second order perturbation will be enough, since it this is sufficient to explain the motion of mater and light particles at the non-relativistic level (Will, 1993). Taking as base the work of (Mendoza and Olmo, 2014), the metric coefficients at second perturbation order are given by eq. (2.24).

3.3. Warming up attempts

3.3.1. $f(\tilde{R}, T)$

Let us now make the assumption that the MONDian behavior of gravity is a physical effect due to the existence of torsion. The way to express this assumption is by the addition of torsion terms to the Hilbert action. Using eq. (3.8) as base, we propose the following action:

$$\mathcal{S}_2 = \frac{c^3}{16\pi G L_M^2} \int \sqrt{-g} \left[\tilde{R} + \kappa \left(\tilde{\nabla}_\alpha T^\alpha + T^\alpha T_\alpha \right)^b \right] \mathrm{d}^4 x + \frac{1}{c} \int \sqrt{-g} \mathcal{L}_\mathrm{m} \mathrm{d}^4 x, \quad (3.11)$$

where κ is a coupling constant. In this case the null variations are calculated with respect to the metric $g_{\mu\nu}$ and the modified torsion tensor T^μ. The field equations derivated from the action (3.11) are:

$$\begin{aligned} &\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} - \frac{1}{2} g_{\mu\nu} \kappa (\tilde{\nabla}_\alpha T^\alpha + T^\alpha T_\alpha)^b + \kappa b (\tilde{\nabla}_\alpha T^\alpha + T^\alpha T_\alpha)^{b-1} T_\mu T_\nu \\ &- \kappa b T_\nu \tilde{\nabla}_\mu \left[(\tilde{\nabla}_\alpha T^\alpha + T^\alpha T_\alpha)^{b-1} \right] = \frac{8\pi G}{c^4} \Sigma_{\mu\nu}, \end{aligned} \quad (3.12)$$

for the null variations with respect to the metric, and

$$2 T_\mu (\tilde{\nabla}_\alpha T^\alpha + T^\alpha T_\alpha)^{b-1} = \tilde{\nabla}_\mu \left[(\tilde{\nabla}_\alpha T^\alpha + T^\alpha T_\alpha)^{b-1} \right], \quad (3.13)$$

for the null variations with respect to the modified torsion. Eq.(3.13) is a differential equation for the torsion T^α and it can be substituted into (3.12), yielding a single field equation:

$$\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}-\frac{1}{2}g_{\mu\nu}\kappa(\tilde{\nabla}_\alpha T^\alpha+T^\alpha T_\alpha)^b-\kappa b(\tilde{\nabla}_\alpha T^\alpha+T^\alpha T_\alpha)^{b-1}T_\mu T_\nu=\frac{8\pi G}{c^4}\Sigma_{\mu\nu}. \quad (3.14)$$

From the latter equation we conclude that a relation between $\tilde{R}$ and Σ is not possible because eq.(3.13) is a differential equation involving only T, and not $\mathcal{L}_{\mathrm{m}}$).

Thus, in order to continue analysing this proposal, we need to make an extra assumption for the functional relation between T and Σ[1]. Let us assume the following:

$$\tilde{\nabla}_\alpha T^\alpha=0, \qquad \text{and} \qquad T_\alpha=\kappa'\tilde{\nabla}_\alpha\Sigma, \quad (3.15)$$

where κ' is a constant of proportionality. At first view, it seems that these assumptions are very arbitrary, but the first one is for simplicity and the second one is based on an order of magnitude analysis that will recover MONDian acceleration as will be further discussed.

With eqs. (3.15), expression (3.14) takes the following form:

$$\tilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tilde{R}=\frac{1}{2}g_{\mu\nu}\kappa\kappa'^{2b}(\partial_\alpha\Sigma\,\partial^\alpha\Sigma)^b+\kappa b\kappa'^{2b}(\partial_\alpha\Sigma\,\partial^\alpha\Sigma)^{b-1}\partial_\mu\Sigma\,\partial_\nu\Sigma, \quad (3.16)$$

where we have changed $\tilde{\nabla}$ by ∂_α and dropped the Newtonian-like $4\pi G\rho$ term since we are only interested in the MONDian regime of gravity. Contracting the previous equation and substituting the trace of the energy-momentum for dust, we obtain:

$$-\tilde{R}=(b+2)\kappa\kappa'^{2b}c^{4b}(\partial_\alpha\rho\,\partial^\alpha\rho)^b. \quad (3.17)$$

So far, we have not said anything about the constants κ and κ'. Due this freedom, we propose the the following constraint:

[1] By making this assumption, we are introducing additional information to the proposal, which makes it somewhat nonviable, but it will give us a good idea on to the correct path to follow.

$$\kappa\kappa'^{2b}c^{4b} \approx \frac{1}{c^2}. \tag{3.18}$$

This assumption implies that to second order perturbation, the term in parenthesis in equation (3.17) is a zeroth order term. For the metric (2.24), the Ricci scalar at second perturbation order and the term involving the matter density are respectively given by:

$$\tilde{R} = -\frac{2\nabla^2\phi}{c^2} \qquad \text{and} \qquad \partial_\alpha\rho\,\partial^\alpha\rho = -\nabla\rho\cdot\nabla\rho. \tag{3.19}$$

Thus, eq.(3.17) to second perturbation order is:

$$-\nabla\cdot\mathbf{a} = (b+2)(-1)^b\kappa\kappa'^{2b}c^{2(2b+1)}(\nabla\rho\cdot\nabla\rho)^b, \tag{3.20}$$

for the acceleration $\boldsymbol{a} = -\nabla\phi$.

To order of magnitude $\rho \approx M/r^3$ and $\nabla \approx 1/r$ and so, the previous equation is:

$$a \approx \kappa\kappa'^{2b}c^{2(2b+1)}M^{2b}r^{1-8b}. \tag{3.21}$$

MONDian acceleration has a r^{-1} dependence. In order to obtain that, the parameter

$$b = \frac{1}{4}. \tag{3.22}$$

With this value, the acceleration (3.21) is given by:

$$a \approx \kappa\kappa'^{1/2}c^3M^{1/2}r^{-1}. \tag{3.23}$$

The previous equation is important to our analysis. We have already obtained the correct dependence on M and r of the MONDian acceleration. Therefore, the constants κ and κ' depend exclusively on c, a_0 and G in the following form:

$$\kappa\kappa'^{1/2} \approx \frac{(a_0G)^{1/2}}{c^3}. \tag{3.24}$$

This approach represents an entirely local and covariant relativistic formulation of MOND. However, it cannot be an option to become a correct relativistic formulation of MOND because the assumptions (3.15) have no physical or mathematical support.

Despite this, the proposal gives us some clues towards the correct path to follow in order to enhance our theory.

3.3.2. $f(\tilde{R}, \tilde{\nabla}_\mu \mathcal{L}_\mathrm{m})$

The next logical step in order to construct a relativistic formulation of MOND consist in substituting the assumptions (3.15) on the action (3.11). As such, we propose the following action:

$$\mathcal{S}_3 = \frac{16\pi G}{c^3} \int \sqrt{-g} \left[\tilde{R} + \lambda \tilde{\nabla}_\mu \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}^\mu \mathcal{L}_\mathrm{m} \right) \right]^\gamma \mathrm{d}^4 x, \tag{3.25}$$

where λ is a coupling constant. This formulation, unlike the two previous, has only the metric as a dynamical variable. The field equations obtained from the null variations of the previous action with respect to the metric are given by:

$$\begin{aligned}
\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} = &\frac{1}{2} g_{\mu\nu} \lambda \left[\tilde{\nabla}_\alpha \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right) \right]^\gamma \\
&- \gamma\lambda \left[\tilde{\nabla}_\alpha \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right) \right]^{\gamma-1} \tilde{\nabla}_\mu \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} \right) \\
&- \frac{\gamma}{2} \lambda \left(\mathcal{L}_\mathrm{m} g_{\mu\nu} - \Sigma_{\mu\nu} \right) \mathcal{L}_\mathrm{m} \tilde{\Delta} \left[\left[\tilde{\nabla}_\alpha \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right) \right]^{\gamma-1} \right],
\end{aligned} \tag{3.26}$$

where the Laplace-Beltrami operator $\Delta := \nabla_\mu \nabla^\mu$. Contracting the latter expression with the metric $g^{\mu\nu}$ yields:

$$\tilde{R} = \lambda(\gamma - 2) \left[\tilde{\nabla}_\alpha \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right) \right]^\gamma + \frac{\gamma}{2} \lambda \left(4\mathcal{L}_\mathrm{m} - \Sigma \right) \mathcal{L}_\mathrm{m} \tilde{\Delta} \left[\left[\tilde{\nabla}_\alpha \left(\mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right) \right]^{\gamma-1} \right]. \tag{3.27}$$

For the case of dust, the previous equation yields:

$$\tilde{R} = \lambda(\gamma - 2) c^{4\gamma} \left[\tilde{\nabla}_\alpha \left(\rho \tilde{\nabla}^\alpha \rho \right) \right]^\gamma + \frac{3}{2} \gamma \lambda c^{4\gamma} \rho^2 \tilde{\Delta} \left[\left[\tilde{\nabla}_\alpha \left(\rho \tilde{\nabla}^\alpha \rho \right) \right]^{\gamma-1} \right]. \tag{3.28}$$

In order not to obtain dependence on the speed of light at second perturbation order on the terms in between parenthesis in the previous equation it is required that:

$$\lambda c^{4\gamma} \approx \frac{1}{c^2}. \tag{3.29}$$

At the same perturbation order, the terms involving ρ are of the zeroth order. Using the metric (2.24), such terms are:

$$\tilde{\nabla}_\alpha \left(\rho \tilde{\nabla}^\alpha \rho\right) = -\nabla \cdot (\rho \nabla \rho) \qquad \text{and} \qquad \tilde{\Delta}\psi = -\nabla^2 \psi. \tag{3.30}$$

Direct substitution of these last two expressions and relation (3.19), in eq. (3.28) yields:

$$\frac{2\nabla^2 \phi}{c^2} = -\lambda(\gamma - 2)c^{4\gamma}(-1)^\gamma \left[\nabla \cdot (\rho\nabla\rho)\right]^\gamma - \frac{3}{2}\gamma\lambda c^{4\gamma}(-1)^\gamma \rho^2 \nabla^2 \left[\left[\nabla \cdot (\rho\nabla\rho)\right]^{\gamma-1}\right]. \tag{3.31}$$

Based on the results of subsection 3.3.1, particularly on the ones in eqs. (3.22) and (3.24), we take the following values:

$$\gamma = \frac{1}{4}, \qquad \text{and} \qquad \lambda = \zeta \frac{(a_0 G)^{1/2}}{c^3}, \tag{3.32}$$

in order to obtain the following formula for the acceleration (given by eq.(3.31)):

$$-\nabla \cdot \mathbf{a} = \frac{(-1)^{1/4}}{4}\zeta (a_0 G)^{1/2} \left[\left(\frac{1}{2}\nabla^2 \rho^2\right)^{1/4} - \frac{3}{4}\rho^2 \nabla^2 \left[\left(\frac{1}{2}\nabla^2\rho^2\right)^{-3/4} \right] \right], \tag{3.33}$$

An order of magnitude calculation of the previous equation yields:

$$a \approx \frac{(a_0 G M)^{1/2}}{r}, \tag{3.34}$$

which is the right MONDian dependence for acceleration. For completeness, we must adjust the numerical value of ζ. This is accomplished solving analytically eq.(3.33), but this expression is very complicated to handle and so, we will not to solve eq.(3.33) directly. Instead in the following section we put together what we have learnt from subsections 3.3.1 and 3.3.2, in order to build a theory which in its weakest field limit yields a Poisson-like equation less complicated than the one of (3.33).

3.4. The final proposal

3.4.1. Field equations

With all the knowledge acquired from the previous attempts, let us start with the following action:

$$\mathcal{S}_4 = \omega \int \sqrt{-g} f(R)\, \mathrm{d}^4x + \omega' \int \sqrt{-g} \left[A \left(\tilde{\nabla}_\mu \mathcal{L}_\mathrm{m} \tilde{\nabla}^\mu \mathcal{L}_\mathrm{m} \right)^\eta + B \left(\mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} \right)^\eta \right] \mathrm{d}^4x. \tag{3.35}$$

where ω and ω' are the action's coupling constants. Since the action is a $f(R)$ function, there are two variables again, the connection (via torsion) and the metric. The resulting field equations are:

$$\begin{aligned}
&\omega \left(f' R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f \right) = \frac{1}{2} g_{\mu\nu} \omega' \left[A \left(\tilde{\nabla}_\mu \mathcal{L}_\mathrm{m} \tilde{\nabla}^\mu \mathcal{L}_\mathrm{m} \right)^\eta + B \left(\mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} \right)^\eta \right] \\
&- A\omega'\eta \left(\tilde{\nabla}_\alpha \mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right)^{\eta-1} \tilde{\nabla}_\mu \mathcal{L}_\mathrm{m} \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} - B\omega' \frac{\eta}{2} \left(\mathcal{L}_\mathrm{m} g_{\mu\nu} - \Sigma_{\mu\nu} \right) \tilde{\Delta} \left[\left(\mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} \right)^{\eta-1} \mathcal{L}_\mathrm{m} \right] \\
&- B\omega'\eta \left(\mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} \right)^{\eta-1} \left[\mathcal{L}_\mathrm{m} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} + \frac{1}{2} \left(\mathcal{L}_\mathrm{m} g_{\mu\nu} - \Sigma_{\mu\nu} \right) \tilde{\Delta} \mathcal{L}_\mathrm{m} \right] \\
&+ A\omega'\eta \left(\mathcal{L}_\mathrm{m} g_{\mu\nu} - \Sigma_{\mu\nu} \right) \tilde{\nabla}_\epsilon \left[\left(\tilde{\nabla}_\alpha \mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} \right)^{\eta-1} \tilde{\nabla}^\epsilon \mathcal{L}_\mathrm{m} \right],
\end{aligned} \tag{3.36}$$

for the null variations with respect to the metric, and:

$$\partial_\lambda f' \left(\delta^\mu_\tau \delta^\lambda_\sigma - \delta^\mu_\sigma \delta^\lambda_\tau \right) + 2 f' T^\mu{}_{\tau\sigma} = 0, \tag{3.37}$$

for the null variations with respect to the connection and $f' := \partial f / \partial R$. The corresponding traces of the previous equations are given by:

$$\begin{aligned}
&\omega\left(f'R-2f\right)=\omega'(2-\eta)\left[A\left(\tilde{\nabla}_\mu\mathcal{L}_\mathrm{m}\tilde{\nabla}^\mu\mathcal{L}_\mathrm{m}\right)^\eta+B\left(\mathcal{L}_\mathrm{m}\tilde{\Delta}\mathcal{L}_\mathrm{m}\right)^\eta\right]\\
&+A\omega\eta\left(4\mathcal{L}_\mathrm{m}-\Sigma\right)\tilde{\nabla}_\epsilon\left[\left(\tilde{\nabla}_\alpha\mathcal{L}_\mathrm{m}\tilde{\nabla}^\alpha\mathcal{L}_\mathrm{m}\right)^{\eta-1}\tilde{\nabla}^\epsilon\mathcal{L}_\mathrm{m}\right]\\
&-\frac{1}{2}B\omega\eta\left(4\mathcal{L}_\mathrm{m}-\Sigma\right)\left[\left(\mathcal{L}_\mathrm{m}\tilde{\Delta}\mathcal{L}_\mathrm{m}\right)^{\eta-1}\tilde{\Delta}\mathcal{L}_\mathrm{m}\tilde{\Delta}\left[\left(\mathcal{L}_\mathrm{m}\tilde{\Delta}\mathcal{L}_\mathrm{m}\right)^{\eta-1}\mathcal{L}_\mathrm{m}\right]\right]
\end{aligned}\tag{3.38}$$

and:

$$\partial_\sigma f'=\frac{2}{3}f'T_\sigma.\tag{3.39}$$

For the dust case, eq.(3.39) remains the same, while eq.(3.38) turns into:

$$\begin{aligned}
&\omega\left(f'R-2f\right)=\omega'(2-\eta)c^{4\eta}\left[A\left(\tilde{\nabla}_\mu\rho\tilde{\nabla}^\mu\rho\right)^\eta+B\left(\rho\tilde{\Delta}\rho\right)^\eta\right]\\
&+3A\omega\eta c^{4\eta}\rho\tilde{\nabla}_\epsilon\left[\left(\tilde{\nabla}_\alpha\rho\tilde{\nabla}^\alpha\rho\right)^{\eta-1}\tilde{\nabla}^\epsilon\rho\right]\\
&-\frac{3}{2}B\omega'\eta c^{4\eta}\rho\left[\left(\rho\tilde{\Delta}\rho\right)^{\eta-1}\tilde{\Delta}\rho+\tilde{\Delta}\left[\left(\rho\tilde{\Delta}\rho\right)^{\eta-1}\rho\right]\right].
\end{aligned}\tag{3.40}$$

Let us make the following assumption:

$$f(R)=R^d.\tag{3.41}$$

With this explicit relation, the traces (eqs.(3.40) and (3.39)) are given by:

$$\begin{aligned}
&\omega(d-2)R^d=\omega'(2-\eta)c^{4\eta}\left[A\left(\tilde{\nabla}_\mu\rho\tilde{\nabla}^\mu\rho\right)^\eta+B\left(\rho\tilde{\Delta}\rho\right)^\eta\right]\\
&+3A\omega\eta c^{4\eta}\rho\tilde{\nabla}_\epsilon\left[\left(\tilde{\nabla}_\alpha\rho\tilde{\nabla}^\alpha\rho\right)^{\eta-1}\tilde{\nabla}^\epsilon\rho\right]\\
&-\frac{3}{2}B\omega'\eta c^{4\eta}\rho\left[\left(\rho\tilde{\Delta}\rho\right)^{\eta-1}\tilde{\Delta}\rho+\tilde{\Delta}\left[\left(\rho\tilde{\Delta}\rho\right)^{\eta-1}\rho\right]\right]
\end{aligned}\tag{3.42}$$

and

$$T_\sigma = \frac{3}{2}(d-1)\frac{\partial_\sigma R}{R}. \tag{3.43}$$

3.4.2. MOND

Based on the results of subsections 3.3.1 and 3.3.2, we choose the following values:

$$d = 4, \qquad \eta = 1. \tag{3.44}$$

Direct substitution of these values into eqs. (3.42) and (3.43) yields:

$$2\omega R^4 = \omega' c^4 \left[A\tilde{\nabla}_\alpha \rho \tilde{\nabla}^\alpha \rho + (3A - 2B)\rho\tilde{\Delta}\rho \right], \tag{3.45}$$

and

$$T_\sigma = \frac{9}{2}\frac{\partial_\sigma R}{R}. \tag{3.46}$$

Let us analyse in more detail these expressions. From eq.(3.45) we obtain a relation $R = R(\rho)$ and substitution of this into eq.(3.46) yields $T = T(\rho)$. Thus, for a vectorial torsion (3.8) we find a relation $\tilde{R} = \tilde{R}(\rho)$. The end result of performing these substitutions yields a complicated expression and so, instead we perform an analogous procedure to the one followed by Barrientos and Mendoza (2016) and write eq.(3.8) as:

$$R = \tilde{R} + H(R), \tag{3.47}$$

in which we have used eq.(3.46) which allow us to express express $T_\mu = T_\mu(R)$. By performing Taylor expansion for $H(R)$, and keeping only terms up to the linear term in R, it follows that:

$$H(R) = \vartheta R + \mathcal{O}(R^2), \tag{3.48}$$

where ϑ is a constant. Substitution of this result in eq. (3.8) gives:

$$\tilde{R} = \vartheta' R \qquad \text{where} \qquad \vartheta' := 1 - \vartheta. \tag{3.49}$$

Direct substitution of this equation into eq. (3.45) yields:

$$\tilde{R} = \vartheta' c \left[\frac{\omega'}{2\omega} \right]^{1/4} \left[A \tilde{\nabla}_\alpha \rho \tilde{\nabla}^\alpha \rho + (3A - 2B) \rho \tilde{\Delta} \rho \right]^{1/4}. \tag{3.50}$$

Since we are only interested in second order terms of $1/c$, we require that the coupling constants ω and ω' must satisfy the following constraint:

$$\left[\frac{\omega'}{\omega} \right]^{1/4} \propto \frac{1}{c^3}. \tag{3.51}$$

From this restriction and using eqs.(3.19) and (3.30), the acceleration derived from eq.(3.50) to second perturbation order is given by:

$$\nabla \cdot \mathbf{a} = \vartheta' \frac{c^3}{2^{5/4}} \left[- \frac{\omega'}{\omega} \right]^{1/4} \left[A \nabla \rho \cdot \nabla \rho + (3A - 2B) \rho \nabla^2 \rho \right]^{1/4}, \tag{3.52}$$

which, to order of magnitude yields:

$$a \approx \left[\frac{\omega'}{\omega} \right]^{1/4} c^3 \frac{M^{1/2}}{r}. \tag{3.53}$$

In order to recover a MONDian acceleration, the coupling constants ω and ω' must satisfy the following condition:

$$\left[\frac{\omega'}{\omega} \right]^{1/4} \propto \frac{(a_0 G)^{1/2}}{c^3}. \tag{3.54}$$

Using Buckingham's theorem of dimensional analysis (see e.g. Sedov, 1959) with a_0, G and c as the independent variables, it follows that:

$$\omega = \Lambda \frac{c^{15}}{a_0^6 G}, \qquad \omega' = \Lambda' \frac{c^3 G}{a_0^4}, \tag{3.55}$$

which satisfy the requirement (3.54). Defining $\Lambda'/\Lambda := \Xi$ and using the previous expression for the coupling constants, eq.(B.12) is:

$$\nabla \cdot \mathbf{a} = \vartheta' \frac{(-\Xi)^{1/4}}{2^{5/4}} (G a_0)^{1/2} \left[A \nabla\rho \cdot \nabla\rho + (3A - 2B)\rho \nabla^2 \rho \right]^{1/4}. \tag{3.56}$$

Since we are looking for a Poisson-like equation as simply as possible, we choose $A = 1$ and $B = 3/2$, so that eq.(3.56) turns into:

$$\nabla \cdot \mathbf{a} = \vartheta \frac{(-\Xi)^{1/4}}{2^{5/4}} (G a_0)^{1/2} \left[\nabla\rho \cdot \nabla\rho \right]^{1/4}, \tag{3.57}$$

Solving analytically the last relation (see appendix A), the following value of Ξ is founded:

$$\Xi = -\frac{128\pi^2}{9\vartheta'^4}. \tag{3.58}$$

3.4.3. PPN consistency

In this analysis, we expand the metric $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.59}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowskian metric and $h_{\mu\nu}$ is a small perturbation. To first order on $h_{\mu\nu}$ (second order in $1/c^2$), the components of the Ricci tensor are given by:

$$^{(2)}\tilde{R}_{00} = \frac{1}{2}\nabla^2 h_{00}, \qquad \text{and} \qquad ^{(2)}\tilde{R}_{ij} = \frac{1}{2}\nabla^2 h_{ij}, \tag{3.60}$$

for the PPN gauge (see e.g Will, 1993).

Substituting the value $\eta = 1$, the functional form $f(R) = R^4$, the definition of Ξ and eqs. (3.55) into the full field eqs. (3.36), the following equation is obtained:

$$\begin{aligned} 4R^3 R_{\mu\nu} - \frac{1}{2}R^4 = \Xi \frac{(G a_0)^2}{c^{12}} \Bigg[& \frac{1}{2} g_{\mu\nu} \left(\tilde{\nabla}_\alpha \mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} + \frac{3}{2} \mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} \right) - \tilde{\nabla}_\mu \mathcal{L}_\mathrm{m} \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} \\ & - \frac{3}{2} \mathcal{L}_\mathrm{m} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} - \frac{1}{2} \left(\mathcal{L}_\mathrm{m} g_{\mu\nu} - \Sigma_{\mu\nu} \right) \tilde{\Delta} \mathcal{L}_\mathrm{m} \Bigg], \end{aligned} \tag{3.61}$$

with a trace given by:

$$2R^4 = \Xi \frac{(Ga_0)^2}{c^{12}} \left[\tilde{\nabla}_\alpha \mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} + \frac{3}{2} \mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} - \frac{1}{2} \left(4\mathcal{L}_\mathrm{m} - \Sigma \right) \tilde{\Delta} \mathcal{L}_\mathrm{m} \right]. \tag{3.62}$$

Using this relation in eq.(3.61), the field equations are:

$$\begin{aligned} 4R_{\mu\nu} = & \left(\Xi \frac{(Ga_0)^2}{c^{12}} \right)^{1/4} \left[\frac{3}{4} g_{\mu\nu} \left(\tilde{\nabla}_\alpha \mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} + \frac{3}{2} \mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} \right) - \tilde{\nabla}_\mu \mathcal{L}_\mathrm{m} \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} \right. \\ & \left. - \frac{3}{2} \mathcal{L}_\mathrm{m} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \mathcal{L}_\mathrm{m} - \frac{1}{2} \left(\mathcal{L}_\mathrm{m} g_{\mu\nu} - \Sigma_{\mu\nu} \right) \tilde{\Delta} \mathcal{L}_\mathrm{m} - \frac{1}{8} g_{\mu\nu} (4\mathcal{L}_\mathrm{m} - \Sigma) \tilde{\Delta} \mathcal{L}_\mathrm{m} \right] \\ & \times \left[\tilde{\nabla}_\alpha \mathcal{L}_\mathrm{m} \tilde{\nabla}^\alpha \mathcal{L}_\mathrm{m} + \frac{3}{2} \mathcal{L}_\mathrm{m} \tilde{\Delta} \mathcal{L}_\mathrm{m} - \frac{1}{2} \left(4\mathcal{L}_\mathrm{m} - \Sigma \right) \tilde{\Delta} \mathcal{L}_\mathrm{m} \right]^{-3/4}. \end{aligned} \tag{3.63}$$

Based on eq.(3.47), we can express $R_{\mu\nu}$ as:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + H_{\mu\nu}(R). \tag{3.64}$$

From eq.(3.48), we conclude:

$$H_{\mu\nu} = \vartheta_{\mu\nu} R, \tag{3.65}$$

where $\vartheta_{\mu\nu}$ is a second rank tensor.

Using eq.(3.64) for dust, the 00 component of eq.(3.63) at second order of approximation is given by:

$${}^{(2)}\tilde{R}_{00} + {}^{(2)}H_{00} = \frac{3(\Xi)^{1/4}}{2^{13/4}} \frac{(a_0 G)^{1/2}}{c^2} \left[\tilde{\nabla}_\alpha \rho \tilde{\nabla}^\alpha \rho + \rho \tilde{\Delta} \rho \right] \left[\tilde{\nabla}_\alpha \rho \tilde{\nabla}^\alpha \rho \right]^{-3/4}, \tag{3.66}$$

where we have used the fact that the derivatives with respect to the coordinate x^0 are of order $1/c$. Comparing this latter equation with eq. (3.57), we find the following relation:

$$\frac{1}{2}\nabla^2 h_{00} + {}^{(2)}H_{00} = -\frac{3}{4}\frac{\nabla^2\phi}{c^2\vartheta'} + G(\phi), \tag{3.67}$$

where we have already substituted eqs.(3.60), (3.19) and (3.30), and define $G(\phi)$ as:

$$G(\phi) = \frac{3(-\Xi)^{1/4}}{2^{13/4}}\frac{(a_0 G)^{1/2}}{c^2}\rho\nabla^2\rho\left[\nabla\rho\cdot\nabla\rho\right]^{-3/4}. \tag{3.68}$$

The explicit dependence in ϕ is given for the solution $\rho = \rho(\phi)$ obtained by solving eq.(3.57).

In order to be in agreement with the metric (2.24) employed in our exploration examples, the following relation must hold: $h_{00} = 2\phi/c^2$, and so:

$$^{(2)}H_{00} = -\frac{\nabla^2\phi}{c^2}\left(\frac{3}{4\vartheta'} + 1\right) + G(\phi). \tag{3.69}$$

Using eqs. (3.60) and (3.64), the spatial components of eq.(3.63) for dust are:

$$\begin{aligned}\frac{1}{2}\nabla^2 h_{ij} + {}^{(2)}H_{ij} = \frac{\Xi^{1/4}}{2^{5/4}}\frac{(a_0G)^{1/2}}{c^2}&\left[\frac{1}{4}g_{ij}\left(3\tilde{\nabla}_\alpha\rho\tilde{\nabla}^\alpha\rho + \rho\tilde{\Delta}\rho\right) - \tilde{\nabla}_i\rho\tilde{\nabla}_j\rho \;- \frac{3}{2}\rho\tilde{\nabla}_i\tilde{\nabla}_j\rho\right]\\ &\times\left[\tilde{\nabla}_\alpha\rho\tilde{\nabla}^\alpha\rho\right]^{-3/4},\end{aligned} \tag{3.70}$$

To handle this equation in a better way, we contract it with η^{ij}. Defining $H_3 := \eta^{ij}H_{ij}$ and $h_3 := \eta^{ij}h_{ij}$, eq.(3.70) turns into:

$$\begin{aligned}\frac{1}{2}\nabla^2 h_3 + {}^{(2)}H_3 = &\frac{\Xi^{1/4}}{2^{5/4}}\frac{(a_0G)^{1/2}}{c^2}\left[\frac{3}{4}\left(3\tilde{\nabla}_\alpha\rho\tilde{\nabla}^\alpha\rho + \rho\tilde{\Delta}\rho\right) - \tilde{\nabla}_i\rho\tilde{\nabla}^i\rho - \frac{3}{2}\rho\tilde{\nabla}_i\tilde{\nabla}^i\rho\right]\\ &\times\left[\tilde{\nabla}_\alpha\rho\tilde{\nabla}^\alpha\rho\right]^{-3/4}.\end{aligned} \tag{3.71}$$

Using eqs. (3.19), (3.30) and (3.68) and comparing with eq.(3.56), the latter expression can be expressed as:

$$\frac{1}{2}\nabla^2 h_3 + {}^{(2)}H_3 = -\frac{5}{4}\frac{\nabla^2\phi}{c^2\vartheta'} - G(\phi). \tag{3.72}$$

Since we are looking for H_{ij} in order to have $h_{ij} = (2\phi/c^2)\,\delta_{ij}$, therefore:

$$^{(2)}H_3 = \frac{\nabla^2\phi}{c^2}\left(3 - \frac{5}{4\vartheta'}\right) - G(\phi), \tag{3.73}$$

and because we are working in an isotropic frame, we conclude that:

$$^{(2)}H_{ij} = -\frac{\nabla^2\phi}{c^2}\left(1 - \frac{5}{12\vartheta'}\right)\delta_{ij} + \frac{1}{3}G(\phi)\delta_{ij}. \tag{3.74}$$

In order to keep the contribution of $H_{\mu\nu}$ as small as possible, we choose the following values:

$$\vartheta' = \frac{5}{12} \qquad \text{and} \qquad \vartheta = \frac{7}{12}, \tag{3.75}$$

which guarantee a sufficiently small value of $^{(2)}H_{ij}$ given by the second term on the right hand side of equation (3.74).

3.5. Discussion

As mentioned in the introduction, many proposals of extended theories of gravity have been constructed. Recently, a new approach by Verlinde (2016) yields an estimate of the excess gravity in terms of the baryonic mass distribution and the Hubble parameter. In a first astrophysical test, this approach has been able to account reasonably well for the expected lens signal of low redshift galaxies (Brouwer *et al.*, 2016). Despite this, it is not very clear from the theoretical developments of the theory how to apply such results to an extended system such as a cluster of galaxies.

From the very early stages in the introduction of torsion onto gravitational phenomena, it has never been thought as to which effect it can produce. Furthermore, it has never become clear how it can affect standard gravitational interactions. In this work, we have shown that if we want to understand MONDian phenomenology in the relativistic regime, we require to extend gravity in such a way that the functional action $f(R, \mathcal{L}_\mathrm{m})$ has the following form -see eq.(3.35):

$$f(R,\mathcal{L}_{\mathrm{m}}) = \omega R^4 - \omega' \left[\left(\tilde{\nabla}_\mu \mathcal{L}_{\mathrm{m}} \tilde{\nabla}^\mu \mathcal{L}_{\mathrm{m}} \right) + \frac{3}{2} \left(\mathcal{L}_{\mathrm{m}} \tilde{\Delta} \mathcal{L}_{\mathrm{m}} \right) \right], \tag{3.76}$$

where:

$$\omega = \frac{5^4 c^{15}}{2^{15} a_0^6 G} \approx \frac{0{,}02 c^{15}}{a_0^6 G}, \qquad \text{and} \qquad \omega' = \frac{9\pi^2 c^3 G}{a_0^4}. \tag{3.77}$$

This formalism is fully covariant and local and so, unlike many of the previous attempts built to generalise MOND to a relativistic regime it can be tested in many astrophysical systems, such as weak and strong lensing of individual, groups and clusters of galaxies. It can also be applied for a Friedmann-Lemaître-Robertson-Walker universe and test the behaviour of the large-scale universe at the present epoch. We intend to deal with all these problems elsewhere.

The departures introduced in the matter sector of the action (3.76) with respect to the classical matter action $\mathcal{L}_{\mathrm{m}}$, brings with it some theoretical concerns since it is not clear that such a choice would lead e.g. to geodesic trayectories, but this is a much broader subject to discuss in the present article. However, the motivation of choosing this particular action comes from the field equations at the non-relativistic level, since at this level of approximation, the field equations can be expressed as:

$$\left(\nabla^2 \phi \right)^4 \approx \left(\nabla \rho \right)^2. \tag{3.78}$$

In terms of the mass M, the radial coordinate r and the acceleration a, at order of magnitude, the previous equation can be written as:

$$\left(\frac{a}{r} \right)^4 \approx \left(\frac{M}{r^4} \right)^2. \tag{3.79}$$

This last expression yields the correct mass and radial dependence for the MONDian acceleration. Therefore, our choice (3.76) was made in order to recover the dependence (3.79). From the above simple calculation, this choice is not unique and others actions can be built in order to achieve (3.79). Such actions may in principle contain the theoretical issues that the approach introduced in this work presents.

The fact that the matter Lagrangian appears inside the gravitational action contradicts the precise measurements performed on Earth and on the solar system with respect to this fact. As it has been noted all throughout the article, the MONDian behaviour of gravity occurs at mass to lenght ratios quite different from the characteristic ones associated to the solar system. In this respect, the proposal constructed in this article cannot be applied to any mass to length ratio system similar to those of the solar system. It can only be applied to systems where that ratio is much less than one, in which essentially the equivalent Newtonian gravitational acceleration is $\lesssim a_0$. It is precisely on these systems where the matter Lagrangian will appear inside the gravitational action.

The main conclusion that we can derive from this work is that in order to recover a MONDian acceleration from a $F(R)$ theory, derivatives of the matter Lagrangian must be present in the field equations. The proposal of a matter Lagrangian function appearing on the gravitational action is not new and has been studied previously (Harko *et al.*, 2014b; Pani *et al.*, 2013). The posibility of building similar field equations from a gravitational action that does not involve derivatives of the matter Lagrangian and satisfies standard conservation laws is beyond the scope of this work, but will be studied by us in future research.

Chapter 4

Metric-affine $f(R,T)$ theories of gravity and their applications

4.1. Introduction

Modified theories of gravity are a mainstream topic in modern cosmology, essentially due to the discovery of the late-time cosmic accelerated expansion (Perlmutter *et al.*, 1999; Riess *et al.*, 1998). These theories assume that Einstein's General Relativity (GR) breaks down at large scales and that an extension of the Einstein-Hilbert action describing the gravitational field is necessary, offering an alternative paradigm fundamentally distinct from dark energy models of cosmic acceleration (Copeland *et al.*, 2006; Nojiri *et al.*, 2017). Further physical motivations for these theories include a more realistic representation of quantum and gravitational fields at high-energy densities near curvature singularities, and the possibility to create some effective first order approximation of quantum gravity (Parker and Toms, 2009; Buchbinder *et al.*, 1992). The simplest such extension of GR is perhaps to consider a Lagrangian density given by a certain function $f(R)$, where R is the scalar curvature, whose phenomenology has been largely explored in the literature (Sotiriou and Faraoni, 2010; Nojiri and Odintsov, 2011; De Felice and Tsujikawa, 2010).

An interesting generalization of $f(R)$ gravity involves the inclusion of a nonminimal coupling between the scalar curvature and matter (Goenner, 1984; Bertolami *et al.*, 2007; Nojiri and Odintsov, 2004; Allemandi *et al.*, 2005). One of the original motivations to implement this coupling was to establish a link with MOND and the flat galactic rotation curves. It was further shown that this curvature-matter coupling induces a non-vanishing covariant derivative of the energy-momentum tensor, which implies nongeodesic motion and consequently leads to the appearance of an extra force (Bertolami *et al.*, 2007). Thus, these models allow for an explicit violation of the equivalence principle (EP), which is tightly constrained by solar system experimental tests (Bertolami *et al.*, 2006), by imposing a matter-dependent deviation from geodesic motion. Low-energy features of specific compactified versions of higher-dimensional theories also imply the EP violation (Overduin, 2000). However, it has been argued that the EP is not one of the "universal" principles of physics (Damour, 2001), but rather it is a heuristic hypothesis introduced by Einstein, and used to construct his theory of GR. Further tests of the EP are relevant for new physics and strongly constrain the parameters of the theory (Damour, 1996; Damour and Donoghue, 2010). However, it is important to note that the violation of the EP does not in principle rule out the specific theory.

The linear nonminimal curvature-matter coupling (Bertolami *et al.*, 2007) was further generalized by considering a maximal extension of the Einstein-Hilbert action, namely, $f(R, \mathcal{L}_m)$ gravity (Harko and Lobo, 2010), where $\mathcal{L}_m$ is the matter Lagrangian. A related theory is $f(R, T)$ gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar and the trace T of the energy-momentum tensor $T_{\mu\nu}$ (Harko *et al.*, 2011a). All of these theories induce the presence of an extra force and consequently nongeodesic motion. An interesting cosmological motivation for $f(R, T)$ gravity is that it may be considered a relativistically covariant model of interacting dark energy (Harko *et al.*, 2011a). Note that the dependence from T may be induced by exotic imperfect fluids or quantum effects (conformal anomaly). A physical interpretation consists on the possibility that the curvature-matter coupling is related to the thermodynamics of open systems, and is responsible for matter creation irreversible processes that may take place at a cosmological scale (Harko, 2014; Harko *et al.*, 2015). Fundamental applications of the curvature-matter couplings in the study of quantum gravitational theories with first order quantum corrections induced by a stochastically fluctuating metric have also been analysed (Liu *et al.*, 2016). It is interesting to note that in recent work (Avelino and Sousa, 2018; Avelino and Azevedo, 2018), it was argued that the on-shell Lagrangian of a perfect fluid depends on microscopic properties of the fluid, and consequently it was shown that if the fluid is constituted by localized concentrations of energy with fixed rest mass and structure (solitons) then the average on-shell Lagrangian of a perfect fluid is given by $\mathcal{L}_m = T$. Thus, this seems to indicate that, in this context, $f(R,\mathcal{L}_m)$ theories may be regarded as a subclass of $f(R,T)$ gravity. Further arguments in favor of these theories are found on the fact that the relativistic behavior of a Tully-Fisher law observed in the rotation of galaxies can be modelled with a $f(R,T)$ or $f(R,\mathcal{L}_m)$ description, as shown in (Mendoza, 2015), which is coherent with lensing observations of individual, groups and clusters of galaxies. The literature of $f(R,T)$ gravity is extremely vast and we refer the reader to the review (Harko and Lobo, 2014) for further motivations and applications.

The current approach to $f(R,T)$ theories is framed within the so-called metric formulation, where the affine structure of the spacetime geometry is dictated by the metric tensor[1]. Other approaches, however, are possible. In fact, if one allows the connection to vary independently of the metric tensor, the so-called metric-affine or Palatini approach, the resulting field equations typically lead to different dynamics, offering alternative avenues to explore new gravitational physics. The curvature-matter coupling in metric-affine approach has been scarcely considered in the literature (Harko *et al.*, 2011b), with the main highlight being that the independent connection can be expressed as the Levi-Civita connection of an auxiliary (matter Lagrangian-dependent) metric, which is related with the physical metric by means of a conformal transformation. Analogously to the metric case (Bertolami *et al.*, 2007), the field equations impose the nonconservation of the energy-momentum tensor. In this framework, the FLRW equations for brane-world cosmology and loop quantum cosmology can be derived out of a quadratic $f(R)$ theory plus a nonminimal linear coupling between matter and curvature (Olmo and Rubiera-Garcia, 2015). Let us also point out that generalized descriptions of galaxies rotation curves have been previously implemented in the literature using a metric-affine formalism with torsion included in the description of the gravitational action (Barrientos and Mendoza,

[1]It has been argued in (Alvarenga *et al.*, 2013) that in this approach models of the form $f(R,T) = f_1(R) + f_2(T)$ yield a scale-dependent behavior of scalar cosmological perturbations that is disfavored by observational data, severely limiting the viability of such models.

2016, 2017).

The main aim of this work is to address in detail $f(R,T)$ theories in this, so far quite unexplored, alternative metric-affine view. We will show that the resulting theories are radically different in some aspects from their metric counterparts, though they share many resemblances with their $f(R)$ relatives. In fact, the study of modified theories of gravity in metric-affine scenarios involving torsion and nonmetricity has received a continuous interest in the last two decades, with several review articles focused on those topics (Hehl *et al.*, 1995; Olmo, 2011; Shapiro, 2002). This work will pave the path for future studies of $f(R,T)$ theories in geometric scenarios where torsion and nonmetricity are not *a priori* constrained to vanish. We note in this regard that whether the spacetime structure is Riemannian or otherwise is a foundational question of gravitational physics that must be answered empirically, not decided by convention or on practical terms.

This paper is organized as follows: In Sec. 4.2, we present the formalism of $f(R,T)$ gravity in the metric-affine approach, focussing on the role of the curvature-matter coupling in the equations of motion, the conservation equation, and the geodesic motion and presence of a fifth force. In Sec. 4.3, we trace out the weak field limit and show that the modified Poisson equation is formally identical to that found in Eddington-inspired Born-Infeld gravity. In Sec. 4.4, we present several specific applications, such as the stellar structure equations, and in the presence of electromagnetic fields and scalar fields. Finally, in Sec. 4.5, we summarize our results and depict some future applications.

4.2. Theory, formulation, and equations of motion

To introduce the action of $f(R,T)$ gravity in the metric-affine approach one needs to bear in mind that only the affine connection $\Gamma^{\lambda}_{\mu\nu}$ is needed to define the Ricci tensor, which follows from the Riemann tensor defined by eq. (3.4), as $R_{\mu\nu}(\Gamma) := R^{\alpha}{}_{\mu\alpha\nu}(\Gamma)$ (no indices lowered/raised with the metric). Subsequent contraction with the metric $g_{\mu\nu}$ allows to define the curvature scalar as $R := g^{\mu\nu}R_{\mu\nu}(\Gamma)$. This guarantees that only the symmetric part of the Ricci tensor enters into the action, which significantly simplifies the role of torsion, making it irrelevant if fermions are not considered (Alfonso *et al.*, 2017). Throughout this work, we assumme the $(-,+,+,+)$ signature. With these elements the action considered in this work takes the form

$$\mathcal{S} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R,T) + \int d^4x\sqrt{-g}\mathcal{L}_m(g_{\mu\nu},\psi_m)\ , \tag{4.1}$$

with the following definitions and conventions: κ^2 is some constant with suitable dimensions (in GR, $\kappa^2 = 8\pi G$), g is the determinant of the spacetime metric $g_{\mu\nu}$, the factor $f(R,T)$ is an arbitrary function of the curvature scalar R and the trace of the energy-momentum tensor, $T := g^{\mu\nu}T_{\mu\nu}$, which is defined by eq. (2.7). Finally, the standard matter Lagrangian density $\mathcal{L}_m$ depends on the matter fields ψ_m and the metric $g_{\mu\nu}$, but not on the independent connection $\Gamma^\lambda_{\mu\nu}$.

The variation of the action (4.1) can be conveniently expressed as

$$\begin{aligned}\delta\mathcal{S} &= \int\frac{d^4x\sqrt{-g}}{2\kappa^2}\Big[f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + f_T\frac{\delta T}{\delta g^{\mu\nu}} - \kappa^2 T_{\mu\nu}\Big]\delta g^{\mu\nu} \\ &+ \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f_R g^{\mu\nu}\delta R_{\mu\nu}(\Gamma)\,, \end{aligned}\tag{4.2}$$

where we have defined $f_R := df/dR$ and $f_T := df/dT$ and split the variation into two lines to highlight the variations with respect to the metric and with respect to the affine connection, respectively. Now using the fact that the variation of T with respect to $g_{\mu\nu}$ can be written as

$$\frac{\delta T}{\delta g^{\mu\nu}} = T_{\mu\nu} + \Theta_{\mu\nu}\ , \tag{4.3}$$

where

$$\Theta_{\mu\nu} := g^{\lambda\rho}\frac{\delta T_{\lambda\rho}}{\delta g^{\mu\nu}}\ , \tag{4.4}$$

then the variation of Eq. (4.2) with respect to $g_{\mu\nu}$ can be expressed as

$$f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f = \kappa^2\tau_{\mu\nu}\ , \tag{4.5}$$

where we have introduced the *effective* energy-momentum tensor

$$\tau_{\mu\nu} = T_{\mu\nu}\left(1 - \frac{f_T}{\kappa^2}\right) - \frac{f_T}{\kappa^2}\Theta_{\mu\nu}\ , \tag{4.6}$$

which plays a key role in the dynamics of these models, as shall be clear later. On the other hand, from the variation of the Ricci tensor in (4.2), after integration by parts and a bit of algebra one finds[2]

$$\nabla_\lambda^\Gamma(\sqrt{-g} f_R g^{\mu\nu}) = 0 \,. \tag{4.7}$$

The two sets of Eqs. (4.5) and (4.7) can be written in a more suitable form by noting that the contraction of (4.5) with the metric $g^{\mu\nu}$ yields the result

$$R f_R - 2f = \kappa^2 \tau \,, \tag{4.8}$$

where $\tau := g^{\mu\nu}\tau_{\mu\nu}$. Note that (4.8) is an algebraic equation rather than a differential one and implies that, like in the metric-affine $f(R)$ case, the curvature scalar is a function of the matter sources only. This allows to introduce a new rank-two tensor $h_{\mu\nu}$ such that the connection equations (4.7) can be expressed as $\nabla_\lambda^\Gamma(\sqrt{-h}h^{\mu\nu}) = 0$, which implies the conformal relation

$$h_{\mu\nu} = f_R g_{\mu\nu} \,, \tag{4.9}$$

between these two metrics. This way, the affine connection $\Gamma^\lambda_{\mu\nu}$ is given by the Christoffel symbols of the metric $h_{\mu\nu}$, i.e.,

$$\Gamma^\lambda_{\mu\nu} = \frac{h^{\lambda\alpha}}{2}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}\right) \,. \tag{4.10}$$

Now, contracting Eqs. (4.5) with $h^{\alpha\mu}$, using the conformal relation (4.9), and rearranging terms one arrives at

$$R^\mu{}_\nu(h) = \frac{\kappa^2}{f_R^2}\left(\tau^\mu{}_\nu + \frac{f(R,T)}{2\kappa^2}\delta^\mu{}_\nu\right) \,, \tag{4.11}$$

where $R^\mu{}_\nu(h) := h^{\mu\alpha}R_{\alpha\nu}$. Written in this form, Eqs. (4.11) become (for any $f(R,T)$ function) a system of second-order differential Einstein-like field equations for the metric $h_{\mu\nu}$, with all the terms on the right-hand side being functions of the matter

[2] For a detailed derivation of these equations including torsion, see (Alfonso *et al.*, 2017).

sources, and representing a natural generalization of metric-affine $f(R)$ theories with the f_T-corrections encoded in the effective energy-momentum tensor $\tau^\mu{}_\nu$ of Eq. (4.6). After solving these equations for $h_{\mu\nu}$ one just needs to use the conformal relation (4.9) to find the spacetime metric $g_{\mu\nu}$. A corollary of these features is that, in vacuum, $T_\mu{}^\nu = 0$, all the terms on the right-hand side vanish, one finds that $h_{\mu\nu} = g_{\mu\nu}$ (modulo a trivial rescaling), and the same vacuum solutions of GR (with possibly a cosmological constant term) are recovered. This implies that the propagating degrees of freedom present in these theories are the same as those in GR.

4.2.1. The role of the curvature-matter coupling

To fully specify these theories of gravity one needs not only the particular dependence on the scalar curvature but also the matter Lagrangian density $\mathcal{L}_m$. Once the latter is given, one can compute explicitly the object $\Theta_{\mu\nu}$ in Eq.(4.4) as (Harko *et al.*, 2011a)

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m - 2g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\mu\nu}} . \tag{4.12}$$

This expression allows to rewrite $\tau^\mu{}_\nu$ in Eq. (4.6) into the more suggestive form

$$\tau^\mu{}_\nu = T^\mu_{(I)\nu} + \frac{f_T}{\kappa^2}\left[T^\mu_{(I)\nu} - T^\mu_{(II)\nu}\right] , \tag{4.13}$$

where for convenience we have introduced the tensors

$$T^\mu_{(I)\nu} = -2g^{\mu\rho}\frac{\partial\mathcal{L}_m}{\partial g^{\rho\nu}} + \mathcal{L}_m\delta^\mu_\nu , \tag{4.14}$$

$$T^\mu_{(II)\nu} = -2g^{\mu\rho}g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\rho\nu}} + \mathcal{L}_m\delta^\mu_\nu . \tag{4.15}$$

The first one corresponds to the standard energy-momentum tensor defined in Eq. (2.7), while the second one is a generalization involving second metric derivatives of the matter Lagrangian density. This structure suggests that it should be possible to consider more general theories containing additional couplings between gravity and the matter fields in this context. In particular, a family of $f(R,\tau)$ theories, with $\tau := \tau^\mu{}_\mu$, would lead to an extension involving terms with three derivatives of $\mathcal{L}_m$ with respect to the metric, and so on.

A case of general interest for the matter fields is represented by a perfect fluid, whose energy-momentum tensor is of the form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}\,, \tag{4.16}$$

where u^μ is the unit timelike vector, $u_\mu u^\mu = -1$, while ρ and P are the energy density and pressure of the fluid, respectively. For this matter source, we assume that $\mathcal{L}_m = P$ as the matter Lagrangian density [3] which, from Eq. (4.12), yields

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - P g_{\mu\nu}\,. \tag{4.17}$$

Inserting this result in Eq. (4.11), one finds

$$R^\mu{}_\nu(h) = \frac{1}{f_R^2}\left[(\kappa^2 + f_T)T^\mu{}_\nu + \left(\frac{f}{2} + P f_T\right)\delta^\mu{}_\nu\right]. \tag{4.18}$$

From this expression, it is easy to verify that the limit $P \to 0$ recovers the same dynamics as metric-affine $f(R)$ theories but with a varying effective Newton's constant, namely, $\kappa^2_{eff} = \kappa^2 + f_T$, with f_T a function of ρ. If we further restrict to the case f_T =constant, then the correspondence is exact. This puts forward that the family of models $f(R,T) = f(R) + \epsilon T$ only departs from the $f(R)$ case in scenarios where the fluid pressure becomes relevant as compared to the term $f(R,T)/2$.

4.2.2. Conservation equation

Let us now work out the analogous of the conservation equation in these theories. First we rewrite the field equations (4.11) as

$$G^\mu{}_\nu(h) = \frac{\kappa^2}{f_R^2}\left[\tau^\mu{}_\nu - \frac{\delta^\mu_\nu}{2}\left(\tau + \frac{f}{\kappa^2}\right)\right]. \tag{4.19}$$

Taking a covariant derivative on both sides on this equation and using Bianchi's identities, $\nabla^{(h)}_\mu G^\mu{}_\nu(h) = 0$ (the superindex h indicates covariant derivatives defined

[3] For an extended discussion on the well known problem of whether $\mathcal{L}_m = P$ or $\mathcal{L}_m = -\rho$ is the right Lagrangian of a perfect fluid, and its consequences for nonminimally coupled theories see e.g. (Faraoni, 2009; Bertolami *et al.*, 2008).

with the independent connection $\Gamma^\lambda_{\mu\nu}$), one finds

$$\nabla^{(h)}_\mu \tau^\mu{}_\nu - \frac{1}{2}\partial_\nu \left(\tau + \frac{f}{\kappa^2}\right) - 2\partial_\mu \ln f_R \left[\tau^\mu{}_\nu - \frac{\delta^\mu_\nu}{2}\left(\tau + \frac{f}{\kappa^2}\right)\right] = 0 \,. \tag{4.20}$$

On the other hand, the relation between covariant derivatives defined with the independent connection and those defined with the connection associated to the Christoffel symbols of the metric, $\nabla^{(g)}_\mu$, is obtained as

$$\nabla^{(h)}_\mu \tau^\mu{}_\nu = \nabla^{(g)}_\mu \tau^\mu{}_\nu + C^\mu_{\mu\lambda}\tau^\lambda{}_\mu - C^\lambda_{\mu\nu}\tau^\mu{}_\lambda \,, \tag{4.21}$$

where

$$C^\alpha_{\mu\nu} = \frac{h^{\alpha\rho}}{2}\left[\nabla^{(g)}_\mu h_{\rho\nu} + \nabla^{(g)}_\nu h_{\rho\mu} - \nabla^{(g)}_\rho h_{\mu\nu}\right] \,. \tag{4.22}$$

Now, using the conformal relation (4.9) and after a bit of algebra upon the relation above one arrives at

$$\nabla^{(h)}_\mu \tau^\mu{}_\nu = \nabla^{(g)}_\mu \tau^\mu{}_\nu + 2\tau^\lambda{}_\nu \partial_\lambda \ln f_R - \frac{\tau}{2}\partial_\nu \ln f_R \,. \tag{4.23}$$

Plugging this result into the nonconservation equation (4.20) yields

$$\nabla^{(g)}_\mu \tau^\mu{}_\nu + \left(\frac{\tau}{2} + \frac{f}{\kappa^2}\right)\frac{\partial_\nu f_R}{f_R} - \partial_\nu \left(\frac{\tau}{2} + \frac{f}{2\kappa^2}\right) = 0 \,. \tag{4.24}$$

Using now the trace equation (4.8) to consider the combinations

$$\frac{1}{2}\left(\tau + \frac{f}{\kappa^2}\right) = \frac{1}{2\kappa^2}(Rf_R - f) \,, \tag{4.25}$$

$$\frac{\tau}{2} + \frac{f}{\kappa^2} = \frac{1}{2\kappa^2} R f_R \,, \tag{4.26}$$

and after some manipulations we finally obtain the result

$$\nabla^{(g)}_\mu \tau^\mu{}_\nu = -\frac{f_T}{2\kappa^2}\partial_\nu T \,, \tag{4.27}$$

implying that the effective energy-momentum tensor $\tau^\mu{}_\nu$ is conserved only when the term $f_T \partial_\nu T$ vanishes. This has nontrivial consequences regarding several contexts, in particular, stellar structure, as shall be seen in Sec. 4.4 below.

4.2.3. Geodesic equation and extra force

In order to compute the geodesic equation obtained from the nonconservation equation (4.27), let us substitute the relation (4.12) into the definition of $\tau_{\mu\nu}$ given by Eq. (4.6), to obtain

$$\tau_{\mu\nu} = T_{\mu\nu} + 2\frac{f_T}{\kappa^2}\left(g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\mu\nu}} - \frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}}\right), \tag{4.28}$$

where we have used the expression of the energy-momentum tensor given by Eq. (4.14). Therefore, Eq. (4.27) implies that

$$\nabla^{(g)}_\mu T^\mu{}_\nu = \frac{2}{\kappa^2}\nabla^{(g)}_\mu\left[f_T g^{\mu\lambda}\left(\frac{\partial\mathcal{L}_m}{\partial g^{\lambda\nu}} - g^{\alpha\beta}\frac{\partial^2\mathcal{L}_m}{\partial g^{\alpha\beta}\partial g^{\lambda\nu}}\right)\right] - \frac{f_T}{2\kappa^2}\partial_\nu T. \tag{4.29}$$

On the other hand, since the matter current conservation relation $\nabla^{(g)}_\mu(\rho u^\mu) = 0$ implies that the quantity $u^\mu\rho\sqrt{-g}$ is conserved, therefore the differential of this quantity is null. With this and using the fact that $2\delta u^\mu = u_\nu\delta g^{\mu\nu}$ and $2\delta\sqrt{-g} = \sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ we obtain the following relation:

$$\delta\rho = \frac{1}{2}\rho(g_{\mu\nu} + u_\mu u_\nu)\delta g^{\mu\nu}, \tag{4.30}$$

which facilitates the computation of $\partial\mathcal{L}_m/\partial g^{\lambda\nu}$ and $\partial^2\mathcal{L}_m/\partial g^{\alpha\beta}\partial g^{\lambda\nu}$ on the right-hand side of (4.29). With this last expression, the energy-momentum tensor (4.14) is given by

$$T_{\mu\nu} = -\rho u_\mu u_\nu\frac{d\mathcal{L}_m}{d\rho} + g_{\mu\nu}\left(\mathcal{L}_m - \rho\frac{d\mathcal{L}_m}{d\rho}\right). \tag{4.31}$$

Using Eq. (4.30) to express the derivatives of the matter Lagrangian with respect to the metric as derivatives with respect to ρ in Eq. (4.29) yields

$$\begin{aligned}&\nabla^{(g)}_\mu\left[-\rho u^\mu u_\nu\frac{d\mathcal{L}_m}{d\rho} + \delta^\mu_\nu\left(\mathcal{L}_m - \rho\frac{d\mathcal{L}_m}{d\rho}\right)\right] = \\ &\nabla^{(g)}_\mu\left[\frac{f_T}{2\kappa^2}\left(\rho(u^\mu u_\nu + \delta^\mu_\nu)\left(\frac{d\mathcal{L}_m}{d\rho} - 3\rho\frac{d^2\mathcal{L}_m}{d\rho^2}\right)\right)\right] - \frac{f_T}{2\kappa^2}\partial_\nu T.\end{aligned} \tag{4.32}$$

Now, by taking the divergences in the previous relation and recalling the well-known relation

$$u^\nu \nabla^{(g)}_\nu u^\mu = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\lambda\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds}, \tag{4.33}$$

and expressing: $\partial_\nu T = \frac{\partial T}{\partial\rho}\partial_\nu\rho$ where the trace of the energy-momentum, according to Eq. (4.31) is given by:

$$T = 4\mathcal{L}_m - 3\rho\frac{d\mathcal{L}_m}{d\rho}, \tag{4.34}$$

the geodesic equation of this metric-affine $f(R,T)$ theory is provided by

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\lambda\nu}\frac{dx^\lambda}{ds}\frac{dx^\nu}{ds} = f^\mu, \tag{4.35}$$

where the extra force f^μ is given by

$$f^\mu = -\nabla^{(g)}_\nu \ln\left[\frac{d\mathcal{L}_m}{d\rho} + \frac{f_T}{2\kappa^2}\left(\frac{d\mathcal{L}_m}{d\rho} - 3\rho\frac{d^2\mathcal{L}_m}{d\rho^2}\right)\right](g^{\mu\nu} + u^\mu u^\nu). \tag{4.36}$$

In other words, in this formulation the particles follow geodesic trajectories if and only if $f^\mu = 0$.

To illustrate the above statement, note that for the case of dust, this extra force takes the following expression:

$$f^\mu_{dust} = -(g^{\mu\nu} + u^\mu u^\nu)\nabla^{(g)}_\nu \ln\left(1 + \frac{f_T}{2\kappa^2}\right) \tag{4.37}$$

It is clear from this last relation that the extra force vanishes only for the case $f_T = 0$, i.e., $f(R,T)$ is only a function of R, which coincides with the standard metric-affine approach of $f(R)$ gravity (see for example the direct calculation of this made in (Koivisto, 2006)).

4.3. Weak field, slow-motion limit

To investigate the weak field limit of these theories, we start from the conformal relation (4.9), whose perturbation can be expressed as

$$\delta g_{\mu\nu} = \frac{\delta h_{\mu\nu}}{f_R} - \frac{h_{\mu\nu}}{f_R^2}\delta f_R \,. \tag{4.38}$$

Now let us introduce perturbations upon a Minkowski background, namely, $h_{\mu\nu} \approx \eta_{\mu\nu} + \bar{t}_{\mu\nu}$ and $g_{\mu\nu} \approx \eta_{\mu\nu} + t_{\mu\nu}$, where $\bar{t}_{\mu\nu} \ll \eta_{\mu\nu}$ and $t_{\mu\nu} \ll \eta_{\mu\nu}$. This means that, at the background level via the conformal relation above, one has $f_R \approx 1$ (but $\delta f_R \neq 0$). On the other hand, using the standard gauge choice $\partial_\lambda(\bar{t}^\lambda_\mu - \frac{\bar{t}}{2}\delta^\lambda_\mu) = 0$ one finds that $R_{\mu\nu}(\eta_{\mu\nu} + \bar{t}_{\mu\nu}) \approx -\frac{1}{2}\Box\bar{t}_{\mu\nu}$, where $\Box$ is the DÁlambertian (in flat space). After noting that $\delta R^\mu{}_\nu(h) \approx \eta^{\mu\alpha}\delta R_{\alpha\nu}$, inserting these results into the field equations (4.11) one arrives at

$$-\frac{1}{2}\Box\bar{t}_{\mu\nu} = \kappa^2\left(\tau_{\mu\nu} + \frac{f}{2\kappa^2}\eta_{\mu\nu}\right) \,. \tag{4.39}$$

Limiting ourselves to the nonrelativistic source limit ($P \to 0$), one can compute $\tau_{\mu\nu} \approx \rho\,(1 + f_T/\kappa^2)\,u_\mu u_\nu$, from where the perturbed field equations (4.39) read

$$-\frac{1}{2}\vec{\nabla}\bar{t}_{\mu\nu} \approx \kappa^2\rho\left(1 + \frac{f_T}{\kappa^2}\right)u_\mu u_\nu + \frac{f}{2}\eta_{\mu\nu} \,. \tag{4.40}$$

Given that the background solution is flat Minkowski space and that ρ represents the leading order contribution from the matter sector, the term proportional to f_T in the above expression must be regarded as higher order and, thus, negligible to this order of approximation. Nonetheless, we will keep track of this contribution in the equations by defining the quantity

$$\rho_T = \rho\left(1 + \frac{f_T}{\kappa^2}\right) \,. \tag{4.41}$$

Assuming a standard structure for the metric perturbations

$$\bar{t}_{\mu\nu} = \begin{pmatrix} -2\bar{\phi}_N & \hat{0}_{3\times1} \\ \hat{0}_{1\times3} & \bar{\psi}\delta_{ij}\hat{I}_{3\times3} \end{pmatrix} \,, \tag{4.42}$$

where $\hat{I}$ and $\hat{0}$ are the identity and zero matrices, respectively, then the $(0,0)$ component of the perturbation equations (4.40) reads

$$\vec{\nabla}^2\bar{\phi}_N \approx \kappa^2\rho_T - \frac{f}{2} \ . \tag{4.43}$$

Now, given that $\delta g_{\mu\nu} = \bar{t}_{\mu\nu} - \eta_{\mu\nu}\delta f_R$ and $\delta f_R = f_{RR}\delta R$, one can write the Newtonian potential $\phi_N = -\delta g_{00}/2$ using Eq. (4.8) as

$$\bar{\phi}_N = \phi_N + \lambda\rho \ , \tag{4.44}$$

where $\lambda := (f_R - Rf_{RR})^{-1} f_{RR}\kappa^2/2$ is evaluated in vacuum. This leads to the following modified Poisson equation for metric-affine $f(R,T)$ theories:

$$\vec{\nabla}^2\phi_N \approx \kappa^2\rho_T - \frac{f}{2} - \lambda\vec{\nabla}^2\rho \ . \tag{4.45}$$

Given that in this equation $f(R,T)$ is a function of ρ and P, using the notation $\kappa^2\tilde{\rho}/2 := \kappa^2\rho_T - f/2$, this expression boils down to the usual result in the GR limit, which allows to write

$$\phi_N = \frac{\kappa^2}{8\pi}\int d^3\vec{x}'\,\frac{\tilde{\rho}(t,\vec{x}')}{|\vec{x}-\vec{x}'|} - \lambda\rho \ . \tag{4.46}$$

This modified Newtonian potential is formally identical to that found in the weak field limit of the Eddington-inspired Born-Infeld (EiBI) theory of gravity (see the recent review (Beltrán Jiménez *et al.*, 2018), Sec. 3) and, therefore, the implications derived from it might be similar except, perhaps, due to new effects arising from the redefinitions introduced above. These similarities are expected, in particular, in nonrelativistic stellar models.

4.4. Some applications

4.4.1. Stellar structure equations

The weak field equations derived above were useful to establish some relations between the physics of metric-affine $f(R,T)$ models and other gravity theories such as the EiBI model. In this section we derive the complete Tolman-Oppenheimer-

Volkov (TOV) equations for hydrostatic equilibrium to show that the metric-affine version of $f(R,T)$ theories studied in this work does introduce different physics in the full relativistic regime. For this purpose, we consider the nonconservation equation (4.27) applied to a perfect fluid (4.16) to find

$$\partial_r P = -\frac{(1+\kappa^{-2}f_T)(\rho+P)}{\left[1+\frac{2}{\kappa^2}\left(f_T+P\partial_P f_T+\frac{1}{4}f_T\partial_P T\right)\right]}u^\alpha\nabla_\alpha u_r \ . \tag{4.47}$$

In the $f_T \to 0$ limit, this equation recovers the usual structure equation of GR and of metric theories of gravity with no matter-curvature couplings. For static, spherically symmetric configurations, only the radial derivative equation survives and one finds that $u^\alpha\nabla_\alpha u_r = \Gamma^t_{tr} = A_r/2A$, where $g_{tt} = -A(r)$. The resulting TOV equation thus takes the form

$$\partial_r P = -\frac{(1+\kappa^{-2}f_T)(\rho+P)}{\left[1+\frac{2}{\kappa^2}\left(f_T+P\partial_P f_T+\frac{1}{4}f_T\partial_P T\right)\right]}\frac{A_r}{2A} \ . \tag{4.48}$$

The weak field limit obtained in the general case above follows from this equation by taking

$$(\rho+P)\approx\rho \ , \qquad \kappa^{-2}f_T\to 0 \ , \tag{4.49}$$

and

$$A_r \approx 2\left[\frac{\kappa^2 M(r)}{8\pi r^2}-\lambda\rho_r\right] \ , \tag{4.50}$$

with $M(r) = \int^r d^3\vec{x}x^2\tilde{\rho}(t,\vec{x})$. After setting specific $f(R,T)$ models these equations allow to solve any scenario of interest in this context.

4.4.2. Electromagnetic fields

Let us consider now the case of an electromagnetic field. For a Maxwell field, described by the Lagrangian density $\mathcal{L}_m = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, from Eq. (4.12) one finds that

$$\Theta_{\mu\nu} = -T_{\mu\nu} = -\frac{1}{4\pi}\left(F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \ . \tag{4.51}$$

From Eq. (4.6) this result yields the cancellation of the f_T contributions which, together with the tracelessness of Maxwell's energy-momentum tensor, implies that any solutions for these matter fields will coincide with those of GR regardless of the $f(R,T)$ theory chosen.

In order to find nontrivial new physics associated with electromagnetic fields, one must go beyond Maxwell's theory and consider instead nonlinear electrodynamics theories. In this case, defining the matter sector as

$$\mathcal{S}_m = \frac{1}{8\pi}\int d^4x\sqrt{-g}\,\varphi(X) \ , \tag{4.52}$$

where $\varphi(X)$ is a function of the field invariant $X = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ specifying the model of nonlinear electrodynamics[4] (Maxwell electrodynamics corresponding to $\varphi(X) = X$). The corresponding energy-momentum tensor reads

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\varphi_X F_{\mu\alpha}F_\nu{}^\alpha + \frac{\varphi}{2}g_{\mu\nu}\right) \ , \tag{4.53}$$

where $\varphi_X := \partial\varphi/\partial X$. In this case it is easy to find that

$$\Theta_{\mu\nu} = -T_{\mu\nu} + \frac{1}{2\pi}X\varphi_{XX}F_{\mu\alpha}F_\nu{}^\alpha \,, \tag{4.54}$$

and

$$\tau_{\mu\nu} = T_{\mu\nu} - \frac{f_T}{2\pi\kappa^2}X\varphi_{XX}F_{\mu\alpha}F_\nu{}^\alpha \,. \tag{4.55}$$

The new f_T contributions induce modifications as compared to GR solutions, as we shall see at once with an explicit example.

Let us focus on (electro-)static, spherically symmetric solutions, for which the only nonvanishing component of the field strength tensor is $F_{tr} \neq 0$. In this case, the matter energy-momentum tensor reads

$$T_\mu{}^\nu \;=\; \frac{1}{4\pi}\begin{pmatrix} \left[-X\varphi_X + \frac{\varphi}{2}\right]\hat{I} & \hat{0} \\ \hat{0} & \frac{\varphi}{2}\hat{I} \end{pmatrix} , \tag{4.56}$$

[4] Functions of a second field invariant, $Y = -\frac{1}{2}F_{\mu\nu}F^{\star\mu\nu}$, built out of the dual field strength tensor, $F^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, are also possible, but for simplicity we shall not consider them here.

where now $X = -F_{tr}F^{tr}$, while the conserved energy-momentum tensor takes the form

$$\tau_\mu{}^\nu = \frac{1}{4\pi}\begin{pmatrix} \left[-X\varphi_X + \frac{\varphi}{2} + 2\frac{f_T}{\kappa^2}X^2\varphi_{XX}\right]\hat{I} & \hat{0} \\ \hat{0} & \frac{\varphi}{2}\hat{I}\end{pmatrix}, \tag{4.57}$$

where $\hat{I}$ and $\hat{0}$ are the 2×2 identity and zero matrices, respectively. To proceed further and find solutions we need to specify an $f(R,T)$ model. For simplicity and to illustrate the general procedure to solve the field equations, let us choose the simple model $f(R) = R + \epsilon T$, where ϵ is some parameter[5]. From the trace equation (4.8) one finds that $R = -(\kappa^2 + 2\epsilon)T - \frac{f_T}{\pi\kappa^2}X^2\varphi_{XX}$ and inserting this result into the field equations (4.11), a bit of algebra yields

$$R^\mu{}_\nu(h) = \frac{\kappa^2}{f_R^2}\begin{pmatrix} \bar{\varphi}\,\hat{I} & \hat{0} \\ \hat{0} & (\bar{\varphi} + \bar{\varphi}_X)\,\hat{I}\end{pmatrix}, \tag{4.58}$$

where we have defined the quantities

$$\bar{\varphi} = -\frac{1}{4\pi}\left(\frac{\varphi}{2} + \frac{\epsilon}{\kappa^2}(\varphi - X\varphi_X)\right), \tag{4.59}$$

$$\bar{\varphi}_X = \frac{1}{4\pi}\left(X\varphi_X - \frac{\epsilon}{\kappa^2}\,2X^2\varphi_{XX}\right), \tag{4.60}$$

for notational convenience.

To solve this kind of field equations in metric-affine gravities one usually introduces two different line elements, one for $g_{\mu\nu}$ and another one for $h_{\mu\nu}$, and then makes use of the conformal transformation (4.9) to work out the relations among the functions on each line element. However, for the model chosen here, $f_R = 1$, and such line elements become the same. Let us thus propose an ansatz for a static, spherically symmetric line element of the form

$$ds^2 = -A(r)e^{2\psi(r)}dt^2 - \frac{1}{A(r)}dr^2 + r^2 d\Omega^2, \tag{4.61}$$

[5] Cosmological FRW-type solutions of this model can also be easily worked out, with the result that, for dust, the scale parameter behaves as $a(t) \propto t^\alpha$, where $\alpha = \frac{2}{3}\left(\frac{\kappa^2+3\epsilon/2}{\kappa^2+\epsilon}\right)$. This is a similar result as that obtained in the metric formulation of these theories (Harko *et al.*, 2011a).

where $\{A(r), \psi(r)\}$ are functions of the radial coordinate r and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the angular element on the unit 2-spheres. From the combination $R^t{}_t - R^r{}_r = 0$ of the field equations (4.58) it follows that $\psi(r) =$ constant, which can be set to zero without loss of generality. As for the component $R^\theta{}_\theta = \frac{1}{r^2}[1 - A(r) - rA_r]$ on the left-hand side of Eqs. (4.58), introducing a standard mass ansatz of the form $A(r) = 1 - 2M(r)/r$, it can be solved as (recall that $X = X(r)$)

$$M(r;\epsilon) = M_0 - \frac{\kappa^2}{2}\int_r^\infty dRR^2\left[\bar{\varphi}(X) + \bar{\varphi}_X(X)\right] \ , \tag{4.62}$$

where M_0 is an integration constant identified as Schwarzschild's mass. The next step to provide explicit solutions would be to supply a specific function $\varphi(X)$, i.e., to choose any of the nonlinear models of electrodynamics studied in the literature, for instance, in the context of spherically symmetric solutions in GR, see e.g. (Gibbons and Rasheed, 1995; Hassaïne and Martínez, 2008; Diaz-Alonso and Rubiera-Garcia, 2010; Ruffini *et al.*, 2013; Cembranos *et al.*, 2015; Kruglov, 2016; Hendi *et al.*, 2017; Gulin and Smolić, 2018). Once given, the resolution of the corresponding matter field equations, $\nabla_\mu(\varphi_X F^{\mu\nu}) = 0$, would provide the explicit expression of $X(r)$ needed to carry out the integral in Eq. (4.62), thus closing the problem. The analysis of this kind of models and solutions could open new avenues in the investigation of outstanding problems in this context, such as the singularity avoidance within nonlinear electrodynamics coupled to gravity, paralleling previous analysis carried out in the context of metric-affine $f(R)$ theories, see e.g. (Bambi *et al.*, 2016).

4.4.3. Scalar fields

Scalar fields represent yet another example suitable for investigation within these theories and yielding nontrivial new dynamics. Defining in this case the Lagrangian density as $\mathcal{L}_m = \frac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 2V(\phi))$ where $V(\phi)$ is the potential, one finds $\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m$, and the effective energy-momentum tensor reads

$$\tau^\mu{}_\nu = T^\mu{}_\nu\left(1 + \frac{f_T}{\kappa^2}\right) - \frac{f_T}{\kappa^2}\mathcal{L}_m\delta^\mu{}_\nu \, . \tag{4.63}$$

Likewise the electromagnetic field case above, setting specific $f(R,T)$ models and working out the corresponding field equations one may find f_T-corrections to GR solutions, which brings about new possibilities. For instance, free ($V=0$) geonic solutions of the kind found in Ref. (Afonso *et al.*, 2017) in the context of Eddington-inspired Born-Infeld gravity should also be possible in metric-affine $f(R,T)$ theories.

4.5. Conclusion

In this work we have derived the field equations of $f(R,T)$ theories with independent metric and affine connection (metric-affine approach). We have found that for matter sources not coupled to the connection (for which the torsion degrees of freedom are trivial (Alfonso *et al.*, 2017)), the symmetric part of the connection can be written as the Levi-Civita connection of an auxiliary metric conformally related to $g_{\mu\nu}$ via the matter sources, and that the resulting field equations can be formally written in the same way as those of metric-affine $f(R)$ theories once an effective energy-momentum tensor is defined. These equations impose the nonconservation of the energy-momentum tensor, therefore entailing nongeodesic motion and the appearance of a fifth force, which has a nontrivial impact for the physics of compact objects and relativistic stars. For nonrelativistic stellar objects, the dynamics is qualitatively similar to that found in the EiBI model, for which there exists extensive literature (Beltrán Jiménez *et al.*, 2018).

After having under control the basic framework for metric-affine $f(R,T)$ gravity, we have introduced the main elements for some applications. When coupled to perfect fluids, the nonconservation equation introduces novelties in the hydrodynamical equilibrium equation in the full nonrelativistic regime, with expected non-negligible consequences for compact objects in this context. When coupled to electromagnetic fields, we have shown that these theories yield the same solutions as GR unless a nonlinear theory of electrodynamics is considered, where the problem of non-singular black holes can be tackled from a different perspective, and similar comments apply to scalar fields.

In summary, the primer $f(R,T)$ gravity in the metric-affine formalism developed in this work opens new avenues of research and the possibilities to explore new physics in this context are huge. Further research is expected in these and other directions in the future, on which we hope to report soon.

Chapter 5

MOND as the weak field limit of an extended metric theory of gravity with a matter-curvature coupling.

5.1. Introduction

The non-baryonic dark matter problem constitutes one of the most important unsolved problems in current research (cf. Bertone and Hooper, 2016; Freese, 2017). Despite the huge research and its generally accepted success, the dark matter particle has never been detected. The gravitational anomaly that gives rise to the dark matter and/or energy hypothesis can also be understood as a modification of gravity at certain scales (cf. Mendoza, 2015) as it was first discussed by the pioneer research of Milgrom (1983a,b), with a MOdified Newtonian Dynamics (MOND) approach. A first coherent attempt to find a relativistic version was carried out by Bekenstein (2004) with a TEnsor Scalar Vector (TEVES) theory, this idea has been widely explored (Bekenstein and Sanders, 2012; Zlosnik *et al.*, 2006; Sanders, 1997, 2005; Skordis, 2009), but due to the extreme complexity of the theory and some clear failures, research has continued into finding a relativistic theory of gravity which yields MOND in its non-relativistic, weakest field limit regime.

Bernal *et al.* (2011) showed that MOND acceleration can be accounted by a relativistic $f(\chi) = \chi^{3/2}$ metric theory of gravity described by the action:

$$S = \frac{c^3}{16\pi G L_M^2} \int f(\chi) \sqrt{-g}\, \mathrm{d}^4 x + \frac{1}{c} \int \mathcal{L}_{\mathrm{matt}} \sqrt{-g}\, \mathrm{d}^4 x, \tag{5.1}$$

where $\chi := L^2 R$, R is the Ricci scalar, $L \propto r_{\mathrm{g}}^{1/2} l^{1/2}$, with $r_{\mathrm{g}} := GM/c^2$ the gravitational radius, $l := (GM/a_0)^{1/2}$ the "mass-length" scale of the system and $\mathcal{L}_{\mathrm{matt}}$ is the standard matter Lagrangian, related to the energy-momentum tensor $T_{\alpha\beta}$ by eq.(2.7)

The constant $a_0 \approx 1{,}2 \times 10^{-10} \mathrm{m\,s}^{-2}$ is Milgrom's acceleration constant. This proposal is coherent with the results of gravitational lensing in individual, groups and clusters of galaxies (Mendoza *et al.*, 2013) and at the same second perturbation order is coherent with a Parametrised Post-Newtonian (PPN) description where the parameter $\gamma = 1$ (Mendoza and Olmo, 2014). Another extension of gravity was performed by Barrientos and Mendoza (2016), who analysed the action (5.1) but now using the Palatini approach, obtaining the same functional action $f(\chi) = \chi^{3/2}$ in order to recover the MONDian acceleration, with a mass dependence on the coupling length L.

The problem with action (5.1) is that it can only be applied in regions sufficiently far from the sources that produce the gravitational field, in order to approximate the system as a point mass source. There is however a cosmological attempt by Carranza *et al.* (2013a) in which the mass M was thought of as the causal mass for a particular observer in the cosmic flow, yielding a good description of an accelerated expansion of the universe without the introduction of dark matter and/or energy.

Another recent exploration was carried out by Barrientos and Mendoza (2017) who showed that the mass dependence in the coupling length L can be avoided introducing derivatives of the matter Lagrangian in the action $f(\chi)$. In such proposal the coupling constant depends exclusively on the fundamental constants c, a_0 and G, but the price to pay is in the complexity of the field equations and the theoretical inconvenients that the introduction of the derivatives of the matter Lagrangian produce.

In this article we use an extension of a metric $f(R)$ theory of gravity with matter-curvature couplings $F(R, \mathcal{L}_{\mathrm{matt}})$ following the approach by (Harko *et al.*, 2011a; Lobo and Harko, 2012; Harko *et al.*, 2013, 2014a; Harko and Lobo, 2010) and show that with this generalised action a relativistic theory of MOND can be constructed. The article is presented in the following manner. In Section 5.2 an order of magnitude calculation is performed to show that a specific $F(R, \mathcal{L}_{\text{matt}})$ can reproduce MOND in its simplest form. Section 5.3 shows an exact solution for a point-mass source reproducing these results. In Section 5.4 we use correct dimensional arguments to generalise an action for a $F(R, \mathcal{L}_{\text{matt}})$ and show that with this it is possible to recover either MOND or Newton's gravity at the weakest field limit of the theory. Finally in Section 5.5 we discuss the results of the article and present our conclusions.

5.2. $F(R, L_{\mathbf{matt}})$ approach

The lesson to learn from action (5.1) is that the matter Lagrangian $\mathcal{L}_{\text{matt}}$ needs to be inserted inside the gravitational action (see e.g. Mendoza (2015)). The idea of a non-minimal coupling between the matter and the curvature has been already raised (Goenner, 1984; Bertolami *et al.*, 2007; Nojiri and Odintsov, 2004; Allemandi *et al.*, 2005). To do so, we can use an extension of $f(R)$ of gravity introducing a $F(R, \mathcal{L}_{\text{matt}})$ described by Harko and Lobo (2010):

$$S = \int F(R, \mathcal{L}_{\text{matt}}) \sqrt{-g}\, \mathrm{d}^4 x, \tag{5.2}$$

with the following field equations:

$$F_R R_{\alpha\beta} + \left(g_{\alpha\beta} \nabla^\mu \nabla_\mu - \nabla_\alpha \nabla_\beta \right) F_R - \frac{1}{2} \left(F - F_{\mathcal{L}_{\text{matt}}} \right) g_{\alpha\beta} = \frac{1}{2} F_{\mathcal{L}_{\text{matt}}} T_{\alpha\beta}, \tag{5.3}$$

where $F_R := \partial F / \partial R$ and $F_{\mathcal{L}_{\text{matt}}} := \partial F / \partial \mathcal{L}_{\text{matt}}$. Note that (a) $F(R, \mathcal{L}_{\text{matt}}) = c^3 R / 16\pi G + \mathcal{L}_{\text{matt}}/c$ yields standard general relativity, (b) $F(R, \mathcal{L}_{\text{matt}}) = f(R)/2 + \mathcal{L}_{\text{matt}}$ is standard metric $f(R)$ gravity and (c):

$$F(R, \mathcal{L}_{\text{matt}}) = \frac{c^3}{16\pi G} \frac{f(\chi)}{L^2} + \frac{1}{c} \mathcal{L}_{\text{matt}}, \tag{5.4}$$

is a correct generalisation of (5.1) in which the unknown length function $L = L(\mathcal{L}_{\text{matt}})$ is to be found; and together with the unknown function$f(\chi)$ must yield a correct

MOND behaviour in the limit of low acceleration scales $a \lessapprox a_0$.

5.3. MONDian limit

Let us now show that with the assumptions made in section 5.2 it is possible to obtain the basic MOND relation based on the Tully-Fisher law. To do so, let us substitute equation (5.4) into the field equations (5.3) and take the trace of the resulting relation to yield:

$$f_R(\chi)R + -2f(\chi) + 3L^2\nabla^\alpha\nabla_\alpha\left(\frac{f_R(\chi)}{L^2}\right) = \frac{8\pi G L^2}{c^4}T^\alpha_{\ \alpha}. \tag{5.5}$$

In order to find the correct MONDian limit equation, we follow the procedure by Bernal *et al.* (2011) and so, let

$$f(\chi) = \chi^b, \qquad \text{and} \qquad \mathcal{L}_{\text{matt}} = \rho c^2, \tag{5.6}$$

where we have assumed a point mass source generating the gravitational field, and so $\mathcal{L}_{\text{matt}}$ has a dust-like form. To order of magnitude, i.e. when $R \sim r_{\text{curv}}^{-2}$ -where r_{curv} is the radius of curvature of space- and $\nabla \sim 1/r$, it follows that the first two terms on the left-hand side of equation (5.5) are smaller than the third when $r/r_{\text{curv}} \to 0$, i.e. when the equivalent acceleration a is expected to be $\lesssim a_0$.

Thus, the trace of the field equations that can be adapted to a MONDian regime of low acceleration scales is given by:

$$3L^2\nabla^\alpha\nabla_\alpha\left(\frac{f_R(\chi)}{L^2}\right) = \frac{8\pi G L^2}{c^4}T^\alpha_{\ \alpha}. \tag{5.7}$$

A weak-field limit coherent with bending of light in individual, groups and clusters of galaxies is obtained if the second perturbation order metric is given by (Mendoza and Olmo, 2014):

$$\mathrm{d}s^2 = \left(1 + \frac{2\phi}{c^2}\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2\phi}{c^2}\right)\mathrm{d}\boldsymbol{x}^2, \tag{5.8}$$

for a gravitational scalar potential ϕ and an isotropic space-time with a PPN parameter $\gamma \approx 1$ according to observations of such MONDian systems (Mendoza *et al.*, 2013). With this, the Ricci scalar takes the form: $R \approx -(2/c^2)\nabla^2\phi$, which at order of magnitude yields: $R \sim a/rc^2$, for an acceleration $a = |\nabla\phi|$.

Thus, to order of magnitude, equation (5.7) yields:

$$a \sim G^{1/(b-1)} \, \rho^{1/(b-1)} \, r^{(b+1)/(b-1)} \, c^{(2b-4)/(b-1)} L^{-2}, \tag{5.9}$$

and so, in order to obtain MOND standard equation: $a = \sqrt{G \, a_0 \, M}/r \sim \sqrt{G \, a_0 \, \rho \, r}$, then $b = -3$ together with $L \propto (G\rho)^{-3/8} \, c^{5/4} a_0^{1/4}$, which yields:

$$F(R, \mathcal{L}_{\text{matt}}) \propto R^{-3} \mathcal{L}_{\text{matt}}^3. \tag{5.10}$$

5.4. A dimensionally correct general action

Let us now consider an action motivated by equation (5.1) with the following form:

$$S = \frac{c^3}{16\pi G\alpha} \sqrt{-g} \int f(\chi, \xi) \, \mathrm{d}^4 x + \frac{1}{c} \int \sqrt{-g} \mathcal{L}_{\text{matt}} \, \mathrm{d}^4 x, \tag{5.11}$$

where ψ and ξ are dimensionless quantities given by:

$$\xi := \frac{\mathcal{L}_{\text{matt}}}{\lambda}, \qquad \text{and} \qquad \chi := \alpha R, \tag{5.12}$$

with α and λ unknown "coupling" constants with dimensions of square length and energy density respectively.

The null variations with respect to the metric yields the following field equations:

$$\alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f - \xi f_\xi) = \left(\frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda} \right) T_{\mu\nu} - \alpha (g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) f_\chi. \tag{5.13}$$

with the standard definition of the energy-momentum tensor:

$$T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_{\text{matt}} - 2 \frac{\partial \mathcal{L}_{\text{matt}}}{\partial g^{\mu\nu}}, \tag{5.14}$$

in full agreement with equation (2.7).

The trace of equation (5.13) is given by:

$$\chi f_\chi - 2(f - \xi f_\xi) + 3\alpha\Delta f_\chi = \left(\frac{8\pi G\alpha}{c^4} + \frac{f_\xi}{2\lambda}\right) T. \tag{5.15}$$

Since c, G and a_0 are independent fundamental constants, Buckingham's Π theorem of dimensional analysis implies that:

$$\alpha = \kappa\frac{c^4}{{a_0}^2} \qquad \text{and} \qquad \lambda = \kappa'\frac{{a_0}^2}{G}, \tag{5.16}$$

with κ and κ' pure dimensionless proportionality constants.

Following the previous approach, we can assume that:

$$f(\chi, \xi) = \chi^\gamma \xi^\beta. \tag{5.17}$$

For the case of dust, the perturbation orders in the terms of the field equation are the following:

$$\overbrace{\alpha f_\chi R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(f - \xi f_\xi)}^{\mathcal{O}(-2(\gamma+\beta))} + \overbrace{\alpha(g_{\mu\nu}\Delta - \nabla_\mu\nabla_\nu)f_\chi}^{\mathcal{O}(-2(\gamma+\beta+1))} = \underbrace{\frac{8\pi G\alpha}{c^4}T_{\mu\nu}}_{\mathcal{O}(2)} + \underbrace{\frac{f_\xi}{2\lambda}T_{\mu\nu}}_{\mathcal{O}(2(\gamma+\beta))} . \tag{5.18}$$

5.4.1. Poisson-like equation for MOND

The lowest perturbation order of the previous equation is 2 and so, the choice $\gamma = -\beta$ yields:

$$\left(g_{\mu\nu}\Delta - \nabla_\mu\nabla_\nu\right) f_\chi = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{5.19}$$

Contracting equation (5.19) with $g^{\mu\nu}$ gives:

$$3\Delta f_\chi = \frac{8\pi G}{c^4}T, \tag{5.20}$$

which at the lowest perturbation order for dust takes the following expression:

$$(-2\kappa)^{\gamma-1}\kappa'^\gamma \frac{{a_0}^2}{G^{\gamma+1}}\nabla^2\left(\left\{\nabla^2\phi\right\}^{\gamma-1}\rho^{-\gamma}\right) = \frac{8\pi}{3}\rho. \tag{5.21}$$

To order of magnitude, this last equation implies that:

$$a \approx M^{(1+\gamma)/(\gamma-1)} r^{-2(1+\gamma)/(\gamma-1)}, \tag{5.22}$$

and so, in order to recover a MONDian expression for the acceleration, the following value of γ is found:

$$\gamma = -3. \tag{5.23}$$

With this value, the Poisson-like equation (5.21) is:

$$\frac{3}{8\pi} \frac{(a_0 G)^2}{(2\kappa)^4 \kappa'^3} \nabla^2 \left(\left\{ \nabla^2 \phi \right\}^{-4} \rho^3 \right) = \rho. \tag{5.24}$$

An analytic solution to the previous equation for the case of a point-mass source is given in the appendix B.

5.4.2. Poisson's equation for Newtonian gravity.

Another possible choice for equation (5.18) is $\gamma + \beta = 1$ which yields:

$$\alpha f_\chi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (f - \xi f_\xi) = \left(\frac{8\pi G \alpha}{c^4} + \frac{f_\xi}{2\lambda} \right) T_{\mu\nu}. \tag{5.25}$$

This lowest perturbation order choice means that:

$$(g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) f_\chi = 0. \tag{5.26}$$

Taking the trace of equation (5.25) for dust, a relation between the Ricci scalar and the matter density is obtained:

$$R = \left(-\frac{16\pi}{\gamma + 1} (\kappa\kappa')^{1-\gamma} \right)^{1/\gamma} \frac{G}{c^2} \rho. \tag{5.27}$$

At the lowest perturbation order, when $R = -(2/c^2)\nabla^2\phi$, this previous equation can be constructed -with the appropriate coupling constants- to yield Newtonian gravity (Poisson's equation) for any value of $\gamma \neq -1$.

5.5. Discussion

In this article we have shown that it is possible to show, exactly and by an order of magnitude approach, that a $F(R, \mathcal{L}_{\mathrm{matt}})$ theory of gravity described by:

$$f(\chi, \xi) = \chi^{-3}\xi^{3}, \qquad \chi := \alpha R, \quad \xi := \mathcal{L}_{\mathrm{matt}}/\lambda, \tag{5.28}$$

is a good candidate for a full relativistic extension of MOND, in regions where the acceleration of test particles $\lesssim a_0$. In the weak-field limit of approximation it converges to standard MOND for a point mass source M, with $\rho = M\delta(\boldsymbol{r})$ and $\mathcal{L}_{\mathrm{matt}} = \rho c^2$. It is our intention to explore this interpretation with applications to lensing and dynamics of individual, groups and clusters of galaxies as well as with cosmology. The advantage of this approach is that it is a full metric formalism and does not involve interpretations of gravity using Palatini formalism or torsion as we have previously explored (Barrientos and Mendoza, 2016, 2017). Furthermore, it is a correct generalisation to the first attempts made by Bernal *et al.* (2011).

At first sight, the action given by the Lagrangian density: $R^{-3}\mathcal{L}^{3}_{\mathrm{matt}}$ from which we have proved the MONDian behaviour is obtained, seems to diverge in the Minkowskian regime, namely when $R \to 0$. In order to show that this is not so, we proceed in the following way. Using relations (5.16), (5.17), (5.23), and the fact that $\gamma = -\beta$, expression (5.20) turns into:

$$-\frac{9}{8\pi k^4 k'^3}\left(\frac{a_0 G}{c^6}\right)^2 \Delta(R^{-4}\mathcal{L}^{3}_{\mathrm{matt}}) = T, \tag{5.29}$$

which in the weak-field limit for a point-mass source is:

$$-\frac{9}{8\pi k^4 k'^3}\left(\frac{a_0 G}{c^5}\right)^2 \nabla^2(R^{-4}\mathcal{L}^{3}_{\mathrm{matt}}) = M\delta(\boldsymbol{r}). \tag{5.30}$$

Using the well known result:

$$\nabla^2\left(\frac{1}{\boldsymbol{r}}\right) = -4\pi\delta(\boldsymbol{r}), \tag{5.31}$$

the following relation is satisfied:

$$R^{-4}\mathcal{L}_{\mathrm{matt}}^{3} = \frac{2\pi k^4 k'^3}{9}\left(\frac{c^5}{a_0 G}\right)^2 \frac{M}{r}. \tag{5.32}$$

Therefore, in the weak field limit, this proposal has the following relation: $\mathcal{L}_{\mathrm{matt}}^{3} \propto R^4/r$. This implies that the Lagrangian density for the action that we are interested in converges to $R^{-3}\mathcal{L}_{\mathrm{matt}}^{3} \propto R/r \to 0$ as r increases.

Finally, we discuss the geodesic equation of the theory. Following a similar procedure as the one shown in (Harko and Lobo, 2010; Koivisto, 2006), the geodesic equation is given by:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s^2} + \Gamma^\mu{}_{\nu\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = f^\mu, \tag{5.33}$$

where

$$f_\mu = \left(g_{\mu\nu} - u_\mu u_\nu\right)\nabla^\mu \ln\left[\left(16\pi\kappa\kappa' + f_\xi\right)\frac{\mathrm{d}\mathcal{L}_{\mathrm{matt}}}{\mathrm{d}\rho}\right]. \tag{5.34}$$

As expected, the usual relation $u_\mu f^\mu = 0$ is obtained. This means that the extra force is perpendicular to the four-velocity. For dust, the extra-force takes the following form:

$$f_\nu = \left(g_{\mu\nu} - u_\mu u_\nu\right)\nabla^\mu \ln\left[16\pi\kappa\kappa' + f_\xi\right]. \tag{5.35}$$

This type of extra force has been studied and interpreted in the literature (cf. Harko, 2014). Investigations into its nature and its astrophysical consequences requires further research.

Chapter 6

Conclusions

This thesis took as starting point the work made by Bernal *et al.* (2011), which was developed in the standard pure metric formalism. Through an extended theory $f(\chi)$, the MONDian acceleration was recovered as the weak field limit of this theory. In this work, the term responsible to obtain the correct MONDian acceleration is dominant when the local radius of curvature is greater than a characteristic length where the MONDian regime is valid. Despite obtaining the correct limits, this theory has the characteristic of being non-local due to the presence of the mass in the coupling constant.

In order to make a deeper analysis of this $f(\chi)$ theory, our work started with the change of the formalism in which this theory is thought. We relaxed the metric compatibility condition or the symmetric one for the connection in order to work in the Palatini or torsion formalism. From these analysis, we concluded that the recovering of a MONDian acceleration is possible. Quite interestingly in the Palatini formalism the same functional form of $f(\chi)$ concides with the one of the pure metric formalism, given by: $f(\chi) = \chi^{3/2}$. But these approaches do not have an advantage respect the metric formalism. The inclusion of a general connection brings another frame to the scene, and the relation between the tensors in both frames is complex enough, that does not allow the manipulation of the terms without making extra assumptions. Besides, the non-local problem persists in this approach.

Due to the fact that the coupling constant presents explicit mass dependence in our previous approaches, the next logical step was to replace this mass by the matter lagrangian, which is local and covariant. From this approach, we conclude that in order to keep a theory whose curvature sector in the action is $f(\chi)$ and its weak field limit yields the MONDian acceleration, the matter sector must be modified. An order of magnitude analysis reveals that derivatives of the matter lagrangian are needed. This new modification improves the previous ideas since its coupling constant depends exclusively of the pure physical constants c, G and a_0, but the field equations remain quite complex. On the other hand, the physical meaning of a derivative of the matter lagrangian is not clear.

An option that avoids the use of derivatives of the matter lagrangian is to introduce a coupling between the curvature and the matter, either through the trace of the energy- momentum tensor or the matter lagrangian. These theories have been widely studied and have acquired a great importance in recent years. We introduce this coupling in a $f(\chi,\xi)$ theory in the pure metric formalism, with a previous construction of the coupling constants using dimensional analysis, that depend only of c, G and a_0. We found that the terms involved in the field equations will have different perturbation orders depending of the choice of the power in $f(\chi,\xi) = \chi^\gamma \xi^\beta$. This has as consequence that is possible obtain a MONDian acceleration ($\gamma = -\beta = -3$) or the Newtonian one ($\gamma + \beta = 1$) as the weak field limit of the theory, without the need of making any further assumption or approximations. A characteristic that present these kind of theories is the absence of a geodesic motion. In other words, a fifth force is introduced in the geodesic equation of motion.

Therefore, we can conclude that if we want to recover a MONDian acceleration from a metric theory in the metric formalism, the matter lagrangian or the trace of the energy-momentum tensor must appear in the action in a non-trivial way. The proposal that we introduced in Barrientos and Mendoza (2018) represents a forward step with respect to the idea shown in Bernal *et al.* (2011) not only because is a covariant and local theory due to the absence of the matter in the coupling constants but also in the sense that we do not require a criterion to drop some terms in a specific limit. As such, the MONDian acceleration is a natural consequence when we analyse the terms in perturbation orders of $1/c$. The coupling constants played a main role in the analysis. They were relevant not to discriminate terms but to achieve dimensionless action. Another noteworthy characteristic of this proposal is the fact that it can recover a Newtonian acceleration for another set of values, this can give some clue about the transition between the MONDian regime and the Newtonian one. This is such a complicated issue that requires study in more detail in future works.

Apéndice A

Evaluation of the constant Ξ

Since we are making the assumption that acceleration has only a radial component, the spherical coordinate system is the most suitable one. As such:

$$\mathbf{a} = \alpha r^{\tau} \hat{\boldsymbol{r}}, \tag{A.1}$$

with divergence:

$$\nabla \cdot \mathbf{a} = \alpha(\tau + 2) r^{\tau - 1}. \tag{A.2}$$

Also, the gradient of ρ in this coordinates is given by:

$$\nabla \rho = \frac{\mathrm{d}\rho}{\mathrm{d}r} \hat{\boldsymbol{r}}. \tag{A.3}$$

Squaring eq.(3.57) and substituting on it eqs. (A.2) and (A.3) yields:

$$\alpha^2 (\tau + 2)^2 r^{2(\tau - 1)} = \vartheta'^2 \left(\frac{-\Xi}{2^5} \right)^{1/2} G a_0 \frac{d\rho}{dr}, \tag{A.4}$$

Integrating over r the latter expression gives the following result:

$$\frac{\alpha^2 (\tau + 2)^2}{2\tau - 1} r^{2\tau - 1} = \vartheta'^2 \left(\frac{-\Xi}{2^5} \right)^{1/2} G a_0 \rho, \tag{A.5}$$

with $\tau \neq 1/2$. For a point mass source, the matter density is: $\rho = M\delta(r)/4\pi r^2$. Using this expression and integrating over r, we obtain:

$$\left.\frac{\alpha^2(\tau+2)^2}{2\tau(2\tau-1)}r^{2\tau}\right|_{r=0}^{r=\infty} = \vartheta'^2\left(\frac{-\Xi}{2^5}\right)^{1/2}\left.\frac{GMa_0}{4\pi}\frac{1}{r^2}\right|_{r=0}, \tag{A.6}$$

with the additional condition: $\tau \neq 0$. Since Ξ is just a constant, it does not depend on r and so, in order that eq.(A.6) has meaning, it is necessarily that $\tau = 1$. This value was expected because we built our theory with the requirement that $a \approx r^{-1}$. Using all this, eq. (A.6) can be written as:

$$-\frac{\alpha^2}{6} = \vartheta'^2\left(-\frac{\Xi}{2^5}\right)^{1/2}\frac{GMa_0}{4\pi}. \tag{A.7}$$

MONDiand acceleration sets the value: $\alpha = -(GMa_0)^{1/2}$ and so:

$$-\frac{1}{6} = \vartheta'^2\left(-\frac{\Xi}{2^5}\right)^{1/2}\frac{1}{4\pi}. \tag{A.8}$$

Algebraic manipulation of this expression yields eq.(3.58).

Apéndice B

Poisson-like equation

Let us begin by rewriting equation (5.24) as:

$$K\nabla^2 \left(\left\{\nabla^2\phi\right\}^{-4}\rho^3\right) = \rho, \tag{B.1}$$

where for simplicity we have defined:

$$K := \frac{3}{8\pi}\frac{(a_0 G)^2}{(2\kappa)^4 \kappa'^3}. \tag{B.2}$$

The matter density for a point-mass source is given by:

$$\rho = \frac{M}{4\pi r^2}\delta(r), \tag{B.3}$$

and since the Laplacian for a spherically symmetric problem is:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{d\psi}{dr}\right), \tag{B.4}$$

then, equation (B.1) turns into:

$$4\pi K\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(\left\{\nabla^2\phi\right\}^{-4}\rho^3\right)\right) = M\delta(r). \tag{B.5}$$

Integration of the previous equation yields:

$$4\pi K\frac{\mathrm{d}}{\mathrm{d}r}\left(\left\{\nabla^2\phi\right\}^{-4}\rho^3\right) = \frac{M}{r^2}, \tag{B.6}$$

which after another integration gives:

$$4\pi K \left\{\nabla^2\phi\right\}^{-4}\rho^3 = -\frac{M}{r}. \tag{B.7}$$

Using again eqs. (B.3) and (B.4) and after a some algebraic steps, we obtain:

$$(-K)^{1/4}\left(\frac{M}{4\pi}\right)^{1/2}\left(\frac{r^3}{\delta(r)}\right)^{1/4}\delta(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right), \tag{B.8}$$

which after another integration is written as:

$$(-K)^{1/4}\left(\frac{M}{4\pi}\right)^{1/2}\left(\frac{r^3}{\delta(r)}\right)^{1/4}\Bigg|_0 = r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}. \tag{B.9}$$

Using the fact that the acceleration $a = |\boldsymbol{a}| = |\nabla\phi|$ and the Dirac's delta function is given by:

$$\delta(r=0) = \lim_{r\to 0}\frac{1}{2\pi r}, \tag{B.10}$$

then the relation for the accelerations is given by:

$$\left(-K\frac{M^2}{2^3\pi}\right)^{1/4}\frac{1}{r} = a. \tag{B.11}$$

Substitution of the value of K given in equation (B.2), yields to:

$$\left(-\frac{3}{4^5\kappa'^3\pi^2}\right)^{1/4}\frac{1}{\kappa}\frac{(a_0GM)^{1/2}}{r} = a. \tag{B.12}$$

Thus, the choice $k'^3 = -3/4^5\pi^2k^4$ yields a MONDian acceleration $a = \sqrt{GMa_0}/r$.

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