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SPECTRAL DENSITY OF POISSON RANDOM GRAPHS

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Chapter 1

Introduction

The relation between mathematics and physics is evident. The former provides the language in which the latter develops, but the relation between them is far closer, to the point that they were considered the same science a few centuries ago and nowadays there are some research lines where it is uncertain whether they are part of physics or pure mathematics. In this work, we use two methods developed in physics to obtain analytical expressions of a statistical property of random graphs.

The study of statistical aspects of graphs dates back to the middle of the last century [1, 2], but it was not until the works of Erdős and Rènyi [3] that the Random Graph Theory was established with the study of what we currently know as *Poisson random graphs*. Precisely, this kind of random graphs will be our object of study.

The property in which we are interested is the *limiting spectral density*. This property describes the distribution of eigenvalues of the associated adjacency matrix. It turns out that it also provides information about the structure of the graphs [4] and has encountered applications in chaotic systems [5], biology [6] and physics [7].

The methods that are used to compute this spectral density were developed in statistical physics over the past half century. On the one hand, the physical implications of the *replica method* were not fully understood for many years and there was a lot of work concerning its understanding [8–10]. This led to a very refined method which has encontered applications in a broad spectrum of fields, such as optimization [11, 12], information theory [13] and neural networks [14, 15]. On the other hand, the *cavity method* was developed in an attempt to avoid the using of replicas. It has been applied principally in optimization problems, such as the coloring problem [16] and the k-satisfiability problem [17].

In the present work, the language, concepts and tools of statistical physics are abstracted and applied to compute the spectral density of Poisson random graphs. During this computation, the mathematical problem is transformed into a physical one, with the help of fictitious systems, energies and interactions. This allow us to apply the methods in a statistical physics "environment" and with a physical picture in mind. This work is principally based on [18] and [19].

This work is divided in five chapters, including the present introduction. In the second chapter we establish the notation and present the tools and concepts that will be used throughout the thesis. The computation of the magnetization of the Ising model will help us to introduce both methods in the third chapter. Our main goal, the computation of the spectral density of Poisson random graphs, will be the content of the fourth chapter and finally, the last chapter is devoted to our conclusions.

Chapter 2

Mathematical and Physical Background

In this chapter we review and clarify the concepts and results that appear in this work. It also serves to establish the notation that will be used throughout the thesis. We adopt a pragmatical approach, in particular in the section of functional analysis.

2.1 Probability

In this section we assume that the reader has a basic knowledge of probability. To begin with, some useful definitions and theorems will be given, most of them taken from [20] and [21].

Definition 2.1.1. *A **probability space** is the collection by three elements (Ω,$\mathscr{A}$,P) where Ω is an arbitrary set, $\mathscr{A}$ is a σ-field of subsets of Ω, and P is a measure of probability defined over the σ-field.*

The set Ω is usually known as the *sample space* and the elements of $\mathscr{A}$ are known as *events*.

Definition 2.1.2. *A **random variable** X is a function from the sample space to $\mathbb{R}$, that is*

$$X : \Omega \to \mathbb{R} \,. \tag{2.1}$$

For the sake of clarity we will consider discrete random variables. The case of continuous random variables is treated analogously.

Definition 2.1.3. *The **probability mass function** (or probability function, for short) of a discrete random variable X taken values on an alphabet χ denoted as*

$$P(X = x) = f(x) \,, \qquad x \in \chi \tag{2.2}$$

and fulfills

$$f(x) \geq 0 \qquad \forall x \in \chi \,, \qquad \sum_{x \in \chi} f(x) = 1 \,. \tag{2.3}$$

Henceforth the notation $X \sim f(x)$ will be used to indicate that the random variable X follows the probability function $f(x)$.

2.1.1 Joint and Marginal Probability

When dealing with multiple random variables, a vectorial notation is useful.

Definition 2.1.4. *A N-dimensional **random vector** $\overline{X} = (X_1, X_2, ..., X_N)$ is a collection of N random variables.*

Due to the fact that the vectorial notation stands for multiple random variables, it is necessary to distinguish between the so-called *joint probability function* and the *marginal probability function*.

Definition 2.1.5. *The **joint probability function** of the random vector $\overline{X}$ is*

$$P(\overline{X} = \bar{x}) = f(\bar{x}), \quad \bar{x} \in \chi^N \tag{2.4}$$

and fulfills

$$f(\bar{x}) \geq 0 \qquad \forall x \in \chi^N, \qquad \sum_{\bar{x} \in \chi^N} f(\bar{x}) = 1. \tag{2.5}$$

The notation $\sum_{\bar{x} \in \chi^N}$ (or $\sum_{\bar{x}}$ for short) stands for the sum over the possible values of each entry of the vector $\bar{x}$. For simplicity, we assume that all random variables share the same alphabet χ.

Definition 2.1.6. *Let $\overline{X} \sim f(\bar{x})$ be a random vector. The **marginal probability function** of the random variable X_j is*

$$P(X_j = x) = \sum_{\bar{x}_{/j}} f(\bar{x}) = g(x) \tag{2.6}$$

and fulfills

$$g(x) \geq 0, \qquad \sum_{x} g(x) = 1. \tag{2.7}$$

The notation $\bar{x}_{/j}$ represents the vector $\bar{x}$ without the j-th entry.

The previous definition can be generalized for sets of random variables. For example, to obtain the marginal probability of the set $A = \{X_i, X_j\}$ one only needs to sum over the remaining entries of the random vector $\overline{X} \sim f(\bar{x})$, i.e.

$$P(X_i = y, X_j = w) = \sum_{\bar{x}_{/\{i,j\}}} f(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_{j-1}, w, x_{j+1}, ..., x_N) = h(y, w). \tag{2.8}$$

2.1.2 Expectations

Definition 2.1.7. *The **expected value** of a random variable $X \sim f(x)$ is defined by*

$$\mathbb{E}(X) = \sum_{x} x f(x). \tag{2.9}$$

The words *mean*, *average value* or *expected value* will be used interchangably. Besides, the notation $\langle X \rangle$ will also represent the expected value of the random variable X.

Most of the times we want to obtain the mean of a function. Fortunately, we count with the following theorem

Theorem 2.1.1. *Let $X \sim f(x)$ be a random variable. The expected value of a function $g(x)$, which defines a new random variable $g(X)$, is*

$$\mathbb{E}(g(X)) = \sum_{x} g(x) f(x). \tag{2.10}$$

Let us turn to the interesting field of random graph theory.

2.2 Random Graphs

In this section we introduce some basic concepts and results of random graph theory that are necessary to understand our work. First of all, we establish the notation with the very basic concepts of graph theory.

2.2.1 Graphs

Graph theory is a vast branch of mathematics with many different applications in fields like linguistics [22], biology [23] and social sciences [24]. Some useful references for this section are [25–27]. Let us define what exactly a graph is.

Definition 2.2.1. *A **graph** $G = (\mathscr{V}, \mathscr{E})$ is a family of two sets, where the sets $\mathscr{V} = \{1, 2, ..., N\}$ and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ are called the vertex set and the edge set, respectively.*

Two vertices $i, j \in \mathscr{V}$ are called *adjacent* or *neighbors* if $(i, j) \in \mathscr{E}$. The set of vertices adjacent to the vertex i is the *neighborhood* of i and will be denoted by ∂i; the cardinality of this set will be denoted by $|\partial i|$. A *path* is a sequence of edges that connects a sequence of vertices, which are all distinct from one another. It is said that a graph is a *tree* if any two vertices are connected by exactly one path. If the direction of the edges matters, the graphs are called *directed graphs* or *digraphs*. However, in this work we only deal with *undirected graphs*.

Sometimes it is useful to represent pictorially a graph with the so-called *graph diagrams* [27]. Figure 2.1 shows an example.

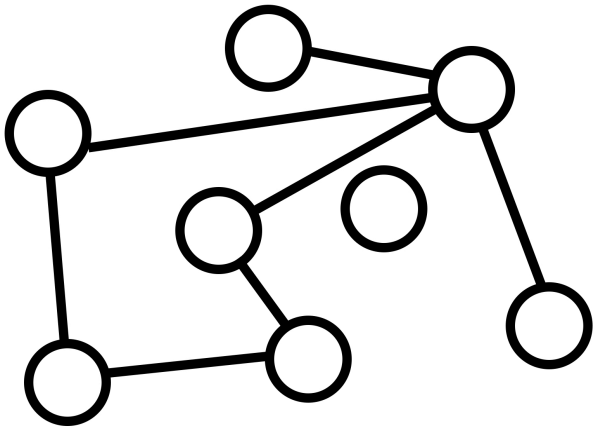


Figure 2.1: Graph diagram: the circles stand for the vertices and edges are represented by lines connecting them.

The *adjacency matrix* is another representation of a graph.

Definition 2.2.2. *The **adjacency matrix** $\boldsymbol{C} = \boldsymbol{C}(G)$ is an $N \times N$ matrix whose entries are given by [26]*

$$C_{ij} = \begin{cases} 1 & \text{if} \quad (i, j) \in \mathscr{E} \\ 0 & \quad \textit{otherwise} \end{cases} . \tag{2.11}$$

The adjacency matrix is a powerful tool in graph theory because it allows us to use many matrix properties in the graph analysis. Besides, we must highlight that the adjacency matrix codify *all* the information of a graph. Another important feature in undirected graphs is that their adjacency matrix is *symmetric*, that is $\mathbf{C} = \mathbf{C}^T$, where the super-index T denotes the transpose operation.

The graphs that we will consider do not have *self-interaction*. This means that $(i,i) \notin \mathscr{E}(G)\ \forall i$.

Now, we turn our attention to random graphs.

2.2.2 Random Graphs

Random graphs are graphs whose adjacency matrix is created through a random process. Their study began with the works of Erdős and Rènyi in the middle of the last century [3].

In this work, we will only consider the simplest model, the so-called *Poisson random graphs* [28]. In these random graphs each pair of vertices are connected with probability p. It is also worthwhile to mention that it is meaningless to associate a dimension, at least in the usual sense, to this kind of random graphs. The probability function of each entry of the adjacency matrix can be written as

$$P(C_{ij}) = p\delta_{C_{ij},1} + (1-p)\delta_{C_{ij},0}\,. \tag{2.12}$$

The name of this kind of random graphs is due to the fact that, if we set the parameter $p = \frac{c}{N}$, the probability that a vertex has degree k tends to a Poisson distribution as the number of vertices increases [29]

$$P(k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{c}{N}\right)^k \left(1 - \left(\frac{c}{N}\right)\right)^{N-k} = \frac{c^k e^{-c}}{k!}\,, \tag{2.13}$$

where c is the mean degree.

On the other hand, a *random matrix ensemble* is defined by the joint probability density function (JPDF) of its entries. Accordingly, the JPDF of the adjacency matrix of the random graphs used throughout the work is

$$P(\mathbf{C}) = \prod_{i<j} \left\{ \frac{c}{N}\delta_{C_{ij},1} + \left(1 - \frac{c}{N}\right)\delta_{C_{ij},0} \right\} \delta_{C_{ij},C_{ji}}\,. \tag{2.14}$$

A property of a random matrix ensemble, which is of interest to us in this work, is its *spectral density*.

2.2.3 Spectral Density

The set of eigenvalues $\left\{\lambda_i^{(\mathbf{M})}\right\}_{i=1,2,\ldots,N}$ of a matrix $\mathbf{M}_{NxN}$ is called the *spectrum* of $\mathbf{M}$. If we are interested in the distribution of these eigenvalues, we can define the *empirical spectral density* of $\mathbf{M}$ by

$$\varphi(\lambda;\mathbf{M}) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\lambda - \lambda_i^{(\mathbf{M})}\right)\,, \tag{2.15}$$

where δ is a δ-function. In the case of symmetric matrices this probability measure is over the real line but, in general, it is over the complex plane.

If $\mathbf{M}$ is drawn from a random matrix ensemble with JPDF $P(\mathbf{M})$, then it is pertinent to ask for the limiting mean empirical spectral density, given by

$$\langle\rho(\lambda;\mathbf{M})\rangle = \lim_{N\to\infty}\left\langle \frac{1}{N}\sum_{i=1}^{N}\delta\left(\lambda - \lambda_i^{(\mathbf{M})}\right)\right\rangle = \lim_{N\to\infty}\frac{1}{N}\sum_{\mathbf{M}} P(\mathbf{M})\sum_{i=1}^{N}\delta\left(\lambda - \lambda_i^{(\mathbf{M})}\right)\,. \tag{2.16}$$

In this context, a good estimator of Eq. (2.16) is the limiting empirical spectral density defined by

$$\rho(\lambda;\mathbf{M}) = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\delta\left(\lambda - \lambda_i^{(\mathbf{M})}\right)\,, \tag{2.17}$$

which fulfills the properties of every probability density [21]. For some specific random matrix ensembles, there are well-known results, such as the Wigner semi-circle distribution [30], the Marčenko-Pastur distribution [31] and the McKay Law [32].

Once we have reviewed the basic concepts and tools in probability and graph theory, we continue with a brief introduction of some concepts and tools of functional analysis.

2.3 Functional Analysis

In this section we review the concepts of *functional derivatives* and *functional integrals*. As it was mentioned before, the exposition is non-rigorous. A more formal discussion of the topic can be found in [33–35].

A *functional* can be seen as a generalization of a function, where the domain is a set of functions.

Definition 2.3.1. *Let S be a set of functions. F is called a **functional** on S if F is a mapping from S into $\mathbb{C}$ (or $\mathbb{R}$), that is [36]*

$$\begin{aligned} F : S &\to \mathbb{C}(\text{or}\,\mathbb{R}) \\ \phi &\mapsto F[\phi]\,. \end{aligned} \tag{2.18}$$

2.3.1 Functional Derivative

The definition that we will use is a practical one, that allows us to easily illustrate its use. This subsection is mainly inspired by [37].

Definition 2.3.2. *Let $F[\phi]$ be a functional from the set of functions $S = \{\phi(x)|x \in \mathbb{R}\}$ to the field of real or complex numbers. The **functional derivative** of F is the limit*

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\varepsilon \to 0} \frac{F[\phi + \varepsilon\delta(x-y)] - F[\phi]}{\varepsilon}\,. \tag{2.19}$$

To exemplify this definition, we prove a result which is analogous to the ordinary differential calculus.

Example 2.3.1. *Consider the following functional*

$$F[\phi] = \int (\phi(x))^n\, dx\,. \tag{2.20}$$

By definition,

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int (\phi(x) + \varepsilon\delta(x-y))^n dx - \int (\phi(x))^n dx \right). \tag{2.21}$$

Expanding the first integrand, we notice that the term $(\phi(x))^n$ cancels out with the second integral, and only the term which is linear with ε will not vanish after taking the limit $\varepsilon \to 0$. Hence

$$\frac{\delta F[\phi]}{\delta \phi(y)} = \int n(\phi(x))^{n-1}\delta(x-y)dx = n(\phi(y))^{n-1}\,. \tag{2.22}$$

One can show that the functional derivative has the following -consistency- property:

$$\frac{\delta\phi(x)}{\delta\phi(y)} = \delta(x-y)\,. \tag{2.23}$$

Furthermore, the functional derivative shares the same properties as partial derivatives. For instance, the functional derivative is a linear operator and the product of two functionals $F(\phi)$ and $G(\phi)$ follows the product rule

$$\frac{\delta}{\delta\phi(x)}(F[\phi]G[\phi]) = \frac{\delta F[\phi]}{\delta\phi(x)}G[\phi] + \frac{\delta G[\phi]}{\delta\phi(x)}F[\phi]\,. \tag{2.24}$$

2.3.2 Functional Integrals

Definition 2.3.3. *Let $F[\phi]$ be a functional on the set of functions $S(K) = \{\phi(x)|x \in K\}$ ($K = \mathbb{R}$ or $K = \mathbb{C}$) and $\phi : K \to K$. The **path integral** of F over $S(K)$ is formulated as [38]*

$$\int F[\phi]\{d\phi\} = \int \cdots \int F[\phi]\prod_x d\phi(x)\,. \tag{2.25}$$

The notation $\{d\phi\}$ stands for the "differential" in the functional integration [1].

A useful functional that will appear frequently in our derivations is the Dirac delta functional [29], denoted by δ_f. If $\phi(x)$ and $\theta(x)$ are functions and $F[\phi]$ is a functional, the functional Dirac delta is defined by

$$F[\phi] = \int \{d\theta\}F[\theta]\delta_f[\theta(x) - \phi(x)]\,. \tag{2.26}$$

The previous equation can be seen as a generalization of the property of the Dirac delta function inside an integral. Similarly, as it happens in the standard case, the δ-functional also has a Fourier representation [39]

$$\delta_f[\theta(x) - \phi(x)] = \int \left\{\frac{d\tilde{\theta}}{2\pi}\right\} \exp\left(i\int \tilde{\theta}(x)(\theta(x) - \phi(x))dx\right)\,, \tag{2.27}$$

where $\tilde{\theta}$ is just the integration variable. Let us finish the mathematical exposition with the following section.

2.4 Other Mathematical Tools

In this section two important mathematical tools are reviewed: the *Laplace method* and the *Hubbard-Stratonovich transformation*.

2.4.1 Laplace Method

This method, originally envisaged by Laplace [40], allows us to estimate the asymptotic behavior of integrals of the form

$$I = \int_a^b e^{-Nf(x)}dx\,, \tag{2.28}$$

[1] The braces avoids confusion with ordinary integration.

where N is a positive parameter and $f(x)$ is a real and smooth function within the interval (a,b). Suppose that there exists a unique point $x^* \in (a,b)$ such that

$$f'(x^*) = 0\,, \qquad f''(x^*) > 0\,, \tag{2.29}$$

that is, a local minimum. As the parameter N increases, the integral value is increasingly dominated by the minimum of $f(x)$. To prove this claim let us perform a power series expansion of $Nf(x)$ around x^*

$$Nf(x) = Nf(x^*) + \frac{N}{2!}f''(x^*)(x-x^*)^2 + \frac{N}{3!}f'''(x^*)(x-x^*)^3 + O((x-x^*)^4)\,. \tag{2.30}$$

Therefore, with the change of variable $x \to y = (x-x^*)/\sqrt{N}$, the integral I transform into

$$I = \frac{1}{\sqrt{N}}\int_{(a-x^*)/\sqrt{N}}^{(b-x^*)/\sqrt{N}} \exp\left(-Nf(x^*) - \frac{1}{2!}y^2 f''(x^*) - \frac{1}{3!}\frac{y^3}{\sqrt{N}}f'''(x^*) - O\left(\frac{y^4}{N}\right)\right)dy\,. \tag{2.31}$$

In this form it is clear that, while N increases, the main contribution to the integral value is given by the first two terms inside the exponential function. Besides, the upper and lower limits tend to ∞ and $-\infty$, respectively. Thus

$$\begin{aligned}\int_a^b e^{-Nf(x)}dx &\asymp e^{-Nf(x^*)}\int_{-\infty}^{\infty} e^{-\frac{f''(x^*)}{2}y^2}\frac{dy}{\sqrt{N}} \\ &= e^{-Nf(x^*)}\sqrt{\frac{2\pi}{Nf''(x_0)}}\,.\end{aligned} \tag{2.32}$$

The factor $\sqrt{\frac{2\pi}{Nf''(x^*)}}$ may be sometimes neglected if we only care about the exponential asymptotic behavior of the integral. In this case we simply write

$$\int_a^b e^{-Nf(x)}dx \asymp e^{-Nf(x^*)}\,. \tag{2.33}$$

The generalization of this method for complex integrands is called the *saddle-point method*.

2.4.2 Hubbard-Stratonovich Transformation

The second mathematical tool is the Hubbard-Stratonovich (H-S) transformation. This transformation was invented by R. L. Stratonovich [41] and popularized by J. Hubbard [42]. In spite of its simplicity, the H-S transformation is a powerful tool widely used in many-body theory [43] and particle physics [44].

To derive this transformation, we begin with a Gaussian integral

$$I = \int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2} - ux\right)du\,. \tag{2.34}$$

The usual method to solve this kind of integral is by completing the square in the argument of the exponential function

$$-\frac{u^2}{2} - ux = -\frac{u^2}{2} - ux - \frac{x^2}{2} + \frac{x^2}{2} = -\frac{(u+x)^2}{2} + \frac{x^2}{2}\,. \tag{2.35}$$

Substituting the previous relation in Eq. (2.34), we obtain

$$\begin{aligned} I &= \exp\left(\frac{x^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(u+x)^2}{2}\right)du \\ &= \exp\left(\frac{x^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{y^2}{2}\right)dy \\ &= \exp\left(\frac{x^2}{2}\right)\sqrt{2\pi}\,. \end{aligned} \tag{2.36}$$

From Eqs. (2.34) and (2.36) it is straightforward to obtain the H-S transformation, which in the present case takes the form

$$\exp\left(\frac{x^2}{2}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2}-ux\right)du\,. \tag{2.37}$$

With this section we have finished the mathematical part of the exposition. We next move to introduce both the cavity and replica methods in a statistical mechanics context, specifically both will be applied to the Ising model. Before doing so we review some basic concepts of statistical mechanics.

2.5 Statistical Physics

Statistical physics aims to explain and predict the macroscopic behavior of systems in terms of their constituent particles and the interactions between them. For this purpose, it makes use of statistical methods and the probability theory.

Particularly, in this work we are interested in the *canonical ensemble*, which can be seen as a collection of identical systems, all in thermal contact with a heat reservoir at temperature T. In this formalism, the probability that the system takes some microscopic configuration k, with corresponding energy E_k, is given by the *Boltzmann distribution*

$$P(k) = \frac{1}{Z}e^{-\beta E_k}\,, \tag{2.38}$$

where $\beta = \frac{1}{Tk_B}$ (k_B is the Boltzmann constant) and the Z is called the *partition function*[2]

$$Z = \sum_k e^{-\beta E_k}\,. \tag{2.39}$$

Despite its simplicity, the partition function plays a major role in the theory, since it is possible to obtain all the thermodynamical properties from it. For instance, the relation between the internal energy U (which is the average energy of the system) and the partition function is straightforward to compute

$$U = \langle E_k\rangle = \sum_k E_k P(E_k) = \frac{1}{Z}\sum_k E_k e^{-\beta E_k} = -\frac{1}{Z}\sum_k \frac{\partial}{\partial\beta}e^{-\beta E_k} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} = -\frac{\partial\left(\log(Z)\right)}{\partial\beta}\,. \tag{2.40}$$

The average over the Boltzmann distribution is usually known as the *thermal average*. In a similar fashion, the entropy S and the Helmholtz free energy F [45] can be obtained from the

[2]The letter Z comes from the German word *Zustandssumme* (sum of states).

partition function

$$\begin{aligned} S &= k_B \log(Z) + k_B \beta U \,, \\ F &= -\frac{1}{\beta} \log(Z) \,. \end{aligned} \tag{2.41}$$

In statistical physics, one colloquially says that a problem is fully solved when the partition function is calculated and from it, the Helmholtz free energy.

We have seen the important concepts (for our purposes) of statistical physics. Now, we want to introduce the Ising model that is going to be our "guinea pig".

2.5.1 Ising Model

The Ising model was proposed by Wilhelm Lenz in 1920 [46] with the purpose of creating a very simple mathematical model which exhibits ferromagnetic behavior. It was solved for the one-dimensional case by Ernst Ising five years later [47]. Although Ising found that in the one-dimensional case there is no phase transition to a ferromagnetic ordered state, nowadays this model is the cornerstone of most of the models which are exactly solved in statistical physics (see for example [48]). The solution for the two-dimensional case was given by Onsager 19 years after the Ising solution [49]. An overview of the curious history of the model can be found in [50].

The Ising model consists of N atoms located in the vertices of a periodic lattice. Each atom is solely characterized by the direction of its spin σ_i, which takes the value 1 if it is upwards or -1 if it is downwards. The configuration of the system is given by the vector $\overline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_N)$; a pictorial representation of the system is given in Figure 2.2. The spin-spin interaction is only between nearest neighbors (n.n.), so that the Hamiltonian reads

$$H(\overline{\sigma}) = -\sum_{n.n.} J_{ij} \sigma_i \sigma_j - \sum_{i=1}^{N} \theta_i \sigma_i \,, \tag{2.42}$$

where J_{ij} are the so-called exchange coupling constants, $\sum_{n.n}$ represents the sum over the nearest neighbors and θ_i stands for the effect of an external magnetic field[3] at the i-th spin.

It is assumed that magnetic properties arise due to the alignment of the spins. In this sense, a relevant property of a magnetic system is its *magnetization*, which is defined as

$$M = \frac{1}{N} \sum_i \langle \sigma_i \rangle \,, \tag{2.43}$$

where the average is taken over the Boltzmann distribution. This quantity tells us how aligned the spins are, depending on the temperature. For example, the more the spins are aligned, the more M tends to its extreme values ± 1.

In thermodynamics the magnetization is defined as

$$M = -\left(\frac{\partial F}{\partial \theta} \right)_T \,, \tag{2.44}$$

where we consider $\theta_i = \theta \; \forall i$. If we compare the previous relation with the Eq. (2.41), we can conclude that

$$M = \frac{1}{\beta} \frac{\partial \log Z}{\partial \theta} \,, \tag{2.45}$$

[3] Normally, it is preferred to use the letter h for the external magnetic field, but we reserve this letter for other purposes.

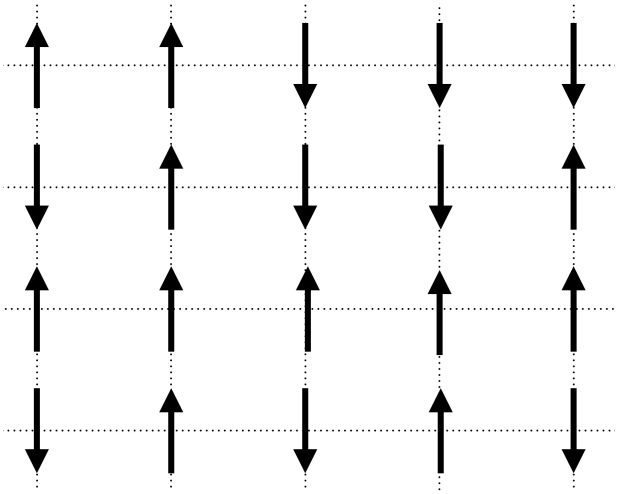


Figure 2.2: Ising model: The arrows represent the direction of the spin of each atom which is localized at the nodes of a lattice.

at constant temperature.

An important remark for our purposes is that, in this model, the marginal probability of each spin can always be written as

$$P_i(\sigma_i) = \frac{e^{-\beta\Theta_i\sigma_i}}{2\cosh(\beta\Theta_i)}\,. \tag{2.46}$$

This equation defines the effective magnetic field Θ_i, which takes into account the external magnetic field and the interactions with other spins. The subindex in P_i is a consequence of the fact that each spin may have a different effective field Θ_i, since the constants $\{J_{ji}\}$ may take different values. With this representation the *local magnetization* is

$$M_i = \langle\sigma_i\rangle = \tanh(\beta\Theta_i)\,. \tag{2.47}$$

Experimentally, it is found that the magnetization of ferromagnetic materials is zero above certain temperature known as the *Curie temperature*, which depends on the material. This disappearance of the magnetization is known as the paramagnetic-ferromagnetic *phase transition*. The usual behavior of the magnetization as a function of the temperature in a magnetic material is shown in the Figure 2.3. The interested reader can see how the magnetization is experimentally measured in [51] and some diagrams estimated from experimental measurements are shown in [52].

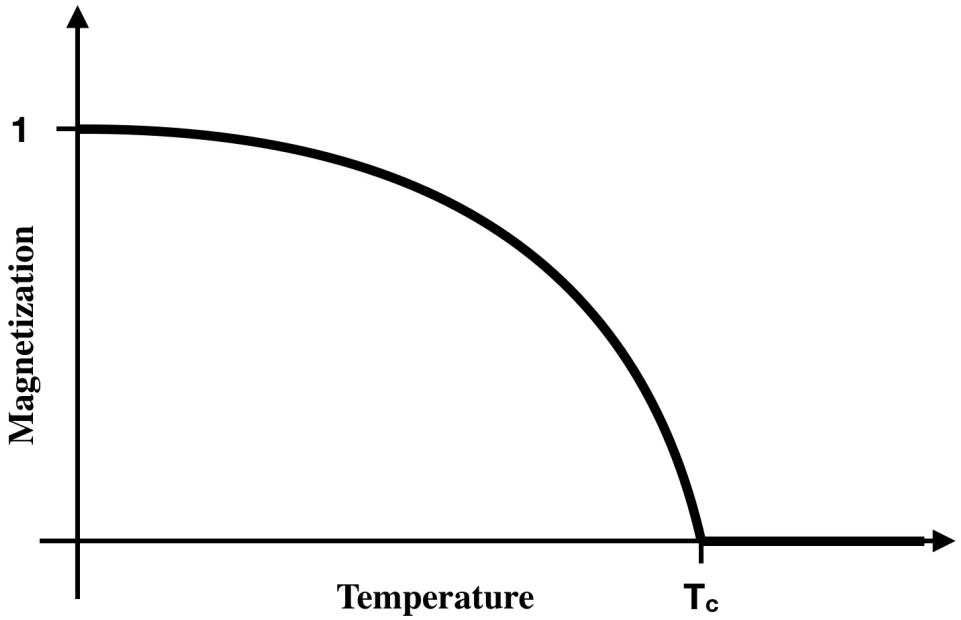


Figure 2.3: Usual behavior of the magnetization as a function of the temperature in a ferromagnetic material.

Chapter 3

Replica and Cavity Method

In this chapter the cavity and replica methods are introduced. Both methods have their origin in statistical physics, specifically, in the study of spin glasses [53].

In this context, the replica method was invented as a simple "trick" to facilitate the computation of thermodynamic properties [54]. On the other hand, the cavity method was originally elucidated [55] to reproduce the results that its peer had achieved in the study of spin glasses but with a more intuitive approach. Therefore we start in the following section with the latter.

As it was mentioned before, both methods are going to be applied to the Ising model to obtain its magnetization. In this case, the atoms will no longer be localized at the nodes of a lattice, but they will be attached to the vertices of a Poisson random graph and the interacting spins are those which are joined by an edge. In other words, the sum $\sum_{n.n}$ in the Hamiltonian (Eq. (2.42)) will be over adjacent spins.

3.1 Cavity Method

Before starting with the method let us introduce a useful approximation that will turn out to be essential for the cavity method, and also reveals a fundamental limitation of its use.

3.1.1 Bethe-Peierls Approximation

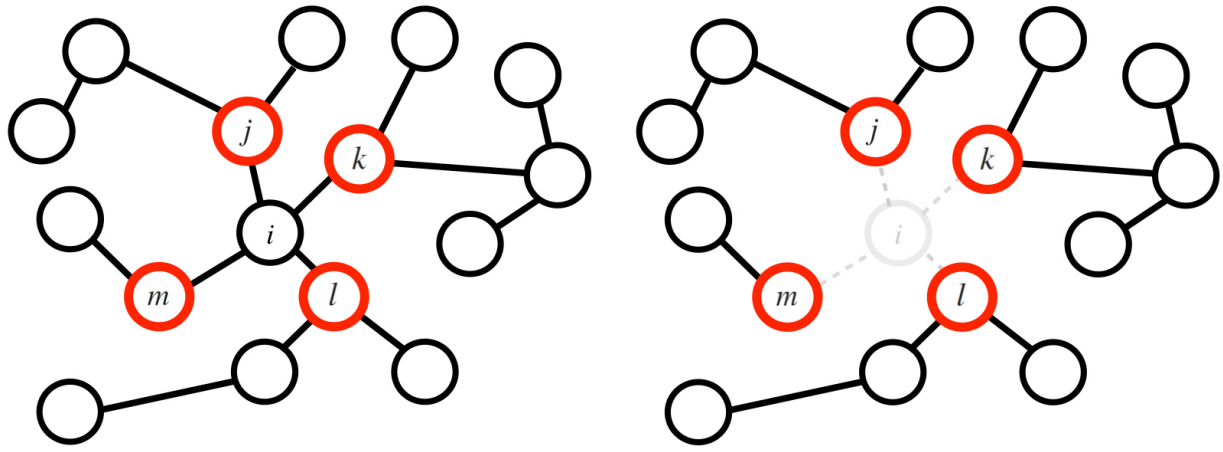
Let us first imagine an Ising model attached to a tree. If we remove a vertex, say the vertex i, then the graph is divided into disconnected subgraphs. In particular, the neighbors of the removed vertex are disconnected, as can be seen in Figure 3.1. Physically, this means that there is no interaction between the spins in ∂i and therefore, their marginal probability factorizes

$$P^{(i)}_{\partial i}(\sigma_{k\in\partial i}) = \prod_{k\in\partial i} P^{(i)}_k(\sigma_k)\,, \tag{3.1}$$

where the super-index in $P^{(i)}$ emphasizes that we are talking about the system without the vertex i and $\sigma_{k\in\partial i}$ is the set of those spins which are neighbors of the vertex i. Evidently, the previous equation is valid for trees, but for more general graphs it is only an approximation, known as the *Bethe-Peierls* (B-P) *approximation* [56, 57].

3.1.2 Cavities in Action

Let us suppose that we are only interested in the marginal probability distribution for each spin $P_i(\sigma_i)$, to calculate for instance the magnetization. In what follows we will use the B-P approxi-



a) b)

Figure 3.1: a) A tree where the set ∂i is highlighted. b) If the vertex i is removed, its neighbors are disconnected.

mation, also known as cavity method, to obtain a closed system of equations for these marginals. Starting from the definition of the marginal probability, we can write

$$P_i(\sigma_i) = \sum_{\overline{\sigma}_{/i}} P(\overline{\sigma}) = \frac{1}{Z} \sum_{\overline{\sigma}_{/i}} \exp\left[-\beta H(\overline{\sigma})\right], \tag{3.2}$$

where, again, the notation $\overline{\sigma}_{/i}$ represents the vector $\overline{\sigma}$ without the i-th entry. Let us separate the Hamiltonian of the system (Eq. (2.42)) into two parts, those terms involving the i-th spin, and those which do not

$$\begin{aligned} P_i(\sigma_i) &= \frac{1}{Z} \sum_{\overline{\sigma}_{/i}} \exp\left[-\beta\left(-\theta_i\sigma_i - \sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j + H^{(i)}(\overline{\sigma})\right)\right] \\ &= \frac{\exp[\beta\theta_i\sigma_i]}{Z} \sum_{\overline{\sigma}_{/i}} \exp\left[\beta\sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j\right] \exp\left[-\beta H^{(i)}(\overline{\sigma})\right], \end{aligned} \tag{3.3}$$

where $H^{(i)}(\overline{\sigma}) = H(\overline{\sigma}) + \theta_i\sigma_i + \sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j$ is the Hamiltonian of the system where the i-th spin has been removed. Next, we separate the sum $\sum_{\overline{\sigma}_{/i}} = \sum_{\sigma_{j\in\partial i}} \sum_{\sigma_{k\notin\partial i}}$, resulting in

$$\begin{aligned} P_i(\sigma_i) &= \frac{\exp[\beta\theta_i\sigma_i]}{Z} \sum_{\sigma_{j\in\partial i}} \exp\left[\beta\sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j\right] \sum_{\sigma_{k\notin\partial i}} \exp\left[-\beta H^{(i)}(\overline{\sigma})\right] \\ &= \frac{\exp[\beta\theta_i\sigma_i]}{Z_i} \sum_{\sigma_{j\in\partial i}} \exp\left[\beta\sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j\right] P^{(i)}_{\partial i}(\sigma_{k\in\partial i}). \end{aligned} \tag{3.4}$$

In the last equation, to obtain the marginal probability of the set ∂i in the system without the vertex i (i.e. $P^{(i)}_{\partial i}(\sigma_{k\in\partial i})$), a normalizing factor has been added and incorporated into Z_i. Up to here, our analysis is rather general, as we have not made any assumption about the graph. Now, we will apply the B-P approximation, yielding

$$P_i(\sigma_i) \simeq \frac{\exp[\beta\theta_i\sigma_i]}{Z_i} \sum_{\sigma_{j\in\partial i}} \exp\left[\beta\sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j\right] \prod_{l\in\partial i} P^{(i)}_l(\sigma_l). \tag{3.5}$$

As it was mention before (cf. Subsection 3.1.1), the B-P approximation is exact for trees. Therefore we will exclusively use *sparse* Poisson random graphs, which are locally tree-like (small *loops*[1] are rare). In our case at hand, the typical size of a loop is of order $\log N$ [58], therefore Eq. (3.5) is an approximation for every finite graph. How this approximation is affected by the presence of loops have been discussed in [59, 60].

Notice that the set of equations given by Eq. (3.5) is not closed yet because the probabilities on the right-hand side correspond to a different physical system from those on the left-hand side. To close the equations let us explore what happen if the vertex $j \in \partial i$ is removed. The marginal probability is then

$$P_i^{(j)}(\sigma_i) = \frac{1}{Z^{(j)}} \sum_{\overline{\sigma}_{/\{i,j\}}} \exp\left[\beta H^{(j)}(\overline{\sigma})\right], \tag{3.6}$$

where $\overline{\sigma}_{/\{i,j\}}$ represents the vector $\overline{\sigma}$ without the entries i and j. As in the previous case, the Hamiltonian and the configuration sum are separated into two, yielding

$$\begin{aligned} P_i^{(j)}(\sigma_i) &= \frac{\exp[\beta\theta_i\sigma_i]}{Z^{(j)}} \sum_{\sigma_{k\in\partial i/j}} \exp\left[\beta\sigma_i \sum_{k\in\partial i/j} J_{ik}\sigma_k\right] \sum_{\sigma_{l\notin\partial i/j}} \exp\left[\beta H^{(i,j)}(\overline{\sigma})\right] \\ &= \frac{\exp[\beta\theta_i\sigma_i]}{Z_i^{(j)}} \sum_{\sigma_{k\in\partial i/j}} \exp\left[\beta\sigma_i \sum_{k\in\partial i/j} J_{ik}\sigma_k\right] P_{\partial i/j}^{(i,j)}(\sigma_{k\in\partial i/j}). \end{aligned} \tag{3.7}$$

In this case the super-index in $P^{(i,j)}$ and $H^{(i,j)}$ indicates that we are talking about the system where the vertices i and $j \in \partial i$ have been removed and $P_{\partial i/j}^{(i,j)}(\sigma_{k\in\partial i/j})$ is the marginal probability of the set $\partial i/j$ in this system. If we use the Bethe-Peierls approximation one more time, we obtain

$$P_i^{(j)}(\sigma_i) = \frac{\exp[\beta\theta_i\sigma_i]}{Z_i^{(j)}} \sum_{\sigma_{k\in\partial i/j}} \exp\left[\beta\sigma_i \sum_{k\in\partial i/j} J_{ik}\sigma_k\right] \prod_{l\in\partial i/j} P_l^{(i,j)}(\sigma_l). \tag{3.8}$$

It seems that we have ended up in a similar problem as in the system of equations given by Eq. (3.5). However, a closer look at the system, whose Hamiltonian is $H^{(i,j)}(\overline{\sigma})$, will shed light on the problem. As we can see in Figure 3.2, the removal of the vertex j (after the vertex i has already been removed) in graphs which are locally tree-like has little impact on the rest of the vertices in ∂i. Therefore, for $l \in \partial i$, we have that

$$P_l^{(i,j)}(\sigma_l) \approx P_l^{(i)}(\sigma_l) \;\;,\;\; j \in \partial i. \tag{3.9}$$

With the help of this remark we can finally obtain a closed set of equations, known as *cavity equations* for the so-called cavity marginals

$$\begin{aligned} P_i^{(j)}(\sigma_i) &= \frac{\exp[\beta\theta_i\sigma_i]}{Z_i^{(j)}} \sum_{\sigma_{k\in\partial i/j}} \exp\left[\beta\sigma_i \sum_{k\in\partial i/j} J_{ik}\sigma_k\right] \prod_{l\in\partial i/j} P_l^{(i)}(\sigma_l) \\ &= \frac{\exp[\beta\theta_i\sigma_i]}{Z_i^{(j)}} \prod_{k\in\partial i/j} \sum_{\sigma_k=\pm1} \exp\left[\beta\sigma_i J_{ki}\sigma_k\right] P_k^{(i)}(\sigma_k). \end{aligned} \tag{3.10}$$

If we manage to solve this closed set of equations[2], we can use Eq. (3.5) to compute the marginals $P_i(\sigma_i)$ of our original problem. In the following we present a way to work out this, proposing a parameterization.

[1] Paths whose initial and final vertex is the same.

[2] Which contains $\sum_i^N |\partial i|$ variables and equations.

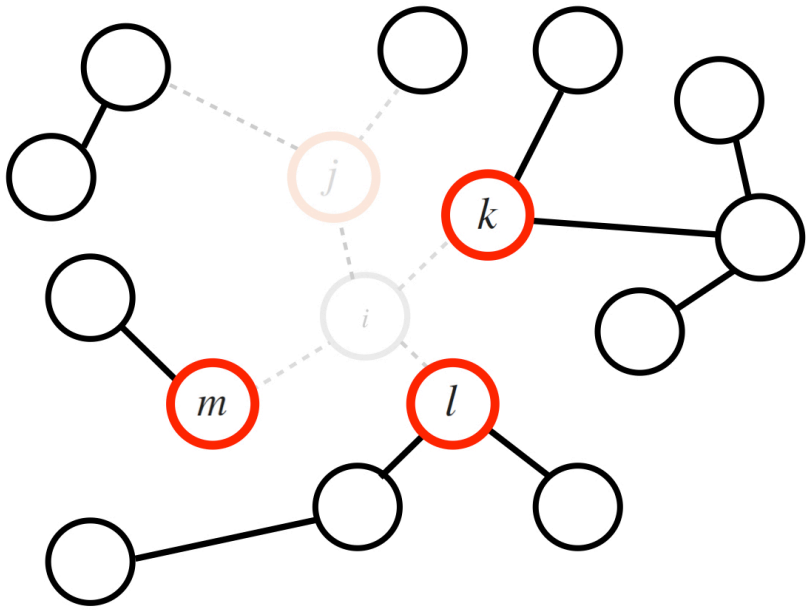


Figure 3.2: The vertices $j \in \partial i$ and i have been removed. If the graph is a tree, the remaining vertices of ∂i are still disconnected as they were in the presence of the vertex j.

Parameterization

It is customary to parameterize the probabilities and to obtain a relation between the parameters using Eq. (3.10). We propose the following parameterization

$$P_i^{(j)}(\sigma_i) = \frac{\exp\left[\beta h_i^{(j)}\right]}{2\cosh\left[\beta h_i^{(j)}\right]}, \qquad P_i(\sigma_i) = \frac{\exp[\beta h_i]}{2\cosh[\beta h_i]}, \tag{3.11}$$

with parameters $\{h_i\}_{i=1}^N$ and $\left\{h_i^{(j)}\right\}_{i=1}^N$ (with $j \in \partial i$), where the latter are known as *cavity fields*. Notice that these parameterization have the same functional form as Eq. (2.46). Therefore, our parameters can be physically interterpreted as effective magnetic fields of our original system and of the system without one vertex, respectively. Now, we will find the value of these parameters.

Isolating[3] the cavity fields from the previous equation, we get

$$h_i^{(j)} = \frac{1}{2\beta} \sum_{\sigma=\pm 1} \sigma_i \log P_i^{(j)}(\sigma), \tag{3.12}$$

and substituting the cavity equations in the previous relation

$$\begin{aligned} h_i^{(j)} &= \frac{1}{2\beta} \sum_{\sigma_i=\pm 1} \sigma_i \log \left\{ \frac{1}{Z_i^{(j)}} \exp[\beta\theta_i\sigma_i] \prod_{k\in\partial i/j} \sum_{\sigma_k=\pm 1} \exp[\beta\sigma_i J_{ki}\sigma_k] P_k^{(i)}(\sigma_k) \right\} \\ &= \frac{1}{2\beta} \left(\underbrace{\sum_{\sigma_i=\pm 1} \sigma_i \log \left\{ \frac{1}{Z_i^{(j)}} \right\}}_{0} + \overbrace{\sum_{\sigma_i=\pm 1} \sigma_i^2 \beta\theta_i}^{2\beta\theta_i} + \sum_{\sigma_i=\pm 1} \sum_{k\in\partial i/j} \sigma_i \log \left\{ \sum_{\sigma_k} \exp[\beta\sigma_i J_{ki}\sigma_k] P_k^{(i)}(\sigma_k) \right\} \right). \end{aligned} \tag{3.13}$$

[3] We take the logarithm in both sides of Eq. (3.11), multiply the resulting equation by σ_i and take the sum over its possible values.

Then, if the parameterization given in Eq. (3.11) is substituted, after some algebraic manipulations[4] the previous equation transforms into the following closed system

$$h_i^{(j)} = \theta_i + \frac{1}{\beta} \sum_{k \in \partial i / j} \operatorname{atanh} \left\{ \tanh \left[\beta h_k^{(i)} \right] \tanh[\beta J_{ki}] \right\} . \tag{3.14}$$

Once the previous system of equations has been numerically solved, using for example the fixed-point iteration, it is possible to obtain the parameters $\{h_i\}_{i=1}^N$ with the help of the relation

$$h_i = \theta_i + \frac{1}{\beta} \sum_{k \in \partial i} \operatorname{atanh} \left\{ \tanh \left[\beta h_k^{(i)} \right] \tanh [\beta J_{ki}] \right\} , \tag{3.15}$$

which is reached in an analogous fashion as Eq. (3.14). Once $\{h_i\}_{i=1}^N$ are obtained, we can use $P_i(\sigma_i)$ to obtain the magnetization

$$M = \frac{1}{N} \sum_i \sigma_i P_i(\sigma_i) = \frac{1}{N} \sum_i \tanh(\beta h_i) . \tag{3.16}$$

Now let us turn our attention to the solution of the same problem by using the replica method.

3.2 Replica Method

Let us reanalyze the previous model with the replica method. For simplicity we will assume coupling constants $J_{ij} = 1 \ \forall i, j$ and the external field $\theta = 0$. Accordingly, we define the Hamiltonian of the system by

$$H_{\mathbf{C}}(\overline{\sigma}) = -\sum_{i<j}^{N} C_{ij} \sigma_i \sigma_j , \tag{3.17}$$

where C_{ij} are the entries of the adjacency matrix of our Poisson random graphs denoted by[5] $\mathbf{C}$, and the subindex in $H_{\mathbf{C}}$ reminds us the dependence on this adjacency matrix. The first evident problem is that the thermodynamic properties of the system depend on $\mathbf{C}$. We will trust in the intuitive assumption (backed up by experience) that for large systems, their thermodynamic properties will be the same, no matter their specific realization. Hence, these properties must be a kind of average over the *disorder* caused by the random graph. This disorder is called *quenched*, because it does not change with time.

With this in mind, if we imagine a large system and divide it into many small (macroscopic) subsystems, then the properties of the former are going to be the average over the properties of the latter, i.e. first we get the properties of each subsystem (thermal average) and then we calculate the average over the quenched disorder. Mathematically, if we are interested in the observable $\mathscr{O}(\overline{\sigma})$ of our system, we need to calculate $\langle \langle \mathscr{O}(\overline{\sigma}) \rangle_T \rangle_C$, where $\langle \cdot \rangle_T$ will represent the thermal average and $\langle \cdot \rangle_C$ the average over the quenched disorder (over the probability function that describes it). Let us see how this quantity is computed.

The thermal average $\langle \cdot \rangle_T$ is usually performed with the help of an external magnetic field s in the Hamiltonian

$$H_{\mathbf{C}}(\overline{\sigma}, s) = -\sum_{i<j}^{N} C_{ij} \sigma_i \sigma_j + s \mathscr{O}(\overline{\sigma}) . \tag{3.18}$$

[4] See Appendix A, for details.

[5] See Definition 2.2.2.

Hence,

$$\left(\frac{\partial}{\partial s}\log\left[Z_{\mathbf{C}}(s)\right]\right)\Big|_{s=0} = -\beta\left\langle \mathscr{O}(\overline{\sigma})\right\rangle_T \tag{3.19}$$

and therefore

$$\left\langle\left\langle \mathscr{O}(\overline{\sigma})\right\rangle_T\right\rangle_C = -\frac{1}{\beta}\left(\frac{\partial}{\partial s}\left\langle\log\left[Z_{\mathbf{C}}(s)\right]\right\rangle_C\right)\Big|_{s=0}. \tag{3.20}$$

In this way, it is clear that we need to compute $\langle \log Z_{\mathbf{C}}\rangle_C$. We can work out this average with a simple mathematical relation [61]

$$\langle \log Z_{\mathbf{C}}\rangle_C = \lim_{n\to 0}\frac{1}{n}\log\left\langle Z_{\mathbf{C}}^n\right\rangle_C. \tag{3.21}$$

This identity is at the heart of the so-called *replica trick*. This trick works as follow: we first assume n to be integer, so that $\left\langle Z_{\mathbf{C}}^n\right\rangle_C$ is easier [62] to compute than $\langle \log Z_{\mathbf{C}}\rangle_C$; then n is assumed to be real and the limit $n\to 0$ is performed. For n integer we can imagine that $Z_{\mathbf{C}}^n$ is the partition function of a new system which consist of n *replicas* of the original system.

3.2.1 Replicas in Action

The first step is to make n replicas of our system

$$Z_{\mathbf{C}}^n = \left(\sum_{\overline{\sigma_1}}\exp\left[-\beta H_{\mathbf{C}}(\overline{\sigma_1})\right]\right)\ldots\left(\sum_{\overline{\sigma_n}}\exp\left[-\beta H_{\mathbf{C}}(\overline{\sigma_n})\right]\right) = \sum_{\overline{\sigma_1}}\ldots\sum_{\overline{\sigma_n}}\left(\exp\left[-\beta\sum_{a=1}^{n}H_{\mathbf{C}}(\overline{\sigma_a})\right]\right). \tag{3.22}$$

Next, we perform the average over the quenched disorder, which in our case is the average over the random graph (over its adjacency matrix $\mathbf{C}$)

$$\left\langle Z_{\mathbf{C}}^n\right\rangle_C = \sum_{\mathbf{C}}P(\mathbf{C})\sum_{\overline{\sigma_1}}\ldots\sum_{\overline{\sigma_n}}\left(\exp\left[-\beta\sum_{a=1}^{n}H_{\mathbf{C}}(\overline{\sigma_a})\right]\right), \tag{3.23}$$

where $\sum_{\mathbf{C}}$ represents the sum over all the possible adjacency matrices. For our random graphs, the JPDF is $P(\mathbf{C}) = \prod_{i<j}\left\{\frac{c}{N}\delta_{C_{ij},1} + \left(1-\frac{c}{N}\right)\delta_{C_{ij},0}\right\}\delta_{C_{ij},C_{ji}}$, as was given by Eq. (2.14), therefore

$$\begin{aligned}\left\langle Z_{\mathbf{C}}^n\right\rangle_C &= \sum_{\mathbf{C}}\left(\prod_{i<j}\left\{\frac{c}{N}\delta_{C_{ij},1} + \left(1-\frac{c}{N}\right)\delta_{C_{ij},0}\right\}\delta_{C_{ij},C_{ji}}\right)\sum_{\overline{\sigma_1}}\ldots\sum_{\overline{\sigma_n}}\left(\exp\left[\beta\sum_{i<j}^{N}C_{ij}\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right]\right)\\ &= \sum_{\overline{\sigma_1}}\ldots\sum_{\overline{\sigma_n}}\prod_{i<j}\sum_{C_{ij}\in\{0,1\}}\left\{\frac{c}{N}\delta_{C_{ij},1} + \left(1-\frac{c}{N}\right)\delta_{C_{ij},0}\right\}\exp\left[\beta C_{ij}\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right].\end{aligned} \tag{3.24}$$

Performing the sum over the possible values of C_{ij} we finally obtain

$$\left\langle Z_{\mathbf{C}}^n\right\rangle_C = \sum_{\overline{\sigma_1}}\ldots\sum_{\overline{\sigma_n}}\prod_{i<j}\left\{\frac{c}{N}\exp\left[\beta\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right] + \left(1-\frac{c}{N}\right)\right\}. \tag{3.25}$$

Let us rewrite the expression between braces in an exponential form

$$\begin{aligned}\langle Z_{\mathbf{C}}^n\rangle_C &= \sum_{\overline{\sigma_1}}\dots\sum_{\overline{\sigma_n}}\exp\left[\sum_{i<j}\log\left\{1+\frac{c}{N}\left(\exp\left[\beta\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right]-1\right)\right\}\right] \\ &= \sum_{\overline{\sigma_1}}\dots\sum_{\overline{\sigma_n}}\exp\left[\frac{1}{2}\sum_{i,j}\log\left\{1+\frac{c}{N}\left(\exp\left[\beta\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right]-1\right)\right\}+\right. \\ &\left.-\frac{N}{2}\log\left\{1+\frac{c}{N}\left(\exp\left[\beta n\right]-1\right)\right\}\right].\end{aligned} \tag{3.26}$$

At the end, we will take the *thermodynamic limit* $N\to\infty$, hence, we can expand the logarithms and take into account only the first terms. It will turn out that only the terms of order N (extensive terms) will be significant. Therefore

$$\langle Z_{\mathbf{C}}^n\rangle_C = \sum_{\overline{\sigma_1}}\dots\sum_{\overline{\sigma_n}}\exp\left[\frac{c}{2N}\sum_{i,j}\left(\exp\left[\beta\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right]-1\right)+O(N^0)\right]. \tag{3.27}$$

The sum $\sum_{i,j}$ is of order N^2, so the explicit term inside the exponential is the only extensive term.

Let us introduce the following useful notation. We denote with $\underline{\sigma}=(\sigma_1,\sigma_2,\dots,\sigma_n)$ a vector in the replica space and, as before, $\overline{\sigma}=(\sigma_1,\sigma_2,\dots,\sigma_N)$. A visual representation of these vectors is shown in the Figure 3.3.

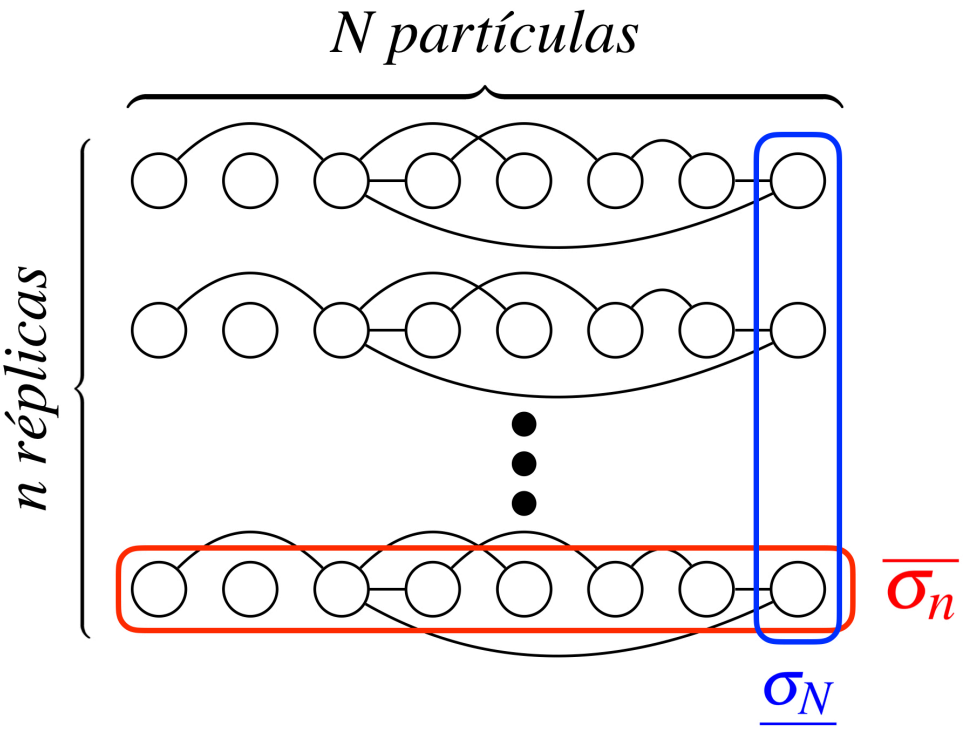


Figure 3.3: Visual representation of the replicated system and the different vectors in the system.

Taking advantage of the fact that $f(x_i)=\sum_m \delta_{m,x_i} f(m)$, we can use a vector $\underline{\tau}$ whose entries,

like $\underline{\sigma}$, can take the values 0 and 1 to write

$$\begin{aligned}\sum_{i,j}\exp\left[\beta\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right] &= \sum_{i,j}\sum_{\underline{\tau}}\delta_{\underline{\tau},\underline{\sigma_i}}\exp\left[\beta\sum_{a=1}^{n}\tau_a\sigma_{ja}\right] \\ &= \sum_{i,j}\sum_{\underline{\tau},\underline{\gamma}}\delta_{\underline{\tau},\underline{\sigma_i}}\delta_{\underline{\gamma},\underline{\sigma_j}}\exp\left[\beta\sum_{a=1}^{n}\tau_a\gamma_a\right].\end{aligned} \tag{3.28}$$

Defining the function $Q(\underline{\tau},\{\underline{\sigma_i}\}_{i=1}^{N}) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\tau},\underline{\sigma_i}}$, which is the probability that a vector $\underline{\sigma}$ takes the value $\underline{\tau}$ in the replicated system [29], we can rewrite the previous equality more conveniently as

$$\sum_{i,j}\exp\left[\beta\sum_{a=1}^{n}\sigma_{ia}\sigma_{ja}\right] = N^2\sum_{\underline{\tau},\underline{\gamma}}Q(\underline{\tau},\{\underline{\sigma_i}\})Q(\underline{\gamma},\{\underline{\sigma_i}\}). \tag{3.29}$$

Substituting it in Eq. (3.27) and ignoring the terms $O(N^0)$, we obtain

$$\langle Z_{\mathbf{C}}^n\rangle_C = \sum_{\overline{\sigma_1}}\cdots\sum_{\overline{\sigma_n}}\exp\left[\frac{cN}{2}\sum_{\underline{\tau},\underline{\gamma}}Q(\underline{\tau},\{\underline{\sigma_i}\})Q(\underline{\gamma},\{\underline{\sigma_i}\})(\exp\left[\beta\underline{\tau}\cdot\underline{\gamma}\right]-1)\right], \tag{3.30}$$

and after introducing a δ-functional we get

$$\langle Z_{\mathbf{C}}^n\rangle_C = \sum_{\overline{\sigma_1}}\cdots\sum_{\overline{\sigma_n}}\int\{dP\}\exp\left[\frac{cN}{2}\sum_{\underline{\tau},\underline{\gamma}}P(\underline{\tau})P(\underline{\gamma})\left(\exp\left[\beta\underline{\tau}\cdot\underline{\gamma}\right]-1\right)\right]\delta_f\left[P(\underline{\tau})-\frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\tau},\underline{\sigma_i}}\right], \tag{3.31}$$

where P is simply the integral variable of the δ-functional. Let us perform the Fourier transform of this δ-functional

$$\delta_f\left[P(\underline{\tau})-\frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\tau},\underline{\sigma_i}}\right] = \int\left\{\frac{d\tilde{P}}{2\pi}\right\}\exp\left[i\sum_{\underline{\tau}}\tilde{P}(\underline{\tau})\left(P(\underline{\tau})-\frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\tau},\underline{\sigma_i}}\right)\right]. \tag{3.32}$$

Substituting in Eq. (3.31)

$$\begin{aligned}\langle Z_{\mathbf{C}}^n\rangle_C = \int\left\{\frac{dPd\tilde{P}}{2\pi}\right\}\exp\left[\frac{cN}{2}\sum_{\underline{\tau},\underline{\gamma}}P(\underline{\tau})P(\underline{\gamma})(\exp\left[\beta\underline{\tau}\cdot\underline{\gamma}\right]-1)+i\sum_{\underline{\tau}}\tilde{P}(\underline{\tau})P(\underline{\tau})\right] \\ \times\sum_{\overline{\sigma_1}}\cdots\sum_{\overline{\sigma_n}}\exp\left[-\frac{i}{N}\sum_{i=1}^{N}\sum_{\underline{\tau}}\delta_{\underline{\tau},\underline{\sigma_i}}\tilde{P}(\underline{\tau})\right].\end{aligned} \tag{3.33}$$

To simplify the previous equation, we notice that we can count all the possible configurations in a different way[6], thus, the last term of the previous expression can be rewritten as

$$\sum_{\overline{\sigma_1}}\cdots\sum_{\overline{\sigma_n}}\exp\left[-\frac{i}{N}\sum_{i=1}^{N}\sum_{\underline{\tau}}\delta_{\underline{\tau},\underline{\sigma_i}}\tilde{P}(\underline{\tau})\right] = \sum_{\underline{\sigma_1}}\cdots\sum_{\underline{\sigma_N}}\exp\left[-\frac{i}{N}\sum_{i=1}^{N}\tilde{P}(\underline{\sigma_i})\right] = \left(\sum_{\underline{\sigma}}\exp\left[-\frac{i}{N}\tilde{P}(\underline{\sigma})\right]\right)^N. \tag{3.34}$$

[6]Instead of counting the possible states of the system row by row in Fig. 3.3, i.e. $\sum_{\overline{\sigma_1}}\cdots\sum_{\overline{\sigma_n}}$, we can count them column by column, i.e. $\sum_{\underline{\sigma_1}}\cdots\sum_{\underline{\sigma_N}}$.

Taking this into account and with the change of variable $\tilde{P} \to N\hat{P}$, we are able to rewrite Eq. (3.33) in the suggestive form

$$\langle Z_{\mathbf{C}}^n \rangle_C = \int D[P,\hat{P}] e^{-NS[P,\hat{P}]}. \tag{3.35}$$

where $D\left[P,\hat{P}\right] \equiv N\left\{\frac{dPd\hat{P}}{2\pi}\right\}$ and we have defined the functional

$$S[P,\hat{P}] = -\frac{c}{2}\sum_{\underline{\tau},\underline{\gamma}} P(\underline{\tau})P(\underline{\gamma})\left(\exp\left[\beta\underline{\tau}\cdot\underline{\gamma}\right]-1\right) - i\sum_{\underline{\tau}}\tilde{P}(\underline{\tau})P(\underline{\tau}) - \log\left\{\sum_{\underline{\sigma}}\exp\left[-i\hat{P}(\underline{\sigma})\right]\right\}. \tag{3.36}$$

Expressing $\left\langle Z_{\mathbf{C}}^n \right\rangle_C$ as in Eq. (3.35) allows us to use the saddle-point method to estimate the integral. Thus, we want to find the extremal value of the functional $S[P,\hat{P}]$, which is usually known as the *action*, in analogy to the Lagrangian formalism of classical mechanics. The equations that must be fulfilled[7] to apply the method are

$$\frac{\delta S[P,\hat{P}]}{\delta P(\underline{\sigma})} = 0, \qquad \frac{\delta S[P,\hat{P}]}{\delta \hat{P}(\underline{\sigma})} = 0. \tag{3.37}$$

These conditions are known as the *saddle-point equations*. From the first condition, performing the functional derivative we obtain

$$\begin{aligned} 0 &= -\frac{c}{2}\sum_{\underline{\tau},\underline{\gamma}}\left(\exp\left[\beta\underline{\tau}\cdot\underline{\gamma}\right]-1\right)\left(\delta_{\underline{\tau},\underline{\sigma}}P(\underline{\gamma})+\delta_{\underline{\gamma},\underline{\sigma}}P(\underline{\tau})\right) - i\sum_{\underline{\tau}}\hat{P}(\underline{\tau})\delta_{\underline{\tau},\underline{\sigma}} \\ &= -\frac{c}{2}\sum_{\underline{\gamma}}\left(\exp\left[\beta\underline{\sigma}\cdot\underline{\gamma}\right]-1\right)P(\underline{\gamma}) - \frac{c}{2}\sum_{\underline{\tau}}\left(\exp[\beta\underline{\tau}\cdot\underline{\sigma}]-1\right)P(\underline{\tau}) - i\hat{P}(\underline{\sigma}). \end{aligned} \tag{3.38}$$

Hence, the first condition reads

$$-i\hat{P}(\underline{\sigma}) = c\sum_{\underline{\tau}}(\exp[\beta\underline{\tau}\cdot\underline{\sigma}]-1)P(\underline{\tau}). \tag{3.39}$$

The second condition in Eq. (4.22) yields

$$0 = -i\sum_{\underline{\tau}}\delta_{\underline{\tau},\underline{\sigma}}P(\underline{\tau}) - \frac{-i\sum_{\underline{\tau}}\delta_{\underline{\tau},\underline{\sigma}}\exp\left[-i\hat{P}(\underline{\tau})\right]}{\sum_{\underline{\tau}}\exp\left[-i\hat{P}(\underline{\tau})\right]}, \tag{3.40}$$

and therefore

$$P(\underline{\sigma}) = \frac{\exp\left[-i\hat{P}(\underline{\sigma})\right]}{\sum_{\underline{\sigma}}\exp\left[-i\hat{P}(\underline{\sigma})\right]}. \tag{3.41}$$

Using the first condition to eliminate the dependence on $\hat{P}(\underline{\sigma})$, the probability $P(\underline{\sigma})$ must satisfy the saddle-point equation

$$\begin{aligned} P(\underline{\sigma}) &= \frac{\exp\left[c\sum_{\underline{\tau}}(\exp[\beta\underline{\tau}\cdot\underline{\sigma}]-1)P(\underline{\tau})\right]}{\sum_{\underline{\sigma}}\exp\left[c\sum_{\underline{\tau}}(\exp[\beta\underline{\tau}\cdot\underline{\sigma}]-1)P(\underline{\tau})\right]} \\ &= \frac{\sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\left(\sum_{\underline{\tau}}\exp\left[\beta\underline{\sigma}\cdot\underline{\tau}\right]P(\underline{\tau})\right)^l}{\sum_{\underline{\sigma}}\exp\left[c\sum_{\underline{\tau}}(\exp[\beta\underline{\tau}\cdot\underline{\sigma}]-1)P(\underline{\tau})\right]}. \end{aligned} \tag{3.42}$$

[7] We skip the second condition in Eq. (2.29).

Now, before performing the limit $n \to 0$, an assumption about the functional form of $P(\underline{\sigma})$ in the replica space must be considered. Intuitively, as the original system is invariant under the permutation of any pair of replicas, we expect $P(\underline{\sigma})$ to also fulfill that symmetry [8]. This is known as *replica symmetric ansatz*. For our case at hand, this implies that

$$P_{RS}(\underline{\sigma}) = \int w(h) \prod_{a=1}^{n} q(\sigma_a|h) dh\,, \tag{3.43}$$

where $w(h)$ is the probability density function of the effective magnetic field of the system and $q(\sigma|h)$ is just the probability that σ takes the value 1 or -1 given that its effective magnetic field is h. Notice that with this ansatz an expression for the magnetization is given by[9]

$$M = \int w(h) \tan(\beta h) dh\,. \tag{3.44}$$

Substituting our ansatz given by Eq. (3.43) into both sides of Eq. (3.42), it is possible to derive an equation for $w(h)$. Considering that the denominator in the latter tends to the unit as n tends to zero, for simplicity, we ignore it altogether in the following

$$\begin{aligned} P_{RS}(\underline{\sigma}) &= \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \left(\sum_{\underline{\tau}} \exp\left[\beta \underline{\sigma} \cdot \underline{\tau}\,\right] \int w(h) \prod_{a=1}^{n} q(\tau_a|h) dh \right)^l \\ &= \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} dh_k w(h_k) \right) \left(\prod_{k=1}^{l} \sum_{\underline{\tau}_l} \exp\left[\beta \sum_{a=1}^{n} \sigma_a \tau_{al} \right] \prod_{a=1}^{n} q(\tau_{al}|h_k) \right) \\ &= \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} dh_k w(h_k) \right) \prod_{a=1}^{n} \prod_{k=1}^{l} \sum_{\tau=\pm 1} \exp\left[\beta \sigma_a \tau\right] q(\tau|h_k)\,. \end{aligned} \tag{3.45}$$

It will turn out that the introduction of a Dirac delta function is going to be useful to rewrite the right-hand side of the previous equation in a similar fashion to the left-hand side, facilitating the identification of $w(h)$. Hence,

$$\begin{aligned} P_{RS}(\underline{\sigma}) = \int dh \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} dh_k w(h_k) \right) \prod_{a=1}^{n} q(\sigma_a|h) \\ \delta \left(q(\sigma|h) - \frac{\prod_{k=1}^{l} \sum_{\tau} \exp\left[\beta \sigma \tau\right] q(\tau|h_k)}{Z\left(\{h_k\}_{k=1}^{l}\right)} \right) Z^n \left(\{h_k\}_{k=1}^{l} \right), \end{aligned} \tag{3.46}$$

where $Z\left(\{h_k\}_{k=1}^{l}\right) = \sum_{\sigma_a} \prod_{k=1}^{l} \sum_{\tau} \exp\left[\beta \sigma_a \tau\right] q(\tau|h_k)$ is just a normalizing factor [10].

To carry out the derivation, we consider the following parameterization

$$q(\sigma|h) = \frac{\exp[\beta \sigma h]}{2\cosh[\beta h]}\,. \tag{3.47}$$

Isolating the field h as a function of $\rho(\sigma|h)$ we find that

$$h = \frac{1}{2\beta} \sum_{\sigma} \sigma \log[q(\sigma|h)]\,. \tag{3.48}$$

[8] This assumption can lead to unphysical conclusions; thus, the theory of *replica symmetry breaking* arises [63]. Fortunately, in our context the method has shown to lead to valid results, as we will see.

[9] See Eq. (2.47).

[10] Remember that $q(\sigma_a|h)$ is a probability and must be normalized.

With this relation and taking advantage of the fact that under the integral in Eq. (3.46) $q(\sigma|h)$ must be evaluated as

$$q(\sigma|h) \to \frac{1}{Z\left(\{h_k\}_{k=1}^{l}\right)} \prod_{k=1}^{l} \sum_{\tau} \exp\left[\beta\sigma\tau\right] q(\tau|h_k)\,, \tag{3.49}$$

we can obtain a relation between the parameters, namely

$$h \to \frac{1}{\beta} \sum_{k=1}^{l} \operatorname{atanh}[\tanh[\beta J] \tanh[\beta h_k]]\,, \tag{3.50}$$

cf. Appendix A for details. With this relation we can finally have an expression for $w(h)$ from Eq. (3.46)

$$\begin{aligned} \int dh w(h) \prod_{a=1}^{n} q(\sigma_a|h) = \int dh \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} dh_k w(h_k) \right) \prod_{a=1}^{n} q(\sigma_a|h) \\ \delta\left(h - \frac{1}{\beta} \sum_{k=1}^{l} \operatorname{atanh}[\tanh[\beta J] \tanh[\beta h_k]] \right) Z^n\left(\{h_k\}_{k=1}^{l}\right)\,. \end{aligned} \tag{3.51}$$

Thus in the limit $n \to 0$ we arrive to a self-consistent equation for $w(h)$

$$w(h) = \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} dh_k w(h_k) \right) \delta\left(h - \frac{1}{\beta} \sum_{k=1}^{l} \operatorname{atanh}[\tanh[\beta J] \tanh[\beta h_k]] \right)\,. \tag{3.52}$$

With the help of the *population dynamics algorithm* (cf. Appendix C) we can use the previous equation together with Eq. (3.44) to obtain the desired magnetization.

3.3 Numerical Results

In the previous sections we have reviewed two different ways of obtaining the magnetization of the Ising model attached to a Poisson random graph. In order to prove the validity of our theory, in this section we make a comparison between the theory and simulations.

In Figure 3.4 the numerical solution of both methods and Monte Carlo simulations, obtained through the *Metropolis Algorithm*[11], are presented. We have assumed a system with exchange coupling constants $J_{ij} = 1\ \forall i, j$ and external magnetic field $\theta = 0$. We can see a very good agreement.

To explain the implementation of the Metropolis algorithm (with $k_B = 1$), let us define our *algorithmic time* as the switch proposal of only one (orderly selected) spin, and a *Monte Carlo* (M-C) *step* as N (the size of the system) algorithmic times. The size of the system in every simulation was $N = 4000$ spins. For each temperature all of them were pointing up at the initial condition and we let the system 400 M-C steps to thermalize. We calculated the average of the magnetization every 20 M-C steps after we reached the thermalization; our average was over 135 values.

Once the methods have been presented, we are able to compute the spectral density of Poisson random graphs with their help.

[11] Cf. Appendix B for a review of this algorithm.

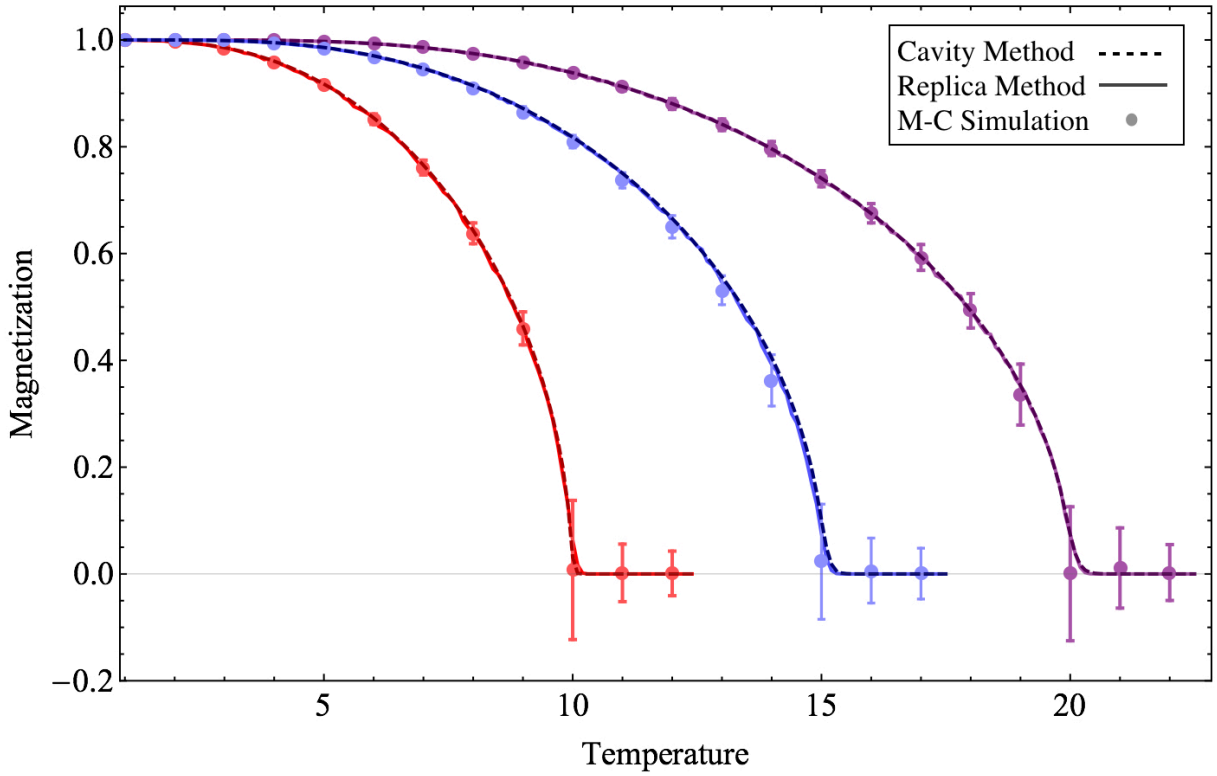


Figure 3.4: Magnetization of an Ising Model. Comparison between the solution obtained from the replica method through a population dynamics algorithm (solid lines), the solution obtained from the cavity method (dashed line) and Monte Carlo simulation for $c \in \{10, 15, 20\}$.

Chapter 4

Spectral Density of Poisson Random Graphs

In this chapter the spectral density of Poisson random graphs is derived using the cavity and replica methods, based on [18] and [19], respectively. The main idea to achieve this is to relate this mathematical object with the partition function of a system of interacting particles. Let us see how this is done.

We already know that the limiting empirical spectral density of a matrix $\mathbf{M}_{N\times N}$ is

$$\rho(\lambda;\mathbf{M}) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i^{(\mathbf{M})}), \tag{4.1}$$

where $\{\lambda_i^{(\mathbf{M})}\}_{i=1,2,\ldots,N}$ are the eigenvalues of the matrix $\mathbf{M}$. In physics, for example, $\mathbf{M}$ could be the Hamiltonian operator of a system, in which case the empirical spectral density is interpreted as the density of states [7]. Our approach here explained is valid for any symmetric matrix $\mathbf{M}$.

We can rewrite this equation with the help of the *Sokhotski-Plemelj theorem* [64], which states that

$$\lim_{\eta\to 0^+} \frac{1}{x - i\eta} = \mathscr{P}\frac{1}{x} + i\pi\delta(x), \tag{4.2}$$

where $\mathscr{P}$ is the *Cauchy principal value*. The corollary that matters for us, is the representation of the Dirac delta that is obtained from this theorem,

$$\delta(x) = \lim_{\eta\to 0^+} \frac{1}{\pi}\mathrm{Im}\left(\frac{1}{x - i\eta}\right). \tag{4.3}$$

Using this result to rewrite the empirical spectral density for $x = \lambda - \lambda_i^{(\mathbf{M})}$, we obtain

$$\rho(\lambda;\mathbf{M}) = \lim_{N\to\infty}\lim_{\eta\to 0^+} \frac{1}{N\pi}\mathrm{Im}\left(\sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i^{(\mathbf{M})} - i\eta}\right). \tag{4.4}$$

We aim to change the dependence on the eigenvalues of the matrix for a dependence on its entries. For this purpose, let us focus on the sum inside the parentheses of the previous equation

$$\begin{aligned}\sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i^{(\mathbf{M})} - i\eta} &= \sum_{i=1}^{N} \frac{\partial}{\partial\lambda}\log\left(\lambda - \lambda_i^{(\mathbf{M})} - i\eta\right) \\ &= \frac{\partial}{\partial\lambda}\log\left[\prod_{i=1}^{N}\left(\lambda - \lambda_i^{(\mathbf{M})} - i\eta\right)\right].\end{aligned} \tag{4.5}$$

Since $\mathbf{M}$ is a symmetric matrix, it can be decomposed as $\mathbf{M} = \mathbf{Q}^T\mathbf{D}\mathbf{Q}$, where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{D}$ is a diagonal matrix of eigenvalues. Hence, the previous sum takes the form

$$\sum_{i=1}^{N} \frac{1}{\lambda - \lambda_i^{(\mathbf{M})} - i\eta} = \frac{\partial}{\partial \lambda} \log\left[\det\left(\mathbb{I}(\lambda - i\eta) - \mathbf{M}\right)\right], \tag{4.6}$$

and its spectral density will be

$$\rho(\lambda;\mathbf{M}) = -\lim_{N\to\infty}\lim_{\eta\to 0^+} \frac{2}{N\pi} \operatorname{Im}\left(\frac{\partial}{\partial z}\log K(z,\mathbf{M})\right)\Big|_{z=\lambda-i\eta} \tag{4.7}$$

where

$$K(z,\mathbf{M}) = \frac{1}{\sqrt{\det(\mathbb{I}z - \mathbf{M})}}. \tag{4.8}$$

The function $K(z,\mathbf{M})$ can be written in a more suggestive form to use the language of statistical mechanics, as it is explained in Appendix A. This form is

$$\begin{aligned} K(z,\mathbf{M}) &= \frac{1}{\sqrt{(2\pi)^N}} \int_{\mathbb{R}^N} d\bar{x} \exp\left[-\frac{1}{2}\sum_{i,j} x_i(\mathbb{I}z - \mathbf{M})_{ij}x_j\right] \\ &= \frac{1}{\sqrt{(2\pi)^N}} Z_{\mathbf{M}}(z), \end{aligned} \tag{4.9}$$

where we have defined

$$Z_{\mathbf{M}}(z) = \int_{\mathbb{R}^N} d\bar{x} \exp\left[-\frac{1}{2}\sum_{i,j} x_i(\mathbb{I}z - \mathbf{M})_{ij}x_j\right]. \tag{4.10}$$

In this occasion, for the sake of consistency, the dependence of the function Z on the matrix $\mathbf{M}$ is pointed out with a sub-index.

Because the limiting empirical spectral density, given by Eq. (4.7), depends only on $K(z,\mathbf{M})$ through the derivative of its logarithm, the constant $\frac{1}{\sqrt{(2\pi)^N}}$ that appears in $K(z,\mathbf{M})$ will cancel out and, therefore, we end up writing $\rho(\lambda;\mathbf{M})$ as

$$\rho(\lambda;\mathbf{M}) = -\lim_{N\to\infty}\lim_{\eta\to 0^+} \frac{2}{N\pi} \operatorname{Im}\left(\frac{\partial}{\partial z}\log Z_{\mathbf{M}}(z)\right)\Big|_{z=\lambda-i\eta}. \tag{4.11}$$

Looking at Eq. 4.10, it is possible to interpret $Z_{\mathbf{M}}(z)$ as a partition function (even though z is a complex variable) with a Hamiltonian over unit temperature given by

$$\mathscr{H}_{\mathbf{M}}(\bar{x};z) = -\frac{1}{2}\sum_{i,j} x_i(\mathbb{I}z - \mathbf{M})_{ij}x_j. \tag{4.12}$$

Continuing with the analogy, the probability that this "system" takes a specific configuration is given by

$$P(\bar{x};z,\mathbf{M}) = \frac{1}{Z_{\mathbf{M}}(z)} e^{-\mathscr{H}_{\mathbf{M}}(\bar{x};z)}. \tag{4.13}$$

In this interpretation the continuous random variable $\bar{x}$ plays the role that the discrete random variable $\overline{\sigma}$ does in the Ising model. Indeed, the Hamiltonian given by Eq. (4.12), ignoring the imginary units, remind us the Gaussian ferromagnetic model [65]. Therefore, we can interpret the variable x_i as the local magnetization, which can take continuos values, in a system with a complex magnetic field. We will apply to this fictitious system the statistical mechanics' methods that have been reviewed in the previous chapter.

4.1 Using Cavities

Using the statistical mechanics interpretation of $Z(z)$, the spectral density given by Eq. (4.11) can be written as

$$\rho(\lambda;\mathbf{C}) = \lim_{N\to\infty}\lim_{\eta\to 0^+}\frac{1}{N\pi}\,\mathrm{Im}\left(\sum_{i=1}^{N}\left\langle x_i^2\right\rangle_z\right)\Big|_{z=\lambda-i\eta}, \tag{4.14}$$

where the expected value $\langle\cdot\rangle_z$ is with respect to $P(\bar{x};z,\mathbf{C})$ given by Eq. (4.13). Here is where the cavity method enters the scene, since the average of a random variable can be computed using its marginal probability, namely $\left\langle x_i^2\right\rangle_z = \int x_i^2 P_i(x_i;z,\mathbf{C})dx_i$.

Let us apply the method. First of all, we separate the terms in the Hamiltonian which involve the random variable x_i from those which do not, and we also separate the integral over the neighbors ∂i and the rest of the system

$$\begin{aligned} P_i(x_i;z,\mathbf{C}) &= \frac{1}{Z_{\mathbf{C}}(z)}d\bar{x}_{/i}\exp\left[\mathscr{H}_{\mathbf{C}}(\bar{x};z)\right] \\ &= \frac{\exp\left[-x_i^2 z/2\right]}{Z_{\mathbf{C}}(z)}\int\left[\prod_{j\in\partial i}dx_j\right]\exp\left[-x_i\sum_{j\in\partial i}x_j\right]\int\left[\prod_{k\notin\partial i}dx_k\right]\exp\left[-\mathscr{H}_{\mathbf{C}}^{(i)}(\bar{x};z)\right]. \end{aligned} \tag{4.15}$$

For the sake of simplicity, we will obviate the dependence of P_i on z and $\mathbf{C}$. If we notice that the second integral of the last equality is just the marginal probability (up to a normalizing constant) of the set ∂i, we can use the Bethe-Peierls approximation to obtain

$$\begin{aligned} P_i(x_i) &= \frac{\exp\left[-x_i^2 z/2\right]}{Z_i(z)}\int\left[\prod_{j\in\partial i}dx_j\right]\exp\left[-x_i\sum_{j\in\partial i}x_j\right]\prod_{j\in\partial i}P_j^{(i)}(x_j) \\ &= \frac{\exp\left[-x_i^2 z/2\right]}{Z_i(z)}\prod_{j\in\partial i}\int dx_j\exp\left[-x_i x_j\right]P_j^{(i)}(x_j), \end{aligned} \tag{4.16}$$

where the normalizing constant of the marginal of ∂i has been added to $Z_i(z)$. In a similar fashion the cavity equations can be obtained

$$P_i^{(j)}(x_i) = \frac{\exp\left[-x_i^2 z/2\right]}{Z_i^{(j)}(z)}\prod_{k\in\partial i/j}\int dx_k\exp\left[-x_i x_k\right]P_k^{(i)}(x_k), \tag{4.17}$$

4.1.1 Parameterization

In principle, $P_i^{(j)}$ could be any possible probability function, but there is a family of functions which are "fixed points" of the cavity equations. These fixed points are the normal distributions. Consequently, a suitable parametrization is

$$P_i^{(j)}(x_i) = \frac{1}{\sqrt{2\pi\Delta_i^{(j)}}}\exp\left[-\frac{x_i^2}{2\Delta_i^{(j)}}\right], \qquad P_i(x_i) = \frac{1}{\sqrt{2\pi\Delta_i}}\exp\left[-\frac{x_i^2}{2\Delta_i}\right], \tag{4.18}$$

with parameters $\{\Delta_i\}_{i=1}^N$ and $\{\Delta_i^{(j)}\}_{i=i}^N$ with $j\in\partial i$ which depend on z and $\mathbf{C}$, as we will see below. If we substitute this parametrization in our cavity equations, we obtain

$$P_i^{(j)}(x_i) = \frac{\exp\left[-x_i^2 z/2\right]}{Z_i^{(j)}(z)}\prod_{k\in\partial i/j}\frac{1}{\sqrt{2\pi\Delta_k^{(i)}}}\int dx_k\exp\left[-x_i x_k - \frac{x_k^2}{2\Delta_k^{(i)}}\right]. \tag{4.19}$$

Let us focus on the integral. This integral can be solved with a change of variable $u = \frac{x_k}{\sqrt{\Delta_k^{(i)}}}$ and with the help of the H-S transformation as follows

$$\begin{aligned}\int dx_k \exp\left[-x_i x_k - \frac{x_k^2}{2\Delta_k^{(i)}}\right] &= \sqrt{\Delta_k^{(i)}} \int du \exp\left[-\sqrt{\Delta_k^{(i)}} x_i u - \frac{u^2}{2}\right] \\ &= \sqrt{2\pi\Delta_k^{(i)}} \exp\left[\frac{x_i\sqrt{\Delta_k^{(i)}}}{2}\right].\end{aligned} \tag{4.20}$$

Substituting back in Eq. (4.19) we end up with

$$P_i^{(j)}(x_i) = \frac{1}{Z_i^{(j)}} \exp\left[-\frac{x_i^2}{2}\left(z - \sum_{k\in\partial i/j} \Delta_k^{(i)}\right)\right]. \tag{4.21}$$

Finally, comparing the left-hand side of the previous equation (taking into account the parameterization given in Eq. (4.18)) with its right hand side, we can deduce a relation between the "cavity" parameters, namely

$$\Delta_i^{(j)}(z,\mathbf{C}) = \frac{1}{z - \sum_{k\in\partial i/j} \Delta_k^{(i)}}. \tag{4.22}$$

Once we work out a solution of this system of equations, using, for example, the fixed-point iteration, the parameters $\{\Delta_i\}_{i=1}^N$ can be computed from

$$\Delta_i(z,\mathbf{C}) = \frac{1}{z - \sum_{k\in\partial i} \Delta_k^{(i)}}. \tag{4.23}$$

After the parameters $\{\Delta_{i=1}^N\}$ are known, the limiting empirical spectral can be obtained from the following expression

$$\rho(\lambda;\mathbf{C}) = \lim_{N\to\infty} \lim_{\eta\to 0^+} \frac{1}{N\pi} \operatorname{Im}\left(\sum_i^N \Delta_i(z,\mathbf{C})\right)\Big|_{z=\lambda-i\eta}. \tag{4.24}$$

The Eq. 4.24 is the final result of the cavity method. Regarding to our statistical mechanics interpretation, the parameters $\{\Delta_i(z,\mathbf{C})\}_{i=1}^N$ are nothing else than the variance of the probability distribution that describes the random variables $\{x_i\}_{i=1}^N$, the local magnetization.

4.2 Using Replicas

To use the replica method, we start from Eq. (4.11) and rewrite $\langle \log Z_{\mathbf{C}} \rangle_C$ that appears in this equation as follows

$$\langle \rho(\lambda;\mathbf{C}) \rangle_C = -\lim_{N\to\infty} \lim_{n\to 0} \lim_{\eta\to 0^+} \frac{2}{Nn\pi} \operatorname{Im}\left(\frac{\partial}{\partial z} \log \langle Z_{\mathbf{C}}^n(z) \rangle_C\right)\Big|_{z=\lambda-i\eta}, \tag{4.25}$$

The first step is to make n replicas of the partition function given by Eq. (4.10), obtaining

$$Z_{\mathbf{C}}^n(z) = \int \left(\prod_{a=1}^n d\bar{x}_a\right) \exp\left[-\frac{1}{2}\sum_{i,j;a} x_{ia}(z\delta_{ij} - C_{ij})x_{ja}\right]. \tag{4.26}$$

The average over the disorder is applied to this replicated partition function

$$\begin{aligned}\langle Z_{\mathbf{C}}^n(z)\rangle_C = \sum_{\mathbf{C}} \left(\prod_{i<j} \left\{ \frac{c}{N}\delta_{C_{ij},1} + \left(1-\frac{c}{N}\right)\delta_{C_{ij},0} \right\} \delta_{C_{ij},C_{ji}} \right) \\ \times \int \left(\prod_{a=1}^{n} d\bar{x}_a \right) \exp\left[-\frac{z}{2}\sum_{i;a}(x_{ia})^2 \right] \exp\left[\frac{1}{2}\sum_{i,j;a} x_{ia}C_{ij}x_{ja} \right],\end{aligned} \tag{4.27}$$

where, as a reminder, in our Poisson random graphs, $p = \frac{c}{N}$ is the probability of two vertices to be adjacents.

By taking into account that the adjacency matrix is symmetric, the last exponential function in the previous equation can be written more conveniently as

$$\exp\left[\frac{1}{2}\sum_{i,j;a} x_{ia}C_{ij}x_{ja} \right] = \exp\left[\sum_{i<j;a} x_{ia}C_{ij}x_{ja} \right] = \prod_{i<j} \exp\left[\sum_{a} x_{ia}C_{ij}x_{ja} \right]. \tag{4.28}$$

Plugging the previous equation in Eq. (4.26) and rearranging terms, the average over the disorder takes the form

$$\begin{aligned}\langle Z_{\mathbf{C}}^n(z)\rangle_C = \int \left(\prod_{a=1}^{n} d\bar{x}_a \right) \exp\left[-\frac{z}{2}\sum_{i;a}(x_{ia})^2 \right] \prod_{i<j} \sum_{C_{ij}\in\{1,0\}} \left(\frac{c}{N}\delta_{C_{ij},1} + \left(1-\frac{c}{N}\right)\delta_{C_{ij},0} \right) \times \\ \exp\left[\sum_{a} x_{ia}C_{ij}x_{ja} \right] \\ = \int \left(\prod_{a=1}^{n} d\bar{x}_a \right) \exp\left[-\frac{z}{2}\sum_{i;a}(x_{ia})^2 \right] \prod_{i<j} \left\{ \frac{c}{N}\left(\exp\left[\sum_{a} x_{ia}x_{ja} \right] - 1 \right) + 1 \right\}.\end{aligned} \tag{4.29}$$

As in the case with the Ising model, let us rewrite the product of the previous equation in an exponential form as follows

$$\begin{aligned}\prod_{i<j} \left\{ \frac{c}{N}\left(\exp\left[\sum_{a} x_{ia}x_{ja} \right] - 1 \right) + 1 \right\} = \exp\left[\sum_{i<j} \log\left\{ \frac{c}{N}\left(\exp\left[\sum_{a} x_i^a x_j^a \right] - 1 \right) + 1 \right\} \right] \\ = \exp\left[\frac{1}{2}\sum_{i,j} \log\left\{ \frac{c}{N}\left(\exp\left[\sum_{a} x_i^a x_j^a \right] - 1 \right) + 1 \right\} - \frac{1}{2}\sum_{i} \log\left\{ \frac{c}{N}\left(\exp\left[\sum_{a} (x_i^a)^2 \right] - 1 \right) + 1 \right\} \right] \\ = \exp\left[\frac{c}{2N}\sum_{i,j}\left(\exp\left[\sum_{a} x_i^a x_j^a \right] - 1 \right) + O(N^0) \right],\end{aligned} \tag{4.30}$$

where only the extensive term is written explicitly. Summing up, the average over the disorder of the replicated system is

$$\langle Z_{\mathbf{C}}^n(z)\rangle_C = \int \left(\prod_{a=1}^{n} d\bar{x}_a \right) \exp\left[\frac{c}{2N}\sum_{i,j}\left(\exp\left[\sum_{a} x_{ia}x_{ja} \right] - 1 \right) - \frac{z}{2}\sum_{i,a}(x_{ia})^2 \right]. \tag{4.31}$$

At this stage, following the notation of the previous chapter[1], we introduce the function $Q(\underline{\tau}, \{\underline{x}_i\}_{i=1}^N) = \frac{1}{N}\sum_{i=1}^N \delta(\underline{\tau} - \underline{x}_i)$. Using this function, the first sum inside the exponential function can be rewritten as

$$\frac{c}{2N}\sum_{i,j}\left(\exp\left[\sum_{a} x_{ia}x_{ja} \right] - 1 \right) = \frac{Nc}{2}\int d\underline{\tau}\, d\underline{\gamma}\, Q(\underline{\tau}, \{\underline{x}_i\}) Q(\underline{\gamma}, \{\underline{x}_i\}) \left(\exp\left[\underline{\gamma}\cdot\underline{\tau} \right] - 1 \right). \tag{4.32}$$

[1] See the Figure 3.3 for a quick reminder.

With this expression and introducing the Fourier transform of a Dirac delta functional in Eq. (4.31), we end up with

$$\langle Z^n_{\mathbf{C}}(z)\rangle_C = \int \left\{\frac{dPd\tilde{P}}{2\pi}\right\} \exp\left[i\int d\underline{\tau}\tilde{P}(\underline{\tau})P(\underline{\tau}) + \frac{Nc}{2}\int_{R^n} d\underline{\tau}d\underline{\gamma}P(\underline{\tau})P(\underline{\gamma})\left(\exp\left[\underline{\gamma}\cdot\underline{\tau}\right]-1\right)\right] \int\left(\prod_{a=1}^{n} d\underline{x}_a\right)\exp\left[-\frac{z}{2}\sum_{i,a}(x_{ia})^2 - i\frac{1}{N}\sum_{i=1}^{N}\tilde{P}(\underline{x}_i)\right]. \tag{4.33}$$

To simplify this equation we notice that the last integral can be re-expressed as follows (just like in Eq. (3.34))

$$\begin{aligned}\int\left(\prod_{a=1}^{n} d\underline{x}_a\right)\exp\left[-\frac{z}{2}\sum_{i,a}(x_{ia})^2 - i\frac{1}{N}\sum_{i=1}^{N}\tilde{P}(\underline{x}_i)\right] &= \int\left(\prod_{i=1}^{N} d\underline{x}_i\right)\exp\left[-\frac{z}{2}\sum_{i,a}(x_{ia})^2 - i\frac{1}{N}\sum_{i=1}^{N}\tilde{P}(\underline{x}_i)\right]\\ &= \prod_{i=1}^{N}\int d\underline{x}\exp\left[-\frac{z}{2}\sum_{a}(x_a)^2 - i\frac{1}{N}\tilde{P}(\underline{x})\right]\\ &= \left(\int d^n\underline{x}\exp\left[-\frac{z}{2}\sum_{a}(x_a)^2 - i\frac{1}{N}\tilde{P}(\underline{x})\right]\right)^N.\end{aligned} \tag{4.34}$$

This transformation, together with the functional change of variable $\tilde{P}\to N\hat{P}$, yields the following expression for the average over the disorder

$$\langle Z^n_{\mathbf{C}}(z)\rangle_C = \int D[P,\hat{P}]e^{-NS[P,\hat{P};z]}, \tag{4.35}$$

with action

$$\begin{aligned}S[P,\hat{P};z] = -i\int d\underline{\tau}\hat{P}(\underline{\tau})P(\underline{\tau}) - \frac{c}{2}\int d\underline{\tau}d\underline{\gamma}P(\underline{\tau})P(\underline{\gamma})\left(\exp\left[\underline{\gamma}\cdot\underline{\tau}\right]-1\right) +\\ -\log\left\{\int d\underline{x}\exp\left[-\frac{z}{2}\sum_{a}(x_a)^2 - i\hat{P}(\underline{x})\right]\right\}.\end{aligned} \tag{4.36}$$

Thus our original problem transforms into

$$\langle\rho(\lambda;\mathbf{C})\rangle_C = -\lim_{N\to\infty}\lim_{n\to 0}\lim_{\eta\to 0^+}\frac{2}{Nn\pi}\mathrm{Im}\left(\frac{\partial}{\partial z}\log\left\{\int D[P,\hat{P}]\exp\left[-NS[P,\hat{P};z]\right]\right\}\right)\Big|_{z=\lambda-i\eta}. \tag{4.37}$$

If we apply the derivative of the logarithm and use the saddle-point method[2] to solve the integral, the result of the derivative is

$$\frac{\partial}{\partial z}\log\left\{\int D[P,\hat{P}]\exp\left[-NS[P,\hat{P},z]\right]\right\} \asymp -N\frac{\partial S[P,\hat{P};z]}{\partial z}, \tag{4.38}$$

where the functions P and $\hat{P}$ fulfill the saddle-point equations

$$\frac{\delta S[P,\hat{P}]}{\delta P(\underline{x})} = 0, \qquad \frac{\delta S[P,\hat{P}]}{\delta\hat{P}(\underline{x})} = 0. \tag{4.39}$$

[2]We assume that the limits $N\to\infty$ and $n\to 0$ can be switched.

Hence, our problem boils down to the computation of $\frac{\partial S[P,\hat{P};z]}{\partial z}$ where P and $\hat{P}$ fulfill the previous conditions.

From the first condition we obtain

$$-i\hat{P}(\underline{x}) = c\int d\underline{\tau}P(\underline{\tau})(\exp[\underline{\tau}\cdot\underline{x}]-1)\,, \tag{4.40}$$

and from the second condition

$$P(\underline{x}) = \frac{\exp\left[-\frac{z}{2}\sum_a (x_a)^2 - i\hat{P}(\underline{x})\right]}{\int d\underline{x}\exp\left[-\frac{z}{2}\sum_a (x_a)^2 - i\hat{P}(\underline{x})\right]}\,. \tag{4.41}$$

Eliminating the function $\hat{P}$ from both saddle-point equations, we can obtain only one condition that must be fulfilled, which is

$$\begin{aligned} P(\underline{x}) &= \frac{\exp\left[-\frac{z}{2}\sum_a (x_a)^2 + c\int d\underline{\tau}P(\underline{\tau})(\exp[\underline{\tau}\cdot\underline{x}]-1)\right]}{\int d\underline{x}\exp\left[-\frac{z}{2}\sum_a (x_a)^2 + c\int d\underline{\tau}P(\underline{\tau})(\exp[\underline{\tau}\cdot\underline{x}]-1)\right]} \\ &= \frac{\exp\left[-\frac{z}{2}\sum_a (x_a)^2\right]\sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\left(\int d\underline{\tau}\exp[\underline{\tau}\cdot\underline{x}]P(\underline{\tau})\right)^l}{\int d\underline{x}\exp\left[-\frac{z}{2}\sum_a (x_a)^2 + c\int d\underline{\tau}P(\underline{\tau})(\exp[\underline{\tau}\cdot\underline{x}]-1)\right]}\,. \end{aligned} \tag{4.42}$$

Using this saddle-point equation the derivative given in Eq. (4.38) can be written as

$$\frac{\partial S[P,\hat{P};z]}{\partial z} = \frac{1}{2}\int d\underline{x}\left(\sum_a (x_a)^2\right)P(\underline{x})\,. \tag{4.43}$$

Let us propose a replica symmetric ansatz for $P(\underline{x})$ given by

$$P_{RS}(\underline{x}) = \int w(\mathbf{h})\prod_{a=1}^{n} q(x_a|\mathbf{h})d\mathbf{h}\,, \tag{4.44}$$

where, in principle, the parameter vector $\mathbf{h}$ could have an infinite number of entries (therefore the new notation), which characterize the probability distribution $q(x|\mathbf{h})$. Substituting this ansatz in Eq. (4.43), the derivative of the action with respect to z is

$$\begin{aligned} \frac{\partial S[P,\hat{P};z]}{\partial z} &= \frac{1}{2}\int d\underline{x}\left(\sum_a (x_a)^2\right)\int w(\mathbf{h})\prod_{a=1}^{n} q(x_a|\mathbf{h})d\mathbf{h} \\ &= \frac{n}{2}\int d\mathbf{h}w(\mathbf{h})\int dx q(x|\mathbf{h})x^2\,. \end{aligned} \tag{4.45}$$

Aiming to obtain an expression for $w(\mathbf{h})$, let us plug our replica symmetric ansatz in the left-hand side of Eq. (4.42) and work out the right-hand side as follows

$$P_{RS}(\underline{x}) = \exp\left[-\frac{z}{2}\sum_a (x_a)^2\right]\sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\left(\int d\underline{\tau}\exp[\underline{\tau}\cdot\underline{x}]\int w(\mathbf{h})\prod_{a=1}^{n} q(\tau_a|\mathbf{h})d\mathbf{h}\right)^l\,, \tag{4.46}$$

where we have considered that the normalizing factor in Eq. (4.42) tends to the unit as n tends to zero and so has been taken. After an algebraic manipulation the previous equation transforms into

$$P_{RS}(\underline{x}) = \sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\int\left(\prod_{k=1}^{l} d\mathbf{h}_k w(\mathbf{h}_k)\right)\prod_{a=1}^{n}\exp\left[-\frac{z}{2}(x_a)^2\right]\prod_{k=1}^{l}\int d\tau\exp[\tau x_a]\,q(\tau|\mathbf{h}_k)\,. \tag{4.47}$$

To identify $w(\mathbf{h})$, which appears in the left hand side of this equation, we introduce a Dirac delta function as follows

$$\begin{aligned} P_{RS}(\underline{x}) = \int d\mathbf{h} \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} d\mathbf{h}_k w(\mathbf{h}_k) \right) \prod_{a=1}^{n} q(x_a|\mathbf{h}) \\ \delta \left[q(x|\mathbf{h}) - \frac{\exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \int d\tau \exp[\tau x]\, q(\tau|\mathbf{h}_k)}{Z(\{\mathbf{h}_k\}_{k=1} l)} \right] Z^n \left(\{\mathbf{h}_k\}_{k=1}^{l} \right), \end{aligned} \tag{4.48}$$

where $Z\left(\{\mathbf{h}_k\}_{k=1}^{l}\right)$ is just a normalizing factor. Therefore, comparing both sides of this equation in the limit $n \to 0$, we can recognize $w(\mathbf{h})$ as

$$w(\mathbf{h}) = \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} d\mathbf{h}_k w(\mathbf{h}_k) \right) \delta \left(q(x|\mathbf{h}) - \frac{\exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \int d\tau \exp[\tau x]\, q(\tau|\mathbf{h}_k)}{Z(\{\mathbf{h}_k\}_{k=1}^{l})} \right). \tag{4.49}$$

According to this equation, under the integral operator, the function $q(x|\mathbf{h})$ must be evaluated as

$$q(x|\mathbf{h}) \to \frac{\exp\left[-\frac{z}{2}x^2\right]}{Z\left(\{\mathbf{h}_k\}_{k=1}^{l}\right)} \prod_{k=1}^{l} \int d\tau \exp[\tau x]\, q(\tau|\mathbf{h}_k). \tag{4.50}$$

It turns out that the functions which conserve their functional form after this association[3] are the normal distributions. Therefore, we propose the following parametrization[4]

$$q(x|\Delta) = \frac{1}{\sqrt{2\pi\Delta}} \exp\left[-\frac{x^2}{2\Delta}\right]. \tag{4.51}$$

Using this parameterization in Eq. (4.50) it is possible to obtain a relation between the parameters in the following way

$$\begin{aligned} q(x|\Delta) &\to \frac{1}{Z\left(\{\Delta_k\}_{k=1}^{l}\right)} \exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \int d\tau \exp[\tau x]\, q(\tau|\Delta_k) \\ &= \frac{1}{Z\left(\{\Delta_k\}_{k=1}^{l}\right)} \exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \int d\tau \exp[\tau x] \frac{1}{\sqrt{2\pi\Delta_k}} \exp\left[-\frac{\tau^2}{2\Delta_k}\right] \\ &= \frac{1}{Z\left(\{\Delta_k\}_{k=1}^{l}\right)} \exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \frac{1}{\sqrt{2\pi\Delta_k}} \int d\tau \exp\left[\tau x - \frac{\tau^2}{2\Delta_k}\right]. \end{aligned} \tag{4.52}$$

To continue, we perform the change of variable $u = \frac{\tau}{\sqrt{\Delta_k}}$ and the Hubbard-Stratonovich transformation, hence

$$\begin{aligned} q(x|\Delta) &\to \frac{1}{Z\left(\{\Delta_k\}_{k=1}^{l}\right)} \exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \frac{\sqrt{\Delta_k}}{\sqrt{2\pi\Delta_k}} \int du \exp\left[ux\sqrt{\Delta_k} - \frac{u^2}{2}\right] \\ &= \frac{1}{Z\left(\{\Delta_k\}_{k=1}^{l}\right)} \exp\left[-\frac{z}{2}x^2\right] \prod_{k=1}^{l} \exp\left[\frac{x^2\Delta_k}{2}\right] \\ &= \frac{1}{Z\left(\{\Delta_k\}_{k=1}^{l}\right)} \exp\left[\frac{x^2}{2}\left(-z + \sum_{k=1}^{l} \Delta_k\right)\right]. \end{aligned} \tag{4.53}$$

[3] Compare Eq. (4.50) with Eq. (4.17).

[4] This parameterization is not general, it works for our case at hand [18].

Comparing the left-hand side of Eq. (4.53) (taking into account our parametrization) with its left-hand side, the relation between the parameters can be obtained, namely

$$\Delta(z) \to \frac{1}{z - \sum_{k=1}^{l} \Delta_k}. \tag{4.54}$$

To bear in mind the dependence on z maintaining a readable notation, let us denote $\Delta_z \equiv \Delta(z)$. With the help of the previous relation, Eq. (4.48) transforms into

$$P_{RS}(\underline{x}) = \int d\Delta_z \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} d\Delta_k w(\Delta_k) \right) \prod_{a=1}^{n} q(x_a|\Delta_z) \delta \left(\Delta_z - \frac{1}{z - \sum_{k=1}^{l} \Delta_k} \right) Z^n \left(\{\Delta_k\}_{k=1}^{l} \right). \tag{4.55}$$

In this way, it is straightforward to identify $w(\Delta_z)$ if we compare the left hand side and the right hand side of Eq. (4.55), which in the limit $n \to 0$ yields

$$w(\Delta_z) = \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{l} d\Delta_k w(\Delta_k) \right) \delta \left(\Delta_z - \frac{1}{z - \sum_{k=1}^{l} \Delta_k} \right). \tag{4.56}$$

With this relation and our parametrization of $q(x|\Delta_z)$, we can finally complete the computation of $\frac{\partial S[P,\tilde{P};z]}{\partial z}$. From Eq. (4.53)

$$\begin{aligned} \frac{\partial S[P,\hat{P};z]}{\partial z} &= \frac{n}{2} \int d\Delta_z w(\Delta_z) \int dx q(x|\Delta_z) x^2 \\ &= \frac{n}{2} \int d\Delta_z w(\Delta_z) \int dx x^2 \frac{1}{\sqrt{2\pi\Delta_z}} \exp\left[-\frac{x^2}{2\Delta_z} \right] \\ &= \frac{n}{2} \int d\Delta_z w(\Delta_z) \Delta_z . \end{aligned} \tag{4.57}$$

Finally, with the previous equation and Eqs. (4.37) and (4.38), the limiting spectral density turns out to be

$$\langle \rho(\lambda; \mathbf{C}) \rangle = \lim_{\eta \to 0^+} \frac{1}{\pi} \operatorname{Im} \left(\int d\Delta_z w(\Delta_z) \Delta_z \right) \Big|_{z=\lambda - i\eta}, \tag{4.58}$$

which, together with Eq. (4.56), is our final result. The population dynamics algorithm (cf. Appendix C) can help us to obtain a numerical result from these pair of equations. Again, in our statistical mechanics interpretation, the parameter Δ_z is the variance of the random variable x, a local magnetization (see Eqs. (4.44) and (4.51)). The variace of the magnetization[5] is relevant in the study of magnetic systems because it is proportional to the magnetic susceptibility, $\chi = \beta \left(\langle M^2 \rangle - \langle M \rangle^2 \right)$ [52], a quatity that can be measured in the lab.

4.3 Numerical Results

In the previous sections the cavity and replica methods were used to obtain the limiting spectral density of Poisson random graphs. In this section we compare their results with numerical diagonalization.

To obtain numerical results the limits $\eta \to 0^+$ and $N \to \infty$ in Eqs. (4.58) and (4.24) were taken numerically (see the caption of the figures). We compare our theoretical results with numerical diagonalization over an ensamble of 10000 adjacency matrices of Poisson random graphs

[5]The magnetization was defined in Eq. (2.43).

$\mathbf{C}_{1000\times1000}$ with parameters $c \in \{10, 15, 20\}$. To create these adjacency matrices, we draw for each entry a random number from a uniform distribution with minimum 0 and maximum 1. Each entry takes the value 1 if its corresponding random number is less than or equal to $p = \frac{c}{N}$, and takes the value 0 otherwise. In Figures 4.1, 4.2 and 4.3 our comparisons are shown; we see a very good agreement between the theory and numerical diagonalization.

The small diferences between theory and numerical diagonalization may be due to the election of the numerical limits (N and η), as well as the stopping criteria selected during our numerical computations.

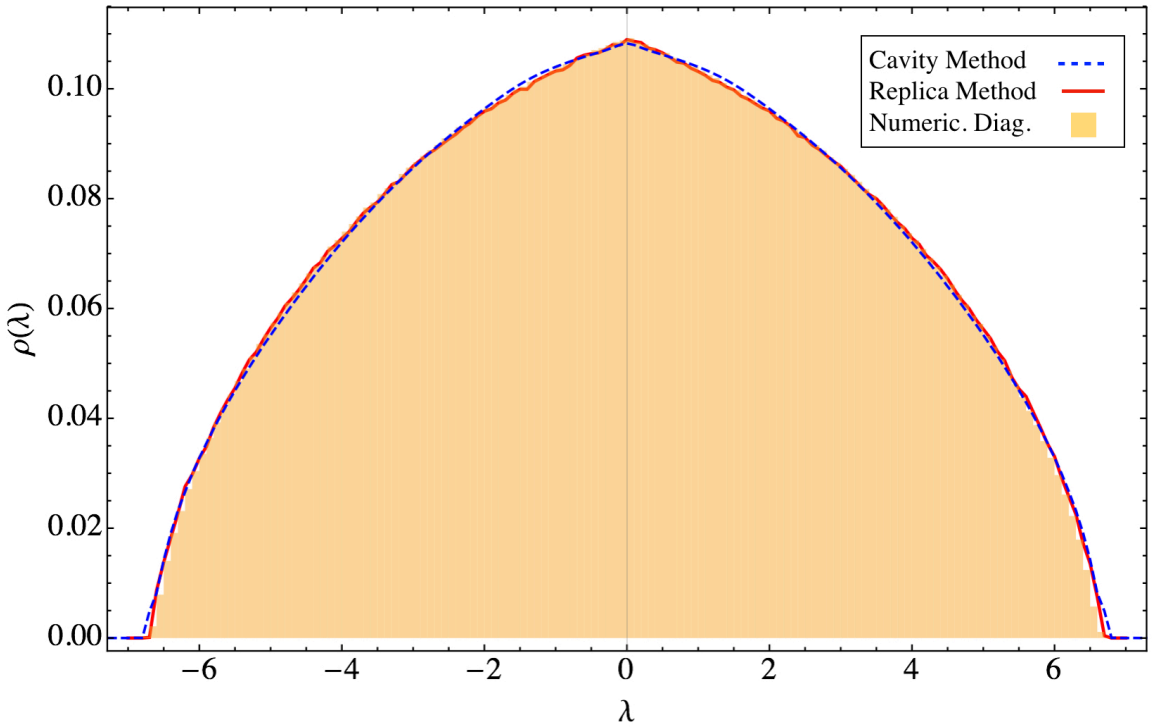


Figure 4.1: Spectral density of a Poisson random graphs with average degree $c = 10$. Comparison between numerical diagonalization and the cavity and replica methods with numerical limits $\eta = 1 \times 10^{-8}$ and $N = 1000$.

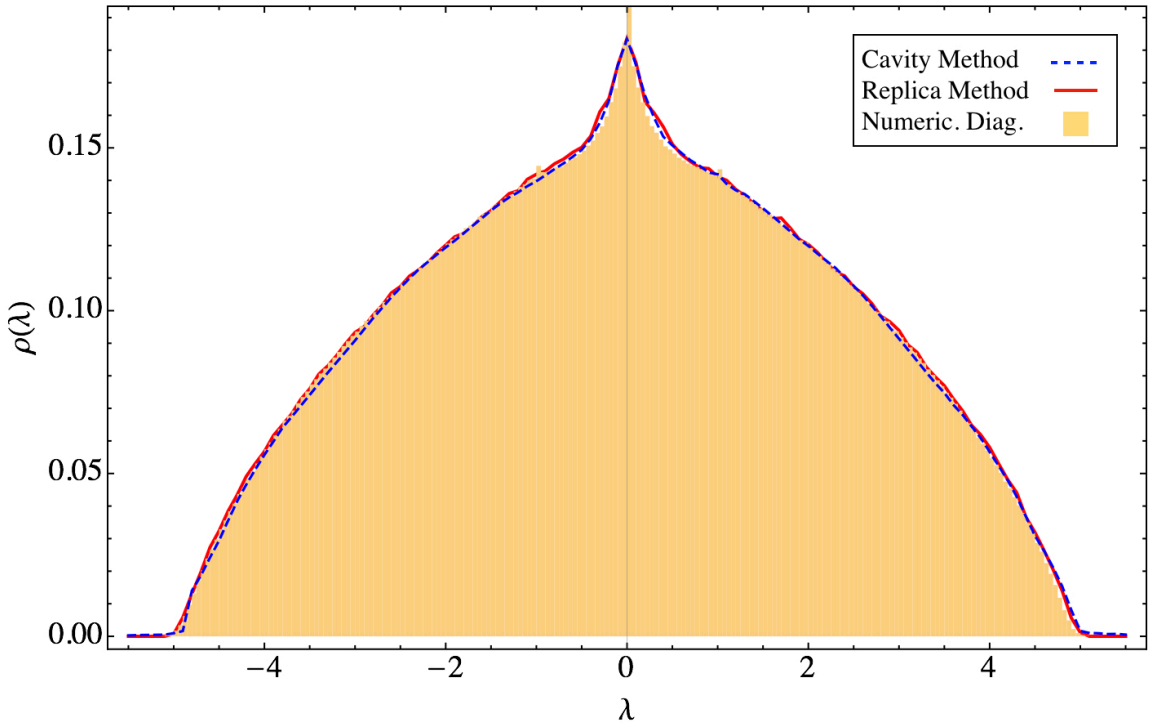


Figure 4.2: Spectral density of a Poisson random graphs with average degree $c = 5$. Comparison between numerical diagonalization and the cavity and replica methods with numerical limits $\eta = 1 \times 10^{-8}$ and $N = 1000$.

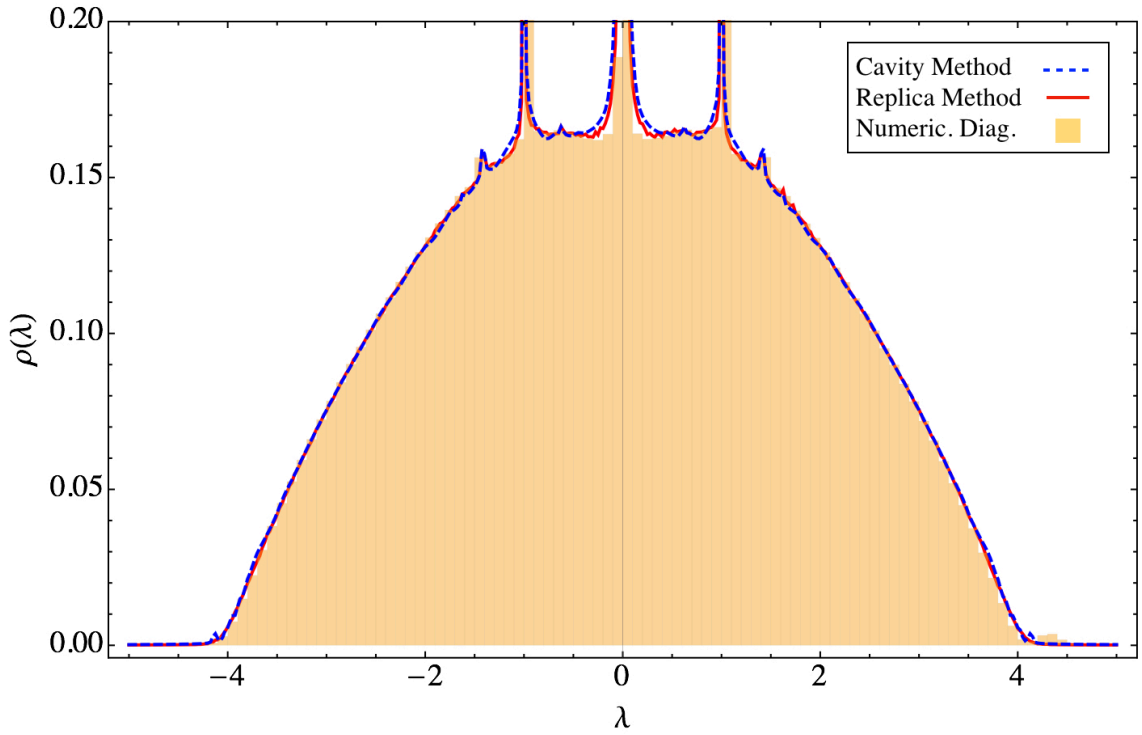


Figure 4.3: Spectral density of a Poisson random graphs with average degree $c = 3$. Comparison between numerical diagonalization and the cavity and replica methods with numerical limits $\eta = 1 \times 10^{-3}$ and $N = 10000$.

Chapter 5

Conclusions

In this bachelor's thesis the cavity and the replica methods have been presented. With them, the magnetization of an Ising model attached to an Poisson random graph was computed with excellent agreement with Monte Carlo simulations.

Along their introduction, some of their limitations and subtleties were highlighted. For example, the factorization of the marginal probability in the Bethe-Peierls approximation strongly depends on the locally tree-like property of the graphs. On the other hand, during the mathematical manipulations of the replica method, complications which are difficult to overcome could arise. Besides, the numerical methods commonly used in these computations were explicitly explained in the appendices.

The main goal of this work has been achieved. We have obtained analytical expressions for the spectral density of Poisson random graphs using the cavity and replica methods. This expressions can help us to study the behavior of the spectral density, particularly when the size of the system tend to infinity. Besides, since we have obtained analytical expressions, these can serve to further mathematical treatment. The very good agreement between our numerical results and numerical diagonalization proves the validity of the theory.

A further work could be a similar treatment but with constricted random graphs, which turn out to present many interesting and exotic properties. Indeed, an onging work in which the constriction is related with topological properties of the graphs has proved to be a true challenge.

Appendix A

Some Derivations

In this appendix we present explicitly some mathematical derivations that were omitted in the main text.

Derivation of Eq. (3.14)

If we plug the parametrization given by Eq. (3.11) into Eq. (3.13), then

$$\begin{aligned} h_i^{(j)} &= \theta_i + \frac{1}{2\beta} \sum_{k\in\partial i/j} \sum_{\sigma_i=\pm 1} \sigma_i \log \left\{ \sum_{\sigma_k=\pm 1} \exp\left[\beta\left(\sigma_i J_{ik} + h_k^{(i)}\right)\sigma_k\right] \right\} \\ &= \theta_i + \frac{1}{2\beta} \sum_{k\in\partial i/j} \log \left\{ \frac{\sum_{\sigma_k=\pm 1} \exp\left[\beta\left(h_k^{(i)} + J_{ki}\right)\sigma_k\right]}{\sum_{\sigma_k=\pm 1} \exp\left[\beta\left(h_k^{(i)} - J_{ki}\right)\sigma_k\right]} \right\}. \end{aligned} \tag{A.1}$$

Performing the sum over σ_k, we obtain

$$h_i^{(j)} = \theta_i + \frac{1}{2\beta} \sum_{k\in\partial i/j} \log \left\{ \frac{\cosh\left[\beta h_k^{(i)} + \beta J_i^{(k)}\right]}{\cosh\left[\beta h_k^{(i)} - \beta J_i^{(k)}\right]} \right\}, \tag{A.2}$$

and after expanding the hyperbolic cosine of a sum, we end up with

$$\begin{aligned} h_i^{(j)} &= \theta_i + \frac{1}{2\beta} \sum_{k\in\partial i/j} \log \left\{ \frac{\cosh\left[\beta h_k^{(i)}\right]\cosh[\beta J_{ki}] + \sinh\left[\beta h_k^{(i)}\right]\sinh[\beta J_{ki}]}{\cosh\left[\beta h_k^{(i)}\right]\cosh[\beta J_{ki}] - \sinh\left[\beta h_k^{(i)}\right]\sinh[\beta J_{ki}]} \right\} \\ &= \theta_i + \frac{1}{2\beta} \sum_{k\in\partial i/j} \log \left\{ \frac{1 + \tanh\left[\beta h_k^{(i)}\right]\tanh[\beta J_{ki}]}{1 - \tanh\left[\beta h_k^{(i)}\right]\tanh[\beta J_{ki}]} \right\}. \end{aligned} \tag{A.3}$$

Finally, we can make use of the following relation

$$\operatorname{atanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right), \tag{A.4}$$

and we obtain Eq. (3.14), namely

$$h_i^{(j)} = \theta_i + \frac{1}{\beta} \sum_{k\in\partial i/j} \operatorname{atanh}\left\{\tanh\left[\beta h_k^{(i)}\right]\tanh\left[\beta J_{ki}\right]\right\}.$$

Derivation of Eq. (3.50)

The derivation of Eq. (3.50) is somewhat similar to the Eq. (3.14). If we plug Eq. (3.49) into Eq. (3.48), we get

$$h \to \frac{1}{2\beta} \sum_{\sigma} \sigma \log \left\{ \frac{\prod_{k=1}^{l} \sum_{\tau} \exp\left[\beta J \sigma_a \tau\right] q(\tau|h_k)}{Z(\{h_k\}_{k=1}^{l})} \right\}. \tag{A.5}$$

Because the normalizing factor does not depend on σ, it will be added and subtracted when we perform the sum $\sum_{\sigma}$. Therefore it is omitted in the following

$$h \to \frac{1}{2\beta} \sum_{\sigma} \sigma \log \left\{ \prod_{k=1}^{l} \sum_{\tau} \exp\left[\beta J \sigma \tau\right] q(\tau|h_k) \right\}. \tag{A.6}$$

Substituting the parameterization given by Eq. (3.47) in the previous relation yields

$$\begin{aligned} h &\to \frac{1}{2\beta} \sum_{k=1}^{l} \sum_{\sigma} \sigma \log \left\{ 2\cosh\left[\beta J \sigma + \beta h_k\right] \right\} \\ &= \frac{1}{2\beta} \sum_{k=1}^{l} \log \left\{ \frac{\cosh[\beta J + \beta h_k]}{\cosh[\beta J - \beta h_k]} \right\} \\ &= \frac{1}{2\beta} \sum_{k=1}^{l} \log \left\{ \frac{1 + \tanh[\beta J] \tanh[\beta h_k]}{1 - \tanh[\beta J] \tanh[\beta h_k]} \right\}, \end{aligned} \tag{A.7}$$

and thus we recover the Eq. (3.50) of the main text, namely

$$h \to \frac{1}{\beta} \sum_{k=1}^{l} \operatorname{atanh}[\tanh[\beta J] \tanh[\beta h_k]].$$

Derivation of Eq. (4.9)

For this derivation we start from the integral

$$I = \int_{\mathbb{R}^N} d\bar{x} \exp\left[-\frac{1}{2} \sum_{i,j} x_i A_{ij} x_j \right] = \int_{\mathbb{R}^N} d\bar{x} \exp\left[-\frac{1}{2} \bar{x}^T \mathbf{A} \bar{x} \right], \tag{A.8}$$

where $\bar{x}$ is a row vector and $\mathbf{A}$ is a symmetric, positive-definite $N \times N$ matrix. Performing the change of variable $\bar{x} \to \mathbf{Q}\bar{y}$, where $\mathbf{Q}$ is an orthogonal matrix whose determinant is 1, the integral I takes the form

$$I = \int_{\mathbb{R}^N} d\bar{y} \exp\left[-\frac{1}{2} \bar{y}^T \mathbf{Q^{-1}} \mathbf{A} \mathbf{Q} \bar{y} \right]. \tag{A.9}$$

We will demand that the matrix $\mathbf{Q}$ is a diagonalizing matrix, this means

$$\mathbf{Q^{-1}} \mathbf{A} \mathbf{Q} = \operatorname{diag}(\lambda_1^{(\mathbf{A})}, \lambda_2^{(\mathbf{A})}, ..., \lambda_N^{(\mathbf{A})}), \tag{A.10}$$

where $\{\lambda_i^{(\mathbf{A})}\}_{i=1}^{N}$ are the eigenvalues of the matrix $\mathbf{A}$. This leads to

$$I = \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} (\lambda_1^M y_1^2 + \lambda_2^M y_2^2 + ... + \lambda_N^M y_N^2) \right] dy_1 dy_2 ... dy_N. \tag{A.11}$$

Having separated the variables, we can perform N times a Gaussian integral, obtaining

$$I = \int_{\mathbb{R}^N} d\bar{x} \exp\left[-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j\right] = \sqrt{\frac{2\pi}{\lambda_1^{(\mathbf{A})}}}\sqrt{\frac{2\pi}{\lambda_2^{(\mathbf{A})}}}\cdots\sqrt{\frac{2\pi}{\lambda_N^{(\mathbf{A})}}} = \sqrt{\frac{(2\pi)^N}{\det \mathbf{A}}}\,. \tag{A.12}$$

Hence, if $\mathbf{A} = \mathbb{I}z - \mathbf{M}$ where $\mathbf{M}$ is a symetric matrix and z a complex number, then we recover Eq. (4.9), namely

$$K(z,\mathbf{M}) = \frac{1}{\sqrt{(2\pi)^N}} \int_{\mathbb{R}^N} d\bar{x} \exp\left[-\frac{1}{2}\sum_{i,j} x_i (\mathbb{I}z - \mathbf{M})_{ij} x_j\right]\,.$$

Appendix B

Monte Carlo Methods

In this appendix we succinctly review Monte Carlo methods (MCm). This section is mainly based on [66].

MCm are a set of techniques whose principal characteristic is the use of random number as "raw material". The first time that the term *Monte Carlo* appears in reference to computational methods, was in the seminal paper of Metropolis and Ulam in 1949 [67]. However, given its nature, some authors prefer the term *stochastic simulations* [68] instead.

When we deal with MCm we usually want to calculate the expected value of a property, say $A(\mathbf{x})$, which depends on the state of the system $\mathbf{x} \sim \frac{1}{\sum_{\mathbf{x}} f(\mathbf{x})} f(\mathbf{x})$. This state could be, for example, the direction of the spins in the Ising Model or a random matrix from a given ensemble, depending what our system is. This expected value, as we know, is

$$\langle A(\mathbf{x}) \rangle = \frac{1}{\sum_{\mathbf{x}} f(\mathbf{x})} \sum_{\mathbf{x}} f(\mathbf{x}) A(\mathbf{x}) \,. \tag{B.1}$$

In many cases, the exact computation of (B.1) is impossible to perform analytically or simply impracticable to perform numerically. Fortunately, MCm may help us to estimate it.

B.1 Simple Sampling

Instead of performing the sum in Eq. (B.1) over all the possible states of the system, a first attempt could be to choose randomly (with equal probability) only n of them $\{\mathbf{x}_i\}_{i=1}^{n}$. Thus a first estimation of the expected value is

$$\langle A(\mathbf{x}) \rangle \approx \frac{\sum_{l=1}^{n} f(\mathbf{x}_l) A(\mathbf{x}_l)}{\sum_{l=1}^{n} f(\mathbf{x}_l)} \,. \tag{B.2}$$

This first approximation is known as *simple sampling*. Unfortunately, in most cases this estimation is very poor. To illustrate this, one can imagine an Ising Model at very low temperature. In this case, the system will spend most of the time in very few states, which will represent the main contribution to the expected value in which we are interested. Nevertheless, it would be very unlikely that one of the randomly chosen $\{\mathbf{x}_i\}_{i=1}^{n}$ coincides with one of these "important" states. One possible improvement of this method is the so-called *importance sampling*.

B.2 Importance Sampling

In the importance sampling method, we draw n states from the probability function that describes the states of the system, which has been denoted by $\frac{1}{\sum_{\mathbf{x}} f(\mathbf{x})} f(\mathbf{x})$. If we denote this sample by

$\left\{x_i^{(f)}\right\}_{i=1}^{n}$, then we can approximate our expected value by

$$\langle A(\mathbf{x})\rangle \approx \frac{1}{n}\sum_{l=1}^{n} A(\mathbf{x}_l^{(f)})\,. \tag{B.3}$$

If we apply this method to the previous example of the Ising Model at low temperature, we would draw the states $\left\{\mathbf{x}_i^{(f)}\right\}_{i=1}^{n}$ from the Boltzmann distribution, which favors those states in which the system spends most of the time, thus improving our estimation.

For some random matrices, like the adjacency matrix of Poisson random graphs, it is straightforward to take a sample from its JPDF. However, this is not the case of the thermal probability due to the normalizing factor, which in most cases is not feasible to compute. The standard solution in statistical physics is the use of a *Markov Chain* to obtain the sample.

B.2.1 Markov Chains

A Markov chain is an example of a *stochastic process*. The following definitions were taken from [69].

Definition B.2.1. *A **stochastic process** is a collection of random variables $\{X_t|t\in T\}$ parameterized by the set T, known as the index set. Each random variable in the collection takes values from a set $\mathbb{S}$, called state space.*

The index set is usually interpreted as the time. With this interpretation we can talk about past and future states of the system.

Definition B.2.2. *A **Markov chain** is a stochastic process $\{X_t|t\in\mathbb{N}\}$ such that the probability distribution of the future state X_{t+1} depends only on the current state X_t, i.e.*

$$P(X_{t+1}=x_{t+1}|X_0=x_0,...,X_t=x_t)=P(X_{t+1}=x_{t+1}|X_t=x_t)\,. \tag{B.4}$$

Definition B.2.3. *Let $\mathbf{y}$ and $\mathbf{z}$ be two states of a Markov chain. The **transition probability** from the state $\mathbf{y}$ to the state $\mathbf{z}$ is $P(X_{t+1}=\mathbf{z}|X_t=\mathbf{y})$ and will be denoted $P(\mathbf{y}\to\mathbf{z})$.*

Naturally, this probability fulfills

$$P(\mathbf{y}\to\mathbf{z})\leq 1\,,\qquad \sum_{\mathbf{z}} P(\mathbf{y}\to\mathbf{z})=1\,. \tag{B.5}$$

Additionally, our Markov chain must satisfy two conditions in order to be a sample of our target probability. These are *ergodicity* and the *detailed balance* condition. We will mention what these conditions are and for further reading the reader can look at [66, 70, 71].

Ergodicity means that if the system begins in one state, it can reach any other state in a finite number of steps. Thus this condition assures that after a sufficiently long time, the process will visit all the set of states the system can occupy.

The detailed balance condition is

$$f(\mathbf{y})P(\mathbf{y}\to\mathbf{z})=f(\mathbf{z})P(\mathbf{z}\to\mathbf{y})\,. \tag{B.6}$$

In words, the system should on average go from $\mathbf{y}$ to $\mathbf{z}$ as many times as it goes from $\mathbf{z}$ to $\mathbf{y}$. Physically, the detailed balance can be interpreted as the condition of equilibrium for our system.

Therefore, if we can get a set of transition probabilities which fulfill Eqs. (B.5) and (B.6), it only remains to contrive an ergodic algorithm to create a new state $\mathbf{z}$ from the current state $\mathbf{y}$ with exactly these probabilities. To facilitate this task, we can make use of the so called *acceptance ratios*.

B.2.2 Acceptance ratios

The idea is to propose a new state to add it to our Markov chain but it will be added only with a certain probability. Let us split the transition probabilities into two parts

$$P(\mathbf{y} \to \mathbf{z}) = S(\mathbf{y} \to \mathbf{z})A(\mathbf{y} \to \mathbf{z}), \tag{B.7}$$

where $S(\mathbf{y} \to \mathbf{z})$ is the *selection probability*, which is the probability that our algorithm propose the state $\mathbf{z}$ if our system is in the state $\mathbf{y}$, and $A(\mathbf{y} \to \mathbf{z})$ is the *acceptance probability* which is the fraction of times that this proposed state will be accepted and added to our Markov chain. If the state $\mathbf{z}$ is rejected, we simply add the current state $\mathbf{y}$ to our Markov chain. After all, $P(\mathbf{y} \to \mathbf{y})$ always fulfills the detailed balance condition.

With this separation, it is virtually possible to obtain a sample of our target probability with any ergodic algorithm which proposes a new state from the current one. We only need to take care that the acceptance probabilities are such that the detailed balance condition is fulfilled. In the following section we review the well-known *Metropolis algorithm*.

B.2.3 Metropolis Algorithm

In the Metropolis algorithm [72] we choose $S(\mathbf{y} \to \mathbf{z})$ to be equal for all the states[1]. Thus the detailed balance condition imposes that

$$\frac{P(\mathbf{y} \to \mathbf{z})}{P(\mathbf{z} \to \mathbf{y})} = \frac{A(\mathbf{y} \to \mathbf{z})}{A(\mathbf{z} \to \mathbf{y})} = \frac{f(\mathbf{z})}{f(\mathbf{y})} \tag{B.8}$$

Hence, only the quotient of the acceptance probabilities matters in the detailed balance condition, so we have certain freedom to choose the exact form of these probabilities. In order to explore efficiently the set of states, we want to maximize these probabilities. In the Metropolis algorithm this is done by setting

$$A(\mathbf{y} \to \mathbf{z}) = \min\left\{1, \frac{f(\mathbf{z})}{f(\mathbf{y})}\right\}. \tag{B.9}$$

This is known as the *Metropolis probability*. Summing up, the scheme of the Metropolis algorithm is the following:

Metropolis algorithm scheme
If we have want to sample the probability $P(\mathbf{x})$ with sample space Ω, then

1. Select a initial state $\mathbf{y} \in \Omega$.
2. Propose a state $\mathbf{z} \in \Omega$ through a Markov chain with symmetric selection probability.
3. Accept the change with acceptance probability

$$A(y \to z) = \min\left\{1, \frac{P(z)}{P(y)}\right\},$$

   if the state is rejected, add to the chain the state y.
4. Return to the second step and repeat (until we have a sufficiently long chain).

[1] This condition can be weaker; we could merely demand a *symmetric selection probability*, i.e. $S(\mathbf{y} \to \mathbf{z}) = S(\mathbf{z} \to \mathbf{y})$

Appendix C

Population Dynamics Algorithm

In this appendix the population dynamics algorithm, which has helped us to obtain numerical result after applying the replica method, is described.

The idea of the algorithm is to approximate a distribution $w(h)$ through a sample $\{h_i\}_{i=1}^{N}$, which is known as the *population*. In the present work, the distribution has taken the following form[1]

$$w(h) = \sum_{l=0}^{\infty} \frac{c^l e^{-c}}{l!} \int \left(\prod_{k=1}^{\infty} dh_k w(h_k) \right) \delta(h - f(\{h_k\}_{k=1}^{l})) \,. \tag{C.1}$$

We will limit ourselves to explain how the previous equation can be solved numerically with the population dynamics algorithm. A deeper analysis can be checked out in [61].

Population dynamics algorithm scheme
If we have want to sample the probability $w(h)$ given in Eq. (C.1), then:

1. Select an initial population $\{h_i\}_{i=1}^{N}$.
2. Draw an integer r with distribution $\frac{c^l e^{-c}}{l!}$.
3. Choose r h's uniformly from $\{h_i\}_{i=1}^{N}$, this new set will be denoted by $\{h_j\}_{j=1}^{r}$.
4. Select uniformly one index α from 1 to N and replace the element h_α from the initial population with the value

$$h_\alpha \to f(\{h_j\}_{j=1}^{r}) \,.$$

5. Return to the second step and repeat (until convergence).

[1] See Eqs. (3.52) and (4.56).

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