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GEOMETRIC AND PHYSICAL PROPERTIES OF CLOSED EVER EXPANDING DUST MODELS

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Para mi madre, Lisset y el resto de mi familia.

Abstract

Current observations suggest that our Universe is not incompatible with a small positive spatial curvature that can be associated with rest frames having a "closed" standard topology. We examine a toy model generalisation of the ΛCDM model in the form of ever expanding Lemaître-Tolman-Bondi (LTB) models with positive spatial curvature. It is well known that such models with $\Lambda = 0$ exhibit a thin layer distribution at the turning values of the area distance that must be studied through the Israel-Lanczos formalism. We find that this distributional source exhibits an unphysical behaviour for large cosmic times and its presence can be detected observationally. However, these unphysical features can always be avoided by assuming $\Lambda > 0$. While these LTB models are very simplified, we believe that these results provide a simple argument favouring the assumption of a nonzero positive cosmological constant in cosmological models.

Resumen

Las observaciones actuales sugieren que nuestro Universo no es incompatible con una pequeña curvatura espacial positiva que puede ser asociada con marcos de reposo que tengan topología estándar cerrada. Examinamos una generalización al modelo de juguete ΛCDM en la forma de modelos Lemaître-Tolman-Bondi (LTB) en expansión perpetua con curvatura espacial positiva. Es sabido que dichos modelos para $\Lambda = 0$ presentan una distribución de capa delgada en los valores críticos de la distancia de área que debe ser estudiada mediante el formalismo de Israel-Lanczos. Encontramos que esta fuente distribucional muestra un comportamiento no físico para valores grandes del tiempo cósmico y su presencia puede ser detectada observacionalmente. Sin embargo, estas características no físicas siempre pueden evitarse asumiendo $\Lambda > 0$. Aún cuando estos modelos LTB son simplificaciones, creemos que estos resultados proveen un argumento simple que favorece la suposición de una constante cosmológica positiva en modelos cosmológicos.

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Introduction

The spherically symmetric exact solutions of Einstein's equations known as the Lemaître-Tolman-Bondi (LTB) dust models are useful *toy models* to study observational issues and structure formation in a Friedman Lemaître Robertson Walker (FLRW) background. If we assume $\Lambda > 0$ these models provide a simple inhomogeneous generalisation of the ΛCDM model favoured by current observations. In fact, models with $\Lambda = 0$ and $\Lambda > 0$ provide simple descriptions of a single CDM structure (overdensity or density void) in an FLRW background. The evolution of such structures can always be mapped rigorously to the formalism of gauge invariant cosmological perturbations (see comprehensive discussion in [1, 2]). As shown in [1, 2] (see also [3, 4]), LTB inhomogeneities can be described as covariant exact fluctuations that in their linear regime reduce to linear cosmological perturbations in the isochronous comoving gauge.

Models with $\Lambda = 0$ and $\Lambda > 0$ provide also provide simple relativistic generalisations of the Newtonian *spherical collapse model*, which provide order of magnitude estimations of collapsing times and density contrasts that are useful in the design of numerical N–body simulations. See discussion and examples in [5, 6, 7].

Ever expanding FLRW models with a closed topology (rest frames difeomorphic to the 3–sphere $\mathbf{S^3}$) and a dust source are not possible unless we assume that $\Lambda > 0$. If $\Lambda = 0$ then all closed FLRW dust models must have positive spatial curvature and must bounce and re–collapse. However, for LTB models the extra degrees of freedom decouple kinematic evolution and the topology of the rest frames, allowing (in principle) for ever expanding closed models even if $\Lambda = 0$. In the 1980's when a nonzero cosmological constant was not favoured, Bonnor [8] showed interest in looking at ever expanding LTB models with $\Lambda = 0$ and a closed topology. He showed that these models exhibit a thin layer surface matter distribution at a timelike hypersurface marked by the *turning value* of the area radius (the "equator" of $\mathbf{S^3}$). Using the Israel-Lanczos formalism, Bonnor derived the equation of state for this surface layer matter–energy distribution, regarding it in a pointblank manner as unphysical because it involved negative surface pressure (these were the times before dark energy). Hence, Bonnor concluded that full regularity of closed LTB models with $\Lambda = 0$ required re-collapse and thus excluded ever expanding kinematics. More recent research allows for the interpretation of the negative surface layer pressure as surface tension [9].

In the present article we extend Bonnor's work by (i) showing that fully regular closed and ever expanding LTB models are possible once we consider $\Lambda > 0$ and (ii) by looking for the case $\Lambda = 0$ at the time evolution of the distributional surface source in comparison with the evolution of the continuous density. We show for models with zero and negative spatial curvature that the behaviour of this source is unphysical, since for large times the continuous dust density surface density decays at a much faster rate than the distributional surface density (which has no contribution to the quasilocal mass integral). In particular, we show that the presence of such distributional source would be detectable by observations through the redshift from sources connected by radial null geodesics that cross the equatorial hypersurface of $\mathbf{S^3}$. While the redshift as a function of comoving radius is continuous, its derivative is not, with the abrupt change of rate occurring precisely at this hypersurface. We show that this effect does not occur for re-collapsing LTB models with closed topology (for which there is no distributional source at the equator of $\mathbf{S^3}$).

Since observations do not rule out a Universe whose rest frames have a closed $\mathbf{S^3}$ topology associated with a very small positive spatial curvature, then a LTB model with $\Lambda > 0$ is a viable toy model approximation to a ΛCDM model that is favoured by observations. Hence, we argue that the results of the present article provide another argument to support the need for a positive cosmological constant, since without the latter all ever expanding CDM dominated models would be incompatible with a closed $\mathbf{S^3}$ topology.

The section by section description of this thesis is as follows. In chapter 1 we present the Israel-Lanczos formalism for thin shells and a brief survey of the differential geometry concepts needed in the construction. Chapter 2 provides an introduction to generic LTB models presenting the relevant equations needed in the analysis, section 2.2 focuses to generic LTB models with $\Lambda = 0$, while in section 2.3 we examine the specific case of closed models. In section 2.3.1 we review Bonnor's work and in section 2.4 we apply the Israel-Lanczos formalism to closed LTB models. In chapter 3 sections 3.1 and 3.2 provide an example of ever expanding closed models with zero and negative spatial curvature, respectively. The surface layer density is evaluated for these models, showing that in the large time regime the continuous density decays much faster than the surface density, which is an unphysical behaviour. We show in section 3.3 that fully regular ever expanding closed models with $\Lambda > 0$ are always possible. In sections 3.4 and 3.5 we compute null radial geodesics for the spatially flat case in order to examine the observational detection of the thin shell distribution. Finally, in section 3.6, we show that no observational effects occur in the the case of re–collapsing models with positive spatial curvature and $\Lambda = 0$, for which no thin shell distributional source arise.

Chapter

1

Israel-Lanczos formalism for thin shells and related subjects

General Relativity (GR) is nowadays the most successful physical theory to describe the universe in a large scale. GR assumes that space-time is a Lorentzian manifold of four dimensions equipped with a metric tensor g_{ab} which is a solution to Einstein's field equations (EFE's)

$$G_{ab} = 8\pi T_{ab} - \Lambda g_{ab}, \tag{1.1}$$

where T_{ab} is the energy-momentum tensor and G_{ab} is the Einstein tensor. The Einstein tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab},$$

with R_{ab} the Ricci tensor and R the Ricci scalar, both related to the intrinsic curvature of the manifold.

The energy-momentum tensor T_{ab} is a tensor that depends on the fields of matter, their covariant derivatives and the metric. It satisfies the following properties c.f. [10]

1. T_{ab} is vanishes on an open set U if and only if all matter fields vanish on U,

2. T_{ab} obeys the conservation equation

$$T^{ab}{}_{;b} = 0, \tag{1.2}$$

the covariant derivative of the right hand side of the Einstein field equations is also zero in virtue of Bianchi's identities. Local causality has to be introduced for the matter fields included in the energy-momentum tensor. The postulate of local causality is that [10] "the equations governing the matter fields must be such that if U is a convex normal neighborhood and p and q are points in U then a signal can be sent in U between p and q if and only if p and q can be joined by a C^1 curve lying entirely in U" and the curve is non-spacelike. This postulate can be also written in terms of the Cauchy problem for the matter fields.

1.1 Israel-Lanczos formalism

In the usual formulation of GR from the EFE's and the conservation equation it is usually assumed that the metric is at least $C^2(\mathscr{M})$. Therefore one of the postulates in this formulation is that the manifold admits a C^3 atlas in which g_{ab} is at least twice continuously differentiable. Nevertheless if one has a C^r atlas ($r \geq 1$) it is a differential topology result that an analytic subatlas can be found [11] so there is no loss of generality in our study to assume from the start that the atlas used in the GR standard formulation is analytic. This imposes certain restrictions to the space-times that can be analyzed under thess hypotheses as many space-times considered physically relevant are not of this differentiable class on the whole manifold. In the late 1930's Oppenheimer and Snyder obtained an exact solution to the EFE's that described the process of gravitational collapse to a black hole by considering a star as a spherical distribution of dust with uniform density. The solution to the problem involves considering space-time as two regions that are matched on a common boundary, i.e. the surface of the star. It is not a trivial problem to match two solutions of the EFE's and obtain a new solution as in general the differentiability conditions on this new *solution* need not be satisfied on the boundary. Therefore the differentiability conditions have to be weakened in order not to discard relevant solutions. Regardless of this consideration, conservation equations make sense in the theory of distributions if the second derivatives of the components of the metric tensor have, at most, a simple discontinuity across a smooth hypersurface [12], hence tensors have to be considered as distributions.

1.1.1 Differential geometry preliminaries

As the formulation of the junction conditions is across a hypersurface, which is regarded as a common boundary between two space-times, it is necessary that we begin with some differential geometry definitions and results to establish terminology we will use henceforth. First we introduce the definition of hypersurface with some related concepts.

Definition 1. *Let $\mathscr{M}$ and $\mathscr{N}$ be C^r differentiable manifolds and let $f : \mathscr{M} \to \mathscr{N}$ be a C^r mapping. We call f an* immersion *if its derivative is injective at each point. An embedding is defined as an immersion which is a homeomorphism onto its image in the induced topology.*

From the injective function theorem [13] the function f is locally injective, and therefore is locally an embedding. With this concepts one can define a hypersurface.

Definition 2. *Let $\mathscr{M}$ be an n-dimensional manifold, Σ an $(n-1)$-dimensional manifold, and let $\Phi : \Sigma \to \mathscr{M}$ an embedding, we say that the image $\Phi(\Sigma)$ of Σ is a* hypersurface *in $\mathscr{M}$.*

Locally, by the implicit function theorem, the hypersurface $\Phi(\Sigma) \in \mathscr{M}$ can be defined through a function $F : \mathscr{M} \to \mathbb{R}$ by the equation $F(x) = 0$. Henceforward, unless specifically stated, we consider $(\mathscr{M}, g)$ to be a 4-dimensional orientable Lorentzian manifold, and $\Phi(\Sigma)$ to be a hypersurface defined on the manifold.

As is usual in differential topology, as soon as a differentiable manifold is defined the differentiable structure allows us to develop calculus on the manifold. As the hypersurface previously defined is a differentiable manifold one can procede as usual to define tangent vectors and tangent spaces to the hypersurface. Given that the hypersurface is embedded in space-time we can use a relation as follows to bring the structure already defined in $\mathscr{M}$ to Σ. We follow [14] in this construction.

Definition 3. *For each $p \in \Sigma$, the embedding Φ naturally defines a differential mapping, called the* push-forward *from $T_p\Sigma$ to $T_{\Phi(p)}\mathscr{M}$,*

$$\begin{aligned} d\Phi|_p : T_p(\Sigma) &\to T_{\Phi(p)}\mathscr{M} \\ \boldsymbol{V} &\mapsto d\Phi|_p\boldsymbol{V}, \end{aligned}$$

which has rank 3 for each $p \in \Sigma$ given that the mapping is injective. The push-forward can be generalized for contravariant tensors of any order in Σ. In a similar manner, the pull-back *maps 1-forms in $T^*_{\Phi(p)}\mathscr{M}$ onto 1-forms in*

$T_p^*\Sigma$,

$$\begin{aligned} \Phi^*|_p : T^*_{\Phi(p)}\mathscr{M} &\to T_p^*(\Sigma) \\ \boldsymbol{\omega} &\mapsto \Phi^*|_p\boldsymbol{\omega}. \end{aligned}$$

Given that the rank of the push-forward is 3, $d\Phi|_p(T_p\Sigma)$ must be a 3-dimensional subspace of $T_{\Phi(p)}\mathscr{M}$, which is the tangent space to the hypersurface at $\Phi(p)$. For convenience in the notation we denote this tangent space in $\Phi(p)$ by $T_p\Sigma$. We now choose a coordinate system in both Σ and $\mathscr{M}$. The coordinate system taken in Σ, $\{y^a\}$ (Greek indexes run from 0 to 3, while Latin indexes go from 1 to 3), is defined in a neighborhood of a point $p \in \Sigma$, while the coordinate system in $\mathscr{M}$, $\{x^\alpha\}$, is taken in a neighborhood of $\Phi(p) \in \mathscr{M}$. We do not use abstract index notation to avoid confusion with indexes which denote coordinates at the hypersurface, instead we denote tensors by bold letters. Even though we choose coordinate systems in Σ and $\mathscr{M}$ the Israel-Lanczos formalism is coordinate independent.

We know that the vectors $\partial_a|_p$ form a basis for $T_p\Sigma$ and that the push-forward maps these basis into three linearly independent vectors in $\Phi(p)$ so they form a basis of the tangent space to the hypersurface. Hence

$$d\Phi|_p\left(\partial_a|_p\right) = \frac{\partial\Phi^\mu}{\partial y^a}\frac{\partial}{\partial x^\mu}\bigg|_{\Phi(p)} \equiv e^\mu{}_a\frac{\partial}{\partial x^\mu}\bigg|_{\Phi(p)} \equiv \vec{e}_a|_{\Phi(p)}.$$

By definition the vector fields $\vec{e}_a$ are defined only at the hypersurface $\Phi(\Sigma)$. As Φ is an homeomorphism between Σ and the hypersurface $\Phi(\Sigma)$, from now on we identify (unless specifically stated) the points p and $\Phi(p)$, and Σ and $\Phi(\Sigma)$ as usual.

Definition 4. *Let $T_p\Sigma$ be the tangent space to the hypersurface, the orthogonal complement in the dual space $T_p^*\mathscr{M}$ is a 1-dimensional vector space. This space is spanned by a non-zero 1-form at p which we denote as $\boldsymbol{n}|_p$, which is* *unique up to scalar multiplication. We call this form the* normal form to the hypersurface. *The vector normal to the hypersurface is defined as the vector obtained by raising the index of $\boldsymbol{n}$ with the metric of $\mathscr{M}$.*

Due to its definition, $\boldsymbol{n}$ is defined only in Σ and $\boldsymbol{n}(\vec{e}_a)|_p = 0$. In components, the form is given by $\boldsymbol{n} = n_\mu dx^\mu$. The hypersurface we previously defined can be equipped with a natural metric inherited by the metric in the manifold $\mathscr{M}$. This metric can have different signatures which are associated to the normal form to the hypersurface.

Definition 5. *If $\boldsymbol{g}$ is a metric in $\mathscr{M}$, the embedding Φ induces a metric $\Phi^*\boldsymbol{g}|_p$ in Σ, such that for each $\boldsymbol{X}, \boldsymbol{Y} \in T_p\Sigma$, $\Phi^* g(\boldsymbol{X}, \boldsymbol{Y})|_p = g(d\Phi(\boldsymbol{X}), d\Phi(\boldsymbol{Y}))|_{\Phi(p)}$. This metric is called the* first fundamental form *of Σ. We denote $\tilde{g} = \Phi^*\boldsymbol{g}|_p$.*

Theorem 1. *Let $\tilde{\boldsymbol{g}}$ be the first fundamental form of Σ and let $\boldsymbol{g}$ be a Lorentzian metric in $\mathscr{M}$, then $\tilde{\boldsymbol{g}}$ is*

1. *Lorentzian if $g^{ab}n_a n_b > 0$, in which case we say Σ is a timelike hypersurface.*
2. *Degenerate if $g^{ab}n_a n_b = 0$, in which case we say the hypersurface is null.*
3. *Positive definite if $g^{ab}n_a n_b < 0$, in which case we say that Σ is a spacelike hypersurface.*

The proof of this theorem can be found in [10]. It is important to notice that the first fundamental form's signature can change from point to point nevertheless we will restrict our study to hypersurfaces with definite signature as the thin layer surface studied in chapter 3 is a time-like hypersurface. Mars and Senovilla have dealt with general hypersurfaces in [14].

If $g^{ab}n_a n_b \neq 0$, we can normalize the normal form $\boldsymbol{n}$ to have unit magnitude. In this case, the mapping $\Phi^* : T^*_{\Phi(p)}\mathscr{M} \to T^*_p\Sigma$ will be injective in the 3-dimensional subspace $H^*_{\Phi(p)}$ of $T^*_{\Phi(p)}\mathscr{M}$ that consists in all forms $\boldsymbol{\omega} \in \Phi(p)$ such that $g^{ab}n_a\omega_b = 0$, given that $\boldsymbol{\Phi}^*\boldsymbol{n} = 0$ and $\boldsymbol{n}$ does not lie in H^*. Consequently, the inverse $(\boldsymbol{\Phi}^*)^{-1}$ will be a mapping $\tilde{\boldsymbol{\Phi}}_* : T^*_p\Sigma$ from $T^*_p\Sigma$ to $H^*_{\Phi(p)}$ and therefore in $T^*_{\Phi(p)}$. We can extend this mapping to a mapping of covariant tensors in Σ to covariant tensors in $\Phi(\Sigma)$; as there already exists a mapping $d\Phi$ of covariant tensors in Σ to $\Phi(\Sigma)$, we extend $d\Phi$ to a mapping $\tilde{\boldsymbol{\Phi}}_*$ of arbitrary tensors in Σ to $\Phi(\Sigma)$, this mapping has the property that $\tilde{\boldsymbol{\Phi}}_*\boldsymbol{T}$ has null contraction with $\boldsymbol{n}$ in all its indexes, i.e.

$$(\tilde{\boldsymbol{\Phi}}_*T)^{a\cdots b}{}_{c\cdots d}n_a = 0 \quad \wedge \quad (\tilde{\boldsymbol{\Phi}}_*T)^{a\cdots b}{}_{c\cdots d}g^{ce}n_e = 0$$

for any tensor defined in Σ. Hence, the tensor h in $\Phi(\Sigma)$ is defined by $h = \tilde{\boldsymbol{\Phi}}_*(\boldsymbol{\Phi}^*\boldsymbol{g})$. In terms of the normalized form $\boldsymbol{n}$,

$$h_{ab} = g_{ab} \mp n_a n_b,$$

given that this implies $\boldsymbol{\Phi}^*\boldsymbol{h} = \boldsymbol{\Phi}^*\boldsymbol{g}$ and $h_{ab}g^{bc}n_c = 0$.

Definition 6. *Let $\tilde{\boldsymbol{n}}$ be any extension of the* 1*-form $\boldsymbol{n}$ to an open neighborhood of* $\Phi(\Sigma)$ *then the tensor $\boldsymbol{K}$ defined in* $\Phi(\Sigma)$ *by*

$$K_{ab} = h^c{}_a h^d{}_b \nabla_d \tilde{n}_c$$

is called the second fundamental form *of* Σ.

This definition is independent of the extension, given that the projections by $h^a{}_b = g^{ac}h_{cb}$ restrict the covariant derivative to tangent directions to $\Phi(\Sigma)$.

1.1.2 Algebra and calculus of tensorial distributions

As mentioned before, in the usual formulation of GR the metric is taken to be at least of class C^2 in order that the EFE's are well-defined. The choice of the differentiability class can be circumvented by defining them in a distributional sense, i.e. taking generalized functions instead of continuous functions. We now proceed to lay the framework in which we will work to understand what we mean by a distributional sense. We follow the standard construction of the space of distributions c.f. [15, 12, 16], for additional details see [17].

Definition 7. *Let $\boldsymbol{T}$ and $\boldsymbol{U}$ be two tensors of type (p,q) and (q,p), respectively, we denote by $(\boldsymbol{T},\boldsymbol{U})$ their scalar product at a point $x \in \mathscr{M}$. Let $\mathscr{D}(\mathscr{M})$ be the set of C_0^∞ tensor fields on $\mathscr{M}$, i.e. smooth tensor fields with compact support on $\mathscr{M}$. We call $\mathscr{D}(\mathscr{M})$ the set of test tensor fields. We denote by ${\mathscr{D}^p}_q$ the subset of $\mathscr{D}(\mathscr{M})$ of tensor fields of type (p,q).*

Definition 8. *Given $\boldsymbol{U} \in {\mathscr{D}^p}_q$, for each locally integrable (p,q) tensor, i.e. integrable in each compact subset of its domain, $\boldsymbol{T}$ we define*

$$< \boldsymbol{T},\boldsymbol{U} >= \int_{\mathscr{M}} (\boldsymbol{T},\boldsymbol{U})\sqrt{-g}\, d^4x.$$

We can now define tensorial distributions by means of functionals.

Definition 9. *Let ${\chi^q}_p$ be a linear continuous functional*

$$\begin{aligned} {\chi^q}_p : {\mathscr{D}^p}_q &\to \mathbb{R} \\ {T^p}_q &\mapsto {\chi^q}_p({T^p}_q) \equiv < {\chi^q}_p, {T^p}_q > . \end{aligned}$$

We call ${\chi^q}_p$ a tensorial distribution of type (p,q).

Henceforth we will refer to tensorial distributions by distributions.

We define the sum of distributions and product by a scalar in the usual manner, giving the set of distributions a vector space structure. Notice that the space of distributions is simply the dual space of $\mathscr{D}(\mathscr{M})$.

Definition 10. *Let* $S^q{}_p$ *be a tensor field of type* (q,p)*, we define a* distribution of type (p,q) associated to $S^q{}_p$ *in the following manner*

$$\underline{S}^q{}_p : \mathscr{D}^p{}_q \to \mathbb{R}$$
$$T^p{}_q \mapsto < \underline{S}^q{}_p, T^p{}_q > \equiv \int_{\mathscr{M}} (S,T)\boldsymbol{\eta}$$

where $\boldsymbol{\eta}$ *if the volume element of the manifold.*

Notice that it is not necessary that distributions act over test tensor fields, this conditions can be weakened considering the action of the distributions whenever it can be defined.

The components of a tensorial distribution $\chi^q{}_p$ in a coordinate system are scalar distributions $\chi^{\alpha_1\cdots\alpha_q}{}_{\beta_1\cdots\beta_p}$ defined by

$$< \chi^{\alpha_1\cdots\alpha_q}{}_{\beta_1\cdots\beta_p}, T > \equiv \left\langle \chi^q{}_p, T dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_p} \otimes \frac{\partial}{\partial x^{\beta_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\beta_q}} \right\rangle,$$

where T is a function of compact support.

Definition 11. *Let* $\chi^q{}_p$ *be distribution of type* (p,q) *and* $S^r{}_s$ *a tensor field of type* (r,s)*, we define their tensor product as the distribution of type* $(s+p, r+q)$ *that acts on the following manner*

$$< S^r{}_s \otimes \chi^q{}_p, T^{s+p}{}_{r+q} > \equiv < \chi^q{}_p, (\boldsymbol{S},\boldsymbol{T})^p{}_q >,$$

where $(\boldsymbol{S},\boldsymbol{T})^p{}_q \in \mathscr{D}^p{}_q$ *is obtained by contracting the corresponding indexes of* $\boldsymbol{S}$ *with those of* $\boldsymbol{T}$*.*

We now define the covariant derivative of distributions by means of the covariant derivative defined on the manifold.

Definition 12. *Let* $(\mathscr{M}, g)$ *be a Lorentzian manifold,* $\chi^q{}_p$ *a distribution of type* (p,q)*, and* $T^{p+1}{}_q$ *a tensor field of type* $(p+1,q)$*. We define the covariant derivative of* $\chi^q{}_p$*,* $\nabla\chi^q{}_p$*, by the following relation*

$$< \nabla\chi^q{}_p, T^{p+1}{}_q > \equiv - < \chi^q{}_p, (DT)^p{}_q >$$

where $(DT)^{a_1\cdots a_p}{}_{b_1\cdots b_q} = \nabla_c T^{ca_1\cdots a_p}{}_{b_1\cdots b_q}$.

With this definition the components of the covariant derivative are the scalar distributions that act in the usual way, as the components of the covariant derivative of the tensor upon which the distribution acts.

1.1.3 Junction conditions

Let $\mathscr{M}^+$ and $\mathscr{M}^-$ be two manifolds with boundaries Σ^+ and Σ^-, respectively. Let $\psi : \Sigma^+ \to \Sigma^-$ be a diffeomorphism, we identify the points on both boundaries and denote both boundaries by Σ. Let $\mathscr{M}$ be the disjoint union of $\mathscr{M}^+$ and $\mathscr{M}^-$. As we are interested in the study of tensors in the hypersurface we proceed to define the associated distributions in a piecewise manner.

Definition 13. *Let Σ be a hypersurface in $(\mathscr{M}, g)$, we define the Heaviside function of Σ $\theta : \mathscr{M} \to \mathbb{R}$ by*

$$\theta(x) = \begin{cases} 1 \ \textit{if} \ \ x \in \mathscr{M}^+, \\ a \ \ \textit{if} \ \ x \in \Sigma, \\ 0 \ \ \textit{if} \ \ x \in \mathscr{M}^-, \end{cases}$$

where $a \in \mathbb{R}$ is arbitrary.

Given that θ is locally integrable, it defines a scalar distribution $\underline{\theta}$ as previously defined

$$< \underline{\theta}, \boldsymbol{T} >= \int_{\mathscr{M}^+} \boldsymbol{T}\boldsymbol{\eta}.$$

Given that Σ has measure zero on $\mathscr{M}$, the distribution $\underline{\theta}$ is independent of the value of a, therefore it won't be fixed.

Let $f : \mathscr{M} \to \mathbb{R}$ be a locally integrable function which is differentiable in all $\mathscr{M}$, except possibly in Σ where it might be discontinuous. Therefore, f is differentiable in both $\mathscr{M}^+$ and $\mathscr{M}^-$. Additionally we demand that both limits of the first derivatives are well defined over $\mathscr{M}^+$ and $\mathscr{M}^-$. We define $f^+ = f|_{\mathscr{M}^+}$ and $f^- = f|_{\mathscr{M}^-}$, being f locally integrable the associated distribution to this function is

$$\underline{f} = f^+ \cdot \underline{\theta} + f^- \cdot (\underline{1} - \underline{\theta}).$$

We calculate the derivative of $\underline{f}$, integrating by parts in $\mathscr{M}^+$ and $\mathscr{M}^-$ we obtain for an arbitrary vector V

$$< \nabla \underline{f}, V >= \int_{\mathscr{M}^+} V^\mu \partial_\mu f^+ \boldsymbol{\eta} + \int_{V^-} V^\mu \partial_\mu f^- \boldsymbol{\eta} + \int_\Sigma [f] V^\mu d\sigma_\mu \tag{1.3}$$

where $d\sigma_\mu$ has an orientation from $\mathscr{M}^-$ to $\mathscr{M}^+$ and $[f]$ is the function defined in Σ called the *jump* of f in Σ, and it is defined for all $q \in \Sigma$ by

$$[f](q) \equiv \lim_{\substack{x \to q \\ \mathscr{M}^+}} f^+(x) - \lim_{\substack{x \to q \\ \mathscr{M}^-}} f^-(x).$$

Definition 14. *Let $\boldsymbol{V} \in \mathscr{D}^1{}_0$, we define the distribution $\boldsymbol{\delta}$ of type $(1,0)$ by*

$$< \boldsymbol{\delta}, \boldsymbol{V} >\equiv \int_\Sigma V^\mu d\sigma_\mu = \int_\Sigma V^\mu n_\mu d\sigma.$$

From equation (1.3) taking $f^+ = 1$ and $f^- = 0$ it is straightforward that $\boldsymbol{\delta} = \nabla \underline{\theta}$. It is possible to define a scalar distribution δ by

$$< \delta, \boldsymbol{V} >\equiv \int_\Sigma \boldsymbol{V} d\sigma.$$

This δ distribution is dependent of the choice of the normal form $\boldsymbol{n}$, $\boldsymbol{\delta} = \boldsymbol{n} \cdot \delta$, in components $\delta_\mu = n_\mu \cdot \delta$. Therefore, (1.3) is

$$\partial_\mu \underline{f} = \partial_\mu f^+ \cdot \underline{\theta} + \partial_\mu f^- \cdot (\underline{1} - \underline{\theta}) + [f] \cdot \delta_\mu. \tag{1.4}$$

N.B. 1. *The most important properties of θ y $\underline{\theta}$ are the following (As this properties are held almost everywhere we omit writing the distributional relations* *as they are analogous to the non-distributional ones.)*

$$
\begin{aligned}
\theta(x)\theta(x) &= \theta(x), \\
\underline{\theta} \cdot \underline{\theta} &= \underline{\theta}, \\
\theta(x)(1-\theta(x)) &= 0, \\
(1-\theta(x)) &= \theta(-x), \\
(1-\theta(x))(1-\theta(x)) &= (1-\theta(x)), \\
\nabla\underline{\theta} &= \boldsymbol{\delta},
\end{aligned}
$$

where $\boldsymbol{\delta}$ *is the Dirac distribution.*

In the same manner that it was done for a scalar function, given a tensor field $\boldsymbol{T} \in \mathscr{D}^p{}_q$ we have that the associated distribution is

$$\underline{\boldsymbol{T}} = \boldsymbol{T}^+ \cdot \underline{\theta} + \boldsymbol{T}^- \cdot (\underline{1} - \underline{\theta}).$$

1.1.3.1 First junction condition

We regard space-time as the disjoint union of two separate Lorentzian manifolds with boundary $\mathscr{M}^+$ and $\mathscr{M}^-$, each a solution to Einstein's equations with metrics $\boldsymbol{g}^+$ and $\boldsymbol{g}^-$, respectively. The boundaries of $\mathscr{M}^+$ and $\mathscr{M}^-$ will be denoted Σ^+ and Σ^-, respectively. We will assume that there exists a C^3 diffeomorphism between both boundaries so henceforth we will denote both boundaries by Σ unless specifically stated. In 1987, Clarke and Dray [18] proved that under this assumptions "if a spacetime is constructed by identifying the boundaries of two spacetimes in such a way that the intrinsic metrics on the boundaries agree (and have a constant signature) then there exists a unique choice of a C^1 atlas in which the (four-dimensional) metric of the spacetime is continuous". As pointed out by Mars and Senovilla [14], the assumption of a constant signature is superflous. Therefore we assume that $\boldsymbol{g}$

is a continuous tensor on a hypersurface, this is called the *first junction condition*. If one assumes that $\boldsymbol{g}$ is discontinuous across the hypersurface then the Christoffel symbols contain terms proportional to a product of distributions which is ill-defined thus the assumption of continuity of the metric across Σ can also be deduced from this argument. The first junction condition is also known as the *preliminary junction conditions*.

We can decompose the metric tensor as

$$\boldsymbol{g} = \theta \cdot \boldsymbol{g}^+ + (1-\theta) \cdot \boldsymbol{g}^-,$$

and from our hypothesis it has an associated distribution which would be

$$\underline{\boldsymbol{g}} = \underline{\theta} \cdot \boldsymbol{g}^+ + (\underline{1} - \underline{\theta}) \cdot \boldsymbol{g}^-.$$

We now define the connection coefficients associated to the metric in the manifold. Denoting by ${\Gamma^{+\alpha}}_{\beta\gamma}$ the Christoffel symbols associated to $\boldsymbol{g}^+$ and defined in $\mathscr{M}^+ \cup \Sigma$, and ${\Gamma^{-\alpha}}_{\beta\gamma}$ the Christoffel symbols associated to the metric defined in $\mathscr{M}^- \cup \Sigma$, $\boldsymbol{g}^-$. We denote by ${\underline{\Gamma}^{\alpha}}_{\beta\gamma}$ the Christoffel symbols associated with the distributional metric $\underline{\boldsymbol{g}}$. For convenience in the notation we denote $(\underline{1} - \underline{\theta}) = \underline{\tilde{\theta}}$. So we obtain

$$\begin{aligned}
{\underline{\Gamma}^{\alpha}}_{\beta\gamma} =& \frac{1}{2}\underline{g}^{\alpha\lambda}\left(\underline{g}_{\lambda\beta,\gamma} + \underline{g}_{\lambda\gamma,\beta} - \underline{g}_{\beta\gamma,\lambda}\right) \\
=& \frac{1}{2}\left(g^{\alpha\lambda+}\underline{\theta} + g^{\alpha\lambda-}\underline{\tilde{\theta}}\right) \\
& (g_{\lambda\beta,\gamma}\underline{\theta} + g_{\lambda\gamma,\beta}\underline{\theta} - g_{\beta\gamma,\lambda}\underline{\theta} \\
& + g_{\lambda\beta,\gamma}\underline{\tilde{\theta}} + g_{\lambda\gamma,\beta}\underline{\tilde{\theta}} - g_{\beta\gamma,\lambda}\underline{\tilde{\theta}}) \\
=& {\Gamma^{+\alpha}}_{\beta\gamma}\,\underline{\theta} + {\Gamma^{-\alpha}}_{\beta\gamma}\,(\underline{1} - \underline{\theta}).
\end{aligned}$$

Consequently we have

$${\underline{\Gamma}^{\alpha}}_{\beta\gamma} = {\Gamma^{+\alpha}}_{\beta\gamma}\,\underline{\theta} + {\Gamma^{-\alpha}}_{\beta\gamma}\,(\underline{1} - \underline{\theta}). \tag{1.5}$$

In this way we now know the Christoffel symbols as distributions so we define the connection coefficients as functions defined on the manifold in the natural way,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \Gamma^{+\alpha}{}_{\beta\gamma}\,\theta + \Gamma^{-\alpha}{}_{\beta\gamma}\,(1-\theta).$$

Notice that this definition produces the associated Christoffel distributions as needed.

Given the relation between the Riemann tensor and the Christoffel symbols we compute the Riemann tensor,

$$R^{\alpha}{}_{\beta\gamma\mu} = \Gamma^{\alpha}{}_{\beta\mu,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\mu} + \Gamma^{\alpha}{}_{\gamma\rho}\,\Gamma^{\rho}{}_{\beta\mu} - \Gamma^{\alpha}{}_{\mu\rho}\,\Gamma^{\rho}{}_{\beta\gamma}$$

and we will treat this relation in a distributional sense. First, given equations (1.4) and (1.5), we have

$$\underline{\Gamma}^{\alpha}{}_{\beta\mu,\gamma} = \Gamma^{+\alpha}{}_{\beta\mu,\gamma}\cdot\underline{\theta} + \Gamma^{-\alpha}{}_{\beta\mu,\gamma}\cdot\underline{\tilde{\theta}} + \delta\cdot n_{\gamma}[\Gamma^{\alpha}{}_{\beta\mu}].$$

As we have defined the functions $\Gamma^{\alpha}{}_{\beta\mu}$, we have that the distributions $\underline{\Gamma}^{\alpha}{}_{\beta\mu}$ are associated to these functions and therefore the product $\underline{\Gamma}^{\alpha}{}_{\gamma\rho}\underline{\Gamma}^{\rho}{}_{\beta\mu}$ is well-defined as it is the distribution associated to $\Gamma^{\alpha}{}_{\gamma\rho}\,\Gamma^{\rho}{}_{\beta\mu}$. From this facts, we obtain the distributional Riemann tensor which has the following expression

$$\underline{R}^{\alpha}{}_{\beta\gamma\mu} = R^{+\alpha}{}_{\beta\gamma\mu}\cdot\underline{\theta} + R^{\alpha}{}_{\beta\gamma\mu}\cdot\underline{\tilde{\theta}} + \delta\cdot n_{\gamma}[\Gamma^{\alpha}{}_{\beta\mu}] - \delta\cdot n_{\mu}[\Gamma^{\alpha}{}_{\beta\gamma}]. \tag{1.6}$$

The term on the right hand side of the equation proportional to δ is called the singular part of the Riemann tensor.

1.1.3.2 Second junction condition

We now seek the conditions to avoid the singular part of the Riemann tensor. As the metric is assumed to be continuous across Σ, any discontinuity of the derivative must be in the normal direction to the hypersurface, i.e.

$$[g_{\alpha\beta,\gamma}] = c_{\alpha\beta}n_{\gamma}, \tag{1.7}$$

where $c_{\alpha\beta}$ is a tensor field. With the jump of the ordinary derivative of the metric in a coordinate system we can calculate the jump of the Christoffel symbols. From (1.7) one obtains $[n_{\alpha;\beta}] = -[\Gamma^{\rho}{}_{\alpha\beta}]n_\rho$ and a direct calculation yields that the singular term of the Riemann distribution is actually a tensor field $\mathscr{R}^{\alpha}{}_{\beta\mu\nu}$ which has the following expression

$$\mathscr{R}^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2}(-n^\alpha[K_{\beta\nu}]n_\mu + n^\alpha[K_{\beta\mu}]n_\nu - n_\beta[K^{\alpha}{}_{\mu}]n_\nu + n_\beta[K^{\alpha}{}_{\nu}]n_\mu).$$

Contracting $\mathscr{R}^{\rho}{}_{\mu\rho\nu} = \mathscr{R}_{\mu\nu}$ one obtains the so called singular part of the Ricci distribution, i.e.

$$\underline{R}_{\mu\nu} = R^+_{\mu\nu}\underline{\theta} + R^-_{\mu\nu}(\underline{1} - \underline{\theta}) + \delta\mathscr{R}_{\mu\nu},$$

where the explicit formula for the singular part is

$$\mathscr{R}_{\mu\nu} = -[K_{\mu\nu}] - [K^{\rho}{}_{\rho}]n_\mu n_\nu.$$

Notice that the singular part of the Ricci distribution vanishes if and only if the jump of the second fundamental form vanishes, and therefore if and only if the singular part of the Riemann distribution does. Contracting one again

$$\underline{R} = R^+\underline{\theta} + R^-(\underline{1} - \underline{\theta}) + \delta\mathscr{R},$$

where $\mathscr{R} = -2[K^{\mu}{}_{\mu}]$. Therefore the Einstein distribution is piecewise defined by

$$\underline{G}_{\mu\nu} := \underline{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\underline{R} = G^+_{\mu\nu}\underline{\theta} + G^-_{\mu\nu}(\underline{1} - \underline{\theta}) + \delta\mathscr{G}_{\mu\nu}, \tag{1.8}$$

where the singular part is

$$\begin{aligned}\mathscr{G}_{\mu\nu} = \mathscr{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathscr{R} &= -[K_{\mu\nu}] - [K^{\rho}{}_{\rho}]n_\mu n_\nu - \frac{1}{2}g_{\mu\nu}(-2[K^{\rho}{}_{\rho}]) \\ &= -[K_{\mu\nu}] + [K^{\rho}{}_{\rho}](g_{\mu\nu} - n_\mu n_\nu) = -[K_{\mu\nu}] + h_{\mu\nu}[K^{\rho}{}_{\rho}], \end{aligned} \tag{1.9}$$

which is tangent to Σ. As our initial assumption was that both metrics are solutions in $\mathscr{M}^+$ and $\mathscr{M}^-$, respectively, we can define an energy-momentum tensor

$$\underline{T}_{\mu\nu} := T^+_{\mu\nu}\underline{\theta} + T^-_{\mu\nu}(\underline{1} - \underline{\theta}) + \delta\mathscr{T}_{\mu\nu}. \tag{1.10}$$

which is a solution to the EFE's in a distributional sense. The singular part of the energy-momentum tensor is interpreted as the surface energy-momentum tensor of a *surface layer* or *thin shell.* As the surface energy-momentum tensor is tangent to Σ we can do a decomposition

$$\mathscr{T}^{\mu\nu} = \mathscr{T}^{ab} e^{\mu}{}_{a} e^{\nu}{}_{b},$$

to arrive to the expression

$$\mathscr{T}_{ab} = -[K_{ab}] + [K]h_{ab}.$$

From this expression it is immediate that to eliminate the singular term from the curvature distributions it is sufficient to demand $[K_{ab}] = 0$, which is the *second junction condition* or occasionally known as the *junction conditions* as one assumes the first junction condition to obtain the second.

1.2 Bianchi's second identity in distributional sense

A lengthy calculation shows [14] that Bianchi's second identity holds in a distributional sense, i.e.

$$\frac{1}{2}\delta^{\alpha\beta\gamma}{}_{\lambda\mu\nu}\underline{R}_{\rho\sigma\alpha\beta;\gamma} = \underline{R}_{\rho\sigma\lambda\mu;\nu} + \underline{R}_{\rho\sigma\mu\nu;\lambda} + \underline{R}_{\rho\sigma\nu\lambda;\mu} = 0, \tag{1.11}$$

which as an immediate consequence has the equation $\nabla^{\mu}\underline{G}_{\mu\nu} = 0$. We now analyze this last equation

$$\nabla^{\mu}\underline{G}_{\mu\nu} = \nabla^{\mu}(G^{+}_{\mu\nu}\underline{\theta} + G^{-}_{\mu\nu}(\underline{1} - \underline{\theta}) + \delta\mathscr{G}_{\mu\nu}) = [G_{\mu\nu}]n^{\mu}\delta + \nabla^{\mu}(\mathscr{G}_{\mu\nu}\delta) \tag{1.12}$$

Using equation (A.3) the last term can be expanded as

$$g^{\mu\rho}\nabla_{\rho}(\mathscr{G}_{\mu\nu}\delta) = g^{\mu\rho}\nabla_{\sigma}(\mathscr{G}_{\mu\nu}n_{\rho}n^{\sigma}\delta) + g^{\mu\rho}h_{\rho}{}^{\lambda}\nabla_{\lambda}\mathscr{G}_{\mu\nu}\delta = h^{\rho\mu}\nabla_{\mu}\mathscr{G}_{\rho_{\nu}}\delta. \tag{1.13}$$

Using (A.2),

$$\nabla^{\mu}(\mathscr{G}_{\mu\nu}\delta) = (\overline{\nabla}^{\mu}\mathscr{G}_{\mu\nu} - K^{\Sigma}_{\alpha\beta}\mathscr{G}^{\alpha\beta}n_{\nu})\delta. \tag{1.14}$$

Substituting in (1.12) we have

$$\left(n^{\mu}[G_{\mu\nu}] + \overline{\nabla}^{\mu}\mathcal{G}_{\mu\nu} - \frac{1}{2}n_{\nu}\mathcal{G}^{\alpha\beta}\left(K^{+}_{\alpha_\beta} + K^{-}_{\alpha_\beta}\right)\right)\delta = 0.$$

Separating in normal and tangential components we get

$$(K^{+}{}_{\mu\nu} + K^{-}{}_{\mu\nu})\mathcal{G}^{\mu\nu} = 2n^{\mu}n^{\nu}[\underline{G}_{\mu\nu}] \tag{1.15}$$

$$\overline{\nabla}^{\mu}\mathcal{G}_{\mu\nu} = -n^{\mu}h^{\rho}{}_{\nu}[\underline{G}_{\mu\rho}]. \tag{1.16}$$

Notice that all relations are coordinate independent.

Chapter

2

Closed LTB Models

Given a 4-dimensional Lorentzian manifold $(\mathscr{M}, g_{ab})$, it is called a *cosmological model* if it admits a congruence of timelike curves $\mathscr{C}^a$, called the word lines of the fundamental observers (WFO's), such that each $\mathbf{p} \in \mathscr{M}$ belongs to one and only one WFO. One also assumes that $(\mathscr{M}, g_{ab})$ is a solution to Einstein's field equations and is asymptotically an FLRW model. The tangent vector field to the WFO's is called the 4-velocity field. Sometimes it is convenient to adapt coordinates to the WFO's assuming that each WFO is parametrized as

$$x^0 = x^0(\tau), \qquad x^i = \text{const.}, \tag{2.1}$$

and that each WFO is labeled by one and only one set of constant values of x^i. These coordinates are known as comoving coordinates.

2.1 LTB Models

LTB models are exact spherically symmetric solutions of Einstein's equations with an inhomogeneous dust source with or without cosmological constant. We now follow the standard derivation of the LTB metric, beginning by considering the general spherically symmetric metric in comoving coordinates

$$ds^2 = -N^2\, dt^2 + B^2\, dr^2 + R^2 d\Omega^2, \tag{2.2}$$

where $N = N(t,r)$, $B = B(t,r)$ and $R = R(t,r)$.

From the EFE's

$$\begin{aligned} G^0{}_0 &= \frac{2R''}{RB^2} - \frac{2R'B'}{RB^3} + \frac{R'^2}{B^2R^2} \\ &\quad - \frac{1}{R^2} - \frac{2\dot{R}\dot{B}}{N^2RB} - \frac{\dot{R}^2}{R^2N^2} = -\kappa\rho - \Lambda, \end{aligned} \tag{2.3}$$

$$\begin{aligned} G^1{}_1 &= -\frac{2\ddot{R}}{RN^2} + \frac{R'^2}{B^2R^2} - \frac{1}{R^2} + \frac{2N'R'}{B^2RN} \\ &\quad - \frac{\dot{R}^2}{R^2N^2} + \frac{2\dot{N}\dot{R}}{RN^3} = -\Lambda, \end{aligned} \tag{2.4}$$

$$G^1{}_0 = \frac{2\dot{R}'}{B^2R} - \frac{2\dot{B}R'}{B^3R} - \frac{2N'\dot{R}}{B^2RN} = 0, \tag{2.5}$$

$$\begin{aligned} G^2{}_2 = G^3{}_3 = \frac{1}{RB^3N^3} \Big(& R''BN^3 - B^2\ddot{B}RN + N''RBN^2 - \ddot{R}B^3N \\ & - B'R'N^3 - B'N'RN^2 + N^2N'R'B - B^2\dot{B}\dot{R}N + B^2\dot{B}\dot{N}R \\ & + \dot{N}\dot{R}B^3 \Big) = -\Lambda, \end{aligned} \tag{2.6}$$

where $R' = \partial R/\partial r$, and $\dot{R} = u^a \nabla_a R = \partial R/\partial t$. We know the fluid moves in timelike geodesics, so the acceleration is zero $\dot{u}^a = u^b \nabla_b u^a = N'/NB^2 = 0$, which implies $N' = 0$ and one can rescale the time coordinate to make $N = 1$. So (2.5) is

$$(BR')_{,t} = 0, \tag{2.7}$$

if $R' = 0$ one obtains a Kantowski-Sach's-like metric, Bondi integrated this equation for $R' \neq 0$ and obtained the result

$$B^2 = \frac{R'^2}{1-K}, \tag{2.8}$$

where $K = K(r)$ is an arbitrary function. Therefore, the LTB metric in comoving coordinates is

$$ds^2 = -dt^2 + \frac{R'^2}{1-K}dr^2 + R^2 d\Omega^2. \tag{2.9}$$

To avoid having a degenerate metric in (2.9) if there is a point in spacetime in which $R'(t_*, r_*) = 0$ then by L'Hôpital's rule $K(r_*) = 1$ in such a way that the limit $R'/(1-K)$ is finite and non-zero. From this limit and that K is a function of one variable one must also have that $R(t, r_*) = 0$ for all t.

As $N' = 0$, using (2.8) we can eliminate R'^2 from the $G^1{}_1$ component, obtaining

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{K}{R^2} + \Lambda = 0. \tag{2.10}$$

Assuming $\dot{R} \neq 0$ we multiply this equation by $R^2\dot{R}$ and integrate the resulting equation to obtain a Friedmann-like equation

$$\dot{R}^2 = \frac{2M}{R} - K - \frac{1}{3}\Lambda R^2, \tag{2.11}$$

with M an arbitrary function which will be discussed in the following paragraph.

Multiplying the $G^0{}_0$ component by R^2R' we obtain by rearrangement

$$\left(R - \frac{R\dot{R}^2}{N^2} + \frac{RR'^2}{B^2} + \frac{1}{3}\Lambda R^3\right)_{,r} + R\left(\frac{\dot{R}^2}{N^2}\right)_{,r} - \frac{2B'R\dot{R}R'}{N^2B} = \kappa\rho R^2R'. \tag{2.12}$$

By equation (2.5) the second and third terms on the left hand side sum up to zero, so (2.12) is in fact

$$\left(R - \frac{R\dot{R}^2}{N^2} + \frac{RR'^2}{B^2} + \frac{1}{3}\Lambda R^3\right)_{,r} = \kappa\rho R^2R'. \tag{2.13}$$

The term in the left hand side of the previous equation is the radial derivative of the Misner–Sharp quasi–local mass–energy function, a well known invariant in a spherically symmetric spacetime, so we obtain the relation

$$2M' = 8\pi\rho R^2 R'. \tag{2.14}$$

In an analogous way, the ${G^1}_1$ component takes the form

$$\left(R - \frac{R\dot{R}^2}{N^2} + \frac{RR'^2}{B^2} + \frac{1}{3}\Lambda R^3\right)_{,t} = 0, \tag{2.15}$$

which implies that $M = M(r)$.

Notice that the density diverges whenever R or R' do not have a zero of the same order as M'. The first divergence is the Big Bang singularity while the second is know as a shell crossing singularity. In this type of singularity the density goes to infinity and changes sign whenever the critical point is also a turning value for R'. The radial geodesic distance between two infinitesimal points is $\sqrt{|g_{rr}|}\, dr$, which is zero for a shell crossing singularity.

Misner–Sharp quasi–local mass–energy function is in general not equal to the proper mass, which is given by

$$M_P = \int_V \rho\sqrt{-g}\, dV = 4\pi \int \rho \frac{R'R^2}{\sqrt{1-K}}\, dr. \tag{2.16}$$

Notice that if $K = 0$ then $M = M_P$. This gives K a physical interpretation as if $K > 0$ then $M < M_P$ and if $K < 0$ then $M > M_P$. Therefore $-K$ determines the local energy per unit mass of the dust particles.

2.2 LTB Models with $\Lambda = 0$

Henceforth we consider $\Lambda = 0$ unless specifically stated. The equation (2.11), a Friedman–like evolution equation, leads to a classification of the models in three kinematic classes according to the sign of $K = K(r)$, which determines the existence of a zero of $\dot{R}^2$, and thus, the kinematic evolution: for $K > 0$ the models expand initially $\dot{R} > 0$, reach a maximal expansion value $R_{\max} = 2M/K$ where $\dot{R} = 0$ and then collapse $\dot{R} < 0$, while for $K \leq 0$ the models are ever expanding. Since $K = K(r)$, it is possible to have in a single model regions with different kinematic class.

The solutions of the Friedman–like equation in (2.11) define the kinematic classes as elliptic ($K > 0$), hyperbolic ($K < 0$) and parabolic ($K = 0$) solutions given by

$$K > 0: \qquad R = \frac{M}{K}(1 - \cos\eta), \quad \eta - \sin\eta = \frac{K^{\frac{3}{2}}}{M}(t - t_{bb}(r)), \tag{2.17}$$

$$K < 0: \qquad R = \frac{M}{|K|}(\cosh\eta - 1), \quad \eta - \sinh\eta = \frac{|K|^{\frac{3}{2}}}{M}(t - t_{bb}(r)), \tag{2.18}$$

$$K = 0: \qquad R = \left[\frac{9}{2}M(t - t_{bb}(r))^2\right]^{\frac{1}{3}}, \tag{2.19}$$

with $t_{bb}(r)$ denoting the Big Bang time function such that $R(t_{bb}(r), r) = 0$ for variable r (notice that in general $t'_{bb} \neq 0$).

To fully determine an LTB model we need to prescribe the three free functions $M(r)$, $K(r)$ and $t_{bb}(r)$. Since the metric is invariant under rescalings of r, it is always possible to reduce this set of free functions to a pair of independent irreducible free functions. Given a choice of free functions, all relevant quantities of the models can be computed from the solutions for (2.17) to (2.19).

2.3 Ever expanding closed models

Closed LTB models are characterised by rest frames that are compact 3–dimensional submanifolds without a boundary and with finite proper volume, which implies two possibilities: the rest frames are diffeomorphic to S^3 or to a 3-torus (an example of how to select the free functions latter case is given in [19]). Since LTB models are spherically symmetric, the topological class of the rest frames is directly connected with the existence of symmetry centers, which are regular timelike comoving worldlines $r = r_c$ generated by the fixed points of $SO(3)$, and thus comply with

$$R(t, r_c) = \dot{R}(t, r_c) = 0. \tag{2.20}$$

Closed models diffeomorphic to $\mathbf{S}^3$ admit two symmetry centers, while rest frames with toroidal topology admit no symmetry centers. In closed LTB models the condition (2.20) holds for two values of r, which can be denoted by $r = 0$ and $r = r_c$. Since $\mathbf{S}^3$ is smooth there must exist a turning value $r = r_*$ such that $R'(t, r_*) = 0$. Regularity conditions implies that $M = K = 0$ and all radial gradients vanish at both symmetry centers.

Using the orthonormal tetrad $\{(e_\mu)^a\}_{a=t,r,\theta,\phi}$ the components of the Riemann tensor are

$$\begin{aligned}
R_{trtr} &= \frac{2M}{R^3} - \frac{M'}{R^2 R'}, \\
R_{r\theta r\theta} = R_{r,\phi,r,\phi} &= \frac{M}{R^3} - \frac{M'}{R^2 R'}, \\
R_{r\theta r\theta} = R_{r,\phi,r,\phi} &= -\frac{2M}{R^3}, \\
R_{\theta\phi\theta\phi} &= -\frac{2M}{R^3}.
\end{aligned}$$

The requirement of a nonsingular curvature at $r = r_c$ is having a finite limit of M/R^3 and having a finite value for the density ρ as

$$\lim_{r\to r_c} \frac{M}{R^3} = \lim_{r\to r_c} \frac{M'}{3R^2 R'} = \frac{4\pi\rho}{3}. \tag{2.21}$$

2.3.1 Regularity of ever expanding closed models

After looking at closed LTB models with zero cosmological constant, Bonnor [8] concluded that all "physically acceptable closed models" (PACM) must be elliptic everywhere and eventually, collapse. Bonnor defined a PACM by the following conditions:

1. ρ is finite and non–negative
2. There are no comoving surface layers nor shell–crossing singularities.
3. K, M and R are C^1.
4. K satisfies extra regularity conditions at the symmetry centers, see [8].

These conditions imply

$$\mathrm{sgn}(R') = \mathrm{sgn}(M') = \mathrm{sgn}(\sqrt{1-K}), \tag{2.22}$$

and, as an immediate consequence, if zeroes of R', M', $\sqrt{1-K}$ exist, they must all be common and of the same order. If the zeroes of R' are different from the zeros of the other quantities, then shell crossings occur where the density and curvature scalars diverge with $R > 0$. The equation (2.14) together with (2.22) imply that the density is non–negative and bounded everywhere, except at the coordinate locus of a central singularity. The necessary and sufficient conditions to avoid shell–crossing singularities, as required by (2.22), are given by the Hellaby–Lake conditions given explicitly in [20, 4].

2.4 Lanczos–Israel formalism for closed models

Applying to the LTB metric the Lanczos-Israel-formalism yields as the only nonzero component of the Einstein tensor: G_{tt}, given by (2.11), while the extrinsic curvature at the hypersurface marked locally by $r = \pi/2$ is given by

$$K^a{}_b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{1-K}|R'|}{RR'} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{1-K}|R'|}{RR'} \end{bmatrix}. \tag{2.23}$$

The Darmois junction conditions demand the continuity of $K^a{}_b$ at the hypersurface. Hence, if $R' > 0$ for all r, then $|R'|/R' = 1$, so that the junction conditions are equivalent to the continuity of R and K. If there exists a zero of R' in some fixed value $r = r_0$, then there exists a discontinuity of $K^a{}_b$ unless $K(r_0) = 1$. At the turning value $r_0 = r^* = \pi/2$ there is clearly a discontinuity of $K^a{}_b$.

Bonnor proved that a PACM must be an elliptic model. First, he proved that if R' changes sign (turning value) on a hypersurface $r = r^*$, with r^* constant, and $1 - K \neq 0$ on the hypersurface, then there is a surface layer. The proof is straightforward. Since R has two zeros (two symmetry centers) in closed LTB models, the continuity of R implies the existence of a turning value marked by a zero of R' in some value $r = r^*$ within the radial coordinate range between the centers. Bonnor's condition 2 implies that $r = r^*$ must lie within an elliptic region ($K > 0$), since the regularity condition $1 - K = 0$ at $r = r^*$ cannot be satisfied for a turning value in parabolic or hyperbolic regions ($K \leq 0$). Turning values in such regions necessarily exhibit a surface layer, which is not contemplated in the definition of a PACM.

The equation of state of the surface layer that follows from (1.9) is $\sigma + \Pi_1 + \Pi_2 = 0$, where σ is the surface density and Π_i are the surface pressures that follow from the right hand side of (1.9) (the distributional energy–momentum tensor). Bonnor considered this equation of state unphysical, not only for having negative pressure, but also because of:

$$M_{TS} = \int (T^1{}_1 + T^2{}_2 + T^3{}_3 - T^4{}_4)\sqrt{h}\, d^3x = \int (\sigma + \Pi_1 + \Pi_2)\sqrt{h}\, d^3x = 0,$$

which means that the surface layer energy–momentum tensor produces zero active gravitational mass.

To choose the appropriate free functions M, K, t_{bb} for a closed model we must demand that their radial gradients vanish at turning values and at the symmetry centers. In the following chapter we re-examine and extend Bonnor's results, looking at the spatially flat ($K = 0$) and negatively curved ($K < 0$) cases separately.

2.4.1 Surface tension

The presence of distributional sources in thin layers can be associated with surface tension through the relativistic generalisation of the Kelvin relation of Newtonian physics [9]

$$\Delta P = -2\mathscr{K} A \tag{2.24}$$

where the surface tension A depends on the material, ΔP the difference of pressures in both sides of the surface layer and $\mathscr{K}$ is the mean curvature given by $\mathscr{K} = 1/R_1 + 1/R_2$, with R_1, R_2 the principal curvature radii. As proven in [9], the relativistic generalisation of (2.24) is connected to a thin shell in the framework of the Israel–Lanczos formalism:

$$\Delta P = \frac{1}{2}\left(K^+{}_{ab} + K^-{}_{ab}\right)\mathscr{T}^{ab}. \tag{2.25}$$

where $\mathscr{T}^{ab}$ is the projected energy–momentum tensor in (1.9)

$$\mathscr{T}^{ab} = h^a_\alpha h^b_\beta \mathscr{T}^{\alpha\beta}, \qquad 8\pi\mathscr{T}^{ab} = -[K^{ab}] + h^{ab}[K^c{}_c] \tag{2.26}$$

where $h^a_\alpha = \delta^a_\alpha \delta^b_\beta + n^a_\alpha n^b_\beta$ with $a, b = t, \theta, \phi$ the hypersurface intrinsic coordinates.

Chapter

3

Analysis of ever expanding closed LTB Models

3.1 The spatially flat case $K = 0$

A convenient choice for the free functions M and t_{bb} is

$$M = M_0 \sin^3 \bar{r}, \quad t_{bb} = -T_0 \sin^2 \bar{r}, \quad \Rightarrow \quad R = \left(\frac{9}{2} M_0\right)^{1/3} \sin \bar{r} \, \left[\bar{t} + T_0 \sin^2 \bar{r}\right]^{2/3}, \tag{3.1}$$

where $M_0 = \frac{3}{2} H_0^{-1}$, T_0 is an arbitrary constant, $\bar{r} = \pi H_0 r$ and $\bar{t} = H_0 t$ are the radial and time dimensionless coordinates respectively. However, to simply notation henceforth we will drop the bars on top of t and r, understanding henceforth that (unless specifically stated) t and r without overbars denote these dimensionless rescaled coordinates.

The parameters in (3.1) have been selected so that the kinematic evolution of the model at the symmetry centres $r = 0, \pi$ coincides with that of the Einstein–de Sitter spatially flat FLRW model, whose Big Bang time is given by $t = 0$. Hence, the constant T_0 can be identified with the Big Bang time of the LTB model at $\bar{r} = \pi/2$ (or equivalently $r = \frac{1}{2}H_0^{-1}$), that is: $t_{bb}(\pi/2) = -T_0 < 0$. For a more realistic cosmological scenario in the context of an inhomogeneous model with small deviation from an FLRW background, we shall assume that $|T_0| \ll t_0$, with present cosmic age given by $t_0 \sim 13.7\times10^9$ years (a convenient bound value is $|T_0| \sim 10,000$ years). With this choice of free functions we have

$$R' = \frac{(M_0/6)^{1/3} \cos r \; [7\,T_0 \sin^2 r + 3t]}{[t + T_0 \sin^2 r]^{1/3}}$$

while the density and the components of the extrinsic curvature follows from (2.11) and (2.23) for $K = 0$:

$$8\pi\rho = \frac{16}{3(t + T_0 \sin^2 r)(7T_0 \sin^2 r + 3t)}. \qquad (3.2)$$

$$K^\theta{}_\theta = K^\phi{}_\phi \quad = \quad -\frac{2}{3}\frac{\mathscr{H}\left(r - \frac{\pi}{2}\right)}{M_0^{\frac{1}{3}} \sin r \;\; [t + T_0 \sin^2 r]^{2/3}},$$

where $\mathscr{H}(r)$ is the Heaviside function and we used the fact that $t+t_0 \sin^2 r \geq 0$ in the full domain $0 \leq r \leq \pi$.

Since these expressions allow us to compute $K_{ab}{}^+ + K_{ab}{}^- = 0$, while G_{ab} is continuous on $\mathscr{S}$, then (1.15) is satisfied identically everywhere. On the other hand, the right hand side of (1.16) is zero, but computing its covariant derivative and evaluating on $\mathscr{S}$ yields the following result: the singular part of the Einstein tensor, $\mathscr{G}_{ab}$, is constant on $\mathscr{S}$. Notice that from (2.25) there is no surface pressure due to surface tension.

At $\mathscr{S}$ the only nonzero components of the distributional energy-momentum tensor are:

$$8\pi\sigma = \frac{1}{(36M_0)^{1/3}(t + T_0)^{2/3}}, \qquad 8\pi\Pi_1 = 8\pi\Pi_2 = -4\pi\sigma,$$

where σ is the distributional density, while Π_1 and Π_2 are the distributional pressures, with the equation of state given (as found by Bonnor) by $\sigma + \Pi_1 + \Pi_2 = 0$. As the units of the distributional and continuous (non–distributional) density are not the same we obtain the quasi-local mass from each density to obtain a quantity that can be compared. The energy momentum tensor is divided into a continuous (non-distributional) part and a distributional part as: $\overline{T}_{ab} = T_{ab} + \mathscr{T}_{ab}\delta(\mathscr{S})$, where δ is the Dirac delta function, and in our case $T_{ab} = \rho u_a u_b$. From the expression of the full energy-momentum tensor it is clear that it makes sense to compare the quantities $T_{ab}u^a u^b$ and $\mathscr{T}_{ab}u^a u^b \delta(\mathscr{S})$, which have the same energy density units, by means of integration over a domain that contains the hypersurface.

We integrate $\rho = T_{ab}u^a u^b$ in a domain $0 < r_1 < \pi/2 < r_2 < \pi$,

$$\mathscr{M}_\rho = 4\pi \int_{r_1}^{r_2} \rho R^2 R' dr = M_0[\sin^3 r_2 - \sin^3 r_1], \tag{3.3}$$

from this expression we obtain an upper and lower bound, $0 \le \mathscr{M}_\rho \le M_0$.

For the distributional matter at the thin shell we obtain the contribution of σ to the active gravitational mass as the integral of $\mathscr{T}_{ab}u^a u^b \delta(\mathscr{S})$,

$$\begin{aligned} \mathscr{M}_\sigma &= \int_{r_1}^{r_2} \sigma R^2|_{\mathscr{S}} \delta\left(r - \frac{\pi}{2}\right) \int d\Omega \, dr = \frac{1}{2}\int_{r_1}^{r_2} \frac{\left(\frac{9}{2}M_0\right)^{\frac{2}{3}} \, (t+T_0)^{\frac{4}{3}}}{6^{\frac{2}{3}} M_0^{\frac{1}{3}} (t+T_0)^{\frac{2}{3}}} \delta\left(r - \frac{\pi}{2}\right) \, dr \\ &= \frac{1}{2}\left(\frac{9M_0}{16}\right)^{1/3} (t+T_0)^{2/3}. \end{aligned}$$

Considering the arbitrary $0 \le r_1, r_2 \le \pi$ which give the upper bound for $\mathscr{M}_\rho$, and from the ratio of the latter and $\mathscr{M}_\sigma$ we obtain a comparison of the continuous mass and the contribution of the distributional density to the quasilocal mass

$$\xi(t) = \frac{\mathscr{M}_\rho}{\mathscr{M}_\sigma} = 2\left(\frac{4}{3}\right)^{\frac{2}{3}} \frac{M_0^{\frac{2}{3}}}{(t+T_0)^{\frac{2}{3}}}. \tag{3.4}$$

To obtain a numerical result we evaluate this ratio at present day cosmic time $t_0 \approx 13.7 \times 10^9$ years and use $M_0 = 3/2H_0^{-1}$, where H_0 is the Hubble constant

($\sim$ 70km/(sec Mpc)). We obtain for these values

$$\xi(t_0) \approx \frac{2^{\frac{2}{3}}2}{H_0^{\frac{2}{3}}(t_0+T_0)^{\frac{2}{3}}} \approx \frac{2^{\frac{2}{3}}2}{\left(70\frac{\mathrm{km/s}}{\mathrm{Mpc}}\right)(13.7\times 10^9+10^5)\mathrm{years}} \approx 3.2176, \quad (3.5)$$

while for ten times the current age of the universe we have

$$\xi(10\,t_0) \approx \frac{2^{\frac{5}{3}}}{\left(70\frac{\mathrm{km/s}}{\mathrm{Mpc}}\right)(13.7\times 10^{10}+10^5)\mathrm{years}} \approx 0.69. \quad (3.6)$$

Thus, for the asymptotic evolution range of large cosmic times the contribution to the quasi–local mass from the distributional surface density dominates the contribution from the continuous dust source. This behaviour is clearly unphysical, since the distributional source does not generate effective gravitational mass (from the quasi–local mass definition), yet it ends up overwhelmingly dominating over the quasilocal mass obtained from the continuous (and physical) dust density. In section 3.4 we further examine the physical implications of this model.

3.2 The case $K < 0$

We select the same free functions as in (3.1), together with $K(r) = -K_0\sin^2 r$. The only non-vanishing components of the extrinsic curvature are

$$K_{\theta\theta} = -\frac{\sqrt{1-K_0\sin^2 r}R|R'|}{R'}, \qquad K_{\phi\phi} = K_{\theta\theta}\sin^2\theta,$$

where

$$R(t,r) = \frac{M_0\sin r(\cosh\eta-1)}{K_0}, \qquad \eta-\sinh\eta = \frac{K_0^{\frac{3}{2}}(t+t_0\sin^2 r)}{M_0}.$$

Once again, $[\underline{G}^a{}_b] = 0$, and $K_{ab}{}^+ + K_{ab}{}^- = 0$, so (1.15) is satisfied. Taking the covariant derivative of $\mathscr{G}_{ab}$ on $\mathscr{S}$ leads to a zero vector and thus (1.16) is identically satisfied once again. At $\mathscr{S}$ the distributional density and pressures are

$$8\pi\sigma = 4\frac{\sqrt{1-K_0}}{R\,(t,r)\,|_{r=\frac{1}{2H_0}}}, \qquad 8\pi\Pi_1 = 8\pi\Pi_2 = -4\pi\sigma,$$

while the non-distributional density takes the form

$$8\pi\rho = \frac{3M_0 \sin^2 r \cos r}{4\pi R^2 R'}. \tag{3.7}$$

To obtain a comparison one would proceed as in the case $K = 0$ but taking into account the proper mass instead of the quasilocal mass, as in this case both masses are not equal. These comparison yields a similar result as in the case studied previously, which we considered to be unphysical.

3.3 Case $\Lambda > 0$

If $\Lambda > 0$, Einstein's field equations yield the same form for the density ρ given in (2.11), but the Friedman–like evolution equation is now:

$$\dot{R}^2 = \frac{Q(R)}{R}, \qquad Q(R) = 2M - KR + \lambda R^3 \tag{3.8}$$

where $\lambda = \frac{1}{3}\Lambda$. The kinematic evolution is governed by the zeroes of the cubic polynomial $Q(R)$ for different values of K. Ever expanding regions or models are characterized by configurations with those choices K and M for which Q has no zeros for a specific range of r. In particular, fully regular closed ever expanding models without thin layer distributional sources require configurations with $K > 0$ for which $Q(R)$ has no zeroes for all the range of r.

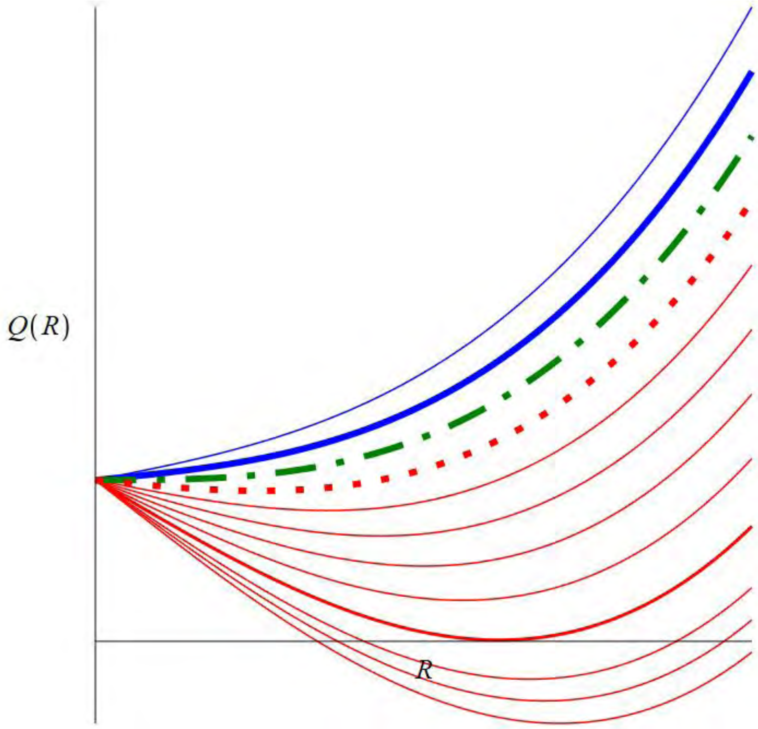


Figure 3.1: Plot of Q(R) in (3.8), see text for explanation.

To look at the sign of $Q(R)$ we plot this cubic polynomial for fixed positive values of M and λ and letting vary K for $R > 0$. As shown in Fig. 3.1, all curves above the lowest red thick curve (colors appear in the online version), which are configurations of a generic LTB dust solution, represent ever expanding universes. The dot-dash green curve represents spatially flat models, below this curve are models with $K > 0$, and above the dash-dot green curve there are negative spatial curvature models. In this case we can choose $K > 0$ so that the condition $K(r^*) = 1$ for $R'(t, r^*) = 0$ holds and thus, we have ever expanding models for which the regularity conditions for a PACM hold: the metric coefficient $\sqrt{g_{rr}} = \pm R'/\sqrt{1-K}$ is well defined at r^* and K^a_b is continuous, which eliminates the surface distributional source at $r = r^*$. This is an important result, since it proves that LTB models that approximate the Λ-CDM model can have rest frames with a closed topology.

3.4 Radial null geodesics at the interface

While the thin shell distributional source at the hypersurface $r = r^*$ in ever expanding closed LTB models does not generate effective mass, it is interesting to find out if the existence of such source could be detected observationally. To explore this question we need to find null geodesics that cross this hypersurface and compute the redshift from light emitted along these curves by distant observers in these models.

Photon trajectories (null geodesics) follow from the solutions of the geodesic equation,

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{\ bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0, \tag{3.9}$$

with the constraint $k^a k_a = 0$, for $k^a = dx^a/d\lambda$ is the tangent vector of these curves and λ is an affine parameter. We will consider only radial null geodesics $k^a = [k^t(\lambda), k^r(\lambda), 0, 0]$, where k^t and k^r are obtained from (3.9)

$$\frac{d^2t}{d\lambda^2} + \frac{\dot{R}'}{R'}\left(\frac{dt}{d\lambda}\right)^2 = 0, \tag{3.10}$$

$$\frac{d^2r}{d\lambda^2} + \left(\frac{R''}{R'} - \frac{K'}{2(1-K)}\right)\left(\frac{dr}{d\lambda}\right)^2 \pm \frac{2\dot{R}'}{\sqrt{1-K}}\frac{|R'|}{R'}\left(\frac{dr}{d\lambda}\right)^2 = 0, \tag{3.11}$$

subjected to the constraint $k_a k^a = 0$

$$-\left(\frac{dt}{d\lambda}\right)^2 + \frac{R'^2}{1-K}\left(\frac{dr}{d\lambda}\right)^2 = 0, \;\; \Rightarrow \;\; \frac{dt}{d\lambda} = \pm\frac{R'}{\sqrt{1-K}}\frac{dr}{d\lambda}. \tag{3.12}$$

The metric functions R, K and their derivatives in the coefficients follow from the closed ever expanding models we have examined in previous sections (with $\Lambda = 0$).

It is well-known that a non-degenerate C^{r+1} metric determines the C^r Levi-Civita connection. For $K = 0$ the metric is C^∞, for $K \neq 0$ in general it can only state that the metric is C^0. For convinience we will analyze the case

$K = 0$ in which the connection is C^r almost everywhere, i.e. it is C^r except on a set of measure zero, namely the symmetry centers and at the turning value of R'. Therefore there exists a convex normal neighborhood at each $p \in M$, i.e. an open set U with $p \in U$ such that for all $q, r \in U$ there exists a unique geodesic γ which stars at q and ends at r and is totally contained in U, see [10]. The connection is not C^r at the symmetry centers and at the hypersurface $r = r_*$, nevertheless the radial geodesic equation is C^r in all the space-time except at the hypersurface $r = r_*$. By the standard existence and uniqueness theorem for ODE's there exists a unique geodesic from a symmetry point to any point arbitrarily near the hypersurface, in comoving coordinates this guarantees the existence of a null geodesic that starts at $r = 0$ and ends at $r = r_* - \epsilon_1$ for any $\epsilon_1 > 0$ and a null geodesic with endpoints at $r = r_* + \epsilon_2$ and $r = r_c$ for all $\epsilon_2 > 0$.

In order to check if the geodesic equation is well defined at $r = \pi/2$, we consider the choice of functions of section 3.1, leading to:

$$\frac{d^2t}{d\lambda^2} + \frac{2}{3}\Phi(t,r)\left(\frac{dt}{d\lambda}\right)^2 = 0, \tag{3.13}$$

$$\frac{d^2r}{d\lambda^2} + \frac{\Psi(t,r) - \Omega(t,r)}{21\cos r\left(t_0\sin^2 r + \frac{3}{7}t\right)\left(t + t_0\sin^2 r\right)}\Psi(t,r)\left(\frac{dr}{d\lambda}\right)^2 = 0. \tag{3.14}$$

where

$$\Phi(t,r) = \frac{3t + t_0\sin^2 r}{\left(t + t_0\sin^2 r\right)\left(3t + 7t_0\sin^2 r\right)}, \tag{3.15}$$

$$\Psi(t,r) = \pm\frac{2(36M_0)^{\frac{1}{3}}}{3}\cos r(t_0\sin^2 r + 3t)\left|\frac{\cos r(3t + 7t_0\sin^2 r)}{(t + t_0\sin^2 r)^{\frac{1}{3}}}\right|, \tag{3.16}$$

$$\Omega(t,r) = 21\left(t_0^2\sin^4 r + \left(\frac{10}{7}tt_0 - \frac{4}{3}\cos^2 r\right)\sin^2 r + \frac{3}{7}t(t - 4t_0\cos^2 r)\right)\sin r, \tag{3.17}$$

and the plus minus sign in the square root from equation (3.12) will distinguish between "ingoing" past directed curves and "outgoing" future directed curves.

Since (3.14) is not well-defined near $r(\lambda) = \pi/2$, we introduce the change of variable: $t(\lambda) = 10^{w(\lambda)}$ and solve numerically the geodesic equations above for generic values of M_0 and t_0. In what follows we consider $M_0 = 10$ and $t_0 = 0.5$. The absolute value needs to be evaluated in a piecewise manner $|x| = x$ for $x > 0$ and $|x| = -x$ for $x < 0$ for any x. For generic initial conditions and working with both signs, each considered also in the geodesic equations (see (3.16)) we solve numerically (3.13) and (3.14) for several initial conditions, leading to the curves plotted in figure 3.2.

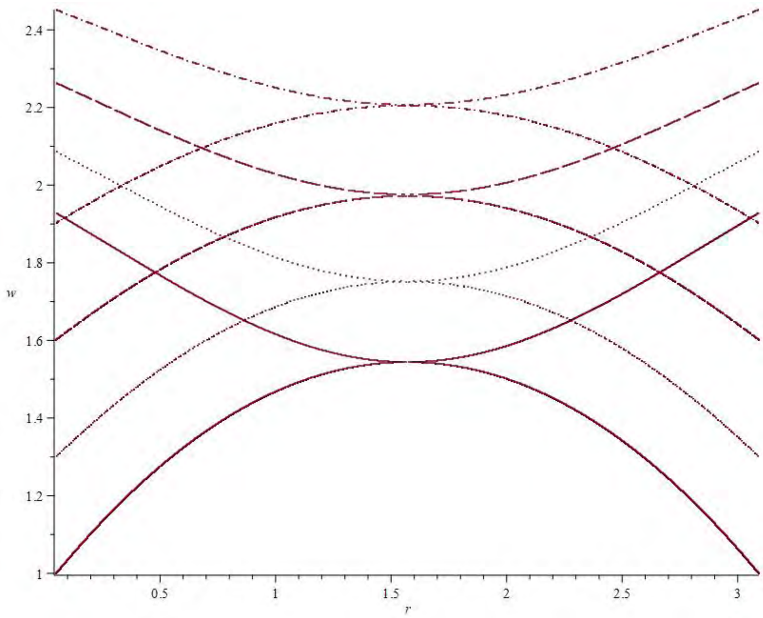
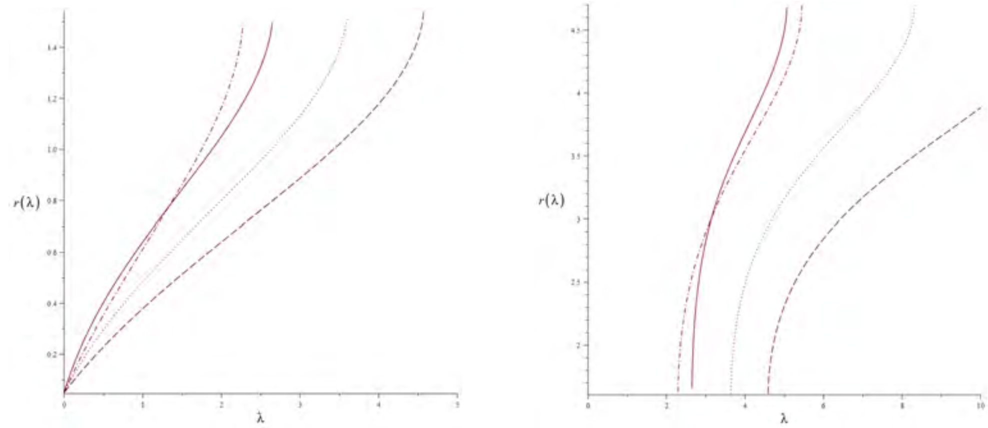


Figure 3.2: Plot to numerical solution of equations (3.13) and (3.14) for four different geodesics with generic initial conditions. As it can be noticed, the curves are smooth when considered as $w(r)$ as opposed to the discontinuity that is examined when w and r are considered as functions of the parameter λ.

The numerical solution for $r \in [0, \pi/2)$ shows that near $\pi/2$ the derivative $dr/d\lambda$ does not tend to zero. The graphs for $r(\lambda)$ and $t(\lambda)$ for some of the geodesics obtained are shown in figure 3.3. It can be seen from the solutions that (3.12) restricts the solutions for $t(\lambda)$ and $r(\lambda)$ to be such that the



(a) Plot for $r(\lambda) \in [0, \pi/2)$ (b) Plot for $r(\lambda) \in (\pi/2, \pi]$

Figure 3.3: Plot to numerical solutions for $r(\lambda)$. The plot on the left is the plot for the values of $r \in [0, \pi/2)$, while the second graph represents the values $r \in (\pi/2, \pi]$. From the plots it can be seen that the values of the derivative $dr/d\lambda$ diverge, which has implications on $dt/d\lambda$, see fig. 3.4.

product $R' dr/d\lambda$ be finite. In this cases the product is not zero which implies that $dr/d\lambda$ must diverge. Also, equation (3.12) reveals that solutions that are not C^1 can be obtained, as arbitrary initial conditions can be chosen over r as a function of λ to obtain a C^0 curve, defining $r(\pi/2) = \lim_{r\to\pi/2^+} r(\lambda) = \lim_{r\to\pi/2^-} r(\lambda)$ that satisfies (3.13) and (3.14) for $r \in [0, \pi/2) \cup (\pi/2, \pi]$. Some of these solutions are shown in figure 3.2. Therefore, there exists a jump in the first derivative of the curve which could be used to probe the existence of thin shells.

Although there is a discontinuity in the first derivative of the coordinates of the geodesics, each value of $r \in [0, \pi/2) \cup (\pi/2, \pi]$ is reached in a finite value of the affine parameter.

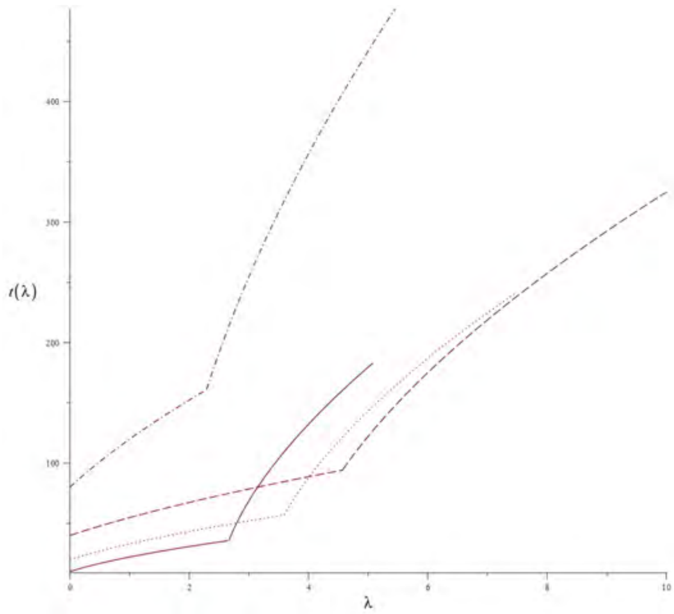


Figure 3.4: Plot to numerical solutions for t, from the plots it can be seen that the values of the derivative of the t curve are always finite due to the divergence of $dr/d\lambda$, see fig. 3.3, as required by equation (3.12).

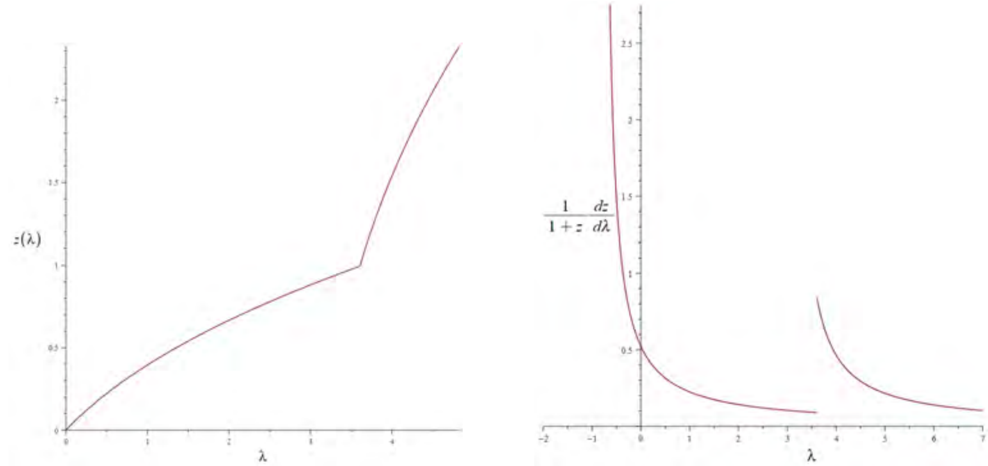
3.5 Redshift

The redshift for a $K = 0$ model is calculated through the following integral, c.f. (A.12),

$$\ln(1 + z(r(\lambda))) = \int_0^\lambda \dot{R}'(t(\lambda), r(\lambda)) \frac{dr}{d\lambda} d\lambda. \tag{3.18}$$

Note that as $dr/d\lambda$ is discontinuous at $r = \pi/2$, the integrand is not continuous but the integral is. Figures 3.5 and 3.6 represent the redshift and the plot for $1/(1 + z)dz/d\lambda$ for two different geodesics.

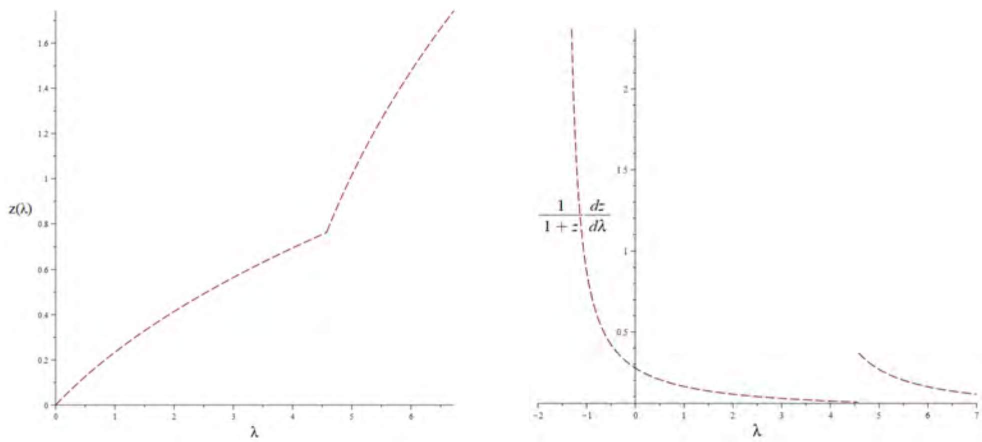
As there is a discontinuity in the derivative of the redshift it is possible probe the existence of a thin shell by measuring the redshift of radial photons. Nevertheless notice that the magnitude of the discontinuity depends on the parametrization chosen.



(a) Plot to numerical solution for $z(\lambda)$

(b) Plot to numerical solution for $\frac{1}{1+z}\frac{dz}{d\lambda}$

Figure 3.5: Plot to numerical solutions for first geodesic, where the disconinuity of the redshift can be apreciated. See text for details.



(a) Plot to numerical solution for $z(\lambda)$

(b) Plot to numerical solution for $\frac{1}{1+z}\frac{dz}{d\lambda}$

Figure 3.6: Plot to numerical solutions for second geodesic, where the disconinuity of the redshift can be apreciated. See text for details.

3.6 A model with $K > 0$

We now analyze the case $K > 0$ and show that in this case observers at the turning value would not detect any thin layers. Analyzing a model with positive K is easier with a change of variables in the metric, where we obtain a FLRW-like metric with line element

$$ds^2 = -dt^2 + a^2 \left[\frac{\Gamma^2 R_i'^2}{1 - k_{qi} R_i^2} dr^2 + R_i^2 d\Omega^2 \right], \tag{3.19}$$

where $a(t, r) \equiv R/R_i$, $R_i \equiv R(t_i, r)$ and $t = t_i$ determines a fiducial initial hypersurface. Henceforth, all quantities evaluated at $t = t_i$ will be denoted by a the subindex i. The dimensionless metric function Γ is

$$\Gamma \equiv \frac{R'/R}{R_i'/R_i} = 1 + \frac{a'/a}{R_i'/R_i},$$

where $\Gamma_i = 1$, while $k_{qi} = K/R_i^2$. Note that the regularity condition on this metric is $R' = 0$ which implies $k_{qi} R_i^2 = 1$.

We now consider the functions a, Γ and R_i taking into account a closed model. We have, c.f. [21], that along turning values regularity conditions on the density, ρ, density at the fiducial time, ρ_i, Ricci scalar of the hypersurfaces at the fiducial time ${}^{(3)}\mathscr{R}_i$ and the metric imply that R_i', M' and $(k_i R_i)'$ must have common zeros along the turning values of the same order in $r - \pi/2$. The function Γ must not have a zero due to the fact that $\Gamma = 0$ and $\rho_i > 0$ imply a shell-crossing singularity. So, with this considerations

$$R' = a' R_i + \frac{R R_i'}{R_i}, \qquad K' = k_{qi}' R_i^2 + 2 k_{qi} R_i R_i'.$$

Evaluating both equations along $\mathscr{S}$ it is obtained that along the hypersurface $a' = k_i' = 0$.

The null geodesic constraint is

$$\frac{dt}{d\lambda} = \pm a \frac{\Gamma R_i'}{\sqrt{1 - k_{qi} R_i^2}} \frac{dr}{d\lambda}. \tag{3.20}$$

The radial null geodesic equations are

$$\frac{d^2t}{d\lambda^2} + \frac{a^2\dot{\Gamma} + a\dot{a}\Gamma}{2\Gamma^2}\left(\frac{dt}{d\lambda}\right)^2 = 0, \qquad \frac{d^2r}{d\lambda^2} + A(B+D)\left(\frac{dr}{d\lambda}\right)^2 = 0,$$

where

$$\begin{aligned}
A &= \frac{1}{2aR_i'^2\Gamma(-1+k_{qi}R_i^2)},\\
B &= 2aR_i'\left(C\Gamma + \left(\pm\frac{\Gamma R_i'^2\dot{\Gamma}}{\sqrt{1-k_{qi}R_i^2}} + \frac{1}{2}R_i'\Gamma'\right)(-1+k_{qi}R_i^2)\right),\\
C &= -\frac{1}{2}k_{qi}'R_i'R_i^2 - R_i'^2k_{qi}R_i + R_i''k_{qi}R_i^2 - R_i',\\
D &= \left(2a'R_i'^2\Gamma \pm \frac{4\dot{a}R_i'^3\Gamma^2}{\sqrt{1-k_{qi}R_i^2}}\right)(-1+k_{qi}R_i^2).
\end{aligned}$$

We check whether the geodesic equation in this variables is well defined. We have that

$$\lim_{r\to\frac{\pi}{2}}\Gamma = \lim_{r\to\frac{\pi}{2}}\frac{R'}{R_i'}\frac{R_i}{R} = \left(\lim_{r\to\frac{\pi}{2}}\frac{R'}{R_i'}\right)\frac{R_i}{R}\bigg|_{r=\frac{\pi}{2}}$$

due to the fact that $R_i' = dR_i/dr$ is the derivative of a radial profile at a given time, and the partial derivative $R' = \partial R/\partial r$ is taken as a limit at a constant time, so the limit in the last parenthesis must be a finite function of time. We now check the product AB

$$AB = -\frac{C}{R_i'(1-k_{qi})} \pm \frac{R_i'\dot{\Gamma}}{\sqrt{1-k_{qi}R_i^2}} + \frac{1}{2}\frac{\Gamma'}{\Gamma} \tag{3.21}$$

The limit of the second term is

$$\lim_{r\to\frac{\pi}{2}}\frac{R_i'\dot{\Gamma}}{\sqrt{1-k_{qi}R_i^2}} = \left(\lim_{r\to\frac{\pi}{2}}\dot{\Gamma}\right)\left(\lim_{r\to\frac{\pi}{2}}\frac{R_i'}{\sqrt{1-k_{qi}R_i^2}}\right).$$

The second limit of the right hand side is finite by regularity conditions, while the first

$$\dot{\Gamma} = \left(\frac{\dot{R}'R + R'\dot{R}}{R^2}\right)\frac{R_i}{R_i'} = R_i\left(\frac{\dot{R}'R + R'\dot{R}}{R_i'R^2}\right)$$

has a finite limit at $r = \frac{\pi}{2}$ as long as the limit $\dot{R}'/R_i'$ exists. We now analyze the first term of the product AB,

$$\frac{C}{R_i'(1 - k_{qi}R_i^2)} = \frac{k_{qi}'R_i^2 + 2R_i'k_{qi}R_i}{2 - 2k_{qi}R_i^2} - \frac{R_i''}{R_i'}.$$

The first limit term of the last equality is nonexistent as $1 - k_{qi}R_i^2$ has a zero of the same order as $R_i'^2$ and $k_{qi}'^2$. Even though the limit does not exist, this is consistent with the non-defined geodesic equation at $\mathscr{S}$ in the comoving variables.

We now check the product AD,

$$AD = \frac{a'}{a} \pm \frac{2\dot{a}R_i'\Gamma}{a\sqrt{1 - k_{qi}R_i^2}}.$$

The limit of the first term is zero from the definition of a and as R' and R_i' are continuous and zero at $\mathscr{S}$. The second term is constant by previous calculations.

Regularity conditions for a closed model require that $K = 1$ at $\mathscr{S}$, where $R' = 0$ and that the following limit be finite and not null,

$$\lim_{\mathbf{x} \to \mathscr{S}} \frac{R'^2}{1 - K}, \tag{3.22}$$

where $\mathbf{x}$ denotes a generic point in the manifold. It is straightforward to prove that the metric component g_{rr} is continuous but does not have a continuous partial derivative g_{rr}' which immediately implies that the connection will not be C^1 in a set of measure zero, $\mathscr{S}$. On the contrary, in the case $K = 0$ the connection is not C^1 due to the fact that the metric is degenerate at $\mathscr{S}$, as opposed to the model with $K > 0$ which is not degenerate by regularity conditions.

Nevertheless, if a solution to the geodesic equation where to exist the derivative of the radial and temporal coordinates should be continuous as they must satisfy the null geodesic constraint (3.20), which relates both derivatives by the following relation

$$\frac{dt}{d\lambda} = \frac{\pm R'}{\sqrt{1-K}}\frac{dr}{d\lambda}. \tag{3.23}$$

Both derivatives are related by a function which is continuous due to the regularity conditions, where it is used that the square root is a continuous function so the passage to the limit under the square root can be taken, and by hypothesis was assumed to be invertible, which completely determines both coordinates, unlike the case $K = 0$ which gives an infinite number of choices of the derivative of the radial coordinate. Therefore LTB models with $K > 0$ present no issues in geodesics and as there are no surface layers, and by the analysis of equation (A.4) there is no effect on the redshift nor on the derivatives of the coordinates.

Chapter

4

Conclusions

We have examined the dynamics and geometric properties of ever expanding "closed" LTB dust models, where by "closed" we mean models whose rest frames (hypersurfaces orthogonal to the 4–velocity marked by constant time) are diffeomorphic to the standard 3–sphere $\mathbf{S^3}$. We considered both cases, with $\Lambda = 0$ and $\Lambda > 0$. Since observations do not rule out a small positive curvature, the case $\Lambda > 0$ can be thought of as a toy model inhomogeneous generalisation of the ΛCDM model.

Ever expanding closed LTB models with $\Lambda = 0$ where examined long time ago by Bonnor [8], who showed that fulfillment of regularity conditions require these models to admit a thin surface layer at the equator of the 3–sphere ("turning value" of the area radius), which must be examined by means of the Israel–Lanczos thin shell formalism. Bonnor found the equation of state state satisfied by this distributional source, which he regarded as unphysical because it does not contribute to the effective quasi–local mass and because of the negative surface pressure (this was before negative pressures were acceptable in connection with dark energy).

In the present thesis we extended Bonnor's work by looking at the time evolution of the distributional source, in comparison with the time evolution of the continuous dust source. We also show that assuming $\Lambda > 0$ allows for perfectly regular closed LTB models, an option not contemplated by Bonnor. By looking first at the spatially flat case $K = 0 = \Lambda$, we found that the distributional density (which does not contribute to the effective mass) dominates the continuous density in the asymptotic time range, which is an unphysical effect. This same effect occurs for the negatively curved case $(\Lambda = 0,\ K < 0)$.

Furthermore, we raised the issue of whether the presence of this unphysical distributional source could be detected by observations based on light rays crossing the timelike hypersurface made by the time evolution of the 3–sphere equator. By looking at radial null geodesics in the case $K = 0 = \Lambda$ and placing the observer at the symmetry center $r = 0$, we showed that the presence of the distributional source causes a discontinuous radial derivative of redshifts from observers beyond the equatorial hypersurface of $\mathbf{S^3}$. Hence, we proved that this type of distributional source would be detectable by observations, even if it does not contribute to the effective quasi–local mass. Finally, and for the purpose of comparison, we showed that this discontinuity of the redshifts does not occur in re-collapsing closed LTB models (for which there is no distributional source at the 3–sphere equator).

Appendix

A

Appendix

A.1 Distributional derivative of terms containing δ

We calculate the derivative of a distribution of the type $\boldsymbol{T}\delta$ where $\boldsymbol{T} \in \mathscr{D}(\mathscr{M})$ is defined at least on Σ. We use the rule for derivating distributions

$$
\begin{aligned}
< \nabla_\mu (T_{\alpha_1 \cdots \alpha_p} \delta), V^{\mu\alpha_1 \cdots \alpha_p} > &= - < T_{\alpha_1 \cdots \alpha_p} \delta, \nabla_\mu Y^{\mu\alpha_1 \cdots \alpha_p} > \\
&= - \int_\Sigma T_{\alpha_1 \cdots \alpha_p} \nabla_\mu Y^{\mu\alpha_1 \cdots \alpha_p} d\sigma \\
&= - \int_\Sigma T_{\alpha_1 \cdots \alpha_p} (n_\mu n^\rho + h_\mu{}^\rho) \nabla_\rho Y^{\mu\alpha_1 \cdots \alpha_p} d\sigma,
\end{aligned}
$$

where, in terms of inner products, the first term is

$$
- < T_{\alpha_1 \cdots \alpha_p} n_\mu n^\rho \delta, \nabla_\rho Y^{\mu\alpha_1 \cdots \alpha_p} > = < \nabla_\rho (T_{\alpha_1 \cdots \alpha_p} n_\mu n^\rho \delta), Y^{\mu\alpha_1 \cdots \alpha_p} > . \tag{A.1}
$$

The second term is expanded as

$$
\begin{aligned}
- \int_\Sigma T_{\alpha_1 \cdots \alpha_p} h_\mu{}^\rho \nabla_\rho Y^{\mu\alpha_1 \cdots \alpha_p} d\sigma = &- \int_\Sigma h_\mu{}^\rho \nabla_\rho (T_{\alpha_1 \cdots \alpha_p} Y^{\mu\alpha_1 \cdots \alpha_p}) d\sigma \\
&+ \int_\Sigma Y^{\mu\alpha_1 \cdots \alpha_p} h_\mu{}^\rho \nabla_\rho T_{\alpha_1 \cdots \alpha_p} d\sigma.
\end{aligned}
$$

Using that

$$h^{\gamma}{}_{\beta} h^{\alpha}{}_{\delta} h^{\rho}{}_{\sigma} \nabla_{\rho} S^{\beta}{}_{\alpha} = \overline{\nabla}_{\sigma}(\overline{S^{\gamma}{}_{\delta}}) + (h^{\beta}{}_{\delta} S^{\rho}{}_{\beta} n_{\rho}) K^{\Sigma\gamma}{}_{\sigma} + (h^{\gamma}{}_{\alpha} S^{\alpha}{}_{\rho} n^{\rho}) K^{\Sigma}{}_{\sigma\delta}, \quad \text{(A.2)}$$

where $\overline{S}^{\alpha\beta} = h_{\gamma}{}^{\alpha} h_{\delta}{}^{\beta} S^{\gamma\delta}$ and $K^{\Sigma}{}_{\alpha\beta} = 1/2(K^{+}{}_{\alpha\beta} + K^{-}{}_{\alpha\beta})$, one arrives to the following expression

$$\begin{aligned} -\int_{\Sigma} T_{\alpha_1\cdots\alpha_p} h_{\mu}{}^{\rho} \nabla_{\rho} Y^{\mu\alpha_1\cdots\alpha_p} d\sigma = &-\int_{\Sigma} \overline{\nabla}_{\mu}(\overline{T_{\alpha_1\cdots\alpha_p} Y^{\mu\alpha_1\cdots\alpha_p}}) d\sigma \\ &-\int_{\Sigma} K^{\Sigma\rho}{}_{\rho} n_{\mu} T_{\alpha_1\cdots\alpha_p} Y^{\mu\alpha_1\cdots\alpha_p} \\ &+\int_{\Sigma} Y^{\mu\alpha_1\cdots\alpha_p} h_{\mu}{}^{\rho} \nabla_{\rho} T_{\alpha_1\cdots\alpha_p} d\sigma. \end{aligned}$$

As $Y \in \mathscr{D}(\mathscr{M})$ the first integral is zero, therefore

$$-\int_{\Sigma} T_{\alpha_1\cdots\alpha_p} h_{\mu}{}^{\rho} \nabla_{\rho} Y^{\mu\alpha_1\cdots\alpha_p} d\sigma = < (h^{\rho}{}_{\mu} \nabla_{\mu} T_{\alpha_1\cdots\alpha_p} - K^{\Sigma\rho}{}_{\rho} n_{\mu} T_{\alpha_1\cdots\alpha_p})\delta, Y^{\mu\alpha_1\cdots\alpha_p} > .$$

Consequently

$$\nabla_{\mu}(T_{\alpha_1\cdots\alpha_p}\delta) = \nabla_{\rho}(T_{\alpha_1\cdots\alpha_p} n_{\mu} n^{\rho}\delta) + (h^{\rho}{}_{\mu} \nabla_{\mu} T_{\alpha_1\cdots\alpha_p} - K^{\rho}{}_{\rho} n_{\mu} T_{\alpha_1\cdots\alpha_p})\delta. \quad \text{(A.3)}$$

A.2 Calculation of limits

A.2.1 $K > 0$

From (2.17) we have the following relations

$$\eta = \arccos(1-\alpha R), \quad \sin\eta = \sqrt{1-\cos\eta} = \sqrt{\alpha R}\sqrt{2-\alpha R},$$

$$t - t_{bb} = \frac{1}{\beta}\left\{\arccos(1-\alpha R) - \sqrt{\alpha R}\sqrt{2-\alpha R}\right\},$$

where $\alpha = \frac{K}{M}$ and $\beta = \frac{K^{\frac{3}{2}}}{M} = \alpha K^{\frac{1}{2}}$.

Derivating respect to r and isolating R' we obtain an expression which involves gradients which vanish at $\mathscr{S}$, so R' vanishes also along the hypersurface.

Derivating once again and substituing

$$R'' = \frac{1}{4}\frac{15M * D(F * B + G) + H + I + J}{K^4RM(KR - 2M)}$$

where

$$\begin{aligned}
F &= \left(K'^2M - \frac{4}{5}K'M'K - \frac{2}{5}K''KM + \frac{4}{15}M''K^2\right)(KR - 2M),\\
G &= \frac{4}{15}t''_{bb}\left(2MK^{\frac{7}{2}} - RK^{\frac{9}{2}}\right),\\
H &= 2K^4MR^3K'' + 8K^4MR^2M'' - 16K^3M^2R^2K'' - 16K^3M^2RM''\\
&\quad +24K^2M^3RK'',\\
I &= 4K^4MR^2K'R' - 3K^3MR^3K'^2 + 4K^4M^2R'^2 - 8K^4MRM'R'\\
&\quad +4K^4R^2M'^2,\\
J &= 48K^2M^2RK'M' - 60KM^3RK'^2 - 32K^3MR^2K'M' + 40K^2M^2R^2K'^2.
\end{aligned}$$

Note that at $\mathscr{S}$, I and J vanish. As not all functions vanish at $\mathscr{S}$, R'' is not necessarily of the form $0/0$. In general, in radial profiles $KR \neq 2M$ so R'' is finite. From our choice of free functions H doesn't vanish at $\mathscr{S}$.

We now analyze the term $K'/(2-2K)$, as the numerator and denominator are zero at the hypersurface, using the choice of free functions previously used we obtain that there is no limit at $\mathscr{S}$, so the term is singular. In the general case, L'Hôpital's rule gives

$$\lim_{r\to\frac{\pi}{2}} \frac{K'}{2-2K} = \lim_{r\to\frac{\pi}{2}} \frac{K''}{K'} \tag{A.4}$$

which necessarily gives a $0/0$ form, $\infty/0$ form or no limit as K' is 0 at the hypersurface.

The term

$$\frac{\dot{R}'}{\sqrt{1-K}} = \frac{1}{2}\frac{\frac{2M'}{R} - \frac{2MR'}{R^2} - K'}{\sqrt{1-K}\sqrt{\frac{2M}{R} - K}}$$

clearly is of the form $0/0$ at $\mathscr{S}$. As R', M', K' and $\sqrt{1-K}$ have zeros of the same order, this limit is well defined.

A.3 Redshift

We include the derivation from [22]. Let there be two light rays emitted the same direction by an observer $\mathscr{O}_1$, where the second light ray is emitted a time-interval later τ. We denote the equation of the first ray as

$$t = T(r). \tag{A.5}$$

We denote the second ray as $t = T(r) + \tau(r)$. As both rays must obey

$$\frac{dt}{dr} = \frac{R'}{\sqrt{1-K}} \tag{A.6}$$

we have

$$\frac{dT}{dr} = \frac{dt}{dr} = \frac{R_{,r}(T(r), r)}{\sqrt{1-K(r)}}, \quad \frac{d(T+\tau)}{dr} = \frac{R_{,r}(T(r)+\tau(r), r)}{\sqrt{1-K(r)}}. \tag{A.7}$$

To first order in τ and as τ was assumed to be small,

$$R_{,r}(T(r)+\tau(r), r) = R_{,r}(T(r), r) + \tau(r) R_{,tr}(T(r), r). \tag{A.8}$$

Using (A.8) and (A.7) we obtain

$$\frac{d\tau}{dr} = \tau(r) \frac{R_{,tr}(T(r), r)}{\sqrt{1-K(r)}}. \tag{A.9}$$

The redshift is defined by

$$\frac{\tau(r_{obs})}{\tau(r_{em})} = 1 + z(r_{em}). \tag{A.10}$$

Considering a fixed observer and the sources separated an infinitesimal distance, one obtains by differentiating $(d\tau/dr)/\tau = -(dz/dr)/(1+z)$. So, one obtains

$$\frac{1}{1+z}\frac{dz}{dr} = -\frac{R_{,tr}(T(r), r)}{\sqrt{1-K(r)}}. \tag{A.11}$$

Integrating one obtains

$$\ln(1 + z(r)) = \int_{r_{em}}^{r_{obs}} \frac{R_{,tr}(T(r), r)}{\sqrt{1 - K(r)}}\, dr. \tag{A.12}$$

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