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DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

THE MCKAY CORRESPONDENCE AND THE η–INVARIANT

TESIS

QUE PARA OPTAR POR EL GRADO DE
MAESTRO EN CIENCIAS

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Contents

Resumen

El presente trabajo puede dividirse en dos partes principales. En la primera vamos a desarrollar las herramientas matemáticas necesarias para establecer la correspondencia de McKay, la cual fue observada por primera vez por John McKay en [23]. Para ello es necesario introducir las singularidades de tipo ADE (que son singularidades aisladas de superficies complejas), dar su resolución minimal y la gráfica dual asociada a dicha resolución.

Luego veremos un resultado clásico demostrado por Klein en [20], el cual nos dice que las singularidades de tipo ADE son homeomorfas a espacios cociente de la forma $\mathbb{C}^2/\Gamma$, donde Γ es un subgrupo finito de $SL(2, \mathbb{C})$.

Posteriormente estudiaremos la teoría de representaciones de los subgrupos Γ, lo cual nos permitirá definir la gráfica de McKay de cada uno de ellos.

Hecho esto haremos la observación de McKay, la cual nos dice que la gráfica dual de la resolución minimal de $\mathbb{C}^2/\Gamma$ coincide con la gráfica de McKay asociada a la teoría de representaciones de Γ.

Terminamos la primera parte del trabajo mencionando que la aureola de las singularidades $\mathbb{C}^2/\Gamma$ son difeomorfas a espacios de órbitas de la forma S^3/Γ, con S^3 la 3-esfera.

En lo que resta de la tesis vamos desarrollar la teoría necesaria para calcular el invariante η del operador de Dirac de las aureolas S^3/Γ, torcido por una representación. Para ello debemos estudiar geometría spin, la cual nos va a permitir definir explícitamente el operador de Dirac y calcularlo para S^3 y S^3/Γ. Luego consideramos una representación irreducible del grupo fundamental $\pi_1(S^3/\Gamma)$ y la usamos para "torcer" el operador de Dirac de S^3/Γ.

A continuación definimos el invariante espectral η. Para calcularlo vamos a utilizar los eigenvalores del operador de Dirac torcido, así como sus multiplicidades.

Concluímos este trabajo con el cálculo del invariante η y plánteándonos la pregunta de si este invariante tiene alguna interpretación en el contexto de la resolución minimal de las singularidades de tipo ADE.

Introduction

There is an almost philosophical question that has bothered mathematicians (and other interested people) for a long time: are mathematics invented or discovered?

I personally believe that mathematics are a human construct, however every once in a while one can encounter some intriguing coincidences between different areas that make us understand why there is still no consensus on that matter.

The present work deals with one of such cases and ends up asking a (possibly) interesting question. Given the extension of the material reviewed, there will be many results cited without proof.

The thesis is divided in two parts. The first one contains three chapters which have the following outline:

The first chapter of this thesis requires an algebraic geometry background. We will introduce a special kind of isolated singularities of complex surfaces which were studied by Du Val [9], Klein [20], Artin [1] and many others. We will refer to them as *ADE singularities* and present them by its defining equations, although they receive various names and have several characterizations (see for example [10]).

Heisuke Hironaka proved in [14] that every algebraic variety over a field of characteristic zero admits a resolution or desingularization. In general a resolution is not unique, but we will see that for singularities of complex surfaces there is, up to isomorphism, a unique *minimal resolution.*

In order to find the minimal resolutions of our singularities, we will describe the fundamental transformation of *blow up* and learn how to use it repeatedly to find the desired results. As an example, we compute explicitly the minimal resolution of one of them, called E_8.

It is possible to encode the information of the minimal resolution on a graph called the *dual graph of the resolution.* In the case of ADE singularities the graphs obtained are known Dynkin diagrams of type ADE.

In Chapter 2 we give a result of Klein proved in [20], which characterizes the ADE singularities as *quotient singularities.* This means that they are homeomorphic to a quotient space of the form

$\mathbb{C}^2/\Gamma$ with Γ a finite subgroup of $SL(2,\mathbb{C})$. To do so, it is necessary to find the finite subgroups of $SL(2,\mathbb{C})$.

It happens that every finite subgroup of $SL(2,\mathbb{C})$ is conjugated to one of $SU(2)$ and at the same time, those of $SU(2)$ my be found using the classification of the finite subgroups of $SO(3)$ (the group of rotations of $\mathbb{R}^3$). This groups are the cyclic groups or order $n \geq 2$, the groups of symmetries of regular n-gons and the rotation groups of the platonic solids.

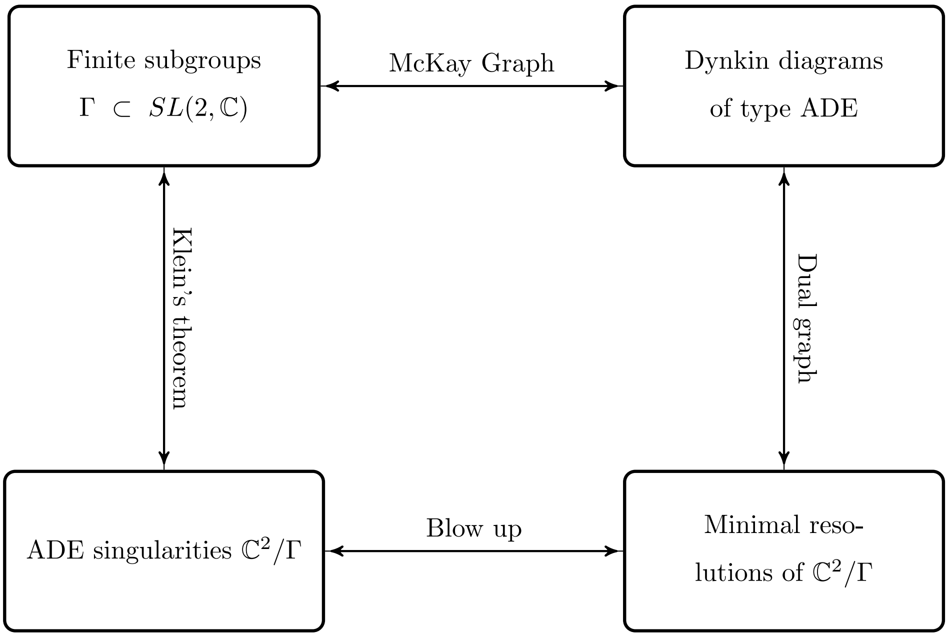
The subgroups Γ, being matrices groups, act naturally on the space of complex polynomials on two variables $\mathbb{C}[z_1, z_2]$ and the Γ-invariant polynomials are generated by three homogeneous invariant polynomials f_1, f_2, f_3. Using these polynomials we give explicitly a map $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ which induces an homeomorphism between $\mathbb{C}^2/\Gamma$ and a singular variety ADE.

Then we will see that the *link* of a quotient singularity $\mathbb{C}^2/\Gamma$ is diffeomorphic to the orbit space S^3/Γ, with S^3 the 3-sphere.

Chapter 3 has the objective of studying the representation theory of the finite subgroups Γ. This analysis allow us to define a graph known as the *McKay graph*, which is constructed taking a vertex for each complex *irreducible representation* and drawing a line between two vertices if some condition regarding *characters* is satisfied.

Here happens an astonishing thing: for the case where Γ is a finite subgroup of $SU(2)$, this graph is exactly the same as the dual graph of the resolution we found in Chapter 1 for the ADE singularity $\mathbb{C}^2/\Gamma$. This interesting coincidence was first noticed by John McKay in [23], and that is why it is known as the *McKay correspondence*.

In summary, we have the scheme:



Here starts the second part of the thesis, containing two chapters that will focus on the study of the links of the ADE singularities from a differential geometry view point.

We have stated that such links are spaces of the form S^3/Γ. In chapter 4 we will see that they are *spin manifolds* and therefore one can define its *Dirac operator*, which is a differential, self-adjoint elliptic operator denoted by D.

We compute the Dirac operator of the sphere $D(S^3)$. Then we use it to obtain the Dirac operator of quotients spaces $D(S^3/\Gamma)$.

Since S^3 is the universal covering of S^3/Γ, the fundamental group $\pi_1(S^3/\Gamma)$ is just Γ.

Considering a representation of $\pi_1(S^3/\Gamma) = \Gamma$ we define the "twisted Dirac operator" and compute it for the spaces S^3/Γ, together with their eigenvalues and multiplicities.

In Chapter 5 we introduce the η and ξ*-invariants*, which are spectral invariants defined for self adjoint elliptic operators as D. We want to compute them for the twisted Dirac operator. In order to do so we need to use the eigenvalues of the operator, together with their multiplicities. We finish this work with the computation of the invariants for the twisted case.

Of course, it is not very clear how the results from the second part are related to those of the first. We naturally wonder if the computed invariants have an interpretation in the context of minimal resolutions. This question remains open for future research.

Part I

The McKay correspondence

Chapter 1

Resolutions of ADE singularities and the dual graphs

In this chapter, we present a special kind of isolated singularities of complex surfaces, called ADE singularities.

Then we introduce the concept of resolution of singularities and review the technique of blowing up singular points to get what is called a minimal resolution.

As an example, we give the explicit calculations for resolving the singularity of type E_8.

After finding the minimal resolution of a singular variety of type ADE, we end up with a series of curves, called the components of the exceptional divisor of the blow up, intersecting transversally at each point they meet. The arrangement of those curves can be represented in a graph, called the dual graph of the resolution.

At the end of the chapter we will give the dual graphs of the resolutions of all the ADE singularities.

1.1 ADE singularities

The *affine space* of dimension n over $\mathbb{C}$ is the set

$$\mathbb{A}^n := \{(a_1, \cdots, a_n) \,|\, a_i \in \mathbb{C}\}.$$

If S is any subset of the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ we define the zero set of S

$$\mathcal{V}(S) := \{P = (a_1, \cdots, a_n) \in \mathbb{A}^n | f(P) = 0, \text{ for all } f \in S\}.$$

We observe that if $I =< S >$ is the ideal generated by S, then $\mathcal{V}(I) = \mathcal{V}(S)$.

The Hilbert's basis theorem says that if $\mathbb{K}$ is a Noetherian ring then the polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$ is also Noetherian ([2, Theorem 5.18, p.469]), that is, any ideal has a finite set of generators. Thus $\mathcal{V}(I)$ can be expressed as the common zeros of this set of generators.

Definition 1. A subset Y of $\mathbb{A}^n$ is an *algebraic set* if there exist a subset $X \subset \mathbb{C}[x_1, \cdots, x_n]$ such that $Y = \mathcal{V}(X)$.

Proposition 2. The union of two algebraic sets, the intersection of any family of algebraic sets, the empty set and the whole affine space are algebraic sets.

Proof. The proof can be found in [11, Proposition 1.1]. □

Remark 3. The previous proposition shows that the algebraic sets, considered as closed sets, define a topology on $\mathbb{A}^n$, called the *Zariski topology.*

Definition 4. A subset $Z \neq \emptyset$ of a topological space is said to be *irreducible* if cannot be expressed as the union of two closed proper subsets.

Definition 5. An irreducible algebraic set $\mathcal{V} \subset \mathbb{A}^n$ is called an *algebraic variety.*

In the following we will be dealing with varieties defined by only one polynomial $h : \mathbb{C}^n \longrightarrow \mathbb{C}$. Such varieties are known as *hypersurfaces* and its gradient $\nabla(h)$ has rank either 1 or 0.

Definition 6. A point $p \in \mathcal{V}$ is called *regular* if

$$\operatorname{rank}\Big(\frac{\partial h}{\partial x_1}(p), \cdots, \frac{\partial h}{\partial x_n}(p)\Big) = 1.$$

Otherwise, the point is *singular.*

A variety with singular points is called singular and a variety without singular points is called smooth.

Remark 7. The set of singular points, denoted by $Sing(\mathcal{V})$ is an algebraic subset of $\mathcal{V}$ (this is shown in [24, Lemma 2.2 p.10]).

Theorem 8. The set $\mathcal{V} \setminus Sing(\mathcal{V})$ of regular points of $\mathcal{V}$ is a non empty smooth manifold. In fact it is complex analytic of dimension $n-1$.

Proof. See [24, Theorem 2.3 p.10]. □

Definition 9. The *ADE singularities* are the isolated singularities of the following complex surfaces:

- $A_n : x^2 + y^2 + z^n = 0$, $(n > 1)$

- $D_n : x^2 + y^2 z + z^{n-1} = 0$, $(n \geq 4)$
- $E_6 : x^2 + y^3 + z^4 = 0$
- $E_7 : x^2 + y^3 + yz^3 = 0$
- $E_8 : x^2 + y^3 + z^5 = 0.$

The ADE singularities are also known as *rational double point singularities*, *Kleinian singularities* or *Du Val singularities.* There are several characterizations for them, see for example [10].

1.2 Resolution of singularities

Definition 10. Given an algebraic set $\mathcal{V} \subset \mathbb{A}^n$, we define $\mathcal{I}(\mathcal{V})$ to be the ideal of all polynomial functions vanishing in $\mathcal{V}$:

$$\mathcal{I}(\mathcal{V}) = \{f \in \mathbb{C}[x_1 \cdots , x_n] \mid f(x) = 0 \text{ for all } x \in \mathcal{V}\}.$$

Definition 11. A function between two algebraic varieties $f : \mathcal{V} \longrightarrow \mathcal{W}$ with $\mathcal{V} \subset \mathbb{C}^n$, $\mathcal{W} \subset \mathbb{C}^m$ is *regular* if it has the form

$$f = (f_1, \cdots , f_m),$$

where $f_i \in \mathbb{C}[x_1, \cdots , x_n]/\mathcal{I}(\mathcal{V})$ for all i and $f(x) \in \mathcal{W}$ for all $x \in \mathcal{V}$.

Regular maps are the morphisms in the category of algebraic varieties.

A regular map whose inverse is also regular is an *isomorphism* of algebraic varieties.

Definition 12. The *field of rational functions over* $\mathcal{V}$ is the set of functions of the form f/g, with f, g classes of functions in $\mathbb{C}[x_1, \cdots , x_n]/\mathcal{I}(\mathcal{V})$ such that g is not the in the class of the function 0.

Definition 13. A function $f : \mathcal{V} \longrightarrow \mathcal{W}$ between algebraic varieties $\mathcal{V} \subset \mathbb{C}^n$ and $\mathcal{W} \subset \mathbb{C}^m$ is called *rational* if it has the form

$$f = (f_1, \cdots , f_m),$$

where each f_i belongs to the field of rational functions of $\mathcal{V}$ and $f_i(x) \in \mathcal{W}$ for all $x \in \mathcal{V}$ and for all i.

An invertible rational function whose inverse is also rational is called *birational function.*

Definition 14. A *resolution* of a singular variety $\mathcal{V}$ is a smooth variety $\tilde{\mathcal{V}}$ and a birational proper function $\psi : \tilde{\mathcal{V}} \longrightarrow \mathcal{V}$ such that ψ is an isomorphism over $\mathcal{V} \setminus Sing(\mathcal{V})$.

Definition 15. The *exceptional divisor* of a resolution $\psi : \tilde{\mathcal{V}} \longrightarrow \mathcal{V}$ is the set $\psi^{-1}(Sing(\mathcal{V}))$.

Remark 16. In 1964 Heisuke Hironaka proved that every algebraic variety over a field of characteristic zero has a resolution. This result appears in [14] and awarded Hironaka the Fields medal in 1970.

Definition 17. Let $\mathcal{V}$ be a curve or surface with an isolated singularity. A resolution $\tilde{\mathcal{V}}$ is called *good* if its exceptional divisor is the union of smooth curves E_i intersecting transversely (that is, the direct sum of their tangent spaces is the tangent space of $\tilde{\mathcal{V}}$) wherever they intersect and such that no three of them meet in one point.

Remark 18. In general a good resolution $\psi : \tilde{\mathcal{V}} \longrightarrow \mathcal{V}$ with exceptional divisor E is not unique.

To avoid this ambiguity, let us introduce the following concept:

Definition 19. A resolution $\psi : \tilde{\mathcal{V}} \longrightarrow \mathcal{V}$ is called *minimal* if given any other resolution $\phi : \tilde{\mathcal{V}_1} \longrightarrow \mathcal{V}$ there is a rational function $\rho : \tilde{\mathcal{V}_1} \longrightarrow \tilde{\mathcal{V}}$ such that $\phi = \psi \circ \rho$.

Theorem 20. Let V be a complex surface with an isolated singular point. Then, up to isomorphism, there exist a unique minimal resolution of V.

Proof. Please see [29, Theorem 5.7, p. 112] □

1.3 Blow up of singular points

In this section we introduce the technique of blowing up, used to resolve some singular varieties.

Specifically for the ADE singularities, it is shown in [19] that we can perform a series of blow ups until we obtain a resolution.

Definition 21. The *n-dimensional complex projective space* denoted by $\mathbb{CP}^n$ or simply by $\mathbb{P}^n$ is the quotient space

$$\mathbb{C}^{n+1} - (\bar{O})/\sim,$$

where the equivalence relation is given by

$$(t_1, \cdots, t_{n+1}) \sim (\lambda t_1, \cdots, \lambda t_{n+1}); \lambda \in \mathbb{C}, \lambda \neq 0.$$

The equivalence classes are usually denoted by $[t_1 : \cdots : t_{n+1}]$.

Remark 22. The complex projective space $\mathbb{P}^n$ is a complex manifold: It is covered by the charts

$$U_i = \{[t_1 : \cdots : t_{n+1}] | t_i \neq 0\}$$

with homeomorphisms

$$\varphi_i : U_i \longrightarrow \mathbb{C}^n$$
$$\varphi_i[t_1 : \cdots : t_{n+1}] = \left(\frac{t_1}{t_i}, \cdots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \cdots, \frac{t_{n+1}}{t_i}\right).$$

The transition maps between charts are given by

$$\tau_{i,j} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$
$$\tau_{i,j} = \varphi_j \circ \varphi_i^{-1}.$$

Definition 23. Let $\mathbb{P}^n$ be the complex projective space and $S \subset \mathbb{C}[x_1, \cdots x_n]$ a subset of homogeneous polynomials. The *zero set* of S is

$$\mathcal{V}(S) = \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

A subset Y of $\mathbb{P}^n$ is called a *projective algebraic set* if $Y = \mathcal{V}(S)$ for some subset of homogeneous polynomials S.

An irreducible projective algebraic set is called a *projective algebraic variety.*

Definition 24. The *blow up at the origin in* $\mathbb{C}^n$ is the following algebraic variety in $\mathbb{C}^n \times \mathbb{P}^{n-1}$:

$$B = \{((x_1, \cdots, x_n), [t_1 : \cdots : t_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | x_i t_j = x_j t_i\}.$$

Together with the projection

$$\pi : B \longrightarrow \mathbb{C}^n$$
$$\pi((x_1, \cdots, x_n), [t_1 : \cdots : t_n]) = (x_1, \cdots, x_n).$$

Definition 25. Let $\mathcal{V} \subset \mathbb{C}^n$ be an affine algebraic variety containing the origin $\bar{O}$. Let B be the blow up at $\bar{O}$ in $\mathbb{C}^n$ and $\pi : B \longrightarrow \mathbb{C}^n$ the previous function. The *blow up of* $\bar{O}$ *at* $\mathcal{V}$ is given by:

$$\tilde{\mathcal{V}} = \overline{\pi^{-1}(\mathcal{V} - \bar{O})} \subset B.$$

Proposition 26. The restriction of the function $\pi : \tilde{\mathcal{V}} \longrightarrow \mathcal{V}$ defines an isomorphism of algebraic varieties between $\mathcal{V} - \bar{O}$ and $\tilde{\mathcal{V}} - \pi^{-1}(\bar{O})$.

Proof. We define the inverse of π as follows:

$$\rho : \mathbb{C}^n - \bar{O} \longrightarrow \mathbb{C}^n \times \mathbb{P}^{n-1}$$
$$\rho(x_1, \cdots, x_n) = ((x_1, \cdots, x_n), [x_1 : \cdots : x_n]).$$

□

Definition 27. Let $\mathcal{V} \subset \mathbb{C}^n$ be an affine algebraic variety, B the blow up at the origin in $\mathbb{C}^n$ and $\tilde{\mathcal{V}}$ the blow up at the origin in $\mathcal{V}$. The *exceptional divisor of the blow up* is the set $E = \pi^{-1}(\bar{O}) \cap \tilde{V}$.

When computing a resolution, with each blow up we get a new component of the exceptional divisor which may consist of several irreducible components, called *exceptional curves.*

The union of all the exceptional curves is the *exceptional divisor of the resolution* and is denoted by E.

Definition 28. The information of a resolution of a singularity after a series of blow ups can be encoded in a graph as follows: for each irreducible component of the exceptional divisor we consider a vertex and two vertices are joined by a line if and only if the two corresponding curves intersect.

This is called the *dual graph of the resolution.*

Later we will give the dual graphs of the resolutions of ADE singularities and resolve explicitly one of them.

1.4 Charts of the blow up

At this point, our approach has been very theoretical. We are looking forward to give an example, but before that we need to give explicit charts for the blow up B.

Let us consider the case $n+1=3$. The charts U_i of the complex projective plane $\mathbb{P}^2$ given in remark 22 induce charts $\mathbb{C}^3 \times U_i$ for $\mathbb{C}^3 \times \mathbb{P}^2$, which we can identify with $\mathbb{C}^5$.

Moreover, we take the intersections with the blow up at the origin

$$B = \{((x,y,z),[t_1 : t_2 : t_3]) \in \mathbb{C}^3 \times \mathbb{P}^2 \,|\, t_2x = t_1y, t_3x = t_1z, t_3y = t_2z\} \tag{1.1}$$

and the relations $t_2x = t_1y$, $t_3x = t_1z$ and $t_3y = t_2z$ allow us to identify the set $(\mathbb{C}^3 \times U_i) \cap B$, called the *strict transform of the i-th chart*, with $\mathbb{C}^3$, via the following maps:

$$
\begin{tikzcd}
(\mathbb{C}^3 \times U_1) \cap B \arrow[r] & \mathbb{C}^5 \arrow[r] & \mathbb{C}^3 \\
((x,y,z),[t_1 : t_2 : t_3]) \arrow[r, mapsto] & (x,y,z,\frac{t_2}{t_1},\frac{t_3}{t_1}) \arrow[r, mapsto] & (x,\frac{t_2}{t_1},\frac{t_3}{t_1}), \\
(\mathbb{C}^3 \times U_2) \cap B \arrow[r] & \mathbb{C}^5 \arrow[r] & \mathbb{C}^3 \\
((x,y,z),[t_1 : t_2 : t_3]) \arrow[r, mapsto] & (x,y,z,\frac{t_1}{t_2},\frac{t_3}{t_2}) \arrow[r, mapsto] & \left(\frac{t_1}{t_2},y,\frac{t_3}{t_2}\right), \\
(\mathbb{C}^3 \times U_3) \cap B \arrow[r] & \mathbb{C}^5 \arrow[r] & \mathbb{C}^3 \\
((x,y,z),[t_1 : t_2 : t_3]) \arrow[r, mapsto] & (x,y,z,\frac{t_1}{t_3},\frac{t_2}{t_3}) \arrow[r, mapsto] & \left(\frac{t_1}{t_3},\frac{t_2}{t_3},z\right).
\end{tikzcd}
\tag{1.2}
$$

Which induce the following transition maps:

$$(\tilde{x},\tilde{y},\tilde{z}) \xmapsto{\tau_{1,2}} \left(\frac{1}{\tilde{y}},\tilde{x}\tilde{y},\frac{\tilde{z}}{\tilde{y}}\right)$$

$$(\tilde{x},\tilde{y},\tilde{z}) \xmapsto{\tau_{2,3}} \left(\frac{\tilde{x}}{\tilde{z}},\frac{1}{\tilde{z}},\tilde{z}\tilde{y}\right)$$

$$(\tilde{x},\tilde{y},\tilde{z}) \xmapsto{\tau_{1,3}} \left(\frac{1}{\tilde{z}},\frac{\tilde{y}}{\tilde{z}},\tilde{x}\tilde{z}\right)$$

$$(\tilde{x},\tilde{y},\tilde{z}) \xmapsto{\tau_{3,1}} (\tilde{x}\tilde{z},\frac{\tilde{y}}{\tilde{x}},\frac{1}{\tilde{x}}))$$

1.5 Resolving E_8

Now we are ready to give a resolution of a singularity using a sequence of blow ups. We choose the surface $\mathcal{V} \subset \mathbb{C}^3$ corresponding to the equation $x^2 + y^3 + z^5 = 0$, which is smooth everywhere except at the origin. To simplify notation we identify the strict transform $(\mathbb{C}^3 \times U_i) \cap \tilde{\mathcal{V}}$ with its image in $\mathbb{C}^3$ under the composition given by (1.2).

Blow up 1

First chart. Here $t_1 \neq 0$. By (1.1) we have $y = \frac{t_2}{t_1}x$ and $z = \frac{t_3}{t_1}x$. Substituting in the equation:

$$x^2 + \frac{t_2^3}{t_1^3}x^3 + \frac{t_3^5}{t_1^5}x^5 = x^2\left(1 + \frac{t_2^3}{t_1^3}x + \frac{t_3^5}{t_1^5}x^3\right) = 0$$

The equality holds when $x = 0$ or $1 + \frac{t_2^3}{t_1^3}x + \frac{t_3^5}{t_1^5}x^3 = 0$. The equation $1 + \frac{t_2^3}{t_1^3}x + \frac{t_3^5}{t_1^5}x^3 = 0$ corresponds to the strict transform. We notice that the intersection of the exceptional divisor with the strict transform is empty in this chart.

Let us compute the gradient of the strict transform in this chart, using a variable change to simplify notation:

$$\nabla(1 + y^3x + z^5x^3) = (y^3 + 3z^5x^2, 3xy^2, 4x^3z^4) \neq (0,0,0)$$

at every point of the strict transform. Therefore, it is smooth.

Second chart. Here $t_2 \neq 0$. By (1.1) we have $x = \frac{t_1}{t_2}y$ and $z = \frac{t_3}{t_2}y$. Substituting:

$$\frac{t_1^2}{t_2^2}y^2 + y^3 + \frac{t_3^5}{t_2^5}y^5 = y^2\left(\frac{t_1^2}{t_2^2} + y + \frac{t_3^5}{t_2^5}y^3\right) = 0$$

Now the equality holds when $y = 0$ or $\frac{t_1^2}{t_2^2} + y + \frac{t_3^5}{t_2^5}y^3 = 0$. Thus, on the intersection of the exceptional divisor with the strict transform, we have $\frac{t_1}{t_2} = 0$ and $\frac{t_3}{t_2}$ is free. Therefore the exceptional curve in this chart has the form $\{(0, 0, \frac{t_3}{t_2})\}$.

To see if the surface still has any singularity in this chart, we take the gradient of the strict transform. Again, variable change is made to simplify notation:

$$\nabla(x^2 + y + z^5y^3) = (2x, 1 + 3z^5y^2, 5y^3z^4) \neq (0,0,0)$$

at every point of the surface. We conclude it is smooth.

Third chart. Now $t_3 \neq 0$. By (1.1) we have $x = \frac{t_1}{t_3}z$ and $y = \frac{t_2}{t_3}z$. Substituting:

$$\frac{t_1^2}{t_3^2}z^2 + \frac{t_2^3}{t_3^3}z^3 + z^5 = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2^3}{t_3^3}z + z^3\right) = 0. \tag{1.3}$$

The equality holds when $z = 0$ or $\frac{t_1^2}{t_3^2} + \frac{t_2^3}{t_3^3}z + z^3 = 0$. Then in the intersection of the exceptional divisor with the strict transform we have $\frac{t_1}{t_3} = 0$ and $\frac{t_2}{t_3}$ is free. Therefore the exceptional curve in this chart has the form $\{(0, \frac{t_2}{t_3}, 0)\}$.

Again we want to see if the strict transform is smooth:

$$\nabla(x^2 + y^3z + z^3) = (2x, 3zy^2, y^3 + 3z^2) = (0,0,0)$$

at the origin, so we need to blow up again.

Apparently in this blow up we obtained two components of the exceptional divisor, but applying a transition map from second to third chart we see that:

$$\{(0,0,\tfrac{t_3}{t_2})\} \xmapsto{\tau_{2,3}} \{(0,\tfrac{t_2}{t_3},0)\}$$

so $\{(0,0,\frac{t_3}{t_2})\}$ and $\{(0,\frac{t_2}{t_3},0)\}$ are in fact the same component of the exceptional divisor. We will call this component E_1.

For simplicity, in the following blow ups we will not compute the gradient of the strict transform when the intersection of the exceptional divisor with the strict transform is empty. In all those cases the surface is smooth.

Blow up 2

Now we resolve the singularity that appeared in the strict transform of (1.3), namely $x^2+y^3z+z^3 = 0$. On each chart, we will make the same substitutions as above.

First chart. $t_1 \neq 0$. We have:

$$x^2 + \frac{t_2^3}{t_1^3}x^3\frac{t_3}{t_1}x + \frac{t_3^3}{t_1^3}x^3 = x^2\left(1 + \frac{t_2^3t_3}{t_1^4}x^2 + \frac{t_3^3}{t_1^3}x\right) = 0.$$

The equality holds when $x = 0$ or $1 + \frac{t_2^3t_3}{t_1^4}x^2 + \frac{t_3^3}{t_1^3}x = 0$. Then the intersection of the exceptional divisor with the strict transform in the first chart is empty.

Second chart. $t_2 \neq 0$. We have:

$$\frac{t_1^2}{t_2^2}y^2 + y^3\frac{t_3}{t_2}y + \frac{t_3^3}{t_2^3}y^3 = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2}y^2 + \frac{t_3^3}{t_2^3}y\right) = 0. \tag{1.4}$$

If $y = 0$ then the strict transform is zero when $\frac{t_1}{t_2} = 0$ and $\frac{t_3}{t_2}$ is free, so the intersection of the exceptional divisor with the strict transform has the form $\{(0,0,\frac{t_3}{t_2})\}$.

We investigate if the strict transform still has any singularity:

$$\nabla(x^2 + zy^2 + z^3y) = (2x, 2zy + z^3, y^2 + 3yz^2) = (0,0,0)$$

at the origin. We will blow up again this curve.

Third chart. $t_3 \neq 0$. We have:

$$\frac{t_1^2}{t_3^2}z^2 + \frac{t_2^3}{t_3^3}z^3z + z^3 = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2^3}{t_3^3}z^2 + z\right) = 0$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_1}{t_3} = 0$ and $\frac{t_2}{t_3}$ is free, then the exceptional curve in this chart has the form $\{(0, \frac{t_2}{t_3}, 0)\}$.

Again, we take the gradient:

$$\nabla(x^2 + y^3z^2 + z) = (2x, 3z^2y^2, 2y^3z + 1)$$

which is different from $(0, 0, 0)$ at every point. We conclude that the curve is smooth.

As in the first blow up, we got two exceptional curves, and applying the same transition map we see that they actually correspond to the same component, which we call E_2.

Blow up 3

We proceed to blow up the strict transform of (1.4): $x^2 + zy^2 + z^3y = 0$.

First chart. Taking $t_1 \neq 0$ and substituting:

$$x^2 + \frac{t_3}{t_1}x\frac{t_2^2}{t_1^2}x^2 + \frac{t_3^3}{t_1^3}x^3\frac{t_2}{t_1}x = x^2\left(1 + \frac{t_3t_2^2}{t_1^3}x^2 + \frac{t_3^3}{t_1^3}\frac{t_2}{t_1}x^2\right) = 0.$$

We notice that the intersection of the exceptional divisor with the strict transform is empty in this chart.

Second chart. When $t_2 \neq 0$ we have:

$$\frac{t_1^2}{t_2^2}y^2 + \frac{t_3}{t_2}yy^2 + \frac{t_3^3}{t_2^3}y^3y = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2}y + \frac{t_3^3}{t_2^3}y^2\right) = 0. \tag{1.5}$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_1}{t_2} = 0$ and $\frac{t_3}{t_2}$ is free, so the exceptional curve in this chart has the form $\{(0, 0, \frac{t_3}{t_2})\}$.

We calculate the gradient:

$$\nabla(x^2 + zy + z^3y^2) = (2x, z + 2z^3y, y + 3y^2z^2) = (0, 0, 0)$$

at the origin. We will blow up again this curve.

Third chart. When $t_3 \neq 0$ we have:

$$\frac{t_1^2}{t_3^2}z^2 + z\frac{t_2^2}{t_3^2}z^2 + z^3\frac{t_2}{t_3}z = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2^2}{t_3^2}z + \frac{t_2}{t_3}z^2\right) = 0. \tag{1.6}$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_1}{t_3} = 0$ and $\frac{t_2}{t_3}$ is free. This implies that the exceptional curve in this chart has the form $\{(0, \frac{t_2}{t_3}, 0)\}$.

Let us see if there is any other singularity:

$$\nabla(x^2 + y^2z + yz^2) = (2x, 2zy + z^2, y^2 + 2yz) = (0, 0, 0)$$

at the origin. To resolve the curve we will perform another blow up.

Just as in the previous two blow ups, the exceptional curves obtained are the same under a transition map. We call it E_3.

Blow up 4

Now we resolve the strict transform of (1.5): $x^2 + zy + z^3y^2 = 0$.

First chart. $t_1 \neq 0 \Rightarrow$ the equation becomes

$$x^2 + \frac{t_3}{t_1}x\frac{t_2}{t_1}x + \frac{t_3^3}{t_1^3}x^3\frac{t_2^2}{t_1^2}x^2 = x^2\left(1 + \frac{t_3}{t_1}\frac{t_2}{t_1} + \frac{t_3^3}{t_1^3}\frac{t_2^2}{t_1^2}x^3\right) = 0.$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_3}{t_1} = -\frac{t_1}{t_2}$. The exceptional curve in this chart has the form $\{(0, -\frac{t_1}{t_3}, \frac{t_3}{t_1})\}$.

Taking the gradient:

$$\nabla(1 + zy + z^3y^2x^3) = (3z^3y^2x^2, z + 2z^3x^3y, y + 3y^2x^3z^2) = (0, 0, 0)$$

at the origin. We may think that another blow up is necessary to resolve this curve, but in fact the origin does not satisfy the equation $1 + zy + z^3y^2x^3 = 0$, so the strict transform is smooth.

Second chart. $t_2 \neq 0 \Rightarrow$ substituting we get

$$\frac{t_1^2}{t_2^2}y^2 + \frac{t_3}{t_2}y^2 + \frac{t_3^3}{t_2^3}y^3y^2 = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2} + \frac{t_3^3}{t_2^3}y^3\right) = 0.$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_3}{t_2} = -\frac{t_1^2}{t_2^2}$. The exceptional curve in this chart has the form $\{(\frac{t_1}{t_2}, 0, -\frac{t_1^2}{t_2^2})\}$.

Now we check smoothness:

$$\nabla(x^2 + z + z^3y^3) = (2x, 3z^3y^2, 1 + 3y^3z^2) \neq (0, 0, 0)$$

at every point. No further blow up is required.

Third chart. $t_3 \neq 0 \Rightarrow$ the equation becomes:

$$\frac{t_1^2}{t_3^2}z^2 + z\frac{t_2}{t_3}z + z^3\frac{t_2^2}{t_3^2}z^2 = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2}{t_3} + \frac{t_2^2}{t_3^2}z^3\right) = 0$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_2}{t_3} = -\frac{t_1^2}{t_3^2}$. Hence the exceptional curve in this chart is $\{(\frac{t_1}{t_3}, -\frac{t_1^2}{t_3^2}, 0)\}$.

Checking smoothness:

$$\nabla(x^2 + y + y^2z^3) = (2x, 1 + 2z^3y, 3y^2z^2) \neq (0, 0, 0)$$

at every point of the curve, so it is smooth.

The exceptional curves obtained were:

1. First chart: $\left\{\left(0, -\frac{t_1}{t_3}, \frac{t_3}{t_1}\right)\right\}$

2. Second chart: $\left\{\left(\frac{t_1}{t_2}, 0, -\frac{t_1^2}{t_2^2}\right)\right\}$

3. Third chart: $\left\{\left(\frac{t_1}{t_3}, -\frac{t_1^2}{t_3^2}, 0\right)\right\}$.

We apply the following transition maps:

$$\left\{\left(0, -\tfrac{t_1}{t_3}, \tfrac{t_3}{t_1}\right)\right\} \xmapsto{\tau_{1,2}} \left\{\left(-\tfrac{t_3}{t_1}, 0, -\tfrac{t_3^2}{t_1^2}\right)\right\}$$

$$\left\{\left(\tfrac{t_1}{t_2}, 0, -\tfrac{t_1^2}{t_2^2}\right)\right\} \xmapsto{\tau_{2,3}} \left\{\left(-\tfrac{t_2}{t_1}, -\tfrac{t_2^2}{t_1^2}, 0\right)\right\}$$

and we conclude that all of them are the same component of the exceptional divisor, E_4.

Blow up 5

Now we resolve the strict transform of (1.6): $x^2 + y^2z + yz^2 = 0$.

First chart. $t_1 \neq 0$, then

$$x^2 + \frac{t_2^2}{t_1^2}x^2\frac{t_3}{t_1}x + \frac{t_2}{t_1}x\frac{t_3^2}{t_1^2}x^2 = x^2\left(1 + \frac{t_2^2}{t_1^2}\frac{t_3}{t_1}x + \frac{t_2}{t_1}\frac{t_3^2}{t_1^2}x\right) = 0.$$

The intersection of the exceptional divisor with the strict transform is empty in this chart.

Second chart. $t_2 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_2^2}y^2 + y^2\frac{t_3}{t_2}y + y\frac{t_3^2}{t_2^2}y^2 = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2}y + \frac{t_3^2}{t_2^2}y\right) = 0 \tag{1.7}$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_1}{t_2} = 0$ and $\frac{t_3}{t_2}$ is free. Thus, the exceptional curve in this chart has the form $\{(0, 0, \frac{t_3}{t_2})\}$.

Let us see the gradient:

$$\nabla(x^2 + zy + z^2y) = (2x, z + z^2, y + 2yz) = (0, 0, 0)$$

at the origin and at $(0, 0, -1)$. Both points lay in the curve.

Third chart. $t_3 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_3^2}z^2 + \frac{t_2^2}{t_3^2}z^3 + \frac{t_2}{t_3}z^3 = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2^2}{t_3^2}z + \frac{t_2}{t_3}z\right) = 0. \tag{1.8}$$

Then in the intersection of the exceptional divisor with the strict transform we have $\frac{t_1}{t_3} = 0$ and $\frac{t_2}{t_3}$ is free. Hence, the exceptional curve in this chart has the form $\{(0, \frac{t_2}{t_3}, 0)\}$.

Now we check smoothness:

$$\nabla(x^2 + y^2z + yz) = (2x, 2zy + z, y^2 + y) = (0,0,0)$$

at the origin and at the point $(0,-1,0)$, which are in the curve.

This blow up gave us four singular points. Applying a transition map from second to third chart:

$$(0,0,-1) \xmapsto{\tau_{2,3}} (0,-1,0) \tag{1.9}$$

therefore, there are only three of them.

Furthermore, just as in the first blow up, performing the same transition map from second to third chart we see that only one additional component of the exceptional divisor appeared in this blow up. We call it E_5.

Blow up 6

Now we resolve the singularity that appeared at the origin of the second chart of the previous blow up. The strict transform is given by (1.7): $x^2 + zy + z^2y = 0$.

First chart. If $t_1 \neq 0$ we have:

$$x^2 + \frac{t_3}{t_1}x\frac{t_2}{t_1}x + \frac{t_3^2}{t_1^2}x^2\frac{t_2}{t_1}x = x^2\left(1 + \frac{t_3}{t_1}\frac{t_2}{t_1} + \frac{t_3^2}{t_1^2}\frac{t_2}{t_1}x\right) = 0$$

Then in the intersection of the exceptional divisor with the strict transform we have $\frac{t_2}{t_1} = -\frac{t_1}{t_3}$. The exceptional curve in this chart has the form $\{(0, -\frac{t_1}{t_3}, \frac{t_3}{t_1})\}$.

We check the gradient:

$$\nabla(1 + zy + z^2yx) = (z^2y, z + z^2x, y + 2yxz) = (0,0,0)$$

at the origin, which does not satisfy the equation. We conclude that the curve is smooth in this chart.

Second chart. $t_2 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_2^2}y^2 + \frac{t_3}{t_2}y^2 + \frac{t_3^2}{t_1^2}y^3 = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2} + \frac{t_3^2}{t_1^2}y\right) = 0$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_3}{t_2} = -\frac{t_1^2}{t_2^2}$. Then the exceptional curve in this chart has the form $\{(\frac{t_1}{t_2}, 0, -\frac{t_1^2}{t_2^2})\}$.

$$\nabla(x^2 + z + z^2y) = (2x, z^2, 1 + 2zy) \neq (0,0,0)$$

Then our curve is smooth in this chart too.

Third chart. $t_3 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_3^2}z^2 + z\frac{t_2}{t_3}z + z^2\frac{t_2}{t_3}z = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2}{t_3} + \frac{t_2}{t_3}z\right) = 0 \tag{1.10}$$

In the intersection of the exceptional divisor with the strict transform we have $\frac{t_2}{t_3} = -\frac{t_1^2}{t_3^2}$. The exceptional curve in this chart has the form $\{(\frac{t_1}{t_3}, -\frac{t_1^2}{t_3^2}, 0)\}$.

$$\nabla(x^2 + y + yz) = (2x, 1 + z, y) = (0,0,0)$$

at $(0, 0, -1)$, which is the same singular point found in the second chart of blow up 5. We will resolve this point in the next blow up.

As in the fourth blow up, all the exceptional curves obtained are the same under a transition map. We name this one component of the exceptional divisor E_6.

Blow up 7

To resolve the singularity of (1.10), given by the equation $x^2 + y + yz = 0$, at the point $(0, 0, -1)$ we perform the variable change $z = z - 1$, that moves the singular point to the origin and later we proceed to blow up as usual. The polynomial becomes:

$$x^2 + y + yz = 0 \longmapsto x^2 + yz = 0.$$

First chart. $t_1 \neq 0 \Rightarrow$

$$x^2 + \frac{t_3}{t_1}x\frac{t_2}{t_1}x = x^2\left(1 + \frac{t_3}{t_1}\frac{t_2}{t_1}\right) = 0.$$

Thus, the exceptional curve in this chart has the form $\{(0, \frac{t_2}{t_1}, -\frac{t_1}{t_2})\}$. Besides, the curve is smooth in this chart, since

$$\nabla(1 + zy) = (0, z, y) = (0,0,0)$$

at the origin, which is not in the curve.

Second chart. $t_2 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_2^2}y^2 + \frac{t_3}{t_2}y^2 = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2}\right) = 0$$

Therefore the exceptional curve in this chart has the form $\{(\frac{t_1}{t_2}, 0, -\frac{t_1^2}{t_2^2})\}$. Now we compute the gradient

$$\nabla(x^2 + z) = (2x, 0, 1) \neq (0, 0, 0).$$

Third chart. $t_3 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_3^2}z^2 + \frac{t_2}{t_3}z^2 = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2}{t_3}\right) = 0$$

Then the exceptional curve in this chart has the form $\{(\frac{t_1}{t_3}, -\frac{t_1^2}{t_3^2}, 0)\}$.

$$\nabla(x^2 + y) = (2x, 1, 0) \neq (0, 0, 0)$$

at every point. Then the curve is smooth in every chart.

Blow up 8

There is only one singularity left to resolve: the origin of the third chart of the fifth blow up, given by (1.8). The polynomial corresponding to this singularity is $x^2 + y^2z + yz = 0$.

First chart. $t_1 \neq 0 \Rightarrow$

$$x^2 + \frac{t_2^2}{t_1^2}x^2\frac{t_3}{t_1}x + \frac{t_2}{t_1}\frac{t_3}{t_1}x^2 = x^2\left(1 + \frac{t_2^2}{t_1^2}\frac{t_3}{t_1}x + \frac{t_2}{t_1}\frac{t_3}{t_1}\right) = 0.$$

In this chart the exceptional curve has the form $\{(0, \frac{t_2}{t_1}, -\frac{t_1}{t_2})\}$.

$$\nabla(1 + y^2zx + yz) = (y^2z, 2zxy + z, y^2x + y) \neq (0, 0, 0)$$

at every point.

Second chart. $t_2 \neq 0 \Rightarrow$

$$\frac{t_1^2}{t_2^2}y^2 + y^2\frac{t_3}{t_2}y + \frac{t_3}{t_2}y^2 = y^2\left(\frac{t_1^2}{t_2^2} + \frac{t_3}{t_2}y + \frac{t_3}{t_2}\right) = 0$$

Therefore the exceptional curve in this chart has the form $\{(\frac{t_1}{t_2}, 0, -\frac{t_1^2}{t_2^2})\}$.

$$\nabla(x^2 + zy + z) = (2x, z, y + 1) = (0, 0, 0)$$

at $(0, -1, 0)$. This is the same singular point we observed and resolved on the previous blow up.

Third chart. $t_3 \neq 0 \Rightarrow$

$$\frac{t_1^3}{t_3^2}z^2 + \frac{t_2^2}{t_3^2}z^3 + \frac{t_2}{t_3}z^2 = z^2\left(\frac{t_1^2}{t_3^2} + \frac{t_2^2}{t_3^2}z + \frac{t_2}{t_3}\right) = 0$$

And the exceptional curve in this chart has the form $\{(\frac{t_1}{t_3}, -\frac{t_1^2}{t_3^2}, 0)\}$.

$$\nabla(x^2 + y^2 z + y) = (2x, 2zy + 1, y^2) \neq (0, 0, 0)$$

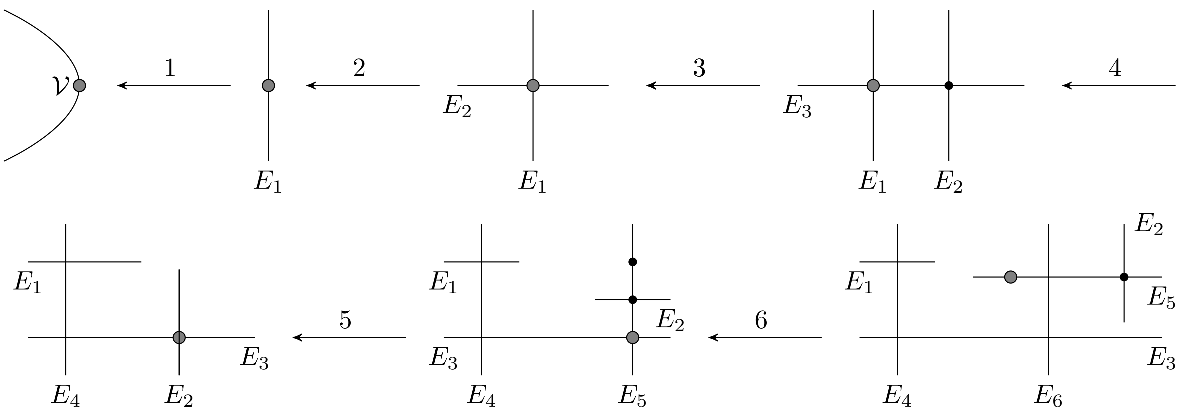
at every point. All the exceptional curves are smooth and correspond to the same component of the exceptional divisor under a transition map. We call this component E_8.

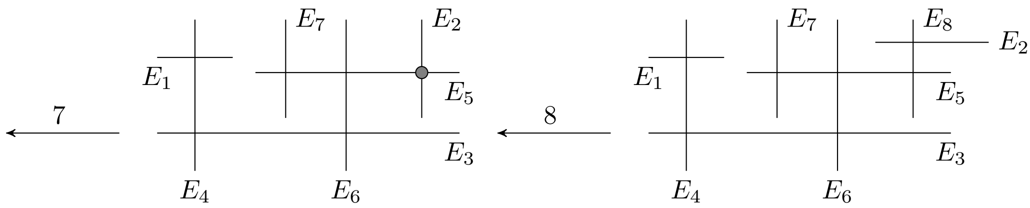
1.6 Drawing the graph of the resolution of E_8

To draw the dual graph of the resolution we need to keep track of the intersections between the components of the exceptional divisor. We use the relations given by (1.1) to identify the components that appeared on the same chart as each singularity and check with which other components of the next blow up they meet. Such information is arranged in the following table, where the $\star$ symbol indicates that a singularity appeared there.

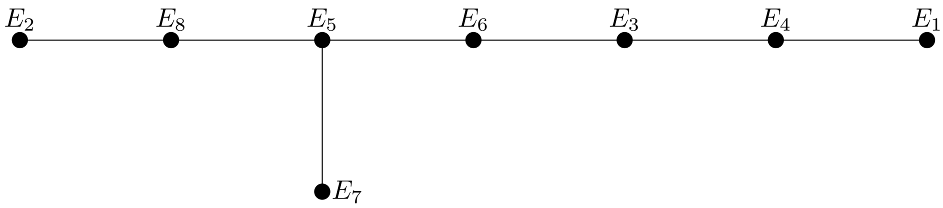
Blow up	E component	First chart	Second chart	Third chart
1	E_1	$\emptyset$	$\{(0,0,z)\}$	$\{(0,y,0)\}\star$
2	E_2	$\emptyset$	$\{(0,0,z)\}\star$	$\{(0,y,0)\}$
	E_1	$\emptyset$	$\{(0,y,0)\}$	$\emptyset$
3	E_3	$\emptyset$	$\{(0,0,z)\}\star$	$\{(0,y,0)\}\star$
	E_2	$\emptyset$	$\emptyset$	$\{(0,0,z)\}$
	E_1	$\emptyset$	$\{(0,y,0)\}$	$\emptyset$
4	E_4	$\{(0,-\frac{1}{z},z)\}$	$\{(x,0,-x^2)\}$	$\{(x,-x^2,0)\}$
	E_3	$\emptyset$	$\emptyset$	$\{(0,0,z)\}$
	E_1	$\emptyset$	$\{(0,y,0)\}$	$\emptyset$
5	E_5	$\emptyset$	$\{(0,0,z)\}\star$	$\{(0,y,0)\}\star$
	E_3	$\emptyset$	$\{(0,y,0)\}$	$\emptyset$
	E_2	$\emptyset$	$\emptyset$	$\{(0,0,z)\}$
6	E_6	$\{(0,-\frac{1}{z},z)\}$	$\{(x,0,-x^2)\}$	$\{(x,-x^2,0)\}\star$
	E_5	$\emptyset$	$\emptyset$	$\{(0,0,z)\}$
	E_3	$\emptyset$	$\{(0,y,0)\}$	$\emptyset$
7	E_7	$\{(0,y,-\frac{1}{y})\}$	$\{(x,0,-x^2)\}$	$\{(x,-x^2,0)\}$
	E_5	$\emptyset$	$\emptyset$	$(0,0,z)$
8	E_8	$\{(0,y,\frac{1}{y})\}$	$\{(x,0,-x^2)\}$	$\{(x,-x^2,0)\}$
	E_5	$\emptyset$	$(0,y,0)$	$\emptyset$
	E_2	$\emptyset$	$\emptyset$	$\{(0,0,z)\}$

Schematically, the behaviour of the exceptional curves is as follows. The numbers above each arrow indicate the number of the blow up and the bullets represent the singularities. The gray bullet is the point we blow up on each step. For simplicity, the surface is not depicted.





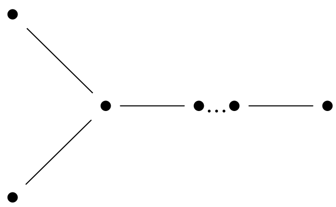
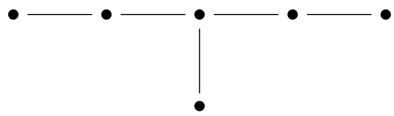
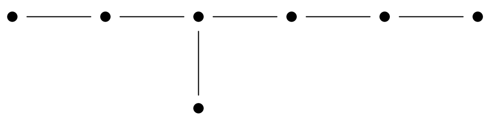
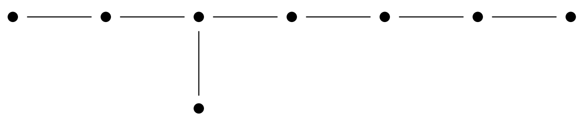
The dual graph of the resolution is:



Remark 29. Please notice that we did not prove that the resolution we gave is the minimal resolution. There is a way to determine whether a resolution is minimal or not, known as the *Castelnuovo criterion*. This method uses *intersection numbers*, which are not within the interests of the present work. We invite the reader to check [4, p. 106] for more details.

1.7 Dual graphs of resolution for ADE singularities

In the following table, which was taken from [10, Table 1, p. 158], we present the dual graphs of the minimal resolution of ADE singularities:

$\boldsymbol{f(x,y,z)}$	Dual graph of resolution	Name
$x^2+y^2+z^n$		A_n
$x^2+y^2z+z^{n-1}$		D_n
$x^2+y^3+z^4$		E_6
$x^2+y^3+yz^3$		E_7
$x^2+y^3+z^5$		E_8

The names of the graphs come from the *ADE classification of Dynkin diagrams*.

These graphs not only appear in the context of singularities, but as we will see in the third chapter, they are also very important in the study of group representations (see, for instance, [13]).

Chapter 2

ADE singularities as quotient singularities

In this chapter we will see that the ADE singularities can be characterized as *quotient singularities*, that is, they are homeomorphic to a quotient space $\mathbb{C}^2/\Gamma$, with Γ a finite subgroup of $SL(2,\mathbb{C})$. This is a classic theorem first proved by Klein in [20].

The first thing we have to do is to prove that every finite subgroup of $SL(2,\mathbb{C})$ is conjugated to a subgroup of $SU(2)$. Then we prove that the finite subgroups of $SU(2)$ are the preimages of those of $SO(3)$ under an explicit homomorphism. After that, we classify the finite subgroups of $SO(3)$ to obtain the classification of finite subgroups of $SL(2,\mathbb{C})$. Finally, we state that these groups act on $\mathbb{C}^2$ and its corresponding orbit space is homeomorphic to an ADE singularity.

We will also see that the *link* of the quotient singularity $\mathbb{C}^2/\Gamma$ is diffeomorphic to S^3/Γ, with S^3 the 3-sphere.

Let us start defining the aforementioned groups.

2.1 The linear groups

Definition 30. Let $\mathbb{K}^n$ be the n-dimensional vector space over the field $\mathbb{K}$. The *general linear group* of $\mathbb{K}^n$, $GL(n,\mathbb{K})$ is the group of linear automorphisms of $\mathbb{K}^n$ where the product is given by composition.

If $\{v_1,\cdots,v_n\}$ is a basis for $\mathbb{K}^n$ and f is a linear automorphism, for every $v=c_1v_1+\cdots+c_nv_n\in\mathbb{K}^n$ we have

$$f(c_1v_1+\cdots+c_nv_n)=c_1f(v_1)+\cdots+c_nf(v_n)$$

which implies that the function f is entirely determined by the vectors

$$f(v_j) = a_{1j}v_1 + \cdots + a_{nj}v_n$$

and f can be represented by a matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Therefore, $GL(n, \mathbb{K})$ can be seen as the group of $n \times n$ invertible matrices with entries in $\mathbb{K}$ and matrix product.

Theorem 31. $GL(n, \mathbb{R})$ has two connected components.

Proof. This is shown in [32, Theorem 3.68 p. 131]. $\square$

Definition 32. The *special linear group* is the subgroup:

$$SL(n, \mathbb{K}) := \{Q \in GL(n, \mathbb{K}) \,|\, \det Q = 1\}.$$

Definition 33. Let V be a vector space over $\mathbb{R}$. An *inner product* over V is a map

$$\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$$

satisfying the following axioms for $u, v, w \in V$; $\lambda \in \mathbb{R}$:

- $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle u, u \rangle \geq 0$
- $\langle u, u \rangle = 0 \Leftrightarrow u = 0.$

Inner product spaces have a naturally defined *norm*:

$$||u|| = \sqrt{\langle u, u \rangle}$$

which induces a *metric* or *distance*

$$d(u, v) = ||v - u||.$$

Definition 34. The *orthogonal group* in dimension n is the group of distance preserving transformations of $\mathbb{R}^n$ that preserve a fixed point, where the group operation is composition. Equivalently, is the subgroup of $GL(n, \mathbb{R})$ given by

$$O(n, \mathbb{R}) = \{Q \in GL(n, \mathbb{R}) \,|\, Q^{\mathrm{T}}Q = QQ^{\mathrm{T}} = I\}.$$

Definition 35. The *orthogonal special group* is defined as:

$$SO(n, \mathbb{R}) = O(n, \mathbb{R}) \cap SL(n, \mathbb{R}).$$

It is also called the *rotation group* of $\mathbb{R}^n$. In the following we denote it simply by $SO(n)$.

Theorem 36. The Gram-Schmidt orthogonalization process applied to the columns of matrices in $GL(n, \mathbb{R})$ provides a retraction, that is, a continuous map

$$r : GL(n, \mathbb{R}) \longrightarrow O(n, \mathbb{R})$$

such that $r \circ i = \mathrm{Id}_{O(n,\mathbb{R})}$, where i is the inclusion.

Proof. Please check [12, p.293]. □

Corollary 37. The group $O(n, \mathbb{R})$ has two connected components. The component which contains the identity is $SO(n)$.

Definition 38. A *Hermitian inner product* over a complex vector space V is a map $\langle -, - \rangle : V \times V \longrightarrow \mathbb{C}$ such that for $u, v, w \in V$, $\lambda \in \mathbb{C}$ the following holds

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$,
3. $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$,
4. $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$,
5. $\langle u, v \rangle = \overline{\langle v, u \rangle}$,
6. $\langle u, u \rangle \geq 0$ and the equality holds if and only if $u = 0$.

Definition 39. Let $A \in GL(n, \mathbb{C})$ be a $n \times n$ invertible matrix and A^* its conjugate transpose. The *unitary group* of degree n is

$$U(n) = \{A \in GL(n, \mathbb{C}) | A^* = A^{-1}\}.$$

Equivalently, it can be defined as the group of matrices A preserving the Hermitian inner product, that is, if

$$\langle Au, Av\rangle = \langle u, v\rangle.$$

Definition 40. The *special unitary group* of degree n is defined as

$$SU(n) = U(n) \cap SL(n, \mathbb{C}).$$

2.2 Conjugacy classes of finite subgroups of $SL(2, \mathbb{C})$

Let us remember that the first purpose of the chapter is to classify the finite subgroups of $SL(2, \mathbb{C})$. In this section we reduce the problem to calculate the finite subgroups of $SU(2)$.

Definition 41. Let G be a group. A vector space V over a field F is called a G*-module* if there is a function $G \times V \longrightarrow V$ satisfying the following axioms: for all $g_1, g_2 \in G\,;\, v, w \in V$ and $\lambda \in F$.

1. $(g_1 + g_2)v = g_1v + g_2v$,
2. $g_1(v + w) = g_1v + g_1w$,
3. $g_1(g_2v) = (g_1g_2)v$,
4. $1v = v$,
5. $g_1(\lambda v) = \lambda(g_1v) = (\lambda g_1)v$.

Definition 42. Let V be a real (resp. complex) G-module. An inner product (resp. Hermitian inner product) is said to be G*-invariant* if

$$\langle gu, gv\rangle = \langle u, v\rangle$$

for all $g \in G$ and $u, v \in V$. In this case we will denote the inner product by $\langle -, -\rangle_G$.

Lemma 43. Let G be a finite group and V a real (resp. complex) G-module. Then there is a G-invariant (resp. Hermitian) inner product $\langle -, -\rangle_G$ on V.

Proof. We will prove the complex case, the real one is analogous.

Let G be a finite group and let $\langle -, -\rangle : V \times V \longrightarrow \mathbb{C}$ be a Hermitian inner product on V. We define

$$\begin{aligned} \langle -, -\rangle_G : V \times V &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto \langle u, v\rangle_G := \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle. \end{aligned}$$

We now prove that $\langle -,-\rangle_G$ is a G-invariant Hermitian inner product.

Let us consider $u, v \in V$, $g \in G$ and $\lambda \in \mathbb{C}$.

1. $\langle u+v, w\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle g(u+v), gw\rangle = \frac{1}{|G|}\sum_{g\in G}\langle gu+gv, gw\rangle$
   $= \frac{1}{|G|}\sum_{g\in G}(\langle gu, gw\rangle + \langle gv, gw\rangle) = \frac{1}{|G|}\sum_{g\in G}\langle gu, gw\rangle + \frac{1}{|G|}\sum_{g\in G}\langle gv, gw\rangle = \langle u, w\rangle_G + \langle v, w\rangle_G$

2. $\langle \lambda u, v\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle g\lambda u, gv\rangle = \frac{1}{|G|}\sum_{g\in G}\lambda\langle gu, gv\rangle = \lambda\langle gu, gv\rangle_G$

3. $\overline{\langle v, u\rangle_G} = \overline{\frac{1}{|G|}\sum_{g\in G}\langle gv, gu\rangle} = \frac{1}{|G|}\sum_{g\in G}\overline{\langle gv, gu\rangle} = \frac{1}{|G|}\sum_{g\in G}\langle gu, gv\rangle = \langle u, v\rangle_G$

4. $\langle u, u\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle gu, gu\rangle \geq 0$ and the equality holds if and only if $0 = gu = u$.

5. On the other hand, for $h \in G$, we see that

$$\langle hu, hv\rangle_G = \frac{1}{|G|}\sum_{g\in G}\langle ghu, ghv\rangle = \frac{1}{|G|}\sum_{g'\in G}\langle g'u, g'v\rangle = \langle u, v\rangle_G.$$

□

Proposition 44. Every finite subgroup of $SL(2,\mathbb{C})$ is conjugated to a subgroup of $SU(2)$.

Proof. Let Γ be a finite subgroup of $SL(2,\mathbb{C})$, $\langle -,-\rangle$ be a Hermitian inner product on $\mathbb{C}^2$ and consider $\langle -,-\rangle_\Gamma$ the Γ-invariant Hermitian inner product given by lemma 43.

Given two bases, $\mathcal{B} = \{e_1, e_2\}$ orthonormal with respect to $\langle -,-\rangle$, and $\mathcal{B}' = \{e'_1, e'_2\}$ orthonormal with respect to $\langle -,-\rangle_\Gamma$; there exist a change of basis matrix $T \in SL(2,\mathbb{C})$ from $\mathcal{B}'$ to $\mathcal{B}$, such that for every i, j

$$\langle e'_i, e'_j\rangle_\Gamma = \delta_{ij} = \langle Te'_i, Te'_j\rangle$$

with δ_{ij} the Kronecker delta. By linear extension we have that $\langle u, v\rangle_\Gamma = \langle Tu, Tv\rangle$ for every $u, v \in \mathbb{C}^2$. Using this fact we have

$$\langle u, v\rangle = \langle T^{-1}u, T^{-1}v\rangle_\Gamma = \langle gT^{-1}u, gT^{-1}v\rangle_\Gamma = \langle TgT^{-1}u, TgT^{-1}v\rangle.$$

Since the previous equality does not depend on the element g, it follows that $T\Gamma T^{-1}$ is unitary, thus Γ is conjugated to a subgroup of $SU(2)$. □

2.3 Some useful characterizations of $SU(2)$

Let us remember that by Proposition 44, the problem of computing the finite subgroups of $SL(2, \mathbb{C})$ is now reduced to compute those of $SU(2)$. In this section we study some properties of $SU(2)$ that, together with the classification of finite subgroups of $SO(3)$ (which will be given in the next section), will help us to achieve our goal.

2.3.1 Quaternions

Definition 45. The *quaternions* are the algebra over $\mathbb{R}$ whose underlying set is:

$$\mathbb{H} = \{q = a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}.$$

The sum of two quaternions is

$$(a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k) = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 - d_2)k$$

and for $\alpha \in \mathbb{R}$ the scalar product is just

$$\alpha(a_1 + b_1 i + c_1 j + d_1 k) = \alpha a_1 + \alpha b_1 i + \alpha c_1 j + \alpha d_1 k.$$

The fundamental formula for quaternion multiplication is $i^2 = j^2 = k^2 = ijk = -1$. Using it and the distributive law we get the formula for the product of two quaternions.

$$\begin{aligned}(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) &= a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ &+ (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)i \\ &+ (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)j \\ &+ (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)k.\end{aligned}$$

For a quaternion q as above, a is called its *real* or *scalar part* and $bi + cj + dk$ is its *pure* or *vector part*. The *conjugate* of q is the element $\bar{q} = a - bi - cj - dk$ and for two quaternions q and p, it holds that $\overline{qp} = \bar{p}\bar{q}$.

The inner product of two quaternions is

$$\langle q, p \rangle = \frac{1}{2}(q\bar{p} + p\bar{q}) = \frac{1}{2}(\bar{q}p + \bar{p}q)$$

or equivalently

$$\langle q, p \rangle = \langle a_1 + b_1 i + c_1 j + d_1 k, a_2 + b_2 i + c_2 j + d_2 k \rangle = a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2.$$

The norm of $q = a + bi + cj + dk$ is then

$$|q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

If $|q| = 1$ the quaternion is said to be *unitary.*

The inverse element of a nonzero quaternion q is $q^{-1} = \bar{q}/|q|^2$.

Observation 46. Notice that these are the usual inner product and norm in $\mathbb{R}^4$. Then $\mathbb{H}$ and $\mathbb{R}^4$ are isomorphic as real vector spaces with inner product.

Lemma 47. Let q and p be two quaternions. Then $|qp| = |q||p|$.

Proof.

$$|qp|^2 = qp\overline{qp} = qp\bar{p}\bar{q} = q|p|^2\bar{q} = |q|^2|p|^2.$$

$\square$

Remark 48. The previous lemma implies that the product of two unitary quaternions is an unitary quaternion. Besides, 1 and the inverse of an unitary quaternion are unitary. Therefore unitary quaternions form a group.

Moreover, the unit 3-sphere can be regarded as $S^3 = \{q \in \mathbb{H} \,|\, |q| = 1\}$. This says that unitary quaternions form a compact, connected Lie group, whose corresponding Lie algebra is generated by the pure quaternions $\{i, j, k\}$.

2.3.2 The algebra $\mathcal{M}$

Definition 49. Let us denote by $\mathcal{M}$ algebra over $\mathbb{R}$ given by 2×2 matrices of the form

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

with the usual sum and multiplication of matrices.

Observation 50. We notice that $\mathcal{M}$ is indeed closed under multiplication:

$$\begin{pmatrix} a_1 + b_1 i & c_1 + d_1 i \\ -c_1 + d_1 i & a_1 - b_1 i \end{pmatrix} \begin{pmatrix} a_2 + b_2 i & c_2 + d_2 i \\ -c_2 + d_2 i & a_2 - b_2 i \end{pmatrix} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

with

$$\begin{aligned} a &= a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ b &= a_2b_1 + a_1b_2 - c_2d_1 + c_1d_2 \\ c &= a_2c_1 + a_1c_2 + b_2d_1 - b_1d_2 \\ d &= -b_2c_1 + b_1c_2 + a_2d_1 + a_1d_2. \end{aligned}$$

The algebra $\mathcal{M}$ can also be equipped with the following inner product

$$\langle A, B\rangle = \frac{1}{2}\mathrm{Tr}(AB^*)$$

where Tr denotes the trace of a matrix.

Lemma 51. Let us consider $A \in \mathcal{M}$, then $\langle A, A\rangle = \det(A)$.

Proof. If $A \in \mathcal{M}$ then it has the form

$$\begin{pmatrix} a+ib & c+di \\ -c+di & a-bi \end{pmatrix}$$

and its square norm is

$$\begin{aligned}
\langle A, A\rangle &= \frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} a+ib & c+di \\ -c+di & a-bi \end{pmatrix}\begin{pmatrix} a-ib & -c-di \\ c-di & a+bi \end{pmatrix}\right) \\
&= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix} = \frac{1}{2}(2a^2+2b^2+2c^2+2d^2) \\
&= a^2+b^2+c^2+d^2 = \det(A)
\end{aligned}$$

□

Observation 52. If A and B are matrices in $\mathcal{M}$, then

$$|AB|^2 = \det(AB) = \det(A)\det(B) = |A|^2|B|^2.$$

Remark 53. The elements of $SU(2)$ are characterized by having determinant equal to 1 and its conjugated transpose coincides with its inverse. Then a matrix

$$\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$$

has an inverse of the form

$$\begin{pmatrix} \gamma & -\beta \\ -\delta & \alpha \end{pmatrix} = \begin{pmatrix} \overline{\alpha} & \overline{\delta} \\ \overline{\beta} & \overline{\gamma} \end{pmatrix}$$

and thus the matrices of $SU(2)$ have the form

$$\begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Using the previous observation, we have that $SU(2)$ is the group of elements with norm 1 in $\mathcal{M}$.

Proposition 54. Quaternions are isomorphic to $\mathcal{M}$ as real algebras via an inner product preserving isomorphism.

Proof. Let us define the correspondence

$$\begin{aligned} L : \mathbb{H} &\longrightarrow \mathcal{M} \\ a + bi + cj + dk &\longmapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \end{aligned}$$

The map L is a bijective vector space homomorphism:

$$L(q+p) = L((a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k))$$
$$= \begin{pmatrix} (a_1+a_2)+(b_1+b_2)i & (c_1+c_2)+(d_1+d_2)i \\ -(c_1+c_2)+(d_1+d_2)i & (a_1+a_2)-(b_1+b_2)i \end{pmatrix} = L(q) + L(p).$$

It preserves multiplication:

$$\begin{aligned} L(qp) &= L((a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k)) \\ &= L(a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i \\ &+ (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k) \\ &= \begin{pmatrix} a_1+b_1i & c_1+d_1i \\ -c_1+d_1i & a_1-b_1i \end{pmatrix} \begin{pmatrix} a_2+b_2i & c_2+d_2i \\ -c_2+d_2i & a_2-b_2i \end{pmatrix} \end{aligned}$$

this last equality follows from observation 50.

It preserves inner product:

$$\langle L(q), L(p) \rangle = \frac{1}{2}\mathrm{Tr}(L(q)L(p)^*) = \frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} a_1+b_1i & c_1+d_1i \\ -c_1+d_1i & a_1-b_1i \end{pmatrix} \begin{pmatrix} a_2-b_2i & -c_2-d_2i \\ c_2-d_2i & a_2+b_2i \end{pmatrix} \right)$$
$$= \frac{1}{2}\mathrm{Tr}\begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} = \frac{1}{2}(\xi_1 + \xi_4) = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 = \langle q, p \rangle$$

where

$$\begin{aligned} \xi_1 &= a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + i(a_2b_1 - a_1b_2 + c_2d_1 - c_1d_2) \\ \xi_2 &= a_2c_1 - a_1c_2 - b_2d_1 + b_1d_2 + i(b_2c_1 - b_1c_2 + a_2d_1 - a_1d_2) \\ \xi_3 &= -a_2c_1 + a_1c_2 + b_2d_1 - b_1d_2 + i(b_2c_1 - b_1c_2 + a_2d_1 - a_1d_2) \\ \xi_4 &= a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 + i(-a_2b_1 + a_1b_2 - c_2d_1 + c_1d_2) \end{aligned}$$

□

Corollary 55. Unitary quaternions S^3 and $SU(2)$ are isomorphic groups.

Proof. Since the map L preserves multiplication and inner product, we have $L(S^3) = SU(2)$. $\square$

Remark 56. By Corollary 55, $SU(2)$ is a compact connected Lie group, whose associated Lie algebra is denoted by $\mathfrak{su}(2)$([5, p. 62]) and generated by the images of pure quaternions i, j, and k under the map L, namely

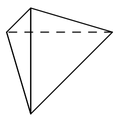
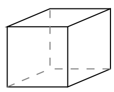
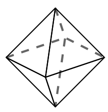
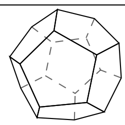
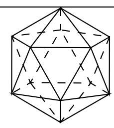
$$\left\{ E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}. \tag{2.1}$$

By observation 46, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ as real vector spaces. Then proposition 54 has the following consequence:

Corollary 57. $\mathfrak{su}(2)$ and pure quaternions, seen as $\mathbb{R}^3$ are isomorphic.

2.4 Classification of finite subgroups of $SO(3)$

Definition 58. In $\mathbb{R}^3$, the *platonic solids* are the regular, convex polyhedra. There are five of them, namely:

Name	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Faces	4	6	8	12	20
Picture					

Definition 59. Let P be a platonic solid inscribed in the unit sphere S^2 in $\mathbb{R}^3$. The rotation group of P consists of the elements of $SO(3)$ which transform P into itself.

Every polyhedron has an associated *dual polyhedron*, which is constructed by putting a point at the barycenter of each face an then taking the convex hull of such points.

The dual of a cube is an octahedron and the dual of a dodecahedron is an icosahedron, while tetrahedron is self-dual.

Duality preserves symmetries of a polyhedron, which implies that the rotation group of a platonic solid P is equal to the rotation group of its dual.

Definition 60. Let X be a set and G a group. We say that G *acts* on X (or X is a G-set) if there is a function $\alpha : G \times X \longrightarrow X$ (called an *action*), denoted by $\alpha : (g, x) \longmapsto gx$, such that:

1. $1x = x$ for all $x \in X$; and

2. $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Definition 61. Let G be a group acting on X. For each point $x \in X$, we define the *isotropy group* in x as follows

$$G_x = \{g \in G | gx = x\}.$$

The isotropy group is subgroup of G. We will use the notation $G_x < G$.

Definition 62. Let G be a group acting continuously on a topological space X and $x \in X$. The *orbit* of x is the subset of X given by

$$O(x) = \{gx \,|\, g \in G\}.$$

The set of all orbits of X under the action of G with the quotient topology is written as X/G and is called the *orbit space*.

Observation 63. Every group G acts on the set of all its subgroups by conjugation.

Let G be a finite subgroup of $SO(3)$, then G acts on the unit sphere S^2 in $\mathbb{R}^3$ by rotations. This action is well defined since the elements of $SO(3)$ act on $\mathbb{R}^3$ fixing the origin and preserving distances. Such action can be used to prove the following:

Theorem 64. The finite subgroups of $SO(3)$, up to conjugation, are:

1. Cyclic groups of order n, with $2 \leq n < \infty$.

   This is the group of rotations of a regular n-gon. It is denoted by C_n and has the presentation: $\langle \beta | \beta^n = 1 \rangle$.

2. Dihedral groups of order $2n$, with $2 \leq n < \infty$.

   This is the group of symmetries of a regular n-gon. It has order $2n$, is denoted by D_n and its presentation is: $\langle \beta, \gamma | \beta^2 = \gamma^n = (\beta\gamma)^2 = 1 \rangle$.

3. The rotation group of the tetrahedron.

   The presentation of this group is: $\langle \gamma, \beta | \beta^3 = \gamma^3 = (\beta\gamma)^2 = 1 \rangle$ and is isomorphic to the alternating group A_4.

4. The rotation group of the octahedron, whose presentation is:

   $\langle \gamma, \beta | \beta^3 = \gamma^4 = (\beta\gamma)^2 = 1 \rangle$ and is isomorphic to the permutation group S_4.

5. The rotation group of the icosahedron, with presentation:

   $\langle \gamma, \beta | \beta^3 = \gamma^5 = (\beta\gamma)^2 = 1 \rangle$ and is isomorphic to the alternating group A_5.

Proof. It can be found in [21, §6 p.20]. $\square$

2.5 Finite subgroups of $SL(2, \mathbb{C})$

In this section the results previously obtained are used to give a classification of the finite subgroups of $SL(2, \mathbb{C})$.

Let us define the following map:

$$\begin{aligned} \varphi : SU(2) &\longrightarrow SO(3) \\ A &\longmapsto \varphi(A), \end{aligned}$$

where the orthogonal transformation $\varphi(A)$ is defined for $X \in \mathfrak{su}(2)$ as

$$\begin{aligned} \varphi(A)(X) : \mathfrak{su}(2) &\longrightarrow \mathfrak{su}(2) \\ X &\longmapsto AXA^{-1}. \end{aligned}$$

Observation 65. $\varphi(A)$ has the following properties:

1. The product AXA^{-1} is indeed in $\mathfrak{su}(2)$, since the elements of $\mathfrak{su}(2)$ are characterized by having zero trace and $\mathrm{Tr}((AXA^{-1})) = \mathrm{Tr}(X) = 0$.

2. Let us remember that by Corollary 57 $\mathfrak{su}(2) \cong \mathbb{R}^3$, so $\varphi(A)$ as it is defined gives us an actual transformation of $\mathbb{R}^3$.

3. By Lemma 51 we have

   $$|X|^2 = \det(X) = \det(AXA^{-1}) = |AXA^{-1}|^2$$

   that is, $\varphi(A)$ is orthogonal.

Theorem 66. We have the following isomorphism:

$$\frac{SU(2)}{\{\pm I\}} \cong SO(3).$$

Sketch of the proof. By remark 56, $SU(2)$ is connected. The map φ is continuous, so $\varphi(SU(2))$ is contained in the connected component of $O(3)$ that contains the identity. This component is clearly $SO(3)$. Moreover, φ is surjective (for details please check [28, Lemma 2B p. 34]).

Besides, its kernel is $\ker(\varphi) = \{A \in SU(2) \mid AE_iA^{-1} = E_i, \text{for}\, i = 1,2,3\} = \{\pm I\}$. (again, see [28, Lemma 2B p. 34]).

From the last two observations and applying the first isomorphism theorem, we have the result.

□

As a consequence of the previous theorem, the finite subgroups of $SU(2)$ are the preimages of the ones of $SO(3)$ (which were listed in Theorem 64 and have even order), and odd-order cyclic groups.

Definition 67. If $G < SO(3)$ is a finite subgroup, then the subgroup $\varphi^{-1}(G) < SU(2)$ and any $\Gamma < SL(2,\mathbb{C})$ conjugated to $\varphi^{-1}(G)$ is called the *binary subgroup* corresponding to G.

Observation 68. These groups Γ have twice as elements as G, since φ is 2 to 1. This is reflected in the names of the groups in the following theorem.

Theorem 69. Let Γ be a finite subgroup of $SL(2,\mathbb{C})$, then Γ is one of the following (up to conjugacy):

Name	Order	Notation	Generators	Relation
Cyclic group	n	C_n	b	$b^n = 1$
Binary dihedral group	$4n$	$\tilde{D}_n$	b, c	$b^2 = c^n = (bc)^2 = -1$
Binary tetrahedral group	24	$\tilde{T}$	b, c	$b^3 = c^3 = (bc)^2 = -1$
Binary octahedral group	48	$\tilde{O}$	b, c	$b^3 = c^4 = (bc)^2 = -1$
Binary icosahedral	120	$\tilde{I}$	b, c	$b^3 = c^5 = (bc)^2 = -1$

where $\varphi(b) = \beta$ and $\varphi(c) = \gamma$; with β, γ the generators in presentations given by Theorem 64.

Proof. It can be found in [21, II§I p.35]. □

2.6 Quotient singularities

In this section we want to use the classification of finite subgroups of $SL(2,\mathbb{C})$ to give a relation between the ADE singularities studied in Chapter 1 and quotient spaces $\mathbb{C}^2/\Gamma$, with Γ one of such subgroups.

Given Γ a finite subgroup of $SL(2,\mathbb{C})$, we can define a right action of Γ on $\mathbb{C}[z_1,z_2]$ as follows. If $z = (z_1,z_2) \in \mathbb{C}^2$, $g \in \Gamma$, $f \in \mathbb{C}[z_1,z_2]$, then

$$(fg)(z) := f(g(z)).$$

Definition 70. A polynomial $f \in \mathbb{C}[z_1, z_2]$ is called Γ-*invariant* if

$$fg = f$$

for every $g \in \Gamma$. The set of Γ-invariant polynomials is denoted by S^Γ and is a subalgebra of $\mathbb{C}[z_1, z_2]$.

Lemma 71. Let $f \in \mathbb{C}[z_1, z_2]$ be a homogeneous polynomial of degree n, then fg is homogeneous of degree n for all $g \in \Gamma$.

Proof. By definition, the polynomial f has the form

$$f = \sum_{i+j=n} a_{ij} z_1^i z_2^j.$$

If Γ is a subgroup of $SL(2, \mathbb{C})$ then, up to conjugation, we can suppose its a subgroup of $SU(2)$. Then if $g \in \Gamma$ it holds

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{C}$.

$$\begin{aligned}
(fg)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= f(g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = f(\begin{pmatrix} \alpha z_1 + \beta z_2 \\ -\bar{\beta} z_1 + \bar{\alpha} z_2 \end{pmatrix}) = \sum_{i+j=n} a_{ij} (\alpha z_1 + \beta z_2)^i (-\bar{\beta} z_1 + \bar{\alpha} z_2)^j \\
&= \sum_{i+j=n} a_{ij} (\sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} (\alpha z_1)^{i-k} (\beta z_2)^k \sum_{l=0}^{j} \begin{pmatrix} j \\ l \end{pmatrix} (-\bar{\beta} z_1)^{j-l} (\bar{\alpha} z_2)^l) \\
&= \sum_{i+j=n} a_{ij} \sum_{k=0}^{i} \sum_{l=0}^{j} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} (\alpha z_1)^{i-k} (\beta z_2)^k (-\bar{\beta} z_1)^{j-l} (\bar{\alpha} z_2)^l \\
&= \sum_{i+j=n} a_{ij} \sum_{k=0}^{i} \sum_{l=0}^{j} \begin{pmatrix} i \\ k \end{pmatrix} \begin{pmatrix} j \\ l \end{pmatrix} (\alpha)^{i-k} (-\bar{\beta} z_1)^{j-l} (z_1)^{i-k+j-l} (\beta)^k (\bar{\alpha})^l (z_2)^{k+l}.
\end{aligned}$$

This last expression has degree $i - k + j - l + k + l = i + j = n$. ☐

A polynomial can be decomposed in a sum of homogeneous polynomials. It happens that a polynomial is Γ-invariant if and only if each one of its homogeneous components is Γ-invariant (see [21, II§6, p.54]).

Therefore, if we want to find the Γ-invariant polynomials it is enough to know the homogeneous Γ-invariant ones.

Theorem 72. Let Γ be a finite subgroup of $SU(2)$. Then S^Γ is generated by three homogeneous invariant polynomials $f_1, f_2, f_3 \in \mathbb{C}[z_1, z_2]$ which satisfy a polynomial equation

$$h(f_1, f_2, f_3) = 0.$$

Group	Equation h
Cyclic of order n	$f_1^2 + f_2^2 + f_3^n = 0$
Binary dihedral	$f_1^2 + f_2^2 f_3 + f_3^{n-1} = 0$
Binary tetrahedral	$f_1^2 + f_2^3 + f_3^4 = 0$
Binary octahedral	$f_1^2 + f_2^3 + f_2 f_3^3 = 0$
Binary icosahedral	$f_1^2 + f_2^3 + f_3^5 = 0$

Proof. May be found in [20, II§9-13, p.50]. $\square$

Theorem 73. The algebra homomorphism

$$\Phi : \mathbb{C}[x, y, z] \longrightarrow \mathbb{C}[z_1, z_2]$$
$$\Phi(x) = f_1, \Phi(y) = f_2, \Phi(z) = f_3$$

induces an isomorphism

$$\mathbb{C}[x, y, z]/\langle h\rangle \cong S^\Gamma$$

where $\langle h\rangle$ is the ideal generated by h.

Proof. The polynomials f_1, f_2, f_3 generate S^Γ, then Φ is surjective and its kernel is precisely $\langle h\rangle$. Applying the first isomorphism theorem we have the result. $\square$

We define the following affine variety

$$\mathcal{V} := \{(x, y, z) \in \mathbb{C}^3 \mid h(x, y, z) = 0\}.$$

Let us consider the map

$$F : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$$
$$(z_1, z_2) \longmapsto (f_1(z_1, z_2), f_2(z_1, z_2), f_3(z_1, z_2)).$$

This map has the following properties.

Proposition 74.
1. $F(\mathbb{C}^2) = \mathcal{V}$.
2. $F(z) = F(w)$ if and only if $w = gz$ for some $g \in \Gamma$.
3. F is proper and closed.
4. The map $F : \mathbb{C}^2 \longrightarrow \mathcal{V}$ with image restricted to $\mathcal{V}$ is open.
5. F is continuous.

Proof. Please see [21, II§9 p.66]. $\square$

This proposition implies the following:

Theorem 75 (Klein)**.** The polynomial map F induces an homeomorphism $\tilde{F} : \mathbb{C}^2/\Gamma \longrightarrow \mathcal{V}$ between the orbit space $\mathbb{C}^2/\Gamma$ and the affine variety $\mathcal{V}$.

From now, $\mathcal{V}$ will be called the *affine orbit variety.*

Proposition 76. Let Γ be a non trivial finite subgroup of $SL(2, \mathbb{C})$ acting on $\mathbb{C}^2$ by multiplication. Then $\mathbb{C}^2/\Gamma$ is isomorphic to $h^{-1}(0)$, where h is the corresponding polynomial in column 1 of the following table:

$h(x, y, z)$	**Subgroup of** $SU(2)$
$x^2 + y^2 + z^n$	Cyclic of order n
$x^2 + y^2z + z^{n-1}$	Binary dihedral of order $4(n-2)$
$x^2 + y^3 + z^4$	binary tetrahedral
$x^2 + y^3 + yz^3$	binary octahedral
$x^2 + y^3 + z^5$	binary icosahedral

We observe that the polynomials h are the same polynomials defining the singularities of type ADE in Chapter 1 (please see Definition 9).

2.7 The link of ADE singularities

We end up this chapter establishing a relationship between the ADE singularities seen as quotient singularities and spaces of the form S^3/Γ, with S^3 the 3-sphere.

Theorem 77. Let $\mathcal{V}$ be a hypersurface with an isolated singular point at the origin. Then $\mathcal{V}$ intersects transversally every sufficiently small sphere around 0 with radius $\varepsilon > 0$, $S_\varepsilon(0)$.

The intersection $\mathcal{V} \cap S_\varepsilon(0)$ is a smooth manifold.

Proof. May be found in [24, Theorem 5.11, p.53]. $\square$

Definition 78. The manifold $\mathcal{V} \cap S_\varepsilon(0)$ is called the *link* of the singular point.

Proposition 79. The *link* of $\mathbb{C}^2/\Gamma$ is a smooth compact 3-manifold diffeomorphic to S^3/Γ.

Proof. Please see [29, p.50]. $\square$

Observation 80. Since S^3 is simply connected and the action of Γ in S^3 is free, S^3 is the universal cover of the spaces S^3/Γ and therefore the fundamental group $\pi_1(S^3/\Gamma)$ is just Γ.

We will return to the spaces S^3/Γ in Chapter 4.

Chapter 3

The McKay correspondence

In this chapter we will study some theory of representations of groups to define a directed graph encoding the structure of the representation theory of the group.

In the particular case where Γ is a finite subgroup of $SU(2)$, this graph is undirected and it happen to coincide with the dual graph of the minimal resolution of the quotient singularity $\mathbb{C}^2/\Gamma$ studied in Chapter 2. This is the so called McKay correspondence, named after John McKay who was the first one to study this phenomenon.

3.1 Group representations

Definition 81. Let us consider a group G, a vector space V over $\mathbb{R}$ or $\mathbb{C}$ and $GL(V)$ the group of automorphisms of V. A *representation* of G on V is a homomorphism $\rho : G \longrightarrow GL(V)$. The *degree* of ρ is $\deg(\rho) = \dim(V)$.

A representation of G on V gives rise to a linear action of G on V

$$\begin{aligned} G \times V &\longrightarrow V \\ gv &= \rho(g)v. \end{aligned} \tag{3.1}$$

The action previously presented gives V the structure of G-module.

Conversely, if V is a G- module then we can define a representation of G on V:

$$\begin{aligned} \rho : G &\longrightarrow GL(V) \\ \rho(a)(v) &= av. \end{aligned}$$

Definition 82. Let $\rho : G \longrightarrow \mathrm{GL}(V)$ and $\sigma : G \longrightarrow GL(W)$ be two representations of G. A linear morphism $\vartheta : V \longrightarrow W$ is said to be *G-equivariant* is for every $g \in G$ and for every $v \in V$ the following equality holds:

$$\sigma(g)(\vartheta(v)) = \vartheta(\rho(g)(v)).$$

Definition 83. Let $\rho : G \longrightarrow GL(V)$ and $\sigma : G \longrightarrow GL(W)$ be two representations of G. We say that ρ is *equivalent* to σ if there exist a G-equivariant isomorphism $\vartheta : V \longrightarrow W$.

3.2 Irreducible representations

Definition 84. Let $\rho : G \longrightarrow GL(V)$. A subspace $W \subset V$ is said to be G-*invariant* if for all $g \in G$ it holds that

$$\rho(g)(W) \subset W.$$

Definition 85. A representation of non-zero degree $\rho : G \longrightarrow GL(V)$ is said to be *irreducible* if the only invariant subspaces are $\{0\}$ and V.

Otherwise, if there exist a non trivial invariant subspace, the representation is called *reducible.*

Definition 86. Let $\rho : G \longrightarrow GL(V)$ be a representation, with V of finite dimension, and $U \subset V$ a G-invariant subspace. U is said to be *irreducible* if the restriction

$$\rho|_U : G \longrightarrow GL(U)$$
$$\rho|_U(g)(u) = \rho(g)(u)$$

is an irreducible representation.

Definition 87. Given a family of representations $\rho_i : G \longrightarrow GL(V_i)$ with $i \in \{1, \cdots, r\}$ and $V = V_1 \oplus \cdots \oplus V_r$ we take $v = v_1 + \cdots + v_r \in V$ with $v_i \in V_i$ and define the *direct sum of representations*:

$$\rho_1 \oplus \cdots \oplus \rho_r : G \longrightarrow GL(V)$$
$$(\rho_1 \oplus \cdots \oplus \rho_r)(g)(v) = \rho_1(g)(v_1) + \cdots + \rho_r(g)(v_r).$$

Example 88. Let $\rho : G \longrightarrow GL(n, \mathbb{F})$ and $\sigma : G \longrightarrow GL(m, \mathbb{F})$ be two representations of G. Then it holds that $\rho \oplus \sigma$ is equivalent to the representation

$$\tau : G \longrightarrow GL(n + m, \mathbb{F})$$
$$\tau = \begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}.$$

Observation 89. The direct sum of representations $\rho_1 \oplus \cdots \oplus \rho_r$ is reducible due to the fact that each V_i is G-invariant.

Theorem 90 (Maschke)**.** Let G be a finite group and $\rho : G \longrightarrow GL(V)$ a representation, with V a vector space over $\mathbb{R}$ or $\mathbb{C}$. If $U \subset V$ is a G-invariant subspace then there exist $W \subset V$ G-invariant subspace such that

$$V = U \oplus W.$$

Proof. Consider $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Let V be a vector space over $\mathbb{F}$ and n its dimension. Then V and $\mathbb{F}^n$ are isomorphic as vector spaces. Let $\langle -, - \rangle$ be an inner product on V. By Lemma 43 of Chapter 2, there exist a $\rho(G)$-invariant inner product $\langle -, - \rangle_G$.

Let W be the orthogonal complement of U with respect to $\langle -, - \rangle_G$, then $V = U \oplus W$. By hypothesis, U is G-invariant, which implies that for $g \in G$ and $u \in U$, $\rho(g)(u) \in U$. We take $w \in W$, it holds that

$$0 = \langle w, u \rangle_G = \langle \rho(g)(w), \rho(g)(u) \rangle_G.$$

Therefore, $\rho(g)(w)$ is orthogonal to U with respect to $\langle -, - \rangle_G$, that is, $\rho(g)(w) \in W$ and W is a G-invariant subspace such that $V = U \oplus W$. □

Observation 91. Under the assumptions of Maschke's theorem, ρ is equivalent to the representation $\sigma \oplus \tau$, where $\sigma : G \longrightarrow GL(U)$ and $\tau : G \longrightarrow GL(W)$ are given by

$$\sigma(g) := \rho(g)|_U \quad , \quad \tau(g) := \rho(g)|_W.$$

Definition 92. A representation $\rho : G \longrightarrow V$ is said to be *completely reducible* if it is equivalent to a sum $\rho_1 \oplus \cdots \oplus \rho_r$, where each ρ_i is an irreducible representation.

Theorem 93. Let G be a finite group and $V \neq \{0\}$, then every representation $\rho : G \longrightarrow GL(V)$ is completely reducible.

Proof. We proceed by induction over the degree n of the representation. If $\deg(\rho) = \mathrm{V} = 1$ then $\{0\}$ and V are the only vector subspaces and hence ρ is irreducible.

Let us assume, by induction hypothesis, that the result holds for representations of degree smaller than n. If the representation is irreducible, there is nothing to prove, so we suppose that ρ is reducible. This implies that V has a non trivial G-invariant subspace U and by theorem 90 there exist a G-invariant subspace W such that $V = U \oplus W$ and ρ is equivalent to the direct sum of representations $\sigma \oplus \tau$, given by observation 91.

Clearly $\deg(\sigma) = \dim(U) < \dim(V)$, and therefore σ is equivalent to

$$\sigma_1 \oplus \cdots \oplus \sigma_r$$

where each σ_i is irreducible.

Analogously, $\deg(\tau) = \dim(W) < \dim(V)$ and τ is equivalent to a direct sum of irreducible representations

$$\tau_1 \oplus \cdots \oplus \tau_s.$$

Therefore, ρ is equivalent to

$$\sigma_1 \oplus \cdots \oplus \sigma_r \oplus \tau_1 \oplus \tau_s.$$

□

Proposition 94. If G is a finite group then there exist a finite number of complex irreducible representations, up to equivalence, and it is equal to the number of conjugacy classes in G.

Proof. Can be found in [17, Corollary 10.7, p. 91]. □

3.3 Character of a representation

In this section we will consider every representation to be complex.

Definition 95. Let V be a vector space over $\mathbb{C}$ with basis B, G a group and $\rho : G \longrightarrow GL(V)$ a representation of G. The *character of* ρ, also called a character of G, is the function

$$\chi : G \longrightarrow \mathbb{C}$$
$$\chi(g) = \mathrm{Tr}(M_\rho(g))$$

where $M_\rho(g)$ is the matrix associated to the automorphism $\rho(g)$ with respect to the basis B. If ρ is an irreducible representation, its character is also said to be irreducible, otherwise χ is called reducible.

The conjugate of the character χ is

$$\overline{\chi} : G \longrightarrow \mathbb{C}$$
$$\overline{\chi}(g) = \overline{\chi(g)}.$$

Proposition 96. The character of a representation does not depend on the choice of B.

Proof. Let $\tilde{B}$ be another basis and $\tilde{M_\rho}(g)$ the matrix of ρ with respect to $\tilde{B}$. Then it holds that

$$\tilde{M_\rho}(g) = T^{-1} M_\rho(g) T$$

where T is the change of basis matrix. Therefore

$$\mathrm{Tr}(\tilde{M_\rho}(g)) = \mathrm{Tr}(T^{-1} M_\rho(g) T) = \mathrm{Tr}(M_\rho(g)).$$

□

Proposition 97. Equivalent representations have the same character.

Proof. Let $\rho : G \longrightarrow GL(V)$ and $\sigma : G \longrightarrow GL(W)$ be two equivalent representations of G with characters χ_ρ and χ_σ, respectively. Then there is a G-equivariant isomorphism $\vartheta : V \longrightarrow W$, which has an inverse $\vartheta^{-1} : W \longrightarrow V$. For every $g \in G$ the following holds

$$\vartheta^{-1} \circ \sigma(g) \circ \vartheta = \rho(g).$$

Therefore, we have

$$\chi_\rho = \mathrm{Tr}(M_{\rho(g)}) = \mathrm{Tr}(M_{\vartheta^{-1}} M_\sigma M_\vartheta) = \mathrm{Tr}(M_\vartheta M_{\vartheta^{-1}} M_\sigma) = \mathrm{Tr}(M_\sigma) = \chi_\sigma.$$

□

Definition 98. Let χ be the character of a representation $\rho : G \longrightarrow GL(V)$. The dimension of V is called the *degree* of χ.

Proposition 99. Let χ be the character of a representation $\rho : G \longrightarrow GL(V)$. If $g \in G$ has order m and $e \in G$ is the identity element of G then the following holds:

1. $\chi(e) = \dim(V)$
2. $\chi(g)$ is a sum of m-roots of unity
3. $\chi(g^{-1}) = \overline{\chi(g)}$
4. $\chi(g)$ is a real number if g and g^{-1} are conjugated.

Proof. Please see [17, Proposition 13.9, p.122]. □

Proposition 100. Let $\rho : G \longrightarrow GL(V)$ be a representation of G over $\mathbb{C}$ whose decomposition into direct sum of irreducible representations is

$$\rho = \rho_1 \oplus \cdots \oplus \rho_r$$

with $\rho_i : G \longmapsto GL(U_i)$. Then the character of ρ is equal to the sum of the characters of $\rho_1, \cdots, \rho_r$.

Proof. Please see [17, Proposition 13.18, p.127]. □

Proposition 101. Let $\rho : G \longrightarrow GL(V)$ be a representation of G on $\mathbb{C}$ written as a direct sum of irreducible representations $\rho_i : G \longrightarrow GL(U_i)$,

$$\rho = \rho_1 \oplus \cdots \oplus \rho_r.$$

If U is any irreducible G-submodule on V, then $U \cong U_i$ for some i.

Proof. Please see [17, Proposition 10.2, p.90]. □

Example 102. Let G be a cyclic group of order n and g its generator. Then G has n irreducible representations of the form

$$\rho_i(g)(\omega^i)$$

with $\omega \in \mathbb{C}$ an n-th root of unity. Therefore there are n irreducible characters $\chi_1, \cdots, \chi_n$ of the form

$$\chi_i(g) = \omega^i.$$

3.4 Inner product of characters

Let G be a finite group. The set $\mathbb{C}^G := \{\vartheta : G \longrightarrow \mathbb{C}\}$ is a vector space over $\mathbb{C}$ with the operations

$$(\vartheta + \psi)(g) = \vartheta(g) + \psi(g)$$
$$(\lambda\vartheta)(g) = \lambda(\vartheta(g))$$

where $g \in G$, $\lambda \in \mathbb{C}$.

It is also possible to equip $\mathbb{C}^G$ with an inner product given by

$$\langle \vartheta, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \vartheta(g)\overline{\psi(g)}.$$

Proposition 103. Let G be a finite group and χ, ψ characters of G. Then

$$\langle \chi, \psi \rangle = \langle \psi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Proof. By proposition 99 we have that $\overline{\psi(g)} = \psi(g^{-1})$ for all $g \in G$, therefore

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Now, since $\{g^{-1} : g \in G\} = G$, we have

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\psi(g) = \langle \psi, \chi \rangle.$$

□

We observe that since $\langle \psi, \chi \rangle = \overline{\langle \chi, \psi \rangle}$, the previous proposition implies that $\langle \chi, \psi \rangle$ is a real number.

3.5 The McKay quiver

Definition 104. Let V and W be vector spaces over $\mathbb{C}$ with bases $v_1, \cdots, v_m$ and $w_1, \cdots, w_n$ respectively. The *tensor product space* $V \otimes W$ is defined to be the mn-dimensional vector space over $\mathbb{C}$ with basis given by $\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

Thus, $V \otimes W$ consists of all expressions of the form

$$\sum_{i,j} \lambda_{ij}(v_i \otimes w_j), \ (\lambda_{ij} \in \mathbb{C}).$$

For $v = \sum_{i=1}^{m} \lambda_i v_i \in V$ and $w = \sum_{j=1}^{n} \mu_j w_j \in W$, with $\lambda_i, \mu_j \in \mathbb{C}$, we define

$$v \otimes w = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j).$$

Definition 105. If $\rho : G \longrightarrow GL(V)$ and $\sigma : G \longrightarrow GL(W)$ are two representations of G over $\mathbb{C}$, the *tensor product* of the representations $\rho \otimes \sigma : G \longrightarrow GL(V \otimes W)$ is defined as

$$(\rho \otimes \sigma)(g)(v \otimes w) := \rho(g)(v) \otimes \sigma(g)(w).$$

Proposition 106. Let $\rho : G \longrightarrow GL(V)$ and $\sigma : G \longrightarrow GL(W)$ be two representations of G over $\mathbb{C}$ with characters χ and ψ, respectively. Then the character of $\rho \otimes \sigma$ is the product character $\chi\psi$, where for all $g \in G$

$$\chi\psi(g) = \chi(g)\psi(g).$$

Proof. Please see [17, Proposition 19.6, p.192] □

Theorem 107. Let $\chi_1, \cdots, \chi_k$ be the characters corresponding to the irreducible complex representations of a finite group G. If ψ is any other character of G, then

$$\psi = d_1\chi_1 + \cdots + d_k\chi_k$$

with $d_i = \langle \psi, \chi_i \rangle$ for $1 \leq i \leq k$.

Proof. Please see [17, Theorem 14.17, p. 142]. □

Observation 108. Let $\rho : G \longrightarrow GL(V)$ be a representation of a finite group G over $\mathbb{C}$ with character χ. Let $\rho_1, \cdots, \rho_d$ be the irreducible, representations of G and $\chi_1, \cdots, \chi_d$ its irreducible characters. Then by the previous theorem we have

$$\rho \otimes \rho_i = \sum_j n_{ij}\rho_j$$

where

$$n_{ij} = \langle \rho \otimes \chi_i, \chi_j \rangle = \frac{1}{|G|}\sum_{g \in G} \chi(g)\chi_i(g)\chi_j(g^{-1}). \tag{3.2}$$

Definition 109. Let $\rho : G \longrightarrow GL(V)$ be a representation of a finite group G over $\mathbb{C}$ with character χ. Let $\chi_1, \cdots, \chi_d$ be its irreducible characters. We define the *Mckay quiver* of G with respect to the representation ρ, denoted by $\mathcal{G}_\rho$ as follows:

- To each irreducible representation of G corresponds a vertex in $\mathcal{G}_\rho$.
- There is an arrow from ρ_i to ρ_j if and only if $n_{ij} > 0$ and n_{ij} is called the *weight* of the arrow.
- If $n_{ij} = n_{ji}$ then we put an edge between ρ_i and ρ_j instead of a double arrow.

3.6 The McKay graph for finite subgroups of $SL(2, \mathbb{C})$

In this section we will prove that for the finite subgroups of $SL(2, \mathbb{C})$ the McKay quivers are graphs and we will draw them. In such graphs we will put a dotted line instead of an edge joining the vertex corresponding to the trivial representation. The reason for this is the *McKay correspondence*, which will be properly enounced in Theorem 115.

Proposition 110. Let Γ be a finite subgroup of $SL(2, \mathbb{C})$, $\rho_0 : \Gamma \longrightarrow GL(2, \mathbb{C})$ be the representation of Γ given by the inclusion and χ_0 its character. If χ_i and χ_j are irreducible characters of Γ, then $\langle \chi_0\chi_i, \chi_j \rangle = \langle \chi_0\chi_j, \chi_i \rangle$.

Proof. We saw in Remark 53 of the previous chapter that every element of $SU(2)$ has the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

therefore, its trace χ_0 is a real number. This implies

$$\begin{aligned} \langle \chi_0\chi_i, \chi_j \rangle &= \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \chi_0(g)\chi_i(g)\overline{\chi_j(g)} \\ &= \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \chi_i(g)\overline{\chi_0(g)\chi_j(g)} \\ &= \langle \chi_i, \chi_0\chi_j \rangle = \langle \chi_0\chi_j, \chi_i \rangle. \end{aligned} \tag{3.3}$$

The commutativity of the last equality was proven in proposition 103. □

The following proposition shows that the McKay quiver of the finite subgroups of $SU(2)$ is a graph.

Proposition 111. Let Γ be a finite subgroup of $SL(2,\mathbb{C})$ and $\rho_0 : \Gamma \longrightarrow GL(2,\mathbb{C})$ be the representation of Γ given by the inclusion. Then for every pair of indices i, j it holds

$$n_{ij} = n_{ji} \leq 1, \text{ and}$$
$$n_{ii} = 0$$

Proof. By (3.3) of the previous proposition, $n_{ij} = n_{ji}$.

Since every finite subgroup of $SL(2,\mathbb{C})$ is conjugated to a subgroup of $SU(2)$ and the trace is invariant under conjugation, we can suppose without loss of generality that $\chi_0(g)$ is the trace of a unitary matrix with determinant equal to 1, which have the form

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

Moreover, since $a^2+b^2+c^c+d^2 = 1$, the maximum value of a is 1 and therefore, $\chi_0(g) = 2a \leq 2$.

Using the Cauchy–Schwarz inequality and the previous observation we have:

$$\begin{aligned} n_{ij} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} \chi_0(g)\chi_i(g)\overline{\chi_j(g)}) &\leq \frac{1}{|\Gamma|}\sqrt{\sum_{g\in\Gamma}|\chi_0(g)|^2|\chi_i(g)|^2}\sqrt{\sum_{g\in\Gamma}|\chi_j(g)|^2} \\ < \frac{1}{|\Gamma|}\sqrt{4\sum_{g\in\Gamma}|\chi_i(g)|^2}\sqrt{\sum_{g\in\Gamma}|\chi_j(g)|^2} &= \frac{2}{|\Gamma|}\sqrt{|\Gamma|\langle\chi_i,\chi_i\rangle}\sqrt{|\Gamma|\langle\chi_j,\chi_j\rangle} \\ = \frac{2}{|\Gamma|}\sqrt{|\Gamma|}\sqrt{|\Gamma|} = 2 \end{aligned}$$

Applying formula (3.2) we have

$$n_{ii} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)\chi_i(g)\overline{\chi_i(g)} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)|\chi_i(g)|^2. \tag{3.4}$$

By the list given in theorem 69 of Chapter 2, if Γ has even order, then it is binary and therefore contains the element $-I$. Then for every $g \in \Gamma$, $-g$ is also in Γ.

We observe that $\chi_0(g) = -\chi_0(-g)$, which implies

$$n_{ii} = -\frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(-g)|\chi_i(g)|^2 = -\frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)|\chi_i(g)|^2. \tag{3.5}$$

Adding equations (3.4) and (3.5) we have

$$2n_{ii} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)|\chi_i(g)|^2 - \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)|\chi_i(g)|^2 = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)(|\chi_i(g)|^2 - |\chi_i(g)|^2) = 0$$

thus, $n_{ii} = 0$, as we wanted.

Otherwise, if Γ has odd order n, it is a cyclic group. Let $A \in \Gamma$ be its generator, where $A \neq I$ and $A^n = I$. Then we have

$$(I + A + A^2 + \cdots + A^{n-1})(I - A) = 0$$

and one factor of the product must be zero. Since we supposed $A \neq I$, then $(I - A) \neq 0$. We conclude that $(I + A + A^2 + \cdots + A^{n-1}) = 0$.

Let us remember that by example 102 the characters of cyclic groups are powers of roots of unity. Using this and the previous result we have:

$$n_{ii} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g)|\chi_i(g)|^2 = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\chi_0(g) = \frac{1}{|\Gamma|}\mathrm{Tr}(I + A + A^2 + \cdots + A^{n-1}) = 0.$$

□

3.7 Character table for finite subgroups of $SU(2)$

Proposition 94 stands that the number of irreducible representations of a finite group Γ is equal to the number of conjugacy classes in Γ. Therefore, it is convenient to display all the values of irreducible characters of Γ in a square matrix; whose entries are related to each other via some *orthogonality relations.*

This matrix is called the *character table* and will be used to get the necessary information to compute the McKay graph of the finite subgroups of $SU(2)$.

Definition 112. Let $\chi_1, \cdots \chi_k$ be the irreducible characters of Γ and let $g_1, \cdots, g_k$ be representatives of the conjugacy classes of Γ. The $k \times k$ matrix whose ij-entry is $\chi_i(g_j)$, for all i, j with $1 \leq i \leq k$, $1 \leq j \leq k$, is called the *character table* of Γ.

As an example, we will show the character table of the binary icosahedral group $\tilde{I}$ and some of the calculations that lead to the McKay graph.

Each conjugacy class is denoted by the order of its elements. If there are more than one class of element of a given order, they will be distinguished by subscripts. The "size" row shows the number of elements on the conjugacy class.

Table 3.1: Character table of binary icosahedral group $\tilde{I}$. It may be found in [17, p. 445].

Class	1	2	4	6	3	10_1	5_1	5_2	10_2
size	1	1	30	20	20	12	12	12	12
χ_0	2	-2	0	1	-1	μ	ν	$-\mu$	$-\nu$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	0	1	-1	$-\nu$	$-\mu$	ν	μ
χ_3	3	3	-1	0	0	$-\nu$	μ	$-\nu$	μ
χ_4	3	3	-1	0	0	μ	$-\nu$	μ	$-\nu$
χ_5	4	4	0	1	1	-1	-1	-1	-1
χ_6	4	-4	0	-1	1	1	-1	-1	1
χ_7	5	5	1	-1	-1	0	0	0	0
χ_8	6	-6	0	0	0	-1	1	1	-1

Here $\mu = \frac{\sqrt{5}+1}{2}$, $\nu = \frac{\sqrt{5}-1}{2}$.

In this case, we take the representation $\rho_0 : \tilde{I} \longrightarrow GL(2, \mathbb{C})$ induced by the inclusion of $\tilde{I}$ in $SU(2)$. The character of this representation is χ_0. We notice that all the characters take real values.

In order to make the computations, let us observe the following:

Definition 113. A *class function* on G is a function $\psi : G \longrightarrow \mathbb{C}$ such that $\psi(x) = \psi(y)$ whenever x and y are conjugate elements of G; that is, ψ is constant on conjugacy classes.

Proposition 114. Suppose that $g,\ h \in G$. Then g is conjugate to h if and only if $\chi(g) = \chi(h)$ for all characters χ of G.

Proof. Please see [17, Proposition 15.5, p.154]. $\square$

Proposition 114 says that characters are class functions. Therefore the formula 3.2 becomes

$$n_{ij} = \frac{1}{|\tilde{I}|} \sum_{k=1}^{\alpha} |Cl_k| \chi_2(g_k) \chi_i(g_k) \chi_j(g_k)$$

where α is the number of conjugacy classes of $\tilde{I}$, g_k is a representative of the k-th conjugacy class and Cl_k is the size of the conjugacy class of g_k.

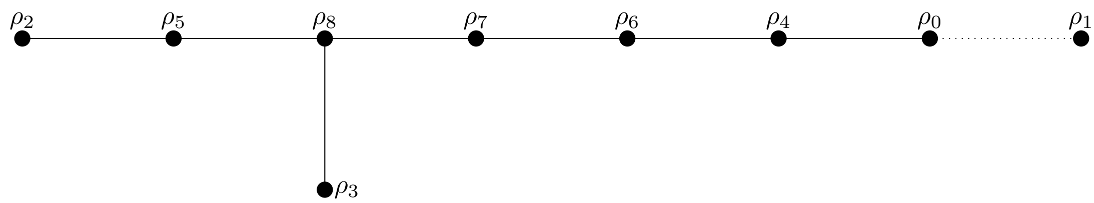
We calculate explicitly, as an example, the value $n_{1,2}$:

$$\begin{aligned} n_{1,2} = & \frac{1}{|\tilde{I}|}[1(2 \cdot 1 \cdot 2) + 1(-2 \cdot 1 \cdot -2) + 30(0 \cdot 1 \cdot 0) + 20(1 \cdot 1 \cdot 1) + 20(-1 \cdot 1 \cdot -1) \\ & + 12(\mu \cdot 1 \cdot -\nu) + 12(\nu \cdot 1 \cdot -\mu) + 12(-\mu \cdot 1 \cdot \nu) + 12(-\nu \cdot 1 \cdot \mu)] \\ & = \frac{1}{120}[4 + 4 + 0 + 20 + 20 + 12(-1) + 12(-1) + 12(-1) + 12(-1)] = \frac{1}{120}[48 - 48] = 0. \end{aligned}$$

Using this method we obtain the remaining n_{ij}. We display the results in the following table:

$i \setminus j$	0	1	2	3	4	5	6	7	8
0	0	1	0	0	1	0	0	0	0
1	1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	0	0	0
3	0	0	0	0	0	0	0	0	1
4	1	0	0	0	0	0	1	0	0
5	0	0	1	0	0	0	0	0	1
6	0	0	0	0	1	0	0	1	0
7	0	0	0	0	0	0	1	0	1
8	0	0	0	1	0	1	0	1	0

Thus, the McKay graph of $\tilde{I}$ is:



As we mentioned before, the line joining the representation ρ_0 with ρ_1 is dotted. The reason for this is the *McKay correspondence*, which will be properly stated in Theorem 115.

Now we show the character tables of all the remaining finite subgroups of $SU(2)$, together with their McKay graphs. As in the previous example, we consider ρ_0 the representation given by the inclusion and ρ_1 the trivial representation. Again, a dotted line will appear joining the vertex corresponding to the trivial representation ρ_1.

Table 3.2: Character table of cyclic group C_n. It may be found in [17, Example 1, p. 82].

Class	0	1	$\cdots$	$n-1$
size	1	1	$\cdots$	1
χ_1	1	1	$\cdots$	1
χ_2	1	ζ	$\cdots$	ζ^{n-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_n	1	ζ^{n-1}	$\cdots$	$\zeta^{(n-1)^2}$

Here $\zeta = \frac{2\pi i}{n}$.

Let us notice that the character χ_0 does not appear in the table. The reason for this is because the representation ρ_0 given by the inclusion is not irreducible.

The correspondent McKay graph of the cyclic groups is the following:

Figure 3.1: McKay graph of C_n

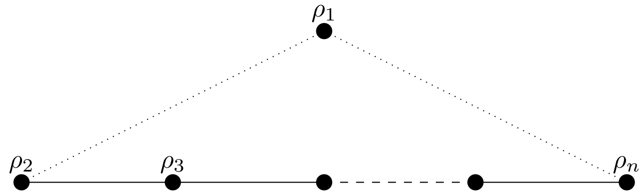


Table 3.3: Character table of binary dihedral group $\tilde{D}_n$, with n odd. It may be found in [17, p. 183].

Class	1_+	1_-	c^r $(1 \le r \le (n-1)/2)$	b	cb
size	1	1	2	n	n
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	$(-1)^r$	1	-1
χ_4	1	-1	$(-1)^r$	-1	1
ψ_j	2	$(-1)^j 2$	$\omega^{jr} + \omega^{-jr}$	0	0

Here ω is an $2n$-th primitive root of unity.

Table 3.4: Character table of binary dihedral group $\tilde{D}$, with $n = 2m$. It may be found in [17, p. 183].

Class	1_+	1_-	c^r $(1 \le r \le n-1)$	b	cb
size	1	1	2	n	n
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	$(-1)^r$	i	$-i$
χ_4	1	-1	$(-1)^r$	$-i$	i
ψ_j	2	$(-1)^j 2$	$\omega^{jr} + \omega^{-jr}$	0	0

Here ω is an $2n$-th root of unity and $1 \le j \le m-1$.

Again, we notice that χ_0 does not appear in the tables 3.3 and 3.4. This is due to the fact that ρ_0 is not irreducible. The McKay graph for the binary dihedral groups is the following:

Figure 3.2: McKay graph of $\tilde{D}_n$

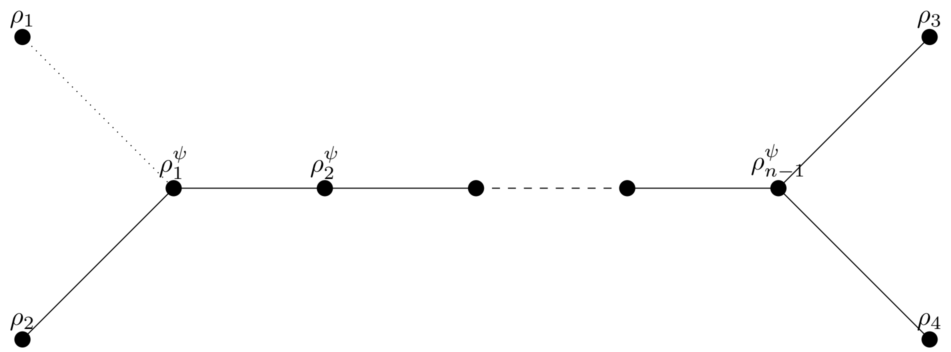


Table 3.5: Character table of binary tetrahedral group $\tilde{T}$. It may be found in [17, p. 440].

Class	1	2	4	3_1	3_2	6_1	6_2
size	1	1	6	4	4	4	4
χ_0	2	-2	0	-1	-1	1	1
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	ω	ω^2	ω	ω^2
χ_3	1	1	1	ω^2	ω	ω^2	ω
χ_4	3	3	-1	0	0	0	0
χ_5	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω
χ_6	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2

Here $\omega = e^{2\pi i/3}$.

Figure 3.3: McKay graph of $\tilde{T}$

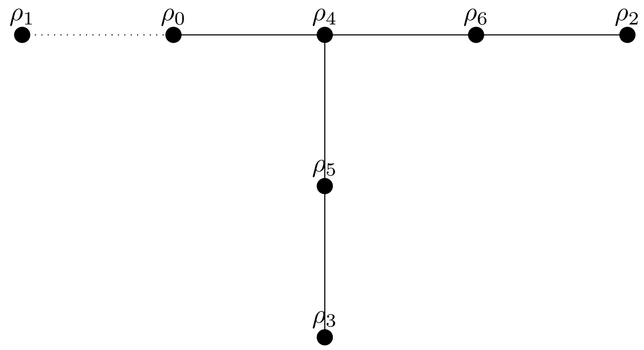
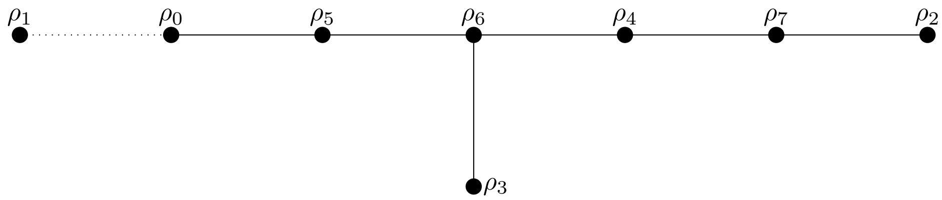


Table 3.6: Character table of binary octahedral group $\tilde{O}$. It may be found in [17, p. 444].

Class	1	2	8_1	4	8_2	6	3	4_2
size	1	1	6	6	6	8	8	12
χ_0	2	-2	$-\sqrt{2}$	0	$\sqrt{2}$	1	-1	0
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	-1	1	1	-1
χ_3	2	2	0	2	0	-1	-1	0
χ_4	3	3	-1	-1	-1	0	0	1
χ_5	3	3	1	-1	1	0	0	-1
χ_6	4	-4	0	0	0	-1	1	0
χ_7	2	-2	$\sqrt{2}$	0	$-\sqrt{2}$	1	-1	0

Figure 3.4: McKay graph of $\tilde{O}$



3.8 The McKay observation

In Chapter 1 we studied the resolutions of ADE singularities and the graphs (Dynkin diagrams) that encode the information of the intersection of components of the exceptional divisors of the blow up.

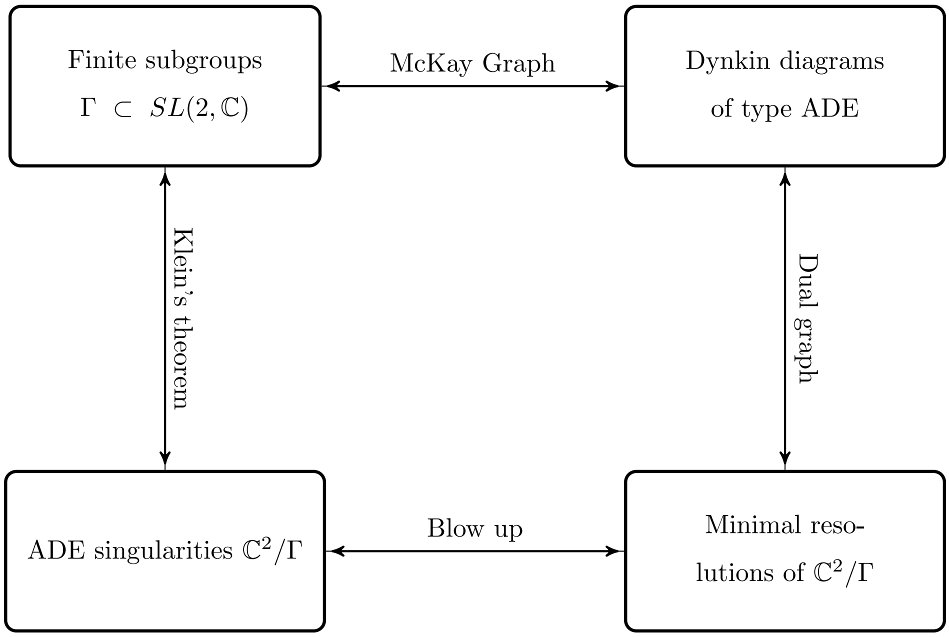
Later, in Chapter 2 we saw that such singularities can be thought as quotient singularities $\mathbb{C}^2/\Gamma$, with Γ a finite subgroup of $SL(2,\mathbb{C})$.

Now we have seen that the irreducible representations of the finite subgroups $\Gamma \subset SL(2,\mathbb{C})$ can be displayed in the McKay graph. The reason why we studied all the previous material is to state the following

Theorem 115. The McKay graph of the non trivial irreducible representations of Γ coincide with the dual graph of the resolution of the surface $\mathbb{C}^2/\Gamma$.

This is the reason why we "removed" the line joining the trivial representation to the rest of the graphs.

Theorem 115 was first observed by John McKay in [23] and that is why the relationship among this theories is now known as the *McKay correspondence*. This relationship is described in the following scheme.



Part II

The Dirac operator and its η-invariant

Chapter 4

Spin Geometry and the Dirac Operator

At this point we have studied the ADE singularities as quotient singularities $\mathbb{C}^2/\Gamma$ and by Proposition 79 we know that its link is is a smooth compact 3-manifold diffeomorphic to S^3/Γ, where Γ is a finite subgroup of S^3.

Now we will study further the spaces S^3/Γ, applying some results from representation theory in the context of differential geometry.

The Dirac operator D is a differential self-adjoint elliptic operator that will be properly defined in section 4.2. The main goal of this chapter is to develop the theory needed to compute the Dirac operator of S^3, of the spaces S^3/Γ and finally consider a representation of the fundamental group $\pi_1(S^3/\Gamma)$ to "twist" the Dirac operator of S^3/Γ. That requires the study of spinor and Clifford bundles (which will be done in section 4.1) and then spin manifolds (reviewed in section 4.1.4).

In the next chapter we will define the η-invariant and compute it for the twisted Dirac operator.

4.1 Spin Geometry

4.1.1 Clifford Algebras

Definition 116. Let V be a vector space over $\mathbb{R}$. For any nonnegative integer k we define the *k-th tensor power of* V as the tensor product of V with itself k times:

$$T^k V = V^{\otimes k} = V \otimes V \otimes \cdots \otimes V.$$

By convention $V^{\otimes 0} = \mathbb{R}$.

The *tensor algebra of* V is constructed as the direct sum

$$T(V) = \bigoplus_{k=0}^{\infty} T^k V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus \cdots$$

Definition 117. Let V be a finite dimensional real vector space equipped with a nondegenerate bilinear form $q(v, w) \in \mathbb{R}$, where $v,\ w \in V$. The *Clifford algebra* of V is

$$C(V, q) := T(V)/I(q)$$

where $I(q)$ is the ideal generated by elements of the form

$$v \otimes v + q(v, v)1, \qquad v \in V.$$

An alternative construction of the Clifford algebra is as follows. We pick an orthonormal basis $(e_1, \cdots, e_n)$ for V, then a basis for $C(V, q)$ is formed by the vectors

$$e_{i_1} e_{i_2} \cdots e_{i_k} \qquad (i_1 < i_2 < \cdots < i_k)$$

under the relations

$$e_i^2 = -1 \qquad e_i e_j = -e_j e_i.$$

This means that $C(V, q)$ has dimension 2^n.

If $V = \mathbb{R}^n$ and q is the standard inner product we denote the algebra $C(V, q)$ by C_n.

Theorem 118. 1. The first eight Clifford algebras are the following

n	0	1	2	3	4	5	6	7	8
C_n	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathrm{Mat}(2, \mathbb{H})$	$\mathrm{Mat}(4, \mathbb{C})$	$\mathrm{Mat}(8, \mathbb{R})$	$\mathrm{Mat}(8, \mathbb{R}) \oplus \mathrm{Mat}(8, \mathbb{R})$	$\mathrm{Mat}(16, \mathbb{R})$

where $\mathrm{Mat}(n, \mathbb{K})$ denotes the algebra of $n \times n$ matrices over $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

2. Clifford algebras satisfy an 8-fold periodicity given by

$$C(\mathbb{R}^{n+8}) \cong C(\mathbb{R}^n) \otimes C(\mathbb{R}^8).$$

Proof. May be found in [31, p.96] □

The previous theorem tells us that we can know C_n for every n. Now we stand a property of these algebras.

The algebra C_n can be divided into *even* and *odd* parts: C_n^0 and C_n^1 respectively. The part C_n^0 is spanned by the elements of the form $e_{i_1} \cdots e_{i_{2r}}$ and C_n^1 is spanned by the elements of the form $e_{i_1} \cdots e_{i_{2r+1}}$.

4.1.2 Spin groups and Spin representations

Observation 119. The following involution on a basis of $\mathbb{R}^n$

$$\varepsilon : C_n \longrightarrow C_n$$
$$\varepsilon(e_{i_1} \cdots e_{i_k}) = (-1)^k e_{i_k} \cdots e_{i_1}.$$

allow us to identify $\mathbb{R}^n$ with the subspace of C_n spanned by the elements e_i.

We define the group

$$P(n) := \{x \in C_n^* \mid xvx^{-1} \in \mathbb{R}^n, \text{ for all } v \in \mathbb{R}^n\}$$

where C_n^* is the *group of units* of C_n.

Now we consider the following subgroup of $P(n)$:

$$\mathrm{Pin}(n) := \{x \in P(n) \mid x\varepsilon(x) = 1\}.$$

The group $\mathrm{Pin}(n)$ is not connected, its components are $\mathrm{Pin}(n) \cap C_n^0$ and $\mathrm{Pin}(n) \cap C_n^1$. We define the *spinor group* to be the connected component of the identity, that is:

$$\mathrm{Spin}(n) := \mathrm{Pin}(n) \cap C_n^0.$$

Observation 120. An alternative way of defining the spinor group is as a double covering space of the special orthonormal group of dimension n:

$$\omega : \mathrm{Spin}(n) \longrightarrow SO(n).$$

This double covering identifies the Lie algebra of $\mathrm{Spin}(n)$ with $\mathfrak{so}(n)$.

It is shown in [6, Theorem 13.6, p.78] that $\pi_1(SO(n)) = \mathbb{Z}_2$, for $n > 2$. Besides, the only subgroups of $\mathbb{Z}_2$ are the whole group and the trivial one. This implies that, up to isomorphism, $SO(n)$ has only two covers, the universal cover $\mathrm{Spin}(n)$ and $SO(n)$ itself.

For the case $n = 3$, we already know that $SU(2)$ is a double cover of $SO(3)$, thus we conclude that $\mathrm{Spin}(3)$ is isomorphic to $SU(2)$.

Definition 121. The *complexification* of C_n is

$$C_n^{\mathbb{C}} := C_n \otimes \mathbb{C}$$

where the tensor product is taken over $\mathbb{R}$.

Proposition 122. We have the following algebra isomorphisms:

$$\mathbb{C}_0^{\mathbb{C}} \cong \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}$$
$$\mathbb{C}_1^{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$$
$$\mathbb{C}_2^{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Mat}(2, \mathbb{C}).$$

Proof. Please see [18, Proposition 4.2, p. 26]. □

In fact, there is a periodicity isomorphism that classifies complexified Clifford algebras:

Theorem 123. There is an algebra isomorphism

$$C_{n+2}^{\mathbb{C}} \cong C_n^{\mathbb{C}} \otimes_{\mathbb{C}} C_2^{\mathbb{C}}$$

Proof. May be found in [18, Theorem 4.3, p. 27]. □

Therefore, the complexifications are

$$\begin{aligned} C_{2k}^{\mathbb{C}} &\cong \mathrm{Mat}(2^k, \mathbb{C}) \\ C_{2k+1}^{\mathbb{C}} &\cong \mathrm{Mat}(2^k, \mathbb{C}) \oplus \mathrm{Mat}(2^k, \mathbb{C}). \end{aligned} \tag{4.1}$$

Theorem 124. The following holds:

1. The natural representation ρ of $\mathrm{Mat}(n, \mathbb{C})$ on $\mathbb{C}^n$ is, up to equivalence, the only irreducible representation of $\mathrm{Mat}(n, \mathbb{C})$.

2. The algebra $\mathrm{Mat}(n, \mathbb{C}) \oplus \mathrm{Mat}(n, \mathbb{C})$ has exactly two equivalence classes of irreducible representations. They are given by

$$\rho_1(\varphi_1, \varphi_2) = \rho(\varphi_1) \quad \text{and} \quad \rho_2(\varphi_1, \varphi_2) = \rho(\varphi_2).$$

Proof. Please check [18, Theorem 5.6, p.31]. □

Using the equivalences (4.1) and Theorem 124, we can know the complex representations of Clifford algebras $C_n^{\mathbb{C}}$: if $n = 2k$, then the algebra $C_{2k}^{\mathbb{C}}$ has only one irreducible representation of dimension 2^k; and if $n = 2k+1$ then $C_{2k+1}^{\mathbb{C}}$ has two irreducible representations, both of dimension 2^k.

Since $\mathrm{Spin}(n)$ is a subgroup of C_n^*, the restriction of the irreducible representations of $C_n^{\mathbb{C}}$ give us the complex representations of $\mathrm{Spin}(n)$, called the *spin representations* (for details please see [22, Appendix A, p. 84]):

1. When $n = 2k$ the corresponding complex representation of $\mathrm{Spin}(n)$, given by the restriction and denoted $\mathcal{S}$ is not irreducible; it decomposes as the direct sum of two irreducible representations, both of dimension 2^k
$$\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-.$$
2. If $n = 2k+1$, the two irreducible representations of $C^{\mathbb{C}}_{2k+1}$ restricted to $\mathrm{Spin}(2k+1)$ are isomorphic. This unique representation of $\mathrm{Spin}(2k+1)$ is denoted by $\mathcal{S}$ and has dimension 2^k.

4.1.3 Fiber bundles

Vector bundles

Definition 125. An n-dimensional smooth *vector bundle* is a smooth map $\pi : E \longrightarrow M$, with M and E connected smooth manifolds together with a vector space structure over the field $\mathbb{K}$ on $\pi^{-1}(x)$ for each $x \in M$, such that the following local triviality condition is satisfied: There is an atlas of M by open sets U_i for each of which there exists a diffeomorphism $\varphi_{U_i} : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{K}^n$ taking $\pi^{-1}(x)$ to $\{x\} \times \mathbb{K}^n$ by a vector space isomorphism for each $x \in U_i$. Such φ_{U_i} is called a *local trivialization* of the vector bundle, n is called its *rank* and $\pi^{-1}(x)$ is the *fiber* of x.

Sometimes we will denote a vector bundle by its total space E.

Given a vector bundle $\pi : E \longrightarrow M$ of rank n and two open sets U and V with their respective local trivializations
$$\varphi_U : \pi^{-1}(U) \longrightarrow U \times \mathbb{K}^n$$
$$\varphi_V : \pi^{-1}(V) \longrightarrow V \times \mathbb{K}^n$$
the composite function, called *glueing map*
$$\varphi_V \circ \varphi_U^{-1} : U \cap V \times \mathbb{K}^n \longrightarrow U \cap V \times \mathbb{K}^n$$
$$\varphi_V \circ \varphi_U^{-1}(x, v) = (x, g_{UV}(x)v)$$
is well defined on the intersection, and uniquely determined by some function
$$g_{UV} : U \cap V \longrightarrow GL(n, \mathbb{K})$$
called the *transition function* from U to V. In this case, $GL(n, \mathbb{K})$ is called the *structure group* of the bundle.

The following is an important example of a vector bundle.

Example 126. Let M be a smooth sub-manifold of $\mathbb{R}^N$. The *tangent bundle* of M is a vector bundle whose total space is given by the collection of all the tangent spaces T_pM at all points $p \in M$

$$TM := \{(p, v) \in M \times \mathbb{R}^N \mid p \in M,\ v \in T_pM\}$$

with the projection

$$\begin{aligned} T : TM &\longrightarrow M \\ v &\longmapsto p \end{aligned}$$

where $v \in T_pM$.

Definition 127. Let $\pi : E \longrightarrow M$ be a smooth real vector bundle over a manifold M. A *Riemannian metric on* E is a positive definite inner product on each fiber $p^{-1}(x)$ that varies smoothly with $x \in M$.

Definition 128. Let M be a manifold equipped with an inner product $\langle -, - \rangle_p$ on the tangent space T_pM at each point, which varies smoothly with p. The collection of all these inner products is called a *Riemannian metric* of M.

Definition 129. Let M be a manifold with an atlas $\{U_i\}$ such that each change of coordinates is orientation preserving (that is, its Jacobian determinant is positive). Then M is said to be an *oriented manifold* and $\{U_i\}$ is an *oriented atlas.*

Remark 130. If M is an n-dimensional oriented manifold, we have a reduction of the structure group of the tangent bundle TM of M to $SL(n, \mathbb{R})$. Moreover, if M is equipped with a Riemannian metric, then the structure group of TM is reduced to $SO(n)$ (please see [18, p. 5]).

Fiber bundles

Remark 131. One can also consider a *fiber bundle*, which is an object very similar to the vector bundle, but without the vector space structure on the fibers. In this case each fiber is a differentiable manifold, the local trivializations are diffeomorphisms and the structure group is a subgroup of the group of self-diffeomorphisms of the fiber.

Definition 132. Let G be a Lie group. A *principal G-bundle* is a fiber bundle $\pi : P_G \longrightarrow M$ together with a smooth right action $\alpha : P_G \times G \longrightarrow P_G$ such that G preserves the fibers of P_G, that is, if $y \in \pi^{-1}(x)$ then $yg \in \pi^{-1}(x)$ for all $g \in G$.

Observation 133. The previous definition implies that each fiber of the bundle π is diffeomorphic to G (please see [16, Proposition 2.6, p. 43]).

Example 134. Let M be an oriented, n-dimensional, Riemannian manifold and $T : TM \longrightarrow M$ its tangent bundle. The *orthonormal frame bundle* is a principal $SO(n)$-bundle denoted by $P_{SO}(M)$. It is constructed in the following way: Its total space is given by

$$P_{SO}(M) = \{(v_1, \cdots, v_n) \in (TM)^n \mid T(v_1) = \cdots = T(v_n) \text{ and } \{v_1, \cdots, v_n\} \text{ is an orthonormal basis}\}.$$

and we define the continuous function

$$\begin{aligned} p : P_{SO}(M) &\longrightarrow M \\ (v_1, \cdots, v_n) &\longmapsto T(v_1). \end{aligned}$$

We notice that the vector $(v_1, \cdots, v_n)$, seen as a matrix, is orthogonal and any other orthogonal matrix has the same form. Then the fibers are indeed diffeomorphic to $SO(n)$.

Now we describe a very useful construction regarding principal bundles. Let $\pi : P_G \longrightarrow M$ be a principal G-bundle. Consider a vector space V, equipped with a linear left action of G:

$$\beta : G \times V \longrightarrow V.$$

We can define a left action of G on $P_G \times V$ using the actions of G on each factor:

$$\begin{aligned} G \times (P_G \times V) &\longrightarrow (P_G \times V) \\ (g, p, f) &\longmapsto g \cdot (p, v) = (p \cdot g^{-1}, g \cdot v). \end{aligned}$$

We take the quotient of the action and denote it by $P_G \times_G V$. The elements of $P_G \times_G V$ are equivalence classes satisfying $[p, f] \sim [p \cdot g^{-1}, g \cdot v]$. Then we have the *vector bundle with fiber V associated to the principal bundle P_G* with projection given by:

$$\begin{aligned} P_G \times_G V &\longrightarrow M \\ [p, f] &\longmapsto \pi(p). \end{aligned}$$

4.1.4 Spin manifolds

Let us consider an oriented Riemannian manifold M of dimension n. Again, by Remark 130 there is a reduction of the structure group of its tangent bundle TM to $SO(n)$.

The transition functions of the bundle TM satisfy an important property, known as the *cocycle condition*:

$$g_{ij} g_{jk} g_{ki} = Id.$$

Let us remember that the group Spin(n) is the double covering of $SO(n)$, then every transition function has a lifting:

$$\begin{tikzcd} & \mathrm{Spin}(n) \arrow[d] \\ U \cap V \arrow[ur, "\tilde{g}_{UV}"] \arrow[r, "g_{UV}"'] & SO(n). \end{tikzcd}$$

If these liftings $\tilde{g}_{UV}$ satisfy the cocycle condition

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = Id$$

then they can be seen as the transition maps of some principal bundle $P_{Spin}(M) \longrightarrow M$ whose structure group is Spin(n). In this case, the liftings are called a *spin structure* on M and $P_{Spin}(M)$ is called the *spin bundle on* M.

Observation 135. Spin structures not always exists, there is a cohomological obstruction called the second Stiefel-Whitney class

$$w_2(TM) \in H^2(M, \mathbb{Z}_2)$$

which vanishes if and only if M has a spin structure. For details please check [18, Theorem 2.1, p.86].

Observation 136. The double covering $\omega : \mathrm{Spin}(n) \longrightarrow SO(n)$ induces a double cover

$$\overline{\omega} : P_{Spin}(M) \longrightarrow P_{SO}(M).$$

Definition 137. Let M be a manifold and consider the spin representation $\mathcal{S}$ defined in section 4.1.2. The vector bundle over M associated to $P_{Spin}(M)$ with fiber $\mathcal{S}$

$$\mathcal{S}(M) := P_{Spin}(M) \times_{Spin(n)} \mathcal{S}$$

is called the *spinor bundle* of M.

Definition 138. Let $\pi : E \longrightarrow M$ be a fiber bundle. A *section* of E is a continuous map $\sigma : M \longrightarrow E$ such that $\pi(\sigma(x)) = x$ for all $x \in M$. We denote the space of sections of the bundle E by $C^\infty(M, E)$.

Definition 139. The sections of the spinor bundle, $C^\infty(M, \mathcal{S}(M))$, are called *spinor fields.*

Proposition 140. Let V be a vector space over $\mathbb{R}$ and q a quadratic form on V. Let $f : V \longrightarrow \mathcal{A}$ be a linear map into an associative real algebra with unit, such that

$$f(v) \cdot f(v) = -q(v, v)1$$

for all $v \in V$. Then f extends uniquely to a real algebra homomorphism $\tilde{f} : C(V, q) \longrightarrow \mathcal{A}$. Furthermore, $C(V, q)$ is the unique associative real algebra with this property.

Proof. May be found in [18, Proposition 1.1, p.8]. □

Let us consider an orthogonal transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. By observation 119, there is an embedding $\phi : \mathbb{R}^n \longrightarrow C_n$, composing we get a map $\phi \circ f : \mathbb{R}^n \longrightarrow C_n$ and the previous proposition implies that there is an induced automorphism $\tilde{f} : C_n \longrightarrow C_n$.

Hence we get a representation

$$cl : SO(n) \longrightarrow GL(C_n).$$

In Chapter 3 we studied group representations, let us remember that every representation $\rho : G \longrightarrow GL(V)$ induces an action

$$G \times V \longrightarrow V$$
$$g \cdot v = \rho(g)v$$

Therefore, from the representation cl we can induce the left action:

$$SO(n) \times C_n \longrightarrow C_n$$
$$g \cdot x = cl(g) \cdot x.$$

Definition 141. The *Clifford bundle* of M is the vector bundle over M associated to the frame bundle $P_{SO}(M)$ using the representation cl:

$$C(M) := P_{SO}(M) \times_{cl} C_n$$

Since for $p \in M$ we have that $\mathbb{R}^n \cong T_p^*M$. Thus, the fiber of p on the Clifford bundle can be thought as the Clifford algebra of the cotangent space with the inner product given by the Riemannian metric $C(T_p^*M)$.

Definition 142. There is a homomorphism, called the *adjoint representation*, given by

$$\mathrm{Ad}_n : C_n^* \longrightarrow GL(C_n)$$
$$\mathrm{Ad}_n(x)(v) = xvx^{-1}.$$

We restrict the adjoint representation to $\mathrm{Spin}(n)$. Since $\mathrm{Ad}_n(-1)$ is the identity, $\pm 1 \in \ker(\mathrm{Ad_n})$ and therefore this representation descends to cl:

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r, "\mathrm{Ad_n}"] \arrow[d] & GL(C_n) \\
SO(n) \arrow[ur, dotted] &
\end{tikzcd}$$

and therefore the Clifford bundle is isomorphic to

$$C(M) \cong P_{Spin}(M) \times_{\mathrm{Ad_n}} C_n.$$

Since $\mathcal{S}$ is also a representation of $C(T_p^*M)$, $\mathcal{S}(M)$ is a bundle of left modules over $C(M)$ and there is a pairing called *Clifford multiplication* given by

$$\begin{aligned} C : C(M) \otimes \mathcal{S}(M) &\longrightarrow \mathcal{S}(M) \\ [\tilde{p}, v] \cdot [\tilde{p}, \psi] &= [\tilde{p}, v\psi]. \end{aligned}$$

If we consider the inclusion $T^*M \longrightarrow C(T^*M)$ then we get the pairing

$$T^*M \otimes \mathcal{S}(M) \longrightarrow \mathcal{S}(M).$$

4.2 The Dirac operator

4.2.1 Connections

Definition 143. Let $\pi : P \longrightarrow M$ be a smooth principal bundle over a manifold M of dimension n with structural group G. A *connection* on P is a G-invariant field of tangent n-planes τ on P such that the linear map

$$\pi_* : \tau_p \longrightarrow T_{\pi(p)}(M)$$

is an isomorphism for all $p \in P$.

This is equivalent to define a 1-form on P with values on the Lie algebra associated with G which satisfies some properties. For details please refer to [18, Definition 4.1, p. 101].

If $p : E \longrightarrow M$ is a Riemannian vector bundle we already know that there is a reduction of the structure group of its orthonormal frame bundle $P_{SO}(E)$ to $SO(n)$. Therefore the 1-form corresponding to the connection takes values on the Lie algebra $\mathfrak{so}(n)$. For details please see [18, Example 4.2, p. 102].

Definition 144. A *covariant derivative* on a vector bundle $\pi : E \longrightarrow M$ is a linear map

$$\nabla : C^\infty(M, E) \longrightarrow C^\infty(M, T^*M \otimes E)$$

such that

$$\nabla(fe) = df \otimes e + f\nabla e$$

for all $f \in C^\infty(M)$ and all $e \in C^\infty(M, E)$.

Given a smooth vector field V on M we obtain a map $\nabla_V : C^\infty(M, E) \longrightarrow C^\infty(M, E)$ called the *covariant derivative with respect to* V.

Proposition 145. Let ω be a connection 1-form on $P_{SO}(E)$ as above. Then ω determines a unique covariant derivative on E by the rule

$$\nabla e_i = \sum_{j=1}^{n} \tilde{\omega}_{ji} \otimes e_j \tag{4.2}$$

where $\mathcal{E} = (e_1, \cdots, e_n)$ is a local family of pointwise orthonormal sections of E, i. e., a local section of $P_{SO}(E)$ and where $\tilde{\omega} = \mathcal{E}^*\omega$. This covariant derivative satisfies the rule

$$V\langle e, e'\rangle = \langle \nabla_V e, e'\rangle + \langle e, \nabla_V e'\rangle \tag{4.3}$$

for all $V \in T(M)$ and e, $e' \in C^\infty(M, E)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in E.

Conversely, any covariant derivative on E satisfying equation (4.3) determines a unique connection 1-form by equations (4.2).

Proof. Please see [18, Proposition 4.4, p. 103]. $\square$

A covariant derivative with property (4.3) will be called *Riemannian.*

Definition 146. Let M be a smooth manifold and let $C^\infty(M, TM)$ be the space of vector fields on M. An *affine connection on* M is a bilinear map

$$C^\infty(M, TM) \times C^\infty(M, TM) \longrightarrow C^\infty(M, TM)$$
$$(X, Y) \longmapsto \nabla_X Y$$

such that for all smooth functions $f \in C^\infty(M, \mathbb{R})$ and all vector fields X, Y on M it holds:

1. $\nabla_{fX} Y = f\nabla_X Y$

2. $\nabla_X(fY) = df(X)Y + f\nabla_X Y$

Definition 147. An affine connection ∇ is called the *Levi-Civita connection* if for any vector fields X, Y, Z

1. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (Riemannian)

2. $\nabla_X Y - \nabla_Y X = [X, Y]$ (Torsion free)

Where $\langle\ ,\ \rangle$ is the Riemannian metric and $[\ ,\ ]$ is the Lie bracket of vector fields, defined as

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for $f \in C^\infty(M, \mathbb{R})$.

Theorem 148. Given a Riemannian manifold M, there exist a unique Levi-Civita connection.

Proof. May be found in [8, Theorem 3.6, p. 55]. $\square$

By observation 120, the Lie algebras of Spin(n) and $SO(n)$ are isomorphic. Therefore, the double covering $\overline{\omega} : P_{Spin}(M) \longrightarrow P_{SO}(M)$ given by the spin structure on M gives us a 1-form $\overline{\omega}^*(\beta)$ which defines a connection on $P_{Spin}(M)$, called the *spin connection*.

This connection induces a covariant derivative

$$\nabla^{\mathcal{S}} : C^\infty(M, \mathcal{S}(M)) \longrightarrow C^\infty(M, T^*M \otimes \mathcal{S}(M))$$

on spinor fields.

Definition 149. The *Dirac operator* is the composite of $\nabla^{\mathcal{S}}$ with Clifford multiplication $C : C^\infty(M, T^*M \otimes \mathcal{S}(M)) \longrightarrow C^\infty(M, \mathcal{S}(M))$:

$$D : C^\infty(M, \mathcal{S}(M)) \longrightarrow C^\infty(M, \mathcal{S}(M)).$$

It is possible to define new operators from the Dirac operator in the following way. Let E be a vector bundle over M equipped with a covariant derivative ∇^E. We take the tensor product $\mathcal{S}(M) \otimes E$ and endow it with a covariant derivative $\nabla^{\mathcal{S}\otimes E}$, defined by the formula

$$\nabla^{\mathcal{S}\otimes E}(\psi \otimes \phi) := \nabla^{\mathcal{S}}(\psi) \otimes \phi + \psi \otimes \nabla^E(\phi)$$

for every $\psi \in C^\infty(M, \mathcal{S}(M))$ and $\phi \in C^\infty(M, E)$. Now we can give the following:

Definition 150. The *twisted Dirac operator* D_E is given by the composite

$$C^\infty(M, \mathcal{S}(M) \otimes E) \xrightarrow{\nabla^{\mathcal{S}\otimes E}} C^\infty(M, T^*M \otimes \mathcal{S}(M) \otimes E) \xrightarrow{C \otimes 1_E} C^\infty(M, \mathcal{S}(M) \otimes E)$$

where 1_E denotes the identity on E.

4.2.2 Local expressions (Spin connection)

To compute the Dirac operator on S^3 first we have to know the spin connection.

Proposition 151. Let $\varepsilon = \{e_1, \cdots, e_n\}$ be a n-tuple of pointwise orthonormal sections of the tangent bundle of M over a contractible open set $U \subset M$. With respect to this basis the Riemannian connection is locally given by the *connection matrix* ω_{ij} of 1-forms on U defined by

$$\nabla e_i = \sum_{j=1}^{n} \omega_{ji} e_j$$

where n is the dimension of M.

Proof. This is a consequence of proposition 145. $\square$

The basis ε of the previous proposition can be seen as a section of the frame bundle $P_{SO}(M)$ over U. Using the spin structure we can lift it to a section $\tilde{\varepsilon}$ of $P_{Spin}(M)$ over U.

Observation 152. Since there is a double cover $\overline{\omega} : P_{Spin}(M) \longrightarrow P_{SO}(M)$, we have two possible liftings which must satisfy the relation $\overline{\omega} \circ \tilde{\varepsilon} = \varepsilon$.

The section ε of $P_{Spin}(M)$ determines an m-tuple $\{\Psi_1, \cdots, \Psi_m\}$ of pointwise orthonormal sections of the spin bundle $\mathcal{S}(M)$ over U, where m is the dimension of the spin representation $\mathcal{S}$.

This is true in general, however for the purposes of this work we will only show it for the case $M = S^3$. In this case $\mathcal{S} = \mathbb{C}^2$ and $\mathrm{Spin}(3) \subset C_3^{\mathbb{C}} \cong \mathrm{Mat}(2, \mathbb{C}) \oplus \mathrm{Mat}(2, \mathbb{C})$.

Let us remember that the Clifford algebra $C_3^{\mathbb{C}}$ has two irreducible representations

$$\begin{aligned} C_3^{\mathbb{C}} \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ \big((A,B),(z_1,z_2)\big) &\longmapsto A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ \big((A,B),(z_1,z_2)\big) &\longmapsto B \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}$$

However, these representations are isomorphic on Spin(3). Moreover, by observation 120, $\mathrm{Spin}(3) \cong SU(2)$. Therefore we have the action (by standard matrix multiplication)

$$\begin{aligned} SU(2) \times \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (u,(z_1,z_2)) &\longmapsto u(z_1,z_2) \end{aligned}$$

and since unitary matrices preserve orthonormal bases, we can define a correspondence

$$\begin{aligned} SU(2) &\longrightarrow \{\text{Orthonormal bases of } \mathcal{S} = \mathbb{C}^2\} \\ u &\longmapsto u(e_1, e_2) \end{aligned}$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{C}^2$.

We call the orthonormal basis $\{\Psi_\alpha\}$ *spinor basis* of the spin bundle $\mathcal{S}(M)$ corresponding to the local basis ε of $C^\infty(M, TM)$.

Theorem 153. Respect to the spinor basis the spin connection $\nabla^{\mathcal{S}}$ can be expressed locally by the formula

$$\nabla^{\mathcal{S}} \Psi_\alpha = \frac{1}{2} \sum_{i<j} \omega_{ji} \otimes e_i e_j \cdot \Psi_\alpha$$

Proof. Please see [18, Theorem 4.14, p. 110]. □

Proposition 154. The Dirac operator may be expressed locally in the basis ε by

$$D(\Psi) = \sum_{i=1}^{n} e_i \cdot \nabla^{\mathcal{S}}_{e_i} \Psi$$

where $\cdot$ stands for Clifford multiplication.

Proof. May be found in [18, p. 113]. □

Proposition 155. For any function $a \in c^{\infty}(M)$ and any spinor field $\Psi \in C^{\infty}(M, \mathcal{S}(M))$ we have that

$$D(a\Psi) = \sum_{i=1}^{n} (e_1 \cdot a) e_i \Psi + aD\Psi$$

where $e_i \cdot a = da(e_i)$, that is, the differential of the function a evaluated in the vector field e_i and e_i acts on Ψ by Clifford multiplication.

Proof. Again, please see [18, Lemma 5.5, p.116]. □

4.3 The Dirac operator on S^3

4.3.1 Important bundles over S^3

The tangent bundle

We are ready to start applying the ideas of the previous sections of the chapter to the particular case we are interested in, the unitary sphere S^3.

First we will give explicitly a description of its tangent bundle.

Let us remember that S^3 can be regarded as the subspace of unitary quaternions

$$S^3 = \{p \in \mathbb{H} \mid |p| = 1\}$$

and inherits the inner product from $\mathbb{H}$

$$\langle p, q \rangle = \frac{1}{2}(\overline{p}q + \overline{q}p).$$

By differentiating we obtain its tangent bundle

$$TS^3 = \{(p, q) \in \mathbb{H}^2 \mid |p| = 1, \ \langle p, q \rangle = 0\}$$

and its Lie algebra (tangent space at the identity)

$$T_1 S^3 = \{(1,q) \in \mathbb{H}^2 \mid q + \bar{q} = 0\}.$$

Note that the last equation means that the quaternion q is pure, therefore by corollary 57 of chapter two, we have $T_1 S^3 \cong \mathfrak{su}(2) \cong \mathbb{R}^3$.

Using this identification we find the following trivialization of TS^3:

$$\begin{aligned} S^3 \times \mathbb{R}^3 &\cong TS^3 \\ (p,v) &\longrightarrow (p,pv) \\ (p,p^{-1}q) &\longleftarrow (p,q) \qquad \langle p,q\rangle = 0. \end{aligned} \tag{4.4}$$

The first map is well defined since for $p \in S^3$ and $v \in \mathbb{R}^3$ seen as pure quaternions we have:

$$\langle p, pv\rangle = \frac{1}{2}(\overline{p}pv + \overline{pv}p) = \frac{1}{2}(v + \overline{v}\,\overline{p}p) = \frac{1}{2}(v + \overline{v}) = 0.$$

The frame bundle

The frame bundle $P_{SO}(S^3)$ is isomorphic to the $SO(3)$-bundle

$$\begin{aligned} p : SO(4) &\longrightarrow S^3 \\ A &\longmapsto Af_1 \end{aligned} \tag{4.5}$$

where $f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and the action of $SO(3)$ on $SO(4)$ is given by:

$$\begin{aligned} SO(4) \times SO(3) &\longrightarrow SO(4) \\ (A,B) &\longmapsto Ai(B) \end{aligned}$$

where i is the inclusion

$$\begin{aligned} i : SO(3) &\longrightarrow SO(4) \\ i(B) &= \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \end{aligned} \tag{4.6}$$

Indeed, an element of $SO(4)$ is an orthogonal matrix with column vectors

$$(\overrightarrow{p}, \overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w})$$

then $\overrightarrow{p}$ represents an element of S^3 and $\overrightarrow{u}$, $\overrightarrow{v}$, $\overrightarrow{w}$ form an orthonormal frame of the tangent space $T_p S^3$.

The spin structure bundle

Let us remember that observation 135 gave us a cohomological obstruction for a spin structure to exist. Since $H^1(S^3, \mathbb{Z}_2) = H^2(S^3, \mathbb{Z}_2) = 0$, we conclude that S^3 has a unique spin structure and hence a principal Spin(3)-bundle $P_{Spin}(S^3)$, which must be the double cover of $P_{SO}(S^3) = SO(4)$. That is, $P_{Spin}(S^3) \cong \mathrm{Spin}(4) \cong S^3 \times S^3$. Hence we have the following commutative diagram:

$$\begin{tikzcd}
S^3 \arrow[r, "Ad_3"] \arrow[d, "d"'] & SO(3) \arrow[d, "i"] \\
S^3 \times S^3 \arrow[r, "Ad_4"] \arrow[d, "P"'] & SO(4) \arrow[d, "p"] \\
S^3 \arrow[r, equal] & S^3
\end{tikzcd}$$

where the maps Ad_3 and Ad_4 are the adjoint representations given in definition 142:

$$\begin{aligned}
Ad_3 &: S^3 \longrightarrow SO(3) \\
Ad_3(q)v &= pvp^{-1} \qquad v \in \mathbb{R}^3 \cong \text{pure quaternions} \\
Ad_4 &: S^3 \longrightarrow SO(4) \\
Ad_4(p,q)w &= pwq^{-1} \qquad w \in \mathbb{H} \cong \mathbb{R}^4,
\end{aligned}$$

the homomorphism d is the diagonal map

$$\begin{aligned}
d : S^3 &\longrightarrow S^3 \times S^3 \\
q &\longmapsto (q,q),
\end{aligned}$$

the map P is the projection

$$\begin{aligned}
P : S^3 \times S^3 &\longrightarrow S^3 \\
(p,q) &\longmapsto pq^{-1}
\end{aligned}$$

and p and i are the maps given by (4.6) and (4.5) respectively.

Proposition 156. The following are equivalent for a principal G-bundle.

1. The bundle has a global section.

2. The bundle is trivial

Proof. Please see [16, Corollary 8.3, p. 49]. □

Using the previous proposition and considering the global section of P

$$s : S^3 \longrightarrow S^3 \times S^3$$
$$s(p) = (p, 1)$$

we conclude that the bundle P is trivial.

The spinor bundle

We have already seen in chapter three that, as a Lie group, S^3 is isomorphic to $SU(2) \cong \mathrm{Spin}(3)$. There is a natural action of $SU(2)$ on $\mathbb{C}^2$, by matrix multiplication, that is, the *standard representation* of Spin(3) on $\mathbb{C}^2$.

The spinor bundle $\mathcal{S}(S^3)$ is the vector bundle associated with $P_{Spin}(S^3)$ using the standard spin representation. Since the bundle $P_{Spin}(S^3) = P$ is trivial, the spin bundle $\mathcal{S}(S^3)$ is also trivial, with a trivialization given by

$$\begin{aligned} (S^3 \times S^3) \times_{S^3} \mathbb{C}^2 &\cong S^3 \times \mathbb{C}^2 \\ [(p,q), w] \sim [(pq^{-1}, 1), \sigma(q)w] &\longrightarrow (pq^{-1}, \sigma(q)w) \\ [(p,1), w] &\longleftarrow (p, w). \end{aligned} \tag{4.7}$$

Hence, the space of spinor fields $C^\infty(S^3, \mathcal{S}(S^3))$ is just the space of smooth functions $C^\infty(S^3, \mathbb{C}^2)$.

4.3.2 Computing the Dirac operator on S^3

Now we will compute the Levi-Civita connection, then the spin connection and lastly the Dirac operator on S^3.

Let us remember that the Levi-Civita connection satisfies:

1. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$ (Riemannian)
2. $\nabla_X Y - \nabla_Y X = [X, Y]$ (Torsion free).

Combining the previous two properties we get the formula

$$2\langle Y, \nabla_X Z\rangle + \langle X, [Y, Z]\rangle + \langle Z, [Y, X]\rangle - \langle Y, [Z, X]\rangle. \tag{4.8}$$

Let us remember that the orthonormal basis of the Lie algebra $\mathfrak{su}(2)$ is given by the matrices

$$\left\{ E_1 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_2 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \right\}.$$

Lemma 157. The orthonormal basis $\{E_1, E_2, E_3\}$ satisfies the relations:

$$[E_1, E_2] = 2E_3 \qquad [E_2, E_3] = 2E_1 \qquad [E_3, E_1] = 2E_2$$
$$[E_2, E_1] = -2E_3 \qquad [E_3, E_2] = -2E_1 \qquad [E_1, E_3] = -2E_2$$

Proof. We will compute the fist case, the others are analogous.

$$\begin{aligned}[E_1, E_2] = E_1E_2 - E_2E_1 &= i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} i \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} i \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix} = 2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = 2E_3\end{aligned}$$

□

We can compute the Levi-Civita connection for the basis $\{E_1, E_2, E_3\}$ as

$$\nabla_{E_i} E_j = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_2 E_3$$

where the coefficients are given by

$$\alpha_1 = \langle E_1, \nabla_{E_i} E_j\rangle \qquad \alpha_2 = \langle E_2, \nabla_{E_i} E_j\rangle \qquad \alpha_3 = \langle E_3, \nabla_{E_i} E_j\rangle.$$

and can be obtained using the previous lemma and equation (4.8).

For example, let us calculate $\nabla_{E_1} E_1 = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_2 E_3$

$$\begin{aligned}2\alpha_1 &= 2\langle E_1, \nabla_{E_1} E_1\rangle = \langle E_1, [E_1, E_1]\rangle + \langle E_1, [E_1, E_1]\rangle + \langle E_1, [E_1, E_1]\rangle = 0\\ 2\alpha_2 &= 2\langle E_2, \nabla_{E_1} E_1\rangle = \langle E_1, [E_2, E_1]\rangle + \langle E_1, [E_2, E_1]\rangle + \langle E_2, [E_1, E_1]\rangle\\ &= 2\langle E_1, [E_2, E_1]\rangle = -4\langle E_1, E_3\rangle = 0\\ 2\alpha_3 &= 2\langle E_3, \nabla_{E_1} E_1\rangle = \langle E_1, [E_3, E_1]\rangle + \langle E_1, [E_3, E_1]\rangle + \langle E_3, [E_1, E_1]\rangle\\ &= 2\langle E_1, [E_2, E_1]\rangle = -4\langle E_1, E_3\rangle = 0\end{aligned}$$

And therefore $\nabla_{E_1} E_1 = 0$.

The Levi-Civita connection on the basis is

$$\begin{array}{lll}\nabla_{E_1} E_1 = 0 & \nabla_{E_1} E_2 = E_3 & \nabla_{E_1} E_3 = -E_2\\ \nabla_{E_2} E_1 = -E_3 & \nabla_{E_2} E_2 = 0 & \nabla_{E_2} E_3 = E_1\\ \nabla_{E_3} E_1 = E_2 & \nabla_{E_3} E_2 = -E_1 & \nabla_{E_3} E_3 = 0\end{array}$$

Now we compute the connection matrix ω using proposition 151 and the Levi-Civita connection:

$$\nabla_X e_i = \sum_{j=1}^{n} \omega_{ji}(X) e_j.$$

For example, we have

$$\nabla_{E_1}E_1 = \omega_{11}(E_1)E_1 + \omega_{21}(E_1)E_2 + \omega_{31}(E_1)E_3$$
$$\omega_{11}(E_1) = \omega_{21}(E_1) = \omega_{31}(E_1) = 0$$

$$\nabla_{E_1}E_2 = \omega_{11}(E_2)E_1 + \omega_{21}(E_2)E_2 + \omega_{31}(E_2)E_3$$
$$\omega_{11}(E_2) = \omega_{21}(E_2) = 0\ ;\ \omega_{31}(E_2) = 1.$$

Just as above, we get the values of the ω_{ji} evaluated in the basis:

$$\begin{array}{lll}
\omega_{11}(E_1) = 0 & \omega_{21}(E_1) = 0 & \omega_{31}(E_1) = 0 \\
\omega_{11}(E_2) = 0 & \omega_{21}(E_2) = 0 & \omega_{31}(E_2) = -1 \\
\omega_{11}(E_3) = 0 & \omega_{21}(E_3) = 1 & \omega_{31}(E_3) = 0 \\
 & & \\
\omega_{12}(E_1) = 0 & \omega_{22}(E_1) = 0 & \omega_{32}(E_1) = 1 \\
\omega_{12}(E_2) = 0 & \omega_{22}(E_2) = 0 & \omega_{32}(E_2) = 0 \\
\omega_{12}(E_3) = -1 & \omega_{22}(E_3) = 0 & \omega_{32}(E_3) = 0 \\
 & & \\
\omega_{13}(E_1) = 0 & \omega_{23}(E_1) = -1 & \omega_{33}(E_1) = 0 \\
\omega_{13}(E_2) = 1 & \omega_{23}(E_2) = 0 & \omega_{33}(E_2) = 0 \\
\omega_{13}(E_3) = 0 & \omega_{23}(E_3) = 0 & \omega_{33}(E_3) = 0.
\end{array}$$

Substituting these values in the formula for the spin connection $\nabla^{\mathcal{S}}$ given by theorem 153 we get:

$$
\begin{aligned}
\nabla^{\mathcal{S}}_{E_1}\Psi_\alpha &= \frac{1}{2}\sum_{i<j}\omega_{ji}(E_1)\otimes E_iE_j\cdot\Psi_\alpha \\
&= \frac{1}{2}\big(\omega_{21}(E_1)\otimes E_1E_2\cdot\Psi_\alpha + \omega_{31}(E_1)\otimes E_1E_3\cdot\Psi_\alpha + \omega_{32}(E_1)\otimes E_2E_3\cdot\Psi_\alpha\big) \\
&= \frac{1}{2}E_2E_3\cdot\Psi_\alpha \\
\nabla^{\mathcal{S}}_{E_2}\Psi_\alpha &= \frac{1}{2}\sum_{i<j}\omega_{ji}(E_2)\otimes E_iE_j\cdot\Psi_\alpha \\
&= \frac{1}{2}\big(\omega_{21}(E_2)\otimes E_1E_2\cdot\Psi_\alpha + \omega_{31}(E_2)\otimes E_1E_3\cdot\Psi_\alpha + \omega_{32}(E_2)\otimes E_2E_3\cdot\Psi_\alpha\big) \\
&= -\frac{1}{2}E_1E_3\cdot\Psi_\alpha \\
\nabla^{\mathcal{S}}_{E_3}\Psi_\alpha &= \frac{1}{2}\sum_{i<j}\omega_{ji}(E_3)\otimes E_iE_j\cdot\Psi_\alpha \\
&= \frac{1}{2}\big(\omega_{21}(E_3)\otimes E_1E_2\cdot\Psi_\alpha + \omega_{31}(E_3)\otimes E_1E_3\cdot\Psi_\alpha + \omega_{32}(E_3)\otimes E_2E_3\cdot\Psi_\alpha\big) \\
&= \frac{1}{2}E_1E_2\cdot\Psi_\alpha
\end{aligned}
\tag{4.9}
$$

Now we are ready to compute the Dirac operator of S^3:

Theorem 158 (Hitchin)**.** Let $\{E_1, E_2, E_3\}$ be an orthonormal basis of the bi-invariant metric in S^3. Then relative to the corresponding spinor basis the Dirac operator D^{S^3} may be written as

$$
D^{S^3} = i\begin{pmatrix} E_1 & E_2 + iE_3 \\ E_2 + iE_3 & -E_1 \end{pmatrix} - \frac{3}{2}
$$

Proof. From proposition 154, the Dirac operator is locally given by

$$
D^{S^3}\Psi = \sum_{i=1}^{3} E_i\cdot\nabla^{\mathcal{S}}_{E_i}\Psi
$$

substituting the values of the spin connection given by equations (4.9) we get

$$
\begin{aligned}
D^{S^3}\Psi &= E_1\cdot\left(\frac{1}{2}E_2E_3\Psi_\alpha\right) + E_2\cdot\left(-\frac{1}{2}E_1E_3\Psi_\alpha\right) + E_3\cdot\left(\frac{1}{2}E_1E_2\Psi_\alpha\right) \\
&= \frac{1}{2}\big(E_1E_2E_3\Psi_\alpha - E_2E_1E_3\Psi_\alpha + E_3E_1E_2\Psi_\alpha\big) \\
&= \frac{1}{2}\big(E_1E_2E_3\Psi_\alpha + E_1E_2E_3\Psi_\alpha + E_1E_2E_3\Psi_\alpha\big) \\
&= \frac{3}{2}\big(E_1E_2E_3\Psi_\alpha\big) \qquad \alpha = 1,\ 2
\end{aligned}
$$

Now, if $\Psi \in C^\infty(S^3, \mathbb{C}^2)$ and $a \in C^\infty(S^3, \mathbb{C})$, by proposition 155 we have

$$\begin{aligned} D^{S^3}(a\Psi) &= \sum_{i=1}^{3}(E_i \cdot a)E_i\Psi + aD^{S^3}\Psi \\ &= (E_1 \cdot a)E_1\Psi + (E_2 \cdot a)E_2\Psi + (E_3 \cdot a)E_3\Psi + a\left(\frac{3}{2}(E_1E_2E_3\Psi_\alpha)\right) \qquad \alpha = 1,\ 2 \\ &= da(E_1)E_1\Psi + da(E_2)E_2\Psi + da(E_3)E_3\Psi + a\left(\frac{3}{2}(E_1E_2E_3\Psi_\alpha)\right) \qquad \alpha = 1,\ 2 \end{aligned}$$

we write the spinor field in terms of the basis $\Psi = a_1\Psi_1 + a_2\Psi_2$ and substitute:

$$\begin{aligned} D^{S^3}(a_1\Psi_1 + a_2\Psi_2) &= D^{S^3}(a_1\Psi_1) + D^{S^3}(a_2\Psi_2) \\ &= (E_1a_1)E_1\Psi_1 + (E_2a_1)E_2\Psi_1 + (E_3a_1)E_3\Psi_1 + a\left(\frac{3}{2}(E_1E_2E_3\Psi_1)\right) \\ &+ (E_1a_2)E_1\Psi_2 + (E_2a_2)E_2\Psi_2 + (E_3a_2)E_3\Psi_2 + a\left(\frac{3}{2}(E_1E_2E_3\Psi_2)\right) = * \end{aligned}$$

in order to finish the computations, let us calculate the following terms:

$$E_1\Psi_1 = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}1\\0\end{pmatrix} = i\Psi_1, \qquad E_1\Psi_2 = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = i\begin{pmatrix}0\\-1\end{pmatrix} = -i\Psi_2$$

$$E_2\Psi_2 = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = i\begin{pmatrix}1\\0\end{pmatrix} = i\Psi_1, \qquad E_3\Psi_2 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} = -\Psi_1$$

$$E_3\Psi_1 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix} = \Psi_2, \qquad E_2\Psi_1 = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = i\begin{pmatrix}0\\1\end{pmatrix} = i\Psi_2$$

$$E_1E_2E_3\Psi_1 = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} = -\Psi_1$$

$$E_1E_2E_3\Psi_2 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -\Psi_2$$

we are ready to continue:

$$\begin{aligned} * &= i(E_1 \cdot a_1)\Psi_1 + i(E_2 \cdot a_1)\Psi_2 + (E_3 \cdot a_1)\Psi_2 - \frac{3}{2}a_1\Psi_1 - i(E_1 \cdot a_2)\Psi_2 + i(E_2 \cdot a_2)\Psi_1 + (E_3 \cdot a_2)\Psi_1 + \frac{3}{2}a_2\Psi_2 \\ &= \left(i(E_2 \cdot a_1) + i(E_2 \cdot a_2) + (E_3 \cdot a_2) - \frac{3}{2}a_1\right)\Psi_1 + \left(i(E_2 \cdot a_1) + (E_3 \cdot a_1) - i(E_1 \cdot a_2) - \frac{3}{2}a_2\right)\Psi_2 \\ &= \left((iE_1 \cdot a_1) + ((iE_2 - E_3) \cdot a_2 - \frac{3}{2}a_1\right)\Psi_1 + \left((iE_2 + E_3) \cdot a_1 - iE_1 \cdot a_2 + \frac{3}{2}a_2\right)\Psi_2 \\ &= \begin{pmatrix} iE_1 \cdot a_1 + (iE_2 - E_3) \cdot a_2 + \frac{3}{2}a_1 \\ (iE_2 + E_3) \cdot a_1 - iE_1 \cdot a_2 - \frac{3}{2}a_2 \end{pmatrix} = \begin{pmatrix} iE_1 \cdot a_1 + (iE_2 + E_3) \cdot a_2 \\ (iE_2 + E_3) \cdot a_1 - iE_1 \cdot a_2 \end{pmatrix} - \begin{pmatrix} \frac{3}{2}a_1 \\ \frac{3}{2}a_2 \end{pmatrix} \\ &= i\begin{pmatrix} E_1 \cdot a_1 + (E_2 + iE_3) \cdot a_2 \\ (E_2 - iE_3) \cdot a_1 - E_1 \cdot a_2 \end{pmatrix} - \begin{pmatrix} \frac{3}{2}a_1 \\ \frac{3}{2}a_2 \end{pmatrix} = i\begin{pmatrix} E_1 & E_2 + iE_3 \\ E_2 + iE_3 & -E_1 \end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} - \frac{3}{2}\begin{pmatrix}a_1\\a_2\end{pmatrix} \end{aligned}$$

therefore we have

$$D^{S^3} = i\begin{pmatrix} E_1 & E_2 + iE_3 \\ E_2 - iE_3 & -E_1 \end{pmatrix} - \frac{3}{2}.$$

□

The Dirac operator on S^3 was computed by Hitchin in [15, Proposition 3.1]. He also proved in that same article [15, Proposition 3.2] the following:

Proposition 159. The eigenvalues of the Dirac operator on S^3 with respect to the standard metric are

$$\begin{aligned} &-\frac{1}{2} - (k+1) \quad \text{with multiplicity} \quad (k+2)(k+1), \qquad k \geq 0, \\ &-\frac{1}{2} + (k+1) \quad \text{with multiplicity} \quad k(k+1), \qquad k \geq 1. \end{aligned}$$

In the next chapter we will give a result that generalizes proposition 159, considering the Dirac operator on S^3 "twisted" by a representation of its fundamental group.

4.4 The Dirac operator on S^3/Γ

In this section we will learn how to compute the Dirac operator for quotient spaces S^3/Γ.

4.4.1 G-bundles

Definition 160. Let M be a smooth manifold and let G be a Lie group.

1. We say that M is a *G-manifold* if it is equipped with a smooth right action $\mu : M \times G \longrightarrow M$.

2. Let $p : E \longrightarrow M$ be a vector bundle; it is called a *G-bundle* if there is a right action of G on the total space E which carries fibers to fibers, that is, such that for every $g \in G$ and $x \in M$ it holds $E_x g \subset E_{xg}$.

Example 161. The tangent bundle of a G-manifold can be seen as a G-bundle considering the following diffeomorphism, called the *right translation*:

$$R_g : M \longrightarrow M$$
$$R_g(x) = \mu(x, g) = xg.$$

Then for each $x \in M$ the differential $dR_g|_x$ takes the tangent space at x into the tangent space at xg inducing a right action of G on the tangent bundle as follows. Let v be a tangent vector at x, then

$$vg = dR_g|_x v.$$

This action projects to the action μ making the tangent bundle a G-bundle.

Remark 162. We notice that the action of G on a G-bundle E induces a right action of G on the sections of E defined by

$$\begin{aligned} C^\infty(M,E) \times G &\longrightarrow C^\infty(M,E) \\ (s(x), g) &\longmapsto (s \cdot g)(x) = s(xg^{-1})g \end{aligned}$$

for every section $\psi \in C^\infty(M,E)$ and every $g \in G$.

Definition 163. Let E and F be G-bundles over M. A differential operator $A : C^\infty(M,E) \longrightarrow C^\infty(M,F)$ is called a *G-operator* if $A(\psi g) = A(\psi)g$ for all $g \in G$ and all $\psi \in C^\infty(M,E)$.

Proposition 164. The Dirac operator is a G-operator.

Proof. Please see [18, p.221]. $\square$

Let M be a spin manifold with a given spin structure and let G be a Lie group acting on M by orientation-preserving isometries. The action of G *preserves the spin structure of* M or is said to be a *spin action* if the induced action of G on the frame bundle $P_{SO}(M)$ of M lifts to an action on the bundle $P_{Spin}(M)$ given by the spin structure.

4.4.2 The Dirac operator for homogeneous spaces

Let us consider a Lie group G together with a closed subgroup Γ. The space of right Γ-cosets $X = G/\Gamma$ can be endowed with a differential structure such that the projection $G \longrightarrow G/\Gamma$ is C^∞ and it has local sections. In fact, it is a Γ-principal bundle over X. This kind of manifolds are called *homogeneous spaces.*

Suppose that G is equipped with a right invariant metric. Then Γ acts on G by right multiplication preserving the metric. If Γ is a finite subgroup then its action on G is free and properly discontinuous; in this case the projection $\pi : G \longrightarrow G/\Gamma$ is a covering map and therefore a local diffeomorphism.

For a fixed $g \in \Gamma$, the action of Γ on G can be thought as a function

$$\begin{aligned} f_g : G &\longrightarrow G \\ x &\longmapsto x \cdot g \end{aligned}$$

whose differential on a given point $x \in G$

$$(df_g)_x : T_xG \longrightarrow T_{(f_g(x))}G = T_{x \cdot g}G$$

induces an action on the tangent space as follows:

$$\begin{aligned} T_xG \times \Gamma &\longrightarrow T_{x \cdot g}G \\ (v, g) &\longmapsto v \cdot g = (df_g)_x(v) \end{aligned}$$

which can be extended to the whole tangent bundle TG.

Then we also have an induced action of Γ on the frame bundle $p : P_{SO}(G) \longrightarrow G$ as follows

$$\begin{aligned} P_{SO}(G) \times \Gamma &\longrightarrow P_{SO}(G) \\ ((v_1, \cdots, v_n), g) &\longmapsto (v_1 \cdot g, \cdots, v_n \cdot g) \end{aligned}$$

The induced action of Γ makes the frame bundle Γ-equivariant:

$$p((v_1, \cdots, v_n) \cdot g) = p(v_1, \cdots, v_n) \cdot g = T(v_1) \cdot g = p((df_g)_{T(v_1)}(v_1), \cdots, (df_g)_{T(v_n)}(v_n))$$

therefore we have the isomorphism

$$\begin{aligned} P_{SO}(G/\Gamma) &\cong P_{SO}(G)/\Gamma \\ (v_1\Gamma, \cdots, v_n\Gamma) &\longmapsto (v_1, \cdots, v_n)\Gamma \end{aligned}$$

that is:

$$\begin{tikzcd} P_{SO}(X) \arrow[d] \arrow[r, phantom, "\cong"] & P_{SO}(G)/\Gamma \arrow[d] \\ X \arrow[r, phantom, "\cong"] & G/\Gamma. \end{tikzcd}$$

Let us remember that in general $\mathcal{S}(X)$ is the vector bundle associated with the principal Spin(n)-bundle $P_{Spin}(X)$ through the spin representation $\mathcal{S}$, therefore its elements are equivalence classes of the form $[\tilde{p}, \psi]$ where $\tilde{p} \in P_{Spin}(X)$ and $\psi \in \mathcal{S}$.

If the group G is spin, then the action of Γ is a spin action and it lifts to the principal spin bundle $P_{Spin}(G)$ given by the spin structure and induces an action on the spinor bundle $\mathcal{S}(G)$.

$$\begin{aligned} \mathcal{S}(G) \times \Gamma &\longrightarrow \mathcal{S}(G) \\ ([\tilde{p}, \psi], g) &\longmapsto [\tilde{p}, \psi] \cdot g = [\tilde{p} \cdot g, \psi] \end{aligned}$$

Hence, the corresponding spinor bundle $\mathcal{S}(X)$ of X is given by the bundle $\mathcal{S}(G)$ modulo the action of Γ:

$$
\begin{array}{ccccc}
\mathcal{S}(X) & \cong & \mathcal{S}(G)/\Gamma & & [\tilde{p},\psi]\Gamma \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
X & \cong & G/\Gamma & & \pi([\tilde{p},\psi])\Gamma.
\end{array}
$$

Lemma 165. The sections $C^\infty(X, \mathcal{S}(X))$ of the bundle $\mathcal{S}(X)$ are the Γ-equivariant sections of the bundle $\mathcal{S}(G)$

Proof. Let us consider a section s of the bundle $\pi : \mathcal{S}(G)/\Gamma \longrightarrow G/\Gamma$. Then for $x\Gamma \in G/\Gamma$ it holds

$$\pi(s(x\Gamma)) = x\Gamma = \pi([\tilde{p},\psi]\Gamma) = \pi([\tilde{p},\psi])\Gamma$$

therefore $\pi([\tilde{p},\psi]) = x$, which implies that $[\tilde{p},\psi] \in \pi^{-1}(x)$, that is, s is a Γ-invariant section of the bundle $\mathcal{S}(G) \longrightarrow G$. $\square$

We denote by $C^\infty(G, \mathcal{S}(G))^\Gamma$ the space of Γ-equivariant sections of $\mathcal{S}(G)$.

Lemma 166. The space $C^\infty(G, \mathcal{S}(G))^\Gamma$ is the subspace of sections which are invariant under the action of Γ on the space of sections $C^\infty(G, \mathcal{S}(G))$ defined in remark 162.

Proof. Let us remember the aforementioned action

$$
\begin{aligned}
C^\infty(G, \mathcal{S}(G)) \times \Gamma &\longrightarrow C^\infty(G, \mathcal{S}(G)) \\
(s(x), g) &\longmapsto (s \cdot g)(x) = s(xg^{-1})g.
\end{aligned}
$$

We have

$$
\begin{aligned}
(s \cdot g)(x) = s(xg^{-1})g &= s(x) \\
s(x \cdot g) &= s(x) \cdot g
\end{aligned}
$$

which means that s is Γ-equivariant. $\square$

The previous lemma is telling us that the action of Γ on $C^\infty(G, \mathcal{S}(G))$ preserves the subspace $C^\infty(G, \mathcal{S}(G))^\Gamma$.

Lemma 167. The subspace $C^\infty(G, \mathcal{S}(G))^\Gamma$ is invariant under the Dirac operator.

Proof. Taking s a fixed point of the action and using the fact that by proposition 164 the Dirac operator is a Γ-operator, we have:

$$D(s) = D(s \cdot g) = D(s) \cdot g$$

therefore $D(s)$ is a fixed point too. $\square$

Hence, the Dirac operator on X (denoted by D^X) is the Dirac operator D^G restricted to the Γ-equivariant sections of the spinor bundle $\mathcal{S}(G)$:

$$D^X = D^G| : C^\infty(G, \mathcal{S}(G))^\Gamma \cong C^\infty(X, \mathcal{S}(X)) \longrightarrow C^\infty(G, \mathcal{S}(G))^\Gamma \cong C^\infty(X, \mathcal{S}(X)).$$

Remark 168. From the discussion above, to compute the Dirac operator D^X on the homogeneous space $X = G/\Gamma$ together with its eigenvalues, it is enough to compute the Dirac operator D^G on G and its eigenvalues and count their multiplicities in the subspace $C^\infty(G, \mathcal{S}(G))^\Gamma$.

We are interested in applying the ideas of the previous sections to the homogeneous spaces of the form S^3/Γ; regarding S^3 as the unitary quaternions and taking Γ as a finite subgroup.

4.4.3 The Dirac operator on S^3/Γ

The induced actions of S^3

Since S^3 is a Lie group, it can be regarded as a group and as a manifold. In this subsection we will denote the elements of the S^3 seen as a manifold by p or q and the elements of S^3 seen as a group by g.

The sphere S^3 acts on itself by isometries by right multiplication, denoted by pg. We will see that this is a spin action and write explicitly the actions induced on the bundles TS^3, $P_{SO}(S^3)$, $P_{Spin}(S^3)$ and $\mathcal{S}(S^3)$ described in subsection 4.3.1.

The action of S^3 on TS^3 is given by

$$\begin{aligned} TS^3 \times S^3 &\longrightarrow TS^3 \\ (p,q)g &= (pg, qg) \end{aligned}$$

where g acts on p and q by quaternionic multiplication. Using the trivialization of the tangent bundle given by equation (4.4), this action corresponds to

$$\begin{aligned} (S^3 \times \mathbb{R}^3) \times S^3 &\longrightarrow S^3 \times \mathbb{R}^3 \\ (p,v)g &= (pg, \mathrm{Ad}_3(g^{-1}v)) \end{aligned}$$

identifying $\mathbb{R}^3$ with pure quaternions.

The induced action on the frame bundle $P_{SO}(S^3)$ is

$$\begin{aligned} SO(4) \times S^3 &\longrightarrow SO(4) \\ Ag &= (pg, v_1 g, v_2 g, v_3 g) \end{aligned}$$

where $A \in SO(4)$ is given by the column vectors p, v_1, v_2, v_3.

Lifting the action on the frame bundle we get the action on the spin structure bundle $P_{Spin}(S^3)$, namely

$$\begin{aligned}(S^3 \times S^3) \times S^3 &\longrightarrow S^3 \times S^3 \\ (p,q)g &= (p, g^{-1}q)\end{aligned}$$

since $\mathrm{Ad}_4((p,q)g)w = \mathrm{Ad}_4(p, g^{-1}q)w = pwq^{-1}g$.

The action on the spinor bundle $\mathcal{S}(S^3)$ is given by

$$\begin{aligned}((S^3 \times S^3) \times_{S^3} \mathbb{C}^2) \times S^3 &\longrightarrow (S^3 \times S^3) \times_{S_3} \mathbb{C}^2 \\ [(p,q),w]g &= [(p, g^{-1}q), w]\end{aligned} \tag{4.10}$$

and by the trivialization given in equation (4.7) this correspond to the action

$$\begin{aligned}(S^3 \times \mathbb{C}^2) \times S^3 &\longrightarrow S^3 \times \mathbb{C}^2 \\ (p,w)g &= (pg, \sigma(g)^{-1}w)\end{aligned}$$

where σ is the standard representation of S^3 on $\mathbb{C}^2$.

The induced action on the space of spinor fields is

$$\begin{aligned}C^\infty(S^3, \mathbb{C}^2) \times S^3 &\longrightarrow C^\infty(S^3, \mathbb{C}^2) \\ (\Psi g)(p) &= \sigma(g)^{-1}\Psi(pg^{-1}).\end{aligned} \tag{4.11}$$

Let Γ be a finite subgroup of S^3. Since S^3 acts on itself by isometries, we can restrict the action and conclude that Γ also acts on S^3 by isometries. This action is also spin and we can consider the induced actions in the bundles as in the previous subsection. In particular, the action in the spinor bundle described in equation (4.10) restricted to Γ is

$$\begin{aligned}(S^3 \times \mathbb{C}^2) \times S^3 &\longrightarrow S^3 \times \mathbb{C}^2 \\ (p,w)g &= (pg, \sigma(g)^{-1}w) \qquad \forall g \in \Gamma.\end{aligned}$$

Spinors on S^3/Γ

As we saw in section 4.4.2, the spin bundle for the homogeneous space S^3/Γ is given by the quotient of the bundle $\mathcal{S}(S^3) \cong S^3 \times \mathbb{C}^2$ by the action of Γ, which is the definition of the vector bundle associated with the universal covering $\pi : S^3 \longrightarrow S^3/\Gamma$ using the standard representation σ restricted to Γ:

$$S^3 \times_\sigma \mathbb{C}^2 \longrightarrow S^3/\Gamma.$$

Thus, the spinor fields are just the Γ- equivariant spinors on S^3:

$$\{\Psi \in C^\infty(S^2, \mathbb{C}^2) \mid \Psi(pg) = \sigma(g)^{-1}\Psi(p), \text{ for every } p \in S^3 \text{ and } g \in \Gamma\} = C^\infty(S^3, \mathbb{C}^2)^\Gamma$$

and can also be seen as the subspace of spinors on S^3 which are invariant under the action given by equation (4.11). Since this action commutes with the Dirac operator D on S^3, the subspace $C^\infty(S^3, \mathbb{C}^2)^\Gamma$ is mapped to itself under D and therefore we have the following result:

Theorem 169. The Dirac operator on the homogeneous space S^3/Γ, D^Γ, is just the restriction of D to the Γ-equivariant spinors $C^\infty(S^3, \mathbb{C}^2)^\Gamma$.

4.5 The twisted Dirac operator on S^3/Γ

Since $\pi : S^3 \longrightarrow S^3/\Gamma$ is the universal cover, we have that $\pi_1(S^3/\Gamma) = \Gamma$. Moreover, π can be regarded as a $\pi_1(S^3/\Gamma)$-principal bundle. If we consider a unitary representation of $\pi_1(S^3/\Gamma)$, $\alpha : \Gamma \longrightarrow U(N)$ we can construct the associated principal bundle

$$V_\alpha = S^3 \times_\alpha \mathbb{C}^N$$

As we saw in observation 185, V_α has a canonical flat connection and therefore we can "twist" the Dirac operator D^Γ by this bundle. The twisted spinor bundle is

$$\mathcal{S}(S^3/\Gamma) \otimes V_\alpha \cong (S^3 \times_\sigma \mathbb{C}^2) \otimes (S^3 \times_\alpha \mathbb{C}^N) \cong S^3 \times_{\sigma\otimes\alpha} \mathbb{C}^2 \otimes \mathbb{C}^N$$

and the space of twisted spinors is isomorphic to the space of functions $f : S^3 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^N$ which are Γ-equivariant, where Γ acts on S^3 by right multiplication and in $\mathbb{C}^2 \otimes \mathbb{C}^N$ via the representation $\sigma \otimes \alpha$. From now on we denote this space by $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha}$.

Since $\pi_1(S^3)$ is trivial, every irreducible representation is also trivial. Therefore we can only twist the spinor bundle using trivial bundles of rank N.

Consider the Dirac operator D twisted by the trivial bundle $S^3 \times \mathbb{C}^N \longrightarrow S^3$ with the standard flat connection. We get the *twisted Dirac operator*

$$D_N : C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N) \longrightarrow C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N). \tag{4.12}$$

Lemma 170. The Dirac operator D_N is given by the direct sum of the Dirac operator D with itself N times, that is:

$$D_N = \left.\begin{pmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{pmatrix}\right\} N \text{ times.}$$

Proof. Please see [27, p. 76]. ☐

Proposition 171. The Dirac operator on S^3/Γ twisted by the representation α is the operator D_N restricted to the subspace of twisted spinors $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha}$.

Proof. Please see [27, p. 76]. ☐

The twisted Dirac operator on S^3/Γ will be denoted by D_α^Γ.

4.5.1 The eigenvalues of the twisted Dirac operator D_α^Γ

We start this section reviewing some facts about irreducible representations of $SU(2) \cong S^3$.

Proposition 172. For each $k = 0, 1, \cdots$, there is an irreducible representation E_k of $SU(2)$ of dimension $k+1$. They are described as follows. Firstly, $E_0 = \mathbb{C}$ is the trivial representation, E_1 is the standard representation on $\mathbb{C}^2$ given by matrix multiplication and for $k \geq 2$ the representation space of E_k is the space of homogeneous polynomials of degree k in two variables x and y. Moreover, every unitary irreducible representation of $SU(2)$ is isomorphic to one of the E_k.

Proof. May be found in [5, Proposition 5.3, p. 86]. ☐

In order to compute the eigenvalues of the twisted Dirac operator D_α^Γ it is enough to know the eigenvalues of D (which we already know by proposition 159) and count their multiplicities on $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha}$.

To do so, we first identify the subspace $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha}$ with something easier to work with.

Let V be a finite dimensional complex representation of G and V^* its dual representation. The group G acts on V^* by $(gf)(v) = f(g^{-1}v)$. Given $v \in V$ and $f \in V^*$ we define

$$\begin{aligned} \mathrm{d}_{v,f} &: G \longrightarrow \mathbb{C} \\ \mathrm{d}_{v,f}(g) &= f(g^{-1}v) \end{aligned}$$

which allow us to obtain a linear map

$$\begin{aligned} S_V : V \otimes V^* &\longrightarrow C^\infty(G, \mathbb{C}) \\ v \otimes f &\longrightarrow \mathrm{d}_{v,f}. \end{aligned}$$

The following result is in some sense, analogous to the regular representation studied in chapter 3.

Theorem 173 (Peter-Weyl)**.** The map

$$(S_V) : \bigoplus_{V \in \mathrm{Irr}(G)} V \otimes V^* \longrightarrow C^\infty(G, \mathbb{C})$$

induces an isomorphism of representations of $G \times G$ on the Hilbert completions

$$\bigoplus_{V \in \mathrm{Irr}(G)} V \otimes V^* \cong L^2(G, \mathbb{C}).$$

Proof. Please see [5, Theorem 3.1, p. 134]. □

The previous theorem gives the isomorphism

$$C^\infty(S^3, \mathbb{C}) \cong \bigoplus_k E_k \otimes E_k^*$$

after Hilbert completions, where the E_k are the irreducible representations of $S^3 \cong SU(2)$.

Since $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N) \cong C^\infty(S^3, \mathbb{C}) \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$, we have

$$C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N) \cong \bigoplus_k E_k \otimes E_k^* \otimes \mathbb{C}^2 \otimes \mathbb{C}^N.$$

Lemma 174. The operator D restricted to $E_k \otimes E_k^* \oplus E_k \otimes E_k^* \cong E_k \otimes E_k^* \otimes \mathbb{C}^2$ decomposes as

$$D = \mathrm{Id}_{E_k} \otimes D_k$$

where

$$D_k : E_k^* \otimes \mathbb{C}^2 \longrightarrow E_K^* \otimes \mathbb{C}^2.$$

Proof. Please see [27, Proof of proposition 4.2, p. 74]. □

By the lemma, the operator D_N decomposes as

$$D_N = \bigoplus \mathrm{Id}_{E_k} \otimes D_k \otimes \mathrm{Id}_{\mathbb{C}^\mathrm{N}}.$$

The space $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha}$ is the representation of S^3 induced by the representation $\sigma \otimes \alpha$ of Γ. We want to simplify this space using the following theorem.

Theorem 175 (Frobenius reciprocity)**.** Consider a compact Lie group G, Γ a closed subgroup of G and $i : \Gamma \longrightarrow G$ the inclusion. Given two representations

$$\alpha : \Gamma \times V \longrightarrow V$$

$$\beta : G \times W \longrightarrow W$$

of Γ and G respectively, there is a canonical isomorphism

$$\begin{aligned} \mathrm{Hom}_G(W, C^\infty(G,V)^\alpha) &\cong \mathrm{Hom}_\Gamma(i^*W, V) \\ F &\longleftarrow f \end{aligned}$$

where $F(w)(g) = f(g^{-1}w)$, $i^* = \beta|_\Gamma$, and

$$F \longmapsto e \circ F$$

where

$$\begin{aligned} e : C^\infty(G,V)^\alpha &\longrightarrow V \\ f &\longmapsto f(1). \end{aligned}$$

Proof. Please see [5, Proposition 6.2, p. 144]. □

Using Frobenius reciprocity we define the map

$$\begin{aligned} A_W : W \otimes \mathrm{Hom}_\Gamma(i^*W, V) &\longrightarrow C^\infty(G,V)^\alpha \\ w \otimes A &\longmapsto A_W(w \otimes A) \end{aligned}$$

where $A_W(w \otimes A)(g) = A(g^{-1}w)$.

Thus, we have the following

Theorem 176. The map

$$(A_W) : \bigoplus_{W \in \mathrm{Irr}(G)} W \otimes \mathrm{Hom}_\Gamma(i^*W, V) \longrightarrow C^\infty(G,V)^\alpha$$

induces an isomorphism of representations.

Proof. May be found in [5, Equation 6.4, p. 145]. □

Using Frobenius reciprocity we have the equivalence

$$C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha} \cong \bigoplus_k E_k \otimes \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$$

therefore, restricting the operator D_N to the subspace $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^{\Gamma,\alpha}$ is equivalent to restrict the operator $D_k \otimes \mathrm{Id}_{\mathbb{C}^N}$ to the subspace $\mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$. To know the dimension of this last subspace we use the following result of character theory

Proposition 177. Let V and W be representations of G. Then

$$\langle \chi_V, \chi_W \rangle_\Gamma = \dim \mathrm{Hom}_G(V, W).$$

By the proposition we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N) = \langle \chi_{E_k}, \chi_{\sigma\otimes\alpha} \rangle_\Gamma$$

where χ_{E_k} and $\chi_{\sigma\otimes\alpha}$ are the characters of the respective representations and the notation $\langle\ ,\ \rangle_\Gamma$ is used to emphasize that we are taking the inner product of characters of representations of Γ .

Now we have reduce the problem of counting the total number of eigenvalues in the subspace $\mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$ to computing $\langle \chi_{E_k}, \chi_{\sigma\otimes\alpha} \rangle_\Gamma$.

By proposition 159 of chapter four there are only two different eigenvalues in $\mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$: $-\frac{1}{2} - (k+1)$ which is negative for any value of k and $-\frac{1}{2} + (k+1)$ which is always positive. Hence, what is left is to count how many of the $\langle \chi_{E_k}, \chi_{\sigma\otimes\alpha} \rangle_\Gamma$ are negative and how many positive.

We have that the operator D restricts to the operator

$$D_k : \mathrm{Hom}(E_k, \mathbb{C}^2) \longrightarrow \mathrm{Hom}(E_k, \mathbb{C}^2).$$

We denote by V_k^+ and V_k^- the eigenspaces corresponding to the the positive and negative eigenvalues of D_k, respectively. Since the action of Γ commutes with D_k, V_k^+ and V_k^- are subrepresentations of $\mathrm{Hom}(E_k, \mathbb{C}^2)$.

We have that Γ acts on $\mathrm{Hom}(E_k, \mathbb{C}^2)$ by $(gL)(P) = gL(g^{-1}P)$, for $L \in \mathrm{Hom}(E_k, \mathbb{C}^2)$, $g \in \Gamma$ and $P \in E_k$. The set of fixed point of this action, denoted by $\mathrm{Hom}(E_k, \mathbb{C}^2)^\Gamma$ is equal to $\mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2)$. Therefore, the number of positive eigenvalues is given by the dimension of the set of fixed points of V_k^+, denoted by $\dim_{\mathbb{C}}(V_k^+)^\Gamma$; and the number of negative eigenvalues is given by the dimension of V_k^-, denoted by $\dim_{\mathbb{C}}(V_k^-)^\Gamma$.

The following proposition will be used to give two useful isomorphisms:

Proposition 178 (Glebsch-Gordan formula)**.** The tensor product of two irreducible representations of $SU(2)$ may be expressed as the direct sum of irreducible representations, as follows

$$E_k \otimes E_l = \bigoplus_{j=0}^{q} E_{k+l-2j}, \qquad \text{with } q = \min\{k, l\}.$$

Proof. Please see [5, Proposition 5.5, p.87] □

Proposition 179. We have the isomorphisms

$$V_k^- \cong E_{k+1}, \qquad V_k^+ \cong E_{k-1}.$$

Proof. Let us remember that S^3 acts on E_k^* since E_k^* is an irreducible representation and also acts on $\mathbb{C}^2$ by multiplication. Then S^3 acts on $E_k^* \otimes \mathbb{C}^2 \cong \mathrm{Hom}(E_k, \mathbb{C}^2) \cong V_k^+ \oplus V_k^-$ and since D^k

commutes with this action, V_k^- and V_k^+ are sub-representations of S^3. By the Clebsch-Gordan formula we have the decomposition

$$\mathrm{Hom}(E_k, \mathbb{C}^2) \cong E_k^* \otimes \mathbb{C}^2 \cong E_k \otimes E_1 \cong E_{k+1} \oplus E_{k-1} \cong V_k^+ \oplus V_k^-$$

and matching dimensions we get the desired isomorphisms. □

Theorem 180. Let $\alpha : \Gamma \longrightarrow GL_N(\mathbb{C})$ be a representation of Γ. Then the eigenvalues of the twisted Dirac operator D_α^Γ on S^3/Γ are

$$\begin{aligned} &-\frac{1}{2} - (k+1) \quad \text{with multiplicity } \langle \chi_{E_{k+1}}, \chi_\alpha \rangle_\Gamma (k+1), \quad k \geq 0, \\ &-\frac{1}{2} + (k+1) \quad \text{with multiplicity } \langle \chi_{E_{k-1}}, \chi_\alpha \rangle_\Gamma (k+1), \quad k \geq 1. \end{aligned}$$

Proof. First we decompose the space $\mathrm{Hom}(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N) \cong E_k^* \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$ as the direct sum of eigenspaces of the twisted Dirac operator. Denote by $V_{k,\alpha}^+$ and $V_{k,\alpha}^-$ the positive and negative eigenspaces, respectively.

Since D_N is the direct sum of N copies of D we have that the eigenspaces of D_N are also the sum of N copies of the eigenspaces of D, therefore $V_{k,\alpha}^+ \cong V_k^+ \otimes \mathbb{C}^N$. We want to know the dimension of the fixed point set of $V_{k,\alpha}^+$.

For a finite group, every representation is equivalent to a unitary representation and therefore $\chi_\alpha = \overline{\chi_\alpha}$.

Using proposition 179 and elemental character properties we have

$$\chi_{V_{k,\alpha}^-} = \chi_{V_k^- \otimes \mathbb{C}^N} = \chi_{E_{k+1} \otimes \mathbb{C}^N} = \chi_{E_{k+1}} \chi_\alpha = \chi_{E_{k+1}} \overline{\chi_\alpha}$$

Hence it holds

$$\dim_{\mathbb{C}} (V_{k,\alpha}^-)^\Gamma = \langle \chi_{E_{k-1}}, \chi_\alpha \rangle_\Gamma .$$

The other case is analogous. □

Chapter 5

The η invariant

In the previous chapter we computed the Dirac operator, both regular and twisted, for spaces S^3/Γ, where Γ is a finite subgroup of S^3.

Now we will define two spectral invariants η and ξ. In order to compute them, we will need to find the eigenvalues and multiplicities of the Dirac operator on S^3/Γ, twisted by a unitary representation α of its fundamental group.

We start this chapter giving the definitions of the aforementioned invariants.

5.1 Definition of the η and ξ invariants

Let us consider M a Riemannian odd-dimensional manifold and E a smooth vector bundle over M.

Proposition 181. Let $A : C^\infty(M, E) \longrightarrow C^\infty(M, E)$ be a *self adjoint elliptic operator*. Then A has discrete spectrum with real eigenvalues $\{\lambda\}$.

Proof. May be found in [18, Theorem 5.8, p. 196]. □

Definition 182. Let A be an operator as above with eigenvalues $\{\lambda\}$. The *η-series of A* is a function of the complex variable s given by

$$\eta(s; A) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)|\lambda|^{-s}$$

where the sum is taken over the non-zero eigenvalues of A.

Observation 183. This series converges when the real part $\Re(s)$ is sufficiently large; it extends by analytic continuation to a meromorphic function on the whole s-plane and it is finite at $s = 0$ (for more details please see [30]).

The number $\eta(0;A)$ is called the *η-invariant of A* and denoted simply by $\eta(A)$.

Definition 184. Let h be the dimension of the kernel of A. We define the *ξ-series of A* by

$$\xi(s;A) = \frac{h + \eta(s;A)}{2}.$$

We notice that ξ is a refinement of the η-series which takes into account the zero eigenvalues of A.

Let $\pi : \tilde{M} \longrightarrow M$ be the universal covering of M. This covering is a principal $\pi_1(M)$-bundle (for details please see [18, Appendix A, p.370]). Taking a unitary representation of the fundamental group of M: $\alpha : \pi_1(M) \longrightarrow U(N)$ as an action of $\pi_1(M)$ on $\mathbb{C}^N$, we can construct the $\pi_1(M)$-principal bundle associated with the universal covering:

$$V_\alpha = \tilde{M} \times_\alpha \mathbb{C}^N.$$

Observation 185. The bundle V_α has a canonical flat connection ∇^α given by the exterior derivative as follows.

As usual, we denote by $C^\infty(\tilde{M}, \mathbb{C}^N)^\alpha$ the space of equivariant functions under the action of $\pi_1(M)$ and by $\Omega^1(\tilde{M}, \mathbb{C}^N)^\alpha$ the space of equivariant 1-forms under the action of $\mathbb{C}^N$.

By the previous construction of the bundle V_α we have that $C^\infty(M, V_\alpha) \cong C^\infty(\tilde{M}, \mathbb{C}^N)^\alpha$ and $\Omega^1(M, V_\alpha) \cong \Omega^1(\tilde{M}, \mathbb{C}^N)^\alpha$.

Since the exterior derivative

$$d : C^\infty(\tilde{M}, \mathbb{C}^N) \longrightarrow \Omega^1(\tilde{M}, \mathbb{C}^N)$$

maps equivariant functions to equivariant 1-forms, we have the connection on V_α

$$\nabla^\alpha : C^\infty(M, V_\alpha) \cong C^\infty(\tilde{M}, \mathbb{C}^N)^\alpha \xrightarrow{d} \Omega^1(\tilde{M}, \mathbb{C}^N)^\alpha \cong \Omega^1(M, V_\alpha).$$

We can use this connection to "couple" the operator A to V_α to get an operator

$$A_\alpha : C^\infty(M, E \otimes V_\alpha) \longrightarrow C^\infty(M, E \otimes V_\alpha)$$

and define the functions

$$\eta(s;\alpha, A) := \eta(s; A_\alpha), \qquad \xi(s;\alpha, A) := \xi(s; A_\alpha)$$

and their *reduced forms*

$$\tilde{\eta}(s;\alpha, A) := \eta(s;\alpha, A) - N\eta(s;A), \qquad \tilde{\xi}(s;\alpha, A) := \xi(s;\alpha, A) - N\xi(s;A)$$

where N is the dimension of the representation α.

Proposition 186. The functions $\tilde{\eta}(s;\alpha,A)$ and $\tilde{\xi}(s;\alpha,A)$ are finite at $s=0$ and if we reduce modulo $\mathbb{Z}$ then

$$\tilde{\eta}(\alpha,A)=\tilde{\eta}(0;\alpha,A)\in\mathbb{C}/\mathbb{Z},\qquad \tilde{\xi}(\alpha,A)=\tilde{\xi}(0;\alpha,A)\in\mathbb{C}/\mathbb{Z}$$

are homotopy invariants of A.

Proof. Please check [3, Proposition 2.14, p. 79]. $\square$

Now we apply the previous construction to the case when $M=S^3$ and $A=D$.

The hypotheses for the definition of the η and ξ invariants require the operator to be *elliptic* and *self adjoint*. We will not enter into the detail of those definitions but it is known that the Dirac operator satisfy these conditions. It is shown in [18, Lemma 5.1,p.113] that it is elliptic and in [18, Proposition 5.3, p.114] that it is self adjoint.

We want to compute the η and ξ-invariants for the twisted Dirac operator D_N given by (4.12). In order to do so we need to compute its eigenvalues and multiplicities. That will be done in the next section.

5.1.1 Multiplicities of eigenvalues of D_α^Γ

To compute the multiplicities of the eigenvalues of D_α^Γ twisted by a representation $\alpha:\Gamma\longrightarrow U(N)$ we need to know the characters χ_{E_k} and then obtain the inner products $\langle\chi_{E_k},\chi_\alpha\rangle_\Gamma$ for every k.

In this section we will perform such computations. For that matter, we will start reviewing some useful properties of matrices.

Given a matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $b\neq 0$ and denote its k-th power by $A^k=\begin{pmatrix}a_k & b_k\\ c_k & d_k\end{pmatrix}$. Let $\mathrm{Mat}(2,\mathbb{K})$ be the vector space of 2×2 matrices over a field $\mathbb{K}$. Let $A\in\mathrm{Mat}(2,\mathbb{K})$, then A acts naturally on $\mathbb{K}^2$ by matrix multiplication.

The k-symmetric power $S^k\mathbb{K}^2$ is isomorphic to the space of homogeneous polynomials of degree k in two variables x and y. The monomials

$$P_j(x,y)=x^{k-j}y^j,\qquad 0\leq j\leq k,\tag{5.1}$$

give a basis for the space $S^k\mathbb{K}^2$.

A matrix $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ gives us a linear map

$$\begin{aligned}\mathbb{K}^2&\longrightarrow\mathbb{K}^2\\ z=(x,y)&\longmapsto Az=(ax+by,cx+dy)\end{aligned}$$

which induces another linear map

$$\begin{aligned} S^k\mathbb{K}^2 &\longrightarrow S^k\mathbb{K}^2 \\ P &\longmapsto (A\cdot P)(z) = P(Az). \end{aligned}$$

The *k-th symmetric power of A* is the matrix corresponding to the last transformation, respect to the basis given by (5.1) and is denoted by S^kA.

Theorem 187 (Cisneros-Molina)**.** Let b be the coordinate function on $\mathrm{Mat}(2,\mathbb{K})$ on the 2,1 entry and $b_k(A) = b(A^k)$, then

$$b_{k+1}(A)/b(A) = \mathrm{Tr}(S^kA)$$

Proof. Please see [26, Proposition 2.2.2, p. 147] □

We will use the previous proposition to prove an important property regarding periodicity of characters.

Definition 188. Let Γ be a finite group. The least common multiple of the different orders of the elements of Γ is called the *exponent of the group* and denoted by c_Γ.

Proposition 189 (Periodicity property)**.** Let $g \in \Gamma \subset S^3$ with $g \neq 1$ and $g \neq -1$. If $k = l \bmod c_\Gamma$, then $\chi_{E_k}(g) = \chi_{E_l}(g)$.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ of finite order $|g|$ with $g \neq \pm I$.

If g were diagonal, we conjugate it by a non-diagonal matrix and get another non-diagonal matrix with the same character (this is possible since characters are invariant under conjugation). Therefore, without loss of generality, we can suppose that g is not diagonal.

If $k = l \bmod c_\Gamma$, then $k = l \bmod |g|$ which implies $b_{k+1} = b_{l+1}$. Therefore, by proposition 187 we have

$$\chi_k(g) = b_{k+1}/b = b_{l+1}/b = \chi_l(g).$$

□

We have made a step forward towards computing the desired characters. Next we will use the previous proposition and the following lemma to obtain such values.

Lemma 190. Let Γ be a finite subgroup of S^3. Then

$$\chi_{E_{k+c_\Gamma}}(1) = \chi_{E_k}(1) + c_\Gamma.$$

If $-1 \in \Gamma$, then

$$\chi_{E_{k+c_\Gamma}}(-1) = \begin{cases} \chi_{E_k}(-1) + c_\Gamma & \text{if } k \text{ is even,} \\ \chi_{E_k}(-1) - c_\Gamma & \text{if } k \text{ is odd.} \end{cases}$$

Proof. May be found in [27, Lemma 4.17, p.85]. □

Proposition 191. Let Γ be a finite subgroup of S^3 and let $\alpha : \Gamma \longrightarrow U(N)$ be an unitary representation of dimension N. Then we have the following two cases

1. If $-1 \in \Gamma$ (this case corresponds to binary groups and cyclic groups of even order), then

$$\langle \chi_{E_{k+c_\Gamma}}, \chi_\alpha \rangle_\Gamma = \begin{cases} \frac{c_\Gamma}{|\Gamma|}(\chi_\alpha(1) + \chi_\alpha(-1)) + \langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma & \text{if } k \text{ is even,} \\ \frac{c_\Gamma}{|\Gamma|}(\chi_\alpha(1) - \chi_\alpha(-1)) + \langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma & \text{if } k \text{ is odd.} \end{cases}$$

2. If $-1 \notin \Gamma$ (this case corresponds to cyclic groups of odd order) then

$$\langle \chi_{E_{k+c_\Gamma}}, \chi_\alpha \rangle_\Gamma = \frac{N c_\Gamma}{|\Gamma|} + \langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma$$

Proof. We will prove the case 1. for k even. The other cases are similar.

Let $\mathcal{O}$ be an arbitrary conjugacy class of Γ, $|\mathcal{O}|$ its order and $g_{\mathcal{O}}$ a representative of such class. Since characters are class functions (remember proposition 114 chapter three) we have

$$\begin{aligned} \langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma &= \frac{1}{|\Gamma|} \sum_{\mathcal{O}} |\mathcal{O}| \chi_{E_k}(g_{\mathcal{O}}) \overline{\chi_\alpha(g_{\mathcal{O}})} \\ &= \frac{1}{\Gamma}\Big(\chi_{E_k}(1)\overline{\chi_\alpha(1)} + \chi_{E_k}(-1)\overline{\chi_\alpha(-1)}\Big) + \sum_{\mathcal{O} \neq \pm 1} |\mathcal{O}| \chi_{E_k}(g_{\mathcal{O}}) \overline{\chi_\alpha(g_{\mathcal{O}})} \\ &= \frac{1}{|\Gamma|}(k+1)\Big(\overline{\chi_\alpha(1)} + \overline{\chi_\alpha(-1)}\Big) + \frac{1}{|\Gamma|} \sum_{\mathcal{O} \neq \pm 1} |\mathcal{O}| \chi_{E_k}(g_{\mathcal{O}}) \overline{\chi_\alpha(g_{\mathcal{O}})}. \end{aligned}$$

substituting in the last equation k by $k + c_\Gamma$ and applying proposition 189 we get

$$\begin{aligned} \langle \chi_{E_{k+c_\Gamma}}, \chi_\alpha \rangle_\Gamma &= \frac{1}{|\Gamma|}(k + c_\Gamma + 1)(\overline{\chi_\alpha(1)} + \overline{\chi_\alpha(-1)}) + \frac{1}{|\Gamma|} \sum_{\mathcal{O} \neq \pm 1} |\mathcal{O}| \chi_{E_{k+c_\Gamma}}(g_{\mathcal{O}}) \overline{\chi_\alpha(g_{\mathcal{O}})} \\ &= \frac{1}{|\Gamma|}(k + c_\Gamma + 1)\Big(\overline{\chi_\alpha(1)} + \overline{\chi_\alpha(-1)}\Big) - \frac{1}{|\Gamma|}(k+1)\Big(\overline{\chi_\alpha(1)} + \overline{\chi_\alpha(-1)}\Big) + \langle \chi_{E_k}, \chi_\alpha \rangle \\ &= \frac{c_\Gamma}{|\Gamma|}\overline{(\chi_\alpha(1) + \chi_\alpha(-1))} + \langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma \end{aligned}$$

□

Applying the previous proposition recursively we finally get the desired result:

Theorem 192. Let $k = c_\Gamma l + m$ with $0 \leq m < c_\Gamma$.

1. If $-1 \in \Gamma$, then

$$\langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma = \begin{cases} \frac{c_\Gamma l}{|\Gamma|}(\chi_\alpha(1) + \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma & \text{if } k \text{ is even,} \\ \frac{c_\Gamma l}{|\Gamma|}(\chi_\alpha(1) - \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma & \text{if } k \text{ is odd.} \end{cases}$$

2. If $-1 \notin \Gamma$ then

$$\langle \chi_{E_k}, \chi_\alpha \rangle_\Gamma = \frac{N c_\Gamma l}{|\Gamma|} + \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma$$

5.2 Computation of the η and ξ invariants

Now that we know the eigenvalues of the twisted Dirac operator D^Γ_α, together with their multiplicities we can compute the η-invariant. From now on we shall denote the η-series $\eta(s; \alpha, D^\Gamma)$ simply by $\eta(s; \alpha, \Gamma)$ and the η-invariant $\eta(\alpha, D^\Gamma)$ by just $\eta(\alpha, \Gamma)$.

We start with a theorem that will allow us to perform the computations.

Theorem 193. Let Γ be a finite subgroup of S^3. Let $\alpha : \Gamma \longrightarrow U(N)$ be an unitary representation of dimension N and let c_Γ be the exponent of Γ. Then

1. If $-1 \in \Gamma$ we have the following two cases

   (a) If $\chi_\alpha(1) = \chi_\alpha(-1)$

$$\eta(\alpha, \Gamma) = \frac{N}{6|\Gamma|}(c_\Gamma^2 - 12c_\Gamma + 23) + \frac{c_\Gamma - 2}{c_\Gamma} \sum_{j=0}^{\frac{c_\Gamma}{2}-1} \langle \chi_{E_{2j+2}}, \chi_\alpha \rangle_\Gamma - \frac{4}{c_\Gamma} \sum_{j=0}^{\frac{c_\Gamma}{2}-1} j \langle \chi_{E_{2j}}, \chi_\alpha \rangle_\Gamma.$$

   (b) If $\chi_\alpha(1) = -\chi_\alpha(-1)$

$$\eta(\alpha, \Gamma) = \frac{N}{6|\Gamma|}(c_\Gamma^2 + 6c_\Gamma + 5) + \sum_{j=0}^{\frac{c_\Gamma}{2}-1} \langle \chi_{E_{2j+2}}, \chi_\alpha \rangle_\Gamma - \frac{4}{c_\Gamma} \sum_{j=0}^{\frac{c_\Gamma}{2}} j \langle \chi_{E_{2j-1}}, \chi_\alpha \rangle_\Gamma.$$

2. If $-1 \notin \Gamma$

$$\eta(\alpha, \Gamma) = \frac{N}{6|\Gamma|}(c_\Gamma^2 + 3c_\Gamma + 2) + \frac{c_\Gamma - 2}{c_\Gamma} \sum_{m=0}^{c_\Gamma - 1} \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma - \frac{2}{c_\Gamma} \sum_{m=0}^{c_\Gamma - 1} m \langle \chi_{E_m}, \chi_\alpha \rangle_\Gamma.$$

Proof. May be found in [25, Theorem 5.1, p. 217]. □

The following lemma allow us to write the characters χ_{E_k} in terms of the character $\chi_{\mathbb{C}^2}$ of the standard representation.

Lemma 194.

$$\chi_{E_k}(g) = \begin{cases} \sum_{j=1}^r \chi_{\mathbb{C}^2}(g^{2j}) + 1 & \text{if } k = 2r \text{ is even,} \\ \sum_{j=0}^r \chi_{\mathbb{C}^2}(g^{2j+1}) & \text{if } k = 2r+1 \text{ is odd.} \end{cases}$$

Proof. Please see [27, Lemma 4.11, p. 81]. □

Before we continue, let us remember that in chapter two we found the classification of the finite subgroups of $S^3 \cong SU(2)$ (please see theorem 69). These are the cyclic groups C_n, the binary dihedral groups $\tilde{D_n}$, the binary tetrahedral group $\tilde{T}$, the binary octahedral group $\tilde{O}$ and the binary icosahedral $\tilde{I}$.

Let $\langle p, q, r\rangle$ denote the group given by the presentation $\langle x, y, z \,|\, x^p = y^q = z^= xyz$. Using this notation we write the binary dihedral groups as $\langle 2, 2, r\rangle$ with $r \geq 2$, the binary tetrahedral as $\langle 2, 3, 3\rangle$, the binary octahedral as $\langle 2, 3, 4\rangle$ and the binary icosahedral as $\langle 2, 3, 5\rangle$.

Observation 195. Let Γ be a finite subgroup of S^3 and let $\alpha : \Gamma \longrightarrow U(N)$ be a representation of Γ.

If $a+bi+cj+dk$ is an unitary quaternion, its image under the isomorphism given by proposition 54 of chapter two is the matrix

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

We compute its characteristic polynomial

$$\det\left(\begin{pmatrix} a+bi-\lambda & c+di \\ -c+di & a-bi-\lambda \end{pmatrix}\right) = (a+bi-\lambda)(a-bi-\lambda) - (c+di)(-c+di)$$

$$= a^2 - abi - a\lambda + abi - (bi)^2 - bi\lambda + a\lambda + \lambda bi + \lambda^2 - \big(-c^2 + cdi - cdi + (di)^2\big)$$

$$= a^2 + 2a\lambda + b^2 + \lambda^2 + c^2 + d^2 = \lambda^2 - 2a\lambda + 1 = 0$$

and its roots are the eigenvalues of the matrix. Therefore if two quaternions have the same real part, their associated matrices have the same eigenvalues.

Since $\eta(\alpha \oplus \beta, \Gamma) = \eta(\alpha, \Gamma) + \eta(\beta, \Gamma)$ for any two representations of Γ, it is enough to compute the invariants for the irreducible representations.

The computation of the invariants is made according to the following outline:

The fists step is to compute the character $\chi_{\mathbb{C}^2}$ of the standard representation of S^3 on $\mathbb{C}^2$ restricted to the subgroup Γ, using observation 195.

Then it is necessary to use lemma 194 to compute the characters χ_{E_k} and their inner products $\langle \chi_{E_k}, \chi_\alpha\rangle_\Gamma$ for $0 \leq k \leq c_\Gamma$, where χ_α is the character of the representation α and c_Γ is the exponent of Γ.

Next, by theorem 193 it is possible to obtain the value of $\eta(\alpha, \Gamma)$.

Finally one can get the reduced ξ-invariant by

$$\tilde{\xi}(\alpha, \Gamma) = \frac{1}{2}(\eta(\alpha, \Gamma) - N\eta(\Gamma)),$$

where N is the dimension of α.

For details about the steps previously mentioned, please refer to [27, Section 4.7].

In this work we will limit to list the values of the invariants $\eta(\alpha,\Gamma)$ and $\tilde{\xi}(\alpha,\Gamma)$ for the irreducible representations of each of the finite subgroups of S^3. Such representations will be denoted by α_t and their respective character by χ_t.

1. Cyclic group C_n. Order n. Exponent n. It is generated by the quaternion

$$\gamma_n = \cos\frac{2\pi}{n} + \left(\sin\frac{2\pi}{n}\right)i.$$

Let $\zeta_n = e^{\frac{2\pi i}{n}}$ be the standard primitive n-root of unity. The group C_n has n irreducible representations of dimension 1 given by

$$\chi_t(\gamma_n^l) = (\zeta_n^l)^{t-1} \qquad 1 \leq t \leq n.$$

Here we show an example for $n = 4$:

	$\chi_\alpha(1) = \chi_\alpha(-1)$		$\chi_\alpha(1) = -\chi_\alpha(-1)$	
rep.	α_1	α_2	α_3	α_4
$\eta(\alpha,\Gamma)$	$\frac{5}{8}$	$-\frac{3}{8}$	$-\frac{1}{8}$	$-\frac{1}{8}$
$\tilde{\xi}(\alpha,\Gamma)$	0	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{5}{8}$

2. Binary dihedral group $\tilde{D}$. Order $4r$. Exponent $2r$ if r is even and $4r$ if r is odd. It has $r+3$ conjugacy classes. Its character table was given in table 3.4 for even order and 3.3 for odd order. Here is an example for $r = 4$:

	$\chi_\alpha(1) = \chi_\alpha(-1)$					$\chi_\alpha(1) = -\chi_\alpha(-1)$	
rep.	α_1	α_2	α_3	α_4	ψ_2	ψ_1	ψ_3
$\eta(\alpha,\Gamma)$	$\frac{37}{32}$	$\frac{5}{32}$	$-\frac{11}{32}$	$-\frac{11}{32}$	$-\frac{3}{16}$	$\frac{7}{16}$	$-\frac{9}{16}$
$\tilde{\xi}(\alpha,\Gamma)$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{16}$	$\frac{9}{16}$

3. Binary tetrahedral group $\langle 2,3,3\rangle$. Order 24. Exponent 12. It has 7 conjugacy classes, its character table was given in table 3.5 of chapter three. The η and $\tilde{\xi}$-invariants are

	$\chi_\alpha(1) = \chi_\alpha(-1)$					$\chi_\alpha(1) = -\chi_\alpha(-1)$	
rep.	α_1	α_2	α_3	α_7	α_4	α_5	α_6
$\eta(\alpha,\Gamma)$	$\frac{167}{144}$	$-\frac{25}{144}$	$-\frac{25}{144}$	$-\frac{3}{16}$	$\frac{29}{72}$	$-\frac{19}{72}$	$-\frac{19}{72}$
$\tilde{\xi}(\alpha,\Gamma)$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{24}$	$\frac{17}{24}$	$\frac{17}{24}$

4. Binary octahedral group $\langle 2,3,4\rangle$. Order 48. Exponent 24. It has 8 conjugacy classes, its character table is table 3.6 of chapter three. The η and $\tilde{\xi}$-invariants are

	$\chi_\alpha(1)=\chi_\alpha(-1)$					$\chi_\alpha(1)=-\chi_\alpha(-1)$		
rep.	α_1	α_2	α_3	α_6	α_7	α_4	α_5	α_8
$\eta(\alpha,\Gamma)$	$\frac{383}{288}$	$-\frac{49}{288}$	$-\frac{25}{144}$	$-\frac{11}{32}$	$\frac{5}{32}$	$\frac{101}{144}$	$-\frac{43}{144}$	$-\frac{19}{72}$
$\tilde{\xi}(\alpha,\Gamma)$	0	$\frac{1}{4}$	$\frac{7}{12}$	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{48}$	$\frac{25}{48}$	$\frac{5}{24}$

5. Binary icosahedral group $\langle 2,3,5\rangle$. Order 120. Exponent 60. It has 9 conjugacy classes, its character table is table 3.1 of chapter three. The η and $\tilde{\xi}$-invariants are

	$\chi_\alpha(1)=\chi_\alpha(-1)$					$\chi_\alpha(1)=-\chi_\alpha(-1)$			
rep.	α_1	α_4	α_5	α_6	α_8	α_2	α_3	α_7	α_9
$\eta(\alpha,\Gamma)$	$\frac{1079}{720}$	$-\frac{19}{80}$	$\frac{9}{16}$	$-\frac{61}{180}$	$-\frac{25}{144}$	$\frac{73}{72}$	$-\frac{67}{360}$	$\frac{29}{180}$	$-\frac{17}{40}$
$\tilde{\xi}(\alpha,\Gamma)$	0	$\frac{19}{30}$	$\frac{1}{30}$	$\frac{5}{6}$	$\frac{1}{6}$	$\frac{1}{120}$	$\frac{49}{120}$	$\frac{1}{12}$	$\frac{7}{24}$

Conclusions and future work

Let us make a summary of what we have already done.

We started Chapter 1 introducing the *ADE* singularities and learned to use the blow up technique to resolve them, as well as a way to encode the arrangement of the components of the minimal resolution known as the dual graph of the resolution.

Later, in chapter two we listed the finite subgroups Γ of $SL(2,\mathbb{C})$ and proved that ADE singularities can be characterized as quotient singularities of the form $\mathbb{C}^2/\Gamma$.

By the McKay correspondence studied in Chapter 3, each irreducible component of the exceptional divisor of the minimal resolution of an ADE singularity has a corresponding irreducible representation of a finite subgroup Γ of $SL(2,\mathbb{C})$.

In Chapter 4 we developed the theory needed to define the Dirac operator and we computed it for S^3.

In Chapter 5 we introduced the η and ξ-invariants and computed them for the Dirac operator of a quotient S^3/Γ twisted by a representation α of its fundamental group Γ.

It is natural to wonder if this number has any interpretation in the context of the minimal resolution of the singularity.

This interesting question is beyond the reach of this work and remains open for future research. One possible way to answer it may be regarding the minimal resolution of a singularity as a *manifold with fibred corners* and follow the techniques given in [7] to "lift" the twisted Dirac operator and try to give an interpretation of the η-invariant.

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