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CONSTRUCTION AND STRUCTURE OF LIE SUPERGROUPS

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The past knowledge is always with us, and it's our main ingredient for understanding. The theoretical ideas which are based on "let's imagine that this may happen because why not" are not taking us anywhere.

Carlo Rovelli, *Science is not about certainty: a philosophy of physics* (interview on Edge, 30/05/2012).

Mi timidez es la causa de todos mis fracasos. Yo no soy, precisamente, un fracasado. Pero he tenido algunos fracasos, de que quizás sólo yo me doy cuenta. Sin mi timidez, de que también sólo yo me doy cuenta, yo sería un grande hombre.

Alfonso Reyes, *La casa del grillo*.

Quid non intellegit aut tasceat aut discat.

John Dee, *Monas Hieroglyphica*.

Contents

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Abstract

The purpose of this work is to give an account of the theory of smooth Lie supergroups. We do so using a completely differential approach to the subject, based on our previous work [Gua16].

The first chapter gives a summary of the mentioned work and introduces what we call the contact relation between two supermanifold morphisms.

The second chapter offers a complete classification of split Lie supergroups. Any Lie supergroup is in contact of order 1 to one of these (in a very precise sense defined in the first chapter) so they turn out to be important for the general classification result.

In the third chapter we study multiplication maps that are more general on split supergroups; to wit, they are differential operators of positive order.

The fourth and last chapter deals with the structure of Lie supergroups in general. The main result is tantamount to approximating an abstract smooth Lie supergroup up to arbitrary order of contact with multiplication maps on a split Lie supergroup.

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Introduction

The objects of study of this work are real Lie supergroups in the broadest sense: smooth real supermanifolds with a multiplication, inversion and neutral-element map. The theory of these objects was first proposed by Gol'fand and Likhtman in [GL89] to extend the symmetries of spacetime to include the recently discovered supersymetries. Somewhat later Wess and Zumino wrote their very celebrated paper [WZ74]. In essence these two papers studied the object

$$\mathfrak{q} = \mathfrak{a} \oplus \mathrm{S}$$

where $\mathfrak{a}$ is the Lie algebra of the Lorenz group $\mathbb{R}^{1,3} \rtimes \mathrm{SO}(1,3)$ and S is a real spin representation of $\mathrm{SO}(1,3)$; for this group the representation S has a natural equivariant symmetric bilinear form

$$\Gamma\colon \operatorname{Sym}^2 \mathrm{S} \to \mathbb{R}^4$$

(cf. [Var04, sections 6.6 and 6.7]). This turns the space $\mathfrak{q}$ into a Lie superalgebra by defining

$$[\![A,B]\!] = \begin{cases} [A,B], & A,B \in \mathfrak{a} \\ \rho(A)B := -B\rho(A), & A \in \mathfrak{a}, B \in \mathrm{S} \\ \Gamma(A,B), & A,B \in \mathrm{S} \end{cases} \tag{*}$$

where $\rho\colon \mathfrak{a} \to \operatorname{End} \mathrm{S}$ is the representation; it is also declared that $\rho(A)B = -B\rho(A)$. What this means is the following: assign a $\mathbb{Z}_2$-grading to the space $\mathfrak{q}$ by declaring the elements of $\mathfrak{a} - \{0\}$ to be even and those of $\mathrm{S} - \{0\}$ to be odd; call these elements *homogeneous*. The *parity* of a homogeneous element is given by:

$$\lceil A \rfloor = \begin{cases} 0, & A \in \mathfrak{a} \\ 1, & A \in \mathrm{S} \end{cases}$$

Then the operation $[\![\cdot,\cdot]\!]$ above satisfies

$$[\![A,B]\!] = -(-1)^{\lceil A \rfloor \lceil B \rfloor}[\![B,A]\!]$$

and

$$[\![A,[\![B,C]\!]]\!] = [\![[\![A,B]\!],C]\!] + (-1)^{\lceil A \rfloor \lceil B \rfloor}[\![B,[\![A,C]\!]]\!]$$

for all homogeneous A, B and C. Notice that if all elements are even we recover the skew-symmetry and the Jacobi identity characteristic of a Lie algebra. Simple Lie superalgebras were classified in 1977 by Victor Kac on his famous [Kac77].

Once Lie superalgebras became interesting objects there arose a natural question: *given a Lie superalgebra is there a Lie supergroup that stands to it in the same relation as a Lie group stands to a Lie algebra?* This is equivalent to having a "super" version of Lie's Theorems. The obvious answer for this question is yes: a Lie supergroup is a group object in the category of supermanifolds. This raises another question: *what is a supermanifold?*

The earliest work on supergroups was published by Felix Berezin and Victor Kac in 1970 ([BK70]. They were motivated by physics as well, although it is not clear that they knew of the then-recent supersymmetry. In that work the authors introduce "Grassman coordinates" to account for the anticommuting variables expected from their considerations.

The first widely-circulated works on supermanifolds were the monographs [Lei80] and [Kos77]. In them, the authors proposed an approach to the newly-founded field of supergeometry using sheaves of supercommutative algebras over differentiable or complex manifolds.

An important step towards a geometric theory of supermanifolds was given by Batchelor's [Bat79]; in that work it is proved that for any supermanifold $(M, \mathcal{A})$ the sheaf of supercommutative algebras A can be rendered to be isomorphic to the sheaf of sections $\Gamma(\Lambda E)$ for a vector bundle E over M; this isomorphism is highly non-natural in the sense that many choices have to be made that depend on local trivialisations of $\mathcal{A}$. Nevertheless this paper was very fruitful for it allowed a rapid development of the theory. Important examples of this development are [SV86] and [SVB91], which laid strong geometric foundations for the theory of supermanifolds.

In this work we extend the geometric trend of the theory by relying on our previous work [Gua16] and proving a classification theorem for Lie supergroups using this geometric approach. More precisely we prove that for a supergroup of the form $(G|\Lambda\mathbf{S}^*G)$ there is a very straightforward way of constructing the multiplication map using the Lie superbracket (*); another result is that for the case $B = 0$ the Lie supergroups arising from such a Lie superalgebra turn out to be essentialy equivalent to representations of the underlying Lie group G. Finally in the last chapter we prove a "straightening" result for general Lie supergroups; this entails a natural isomorphism between a general Lie supergroup and one of the form $(G|\Lambda\mathbf{S}^*G)$ which are completely classified in chapter 3.

It is our hope that our results will prove to be useful for people developing the theory of supermanifolds with the physical applications in mind.

Chapter 1

Preliminaries

Quia parvus error in principio magnus est in fine...

Thomas Aquinas, *De ente et essentia*, proemium.

In this chapter we work out the preliminaries of our study; in the first three sections we summarise the work done on [Gua16]. We state most results without their proofs and refer to the cited work for them. We also study products in the category of smooth supermanifolds on section 1.4. On section 1.5 we state the characterisation of Lie superalgebras for reference. Finally, on section 1.6 we study an important relation between morphisms of smooth supermanifolds that will be useful later on.

§ 1.1 Supermanifolds

1.1 Definition. A **smooth supermanifold** of dimension $(m|k)$ is a pair $(M|\mathcal{R}M)$ such that

- M is a m-dimensional smooth manifold.
- $\mathcal{R}M$ is a smooth unital superalgebra bundle of rank k; i.e. for each p of M the fibre $\mathcal{R}_pM \cong \Lambda\mathrm{S}^*$ for some vector space S such that $\dim \mathrm{S} = k$.

The unital supercommutative algebra $\Gamma(\mathcal{R}M)$ is the algebra of **smooth superfunctions**. The integers m and k are, respectively, the **even dimension** and the **odd dimension** of $(M|\mathcal{R}M)$.

An obvious albeit important family of examples are the exterior algebra bundles: if $\pi\colon E \to M$ is a vector bundle of rank k then $(M|\Lambda E^*)$ is a supermanifold of dimension $(\dim M|k)$.

1.2 Remark. The reason for considering the fibres of $\mathcal{R}M$ isomorphic to the dual exterior algebra $\Lambda\mathrm{S}^*$ is heuristic: one would like to think of superfunctions as legitimate functions of the odd coordinates; in particular, if $\{s_1, \ldots, s_k\}$ is a basis of S then the simplest supercommutative functions of these coordinates are given by $\Lambda\mathrm{S}^*$, with generators $\{ds_1, \ldots, ds_k\}$, the dual basis of $\{s_1, \ldots, s_k\}$.

Because of the point-wise isomorphism $\mathcal{R}_p M \cong \Lambda S^*$ there is a unital algebra morphism $\varepsilon_p \colon \mathcal{R}_p M \to \mathbb{R}$ that "forgets" the nilpotent part of $r(p) \in \mathcal{R}_p M$. If we extend this to sections $r \in \Gamma(\mathcal{R}M)$ we get a morphism $\varepsilon_M \colon \Gamma(\mathcal{R}M) \to \mathcal{C}^\infty(M)$ that is surjective. This is called the **augmentation map**. The kernel of this map $\Gamma\left(\mathcal{R}^{\geq 1}M\right)$ is the **nilpotent ideal** of $\Gamma(\mathcal{R}M)$; it is, as notation suggests, the space of sections of a bundle $\mathcal{R}^{\geq 1}M$ which we call the **nilpotent bundle** of $(M|\mathcal{R}M)$. Since the exterior algebra is $\mathbb{Z}_2$-graded so is each fibre and we get a bundle direct sum decomposition

$$\mathcal{R}M = \mathcal{R}_+ M \oplus \mathcal{R}_- M$$

whose sections are the **even** $(+)$ and **odd** $(-)$ superfunctions respectively. To emphasise the $\mathbb{Z}_2$-grading of the bundle $\mathcal{R}M$ we will use the notation $\mathcal{R}_\bullet M$.

1.1.1. Morphisms

Supermanifolds morphisms have, under our approach, a special geometric significance. First we give the

1.3 Definition. A **supersmooth map** is a pair

$$(\phi|\Phi) : \ (M|\mathcal{R}M) \to (N|\mathcal{R}N)$$

such that $\phi \colon M \to N$ is a smooth map in the usual sense and $\Phi \colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ is a unital homomorphism of superalgebras.

The above definition entails the commutativity of

$$\begin{array}{ccc} \Gamma(\mathcal{R}N) & \xrightarrow{\Phi} & \Gamma(\mathcal{R}M) \\ {\scriptstyle \varepsilon_N}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_M} \\ \mathcal{C}^\infty(N) & \xrightarrow{\phi^*} & \mathcal{C}^\infty(M) \end{array} \tag{1.1}$$

Let η be in $\Gamma(\mathcal{R}N)$ and g be a smooth function on N; the **twisted commutator** of Φ and g is

$$[\Phi|\phi|g](\eta) := \Phi(g\eta) - (g \circ \phi)\Phi(\eta) \tag{1.2}$$

Since Φ is a superalgebra morphism we get

$$[\Phi|\phi|g] = (\Phi(g) - g \circ \phi)\Phi(\eta).$$

With the above identity and an elementary computation one can prove:

1.4 Theorem ([Gua16], Proposition 1.19)**.** *For every $(\phi|\Phi) : \ (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ there is a non-negative integer d such that if $f_0, \dots, f_d$ are smooth functions on N then*

$$[\Phi|\phi|f_0, \dots, f_d] = 0 \tag{1.3}$$

where $[\Phi|\phi|f_0, \dots, f_d]$ is the twisted commutator (1.2) iterated $d+1$ times.

Naturally, the first thing to prove is that (1.2) is independent of the order of the smooth functions under iteration, but that is also an elementary computation (cf. [Gua16, Proposition A.3]). Once this is proved the following caracterisation is justified:

1.5 Corollary. *Let $(\phi|\Phi): (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ and $d \geq 0$ the least integer for which (1.3) hold for $f_0, \dots, f_d$ smooth functions on N; then there exists a unique bundle homomorphism $\sigma^{\text{total}}(\Phi)\colon \phi^* \operatorname{Jet}^d \mathcal{R}N \to \mathcal{R}M$ such that*

$$\begin{tikzcd} \Gamma(\mathcal{R}N) \arrow[r, "\Phi"] \arrow[d, "\phi^*\circ \mathrm{jet}^d"'] & \Gamma(\mathcal{R}M) \\ \Gamma\left(\phi^*(\operatorname{Jet}^d \mathcal{R}N)\right) \arrow[ur, "\Gamma(\sigma^{\text{total}}\Phi)"'] & \end{tikzcd} \tag{1.4}$$

and therefore Φ is a ***differential operator of order d along ϕ****.*

Sketch of proof. We define the space of d-jets over the point p of a vector bundle $\pi\colon E \to M$ as

$$\operatorname{Jet}^d_p E := \Gamma(E) \Big/ \left(I_p^{d+1}\Gamma(E)\right)$$

where I_p is the ideal in $\mathcal{C}^\infty(M)$ of functions vanishing on p. This definition makes sense inasmuch we think of the d-jet of a section η as the Taylor polynomial of η in some coordinate chart $(x_1, \dots, x_m)$ around p, and therefore it just depends on the partial derivatives of η up to order d. To wit let $k \leq d$ and $\mu_1, \dots, \mu_d$ non-negative integers such tat $\mu_1 + \cdots + \mu_d = k$. The operator

$$\frac{\partial^k}{\partial x_{\nu_1}^{\mu_1} \cdots \partial x_{\nu_m}^{\mu_m}}$$

is a differential operator of order k. Inserting as argument the product $f_0 \cdots f_d$ of functions in $I_p^{d+1}\Gamma(E)$ and evaluating at p the result vanishes, since at least one of the functions will show up in the resulting expresion with no derivatives and evaluated at p.

The same remarks apply to the operator defined by (1.3) and therefore $\Phi(\eta)$ just depends on derivatives up to order d of η. The map $\sigma^{\text{total}}(\Phi; f_1, \dots, f_d)$ is given by

$$\sigma^{\text{total}}(\Phi)(\operatorname{jet}^d \eta) = \Phi(\eta)$$

which is $\mathbb{R}$-linear with appropriate domain and codomain; it is well-defined since the d-jet of η does not depend on the representative $I_p^{d+1}\Gamma(\mathcal{R}N)$ and Φ only depends on the derivatives (in any local chart) of order at most d of η. This proves the commutativity of (1.4). The definition of a differential operator along a smooth map is exactly this commutativity. □

If we set $\eta = 1$ in (1.2) the function defined by this equality is even and nilpotent, and so we get

1.6 Corollary. *If $\dim(N|\mathcal{R}N) = (n|q)$ then the map Φ is a differential operator of order at most $\lfloor \frac{q}{2} \rfloor$.*

§ 1.2 The supertangent bundle

Recall that to every smooth map $\phi\colon M \to N$ correspond two smooth maps

$$\phi_*\colon TM \to TN$$
$$\phi^*\colon T^*N \to T^*M$$

that are fibrewise linear. The map ϕ^* is defined point-wise as $(\phi^*\alpha)_{f(p)} = \alpha \circ \phi(p)$ (precomposition with ϕ). The map ϕ_* is point-wise dual to ϕ^*; that is, if α is a 1-form on N and X is a vector field on M then for each p in M

$$\langle \phi_* X, \alpha \rangle_{f(p)} = \langle X, \phi^*\alpha \rangle_p.$$

To extend the above constructions to a smooth supermap $(\phi|\Phi)$ we first have to define the supertangent bundle. The smooth supermap $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ will be fixed throughout this section.

Recall the point-wise decomposition

$$\mathcal{R}_p M \cong \mathbb{R} \oplus \mathcal{R}_p^{\geq 1} M$$

and that the spaces $\mathcal{R}_p^{\geq 1} M$ are the fibres of a vector bundle, the nilpotent bundle of $(M|\mathcal{R}M)$, which are ideals of the corresponding fibres; it therefore makes sense to take powers of this space: if $k \geq 2$ we define

$$\mathcal{R}_p^{\geq k} M = \left(\mathcal{R}_p^{\geq 1} M\right)^k. \tag{1.5}$$

Observe that if $\dim(M|\mathcal{R}M) = (m|r)$ then for every $k \geq r$ we obtain $\mathcal{R}_p^{\geq k} M = \{0\}$; this gives rise to a filtration of $\mathcal{R}_p M$:

$$\mathcal{R}_p M = \mathcal{R}_p^{\geq 0} M \supset \mathcal{R}_p^{\geq 1} M \supset \mathcal{R}_p^{\geq 2} M \supset \cdots \supset \mathcal{R}_p^{\geq r} M \supset \{0\}. \tag{1.6}$$

These spaces are then fibres of a vector bundle $\mathcal{R}^{\geq k} M$.

1.7 Definition. For a point p in M the space

$$\mathbf{S}_p M := \left(\mathcal{R}_p^{\geq 1} M / \mathcal{R}_p^{\geq 2} M\right)^*$$

is the **space of odd directions** over p; the resulting bundle $\mathbf{S}M$ is the bundle of odd directions. The dual bundle $\mathbf{S}^*M$ is the **bundle of odd codirections**.

This definition is akin to the algebraic definition of the tangent and cotangent bundles of a manifold (cf. [War71, 1.16]).

1.8 Remark. For each k the quotient $\mathcal{R}_p^{\geq k} M / \mathcal{R}_p^{\geq k+1} M$ is isomorphic to $\Lambda^k \mathbf{S}_p^* M$ since the exterior algebra is also filtered (since it is graded); nevertheless this doesn't imply $\mathcal{R}_p M$ is $\mathbb{Z}$-graded.

Recall that the tangent bundle of a manifold M is characterised as the bundle whose sections are the derivations of the smoooth functions on M. The tangent superbundle will accordingly be defined as that which has as sections the superderivations of $\Gamma(\mathcal{R}M)$.

Derivations of exterior algebras

If A is an algebra denote by $\operatorname{der} A$ its space of derivations. On a smooth manifold the algebra $\mathcal{C}^\infty(M)$ is the space of sections of the trivial bundle $M \times \mathbb{R}$ and it is known that $\operatorname{der}\mathbb{R} = \{0\}$. On a supermanifold $(M|\mathcal{R}M)$ the fibres are algebras with non-trivial derivations. This space is very easily characterised:

1.9 Theorem ([Gua16], Theorem 1.10)**.** *The space of derivations of $\Lambda \mathsf{S}^*$ is naturally isomorphic to $\Lambda_- \mathsf{S}^* \otimes \mathsf{S}$, where $\Lambda_- \mathsf{S}^*$ is the space of odd forms.*

Since derivations are first and foremost linear maps and we are working on supervector spaces, there are even endomorphisms (those that preserve the $\mathbb{Z}_2$-grading) and odd ones (those that invert it); this gives a $\mathbb{Z}_2$-grading to the space of linear endomorphisms of a supervector space and it is reasonable to expect a similar grading on the space of superderivations. Recall that a homogeneous superderivation of a superalgebra $\mathcal{A}$ is an endomorphism D that satisfies the graded Leibniz identity:

$$D(ab) = D(a)b + (-1)^{\lceil D \rfloor \lceil a \rfloor} aD(b) \tag{1.7}$$

for any homogeneous a, b in $\mathcal{A}$; they symbol $\lceil \cdot \rfloor$ denotes the parity of a homogeneous element.

1.10 Theorem ([Gua16], Theorem 1.11)**.** *The space* $\operatorname{sder}(\Lambda \mathsf{S}^*)$ *of superderivations of* $\Lambda \mathsf{S}^*$ *satisfies*

$$\operatorname{sder}_\bullet(\Lambda \mathsf{S}^*) \cong \Lambda_{-\bullet} \mathsf{S}^* \otimes \mathsf{S}$$

where $\bullet$ denotes the $\mathbb{Z}_2$-grading and $-\bullet$ the change of parity. The isomorphism is natural.

1.11 Remark. The above theorem is equivalent to endowing the vector space S with the structure of a left $\Lambda \mathsf{S}^*$-module.

1.2.1. Characterization of the tangent superbundle

Since each fibre $\mathcal{R}_p M$ is an algebra with a non-trivial space of derivations we can bundle each of these spaces together:

1.12 Definition. The bundle denoted $\operatorname{sder}(M|\mathcal{R}M)$ of the supermanifold **point-wise derivations** of $(M|\mathcal{R}M)$ is the vector bundle over M whose fibres are $\operatorname{sder}_p(M|\mathcal{R}M) = \operatorname{sder} \mathcal{R}_p M$.

These derivations are very simple operators:

1.13 Proposition ([Gua16], Proposition 2.3)**.** *A section D of* $\operatorname{sder}(M|\mathcal{R}M)$ *is a differential operator of order* 0*, i.e. it is an endomorphism of the bundle $\mathcal{R}M$.*

Classically, derivations are differential operators of order 1, so we need more operators to obtain all the possible superderivations. The following result characterises all the possibilities:

1.14 Theorem ([Gua16], Theorem 2.4)**.** *The space* $\operatorname{sder}(\Gamma(\mathcal{R}M))$ *of superderivations of the algebra of smooth superfunctions is the space of sections of a vector bundle* $\operatorname{Der}(M|\mathcal{R}M)$ *over* M *that fits into the exact sequence*

$$0 \longrightarrow \operatorname{sder}(M|\mathcal{R}M) \xhookrightarrow{\iota} \operatorname{Der}(M|\mathcal{R}M) \xrightarrow{\sigma} \mathcal{R}M \otimes TM \longrightarrow 0$$

where σ *denotes the principal symbol.*

1.15 Remark. The notations $\operatorname{Der}(M|\mathcal{R}M)$ and $T(M|\mathcal{R}M)$ will appear indistinctively to refer to the above bundle.

Remark 1.11 has the following consequence:

1.16 Corollary. *For every* p *the fibre* $\operatorname{Der}_p(M|\mathcal{R}M)$ *is isomorphic to* $\mathcal{R}_pM \otimes (T_pM \oplus \mathbf{S}_pM)$*; i.e. the bundle* $\operatorname{Der}(M|\mathcal{R}M)$ *is a bundle of left modules of* $\mathcal{R}M$ *generated by* $TM \oplus \mathbf{S}M$*.*

1.2.2. Tangential maps

We can now turn our attention to extending the natural tangential maps to supermanifolds.

1.17 Proposition ([Gua16], Proposition 1.20)**.** *The map* Φ *preserves the filtration* (1.6)*; i.e. for every* $k \geq 0$ *the inclusion*

$$\Phi(\Gamma\left(\mathcal{R}^{\geq k}N\right)) \subseteq \Gamma\left(\mathcal{R}^{\geq k}M\right)$$

holds.

1.18 Corollary ([Gua16], Corollary 1.21)**.** *Given* Φ *there are vector bundle morphisms*

$$\Phi^{(k)} \colon \Lambda^k \mathbf{S}^* N \to \Lambda^k \mathbf{S}^* M$$

for every $k \geq 0$*.*

Now we focus on the case $k = 1$ to finally construct tangential maps:

1.19 Lemma. *The map* $\Phi_{(1)} = \Phi^{(1)*}$ *satisfies the following identities:*

1. $(\phi|\Phi)_* := \phi_* \oplus \Phi_{(1)} \colon TM \oplus \mathbf{S}M \to TN \oplus \mathbf{S}N$*, i.e. it maps generators of* $\operatorname{Der}(M|\mathcal{R}M)$ *to generators of* $\operatorname{Der}(N|\mathcal{R}N)$*.*
2. $((\phi|\Phi) \circ (\psi|\Psi))_* = (\phi|\Phi)_* \circ (\psi|\Psi)_*$ *for every* $(\psi|\Psi) \colon (P|\mathcal{R}P) \to (M|\mathcal{R}M)$*.*

Proof. The first statement is obvious. As for the second statment first observe $\Psi_{(1)} \colon TP \oplus \mathbf{S}P \to TM \oplus \mathbf{S}M$ and apply point 1 of this lemma. □

The above lemma allows the following definition:

1.20 Definition. The **diferential** of $(\phi|\Phi)$ is the map $(\phi|\Phi)_* = \phi_* \oplus \Phi_{(1)}$. The dual map $(\phi|\Phi)^*$ is the **codiferential**.

1.2.3. The auxiliary differential

In view of corollary 1.5 the operator

$$[\Phi|\phi|\cdot] : \mathcal{C}^\infty(N) \to \mathrm{sEnd}_+\left(\Gamma\left(\mathcal{R}M\right)\right)$$

is a differential operator along ϕ (sEnd_+ denotes the even endomorphisms of a super-vector space); observe that $[\Phi|\phi|f]\,(\mathbf{1})$ takes values on $\Gamma\left(\mathcal{R}_+^{\geq 2}M\right)$ ($\mathbf{1}$ is the function identically equal to 1), since Φ is even. We claim that $[\Phi|\phi|f]$ is a derivation modulo $\Gamma\left(\mathcal{R}_+^{\geq 4}M\right)$; i.e. if f and g are smooth functions on N we get

$$[\Phi|\phi|fg] \equiv [\Phi|\phi|f]\,(g\circ\phi) + (f\circ\phi)\,[\Phi|\phi|g] \mod \Gamma\left(\mathcal{R}_+^{\geq 4}M\right). \tag{1.8}$$

From this identity we get, for every f smooth on N,

$$\Phi^{\mathrm{aux}} := [\Phi|\phi|\cdot] \mod \Gamma\left(\mathcal{R}^{\geq 4}M\right) : \mathcal{C}^\infty(N) \to \Gamma\left(\Lambda^2\mathbf{S}^*M\right)$$

after modding out. Put another way, this operator transforms smooth functions on N into even superfunctions on $(M|\mathcal{R}M)$.

1.21 Proposition ([Gua16], section 2.2)**.** *The operator $[\Phi|\phi|\cdot]$ is a differential operator of order 1 along ϕ and therefore factorises*

$$\begin{tikzcd} \mathcal{C}^\infty(N) \arrow[r, "\Phi^{aux}"] \arrow[d, "\mathrm{jet}^1"'] & \Gamma\left(\Lambda^2\mathbf{S}^*M\right) \\ \Gamma\left(\phi^*\,\mathrm{Jet}^1\,\mathbb{R}N\right) \arrow[ur, "\Gamma\left(\sigma^{\mathrm{total}}(\Phi^{aux})\right)"'] & \end{tikzcd} \tag{1.9}$$

where $\mathbb{R}N$ is the trivial bundle $N\times\mathbb{R}$.

Taking the principal symbol $\sigma([\Phi|\phi|\cdot])$ we obtain a bundle morphism $\Phi^{\mathrm{aux}}\colon T^*N \to \Lambda^2\mathbf{S}^*M$.

1.22 Definition. The **auxiliary differential** of $(\Phi|\phi)$ is the map

$$\Phi_{\mathrm{aux}} = \Phi^{\mathrm{aux}*}\colon \Lambda^2\mathbf{S}M \to TN.$$

The **auxiliary codifferential** is the map Φ^{aux}.

Puncutally the auxiliary codifferential is given by

$$\Phi^{\mathrm{aux}}(df_p) = [\Phi|\phi|f]_p \mod \mathcal{R}_{+,p}^{\geq 4}M.$$

From proposition 1.18 we get:

1.23 Proposition. *If Φ is of order 0 then $\Phi_{aux} \equiv 0$.*

§ 1.3 Infinitesimal supermanifolds

Let $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$. By proposition 1.18 the bundle map $\Lambda\Phi^{(1)}$ extends to a supersmooth map

$$\left(\phi\middle|\Gamma(\Lambda\Phi^{(1)})\right) : (M|\Lambda\mathbf{S}^*M) \to (N|\Lambda\mathbf{S}^*N)$$

whose differential is identically equal to the differential of $(\phi|\Phi)$ (by construction). With these maps as morphisms we obtain a special category of supermanifolds:

1.24 Definition. An **infinitesimal or split supermanifold** is a $(M|\mathcal{R}M)$ such that $\mathcal{R}M = \Lambda\mathbf{S}^*M$; we denote these objects by $(M|\mathbf{S}M)$.

Observe that in general it's not possible to form an arbitrary supermap just from $\Phi^{(1)}$; the reason is corollary 1.5: a bundle map is necessarily an operator of order 0. By the remarks made at the beginning of this section we conclude:

1.25 Proposition ([Gua16], Proposition 1.25)**.** *For a supermap $(\phi|F)$ of split supermanifolds the map F is a differential operator of order 0; that is, all morphisms of split supermanifolds are completely characterised by bundle maps.*

§ 1.4 Products

We now study the behaviour of products of supermanifolds. First observe that the product of $(M|\mathcal{R}M)$ on $(N|\mathcal{R}N)$ is necessarily of the form

$$(M|\mathcal{R}M) \times (N|\mathcal{R}N) = (M \times N|\mathcal{R}(M \times N))$$

if we want the categorical properties of a product to hold. The first step is to define the appropriate tensor product for superalgebras:

1.26 Definition. Let $\mathcal{A}$ and $\mathcal{B}$ be unital supercommutative algebras. The **supertensor product** of $\mathcal{A}$ and $\mathcal{B}$ is the usual tensor prodcut $\mathcal{A} \otimes \mathcal{B}$ endowed with the multiplication

$$a \otimes b)(\tilde{a} \otimes \tilde{b}) - (-1)^{\lceil \tilde{a} \rfloor \lceil b \rfloor} a\tilde{a} \otimes b\tilde{b}$$

for homogeneous elements $a, \tilde{a}$ of $\mathcal{A}$ and $b, \tilde{b}$ of $\mathcal{B}$.

Recall that if E and F are vector bundles over M then the direct sum $E \oplus F$ is the vector bundle over M whose fibre over p is precisely $E_p \oplus F_p$; if E and $\widetilde{E}$ are bundles over M y N respectively, $M \neq N$, then we can define the bundle $E \boxplus \widetilde{E}$ over $M \times N$ as follows: if $p \in M$ and $\tilde{p} \in N$ then

$$(E \boxplus \widetilde{E})_{(p,\tilde{p})} := E_p \oplus \widetilde{E}_{\tilde{p}} \tag{1.10}$$

and it can be proved that there exist P_1 and P_2 such that the diagram

$$\begin{tikzcd} E \arrow[d, "\pi"'] & E \boxplus \widetilde{E} \arrow[l, dashed, "P_1"'] \arrow[r, dashed, "P_2"] \arrow[d, "\pi \times \tilde{\pi}"] & \widetilde{E} \arrow[d, "\tilde{\pi}"] \\ M & M \times N \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & N \end{tikzcd}$$

commutes.

Finally we state:

1.27 Proposition. *If V and W are vector spaces then there is a natural isomorphism*

$$\psi\colon \Lambda(V \oplus W) \to \Lambda V \widehat{\otimes} \Lambda W$$

of superalgebras.

Now we can state:

1.28 Proposition. *In the product $(M \times N|\mathcal{R}(M \times N))$ the superalgebra bundle is naturally isomorphic to $\mathcal{R}M \widehat{\boxtimes} \mathcal{R}N$.*

Sketch of proof. If U and $\widetilde{U}$ are trivialising neighbourhoods for E and $\widetilde{E}$ around p and $\tilde{p}$ respectively, then $U \times \widetilde{U}$ is a trivialising neighbourhood of $\mathcal{R}M \widehat{\boxtimes} \mathcal{R}N$ with projection $\pi \otimes \widetilde{\pi}$; the fibre over $(p, \tilde{p})$ is then $\mathcal{R}_p M \widehat{\otimes} \mathcal{R}_{\tilde{p}} N$. □

§ 1.5 Lie superalgebras

Lie's theory of groups rests on the correspondence between Lie algebras (infinitesimal, linear objects) and germs of Lie groups (global, non-linear objects). Here we state the fundamental proposition of the theory of Lie superalgebras. The proof is a straightforward verification of the axioms of a Lie superalgebra.

1.29 Definition. A **Lie superalgebra** is a supervector space $(\mathfrak{g}|\mathfrak{s})$ with a binary operation $[\![\cdot,\cdot]\!]$ called *Lie superbracket*, such that for every homogeneous L, L' and L'' the following are satisfied:

1. $[\![L, L']\!] = (-1)^{\lceil L \rfloor \lceil L' \rfloor} [\![L', L]\!]$ (graded skew-symmetry of the bracket).
2. $[\![L, [\![L', L'']\!]]\!] = [\![[\![L, L']\!], L'']\!] + (-1)^{\lceil L \rfloor \lceil L' \rfloor} [\![L, [\![L', L'']\!]]\!]$ (graded Jacobi identity).

1.30 Theorem. *A Lie superalgebra is completely characterised by the following data:*

1. *A supervector space $(\mathfrak{g}|\mathfrak{s})$ such that $\mathfrak{g}$ is a Lie algebra;*
2. *a representation $\rho\colon \mathfrak{g} \to \operatorname{End} \mathfrak{s}$;*
3. *A symmetric, bilinear, $\mathfrak{g}$-equivariant form $B\colon \operatorname{Sym}^2 \mathfrak{s} \to \mathfrak{g}$ in the kernel of the composition*

$$\left(\operatorname{Sym}^2 \mathfrak{s}^* \otimes \mathfrak{g}\right)^{\mathfrak{g}} \xrightarrow{\mathrm{id} \otimes \rho} \left(\operatorname{Sym}^2 \mathfrak{s}^* \otimes \mathfrak{s}^* \otimes \mathfrak{s}\right)^{\mathfrak{g}} \xrightarrow{c \otimes \mathrm{id}} \left(\operatorname{Sym}^3 \mathfrak{s}^* \otimes \mathfrak{s}\right)^{\mathfrak{g}}$$

where $c\colon \operatorname{Sym}^2 \mathfrak{s}^* \otimes \mathfrak{s}^* \to \operatorname{Sym}^3 \mathfrak{s}^*$ is the multiplication of forms.

6

§ 1.6 Contact relation

We finish this chapter with a useful relation between supermaps. On corollary 1.5 we sketched the constructions of the space of d-jets in terms of the space of sections of a vector bundle $\pi\colon E \to M$. It can be proved (cf. [Gua16, Proposition A.8]) that it is equivalent to the following: define the relation

$$\eta \underset{d,p}{\sim} \widetilde{\eta} \quad \text{if and only if} \quad \frac{\partial^k(\eta - \widetilde{\eta})}{\partial x_{\nu_1}^{\mu_1} \cdots \partial x_{\nu_m}^{\mu_m}} = 0$$

where $\mu_1 + \cdots + \mu_m = k \leq d$ and the relation holds on some coordinate chart $(x_1, \ldots, x_m)$ around p; then $\eta \underset{d,p}{\sim} \widetilde{\eta}$ if and only if $\eta - \widetilde{\eta} \equiv 0$ modulo $I_p^{d+1}\Gamma(E)$ (in the notation of the proof of corollary 1.5). In this section we define a similar relation for supermaps.

1.31 Definition. Two supermaps $\Phi, \Phi'\colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ over the smooth map $\phi\colon M \to N$ are in **contact up to order** $k \geq 0$, denoted $\Phi \underset{k}{\sim} \Phi'$, if $\operatorname{im}(\Phi - \Phi') \subseteq \Gamma\left(\mathcal{R}^{\geq k+1}N\right)$. The k-contact class of Φ is denoted by $\operatorname{cont}_\phi^k \Phi$ and the set of these classes by $\operatorname{Cont}_\phi^k(\mathcal{R}M, \mathcal{R}N)$.

The following is just a reformulation of the above definition:

1.32 Corollary. *Two supermaps $\Phi, \widetilde{\Phi}\colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ over the same smooth map ϕ are in contact up to order k if and only if $\operatorname{cont}_\phi^k(\widetilde{\Phi}) = \operatorname{cont}_\phi^k(\Phi)$.*

Observe that the relation is not punctual but global, meaning it is defined for the whole space of sections. The following property of this relation is an immediate consequence of proposition 1.17:

1.33 Proposition. *Two maps Φ and Φ' are in contact up to order k if and only if*

$$(\Phi - \Phi')\colon \Gamma(\mathcal{R}^{\geq d}N) \to \Gamma(\mathcal{R}^{\geq d+k}M)$$

for every $d \geq 0$. If $\Phi \underset{k}{\sim} \Phi'$ then for all $r \leq k$ also $\Phi \underset{r}{\sim} \Phi'$.

The most useful property of this relation is given by the following

1.34 Lemma. *Let k be even. Two maps Φ, Φ' are in contact up to order k if and only if there exists a bundle map*

$$D\colon \mathbf{S}^*N \to \Lambda^{k+1}\mathbf{S}^*M$$

such that $\Phi - \Phi' \equiv D \mod \Gamma(\mathcal{R}^{\geq k+2}M)$. If k is odd then the two maps are in contact up to order k if and only if there exists a derivation D along ϕ, i.e.

$$D\colon \Gamma(\phi^*T^*N) \to \Gamma(\Lambda^{k+1}\mathbf{S}^*M)$$

such that $\Phi - \Phi' \equiv D \mod \Gamma(\mathcal{R}^{\geq k+2}M)$.

Proof. If k is even then for any section σ of $\mathbf{S}^*N$ we get $(\Phi - \Phi')\sigma \in \Gamma(\mathcal{R}^{\geq k+2}M)$ by proposition 1.33; if $\mathrm{pr}\colon \mathcal{R}^{\geq 1}M \to \mathbf{S}^*M$ then $D := (\Phi - \Phi') \circ \mathrm{pr}$ is a bundle map with values in $\Lambda^{k+1}\mathbf{S}^*M$. The converse is immediate, again by proposition 1.33.

Let $\Psi = \Phi - \Phi'$ and f and g be smooth functions on N. Then it follows

$$
\begin{aligned}
\Psi(fg) &= \Phi(f)\Phi(g) - \Phi'(f)\Phi'(g) \\
&= \Phi(f)(\Phi(g) - \Phi'(g)) \\
&+ (\Phi(f) - \Phi'(f))\Phi(g) \\
&= \Phi(f)\Psi(g) + \Psi(f)\Phi(g).
\end{aligned}
$$

which is very similar to the Leibniz identity for a derivation. Observe that if $\Phi \underset{k}{\sim} \Phi'$ then $\Psi(fg)$ is a section of $\mathcal{R}^{k+1}M$ and therefore, passing to the quotient, we obtain a map $D\colon \mathcal{C}^\infty(N) \to \Gamma(\Lambda^{k+1}\mathbf{S}^*M)$ and this class is nonzero if and only if k is odd; in this case the equation above results in $\Psi(fg) = \phi^*(f)D(g) + D(f)\phi^*(g)$, which is exactly the definition of a derivation along ϕ. □

Note that for any k we have $\lceil k \rfloor = \lceil k+2 \rfloor$, so the result tells us that the map associated to a pair of maps which are in contact up to order k takes values in an even exterior power of $\mathbf{S}^*M$ if k is odd and viceversa.

1.6.1. Contact classes and bundle maps

It is possible to codify certain subset of the contact classes by bundle maps. We will consider supermaps $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ with a fixed smooth map ϕ. The bundle maps we will consider are elements of the fibres of the vector bundle over M

$$\phi^*(TN \oplus \mathbf{S}N)_{\lceil k \rfloor} \otimes \Lambda^k\mathbf{S}^*M \cong \mathrm{Hom}\left(\phi^*(T^*N \oplus \mathbf{S}^*N)_{\lceil k \rfloor}, \Lambda^k\mathbf{S}^*M\right)$$

of even bundle maps $\phi^*(T^*M \oplus \mathbf{S}^*N)_{\lceil k \rfloor} \to \Lambda^k\mathbf{S}^*M$; here the notation $(TN \oplus \mathbf{S}N)_{\lceil k \rfloor}$ means the even or odd subspace if k is even or odd respectively. Define

$$\mathrm{cont}_\phi^{k|k-1}\Phi := \left\{\mathrm{cont}_\phi^k \Phi' \,|\, \mathrm{cont}_\phi^{k-1}\Phi = \mathrm{cont}_\phi^{k-1}\Phi'\right\}.$$

That is, the class of k-contact all of whose representatives have the same class of $(k-1)$-contact. Denote by $\mathrm{Cont}_\phi^{k|k-1}(\mathcal{R}M, \mathcal{R}N)$ the set of all these classes. Let Φ' be a representative of $\mathrm{cont}_\phi^{k-1}\Phi$ and consider the map:

$$H := (\Phi - \Phi') \mod \Gamma\left(\mathcal{R}^{\geq k+1}M\right) \tag{1.11}$$

If r is a smooth function on N and k is even, lemma 1.34 tells us that $H(r)$ is a section of $\mathcal{R}^{\geq k+1}M$ and that the map H is a differential operator of order 1 along ϕ; the same lemma implies there is a bundle map $F\colon T^*M \to \Lambda^k\mathbf{S}^*M$ that effectively codifies the difference $\Phi - \Phi'$ modulo sections of $\mathcal{R}^{\geq k+1}M$. If k is odd then lemma 1.34 implies that the map $F := \mathrm{pr} \circ H$ is actually a bundle map $\mathbf{S}^*N \to \Lambda^k\mathbf{S}^*M$.

Note that the previous paragraph effectively defines a map

$$-\colon \operatorname{Cont}^{k|k-1}_{\phi}(\mathcal{R}M, \mathcal{R}N) \times \operatorname{Cont}^{k|k-1}_{\phi}(\mathcal{R}M, \mathcal{R}N) \to \phi^*(TN \oplus \mathbf{S}N) \otimes \Lambda^k \mathbf{S}^* M$$

by $\operatorname{cont}^{k|k-1}_{\phi} \Phi - \operatorname{cont}^{k|k-1}_{\phi} \Phi' = F$ with notation as above. Furthermore if we fix Φ and vary the map Φ' we obtain a bijection:

$$\operatorname{cont}^{k|k-1}_{\phi} \Phi - \cdot\colon \operatorname{Cont}^{k|k-1}_{\phi}(\mathcal{R}M, \mathcal{R}N) \to \phi^*(TN \oplus \mathbf{S}N)_{\lceil k \rfloor} \otimes \Lambda^k \mathbf{S}^* M$$

Indeed, the map F defined as above is the zero map if and only if $\Phi' \underset{k}{\sim} \Phi$; this implies the map is injective. To see that the map is surjective fix a bundle map:

$$F^{\vee}\colon \Lambda^k \mathbf{S}M \to \phi^*(TN \oplus \mathbf{S}N)_{\lceil k \rfloor}$$

To show there exists a Φ' such that its k-contact class corresponds to $F := (F^{\vee})^*$ observe that to determine $F^{\vee}$ it is sufficient to know what happens to sections of the form $s_{\mu_1} \wedge \cdots \wedge s_{\mu_k}$, for sections $s_{\mu_1}, \dots, s_{\mu_k}$ of $\mathbf{S}M$, since these sections span $\Lambda^k \mathbf{S}M$; since it is possible to fix the images of these sections arbitrarily the dual map is well defined; to wit, let $\alpha \oplus \sigma$ be a section of $\phi^*(T^*N \oplus \mathbf{S}^*N)$, then F satisfies:

$$\langle F^{\vee}(s_{\mu_1} \wedge \cdots \wedge s_{\mu_k}), \alpha \oplus \sigma \rangle = \langle s_{\mu_1} \wedge \cdots \wedge s_{\mu_k}, F(\alpha \oplus \sigma) \rangle$$

Then one can define Φ' by setting

$$\langle s_{\mu_1} \wedge \cdots \wedge s_{\mu_k}, (\Phi - \Phi')(\alpha \oplus \sigma) \rangle = \langle F^{\vee}(s_{\mu_1} \wedge \cdots \wedge s_{\mu_k}), \alpha \oplus \sigma \rangle$$

which is well defined by lemma 1.34.

Now let $\Phi' \in \operatorname{cont}^{k-1}_{\phi} \Phi$. Since both Φ and Φ' are differential operators along ϕ they are punctually algebra homomorphisms

$$\Phi_p, \Phi'_p\colon \phi^*(\operatorname{Jet}^{\infty} \mathcal{R}N)_p \to \mathcal{R}_p M$$

where, although the object $\operatorname{Jet}^{\infty} \mathcal{R}N$ is not properly a fibre bundle,[1] the map actually factorises through $\operatorname{Jet}^{r} \mathcal{R}N$ for sufficiently large r (cf. [Gua16, Theorem A.11]). For any point $q \in N$ the algebra $\operatorname{Jet}^{\infty}_q \mathcal{R}N$ is isomorphic to $\overline{\operatorname{Sym}} T^*_q N \otimes \mathcal{R}_q N$ (that is the algebra of formal power series with values in $\mathcal{R}_q N$; cf. [Gua16, Prop. A.14]). Because both Φ and Φ' are differential operators of finite order they are actually completely determined by what happens up to a sufficiently large degree r, so it suffices to work with the algebra $\operatorname{Sym} T^*_q N \otimes \mathcal{R}_q N$; since this algebra is freely generated by $T^*_q N \oplus \mathbf{S}^*_q N$, both Φ and Φ' are completely determined by their action on $\phi^*(T^*N \oplus \mathbf{S}^*N)$. Furthermore, since they are in contact up to order $k-1$ they both coincide modulo $\mathcal{R}^{\geq k} M$. If $F\colon \phi^*(TN \oplus \mathbf{S}^*N)_{\lceil k \rfloor} \to \Lambda^k \mathbf{S}^* M$ it is possible, with all of the above setting, to consider the class $\operatorname{cont}^k_{\phi}(\Phi' + F)$.

[1] The "fibres" would be infinite-dimensional in this case; neverhteless it is the direct limit of bundles (cf. [KMS93, 12.18]).

Notice that F takes values on forms with exact degree k and therefore $\operatorname{cont}_\phi^{k-1}(\Phi' + F) = \operatorname{cont}_\phi^{k-1}\Phi'$. We have now defined a map

$$+\colon \operatorname{Cont}_\phi^{k|k-1}(\mathcal{R}M, \mathcal{R}N) \times \left(\phi^*(TN \oplus \mathbf{S}N) \otimes \Lambda^k \mathbf{S}^* M\right) \to \operatorname{Cont}_\phi^{k|k-1}(\mathcal{R}M, \mathcal{R}N)$$

by $\operatorname{cont}_\phi^{k|k-1}\Phi' + F = \operatorname{cont}_\phi^{k|k-1}(\Phi' + F)$; this is well defined since F can be thought of as the lift to degree k of a split morphism by corollary 1.18. By fixing Φ and varying Φ' the proof above of the bijectivity of $\operatorname{cont}_\phi^{k|k-1}\Phi - \cdot$ furnishes a proof of the bijectivity of $\operatorname{cont}_\phi^{k|k-1}\Phi' + \cdot$. We have thus proved:

1.35 Lemma. *The set* $\operatorname{Cont}_\phi^{k|k-1}(\mathcal{R}M, \mathcal{R}N)$ *is naturally an affine bundle over M modeled on*

$$\phi^*(TN \oplus \mathbf{S}N)_{\lceil k \rfloor} \otimes \Lambda^k \mathbf{S}^* M \cong \operatorname{Hom}\left(\phi^*(T^*N \oplus \mathbf{S}^*N)_{\lceil k \rfloor}, \Lambda^k \mathbf{S}^* M\right)$$

Another way of interpreting the above result is by saying that the set of contact classes $\operatorname{Cont}_\phi^{k|k-1}(\mathcal{R}M, \mathcal{R}N)$ becomes a vector bundle by fixing morphism Φ such that $\operatorname{cont}_\phi^{k-1}\Phi$ acts as the zero element on each fibre. The difference of two such contact classes allows then the following:

1.36 Definition. Let $C \in \operatorname{Cont}_\phi^{k-1}(\mathcal{R}M, \mathcal{R}N)$ and $\operatorname{Cont}_\phi^k(C) := \{\operatorname{cont}_\phi^k \Phi | \operatorname{cont}_\phi^{k-1} \in C\}$. The bundle map associated to the difference $\operatorname{cont}_\phi^k \Phi - \operatorname{cont}_\phi^k \widetilde{\Phi}$ is given by a map

$$\Delta^k\colon \operatorname{Cont}_\phi^k(C) \times \operatorname{Cont}_\phi^k(C) \to \operatorname{Hom}\left(\phi^*(T^*N \oplus \mathbf{S}^*N)_{\lceil k \rfloor}, \Lambda^k \mathbf{S}^* M\right)$$

and it is defined as $\Delta^k(\Phi, \widetilde{\Phi}) = \Phi - \widetilde{\Phi}$.

The following result is then an easy consequence of corollary 1.32:

1.37 Proposition. *Two maps Φ and $\widetilde{\Phi}$ are in contact up to order k if and only if*

$$\operatorname{cont}^{k-1|k}\Phi - \operatorname{cont}^{k-1|k}\widetilde{\Phi} = 0$$

Finally we show how this contact relation behaves under compositions:

1.38 Lemma. *Let $k \geq 2$, $(\phi|\Phi)\colon (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ and $(\psi|\Psi)\colon (N|\mathcal{R}N) \to (P|\mathcal{R}P)$; that is $\Phi\colon \Gamma(\mathcal{R}N) \to \Gamma(\mathcal{R}M)$ and $\Psi\colon \Gamma(\mathcal{R}P) \to \Gamma(\mathcal{R}N)$. If $\Phi \underset{k-1}{\sim} \widetilde{\Phi}$ and $\Psi \underset{k-1}{\sim} \widetilde{\Psi}$ then $\Phi \circ \Psi \underset{k-1}{\sim} \widetilde{\Phi} \circ \widetilde{\Psi}$ and*

$$\Delta^k(\Phi \circ \Psi, \widetilde{\Phi} \circ \widetilde{\Psi}) = \Delta^k(\Phi, \widetilde{\Phi}) \circ \Lambda^k \Psi^* + (\phi|\Phi)^* \circ \Delta^k(\Psi, \widetilde{\Psi}) \tag{1.12}$$

Proof. The relation $\Phi \circ \Psi \underset{k}{\sim} \widetilde{\Phi} \circ \widetilde{\Psi}$ is trivial from (1.17).

Let $\eta = \psi \circ \phi$, $\mathbf{H} = \Phi \circ \Psi$ and $\widetilde{\mathbf{H}} = \widetilde{\Phi} \circ \widetilde{\Psi}$. From lemma 1.34 there are operators

$$\begin{aligned} D\colon \Gamma\left(\phi^*(T^*N \oplus \mathbf{S}^*N)_{\lceil k \rfloor}\right) &\to \Gamma\left(\Lambda^{k+1}\mathbf{S}^*M\right) \\ \tilde{D}\colon \Gamma\left(\psi^*(T^*P \oplus \mathbf{S}^*P)_{\lceil k \rfloor}\right) &\to \Gamma\left(\Lambda^{k+1}\mathbf{S}^*N\right) \\ \hat{D}\colon \Gamma\left(\eta^*(T^*P \oplus \mathbf{S}^*P)_{\lceil k \rfloor}\right) &\to \Gamma\left(\Lambda^{k+1}\mathbf{S}^*M\right) \end{aligned}$$

corresponding to $\Phi - \widetilde{\Phi}$, $\Psi - \widetilde{\Psi}$ and $\mathbf{H} - \widetilde{\mathbf{H}}$ respectively. By hypothesis the images of the operators

$$\mathbf{H} - \widetilde{\mathbf{H}} = \Phi \circ \Psi - \widetilde{\Phi} \circ \widetilde{\Psi} = (\Phi - \widetilde{\Phi}) \circ \widetilde{\Psi} + \Phi \circ (\Psi - \widetilde{\Psi})$$

and $\Phi - \widetilde{\Phi}$ lie in $\Gamma(\mathcal{R}^{\geq k+1}M)$, while $\operatorname{im}(\Psi - \widetilde{\Psi}) \subseteq \Gamma(\mathcal{R}^{\geq k+1}N)$. By taking the quotient modulo sections of $\mathcal{R}^{\geq k+2}M$ the identity for the corresponding operators is

$$\hat{D} = (\phi|\Phi)^* \circ \tilde{D} + D \circ \Lambda^{k+1}\Psi^*$$

since $\Phi^* = \widetilde{\Phi}^*$ and $\Psi^* = \widetilde{\Psi}^*$ by proposition 1.33. By taking the corresponding bundle maps the identity stated follows. $\square$

The above maps can be dualised, just as in the case of the auxilliary differential, to yield bundle morphisms $\Delta_k(\Phi, \widetilde{\Phi})\colon \Lambda^k\mathbf{S}M \to \phi^*(TN \oplus \mathbf{S}N)_{\lceil k \rfloor}$. These shall be useful later on.

1.39 Remark. Observe that dualising identity (1.12) yields

$$\Delta_k(\Phi \circ \Psi, \widetilde{\Phi} \circ \widetilde{\Psi}) = \Delta_k(\Psi, \widetilde{\Psi}) \circ \Lambda^k (\phi|\Phi)_* + (\psi|\Psi)_* \circ \Delta_k(\Phi, \widetilde{\Phi}) \tag{1.13}$$

1.40 Corollary. *If $(\phi|\Phi)$ is a diffeomorphism and $\widetilde{\Phi} \underset{k}{\sim} \Phi$ then $\widetilde{\Phi}$ is a diffeomorphism and*

$$\Delta_k(\Phi^{-1}, \widetilde{\Phi}^{-1}) = -\left(\phi^{-1}\middle|\Phi^{-1}\right)_* \circ \Delta_k(\Phi, \widetilde{\Phi}) \circ \Lambda^k(\Phi_*^{-1})$$

(and similarly for the dual) for all $k \geq 2$.

Proof. It suffices to apply the lemma 1.38 to the identity $\Phi \circ \Phi^{-1} = \mathrm{id}$ since id is a bundle map. $\square$

We conclude with the behaviour of differentials under products for $k \geq 1$.

1.41 Lemma. *Let $(\phi|\Phi) : (M|\mathcal{R}M) \to (N|\mathcal{R}N)$ and $(\psi|\Psi) : (P|\mathcal{R}P) \to (Q|\mathcal{R}Q)$. If $\Phi \underset{k}{\sim} \widetilde{\Phi}$ and $\Psi \underset{k}{\sim} \widetilde{\Psi}$ then $\Psi \hat{\otimes} \Phi \underset{k}{\sim} \widetilde{\Phi} \hat{\otimes} \widetilde{\Psi}$ and $\Delta_k(\Phi \hat{\otimes} \Psi, \widetilde{\Psi} \hat{\otimes} \widetilde{\Phi}) = \Delta_k(\Phi, \widetilde{\Phi}) \boxplus \Delta_k(\Psi, \widetilde{\Psi})$*

Proof. Since it is assumed that the $(k-1)$-contact class is fixed it suffices to observe that $\mathcal{R}^k(M \times P)/\mathcal{R}^{k+1}(M \times P) = \Lambda^k(\mathbf{S}^*M \boxplus \mathbf{S}^*P)$ contains $\Lambda^k\mathbf{S}^*M \oplus \Lambda^k\mathbf{S}^*P$ and the k-th order differential is purely alternating in its arguments. $\square$

Chapter 2

Split Lie supergroups

> Are we for ever to be twisting, and untwisting the same rope? for ever in the same track—for ever at the same pace?
>
> Laurence Sterne, *The Life and Opinions of Tristram Shandy, Gentleman*, vol. v, ch. i.

An efficient way of defining a Lie group is as a group in the category of smooth manifolds. That is, a Lie group is a smooth manifold G along with two smooth maps $\mu\colon G \times G \to G$ and $\iota\colon G \to G$ that satisfy the group axioms expressed in terms of commutative diagrams. For instance, the diagram

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"'] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"] & G \end{tikzcd} \tag{2.1}$$

is equivalent to the associativity of the multiplication. Furthermore there exists a (unique) map

$$e\colon \{*\} \to G$$

whose associated diagram expresses the existence of the identity element of G. All of this motivates the following

2.1 Definition. A **Lie supergroup** $(G|\mathcal{R}G)$ is a group in the category of supermanifolds.

This means there are three supermaps

$$\begin{aligned} (\mu|\mathcal{M}):\ & (G|\mathcal{R}G) \times (G|\mathcal{R}G) \to (G|\mathcal{R}G) \\ (\iota|\mathcal{I}):\ & (G|\mathcal{R}G) \to (G|\mathcal{R}G) \\ (e|\mathcal{E}):\ & (*|\mathbb{R}) \to (G|\mathcal{R}G) \end{aligned}$$

that satisfy commutative diagrams identical with the group axioms in this category. Since morphims are pairs of maps, each of these diagrams are actually two diagrams: one corresponding to the underlying smooth manifolds and the other, with arrows reversed, corresponding to the algebras of superfunctions. For instance, the diagram corresponding to (2.1) is

$$\begin{array}{ccc} \Gamma\left(\mathcal{R}G\right) & \xrightarrow{\mathcal{M}} & \Gamma\left(\mathcal{R}(G\times G)\right) \\ {\scriptstyle\mathcal{M}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\times\mathcal{M}} \\ \Gamma\left(\mathcal{R}(G\times G)\right) & \xrightarrow{\mathcal{M}\times \mathrm{id}} & \Gamma\left(\mathcal{R}(G\times G\times G)\right) \end{array} \tag{2.2}$$

for the operator $\mathcal{M}$ covering the multiplication. From all these ideas the next result follows:

2.2 Proposition. *On a Lie supergroup $(G|\mathcal{R}G)$ the manifold G along with the smooth maps μ, ι and e is a Lie group.*

In this chapter we will classify split Lie supergroups. From proposition 1.25 the supermaps that comprise the supergroup structure are completely determined by the following bundle maps:

$$\begin{aligned} \mathrm{M}\colon \mathbf{S}G \boxplus \mathbf{S}G &\to \mathbf{S}G \\ \mathrm{I}\colon \mathbf{S}G &\to \mathbf{S}G \\ O\colon \{0\} &\to \mathbf{S}G. \end{aligned}$$

Here $\mathbf{S}G \boxplus \mathbf{S}G$ has the same meaning as in (1.10). From this fact we can characterise Lie supergroups as a certain kind of vector bundle over G, which we call bihomogeneous. Lastly we classify all possible multiplications on a supergroup in which $\mathcal{R}G = \Lambda S^*G$ and study the relation between the superbracket in the tangent superspace at the neutral element of G and the multiplication supermap on $\Gamma\left(\Lambda\mathbf{S}^*G\right)$.

§ 2.1 Bihomogeneous vector bundles

If V is a finite-dimensional representation of a Lie group G it is possible to form a bundle VG which is *homogeneous*; that is the bundle has an action $\gamma\colon G\times VG \to VG$ which is linear on each fibre and permutes them covering the left action of G on itself, i.e. $\gamma(g, V_hG) = V_{gh}G$; we use the following notation

$$\pi^{-1}(g) = V_g.$$

This idea, slightly modified, allows us to classify split Lie supergroups.

2.3 Definition. Let $\pi\colon E\to G$ be a vector bundle over the Lie group G. E is **bihomogeneous** if there is a smooth map

$$\beta\colon G\times E\times G \to E$$

that satisfies the following:

1. $\beta(g,\cdot,h)\colon E_x \to E_{gxh}$ is a linear isomorphism.

2. $\beta(e, \cdot, e)\colon E_x \to E_x$ is the identity map for each x in G.
3. $\beta(g, \beta(h, \cdot, \tilde{h}), \tilde{g}) = \beta(gh, \cdot, \tilde{h}\tilde{g})$ for all $g, h, \tilde{g}, \tilde{h}$ in G.

The map β is called a **biaction**.

2.4 Remark. Let $\mathbf{b}\colon G \times G \times G \to G$ the natural biaction of G on itself, i.e. $\mathbf{b}(g, h, k) = ghk$. Then the above definition is equivalent to the diagram

$$\begin{tikzcd} G \times E \times G \arrow[r, "\beta"] \arrow[d, "\mathrm{id}\times\pi\times\mathrm{id}"'] & E \arrow[d, "\pi"] \\ G \times G \times G \arrow[r, "\mathbf{b}"] & G \end{tikzcd} \tag{2.3}$$

being commutative. In the following we shall use the notation $g * s * h := \beta(g, s, h)$ when convenient.

Observe that a G-biaction on a set X (properly defined) is equivalent to having two actions, a left one and a right one, that commute. The fundamental property of bihomogeneous vector bundles is the following:

2.5 Lemma. *For each finite-dimensional representation $\rho\colon G \to GL(V)$ there is a bihomogeneous vector bundle. Conversely, every bihomogeneous vector bundle gives rise to a finite-dimensional representation of G.*

Proof. Let $\pi\colon E \to G$ be a bihomoegenous vector bundle with biaction β and let $V = E_e$, the fibre of E over the identity element of G. Define

$$\begin{aligned} \rho\colon G \times V &\to V \\ (g, v) &\mapsto \beta(g, v, g^{-1}). \end{aligned} \tag{2.4}$$

This map is a genuine representation of G because by definition β is fibre-wise linear and, since β permutes the fibres, the vector $\rho(g, v)$ is an element in E_e, again by definition. This furnishes the representation associated with a bihomogeneous vector bundle E.

Let now (V, ρ) be a representation of G and define the space

$$E := G \times V \times G \Big/ \Big\{ (g, v, h) \sim (g\gamma, \rho(\gamma)v, \gamma^{-1}h) \mid \gamma \in G \Big\} \tag{2.5}$$

and the map $\pi\colon E \to G$ as $\pi[g, v, h] = gh$. This is the associated vector bundle[1] to the representation ρ and the principal bundle $m\colon G \times G \to G$ whose right action is given by $(g, \tilde{g}) \cdot h = (gh, h^{-1}g)$. The biaction is given by

$$\beta(\gamma, [g, v, \tilde{g}], \tilde{\gamma}) := [\gamma g, \rho(\gamma)v, \tilde{g}\tilde{\gamma}].$$

Thus from a representation ρ of G we have constructed a bihomogeneous vector bundle E. □

[1] This is defined as follows: if P is a G-principal bundle and V a finite-dimensional representation of G then $V \times_G P := (P \times V)/G$.

From the above representation of G we obtain a representation of $\mathfrak{g}$ on E_e as follows: let $X \in \mathfrak{g}$ and $v \in E_e$ and define

$$X \star v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \rho(\exp(tX))(v) \tag{2.6}$$

This formula is ρ-equivariant, because

$$\begin{aligned} \rho(g)(X \star v) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \rho(g)\rho(\exp(tX))\rho(g^{-1})\rho(g)v \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \rho(g\exp(tX)g^{-1})\rho(g)v \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \rho(\exp(t\,\mathrm{Ad}_g(X))) \\ &= (\mathrm{Ad}_g X) \star (\rho(g)v). \end{aligned} \tag{2.7}$$

This action can be extended to each fibre as follows: given an arbitrary element g of G suppose it is expressed as $g = ab$ and let $v \in E_g$; let $X \in T_g G$ then there is a fibre-wise action

$$(X \star v)_{ab} = a * \left((a^{-1} * X * b^{-1}) \star (a^{-1} * v * b^{-1})\right) * b \tag{2.8}$$

In order to see this is independent of the expression $g = ab$ let g now be writen as abc and to show that

$$(X \star v)_{a(bc)} = (X \star v)_{(ab)c}$$

we use (2.7), that is that $\star$ is ρ-equivariant.

We now prove some corollaries of lemma 2.5.

2.6 Corollary. *Every bihomogeneous vector bundle E can be trivialised in at least two ways.*

Proof. From the definition of a biaction it follows that

$$\begin{aligned} \tau_L : E &\to G \times E_e \\ v_g &\mapsto (g, \beta(g^{-1}, v_g, e)) \end{aligned}$$

is a bundle isomorphism whose inverse is $(g, v) \mapsto \beta(g, v, e)$. Likewise we get an isomorphism by defining $\tau_R(v_g) = (g, \beta(e, v_g, g^{-1}))$ whose inverse is given by $(g, v) \mapsto \beta(e, v, g)$. $\square$

2.7 Corollary. *The space of sections $\Gamma(E)$ of a bihomogeneous vector bundle is isomorphic to $\mathcal{C}^\infty(G, V)$, where $V = E_e$, in at least two ways.*

Proof. Let ω be a section of E. Define $\theta_L(\omega)(x) = \beta(x^{-1}, \omega_x, e)$; this map is a fibre-wise isomorphism with E_e and by definition the result is a function on G with values in E_e.

Defining $\theta_R(\omega)(x) = \beta(e, \omega_x, x^{-1})$ we obtain another isomorphism. $\square$

2.8 Corollary. *Both trivialisations are related by $\theta_R(v)(g) = \rho(g)\theta_L(v)(g)$.*

2.9 Corollary. *There are two naturally defined flat connections ∇^L and ∇^R in any bihomogeneous vector bundle E.*

Proof. Given a section η of E and a vector field X on G the map $\theta_L(\eta)(g)$ allows one to apply X as a derivation on $\mathcal{C}^\infty(G, E_e)$ (for instance by applying X to each component function relative to a basis of E_e); therefore (setting $g = g_0$ for a smooth curve g_t in G)

$$\left(\nabla^L_X \eta\right)_g = \theta_L^{-1}(X_g \cdot (\theta_L(\eta))) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \beta(g_0 g_t^{-1}, \eta(g_t), e)$$

defines a connection on E, necessarily flat because it is defined through the left trivialisation; another flat connection ∇^R is defined by using θ_R. □

Parallel sections with respect to these connections are very simple:

2.10 Proposition. *There is a biyection between E_e and sections of E that are parallel with respect to either ∇^R or ∇^L.*

Proof. By the definition of the connections it is straightforward that a section is left (resp. right) parallel if and only if it is taken to a constant function by θ_L (resp. by θ_R). □

2.11 Corollary. *There is a biyection between the space $(E_e)^G$ of G-invariants of the action and parallel sections of E with respect to both connections.*

Proof. If a section v is constant with respect to both θ_L and θ_R corollary 2.8 implies $\theta_R(v) = \rho \circ \theta_L(v)$ is constant and therefore its value at the identity is ρ-invariant.

Conversely, let $v \in E_e$ be an invariant and let $f_v \in \mathcal{C}^\infty(G, E_e)$ be the constant function $f(g) = v$; this function is the same regardless of the trivialisation chosen. Then $\rho(g) f_v(g) = \rho(g) v = v$ and therefore f_v is parallel with respect to both connections. □

Another feature of these connections is that they are related through the infinitesimal representation associated to the biaction. From the definition (2.8) we consider a smooth curve g_t on G such that $g_0 = ab$ and $X = \dot{g}_0$. If $v_t \in E_{g_t}$ we compute

$$\begin{aligned}
(X \star v)_{ab} &= a * \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (a^{-1} g_t b^{-1}) \star (a^{-1} * v_t * b^{-1}) \right) * b \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (a a^{-1} g_t b^{-1} a^{-1}) * v_t * (b^{-1} b g_t^{-1} a b) && \text{(because a and b are constant w.r.t. t)} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g_t g_0^{-1} * v_t * g_t^{-1} g_0 \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e * v_t * g_t^{-1} g_0 - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g_t^{-1} g_0 * v_t * e && \text{(Leibniz's rule for the connection)}
\end{aligned}$$

and finally

$$X \star = \nabla^R_X - \nabla^L_X. \tag{2.9}$$

This defines a section $X\star$ of $\mathrm{End}(E)$ for any vector field X of G; we call it the **algebraic infinitesimal action**.

2.1.1. The twist map

It turns out that the multiplication map can "untangle" the vector bundle $E \boxplus E$ (see (1.10) for this notation). First, the map $m\colon G \times G \to G$ allows the construction of the pullback bundle $m^*(E \oplus E)$ which is a bundle over $G \times G$; however it does not have the appropriate fibres, since

$$m^*(E \oplus E)_{(g,h)} = E_{gh} \oplus E_{gh}$$

whereas $(E \boxplus E)_{(g,h)} = E_g \oplus E_h$. Nevertheless we can use the biaction to map $m^*(E \oplus E)$ to $E \boxplus E$:

$$\eta_{gh} \oplus \tilde{\eta}_{gh} \mapsto \beta(e, \eta_{gh}, h^{-1}) \oplus \beta(g^{-1}, \tilde{\eta}_{gh}, e) \tag{2.10}$$

and it turns out this is a bundle isomorphism. To prove this observe that it has a left and right inverse, again using the biaction:

$$\eta_g \oplus \tilde{\eta}_h \mapsto \beta(e, \eta_g, h) \oplus \beta(g, \tilde{\eta}, e)$$

so we have proved:

2.12 Proposition. *The map*

$$\mathrm{twist}\colon m^*(E \oplus E) \to E \boxplus E$$

defined by (2.10) *is a bundle isomorphism.*

Observe that the sections of m^*E (only one factor) are maps

$$\eta\colon G \times G \to E$$

such that $\pi \circ \eta = m$; that is $\phi(\eta(g, \tilde{g}) = g\tilde{g}$. Therefore the trivialisation maps in this bundle are

$$\begin{aligned} \theta_L &= \beta(\tilde{g}^{-1} g^{-1}, \cdot, e) \\ \theta_R &= \beta(e, \cdot, \tilde{g}^{-1} g^{-1}). \end{aligned} \tag{2.11}$$

We shall make use of these (and other) trivialisation maps and their associated connections on section 3.1.

§ 2.2 Classification of split Lie supergroups

As we remarked at the beginning of this chapter the supermaps that endow a split supermanifold with a group structure are bundle maps. Let's first analyse these morphisms. Since they have to cover both the multiplication and the inversion maps on G we obtain the following diagrams:

$$\begin{tikzcd} \mathbf{S}_x G \oplus \mathbf{S}_y G \oplus \mathbf{S}_z G \arrow[r, "\mathrm{M}_{(x,y)} \oplus \mathrm{id}"] \arrow[d, "\mathrm{id} \oplus \mathrm{M}_{(y,z)}"'] & \mathbf{S}_{xy} \oplus \mathbf{S}_z G \arrow[d, "\mathrm{M}_{(xy,z)}"] \\ \mathbf{S}_x G \oplus \mathbf{S}_{yz} G \arrow[r, "\mathrm{M}_{(x,yz)}"'] & \mathbf{S}_{xyz} G \end{tikzcd} \tag{2.12}$$

which codifies the associativity of M; also we have

$$
\begin{tikzcd}
\mathbf{S}_x G \oplus \mathbf{S}_x G \arrow[rr, "\mathrm{id} \oplus \mathrm{I}_x"] \arrow[dr, "0"] \arrow[dd, "\mathrm{I}_x \oplus \mathrm{id}"'] & & \mathbf{S}_x G \oplus \mathbf{S}_{x^{-1}} G \arrow[dl, "0"'] \arrow[dd, "\mathrm{M}_{(x,x^{-1})}"] \\
& \{0\} \arrow[dr, "O"] & \\
\mathbf{S}_{x^{-1}} G \oplus \mathbf{S}_x G \arrow[ur, "0"] \arrow[rr, "\mathrm{M}_{(x^{-1},x)}"'] & & \mathbf{S}_e G
\end{tikzcd}
\tag{2.13}
$$

for the inversion; the map O corresponds to the identity of the group at the level of bundles.

Applying induction to diagram (2.12) we get a map

$$
\mathrm{M}_{(g_1,\ldots,g_r)} \colon \mathbf{S}_{g_1} G \oplus \cdots \oplus \mathbf{S}_{g_r} G \to \mathbf{S}_{g_1 \cdots g_r} G \tag{2.14}
$$

which is well-defined for any $r \geq 3$. From all these facts we can now state and prove the main result of this chapter:

2.13 Theorem. *A split supermanifold $(G|\mathbf{S}G)$ is a Lie supergroup if and only if G is a Lie group and $\mathbf{S}G$ is a bihomogeneous vector bundle over G. Moreover β is completely determined by* M *and viceversa.*

Proof. Let $(G|\mathbf{S}G)$ be a split Lie supergroup. From proposition 2.2 follows that G is a Lie group. Identity (2.14) allows the definition

$$
\begin{gathered}
\beta \colon G \times \mathbf{S}G \times G \to \mathbf{S}G \\
\beta(g, s_x, h) = \mathrm{M}_{(g,x,h)}(0 \oplus s_x \oplus 0).
\end{gathered}
\tag{2.15}
$$

The symbol s_x denotes an element on the fibre $\mathbf{S}_x G$. To see that diagram (2.3) commutes in this case it suffices to observe that $\mathrm{M}_{(\tilde{g},(g,x,h),\tilde{h})} = \mathrm{M}_{(\tilde{g},g,x,h,\tilde{h})} = \mathrm{M}_{(\tilde{g}g,x,\tilde{h}h)}$ due to identity (2.14).

Let now $\mathbf{S}G$ be a bihomogeneous vector bundle over G with biaction β. Define

$$
\mathrm{M}_{(x,y)}(s_x \oplus s_y) = \beta(e, s_x, y) + \beta(x, s_y, e) \tag{2.16a}
$$

$$
\mathrm{I}_x(s_x) = -\beta(x^{-1}, s_x, x^{-1}). \tag{2.16b}
$$

To prove the commutativity of diagrams (2.12) and (2.13) first observe that the right hand side of equations (2.16) belong to the appropriate fibres. Commutativity of (2.12) is a direct consequence of the biassociativity of β, i.e. $\beta(g, \beta(x, \cdot, y), h) = \beta(gx, \cdot, yh)$ and of fibre-wise linearity. Let $s_x \in \mathbf{S}_x G$; compute

$$
\begin{aligned}
\mathrm{M}_{(x^{-1},x)} \circ \mathrm{I}_x \oplus \mathrm{id}(s_x \oplus s_x) &= \mathrm{M}_{(x^{-1},x)}(-\beta(x^{-1}, s_x, x^{-1}) \oplus s_x) \\
&= \beta(e, -\beta(x^{-1}, s_x, x^{-1}), x) + \beta(x^{-1}, s_x, e) = 0
\end{aligned}
$$

whereas

$$\begin{aligned}\mathrm{M}_{(x,x^{-1})} \circ \mathrm{id} \oplus \mathrm{I}_x(s_x \oplus s_x) &= \mathrm{M}_{(x,x^{-1})}(s_x \oplus -\beta(x^{-1}, s_x, x^{-1})) \\ &= \beta(e, s_x, x^{-1}) + \beta(x, -\beta(x^{-1}, s_x, x^{-1}), e) = 0\end{aligned}$$

and from the definition of the two maps we obtain the vector 0 in $\mathbf{S}_e G$. This proves diagram (2.13) commutes. Now we have shown that the operations defined by (2.16) on a bihomogeneous vector bundle transform it into a split Lie supergroup. $\square$

Although they furnish the simplest class of Lie supergroups, the split ones are far from exhausting the whole class of them. In view of diagram (2.2) it is necessary to classify all possible mappings

$$\mathcal{M} \colon \Gamma\left(\mathcal{R}G\right) \to \Gamma\left(\mathcal{R}G \widehat{\boxtimes} \mathcal{R}G\right)$$

for an *arbitrary* superalgebra bundle $\mathcal{R}G$. We will deal with an intermediate step in the next chapter.

Chapter 3

The multiplication and the Lie superalgebra

1

> El panorama, la cara del mundo, sólo se [nos] revela ante lo infinitamente pequeño y lo infinitamente grande.
>
> ALFONSO REYES, *Los estudios y los juegos*.

From everything done in the last chapter we can form the map

$$\mathcal{M}^{(0)} := \Gamma(\Lambda \mathbf{M}^*) \colon \Gamma\left(\Lambda \mathbf{S}^* G\right) \to \Gamma\left(\Lambda(\mathbf{S}^* G \boxplus \mathbf{S}^* G)\right)$$

which is a bona fide multiplication map over $m\colon G \times G \to G$. What is missing is the relation between this multiplication map and the Lie superalgebra structure on $T_e G \oplus \mathbf{S}_e G$. Theorem 1.30 tells us exactly what to look for. From proposition 2.2 we already know G to be a Lie group and therefore $T_e G = \mathfrak{g}$ is a Lie algebra; also, lemma 2.5 implies that $\mathfrak{s} := \mathbf{S}_e G$ is a representation space for G and therefore also for $\mathfrak{g}$. Thus for split Lie supergroups we get items 1 and 2 of theorem 1.30. The symmetric equivariant map B is the missing ingredient so far. We will now prove that this map can be read from the multiplication supermap. In all that follows we assume $\mathcal{R}G = \Lambda \mathbf{S}^* G$ (the structure bundle *is* actually an exterior bundle).

Recall that

$$A := \mathcal{M}^{(0)}_{\text{aux}} \colon \Lambda^2(\mathbf{S}G \boxplus \mathbf{S}G) \to TG$$

so it follows that $A_{(g,\tilde{g})}(s \oplus \tilde{s}, t \oplus \tilde{t})$ is on the tangent space $T_g G$. Associativity of the multiplication supermap entails

$$\begin{aligned} &A_{(g\tilde{g},\hat{g})}\big((e * s * \tilde{g} + g * \tilde{s} * e) \oplus \hat{s}, (e * t * \tilde{g} + g * \tilde{t} * e) \oplus \hat{t}\big) + e * A_{(g,\tilde{g})}(s \oplus \tilde{s}, t \oplus \tilde{t}) * \hat{g} = \\ &A_{(g,\tilde{g}\hat{g})}\big(s \oplus (e * \tilde{s} * \hat{g} + \tilde{g} * \hat{s} * e), t \oplus (e * \tilde{t} * \hat{g} + \tilde{g} * \hat{t} * e)\big) + g * A_{(\tilde{g},\hat{g})}(\tilde{s} \oplus \hat{s}, \tilde{t} \oplus \hat{t}) * e \end{aligned} \tag{3.1}$$

This formula follows from lemmas 1.38 and 1.41, and proposition 1.23, since

$$(\mathcal{M}^{(0)} \otimes \text{id})_{\text{aux}} = A \oplus 0 \quad \text{and} \quad (\text{id} \otimes \mathcal{M}^{(0)})_{\text{aux}} = 0 \oplus A$$

and $\mathcal{M}_*^{(0)} = \mathrm{M}$. If in the above formula we set some of the group elements equal to the identity and some of the fibre elements equal to zero we obtain

$$
\begin{aligned}
A_{(g\tilde{g},\hat{g})}(e * s * \tilde{g} \oplus 0, 0 \oplus \hat{t}) &= A_{(g,\tilde{g}\hat{g})}(s \oplus 0, 0 \oplus \tilde{g} * \hat{t} * e) && \text{(3.2a)}\\
A_{(g,\hat{g})}(s \oplus 0, 0 \oplus e * \tilde{t} * \hat{g}) &= A_{(g,\hat{g})}(s \oplus 0, g * \tilde{t} * e \oplus 0)\\
&\quad + e * A_{(g,e)}(s \oplus 0, 0 \oplus \tilde{t}) * \hat{g} && \text{(3.2b)}\\
A_{(g,\hat{g})}(0 \oplus \hat{s}, g * \tilde{t} * e \oplus 0) &= A_{(g,\hat{g})}(0 \oplus \hat{s}, 0 \oplus e * \tilde{t} * \hat{g})\\
&\quad + g * A_{(e,\hat{g})}(0 \oplus \hat{s}, \tilde{t} \oplus 0) * e && \text{(3.2c)}
\end{aligned}
$$

as follows: (3.2a) follows from setting $t, \tilde{s}$ and $\tilde{t}$ equal to zero in (3.1); formula (3.2b) from setting $\tilde{g} = e$ as well, whereas (3.2c) follows by setting $s = 0$ and $\hat{t} = 0$.

Formula (3.2a) allows the following definition: set $g = hh'$ and

$$\mathbf{B}_g(s,t) := A_{(h,\tilde{h})}(e * s * \tilde{h}^{-1} \oplus 0, 0 \oplus h^{-1} * t * e) \qquad (3.3)$$

since the former implies that the latter is well-defined. So B is a bilinear bundle map

$$\mathbf{B} \colon \mathbf{S}G \otimes \mathbf{S}G \to TG$$

Observe that this map is defined as a bundle map over G not over $G \times G$ but nonetheless, in view of formulas (3.2), it completely determines the auxiliary differential of the multiplication, since equations (3.2) now read

$$
\begin{aligned}
A_{(g,\hat{g})}(s \oplus 0, 0 \oplus \hat{t}) &= \mathbf{B}_{g\hat{g}}(e * s * \hat{g}, g * \hat{t} * e) && \text{(3.4a)}\\
A_{(g,\hat{g})}(s \oplus 0, t \oplus 0) &= \mathbf{B}_{g\hat{g}}(e * s * \hat{g}, e * t * \hat{g}) - e * \mathbf{B}_g(s,t) * \hat{g} && \text{(3.4b)}\\
A_{(g,\hat{g})}(0 \oplus \hat{s}, 0 \oplus \hat{t}) &= \mathbf{B}_{g\hat{g}}(g * \hat{s} * e, g * \hat{t} * e) - g * \mathbf{B}_{\hat{g}}(\hat{s}, \hat{t}) * e. && \text{(3.4c)}
\end{aligned}
$$

Indeed: (3.4a) follows from the definition of B; to obtain (3.4b) set $\hat{t} := g^{-1} * t * \hat{g}$ in (3.2b) and for (3.4c) substitute s for $k * \hat{s} * \hat{k}^{-1}$ in (3.2c).

If we define $\mathbf{B}^{\pm}(s,t) := \frac{1}{2}\big(\mathbf{B}(s,t) \pm \mathbf{B}(t,s)\big)$ then the expressions corresponding to $\mathbf{B}^+$ in formulas (3.4b) and (3.4c) vanish since the auxiliary differential is alternating, therefore

$$
\begin{aligned}
\mathbf{B}^+_{g\hat{g}}(e * s * \hat{g}, e * t * \hat{g}) &= e * \mathbf{B}^+_g(s,t) * \hat{g}\\
\mathbf{B}^+_{g\hat{g}}(g * s * e, g * t * e) &= g * \mathbf{B}^+_{\hat{g}}(s,t) * e
\end{aligned} \qquad (3.5)
$$

which means $B := \mathbf{B}^+$ is a symmetric, G-biequivariant map and setting $g = \hat{g} = e$ we obtain a map

$$B \colon \operatorname{Sym}^2 \mathfrak{s} \to \mathfrak{g}$$

which is half the missing ingredient of theorem 1.30. The other half (which codifies the graded Jacobi identity) requires different techniques. We will study them in the next section. So far we have proved:

3.1 Theorem. *The auxiliary differential of a multiplication map $\mathcal{M}$ on a Lie supergroup $(G|\mathcal{R}G)$ determines a bilinear G-equivariant map $B \colon \operatorname{Sym}^2 \mathfrak{s} \to \mathfrak{g}$.*

Since the bundle map M completely characterises the multiplication supermap for a split supergroup, proposition 1.25 implies that $\mathcal{M}^{(0)}_{\text{aux}} \equiv 0$ and therefore $B \equiv 0$. Nevertheless all of our work above has not been in vain. In the next section we shall endow a supergroup $(G|\Lambda\mathbf{S}^*G)$ with different multiplication maps so as to construct the necessary Lie superalgebra structure on $\mathfrak{g} \oplus \mathfrak{s}$ that has the same dependence to the group structure as in the classical case.

§ 3.1 General multiplications on split supergroups

The motivation for what follows comes from considering a symmetric space $X = K/P$ where K and P form a symmetric pair, i.e. K is a Lie group and P is a closed subgroup fixed by an involutive automorphism Θ of K. The infinitesimal counterpart of one of these objects is a $\mathbb{Z}_2$-graded Lie algebra, i.e. a Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{k}$ is a subalgebra, $\mathfrak{p}$ is a $\mathfrak{k}$-module and the bracket of two vectors in $\mathfrak{p}$ returns one in $\mathfrak{k}$; this means that

$$[\cdot,\cdot]_{\mathfrak{g}} = [\cdot,\cdot]_{\mathfrak{k}} \oplus \mathrm{ad}_{\mathfrak{k}} \oplus [\cdot,\cdot]_{\mathfrak{p}}$$

so setting $\omega = [\cdot,\cdot]_{\mathfrak{p}}$ we have an analogue of theorem 1.30:

3.2 Theorem. *A $\mathbb{Z}_2$-graded Lie algebra is characterised by*

- *A Lie algebra $\mathfrak{k}$;*
- *the representation $\mathrm{ad}_{\mathfrak{k}} \colon \mathfrak{k} \to \mathrm{End}\,\mathfrak{p}$;*
- *the alternating, bilinear, $\mathfrak{k}$-equivariant form $\omega \colon \Lambda^2\mathfrak{p} \to \mathfrak{k}$ in the kernel of the composition*

$$\left(\Lambda^2\mathfrak{p}^* \otimes \mathfrak{k}\right)^{\mathfrak{k}} \xrightarrow{\mathrm{id}\otimes\mathrm{ad}} \left(\Lambda^2\mathfrak{p}^* \otimes \mathfrak{p}^* \otimes \mathfrak{p}\right)^{\mathfrak{k}} \xrightarrow{c\otimes\mathrm{id}} \left(\Lambda^3\mathfrak{p}^* \otimes \mathfrak{p}\right)^{\mathfrak{k}}$$

where $c \colon \Lambda^2\mathfrak{p}^* \otimes \mathfrak{p}^* \to \Lambda^3\mathfrak{p}^*$ is multiplication of forms.

In such an algebra, the Baker–Campbell–Hausdorff formula restricted to $\mathfrak{p}$ reads

$$e^X e^Y = e^{k(X,Y)} e^{p(X,Y)} = e^{\tilde{p}(X,Y)} e^{\tilde{k}(X,Y)} \tag{3.6}$$

such that $p, \tilde{p}$ are formal power series on $\mathfrak{p} \oplus \mathfrak{p}$ which take values on $\mathfrak{p}$ whereas $k, \tilde{k}$ take values on $\mathfrak{k}$. For example, the expansion of k reads

$$k(X,Y) = \tanh\left(\frac{\mathrm{ad}\,X}{2}\right)(Y) + O(Y^2) = -\tanh\left(\frac{\mathrm{ad}\,Y}{2}\right)(X) + O(X^2) = \frac{1}{2}[X,Y] + \dots$$

Multiplying by $\left(e^X e^Y\right)^{-1} = e^{-Y} e^{-X}$ we obtain the identities

$$k = \tilde{k} \quad \text{and} \quad \tilde{p} = (\exp \mathrm{ad}(k))p. \tag{3.7}$$

Moreover p involves only odd bracket terms (e.g. X and $[X,[X,Y]]$) while k involves the even terms (e.g. $[X,Y]$ and $[X,[X,[X,Y]]]$) and therefore

$$p(-X,-Y) = -p(X,Y) \quad \text{and} \quad k(-X,-Y) = k(X,Y) \tag{3.8}$$

Formula (3.6) then provides the multiplication rule for a symmetric space.

In order to extend the above ideas to a Lie superalgebra $\mathfrak{L} = (\mathfrak{g}|\mathfrak{s})$ let $r \geq 2$ be an integer and define

$$\mathfrak{L}^{[r]} := \left(\Lambda(\mathfrak{s}^{*\oplus r}) \otimes \mathfrak{L}\right)_{(+)},$$

that is the even space of the above superspace endowed with the bracket

$$[\omega \otimes L, \eta \otimes L'] := \omega \wedge \eta \otimes [\![L, L']\!]$$

so as to get a $\mathbb{Z}_2$-graded Lie algebra; then $\mathfrak{L}^{[r]}$ is a $\mathbb{Z}_2$-graded Lie algebra and therefore formula (3.6) applies.

Observe there is an action of $\mathfrak{L}^{[r]}$ on $\mathfrak{L}$ by defining the action of the exterior algebra to be zero whereas the action of the even and odd parts of $\mathcal{L}$ on itself are given by the superbracket. Therefore there is an algebra homomorphism

$$\begin{aligned} F \colon \mathcal{U}\mathfrak{L}^{[r]} &\to \Lambda(\mathfrak{s}^{*\oplus r}) \otimes \mathcal{U}\mathfrak{L} \\ (\omega_1 \otimes X_1) \cdots (\omega_r \otimes X_r) &\mapsto (\omega_1 \wedge \cdots \wedge \omega_r) \otimes X_1 \cdots X_r \end{aligned} \tag{3.9}$$

Letting $r = 2$, choosing a basis $\{s_1, \ldots, s_n\}$ for $\mathfrak{s}$ with dual basis $\{ds_1, \ldots, ds_n\}$, and setting

$$\begin{aligned} X &= \sum_{\nu} (0 \oplus ds_\nu) \otimes s_\nu \\ Y &= \sum_{\nu} (ds_\nu \oplus 0) \otimes s_\nu \end{aligned} \tag{3.10}$$

formula (3.6) holds in $\mathcal{U}\mathfrak{L}^{[r]}$; applying the homomorphism (3.9) to (3.6) we obtain the same identity in the superalgebra $\Lambda(\mathfrak{s}^{*\oplus 2}) \otimes \mathcal{U}\mathfrak{L}$ where

$$p, \tilde{p} \in \left(\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*) \otimes \mathfrak{s}\right)^G \qquad \text{and} \qquad k, \tilde{k} \in \left(\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*) \otimes \mathfrak{g}\right)^G$$

are the corresponding expansions in the bracket of X and Y for $e^X e^Y$; observe that X and Y are in $\left(\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*) \otimes \mathfrak{s}\right)^G$ and therefore identities (3.7) hold. Since they are G-invariant proposition 2.11 implies there are parallel sections

$$P, \tilde{P} \in \Gamma\left(\Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G) \otimes \mathbf{S}G\right) \qquad \text{and} \qquad K, \tilde{K} \in \Gamma\left(\Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G) \otimes TG\right)$$

with respect to either ∇^L or ∇^R. Summarising:

3.3 Lemma. *The sections $P, \tilde{P}, K$ and $\tilde{K}$ are parallel with respect to both the left and right connections.*

3.4 Remark. Observe p, $\tilde{p}$, k and $\tilde{k}$ depend only on the superbracket and therefore they depend on the symmetric equivariant map B of theorem 3.1. If $B = 0$ it follows that $k = \tilde{k} = 0$ and $p(s,t) = \tilde{p}(s,t) = s + t$. In general the superbracket satisfies $[\![\cdot,\cdot]\!] = 2B$.

Using the forms p and $\tilde{p}$ one can construct special algebra homomorphisms.

3.5 Proposition. *The map*

$$\mathfrak{s}^* \to \Lambda_{(-)}(\mathfrak{s}^* \oplus \mathfrak{s}^*)$$

given by $\sigma \mapsto \sigma \circ p$ for $\sigma \in \mathfrak{s}^*$ extends to a unital superalgebra homomorphism

$$\Delta_p \colon \Lambda\mathfrak{s}^* \to \Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$$

A similar map can be constructed from $\tilde{p}$.

Proof. Since the map Δ_p is defined on generators it has a unique extension to a superalgebra homomorphism. More precisely: p and $\tilde{p}$ can be written as

$$p = \sum_\mu \eta_\mu \otimes s_\mu \quad \text{and} \quad \tilde{p} = \sum_\mu \tilde{\eta}_\mu \otimes s_\mu \tag{3.11}$$

respectively, with uniquely determined odd forms η_μ and $\tilde{\eta}_\mu$; so defined p and $\tilde{p}$ map the basis $\{ds_1, \dots, ds_n\}$ of $\mathfrak{s}^*$ to the odd forms $\{\eta_1, \dots, \eta_n\}$ and $\{\tilde{\eta}_1, \dots, \tilde{\eta}_n\}$ respectively. Since a homomorphism of exterior algebras is determined by the image of generators which can be chosen arbitrarily (because it is a free superalgebra) equation (3.11) determine unique unital homomorphisms Δ_p and $\Delta_{\tilde{p}}$. □

Since p and $\tilde{p}$ are G-equivariant there exist parallel sections

$$\Delta_P, \Delta_{\tilde{P}} \in \Gamma\left(\mathrm{Hom}\left(\Lambda\mathbf{S}^*G, \Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G)\right)\right).$$

Observe that definition (3.6) implies P is associated with the left trivialisation of $\mathbf{S}G$ and $\tilde{P}$ with the right trivialisation, because $e^{k(X,Y)}$ is an element of G; for this reason we need the left connection for P and the right connection for $\tilde{P}$. We thus denote them by P_L and P_R respectively.

As for the maps k and $\tilde{k}$ observe that upon evaluation on an arbitrary vector $s \oplus \tilde{s}$ the result is an even element of $\mathfrak{L}$. Then k has an expansion of the form

$$k = \sum_\mu \omega_\mu \otimes X_\mu \tag{3.12}$$

where the ω_μ are even forms in $\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$ and the X_μ are vectors in $\mathfrak{g}$. Since $\mathfrak{g}$ acts on $\mathfrak{s}$ and therefore on $\mathfrak{s}^* \oplus \mathfrak{s}^*$ this action can be extended to an action $\star$ on $\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$ by derivations and therefore the derivation associated to k is:

$$k\star = \sum_\mu (\omega_\mu \wedge) \circ X_\mu \star$$

Then the map $\exp(k\star)$ is a superalgebra automorphism of $\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$.

All of the above maps are G-equivariant and therefore by proposition 2.10 expression (3.12) can be extended to a section of $\Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G) \otimes TG$; therefore there are derivations $K\star, \nabla_K^L$ and ∇_K^R of $\Gamma(\Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G))$, using the connections ∇^L and ∇^R. Exponentiating the latter two we obtain differential operators $\exp(\nabla_K^L)$ and $\exp(\nabla_K^R)$ acting on sections of $\Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G)$ as superalebra automorphisms. With these maps we can now form

$$\mathcal{M}_B^{L,R} := \mathrm{twist} \circ m^* \circ \exp \nabla_K^{L,R} \circ \Delta_{P_{L,R}} \tag{3.13}$$

(here m is the multiplication map of G). From remark 3.4 it follows that this map is completely determined by the symmetric equivariant map B which is the superbracket when its two arguments are odd.

3.6 Lemma. *The map $K\star$ commutes with both ∇^L_K and ∇^R_K.*

Proof. From identity (3.12) we obtain the following expansions:

$$\nabla^{L,R}_K = \sum_\mu \omega_\mu \wedge \circ \nabla^{L,R}_{X_\mu} \tag{3.14}$$

$$K\star = \sum_\mu \omega_\mu \wedge \circ X_\mu \star \tag{3.15}$$

Observe first that the map (3.14) is an even superderivation on sections (i.e. a differential operator of order 1) whereas (3.15) is a point-wise derivation. The same is true of $\nabla^{L,R}_X$ and $Y\star$ for any vector fields X and Y on G. Furthermore, for any section ω of $\Lambda \mathbf{S}^* G$ (or any of its tensor powers) the identities

$$\nabla^{L,R}_X (Y \star \omega) = (\nabla^{L,R}_X Y) \star \omega + Y \star (\nabla^{L,R}_X \omega) \tag{3.16}$$

$$(X\star) \circ (\omega\wedge) = (X \star \omega) \wedge + (\omega\wedge) \circ X\star \tag{3.17}$$

$$\nabla^{L,R}_Y \circ (\omega\wedge) = \left(\nabla^{L,R}_Y \omega\right) \wedge + (\omega\wedge) \circ \nabla^{L,R}_Y \tag{3.18}$$

hold, again because of the Leibniz identity for derivations[1]. First we compute:

$$\begin{aligned}
\nabla^L_K \circ K\star &= \sum_{\alpha,\beta} (\omega_\alpha \wedge) \circ \nabla^L_{X_\alpha} \circ (\omega_\beta \wedge) \circ X_\beta \star \\
&= \sum_{\alpha,\beta} \omega_\alpha \wedge (\nabla^L_{X_\alpha} \omega_\beta) \wedge X_\beta \star + \sum_{\alpha,\beta} (\omega_\alpha \wedge \omega_\beta \wedge) \circ \nabla^L_{X_\alpha} \circ X_\beta \star && \text{(by (3.18))} \\
&= \sum_{\alpha,\beta} \left[\omega_\alpha \wedge (\nabla^L_{X_\alpha} \omega_\beta) \wedge X_\beta \star + (\omega_\alpha \wedge \omega_\beta \wedge) \circ \left(\nabla^L_{X_\alpha} X_\beta \right) \star \right] \\
&+ \sum_{\alpha,\beta} (\omega_\alpha \wedge \omega_\beta \wedge) \circ X_\beta \star \circ \nabla^L_{X_\alpha} && \text{(by (3.16))}
\end{aligned}$$

Summing the terms in square brackets over β we obtain

$$\sum_\alpha \omega_\alpha \wedge \nabla^L_{X_\alpha} K = 0$$

since K is parallel with respect to both connections (Corollary 2.11), and thus

$$\nabla^L_K \circ K \star \sum_{\alpha,\beta} (\omega_\beta \wedge \omega_\alpha \wedge) \circ X_\beta \star \circ \nabla^L_{X_\alpha} \tag{3.19}$$

since ω_α and ω_β are all even forms. On the other hand:

$$\begin{aligned}
(K\star) \circ \nabla^L_K &= \sum_{\alpha,\beta} (\omega_\beta \wedge) \circ X_\beta \star \circ (\omega_\alpha \wedge) \circ \nabla^L_{X_\alpha} \\
&= \sum_{\alpha,\beta} \omega_\beta \wedge (X_\beta \star \omega_\alpha \wedge) \circ \nabla^L_{X_\alpha} && \text{(by (3.17))} \\
&+ \sum_{\alpha,\beta} (\omega_\beta \wedge \omega_\alpha \wedge) \circ X_\beta \star \circ \nabla^L_{X_\alpha}
\end{aligned}$$

[1] Identity (3.16) actually holds true on any bihomogeneous vector bundle.

Observe that the last sum is equal to (3.19), therefore:

$$\begin{aligned}\left[\nabla^L_K, K\star\right] &= -\sum_{\alpha,\beta}\omega_\beta \wedge \left(X_\beta \star \omega_\alpha \wedge\right) \nabla^L_{X_\alpha} \\ &= -\sum_{\alpha,\beta}\omega_\beta \wedge \left(X_\beta \star \omega_\alpha \wedge\right) \nabla^L_{X_\alpha} - \sum_{\alpha,\beta}\omega_\beta \wedge \omega_\alpha \wedge \nabla^L_{[X_\alpha,X_\beta]} \\ &+ \sum_{\alpha,\beta}\omega_\beta \wedge \omega_\alpha \wedge \nabla^L_{[X_\alpha,X_\beta]}\end{aligned}$$

By summing over α in the terms with the minus sign we obtain

$$\sum_\beta \omega_\beta \wedge \nabla^L_{X_\beta \star K} = 0$$

since K is $\mathfrak{g}$-invariant, i.e. $X \star K = 0$ for any vector field X (cf. Corollary 2.11). Thus:

$$\left[\nabla^L_K, K\star\right] = \sum_{\alpha,\beta}\omega_\beta \wedge \omega_\alpha \wedge \nabla^L_{[X_\alpha,X_\beta]}$$

and since the sum above is symmetric in $\omega_\alpha \wedge \omega_\beta$ while alternating in $\nabla^L_{[X_\beta,X_\alpha]}$ it equals zero; therefore $\nabla^L_K \circ K\star = K \star \circ \nabla^L_K$. From identity (2.9) which in this case reads:

$$\nabla^R_K = \nabla^L_K + K\star$$

and the above computation it follows that $\nabla^R_K \circ K\star = K \star \circ \nabla^R_K$. □

3.7 Corollary. *The identity* $\exp \nabla^R_K \circ \exp(-K\star) = \exp \nabla^L_K$ *holds.*

Proof. Since ∇^L_K and $K\star$ commute we know $\exp(\nabla^L_K + K\star) = \exp \nabla^L_K \circ \exp(K\star)$; now recall that $\nabla^R_X = \nabla^L_X + X\star$ (cf. (2.9)) and substitute in the previous identity. □

Observe this is the appropriate version of identity (3.7) for K, since we substitute the adjoint representation ad with the infinitesimal representation $\star$ when passing to a superalgebra. The corresponding identity for P also holds:

3.8 Proposition. *Identity* $\exp(-K\star) \circ \Delta_{P_L} = \Delta_{P_R}$ *holds.*

Proof. First of all observe that the homomorphism $\Delta_p \colon \Lambda\mathfrak{s}^* \to \Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$ is completely determined by the image of the generators (proposition 3.5) under the linear map

$$\mathfrak{s}^* \xrightarrow{\circ p} \Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*) \xrightarrow{\exp(-k\star)} \Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$$

Secondly, since $\exp(-k\star)$ is an algebra automorphism of $\Lambda(\mathfrak{s}^* \oplus \mathfrak{s}^*)$ we can deal with the map $\exp(k\star) \circ p$. We have to prove that this equals $\tilde{p}$. Let $\hat{p} = \sum_\nu \exp(-k\star)\eta_\nu \otimes s_\nu$; we will be done if we prove $\tilde{p} = \hat{p}$. Since p is G-invariant we obtain:

$$\sum_\nu \eta_\nu \otimes \exp(\operatorname{ad} k)s_\mu$$

This is exactly identity (3.7), so we obtain $\Delta_{\tilde{p}} = \exp k\star \circ \Delta_p$. The result follows from corollaries 2.11 and 2.8. □

Corollary 3.7 and proposition 3.8 imply:

3.9 Proposition. *The maps $\mathcal{M}_B^L$ and $\mathcal{M}_B^R$ are equal as operators on $\Gamma\left(\Lambda \mathbf{S}^* G\right)$.*

Summarising: so far we know that the auxiliary codifferential of a multiplication supermap $\mathcal{M}$ furnishes a bilinear symmetric biequivariant map $B := \mathbf{B}^+$ given by (3.5) and, conversely, a bilinear symmetric biequivariant map B allows the construction of the map $\mathcal{M}_B = \mathcal{M}_B^L = \mathcal{M}_B^R$. It remains to establish the following two statements:

1. That the map $\mathcal{M}_B$ is "associative"; i.e. it makes diagram (2.2) commute.

2. That the form B obtained from a coassociative multiplication supermap $\mathcal{M}$ satisfies the Jacobi identity.

When these two propositions are proved the supergeometric version of Lie's third theorem for supergroups of the form $(G|\Lambda \mathbf{S}^* G)$ can be established; that is, they imply that the multiplication of a Lie supergroup $(G|\Lambda \mathbf{S}^* G)$ determines and is determined by the superbracket on the supertangent space at the identity of G.

Define the space $\mathfrak{J}(\mathfrak{g}|\mathfrak{s})$ of bilinear equivariant maps $B \in (\operatorname{Sym}^2 \mathfrak{s}^* \otimes \mathfrak{g})^G$ such that condition 3 of theorem 1.30 holds (i.e. the maps that satisfy the Jacobi identity). The theorem we aim to prove is then the following:

3.10 Theorem. *Let $B \in \mathfrak{J}(\mathfrak{g}|\mathfrak{s})$; then the maps given by (3.13) make diagram (2.2) commute. Conversely, given a map $\mathcal{M}$ such that diagram (2.2) commutes the form $B := \mathbf{B}^+$ defined by (3.5) is in the space $\mathfrak{J}(\mathfrak{g}|\mathfrak{s})$.*

We shall divide the proof in several steps.

3.1.1. The Jacobi identity implies coassociativity

The goal of this subsection is proving the following

3.11 Theorem. *If $B \in \mathfrak{J}(\mathfrak{g}|\mathfrak{s})$ then $(\mathcal{M}^L \otimes \mathrm{id}) \circ \mathcal{M}^L = (\mathrm{id} \otimes \mathcal{M}^R) \circ \mathcal{M}^R$.*

3.12 Proposition. *Let X and $\widetilde{X}$ be vector fields on G. The identities*

$$\nabla^L_{(X,\widetilde{X})} \circ \mathrm{twist} = \mathrm{twist} \circ \big(m^* \nabla^L_{(X,0)} + (m_*(0,\widetilde{X})) \star \otimes \mathrm{id} \big)$$
$$\nabla^R_{(X,\widetilde{X})} \circ \mathrm{twist} = \mathrm{twist} \circ \big(m^* \nabla^R_{(0,\widetilde{X})} + \mathrm{id} \otimes (m_*(X,0)) \star \big)$$

hold.

Proof. These are computations; we give the first one explicitly. First observe it suffices to prove it for the following map

$$\begin{aligned} \mathrm{twist}_1 \colon m^* \mathbf{S}^* G &\to \mathrm{pr}_1^* \mathbf{S}^* G \\ \sigma_{(g,\tilde{g})} &\mapsto e * \sigma * \tilde{g}^{-1} \end{aligned} \tag{3.20}$$

since twist: $m^*(\mathbf{S}^*G \oplus \mathbf{S}^*G) \to \mathbf{S}^*G \boxplus \mathbf{S}^*G$ factorises by this map and a similar one whose codomain is $\mathrm{pr}_2^* \mathbf{S}^*G$ and also the maps in the statement are completely determined by this latter map because they act on generators of the appropriate algebras. Then

$$\begin{aligned}\left(\nabla^L_{\frac{\mathrm{d}}{\mathrm{d}t}|_0(g_t,\tilde g_t)}(\mathrm{twist}_1\,\sigma)\right)_{(g,\tilde g)} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g_0 g_t^{-1} * \sigma(g_t,\tilde g_t) * \tilde g_t^{-1}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g_0 g_t^{-1} * \sigma(g_t,\tilde g_0) * \tilde g_0^{-1} + \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 e * \sigma(g_0,\tilde g_t) * \tilde g_t^{-1}\end{aligned}$$

On the other hand by identities (2.11) we have

$$\begin{aligned}\mathrm{twist}_1\left(m^*\nabla^L_{\frac{\mathrm{d}}{\mathrm{d}t}|_0(g_t,\tilde g_t)}\sigma\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g_0\tilde g_0\tilde g_t^{-1} g_t^{-1} * v(g_t,\tilde g_t) * \tilde g_0^{-1}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g_0 g_t^{-1} * v(g_t,\tilde g_0) * \tilde g_0 + \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g_0\tilde g_0\tilde g_t^{-1} g_0^{-1} * v(g_0,\tilde g_t) * \tilde g_0^{-1};\end{aligned}$$

as for $m_*\widetilde{X}$ we compute

$$m_*\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 (g_0,\tilde g_t)\right)\star\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 g_0\tilde g_0\tilde g_t^{-1} g_0^{-1} * \sigma * e - \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 e * \sigma * \tilde g_t^{-1}\tilde g_0.$$

Adding the two latter yields

$$\nabla^L_{(X,\widetilde{X})} \circ \mathrm{twist}_1 = \mathrm{twist}_1 \circ \left(m^*\nabla^L_{(X,\widetilde{X})} + m_*\widetilde{X}\star\right)$$

which is the result for ∇^L and twist_1. □

Since the multiplication map has to take sections over G to sections over $G \times G$ the associativity identity must take sections over G to sections over $G \times G \times G$. The appropriate twist maps are then as follows:

$$\begin{aligned}\mathrm{twist}^{(12)}\colon (m\times\mathrm{id})^*(\mathbf{S}^*G^{\oplus 3}) &\to \mathbf{S}^*G \boxplus \mathbf{S}^*G \boxplus \mathbf{S}^*G\\ \mathrm{twist}^{(23)}\colon (\mathrm{id}\times m)^*(\mathbf{S}^*G^{\oplus 3}) &\to \mathbf{S}^*G \boxplus \mathbf{S}^*G \boxplus \mathbf{S}^*G\end{aligned} \tag{3.21}$$

On our definition of the multiplication supermap we used the fact that $\Delta_{P_{L,R}}$ are defined over G; that is, they are bundle maps over the manifold G. There is a priori no corresponding map over $G \times G$. However we only need Δ_P as defined, since the coassociativity of $\mathcal{M}$ means that one of the factors remains unchanged; this is precisely the identity of theorem 3.11.

3.13 Proposition. *The identities*

$$\Delta_{P_L} \boxtimes \mathrm{id} \circ \mathrm{twist} = \mathrm{twist}^{(12)} \circ \Delta_{P_L} \quad \textit{and} \quad \mathrm{id} \boxtimes \Delta_{P_R} \circ \mathrm{twist} = \mathrm{twist}^{(23)} \circ \Delta_{P_R}$$

hold.

Proof. Since P is biequivariant so is Δ_P and it is therefore left and right invariant; thus it commutes with twist which is a composition of left and right translations. □

The same is true for $\exp \nabla_K^{L,R}$, mutatis mutandis.

3.14 Lemma. *Let $\widetilde{\nabla}^L$ denote the left connection on the bundle $\Lambda(\mathbf{S}^*G \boxplus (\mathbf{S}^*G \oplus \mathbf{S}^*G))$ and ∇^L the connection on $\Lambda(\mathbf{S}^*G \oplus \mathbf{S}^*G)$. Then identity*

$$(\Delta_{P_L} \otimes \mathrm{id}) \circ \exp(\nabla_K^L) = \exp \widetilde{\nabla}^L_{(\Delta_{P_L} \otimes \mathrm{id})(K,0)} \circ (\Delta_{P_L} \otimes \mathrm{id})$$

holds. A similar identity holds for ∇_K^R and Δ_{P_R} on the bundle $\Lambda((\mathbf{S}^*G \boxplus \mathbf{S}^*G) \oplus \mathbf{S}^*G)$.

Proof. Since Δ_P is ∇^L-parallel and an algebra homomorphism, formulas (3.11) and (3.12) imply the identity. □

Proof of theorem 3.11. Observe that formula (3.6) implies

$$\begin{aligned}(e^X e^Y)e^Z &= (e^{k(X,Y)} e^{p(X,Y)})e^Z \\ &= e^{k(X,Y)} e^{k(p(X,Y),Z)} e^{p(p(X,Y),Z)}.\end{aligned} \tag{3.22}$$

This formula corresponds to the map $(m \times \mathrm{id}) \circ m$ on the group level. Observe that the last term can be expressed as a (non-linear) map

$$\begin{aligned}\operatorname{Sym}\mathfrak{s} \otimes \operatorname{Sym}\mathfrak{s} \otimes \operatorname{Sym}\mathfrak{s} &\to \operatorname{Sym}\mathfrak{s} \\ e^X \otimes e^Y \otimes e^Z &\mapsto e^{p(p(X,Y),Z)}\end{aligned}$$

which on dualisation yields

$$\Omega_p \colon \operatorname{Sym}\mathfrak{s}^* \to \operatorname{Sym}\mathfrak{s}^* \otimes \operatorname{Sym}\mathfrak{s}^* \otimes \operatorname{Sym}\mathfrak{s}^* \tag{3.23}$$

analogous to Δ_P. Using the map from (3.9) we obtain Δ_p since it is the only G-equivariant algebra homomorphism associated to B. Applying the same representation to (3.22) we obtain the map $(\mathcal{M}^L \otimes \mathrm{id}) \circ \mathcal{M}^L$ at the level of sections. Analogous reasons relate $e^X(e^Y e^Z)$, $\Delta_{\tilde{p}}$ and $(\mathrm{id} \otimes \mathcal{M}^R) \circ \mathcal{M}^R$. To obtain equality and therefore establishing the result we shall use the operators in the statement of the theorem: $(\mathcal{M}^L \otimes \mathrm{id}) \circ \mathcal{M}^L$ and $(\mathrm{id} \otimes \mathcal{M}^R) \circ \mathcal{M}^R$. Let us write $T = \mathrm{twist} \circ m^*$. The closed expression for the first operator is

$$\big(T^{(12)} \circ \exp(\nabla^L_{(K,0)}) \circ (\Delta_{P_L} \otimes \mathrm{id})\big) \circ (T \circ \exp(\nabla_K^L) \circ \Delta_{P_L}). \tag{3.24}$$

Observe that se have replaced K with $(K,0)$ on the left because the operator in question acts non-trivially only on the left factor; also, the map $T^{(12)}$ is the appropriate twist map

$$T^{(12)} \colon \Lambda\big(m^*(\mathbf{S}^*G \oplus \mathbf{S}^*G) \boxplus \mathbf{S}^*G\big) \to \Lambda(\mathbf{S}^*G \boxplus \mathbf{S}^*G \boxplus \mathbf{S}^*G) \tag{3.25}$$

Since $\Delta_{P_{L,R}}$ are G-equivariant it follows they commute with T. By proposition 3.12 T commutes, in this case, with $\exp \nabla_K^L$. Finally lemma 3.14 allows us to write (3.24) as

$$T^{(3)} \circ \exp(\nabla^L_{(K,0)}) \circ \exp(\nabla^L_{(\Delta_{P_L} \otimes \mathrm{id}) \circ K}) \circ (\Delta_{P_L} \otimes \mathrm{id}) \circ \Delta_{P_L} \tag{3.26}$$

which equals $(\mathrm{id} \otimes \mathcal{M}^R) \circ \mathcal{M}^R$ by exactly the same reasons above. □

3.1.2. Coassociativity implies the Jacobi identity

Now we shall prove:

3.15 Theorem. *If $\mathcal{M}$ is a coassociative multiplication map then the map B obtained in theorem 3.1 is in $\mathfrak{J}(\mathfrak{g}|\mathfrak{s})$.*

First of all observe that

$$\mathcal{M}^*(\sigma) = \sigma \boxplus \sigma \tag{3.27}$$

for any section σ of $\mathbf{S}^*G$. Define the following map:

$$\begin{gathered} \mathrm{PR}\colon \Lambda^3(\mathbf{S}^*G \boxplus \mathbf{S}^*G \boxplus \mathbf{S}^*G) \to m^*(\mathrm{Sym}^3\, \mathbf{S}^*G) \\ (\sigma_1 \boxplus \tilde{\sigma}_1 \boxplus \hat{\sigma}_1) \wedge (\sigma_2 \boxplus \tilde{\sigma}_2 \boxplus \hat{\sigma}_2) \wedge (\sigma_3 \boxplus \tilde{\sigma}_3 \boxplus \hat{\sigma}_3) \mapsto \sum_{\tau \in S_3} \mathrm{sgn}(\tau) \sigma_{\tau(1)} \tilde{\sigma}_{\tau(2)} \hat{\sigma}_{\tau(3)} \end{gathered}$$

For vectors in the fibres of the bundle $(\mathcal{M}^* \otimes \mathrm{id})^*(\Lambda^3(\mathbf{S}^*G \boxplus \mathbf{S}^*G))$ this map evaluates to zero, because applying it to $(\mathcal{M} \times \mathrm{id})^*(\sigma \boxplus \tilde{\sigma}) = \sigma \boxplus \sigma \boxplus \tilde{\sigma}$ is zero. The same applies for the bundle $(\mathrm{id} \otimes \mathcal{M})^*(\Lambda^3(\mathbf{S}^*G \boxplus \mathbf{S}^*G))$. Also, elements of the form $\sigma \boxplus \tilde{\sigma} \boxplus 0$ and $0 \boxplus \sigma \boxplus \tilde{\sigma}$ are annihilated by PR. We now define the operators

$$\begin{aligned} D_L &:= \mathrm{PR} \circ (\mathcal{M} \otimes \mathrm{id}) \\ D_R &:= \mathrm{PR} \circ (\mathrm{id} \otimes \mathcal{M}) \end{aligned}$$

as maps $\Gamma_{G \times G}\big(\Lambda^{\leq 3}_{(-)}(\mathbf{S}^*G \boxplus \mathbf{S}^*G)\big) \to \Gamma_{G \times G \times G}\big(m^*(\mathrm{Sym}^3\, \mathbf{S}^*G)\big)$. They factorise over the quotient

$$\Gamma\left(\Lambda^{\leq 3}_{(-)}(\mathbf{S}^*G \boxplus \mathbf{S}^*G)\right) \to \Gamma\left(\mathbf{S}^*G \boxplus \mathbf{S}^*G\right) \tag{3.28}$$

and are differential operators of order at most 1. Their principal symbol is the wedge product with a 2-form. In the case of D_L the principal symbol is a map

$$(m \times \mathrm{id})^*\left(T^*(G \times G) \otimes (\mathbf{S}^*G \boxplus \mathbf{S}^*G)\right) \to m^*(\mathrm{Sym}^3\, \mathbf{S}^*G)$$

can be computed to be equal to

$$\alpha \otimes (\sigma \boxplus \tilde{\sigma}) \mapsto \mathrm{PR}\left(\frac{1}{2}\sum_{\mu,\nu} \alpha([\![s_\mu, s_\nu]\!])(ds_\mu \boxplus 0 \boxplus 0) \wedge (0 \boxplus ds_\nu \boxplus 0) \wedge (\sigma \boxplus \sigma \boxplus \tilde{\sigma})\right)$$

because of (3.27). By the definition of PR the sum above is mapped to:

$$\frac{1}{2}\sum_{\mu,\nu} \alpha([\![s_\mu, s_\nu]\!]) ds_\mu ds_\nu \tilde{\sigma} \tag{3.29}$$

The same argument applied to D_R results in

$$\frac{1}{2}\sum_{\mu,\nu} \alpha([\![s_\mu, s_\nu]\!]) ds_\mu ds_\nu \sigma \tag{3.30}$$

Since we are restricting everything to the quotient (3.28) the maps $\Delta_P \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Delta_P$ act as

$$\begin{aligned} \theta_{12} \otimes \mathrm{id} \colon \mathbf{S}^*G \oplus \mathbf{S}^*G &\to (\mathbf{S}^*G \oplus \mathbf{S}^*G) \boxplus \mathbf{S}^*G \\ \sigma \boxplus \tilde{\sigma} &\mapsto (\sigma \oplus \sigma) \boxplus \tilde{\sigma} \\ \mathrm{id} \otimes \theta_{23} \colon \mathbf{S}^*G \oplus \mathbf{S}^*G &\to \mathbf{S}^*G \boxplus (\mathbf{S}^*G \oplus \mathbf{S}^*G) \\ \sigma \boxplus \tilde{\sigma} &\mapsto \sigma \boxplus (\tilde{\sigma} \oplus \tilde{\sigma}) \end{aligned} \tag{3.31}$$

They correspond to $\mathcal{M} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \mathcal{M}$ respectively. Also, we will use the following notations

$$\widetilde{\mathrm{PR}}_{12} := \mathrm{PR} \circ \mathrm{twist}^{(12)}, \quad \text{and} \quad \widetilde{\mathrm{PR}}_{23} := \mathrm{PR} \circ \mathrm{twist}^{(23)}$$

Proof of theorem 3.15. We shall apply the operators $D_L \circ \mathcal{M}$ and $D_R \circ \mathcal{M}$ to a left-invariant section ds_α of $\mathbf{S}^*G$ and we will restrict everything to degree 1; that is, the result of appling $\mathcal{M}$ to ds_α will be taken modulo $\Gamma(\Lambda^{\geq 3}(\mathbf{S}G^* \boxplus \mathbf{S}^*G))$. We are allowed this because of (3.28). Using proposition 3.12 we obtain

$$(\nabla^L_K \otimes \mathrm{id})(ds_\alpha \boxplus ds_\alpha) = \nabla^L_{(K,0)}(ds_\alpha \boxplus ds_\alpha) = 0 \tag{*}$$

because $\nabla^L ds_\alpha = 0$. Note that (3.28) implies $\exp \nabla^L_K = \mathrm{id} + \nabla^L_K$ since the rest of the terms map a given $\sigma \in \Gamma(\mathbf{S}^*G)$ to forms of higher degree.

Now we compute:

$$\begin{aligned} D_L \circ \mathcal{M}(ds_\alpha) &= \mathrm{PR}\left(\mathrm{twist}^{(12)} \circ (\exp(\nabla^L_K) \otimes \mathrm{id}) \circ (\Delta_{P_L} \otimes \mathrm{id}) \circ \mathrm{twist} \circ m^*(ds_\alpha)\right) \\ &= \widetilde{\mathrm{PR}}_{12} \circ (\mathrm{id} + \nabla^L_{(K,0)})\Big((\theta_{12} \otimes \mathrm{id}) \circ \mathrm{twist}(m^* ds_\alpha \oplus m^* ds_\alpha)\Big) \\ &= \widetilde{\mathrm{PR}}_{12} \circ (\nabla^L_{(K,0)})\Big(\mathrm{twist}^{(12)}[(m^* ds_\alpha \oplus m^* ds_\alpha) \boxplus m^* ds_\alpha]\Big) \quad \text{(by lemma 3.14)} \\ &= \widetilde{\mathrm{PR}}_{12} \circ \mathrm{twist}\Big((m^*\nabla^L)_{(K,0)}[(m^* ds_\alpha \oplus m^* ds_\alpha) \boxplus m^* ds_\alpha]\Big) \\ &= 0 \qquad \text{(by proposition 3.12.)} \end{aligned}$$

Note that in the third equality we have replaced $\mathrm{id} + \nabla^L$ with ∇^L; this is because PR forgets terms of order 1 and are thus neglible. Computing for $D_R \circ \mathcal{M}$ we obtain:

$$\begin{aligned} D_R \circ \mathcal{M}(ds_\alpha) &= \mathrm{PR} \circ \mathrm{twist}^{(23)} \circ (\mathrm{id} \times m)^* \circ \nabla^L_{(0,K)} \circ (\mathrm{id} \otimes \theta_{23}) \circ \mathrm{twist} \circ m^*(ds_\alpha) \\ &= \widetilde{\mathrm{PR}}_{23} \circ (\mathrm{id} + \nabla^L_{(0,K)}) \circ (\mathrm{id} \otimes \theta_{23}) \circ \mathrm{twist}(m^* ds_\alpha \oplus m^* ds_\alpha) \\ &= \widetilde{\mathrm{PR}}_{23} \circ \nabla^L_{(0,K)}\Big(\mathrm{twist}^{(23)}[m^* ds_\alpha \boxplus (m^* ds_\alpha \oplus m^* ds_\alpha)]\Big) \\ &= \widetilde{\mathrm{PR}}_{23} \circ \mathrm{twist}\Big((m^*\nabla^L)_{(0,K)}[m^* ds_\alpha \boxplus (m^* ds_\alpha \oplus m^* ds_\alpha)] \\ &\quad + \mathrm{id} \boxplus (m_*(0,K))[m^* ds_\alpha \boxplus (m^* ds_\alpha \oplus m^* ds_\alpha)]\Big) \\ &= \frac{1}{2}\sum_{\mu,\nu} ds_\mu ds_\nu(\llbracket s_\mu, s_\nu \rrbracket \star ds_\alpha) \qquad \text{(by proposition 3.12)} \end{aligned}$$

Now, since $\mathcal{M}$ is coassociative we know $(D_R - D_L) \circ \mathcal{M} = 0$, and thus:

$$D_R \circ \mathcal{M}(ds_\alpha) = \frac{1}{2}\sum_{\mu,\nu} ds_\mu ds_\nu (\llbracket s_\mu, s_\nu \rrbracket \star ds_\alpha) = 0 \tag{3.32}$$

Let $B \in (\operatorname{Sym}^2 \mathfrak{s}^* \otimes \mathfrak{g})^G$ be the map constructed in theorem 3.1 from the auxiliary differential of $\mathcal{M}$. Using a basis $\{ds_1, \dots, ds_n\}$ it is written as:

$$B = \frac{1}{2}\sum_{\mu,\nu} ds_\mu ds_\nu \otimes \llbracket s_\mu, s_\nu \rrbracket$$

Define the operator:

$$J(B) := \frac{1}{2}\sum_{\mu,\nu,\lambda} ds_\mu ds_\nu ds_\lambda \otimes \llbracket s_\mu, s_\nu \rrbracket \star s_\lambda \in \operatorname{Sym}^3 \mathfrak{s}^* \otimes \mathfrak{s}$$

This operator can be extended to a bundle map

$$\mathbf{J}(B) \colon m_3^*(\mathbf{S}^* G) \to m_3^*(\operatorname{Sym}^3 \mathbf{S}^* G)$$

over the multiplication $m_3 \colon G \times G \times G \to G$. This map corresponds to the map $(D_R - D_L) \circ \mathcal{M}$ according to the computations above. Since $\mathcal{M}$ is coassociative we know this map $\mathbf{J}(B)$ is zero.

Observe that $J(B)$ maps a generator ds_α of $\mathfrak{s}^*$ to

$$\frac{1}{2}\sum_{\mu,\nu,\lambda} ds_\alpha (\llbracket s_\mu, s_\nu \rrbracket \star s_\lambda) ds_\mu ds_\nu ds_\lambda.$$

Using the dual representation on $\mathfrak{s}^*$ this transforms into

$$-\frac{1}{2}\sum_{\mu,\nu,\lambda} (\llbracket s_\mu, s_\nu \rrbracket \star ds_\alpha)(s_\lambda) ds_\mu ds_\nu ds_\lambda = -\frac{1}{2}\sum_{\mu,\nu} (\llbracket s_\mu, s_\nu \rrbracket \star ds_\alpha) ds_\mu ds_\nu.$$

By (3.32) we know the above is zero, but this is precisely the Jacobi identity. Therefore, starting with a coassociative $\mathcal{M}$ we obtain a $B \in \mathfrak{J}(\mathfrak{g}|\mathfrak{s})$. $\square$

The theorems of the two latter subsections furnish the proof of of theorem 3.10. This theorem classifies all possible Lie supergroup structures in supermanifolds of the form $(G|\Lambda \mathbf{S}^* G)$. For an abstract superalgebra bundle $\mathcal{R}G$ things are not that simple because there is no a priori way to define a biaction on it. It becomes possible, however, by fixing an isomorphism $E \colon \Gamma(\Lambda \mathbf{S}^* G) \to \Gamma(\mathcal{R}G)$; however not any isomorphism will do. Since both supergroups share the supervector space $(\mathfrak{g}|\mathfrak{s})$ as the tangent superspace at the identity the above isomorphism has to intertwine the supermultiplication $\mathcal{M}_B$ given by the superbracket and the one given in $(G|\mathcal{R}G)$, while at the same time inducing the identity on $(\mathfrak{g}|\mathfrak{s})$, so as to extend Lie's second theorem: an isomorphism of Lie superalgebras induces a (local) isomorphism of Lie supergroups. We will do this in the next chapter.

Chapter 4

General Lie supergroups

Hands, do what you're bid;
Bring the balloon of the mind
That bellies and drags in the wind
Into its narrow shed.

W.B. Yeats, *The balloon of the mind*.

In this chapter we prove our main result concerning general Lie supergroups. It states, in essence, that a given multiplication map on a Lie supergroup can be approximated by maps that are in arbitrary contact to it, in the sense of definition 1.31.

We proceed by using lemma 1.35 to inductively construct maps that satisfy the necessary properties of a multiplication operator on each step while preserving the ones that the operators on previous steps satisfied.

First we prove that every Lie supergroup is in contact to order 1 to a split supergroup:

4.1 Proposition. *In every Lie supergroup $(G|\mathcal{R}G)$ the bundle $\mathbf{S}G$ of odd directions is bihomogeneous.*

Proof. The differential $\mathcal{M}_*$ of the multiplication supermap is a bundle map

$$\mathcal{M}_* \colon \mathbf{S}G \boxplus \mathbf{S}G \to \mathbf{S}G$$

that satisfies $(\mathcal{M}_* \oplus \mathrm{id}) \circ \mathcal{M}_* = (\mathrm{id} \oplus \mathcal{M}_*) \circ \mathcal{M}_*$ and therefore theorem 2.13 applies. □

The problem is that for a general supergroup there is no naturally defined biaction on the bundle $\mathcal{R}G$; defining one requieres, for instance, the choice of a bundle isomorphism with $\Lambda\mathbf{S}^*G$.

§ 4.1 The main theorem

Let $(G|\mathcal{R}G)$ be a Lie supergroup with multiplication supermap $\mathcal{M}$ and $\mathbf{S}G$ its (bihomogeneous) bundle of odd directions. Let $\mathcal{M}^B$ be the multiplication on the supergroup $(G|\Lambda\mathbf{S}^*G)$ constructed from the bilinear biequivariant odd bracket in (3.13). The main result of this work is the following:

4.2 Theorem. *There exists a unique unital superalgebra isomorphism* $E\colon \Gamma(\Lambda \mathbf{S}^* G) \to \Gamma(\mathcal{R}G)$ *such that*

1. *E is a differential operator (along* id_G*).*

2. $E_* = \mathrm{id}_{\Gamma(\mathbf{S}G)}$.

3. $E^{\text{-}1} \widehat{\boxtimes} E^{\text{-}1} \circ \mathcal{M} \circ E = \mathcal{M}^B$.

Item 2 is a consequence of proposition 4.1. Item 1 is just a restatement of proposition 4.1. Item 3 is the important one: it tells us that through the operator E it is possible to construct a biaction on the bundle $\mathcal{R}G$ that is equivalent with the given biaction on $\Lambda \mathbf{S}^* G$.

Strategy of proof: Starting with an isomorphism $E_1\colon \Gamma(\Lambda \mathbf{S}^* G) \to \Gamma(\mathcal{R}G)$ satisfying $E_1 = \Gamma(F)$ for an algebra bundle isomorphism $F\colon \mathcal{R}G \to \Lambda \mathbf{S}^* G$, we shall construct isomorphisms E_k for $k \geq 2$ such that

- $(E_k)_* = \mathrm{id}$.
- $E_{k-1} \underset{k-1}{\sim} E_k$ for all $k > 1$ and
- $\mathcal{M}^{(k)} := E_k^{\text{-}1} \widehat{\otimes} E_k^{\text{-}1} \circ \mathcal{M} \circ E_k \underset{k}{\sim} \mathcal{M}^B$.

For this we rely on the work done in section 1.6, more precisely we shall use corollary 1.32 and proposition 1.37 to ensure that the multiplications constructed up to order $k-1$ can be modified so as to be in contact to order k with the multiplication $\mathcal{M}^B$ of (3.13).

§ 4.2 Proof of the main theorem

Observe that by hypothesis we have fixed a class of 1-contact for all the maps E_k, which is the 1-contact class of the identity map; in each step the class of $(k-1)$-contact will be fixed and therefore lemma 1.35 allows us to work with the bundle map $\Delta_k(E_k, E_{k-1})$. We will now prove the following

4.3 Theorem. *Let $k \geq 2$. Given E_{k-1} it is possible to choose E_k in such a way that the map $\Delta_k(E_k, E_{k-1})$ is completely determined by $\Delta_k(\mathcal{M}^B, \mathcal{M}^{(k-1)})$, and $\mathcal{M}^{(k)} \underset{k}{\sim} \mathcal{M}^B$.*

Proof. The proof is by induction on k. The basis of the induction is the case $k = 2$.

Since $E_1 = \Gamma(F)$ for a bundle isomorphism F we know its auxiliary differential to be zero and its differential E_{1*} to be the identity map; this implies $\mathcal{M}^{(1)} \underset{1}{\sim} \mathcal{M}^B$ immediately. Hence, our choice of E_2 must modify $\mathcal{M}^B$ so that the new multiplication $\mathcal{M}^{(2)}$ be in contact to order 2 with $\mathcal{M}^B$ as desired. To achieve this the only possibility there is for such a modification is $\mathcal{M}_{\text{aux}}$; indeed, from $\mathcal{M}^{(2)} = E_2^{\text{-}1} \boxtimes E_2^{\text{-}1} \circ \mathcal{M} \circ E_2$ and formulas (3.3) and (3.4) it follows that with the above definition the obstruction for the two multiplications to be at contact of order 2 is precisely the auxiliary differential of $\mathcal{M}$. Recall that the map $\mathbf{B}$ of (3.3) is decomposed into its symmetric and alternating parts, viz. $\mathbf{B}^+$ and $\mathbf{B}^-$ respectively. Since the map E_2 is only constrained by the requirement $E_{2,*} = \mathrm{id}$, we can choose it in such a way that

$$E_{2,\mathrm{aux}} = -\mathbf{B}^-. \tag{4.1}$$

Under this hypothesis formula (1.13) reduces to

$$\mathcal{M}^{(2)}_{\mathrm{aux}} = -\mathbf{B}^- \circ \Lambda^2 \mathrm{M} + \mathcal{M}_{\mathrm{aux}} + m_* \circ (-\mathbf{B}^- \boxplus -\mathbf{B}^-).$$

Since $\mathcal{M}^{(2)}$ is a multiplication map we can construct a bilinear map $\widetilde{\mathbf{B}}$ using formula (3.3); this map also decomposes in its symmetric and alternating parts and then (3.5) applies. More precisely

$$\widetilde{\mathbf{B}}_{g\tilde{g}}(s,t) := \mathcal{M}^{(2)}_{\mathrm{aux},(g,\tilde{g})}(e * s * \tilde{g}^{-1} \oplus 0, 0 \oplus g^{-1} * t * e)$$

it follows

$$\begin{aligned}\widetilde{\mathbf{B}}^{\pm}_{g\tilde{g}}(s,t) = \frac{1}{2}\Big(& -\mathbf{B}^-_{g\tilde{g}} \circ \Lambda^2\mathrm{M}_{(g,\tilde{g})}(e * s * \tilde{g}^{-1} \oplus 0, 0 \oplus g^{-1} * t * e) \\ & + \mathcal{M}_{\mathrm{aux},(g,\tilde{g})}(e * s * \tilde{g}^{-1} \oplus 0, 0 \oplus g^{-1} * t * e) \\ & + m_{*,(g,\tilde{g})}\Big(-\mathbf{B}^-_{g}(e * s * \tilde{g}^{-1}, 0) \oplus -\mathbf{B}^-_{\tilde{g}}(0, g^{-1} * t * e)\Big) \\ & \pm \Big[-\mathbf{B}^-_{g\tilde{g}} \circ \Lambda^2\mathrm{M}_{(g,\tilde{g})}(e * t * \tilde{g}^{-1} \oplus 0, 0 \oplus g^{-1} * s * e) \\ & + \mathcal{M}_{\mathrm{aux},(g,\tilde{g})}(e * t * \tilde{g}^{-1} \oplus 0, 0 \oplus g^{-1} * s * e) \\ & + m_{*,(g,\tilde{g})}\big(-\mathbf{B}^-_{g}(e * t * \tilde{g}^{-1}, 0) \oplus -\mathbf{B}^-_{\tilde{g}}(0, g^{-1} * s * e)\big)\Big]\Big).\end{aligned}$$

Since $\mathbf{B}$ is bilinear it is zero if any of its arguments is, so the summands involving m_* vanish; the summands involving $\mathcal{M}_{\mathrm{aux}}$ are equal to $\mathbf{B}_{g\tilde{g}}(s,t)$ (left of $\pm$) and $\mathbf{B}_{g\tilde{g}}(t,s)$ (right of $\pm$) by the definition of $\mathbf{B}$; finally the summands involving $\Lambda^2\mathrm{M}$ are equal to $-\mathbf{B}^-_{g\tilde{g}}(s,t)$ (left of $\pm$) and $-\mathbf{B}^-_{g\tilde{g}}(t,s)$ (right of $\pm$). The formula then reduces to

$$\widetilde{\mathbf{B}}^{\pm}_{g\tilde{g}}(s,t) = \frac{1}{2}\Big(-\mathbf{B}^-_{g\tilde{g}}(s,t) + \mathbf{B}_{g\tilde{g}}(s,t) \pm \big(-\mathbf{B}^-_{g\tilde{g}}(t,s) + \mathbf{B}_{g\tilde{g}}(t,s)\big)\Big).$$

And finally it follows that $\widetilde{B} := \widetilde{\mathbf{B}}^+ = \mathbf{B}^+ = B$ and $\widetilde{\mathbf{B}}^- = 0$; this means the auxiliary differential of $\mathcal{M}^{(2)}$ is completely determined by the map B in exactly the same manner as $\mathcal{M}^B_{\mathrm{aux}}$. Theorem 3.10 now implies $\mathcal{M}^{(2)} \underset{2}{\sim} \mathcal{M}^B$. This completes the first step of the induction.

The induction hypothesis can be stated as follows: for $k \geq 3$ there exists a differential operator E_{k-1} such that $\mathcal{M}^{(k-1)} := E^{-1}_{k-1} \widehat{\otimes} E^{-1}_{k-1} \circ \mathcal{M} \circ E_{k-1}$ is in contact up to order $k-1$ with $\mathcal{M}^B$. To complete the induction we have to prove there exists a differential operator E_k such that $\mathcal{M}^{(k)}$ is in contact up to order k with $\mathcal{M}^B$.

Claim. *There exists an alternating k-multilinear map B^* associated to $\Delta_k(\mathcal{M}^B, \mathcal{M}^{(k)})$ in the same way B^- is associated to $\mathcal{M}_{aux}$.*

Proof of the claim: This map is given as follows: set $A^{(k)} := \Delta_k(\mathcal{M}^B, \mathcal{M}^{(k)})$. For $s_1, \dots, s_k$ in the fibre $\mathbf{S}_{g\tilde{g}}G$ we form the map

$$B^{(p)}_{g\tilde{g}}(s_1, \dots, s_k) = \\ A^{(k)}_{(g,\tilde{g})}(e * s_1 * \tilde{g}^{-1} \boxplus 0, \dots e * s_p * \tilde{g}^{-1} \boxplus 0, 0 \boxplus g^{-1} * s_{p+1} * e, \dots 0 \boxplus g^{-1} * s_k * e) \quad (4.2)$$

for $1 \leq p \leq k-1$. We have to show the map above is well-defined regardless of the factorisation in G of $g\tilde{g}$. Since for each $k \geq 2$ the maps $\mathcal{M}^{(k)}$ are coassociative we know

$$\Delta_k\Big((\mathrm{id} \otimes \mathcal{M}^B) \circ \mathcal{M}^B, (\mathrm{id} \otimes \mathcal{M}^{(k)}) \circ \mathcal{M}^{(k)}\Big) = \\ \Delta_k\Big((\mathcal{M}^B \otimes \mathrm{id}) \circ \mathcal{M}^B, (\mathcal{M}^{(k)} \otimes \mathrm{id}) \circ \mathcal{M}^{(k)}\Big) \quad (4.3)$$

Observe that the above maps take arguments of the form

$$(s_1 \boxplus \tilde{s}_1 \boxplus \hat{s}_1, \dots, s_k \boxplus \tilde{s}_k \boxplus \hat{s}_k)$$

where $s_j \boxplus \tilde{s}_j \boxplus \hat{s}_j$ in in the fibre $\mathbf{S}_g G \boxplus \mathbf{S}_{\tilde{g}} G \boxplus \mathbf{S}_{\hat{g}} G$. In analogy with (3.2) the formulas for the above identity are

$$\Delta_k\Big((\mathrm{id} \otimes \mathcal{M}^B) \circ \mathcal{M}^B, (\mathrm{id} \otimes \mathcal{M}^{(k)}) \circ \mathcal{M}^{(k)}\Big)(s_1 \boxplus \tilde{s}_1 \boxplus \hat{s}_1, \dots s_k \boxplus \tilde{s}_k \boxplus \hat{s}_k) = \\ g * A^{(k)}_{(\tilde{g},\hat{g})}(\tilde{s}_1 \boxplus \hat{s}_1, \dots, \tilde{s}_k \boxplus \hat{s}_k) * e \\ + A^{(k)}_{(g,\tilde{g}\hat{g})}\Big(s_1 \boxplus (e * \tilde{s}_1 * \hat{g} + \tilde{g} * \hat{s}_1 * e), \dots, s_k \boxplus (e * \tilde{s}_k * \hat{g} + \tilde{g} * \hat{s}_k * e)\Big) \quad (4.4)$$

and

$$\Delta_k\Big((\mathcal{M}^B \otimes \mathrm{id}) \circ \mathcal{M}^B, (\mathcal{M}^{(k)} \otimes \mathrm{id}) \circ \mathcal{M}^{(k)}\Big)(s_1 \boxplus \tilde{s}_1 \boxplus \hat{s}_1, \dots s_k \boxplus \tilde{s}_k \boxplus \hat{s}_k) = \\ e * A^{(k)}_{(g,\tilde{g})}(s_1 \boxplus \tilde{s}_1, \dots, s_k \boxplus \tilde{s}_k) * \hat{g} \\ + A^{(k)}_{(g\tilde{g},\hat{g})}\Big((e * s_k * \tilde{g} + g * \tilde{s}_k * e) \boxplus \hat{s}_k, \dots, (e * s_k * \tilde{g} + g * \tilde{s}_k * e) \boxplus \hat{s}_k\Big) \quad (4.5)$$

And of course (4.5) and (4.4) are equal. Setting $\tilde{s}_j = 0$ for all $j \in 1, \dots, k$ it follows that (4.4) reduces to

$$g * A_{(\tilde{g},\hat{g})}(0 \boxplus \hat{s}_1, \dots, 0 \boxplus \hat{s}_k) * e + A^{(k)}_{(g,\tilde{g}\hat{g})}\Big(s_1 \boxplus (\tilde{g} * \hat{s}_1 * e), \dots, s_k \boxplus (\tilde{g} * \hat{s}_k * e)\Big) \quad (4.6)$$

whereas (4.5) becomes

$$e * A^{(k)}_{(g,\tilde{g})}(s_1 \boxplus 0, \dots, s_k \boxplus 0) * \hat{g} + A^{(k)}_{(g\tilde{g},\hat{g})}\Big((e * s_k * \tilde{g}) \boxplus \hat{s}_k, \dots, (e * s_k * \tilde{g}) \boxplus \hat{s}_k\Big) \quad (4.7)$$

and we know they are equal. These are computations completely analogous to those made in (3.2) and reduce to them when $k = 2$. Now observe that any two factorisations of a given element of G can be written as $g(\tilde{g}\hat{g}) = (g\tilde{g})\hat{g}$. In addition to setting all $\tilde{s}_j$ equal to zero pick $p \in \{1, \dots, k-1\}$ and set:

$$s_p = \cdots = s_k = 0 = \hat{s}_1 = \cdots = \hat{s}_p$$

Substituting the above into equations (4.6) and (4.7) the first two terms of them vanish and from the resulting equality follows that $B^{(p)}$ is well-defined.

We now want to show that all the $B^{(p)}$ are actually equal. For this purpose let $p \in \{2, \dots, k-1\}$ and set

$$\begin{gathered} s_p = \cdots = s_k = 0 = \hat{s}_1 = \cdots = \hat{s}_p \\ \tilde{s}_1 = \cdots = \tilde{s}_{p-1} = 0 = \tilde{s}_{p+1} = \cdots = \tilde{s}_k \end{gathered}$$

so (4.4) reduces to

$$\begin{aligned} A^{(k)}_{(g,\tilde{g}\hat{g})}\Big(\dots, s_{p-1} \boxplus 0, 0 \boxplus (e * \tilde{s}_p * \hat{g}), 0 \boxplus (\tilde{g} * \hat{s}_{p+1} * e), \dots\Big) &= \\ B^{(p-1)}_{g\tilde{g}\hat{g}}\big(\dots, e * s_{p-1} * \tilde{g}\hat{g}, g * \tilde{s}_p * \hat{g}, g\tilde{g} * \hat{s}_{p+1} * e, \dots\big) & \end{aligned}$$

whereas (4.5) results in

$$\begin{aligned} A^{(k)}_{(g\tilde{g},\hat{g})}\Big(\dots, (e * s_{p-1} * \tilde{g}) \boxplus 0, 0 \boxplus (g * \tilde{s}_p * e,)0 \boxplus \hat{s}_{p+1}, \dots\Big) &= \\ B^{(p)}_{g\tilde{g}\hat{g}}\big(\dots, e * s_{p-1} * \tilde{g}\hat{g}, g * \tilde{s}_p * \hat{g}, e * \hat{s}_{p+1} * \tilde{g}\hat{g} \dots\big) & \end{aligned}$$

which are equal by the coassociativity condition. It follows that $B^{(p)} = B^{(p-1)}$; since for each p the map $B^{(p)}$ is alternating in the first p arguments and the last $k-p$ arguments separately and

$$(\Lambda^p \mathbf{S}G \otimes \Lambda^{k-p}\mathbf{S}G) \cap (\Lambda^{p+1}\mathbf{S}G \otimes \Lambda^{k-p-1}\mathbf{S}G) = \Lambda^k \mathbf{S}G.$$

it follows that

$$B^* = B^{(1)} = \cdots = B^{(k-1)}$$

is completely alternating. Note that it is crucial that in the intersection argument above $k \geq 3$, otherwise the intersection makes no sense. This proves the claim. ■

Now we prove that $A^{(k)}$ is completely determined by B^*. Note that for $1 \leq p < k$ we have

$$\begin{aligned} A^{(k)}_{(g,\tilde{g})}(s_1 \boxplus 0, \dots, s_p \boxplus 0, 0 \boxplus \tilde{s}_{p+1}, \dots, 0 \boxplus \tilde{s}_k) &= \\ B^*_{g\tilde{g}}(e * s_1 * \tilde{g}, \dots, e * s_p * \tilde{g}, g * \tilde{s}_{p+1} * e, \dots, g * \tilde{s}_k * e) & \quad (4.8) \end{aligned}$$

where each s_j is on the fibre $\mathbf{S}_g G$ and each $\tilde{s}_l$ is on $\mathbf{S}_{\tilde{g}} G$. Therefore $A^{(k)}$ is completely determined by B^* up to terms of the form

$$A^{(k)}_{(g,\tilde{g})}\Big(s_1 \boxplus 0, \dots, s_k \boxplus 0\Big) \qquad \text{and} \qquad A^{(k)}_{(g,\tilde{g})}\Big(0 \boxplus s_1, \dots, 0 \boxplus s_k\Big)$$

due to the decomposition:

$$\Lambda^k(\mathbf{S}G \boxplus \mathbf{S}G) \cong \bigoplus_{p+q=k} (\Lambda^p \mathbf{S}G \boxtimes \Lambda^q \mathbf{S}G)$$

In order to deal with the terms above we substitute into (4.4) and (4.5) the conditions

$$s_1 = \cdots = s_k = 0 = \hat{s}_1 = \tilde{s}_2 = \cdots = \tilde{s}_k$$

to obtain (recall (4.4) equals (4.5))

$$A^{(k)}_{(g,\tilde{g}\hat{g})}\Big(0 \boxplus e * \tilde{s}_1 * \hat{g}, 0 \boxplus \tilde{g} * \hat{s}_2 * e, \dots, 0 \boxplus \tilde{g} * \hat{s}_k * e\Big) =$$
$$A^{(k)}_{(g\tilde{g},\hat{g})}\Big(g * \tilde{s}_1 * e \boxplus 0, 0 \boxplus \hat{s}_2, \dots, 0 \boxplus s_k\Big) - g * A^{(k)}_{(\tilde{g},\hat{g})}\Big(\tilde{s}_1 \boxplus 0, 0 \boxplus \hat{s}_2, \dots, 0 \boxplus \hat{s}_k\Big) * e$$

and note that the right hand side of the equality is completely determined by B^* by the proof of the claim above; this handles the terms of the form $A^{(k)}_{(g,\tilde{g})}(0 \boxplus s_1, \dots, 0 \boxplus s_k)$. The terms of the form $A^{(k)}_{(g,\tilde{g})}(s_1 \boxplus 0, \dots, s_k \boxplus 0)$ are taken care of by setting

$$\hat{s}_1 = \cdots = \hat{s}_k = 0 = s_1 = \tilde{s}_2 = \cdots = \tilde{s}_k$$

to obtain

$$A^{(k)}_{(g\tilde{g},\hat{g})}\Big(g * \tilde{s}_1 * e \boxplus 0, e * s_2 * \tilde{g} \boxplus 0, \dots, e * s_k * \tilde{g} \boxplus 0\Big) =$$
$$A^{(k)}_{(g,\tilde{g}\hat{g})}\Big(0 \boxplus e * \tilde{s}_1 * \hat{g}, s_2 \boxplus 0, \dots, s_k \boxplus 0\Big) - e * A\Big(0 \boxplus \tilde{s}_1, s_2 \boxplus 0, \dots, s_k \boxplus 0\Big) * \hat{g}$$

and again the right hand side is completely determined by B^*. Setting

$$\Delta_k(E_{k-1}, E_k) = -B^*$$

it follows that the multiplication map $\mathcal{M}^{(k)}$ is then in contact to order k to $\mathcal{M}^B$. □

Observe that the important step in the proof above is setting the first isomorphism E_1 to be an operator of order 0. This is always possible due to the supergeometric version of the Flowbox Theorem (Theorem 4.2 in [Gua16]) which completely characterises the possibilites of these choices.

Proof of theorem 4.2 from theorem 4.3. Let n be the odd dimension of $(G|\mathcal{R}G)$; this implies that for $r > n$ we have $\Lambda^r \mathbf{S}^* G = \{0\}$. Theorem 4.3 shows that it is possible to construct multiplications on $(G|\Lambda \mathbf{S}^* G)$ that are in arbitrary contact k to $\mathcal{M}^B$, so the definition of the contact relation implies $\mathcal{M}^{(n)} = \mathcal{M}^B$ and so we set $E = E_n$. The naturality of E follows from the fact that in theorem 4.3 the construction of each E_k depended only on the k-th contact class of $\mathcal{M}$ relative to $\mathcal{M}^B$ and these maps are given from the outset. □

Let $(M, \mathcal{O})$ be a supermanifold in the sense of Kostant and Leites (cf. [Gua16, Definition 1.4]). It is well-known that the realisation of a supermanifold as a pair $(M|\mathcal{R}M)$ is equivalent to choosing a section $\iota\colon \mathcal{C}^\infty \hookrightarrow \mathcal{O}$ from the sheaf of smooth functions on M to the structure sheaf $\mathcal{O}$ of the supermanifold so that the exact sequence

$$0 \longrightarrow \mathcal{N} \hookrightarrow \mathcal{O} \xrightarrow{\varepsilon} \mathcal{C}^\infty \longrightarrow 0$$

splits, where $\mathcal{N}$ is the nilpotent sheaf. All of our approach depends on the possibility of choosing this section appropriately (cf. [Gua16, Theorem 3.1]). An immediate consequence of our main theorem is that for Lie supergroups there is a natural section because there is a unique and natural isomorphism

$$E\colon \Gamma\left(\Lambda \mathbf{S}^* G\right) \to \Gamma\left(\mathcal{R}G\right)$$

In other words, a smooth real Lie supergroup, whether thought of as a pair $(G|\mathcal{R}G)$ or as $(G, \mathcal{O})$ is naturally isomorphic to a Lie supergroup of the form $(G|\Lambda\mathbf{S}^*G)$ with a twisted multiplication in the sense of § 3.1.

The structure result proved with everything done so far is the following:

4.4 Theorem. *Let G be a Lie group and $\mathcal{R}G$ a superalgebra bundle; suppose furthermore that $\mathcal{M}$ and $\mathcal{M}'$ are two multiplications supermaps on the supermanifold $(G|\mathcal{R}G)$ over the multiplication m of G. Then:*

1. *$\mathcal{M} \underset{1}{\sim} \mathcal{M}'$ (since the Lie superalgebra $(\mathfrak{g}|\mathfrak{s})$ is independent of either $\mathcal{M}$ or $\mathcal{M}'$).*
2. *$\mathcal{M} \underset{2}{\sim} \mathcal{M}'$ if and only if $\mathbf{B} = \mathbf{B}'$ (i.e. their auxiliary differentials are identical).*
3. *If $\mathcal{M} \underset{2}{\sim} \mathcal{M}'$ then there exists a unique and natural isomorphism*

$$(\mathrm{id}|\Phi) : \Big(\left(G|\mathcal{R}G\right), \mathcal{M}\Big) \to \Big(\left(G|\mathcal{R}G\right), \mathcal{M}'\Big)$$

such that $\Phi^{-1} \widehat{\otimes} \Phi^{-1} \circ \mathcal{M}' \circ \Phi = \mathcal{M}$.

The proof of the last part is identical with the proof of our Main Theorem since for $\mathcal{M}'$ one can construct a differential operator E' in the same way as E is constructed from $\mathcal{M}$, and both E and E' are isomorphisms from $\big(\left(G|\mathcal{R}G\right), \mathcal{M}\big)$ and $\big(\left(G|\mathcal{R}G\right), \mathcal{M}'\big)$ to $\big(\left(G|\Lambda\mathbf{S}^*G\right), \mathcal{M}^B\big)$ respectively. The resulting isomorphism $\Phi := E^{-1} \circ E'$ is then natural since both E and E' are by our Main Theorem. This yields the result.

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