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SAMPLE COPULA PROPERTIES AND WEAK CONVERGENCE OF THE SAMPLE PROCESS

T E S I S

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Introduction

The sample d-copula of order m is a new way to estimate a d copula, see [24], it is an estimator which is already a copula, unlike the empirical copula which is only a subcopula.

In this thesis we will study the main two properties of the sample d-copula, that is, a Glivenko-Cantelli's theorem and the asymptotic properties of its associated empirical process. The main objective of this work is to obtain the same results that exist for the empirical copula. The parameter m, which it is an integer with $2 \leq m \leq n$, where n is the sample size, is not difficult to estimate and in most cases it is less than n. Besides the evaluation of the sample copula is fairly simple and quicker than the empirical copula.

In Chapter 1 we given a review of the basic results of the theory of copulas and we describe the main definitions needed for the rest of the thesis.

In Chapter 2 we provide definitions and results corresponding to the sample and empirical copula and we give the proof of a version of the Glivenko-Cantelli's theorem for the sample d-copula. We also provide a large number of simulations to compare the approximation of the empirical copula and the sample d-copula to the real copula based on samples. At the end of this chapter we provide a method to estimate the order m, with $2 \leq m \leq n$, of the sample d-copula using the estimated value of the Sperman's rho in dimension two, and we also provide a methodology for higher dimensions.

In Chapter 3 we present the behavior of the random variables associated with the counting process generated by the sample d-copula. We obtain similar results to the ones found in [8] for the two dimensional case, and we extend the study of the these random variables for higher dimensions.

In Chapter 4 we found the weak convergence of the process generated by the sample d-copula using the convergence result of the empirical copula process. We obtain the weak convergence to a Gaussian process, and we evaluate its variance-covariance structure. Finally, in this chapter we provide several simulations to test the convergence of this process at each point.

In the final remarks we point out the advantages of using the sample d-copula instead of the empirical copula and we also give several possibles extensions and applications of the sample d-copula for real data.

1 Preliminaries

This chapter describes the basic results about copulas used in this work, we provide some examples related to the Archimedean copulas and the relations between copulas and other dependence measures. Most of the results presented here can be found in Nelsen's book [37].

1.1 Basic definitions and properties of copulas

Definition 1.1 *Let $S_1, S_2 \subset \overline{\mathbb{R}}$ be not empty sets, with $\overline{\mathbb{R}} = [-\infty, \infty]$ the extended real line. A function $H : S_1 \times S_2 \longrightarrow \mathbb{R}$ is* ***grounded*** *if there exists a least element $a_1 \in S_1$ and a least element $a_2 \in S_2$ such that $H(x, a_2) = H(a_1, y) = 0$ for all $(x, y) \in S_1 \times S_2$.*

Definition 1.2 *Let $S_1, S_2 \subset \overline{\mathbb{R}}$ be not empty sets, with $\overline{\mathbb{R}} = [-\infty, \infty]$ the extended real line. Let $B = [x_1, x_2] \times [y_1, y_2]$ be a box with vertices in the domain of the function H, we define the* ***H-volume of*** *B by*

$$V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1).$$

Definition 1.3 *Let $S_1, S_2 \subset \overline{\mathbb{R}}$ be not empty sets and let $H : S_1 \times S_2 \longrightarrow \mathbb{R}$ be a bivariate function. We say that H is* ***2-increasing*** *if $V_H(B) \geq 0$ for every box B with vertices in the domain of the H function.*

Definition 1.4 *A* ***two-dimensional subcopula*** *(or 2-subcopula, or briefly subcopula) is a function C' with the following properties*

1. *$Dom(C') = S_1 \times S_2$, where S_1 y S_2 are subsets of $\mathbf{I} = [0, 1]$ such that $0, 1 \in S_1$ and $0, 1 \in S_2$.*
2. *C' is a grounded function and is 2-increasing.*
3. *For all $u \in S_1$ and for all $v \in S_2$,*

$$C'(u, 1) = u \quad y \quad C'(1, v) = v.$$

The range of the function C' is a subset of $\mathbf{I} = [0, 1]$.

Definition 1.5 *A* ***two-dimensional copula*** *(or 2-copula, or briefly copula) is a 2-subcopula C with domain equal to $\mathbf{I}^2 = [0, 1]^2$. That is, a copula C is a function with domain $\mathbf{I}^2 = [0, 1]^2$ and range $\mathbf{I} = [0, 1]$ that satisfies the following properties*

1. *For all $u, v \in \mathbf{I}$*

$$C(u, 0) = 0 = C(0, v)$$

and

$$C(u, 1) = u \quad and \quad C(1, v) = v.$$

2. *For all* $u_1, u_2, v_1, v_2 \in \mathbf{I}$ *such that* $u_1 \le u_2$ *and* $v_1 \le v_2$

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0.$$

Example 1.6 *An example of a copula is the function* $\Pi^2 : [0,1]^2 \to [0,1]$ *defined by* $\Pi^2(u,v) = u \cdot v$*, this function satisfies the above conditions. If we define* $W^2(u,v) = \max\{0, u+v-1\}$ *and* $M^2(u,v) = \min\{u,v\}$ *then* M^2 *and* W^2 *are also copulas which satisfy that for every subcopula or copula* C *we have that*

$$W^2(u,v) \le C(u,v) \le M^2(u,v) \quad \textit{for every} \quad u,v \in [0,1].$$

The copulas W^2 *and* M^2 *are called the* ***lower and upper Fréchet-Hoeffding bounds***

Lemma 1.7 *For every* C *subcopula and for every* $u_1, u_2, v_1, v_2 \in [0,1]$ *we have that*

$$|C(u_1, v_1) - C(u_2, v_2)| \le |u_1 - u_2| + |v_1 - v_2|,$$

which guarantees that C *is uniformly continuous (Lipschitz).*

Following, the above definitions extends to the case n-dimensional.

Definition 1.8 *Let* $S_1, \cdots, S_n \subset \overline{\mathbb{R}}$ *be not empty sets, with* $\overline{\mathbb{R}} = [-\infty, \infty]$ *the extended real line. A function* $H : S_1 \times \cdots \times S_n \longrightarrow \mathbb{R}$ *is* ***grounded*** *if there exists a least element* $a_k \in S_k$ *for every* $k \in \{1, \cdots, n\}$*, such that* $H(\underline{t}) = 0$ *for all* $\underline{t} \in Dom(H)$*, with* $t_k = a_k$*, for at least one* $k \in \{1, \cdots, n\}$*.*

Definition 1.9 *Let* $S_1, \cdots, S_n \subset \overline{\mathbb{R}}$ *be not empty sets, with* $\overline{\mathbb{R}} = [-\infty, \infty]$ *the extended real line. Let* $H : S_1 \times \cdots \times S_n \longrightarrow \mathbb{R}$ *be a function and let* $B = [\underline{a}, \underline{b}]$ *be a n-box with vertices in the domain of the* H*, we define the* ***H-volume of*** B *by*

$$V_H(B) = \sum sgn(\underline{c}) H(\underline{c}), \tag{1}$$

where the sum is considered on all vertices $\underline{c}$ *of* B *and we define*

$$sgn(\underline{c}) = \begin{cases} 1 & \textit{if } c_k = a_k \textit{ for an even number of values } k \\ -1 & \textit{if } c_k = a_k \textit{ for an odd number of values } k. \end{cases}$$

Example 1.10 *Let* $H : \overline{\mathbb{R}}^3 \longrightarrow \mathbb{R}$ *be a function and let* $B = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ *be a* 3*-box. Then the* H*-volume of* B *is given by*

$$\begin{aligned} V_H(B) &= H(x_2, y_2, z_2) - H(x_2, y_2, z_1) - H(x_2, y_1, z_2) - H(x_1, y_2, z_2) \\ &\quad + H(x_2, y_1, z_1) + H(x_1, y_2, z_1) + H(x_1, y_1, z_2) - H(x_1, y_1, z_1). \end{aligned}$$

Definition 1.11 *Let $S_1, \cdots, S_n \subset \overline{\mathbb{R}}$ be not empty sets and let $H : S_1 \times \cdots \times S_n \longrightarrow \mathbb{R}$ be a n-variate function. We say that H is **n-increasing** if $V_H(B) \geq 0$ for all n-box B with vertices in the domain of the function H.*

Definition 1.12 *Let $S_1, \cdots, S_n \subset \overline{\mathbb{R}}$ be not empty sets and let $H : S_1 \times \cdots \times S_n \longrightarrow \mathbb{R}$ be a n-variate function. If every set S_k has a maximum element b_k, $k \in \{1, \cdots, n\}$, we say that the H function has marginals and we define the **k-th marginal function of one dimension of** H, denoted by $H_k : S_k \longrightarrow \mathbb{R}$ by*

$$H_k(x) = H(b_1, \cdots, b_{k-1}, x, b_{k+1}, \cdots, b_n).$$

for all $x \in S_k$. We can define marginals of higher order that one (k-marginals), setting fewer positions in the domain of H.

Definition 1.13 *A **n-dimensional subcopula** (or n-subcopula) is a function C' with the following properties*

1. *$Dom(C') = S_1 \times \cdots \times S_n$, with $S_k \subset \mathbf{I}$ such that $0, 1 \in S_k$ for every $k \in \{1 \cdots, n\}$.*
2. *C' is grounded and is n-increasing.*
3. *C' has marginals and each marginal C'_k, for every $k \in \{1 \cdots, n\}$, satisfies*

$$C'_k(u) = u \quad \textit{para toda } u \in S_k.$$

The range of the C' function is a subset of $\mathbf{I} = [0, 1]$.

Definition 1.14 *A **n-dimensional copula** (or n-copula) is a n-subcopula C with domain equal to $\mathbf{I}^n = [0, 1]^n$. That is, a n-copula C is a function with domain $\mathbf{I}^n$ and range $\mathbf{I}$ that satisfies the following properties*

1. *For all $\underline{u} \in \mathbf{I}^n$*

$$C(\underline{u}) = 0 \quad \textit{if at least one coordinate of } \underline{u} \textit{ is } 0$$

   and, if every coordinate of $\underline{u}$ are 1 except u_k, then $C(\underline{u}) = u_k$.

2. *For all $\underline{a}, \underline{b} \in \mathbf{I}^n$, such that $\underline{a} \leq \underline{b}$, that is, $a_k \leq b_k$ for every $k \in \{1, \cdots, n\}$, we have that*

$$V_C([\underline{a}, \underline{b}]) \geq 0.$$

Remark 1.15 *For all $2 \leq k < n$, every $k-$marginal function of C is a k-copula.*

Lemma 1.16 *Let* $C : [0,1]^d \to [0,1]$ *be a d-copula. We define for every* $\underline{u} = (u_1, \ldots, u_d) \in \mathbf{I}^d = [0,1]^d$

$$W^d(\underline{u}) = \max(0, \sum_{i=1}^{d} u_i - d + 1) \ \ \textit{and} \ M^d(\underline{u}) = \min(u_1, \ldots, u_d).$$

Then we have that for every d-copula or d-subcopula C ,

$$W^d(\underline{u}) \leq C(\underline{u}) \leq M^d(\underline{u}). \tag{2}$$

In this case M^d *is always a d-copula. However,* W^d *is never a d-copula for* $d > 2$*, because the volume of the box* $R = [1/2, 1]^d$ *is given by* $V_{W^d}(R) = 1 - (d/2) < 0$ *for every* $d > 2$*. However, the inequality (2) is sharp because for every* $\underline{u} \in [0,1]^d$ *there exists a d-copula C depending on* $\underline{u}$*, such that the left inequality in (2) becomes an equality.*

Lemma 1.17 *For every* $\underline{u}, \underline{v} \in [0,1]^d$ *and every d-subcopula C we have that*

$$|C(\underline{u}) - C(\underline{v})| \leq \sum_{i=1}^{d} |u_i - v_i|.$$

Again any d-copula C is a continuous (Lipschitz) function.

Theorem 1.18 (Sklar's Theorem) *Let H be a joint d-distribution function for* $d \geq 2$ *with margins* $F_1, F_2, \ldots, F_d$*. Then there exists a d-copula C such that for every* $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$*,*

$$H(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)). \tag{3}$$

If $F_1, F_2, \ldots, F_d$ *are continuous, then C is unique; otherwise, C is uniquely determined on* $Ran(F_1) \times Ran(F_2) \times \cdots \times Ran(F_d)$*. Conversely, if C is a d-copula and* $F_1, F_2, \ldots, F_d$ *are distribution functions, then the function H defined in equation (3) is a joint d-distribution function.*

Definition 1.19 *Given a copula C and* $a, b \in [0,1]$ *we define the* ***horizontal section of*** C ***at*** a *by* $h_a(t) = C(t, a)$ *and the* ***vertical section of*** C ***at*** b *by* $v_b(t) = C(b, t)$ *for every* $t \in [0,1]$*.*

Remark 1.20 *Using equation (1) we can see that the functions* $h_a(t)$ *and* $v_b(t)$ *are increasing functions.*

Definition 1.21 *We define the* ***diagonal section of*** C *by*

$$\delta_C(t) = C(t, t) \quad \textit{for every} \quad t \in [0,1].$$

Remark 1.22 *We can see that* $\delta_C(t)$ *is also an increasing function and, in the case of d-copulas C, with* $d > 2$*, we have equivalent definitions for each coordinate by fixing* $d-1$ *coordinates, and defining* $\delta_C(t) = C(t, t, \ldots, t)$*.*

Remark 1.23 *Given a d-copula C for any* $(v_1, \ldots v_{i-1}, v_{i+1}, \ldots, v_d) \in [0,1]^{d-1}$ *we have that*

$$\frac{\partial C}{\partial u_i}(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_d)$$

exists for almost every $u_i \in [0,1]$ *and for every* $i \in \{1, \ldots, d\}$*. In fact,*

$$0 \leq \frac{\partial}{\partial u_i} C(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_d) \leq 1.$$

The last inequality follows from the Lipschitz continuity of C. Furthermore the partials are defined and nondecreasing almost everywhere (a.e.) on $[0,1]^{d-1}$ *with respect to Lebesgue measure.*

The mixed partials of C also exist almost everywhere with respect to Lebesgue measure. In fact,

$$c(u_1, \ldots, c_d) = \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C(u_1, \ldots, u_d)$$

also exists a.e. and it is defined as the density function c of the distribution function C.

For example, in the case of Π^d *we have that its density is given simply by* $\pi^d(u_1, \ldots, u_d) = 1$ *for every* $(u_1, \ldots, u_d) \in [0,1]^d$*.*

Definition 1.24 *We define the* ***support of a d-copula*** *C by*

$$supp(C) = \{\underline{u} \in [0,1]^d \,|\, V_C(R_r) > 0 \;\; for\ every\ r > 0\},$$

where $R_r = [\underline{u} - \underline{r}, \underline{u} + \underline{r}] \subset [0,1]^d$ *and* $\underline{r} = (r, r, \ldots, r)$ *(d times).*

Example 1.25 *We have that the support of the copula* M^2 *is given by the main diagonal* $D = \{(u, v) \in [0,1]^2 \,|\, u = v\}$*. Similarly the support of the copula* W^2 *is the secondary diagonal* $D_1 = \{(u, v) \in [0,1]^2 \,|\, u + v = 1\}$*.*

Definition 1.26 *For any d-copula C we define*

$$C_{a.c.}(u_1, \ldots, u_d) = \int_0^{u_1} \cdots \int_0^{u_d} \frac{\partial^d}{\partial v_1 \cdots \partial v_d} C(v_1, \ldots, v_d) dv_d \ldots dv_1$$

where a.c. stands for absolutely continuous. Also define

$$C_{s.}(u_1, \ldots, u_d) = C(u_1, \ldots, u_d) - C_{a.c.}(u_1, \ldots, u_d),$$

where s. stands for singular.

If $C = C_{a.c.}$ we say that C is ***absolutely continuous****, if $C = C_{s.}$ we say C is* ***singular****, in any other case C is* ***hybrid****.*

We can observe that $C_{s.}(1, \ldots, 1)$ is the measure of the singular part if it exists.

1.2 Archimedean copulas and types of dependence

Definition 1.27 *Let $\varphi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing, convex function, that is, if $u, v \in [0,1]$ and $0 \leq \alpha \leq 1$ then $\varphi(\alpha u + (1-\alpha)v) \leq \alpha\varphi(u) + (1-\alpha)\varphi(v)$, such that $\varphi(1) = 0$, in this case we will call φ a* ***generator****. Then, if we define*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{if} \quad 0 \leq t \leq \varphi(0) \\ 0 & \text{if} \quad \varphi(0) \leq t \leq \infty, \end{cases}$$

called ***pseudo-inverse of*** *φ, and we define*

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \ \text{ for every } (u,v) \in [0,1]^2. \tag{4}$$

Then C is always a copula with generator φ, φ is ***strict*** *if $\varphi(0) = \infty$ and* ***non-strict*** *if $\varphi(0) < \infty$. The copulas given in equation (4) are called* ***Archimedean copulas****.*

Lemma 1.28 *Let φ be a generator of C as in equation (4). Then*
i) C is ***symmetric****, that is, $C(u,v) = C(v,u)$ for every $(u,v) \in [0,1]^2$.*
ii) C is ***associative****, that is, $C(C(u,v),w) = C(u, C(v,w))$ for every $u, v, w \in [0,1]$.*
iii) If $c > 0$ then $\psi = c \cdot \varphi$ is also a generator of C.

Theorem 1.29 *C is an Archimedean copula if and only if C is an associative copula such that $\delta_C(t) = C(t,t) < t$, for every $t \in (0,1)$.*

Remark 1.30 *We have the following observations:*
a) If we define $\varphi(t) = -\ln(t)$, then φ is a strict generator with $\varphi^{[-1]}(t) = \varphi^{-1}(t) = \exp(-t)$, and using equation (4) we have that Π^2 is an Archimedean copula.
b) If we define $\varphi(t) = 1 - t$, then φ is a non-strict generator with $\varphi^{[-1]}(t) = \max\{1-t, 0\}$. Hence, W^2 is also Archimedean.
c) However, using Theorem 1.29, M^2 is not Archimedean because $\delta_{M^2}(t) = t$ for every $t \in [0,1]$.

Remark 1.31 *Several families of copulas used in practice for modeling are Archimedean among them we have:*

1. ***Clayton Family**, with generator* $\varphi(t) = \max\{0, \frac{1}{\theta}(t^{-\theta} - 1)$*, where* $\theta \in [-1, \infty)\backslash\{0\}$.

2. ***Ali-Mikhail-Haq (AMH)**, with generator* $\varphi(t) = \ln\left(\frac{1-\theta(1-t)}{t}\right)$ *where* $\theta \in [-1, 1)$.

3. ***Gumbel**, with generator* $\varphi(t) = (-\ln(t))^{\theta}$ *where* $\theta \in [1, \infty)$.

4. ***Frank**, with generator* $\varphi(t) = -\ln\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right)$ *where* $\theta \in (-\infty, \infty)\backslash\{0\}$.

Definition 1.32 *Let* $U_n = \{(x_1, y_1), \ldots (x_n, y_n)\}$ *be a random sample from a bivariate continuous vector* (X, Y)*, for every* $1 \leq i < j \leq n$ *we say that* (x_i, y_i) *and* (x_j, y_j) *are **concordant** if and only if* $(x_i - x_j)(y_i - y_j) > 0$ *and they are **discordant** if* $(x_i - x_j)(y_i - y_j) < 0$*. The **sample Kendall's tau** is defined as*

$$\tau = \frac{c - d}{c + d} = (c - d)/\binom{n}{2},$$

*where c and d are the number of pairs which are concordant and discordant respectively. It can be thought as the probability of concordance minus the probability of discordance. Therefore, the **population version of Kendall's tau** is defined by*

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0],$$

where (X_1, Y_1) *and* (X_2, Y_2) *are independent and identically distributed random vectors with common joint distribution* F.

Theorem 1.33 *Let* (X_1, Y_1) *and* (X_2, Y_2) *be independent vectors of continuous random variables with joint distributions* H_1 *and* H_2*, respectively, with common margins* F_1 *of (*X_1 *and* X_2*) and* F_2 *of (*Y_1 *and* Y_2*). Let* C_1 *and* C_2 *be the respective copulas of* (X_1, Y_1) *and* (X_2, Y_2)*. Let*

$$Q = Q(C_1, C_2) = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Then

$$Q(C_1, C_2) = 4 \int\int_{[0,1]^2} C_2(u, v) dC_1(u, v) - 1. \tag{5}$$

Remark 1.34 *We can see that* $Q(C_1, C_2) = Q(C_2, C_1)$*. We also have that* $Q(M^2, M^2) = 1$*,* $Q(M^2, \Pi^2) = 1/3$*,* $Q(M^2, W^2) = 0$*,* $Q(W^2, \Pi^2) = -1/3$*,* $Q(W^2, W^2) = -1$ *and* $Q(\Pi^2, \Pi^2) = 0$.

Lemma 1.35 *If X and Y are continuous random variables with copula C. Then the population version of Kendall's tau is given by*

$$\tau_{X,Y} = Q(C,C) = 4\int\int_{[0,1]^2} C(u,v)dC(u,v) - 1 = 4E(C(U,V)) - 1.$$

It also denoted by τ_C. Hence, we have that $\tau_{M^2} = 1$, $\tau_{\Pi^2} = 0$ and $\tau_{W^2} = -1.7$

In fact, if $C_1(u,v) \le C_2(u,v)$ for every $(u,v) \in [0,1]^2$ then $\tau_{C_1} \le \tau_{C_2}$, and using the Fréchet-Hoeffding bounds we have that $-1 \le \tau_C \le 1$ for every copula C.

Example 1.36 *For the Clayton family C_θ with parameter $\theta \ge -1$, we have that $\tau_{C_\theta} = \theta/(\theta+2)$. Another example is the Gumbel family with parameter $\theta \ge 1$, in this case, we have that $\tau_{C_\theta} = (\theta-1)/\theta$.*

Example 1.37 *If C is Archimedean with generator φ then we can find Kendall's tau using the formula*

$$\tau_C = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt.$$

Definition 1.38 *Another measure of association is* ***Sperman's rho****. Let (X_1,Y_1), (X_2,Y_2) and (X_3,Y_3) be three independent random vectors with common continuous joint distribution F with margins F_1 and F_2. The* ***population version of Spearman's rho*** *is defined as proportional to the probability of concordance minus the probability of discordance of the vectors (X_1,Y_1) and (X_2,Y_3). In fact,*

$$v\rho = \rho_{X,Y} = 3(P[(X_1-X_2)(Y_1-Y_3) > 0] - P[(X_1-X_2)(Y_1-Y_3) < 0]).$$

Therefore, using (5) we have that if X and Y are continuous random variables with copula C then

$$\begin{aligned}\rho_{x,y} = \rho_C &= 3Q(C,\Pi^2) \\ v = &\ 12\int\int_{[0,1]^2} uvdC(u,v) - 3 \\ &= 12\int\int_{[0,1]^2} C(u,v)dudv - 3 \qquad (6)\end{aligned}$$

Example 1.39 *We have that convex combinations of copulas are copulas. Then, we can consider the Fréchet family of copulas defined as $C_{\alpha,\beta}(u,v) = \alpha\cdot M^2(u,v) + (1-\alpha-\beta)\cdot\Pi^2(u,v) + \beta\cdot W^2(u,v)$ for every $\alpha,\beta \ge 0$ such that $\alpha+\beta \le 1$ and for every $(u,v) \in [0,1]^2$. Using the previous results, we have that $Q(C_{\alpha,\beta}) = \alpha Q(M^2,\Pi^2) + (1-\alpha-\beta)Q(\Pi^2,\Pi^2) + \beta Q(W^2,\Pi^2)$. So,*

$$\rho_{C_{\alpha,\beta}} = \alpha - \beta.$$

We also observe that from the previous observations we have that

$$-1 \leq \rho_C \leq 1 \quad \textit{for every copula } C.$$

Remark 1.40 *If U and V are uniform* $(0,1)$ *random variables with copula C, then*

$$\begin{aligned}
\rho_C &= 12\int\int_{[0,1]^2} uvdC(u,v) - 3 \\
&= 12E(UV) - 3 \\
&= \frac{E(UV) - 1/4}{1/12} \\
&= \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)}\sqrt{Var(V)}}.
\end{aligned}$$

Here, we used that $E(U) = 1/4$ *and* $Var(U) = 1/12$. *Hence, Sperman's rho for a pair of continuous random variables X and Y is identical to Pearson's correlation coefficient for the random variables* $U = F_1(X)$ *and* $V = F_2(Y)$.

Lemma 1.41 *If X and Y are continuous random variables, then*

$$-1 \leq 3\tau - 2\rho \leq 1.$$

$$\frac{1+\rho}{2} \geq \left(\frac{1+\tau}{2}\right)^2 \quad \textit{and} \quad \frac{1-\rho}{2} \geq \left(\frac{1-\tau}{2}\right)^2.$$

For $\tau \geq 0$

$$\frac{3\tau - 1}{2} \leq \rho \leq \frac{1 + 2\tau - \tau^2}{2},$$

and if $\tau \leq 0$

$$\frac{\tau^2 + 2\tau - 1}{2} \leq \rho \leq \frac{1 + 3\tau}{2}.$$

Definition 1.42 *Let X and Y be random variables. Then X and Y are* ***positively quadrant dependent*** *if for every* $(x, y) \in \mathbb{R}^2$

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y). \tag{7}$$

Equivalently,

$$P(X > x, Y > y) \geq P(X > x)P(Y > y).$$

If equation (7) holds we will write $PQD(X, Y)$. *In terms of copulas (7) can be written as*

$$C(u, v) \geq uv = \Pi^2(u, v) \quad \textit{for every} \quad (u, v) \in [0, 1]^2.$$

Negative quadrant dependence *is defined analogously, by reversing the inequalities. If* $PQD(X, Y)$ *then*

$$3\tau_{X,Y} \geq \rho_{X,Y} \geq 0.$$

Definition 1.43 *Let X, Y be two random variables, we will say that Y is **left tail decreasing in** X, denoted* $LTD(Y|X)$ *if and only if* $P(Y \leq y|X \leq x)$ *is a decreasing function of x for all y, equivalently, if and only if* $C(u, v)/u$ *is decreasing in u or if and only if* $\partial C(u, v)/\partial u \leq C(u, v)/u$ *for almost every u.*

*We will say that Y is **right tail increasing in** X, denoted* $RTI(Y|X)$ *if and only if* $P(Y > y|X > x)$ *is a increasing function of x for all y or equivalently, if and only if* $1 - u - v + C(u, v)/(1 - u)$ *is decreasing in u or if and only if* $\partial C(u, v)/\partial u \geq [v - C(u, v)]/(1 - u)$ *for almost every u.*

$LTD(X|Y)$ $RTI(X|Y)$ *are defined by interchanging X and Y and we have that tail monotonicity implies PQD.*

We will use later on the Spearman's rho to define a methodology to establish the order of the sample copula in the two-dimensional case.

2 Comparison between the empirical copula and the sample d-copula of order m

The d-sample copula of order m, C_n^m, is a d-copula which is a sample estimator of $C^{(m)}$ the checkerboard approximation of order m of a given d-copula C, see [36] and [9], as we will see the estimator C_n^m approaches $C^{(m)}$ as the sample size n increases. If m is relatively large, $C^{(m)}$ is a very good approximation of C. Hence, C_n^m can be thought as a quasi-nonparametric method to estimate the real d-copula C, and it becomes a nonparametric estimator when we choose the order m.

In this Chapter we make an extensive comparison of the supremum distance between the empirical copula C_n and the real copula C, and the supremum distance between the real copula and the sample copula of order m, C_n^m which is simply the multilinear interpolation used in the proof of Sklar's Theorem, based on a sample of size n and a regular partition of order m in m^d d-boxes of Lebesgue measure $1/m^d$ of $[0,1]^d$. The same partition is used to define $C^{(m)}$ the checkerboard approximation. We used different samples sizes n, and we consider a large class of frequently used families of copulas, and some interesting families of singular copulas.

We simulated a large number of samples of each copula with sample sizes $n = 20$, $n = 30$ and $n = 50$, and we obtain the basic statistics of the supremum distance when we vary m from 2 up to n. We observed that always there exist values of m, in general far smaller that n, such that the supremum distance between the sample d-copula of order m and the real copula C gives better approximations than using the supremum distance between the empirical copula and the real copula. We also prove a Glivenko-Cantelli's Theorem for the sample copula, and we provide a method to estimate the value of m such that C_m^n the sample d-copula of order m is a good approximation of the real d-copula C based on the simulations.

In the last section we give some remarks and observations which include an important comment on why the case $m = n$ is not a good option. We also see that we can easily simulate from the sample copula C_n^m, and that these simulated samples are quite similar to the original sample. On the other hand, we give strong evidence that it is possible to obtain a Glivenko-Cantelli's Theorem for the total variation distance for the checkerboard approximation $C^{(m)}$ and the sample copula C_n^m.

2.1 Definitions and results about the empirical copula

We start this section by recalling the principal results about the empirical copulas.

Definition 2.1 (Rank Function) *Let $X_1, \cdots, X_n$ be a random sample of size n of a continuous random variables X and let $X_{(1)}, \cdots, X_{(n)}$ be their order statistics. The rank function $r : \{1, \ldots, n\} \times$*

$\mathbb{R}^n \rightarrow \{1, \ldots, n\}$ *is defined by*

$$r(j, X_1, \cdots, X_n) = k, \quad \textit{if and only if } X_j = X_{(k)} \textit{ where } j, k \in \{1, \ldots, n\}.$$

Definition 2.2 (Modified Sample or Pseudosample) *Let $\underline{X}_1, \cdots, \underline{X}_n$ be a random sample of size n of a continuous random vector $\underline{X}$ of dimension d, where $\underline{X}_i = (X_{i,1}, \ldots, X_{i,d}) \in \mathbb{R}^d$, for every $i = 1, \ldots, n$. Let $i \in I_n$, the i-th modified sample $\underline{Y}_i = (Y_{i,1}, \cdots, Y_{i,d})$, is defined by*

$$Y_{i,j} = \frac{1}{n} r(i, X_{1,j}, \ldots, X_{n,j}) \textit{ for every } j \in I_d.$$

Here we observe that the modified sample $\{\underline{Y}_1, \ldots, \underline{Y}_n\}$ is always a subset of $\mathbf{I}^d$

Definition 2.3 (Empirical Copula) *Let $\underline{X}_1, \cdots, \underline{X}_n$ be a random sample of size n of a random vector $\underline{X}$ of dimension d, with continuous joint distribution H, where $\underline{X}_i = (X_{i,1}, \cdots, X_{i,d}) \in \mathbb{R}^d$, for every $i = 1, \ldots, n$. Let $\underline{Y}_1, \cdots, \underline{Y}_n$ be the corresponding modified sample. We define the empirical copula denoted by $C_n : \mathbf{I}^d \rightarrow \mathbf{I}$ by*

$$C_n(u_1, \ldots, u_d) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{Y_{i,1} \le u_1, \ldots, Y_{i,d} \le u_d\}}(u_1, \ldots, u_d) \textit{ for every } (u_1, \ldots, u_d) \in \mathbf{I}^d. \tag{8}$$

Remark 2.4 *The empirical copula C_n is an approximation of the real copula C. The empirical copula C_n given in equation (8) has jumps of magnitude $1/n$ at each $\underline{Y}_i$ of the modified sample for every $i \in I_n$ almost surely. Hence, C_n is not continuous, and therefore C_n is not a d-copula. However, C_n is a d-subcopula if we restrict the domain to be T^d, where $T = \{0, 1/n, 2/n, \ldots (n-1)/n, 1\}$. This follows easily by observing that from the continuity of the joint distribution function H of the random vector $\underline{X}$, the ranks in each coordinate vary from one to n. Using Sklar's Theorem for the continuous joint distribution H in Definition 2.3 there exists a unique d-copula C such that equation (3) holds.*

If we are sampling from a d-copula C instead of a joint distribution function H the Definition 2.3 of the empirical copula still holds using modified samples.

A very important result about empirical copulas is the Glivenko-Cantelli's Theorem which states that the empirical copula C_n approaches the real copula C in supremum norm almost surely.

Theorem 2.5 (Glivenko-Cantelli) *Let C_n the empirical copula constructed from a sample of size n of a continuous joint distribution H with d-copula C, or from a d-copula C, as in Remark (2.4). Then*

$$\lim_{n \to \infty} \sup_{(u_1, u_2, \ldots, u_d) \in \mathbf{I}^d} |C_n(u_1, u_2, \ldots, u_d) - C(u_1, u_2, \ldots, u_d)| = 0 \quad \textit{almost surely.} \tag{9}$$

It is quite important to observe here, that the empirical copula has been used extensively in statistical applications to model multivariate data, see for example [4], [5], [6], [7], [10], [12], [14], [16], [18], [19], [20], [22], [23], [28], [30], [31], [39], [42], [45] and [48], just to cite some of them. However, as observed above the empirical d-copula is not a d-copula. In order to correct this problem some authors have proposed some modifications of the empirical copulas such as the linear B-spline copulas, see [43] and [17]. We will see later on that this approximation corresponds to our sample d-copula of order $m = n$, but we will also see that, in many instances, this approximation of the real copula does not improve the approximation given by the empirical copula. Another well known approximation for a copula is the Bernstein copula, which is based on polynomial approximations of a d-copula, see for example [41] or [36]. We will see that our proposal is far easier to implement specially in higher dimensions and even with large sample sizes.
The empirical d-copula , based on the theory of empirical processes includes two important results which justify its use: The first one is the Glivenko-Cantelli Theorem, stated above, which states that for n large enough it approximates the real copula almost surely, and the second one is the asymptotic theory which states that the normalized process converges to a Gaussian process with a given covariance structure, see for example [8], [13] and [44]. As we will see in this paper, using the checkerboard approximation of order m, see [36], which is a very good approximation of a d-copula C, that when m increases, approaches rapidly C, we can obtain the Glivenko-Cantelli's result for the sample d-copula of order m. See also, [9] and [35].

First, we observe that in dimension one, if we have a univariate distribution function F and a random sample $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ from F of size n. We know that the **empirical distribution function** is defined by

$$F_n(x) = (1/n)\sum_{i=1}^{n} 1_{\{X_i \leq x\}} \quad \text{for every} \quad x \in \mathbb{R}.$$

If F is continuous then with probability one F_n has jumps of magnitude $1/n$ at each point X_i of the sample. If we let F_0 denote the distribution function of the constant random variable $X_0 = 0$, and taking order statistics we can assume that the sample satisfies $-\infty < X_1 < X_2 < \cdots < X_n < \infty$, then $F_n(x) = \sum_{i=1}^{n}(1/n)F_0(x - X_k)$ for every $x \in \mathbb{R}$. So, if we assume that F is continuous, it is easy to see that

$$\begin{aligned} \sup_{x \in I\!R} |F_n(x) - F(x)| &= \max_{1\leq k\leq n}\left(\max\left(|F_n(X_k-) - F(X_k)|, |F_n(X_k) - F(X_k)|\right)\right) \\ &= \max_{1\leq k\leq n}\left(\max\left(\left|\tfrac{k-1}{n} - F(X_k)\right|, \left|\tfrac{k}{n} - F(X_k)\right|\right)\right). \end{aligned} \tag{10}$$

On the other hand, we have that for every $k \in I_n$, $\min(\max(|(k-1)/n - F(X_k)|, |k/n - F(x_k)|))$ is attained when $F(X_k) = (2k-1)/2n$, and in this case $\min(|(k-1)/n - F(X_k)|, |k/n - F(x_k)|) =$

$\max(|(k-1)/n - F(X_k)|\,,\ |k/n - F(x_k)|) = 1/2n$ for every $k \in I_n$. So, if we define $X_k = F^{-1}((2k-1)/2n)$ for every $k \in I_n$, we have, using equation (10), that

$$\sup_{x \in I\!R} |F_n(x) - F(x)| = \frac{1}{2n}. \tag{11}$$

Therefore, we have proved the following Lemma, which we could not find a reference for it

Lemma 2.6 *Let F be a univariate continuous distribution function and let $X = \{X_1, X_2, \ldots, X_n\}$ be a random sample of size $n \geq 1$ from F. Then*

$$\sup_{x \in I\!R} |F_n(x) - F(x)| \geq \frac{1}{2n} \quad a.s.[P_F],$$

where P_F is the probability measure induced by F on $I\!R$.

For an upper bound on the tail probabilities we have the Dvoretzky-Kiefer-Wolfowitz inequality, improved by Massart, see [11] and [34], which states that for every $\epsilon > 0$ and for every $n \geq 1$

$$P\left(\sup_{x \in I\!R} |F_n(x) - F(x)| > \epsilon\right) \leq 2e^{-2n\epsilon^2}. \tag{12}$$

If we take $\epsilon = 1/(2n)$ in (12) we observe that that the minimum of the right hand side is attained at $n = 1$ where it takes the value $2\exp(-1/2) = 1.2103 > 1$ which agrees with Lemma 2.6

Let us return to the case $d \geq 2$, let H be a continuous joint distribution d-dimensional and let C the unique d-copula given in equation (3) of Sklar'sTheorem. Let $\underline{X}_1, \cdots, \underline{X}_n$ be a random sample of size n of a random vector $\underline{X}$ of dimension d, with continuous joint distribution H, and let $\underline{Y}_1, \cdots, \underline{Y}_n$ be the corresponding modified sample. Define the empirical copula C_n as in equation (8), then as observed in Remark 2.4 we have that $C_n : T^d \to [0, 1]$, where $T = \{0, 1/n, \ldots, (n-1)/n, 1\}$ is a d-subcopula, but not a d-copula when defined on $\mathbf{I}^d$. If $0 < \epsilon < 1/n$, since C is a d-copula then $C(\epsilon, 1, \ldots, 1) = \epsilon$. However, $C_n(\epsilon, 1, \ldots, 1) = 0$, because $\epsilon < 1/n$. So, letting ϵ approach $1/n$ from the left we have that

$$\lim_{\epsilon \uparrow (1/n)} |C(\epsilon, 1, \ldots, 1) - C_n(\epsilon, 1, \ldots, 1)| = 1/n.$$

Therefore we have proved the following:

Lemma 2.7 *Let H be a continuous joint distribution d-dimensional for $d \geq 2$, let C the unique d-copula given in equation (3) of Sklar'sTheorem. Let $\underline{X}_1, \cdots, \underline{X}_n$ be a random sample of size n of* *a random vector $\underline{X}$ of dimension d, with continuous joint distribution H, and let $\underline{Y}_1, \cdots, \underline{Y}_n$ be the corresponding modified sample. Define the empirical copula C_n as in equation (8). Then*

$$\sup_{(x_1,x_2,\ldots,x_d)\in\mathbf{I}^d} |C(x_1,\ldots,x_d) - C_n(x_1,\ldots,x_d)| \geq \frac{1}{n} \quad a.s.[P_C], \tag{13}$$

where P_C is the probability function induced by the copula C on $\mathbb{R}^d$.

2.2 Sample d-copula of order m

We start this section by defining the concept of generalized transformation matrix given in [15] and extended in [46] in the construction of fractal copulas.

Definition 2.8 *Let $I_n = \{1, 2, \cdots, n\}$, with $n \geq 1$. For dimension $d \geq 2$, let $m \in \mathbb{N}$, we define $I_m^d = \times_{i=1}^d I_m$. Let τ a probability measure in $(I_m^d, 2^{I_m^d})$, τ is known as a generalized transformation matrix if for all $j \in \{1, \cdots, d\}$ and for all $k \in \{1, \cdots, m\}$*

$$\sum_{\underline{i}\in I_m^d, i_j=k} \tau(\underline{i}) > 0$$

where $\underline{i} = (i_1, \cdots, i_{j-1}, i_j = k, i_{j+1}, \cdots, i_d) \in I_m^d$. τ can be thought as a d-dimensional matrix τ, considering

$$\tau(\underline{i}) = \tau_{i_1,\cdots,i_d} \quad if \quad \underline{i} = (i_1, \cdots, i_d) \in I_m^d.$$

Example 2.9 *Let $d = 2$, $m = 3$. Then*

$$I_3^2 = \{1, 2, 3\}^2.$$

We define two probability measures τ_1 and τ_2 in $(I^2, 2^{I^2})$ given by the following matrices

$$\tau_1 = \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \quad and \quad \tau_2 = \begin{pmatrix} 1/6 & 0 & 1/6 \\ 1/6 & 0 & 1/6 \\ 2/6 & 0 & 0 \end{pmatrix}.$$

We can see by adding the elements in each row and column that τ_1 is a transformation matrix, but τ_2 is not since the sum of the entries in the second column equals zero.

Definition 2.10 *Let $\tau = (\tau_{i,j})_{i,j\in\{1,\cdots,m\}}$ be a generalized transformation matrix where $d = 2$. Define $\{q_{1,0}, q_{1,1}, \cdots, q_{1,m}\}$ and $\{q_{2,0}, q_{2,1}, \cdots, q_{2,m}\}$ two partitions of $[0, 1]$, such that $q_{1,0} = q_{2,0} = 0$ and for $i, j \in I_m$ we have that*

$$q_{1,i} = \sum_{i'=1}^{i} \sum_{j\in I_m} \tau_{i',j} \quad and \quad q_{2,j} = \sum_{j'=1}^{j} \sum_{i\in I_m} \tau_{i,j'},$$

we also define the partition induced by τ on $\mathbf{I}^2$ by

$$Q^m_{i,j} = \langle q_{1,i-1}, q_{1,i}] \times \left\langle q_{2,j-1}, q_{2,j}\right] \quad \textit{for every} \quad (i,j) \in I_m \times I_m,$$

where the $\langle$ notation indicates that the left end of the interval is closed if $i = 1$ or $j = 1$, and open in any other case. Let Π^2 be the product 2-copula, we define the $\tau(\Pi^2)$ transformation by

$$\begin{aligned}\tau(\Pi^2)(u,v) \;\; &= \sum_{i'<i,j'<j} \tau_{i',j'} + \frac{u - q_{1,i-1}}{q_{1,i} - q_{1,i-1}} \sum_{j'<j} \tau_{i,j'} + \frac{v - q_{2,j-1}}{q_{2,j} - q_{2,j-i}} \sum_{i'<i} \tau_{i',j} \\ &\quad + \tau_{i,j}\Pi^2\left(\frac{u - q_{1,i-1}}{q_{1,i} - q_{1,i-1}}, \frac{v - q_{2,j-1}}{q_{2,j} - q_{2,j-1}}\right) \qquad (14)\end{aligned}$$

with $u, v \in Q^m_{i,j}$ for every $i, j \in I_m$.

It is easy to see that $\tau(\Pi^2)$ is always a 2-copula. Evenmore, it is not difficult to see that equation (14) coincides exactly with equation (2.3.2) in Lemma 2.3.5 in the proof of Sklar's Theorem in [37], where $a_1 = q_{1,i-1}$, $a_2 = q_{1,i}$, $b_1 = q_{2,j-1}$, $b_2 = q_{2,j}$, $\lambda_1 = (u - q_{1,i-1})/(q_{1,i} - q_{1,i-1})$, $\mu_1 = (v - q_{2,j-1})/(q_{2,j} - q_{2,j-1})$, $C''(a_1, b_1) = \sum_{i'<i,j'<j} \tau_{i',j'}$, $C''(a_1, b_2) = \sum_{i'<i,j'\le j} \tau_{i',j'}$, $C''(a_2, b_1) = \sum_{i'\le i,j'<j} \tau_{i',j'}$ and $C''(a_2, b_2) = \sum_{i'\le i,j'\le j} \tau_{i',j'}$. Hence equation (14) is simply a bilinear interpolation. This definition can be extended to dimension $d > 2$ using the product d-copula Π^d with a d-linear interpolation.

Definition 2.11 *Let $m \geq 2$ and let $\tau = (\tau_{i_1,\cdots,i_d})_{(i_1,\cdots,i_d)\in I^d_m}$ be a generalized transformation matrix. We define $q_{1,0} = q_{2,0} = \cdots = q_{d,0} = 0$, and for every $j \in \{1, \cdots, d\}$ and for every $k \in \{1, \cdots, m\}$*

$$q_{j,k} = \sum_{i_j=1}^{k} \sum_{i_1=1}^{m} \cdots \sum_{i_{j-1}=1}^{m} \sum_{i_{j+1}=1}^{m} \cdots \sum_{i_d=1}^{m} t_{i_1,\cdots,i_{j-1},i_j,i_{j+1},\cdots,i_d}.$$

Then $0 = q_{j,0} < q_{j,1} < \cdots < q_{j,m-1} < q_{j,m} = 1$ is a partition of the $[0,1]$ interval, induced for the matrix τ in the j-coordinate. For every $\underline{i} = (i_1, \ldots, i_d) \in I^d_m$ we define

$$Q^m_{\underline{i}} = \langle q_{1,(i_1-1)}, q_{1,i_1}] \times \langle q_{2,(i_2-1)}, q_{2,i_2}] \times \cdots \times \langle q_{d,(i_d-1)}, q_{d,i_d}]. \qquad (15)$$

Then the family $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m}$ is a partition of $\mathbf{I}^d$.

Now we can give the definition of the sample copula of order m, based on a random sample of size n, where $2 \leq m \leq n$, coming from a continuous joint d-distribution function H or a d-copula C.

Definition 2.12 (Sample Copula of order m**)** *Let* $2 \le m \le n$ *and let* $\underline{X}_1, \ldots, \underline{X}_n$ *be a random sample of size n of a random vector* $\underline{X}$ *of dimension d, with continuous joint distribution H or d-copula C, where* $\underline{X}_i = (X_{i,1}, \cdots, X_{i,d}) \in I\!R^d$*, for every* $i = 1, \ldots, n$*. Let* $U_n = \{\underline{Y}_1, \ldots, \underline{Y}_n\}$ *be the corresponding modified sample.*
Define the uniform partition of size m of $\mathbf{I}^d$*, where for every* $\underline{i} = (i_1, \ldots, i_d) \in I_m^d$

$$R_{\underline{i}}^m = \left\langle \frac{i_1 - 1}{m}, \frac{i_1}{m} \right] \times \cdots \times \left\langle \frac{i_d - 1}{m}, \frac{i_d}{m} \right]. \tag{16}$$

Define

$$s_{i_1,\ldots,i_d}^{n,(m)} = \frac{card(R_{\underline{i}}^m \cap U_n)}{n} \tag{17}$$

where $card(\cdot)$ *denotes the cardinality of a set. Let*

$$S_m^n = (s_{i_1,\ldots,i_d}^{n,(m)})_{(i_1,\ldots,i_d) \in I_m^d}, \tag{18}$$

then S_m^n *is always a d-dimensional generalized transformation matrix. Let* $(Q_{\underline{i}}^m)_{\underline{i} \in I_m^d}$ *be the partition of* $\mathbf{I}^d$ *induced by the generalized transformation matrix* S_m^n *given in equation (15). Using the partition* $(Q_{\underline{i}}^m)_{\underline{i} \in I_m^d}$*, we define the sample d-copula of order m by*

$$C_m^n(u_1, \ldots, u_d) = S_m^n(\Pi^d)(u_1, \ldots, u_d), \tag{19}$$

as in the generalization of equation (14), where Π^d *is the product copula in* $[0, 1]^d$*.*

To clarify this definition we give a simple example

Example 2.13 *We took a sample of size* $n = 4$ *from a bivariate normal distribution with mean* $\mu = (0, 0)$ *and correlation coefficient* $\rho = 0.5$*. The observations were* $\underline{X}_1 = (0.662, 0.895)$*,* $\underline{X}_2 = (-1.352, -0.174)$*,* $\underline{X}_3 = (1.304, -0.682)$*, and* $\underline{X}_4 = (0.651, 0.137)$*, the corresponding modified sample is* $U_4 = \{\underline{Y}_1 = (3/4, 1), \underline{Y}_2 = (1/4, 2/4), \underline{Y}_3 = (1, 1/4,), \underline{Y}_4 = (2/4, 3/4)\}$*.*
First, let $m = 2$*, then using equation (16) we have that*

$$R_{1,1}^2 \cap U_4 = \{\underline{Y}_2\}, \quad R_{1,2}^2 \cap U_4 = \{\underline{Y}_4\}, \quad R_{2,1}^2 \cap U_4 = \{\underline{Y}_3\}, \ \textit{and} \ R_{2,2}^2 \cap U_4 = \{\underline{Y}_1\}.$$

Then $s_{i,j}^{4,(2)} = 1/4$ *for every* $i, j \in I^2$*, that is,*

$$S_2^4 = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}.$$

Therefore, S_2^4 is clearly a transformation matrix, which induces the partitions $q_{1,0} = q_{2,0} = 0 < q_{1,1} = q_{2,1} = 1/2 < q_{1,2} = q_{2,2} = 1$ on the first and second coordinates, see Definition 2.10. Observe that the partition $(Q^2_{i,j})_{i,j\in I_2}$ induced by the matrix S_2^4, given in Definition 2.10 coincides with the uniform partition of size 2 $(R^2_{i,j})_{i,j\in I_2}$. So, using equation (14), it is easy to see that $C_2^4 = S_2^4(\Pi^2)$ is the copula which has joint density $c_2^4(u,v) = 1$ for every $(u,v) \in \mathbf{I}^2$. Observe that in all four cases $c_2^4(u,v) = s_{i,j}^{4,(2)}/\lambda^2(Q^2_{i,j})$, where λ^2 is the Lebesgue measure in $\mathbb{R}^2$.

Second, let us assume that $m = 3$ and let us take $\{R^3_{i,j}\}_{i,j\in I_3}$ the uniform partition of size 3 of $\mathbf{I}^2$. Then we have that

$$R^3_{1,2} \cap U_4 = \{\underline{Y}_2\}, \quad R^3_{2,3} \cap U_4 = \{\underline{Y}_4\}, \quad R^3_{3,1} \cap U_4 = \{\underline{Y}_3\}, \;\; \textit{and} \;\; R^3_{3,3} \cap U_4 = \{\underline{Y}_1\},$$

and for the remaining boxes $R^3_{i,j} \cap U_4 = \emptyset$. Then

$$s_{i,j}^{4,(3)} = \begin{cases} 1/4 & \textit{if} \quad (i,j) \in \{(1,2),(2,3),(3,1),(3,3)\} \\ 0 & \textit{if} \quad (i,j) \in ((I_3 \times I_3)\backslash\{(1,2),(2,3),(3,1),(3,3)\}). \end{cases}$$

Hence,

$$S_3^4 = \begin{pmatrix} 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \\ 1/4 & 0 & 1/4 \end{pmatrix},$$

which is clearly a transformation matrix. The partitions in $[0,1]$ induced by the matrix S_3^4 are $q_{1,0} = q_{2,0} = 0 < q_{1,1} = q_{2,1} = 1/4 < q_{1,2} = q_{2,2} = 2/4 < q_{1,3} = q_{2,3} = 1$. Define as Definition 2.10

$$Q^3_{i,j} = \langle q_{1,i-1}, q_{1,i}] \times \langle q_{2,j-1}, q_{2,j}] \; \textit{for every} \; i,j \in I_3.$$

So, using Definition 2.10 again, it is easy to see that the sample copula $C_3^4 = S_3^4(\Pi^2)$ of order 3, has a joint density c_3^4 given by

$$c_3^4(u,v) = \begin{cases} 4 & \textit{if} \quad (u,v) \in Q^3_{1,2} \\ 2 & \textit{if} \quad (u,v) \in Q^3_{2,3} \\ 2 & \textit{if} \quad (u,v) \in Q^3_{3,1} \\ 1 & \textit{if} \quad (u,v) \in Q^3_{3,3} \\ 0 & \textit{if} \quad (u,v) \in \mathbf{I}^2\backslash(Q^3_{1,2} \cup Q^3_{2,3} \cup Q^3_{1,2} \cup Q^3_{1,2}) \end{cases}$$

Observe that again, in all cases $c_3^4(u,v) = s_{i,j}^{4,(3)}/\lambda^2(Q^3_{i,j})$.

Finally, assume that $m = n = 4$ and let us take $\{R^4_{i,j}\}_{i,j\in I_4}$ the uniform partition of size 4 of $\mathbf{I}^2$. Then we have that

$$R^4_{1,2} \cap U_4 = \{\underline{Y}_2\}, \quad R^4_{2,3} \cap U_4 = \{\underline{Y}_4\}, \quad R^4_{3,4} \cap U_4 = \{\underline{Y}_3\}, \text{ and } R^4_{4,1} \cap U_4 = \{\underline{Y}_1\},$$

and for the remaining boxes $R^4_{i,j} \cap U_4 = \emptyset$. Then

$$s^{4,(4)}_{i,j} = \begin{cases} 1/4 & \text{if} \quad (i,j) \in \{(1,2),(2,3),(3,4),(4,1)\} \\ 0 & \text{if} \quad (i,j) \in ((I_4 \times I_4)\backslash\{(1,2),(2,3),(3,4),(4,1)\}). \end{cases}$$

Hence,

$$S^4_4 = \begin{pmatrix} 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 \end{pmatrix},$$

which is clearly a transformation matrix. The partitions in $[0,1]$ induced by the matrix S^4_4 are $q_{1,0} = q_{2,0} = 0 < q_{1,1} = q_{2,1} = 1/4 < q_{1,2} = q_{2,2} = 2/4 < q_{1,3} = q_{2,3} = 3/4 < q_{1,4} = q_{2,4} = 1$. Define as in Definition 2.10

$$Q^4_{i,j} = \langle q_{1,i-1}, q_{1,i}] \times \langle q_{2,j-1}, q_{2,j}] \text{ for every } i, j \in I_4,$$

and in this case again, the partition $(Q^4_{i,j})_{i,j\in I_4}$ of $\mathbf{I}^2$ coincides with the uniform partition of size 4 $(R^4_{i,j})_{i,j\in I_4}$ of order 4 of $\mathbf{I}^2$. So, using Definition 2.10 again, it is easy to see that the sample copula $C^4_4 = S^4_4(\Pi^2)$ of order 4, has a joint density c^4_4 given by

$$c^4_4(u,v) = \begin{cases} 4 & \text{if} \quad (u,v) \in Q^4_{1,2} \cup Q^4_{2,3} \cup Q^4_{3,4} \cup Q^4_{4,1} \\ 0 & \text{if} \quad (u,v) \in \mathbf{I}^2\backslash(Q^4_{1,2} \cup Q^4_{2,3} \cup Q^4_{3,4} \cup Q^4_{4,1}) \end{cases}$$

Observe again, that $c^4_3(u,v) = s^{4,(4)}_{i,j}/\lambda^2(Q^4_{i,j})$ for every $i, j \in I_4$.

We will use the previous Example in order to see that even if Definitions 2.10, 2.11 and 2.12 seem to be quite cumbersome, in the case of the sample copula of order m they become quite simple, this will become apparent in the next:

Theorem 2.14 *Let $2 \le m \le n$ and let $\underline{X}_1, \ldots, \underline{X}_n$ be a random sample of size n of a random vector $\underline{X}$ of dimension d, with continuous joint distribution H or d-copula C, where $\underline{X}_i = (X_{i,1}, \cdots, X_{i,d}) \in \mathbb{R}^d$, for every $i = 1, \ldots, n$. Let $U_n = \{\underline{Y}_1, \ldots, \underline{Y}_n\}$ be the corresponding modified sample.*

Let $2 \leq m \leq n$ fixed and define $(R^m_{\underline{i}})_{\underline{i}\in I^d_m}$ the uniform partition of size m of $\mathbf{I}^d$ as in equation (16), $s^{n,(m)}_{i_1,\ldots,i_d}$ as in equation (17), the generalized transformation matrix $S^{\,n}_{\,m}$ as in equation (18), the partition $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m}$ of $\mathbf{I}^d$ induced by $S^{\,n}_{\,m}$ given in equation (15), and C^n_m the sample copula order m as in equation (19). Then

i) For the partitions of $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m}$ we know that $0 = q_{1,0} < q_{1,1} < \cdots < q_{1,m} = 1$, but we also have that

$$q_{j,0} = q_{1,0} = 0, q_{j,1} = q_{1,1}, q_{j,2} = q_{1,2}, \ldots, q_{j,m} = q_{1,m} = 1 \quad \text{for every} \quad j \in \{2, 3, \ldots, d\}, \tag{20}$$

that is, in the d coordinates the partition of $[0, 1]$ does not change. Evenmore, with probability one, the partition $0 = q_{1,0} < q_{1,1} < \cdots < q_{1,m} = 1$ only depends on n and m, and does not depend on the sample, in fact we have that

$$q_{1,j} = \frac{1}{n} \cdot \left\lfloor \frac{j \cdot n}{m} \right\rfloor \quad \text{for every} \quad j \in \{0, 1, 2, \ldots, m\}, \tag{21}$$

where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a.

ii) For every $2 \leq m \leq n$, C^n_m is always a d-copula.

iii) Assume that m divides n, then the partition $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m}$ of $\mathbf{I}^d$ induced by $S^{\,n}_{\,m}$ coincides with the uniform partition $(R^m_{\underline{i}})_{\underline{i}\in I^d_m}$ of size m.

iv) Let λ^d be the Lebesgue measure on the measurable space $(I\!R^d, \mathcal{B}(I\!R^d))$, where $\mathcal{B}(I\!R^d)$ denotes the σ-algebra of Borel. If C^n_m is the sample copula of order m, let us denote by c^n_m its joint density function. Then

$$c^n_m(u_1, \ldots, u_d) = s^{n,(m)}_{i_1,\ldots,i_d} / \lambda^d(Q^m_{i_1,\ldots,i_d}) \text{ for every } (u_1, \ldots, u_d) \in Q^m_{i_1,\ldots,i_d} \text{ and } (i_1, \ldots, i_d) \in I^d_m. \tag{22}$$

Hence, the density is constant on every d-box $Q^m_{i_1,\ldots,i_d}$ of the partition of $\mathbf{I}^d$ induced by $S^{\,n}_{\,m}$. Besides, if $m^d > n$ then there exists at least one d-box $Q^m_{i_1,\ldots,i_d}$ on which the density is zero. In fact, there are at most n d-boxes with positive density.

v) If $m = n$ there are exactly n elements of the partition $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m} = (R^m_{\underline{i}})_{\underline{i}\in I^d_m}$ on which the density equals n^{d-1} and the remaining elements have density zero.

Proof: i) We first observe that the modified sample satisfies that in each coordinate the different values are $1/n, 2/n, \ldots, (n-1)/n, n/n = 1$, then for any $2 \leq m \leq n$ we have that equations (20) and (21) hold. We also have that the matrix $S^{\,n}_{\,m}$ is always a generalized transformation matrix. Hence, by the definition of the sample d-copula of order m given in equation (19) and using the results in [15] and [46] we obtain ii).

Assume that m divides n, to see that iii) holds is enough to observe that we obtain always integers in the expressions $\lfloor\cdot\rfloor$ in equation (21) and that $q_{1,j} = j/m$ for every $j \in \{1, 2, \ldots, m\}$.

iv) We know from equation (19) that the d-sample copula of order m is a multilinear function which spreads uniformly the mass $s^{n,(m)}_{i_1,\ldots,i_d}$ over the d-box $Q^m_{i_1,\ldots,i_d}$ for every $(i_1, \ldots, i_d) \in I^d_m$. Hence the density on each d-box $Q^m_{i_1,\ldots,i_d}$ is a constant, the result now follows directly from evaluating the constant in the integral expression.

v) follows directly from parts iii) and iv). □

Observe that from Theorem 2.14 the two partitions of $\mathbf{I}^d$, that is, the uniform partition of size m $(R^m_{\underline{i}})_{\underline{i}\in I^d_m}$, where $2 \le m \le n$, and the partition induced by the generalized transformation matrix S^n_m, $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m}$ do not always coincide. Using equation (16), if we define $0 = r_{k,0} < 1/m = r_{k,1} < 2/m = r_{k,2} < \cdots < (m-1)/m = r_{k,m-1} < 1 = r_{k,m}$ for every $k \in \{1, 2, \ldots, d\}$, then $(r_{k,j})_{k\in I_d, j\in\{0,1,\ldots,m\}}$ provides the partition in each coordinate induced by the uniform partition of size m. We define the distance between $(R^m_{\underline{i}})_{\underline{i}\in I^d_m}$ and $(Q^m_{\underline{i}})_{\underline{i}\in I^d_m}$ by

$$e_m((R^m_{\underline{i}}), (Q^m_{\underline{i}})) = \max_{j\in\{0,1,\ldots,m\}} |r_{1,j} - q_{1,j}|,$$

where $q_{1,j}$ is given in equation (21). Then e_m measures the "distortion" of the uniform partition of size m, caused by the sample size n. It is easy to see that $\max_{2\le m\le n} e_m((R^m_{\underline{i}}), (Q^m_{\underline{i}})) \le (n-2)/((n-1)n) < 1/n$, that is, the maximum is attained when $m = n - 1$, and this maximum distance is always smaller that $1/n$. Using part iii) of Theorem 2.14, we have that if m divides n, then $e_m((R^m_{\underline{i}}), (Q^m_{\underline{i}})) = 0$.

In Theorem 2.14, part iv) we give the joint density c^n_m associated to C^n_m the sample copula of order m, from this density the evaluation of C^n_m is absolutely trivial, and easily implemented in any computer.

Part v) of Theorem 2.14, implies that most of the d-boxes in the partition of $\mathbf{I}^d$ have zero density. For example if the sample size $n = 100$ and $d = 4$, we have that only one hundred out of $100,000,000$ 4-boxes have positive density. In this example is quite important to notice that the one hundred 4-boxes with positive density include the support of the empirical copula C^n, which is included in T^d, where $T = \{0, 1/n, 2/n, \ldots, 1\}$ given in Definition 2.3 and Remark 2.4. Besides, the sample copula C^n_n is a d-copula unlike the empirical copula C^n, which is only a d-subcopula.

Now we will see, that in some cases, C_m^n the sample copula of order m may coincide with C, the d-copula we are sampling from, that is, $\sup_{(u_1,\ldots,u_d)\in\mathbf{I}^d}|C_m^n(u_1,\ldots,u_d) - C(u_1,\ldots,u_d)| = 0$.

Lemma 2.15 *Let $d \geq 2$ be an integer and let $n \geq 2$ be an even integer. Then there exist $2 \leq m \leq n$, C a d-copula and a sample of size n from C, such that*

$$\sup_{(u_1,\ldots,u_d)\in\mathbf{I}^d} |C_m^n(u_1,\ldots,u_d) - C(u_1,\ldots,u_d)| = 0. \tag{23}$$

Proof: Let $d \geq 2$ be an integer and let $n \geq 2$ be an even integer. Let C be a d-copula with density c given by

$$c(u_1,\ldots,u_d) = \begin{cases} 2^{d-1} & \text{if} \quad (u_1,\ldots,u_d) \in [0,1/2]^d \cup (1/2,1]^d \\ 0 & \text{if} \quad (u_1,\ldots,u_d) \in \mathbf{I}^d \backslash ([0,1/2]^d \cup (1/2,1]^d). \end{cases}$$

Let $m = 2$. The uniform partition of size 2 of $\mathbf{I}^d$ is such that $R_{1,1,\ldots,1} = [0,1/2]^d$ and $R_{2,2,\ldots,2} = (1/2,1]^d$ accumulate all the mass of the d-copula C. Let $\underline{X}_1,\ldots,\underline{X}_n$ be a random sample of size n from the d-copula C, and assume that exactly $n/2$ points of the sample fall in the d-box $R_{1,1,\ldots,1}$, then the remaining $n/2$ points fall in the box $R_{2,2,\ldots,2}$ with probability one, because the density is zero on any other d-box of the uniform partition of size 2. Let $U_n = \{\underline{Y}_1,\ldots,\underline{Y}_n\}$ be the corresponding modified sample, then it is obvious that this modified sample satisfies exactly the same conditions.

Hence, using equation (17), $s_{1,1,\ldots,1}^{n,(2)} = s_{2,2,\ldots,2}^{n,(2)} = 1/2$ and the remaining $s_{i_1,i_2,\ldots,i_d}^{n,(2)} = 0$. Therefore, using Theorem 2.14, part iv), the density of the sample d-copula of order 2 is given by

$$c_m^n(u_1,\ldots,u_d) = \begin{cases} 2^{d-1} & \text{if} \quad (u_1,\ldots,u_d) \in [0,1/2]^d \cup (1/2,1]^d \\ 0 & \text{if} \quad (u_1,\ldots,u_d) \in \mathbf{I}^d \backslash ([0,1/2]^d \cup (1/2,1]^d), \end{cases}$$

which is exactly the density of the d-copula C, and the result follows. □

However, observe that from Lemma 2.7, the empirical copula C_n satisfies that

$$\sup_{(u_1,\ldots,u_d)\in\mathbf{I}^d} |C_n(u_1,\ldots,u_d) - C(u_1,\ldots,u_d)| \geq 1/n$$

almost surely.

Since we are using the product copula in order to define the sample d-copula of order m, see equation (19), we have to see what is the largest error we can incur by doing so. First we observe that for $d = 2$, we have that

$$\sup_{(u,v)\in\mathbf{I}^2} |\Pi^2(u,v) - M^2(u,v)| = \sup_{(u,v)\in\mathbf{I}^2} |\Pi^2(u,v) - W^2(u,v)| = 1/4, \tag{24}$$

where the supremum is attained at $u = v = 1/2$ in both cases, as can be easily checked. Hence, from equation (24) we have that for every 2-copula C, using the Fréchet-Hoeffding's bounds, $\sup_{(u,v)\in\mathbf{I}^2} |\Pi^2(u,v) - C(u,v)| \leq 1/4$.

For the case $d \geq 3$, we have that

$$\sup_{(u_1,\ldots,u_d)\in\mathbf{I}^d} |\Pi^d(u_1,\ldots,u_d) - M^d(u_1,\ldots,u_d)| = \frac{d-1}{d^{d/(d-1)}}, \tag{25}$$

where the supremum is attained at $u_1 = u_2 = \cdots u_d = 1/d^{1/(d-1)}$.

For every C_1, C_2 d-copulas let us define

$$d_{\text{sup}}(C_1,C_2) = \sup_{(u_1,u_2,\ldots,u_d)\in\mathbf{I}^d} |C_1(u_1,u_2,\ldots,u_d) - C_2(u_1,u_2,\ldots,u_d)|. \tag{26}$$

Then clearly d_{sup} is a metric in the set of all d-copulas.

Definition 2.16 *Let $2 \leq m \leq n$ and let $\underline{X}_1,\ldots,\underline{X}_n$ be a random sample of size n of a random vector $\underline{X}$ of dimension d, with continuous joint distribution H with d-copula C, or from the d-copula C where $\underline{X}_i = (X_{i,1},\cdots,X_{i,d}) \in I\!R^d$, for every $i = 1,\ldots,n$. Let $U_n = \{\underline{Y}_1,\ldots,\underline{Y}_n\}$ be the corresponding modified sample.*

Let C_n be the empirical copula defined in equation (8), and let C^n_m be the sample copula of order m defined as in equation (19) of Definition 2.12. We define

$$d_{sup_n}(C_n,C) = \max\left(\sup_{(i_1,i_2,\ldots,i_d)\in {I_n}^d} |C_n(i_1/n,i_2/n,\ldots,i_d/n) - C(i_1/n,i_2/n,\ldots,i_d/n)|\ ,\ \frac{1}{n} \right) \tag{27}$$

and

$$d_{sup_{n,(m)}}(C^n_m,C) = \sup_{(i_1,i_2,\ldots,i_d)\in {I_n}^d} |C^n_m(i_1/n,i_2/n,\ldots,i_d/n) - C(i_1/n,i_2/n,\ldots,i_d/n)|. \tag{28}$$

By equation (13) we now that $d_{\text{sup}}(C_n,C) \geq 1/n$ almost surely, that is why the term $1/n$ appears in equation (27). Hence $d_{\text{sup}_n}(C_n,C)$ is never a metric. However, $d_{\text{sup}_{n,(m)}}(C^n_m,C)$ in equation (28) is a pseudometric in the family of all d-copulas. In order to see that this statement holds just observe that C^n_m is always a d-copula, and for any d-copula C such that $C^n_m(i_1/n,i_2/n,\ldots,i_d/n) = C(i_1/n,i_2/n,\ldots,i_d/n)$ for every $(i_1,i_2,\ldots,i_d) \in {I_n}^d$ we have that $d_{\text{sup}_{n,(m)}}(C^n_m,C) = 0$. In particular, if $C = C^n_m$ we have that $d_{\text{sup}_{n,(m)}}(C^n_m,C) = 0$.

Of course we have from equations (26), (27) and (28)that

$$d_{\sup}(C_n, C) \geq d_{\sup_n}(C_n, C) \quad \text{and} \quad d_{\sup}(C_m^n, C) \geq d_{\sup_{n,(m)}}(C_m^n, C), \tag{29}$$

for every integers $2 \leq m \leq n$ and for every d-copula C. We will show with an easy example that the first inequality in equation (29) can be strict, but the second one is an equality.

Let $d = 2$, $n = 2$ and let $C = \Pi^2$ be the product copula. Then the modified sample is $U_2 = \{(1/2, 2/2 = 1), (2/2 = 1, 1/2)\}$ or $U_2 = \{(1/2, 1/2), (1 = 2/2, 1 = 2/2)\}$ each with probability $1/2$. In the first case, the mass of the empirical copula is concentrated in the two points $(1/2, 1)$ and $(1, 1/2)$, therefore, it is quite easy to see from equation (27) that $d_{\sup_2}(C_2, \Pi^2) = \max\{|0 - 1/4|, 1/2\} = 1/2$. But, if we take any $0 < \epsilon < 1$ then we have that $d_{\sup}(C_2, \Pi^2) \geq |C_2(1-\epsilon, 1-\epsilon) - (1-\epsilon)^2| = (1-\epsilon)^2$, so if we let $\epsilon \downarrow 0$ we have that $d_{\sup}(C_2, \Pi^2) = 1 > 1/2 = d_{\sup_2}(C_2, \Pi^2)$. Observe that in this example we obtain the upper bound, since $d_{\sup}(C, D) \leq 1$ for any two distribution functions C and D on $\mathbf{I}^2$. Since $n = 2$, then we have that $m = 2$ for the sample copula, and in this case we have that the sample copula of order $m = 2$ gives uniform masses $1/2$ to the boxes $R_{1,2} = [0, 1/2] \times (1/2, 1]$ and $R_{2,1} = (1/2, 1] \times [0, 1/2]$, so using equation (28) we have that $d_{\sup_{2,(2)}}(C_2^2, \Pi^2) = |C_2^2(1/2, 1/2) - \Pi^2(1/2, 1/2)| = 1/4 = d_{\sup}(C_2^2, \Pi^2)$.

In the second case, the mass of the empirical copula is concentrated in the two points $(1/2, 1/2)$ and $(1, 1)$, so, from equation (27) we have that $d_{\sup_2}(C_2, \Pi^2) = \max\{|1/2 - 1/4|, 1/2\} = 1/2$. Using (13) we can see that $d_{\sup}(C_2, \Pi^2) = 1/2$. In this case we have that the sample copula of order $m = 2$ gives uniform masses $1/2$ to the boxes $R_{1,1} = [0, 1/2]^2$ and $R_{2,2} = (1/2, 1]^2$, so using equation (28) we have that $d_{\sup_{2,(2)}}(C_2^2, \Pi^2) = |C_2^2(1/2, 1/2) - \Pi^2(1/2, 1/2)| = 1/4 = d_{\sup}(C_2^2, \Pi^2)$.

Let $\underline{X}_1, \ldots, \underline{X}_n$ be a random sample of size n from the d-copula M^d. Let $U_n = \{\underline{Y}_1, \ldots, \underline{Y}_n\}$ be the corresponding modified sample, then $\{\underline{Y}_i = (i/n, i/n, \ldots, i/n)\}_{i=1}^n$ almost surely. So, it is obvious that $\sup_{(i_1, i_2, \ldots, i_d) \in I_n{}^d} |C_n(i_1/n, i_2/n, \ldots, i_d/n) - M^d(i_1/n, i_2/n, \ldots, i_d/n)| = 0$, but from equation (13) $d_{\sup}(C_n, C) = 1/n$.

We finish this section by proving a Glivenko-Cantelli's Theorem for the sample d-copula of order m.

Theorem 2.17 *Let C be a d-copula, let $m \geq 2$ and let n be a multiple of m, let C_m^n be the sample copula built from a modified sample of size n from C. Then*

$$\lim_{m \longrightarrow \infty} \sup_{(u,v) \in \mathbf{I}^2} \left| C_m^n(u, v) - C(u, v) \right| = 0 \quad a.s.$$

Proof: We prove the result for $d = 2$. Using the notation in Lemma 2.3.5 in Nelsen's book [37]: For $(a, b) \in [0, 1]^2$ and a subcopula C' with domain finite $S_1 \times S_2$, let a_1 and a_2 be, respectively, the greatest and least elements of S_1 that satisfy $a_1 \leq a \leq a_2$; and let b_1 and b_2 be, respectively, the greatest and least elements of S_2 that satisfy $b_1 \leq b \leq b_2$. If $a \in S_1$, then $a_1 = a = a_2$, and if $b \in S_2$, then $b_1 = b = b_2$. Let

$$\lambda_1(a,b) = \lambda_1 = \begin{cases} (a - a_1)/(a_2 - a_1) & \text{if} \quad a_1 < a_2 \\ 1 & \text{if} \quad a_1 = a_2 \end{cases}$$

and

$$\mu_1(a,b) = \mu_1 = \begin{cases} (b - b_1)/(b_2 - b_1) & \text{if} \quad b_1 < b_2 \\ 1 & \text{if} \quad b_1 = b_2. \end{cases}$$

We will consider the following representation of the the sample copula C_m^n, see [26].

$$\begin{aligned} C_m^n(u,v) \quad = \quad & \sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[\mathbf{1}_{\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]}(u,v)\Bigg((1-\lambda_1(u,v))(1-\mu_1(u,v))C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right) \\ & +(1-\lambda_1(u,v))\mu(u,v)_1 C_n\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(u,v)(1-\mu_1(u,v))C_n\left(\frac{i}{m},\frac{j-1}{m}\right) \\ & +\ \lambda_1(u,v)\mu_1(u,v)C_n\left(\frac{i}{m},\frac{j}{m}\right)\Bigg)\Bigg] \end{aligned}$$

where C_n is the empirical copula built from the modified sample, and the checkerboard approximation $C^{(m)}$, see [33], given by

$$\begin{aligned} C^{(m)}(u,v) \quad = \quad & \sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[\mathbf{1}_{\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]}(u,v)\Bigg((1-\lambda_1(u,v))(1-\mu_1(u,v))C\left(\frac{i-1}{m},\frac{j-1}{m}\right) \\ & +(1-\lambda_1(u,v))\mu_1(u,v)C\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(u,v)(1-\mu_1(u,v))C\left(\frac{i}{m},\frac{j-1}{m}\right) \\ & +\ \lambda_1(u,v)\mu_1(u,v)C\left(\frac{i}{m},\frac{j}{m}\right)\Bigg)\Bigg] \end{aligned}$$

for all $(u, v) \in \mathbf{I}^2$.

Let $(u^*, v^*) \in \mathbf{I}^2$ and $m \geq 2$, where m divides n, such that $(u^*, v^*) \in R^m_{i^* j^*} = \left\langle \frac{i^*-1}{m}, \frac{i^*}{m}\right] \times \left\langle \frac{j^*-1}{m}, \frac{j^*}{m}\right]$ where the notation "$\langle$" indicates that the left end of interval is closed if $i^* = 1$ or $j^* = 1$ and open in any other case, then

$$|C_m^n(u^*,v^*) - C^{(m)}(u^*,v^*)| \quad \leq \quad (1-\lambda_1(u^*,v^*))(1-\mu_1(u^*,v^*))\left|C_n\left(\frac{i^*-1}{m},\frac{j^*-1}{m}\right) - C\left(\frac{i^*-1}{m},\frac{j^*-1}{m}\right)\right|$$

$$+(1-\lambda_1(u^*,v^*))\mu_1(u^*,v^*)\left|C_n\left(\frac{i^*-1}{m},\frac{j^*}{m}\right)-C\left(\frac{i^*-1}{m},\frac{j^*}{m}\right)\right|$$

$$+\lambda_1(u^*,v^*)(1-\mu_1(u^*,v^*))\left|C_n\left(\frac{i^*}{m},\frac{j^*-1}{m}\right)-C\left(\frac{i^*}{m},\frac{j^*-1}{m}\right)\right|$$

$$+\lambda_1(u^*,v^*)\mu_1(u^*,v^*)\left|C_n\left(\frac{i^*}{m},\frac{j^*}{m}\right)-C\left(\frac{i^*}{m},\frac{j^*}{m}\right)\right|.$$

Hence, using Glivenko-Cantelli's Theorem for the empirical copula, we have that

$$\lim_{m\longrightarrow\infty}\sup_{(u,v)\in\mathbf{I}^2}\left|C_m^n(u,v)-C^{(m)}(u,v)\right|=0\quad a.s.$$

From [33], we have that, for every $m\geq 1$,

$$\sup_{(u,v)\in\mathbf{I}^2}\left|C^{(m)}(u,v)-C(u,v)\right|<\frac{2}{m},$$

and

$$\lim_{m\longrightarrow\infty}\sup_{(u,v)\in\mathbf{I}^2}\left|C^{(m)}(u,v)-C(u,v)\right|=0.$$

From these relations we get that

$$\lim_{m\longrightarrow\infty}\sup_{(u,v)\in\mathbf{I}^2}\left|C_m^n(u,v)-C(u,v)\right|=0\quad a.s.$$

The proof of the case d-dimensional, where $d>2$, is performed similarly. □

2.3 Simulation study: dimension two

For this section we performed a large simulation study in order to compare the supremum distance defined on equation (26), between the real copula and the empirical copula and then the supremum distance between the real copula and the sample copula of order m. Here we used equations (27) and (28), which are good approximations of the supremum distance (26).

In this subsection, we study twenty five different families in dimension $d=2$, which include several of the most used families of copulas such as: Ali-Mikhail-Haq, Clayton, Frank, Gumbel, Joe, Normal, Plackett, Product and t-Student. We also include some copulas that are singular such as M^2, W^2, Examples 3.3, 3.4, 3.5 and the circular distribution copula from Nelsen‘s book [37]. We also include an example of an absolutely continuous copula whose support is $[0,1/2]^2\cup(1/2,1]^2$

and we call it the protocol copula. We also study increasing and decreasing transformations of the coordinates of a copula in the Plackett and the Gumbel families. Finally we study a couple of examples of mixtures of copulas to study the behavior of asymmetric copulas. The study consisted in taking random samples of sizes $n = 20, 30$ and $n = 50$ with $N = 2000$ or $N = 10000$ repetitions for the first two values of n and $N = 1500$ repetitions for $n = 50$. We obtained the sample copulas C_m^n for every $2 \leq m \leq n$ and the empirical copula C_n and we approximate the supremum distance using equations (27) and (28). Finally we report the mean values, minima and maxima of the N iterations in the first figures, the straight lines correspond to the minima, mean values and maxima of $d_{\sup_n}(C_n, C)$ and the other lines correspond to the same statistics for $d_{\sup_{n,(m)}}(C_m^n, C)$. In the second figures we report for some of the cases the comparisons of the variances.

We report the values of Spearman's ρ, which is a common concordance measure, instead of the parameters of the families in order to make comparisons.

From now on, we will say that C_m^n **is a better approximation than** C_n **to the real copula** C **for a given value of** $2 \leq m \leq n$ if the mean value of the iterations satisfy that $d_{\sup_{n,(m)}}(C_m^n, C) \leq d_{\sup_n}(C_n, C)$. We will make some remarks about the variances later on.

First, we start to the case of the product copula Π^2, see Figure 2.11, in which we show that for $n = 20, 30$ and $n = 50$ we have that for every $2 \leq m \leq n$, C_m^n is a better approximation than C_n, and the variances in all three cases are quite similar for $3 \leq m \leq n$, even for $m = 2$ where the variance difference is the greatest we also have that the difference between the mean values is quite remarkable. However, these facts are expected since in the construction of the d-sample copula of order m we used uniform masses.

Second, for the Ali-Mikhail-Haq family we have in Figure 2.1 the results for $\theta \in (-1, 0)$ and in Figure 2.2 for $\theta \in (0, 1)$. As we can see for every $2 \leq m \leq n$ C_{20}^m is a better approximation than C_{20}, and in some instances these mean values are even smaller than the minimum of $d_{\sup_{m,(20)}}(C_{20}, C)$ for $m = 2$. The behavior of this family follows since it is known that its dependence is short of independence, because $-0.27106 \leq \rho \leq 04784$.

Third, the Clayton family is a symmetric one and its results are given in Figures 2.3 and Figure 2.4, the first one includes the results for $-1 \leq \rho < 0$, we know that the limit of this family as θ approaches -1 is W^2, an when θ approaches zero its limit is the product copula Π^2, and for θ close to -1 is quite similar to the case W^2. Observe that for every $m \geq 5$ we have that C_m^{20} is a better approximation than C_{20}. In Figure 2.4 we show the results for $0 < \rho \leq 1$, in this case we know that whenever θ increases to ∞ the copula tends to M^2. Recall that a family which has limits W^2

and M^2 it is called comprehensive, here we also observe that for every $m \geq 5$ we have that C_m^{20} is a better approximation than C_{20}.

Fourth, the Frank family is also a symmetric and comprehensive set of copulas when θ tends to $-\infty$ the copula approaches W^2, and when θ tends to ∞ the copula approaches M^2. We observe that for values of ρ which are not too far from zero we have that C_m^{20} is a better approximation than C_{20} for several values of m, see Figure 2.5 and Figure 2.6. In fact, this holds to be true for any ρ if $m \geq n/2 = 10$.

Fifth, the Gumbel family is a symmetric family such that for $\theta = 1$ we obtain the product copula Π^2 with $\rho = 0$, and when θ tends to ∞ the copula approaches M^2. Again, for values of ρ close to zero C_m^{20} is a better approximation than C_{20} for every $2 \leq m \leq 20$, and for large values of ρ it also holds for $m \geq n/2 = 10$ see Figure 2.7.

Sixth, the Joe family is also symmetric and it behaves quite similar to the Gumbel family, see Figure 2.8.

Seventh, the normal family is symmetric, comprehensive and it is parametrized via its correlation coefficient ρ, it can be seen that for $|\rho| \leq 0.9$, C_m^{20} is a better approximation than C_{20} for every $3 \leq m \leq n$. But for $|\rho|$ close to one it starts to behaves like W^2 or M^2 depending on the sign of ρ.

Eighth, the Plackett family is symmetric with parameter $\rho \in (-1, 1)$, for θ near zero it behaves like W^2, for $\theta = 1$ it is Π^2, and for very large values of θ it behaves like M^2, so this family is also comprehensive. Hence, C_m^{20} is a better approximation than C_{20} for several values of m if $|\rho|$ is relatively larger than zero and not too large, see Figure 2.10.

Ninth, the t-Student family is also symmetric, comprehensive and it is parametrized via its correlation coefficient ρ, it can be seen that for $|\rho| \leq 0.95$, C_m^{20} is a better approximation than C_{20} for every $2 \leq m \leq 20$. The only difference respect to the normal case is the fact that the t-Student has heavier tail behavior, see Figure 2.12.

Tenth, the cases of M^2 and W^2 are of particular importance. To begin with they are the upper and lower bounds for every copula, besides, they represent extreme dependence, because they are the copulas of random variables X and Y for which Y is a strictly increasing or a strictly decreasing function of X, with $\rho = 1$ and $\rho = -1$, respectively. Besides, both copulas are singular with supports the main diagonal and the second diagonal, respectively. From Figure 2.13 left two graphs, we see that they have similar behavior and that for $m \geq 5$ we have that C_m^{20} is a better approximation than C_{20}. As we have seen above, several families of copulas with limit cases M^2 or W^2 have pretty similar behavior to the first two graphs in Figure 2.13. It is also very important to observe that for any given sample of size n, the minimum and the maximum coincide with the mean value, this happens because the modified samples are always the same.

Eleventh, the cases of the circular uniform distribution and Example 3.5 given in Nelsen's book [37] are examples of singular copulas, the second one with support on two quarters of circles of radius one, see Figure 3.5 in [37], as we can see in the right graphs of Figure 2.13 we have that C_m^{20} is a better approximation than C_{20} for every $2 \leq m \leq 20$.

Twelfth, the case of Example 3.3 in [37] is another case of singular copulas with given support the segment of lines joining $(0,0)$ with $(\theta,1)$, and $(\theta,1)$ with $(1,0)$, for $\theta \in [0,1]$. As we can see in Figure 2.14 the behavior of the difference between the supremum distances behaves like the case of M^2 for $\rho = \theta = 1$ and like W^2 for $\rho = \theta = 0$.

Thirteenth, the case of Example 3.4 in Nelsen's book [37], it is simply shuffles of M^2 with support the segment of lines joining $(0,\theta)$ with $(\theta,0)$, and $(\theta,1)$ with $(1,\theta)$, where $\theta \in [0,1]$. for a general definition of shuffle see [35] or [37], for the multivariate case see [9]. In this case, see Figure 2.15, we have that C_m^{20} is a better approximation than C_{20} for $10 = n/2 \leq m \leq n = 20$ with a similar behavior to W^2. This case is the one that presents smaller differences between the mean values for $0.1 \leq \theta \leq 0.9$.

Fourteenth, the case of what we called the protocol copula, which has incomplete support given by $[0,1/2]^2$ and $(1/2,1]^2$ with uniform masses on each square it is quite important. We can see from Figure 2.16 left graph, that C_2^{20} is a better approximation than C_{20}, but C_3^{20} is not a better approximation than C_{20}, in fact, we can see that the behavior of the graph oscillates between even and odd values of m this can be easily explained due to the form of its support. It is also very important to see that the minimum of the observed supremum distances between C_2^{20} and the protocol copula is zero.

Fifteenth, recall that from Theorem 2.4.4 in [37] if two random variables U and V have copula C, then we can give just in terms of C the copulas of the pairs $(U, 1-V)$, $(1-U, V)$ and $(1-U, 1-V)$, we will denote these cases by ID, DI and DD, where I means a strictly increasing transformation and D denotes a strictly decreasing transformation. We studied these transformation for the Gumbel and the Plackett families, we choose these two families to obtain asymmetric copulas with the transformations ID and DI. The result for the Gumbel family are shown in Figure 2.16 last three graphs, Figure 2.17 and Figure 2.18. For the Plackett family we refer to Figure 2.19, Figure 2.20

and Figure 2.21. We can observe that the values of the parameters that we used were the same for each family, and that the graphs look very much alike in each case.

Sixteenth, the last three cases are mixtures; First of Gumbel, GumbelID (GID), and GumbelDI (GDI), second GID, Frank and Joe, in both cases we obtain highly asymmetric copulas; and third a mixture of M^2 and W^2 which is singular. The results are shown in Figure 2.22, Figure 2.23 and Figure 2.24 and the comments are quite similar to some of the previous cases. For the PlackettID we will use the abbreviation (PID), for PlackettDI (PDI) and for PlackettDD (PDD).

Summing up, the results for all the families of copulas absolutely continuous with complete support are quite similar as a function of the Spearman's ρ. However, for the families of singular copulas the results are sometimes comparable to those of M^2 or W^2, and they depend strongly on their supports. When the copula is absolutely continuous, but it does not have complete support, the behavior may be alittle erratic as shown by the protocol copula in Figure 2.16.

We can observe that when $m = n$, the sample copula C_m^n is at least as good an approximation to the real copula C as the empirical copula C_n, and in many cases a better approximation, when $|\rho|$ is close to one. In fact, in many cases the best approximation is given with values of m quite smaller than n, and this value of m depends strongly on the value of Spearman's ρ.

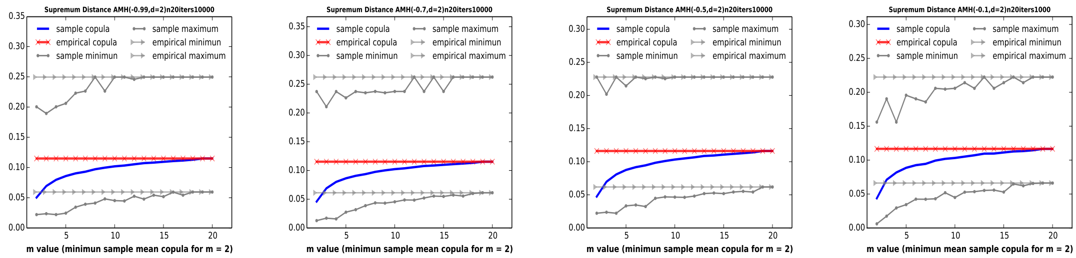

Figure 2.1: Ali-Mikhail-Haq $d = 2$ for $\rho = -0.2688, -0.2004, -0.1489$ and -0.0325 with $n = 20$

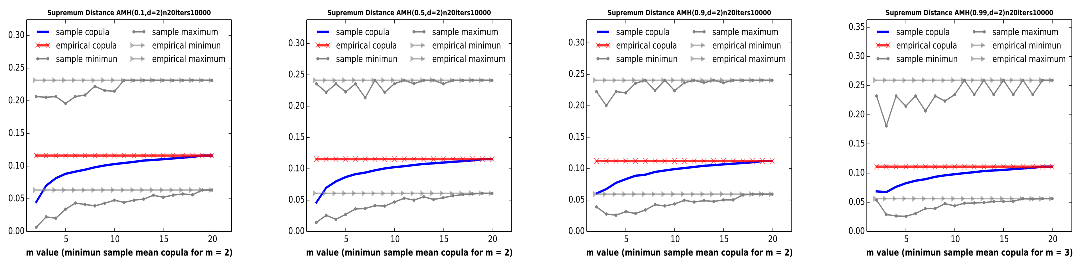

Figure 2.2: Ali-Mikhail-Haq $d = 2$ for $\rho = 0.0342, 0.1924, 0.4070$ and 0.4706 with $n = 20$

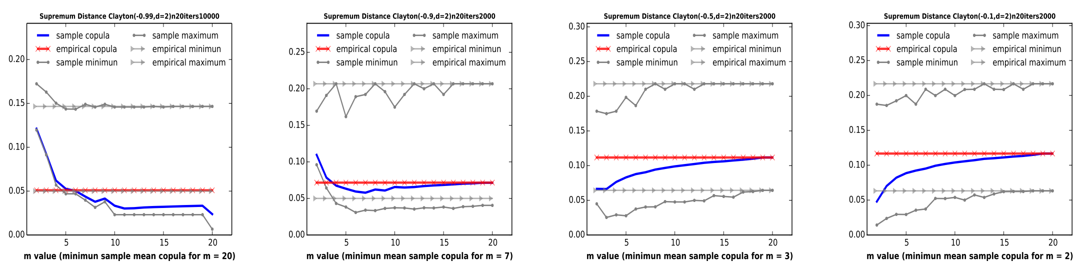

Figure 2.3: Clayton $d = 2$ for $\rho = -0.99, -0.8978, -0.4670$ and -0.0787 with $n = 20$

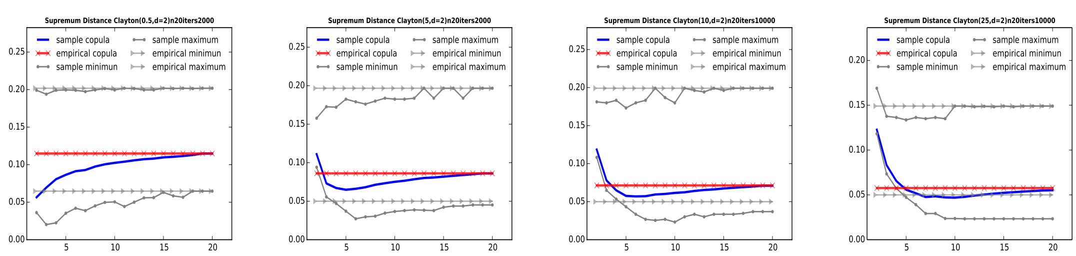

Figure 2.4: Clayton $d = 2$ for $\rho = 0.2955, 0.8848, 0.9582$ and 0.9915 with $n = 20$

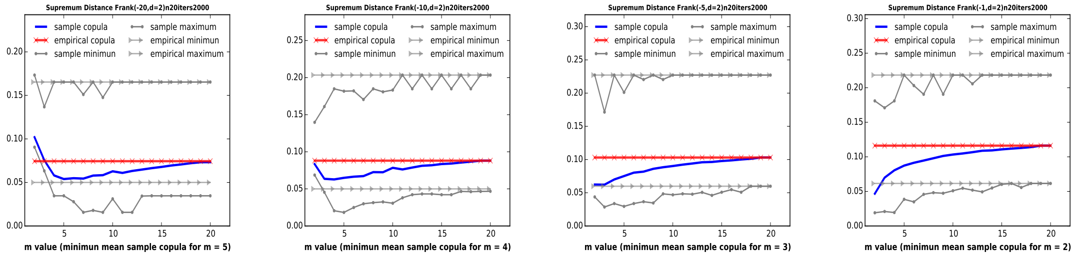

Figure 2.5: Frank $d = 2$ for $\rho = -0.9578, -0.8602, -0.6434$ and -0.1644 with $n = 20$

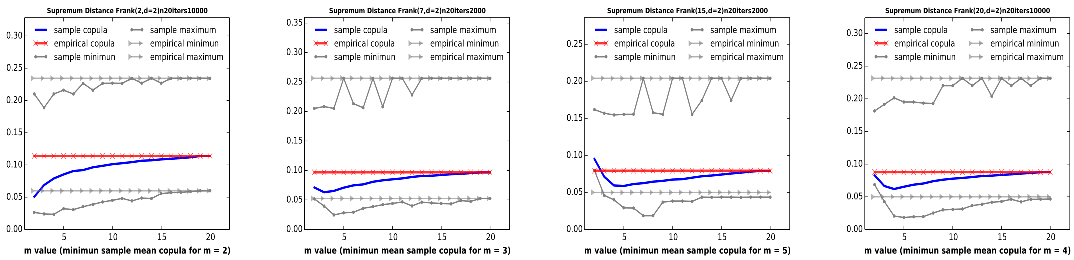

Figure 2.6: Frank $d = 2$ for $\rho = 0.3168, 0.7630, 0.9293$ and 0.9578 with $n = 20$

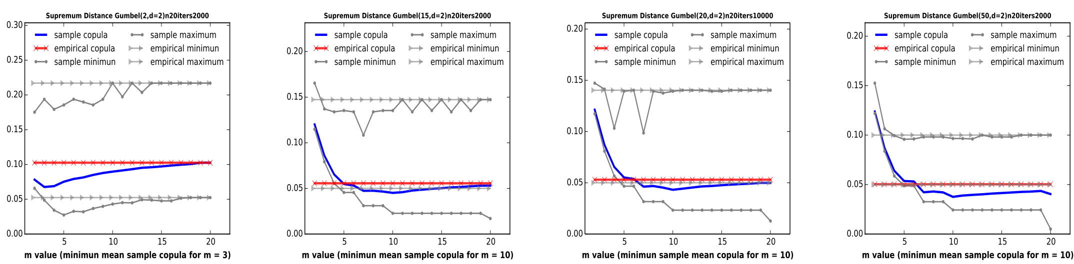

Figure 2.7: Gumbel $d = 2$ for $\rho = 0.6828, 0.9935, 0.9963$ and 0.999 with $n = 20$

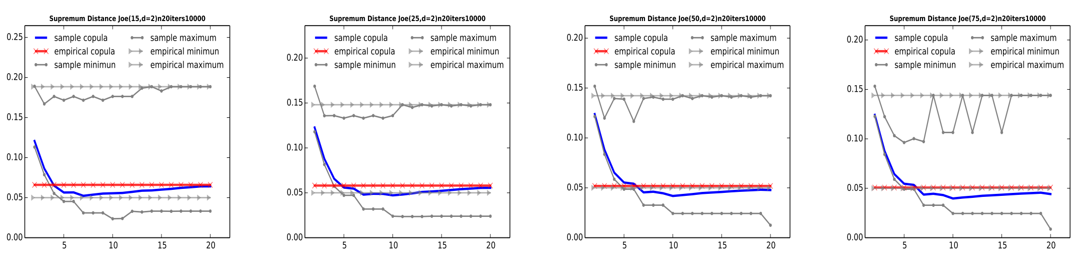

Figure 2.8: Joe $d = 2$ for $\rho = .9766, .9908, .9975$ and $.9988$ with $n = 20$

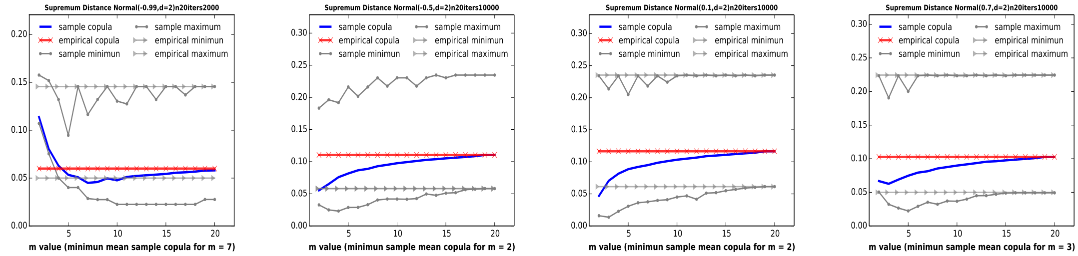

Figure 2.9: Normal $d = 2$ for $\rho = -0.9889, -0.4825, 0.0955$ and 0.6829 with $n = 20$

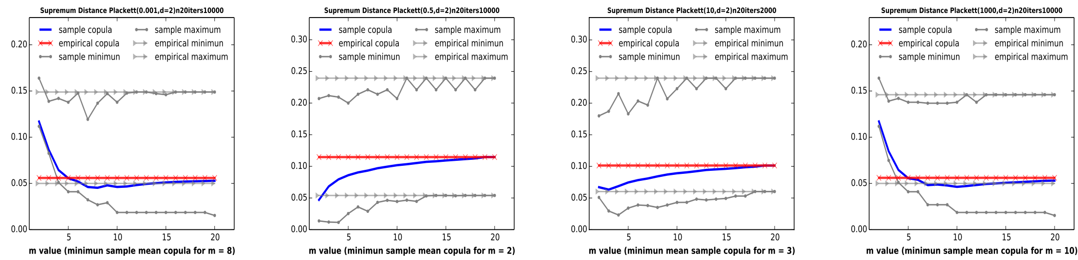

Figure 2.10: Plackett $d = 2$ for $\rho = -0.9881, -0.2274, 0.6536$ and 0.9881 with $n = 20$

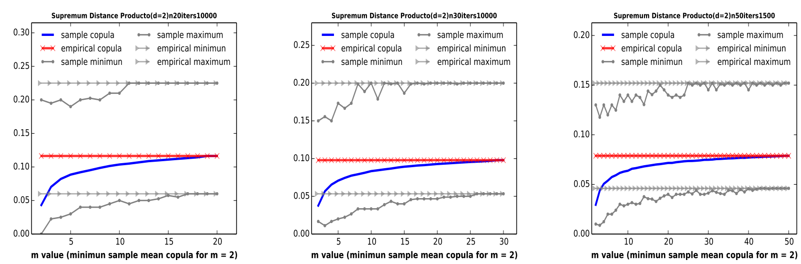

Figure 2.11: Product $d = 2$ with $\rho = 0$ for $n = 20, 30$ and 50

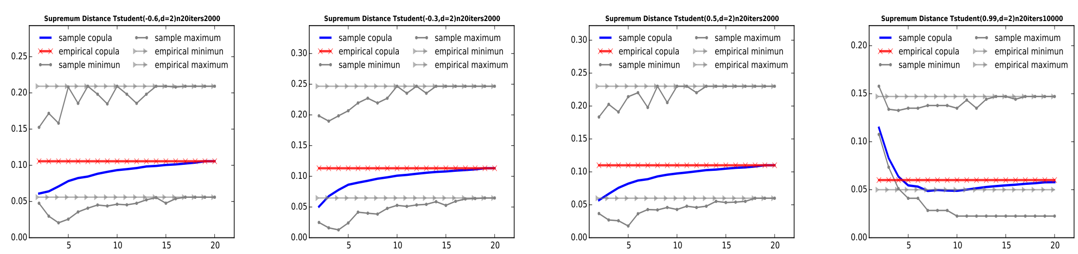

Figure 2.12: t-Student $d = 2$ for $\rho = -0.5819, -0.2875, 0.4825$ and 0.9889 with $n = 20$

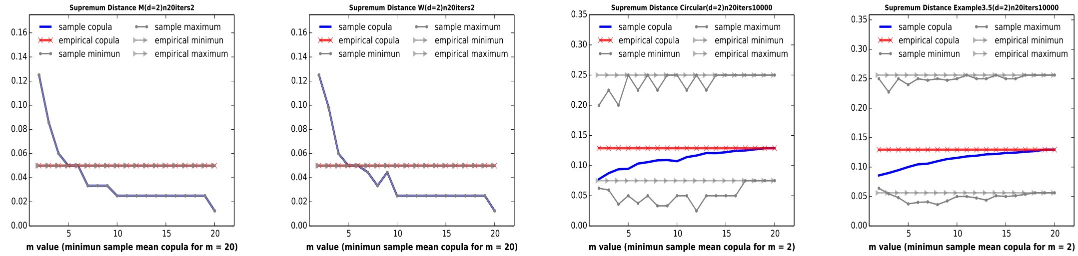

Figure 2.13: $M^2\rho = 1$, $W^2\rho = -1$, Circular $\rho = 0$ and Examp. 3.5 of Nelsen $\rho.286$ with $n = 20$

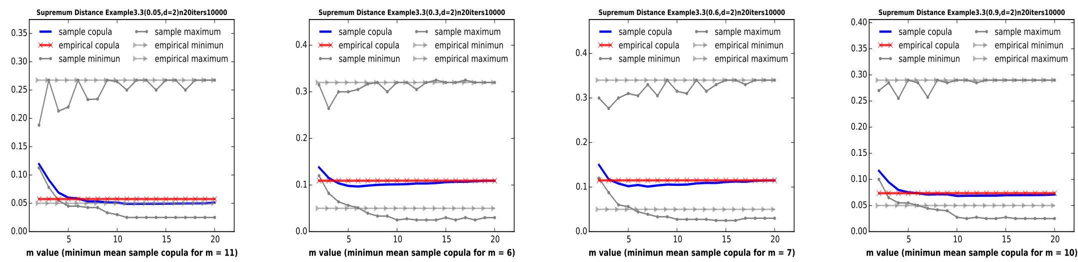

Figure 2.14: Example 3.3 of Nelsen for $\rho = -0.9, -0.4, 0.2$ and 0.8 with $n = 20$

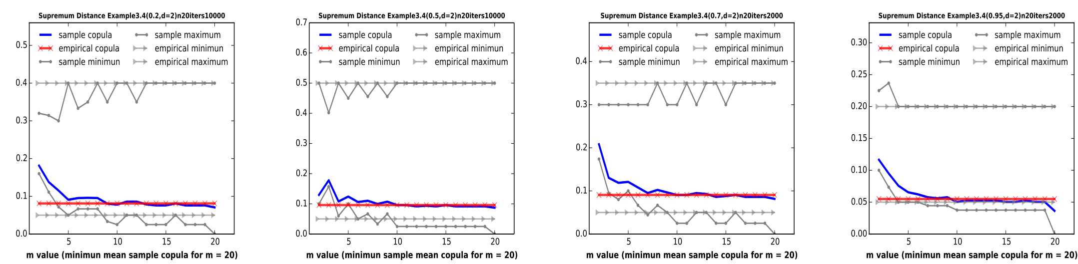

Figure 2.15: Example 3.4 of Nelsen for $\rho = -0.0396, 0.5, 0.2591$ and -0.7151 with $n = 20$

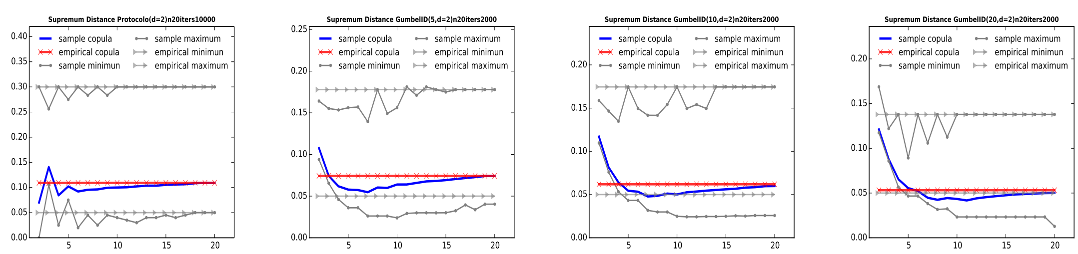

Figure 2.16: Cop, Protocol $\rho = .75$ and GumbelID for $\rho = -.9429, -.9854$ and $-.9963$ with $n = 20$

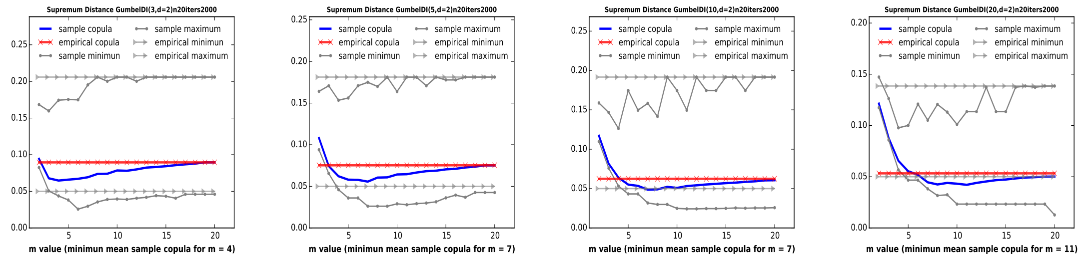

Figure 2.17: Copulas GumbelDI for $\rho = -.8487, -.9431, -.9855$ and $-.9963$ with $n = 20$

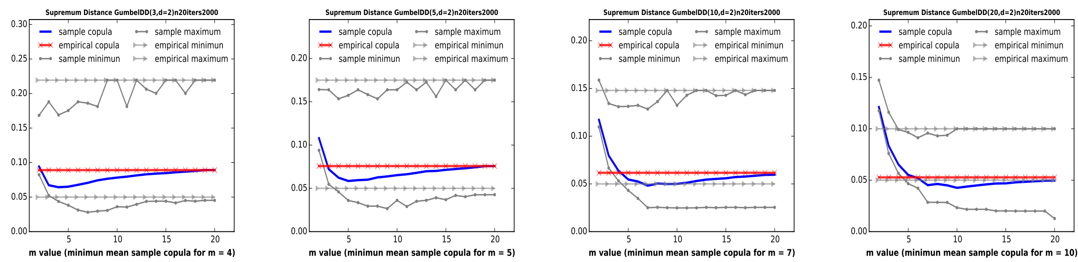

Figure 2.18: Copulas GumbelDD for $\rho = .8494, .9434, .9854$ and $.9963$ with $n = 20$

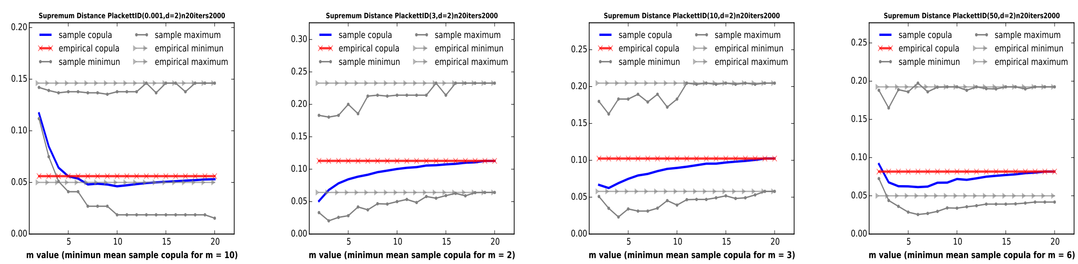

Figure 2.19: Copulas PlackettID for $\rho = 0.9880, -.3521, -.6537$ and $-.8780$ with $n = 20$

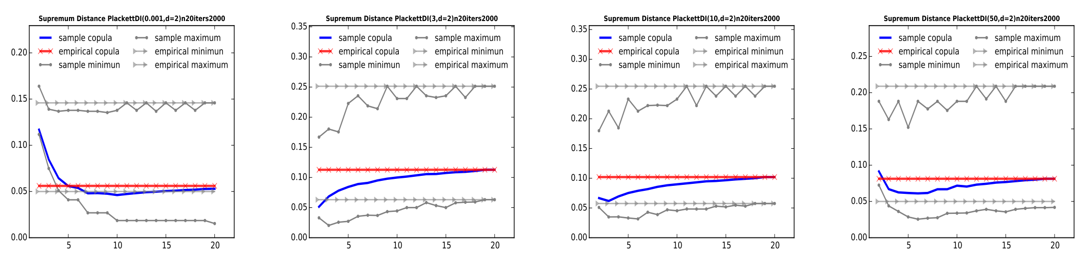

Figure 2.20: Copulas PlackettDI for $\rho = 0.9880, -.3521, -.6537$ and $-.8780$ with $n = 20$

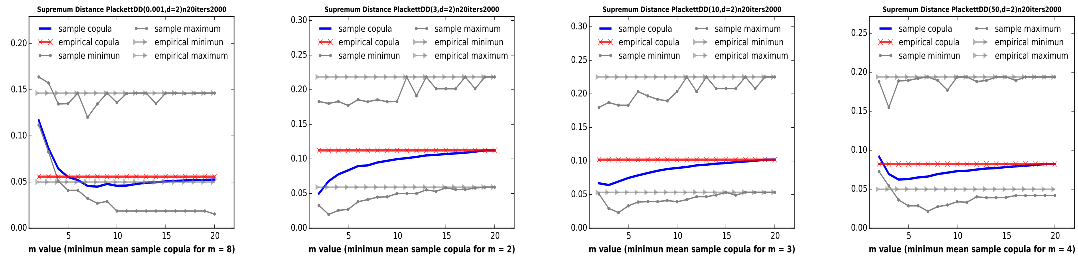

Figure 2.21: Copulas PlackettDD for $\rho = -0.9881, .3517, .6532$ and $.8778$ with $n = 20$

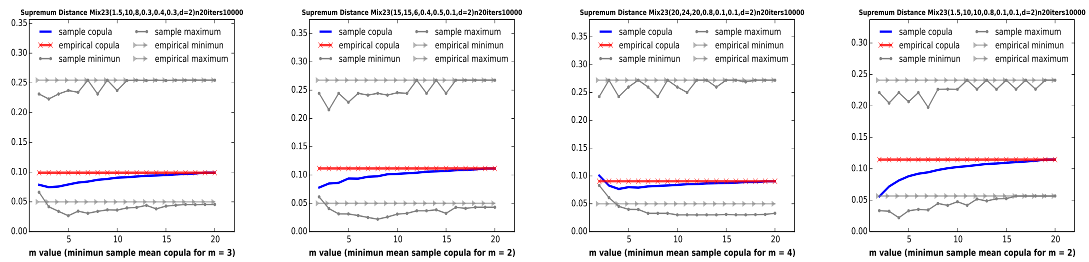

Figure 2.22: Mixt. of G, GID and GDI with ρ= -.54459, -.19608, .59829 and .18460 with $n = 20$

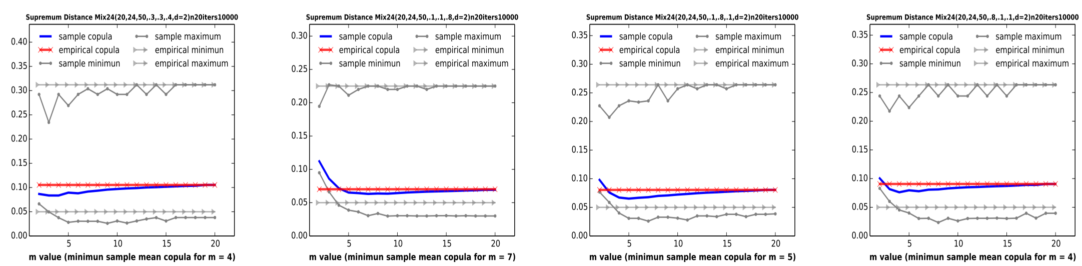

Figure 2.23: Mixt. of GID, Fr. and Joe with ρ = .39085, .79548, .77585 and -.600093 with $n = 20$

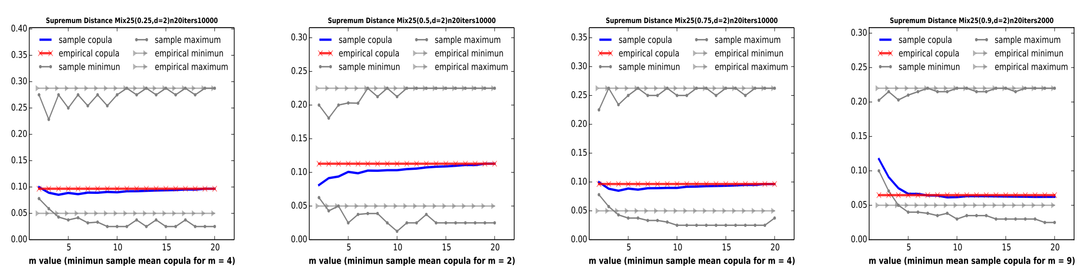

Figure 2.24: Mixt. of M^2 and W^2 with $\rho = -0.5, 0, 0.5$ and 0.8 with $n = 20$

Now we make a quick study of the variances for the twenty five families with $n = 20$ related to all the Figures considerd above. In Figure 2.25, Figure 2.26, $\dots$, Figure 2.30 and Figure 2.31 we include the behavior of the variances of the statistics that measure the differences between the supremum distance between the sample copula of order m and the real copula, and also the supremum distance between the empirical copula and the real copula. We chose only one value of the parameter in everyone of the previous Figures. For each family, the chosen parameter best describes the behavior of the variance, leaving out the limit cases. In all graphs we give the value of ρ.

First we observe that from Figure 2.25 the variances for the Ali-Mikhail-Haq family behave like the variance given for the product copula Π^2 in Figure 2.27, this fact follows from what we explained before for Figures 2.1 and 2.2. Observe that the biggest variance is given for $m = 2$, but the difference of the variance of the sample copula and the variance for the empirical copula in this case is only of magnitude 0.0005. For larger values of m, the sample copula has practically the same variance as the empirical copula.

In the cases of the Clayton, Frank, Gumbel, Joe, Normal, Plackett and t-Student families, the variance of the sample copula of order m for $10 \leq m \leq 20 = n$ is close to the variance of the empirical copula, and in some cases the variances of the sample copula of order m are smaller than the ones reported for the empirical copula for some values of m, see Figure 2.25, Figure 2.26 and Figure 2.27.

In the case of singular copulas such as the circular distribution, Example 3.3, Example 3.4 and Example 3.5 we see that in average the variance of the sample copula of order m is smaller than the variance for the empirical case.

For the protocol copula, see Figure 2.29, we observe that the oscillating behavior of the variances for $2 \leq m \leq n = 20$ follows the pattern of the mean, minimum and maximum given in Figure 2.16, giving greater variances to even values of m and smaller variances for odd values of m.

In the case of considering increasing and decreasing transformations in the Gumbel and Plackett families, we observe that the variances graphs in Figure 2.29 and Figure 2.30 have a similar behavior when we keep the parameter fixed in the cases GumbelID, GumbelDI and GumbelDD, and also fixed in the cases PlackettID, PlackettDI and PlackettDD. If we observe that the cases ID, DD and DI correspond of rotations of 90, 180 and 270 degrees of the original copula around the point $(1/2, 1/2)$, then we have that there is a certain invariance of the graphs, this invariance can also be observed in Figure 2.16, Figure 2.17 and Figure 2.18, and in Figure 2.19, Figure 2.20 and Figure 2.21.

For the case of mixtures we denote Mix23 the mixture of Gumbel, GumbelID and GumbelDI, Mix24 is the mixture of GID, Frank and Joe, in this two cases the first three numbers correspond to the parameters of each member of the mixture, and the last three correspond to the weights. Mix25 is the mixture of M^2 and W^2. In these cases we observe that the variances for the sample copula of order m remain below the variances of the empirical copula almost in all cases.

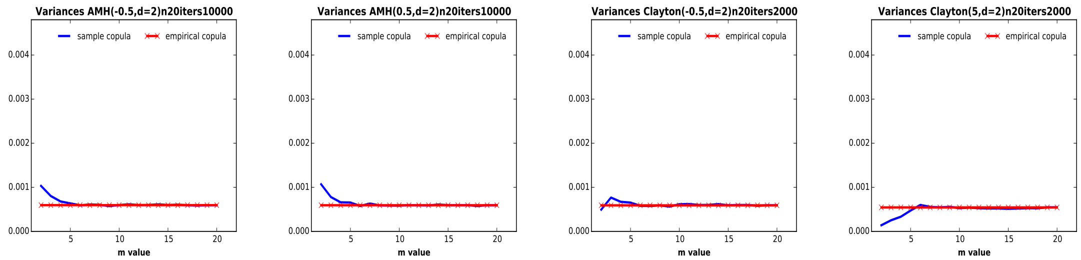

Figure 2.25: Var. of AMH(-0.148), AMH(0.192), Clay.(-0.467) and Clay.(0.844) with $n = 20$

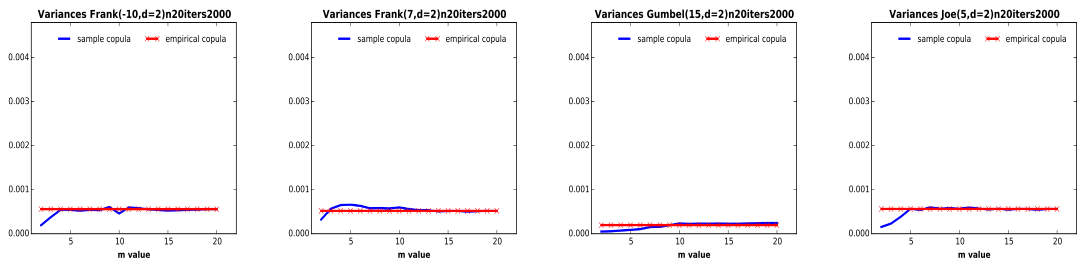

Figure 2.26: Var. of Fr.(-0.860), Fr.(0.763), Gum.(0.993) and Joe(0.854) with $n = 20$

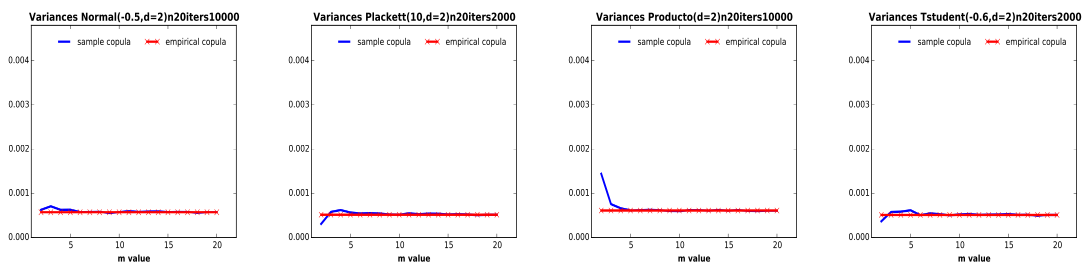

Figure 2.27: Var. of Norm.(-0.482), Plack.(0.654), Prod.(0) and t-Student(-0.582) with $n = 20$

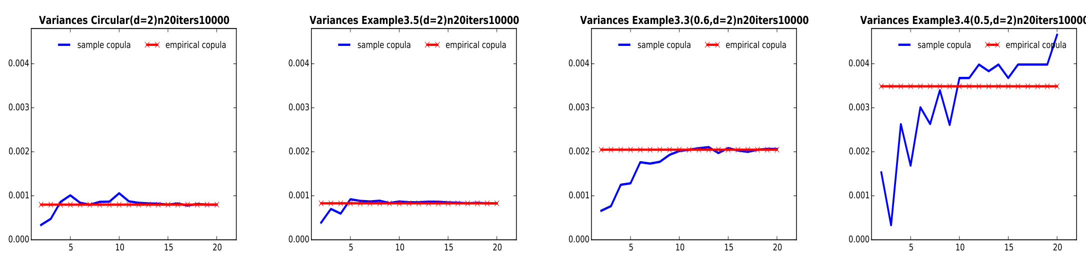

Figure 2.28: Var. of Circ.(0), Ex.3.5(.286), Ex.3.3(0.2) and Ex.3.4(0.5) with $n = 20$

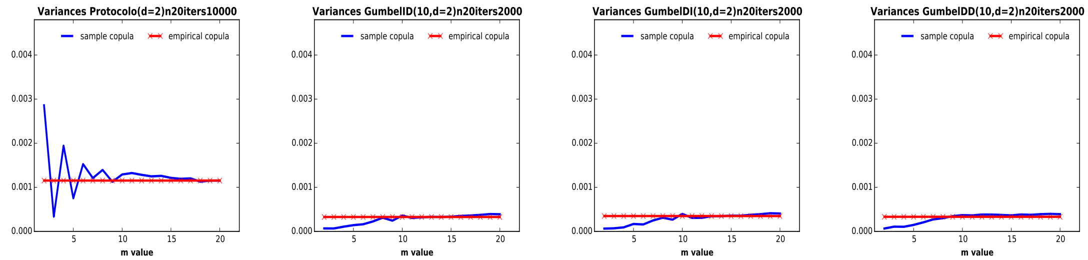

Figure 2.29: Var. of Protocol(.75), GID(-.985), GDI(-.985) and GDD(.985) with $n = 20$

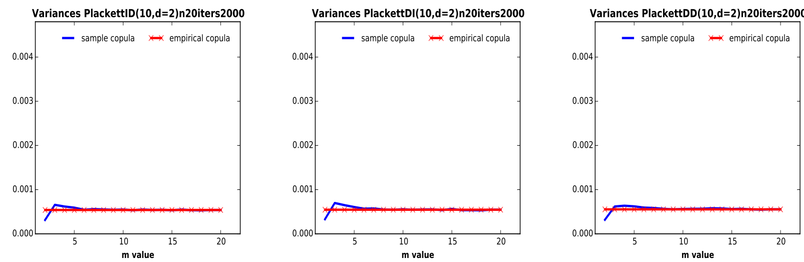

Figure 2.30: Var. of PID(-.653), PDI(-.653) and PDD(.653) with $n = 20$

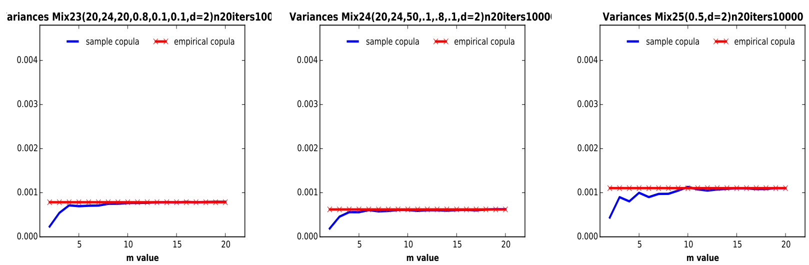

Figure 2.31: Var. of Mix23(0.5977), Mix24(0.7758) and Mix25(0)

We also performed several simulations for sample sizes $n = 30$ and $n = 50$ in these two cases we obtained very similar graphs to the ones obtained above in Figures one to thirty one, but of course in different scales. We do not present all these results because the differences are pretty much negligible.

In the next Subsection we will see that we can extend the previous results for higher dimensions.

2.4 Simulation study: dimension three

In this subsection, we study eleven different families in dimension $d = 3$, which include families of copulas such as: Clayton, Frank, Gumbel, Normal, Product and t-Student. We also included the copula M^3 and its transformations by increasing and decresing families to obtain 3-copulas with support in every diagonal of $\mathbf{I}^3$, these are examples of singular copulas. We also give a

3-dimensional version of the protocol copula to see what happens when we have an absolutely continuous 3-copula with support $[0, 1/2]^3 \cup (1/2, 1]^3$. We also include just as an interesting reference a 3-copula denoted by $C =$"W^3" such that $C(2/3, 2/3, 2/3) = 0$, that is, it behaves like W^3, which is not a 3-copula at the point $(2/3, 2/3, 2/3)$. Finally we include two families of mixtures of 3-copulas to study the behavior under asymmetric 3-copulas.

For the product copula Π^3 in $d = 3$, see Figure 2.34 first graph, the behavior of the difference between the supremum distances is quite similar to the case $d = 2$ with $n = 30$, see Figure 11. Hence, we have that C_m^{30} is a better approximation than C_{30} for every $2 \leq m \leq n = 30$. Observe that the variances have the same behavior as in dimension 2, see Figure 2.27 and Figure 2.40.

As can be seen in Figure 2.32, Figure 2.33, Figure 2.34, Figure 2.35 and Figure 2.36, the cases Clayton, Frank, Gumbel, Normal and Clayton have similar behavior, and at least at these graphs, C_m^{30} is a better approximation than C_{30} for every $6 \leq m \leq n = 30$. The same similarity holds for the graphs of the variances in Figure 2.40, Figure 2.41 and Figure 2.42.

In Figure 2.37 we show the results for singular copulas, the first graph corresponds to the copula M^3, the second, third and fourth graphs for M^3IID, M^3IDI and M^3DII, that is, if the vector (U, V, W) has copula M^3. Then the vector $(U, V, 1 - W)$ has copula M^3IID, the vector $(U, 1 - V, W)$ has copula M^3IDI and the vector $(1 - U, V, W)$ has copula M^3DII, these three copulas have supports on the other three diagonals of the unit cube $[0, 1]^3$, and its existence is guaranteed by an easy extension of Theorem 2.4.4 in [37]. As in the case of dimension $d = 2$, the variances in the four cases are zero.

The case that we called protocol consists of a 3-copula which has incomplete support given by $[0, 1/2]^3 \cup (1/2, 1]^3$, with uniform masses on each cube. We can see from the first graph in Figure 2.38, that C_2^{30} is a better approximation than C_{30} and C_4^{30} is a better approximation than C_{30}, but C_3^{30} is not a better approximation than C_{30} and C_5^{30} is not a better approximation than C_{30}, that is the oscillating pattern is also present in dimension $d = 3$. In this case C_m^{30} is a better approximation than C_{30} for every $6 \leq m \leq 30 = n$. In the case of the variance, see Figure 2.41, this oscillation is also evident with larger variances for even values of m and smaller variances for odd values of m.

The case Mixture 9 is just a convex combination of a Clayton, a Gumbel and a Frank, in Figure 2.38 we present the results when the corresponding parameters are 10, 15 and 20 respectively, and with weights $0.2, 0.3$ and 0.5 in different orders. As we can see C_m^{30} is a better approximation than C_{30} for every $5 \leq m \leq 30 = n$. The variances are almost always below the variance of the empirical copula, see Figure 2.42.

The case Mixture 10 which corresponds to a convex combination of Gumbel, GumbelID, GumbelIDI and GumbelDII, defined as in the case of M^3 above. This family is highly assymetric in all directions. We present two cases the first when the parameters are 5, 10, 15 and 20 with weights $0.1, 0.2, 0.3$ and 0.4 in three different orders, and the second when all the parameters are equal to 10 and all the weights are equal to 0.25. The results are presented in Figure 2.39, and in these cases we have that C_{30}^m is a better approximation than C_{30} for every $2 \leq m \leq 30 = n$. The variances are again almost always below the variance of the empirical copula, see Figures 2.40, 2.41 and 2.42.

The variance of the copula denoted by "W^3" is presented in Figure 2.40 last graph.

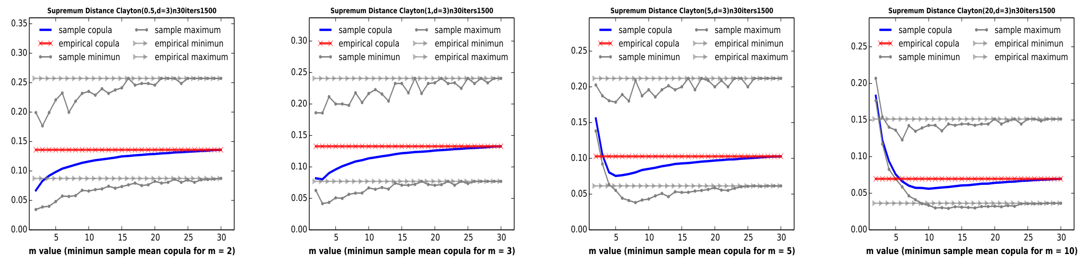

Figure 2.32: Clayton $d = 3$ for $\rho = 0.2955, 0.4784, 0.8848$ and 0.9869 with $n = 30$

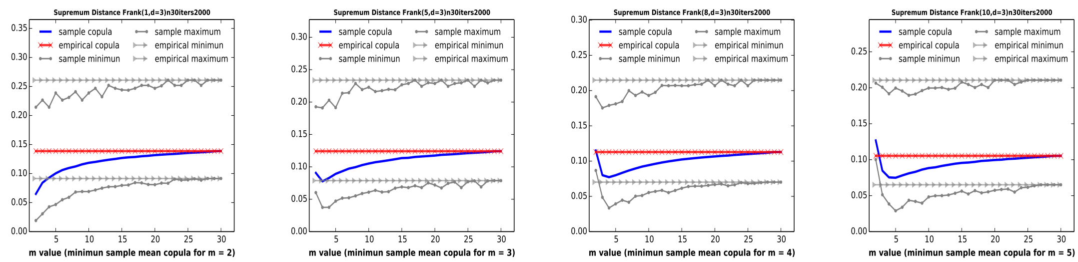

Figure 2.33: Frank $d = 3$ for $\rho = 0.1644, 0.6434, 0.8035$ and 0.8602 with $n = 30$

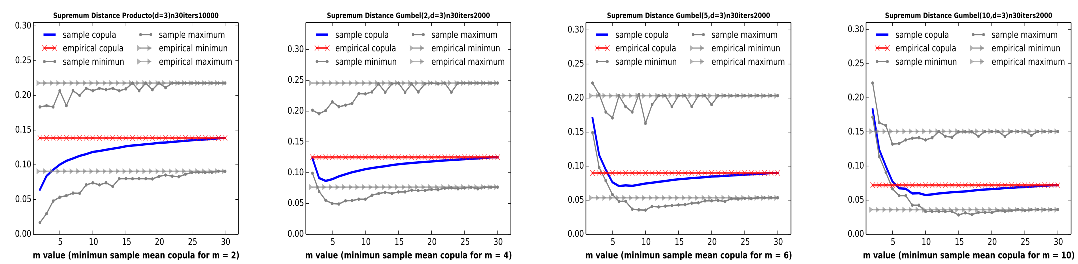

Figure 2.34: Product (0)and Gumbel $d = 3$ for $\rho = 0.6828, 0.9430$ and 0.9855 with $n = 30$

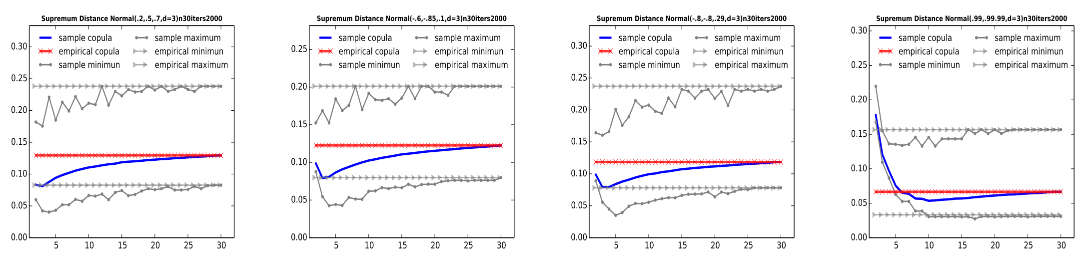

Figure 2.35: Normal $d = 3$ for $\theta = (.2, .5, .7), (-.6, -.85, .1), (-.8, -.8, .29)$ and $(.99, .99, .99)$

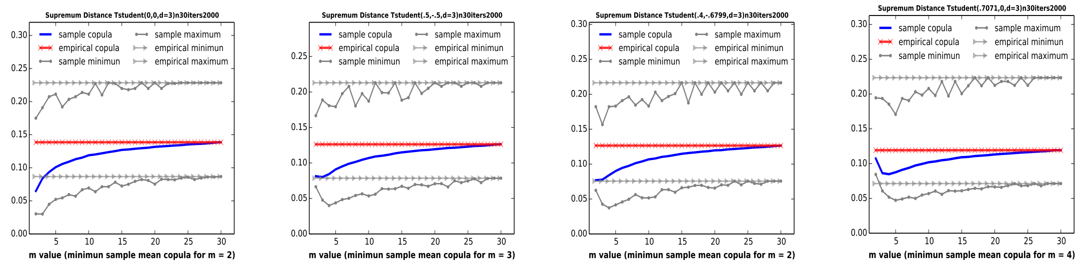

Figure 2.36: t-Student $d = 3$ for $\theta = (0, .0), (.5, -.5), (.4, -.6799)$ and $(.7071, 0)$ with $n = 30$

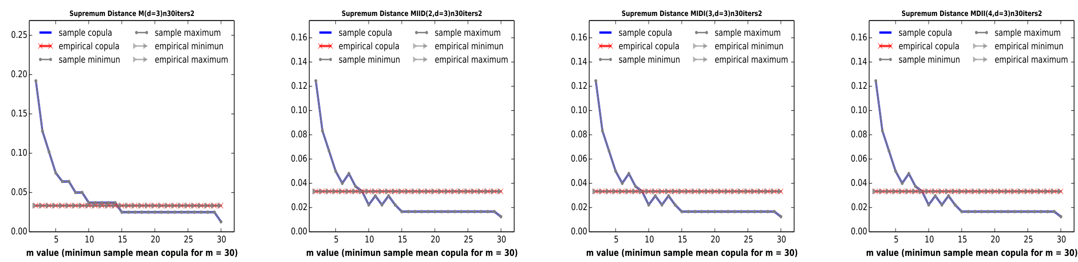

Figure 2.37: M^3, M^3IID, M^3IDI and M^3DII with $n = 30$

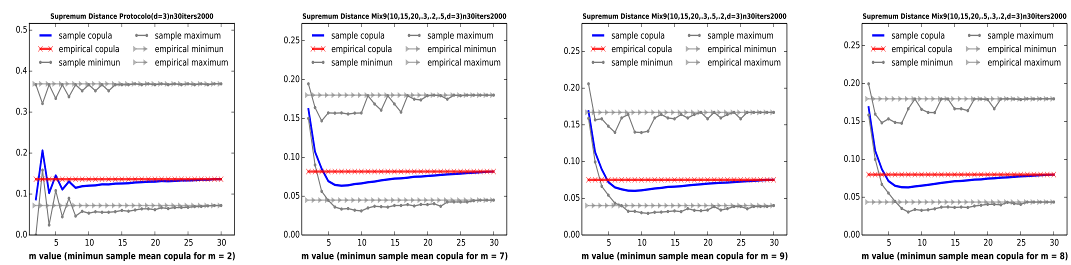

Figure 2.38: Protocol and Mixture 9 $d = 3$ for $\theta = (10, 15, 20)$ for different weights with $n = 30$

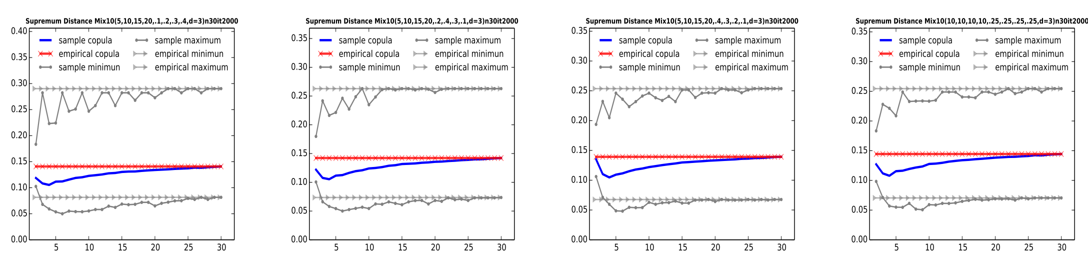

Figure 2.39: Mixture 10 $d = 3$ for $\theta = (5, 10, 15, 20)$ and $(10, 10, 10, 10)$ for different weights

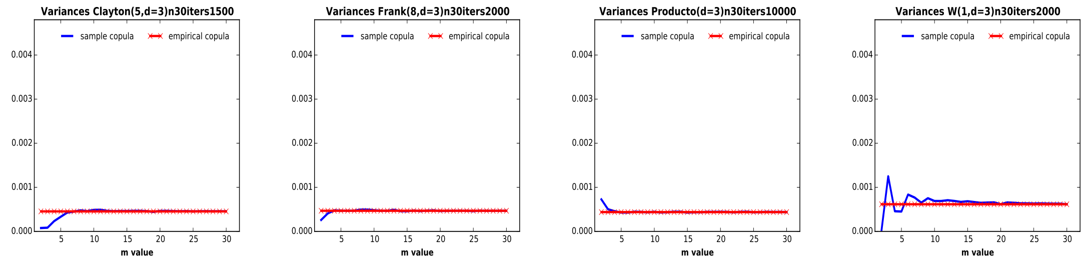

Figure 2.40: Variances of Clayton(ρ = 0.8848), Frank(ρ = 0.8035), Product and "W^3" when $n = 30$

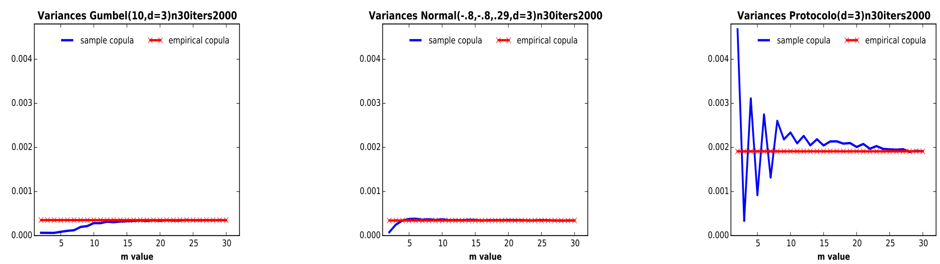

Figure 2.41: Var. of Gumb.(ρ = 0.9855), Normal($-0.8, -0.8, 0.29$) and Protocol copula when $n = 30$

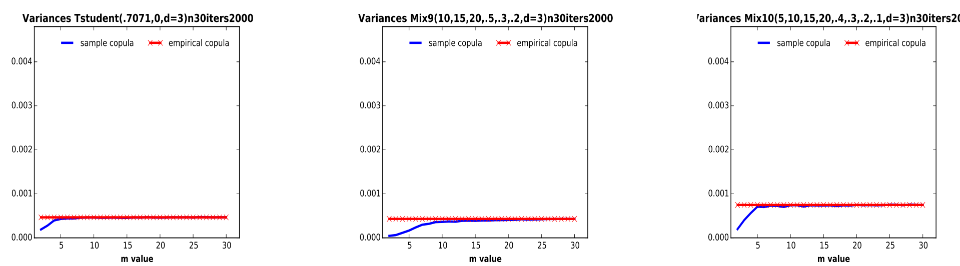

Figure 2.42: Variances of t-St.(5), Mix9(10, 15, 20, .3, .2, .5) and Mix10(5, 10, 15, 20, .4, .3, .2, .1)

As we have seen in all these simulations not only in dimension $d = 3$, but also in dimension $d = 2$, we can say that there exists at least one m such C_m^n is a better approximation than C_n, and that this value of m is smaller than n.

The results that we obtained for $n = 20$, which are not presented here, have many features in common with the case $n = 30$.

It is obvious that these results can be extended to higher dimensions. In fact, the programs that we wrote in language **R** can be easily extended to at least dimension $d = 5$ without any problems.

2.5 A method for estimating an adequate m

From all the examples in Section 3 we can observe that even for a small value of n, that is, $n = 20$ in dimension $d = 2$, we can find an m with $2 \leq m \leq n$, such that C_m^n the sample copula of order m is a better approximation than the empirical copula C_n. In fact, in all cases, we can also find a value of m that minimizes the expected value of the difference between C_m^n and the real copula C, observe all the comments between parenthesis below each of the graphs in Figures 2.1 through Figure 2.24. It is quite important to observe that this minimum is reached at $m = 2$ when the real copula C is "close" to the product copula Π^2, and that this minimum increases as the real copula C approaches the Frechet-Hoeffding bounds M^2 and W^2, in fact, as observed in Figure 2.13, in these two cases the minimum is reached at $m = n$, this also holds for the Example 3.4 of Nelsen's book which is a singular copula, see Figure 2.15.

As observed in Section 3, in the case of dimension $d = 2$, when we have an absolutely continuous copula with complete support, which by the way is the most interesting case in applications, we have a function that relates the Spearman's ρ with the value of m that minimizes the expected value of the difference between C_m^n and the real copula C.

In Figure 2.43 we graph the value of m which minimizes the expected value of the supremum distance between the sample copula C_m^n and the real copula C, for sample sizes $n = 20, 30$ and $n = 50$, for the families of absolutely continuous copulas with complete support which include AMH, Clayton, Gumbel, Frank, Joe, Plackett, Normal, t, Plackett ID, Plackett DI, Plackett DD and Gumbel DD, for different values of ρ between $\rho = 0.1$ and $\rho = .99$. As can be seen from these graphs, for ρ fixed, the value of m is quite stable and it is always smaller than the sample size n. This result provides a natural way to give an estimator of the parameter m based on the sample Spearman's ρ.

Now we propose a method to estimate m for the sample copula of order m based on a random sample of size n:

- Obtain the modified sample.
- Graph the points of the modified sample. If the data does not follow a clear pattern which indicates that the copula C is a discrete copula then
- Estimate the Spearman's rho using the sample value $\tilde{\rho}$ of the Spearman's ρ.
- If $|\tilde{\rho}| \leq 0.25$ we can take $m = 2$ for any sample size.
- If $|\tilde{\rho}| \geq .95$ we have to take a large value of m. Let us say $10 \leq m \leq 15$.

- If $0.25 < |\tilde{\rho}| < 0.95$ take $3 \leq m \leq 10$, with a linear approximation which depends on the value of $\tilde{\rho}$.

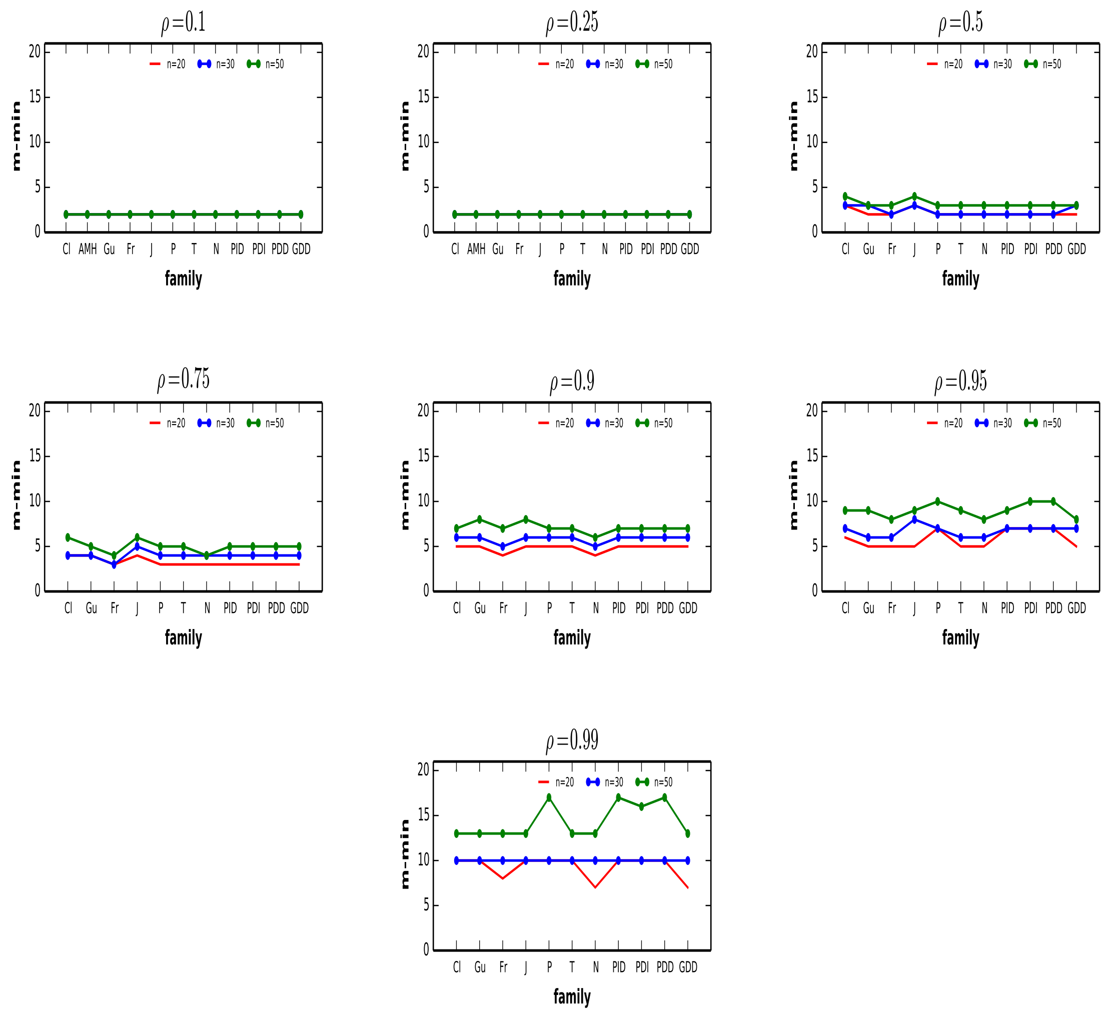

Figure 2.43

If the modified sample shows a pattern which indicates a singular support we may need to take larger values of m, but not larger than 15.

We also observe that for $d \geq 3$ the same methodology can be used for Archimedean copulas, since all of their margins have the same parameter, hence, the same value of Spearman's ρ for each pair of random variables. This behavior was observed in Figure 2.32, Figure 2.33 and Figure 2.34.

For dimension $d \geq 3$ we are still working on the possibility of estimating m using the averages of all the estimated Spearman's ρ for every pair of random variables.

Also, for dimension $d \geq 3$, we know from all the Figures, that there exists a value $2 \leq m_0 \leq n$ such that for every $m_0 \leq m \leq n$, C^n_m is a better approximation than C_n, of course, the minimum value of m_0 is among them. We would like to find a method of estimation of m, given a fix sample $(\underline{X}_1, \ldots, \underline{X}_n)$ of size n from an unknown continuous joint distribution function H with copula C, such that $m_0 \leq m \leq n$. Since the copula C is also unknown, we will use all the remarks that we have made in order to establish a "measure of discrepancy" between the sample and the product copula Π^2, which may be used to estimate the value of m. Besides, from Figure 2.32 through Figure 2.42 we observe that all the previous comments apply to $n = 30$ in dimension $d = 3$, this also holds when $n = 20$.

We start by defining our proposal of the "measure of discrepancy" in any dimension $d \geq 2$ for n a fixed positive integer.

Definition 2.18 *Let $2 \leq m \leq n$ and let $(\underline{X}_1, \ldots, \underline{X}_n)$ be a random sample from a random vector $\underline{X}$ of dimension $d \geq 2$ with continuous joint distribution H or d-copula C. Let $U_n = (\underline{Y}_1, \ldots, \underline{Y}_n)$ be the corresponding modified sample. Define $(Q^m_{\underline{i}})_{\underline{i} \in I^d_m}$, $(R^m_{\underline{i}})_{\underline{i} \in I^d_m}$, $s^{n,(m)}_{i_1,\ldots,i_d}$, S^n_m and C^n_m as in equations (15), (16), (17), (18) and (19) respectively. Let Π^d the product d-copula, since C^n_m the sample d-copula of order m is always a d-copula, define for every $(i_1, \ldots, i_d) \in I^d_m$*

$$V_{C^n_m}(Q^m_{i_1,\ldots,i_d}) = s^{n,(m)}_{i_1,\ldots,i_d} \quad and \quad V_{\Pi^d}(Q^m_{i_1,\ldots,i_d}) = \lambda^d(Q^m_{i_1,\ldots,i_d}), \tag{30}$$

where λ^d is the Lebesgue measure on $(I\!R^d, \mathcal{B}(I\!R^d))$. We define

$$d_{TV}(C^n_m, \Pi^d) = \frac{1}{2}\int_{[0,1]^2} |f_{C^n_m} - f_{\Pi^d}| d\lambda^d = \frac{\sum_{\underline{i} \in (I_m)^d} \left| V_{C^n_m}(Q^m_{\underline{i}}) - V_{\Pi^d}(Q^m_{\underline{i}}) \right|}{2}, \tag{31}$$

We first observe that from equations (30), (17) and (31) we have that

$$d_{\mathrm{TV}}(C^n_m, \Pi^d) = \frac{\sum_{\underline{i} \in (I_m)^d} \left| s^{n,(m)}_{\underline{i}} - \lambda(Q^m_{\underline{i}}) \right|}{2} = \frac{\sum_{\underline{i} \in (I_m)^d} \left| \frac{\mathrm{card}(R^m_{\underline{i}} \cap U_n)}{n} - \lambda(Q^m_{\underline{i}}) \right|}{2}. \tag{32}$$

So, if we assume that $n = m^d$ and that $\mathrm{card}(R^m_{\underline{i}} \cap U_n) = 1$ for every $\underline{i} \in I^d_m$, we have from equation (32) that $d_{\mathrm{TV}}(C^n_m, \Pi^d) = 0$.

Now, we assume that the sample of size n is taken from the the copula M^d, then clearly the modified sample is given by $U_n = (\underline{Y}_1, \ldots, \underline{Y}_n)$, where $Y_j = (j/n, j/n, \ldots, j/n)$ for every $j \in \{1, 2, \ldots, n\}$.

First, if m divides n then we know from Theorem 2.14, part iii) that $(Q^m_{\underline{i}})_{\underline{i} \in I^d_m}$ coincides with $(R^m_{\underline{i}})_{\underline{i} \in I^d_m}$, so, $\lambda^d(Q^m_{\underline{i}}) = (1/m)^d$ for every $\underline{i} \in I^d_m$. Since the modified sample lies on the main diagonal of the unit d cube $\mathbf{I}^d$, then we have from equation (21) that for every $i \in I_m$ if $R^m_{\underline{i}=(i,i,\ldots,i)} = [(i-1)/m, i/m]^d$ then $\frac{\text{card}(R_{\underline{i}} \cap U_n)}{n} = (n/m)/n = 1/m$. Hence, using equation (32) we have that

$$d_{\mathrm{TV}}(C^n_m, \Pi^d) = \left\{ m\left(\frac{1}{m}\left(\frac{m^{d-1}-1}{m^{d-1}}\right)\right) + (m^d - m)\left(\frac{1}{m}\right)^d \right\} \Big/ 2 = 2\left(\frac{m^{d-1}-1}{m^{d-1}}\right) \Big/ 2 = 1 - \frac{1}{m^{d-1}}.$$

Second, if m does not divide m then Theorem 2.14 part i) we have that

$$Q^m_{\underline{i}=(i,i,\ldots,i)} = [(1/n)\lfloor((i-1)\cdot n)/m\rfloor, (1/n)\lfloor(i \cdot n)/m\rfloor]^d$$

for every $i \in I_m$. So, in this case, $\lambda^d(Q^m_{\underline{i}=(i,i,\ldots,i)}) = (1/(n^d))(\lfloor(i\cdot n)/m\rfloor - \lfloor((i-1)\cdot n)/m\rfloor)^d$ and $V_{C^n_m}(Q^m_{\underline{i}=(i,i,\ldots,i)}) = (1/n)(\lfloor i/m\rfloor - \lfloor(i-1)/m\rfloor)$ for every $i \in I_m$. So, using the fact that the sum of all the volumes of the d-boxes in the partition $(Q^m_{\underline{i}})_{\underline{i} \in I^d_m}$ is one, it is easy to see that

$$\begin{aligned} d_{\mathrm{TV}}(C^n_m, \Pi^d) &= \left(\frac{m^{d-1}}{m^{d-1}-1}\right)\left(1 - \sum_{i=1}^{m} \frac{1}{n^d}\left(\lfloor(i\cdot n)/m\rfloor - \lfloor((i-1)\cdot n)/m\rfloor\right)^d\right) \\ &\leq \left(\frac{m^{d-1}}{m^{d-1}-1}\right)\left(1 - m\left(\frac{1}{m^d}\right)\right) = 1, \end{aligned}$$

where the inequality follows from Lagrange multiplier applied to the function $h(x_1, \ldots, x_m) = 1 - (x_1^d + \cdots + x_m^d)$ with conditions $x_1 + x_2 + \ldots + x_m = 1$ and $x_1, \ldots, x_m \geq 0$.

Hence, if we change the denominator 2 by $K_{m,d} = 2(1 - 1/m^{d-1})$ in equation 31, we have that $0 \leq d_{\mathrm{TV}}(C^n_m, \Pi^d) \leq 1$, and this what we will do in what follows.

In order to see how this density difference works, we show the averages of 10000 simulations when the sample size is $n = 100$ and we evaluate $\eta_{m,2}(C^{100}_m)$ for m between 2 and 10 for several of the families studied in Section 3, we use only values of $\rho \geq 0$, but the results for $\rho < 0$ follow the same pattern.

In Figures 2.44 and 2.45, we have the results for the families Clayton,Frank, Gumbel, Plackett, Normal and t-Student. It can be easily observed that for all these families of copulas which are absolutely continuous and with complete support. the graphs are almost identical, which indicates that they can be used to estimate the value of the order m of the sample d-copula. We have similar results for $d = 3$ and we are working in a resampling method to estimate the graph given a fixed sample of size n.

As can be see in Figure 2.46, the graphs for singular copulas can be quite different as the absolutely continuous case, and it depends strongly on the support of the copula. But, as observed above, in these cases we always need larger values of m. Observe that for the last graph which corresponds to the copula M^2, the graph is the constant 1, as expected.

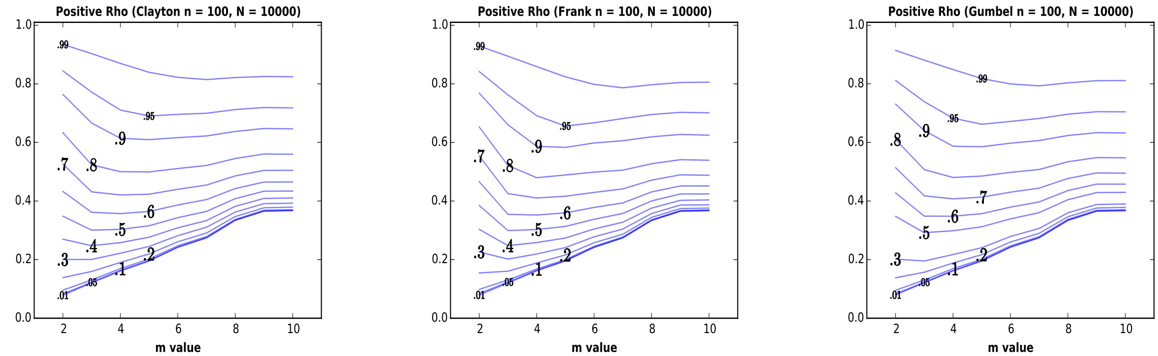

Figure 2.44: Averages of $\eta_{m,2}$ for the Clayton, Frank and Gumbel families with positive ρ

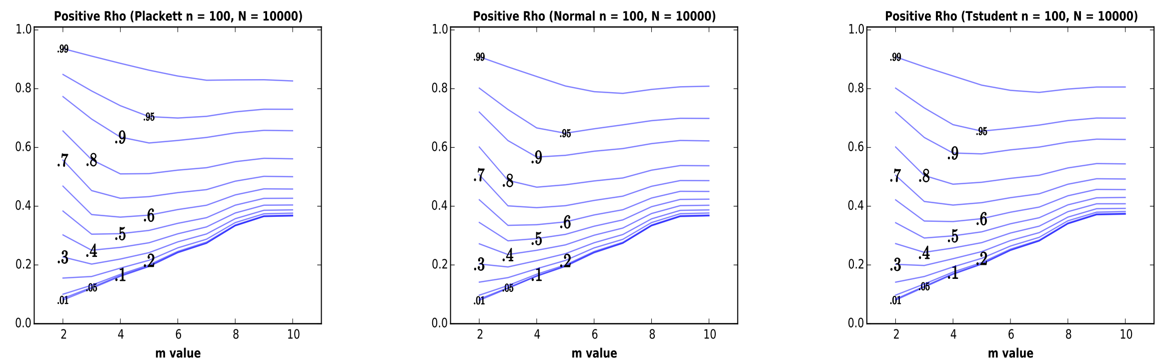

Figure 2.45: Averages of $\eta_{m,2}$ for the Plackett, Normal and tStudent families with positive ρ

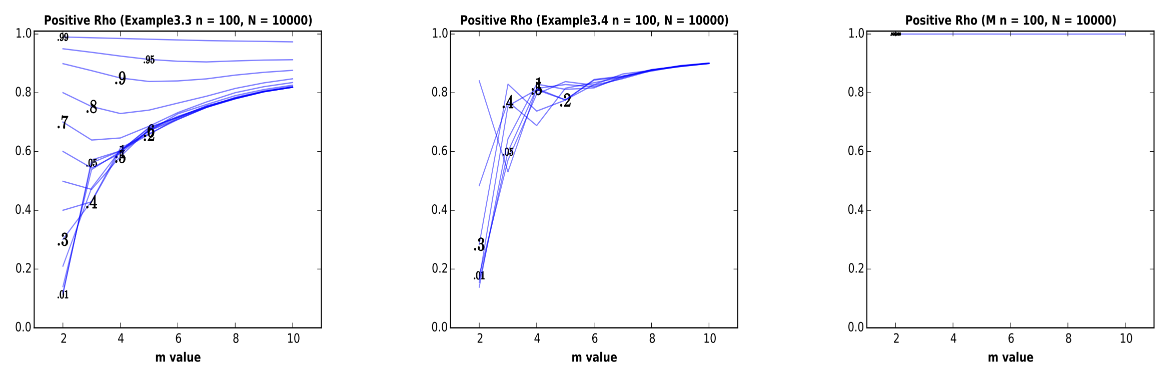

Figure 2.46: Averages of $\eta_{m,2}$ for the Example 3.3, Example 3.4 and M^2 families with positive ρ

A last very important result is the following: Recall that the total variation distance of two probability measures P and Q in $(\mathbb{R}^d, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra, with Radon-Nykodim's derivatives f_P and f_Q is defined by $d_{\mathrm{TV}}(P, Q) = \sup_{A \in \mathcal{B}} |P(A) - Q(A)| = \frac{1}{2} \int_{\mathbb{R}^d} |f_P - f_Q| d\lambda \leq 1$, where λ is the Lebesgue measure in $(\mathbb{R}^d, \mathcal{B})$. Let us assume that we take a random sample from Π^d the independent copula of size $n \geq 2$ and we we take $m = n$ in the definition of the sample copula, which corresponds to the linear B-spline copula construction in [43]. Since in this case, we are considering the uniform partition of order n of $[0, 1]^d$, that is, $(R^n_{\underline{i}})_{\underline{i} \in I^d_n}$, using part v) of Theorem 2.14, we have that only n of the d-boxes in the uniform partition have density n^{d-1} and the remaining boxes have density 0. Let J be the subset of n indices $\underline{i} \in I^d_n$ with positive density. Then, using the above definition we have that

$$
\begin{aligned}
d_{\mathrm{TV}}(\Pi^d, C^n_n) &= \frac{1}{2} \int_{[0,1]^d} |f_{\Pi^d} - f_{C^n_n}| d\lambda \\
&= \frac{1}{2} \left(\sum_{\underline{i} \in J} \int_{R^n_{\underline{i}}} |1 - n^{d-1}| d\lambda + \sum_{\underline{i} \in (I^d_n \setminus J)} \int_{R^n_{\underline{i}}} |1 - 0| d\lambda \right) \\
&= \frac{1}{2} \left(\frac{n(n^{d-1} - 1)}{n^d} + \frac{(n^d - n)}{n^d} \right) \\
&= \frac{1}{2} \left(1 - \frac{1}{n^{d-1}} + 1 - \frac{1}{n^{d-1}} \right) \\
&= 1 - \frac{1}{n^{d-1}}. \qquad (33)
\end{aligned}
$$

The last equality implies that, with probability one, if we let n go to infinity then $d_{\mathrm{TV}}(\Pi^d, C^n_n) \uparrow 1$. Hence, it can be thought as an "anti" Glivenko-Cantelli's result. Even more, if we let the dimension d go to infinity for any fixed $n \geq 2$ then again $d_{\mathrm{TV}}(\Pi^d, C^n_n) \uparrow 1$. This argument tell us that using $m = n$ is not a really good option at all.

On the other hand, if we take $d = 2$, $m = 2$ and we assume that the sample size n is a multiple of $m^d = 4$, and that we are sampling from Π^2. Then with positive probability $((n/2)!)^4/(n! \cdot ((n/4)!)^4)$, see [26], we have that each 2-box of the uniform partition of order $m = 2$ has exactly $n/4$ points. So, using Theorem 2.14, we have that the density of the sample copula of order 2 is one on each 2-box of the uniform partition. But in this case we have that $d_{\mathrm{TV}}(C^2_n, \Pi^2) = 0$. In Figure 11 we can see that when $n = 20$ the minimum value of the simulations attains 0, when $m = 2$. In fact, it is not difficult to see that $0 \leq d_{\mathrm{TV}}(C^2_n, \Pi^2) \leq 1/2$ with probability one. Of course, the above argument works also when $d > 2$, $m \geq 2$ and the sample size n is a multiple of m^d.

Finally, we have observed using simulations, that the total variation distance may be used to measure the distance between C_n^m, the d-sample copula of order m, and $C^{(m)}$, the checkerboard approximation of size m, and that when the sample size increases to infinity as a multiple of m, then we obtain a Glivenko-Cantelli's theorem. Observe that this happens even using the strongest distance, that is, the total variation, which in this case is surprisingly easy to evaluate.

In Tables 1 to 7, we present the results of evaluating the total variation distance between C_n^m the sample copula of order $m = 10$ and $C^{(10)}$ the checkerboard approximation, with 1000 simulations of sample sizes varying between $n = 10$ and $n = 50000$, the rows include basic statistics such as mean, variance, minimum and maximum. We used some of the absolutely continuous families in Section 3, specifically the AMH, Clayton, Frank, Gumbel and Plackett copulas with different values of Spearman's ρ. We also include with $m = 15$ Π^2 the product copula in dimension 2, with n varying from 15 to 60000, and also, Π^3 the product copula in dimension 3 when $m = 2$, when n varies from $n = 2$ up to $n = 49152$. As can be seen in all cases, the statistics mean, minimum and maximum decrease to zero as n increases, and the variance also decrease to zero for n large, which gives evidence of the existence of a Glivenko-Cantelli's theorem for the total variation distance. Observe also, that the first columns in Table 6 and Table 7 agree with the result given in equation (33). Even more, in the two last tables it is obvious that the checkerboard approximations coincides with the real copulas, that is, Π^2 and Π^3, so, we have a real Glivenko-Cantelli's theorem between C_n^m and C the true copula.

Table 1 Total Variation Distance for AMH ($\rho = 0.3451$) with $m = 10$

n	10	100	500	1000	2000	10000	50000
mean	0.8874	0.3532	0.1573	0.1108	0.0786	0.0350	0.0156
variance	0.00015	0.00079	0.00017	0.00008	0.00004	8 e-06	1 e-06
minimum	0.8577	0.2605	0.1189	0.0758	0.0582	0.0248	0.0118
maximum	0.9244	0.4879	0.2037	0.1443	0.1032	0.0436	0.0204

Table 2 Total Variation Distance for Clayton ($\rho = 0.9582$) with $m = 10$

n	10	100	500	1000	2000	10000	50000
mean	0.5915	0.1757	0.0788	0.0555	0.0389	0.0175	0.0078
variance	0.00720	0.00090	0.00014	0.00007	0.00003	7 e-06	1 e-06
minimum	0.4521	0.1013	0.0459	0.0304	0.02236	0.0104	0.0043
maximum	0.8579	0.2835	0.1153	0.0847	0.0616	0.0279	0.0124

Table 3 Total Variation Distance for Frank ($\rho = -0.64348$) with $m = 10$

n	10	100	500	1000	2000	10000	50000
mean	0.8559	0.3316	0.1456	0.1029	0.0730	0.0324	0.0145
variance	0.00050	0.00072	0.00016	0.00008	0.00005	9 e-06	1 e-06
minimum	0.7966	0.2514	0.1019	0.0749	0.0505	0.0240	0.0107
maximum	0.9254	0.4144	0.1873	0.1325	0.0946	0.0433	0.0192

Table 4 Total Variation Distance for Gumbel ($\rho = 0.84816$) with $m = 10$

n	10	100	500	1000	2000	10000	50000
mean	0.7539	0.2658	0.1206	0.0854	0.0604	0.0268	0.0119
variance	0.00268	0.00085	0.00016	0.00007	0.00004	8 e-06	1 e-06
minimum	0.6556	0.1887	0.0841	0.0553	0.0407	0.0188	0.0080
maximum	0.9092	0.3548	0.1659	0.1141	0.0825	0.0363	0.0169

Table 5 Total Variation Distance for Plackett ($\rho = -0.90005$) with $m = 10$

n	10	100	500	1000	2000	10000	50000
mean	0.7110	0.2351	0.1105	0.0782	0.0552	0.0246	0.0109
variance	0.00509	0.00082	0.00016	0.00007	0.00004	8 e-06	1 e-06
minimum	0.5367	0.1460	0.0715	0.0528	0.0391	0.01644	0.0077
maximum	0.8907	0.03555	0.1542	0.1017	0.0896	0.0350	0.01570

Table 6 Total Variation Distance for Product $d = 2$ ($\rho = 0$) with $m = 15$

n	15	225	750	1500	3000	12000	60000
mean	0.9333	0.3421	0.2067	0.1453	0.1017	0.0509	0.0227
variance	0	0.00044	0.00011	0.00005	0.00002	7 e-06	1 e-06
minimum	0.9333	0.2755	0.1746	0.1253	0.0855	0.0415	0.0187
maximum	0.9333	0.4044	0.2382	0.1702	0.1180	0.0603	0.0274

Table 7 Total Variation Distance for Product $d = 3$ ($\rho = 0$) with $m = 2$

n	2	16	48	192	6144	24576	49152
mean	0.7500	0.1890	0.1150	0.0569	0.0101	0.0050	0.0035
variance	0	0.00567	0.00191	0.00045	0.00001	3 e-06	1 e-06
minimum	0.7500	0	0	0.0104	0.00162	0.0010	0.00038
maximum	0.7500	0.3750	0.29166	0.1718	0.0273	0.0112	0.0080

The sample d-copula of order m is already a d-copula which provides a quasi-nonparametric method to estimate a d-copula C. Once m has been chosen, it becomes a nonparametric estimator of the checkerboard approximation $C^{(m)}$, which is a good estimator of C, even for m relatively small. After C_m^n has been constructed, since we know that its density is constant on each of the d-boxes of the partition of $[0,1]^d$ that it generates, it is absolutely trivial to generate random samples of size $N \geq 2$ from it. We have written a program using the package **R**, that generates these samples. Using this program; First, we generated a sample of size n, that we called original sample size, denoted by OSS, from this sample we obtain the original modified sample generated from the absolutely continuous copulas Clayton, AMH, Gumbel, Plackett, Normal, t-Student and Product, respectively, using different values of Spearman's rho . Second, for a given value of m, we obtained C_m^n the sample copula of order m, corresponding to each of the original modified samples. Third, using our program we generated a sample from the copula C_m^n in each case of size N, that we called simulated sample size, denoted by SSS. In Figures 2.47 through Figure 2.53, we took OSS = 5000, SSS = 5000 and $m = 50$, on the left hand side of each Figure we show the original modified sample, and on the right hand side we show the simulated sample, as can be easily seen, both samples are quite similar in all cases, which indicates that C_m^n , the sample copula of order 50 is a really good estimator of the original copula C.

In Figure 2.54, we took OSS = 20, $m = 20$, that is, the case $m = n$, SSS = 1000, and the sample was taken from Π^2 the product copula, in this case the original modified sample on the left hand side looks like an independent sample, but the simulated sample on the right hand side does not look at all, like an independent sample on $[0,1]^2$. In this case the sample copula C_{20}^{20} corresponds to the linear B-spline given in [43], which generates poor samples for the independence copula Π^2. In Figures 2.55, 2.56 2.57 and 2.58, we sample from M^2 , Example 3.3, Example 3.4 and Example 3.5 in Nelsens book [37]. We took OSS = 5000, SSS = 5000 and m = 50, except on Figure 2.58, in which m = 100. As we can see in all these four Figures the simulated samples *replicate* the real supports of these four singular copulas. The only difference is that the supports of the simulated samples are *slightly enlarged* versions (mounted on little squares) of the real support, due to the definition of the sample copula of order m. However, from looking at these simulated samples, it is obvious, that we can deduce if the original copulas are singular.

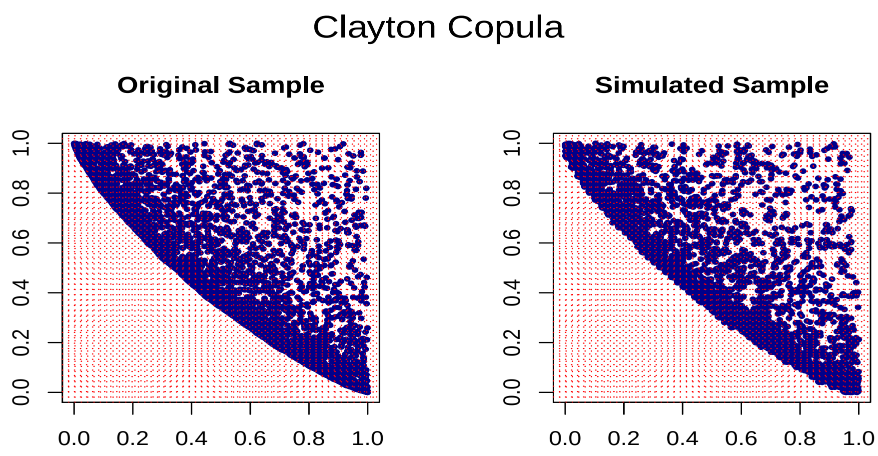

Figure 2.47: $\rho = -0.7921$, OSS=5000, SSS=5000, $m = 50$

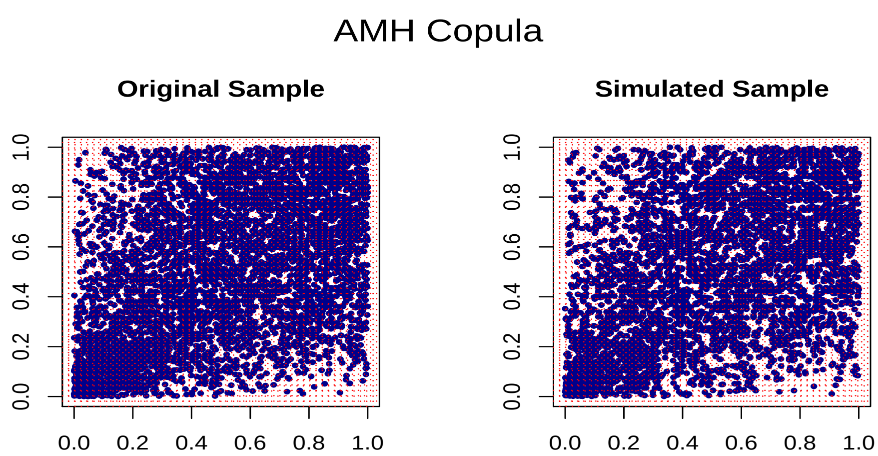

Figure 2.48: $\rho = 0.3451$, OSS=5000, SSS=5000, $m = 50$

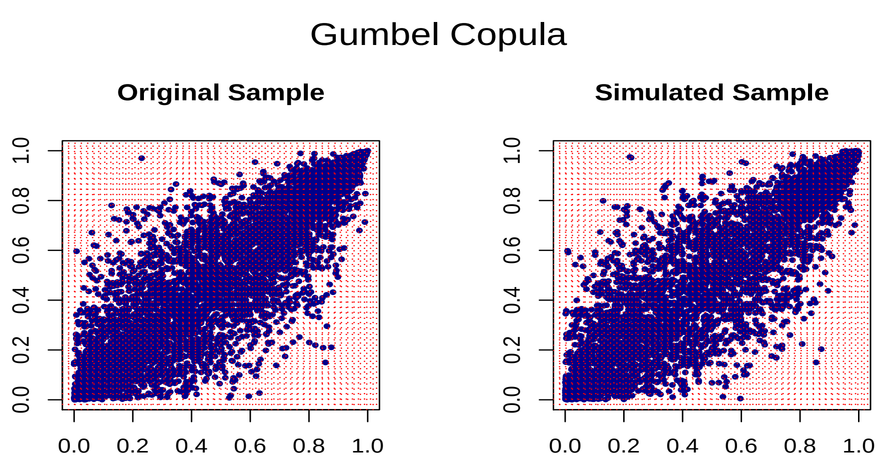

Figure 2.49: $\rho = 0.4412$, OSS=5000, SSS=5000, $m = 50$

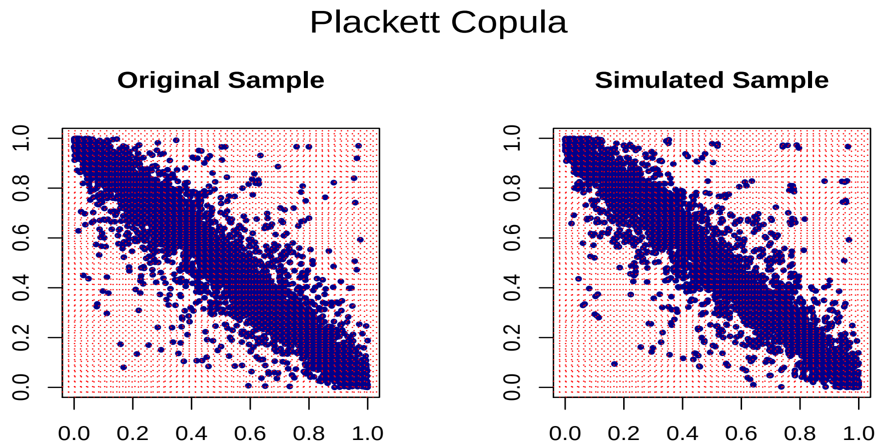

Figure 2.50: $\rho = -0.9262$, OSS=5000, SSS=5000, $m = 50$

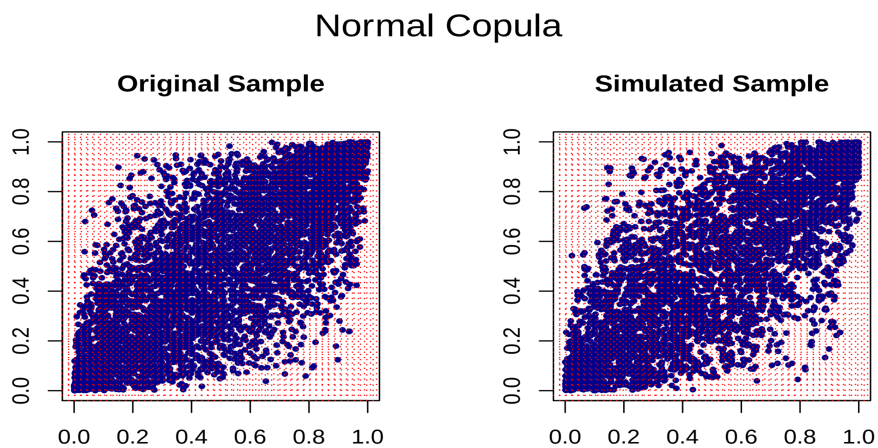

Figure 2.51: $\rho = 0.7341$, OSS=5000, SSS=5000, $m = 50$

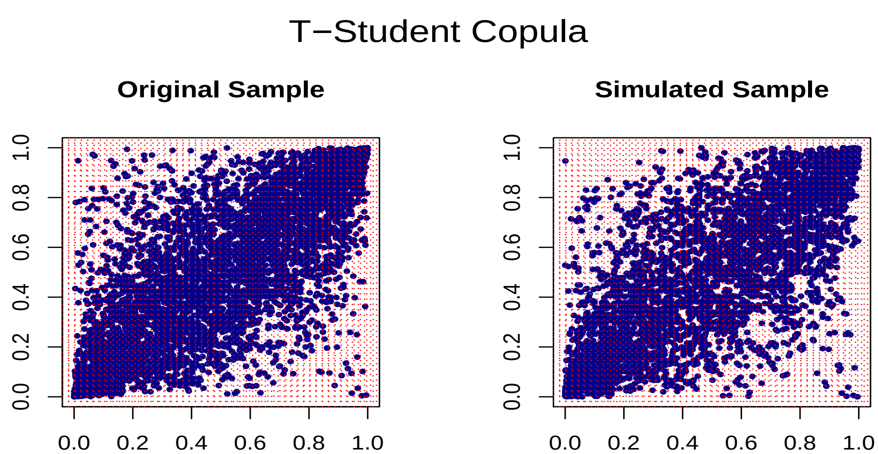

Figure 2.52: $\rho = 0.7341$, OSS=5000, SSS=5000, $m = 50$

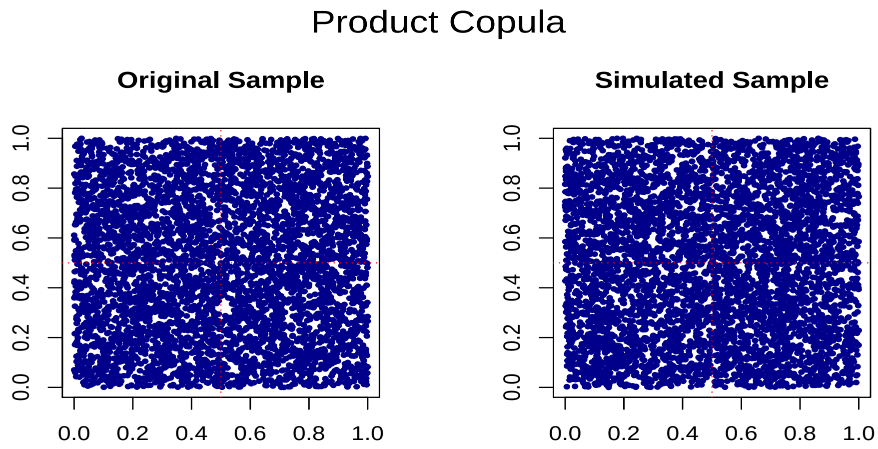

Figure 2.53: $\rho = 0$, OSS=5000, SSS=5000, $m = 50$

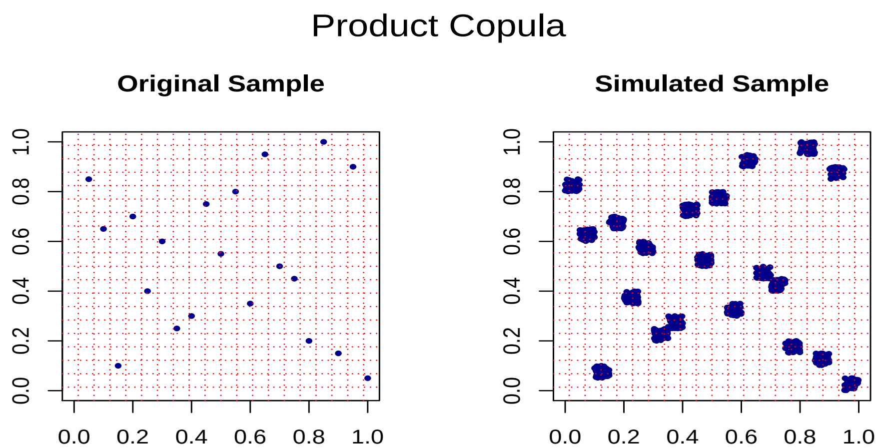

Figure 2.54: $\rho = 0$, OSS=20, SSS=1000, $m = 20$

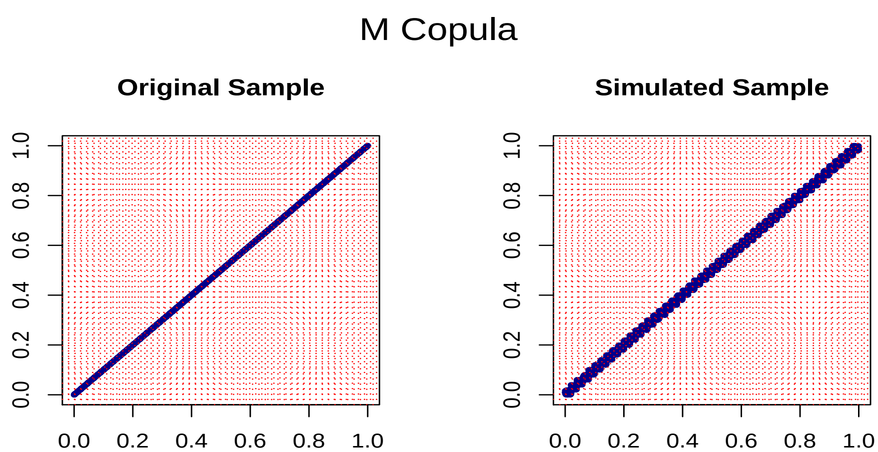

Figure 2.55: $\rho = 1$, OSS=5000, SSS=5000, $m = 50$

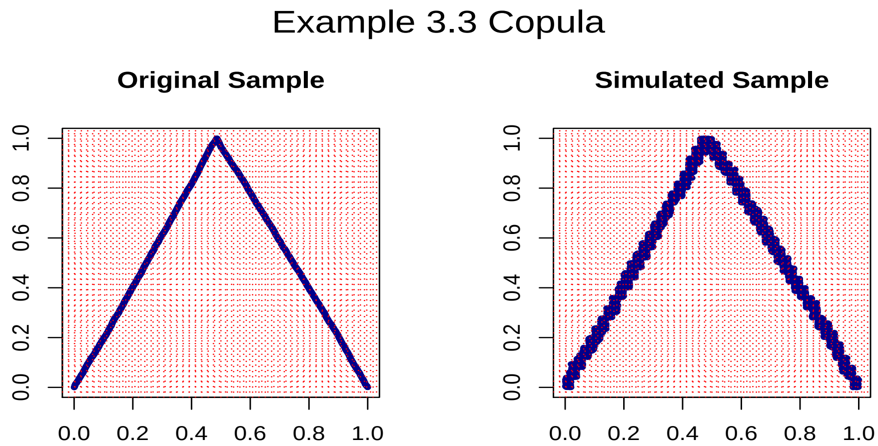

Figure 2.56: ρ = 0, OSS=5000, SSS=5000, m = 50

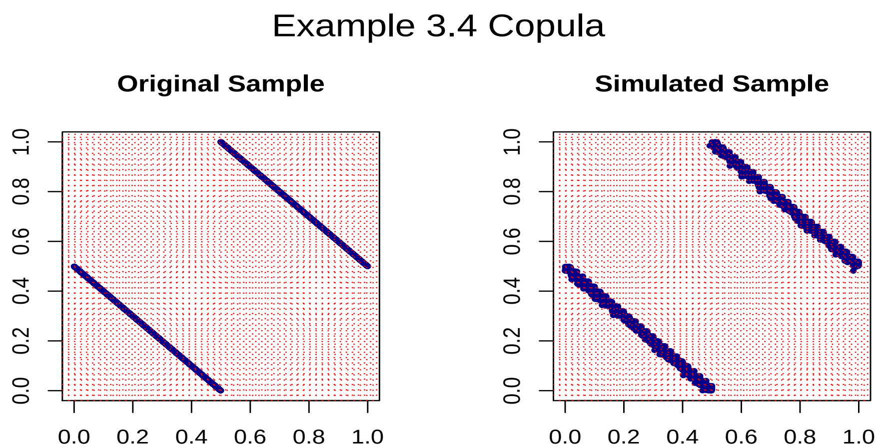

Figure 2.57: ρ = 0.75, OSS=5000, SSS=5000, m = 50

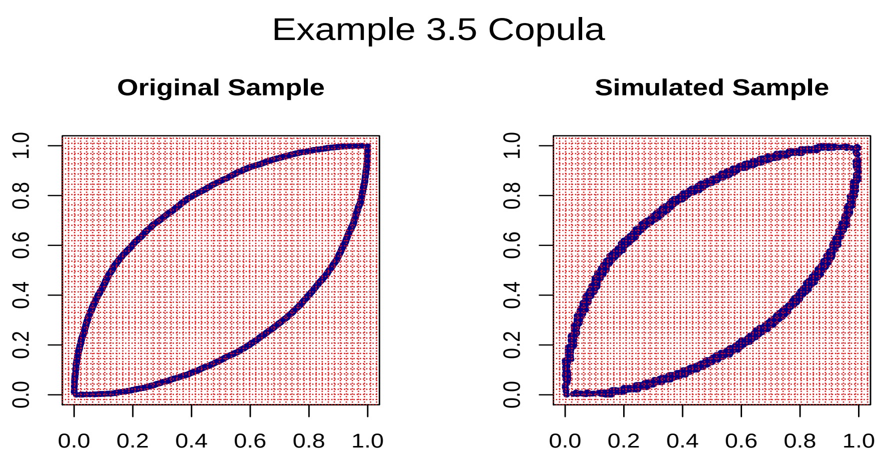

Figure 2.58: ρ = 0.286, OSS=5000, SSS=5000, m = 100

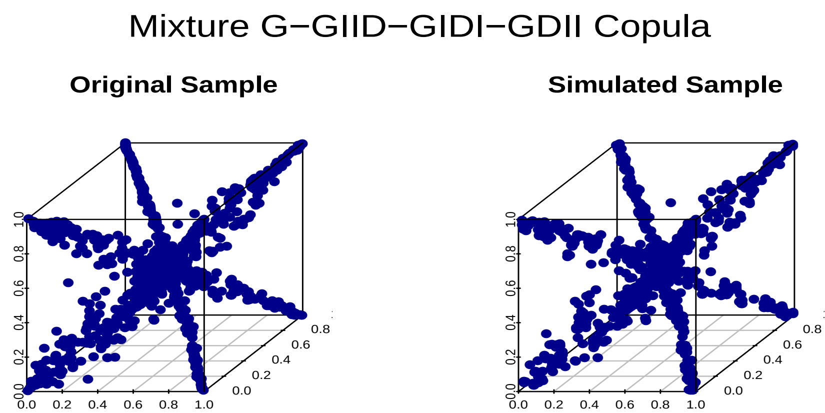

Figure 2.59: Dimension 3, model = 10, OSS=2000, SSS=2000, $m = 50$

Finally, in Figure 2.59 we sample in dimension $d = 3$ from model 10, which is a mixture of Gumbel, GumbelIID, GumbelIDI and GumbelDII, and allows us to have different dependencies on the vertices of the cube $[0, 1]^3$, with OSS = 2000, SSS = 2000 and $m = 50$, by its shape we call this the *pinhata copula*, and both samples look alike. From all the Figures it is clear that the sample copula of order m is a very nice estimator of the original copula C, when the sample size n is not too small, and the value of the order m is not close to n. In general, we recommend to take values of $m \leq n^{1/d}$.

3 Moments of some of the random variables associated with the sample copula

In this section we show the characteristics of the distributions associated to the count of the boxes generated by the uniform partition of order m of $\mathbf{I}^k = [0,1]^k$.

We obtained similar results to the ones in [8], the identities corresponding to permutations and combinations, used in this section, can be found in [27]. We use the following notation to indicate the **k-permutations of** n

$$P_k^n = \frac{n!}{(n-k)!}.$$

To note that our case of study corresponds to observations of the modified sample from a d-copula C or a continuous joint distribution H. Let $I_m = \{1, \cdots, m\}$, if we consider the sample without applying the rank transformation, then by [24] and [25] and considering $m \geq 2$, $d \geq 2$ and for every $\underline{i} = (i_1, \cdots, i_d) \in I_m^d$

$$R_{\underline{i}}^m = \left\langle \frac{i_1 - 1}{m}, \frac{i_1}{m} \right] \times \cdots \times \left\langle \frac{i_d - 1}{m}, \frac{i_d}{m} \right]$$

the uniform partition of size m of $\mathbf{I}^d = [0,1]^d$, where the notation "$\langle$" indicates "(" if $i_j > 1$ and "[" if $i_j = 1$, for all $j \in I_m$, the random vector

$$\left(N_{\underline{i}} \mid \underline{i} = (i_1, \cdots, i_d) \in I_m^d \right)$$

have a multinomial distribution with parameters $p_{\underline{i}} = Vol_C(R_{\underline{i}}^m)$, the C-volume of the region $R_{\underline{i}}$, for all $\underline{i} = (i_1, \cdots, i_d) \in I_m^d$; [25] affirms that it is only necessary to consider $d(m-1)+1$ of these parameters as free, the remaining parameters are determined by the properties that satisfy a copula.

For the case of the original sample, in dimension $d = 2$, we have the following lemma (see [25], for the proof of this result),

Lemma 3.1 *Let $m \geq 2$, we consider the uniform partition R_{ij}, $i, j \in \{1, \cdots, m\}$, of $\mathbf{I}^2 = [0,1]^2$, and given a 2-copula C we define, for all $i, j \in \{1, \cdots, m\}$*

$$p_{ij} = V_C(R_{ij})$$

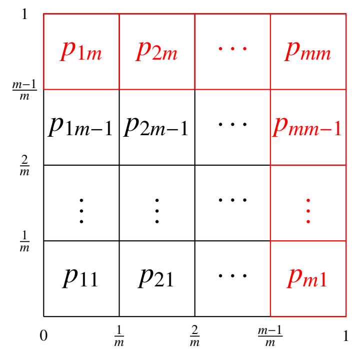

Figure 3.1: Uniform partition in $\mathbf{I}^2 = [0,1]^2$

We have the following relations

$$0 \leq \sum_{i=1}^{m-1} p_{ij} \leq \frac{1}{m} \quad (j \in \{1, \cdots, m-1\})$$

$$0 \leq \sum_{j=1}^{m-1} p_{ij} \leq \frac{1}{m} \quad (i \in \{1, \cdots, m-1\})$$

$$1 - \frac{2}{m} \leq \sum_{i,j=1}^{m-1} p_{ij} \leq 1 - \frac{1}{m}$$

for $i, j \in \{1, \cdots, m-1\}$

$$p_{im} = \frac{1}{m} - \sum_{j=1}^{m-1} p_{ij}, \qquad p_{mj} = \frac{1}{m} - \sum_{i=1}^{m-1} p_{ij}$$

and

$$p_{mm} = \frac{1}{m} - \sum_{j=1}^{m-1} p_{mj} = \frac{1}{m} - \sum_{i=1}^{m-1} p_{im}$$

Let $n, m \geq 2$*, where* m *divides* n*, and* R_{ij}*,* $i, j \in \{1, \cdots, m\}$*, the uniform partition of* $\mathbf{I}^2 = [0,1]^2$*. Let* N_{ij}*,* $i, j \in \{1, \cdots, m\}$*, be the random variables that indicates the number of observation in the regions* R_{ij}*,* $i, j \in \{1, \cdots, m\}$*, respectively, then the distribution of the random vector*

$$(N_{ij}, i, j \in \{1, \cdots, m\})$$

is multinomial with parameters n *and* p_{ij}*,* $i, j \in \{1, \cdots, m\}$*. It is important to note that only* $(m-1)^2$ *values of the probabilities* p_{ij}*,* $i, j \in \{1, \cdots, m-1\}$*, are free, the remaining probabilities* p_{ij}*, can be written in terms of the previous probabilities (in the Figure 3.1 they are appear in red).*

In dimension $d = 3$ we have the following lemma (in [25], we found the proof of this result),

Lemma 3.2 *Let $m \geq 2$, we consider the uniform partition R_{ijk}, $i, j, k \in \{1, \cdots, m\}$, of the unit cube $\mathbf{I}^3 = [0, 1]^3$, given a 3-copula C we define*

$$p_{ijk} = V_C(R_{ijk})$$

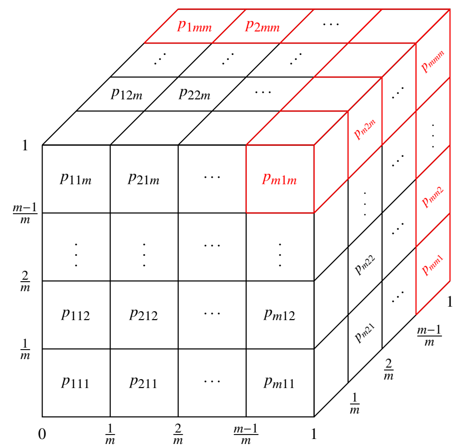

Figure 3.2: Uniform partition in $\mathbf{I}^3 = [0, 1]^3$

We have the following relations,

Let $i \in \{1, \cdots, m-1\}$

$$0 \leq \sum_{(j,k)\in\{1,\cdots,m\}^2\setminus\{(m,m)\}} p_{ijk} \leq \frac{1}{m}.$$

Let $j \in \{1, \cdots, m-1\}$

$$0 \leq \sum_{(i,k)\in\{1,\cdots,m\}^2\setminus\{(m,m)\}} p_{ijk} \leq \frac{1}{m}.$$

Let $k \in \{1, \cdots, m-1\}$

$$0 \leq \sum_{(i,j)\in\{1,\cdots,m\}^2\setminus\{(m,m)\}} p_{ijk} \leq \frac{1}{m}.$$

We define

$$I^d_{m,r} = \{(i_1, \cdots, i_d) \in \{1, \cdots, m\}^d \mid (d-r) \text{ coordinates equal to } m\}$$

also it holds

$$2 - \frac{3}{m} \leq \sum_{r=2}^{3} \left((r-1) \sum_{(i,j,k)\in I^3_{m,r}} p_{ijk} \right) \leq 2 - \frac{2}{m}.$$

We observe that the probabilities

$$p_{1mm}, \cdots, p_{m-1mm}, p_{m1m}, \cdots, p_{mm-1m}, p_{mm1}, \cdots, p_{mmm-1} \quad y \quad p_{mmm}$$

can be written in terms of the other probabilities p_{ijk}, *for example*

$$p_{1mm} = \frac{1}{m} - \sum_{(j,k)\in\{1,\cdots,m\}^2\setminus\{(m,m)\}} p_{1jk} \quad y \quad p_{mmm} = \frac{3}{m} - 2 + \sum_{r=2}^{3}\left((r-1)\sum_{(i,j,k)\in I^3_{m,r}} p_{ijk}\right).$$

Similarly to the case $d = 2$, *let* $n, m \geq 2$, *where* m *divides* n, *and* R_{ijk}, $i, j, k \in \{1, \cdots, m\}$, *the regions corresponding to the uniform partition of the unit cube* $\mathbf{I}^3 = [0,1]^3$. *Let* N_{ijk}, $i, j, k \in \{1, \cdots, m\}$, *be the random variables that indicates the number of observations in the regions* R_{ijk}, $i, j, k \in \{1, \cdots, m\}$, *respectively, then the distribution of de random vector*

$$(N_{ijk}, i, j, k \in \{1, \cdots, m\})$$

is multinomial with parameters n *and* p_{ijk}, $i, j, k \in \{1, \cdots, m\}$. *Regarding to the probabilities* p_{ijk}, $i, j, k \in \{1, \cdots, m\}$ *is only necessary to consider* $m^3 - (3m - 2)$ *free parameters (in the Figure 3.2 they are marked with red color the regions determined for the remaining regions).*

The distributions described in the above lemmas corresponds to the case where we considered the original sample, that is, the sample without the rank transformation. Following we present the procedure to make the count of the observations in the boxes considering the modified sample.

Remark 3.3 *In the grid of* $\mathbf{I}^2 = [0,1]^2$, *generated by the partitions* $P_x = \{0, 1/n, \cdots, (n-1)/n, 1\}$, *in the first coordinate* X *and* $P_y = \{0, 1/n, \cdots, (n-1)/n, 1\}$ *in the second coordinate* Y, *there exist* $n!$ *different ways in which can be observed* n *rank statistics from a sample of size* n, *because points are not considered if they have the same value in some coordinate.*

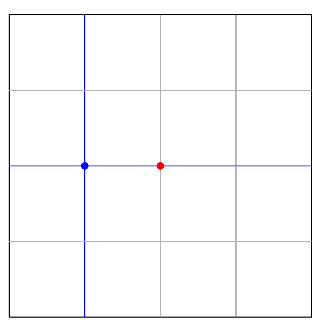

Figure 3.3: Grid in $\mathbf{I}^2 = [0,1]^2$

In the Figure 3.3 we show, in red, a point that should not be considered, because it has the same second coordinate as another point (marked in blue).

We consider $n \geq m \geq 2$, where m divides n, and $l = n/m$. let $R_{11} = [0, l/n]^2$ and let N_{11} be the random variable that indicates the number of the observations k in R_{11}, with $k \in \{0, 1, \cdots, l\}$ (see Figure 3.4).

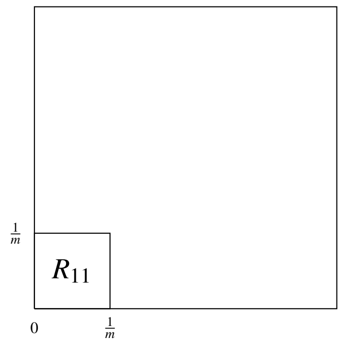

Figure 3.4: Region R_{11}

To calculate $P\{N_{11} = k\}$, with $k \in \{0, 1, \cdots, l\}$, we first select the number of ways in which we can select the coordinates in the first coordinate X, from these k observations, this count corresponds to $\binom{l}{k}$ (see Figure 3.5).

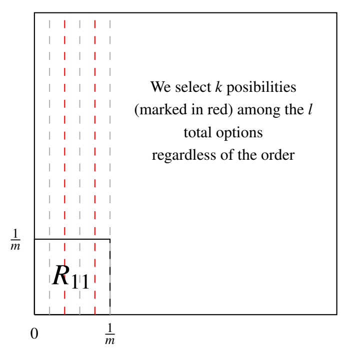

Figure 3.5

subsequently we calculate the number of ways in which we can select the coordinates in the second coordinate Y from these k observations, this count is given by P^l_k (see Figure 3.6).

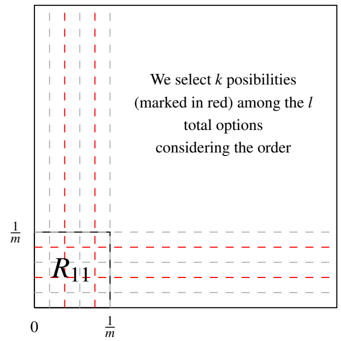

Figure 3.6

Let $\hat{R}_{12} = [0, l/n] \times [l/n, 1]$*, the coordinates in the first coordinate X of the* $l-k$ *points that we must see in the region* $\hat{R}_{12}$ *can be selected of* $\binom{l-k}{l-k} = 1$ *way, the coordinates in the second coordinate Y, can be selected of* P^{n-l}_{l-k} *different ways (see Figure 3.7).*

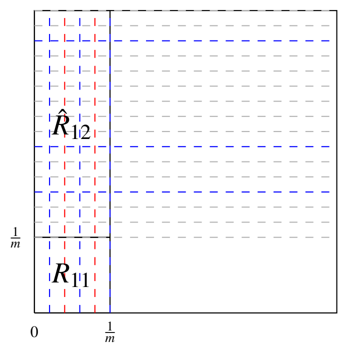

Figure 3.7

Let $\hat{R}_{22} = [l/n, 1] \times [0, 1]$*, the* $n-l$ *points that we must see in this region, they have* $(n-l)!$ *different ways to appear (see Figure 3.8).*

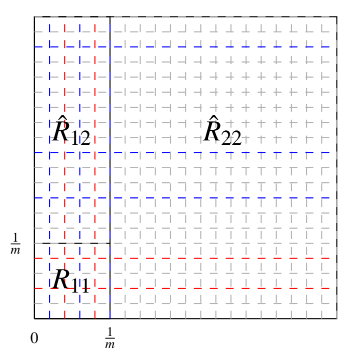

Figure 3.8

From this observations, the number in which k points can be observed in the region $R_{11} = [0, l/n]^2$, $l-k$ points in the region $\hat{R}_{12} = [0, l/n] \times (l/n, 1]$ and $n-l$ points in the region $\hat{R}_{22} = (l/n, 1] \times [0, 1]$, is

$$\binom{l}{k} P^l_k \binom{l-k}{l-k} P^{n-l}_{l-k} (n-l)!$$

from $n!$ possibilities.

The counting procedure described above, it can be generalized to higher dimensions, considering permutations in the counting process of the different coordinates respect to the first coordinate axis.

3.1 Case: dimension two when m divides n

In this part, we present the distribution and moments of the random variables associated to the counting in the boxes generated by the uniform partition of size m, with $m \geq 2$, of $\mathbf{I}^2 = [0, 1]^2$, induced by the modified sample from the product copula.

Definition 3.4 *Let $m \geq 2$, $n \in \mathbb{N}$, where m divides n, and $l = n/m$. We define the following regions in the unit square $\mathbf{I}^2 = [0, 1]^2$*

$$\begin{aligned} R_{11} &= [0, l/n] \times [0, l/n] \\ R_1 &= [0, l/n] \times (l/n, 1] \\ R &= (l/n, 1] \times [0, 1]\,. \end{aligned}$$

Remark 3.5 *In the following results we consider that in the unit square $\mathbf{I}^2$ exists n points corresponding to the rank transformation of a sample of size n from the product copula.*

The following results describe the probability distribution of the random variables associated with the counting of observations in the boxes generated by the uniform partition.

Lemma 3.6 *Let k_1 be the number of points in the region R_{11} and x_1 the number of points in the region R_1, we have that*

$$\sum_{k_1+x_1=l} \binom{l}{k_1} P^l_{k_1} \binom{l-k_1}{x_1} P^{n-l}_{x_1} = P^n_l.$$

Proof: We can observe that

$$\binom{l-k_1}{x_1} = \binom{x_1}{x_1} = 1$$

then

$$\begin{aligned}
\sum_{k_1+x_1=l}\binom{l}{k_1}P^l_{k_1}\binom{l-k_1}{x_1}P^{n-l}_{x_1} &= \sum_{k_1+x_1=l}\binom{l}{k_1}P^l_{k_1}P^{n-l}_{x_1}\\
&= \sum_{k_1+x_1=l}\frac{l!}{k_1!(l-k_1)!}\frac{l!}{(l-k_1)!}\frac{(n-l)!}{(n-l-x_1)!}\\
&= \sum_{k_1+x_1=l}\frac{l!}{k_1!(l-k_1)!}\frac{l!}{x_1!}\frac{(n-l)!}{(n-l-x_1)!}\\
&= l!\sum_{k_1+x_1=l}\binom{l}{k_1}\binom{n-l}{x_1}\\
&= l!\binom{n}{l} \quad \text{(Vandermonde's identity)}\\
&= P^n_l.
\end{aligned}$$

□

Lemma 3.7 *Let N_{11} be the random variable that indicates the number of observations falling in the region R_{11}, then the following equality holds*

$$\begin{aligned}
P\{N_{11}=k_1\} &= \frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{n-l}{l-k_1}\\
&= \frac{\binom{l}{k_1}\binom{n-l}{l-k_1}}{\binom{n}{l}}
\end{aligned}$$

and the random variable N_{11} has a hypergeometric distribution con parameters: n the population size, l the class size and l the sample size.

Proof: From the proof of Lemma 3.6, we have

$$\sum_{k_1+x_1=l}\binom{l}{k_1}P^l_{k_1}\binom{l-k_1}{x_1}P^{n-l}_{x_1} = l!\sum_{k_1+x_1=l}\binom{l}{k_1}\binom{n-l}{x_1},$$

this equality indicates the number of ways in which we can have $k_1 + x_1$ points in the region $R_{11} \cup R_1$; fixing the value k_1, this result is multiplied by $(n-l)!$ (number of ways that we can have $n-l$ points in the region R discarding l possibilities corresponding to the coordinates occupied by the observations in the region $R_{11} \cup R_1$) and divided by $n!$, the total number of ways that we can observe n points in $\mathbf{I}^2 = [0,1]^2$. □

Then

$$\begin{aligned}
\sum_{k_1=1}^{l} P\{N_{11} = k_1\} &= \sum_{k_1=1}^{l} \frac{l!(n-l)!}{n!} \binom{l}{k_1} \binom{n-l}{l-k_1} \\
&= \frac{(n-l)!}{n!} l! \sum_{k_1=1}^{l} \binom{l}{k_1} \binom{n-l}{l-k_1} \\
&= \frac{(n-l)!}{n!} P_l^n \\
&= 1.
\end{aligned}$$

Theorem 3.8 *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$. Let N_{11} be the random variable indicating the number of observations falling in R_{11}, when we take a sample of size n from the product copula, then*

$$E(N_{11}/n) = \frac{1}{m^2}, \quad E((N_{11}/n)^2) = \frac{l^2(l-1)^2}{n^3(n-1)} + \frac{1}{nm^2}$$

and

$$Var(N_{11}/n) = \frac{l^2(l-1)^2}{n^3(n-1)} + \frac{1}{nm^2} - \left(\frac{1}{m^2}\right)^2.$$

Proof: From the Lemma 3.7, we have

$$\begin{aligned}
E(N_{11}) &= \sum_{k_1=0}^{l} k_1 \frac{l!(n-l)!}{n!} \binom{l}{k_1} \binom{n-l}{l-k_1} \\
&= \frac{l!l(n-l)!}{n!} \sum_{k_1=1}^{l} \binom{l-1}{k_1-1} \binom{n-l}{l-k_1} \\
&= \frac{l!l(n-l)!}{n!} \sum_{u_1=0}^{l-1} \binom{l-1}{u_1} \binom{n-l}{l-1-u_1} \quad (u_1 = k_1 - 1) \\
&= \frac{l!l(n-l)!}{n!} \binom{n-1}{l-1} \quad \text{(Vandermonde's identity)} \\
&= \frac{l^2}{n} \\
&= \frac{n}{m^2}
\end{aligned}$$

and

$$E(N_{11}/n) = \frac{1}{m^2}.$$

Then,

$$\begin{aligned}
E((N_{11})^2) &= \sum_{k_1=0}^{l} k_1^2 \frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{n-l}{l-k_1} \\
&= \sum_{k_1=0}^{l} (k_1(k_1-1)+k_1)\frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{n-l}{l-k_1} \\
&= \sum_{k_1=2}^{l} k_1(k_1-1)\frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{n-l}{l-k_1} + \sum_{k_1=0}^{l} k_1\frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{n-l}{l-k_1} \\
&= \sum_{k_1=2}^{l} \frac{l!(n-l)!}{n!} l(l-1)\binom{l-2}{k_1-2}\binom{n-l}{l-k_1} + \frac{n}{m^2} \\
&= \frac{l!(n-l)!}{n!} l(l-1)\sum_{u_1=0}^{l-2}\binom{l-2}{u_1}\binom{n-l}{(l-2)-u_1} + \frac{n}{m^2} \quad (u_1 = k_1 - 2) \\
&= \frac{l!(n-l)!}{n!} l(l-1)\binom{n-2}{l-2} + \frac{n}{m^2} \\
&= \frac{l^2(l-1)^2}{n(n-1)} + \frac{n}{m^2}.
\end{aligned}$$

Therefore

$$E((N_{11}/n)^2) = \frac{l^2(l-1)^2}{n^3(n-1)} + \frac{1}{nm^2}$$

and

$$\begin{aligned}
Var(N_{11}/n) &= E((N_{11}/n)^2 - (E(N_{11}/n))^2 \\
&= \frac{l^2(l-1)^2}{n^3(n-1)} + \frac{1}{nm^2} - \left(\frac{1}{m^2}\right)^2.
\end{aligned}$$

□

Remark 3.9 *To evaluate the covariance, we consider two cases, corresponding to the position of the boxes in the square* $\mathbf{I}^2 = [0,1]^2$*, as illustrated in Figure 3.9.*

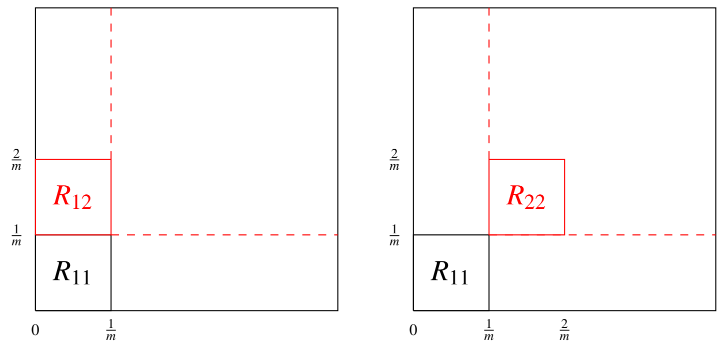

Figure 3.9: Covariances cases $d = 2$.

The results presented below correspond to the calculation of the covariances for the first case.

Definition 3.10 *Let $r \in \{2, \cdots, m\}$, we define the following region in the unit square $\mathbf{I}^2 = [0,1]^2$*

$$R_{1r} \quad = \quad [0, l/n] \times ((l/n)(r-1), (l/n)r]\,.$$

Lemma 3.11 *Let k_1 be the number of points in R_{11}, let k_2 be the number of points in R_{1r}, $r \in \{2, \cdots, m\}$, and let x_1 be the number of points in $([0, l/n] \times [0,1]) \backslash (R_{11} \cup R_{1r})$, then*

$$\sum_{k_1+k_2+x_1=l} \binom{l}{k_1} P^l_{k_1} \binom{l-k_1}{k_2} P^l_{k_2} \binom{l-k_1-k_2}{x_1} P^{n-2l}_{x_1} = P^n_l.$$

Proof: We have that

$$\binom{l-k_1-k_2}{x_1} = \binom{x_1}{x_1} = 1$$

and

$$\begin{aligned}
\sum_{k_1+k_2+x_1=l} \binom{l}{k_1} P^l_{k_1} \binom{l-k_1}{k_2} P^l_{k_2} \binom{l-k_1-k_2}{x_1} P^{n-2l}_{x_1} &= \sum_{k_1+k_2+x_1=l} \binom{l}{k_1} P^l_{k_1} \binom{l-k_1}{k_2} P^l_{k_2} P^{n-2l}_{x_1} \\
&= \sum_{k_1+k_2+x_1=l} \frac{l!}{k_1!(l-k_1)!} \frac{l!}{(l-k_1)!} \frac{(l-k_1)!}{k_2!(l-k_1-k_2)!} \\
&\quad \cdot \frac{l!}{(l-k_2)!} \frac{(n-2l)!}{(n-2l-x_1)!} \\
&= l! \sum_{k_1+k_2+x_1=l} \binom{l}{k_1} \binom{l}{k_2} \binom{n-2l}{x_1} \\
&= l! \binom{n}{l} \\
&= P^n_l.
\end{aligned}$$

$\square$

Lemma 3.12 *Let N_{11} be the random variable indicating the number of observations falling in R_{11} and let N_{1r}, $r \in \{2, \cdots, m\}$, be the random variable indicating the number of observations falling in R_{1r} then*

$$P\{N_{11} = k_1, N_{1r} = k_2\} = \frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{l-k_1-k_2}.$$

Proof: We consider

$$l! \sum_{k_1+k_2+x_1=l} \binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1} = P_l^n$$

from the proof of Lemma 3.11, using the notation of the Definition 3.4, this equality indicates the number of ways in which we can have $k_1 + k_2 + x_1$ points in the region $R_{11} \cup R_1$; similarly to the Lemma 1.4, setting the values k_1 and k_2, this result is multiplied by $(n-l)!$ (number of ways that we can have $n-l$ points in the region R discarding l possibilities corresponding to the coordinates occupied by the observations in the region $R_{11} \cup R_1$) and divided by $n!$, the total number of ways that we can observe n points in $\mathbf{I}^2 = [0,1]^2$. $\square$

We can observe that

$$\begin{aligned}
\sum_{k_1+k_2=l} P\{N_{11} = k_1, N_{1r} = k_2\} &= \sum_{k_1+k_2=l} \frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{l-k_1-k_2} \\
&= \frac{(n-l)!}{n!} l! \sum_{k_1+k_2+x_1=l} \binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1} \\
&= \frac{(n-l)!}{n!} P_l^n \\
&= 1.
\end{aligned}$$

Theorem 3.13 *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$. Let N_{11} be the random variable indicating the number of observations falling in R_{11} (Definition 3.4) and let N_{1r}, $r \in \{2, \cdots, m\}$, be the random variable indicating the number of observations falling in R_{1r} (Definition 3.10), when we consider a sample of size n from the product copula, then*

$$Cov(N_{11}/n, N_{1r}/n) = \frac{l^3(l-1)}{n^3(n-1)} - \left(\frac{1}{m^2}\right)^2.$$

Proof:

$$\begin{aligned}
E(N_{11}N_{1r}) &= \sum_{k_1+k_2=l} k_1k_2\frac{l!(n-l)!}{n!}\binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{l-k_1-k_2} \\
&= \frac{l!(n-l)!}{n!}\sum_{k_1+k_2+x_1=l} k_1k_2\binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1} \\
&= \frac{l!(n-l)!}{n!}\sum_{x_1=0}^{l-2}\binom{n-2l}{x_1}\sum_{k_1+k_2=l-x_1} k_1k_2\binom{l}{k_1}\binom{l}{k_2} \\
&= \frac{l!(n-l)!l^2}{n!}\sum_{x_1=0}^{l-2}\binom{n-2l}{x_1}\sum_{(k_1-1)+(k_2-1)=(l-2)-x_1}\binom{l-1}{k_1-1}\binom{l-1}{k_2-1} \\
&= \frac{l!(n-l)!l^2}{n!}\sum_{x_1=0}^{l-2}\binom{n-2l}{x_1}\binom{2l-2}{(l-2)-x_1} \\
&= \frac{l!(n-l)!l^2}{n!}\binom{n-2}{l-2} \\
&= \frac{l!l^2(n-l)!}{n!}\frac{(n-2)!}{(l-2)!(n-l)!} \\
&= \frac{l^3(l-1)}{n(n-1)}
\end{aligned}$$

and

$$\begin{aligned}
Cov(N_{11}/n, N_{1r}/n) &= E((N_{11}/n)(N_{1r}/n)) - E(N_{11}/n)E(N_{1r}/n) \\
&= \frac{l^3(l-1)}{n^3(n-1)} - \left(\frac{1}{m^2}\right)^2.
\end{aligned}$$

□

The next results correspond to the calculation of the covariances in the second case.

Lemma 3.14 *Let k_1, x_1 and x_2 be the number of observed points in the regions R_{11} (Definition 3.4), $R_{12} = [0, l/n] \times (l/n, 2l/n]$ and $R^*_{13} = [0, l/n] \times (2l/n, 1]$, respectively, let y_1, k_2 and y_2 be the number of observed points in $R_{21} = (l/n, 2l/n] \times [0, l/n]$, $R_{22} = (l/n, 2l/n] \times (l/n, 2l/n]$ and $R^*_{23} = (l/n, 2l/n] \times (2l/n, 1]$, respectively, then*

$$\sum_{k_1+x_1+x_2=l}\sum_{k_2+y_1+y_2=l}\binom{l}{k_1}P^l_{k_1}\binom{l-k_1}{x_1}P^l_{x_1}\binom{l-k_1-x_1}{x_2}P^{n-2l}_{x_2}$$

$$\cdot\binom{l}{y_1}P_{y_1}^{l-k_1}\binom{l-y_1}{k_2}P_{k_2}^{l-x_1}\binom{l-y_1-k_2}{y_2}P_{y_2}^{n-2l-x_2} = P_l^{n-l}P_l^n.$$

Proof: We have that

$$\binom{l-k_1-x_1}{x_2} = 1, \qquad \binom{l-y_1-k_2}{y_2} = 1$$

and

$$\sum_{k_1+x_1+x_2=l}\sum_{k_2+y_1+y_2=l}\binom{l}{k_1}P_{k_1}^{l}\binom{l-k_1}{x_1}P_{x_1}^{l}P_{x_2}^{n-2l}\binom{l}{y_1}P_{y_1}^{l-k_1}\binom{l-y_1}{k_2}P_{k_2}^{l-x_1}P_{y_2}^{n-2l-x_2}$$

$$\begin{aligned}
&= (l!)^2\sum_{k_1+x_1+x_2=l}\sum_{k_2+y_1+y_2=l}\binom{l}{k_1}\binom{l}{x_1}\binom{n-2l}{x_2}\binom{l-k_1}{y_1}\binom{l-x_1}{k_2}\binom{n-2l-x_2}{y_2}\\
&= (l!)^2\sum_{k_1+x_1+x_2=l}\binom{l}{k_1}\binom{l}{x_1}\binom{n-2l}{x_2}\binom{n-k_1-x_1-x_2}{l}\\
&= (l!)^2\binom{n-l}{l}\sum_{k_1+x_1+x_2=l}\binom{l}{k_1}\binom{l}{x_1}\binom{n-2l}{x_2}\\
&= (l!)^2\binom{n-l}{l}\binom{n}{l}\\
&= P_l^{n-l}P_l^n.
\end{aligned}$$

□

Lemma 3.15 *Let N_{11} be the random variable indicating the number of observations falling in R_{11} (Definition 3.4) and N_{22} the random variable indicating the number of observations falling in $R_{22} = (l/n, 2l/n] \times (l/n, 2l/n]$ then*

$$P\{N_{11} = k_1, N_{22} = k_2\} = \frac{(l!)^2(n-2l)!}{n!}\sum_{x_1+x_2=l-k_1}\sum_{y_1+y_2=l-k_2}\binom{l}{k_1}\binom{l}{x_1}\binom{n-2l}{x_2}\binom{l-k_1}{y_1}\binom{l-x_1}{k_2}\binom{n-2l-x_2}{y_2}$$

Proof: This equality is obtained from the proof Lemma 3.14, setting the values k_1 and k_2, and multiplying by $(n-2l)!$ (number of ways that we can have $n-2l$ points in the region $(2l/n, 1]\times[0, 1]$ discarding $2l$ possibilities corresponding to the coordinates occupied by the observations in the region $[0, 2l/n] \times [0, 1]$) and divided by $n!$, the total number of ways that we can observe n points in $I^2 = [0, 1]^2$.

We can note that

$$\sum_{k_1=0}^{l}\sum_{k_2=0}^{l} P\{N_{11}=k_1, N_{22}=k_2\} = \frac{(n-2l)!}{n!}P_l^{n-l}P_l^{n} = 1.$$

□

Theorem 3.16 *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$, with the same hypothesis of the Lemma 3.15 we have*

$$Cov(N_{11}/n, N_{22}/n) = \frac{l^4}{n^3(n-1)} - \frac{1}{m^4}.$$

Proof:

$$\begin{aligned}
E(N_{11}N_{22}) &= \frac{(l!)^2(n-2l)!}{n!}\sum_{k_1+x_1+x_2=l} k_1\binom{l}{k_1}\binom{l}{x_1}\binom{n-2l}{x_2}\sum_{k_2+y_1+y_2=l} k_2\binom{l-k_1}{y_1}\binom{l-x_1}{k_2}\binom{n-2l-x_2}{y_2} \\
&= \frac{(l!)^2 l(n-2l)!}{n!}\sum_{(k_1-1)+x_1+x_2=l-1}\binom{l-1}{k_1-1}\binom{l}{x_1}\binom{n-2l}{x_2} \\
&\qquad \cdot(l-x_1)\sum_{(k_2-1)+y_1+y_2=l-1}\binom{l-k_1}{y_1}\binom{l-x_1-1}{k_2-1}\binom{n-2l-x_2}{y_2} \\
&= \frac{(l!)^2 l(n-2l)!}{n!}\sum_{(k_1-1)+x_1+x_2=l-1}\binom{l-1}{k_1-1}\binom{l}{x_1}\binom{n-2l}{x_2}(l-x_1)\binom{n-1-l}{l-1} \\
&= \frac{(l!)^2 l^2(n-2l)!}{n!}\binom{n-1-l}{l-1}\sum_{(k_1-1)+x_1+x_2=l-1}\binom{l-1}{k_1-1}\binom{l}{x_1}\binom{n-2l}{x_2} \\
&\qquad -\frac{(l!)^2 l^2(n-2l)!}{n!}\binom{n-1-l}{l-1}\sum_{(k_1-1)+(x_1-1)+x_2=l-2}\binom{l-1}{k_1-1}\binom{l-1}{x_1-1}\binom{n-2l}{x_2} \\
&= \frac{(l!)^2 l^2(n-2l)!}{n!}\binom{n-1-l}{l-1}\left[\binom{n-1}{l-1}-\binom{n-2}{l-2}\right] \\
&= \frac{(l!)^2 l^2(n-2l)!}{n!}\frac{(n-1-l)!}{(n-2l)!(l-1)!}\left(\frac{(n-1)!}{(l-1)!(n-l)!}-\frac{(n-2)!}{(l-2)!(n-l)!}\right) \\
&= \frac{(l!)^2 l^2(n-1)!(n-1-l)!}{n!((l-1)!)^2(n-l)!} - \frac{(l!)^2 l^2(n-2)!(n-1-l)!}{n!(l-1)!(l-2)!(n-l)!} \\
&= \frac{l^4}{n(n-l)} - \frac{l^4(l-1)}{n(n-1)(n-l)}
\end{aligned}$$

$$= \frac{l^4}{n(n-1)}$$

and

$$E((N_{11}/n)(N_{22}/n)) = \frac{l^4}{n^3(n-1)}.$$

Therefore

$$\begin{aligned} Cov(N_{11}/n, N_{22}/n) &= E((N_{11}/n)(N_{22}/n)) - E(N_{11}/n)E(N_{22}/n) \\ &= \frac{l^4}{n^3(n-1)} - \frac{1}{m^4}. \end{aligned}$$

□

We finalized this section with a theorem that indicates the joint probability distribution of the boxes generated for the uniform partition of size m, with $m \geq 2$, when m divides n.

Theorem 3.17 *Let $m \geq 2$, $n \in \mathbb{N}$, where m divides n, $l = n/m$ and $I_m = \{1, \cdots m\}$, let R_{ij}, with $i, j \in I_m$, be the boxes of the uniform partition of size m of $\mathbf{I}^2 = [0,1]^2$ and let N_{ij} be the random variables that indicate the number of observations falling in R_{ij}, respectively, for all $i, j \in I_m$, when we consider a sample of size n from the product copula; let n_{ij}, with $i, j \in I_m$, be zero or a positive integer, satisfying the following restrictions*

$$\sum_{j=1}^{m} n_{ij} = l \quad (\text{for all } i \in I_m), \qquad \sum_{i=1}^{m} n_{ij} = l \quad (\text{for all } j \in I_m),$$

then

$$P\left\{\bigcap_{i,j \in I_m} \{N_{ij} = n_{ij}\}\right\} = \frac{(l!)^{2m}}{n! \prod_{ij \in I_m} n_{ij}!}.$$

Proof: Proceeding as in Remark 3.3, first for the region R_{11}, second for the region R_{12} and successively to the regions $R_{13}, \cdots, R_{1m}, R_{21}, \cdots, R_{2m}, R_{m1}, \cdots, R_{mm}$, we have that the number of ways in which we can observe n_{ij} points in each region R_{ij}, $i, j \in I_m$, is given by

$$\binom{l}{n_{11}} P^l_{n_{11}} \binom{l - n_{11}}{n_{12}} P^l_{n_{12}} \cdots \binom{l - \sum_{j=1}^{m-1} n_{1j}}{n_{1m}} P^l_{n_{1m}}$$

$$\cdot \binom{l}{n_{21}} P^{l-n_{11}}_{n_{21}} \binom{l - n_{21}}{n_{22}} P^{l-n_{12}}_{n_{22}} \cdots \binom{l - \sum_{j=1}^{m-1} n_{2j}}{n_{2m}} P^{l-n_{1m}}_{n_{2m}}$$

$$\vdots$$

$$\cdot\binom{l}{n_{m1}}P_{n_{m1}}^{l-\sum_{i=1}^{m-1}n_{i1}}\binom{l-n_{m1}}{n_{m2}}P_{n_{m2}}^{l-\sum_{i=1}^{m-1}n_{i2}}\cdots\binom{l-\sum_{j=1}^{m-1}n_{mj}}{n_{mm}}P_{n_{mm}}^{l-\sum_{i=1}^{m-1}n_{im}}$$

$$= \frac{(l!)^{2m}}{\prod_{i,j\in I_m} n_{ij}!},$$

finally, we divided between $n!$, the total number of possibilities in which we can see n points in $\mathbf{I}^2 = [0,1]^2$, corresponding to the modified sample of the a sample of size n from the product copula, therefore

$$P\left\{\bigcap_{i,j\in I_m}\{N_{ij}=n_{ij}\}\right\} = \frac{(l!)^{2m}}{n!\prod_{i,j\in I_m} n_{ij}!}.$$

□

3.2 Case: dimension three when m divides n

In a similar way to the previous section, we present the distribution and moments of the random variables associated to the counting in the boxes generated by the uniform partition of size m, with $m \geq 2$, of $\mathbf{I}^3 = [0,1]^3$, induced by the modified sample from the product copula.

Theorem 3.18 *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$. Let N_{111} be the random variable that indicates the number of observations falling in $R_{111} = [0, l/n]^3$ when we consider a sample of size n from the product copula, then*

$$E(N_{11}/n) = \frac{1}{m^3}, \quad E((N_{11}/n)^2) = \frac{l^3(l-1)^3}{n^4(n-1)^2} + \frac{1}{nm^3}$$

and

$$Var(N_{111}/n) = \frac{l^3(l-1)^3}{n^4(n-1)^2} + \frac{1}{nm^3} - \left(\frac{1}{m^3}\right)^2.$$

Remark 3.19 *Figure 3.10 shows the related boxes to the count of the number of observations associated to the random variable N_{111}, the Figure 3.11 show the region $R = (l/n, 1]\times[0,1]\times[0,1]$, in this region exist $((n-l)!)^2$ ways in which we can graph the transformed sample data, since that l points are observed in the region $R_1 = [0, l/n]\times[0,1]\times[0,1]$.*

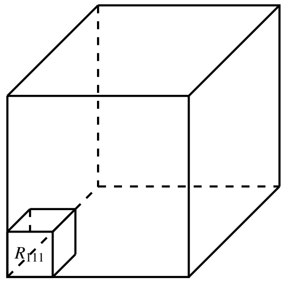

Figure 3.10: Box R_{111}

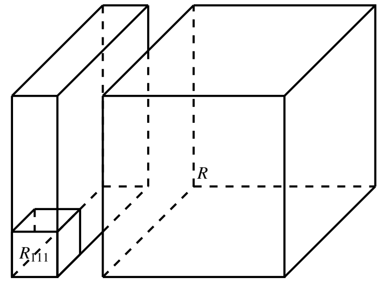

Figure 3.11: Box R

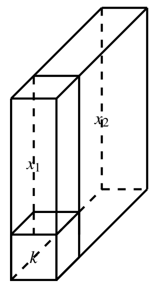

Figure 3.12: Box R_1

Remark 3.20 *The Figure 3.12 it shows the region* $R_1 = [0, l/n]\times[0, 1]\times[0, 1]$*, this region is divided on three parts,* $R_{111} = [0, l/n]^3$*,* $R_{112} = [0, l/n]\times[0, l/n]\times(l/n, 1]$ *and* $R_{122} = [0, l/n]\times(l/n, 1]\times[0, 1]$*, the number of observations in the regions are denoted, respectively, by k,* x_1 *and* x_2*. The number of ways in which we can select the value of k is* $\binom{l}{k}P^l_k P^l_k$*, the number of ways in which we can select the value of* x_1*, given the value k, is* $\binom{l-k}{x_1}P^{l-k}_{x_1}P^{n-l}_{x_1}$ *and the number of possibilities of the value* x_2 *is, given the values k and* x_1*,* $\binom{l-k-x_1}{x_2}P^{n-l}_{x_2}P^{n-k-x_1}_{x_2}$.

Definition 3.21 *Let* $k \in \{0, 1, \cdots, l\}$*, we define*

$$
\begin{aligned}
G^{n,l}_k &= \sum_{x_1+x_2=l-k} \binom{l-k}{x_1} P^{l-k}_{x_1} P^{n-l}_{x_1} \binom{l-k-x_1}{x_2} P^{n-l}_{x_2} P^{n-k-x_1}_{x_2} \\
&= \sum_{x_1=0}^{l-k} \binom{l-k}{x_1} P^{l-k}_{x_1} P^{n-l}_{x_1} \binom{l-k-x_1}{l-k-x_1} P^{n-l}_{l-k-x_1} P^{n-k-x_1}_{l-k-x_1} \\
&= \sum_{x_1=0}^{l-k} \frac{(l-k)!}{x_1!(l-k-x_1)!} \frac{(l-k)!}{(l-k-x_1)!} \frac{(n-l)!}{(n-l-x_1)!} \frac{(n-l)!}{(n-2l+k+x_1)!} \frac{(n-k-x_1)!}{(n-l)!} \\
&= ((l-k)!)^2 \sum_{x_1=0}^{l-k} \binom{n-k-x_1}{l-k-x_1} \binom{n-l}{l-k-x_1} \binom{n-l}{x_1}.
\end{aligned}
$$

Remark 3.22 *We have,*

$$\sum_{k=0}^{l} \binom{l}{k} P_k^l P_k^l G_k^{n,l} = ((P_l^n)!)^2.$$

then

$$\sum_{k=0}^{l} \binom{l}{k} (l!)^2 \left[\sum_{x_1=0}^{l-k} \binom{n-k-x_1}{l-k-x_1} \binom{n-l}{l-k-x_1} \binom{n-l}{x_1} \right] ((n-l)!)^2 = (n!)^2$$

Remark 3.23 *Le* $k \in \{0, 1, \cdots, l\}$*, we have*

$$P\{N = k\} = \binom{l}{k} P_k^l P_k^l G_k^{n,l} ((n-l)!)^2 / (n!)^2$$

Using Remark 3.23, we can proof the Theorem 3.18,

$$\begin{aligned}
E(N_{111}) &= \sum_{k=0}^{l} k \binom{l}{k} (l!)^2 \left[\sum_{x_1=0}^{l-k} \binom{n-k-x_1}{l-k-x_1} \binom{n-l}{l-k-x_1} \binom{n-l}{x_1} \right] ((n-l)!)^2 / (n!)^2 \\
&= l^3 \sum_{k=1}^{l} \binom{l-1}{k-1} ((l-1)!)^2 \left[\sum_{x_1=0}^{(l-1)-(k-1)} \binom{(n-1)-(k-1)-x_1}{(l-1)-(k-1)-x_1} \right. \\
&\qquad \left. \binom{(n-1)-(l-1)}{(l-1)-(k-1)-x_1} \binom{(n-1)-(l-1)}{x_1} \right] (((n-1)-(l-1))!)^2 / (n!)^2 \\
&= l^3 \sum_{u=0}^{l-1} \binom{l-1}{u} ((l-1)!)^2 \left[\sum_{x_1=0}^{(l-1)-u} \binom{(n-1)-u-x_1}{(l-1)-u-x_1} \right. \\
&\qquad \left. \binom{(n-1)-(l-1)}{(l-1)-u-x_1} \binom{(n-1)-(l-1)}{x_1} \right] (((n-1)-(l-1))!)^2 / (n!)^2 \\
&= \frac{l^3 ((n-1)!)^2}{(n!)^2} \qquad \text{(Remark 3.22)} \\
&= \frac{n^3}{m^3} \frac{1}{n^2} \\
&= \frac{n}{m^3}
\end{aligned}$$

and

$$E\left(\frac{N_{111}}{n}\right) = \frac{1}{m^3}.$$

The second moment is obtained from the following equalities

$$
\begin{aligned}
E(N_{111}^2) &= \sum_{k=0}^{l} k^2 \binom{l}{k} (l!)^2 \left[\sum_{x_1=0}^{l-k} \binom{n-k-x_1}{l-k-x_1} \binom{n-l}{l-k-x_1} \binom{n-l}{x_1} \right] ((n-l)!)^2/(n!)^2 \\
&= \sum_{k=0}^{l} [k(k-1)+k] \binom{l}{k} (l!)^2 \left[\sum_{x_1=0}^{l-k} \binom{n-k-x_1}{l-k-x_1} \binom{n-l}{l-k-x_1} \binom{n-l}{x_1} \right] ((n-l)!)^2/(n!)^2 \\
&= \sum_{k=2}^{l} k(k-1) \binom{l}{k} (l!)^2 \left[\sum_{x_1=0}^{l-k} \binom{n-k-x_1}{l-k-x_1} \binom{n-l}{l-k-x_1} \binom{n-l}{x_1} \right] ((n-l)!)^2/(n!)^2 + \frac{n}{m^3} \\
&= (l(l-1))^3 \sum_{k=2}^{l} \binom{l-2}{k-2} ((l-2)!)^2 \left[\sum_{x_1=0}^{(l-2)-(k-2)} \binom{(n-2)-(k-2)-x_1}{(l-2)-(k-2)-x_1} \right. \\
&\qquad \left. \binom{(n-2)-(l-2)}{(l-2)-(k-2)-x_1} \binom{(n-2)-(l-2)}{x_1} \right] (((n-2)-(l-2))!)^2/(n!)^2 + \frac{n}{m^3} \\
&= (l(l-1))^3 \sum_{u=0}^{l-2} \binom{l-2}{u} ((l-2)!)^2 \left[\sum_{x_1=0}^{(l-2)-u} \binom{(n-2)-u-x_1}{(l-2)-u-x_1} \right. \\
&\qquad \left. \binom{(n-2)-(l-2)}{(l-2)-u-x_1} \binom{(n-2)-(l-2)}{x_1} \right] (((n-2)-(l-2))!)^2/(n!)^2 + \frac{n}{m^3} \\
&= \frac{(l(l-1))^3((n-2)!)^2}{(n!)^2} + \frac{n}{m^3} \qquad \text{(Remark 3.22)} \\
&= \frac{l^3(l-1)^3}{n^2(n-1)^2} + \frac{n}{m^3}.
\end{aligned}
$$

and

$$
E\left(\left(\frac{N_{111}}{n}\right)^2\right) = \frac{l^3(l-1)^3}{n^4(n-1)^2} + \frac{1}{nm^3}.
$$

Finally

$$
\begin{aligned}
Var(N_{111}/n) &= E((N_{111}/n)^2) - (E(N_{111}/n))^2 \\
&= \frac{l^3(l-1)^3}{n^4(n-1)^2} + \frac{1}{nm^3} - \left(\frac{1}{m^3}\right)^2.
\end{aligned}
$$

□

Remark 3.24 *To evaluate the covariance, we consider three cases, with respect to the position of the boxes* $R_{112} = [0, l/n] \times [0, l/n] \times (l/n, 2l/n]$, $R_{122} = [0, l/n] \times (l/n, 2l/n] \times (l/n, 2l/n]$ *and* $R_{222} = (l/n, 2l/n]^3$, *relatives to the box* R_{111}.

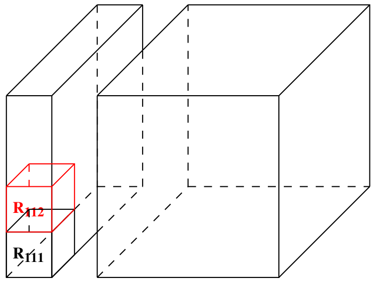

Figure 3.13: Case 1

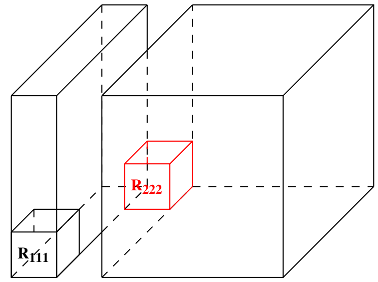

Figure 3.14: Case 2

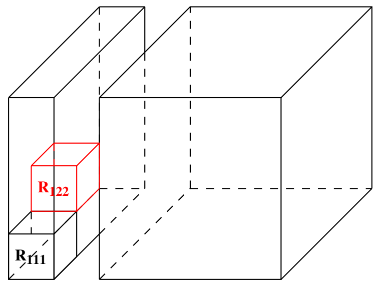

Figure 3.15: Case 3

Lemma 3.25 (Case 1) *Let n and m be integers greater than or equal to two, where m divides n, and* $l = n/m$. *Let* k_1 *be the number of observations falling in* $R_{111} = [0, l/n]^3$, k_2 *the number of observations falling in* $R_{112} = [0, l/n] \times [0, l/n] \times (l/n, 2l/n]$, x_1 *be the number of observations falling in* $R_{113} = [0, l/n] \times [0, l/n] \times (l/n, 1]$ *and* x_2 *the number of observations falling in* $R^* = [0, l/n] \times (l/n, 1] \times [0, 1]$, *when we considered a sample of size n from the product copula, then*

$$\sum_{k_1+k_2+x_1+x_2=l} \binom{l}{k_1} P^l_{k_1} P^l_{k_1} \binom{l-k_1}{k_2} P^{l-k_1}_{k_2} P^l_{k_2} \binom{l-k_1-k_2}{x_1} P^{l-k_1-k_2}_{x_1} P^{n-2l}_{x_1} \cdot \binom{l-k_1-k_2-x_1}{x_2} P^{n-l}_{x_2} P^{n-x_1-k_1-k_2}_{x_2} = (P^n_l)^2 .$$

Proof: We note

$$\binom{l-k_1-k_2-x_1}{x_2} = 1$$

and

$$\sum_{k_1+k_2+x_1+x_2=l} \binom{l}{k_1} P^l_{k_1} P^l_{k_1} \binom{l-k_1}{k_2} P^{l-k_1}_{k_2} P^l_{k_2} \binom{l-k_1-k_2}{x_1} P^{l-k_1-k_2}_{x_1} P^{n-2l}_{x_1} \cdot \binom{l-k_1-k_2-x_1}{x_2} P^{n-l}_{x_2} P^{n-x_1-k_1-k_2}_{x_2}$$

$$
\begin{aligned}
&= \sum_{k_1+k_2+x_1+x_2=l} \binom{l}{k_1} P^l_{k_1} P^l_{k_1} \binom{l-k_1}{k_2} P^{l-k_1}_{k_2} P^l_{k_2} \binom{l-k_1-k_2}{x_1} P^{l-k_1-k_2}_{x_1} P^{n-2l}_{x_1} P^{n-l}_{x_2} P^{n-x_1-k_1-k_2}_{x_2} \\
&= (l!)^2 \sum_{k_1+k_2+x_1+x_2=l} \binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1}\binom{n-l}{x_2}\binom{n-x_1-k_1-k_2}{x_2} \\
&= (l!)^2 \sum_{k_1+k_2+x_1+x_2=l} \binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1}\binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2} \\
&= (l!)^2 \sum_{x_2=0}^{l} \binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2} \sum_{k_1+k_2+x_1=l-x_2} \binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1} \\
&= (l!)^2 \sum_{x_2=0}^{l} \binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2}\binom{n}{l-x_2} \\
&= (l!)^2 n! \sum_{x_2=0}^{l} \frac{1}{x_2!(n-l-x_2)!}\frac{1}{x_2!(l-x_2)!} \\
&= \frac{(l!)^2 n!}{l!(n-l)!} \sum_{x_2=0}^{l} \binom{l}{x_2}\binom{n-l}{(n-l)-x_2} \\
&= \frac{(l!)^2 n!}{l!(n-l)!}\binom{n}{n-l} \\
&= (P^n_l)^2.
\end{aligned}
$$

□

Remark 3.26 *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$. Let N_{111} be the random variable indicating the number of observations falling in $R_{111} = [0, l/n]^3$ and let N_{112} be the random variable indicating the number of observations falling in $R_{112} = [0, l/n] \times [0, l/n] \times (l/n, 2l/n]$, when we take a sample of size n from the product copula, using the proof of the Lemma 3.25 we have*

$$
P\{N_{111} = k_1, N_{112} = k_2\} = \frac{(l!)^2((n-l)!)^2}{(n!)^2} \sum_{x_2=0}^{l-k_1-k_2} \binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2}\binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{l-x_2-k_1-k_2}.
$$

Theorem 3.27 *With the same hypothesis of Remark 3.26, we have*

$$
Cov(N_{111}/n, N_{112}/n) = \frac{l^4(l-1)^2}{n^4(n-1)^2} - \left(\frac{1}{m^3}\right)^2.
$$

Proof: From Lemma 3.25 and Remark 3.26 we have

$$
\begin{aligned}
E(N_{111}N_{112}) &= \frac{(l!)^2((n-l)!)^2}{(n!)^2}\sum_{x_2=0}^{l-2}\binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2}\sum_{k_1+k_2+x_1=l-x_2}k_1k_2\binom{l}{k_1}\binom{l}{k_2}\binom{n-2l}{x_1}\\
&= \frac{l^2(l!)^2((n-l)!)^2}{(n!)^2}\sum_{x_2=0}^{l-2}\binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2}\\
&\qquad \cdot\sum_{(k_1-1)+(k_2-1)+x_1=(l-2)-x_2}\binom{l-1}{k_1-1}\binom{l-1}{k_2-1}\binom{n-2l}{x_1}\\
&= \frac{l^2(l!)^2((n-l)!)^2}{(n!)^2}\sum_{x_2=0}^{l-2}\binom{n-l}{x_2}\binom{n-(l-x_2)}{x_2}\binom{n-2}{(l-2)-x_2}\\
&= \frac{l^2(l!)^2((n-l)!)^2(n-2)!}{(n!)^2(n-l)!(l-2)!}\sum_{x_2=0}^{l-2}\binom{n-l}{x_2}\binom{l-2}{(l-2)-x_2}\\
&= \frac{l^2(l!)^2((n-l)!)^2(n-2)!}{(n!)^2(n-l)!(l-2)!}\binom{n-2}{l-2}\\
&= \frac{l^4(l-1)^2}{n^2(n-1)^2},
\end{aligned}
$$

and

$$
\begin{aligned}
Cov(N_{111}/n, N_{112}/n) &= E((N_{111}/n)(N_{112}/n)) - E(N_{111}/n)E(N_{112}/n)\\
&= \frac{l^4(l-1)^2}{n^4(n-1)^2} - \left(\frac{1}{m^3}\right)^2.
\end{aligned}
$$

□

Remark 3.28 (Case 2) *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$, we consider the following regions*

$$
\begin{aligned}
R_{111} &= [0, l/n] \times [0, l/n] \times [0, l/n]\\
R_{112} &= [0, l/n] \times [0, l/n] \times (l/n, 2l/n]\\
R_{113} &= [0, l/n] \times [0, l/n] \times (2l/n, 1]\\
R_{121} &= [0, l/n] \times (l/n, 2l/n] \times [0, l/n]\\
R_{122} &= [0, l/n] \times (l/n, 2l/n] \times (l/n, 2l/n]\\
R_{123} &= [0, l/n] \times (l/n, 2l/n] \times (2l/n, 1]\\
R_{131} &= [0, l/n] \times (2l/n, 1] \times [0, l/n]
\end{aligned}
$$

$$R_{132} = [0,l/n]\times(2l/n,1]\times(l/n,2l/n]$$
$$R_{133} = [0,l/n]\times(2l/n,1]\times(2l/n,1],$$

let $x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8$ *and* x_9 *be, respectively, the number of observations in the regions* $R_{111},R_{112},R_{113},R_{121},R_{122},R_{123},R_{131},R_{132}$ *and* R_{133}*.*

We define

$$\begin{aligned} x_1+x_2+x_3 &= A_1\\ x_4+x_5+x_6 &= A_2\\ x_7+x_8+x_9 &= A_3\\ x_1+x_4+x_7 &= B_1\\ x_2+x_5+x_8 &= B_2\\ x_3+x_6+x_9 &= B_3. \end{aligned}$$

To count the number of ways in which we can select l from n points in the region $R^* = [0,l/n]\times[0,1]\times[0,1]$ *we consider*

1. *Number of ways to select the first coordinate X*

$$\begin{aligned} CX &= \binom{l}{x_1}\binom{l-x_1}{x_2}\binom{l-x_1-x_2}{x_3}\binom{l-x_1-x_2-x_3}{x_4}\binom{l-x_1-x_2-x_3-x_4}{x_5}\\ &\cdot\binom{l-x_1-x_2-x_3-x_4-x_5}{x_6}\binom{l-x_1-x_2-x_3-x_4-x_5-x_6}{x_7}\\ &\cdot\binom{l-x_1-x_2-x_3-x_4-x_5-x_6-x_7}{x_8}\binom{l-x_1-x_2-x_3-x_4-x_5-x_6-x_7-x_8}{x_9}\\ &= l!\frac{1}{x_1!}\frac{1}{x_2!}\frac{1}{x_3!}\frac{1}{x_4!}\frac{1}{x_5!}\frac{1}{x_6!}\frac{1}{x_7!}\frac{1}{x_8!}\frac{1}{x_9!}. \end{aligned}$$

2. *Number of ways to select the second coordinate Y*

$$\begin{aligned} CY &= P^{l}_{x_1}P^{l-x_1}_{x_2}P^{l-x_1-x_2}_{x_3}P^{l}_{x_4}P^{l-x_4}_{x_5}P^{l-x_4-x_5}_{x_6}P^{n-2l}_{x_7}P^{n-2l-x_7}_{x_8}P^{n-2l-x_7-x_8}_{x_9}\\ &= \frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}\frac{(n-2l)!}{(n-2l-A_3)!}. \end{aligned}$$

3. *Number of ways to select the third coordinate Z*

$$CZ = P^{l}_{x_1}P^{l}_{x_2}P^{n-2l}_{x_3}P^{l-x_1}_{x_4}P^{l-x_2}_{x_5}P^{n-2l-x_3}_{x_6}P^{l-x_1-x_4}_{x_7}P^{l-x_2-x_5}_{x_8}P^{n-2l-x_3-x_6}_{x_9}.$$

Lemma 3.29 *Using the notation of Remark 3.28 we have*

$$\sum_{A_1+A_2+A_3=l}\ \sum_{x_1+x_2+x_3=A_1}\ \sum_{x_4+x_5+x_6=A_2}\ \sum_{x_7+x_8+x_9=A_3} CX\cdot CY\cdot CZ = (P_l^n)^2.$$

Proof: We have the following identities

$$\begin{aligned}
&\sum_{A_1+A_2+A_3=l}\ \sum_{x_1+x_2+x_3=A_1}\ \sum_{x_4+x_5+x_6=A_2}\ \sum_{x_7+x_8+x_9=A_3} CX\cdot CY\cdot CZ\\
=\ &\sum_{A_1+A_2+A_3=l}\ \sum_{x_1+x_2+x_3=A_1}\ \sum_{x_4+x_5+x_6=A_2}\ \sum_{x_7+x_8+x_9=A_3} l!\frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}\frac{(n-2l)!}{(n-2l-A_3)!}\\
&\cdot\binom{l}{x_1}\binom{l}{x_2}\binom{n-2l}{x_3}\binom{l-x_1}{x_4}\binom{l-x_2}{x_5}\binom{n-2l-x_3}{x_6}\binom{l-x_1-x_4}{x_7}\binom{l-x_2-x_5}{x_8}\binom{n-2l-x_3-x_6}{x_9}\\
=\ &\sum_{A_1+A_2+A_3=l} l!\frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}\frac{(n-2l)!}{(n-2l-A_3)!}\binom{n}{A_1}\binom{n-A_1}{A_2}\binom{n-A_1-A_2}{A_3}\\
=\ &\frac{l!n!}{(n-l)!}\sum_{A_1+A_2+A_3=l}\binom{l}{A_1}\binom{l}{A_2}\binom{n-2l}{A_3}\\
=\ &\frac{l!n!}{(n-l)!}\binom{n}{l}\\
=\ &(P_l^n)^2.
\end{aligned}$$

□

Remark 3.30 *Let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$, we consider the following regions*

$$\begin{aligned}
R_{211} &= (l/n, 2l/n]\times[0, l/n]\times[0, l/n]\\
R_{212} &= (l/n, 2l/n]\times[0, l/n]\times(l/n, 2l/n]\\
R_{213} &= (l/n, 2l/n]\times[0, l/n]\times(2l/n, 1]\\
R_{221} &= (l/n, 2l/n]\times(l/n, 2l/n]\times[0, l/n]\\
R_{222} &= (l/n, 2l/n]\times(l/n, 2l/n]\times(l/n, 2l/n]\\
R_{223} &= (l/n, 2l/n]\times(l/n, 2l/n]\times(2l/n, 1]\\
R_{231} &= (l/n, 2l/n]\times(2l/n, 1]\times[0, l/n]\\
R_{232} &= (l/n, 2l/n]\times(2l/n, 1]\times(l/n, 2l/n]\\
R_{233} &= [l/n, 2l/n]\times(2l/n, 1]\times(2l/n, 1],
\end{aligned}$$

let $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8$ and y_9 be, respectively, the number of observations in the regions $R_{211}, R_{212}, R_{213}, R_{221}, R_{222}, R_{223}, R_{231}, R_{232}$ and R_{233}.

We define

$$\begin{aligned}
y_1 + y_2 + y_3 &= \hat{A}_1 \\
y_4 + y_5 + y_6 &= \hat{A}_2 \\
y_7 + y_8 + y_9 &= \hat{A}_3 \\
y_1 + y_4 + y_7 &= \hat{B}_1 \\
y_2 + y_5 + y_8 &= \hat{B}_2 \\
y_3 + y_6 + y_9 &= \hat{B}_3.
\end{aligned}$$

To count the number of ways in which we can select l from $n - l$ points in the region $R^{**} = (l/n, 2l/n] \times [0, 1] \times [0, 1]$ we consider

1. *Number of ways to select the first coordinate X*

$$\begin{aligned}
CX &= \binom{l}{y_1}\binom{l-y_1}{y_2}\binom{l-y_1-y_2}{y_3}\binom{l-y_1-y_2-y_3}{y_4}\binom{l-y_1-y_2-y_3}{y_5} \\
&\cdot\binom{l-y_1-y_2-y_3-y_4-y_5}{y_6}\binom{l-y_1-y_2-y_3-y_4-y_5-y_6}{y_7} \\
&\cdot\binom{l-y_1-y_2-y_3-y_4-y_5-y_6-y_7}{y_8}\binom{l-y_1-y_2-y_3-y_4-y_5-y_6-y_7-y_8}{y_9} \\
&= l!\frac{1}{y_1!}\frac{1}{y_2!}\frac{1}{y_3!}\frac{1}{y_4!}\frac{1}{y_5!}\frac{1}{y_6!}\frac{1}{y_7!}\frac{1}{y_8!}\frac{1}{y_9!}.
\end{aligned}$$

2. *Number of ways to select the second coordinate Y (we consider the notation of Remark 3.28)*

$$CY = P_{y_5}^{l-A_2} P_{y_4}^{l-A_2-y_5} P_{y_6}^{l-A_2-y_4-y_5} P_{y_2}^{l-A_1} P_{y_1}^{l-A_1-y_2} P_{y_3}^{l-A_1-y_2-y_1} P_{y_8}^{n-2l-A_3} P_{y_7}^{n-2l-A_3-y_8} P_{y_9}^{n-2l-A_3-y_8-y_7}.$$

3. *Number of ways to select the third coordinate Z (we consider the notation of Remark 3.28)*

$$\begin{aligned}
CZ &= {}^{l-B_1}P_{y_1}\,{}^{l-B_1-y_1}P_{y_4}\,{}^{l-B_1-y_1-y_4}P_{y_7}\,{}^{l-B_2}P_{y_2}\,{}^{l-B_2-y_2}P_{y_5}\,{}^{l-B_2-y_2-y_5}P_{y_8}\,{}^{n-2l-B_3}P_{y_3}\,{}^{n-2l-B_3-y_3}P_{y_6} \\
&\cdot\,{}^{n-2l-B_3-y_3-y_6}P_{y_9} \\
&= \frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}.
\end{aligned}$$

Lemma 3.31 *Using the notation of Remark 3.30 and Remark 3.28, we have that*

$$\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\ \sum_{y_2+y_5+y_8=\hat{B}_2}\ \sum_{y_1+y_4+y_7=\hat{B}_1}\ \sum_{y_3+y_6+y_9=\hat{B}_3} CX\cdot CY\cdot CZ=(P_l^{n-l})^2.$$

Proof: We have the following identities

$$\begin{aligned}
&\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\ \sum_{y_2+y_5+y_8=\hat{B}_2}\ \sum_{y_1+y_4+y_7=\hat{B}_1}\ \sum_{y_3+y_6+y_9=\hat{B}_3} CX\cdot CY\cdot CZ\\
=\ &\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\ \sum_{y_2+y_5+y_8=\hat{B}_2}\ \sum_{y_1+y_4+y_7=\hat{B}_1}\ \sum_{y_3+y_6+y_9=\hat{B}_3} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\\
&\cdot\binom{l-A_2}{y_5}\binom{l-A_1}{y_2}\binom{n-2l-A_3}{y_8}\binom{l-A_2-y_5}{y_4}\binom{l-A_1-y_2}{y_1}\binom{n-2l-A_3-y_8}{y_7}\\
&\cdot\binom{l-A_2-y_4-y_5}{y_6}\binom{l-A_1-y_2-y_1}{y_3}\binom{n-2l-A_3-y_8-y_7}{y_9}\\
=\ &\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\binom{n-l}{\hat{B}_2}\\
&\qquad\cdot\binom{n-l-\hat{B}_2}{\hat{B}_1}\binom{n-l-\hat{B}_1-\hat{B}_2}{\hat{B}_3}\\
=\ &\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l} l!\frac{(n-l)!}{(n-2l)!}\binom{l-B_1}{\hat{B}_1}\binom{l-B_2}{\hat{B}_2}\binom{n-2l-B_3}{\hat{B}_3}\\
=\ &l!\frac{(n-l)!}{(n-2l)!}\binom{n-l}{l}\\
=\ &(P_l^{n-l})^2.
\end{aligned}$$

□

Remark 3.32 *We use the notation on Remark 3.28 and Remark 3.30, and the results in the proofs of the Lemma 3.29 and the Lemma 3.31. Let n and m be integers greater than or equal to two, where m divides n, and $l=n/m$. Let N_{111} be the random variable indicating the number of observations falling in R_{111} and N_{222} the random variable indicating the number of observations falling in R_{222}, when we considered a sample of size n from the product copula, then*

$$P(N_{111}=x_1,N_{222}=y_5)=\Bigg[\sum_{A_1+A_2+A_3=l}\ \sum_{x_2+x_3=A_1-x_1}\ \sum_{x_4+x_5+x_6=A_2}\ \sum_{x_7+x_8+x_9=A_3} l!\frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}$$
$$\frac{(n-2l)!}{(n-2l-A_3)!}\binom{l}{x_1}\binom{l}{x_2}\binom{n-2l}{x_3}\binom{l-x_1}{x_4}\binom{l-x_2}{x_5}\binom{n-2l-x_3}{x_6}$$

$$\binom{l-x_1-x_4}{x_7}\binom{l-x_2-x_5}{x_8}\binom{n-2l-x_3-x_6}{x_9}\Bigg]\cdot\Bigg[\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\ \sum_{y_2+y_8=\hat{B}_2-y_5} l!$$
$$\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\binom{l-A_2}{y_5}\binom{l-A_1}{y_2}$$
$$\binom{n-2l-A_3}{y_8}\binom{n-l-\hat{B}_2}{\hat{B}_1}\binom{n-l-\hat{B}_1-\hat{B}_2}{\hat{B}_3}\Bigg]\cdot\left[\frac{((n-2l)!)^2}{(n!)^2}\right].$$

Remark 3.33 *If we use the notation on Remark 3.28 and Remark 3.30, then*

$$\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\ \sum_{y_2+y_8+(y_5-1)=\hat{B}_2-1} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\binom{l-A_2-1}{y_5-1}$$
$$\binom{l-A_1}{y_2}\binom{n-2l-A_3}{y_8}\binom{n-l-\hat{B}_2}{\hat{B}_1}\binom{n-l-\hat{B}_1-\hat{B}_2}{\hat{B}_3}$$
$$= \sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\binom{n-l-1}{\hat{B}_2-1}\binom{n-l-\hat{B}_2}{\hat{B}_1}$$
$$\binom{n-l-\hat{B}_1-\hat{B}_2}{\hat{B}_3}$$
$$= \sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!}\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\frac{(n-l-1)!}{(\hat{B}_2-1)!}\frac{1}{\hat{B}_1!}\frac{1}{\hat{B}_3!}\frac{1}{(n-2l)!}$$
$$= \frac{l!(n-l-1)!}{(n-2l)!}(l-B_2)\sum_{\hat{B}_1-1+\hat{B}_2+\hat{B}_3=l-1}\binom{l-B_1}{\hat{B}_1}\binom{l-B_2-1}{\hat{B}_2-1}\binom{n-2l-B_3}{\hat{B}_3}$$
$$= \frac{l!(n-l-1)!}{(n-2l)!}(l-B_2)\binom{n-l-1}{l-1}.$$

Remark 3.34 *Using the notation on Remark 3.28 and Remark 3.30, we have*

$$\binom{l}{x_2}\binom{l-x_2}{x_5}\binom{l-x_2-x_5}{x_8} = \frac{l!}{x_2!(l-x_2)!}\frac{(l-x_2)!}{x_5!(l-x_2-x_5)!}\frac{(l-x_2-x_5)!}{x_8!(l-x_2-x_5-x_8)!}$$
$$= \frac{l}{l-x_2}\frac{(l-1)!}{x_2!(l-x_2-1)!}\frac{l-x_2}{l-x_2-x_5}\frac{(l-x_2-1)!}{x_5!(l-x_2-x_5-1)!}$$
$$\cdot\frac{l-x_2-x_5}{l-x_2-x_5-x_8}\frac{(l-x_2-x_5-1)!}{x_8!(l-x_2-x_5-x_8-1)!}$$
$$= \frac{l}{l-B_2}\binom{l-1}{x_2}\binom{l-1-x_2}{x_5}\binom{l-1-x_2-x_5}{x_8}.$$

Theorem 3.35 *With the same hypothesis as in Remark 3.32, we have*

$$Cov(N_{111}/n, N_{222}/n) = \frac{l^6}{n^4(n-1)^2} - \left(\frac{1}{m^3}\right)^2.$$

Proof: From Remark 3.32, Remark 3.33 and Remark 3.34 we have

$$\begin{aligned}
E\left(N_{111}N_{222}\right) &= \Bigg[\sum_{A_1+A_2+A_3=l}\sum_{x_1+x_2+x_3=A_1}\sum_{x_4+x_5+x_6=A_2}\sum_{x_7+x_8+x_9=A_3} x_1 y_5 l! \frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}\frac{(n-2l)!}{(n-2l-A_3)!} \\
&\binom{l}{x_1}\binom{l}{x_2}\binom{n-2l}{x_3}\binom{l-x_1}{x_4}\binom{l-x_2}{x_5}\binom{n-2l-x_3}{x_6}\binom{l-x_1-x_4}{x_7}\binom{l-x_2-x_5}{x_8} \\
&\binom{n-2l-x_3-x_6}{x_9}\Bigg]\cdot\Bigg[\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\sum_{y_2+y_8+y_5=\hat{B}_2} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!} \\
&\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\binom{l-A_2}{y_5}\binom{l-A_1}{y_2}\binom{n-2l-A_3}{y_8}\binom{n-l-\hat{B}_2}{\hat{B}_1} \\
&\binom{n-l-\hat{B}_1-\hat{B}_2}{\hat{B}_3}\Bigg]\cdot\left[\frac{((n-2l)!)^2}{(n!)^2}\right] \\
&= \Bigg[\sum_{A_1+A_2+A_3=l}\sum_{x_1+x_2+x_3=A_1}\sum_{x_4+x_5+x_6=A_2}\sum_{x_7+x_8+x_9=A_3} l!\frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}\frac{(n-2l)!}{(n-2l-A_3)!} \\
&l\binom{l-1}{x_1-1}\binom{l}{x_2}\binom{n-2l}{x_3}\binom{l-x_1}{x_4}\binom{l-x_2}{x_5}\binom{n-2l-x_3}{x_6}\binom{l-x_1-x_4}{x_7}\binom{l-x_2-x_5}{x_8} \\
&\binom{n-2l-x_3-x_6}{x_9}\Bigg]\cdot\Bigg[\sum_{\hat{B}_1+\hat{B}_2+\hat{B}_3=l}\sum_{y_2+y_8+(y_5-1)=\hat{B}_2-1} l!\frac{(l-B_1)!}{(l-B_1-\hat{B}_1)!}\frac{(l-B_2)!}{(l-B_2-\hat{B}_2)!} \\
&(l-A_2)\frac{(n-2l-B_3)!}{(n-2l-B_3-\hat{B}_3)!}\binom{l-A_2-1}{y_5-1}\binom{l-A_1}{y_2}\binom{n-2l-A_3}{y_8}\binom{n-l-\hat{B}_2}{\hat{B}_1} \\
&\binom{n-l-\hat{B}_1-\hat{B}_2}{\hat{B}_3}\Bigg]\cdot\left[\frac{((n-2l)!)^2}{(n!)^2}\right] \\
&= \frac{(l!)^2 l(n-l-1)!}{(n-2l)!}\binom{n-l-1}{l-1}\sum_{A_1+A_2+A_3=l}\sum_{x_1+x_2+x_3=A_1}\sum_{x_4+x_5+x_6=A_2}\sum_{x_7+x_8+x_9=A_3} \\
&\cdot\frac{l!}{(l-A_1)!}\frac{l!}{(l-A_2)!}\frac{(n-2l)!}{(n-2l-A_3)!}(l-A_2)(l-B_2)\binom{l-1}{x_1-1}\binom{l}{x_2}\binom{n-2l}{x_3} \\
&\cdot\binom{l-x_1}{x_4}\binom{l-x_2}{x_5}\binom{n-2l-x_3}{x_6}\binom{l-x_1-x_4}{x_7}\binom{l-x_2-x_5}{x_8}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\binom{n-2l-x_3-x_6}{x_9}\left[\frac{((n-2l)!)^2}{(n!)^2}\right] \quad \text{(using Remark 3.33)}\\
= \quad & \frac{(l!)^2 l^3 (n-l-1)!}{(n-2l)!}\binom{n-l-1}{l-1}\sum_{A_1+A_2+A_3=l}\ \sum_{x_1+x_2+x_3=A_1}\ \sum_{x_4+x_5+x_6=A_2}\ \sum_{x_7+x_8+x_9=A_3}\\
&\cdot\frac{l!}{(l-A_1)!}\frac{(l-1)!}{(l-A_2-1)!}\frac{(n-2l)!}{(n-2l-A_3)!}(l-A_2)(l-B_2)\binom{l-1}{x_1-1}\binom{l-1}{x_2}\binom{n-2l}{x_3}\\
&\cdot\binom{l-x_1}{x_4}\binom{l-1-x_2}{x_5}\binom{n-2l-x_3}{x_6}\binom{l-x_1-x_4}{x_7}\binom{l-1-x_2-x_5}{x_8}\\
&\cdot\binom{n-2l-x_3-x_6}{x_9}\left[\frac{((n-2l)!)^2}{(n!)^2}\right] \quad \text{(using Remark 3.34)}\\
= \quad & \frac{(l!)^2 l^3 (n-l-1)!}{(n-2l)!}\binom{n-l-1}{l-1}\sum_{A_1+A_2+A_3=l}\frac{l!}{(l-A_1)!}\frac{(l-1)!}{(l-A_2-1)!}\frac{(n-2l)!}{(n-2l-A_3)!}\\
&\cdot\binom{n-2}{A_1-1}\binom{n-1-A_1}{A_2}\binom{n-1-A_1-A_2}{A_3}\left[\frac{((n-2l)!)^2}{(n!)^2}\right]\\
= \quad & \frac{(l!)^2 l^4 (n-l-1)!}{(n-2l)!}\binom{n-l-1}{l-1}\sum_{A_1+A_2+A_3=l}\frac{(n-2)!}{(n-1-l)!}\binom{l-1}{A_1-1}\\
&\cdot\binom{l-1}{A_2}\binom{n-2l}{A_3}\left[\frac{((n-2l)!)^2}{(n!)^2}\right]\\
= \quad & \frac{(l!)^2 l^4 (n-l-1)!}{(n-2l)!}\frac{(n-2)!}{(n-1-l)!}\binom{n-2}{l-1}\binom{n-l-1}{l-1}\left[\frac{((n-2l)!)^2}{(n!)^2}\right]\\
= \quad & \frac{(l!)^2 l^4 (n-l-1)!(n-2)!}{(n-2l)!(n-1-l)!}\frac{(n-2)!}{(l-1)!(n-l-1)!}\frac{(n-l-1)!}{(l-1)!(n-2l)!}\frac{((n-2l)!)^2}{(n!)^2}\\
= \quad & \frac{l^6}{n^2(n-1)^2},
\end{aligned}
$$

therefore

$$
\begin{aligned}
Cov\left(N_{111}/n, N_{222}/n\right) &= E\left((N_{111}/n)(N_{222}/n)\right) - E(N_{111}/n)E(N_{222}/n)\\
&= \frac{l^6}{n^4(n-1)^2} - \left(\frac{1}{m^3}\right)^2.
\end{aligned}
$$

□

Remark 3.36 (Case 3) *Using the notation in Remark 3.28 and the results in Lemma 3.29, let n and m be integers greater than or equal to two, where m divides n, and $l = n/m$. Let N_{111} be the random variable indicating the number of observations falling in R_{111} and let N_{122} be the random* *variable indicating the number of observations falling in R_{122}, when we consider a sample of size n from the product copula, then*

$$\begin{aligned} P\{N_{111}=x_1, N_{122}=x_5\} &= \sum_{A_1+A_2+A_3=l} \sum_{x_2+x_3=A_1-x_1} \sum_{x_4+x_6=A_2-x_5} \sum_{x_7+x_8+x_9=A_3} l! \frac{l!}{(l-A_1)!} \frac{l!}{(l-A_2)!} \\ & \frac{(n-2l)!}{(n-2l-A_3)!} \binom{l}{x_1} \binom{l}{x_2} \binom{n-2l}{x_3} \binom{l-x_1}{x_4} \binom{l-x_2}{x_5} \binom{n-2l-x_3}{x_6} \\ & \binom{l-x_1-x_4}{x_7} \binom{l-x_2-x_5}{x_8} \binom{n-2l-x_3-x_6}{x_9} \frac{((n-l)!)^2}{(n!)^2}. \end{aligned}$$

Theorem 3.37 *If we consider the same hypothesis of Remark 3.36, then*

$$Cov(N_{111}/n, N_{122}/n) = \frac{l^5(l-1)}{n^4(n-1)^2} - \left(\frac{1}{m^3}\right)^2.$$

Proof: From Remark 3.36, we have

$$\begin{aligned} E(N_{111}N_{122}) &= \sum_{A_1+A_2+A_3=l} \sum_{x_1+x_2+x_3=A_1} \sum_{x_4+x_5+x_6=A_2} \sum_{x_7+x_8+x_9=A_3} l! \frac{l!}{(l-A_1)!} \frac{l!}{(l-A_2)!} \frac{(n-2l)!}{(n-2l-A_3)!} \\ & \cdot x_1 x_5 \frac{l(l-1)!}{x_1!(l-x_1)!} \frac{l}{(l-x_2)} \frac{(l-1)!}{x_2!(l-x_2-1)!} \binom{n-2l}{x_3} \binom{l-x_1}{x_4} (l-x_2) \\ & \cdot \frac{(l-x_2-1)!}{x_5!(l-x_2-x_5)!} \binom{n-2l-x_3}{x_6} \binom{l-x_1-x_4}{x_7} \binom{l-x_2-x_5}{x_8} \binom{n-2l-x_3-x_6}{x_9} \\ & \cdot \frac{((n-l)!)^2}{(n!)^2} \\ &= \sum_{A_1+A_2+A_3=l} \sum_{x_1+x_2+x_3=A_1} \sum_{x_4+x_5+x_6=A_2} \sum_{x_7+x_8+x_9=A_3} l! \frac{l!}{(l-A_1)!} \frac{l!}{(l-A_2)!} \frac{(n-2l)!}{(n-2l-A_3)!} \\ & \cdot l^2 \frac{((n-l)!)^2}{(n!)^2} \left[\binom{l-1}{x_1-1} \binom{l-1}{x_2} \binom{n-2l}{x_3} \right] \left[\binom{l-x_1}{x_4} \binom{l-x_2-1}{x_5-1} \binom{n-2l-x_3}{x_6} \right] \\ & \cdot \left[\binom{l-x_1-x_4}{x_7} \binom{l-x_2-x_5}{x_8} \binom{n-2l-x_3-x_6}{x_9} \right] \\ &= \sum_{A_1+A_2+A_3=l} l! \frac{l!}{(l-A_1)!} \frac{l!}{(l-A_2)!} \frac{(n-2l)!}{(n-2l-A_3)!} l^2 \frac{((n-l)!)^2}{(n!)^2} \binom{n-2}{A_1-1} \\ & \cdot \binom{n-1-A_1}{A_2-1} \binom{n-A_1-A_2}{A_3} \end{aligned}$$

$$
\begin{aligned}
&= \frac{l!l^2((n-l)!)^2}{(n!)^2} \sum_{A_1+A_2+A_3=l} \left(\frac{l!}{(l-A_1)!} \frac{l!}{(l-A_2)!} \frac{(n-2l)!}{(n-2l-A_3)!} \right. \\
&\quad \left. \frac{(n-2)!}{(A_1-1)!(n-1-A_1)!} \frac{(n-1-A_1)!}{(A_2-1)!(n-A_1-A_2)!} \frac{(n-A_1-A_2)!}{A_3!(n-l)!} \right) \\
&= \frac{l!l^2((n-l)!)^2}{(n!)^2} \frac{l^2(n-2)!}{(n-l)!} \sum_{A_1+A_2+A_3=l} \left(\frac{(l-1)!}{(A_1-1)!(l-A_1)!} \frac{(l-1)!}{(A_2-1)!(l-A_2)!} \right. \\
&\quad \left. \frac{(n-2l)!}{A_3!(n-2l-A_3)!} \right) \\
&= \frac{l!l^4((n-l)!)^2(n-2)!}{(n!)^2(n-l)!} \sum_{A_1+A_2+A_3=l} \binom{l-1}{A_1-1}\binom{l-1}{A_2-1}\binom{n-2l}{A_3} \\
&= \frac{l!l^4((n-l)!)^2(n-2)!}{(n!)^2(n-l)!} \binom{n-2}{l-2} \\
&= \frac{l!l^4((n-l)!)^2(n-2)!}{(n!)^2(n-l)!} \frac{(n-2)!}{(l-2)!(n-l)!} \\
&= \frac{l^5(l-1)}{n^2(n-1)^2},
\end{aligned}
$$

therefore

$$
Cov(N_{111}/n, N_{122}/n) = \frac{l^5(l-1)}{n^4(n-1)^2} - \left(\frac{1}{m^3}\right)^2.
$$

□

The following theorem describes the joint probability distribution of the boxes generated for the uniform partition of size m, with $m \geq 2$, when m divides n, in the three dimensional case.

Theorem 3.38 *Let $m \geq 2$, $n \in \mathbb{N}$, where m divides n, $l = n/m$ and $I_m = \{1, \cdots m\}$, let R_{ijk}, with $i, j, k \in I_m$, be the uniform partition of size m of $\mathbf{I}^3 = [0,1]^3$ and N_{ijk} the random variables that indicates the number of observations falling in R_{ijk}, respectively, for all $i, j, k \in I_m$, when we consider a sample of size n from the product copula; let n_{ijk}, with $i, j, k \in I_m$, be zero or positive integer satisfying the following restrictions*

$$
\sum_{j,k=1}^{m} n_{ijk} = l \quad (\textit{for all } i \in I_m), \qquad \sum_{i,k=1}^{m} n_{ijk} = l \quad (\textit{for all } j \in I_m), \qquad \sum_{i,j=1}^{m} n_{ijk} = l \quad (\textit{for all } k \in I_m),
$$

then

$$P\left\{\bigcap_{i,j,k\in I_m}\{N_{ijk}=n_{ijk}\}\right\}=\frac{(l!)^{3m}}{(n!)^2\prod_{i,j,k\in I_m}n_{ijk}!}.$$

Proof: Similarly to the proof of the Theorem 3.17, we will use the counting methodology provided on the Remark 3.3, using permutations for the counting of the third coordinate Z. The order count is

$$\begin{gathered}
R_{111}, R_{112}, \cdots, R_{11m}, \cdots, R_{1m1}, R_{1m2}, \cdots, R_{1mm}\\
R_{211}, R_{212}, \cdots, R_{21m}, \cdots, R_{2m1}, R_{2m2}, \cdots, R_{2mm}\\
\vdots\\
R_{m11}, R_{m12}, \cdots, R_{m1m}, \cdots, R_{mm1}, R_{mm2}, \cdots, R_{mmm}.
\end{gathered}$$

The number of possibilities for l observations in the region $\hat{R}_1=[0,l/n]\times[0,1]\times[0,1]$ is given by

$$\begin{aligned}
\hat{N}_1 \;=\; & \left[\binom{l}{n_{111}}P^{l}_{n_{111}}P^{l}_{n_{111}}\binom{l-n_{111}}{n_{112}}P^{l-n_{111}}_{n_{112}}P^{l}_{n_{112}}\cdots\binom{l-\sum_{k=1}^{m-1}n_{11k}}{n_{11m}}P^{l-\sum_{k=1}^{m-1}n_{11k}}_{n_{11m}}P^{l}_{n_{11m}}\right]\\
& \left[\binom{l-\sum_{k=1}^{m}n_{11k}}{n_{121}}P^{l}_{n_{121}}P^{l-n_{111}}_{n_{121}}\binom{l-n_{121}-\sum_{k=1}^{m}n_{11k}}{n_{122}}P^{l-n_{121}}_{n_{122}}P^{l-n_{112}}_{n_{122}}\cdots\right.\\
& \left.\binom{l-\sum_{k=1}^{m-1}n_{12k}-\sum_{k=1}^{m}n_{11k}}{n_{12m}}P^{l-\sum_{k=1}^{m-1}n_{12k}}_{n_{12m}}P^{l-n_{11m}}_{n_{12m}}\right]\cdots\left[\binom{l-\sum_{j=1}^{m-1}\sum_{k=1}^{m}n_{1jk}}{n_{1m1}}P^{l}_{n_{1m1}}\right.\\
& P^{l-\sum_{j=1}^{m-1}n_{1j1}}_{n_{1m1}}\binom{l-n_{1m1}-\sum_{j=1}^{m-1}\sum_{k=1}^{m}n_{1jk}}{n_{1m2}}P^{l-n_{1m1}}_{n_{1m2}}P^{l-\sum_{j=1}^{m-1}n_{1j2}}_{n_{1m2}}\cdots\\
& \left.\binom{l-\sum_{k=1}^{m-1}n_{1mk}-\sum_{j=1}^{m-1}\sum_{k=1}^{m}n_{1jk}}{n_{1mm}}P^{l-\sum_{k=1}^{m-1}n_{1mk}}_{n_{1mm}}P^{l-\sum_{j=1}^{m-1}n_{1jk}}_{n_{1mm}}\right],
\end{aligned}$$

and the number of possibilities for l observations in the region $\hat{R}_s=(l(s-1)/n, ls/n]\times[0,1]\times[0,1]$, for all $s\in 2,\cdots,m$, is given by

$$\begin{aligned}
\hat{N}_s \;=\; & \left[\binom{l}{n_{s11}}P^{l-\sum_{i=1}^{s-1}\sum_{k=1}^{m}n_{i1k}}_{n_{s11}}P^{l-\sum_{i=1}^{s-1}\sum_{j=1}^{m}n_{ij1}}_{n_{s11}}\binom{l-n_{s11}}{n_{s12}}P^{l-n_{s11}-\sum_{i=1}^{s-1}\sum_{k=1}^{m}n_{i1k}}_{n_{s12}}P^{l-\sum_{i=1}^{s-1}\sum_{j=1}^{m}n_{ij2}}_{n_{s12}}\cdots\right.\\
& \left.\binom{l-\sum_{k=1}^{m-1}n_{s1k}}{n_{s1m}}P^{l-\sum_{k=1}^{m-1}n_{s1k}-\sum_{i=1}^{s-1}\sum_{k=1}^{m}n_{i1k}}_{n_{s1m}}P^{l-\sum_{i=1}^{s-1}\sum_{j=1}^{m}n_{ijm}}_{n_{s1m}}\right]\left[\binom{l-\sum_{k=1}^{m}n_{s1k}}{n_{s21}}P^{l-\sum_{i=1}^{s-1}\sum_{k=1}^{m}n_{i2k}}_{n_{s21}}\right.\\
& P^{l-n_{s11}-\sum_{i=1}^{s-1}\sum_{j=1}^{m}n_{ij1}}_{n_{s21}}\binom{l-n_{s21}-\sum_{k=1}^{m}n_{s1k}}{n_{s22}}P^{l-n_{s21}-\sum_{i=1}^{s-1}\sum_{k=1}^{m}n_{i2k}}_{n_{s22}}P^{l-n_{s12}-\sum_{i=1}^{s-1}\sum_{j=1}^{m}n_{ij2}}_{n_{s22}}\cdots
\end{aligned}$$

$$\binom{l-\sum_{k=1}^{m} n_{s1k}-\sum_{k=1}^{m-1} n_{s2k}}{n_{s2m}} P_{n_{s2m}}^{l-\sum_{k=1}^{m-1} n_{s2k}-\sum_{i=1}^{s-1}\sum_{k=1}^{m} n_{i2k}} P_{n_{s2m}}^{l-n_{s1m}-\sum_{i=1}^{s-1}\sum_{j=1}^{m} n_{ijm}}\Bigg] \cdots \Bigg[\binom{l-\sum_{j,k=1}^{m} n_{sjk}}{n_{sm1}}$$

$$P_{n_{sm1}}^{l-\sum_{i=1}^{s-1}\sum_{k=1}^{m} n_{imk}} P_{n_{sm1}}^{l-\sum_{j=1}^{m-1} n_{sj1}-\sum_{i=1}^{s-1}\sum_{j=1}^{m} n_{ij1}} \binom{l-n_{sm1}-\sum_{j,k=1}^{m} n_{sjk}}{n_{sm2}} P_{n_{sm2}}^{l-n_{sm1}-\sum_{i=1}^{s-1}\sum_{k=1}^{m} n_{imk}}$$

$$P_{n_{sm2}}^{l-\sum_{j=1}^{m-1} n_{sj2}-\sum_{i=1}^{s-1}\sum_{j=1}^{m} n_{ij2}} \cdots \binom{l-\sum_{k=1}^{m-1} n_{smj}-\sum_{j,k=1}^{m} n_{sjk}}{n_{smm}} P_{n_{smm}}^{l-\sum_{k=1}^{m-1} n_{smk}-\sum_{i=1}^{s-1}\sum_{k=1}^{m} n_{imk}}$$

$$P_{n_{smm}}^{l-\sum_{j=1}^{m-1} n_{sjm}-\sum_{i=1}^{s-1}\sum_{j=1}^{m} n_{ijm}}\Bigg]$$

we have that

$$\hat{N}_1 \cdot \prod_{s=2}^{m} \hat{N}_s = \frac{(l!)^{3m}}{\prod_{i,j,k\in I_m} n_{ijk}!}.$$

We divided between $(n!)^2$, the total number of possibilities in which we can see n points in $\mathbf{I}^3 = [0,1]^3$, corresponding to the modified sample of the a sample of size n from the product copula, therefore

$$P\left\{\bigcap_{i,j,k\in I_m} \{N_{ijk} = n_{ijk}\}\right\} = \frac{(l!)^{3m}}{(n!)^2 \prod_{i,j,k\in I_m} n_{ijk}!}.$$

$\square$

3.3 Summary of results and generalizations

We present a summary of the precedent results and the generalizations of the previous expressions for the moments in dimension greater than three.

1. **Dimension two**. Let $m \geq 2$, $n \in \mathbb{N}$, where m divides n, and $l = n/m$, let N_{11}, N_{12} and N_{22} be the random variables that indicates the number of observations on the boxes $R_{11} = [0, l/n]^2$, $R_{12} = [0, l/n] \times (l/n, 2l/n]$ and $R_{22} = (l/n, 2l/n]^2$, respectively, when we consider the modified sample of a sample of size n from the product copula.

   (a) First Moment.

$$E(N_{11}/n) = \frac{1}{m^2}$$

   (b) Second Moment.

$$E((N_{11}/n)^2) = \frac{l^2(l-1)^2}{n^3(n-1)} + \frac{1}{nm^2}$$

(c) Variance.

$$Var(N_{11}/n) = \frac{l^2(l-1)^2}{n^3(n-1)} + \frac{1}{nm^2} - \frac{1}{m^4}$$

(d) Covariance I.

$$Cov(N_{11}/n, N_{12}/n) = \frac{l^3(l-1)}{n^3(n-1)} - \frac{1}{m^4}$$

(e) Covariance II.

$$Cov(N_{11}/n, N_{22}/n) = \frac{l^4}{n^3(n-1)} - \frac{1}{m^4}$$

2. **Dimension three**. Let $m \geq 2$, $n \in \mathbb{N}$, where m divides n, and $l = n/m$, let N_{111}, N_{112}, N_{122} and N_{222} be the random variables that indicates the number of observations on the boxes $R_{111} = [0, l/n]^3$, $R_{112} = [0, l/n] \times [0, l/n] \times (l/n, 2l/n]$, $R_{122} = [0, l/n] \times (l/n, 2l/n] \times (l/n, 2l/n]$ and $R_{222} = (l/n, 2l/n]^3$, respectively, when we consider the modified sample of a sample of size n from the product copula.

   (a) First Moment.

$$E(N_{111}/n) = \frac{1}{m^3}$$

   (b) Second Moment.

$$E((N_{111}/n)^2) = \frac{l^3(l-1)^3}{n^4(n-1)^2} + \frac{1}{nm^3}$$

   (c) Variance.

$$Var(N_{111}/n) = \frac{l^3(l-1)^3}{n^4(n-1)^2} + \frac{1}{nm^3} - \frac{1}{m^6}$$

   (d) Covariance I.

$$Cov(N_{111}/n, N_{112}/n) = \frac{l^4(l-1)^2}{n^4(n-1)^2} - \frac{1}{m^6}$$

   (e) Covariance II.

$$Cov(N_{111}/n, N_{122}/n) = \frac{l^5(l-1)}{n^4(n-1)^2} - \frac{1}{m^6}$$

   (f) Covariance III.

$$Cov(N_{111}/n, N_{222}/n) = \frac{l^6}{n^4(n-1)^2} - \frac{1}{m^6}$$

3. **General case.** Let $m \geq 2$, $n \in \mathbb{N}$, where m divides n, $l = n/m$, $d \geq 2$, $\underline{1} = (1, \cdots, 1)$ (d times) and $\underline{2} = (2, \cdots, 2)$ (d times), let $N_{\underline{1}}$ be the random variable that indicates the number of observations on the box $R_{\underline{1}} = [0, l/n]^d$, when we consider the modified sample of a sample of size n from the product copula.

(a) First Moment.

$$E(N_{\underline{1}}/n) = \frac{1}{m^d}$$

(b) Second Moment.

$$E((N_{\underline{1}}/n)^2) = \frac{l^d(l-1)^d}{n^{d+1}(n-1)^{d-1}} + \frac{1}{nm^d}$$

(c) Variance.

$$Var(N_{\underline{1}}/n) = \frac{l^d(l-1)^d}{n^{d+1}(n-1)^{d-1}} + \frac{1}{nm^d} - \frac{1}{m^{2d}}$$

(d) Covariance I.

$$Cov(N_{\underline{1}}/n, N_{\underline{2}}/n) = \frac{l^{2d}}{n^{d+1}(n-1)^{d-1}} - \frac{1}{m^{2d}}$$

where $N_{\underline{2}}$ is the random variable that indicates the number of observations on the box $R_{\underline{2}} = (l/n, 2l/n]^d$.

(e) Covariance II. Let $\underline{j} = (j_1, \cdots, j_d) \in \{1, 2\}^d$, $\underline{j} \neq \underline{1}, \underline{2}$, if we define

$$R_{\underline{j}} = \hat{I}_1 \times \cdots \times \hat{I}_d$$

where

$$\hat{I}_i = \begin{cases} [0, l/n] & if \quad j_i = 1 \\ (l/n, 2l/n] & if \quad j_i = 2 \end{cases}$$

for $i \in \{1, \cdots, d\}$, then

$$Cov(N_{\underline{1}}/n, N_{\underline{j}}/n) = \frac{l^{2d-k}(l-1)^k}{n^{d+1}(n-1)^{d-1}} - \frac{1}{m^{2d}}.$$

where $N_{\underline{j}}$ is the random variable that indicates the number of observations on the box $R_{\underline{j}}$ and k the number of coordinates equals to one in $\underline{j}$.

(f) Joint distribution. Let $d \geq 2$, and let $N_{i_1 \cdots i_d}$ be the random variable that indicates the number of observations in the box $R_{i_1 \cdots i_d} = \langle (i_1 - 1)/m, i_1/m] \times \cdots \times \langle (i_d - 1)/m, i_d/m]$, where

$i_1, \cdots, i_d \in I_m$ and the notation "$\langle$" indicates "(" if $i_k - 1 > 0$ and "[" if $i_k - 1 = 0$, for all $k \in I_d$, when we consider the modified sample of size n from the product copula Π^d. Then

$$P\left\{ \bigcap_{i_1,\cdots,i_d \in I_m} \{N_{i_1\cdots i_d} = n_{i_1\cdots i_d}\} \right\} = \frac{(l!)^{dm}}{(n!)^{d-1} \prod_{i_1,\cdots,i_d \in I_m} n_{i_1\ldots i_d}!}.$$

3.4 Case: m does not divide n

In the next pages we show that we can obtain similar results to what was done previously, now without the hypothesis m divides n. We exemplify with the two dimensional case.

The following results establish the properties of random variables associated with the observations in the boxes from a not uniform partition of order m, that is, a partition of the box $\mathbf{I}^2 = [0,1]^2$ generated by dividing the two intervals $\mathbf{I} = [0,1]$ in the Cartesian product into m parts, where m does not divide n.

Definition 3.39 *Let $n \geq 2$, $0 < l_1 < l_2 < n$ and $0 < j_1 < j_2 \leq n$ be integers with $j_2 - j_1 = l_2$. We define the following regions in the unit square $\mathbf{I}^2 = [0,1]^2$*

$$\begin{aligned} R_{1j} &= [0, l_1/n] \times (j_1/n, j_2/n] \\ R_1 &= ([0, l_1/n] \times [0,1]) \backslash R_{1j} \\ R &= (l_1/n, 1] \times [0,1]\,. \end{aligned}$$

Remark 3.40 *In the following results we consider than in the unit square $\mathbf{I}^2 = [0,1]^2$ there exists n points corresponding to the range statistics from a sample of size n from the product copula.*

Lemma 3.41 *Let k_1 be the number of points in R_{1j} and x_1 the number of points in R_1, we have that*

$$\sum_{k_1+x_1=l_1} \binom{l_1}{k_1} P_{k_1}^{l_2} \binom{l_1-k_1}{x_1} P_{x_1}^{n-l_2} = P_{l_1}^{n}.$$

Proof: We can observe that

$$\binom{l_1-k_1}{x_1} = \binom{x_1}{x_1} = 1$$

then

$$\sum_{k_1+x_1=l_1} \binom{l_1}{k_1} P_{k_1}^{l_2} \binom{l_1-k_1}{x_1} P_{x_1}^{n-l_2} = \sum_{k_1+x_1=l_1} \binom{l_1}{k_1} P_{k_1}^{l_2} P_{x_1}^{n-l_2}$$

$$
\begin{aligned}
&= \sum_{k_1+x_1=l_1} \frac{l_1!}{k_1!(l_1-k_1)!}\frac{l_2!}{(l_2-k_1)!}\frac{(n-l_2)!}{(n-l_2-x_1)!} \\
&= \sum_{k_1+x_1=l_1} \frac{l_1!}{k_1!x_1!}\frac{l_2!}{(l_2-k_1)!}\frac{(n-l_2)!}{(n-l_2-x_1)!} \\
&= l_1! \sum_{k_1+x_1=l_1} \binom{l_2}{k_1}\binom{n-l_2}{x_1} \\
&= l_1!\binom{n}{l_1} \quad \text{(Vandermonde's identity)} \\
&= P^n_{l_1}.
\end{aligned}
$$

□

Lemma 3.42 *Let N_{1j} be the random variable that indicates the number of observations falling in R_{1j}, then the following equality holds*

$$
\begin{aligned}
P\{N_{1j}=k_1\} &= \frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1} \\
&= \frac{\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1}}{\binom{n-l_1}{l_1}}
\end{aligned}
$$

this probability corresponds to a hypergeometric distribution with parameters: n the population size, l_1 the class size and l_2 the sample size.

Proof: We obtained the equality considering

$$
\sum_{k_1+x_1=l_1} \binom{l_1}{k_1} P^{l_2}_{k_1}\binom{l_1-k_1}{x_1} P^{n-l_2}_{x_1} = l_1! \sum_{k_1+x_1=l_1}\binom{l_2}{k_1}\binom{n-l_2}{x_1}
$$

from the proof of Lemma 3.42, this equality indicates the number of ways in which we can have k_1+x_1 points in the region $R_{1j}\cup R_1$; setting the value k_1, this result is multiplied by $(n-l_1)!$ (number of ways in which we can have $n-l_1$ points in the region R discarding l_1 possibilities corresponding to the coordinates occupied by the observations in the region $R_{1j}\cup R_1$) and divided by $n!$, the total number of ways that we can observe n points in $\mathbf{I}^2=[0,1]^2$. □

Theorem 3.43 *Let N_{1j} be the random variable that indicates the number of observations falling in R_{1j} when we consider a sample of size n from the product copula, then*

$$
E(N_{1j}/n) = \frac{l_1 l_2}{n^2}, \quad E((N_{1j}/n)^2) = \frac{l_2(l_2-1)l_1(l_1-1)}{n^3(n-1)} + \frac{l_1 l_2}{n^3}
$$

and

$$Var(N_{1j}/n) = \frac{l_2(l_2-1)l_1(l_1-1)}{n^3(n-1)} + \frac{l_1 l_2}{n^3} - \left(\frac{l_1 l_2}{n^2}\right)^2.$$

Proof:

$$\begin{aligned}
E(N_{1j}) &= \sum_{k_1=0}^{l_1} k_1 \frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1} \\
&= \frac{l_1! l_2 (n-l_1)!}{n!}\sum_{k_1=1}^{l_1}\binom{l_2-1}{k_1-1}\binom{n-l_2}{l_1-k_1} \\
&= \frac{l_1! l_2 (n-l_1)!}{n!}\sum_{u_1=0}^{l_1-1}\binom{l_2-1}{u_1}\binom{n-l_2}{l_1-1-u_1} \quad (u_1 = k_1 - 1) \\
&= \frac{l_1! l_2 (n-l_1)!}{n!}\binom{n-1}{l_1-1} \\
&= \frac{l_1! l_2 (n-l_1)!}{n!}\frac{(n-1)!}{(l_1-1)!(n-l_1)!} \\
&= \frac{l_1 l_2}{n}
\end{aligned}$$

and

$$E(N_{1j}/n) = \frac{l_1 l_2}{n^2}.$$

We have,

$$\begin{aligned}
E(N_{1j}^2) &= \sum_{k_1=0}^{l_1} k_1^2 \frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1} \\
&= \sum_{k_1=0}^{l_1}(k_1(k_1-1)+k_1)\frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1} \\
&= \sum_{k_1=2}^{l_1} k_1(k_1-1)\frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1} + \sum_{k_1=0}^{l_1} k_1\frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{n-l_2}{l_1-k_1} \\
&= \sum_{k_1=2}^{l_1}\frac{l_1!(n-l_1)!}{n!} l_2(l_2-1)\binom{l_2-2}{k_1-2}\binom{n-l_2}{l_1-k_1} + \frac{l_1 l_2}{n} \\
&= \frac{l_1!(n-l_1)!}{n!} l_2(l_2-1)\sum_{u_1=0}^{l_1-2}\binom{l_2-2}{u_1}\binom{n-l_2}{(l_1-2)-u_1} + \frac{l_1 l_2}{n} \quad (u_1 = k_1 - 2)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{l_1!(n-l_1)!}{n!} l_2(l_2-1)\binom{n-2}{l_1-2} + \frac{l_1 l_2}{n} \\
&= \frac{l_2(l_2-1)l_1(l_1-1)}{n(n-1)} + \frac{l_1 l_2}{n}.
\end{aligned}
$$

Therefore

$$
E((N_{1j}/n)^2) = \frac{l_2(l_2-1)l_1(l_1-1)}{n^3(n-1)} + \frac{l_1 l_2}{n^3}
$$

and

$$
\begin{aligned}
Var(N_{ij}/n) &= E((N_{ij}/n)^2) - (E(N_{ij}/n))^2 \\
&= \frac{l_2(l_2-1)l_1(l_1-1)}{n^3(n-1)} + \frac{l_1 l_2}{n^3} - \left(\frac{l_1 l_2}{n^2}\right)^2.
\end{aligned}
$$

□

In a similar way to the case where m divides n, the calculus of the covariances is divided in two cases. The following result corresponds to the first one.

Definition 3.44 *Let $n \geq 2$, $0 < l_1, l_2, l_3 < n$ and $0 < j_1 < j_2 < j_3 \leq n$ be integers with $j_2 - j_1 = l_2$ and $j_3 - j_2 = l_3$. We define the following regions in the unit square $\mathbf{I}^2 = [0,1]^2$*

$$
\begin{aligned}
R_{1j_1} &= [0, l_1/n] \times (j_1/n, j_2/n] \\
R_{1j_2} &= [0, l_1/n] \times (j_2/n, j_3/n] \\
R_1 &= ([0, l_1/n] \times [0,1]) \backslash (R_{1j_1} \cup R_{1j_2}) \\
R &= (l_1/n, 1] \times [0,1].
\end{aligned}
$$

Lemma 3.45 *Let k_1 be the number of points in R_{1j_1}, k_2 the number of points in R_{1j_2} and x_1 the number of points in R_1, then*

$$
\sum_{k_1+k_2+x_1=l_1} \binom{l_1}{k_1} P_{k_1}^{l_2} \binom{l_1-k_1}{k_2} P_{k_2}^{l_3} \binom{l_1-k_1-k_2}{x_1} P_{x_1}^{n-l_2-l_3} = P_{l_1}^{n}.
$$

Proof: We have that

$$
\begin{aligned}
\sum_{k_1+k_2+x_1=l_1} \binom{l_1}{k_1} P_{k_1}^{l_2} \binom{l_1-k_1}{k_2} P_{k_2}^{l_3} \binom{l_1-k_1-k_2}{x_1} P_{x_1}^{n-l_2-l_3} &= \sum_{k_1+k_2+x_1=l_1} \frac{l_1!}{k_1!k_2!x_1!} P_{k_1}^{l_2} P_{k_2}^{l_3} P_{x_1}^{n-l_2-l_3} \\
&= \sum_{k_1+k_2+x_1=l_1} l_1! \binom{l_2}{k_1}\binom{l_3}{k_2}\binom{n-l_2-l_3}{x_1}
\end{aligned}
$$

$$
\begin{aligned}
&= l_1!\binom{n}{l_1} \\
&= P^n_{l_1}.
\end{aligned}
$$

□

Remark 3.46 *Let N_{1j_1} be the random variable that indicates the number of observations falling in R_{1j_1} and let N_{1j_2} be the random variable that indicates the number of observations falling in R_{1j_2} then*

$$P\{N_{1j_1} = k_1, N_{1j_2} = k_2\} = \frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{l_3}{k_2}\binom{n-l_2-l_3}{l_1-k_1-k_2}.$$

Lemma 3.47 *Whit the same hypothesis of Remark 3.46, we have*

$$Cov(N_{ij_1}/n, N_{ij_2}/n) = \frac{l_1(l_1-1)l_2l_3}{n^2(n-1)} - \left(\frac{l_1l_2}{n^2}\right)\left(\frac{l_1l_3}{n^2}\right).$$

Proof:

$$
\begin{aligned}
E(N_{1j_1}N_{1j_2}) &= \sum_{k_1+k_2=0}^{l_1} k_1k_2\frac{l_1!(n-l_1)!}{n!}\binom{l_2}{k_1}\binom{l_3}{k_2}\binom{n-l_2-l_3}{l_1-k_1-k_2} \\
&= \sum_{k_1+k_2=0}^{l_1} \frac{l_1!(n-l_1)!}{n!}l_2l_3\binom{l_2-1}{k_1-1}\binom{l_3-1}{k_2-1}\binom{n-l_2-l_3}{l_1-k_1-k_2} \\
&= \frac{l_1!(n-l_1)!l_2l_3}{n!}\sum_{k_1+k_2=0}^{l_1}\binom{l_2-1}{k_1-1}\binom{l_3-1}{k_2-1}\binom{n-l_2-l_3}{l_1-k_1-k_2} \\
&= \frac{l_1!(n-l_1)!l_2l_3}{n!}\binom{n-2}{l_1-2} \\
&= \frac{l_1(l_1-1)l_2l_3}{n(n-1)},
\end{aligned}
$$

and

$$E((N_{1j_1}/n)(N_{1j_2}/n)) = \frac{l_1(l_1-1)l_2l_3}{n^3(n-1)}.$$

Therefore

$$
\begin{aligned}
Cov(N_{1j_1}/n, N_{1j_2}/n) &= E((N_{1j_1}/n)(N_{1j_2}/n)) - E(N_{1j_1}/n)E(N_{1j_2}/n) \\
&= \frac{l_1(l_1-1)l_2l_3}{n^3(n-1)} - \left(\frac{l_1l_2}{n^2}\right)\left(\frac{l_1l_3}{n^2}\right).
\end{aligned}
$$

$\square$

We use the next results for the calculation of the covariances for the second case.

Definition 3.48 *Let $n \geq 2$, $0 < l_1, l_2, l_3, l_4 < m$, $0 < j_1 < j_2 < j_3 \leq n$ and $0 = i_1 < i_2 < i_3 \leq n$ be integers with $i_2 - i_1 = l_1$, $i_3 - i_2 = l_3$, $j_2 - j_1 = l_2$ and $j_3 - j_2 = l_4$. We define the following regions in the unit square $\mathbf{I}^2 = [0,1]^2$*

$$\begin{aligned}
R_{1j_1} &= [0, l_1/n] \times (j_1/n, j_2/n] \\
R_{1j_2} &= [0, l_1/n] \times (j_2/n, j_3/n] \\
R_{2j_1} &= (i_2/n, i_3/n] \times (j_1/n, j_2/n] \\
R_{2j_2} &= (i_2/n, i_3/n] \times (j_2/n, j_3/n] \\
R_1 &= ([0, l_1/n] \times [0,1]) \backslash (R_{1j_1} \cup R_{1j_2}) \\
R_2 &= ((i_2/n, i_3/n] \times [0,1]) \backslash (R_{2j_1} \cup R_{2j_2}).
\end{aligned}$$

Lemma 3.49 *Let k_1 be the number of points in R_{1j_1}, x_1 the number of points in R_{1j_2}, x_2 the number of points in R_1, y_1 the number of points in R_{2j_1}, k_2 the number of points in R_{2j_2} and y_2 the number of points in R_2, then*

$$\sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} \binom{l_1}{k_1}\binom{l_1-k_1}{x_1}\binom{l_1-k_1-x_1}{x_2} P_{k_1}^{l_2} P_{x_1}^{l_4} P_{x_2}^{n-l_2-l_4}$$
$$\cdot \binom{l_3}{k_2}\binom{l_3-k_2}{y_1}\binom{l_3-k_2-y_1}{y_2} P_{y_1}^{l_2-k_1} P_{k_2}^{l_4-x_1} P_{y_2}^{n-l_2-l_4-x_2} = P_{l_1}^{n} P_{l_3}^{n-l_1}.$$

Proof:

$$\begin{aligned}
&\sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} \binom{l_1}{k_1}\binom{l_1-k_1}{x_1}\binom{l_1-k_1-x_1}{x_2} P_{k_1}^{l_2} P_{x_1}^{l_4} P_{x_2}^{n-l_2-l_4} \\
&\qquad \cdot \binom{l_3}{k_2}\binom{l_3-k_2}{y_1}\binom{l_3-k_2-y_1}{y_2} P_{y_1}^{l_2-k_1} P_{k_2}^{l_4-x_1} P_{y_2}^{n-l_2-l_4-x_2} \\
= &\sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} \left(\frac{l_1!}{k_1!x_1!x_2!} P_{k_1}^{l_2} P_{x_1}^{l_4} P_{x_2}^{n-l_2-l_4}\right)\left(\frac{l_3!}{k_2!y_1!y_2!} P_{y_1}^{l_2-k_1} P_{k_2}^{l_4-x_1} P_{y_2}^{n-l_2-l_4-x_2}\right) \\
= &\sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} l_1!l_3!\binom{l_2}{k_1}\binom{l_4}{x_1}\binom{n-l_2-l_4}{x_2}\binom{l_2-k_1}{y_1}\binom{l_4-x_1}{k_2}\binom{n-l_2-l_4-x_2}{y_2} \\
= &\sum_{k_1+x_1+x_2=l_1} l_1!l_3!\binom{l_2}{k_1}\binom{l_4}{x_1}\binom{n-l_2-l_4}{x_2}\binom{n-l_1}{l_3}
\end{aligned}$$

$$
\begin{aligned}
&= l_1!l_3!\binom{n}{l_1}\binom{n-l_1}{l_3} \\
&= P^n_{l_1} P^{n-l_1}_{l_3}.
\end{aligned}
$$

□

Remark 3.50 *Let $N_{1_1 j_1}$ be the random variable indicating the number of observations falling in $R_{1_1 j_1}$ and $N_{i_2 j_2}$, the random variable indicating the number of observations falling in $R_{i_2 j_2}$, then*

$$
\begin{aligned}
P\{N_{i_1 j_1} = k_1, N_{i_2 j_2} = k_2\} &= \frac{(n-l_1-l_3)!}{n!} \sum_{x_1+x_2=l_1-k_1} \sum_{y_1+y_2=l_3-k_2} l_1!l_3!\binom{l_2}{k_1}\binom{l_4}{x_1}\binom{n-l_2-l_4}{x_2} \\
&\quad \cdot \binom{l_2-k_1}{y_1}\binom{l_4-x_1}{k_2}\binom{n-l_2-l_4-x_2}{y_2}.
\end{aligned}
$$

Lemma 3.51 *With the same hypothesis of Remark 3.50, we have*

$$
Cov(N_{i_1 j_1}/n, N_{i_2 j_2}/n) = \frac{l_1 l_2 l_3 l_4}{n^3(n-1)} - \left(\frac{l_1 l_2}{n^2}\right)\left(\frac{l_3 l_4}{n^2}\right).
$$

Proof:

$$
\begin{aligned}
E(N_{i_1 j_1} N_{i_2 j_2}) &= \frac{(n-l_1-l_3)!}{n!} \sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} k_1 k_2 l_1!l_3!\binom{l_2}{k_1}\binom{l_4}{x_1}\binom{n-l_2-l_4}{x_2} \\
&\quad \cdot \binom{l_2-k_1}{y_1}\binom{l_4-x_1}{k_2}\binom{n-l_2-l_4-x_2}{y_2} \\
&= \frac{(n-l_1-l_3)!}{n!} \sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} l_1!l_3!l_2\binom{l_2-1}{k_1-1}\frac{l_4}{l_4-x_1}\binom{l_4-1}{x_1}\binom{n-l_2-l_4}{x_2} \\
&\quad \cdot \binom{l_2-k_1}{y_1}(l_4-x_1)\binom{l_4-x_1-1}{k_2-1}\binom{n-l_2-l_4-x_2}{y_2} \\
&= \frac{(n-l_1-l_3)!}{n!} \sum_{k_1+x_1+x_2=l_1} \sum_{y_1+k_2+y_2=l_3} l_1!l_3!l_2 l_4\binom{l_2-1}{k_1-1}\binom{l_4-1}{x_1}\binom{n-l_2-l_4}{x_2} \\
&\quad \cdot \binom{l_2-k_1}{y_1}\binom{l_4-x_1-1}{k_2-1}\binom{n-l_2-l_4-x_2}{y_2} \\
&= \frac{(n-l_1-l_3)!}{n!} \sum_{k_1+x_1+x_2=l_1} l_1!l_3!l_2 l_4\binom{l_2-1}{k_1-1}\binom{l_4-1}{x_1}\binom{n-l_2-l_4}{x_2}\binom{n-1-l_1}{l_3-1} \\
&= \frac{(n-l_1-l_3)!}{n!} l_1!l_3!l_2 l_4\binom{n-2}{l_1-1}\binom{n-1-l_1}{l_3-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(n-l_1-l_3)!}{n!} l_1! l_3! l_2 l_4 \frac{(n-2)!}{(l_1-1)!(n-l_1-1)!} \frac{(n-1-l_1)!}{(l_3-1)!(n-l_1-l_3)!} \\
&= \frac{l_1 l_2 l_3 l_4}{n(n-1)},
\end{aligned}
$$

and

$$
E((N_{i_1 j_1}/n)(N_{i_2 j_2}/n)) = \frac{l_1 l_2 l_3 l_4}{n^3(n-1)}.
$$

Finally

$$
\begin{aligned}
Cov(N_{i_1 j_1}/n, N_{i_2 j_2}/n) &= E((N_{i_1 j_1}/n)(N_{i_2 j_2}/n)) - E(N_{i_1 j_1}/n)E(N_{i_2 j_2}/n) \\
&= \frac{l_1 l_2 l_3 l_4}{n^3(n-1)} - \left(\frac{l_1 l_2}{n^2}\right)\left(\frac{l_3 l_4}{n^2}\right).
\end{aligned}
$$

□

Finally, the next theorem describes the joint probability distribution of the boxes generated by the not uniform partition of size m, with $m \geq 2$, of $\mathbf{I}^2 = [0,1]^2$.

Theorem 3.52 *Let $m \geq 2$, $n \in \mathbb{N}$, where m not necessarily divides n, and $I_k = \{1, \cdots k\}$, $k \in \mathbb{N}$, we consider $\{t_0, t_1, \cdots, t_m\} \subset I_n$ with $0 = t_0 < t_1 < \cdots < t_m = n$. We define $l_i = t_i - t_{i-1}$, for all $i \in I_m$ and the partition, not necessarily uniform, of size m of $\mathbf{I}^2 = [0,1]^2$ by*

$$
R_{ij} = \left\langle \frac{t_{i-1}}{n}, \frac{t_i}{n} \right] \times \left\langle \frac{t_{j-1}}{n}, \frac{t_j}{n} \right]
$$

for all $i, j \in I_m$, the notation "$\langle$" indicates "(" if $t_{k-1} > 0$ and "[" if $t_{k-1} = 0$, for all $k \in I_m$.

Let N_{ij} be the random variables that indicates the number of observations falling in R_{ij}, respectively, for all $i, j \in I_m$, when we consider a sample of size n from the product copula; let n_{ij}, with $i, j \in I_m$, be zero or a positive integer that satisfies the following restrictions

$$
\sum_{j=1}^{m} n_{ij} = l_i \ \ (\text{for all } i \in I_m), \qquad \sum_{i=1}^{m} n_{ij} = l_j \ \ (\text{for all } j \in I_m),
$$

and

$$
\sum_{k=1}^{m} l_k = n
$$

then

$$
P\left\{ \bigcap_{i,j \in I_m} \{N_{ij} = n_{ij}\} \right\} = \frac{\prod_{i=1}^{m} (l_i!)^2}{n! \prod_{i,j \in I_m} n_{ij}!}.
$$

Proof: We use the counting methodology provided on the Remark 3.3 for the regions R_{ij}, $i, j \in I_m$, in the following order

$$R_{11}, \cdots, R_{1m}, R_{21}, \cdots, R_{2m}, R_{m1}, \cdots, R_{mm}.$$

The number of ways in which we can observe n_{ij} points in each region R_{ij}, $i, j \in I_m$, is given by

$$\begin{aligned} & \binom{l_1}{n_{11}} P^{l_1}_{n_{11}} \binom{l_1 - n_{11}}{n_{12}} P^{l_2}_{n_{12}} \cdots \binom{l_1 - \sum_{j=1}^{m-1} n_{1j}}{n_{1m}} P^{l_m}_{n_{1m}} \\ & \cdot \binom{l_2}{n_{21}} P^{l_1 - n_{11}}_{n_{21}} \binom{l_2 - n_{21}}{n_{22}} P^{l_2 - n_{12}}_{n_{22}} \cdots \binom{l_2 - \sum_{j=1}^{m-1} n_{2j}}{n_{2m}} P^{l_m - n_{1m}}_{n_{2m}} \\ & \vdots \\ & \cdot \binom{l_m}{n_{m1}} P^{l_1 - \sum_{i=1}^{m-1} n_{i1}}_{n_{m1}} \binom{l_m - n_{m1}}{n_{m2}} P^{l_2 - \sum_{i=1}^{m-1} n_{i2}}_{n_{m2}} \cdots \binom{l_m - \sum_{j=1}^{m-1} n_{mj}}{n_{mm}} P^{l_m - \sum_{i=1}^{m-1} n_{im}}_{n_{mm}} \\ = \ & \frac{\prod_{i=1}^{m} (l_i!)^2}{\prod_{i,j \in I_m} n_{ij}!}, \end{aligned}$$

and we divided between $n!$, the total number of possibilities in which we can see n points in $\mathbf{I}^2 = [0, 1]$, corresponding to the modified sample of the original sample of size n from the product copula, therefore

$$P\left\{ \bigcap_{i,j \in I_m} \{N_{ij} = n_{ij}\} \right\} = \frac{\prod_{i=1}^{m} (l_i!)^2}{n! \prod_{i,j \in I_m} n_{ij}!}.$$

$\square$

4 Convergence results of the sample copula

In this last chapter, we study the weak convergence of the sample copula process $\sqrt{n}(C_m^n - C)$, in this way we will have, for the sample copula, a weak convergence theorem similar to that for the empirical copula. This chapter is divided into three parts: In the first part we give a review of the results in the theory of empirical processes, and the results of the weak convergence to a Gaussian process of the empirical process $\sqrt{n}(C_n - C)$ given in [13] because, using this convergence, we obtain the corresponding convergence of the sample copula process; in the second part, and in order to be able to apply the results given in the first part, we show that the sample copula can be represented as a linear functional evaluated in the empirical copula, under this representation we have Hadamard's differentiation and then we can apply the delta method. In the third part we perform several simulations of the sample copula process at a given point to analyze the properties of the convergence to the Gaussian process with a given variance-covariance structure.

4.1 Weak convergence of empirical process

The weak convergence of an empirical process is studied extensively in Billingsley's book [2] in the case of processes with parameter space $\mathbf{I} = [0, 1]$. The extensions of these results, that is, processes with parameter space equal to $\mathbf{I}^d = [0, 1]^d$, $d \geq 1$, can be found in [38]. We begin this section with some of the main definitions and results of [38].

Definition 4.1 *We define the **unit cube in** $\mathbb{R}^d$ by*

$$E_d = [0, 1] \times \cdots \times [0, 1] \quad \textit{(d times)}$$

for $t \in E_d$, with $t = (t_1, \cdots, t_d)$, it is considered

$$|t| = \max\{|t_i| \mid i = 1, \cdots, d\}.$$

We define

$$\mathcal{P} = \{\rho = (\rho_i, \cdots, \rho_d) \mid \rho_i = 0 \ \text{ó} \ \rho_i = 1, \ i = 1, \cdots, d\}$$

as the set of the 2^d vertices of E_d.

Definition 4.2 *Let $t \in E_d$ and $\rho \in \mathcal{P}$, we define the **quadrants** $Q(\rho, t)$ **and** $\hat{Q}(\rho, t)$ **in** E_d **with vertex** t as follows*

$$Q(\rho, t) = I(\rho_1, t_1) \times \cdots \times I(\rho_d, t_d)$$

where

$$I(r, s) = \begin{cases} [0, s) & si \quad r = 0 \\ (s, 1] & si \quad r = 1 \end{cases}$$

and

$$\hat{Q}(\rho, t) = \hat{I}(\rho_1, t_1) \times \cdots \times \hat{I}(\rho_d, t_d)$$

where

$$\hat{I}(r, s) = \begin{cases} [0, s) & si \quad r = 0 \quad y \quad s < 1 \\ [0, 1] & si \quad r = 0 \quad y \quad s = 1 \\ \emptyset & si \quad r = 1 \quad y \quad s = 1 \\ [s, 1] & si \quad r = 1 \quad y \quad s < 1. \end{cases}$$

Example 4.3 *To exemplify the Definition 4.2, we consider $d = 3$, then*

$$E_3 = [0, 1] \times [0, 1] \times [0, 1],$$

let $t \in E_3$, with $t = (0.5, 0.8, 0)$ and $\rho \in \mathcal{P}$, with $\rho = (0, 1, 1)$. We have

$$Q(\rho, t) = I(0, 0.5) \times I(1, 0.8) \times I(1, 0)$$

with

$$\begin{aligned} I(0, 0.5) &= [0, 0.5) \\ I(1, 0.8) &= (0.8, 1] \\ I(1, 0) &= (0, 1] \end{aligned}$$

therefore,

$$Q(\rho, t) = [0, 0.5) \times (0.8, 1] \times (0, 1].$$

On the other hand

$$\hat{Q}(\rho, t) = \hat{I}(0, 0.5) \times \hat{I}(1, 0.8) \times \hat{I}(1, 0)$$

with

$$\begin{aligned} \hat{I}(0, 0.5) &= [0, 0.5) \\ \hat{I}(1, 0.8) &= [0.8, 1] \\ \hat{I}(1, 0) &= [0, 1] \end{aligned}$$

therefore,

$$\hat{Q}(\rho, t) = [0, 0.5) \times [0.8, 1] \times [0, 1].$$

We note that $\hat{Q}$ is the closure, on the left side, of the intervals that define Q.

Remark 4.4 *The quadrants $Q(\rho, t)$ and $\hat{Q}(\rho, t)$, called **quadrants of continuity**, satisfy the following properties*

1. $Q(\rho,t) \subset \hat{Q}(\rho,t) \subset \overline{Q}(\rho,t)$ *(the notation $\overline{A}$ indicates the closure of set A).*

2. $Q(\rho,t) = \emptyset$ *if and only if* $\hat{Q}(\rho,t) = \emptyset$.

3. $\hat{Q}(\rho,t) \cap \hat{Q}(\rho',t) = \emptyset$ *if* $\rho \neq \rho'$.

4. $\bigcup_{\rho\in\mathcal{P}} \hat{Q}(\rho,t) = E_d \quad (t \in E_d)$,

5. *For each* $t \in E_d$ *exists only a vertex* $\rho = \rho(t)$, *denoted by* σ, *such as* $t \in \hat{Q}(\sigma,t)$.

6. $int(Q(\sigma,t)) \neq \emptyset$ *y* $\hat{Q}(\sigma,t) = \overline{Q}(\sigma,t)$.

Definition 4.5 *Let* $f : E_d \longrightarrow \mathbb{R}$ *be a function. If for* $t \in E_d$, $\rho \in \mathcal{P}$, *with* $Q(\rho,t) \neq \emptyset$, *and for every sequence* $\{t_n\}_{n\geq 1} \subset Q(\rho,t)$, *with* $t_n \longrightarrow t$, *the sequence* $\{f(t_n)\}_{n\geq 1}$ *converges, then the only limit is denoted by* $f(t+0_\rho)$ *and is called* ***ρ-limit or quadrant limit***.

Definition 4.6 ***The*** $D[0,1]^d$ ***space*** *is define as the set of all functions* $f : E_d \longrightarrow \mathbb{R}$ *for which the* ρ*-limit of* f *in* t, *with* $t \in E_d$, *exists for all* $\rho \in \mathcal{P}$ *such that* $Q(\rho,t) \neq \emptyset$ *and they are continuous in the following sense:* $f(t) = f(t+0_\sigma)$.

Definition 4.7 *We denote by* Λ *the set of all functions*

$$\lambda : [0,1] \longrightarrow [0,1]$$

increasing, continuous and surjectives. For $\hat{\lambda} = (\lambda_1,\cdots,\lambda_d) \in \Lambda_d = \Lambda \times \cdots \times \Lambda$ *(d times) and* $t = (t_1,\cdots,t_d) \in E_d$ *we define* $\hat{\lambda}(t) = (\lambda_1(t_1),\cdots,\lambda_d(t_d))$.

Definition 4.8 *For* $\mu \in \Lambda$ *we define*

$$\|\mu\| = \sup_{u,v\in[0,1],u\neq v}\left\{\left|\log\left(\frac{\mu(u)-\mu(v)}{u-v}\right)\right|\right\}$$

and we define the ***metric*** d_0 ***in*** $D[0,1]^d$ *by*

$$d_0(f,g) = \inf\left\{\varepsilon > 0 \mid \textit{there exists } \hat{\lambda} \in \Lambda_d \textit{ with } \|\lambda_i\| \leq \varepsilon,\ i \in I_d,\ \textit{and } \sup_{t\in E_d}|f(t) - g(\lambda(t))| \leq \varepsilon\right\}$$

for $f, g \in D[0,1]^d$.

Remark 4.9 *The metric space* $(D[0,1]^d, d_0)$ *is a* ***Polish space***.

Since the empirical distributions functions are functions that have discontinuities, then these are elements of the metric space $(D[0,1]^d, d_0)$. We have that the space $C[0,1]^d$, that is the space of the real continuous functions with domain $\mathbf{I}^d = [0,1]^d$, is a subset of $D[0,1]^d$, and in this case, from [2] and [38], we can establish the equivalence between the d_0 metric, given in Definition 4.8 and the supreme metric given by $d_{sup}(f,g) = \sum_{t\in[0,1]^d} |f(t) - g(t)|$ for all $f, g \in C[0,1]^d$. Among the advantages of using the sample copula rather than empirical copula, is that the sample copula is a continuous function, that is, for $d \geq 2$, the d-sample copula is an element of the metric spaces $C[0,1]^d$ and $D[0,1]^d$, while the empirical copula in dimension d, being a function with discontinuities of the first order, is only an element of the space $D[0,1]^d$.

The next results given in [38], are the basis for the study of weak convergence of the empirical copula process under the independence case.

Remark 4.10 *We consider $U_j = (U_{j1}, \cdots, U_{jd})$, $j \in \mathbb{N}$, random vector independent and identically distributed, where U_{ji} is an uniform random variable in $(0,1)$, $j \in \mathbb{N}$ and $i \in \{1, \cdots, d\}$. For each $j \in \mathbb{N}$ the distribution function F of U_j satisfies the Lipschitz's condition*

$$|F(t) - F(t')| \leq K|t - t'|$$

with a constant K and $t, t' \in [0,1]^d$.

Definition 4.11 *Let $n \in \mathbb{N}$, we define the random variables Y_n in $D[0,1]^d$ by*

$$Y_n(t) = \frac{1}{n^{\frac{1}{2}}} \left[\sum_{j=1}^{n} \prod_{i=1}^{d} (1_{[0,t_i]}(U_{ji}) - t_i) \right].$$

Theorem 4.12 *Let $r \geq 1$ and let $t_1, \cdots, t_r \in E_d$, under the assumption of independence in the components of the random vector U_j, $j \in \mathbb{N}$, we have*

$$(Y_n(t_1), \ldots, Y_n(t_r)) \xrightarrow{\mathcal{D}} N(\hat{0}, \gamma_Y(t_1, \cdots, t_r))$$

with $\hat{0}$ the zero vector in E_d and

$$\gamma_Y(t_1, \cdots, t_r) = \left(\prod_{i=1}^{d} ((t_{\nu i} \wedge t_{\mu i}) - (t_{\nu i} t_{\mu i})) \right)_{\nu,\mu=1,\cdots,r}$$

The next definitions and results provide the principles to formulate the convergence of the empirical processes using Hadamard's derivative and the delta method, these methodologies allows us to study the convergence of an empirical process using a Taylor's first order approximation applied to a linear functional and the convergence in probability of the residual of the approximation from the Slustky's theorem. The delta method in empirical process with one dimension of the parameter space is shown in [47] and the generalization for empirical process with higher dimension of the parameter space can be found in [49].

Definition 4.13 *A topological vector space V on a field K is a vector space for which addition and scalar multiplication are continuous operations, that is, if $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}, x, y \in V$ and $\{c_n\}_{n\geq 1}, c \in K$ with $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $c_n \longrightarrow c$, then $x_n + y_n \longrightarrow x + y$ and $c_n x_n \longrightarrow cx$.*

Remark 4.14 *In the following results we use some concepts from von Mises calculus like Hadamard differentiability in topological vector spaces, and we consider the hypothesis that space $D[0, 1]^d$ is a topological vector space. We study the convergence of the sample copula from the results of convergence of the empirical copula. However, the convergence of the sample copula can be found in the space $C[0, 1]^d$, because it is a continuous function.*

Definition 4.15 *Let $\mathbb{D}$ and $\mathbb{E}$ be metrizable, topological vector spaces, a map $\varphi : \mathbb{D}_\varphi \subset \mathbb{D} \longrightarrow \mathbb{E}$ is called Hadamard differentiable at $\theta \in \mathbb{D}_\varphi$ if there is a continuous linear map $\varphi'_\theta : \mathbb{D} \longrightarrow \mathbb{E}$ such that*

$$\frac{\varphi(\theta + t_n h_n) - \varphi(\theta)}{t_n} \longrightarrow \varphi'_\theta(h)$$

if $n \longrightarrow \infty$, for all converging sequences $t_n \longrightarrow 0$ and $h_n \longrightarrow h$ such that $\theta + t_n h_n \in \mathbb{D}_\varphi$, for all $n \in \mathbb{N}$. We say that φ is Hadamard differentiable tangentially to a set $\mathbb{D}_0 \subset \mathbb{D}$ by requiring that every $h_n \longrightarrow h$ has $h \in \mathbb{D}_0$.

Definition 4.16 *Let $(\mathcal{M}, \rho)$ be a given separable, pseudometric space and let $(\Omega, \mathcal{A}, P)$ be a probability space. A stochastic process $\{\mathbb{G}(t, \omega) : t \in \mathcal{M}, \omega \in \Omega\}$ is called separable if there exists a null set $N \in \mathcal{A}$ and a countable subset $\mathcal{G} \subset \mathcal{M}$ such that, for all $\omega \notin N$ and $t \in \mathcal{M}$, there exists a sequence $t_n \in \mathcal{G}$, with $t_n \longrightarrow t$ and $\mathbb{G}(t_n, \omega) \longrightarrow \mathbb{G}(t, \omega)$.*

Theorem 4.17 *Let $(\mathcal{M}, \rho)$ be a given separable, pseudometric space and let $(\Omega, \mathcal{A}, P)$ be a probability space. If there is a set $A \in \mathcal{A}$ with probability zero such that for $\omega \notin A$, $\mathbb{G}(t, \omega)$ is continuous in t, then $\{\mathbb{G}(t, \omega) : t \in \mathcal{M}, \omega \in \Omega\}$ is separable, and $\mathcal{G} \subset \mathcal{M}$ can be taken as any countable dense subset of $\mathcal{M}$.*

Theorem 4.18 **(Delta Method)** *Let $\mathbb{D}$ and $\mathbb{E}$ be metrizable, topological vector spaces, let $\varphi : \mathbb{D}_\varphi \subset \mathbb{D} \longrightarrow \mathbb{E}$ be Hadamard differentiable at θ tangentially to $\mathbb{D}_0$. Let $X_n : \Omega_n \longrightarrow \mathbb{D}_\varphi$ be maps with $r_n(X_n - \theta_n) \xrightarrow{\mathcal{D}} X$ for some sequence of constants $r_n \longrightarrow \infty$, where X is separable and takes its values in $\mathbb{D}_0$, and $\theta_n \longrightarrow \theta$, then $r_n(\varphi(X_n) - \varphi(\theta_n)) \xrightarrow{\mathcal{D}} \varphi'_\theta(X)$.*

The following theorem is given in [13], in this result the convergence of the empirical process associated to the empirical copula to a Gaussian process is studied. This theorem is the main result to obtain in the next section the convergence of the process associated to the sample copula, also from this result we can extend the Theorem 4.12 to get weak convergence without the hypothesis of independence, with some restrictions.

Theorem 4.19 *Let H be a bivariate distribution function with continuous marginal distribution functions and associated copula function C with continuous partial derivatives. Then the empirical copula process*

$$\mathbb{Z}_n = \sqrt{n}(C_n - C)$$

with C_n the empirical copula, converge weakly to the centered Gaussian process $\mathbb{G}_C$ in $l^\infty([0,1]^2)$, the set of all uniformly bounded functions from $\mathbf{I}^2 = [0,1]^2$ to $\mathbb{R}$. The limiting Gaussian process can be written as

$$\mathbb{G}_c(u,v) = \mathbb{B}_C(u,v) - \partial_1 C(u,v)\mathbb{B}_C(u,1) - \partial_2 C(u,v)\mathbb{B}_C(1,v),$$

where $\mathbb{B}_C$ is the Brownian sheet on $\mathbf{I}^2 = [0,1]^2$ with covariance function

$$E(\mathbb{B}_C(u,v)\cdot\mathbb{B}_C(u',v')) = C(u\wedge u', v\wedge v') - C(u,v)C(u',v').$$

4.2 Weak convergence of sample copula process

In this section the sample copula is represented as a linear functional evaluated in the empirical copula, in order to apply the results described in the previous section to get the weak convergence of the process associated with the sample copula. In this way, the weak convergence theorem for the empirical copula can be extended for the sample copula.

We begin this section with a simple result which allows us the study of subcopulas with restricted domains, because later we will consider the empirical copula with a restricted domain in the description of the sample copula.

Lemma 4.20 *Let $C' : S_1\times S_2 \longrightarrow [0,1]$ be a subcopula, let $\hat{S}_1 \subset S_1$ and $\hat{S}_2 \subset S_2$ such as $0,1 \in \hat{S}_1$ and $0,1 \in \hat{S}_2$, then $\hat{C} = C'|_{\hat{S}_1\times\hat{S}_2}$, with $\hat{C}(u,v) = C(u,v)$ for all $(u,v) \in \hat{S}_1 \times \hat{S}_2 \subset S_1 \times S_2$, is a subcopula.*

Proof: *We verified that $\hat{C}$ satisfies the subcopulas properties,*

1. *$Dom(\hat{C}) = \hat{S}_1 \times \hat{S}_2 \subset [0,1]^2$ and $0,1 \in \hat{S}_1 \cap \hat{S}_2$.*

2. *Let* $u \in \hat{S}_1$ *and* $v \in \hat{S}_2$*, then*

$$\hat{C}(0,v) = C'(0,v) = 0$$
$$\hat{C}(u,0) = C'(u,0) = 0.$$

Let $a_1, a_2 \in \hat{S}_1$ *and* $b_1, b_2 \in \hat{S}_2$*, with* $a_1 \leq a_2$ *and* $b_1 \leq b_2$*, then*

$$\begin{aligned} Vol_{\hat{C}}([a_1,a_2]\times[b_1,b_2]) &= \hat{C}(a_2,b_2) - \hat{C}(a_1,b_2) - \hat{C}(a_2,b_1) + \hat{C}(a_1,b_1) \\ &= C'(a_2,b_2) - C'(a_1,b_2) - C'(a_2,b_1) + C'(a_1,b_1) \\ &= Vol_{C'}([a_1,a_2]\times[b_1,b_2]) \\ &\geq 0. \end{aligned}$$

3. *Let* $u \in \hat{S}_1$ *and* $v \in \hat{S}_2$*, then*

$$\hat{C}(u,1) = C'(u,1) = u$$
$$\hat{C}(1,v) = C'(1,v) = v.$$

Therefore $\hat{C}$ *is a subcopula.*

Remark 4.21 *In the following results we use the Nelsen's notation* [37]*: For* $(a,b) \in \mathbf{I}^2 = [0,1]^2$ *and a subcopula* C' *with domain* $S_1 \times S_2$*, let* a_1 *and* a_2 *be, respectively, the greatest and least elements of* S_1 *that satisfy* $a_1 \leq a \leq a_2$*; and let* b_1 *and* b_2 *be, respectively, the greatest and least elements of* S_2 *that satisfy* $b_1 \leq b \leq b_2$*. If* $a \in S_1$*, then* $a_1 = a = a_2$*, and if* $b \in S_2$*, then* $b_1 = b = b_2$*. Let*

$$\lambda_1(a,b) = \lambda_1 = \begin{cases} (a-a_1)/(a_2-a_1) & \text{if} \quad a_1 < a_2 \\ 1 & \text{if} \quad a_1 = a_2 \end{cases}$$

and

$$\mu_1(a,b) = \mu_1 = \begin{cases} (b-b_1)/(b_2-b_1) & \text{if} \quad b_1 < b_2 \\ 1 & \text{if} \quad b_1 = b_2. \end{cases}$$

If we define

$$C(a,b) = (1-\lambda_1)(1-\mu_1)C'(a_1,b_1) + (1-\lambda_1)\mu_1 C'(a_1,b_2) + \lambda_1(1-\mu_1)C'(a_2,b_1) + \lambda_1\mu_1 C'(a_2,b_2)$$

then C *is a copula that extends the subcopula* C'*.*

Also, we considered C_n *and* C_m^n $(m \geq 2)$*, respectively, the empirical copula and the sample copula, build from the modified sample of a sample of size n from a copula* C*.*

In the following results the sample copula is represented like a linear functional evaluated in the empirical copula and we proof that this functional is Hadamard differentiable. These properties are required in order to use the delta method for the study of weak convergence of the process associated to the sample copula.

Lemma 4.22 *Let $m \geq 2$ and $\varphi_m : D[0,1]^2 \longrightarrow l^\infty([0,1]^2)$, with*

$$\begin{aligned}\varphi_m(H) &= \sum_{j=1}^{m}\sum_{i=1}^{m}\left[\mathbf{1}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)H\left(\frac{i-1}{m},\frac{j-1}{m}\right)\right.\\ &+ \left.(1-\lambda_1)\mu_1 H\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)H\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1 H\left(\frac{i}{m},\frac{j}{m}\right)\right]\end{aligned}$$

then

$$\varphi_m(C_n) = C_m^n.$$

Proof: *We consider $(a,b) \in \mathbf{I}^2 = [0,1]^2$ with $(a,b) \in ((i-1)/m, i/m] \times ((j-1)/m, j/m]$ and let λ be the Lebesgue's measure in $\mathcal{B}([0,1]^2)$. We define*

$$\begin{aligned}C(a,b) &= (1-\lambda_1)(1-\mu_1)\hat{C}_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)+(1-\lambda_1)\mu_1\hat{C}_n\left(\frac{i-1}{m},\frac{j}{m}\right)\\ &\quad+\lambda_1(1-\mu_1)\hat{C}_n\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1\hat{C}_n\left(\frac{i}{m},\frac{j}{m}\right)\end{aligned}$$

where $\lambda_1 = \lambda_1(a,b) = (a-(i-1)/m)/(i/m-(i-1)/m)$, $\mu_1 = \mu_1(a,b) = (b-(j-1)/m)/(j/m-(j-1)/m)$ and $\hat{C}_n$ is the subcopula obtained from the empirical copula C_n restricted to a domain consisting of points of the form $(i/m, j/m)$, with $i, j \in \{1, \cdots, m\}$, from the Lemma 4.20, $\hat{C}$ is a subcopula and therefore C is a copula. We will denote $\hat{C}_n$ by C_n and we calculate with this definition the C-volumen of the rectangle $((i-1)/m, a] \times ((j-1)/m, b]$,

$$Vol_C\left(\left(\frac{i-1}{m},a\right]\times\left(\frac{j-1}{m},b\right]\right) = C(a,b) - C\left(a,\frac{j-1}{m}\right) - C\left(\frac{i-1}{m},b\right) + C\left(\frac{i-1}{m},\frac{j-1}{m}\right),$$

where

$$\begin{aligned}C\left(a,\frac{j-1}{m}\right) &= (1-\lambda_1)C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)+\lambda_1 C_n\left(\frac{i}{m},\frac{j-1}{m}\right),\\ C\left(\frac{i-1}{m},b\right) &= (1-\mu_1)C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)+\mu_1 C_n\left(\frac{i-1}{m},\frac{j}{m}\right),\end{aligned}$$

and

$$C\left(\frac{i-1}{m}, \frac{j-1}{m}\right) = C_n\left(\frac{i-1}{m}, \frac{j-1}{m}\right),$$

we have that

$$\begin{aligned}
&C(a,b) - C\left(a, \tfrac{j-1}{m}\right) - C\left(\tfrac{i-1}{m}, b\right) + C\left(\tfrac{i-1}{m}, \tfrac{j-1}{m}\right) = \\
&\qquad [(1-\lambda_1)(1-\mu_1) - (1-\lambda_1) - (1-\mu_1) + 1]\, C_n\left(\tfrac{i-1}{m}, \tfrac{j-1}{m}\right) \\
&\qquad + [(1-\lambda_1)\mu_1 - \mu_1]\, C_n\left(\tfrac{i-1}{m}, \tfrac{j}{m}\right) + [\lambda_1(1-\mu_1) - \lambda_1]\, C_n\left(\tfrac{i}{m}, \tfrac{j-1}{m}\right) + \lambda_1\mu_1 C_n\left(\tfrac{i}{m}, \tfrac{j}{m}\right) \\
&\quad = \lambda_1\mu_1\left[C_n\left(\tfrac{i}{m}, \tfrac{j}{m}\right) - C_n\left(\tfrac{i-1}{m}, \tfrac{j}{m}\right) - C_n\left(\tfrac{i}{m}, \tfrac{j-1}{m}\right) + C_n\left(\tfrac{i-1}{m}, \tfrac{j-1}{m}\right)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
Vol_C\left(\left(\frac{i-1}{m}, a\right] \times \left(\frac{j-1}{m}, b\right]\right) &= \lambda_1\mu_1 Vol_{C_n}\left(\left(\frac{i-1}{m}, \frac{i}{m}\right] \times \left(\frac{j-1}{m}, \frac{j}{m}\right]\right) \\
&= \lambda_1\mu_1 s_{ij} \\
&= \frac{\lambda\left(\left(\frac{i-1}{m}, a\right] \times \left(\frac{j-1}{m}, b\right]\right)}{\lambda\left(\left(\frac{i-1}{m}, \frac{i}{m}\right] \times \left(\frac{j-1}{m}, \frac{j}{m}\right]\right)} s_{ij} \\
&= Vol_{C_m^n}\left(\left(\frac{i-1}{m}, a\right] \times \left(\frac{j-1}{m}, b\right]\right).
\end{aligned}$$

Now, we calculate the C-volume of the box $((i-1)/m, i/m] \times ((j-1)/m, b]$,

$$Vol_C\left(\left(\frac{i-1}{m}, \frac{i}{m}\right] \times \left(\frac{j-1}{m}, b\right]\right) = C\left(\frac{i}{m}, b\right) - C\left(\frac{i-1}{m}, b\right) - C\left(\frac{i}{m}, \frac{j-1}{m}\right) + C\left(\frac{i-1}{m}, \frac{j-1}{m}\right)$$

where

$$\begin{aligned}
C\left(\frac{i}{m}, b\right) &= \left(\frac{\frac{j}{m} - b}{\frac{j}{m} - \frac{j-1}{m}}\right) C_n\left(\frac{i}{m}, \frac{j-1}{m}\right) + \left(\frac{b - \frac{j-1}{m}}{\frac{j}{m} - \frac{j-1}{m}}\right) C_n\left(\frac{i}{m}, \frac{j}{m}\right), \\
C\left(\frac{i-1}{m}, b\right) &= \left(\frac{\frac{j}{m} - b}{\frac{j}{m} - \frac{j-1}{m}}\right) C_n\left(\frac{i-1}{m}, \frac{j-1}{m}\right) + \left(\frac{b - \frac{j-1}{m}}{\frac{j}{m} - \frac{j-1}{m}}\right) C_n\left(\frac{i-1}{m}, \frac{j}{m}\right),
\end{aligned}$$

then

$$Vol_C\left(\left(\frac{i-1}{m}, \frac{i}{m}\right] \times \left(\frac{j-1}{m}, b\right]\right) = C\left(\frac{i}{m}, b\right) - C\left(\frac{i-1}{m}, b\right) - C\left(\frac{i}{m}, \frac{j-1}{m}\right) + C\left(\frac{i-1}{m}, \frac{j-1}{m}\right)$$

$$
\begin{aligned}
&= \left[\left(\frac{\frac{j}{m}-b}{\frac{j}{m}-\frac{j-1}{m}}\right)-1\right]C_n\left(\frac{i}{m},\frac{j-1}{m}\right)+\left(\frac{\frac{j-1}{m}-b}{\frac{j}{m}-\frac{j-1}{m}}\right)C_n\left(\frac{i}{m},\frac{j}{m}\right)\\
&\quad+\left[\left(\frac{b-\frac{j}{m}}{\frac{j}{m}-\frac{j-1}{m}}\right)+1\right]C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)-\left(\frac{\frac{j-1}{m}-b}{\frac{j}{m}-\frac{j-1}{m}}\right)C_n\left(\frac{i-1}{m},\frac{j}{m}\right)\\
&= \left(\frac{b-\frac{j-1}{m}}{\frac{j}{m}-\frac{j-1}{m}}\right)\left[C_n\left(\frac{i}{m},\frac{j}{m}\right)-C_n\left(\frac{i}{m},\frac{j-1}{m}\right)-C_n\left(\frac{i-1}{m},\frac{j}{m}\right)\right.\\
&\qquad\left.+C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)\right]\\
&= \left(\frac{b-\frac{j-1}{m}}{\frac{j}{m}-\frac{j-1}{m}}\right)\left(\frac{\frac{i}{m}-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)V_{C_n}\left(\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right)\\
&= \frac{\lambda\left(\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},b\right]\right)}{\lambda\left(\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right)}s_{ij}\\
&= Vol_{C_m^n}\left(\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},b\right]\right).
\end{aligned}
$$

Finally, we calculate the C-volume of the box $((i-1)/m, a]\times((j-1)/m, j/m]$*,*

$$
Vol_C\left(\left(\frac{i-1}{m},a\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right)=C\left(a,\frac{j}{m}\right)-C\left(\frac{i-1}{m},\frac{j}{m}\right)-C\left(a,\frac{j-1}{m}\right)+C\left(\frac{i-1}{m},\frac{j-1}{m}\right)
$$

where

$$
\begin{aligned}
C\left(a,\frac{j}{m}\right) &= \left(\frac{\frac{i}{m}-a}{\frac{i}{m}-\frac{i-1}{m}}\right)C_n\left(\frac{i-1}{m},\frac{j}{m}\right)+\left(\frac{a-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)C_n\left(\frac{i}{m},\frac{j}{m}\right),\\
C\left(a,\frac{j-1}{m}\right) &= \left(\frac{\frac{i}{m}-a}{\frac{i}{m}-\frac{i-1}{m}}\right)C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)+\left(\frac{a-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)C_n\left(\frac{i}{m},\frac{j-1}{m}\right),
\end{aligned}
$$

then

$$
\begin{aligned}
Vol_C\left(\left(\frac{i-1}{m},a\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right) &= C\left(a,\frac{j}{m}\right)-C\left(\frac{i-1}{m},\frac{j}{m}\right)-C\left(a,\frac{j-1}{m}\right)+C\left(\frac{i-1}{m},\frac{j-1}{m}\right)\\
&= \left[\left(\frac{\frac{i}{m}-a}{\frac{i}{m}-\frac{i-1}{m}}\right)-1\right]C_n\left(\frac{i-1}{m},\frac{j}{m}\right)+\left(\frac{a-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)C_n\left(\frac{i}{m},\frac{j}{m}\right)\\
&\quad+\left[\left(\frac{a-\frac{i}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)+1\right]C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)-\left(\frac{a-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)C_n\left(\frac{i}{m},\frac{j-1}{m}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{a-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)\left[C_n\left(\frac{i}{m},\frac{j}{m}\right)-C_n\left(\frac{i}{m},\frac{j-1}{m}\right)-C_n\left(\frac{i-1}{m},\frac{j}{m}\right)\right.\\
&\qquad\left.+C_n\left(\frac{i-1}{m},\frac{j-1}{m}\right)\right]\\
&= \left(\frac{a-\frac{i-1}{m}}{\frac{i}{m}-\frac{i-1}{m}}\right)\left(\frac{\frac{j}{m}-\frac{i-1}{m}}{\frac{j}{m}-\frac{j-1}{m}}\right)V_{C_n}\left(\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right)\\
&= \frac{\lambda\left(\left(\frac{i-1}{m},a\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right)}{\lambda\left(\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]\right)}s_{ij}\\
&= Vol_{C_m^n}\left(\left(\frac{i-1}{m},a\right]\times\left(\frac{j-1}{m},b\right]\right).
\end{aligned}
$$

Because any box can be written as the union of boxes considered in the previous cases we can conclude that

$$\varphi_m(C_n) = C_m^n.$$

□

Lemma 4.23 *Let $m \geq 2$ and $\varphi_m : D[0,1]^2 \longrightarrow l^\infty([0,1]^2)$, with*

$$
\begin{aligned}
\varphi_m(H) &= \sum_{j=1}^{m}\sum_{i=1}^{m}\left[\mathbf{1}_{\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]}(1-\lambda_1)(1-\mu_1)H\left(\frac{i-1}{m},\frac{j-1}{m}\right)\right.\\
&+ \left.(1-\lambda_1)\mu_1 H\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)H\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1 H\left(\frac{i}{m},\frac{j}{m}\right)\right]
\end{aligned}
$$

then φ_m is Hadamard differentiable in every $H \in D[0,1]^2$.

Proof: *Let $\{h_n\}_{n\geq 1} \subset D[0,1]^2$, $H \in D[0,1]^2$, and $\{t_n\}_{n\geq 1} \subset \mathbb{R}$ such as $h_n \longrightarrow h$, with $h \in D[0,1]^2$, and $t_n \longrightarrow 0$. It holds*

$$
\begin{aligned}
\lim_{n\longrightarrow\infty}\frac{\varphi_m(H+t_nh_n)-\varphi_m(H)}{t_n} &= \lim_{n\longrightarrow\infty}\left[\sum_{j=1}^{m}\sum_{i=1}^{m}\left[\mathbf{1}_{\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]}(1-\lambda_1)(1-\mu_1)\left(H\left(\frac{i-1}{m},\frac{j-1}{m}\right)\right.\right.\right.\\
&\quad\left.+t_nh_n\left(\frac{i-1}{n},\frac{j-1}{m}\right)\right)+(1-\lambda_1)\mu_1\left(H\left(\frac{i-1}{m},\frac{j}{m}\right)+t_nh_n\left(\frac{i-1}{m},\frac{j}{m}\right)\right)\\
&\quad+\lambda_1(1-\mu_1)\left(H\left(\frac{i}{m},\frac{j-1}{m}\right)+t_nh_n\left(\frac{i}{m},\frac{j-1}{m}\right)\right)\\
&\quad\left.\left.+\lambda_1\mu_1\left(H\left(\frac{i}{m},\frac{j}{m}\right)+t_nh_n\left(\frac{i}{m},\frac{j}{m}\right)\right)\right]-\varphi_m(H)\right]/t_n
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{n\longrightarrow\infty}\Bigg[\sum_{j=1}^{m}\sum_{i=1}^{m}\Big[\boldsymbol{1}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)h_n\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1 h_n\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)h_n\left(\frac{i}{m},\frac{j-1}{m}\right) \\
&\quad +\lambda_1\mu_1 h_n\left(\frac{i}{m},\frac{j}{m}\right)\Big]\Bigg] \\
&= \sum_{j=1}^{m}\sum_{i=1}^{m}\Big[\boldsymbol{1}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)h\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1 h\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)h\left(\frac{i}{m},\frac{j-1}{m}\right) \\
&\quad +\lambda_1\mu_1 h\left(\frac{i}{m},\frac{j}{m}\right)\Big] \\
&= \varphi_m(h).
\end{aligned}
$$

Let $c\in\mathbb{R}$ and $h_1,h_2\in D[0,1]^2$, we have

$$
\begin{aligned}
\varphi_m(ch_1+h_2) &= \sum_{j=1}^{m}\sum_{i=1}^{m}\Big[\boldsymbol{1}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)(ch_1+h_2)\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1(ch_1+h_2)\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)(ch_1+h_2)\left(\frac{i}{m},\frac{j-1}{m}\right) \\
&\quad +\lambda_1\mu_1(ch_1+h_2)\left(\frac{i}{m},\frac{j}{m}\right)\Big] \\
&= c\sum_{j=1}^{m}\sum_{i=1}^{m}\Big[\boldsymbol{1}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)h_1\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1 h_1\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)h_1\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1 h_1\left(\frac{i}{m},\frac{j}{m}\right)\Big] \\
&\quad +\sum_{j=1}^{m}\sum_{i=1}^{m}\Big[\boldsymbol{1}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)h_2\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1 h_2\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)h_2\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1 h_2\left(\frac{i}{m},\frac{j}{m}\right)\Big] \\
&= c\varphi_m(h_1)+\varphi_m(h_2)
\end{aligned}
$$

and φ_m is linear.

Let $\{h_n\}_{n\geq 1} \subset D[0,1]^2$ *with* $h_n \longrightarrow h \in D[0,1]^2$*, it holds*

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \varphi_m(h_n) &= \lim_{n\longrightarrow\infty} \sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[\boldsymbol{1}_{\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]}(1-\lambda_1)(1-\mu_1)h_n\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1 h_n\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)h_n\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1 h_n\left(\frac{i}{m},\frac{j}{m}\right)\Bigg] \\
&= \sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[\boldsymbol{1}_{\left(\frac{i-1}{m},\frac{i}{m}\right]\times\left(\frac{j-1}{m},\frac{j}{m}\right]}(1-\lambda_1)(1-\mu_1)h\left(\frac{i-1}{n},\frac{j-1}{m}\right) \\
&\quad +(1-\lambda_1)\mu_1 h\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)h\left(\frac{i}{m},\frac{j-1}{m}\right)+\lambda_1\mu_1 h\left(\frac{i}{m},\frac{j}{m}\right)\Bigg] \\
&= \varphi_m(h)
\end{aligned}$$

and φ_m *is continuous.*

Therefore we can define the Hadamard derivative in H *of* φ_m *as* $\varphi'_H(h) = \varphi_m(h)$*, for all* $h \in D[0,1]^2$*.*
□

Remark 4.24 *The functional defined in Lemma 4.22 applied to C is exactly the* ***checkerboard copula of order*** *m defined in* [33]*, and the* ***linear B-spline copula*** *defined in* [43]*, in the case m equals to n. In Chapter 2 we can see that we obtain a better approximation to the real copula if we consider m smaller than n.*

Remark 4.25 *Because in the proof of Lemma 4.23 the relations holds for every sequence* $\{h_n\}_{n\geq 1} \subset D[0,1]^2$ *and* $h \in D[0,1]^2$ *such that* $h_n \longrightarrow h$*, particularly we can take* $h \in C[0,1]^2 \subset D[0,1]^2$*, that is,* φ_m *is Hadamard differentiable in every* $H \in D[0,1]^2$ *tangentially at* $\mathbb{D}_0 = C[0,1]^2$*.*

The following theorem is the main result of this chapter, since it provides the weak convergence of the process associated with the sample copula. It is important to notes that the proof of the following theorem uses the convergence of the process associated to the empirical copula given in the Theorem 4.19.

Theorem 4.26 *Let* $m \geq 2$ *be fixed, let C be a copula with continuous partial derivatives and let* C^n_m *be the sample copula build from the modified sample of a sample of size n from C. Let* $\{r_n\}_{n\geq 1}$ *be a increasing sequence such as* $r_n \longrightarrow \infty$ *and m divides* r_n*, for all* $n \in \mathbb{N}$*, then*

$$\sqrt{r_n}\,(C^{r_n}_m - \varphi_m(C)) \xrightarrow{\mathcal{D}} \varphi'_C(\mathbb{G}_C) = \varphi_m(\mathbb{G}_C)$$

where φ_m is defined in the Lemma 4.23, and $\varphi'_C\mathbb{G}_C) = \varphi_m(\mathbb{G}_C) = G$ is defined by

$$\begin{aligned} G &= \sum_{j=1}^{m}\sum_{i=1}^{m}\Bigg[\boldsymbol{I}_{(\frac{i-1}{m},\frac{i}{m}]\times(\frac{j-1}{m},\frac{j}{m}]}(1-\lambda_1)(1-\mu_1)\mathbb{G}_C\left(\frac{i-1}{m},\frac{j-1}{m}\right) \\ &\quad +(1-\lambda_1)\mu_1\mathbb{G}_C\left(\frac{i-1}{m},\frac{j}{m}\right)+\lambda_1(1-\mu_1)\mathbb{G}_C\left(\frac{i}{m},\frac{j-1}{m}\right) \\ &\quad +\lambda_1\mu_1\mathbb{G}_C\left(\frac{i}{m},\frac{j}{m}\right)\Bigg]. \end{aligned}$$

Proof: Based on the Theorem 4.19 we have that

$$\sqrt{n}\,(C_n - C) \xrightarrow{\mathcal{D}} \mathbb{G}_C,$$

if we define $r_n = mn$, for all $n \in \mathbb{N}$, then the above relation implies that

$$\sqrt{r_n}\,(C_{r_n} - C) \xrightarrow{\mathcal{D}} \mathbb{G}_C,$$

when $n \longrightarrow \infty$, the Lemma 4.23 and Remark 4.25 imply that φ_m is Hadamard differentiable at every $H \in D[0,1]^2$ tangentially at $C[0,1]^2$; we can consider, by Theorem 4.17, that the processes $\mathbb{G}_C \in C[0,1]^2$ is a separable stochastic process, because it is continuous almost surely respect to a probability measure P. Applying the Delta Method (Theorem 4.18) with $\mathbb{D}_\varphi = \mathbb{D} = D[0,1]^2$, $E = l^\infty([0,1]^2)$, $\theta_{r_n} = C$, for all $n \in \mathbb{N}$, $\theta = C$, and $D_0 = C[0,1]^2$, we have that

$$\sqrt{r_n}\,(\varphi_m(C_{r_n}) - \varphi_m(C)) \xrightarrow{\mathcal{D}} \varphi'_C(\mathbb{G}_C) = \varphi_m(\mathbb{G}_C)$$

and, by the Lemma 4.22, it is satisfied that $\varphi_m(C_{r_n}) = C_m^{r_n}$, therefore

$$\sqrt{r_n}\,(C_m^{r_n} - \varphi_m(C)) \xrightarrow{\mathcal{D}} \varphi'_C(\mathbb{G}_C) = \varphi_m(\mathbb{G}_C).$$

□

In the following remark we make observations about the parameters in the Gaussian process.

Remark 4.27 *Let $(a,b) \in \mathbf{I}^2 = [0,1]^2$ with $(a,b) \in (i-1, i/m] \times (j-1, j/m]$, because the linear combination of the components of a Gaussian vector have normal distribution and the process $\mathbb{G}_C$ is centered* [29, Definition 16.1]*, we have that $\varphi'_C(\mathbb{G}_C)(a,b) = \varphi_m(\mathbb{G}_C)(a,b) = G(a,b)$ is normally* *distributed with mean equal to zero and variance given by (using the values $\lambda_1(a,b)$ and $\mu_1(a,b)$, given on the Remark 4.21, and the results from Theorem 4.19)*

$$
\begin{aligned}
Var(G(a,b)) \quad = \quad & (1-\lambda_1)^2(1-\mu_1)^2 Var\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j-1}{m}\right)\right) + (1-\lambda_1)^2\mu_1^2 Var\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j}{m}\right)\right) \\
& +\lambda_1^2(1-\mu_1)^2 Var\left(\mathbb{G}_C\left(\frac{i}{m},\frac{j-1}{m}\right)\right) + \lambda_1^2\mu_1^2 Var\left(\mathbb{G}_C\left(\frac{i}{m},\frac{j}{m}\right)\right) \\
& +2(1-\lambda_1)^2\mu_1(1-\mu_1)E\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j-1}{m}\right)\cdot\mathbb{G}_C\left(\frac{i-1}{m},\frac{j}{m}\right)\right) \\
& +2\lambda_1(1-\lambda_1)(1-\mu_1)^2 E\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j-1}{m}\right)\cdot\mathbb{G}_C\left(\frac{i}{m},\frac{j-1}{m}\right)\right) \\
& +2\lambda_1(1-\lambda_1)\mu_1(1-\mu_1)E\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j-1}{m}\right)\cdot\mathbb{G}_C\left(\frac{i}{m},\frac{j}{m}\right)\right) \\
& +2\lambda_1(1-\lambda_1)\mu_1(1-\mu_1)E\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j}{m}\right)\cdot\mathbb{G}_C\left(\frac{i}{m},\frac{j-1}{m}\right)\right) \\
& +2\lambda_1(1-\lambda_1)\mu_1^2 E\left(\mathbb{G}_C\left(\frac{i-1}{m},\frac{j}{m}\right)\cdot\mathbb{G}_C\left(\frac{i}{m},\frac{j}{m}\right)\right) \\
& +2\lambda_1^2\mu_1(1-\mu_1)E\left(\mathbb{G}_C\left(\frac{i}{m},\frac{j-1}{m}\right)\cdot\mathbb{G}_C\left(\frac{i}{m},\frac{j}{m}\right)\right)
\end{aligned}
$$

with

$$
\begin{aligned}
E\left[\mathbb{G}_C(u,v)\cdot\mathbb{G}_C(u',v')\right] \quad = \quad & E\left[(\mathbb{B}_C(u,v)-\partial_1 C(u,v)\mathbb{B}_C(u,1)-\partial_2 C(u,v)\mathbb{B}_C(1,v))\right. \\
& \left.\cdot(\mathbb{B}_C(u',v')-\partial_1 C(u',v')\mathbb{B}_C(u',1)-\partial_2 C(u',v')\mathbb{B}_C(1,v'))\right] \\
= \quad & E\left[\mathbb{B}_C(u,v)\mathbb{B}_C(u',v')\right]-E\left[\mathbb{B}_C(u,v)\partial_1 C(u',v')\mathbb{B}_C(u',1)\right] \\
& -E\left[\mathbb{B}_C(u,v)\partial_2 C(u',v')\mathbb{B}_C(1,v')\right]-E\left[\partial_1 C(u,v)\mathbb{B}_C(u,1)\mathbb{B}_C(u',v')\right] \\
& +E\left[\partial_1 C(u,v)\mathbb{B}_C(u,1)\partial_1 C(u',v')\mathbb{B}_C(u',1)\right] \\
& +E\left[\partial_1 C(u,v)\mathbb{B}_C(u,1)\partial_2 C(u',v')\mathbb{B}_C(1,v')\right] \\
& -E\left[\partial_2 C(u,v)\mathbb{B}_C(1,v)B_C(u',v')\right] \\
& +E\left[\partial_2 C(u,v)\mathbb{B}_C(1,v)\partial_1 C(u',v')\mathbb{B}_C(u',1)\right] \\
& +E\left[\partial_2 C(u,v)\mathbb{B}_C(1,v)\partial_2 C(u',v')\mathbb{B}_C(1,v')\right] \\
= \quad & C(u\wedge u',v\wedge v')-C(u,v)C(u',v') \\
& -\partial_1 C(u',v')\left[C(u\wedge u',v)-u'C(u,v)\right] \\
& -\partial_2 C(u',v')\left[C(u,v\wedge v')-v'C(u,v)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\partial_1 C(u,v)\left[C(u\wedge u',v') - uC(u',v')\right]\\
&+\partial_1 C(u,v)\partial_1 C(u',v')\left[u\wedge u' - uu'\right]\\
&+\partial_1 C(u,v)\partial_2 C(u',v')\left[C(u,v') - uv'\right]\\
&-\partial_2 C(u,v)\left[C(u',v\wedge v') - vC(u',v')\right]\\
&+\partial_2 C(u,v)\partial_1 C(u',v')\left[C(u',v) - vu'\right]\\
&\partial_2 C(u,v)\partial_2 C(u',v')\left[v\wedge v' - vv'\right]
\end{aligned}
$$

and

$$
\begin{aligned}
Var(\mathbb{G}_C(u,v)) \;=\;& C(u,v) - C(u,v)^2 - \partial_1 C(u,v)\left[C(u,v) - C(u,v)u\right]\\
&-\partial_2 C(u,v)\left[C(u,v) - C(u,v)v\right] - \partial_1 C(u,v)\left[C(u,v) - C(u,v)u\right]\\
&+\partial_1 C(u,v)\partial_1 C(u,v)\left(u - u^2\right) + \partial_2 C(u,v)\partial_1 C(u,v)\left[C(u,v) - uv\right]\\
&-\partial_2 C(u,v)\left[C(u,v) - vC(u,v)\right] + \partial_2 C(u,v)\partial_1 C(u,v)\left[C(u,v) - uv\right]\\
&+\partial_2 C(u,v)\partial_2 C(u,v)\left(v - v^2\right).
\end{aligned}
$$

Remark 4.28 *In the case where the copula C is equal to the product copula, for all* $(a,b)\in \mathbf{I}^2 = [0,1]^2$*, using Nelsen's notation given in the Remark 4.21 and taking* $a_1 = (i-1)/m, a_2 = i/m, b_1 = (j-1)/m$ *and* $b_2 = j/m$*, we have*

$$
\begin{aligned}
C(a,b) \;=\;& (1-\lambda_1)(1-\mu_1)\Pi(a_1,b_1) + (1-\lambda_1)\mu_1\Pi(a_1,b_2) + \lambda_1(1-\mu_1)\Pi(a_2,b_1) + \lambda_1\mu_1\Pi(a_2,b_2)\\
=\;& \left(\frac{a_2-a}{a_2-a_1}\right)\left(\frac{b_2-b}{b_2-b_1}\right)a_1b_1 + \left(\frac{a_2-a}{a_2-a_1}\right)\left(\frac{b-b_1}{b_2-b_1}\right)a_1b_2\\
&+\left(\frac{a-a_1}{a_2-a_1}\right)\left(\frac{b_2-b}{b_2-b_1}\right)a_2b_1 + \left(\frac{a-a_1}{a_2-a_1}\right)\left(\frac{b-b_1}{b_2-b_1}\right)a_2b_2\\
&\left(\frac{a_2-a}{a_2-a_1}\right)a_1\left[\left(\frac{b_2-b}{b_2-b_1}\right)\left(\frac{b-b_1}{b_2-b_1}\right)b_2\right] + \left(\frac{a-a_1}{a_2-a_1}\right)a_2\left[\left(\frac{b_2-b}{b_2-b_1}\right)\left(\frac{b-b_1}{b_2-b_1}\right)b_2\right]\\
=\;& \left(\frac{a_2-a}{a_2-a_1}\right)a_1b + \left(\frac{a-a_1}{a_2-a_1}\right)a_2b\\
=\;& \frac{a_2-a_1}{a_2-a_2}ab\\
=\;& ab\\
=\;& \Pi(a,b).
\end{aligned}
$$

Therefore the conclusion of Theorem 4.26, can we written as

$$
\sqrt{r_n}\,(C_m^{r_n} - \Pi) \xrightarrow{\mathcal{D}} \varphi'_H(\mathbb{G}_C) = \varphi_m(\mathbb{G}_C).
$$

In this case, the covariance and variance of $Z = \mathbb{G}_{\mathbb{C}}$ processes is given by

$$Cov(Z(s,t), Z(u,v)) = (s \wedge u - su)(t \wedge v - tv)$$

and

$$Var(Z(s,t)) = (s - s^2)(t - t^2).$$

Let $(a,b) \in \mathbf{I}^2 = [0,1]^2$ with $(a,b) \in (i-1, i/m] \times (j-1, j/m]$, for the calculus of variance of the $\varphi_m(Z)(a,b)$, normally distributed with mean equal to zero, we have the following relations

$$\begin{aligned}
V_1 &= Var\left((1-\lambda_1)(1-\mu_1)Z\left(\frac{i-1}{n}, \frac{j-1}{m}\right)\right) \\
&= (1-\lambda_1)^2(1-\mu_1)^2\left(\frac{i-1}{m} - \left(\frac{i-1}{m}\right)^2\right)\left(\frac{j-1}{m} - \left(\frac{j-1}{m}\right)^2\right), \\
V_2 &= Var\left((1-\lambda_1)\mu_1 Z\left(\frac{i-1}{n}, \frac{j}{m}\right)\right) \\
&= (1-\lambda_1)^2\mu_1^2\left(\frac{i-1}{m} - \left(\frac{i-1}{m}\right)^2\right)\left(\frac{j}{m} - \left(\frac{j}{m}\right)^2\right), \\
V_3 &= Var\left(\lambda_1(1-\mu_1)Z\left(\frac{i}{n}, \frac{j-1}{m}\right)\right) \\
&= \lambda_1^2(1-\mu_1)^2\left(\frac{i}{m} - \left(\frac{i}{m}\right)^2\right)\left(\frac{j-1}{m} - \left(\frac{j-1}{m}\right)^2\right), \\
V_4 &= Var\left(\lambda_1\mu_1 Z\left(\frac{i}{n}, \frac{j}{m}\right)\right) \\
&= \lambda_1^2\mu_1^2\left(\frac{i}{m} - \left(\frac{i}{m}\right)^2\right)\left(\frac{j}{m} - \left(\frac{j}{m}\right)^2\right), \\
C_{12} &= Cov\left((1-\lambda_1)(1-\mu_1)Z\left(\frac{i-1}{m}, \frac{j-1}{m}\right), (1-\lambda_1)\mu_1 Z\left(\frac{i-1}{m}, \frac{j}{m}\right)\right) \\
&= (1-\lambda_1)^2(1-\mu_1)\mu_1\left(\frac{i-1}{m} - \left(\frac{i-1}{m}\right)^2\right)\left(\frac{j-1}{m} - \frac{j(j-1)}{m^2}\right), \\
C_{13} &= Cov\left((1-\lambda_1)(1-\mu_1)Z\left(\frac{i-1}{m}, \frac{j-1}{m}\right), \lambda_1(1-\mu_1)Z\left(\frac{i}{m}, \frac{j-1}{m}\right)\right) \\
&= \lambda_1(1-\lambda_1)(1-\mu_1)^2\left(\frac{i-1}{m} - \frac{i(i-1)}{m^2}\right)\left(\frac{j-1}{m} - \left(\frac{j-1}{m}\right)^2\right), \\
C_{14} &= Cov\left((1-\lambda_1)(1-\mu_1)Z\left(\frac{i-1}{m}, \frac{j-1}{m}\right), \lambda_1\mu_1 Z\left(\frac{i}{m}, \frac{j}{m}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_1(1-\lambda_1)\mu_1(1-\mu_1)\left(\frac{i-1}{m}-\frac{i(i-1)}{m^2}\right)\left(\frac{j-1}{m}-\frac{j(j-1)}{m^2}\right),\\
C_{23} &= Cov\left((1-\lambda_1)\mu_1 Z\left(\frac{i-1}{m},\frac{j}{m}\right),\lambda_1(1-\mu_1)Z\left(\frac{i}{m},\frac{j-1}{m}\right)\right)\\
&= \lambda_1(1-\lambda_1)\mu_1(1-\mu_1)\left(\frac{i-1}{m}-\frac{i(i-1)}{m^2}\right)\left(\frac{j-1}{m}-\frac{j(j-1)}{m^2}\right),\\
C_{24} &= Cov\left((1-\lambda_1)\mu_1 Z\left(\frac{i-1}{m},\frac{j}{m}\right),\lambda_1\mu_1 Z\left(\frac{i}{m},\frac{j}{m}\right)\right)\\
&= \lambda_1(1-\lambda_1)\mu_1^2\left(\frac{i-1}{m}-\frac{i(i-1)}{m^2}\right)\left(\frac{j}{m}-\left(\frac{j}{m}\right)^2\right),\\
C_{34} &= Cov\left(\lambda_1(1-\mu_1)Z\left(\frac{i}{m},\frac{j-1}{m}\right),\lambda_1\mu_1 Z\left(\frac{i}{m},\frac{j}{m}\right)\right)\\
&= \lambda_1^2\mu_1(1-\mu_1)\left(\frac{i}{m}-\left(\frac{i}{m}\right)^2\right)\left(\frac{j-1}{m}-\frac{j(j-1)}{m^2}\right),
\end{aligned}
$$

and

$$Var(\varphi_m(Z)(a,b)) = V_1 + V_2 + V_3 + V_4 + 2\left(C_{12} + C_{13} + C_{14} + C_{23} + C_{24} + C_{34}\right).$$

From [40], and using similar calculations as above, we can extend the convergence results of the Theorem 4.26 to higher dimensions $d > 2$. This is given in the following theorem:

Theorem 4.29 *Let $d > 2$, $m \geq 2$, let C be a d-copula with continuous partial derivatives and let C_m^n be the sample copula build from the modified sample of a sample of size n from C. Let $\{r_n\}_{n\geq 1}$ be a increasing sequence such as $r_n \longrightarrow \infty$ and m divides r_n, for all $n \in \mathbb{N}$, then*

$$\sqrt{r_n}\left(C_m^{r_n} - \varphi_m(C)\right) \xrightarrow{\mathcal{D}} \varphi'_C(\mathbb{G}_C) = \varphi_m(\mathbb{G}_C)$$

where

$$\mathbb{G}_C = \mathbb{B}_C(u_1,\cdots,u_d) - \sum_{i=1}^{d}\partial_i C(u_1,\cdots,u_d)\mathbb{B}_{\mathbb{C}}(1,1,\cdots,u_i,\cdots,1)$$

and $\mathbb{B}_C$ is a Gaussian process with

$$E(\mathbb{B}_C(u_1,\cdots,u_d)\cdot\mathbb{B}_C(u'_1,\cdots,u'_d)) = C(u_1\wedge u'_1,\cdots,u_d\wedge u'_d) - C(u_1,\cdots,u_d)C(u'_1,\cdots,u'_d).$$

In the following section we given the results of perform several simulations of the associated process of the sample copula given in Theorem 4.26 to exemplify its convergence.

4.3 Simulation study: Gaussian process

In the next tables we present the results when we perform $1,000$ iterations from samples of size $50,000$ for the process associated to some family of copulas with continuous partial derivatives. The tables show the results when we repeat these simulations 100 times. The column θ indicates the parameter value, the column ρ indicates the value of the Spearman's Rho associated to the parameter of the copula, the column **Point** gives the point of evaluation of the process, the column **M. means** is the mean of the means of the 100 simulations of the process (since the processes is a centered one, this value most be close to zero), **M. var** denotes the mean of the variances obtained form the 100 repetitions, **Real var.** denotes the real variance calculated using Remark 4.27, the firsts three columns **0.01**, **0.05** and **0.10** indicates the number of rejections of normality in the 100 simulations at $\alpha-$level using the Anderson-Darling's test, and the second three columns **0.01**, **0.05** and **0.10** indicates the number of rejections of normality in the 100 simulations at $\alpha-$level using the Shapiro-Wilk's test, the column **UAD** indicates the p-value of a Kolmogorov-Smirnov's uniformity test applied to the p-values of Anderson-Darling's tests, and the column **USW** indicates the p-value of a Kolmogorov-Smirnov's uniformity test applied to the p-values of Shapiro-Wilk's tests. In all cases we consider the parameter m of the sample copula equals to 4.

θ	ρ	Point	M. means	M. var.	Real var.	0.01	0.05	0.10	0.01	0.05	0.10	UAD	USW
20	0.9578	(0.8,0.9)	-0.00010	0.001643	0.001638	1	4	11	1	5	14	0.00032	0.02205
20	0.9578	(0.1,0.2)	-0.00023	0.00163	0.00163	2	10	16	3	8	15	0.00036	0.01440
20	0.9578	(0.4,0.6)	-0.00004	0.00220	0.00220	3	11	15	0	5	11	0.34306	0.38131
-20	**-0.9578**	**(0.1,0.9)**	**0.00010**	**0.00040**	**0.00040**	**2**	**9**	**16**	**0**	**6**	**14**	**0.00299**	**0.00692**
-20	-0.9578	(0.8,0.2)	0.00034	0.00654	0.00655	2	7	12	3	7	10	0.01939	0.12012
-20	-0.9578	(0.4,0.6)	0.00046	0.00252	0.00252	0	4	11	3	8	12	0.74344	0.98920
5	0.6434	(0.1,0.2)	-0.00027	0.00378	0.00377	0	4	11	0	5	8	0.06614	0.75172
5	0.6434	(0.8,0.9)	-0.00001	0.00377	0.00377	1	8	14	2	9	13	0.13685	0.27477
5	0.6434	(0.4,0.6)	-0.00047	0.01665	0.01672	0	2	9	0	4	7	0.01777	0.57689
-5	-0.6434	(0.1,0.9)	-0.00002	0.00093	0.00094	2	6	17	1	7	14	0.00298	0.03665
-5	-0.6434	(0.8,0.2)	0.00005	0.01518	0.01508	1	5	10	2	9	16	0.04696	0.15866
-5	-0.6434	(0.4,0.6)	0.00040	0.01919	0.01916	1	2	4	1	3	5	0.017208	0.02306

Table 4.1 Simulations results (Frank Copula).

θ	ρ	Point	M. means	M. var.	Real var.	0.01	0.05	0.10	0.01	0.05	0.10	UAD	USW
1.1	0.1341	(0.8,0.9)	0.00009	0.00400	0.00400	1	3	7	1	3	5	0.13436	0.59930
1.1	0.1341	(0.1,0.2)	-0.00011	0.00383	0.00384	2	5	12	1	3	6	0.07756	0.24805
1.1	0.1341	(0.4,0.6)	-0.00139	0.03150	0.03164	2	5	10	0	7	11	0.44923	0.48465
3	**0.8481**	**(0.8,0.81)**	**-0.00027**	**0.00889**	**0.00891**	**1**	**7**	**20**	**3**	**10**	**19**	**0.00000**	**0.00000**
3	0.8481	(0.1,0.15)	-0.00016	0.00171	0.00171	1	4	13	0	5	13	0.04158	0.44648
3	0.8481	(0.45,0.5)	0.00029	0.02529	0.02537	3	5	11	2	6	8	0.99269	0.41925
8	0.9773	(0.8,0.81)	-0.00035	0.00346	0.00345	2	9	17	1	8	18	0.00000	0.01543
8	0.9773	(0.1,0.13)	-0.00005	0.00058	0.00058	2	8	18	3	8	16	0.00036	0.02517
8	0.9773	(0.45,0.48)	-0.00078	0.00787	0.00785	2	6	10	2	4	6	0.67848	0.46945
20	**0.9963**	**(0.8,0.81)**	**-0.00056**	**0.00138**	**0.00138**	**6**	**23**	**37**	**4**	**15**	**21**	**0.00000**	**0.00000**
20	**0.9963**	**(0.1,0.11)**	**-0.00018**	**0.00017**	**0.00017**	**0**	**14**	**31**	**4**	**11**	**19**	**0.00000**	**0.00000**
20	**0.9963**	**(0.4,0.6)**	**-0.00020**	**0.00076**	**0.00076**	**2**	**10**	**26**	**2**	**10**	**20**	**0.00000**	**0.00000**

Table 4.2 Simulations results (Gumbel Copula).

θ	ρ	Point	M. means	M. var.	Real var.	0.01	0.05	0.10	0.01	0.05	0.10	UAD	USW
-0.9	-0.8978	(0.8,0.3)	0.00005	0.01399	0.01408	2	7	8	2	4	11	0.23918	0.39465
-0.9	**-0.8978**	**(0.8,0.9)**	**0.00007**	**0.00078**	**0.00077**	**0**	**6**	**20**	**3**	**10**	**18**	**0.00000**	**0.00000**
-0.5	-0.9878	(0.5,0.49)	-0.00014	0.05621	0.05624	1	3	8	2	3	7	0.74006	0.51611
-0.1	-0.0787	(0.8,0.9)	-0.00022	0.00347	0.00346	1	3	5	1	4	6	0.00465	0.06234
-0.1	-0.0787	(0.8,0.3)	-0.00052	0.02112	0.02131	0	2	6	0	3	4	0.78381	0.23228
-0.1	-0.0787	(0.2,0.9)	-0.00012	0.00380	0.00378	1	6	11	2	6	11	0.00456	0.06511
5	0.8848	(0.1,0.12)	-0.00019	0.00058	0.00058	0	9	12	2	10	19	0.01646	0.16689
5	0.8848	(0.8,0.78)	-0.00126	0.1646	0.01647	0	4	11	0	5	13	0.08772	0.10310
5	0.8848	(0.4,0.38)	-0.00068	0.00447	0.00446	1	4	8	1	5	10	0.50330	0.8815
20	**0.9869**	**(0.1,0.11)**	**-0.00014**	**0.00013**	**0.00013**	**2**	**15**	**30**	**1**	**7**	**20**	**0.00000**	**0.00000**
20	0.9869	(0.8,0.79)	-0.00052	0.00574	0.00573	4	15	16	3	11	18	0.00142	0.05822
20	0.9869	(0.4,0.39)	-0.00046	0.00109	0.00109	0	0	8	0	2	6	0.06122	0.67889

Table 4.3 Simulations results (Clayton Copula).

θ	ρ	Point	M. means	M. var.	Real var.	0.01	0.05	0.10	0.01	0.05	0.10	UAD	USW
1.1	0.0820	(0.8,0.9)	-0.00016	0.00396	0.00393	2	4	7	2	6	9	0.29601	0.37343
1.1	0.0820	(0.1,0.2)	-0.00001	0.00371	0.00371	3	6	9	0	7	12	0.32767	0.58562
1.1	0.0820	(0.4,0.6)	0.00008	0.03230	0.03214	3	8	13	1	7	13	0.63409	0.85489
3	0.6993	(0.8,0.79)	-0.00035	0.01340	0.01332	0	7	11	1	6	8	0.18609	0.12258
3	0.6993	(0.1,0.12)	0.00014	0.00148	0.00147	2	5	12	3	6	11	0.50472	0.16758
3	0.6993	(0.4,0.41)	-0.00065	0.02382	0.02363	2	5	10	1	4	15	0.56857	0.33761
8	**0.9310**	**(0.8,0.79)**	**-0.00047**	**0.00507**	**0.00506**	**2**	**10**	**19**	**1**	**11**	**20**	**0.00000**	**0.00000**
8	0.9310	(0.1,0.12)	0.00006	0.00104	0.00104	1	4	9	1	1	7	0.01724	0.42950
8	0.9310	(0.4,0.41)	-0.00058	0.00566	0.00568	0	3	9	1	5	8	0.06168	0.12568
20	0.9860	(0.7,0.7)	-0.00064	0.00182	0.00181	2	4	9	1	7	12	0.71708	0.19691
20	0.9860	(0.1,0.1)	-0.00012	0.00033	0.00033	4	10	17	4	8	12	0.00094	0.09628
20	0.9860	(0.4,0.4)	-0.00040	0.00148	0.00148	2	4	9	1	7	12	0.96922	0.27021

Table 4.4 Simulations results (Joe Copula).

θ	ρ	Point	M. means	M. var.	Real var.	0.01	0.05	0.10	0.01	0.05	0.10	UAD	USW
0.1	-0.6536	(0.8,0.2)	0.00004	0.01436	0.01445	2	2	8	2	7	11	0.14030	0.19872
0.1	-0.6536	(0.1,0.9)	0.00015	0.00089	0.00090	3	7	9	2	5	7	0.48364	0.61248
0.1	-0.6536	(0.4,0.41)	0.00016	0.01803	0.01811	1	7	13	3	8	17	0.32480	0.0568
1.5	0.1344	(0.8,0.9)	0.00007	0.00392	0.00390	1	4	12	2	7	13	0.22429	0.58390
1.5	0.1344	(0.1,0.2)	0.00006	0.00391	0.00390	1	6	11	1	3	13	0.00762	0.58638
1.5	0.1344	(0.4,0.6)	0.00036	0.03165	0.03145	2	5	9	0	5	13	0.62387	0.13984
8	0.6067	(0.8,0.79)	-0.00039	0.01646	0.01658	3	10	14	3	8	17	0.22829	0.48114
8	0.6067	(0.1,0.11)	0.00010	0.00113	0.00113	1	5	9	2	5	10	0.53682	0.92631
8	0.6067	(0.4,0.41)	-0.00105	0.02172	0.02170	0	3	9	0	4	8	0.05772	0.04771
30	0.8263	(0.8,0.79)	-0.00065	0.01200	0.01199	1	8	17	0	7	13	0.02233	0.09471
30	0.8263	(0.1,0.11)	-0.00009	0.00082	0.00082	0	2	10	2	5	11	0.40344	0.60288
30	0.8263	(0.4,0.41)	-0.00090	0.01093	0.01095	5	9	11	3	10	12	0.61730	0.82075

Table 4.5 Simulations results (Plackett Copula).

Point	M. means	M. var.	Real var.	0.01	0.05	0.10	0.01	0.05	0.10	UAD	USW
(0.8,0.9)	-0.00003	0.00366	0.00360	0	5	13	1	9	14	0.18762	0.91708
(0.1,0.2)	0.00028	0.00358	0.00352	1	7	15	4	10	15	0.09103	0.18777
(0.4,0.6)	0.00042	0.03242	0.03240	2	6	8	2	7	10	0.67877	0.41021
(0.1,0.9)	0.00002	0.00090	0.00090	3	6	9	1	5	8	0.04456	0.04697
(0.8,0.2)	**0.00015**	**0.01439**	**0.01440**	**1**	**6**	**10**	**3**	**8**	**13**	**0.00066**	**0.00484**

Table 4.6 Simulations results (Product Copula).

In the previous results, it can be seen that in most cases the number of rejections are close to the α-level. However in some cases we have large values in the columns that indicate the number of rejections at α-level (rows in bold type). The problem occur in simulations where the point is located near to the boundary of $\mathbf{I}^2 = [0, 1] \times [0, 1]$ and in simulations where the copula is close to one of the Fréchet-Hoeffding bounds and the point is not in the support of the copula M or copula W. This also happens if the copula is singular. As a possible solution in such cases, we propose to increase the sample size or consider points very close to the support of the copula M or copula W to obtain a better approximation to the normal distribution with the parameters indicated in Remark 4.27. For example, if we consider a sample size of 100,000 instead of a sample size of 50,000 in the simulation of the Gumbel copula with $\theta = 20$ in the point $(0.8, 0.8)$ we obtain a mean of means equal to -0.00039, a mean of variances equal to 0.00149, a real variance equal to 0.00153 and 4, 8 and 12 rejections of normality at α-level 0.01, 0.05 and 0.10, respectively, for the Anderson-Darling's test, and 1, 10 and 15 for the Shapiro-Wilk's test, also we have a p-value of 0.79405 for the Kolmogorov-Smirnov's uniformity test applied to the p-values of the Anderson-Darling's tests and a p-value of 0.77178 for the Kolmogorov-Smirnov's uniformity test applied to the p-values of the Shapiro-Wilk's tests.

Conclusions

We know that Glivenko-Cantelli's Theorem 2.5, is one out of two results which has made the study of the empirical distribution functions a really important task. In our Theorem 2.17 we proved that the sample copula also satisfies a Glivenko-Cantelli's result. It is unquestionable that the empirical distribution function in dimension $d = 1$ is the best possible approximation to the real distribution function, a proof of that is the Dvoretzky-Kiefer-Wolfowitz's inequality, see [11] and [34]. However, for dimension higher than one, even if Glivenko-Cantelli's Theorem still holds, the price we have to pay is that we need very large sample sizes in order to obtain a good approximation of the real distribution function, this needed sample size increases dramatically with the dimension d. Besides, the evaluation of the empirical distribution function needs arrays of order n^d where n is the sample size and this becomes a problem for large sample sizes even in relatively small dimensions. On the other hand, for the sample d-copula or order m, if m is relatively small compare to n the arrays we need for the construction of C_m^n is only of order m^d, which in general is a lot smaller that n^d.

We saw in every simulation of Chapter 2, that there exists a value of m for which C_m^n is a better approximation than the empirical copula C_n. If we define m_0 the value of $2 \leq m \leq n$ such that $C_{m_0}^n$ is a better approximation C_n and it minimizes the mean value of the supremum distance between C_m^n and C the real copula, then we would have had selected the best possible m. Of course, when we have only one sample from an unknown distribution it is not easy to select an appropriate value of m. In Section 5 of the Chapter 2, for dimension $d = 2$ we are proposing a method to estimate the value of m based on the sample Spearman's ρ. This idea is based on the fact that the best approximation of C_m^n for Π^d the product d-copula is always reached when $m = 2$ for any sample size n, that is, when $\rho = 0$. This statement also suggests that we may use the total variation between the sample copula and the product copula using only the densities induced by Π^d and by C_m^n in order to estimate the parameter m.

As a last interesting remark, let us assume that $d = 1$ and that we have a random sample $X_1, X_2, \ldots, X_n$ from a continuous distribution F, let $Y_i = i/n$ for $i \in I_n$ be the modified sample, and take $2 \leq m \leq n$. Then if we define the uniform partition of $\mathbf{I} = [0, 1]$ of size m and we construct C_m^n following the same ideas as in Definition 2.12 for $d \geq 2$, but in dimension one. We have that $C_m^n(u) = u$ for every $u \in \mathbf{I}$. Therefore, $C_m^n(u)$ is the distribution of $F(X)$ independently of the selection of m. This statement clearly indicates that using the ideas of Deheuvels [8] we always find the right distribution independently of the real continuous F, the sample size n and the selected m.

The aim of Chapter 3 was to show that in addition to the comparative advantages shown in Chapter

2 between the sample d-copula of order m and the empirical copula, we can obtain similar results to those existing for the empirical copula given in [8] and that these results can be extended to higher dimensions, in this way we obtain a complete description of the distributions associated to the sample copula under the independence assumption.

As we mentioned before, the second result that makes important the use of F_n, the empirical distribution, or the empirical copula C_n is the central limit theorem. In the Chapter 4 we proved the convergence to a Gaussian process of the sample copula process $\sqrt{n}(C_m^n - C)$, from this result we obtained a Central Limit Theorem similar to the existing one for the empirical copula process.

From the last two chapters of this thesis we can replicate, for the sample copula, the most important results of the convergence of the empirical copula, and along with the results of Chapter 2 we can consider that the sample copula could be a good approximation to the checkerboard copula and therefore to the real copula in comparison to the approximation given for the empirical copula. Among possible applications of the sample copula we have the study of the credit risk of the financial institutions, in [32] a methodology is proposed to calculate the default correlations using the Gaussian copula, although this work suffered several criticism because these models did not provide adequate results under the conditions of the financial crisis of 2008, many efforts have been made by using Gaussian model variants, see [3]. The sample copula can be used to model the overall risk without assuming any specific family of copulas. We have also worked on an algorithm to generate, from the sample copula, simulated samples that preserve the dependence structure which may be used in methodologies related to tests of goodness of fit or parameter estimation.

References

[1] Ash, R. B., Gardner, M. F. (1975), *Topics in Stochastic Processes* Academic Press, Inc., New York.

[2] Billingsley, P. (1968). *Convergence of probability measures*. John Wiley & Sons, Inc. New York.

[3] Bluhm, C., Overbeck, L. and Wagner C. (2010). *Introduction to credit risk modeling*. 2nd ed., *Mathematics Series*, CRC Press.

[4] Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G. and Roncalli, T. (2000). Copulas for Finance: A reading guide and some applications. City Univ. Business School London and Groupe de Recherche Opérationnelle Crédit Lyonnais Paris. Preprint.

[5] Bücher, A., Dette, H. and Volgushev, S. (2011). New estimators of the Pickands dependence function and a test for extreme-value dependence. *Ann. Statist.*, **39**, (4), 1963–2006.

[6] Caillault, C. and Guegan, D. (2005). Empirical estimation of tail dependence using copulas. Application to Asian markets. *Quantitaives Finance*, **5**, 489–501.

[7] Cherubini, U., Luciano, E. and Vechiatto, W. (2004). *Copula Methods in Finance.* John Wiley & Sons, Chichester.

[8] Deheuvels, P. (1979). La fonction de dépendance empirique et ses proprietés. Un test non paramétrique d'indépendance. *Roy. Belg. Bull. Cl. Sci.*, **65**, (5), 274–292.

[9] Durante, F. and Fernández-Sánchez, J. (2010). Multivariate shuffles and approximations of copulas. *Statist. Probab. Lett.*, **80**, 1827–1834.

[10] Durrelman, V., Nikeghbali, A. and Roncalli, T. (2000). Which copula is the right one? Groupe de Recherche Opérationnelle Crédit Lyonnais Paris. Preprint.

[11] Dvoretzky, A., Kiefer,J. and Wolfowitz, J. (1956). Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *Ann. Math. Statist.*, **27**, (3), 642–669.

[12] Embrechts, P., Höing, A. and Juri, A. (2003). Using copulae to bound the value-at-risk for functions of dependent risks. *Finance Stochast.*, **7**, 145–167,

[13] Fermanian,J.D., Radulović, D. and Wegkamp, M. (2004). Weak convergence of empirical copula processes. *Bernoulli*, **10**, (5), 847–860.

[14] Fermanian, J. D., Radulović, D. and Wegkamp, M. (2015). Asymptotic total variation tests for copulas. *Bernoulli*, **21**, (3), 1911–1945.

[15] Fredricks, G.A., Nelsen, R.B. and Rodríguez-Lallena, J. (2005). Copulas with fractal support. *Insurance Math. Econom.*, **37**, 42–48.

[16] Genest, C., Huang, W. and Dufour, J. M. (2013). A regularized goodness-of-fit test for copulas. *J. de la Société Française de Statistique*, **154**, 64–77.

[17] Genest, C., Nešlehová, J. G. and Rémillard B. (2014). On the empirical multilinear copula process for count data. *Bernoulli*, **20**, 1344-1371.

[18] Genest, C., Quessy, J. F. and Rémillard, B. (2007). Asymptotic local efficiency of Cramér-von Mises tests for multivariate independence. *Ann. Statist.*, **35**, (1), 166–191.

[19] Genest, C. and Rémillard, B. (2004). Tests of independence and randomness based on the empirical copula process. *Test*, **13**, 335-369.

[20] Genest, C., Remillard, B. and Beaudoin, D. (2009). Goodness-of-fit tests for copulas: A review and a power study. *Insurance Math. Econom.*, **44**, 199–213.

[21] Genest, C. and Serges J. (2010). On the covariance of the asymptotic empirical copula process. *J. Multivariate Anal.*, **101**, 1837–1845.

[22] Gijbels, I., Omelka, M. and Veraverbeke, N. (2012). Multivariate and functional covariates and conditional copulas. *Electron. J. Stat.*, **6**, 1273–1306.

[23] Gijbels, I., Omelka, M. and Veraverbeke, N. (2015). Partial and average copulas and association measures. *Electron. J. Stat.*, **9**, (2), 2420–2474.

[24] González-Barrios, J.M. and Hernández-Cedillo, M.M. (2013). Sample d-copula of order m. *Kybernetika*, **49**, (5), 663–691.

[25] González-Barrios, J.M., Hernández-Cedillo, M.M. and Rueda, R. (2014). Minimal number of parameters of a d-dimensionally stochastic matrix. In *Modelos en Estadística y Probabilidad III*, Eds. González-Barrios, J.M., León J.A., Navarro, R. and Villa, J., Aportaciones Matemáticas de la SMM, Ser. Comunicaciones, **47**, 123–137.

[26] González-Barrios, J. M. and Hoyos-Argüelles, R. (2016). Asymptotic results for the sample copula or order m. Preprint IIMAS UNAM.

[27] Gould, H. W. (1972). *Combinatorial identities*. Morgantown Printing and Binding Co.

[28] Grégoire, V., Genest, C. and Gendron, M. (2008). Using copulas to model price dependence in energy markets. *Energy Risk*, **5**, (5), 58–64.

[29] Jacod, J. and Protter, P. (2000), *Probability Essentials*. 2nd ed., Springer, New York.

[30] Jaworski, P., Durante, F., Härdle, W. and Rychlik, T. (2010). *Copula Theory and its Applications.* Lect. Notes in Statist.- Proceedings, **198**. Springer, Heidelberg.

[31] Joe, H. (2015). *Dependence Modelling with Copulas.* Monogr. Statist. Appl. Probab., **134**, Chapman & Hall/CRC, Boca Raton.

[32] Li, D. X., (2000), On default correlation: A copula function approach. *Journal of Fixed Econometrics*, **9**, (4), 43–54.

[33] Li, X., Mikusiński, P., Sherwood, H. and Taylor, M. D. (1997), On approximation of copulas. In Benes, V. and Stepán, J., editors, *Distributions with given marginals and moment problems*, 107-116.

[34] Massart, P. (1990). The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *Ann. Probab.*, **18**, (3), 1269–1283.

[35] Mikusiński, P., Sherwood, H. and Taylor, M.D. (1992). Shuffles of Min. *Stochastica*, **13**, (1), 61–74.

[36] Mikusiński, P. and Taylor, M.D. (2010). Some approximation of n-copulas. *Metrika*, **72**, 385-414.

[37] Nelsen, R.B. (2006). *An introduction to copulas*. 2nd ed., *Lect. Notes in Statist.*, **139**, Springer, New York.

[38] Neuhaus, G. (1971), On weak convergence of stochastic processes with multidimensional time parameter. *The Annals of Mathematical Statistics* , **42** (4), 1285-1295.

[39] Quessy, J. F. (2016). A general framework for testing homogeneity hypotheses about copulas. *Electron. J. Stat.*, **10**, (1), 1064–1097.

[40] Rüschendorf L. (1976), Asymptotic Distributions of multivariate rank order statistics. *The Annals of Statistics* , **4** (5), 912-923.

[41] Sancetta, A. and Satchell, S. (2004). The Bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econom. Theory*, **20**, (3), 535–562.

[42] Schefzik, R., Thorarinsdottir, T. L. and Gneiting, T. (2013). Uncertainty Quantification in Complex Simulation Models Using Ensemble Copula Coupling. *Statist. Sci.*, **28**, (4), 616–640.

[43] Shen, X., Zhu, Y. and Song, L. (2008). Linear *B*-spline copulas with applications to non parametric estimation of copulas. *Compu. Statist. Data Anal.*, **52**, (7), 3806–3819.

[44] Stute, W. (1984). The Oscillation Behavior of Empirical Processes: The Multivariate Case. *Ann. Probab.*, **12**, (2), 361–379.

[45] Swanepoel, J. W. H. and Allison, J. S. (2013). Some new results on the empirical copula estimator with applications. *Statist. Probab. Lett.*, **83**, (7), 1731–1739.

[46] Trutschnig, W. and Fernández-Sánchez, J. (2012). Idempotent and multivariate copulas with fractal support. *J. Statist. Plan. Infer.*, **142**, 3086–3096.

[47] Turrin, F. L. (1983). *von Mises Calculus for Statistical Functionals.* Lecture Notes in Statistics, **19**. Springer-Verlag, New York, Berlin, Heidelberg Tokyo.

[48] van den Goorbergh, R.W.J., Genest, C. and Werker, B.J.M. (2005). Bivariate option pricing using dynamic copula models. *Insurance Math. Econom.*, **37**, 101–114.

[49] van der Vaart, A. W. and Wellner, J. A. (1996), *Weak convergence and empirical processes with applications to statistics (In Springer Series in Statistics)* Springer, New York.