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# Minimally Generated Boolean Algebras

## T E S I S

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# RESUMEN

El presente trabajo pretende ser una introducción al estudio de las álgebras booleanas generadas mínimamente. Una álgebra booleana está generada mínimamente si se puede obtener como el último paso en una sucesión de extensiones del álgebra de dos elementos, donde entre cualesquiera dos álgebras consecutivas de dicha sucesión no existe un álgebra booleana intermedia propia.

La propiedad de ser mínimamente generada fue presentada por primera vez por Sabine Koppelberg, quien la obtuvo a partir de lo trabajos de S. Shelah, S. Grigorieff, J. Baumgartner y P. Komjath.

Consideramos que la literatura carece de introducciones al estudio de las álgebras booleanas generadas mínimamente. Además, aunque Koppelberg presentó la clase de las álgebras de Boole que son generadas mínimamente en [11], este artículo está dirigido a los especialistas del área. Así, esperamos contribuir con texto introductorio, en particular, nuestro mayor aporte es presentar con detalle los primeros resultados de dicho artículo.

El primer capítulo trata sobre los preliminares necesarios para la lectura del texto, en primer lugar exponemos los resultados básicos acerca de extensiones simples, filtros e ideales en álgebras booleanas. En esta primera parte también presentamos las clases de las álgebras libres y superatómicas, estos tipos de álgebras aportan valiosos ejemplos para el resto del material. Asimismo, incluimos un breviario sobre la dualidad de Stone: herramienta necesaria para traducir propiedades topológicas de los espacios compactos de Hausdorff cero-dimensionales en sus contrapartes booleanas y viceversa. Finalizamos el capítulo presentando al invariante cardinal $\mathfrak{h}$, los teoremas que traten acerca de este cardinal serán necesarios para la exposición del ejemplo más importante del capítulo 3.

La segunda parte comienza con el estudio de las extensiones simples que son mínimas, es decir, aquellas para las cuales no hay álgebra propia entre el álgebra base y la extensión. Resulta que este tipo de extensiones son determinadas por cierto ideal en la base, al comienzo del segundo capítulo analizamos este ideal a detalle. Posteriormente se muestra la forma

de construir extensiones mínimas por medio de la técnica de forcing del matemático P. Koszmider. El resto del capítulo está dedicado a presentar los resultados básicos sobre las álgebras estudiadas en esta tesis. De manera especial, nos enfocamos en el comportamiento de esta clase bajo las operaciones usuales entre álgebras booleanas (productos, imágenes homomorfas, subalgebras, etc.)

El hilo conductor del tercer capítulo es presentar ejemplos de clases de álgebras que son generadas mínimamente, así como álgebras booleanas específicas que no son generadas mínimamente ($\mathcal{P}(\omega)/\operatorname{fin}$). El texto concluye con una discusión sobre árboles densos en álgebras booleanas.

CONTENTS

# CHAPTER 1: PRELIMINARIES

Most of this chapter is devoted to a presentation of the basic notions and results on set theory and Boolean algebras.

## 1.1 Set Theory

For every set $X$ and every cardinal $\kappa$ we will denote by $[X]^{<\kappa}$ the collection of all subsets of $X$ that have less than $\kappa$ elements. Moreover, $[X]^{\kappa}$ is the set of all subsets of $X$ that have size $\kappa$.

If $f$ is a function, the symbols $f``A$ or $f[A]$ shall denote the image of $A$ under $f$.

In case that $X$ and $Y$ are sets, we will write the collection of all functions from $X$ into $Y$ as ${}^{X}Y$. Moreover, if $\alpha$ is an ordinal, define ${}^{<\alpha}X := \bigcup \left\{ {}^{\beta}X : \beta < \alpha \right\}$.

## 1.2 Boolean Algebras

Here we introduce some basic results about Boolean algebras. We shall often refer to [8] and [10] for further details.

Throughout the text we adopt the definitions of *Boolean algebra* and *subalgebra* as they appear in [8, Section 7]. However, we will use $\wedge$ and $\vee$ to denote the binary operations $\cdot$ and $+$. Also, we will denote by $-a$ the complement of $a$.

Moreover, if $E$ is a subset of a Boolean algebra, then $\bigvee E$ is the least upper bound of $E$ and $\bigwedge E$ is the greatest lower bound of $E$, as long as they exist. Particularly, if $E = \{x_i : i < n\}$ for some $n \in \omega$, then the symbols $\bigwedge_{i=0}^{n} x_i$ and $\bigvee_{i=0}^{n} x_i$ will be used instead of $\bigwedge E$ and $\bigvee E$, respectively.

If $B$ is a Boolean algebra, we will denote by $1_B$ and $0_B$ its maximum and its minimum, respectively. If there is no risk of confusion we will discard the subscripts. Also, $B^+ := B \setminus \{0\}$.

Unless otherwise specified, all our Boolean algebras $A$ will be *non-trivial*, i.e., $1_A \neq 0_A$.

Observe that $\mathcal{P}(X)$, the power set of $X$, is a Boolean algebra with the usual set theoretic operations (union, intersection and complements of sets). Whenever we considerer the power set as a Boolean algebra, it will always be with these operations.

**Definition 1.1.** If $B$ is a Boolean algebra, $x \in B$ will be called an *atom in* $B$ if $x > 0$ and there is no $b \in B$ satisfying $0 < b < x$.

A Boolean algebra $B$ is called *atomic* if for every $b \in B^+$, there is $x$, an atom in $B$, such that $x \leqslant b$. On the other hand, if a Boolean algebra does not have atoms it will be named *atomless.*

Let us note that if $x$ is an atom in $B$ and $b \in B$ is such that $x \nleqslant b$, then $0 < x - b \leqslant x$, where $x - b$ denotes $x \wedge (-b)$. Therefore, $x - b = x$ and in consequence, whenever $x$ is an atom in $B$ and $b \in B$, we conclude that either $x \leqslant b$ or $x \leqslant -b$ (notice that these inequalities cannot happen simultaneously).

If $B$ is a Boolean algebra, we shall denote the collection of all atoms of $B$ by $\mathrm{At}(B)$.

In order to simplify our arguments we will adopt the following notation: given $x$, an arbitrary element of a Boolean algebra, $x^0 = x$ and $x^1 = -x$.

If $A$ is a subalgebra of $B$, we shall write $A \leqslant B$. In case that $A \leqslant B$ and $A \neq B$ we will write $A < B$ to emphasize that $A$ is a proper subset of $B$.

**Definition 1.2.** Given $X$, a subset of a Boolean algebra $B$, we will denote by $\langle X \rangle_B$ the *Boolean algebra generated* by $X$ in $B$, i.e., the smallest subalgebra of $B$ containing $X$.

We will omit the subscript in $\langle X \rangle_B$ whenever we see no risk of confusion regarding our Boolean algebra.

**Definition 1.3.** For every subset $X$ of a Boolean algebra we define $X^* := \{-x : x \in X\}$.

Now we present a useful result whose proof is omitted for being a standard exercise. It can be found as a problem in [8, Exercise 7.18].

**Proposition 1.4.** *Let $X$ be a subset of the Boolean algebra $B$. For every $b \in B$, it follows that $b \in \langle X \rangle$ if and only if there exist $n \in \omega$ and $\{E_i : i < n\} \subseteq [X \cup X^*]^{<\omega}$ such that $b = \bigvee_{i<n} (\bigwedge E_i)$.*

Keeping the notation from the previous proposition, let us argue that if $X$ is infinite, then $|\langle X \rangle| = |X|$. Certainly, $X \subseteq \langle X \rangle$ implies $|X| \leqslant |\langle X \rangle|$, and $|\langle X \rangle| \leqslant \left| \left[ [X \cup X^*]^{<\omega} \right]^{<\omega} \right|$ provides us with the equality.

**Definition 1.5.** Let $A$ be a subalgebra of $B$. For $x \in B$, the Boolean algebra

$$A(x) := \langle A \cup \{x\} \rangle_B$$

will be called the *simple extension of $A$ by $x$.*

**Proposition 1.6.** *If $A$ is a subalgebra of $B$ and $x \in B$, the following statements are equivalent for each $z \in B$.*

1. *$z \in A(x)$.*
2. *There exist $a, b \in A$ such that $z = (a \wedge x) \vee (b - x)$.*
3. *There are $a, b, c \in A$ which are pairwise disjoint (i.e., $a \wedge b = a \wedge c = b \wedge c = 0$) and satisfy $z = (a \wedge x) \vee (b - x) \vee c$.*

*Proof.* $(1) \to (2)$ : Let $z \in A(x)$. We know by Proposition 1.4 that there exist $n \in \omega$ and $\{E_i : i < n\} \subseteq [A \cup \{x, -x\}]^{<\omega}$ such that $z = \bigvee_{i<n} (\bigwedge E_i)$.

Let us define $I_0 := \{i < n : x \in E_i\}$ and $J_0 := \{i < n : -x \in E_i\}$. Notice that if $i \in I_0 \cap J_0$, we have $\bigwedge E_i = 0$. By letting $I := n \setminus J_0$ and $J := n \setminus I_0$ we obtain the following:

$$\begin{aligned} z &= \bigvee_{i<n} \left( \bigwedge E_i \right) = \left( \bigvee_{i \in I} \left( \bigwedge E_i \right) \right) \vee \left( \bigvee_{i \in J} \left( \bigwedge E_i \right) \right) \\ &= \left[ x \wedge \left( \bigvee_{i \in I} \left( \bigwedge (E_i \setminus \{x\}) \right) \right) \right] \vee \left[ \left( \bigvee_{i \in J} \left( \bigwedge E_i \setminus \{-x\} \right) \right) - x \right]. \end{aligned}$$

Thus, $z$ has the desired form.

$(2) \to (3)$ : Assume $a, b \in A$ are such that $z = (a \wedge x) \vee (b - x)$. Notice that

$$\begin{aligned}
(a \wedge x) \vee (b - x) &= ([(a \wedge b) \vee (a - b)] \wedge x) \vee ([(b \wedge a) \vee (b - a)] - x) \\
&= ([(a \wedge b) \wedge x] \vee [(a - b) \wedge x]) \vee ([(b \wedge a) - x] \vee [(b - a) - x]) \\
&= [(a - b) \wedge x] \vee [(b - a) - x] \vee [(b \wedge a) - x] \vee [(a \wedge b) \wedge x] \\
&= [(a - b) \wedge x] \vee [(b - a) - x] \vee (a \wedge b)
\end{aligned}$$

Since $a - b, b - a$, and $a \wedge b$ are pairwise disjoint, we get the desired form for $z$.

$(3) \to (1)$ : Let $a, b, c \in A$ with the property that $a \wedge b = a \wedge c = b \wedge c = 0$ and $z = (a \wedge x) \vee (b - x) \vee c$. Define $E_0 = \{a, x\}$, $E_1 = \{b, -x\}$ and $E_2 = \{c\}$. Then, $E_i \in [A \cup \{x, -x\}]^{<\omega}$ for every $i \in 3$ and $z = \bigvee_{i<3} (\bigwedge E_i)$. Thus, according to Proposition 1.4, $z \in \langle A \cup \{x, -x\} \rangle$. $\square$

Before we finish this section, let us prove that homomorphic images of simple extensions are also simple extensions.

**Proposition 1.7.** *If $h : B \to C$ is a Boolean homomorphism and $A(a) \leqslant B$, then $h[A(a)] = h[A](h(a))$.*

*Proof.* Firstly, if $x \in A(a)$, then $x = (u \wedge a) \vee (v - a)$ with $u, v \in A$. So, $h(x) = (h(u) \wedge h(a)) \vee (h(v) - h(a))$. Therefore, $h[A(a)] \subseteq h[A](h(a))$.

On the other hand, fix $x \in h[A](h(a))$. There are $u, v \in h[A]$ such that $x = (u \wedge h(a)) \vee (v - h(a))$. Hence, there exist $\bar{u}, \bar{v} \in A$ in such a way that $h(\bar{u}) = u$ and $h(\bar{v}) = v$. The reverse inclusion is justified by the following equality, $x = (h(\bar{u}) \wedge h(a)) \vee (h(\bar{v}) - h(a)) = h((\bar{u} \wedge a) \vee (\bar{v} - a))$. $\square$

**Definition 1.8.** If $A$ is a subalgebra of a Boolean algebra $B$ and $x \in B$, we define the set

$$A \upharpoonright x := \{y \in A : y \leqslant x\},$$

where $\leqslant$ is the partial order of $B$.

Notice that $A \upharpoonright x$ is a Boolean algebra with $x = 1_{A\upharpoonright x}$.

## 1.3 Ideals and Filters

In this section, we introduce two important kinds of subsets of a Boolean algebra.

**Definition 1.9.** A subset $I$ of a Boolean algebra $A$ is an *ideal in* $A$ if the following are satisfied:

1. $1 \notin I$.
2. $0 \in I$.
3. If $x \in I$, $y \in A$, and $y \leqslant x$, then $y \in I$.
4. If $x, y \in I$, then $x \vee y \in I$.

According to [10], an ideal is a set $I$ which satisfies (2), (3) and (4) of the previous definition, so, when it also satisfies condition (1), $I$ is called a *proper* ideal. We do not follow this interpretation, in other words, for the rest of the text we only consider ideals in the sense of the Definition 1.9.

**Proposition 1.10.** *Let $X$ be a subset of a Boolean algebra $A$. If for every $E \in [X]^{<\omega}$ we have that $\bigvee E \neq 1$, then the set*

$$\langle X \rangle_A^i := \left\{ y \in A : \exists E \in [X]^{<\omega} \left( y \leqslant \bigvee E \right) \right\}$$

*is an ideal in $A$ which contains $X$. The set $\langle X \rangle_A^i$ will be called the ideal generated by $X$ in $A$.*

*Proof.* We have that $0 \in \langle X \rangle_A^i$ because $\emptyset \in [X]^{<\omega}$ and $\bigvee \emptyset = 0$. Besides, our hypothesis ensures that if $y \in \langle X \rangle_A^i$, then $y < 1$; hence, $1 \notin \langle X \rangle_A^i$.

Now, if $x, y \in \langle X \rangle_A^i$, there are $E_0, E_1 \in [X]^{<\omega}$ such that $x \leqslant \bigvee E_0$ and $y \leqslant \bigvee E_1$. Thus, $x \vee y \leqslant (\bigvee E_0) \vee (\bigvee E_1) = \bigvee (E_0 \cup E_1)$ and therefore, $x \vee y$ belongs to $\langle X \rangle_A^i$.

Condition (3) in Definition 1.9 is an easy consequence of the transitivity of $\leqslant$. $\square$

Now we introduce the dual notion of ideal in a Boolean algebra.

**Definition 1.11.** A subset $F$ of a Boolean algebra $A$ is a *filter in* $A$ if the following are satisfied:

1. $1 \in F$ and $0 \notin F$.

2. If $x \in F$, $y \in A$, and $x \leqslant y$, then $y \in F$.

3. If $x, y \in F$, then $x \wedge y \in F$.

Notice that if $F$ is a filter in $A$, then, for every $a \in F$, we have that $-a \notin F$.

Naturally, there is a similar result to Proposition 1.10 for filters. We omit the proof because it uses analogous arguments.

**Proposition 1.12.** *Let $X$ be a subset of a Boolean algebra $A$. If for every $E \in [X]^{<\omega}$ we have that $\bigwedge E \neq 0$, then the set*

$$\langle X \rangle_A^f := \left\{ y \in A : \exists E \in [X]^{<\omega} \left( \bigwedge E \leqslant y \right) \right\}$$

*is a filter in $A$ which contains $X$. The set $\langle X \rangle_A^f$ will be called the filter generated by $X$ in $A$.*

When a subset $X$ of a Boolean algebra satisfies that for every $E \in [X]^{<\omega}$, $\bigwedge E \neq 0$, we will say that $X$ is *centered* or that it has the finite intersection property. So, every centered set can be extended to a filter.

**Definition 1.13.** Let $F$ be a filter in a Boolean algebra $A$. $F$ is an *ultrafilter in* $A$ if, for each $a \in A$, $a \in F$ or $-a \in F$.

Suppose that $F$ is an ultrafilter in $A$ and $a \vee b \in F$. We claim that $a \in F$ or $b \in F$. Otherwise, $-a, -b \in F$ and so, $-(a \vee b) = (-a) \wedge (-b) \in F$. A contradiction.

**Lemma 1.14.** *Let $A$ be a Boolean algebra and let $X \subseteq A$ be such that $\langle X \rangle = A$. If $F$ is a filter in $A$ with the property that for every $x \in X$, $x \in F$ or $-x \in F$, then $F$ is an ultrafilter.*

*Proof.* By Proposition 1.4, each member of $A^+$ is of the form $\bigvee_{i<n} \left( \bigwedge E_i \right)$ for some $n \in \omega \setminus 1$ and $\{E_i : i < n\} \subseteq [X \cup X^*]^{<\omega}$. Let us prove by induction over $n$ that either $\bigvee_{i<n} (\bigwedge E_i) \in F$ or $-\bigvee_{i<n} (\bigwedge E_i) \in F$.

If $n = 1$, then set $a = \bigwedge E_0$. In case that $E_0 \subseteq F$, we have that $a \in F$. On the other hand, suppose that there is some $x \in E_0$ such that $x \notin F$. Since $x \in E_0$, $x \in X$ or $-x \in X$; either way, $-x \in F$ and in consequence, $-a \in F$ due to the inequality $-x \leqslant -a$.

Now, let us suppose that for every $k < n$, the equality $b = \bigvee_{i<k} (\bigwedge E_i)$ implies that $b \in F$ or $-b \in F$. Fix $a = \bigvee_{i<n+1} (\bigwedge E_i)$. Immediately, $a = \left[\bigvee_{i<n} (\bigwedge E_i)\right] \vee [(\bigwedge E_n)]$. Whenever $F \cap \left\{\bigvee_{i<n} (\bigwedge E_i), \bigwedge E_n\right\} \neq \emptyset$ we get $a \in F$ because $F$ is upward-closed. However, if $F \cap \left\{\bigvee_{i<n} (\bigwedge E_i), \bigwedge E_n\right\} = \emptyset$, by our inductive hypothesis, $-\left[\bigvee_{i<n} (\bigwedge E_i)\right], -[(\bigwedge E_n)] \in F$ and so, $-a \in F$. $\square$

It is proved in [8, Lemma 7.4] that ultrafilters are maximal filters, in the sense of our following result.

**Proposition 1.15.** *Let $F$ be a filter in $A$. $F$ is an ultrafilter in $A$ if and only if there is no filter in $A$ having $F$ as a proper subset.*

We can define maximal ideal in a similar way as we just did with ultrafilters.

**Definition 1.16.** Let $I$ be an ideal in Boolean algebra $A$. $I$ is a *maximal ideal in $A$* if, for each $a \in A$, $a \in I$ or $-a \in I$.

**Proposition 1.17.** *Let $I$ be an ideal in $A$. $I$ is a maximal ideal in $A$ if and only if there is no ideal in $A$ having $I$ as a proper subset.*

Assume that $F$ and $I$ are, respectively, a filter and an ideal in some Boolean algebra $A$. It is straightforward to prove that, actually, $F^*$ is an ideal and $I^*$ is a filter (see Definition 1.3). Moreover, if $F$ is an ultrafilter and $I$ is a maximal ideal, then $F^*$ is a maximal ideal and $I^*$ is an ultrafilter. We will refer to $F^*$ and $I^*$ as the *dual ideal of $F$* and the *dual filter of $I$*, respectively.

Keeping the notation from the previous paragraph: we shall show that $\langle I \rangle_A = I \cup I^*$. Evidently, $\langle I \rangle_A \supseteq I \cup I^*$. To prove the reverse inclusion it is enough to justify that $I \cup I^*$ is a subalgebra of $A$. Let us start by noting that $0 \in I$ and so $1 \in I^*$. If $x, y \in I \cup I^*$, we have to show that $x \wedge y$ and $x \vee y$ also belong to $I \cup I^*$. In case that $x, y \in I$ or $x, y \in I^*$, it follows immediately. Finally, if $x \in I$ and $y \in I^*$, we use that $x \wedge y \leqslant x$ and $y \leqslant x \vee y$.

A useful corollary of the argument given in the preceding paragraph is the following result.

**Proposition 1.18.** *Let $A$ be a Boolean algebra and $I$ be an ideal in $A$. Then, $I$ is maximal if and only if $\langle I \rangle_A = A$.*

Observe that if $A \leqslant B$ and $F$ is a filter in $A$, then $F$ is a centered subset of $B$, so, $F$ generates a filter in $B$. Consequently, by the Axiom of Choice, we have $U$, an ultrafilter in $B$, such that $F \subseteq U$. When this happens we will say that $U$ extends $F$ or that $F$ is extended by $U$.

## 1.4 Free Boolean Algebras

In this section we present the definitions of free Boolean algebra and independent subset. However, we will omit several proofs and recommend the reader to consult [10, Section 9.1].

**Definition 1.19.** If $A$ is a Boolean algebra, then $X \subseteq A$ is *independent in $A$* if for every $F_0, F_1 \in [X]^{<\omega}$ such that $F_0 \cap F_1 = \emptyset$, $(\bigwedge F_1) - \bigvee F_0 > 0$.

Notice that whenever $X$ is independent, it happens that $0 \notin X$ because of the equality $(\bigwedge \{0\}) - \bigvee \emptyset = 0$. A similar argument shows that $1 \notin X$. Moreover, if $x \in X$, $-x \notin X$ because $\bigwedge \{x, -x\} - \bigvee \emptyset = 0$. Therefore, $X \cap X^* = \emptyset$.

**Definition 1.20.** $A$ is a *free Boolean algebra with $\kappa$ generators* if there is $X \subseteq A$, an independent subset of $A$, with $|X| = \kappa$ and such that $\langle X \rangle_A = A$.

It is a remarkable fact that two free Boolean algebras with the same number of generators are isomorphic. The proof can be checked at [10, Lemma 9.2 and Proposition 9.4]. We shall denote the free Boolean algebra with $\kappa$ generators by $\mathrm{Fr}(\kappa)$.

The following propositions are well known results about free Boolean algebras. Their proofs can be checked at [10, Proposition 9.11] and [10, Proposition 9.16], respectively.

**Proposition 1.21.** *If $\kappa$ is an infinite cardinal, then $\mathrm{Fr}(\kappa)$ is atomless.*

**Proposition 1.22.** *Let $\kappa$ be a regular uncountable cardinal. If $A$ is a free Boolean algebra and $X \subseteq A$ is such that $|X| = \kappa$, then $A$ has an independent subset of size $\kappa$ contained in $X$.*

**Definition 1.23.** If $B$ is a Boolean algebra and $A < B$, we say that $x \in B \setminus A$ is *independent of* $A$ if for every $a \in A$ such that $a \wedge x = 0$ or $a - x = 0$, it happens that $a = 0$.

A forthwith consequence of the previous definition is that whenever $x$ is independent of $A$, we have that $A \restriction x = \{0\}$ and $A \restriction (-x) = \{0\}$ (see Definition 1.8).

**Lemma 1.24.** *Let $A$ be a Boolean algebra. If $X \subseteq A$ is independent and $x \in X$, then $x$ is independent of $\langle X \setminus \{x\} \rangle_A$.*

*Proof.* Set $H := X \setminus \{x\}$. Since $X$ is independent and $x \in X$, we get $x \neq 0$. Hence, by showing that $a - x \neq 0$ and $a \wedge x \neq 0$, for each $a \in \langle H \rangle \setminus \{0\}$, we will prove that $x \notin \langle H \rangle$ and $x$ is independent of $\langle H \rangle$.

Fix $n \in \omega$ and $\{E_i : i < n\} \subseteq [H \cup H^*]^{<\omega}$ in such a way that $a = \bigvee_{i<n} \bigwedge E_i$. The assumption $a \neq 0$ implies that, for some $k < n$, $\bigwedge E_k > 0$. Define $F_0 := H \cap E_k$ and $F_1 := H \cap (E_k)^*$ to get a pair of finite subsets of $H$. Observe that if $b \in F_1$, then $-b \in E_k$ and so, $b \notin E_k$; in other words, $F_0 \cap F_1 = \emptyset$. Therefore,

$$\begin{aligned} x \wedge a &\geqslant x \wedge \bigwedge E_k \\ &= x \wedge \left( \left( \bigwedge F_0 \right) \wedge \left( \bigwedge F_1^* \right) \right) \\ &= \bigwedge (F_0 \cup \{x\}) - \bigvee F_1 \neq 0 \end{aligned}$$

and, similarly, $a - x \geqslant (\bigwedge E_k) - x = (-x) \wedge \bigwedge E_k \neq 0$. □

**Lemma 1.25.** *Let $A$ be a subalgebra of $B$. If $x \in B \setminus A$ is independent of $A$, then for each $y \in A(x) \setminus A$ there is $c \in A$ such that $A \restriction y = A \restriction c$.*

*Proof.* Let $y \in A\,(x)$. By Proposition 1.6 there are $a, b, c \in A$, pairwise disjoint and in such a way that $y = (a \wedge x) \vee (b - x) \vee c$. We claim that $A \restriction y = A \restriction c$. Certainly, the inclusion $A \restriction c \subseteq A \restriction y$ is fairly trivial.

Fix $z \in A \upharpoonright y$, and notice that we need to prove that $w := z - c = 0$. Immediately, $w \leqslant (a \wedge x) \vee (b - x)$, which implies that $w \wedge a \leqslant a \wedge x$. Hence, $w \wedge a \in A \upharpoonright x$ and so, $w \wedge a = 0$. In a similar way it can be proved that $w \wedge b = 0$. Additionally, $w \leqslant a \vee b$. Then, $w = w \wedge (a \vee b) = (w \wedge a) \vee (w \wedge b) = 0$. $\square$

For every topological space $X$ let us denote by $\operatorname{co}(X)$ the collection of all subsets of $X$ that are simultaneously closed and open. Straightforward arguments show that $\operatorname{co}(X) \leqslant \mathcal{P}(X)$.

**Proposition 1.26.** *If* $2^\omega := \prod_{n\in\omega} \{0,1\}$, *where* $\{0,1\}$ *is discrete and* $2^\omega$ *has the product topology, then* $\operatorname{co}(2^\omega)$ *is isomorphic to* $\operatorname{Fr}(\omega)$.

*Proof.* For every $s \in {}^{<\omega}2$ define $[s] := \{x \in 2^\omega : s \subseteq x\}$. Straightforward arguments show that $\mathcal{B} := \{[s] : s \in {}^{<\omega}2\}$ is a base for $2^\omega$. Moreover, $\mathcal{B} \subseteq \operatorname{co}(2^\omega)$. Certainly, fix $s \in {}^{<\omega}2$. If $x \in 2^\omega \setminus [s]$, then $s \nsubseteq x$, in other words, there is $n < |s|$ such that $s\,(n) \neq x\,(n)$. Set $t := x \upharpoonright (n+1)$ to get $x \in [t]$. We claim that $[t] \subseteq 2^\omega \setminus [s]$. Indeed, if $y \in [t]$ we have that $x \upharpoonright (n+1) \subseteq y$ and so, $y\,(n) \neq s\,(n)$; therefore, $y \in 2^\omega \setminus [s]$.

We will argue that $\operatorname{co}(2^\omega) = \langle \mathcal{B} \rangle_{\mathcal{P}(2^\omega)}$. Let $U \in \operatorname{co}(2^\omega)$ and get $B \subseteq \mathcal{B}$ such that $U = \bigcup B$. Given that $U$ is closed and $2^\omega$ is compact, there is $E \in [B]^{<\omega}$ in such a way that $U = \bigcup E$. As a consequence, $\operatorname{co}(2^\omega) = \left\{\bigvee F : F \in [\mathcal{B}]^{<\omega}\right\}$. Thus, we can apply Proposition 1.4 to get that $\mathcal{B}$ generates $\operatorname{co}(2^\omega)$.

Let us consider the family $\mathcal{E} := \{[\{(n,0)\}] : n \in \omega\}$. Notice that $[\{(n,0)\}] \neq [\{(m,0)\}]$ whenever $m < n < \omega$. Hence, $|\mathcal{E}| = \omega$.

We claim that $\mathcal{E}$ is independent, so take $F_0, F_1 \in [\omega]^{<\omega} \setminus \{\emptyset\}$ with $F_0 \cap F_1 = \emptyset$. To verify that $\bigwedge_{n\in F_0} [\{(n,0)\}] \setminus \bigvee_{n\in F_1} [\{(n,0)\}] > 0$, consider $x := (F_0 \times \{0\}) \cup ((\omega \setminus F_0) \times \{1\})$. Immediately, $x \in \bigcap_{n\in F_0} [\{(n,0)\}] \setminus \bigcup_{n\in F_1} [\{(n,0)\}]$.

By showing that $\mathcal{B} \subseteq \langle \mathcal{E} \rangle_{\mathcal{P}(2^\omega)}$, we will prove that $\operatorname{co}(2^\omega)$ is generated by a countable independent set and we will be done. To do so, just notice that if $s \in {}^{<\omega}2$, then $[s] = \left(\bigwedge_{n\in s^{-1}\{0\}} [\{(n,0)\}]\right) \wedge \left(\bigwedge_{n\in s^{-1}\{1\}} -[\{(n,0)\}]\right) \in \langle \mathcal{E} \rangle_{\mathcal{P}(2^\omega)}$. $\square$

## 1.5 Stone Duality

Every Boolean algebra can can be used to produce a topological space, its Stone space. This section will mention some useful results about Boolean algebras and their dual topological spaces.

**Definition 1.27.** Let $A$ be a Boolean algebra and denote by $\mathrm{S}(A)$ the collection of all ultrafilters in $A$. Then, for every $a \in A$ define

$$a_A^- := \{F \in \mathrm{S}(A) : a \in F\}.$$

It can be proved that $\left\{a_A^- : a \in A\right\}$ is base for some topology on $\mathrm{S}(A)$. Whenever there is no risk of confusion we shall write simply $a^-$ instead of $a_A^-$.

**Definition 1.28.** The *Stone space of the Boolean algebra* $A$ is the topological space which results of endowing the set $\mathrm{S}(A)$ with the topology which has $\{a^- : a \in A\}$ as a base.

Now let us recall some basic definitions of general topology.

**Definition 1.29.** A topological space $X$ will be called

1. *scattered* if every $Y \in \mathcal{P}(X) \setminus \{\emptyset\}$ has an isolated point in $Y$, i.e., there is $y \in Y$ such that $\{y\}$ is open in $Y$, and

2. *zero-dimensional* if $\mathrm{co}(X)$ is a base for $X$.

It is a well known fact that the Stone space of any Boolean algebra is compact, Hausdorff and zero-dimensional. Another basic fact is that a Boolean algebra $A$ has an atom as long as $\mathrm{S}(A)$ has an isolated point. In a precise way, $c \in \mathrm{At}\,(A)$ if and only if $c^- = \{F\}$, where $F = \{a \in A : c \leqslant a\}$ (see Lemma 1.35 below and [10, Proposition 7.18]).

Recall that ${}^{<\omega}2 = \bigcup_{n\in\omega} {}^{n}\{0,1\}$. Moreover, for every $s \in {}^{<\omega}2$ and $i < 2$ define $s^\frown i := s \cup \{(\mathrm{dom}\, s, i)\}$.

The following results appear in [6, Exercise 3.B.4(c)].

**Lemma 1.30.** *Let $X$ be the Stone space of a Boolean algebra. If there is $Y \subseteq X$ which has no isolated points, then there exist a family $\{F_s : s \in {}^{<\omega}2\} \subseteq \mathrm{co}(X)$ in such a way that the following conditions hold for each $s \in {}^{<\omega}2$.*

1. $F_{\emptyset} = X$,

2. $F_s = F_{s^\frown 0} \cup F_{s^\frown 1}$,

3. $F_{s^\frown 0} \cap F_{s^\frown 1} = \emptyset$, *and*

4. $|F_s \cap Y| \geqslant \omega$.

*Proof.* Proceeding by induction over the levels of ${}^{<\omega}2$. Start by defining $F_{\emptyset} := X$ and notice that $Y$ is a $T_1$ space. It is a well known result of general topology that every $T_1$ space without isolated points is infinite. Therefore, $|F_{\emptyset} \cap Y| = |Y| \geqslant \omega$.

Next, fix an $s \in {}^{<\omega}2$ and assume that we have constructed $F_s$. Given that $|F_s \cap Y| \geqslant \omega$, fix $x, y \in F_s \cap Y$ in such a way that $x \neq y$. Use that $X$ is Hausdorff to get $U \in \mathrm{co}(X)$ such that $x \in U$ and $y \notin U$. Define $F_{s^\frown 1} := U \cap F_s$ and $F_{s^\frown 0} := F_s \setminus U$. Straightforward arguments prove that properties (2) and (3) hold.

Finally, $F_{s^\frown 1}$ is neighborhood of $x$, which is a limit point of $Y$. Then, $|F_{s^\frown 1} \cap Y| \geqslant \omega$ because $Y$ is a $T_1$ space. In a similar way it is proved that $|F_{s^\frown 0} \cap Y| \geqslant \omega$. □

**Proposition 1.31.** *Let $X$ be the Stone space of a Boolean algebra. If $X$ is non-scattered, then $2^\omega$ is a continuous image of $X$.*

*Proof.* Given that $X$ is non-scattered, there is $Y \subseteq X$ without isolated points. Then, there exists $\{F_s : s \in {}^{<\omega}2\} \subseteq \mathrm{co}(X)$ satisfying all properties of Lemma 1.30.

We claim that for every $x, y \in 2^\omega$, with $x \neq y$, $\left(\bigcap_{n\in\omega} F_{x\restriction n}\right) \cap \left(\bigcap_{n\in\omega} F_{y\restriction n}\right) = \emptyset$. Assume that $m = \min\{n \in \omega : x(n) \neq y(n)\}$. Moreover, suppose that $x(m) = 0$. Hence, $F_{x\restriction m^\frown 0} = F_{x\restriction(m+1)}$ and $F_{x\restriction m^\frown 1} = F_{y\restriction(m+1)}$; therefore (see property (3) of Lemma 1.30), $F_{x\restriction(m+1)} \cap F_{y\restriction(m+1)} = \emptyset$. To finish the proof of the claim recall that $\{F_{x\restriction n} : n \in \omega\}$ and $\{F_{y\restriction n} : n \in \omega\}$ are decreasing families.

By the previous paragraph, for every $p \in X$ we can set $f(p)$ to be the only member of $2^\omega$ satisfying that $p \in \bigcap_{n\in\omega} F_{f(p)\restriction n}$. This gives a function $f : X \to 2^\omega$. Moreover, $f$ is onto: if $x \in 2^\omega$, then $\{F_{x\restriction n} : n \in \omega\}$ is a decreasing family of closed sets in a compact space and so, there is $p \in \bigcap_{n\in\omega} F_{x\restriction n}$; hence, $f(p) = x$.

We only need to prove that $f$ is continuous. Given $t \in {}^{<\omega}2$, our concern is to show that $f^{-1}[[t]] = F_t$. If $p \in f^{-1}[[t]]$, then $f(p) \in [t]$, which is equivalent to $f(p) \supseteq t$. Therefore, $f(p) \restriction |t| = t$ and in consequence, $p \in \bigcap_{n\in\omega} F_{f(p)\restriction n} \subseteq F_{f(p)\restriction|t|} = F_t$. For the reverse inclusion set $p \in F_t$ and, seeking a contradiction, assume that there is $m \in \omega$ in such a way that $f(p)(m) \neq t(m)$ and for every $n < m$, $f(p)(n) = t(n)$. Immediately, $F_{f(p)\restriction(m+1)} \cap F_{t\restriction(m+1)} = \emptyset$, however $p \in F_{f(p)\restriction(m+1)}$ and $p \in F_t \subseteq F_{t\restriction(m+1)}$. This contradiction ends the proof. □

The next notable result is known as Stone's Representation Theorem and its proof can be checked at [10, Theorem 7.8].

**Proposition 1.32.** *If $A$ is a Boolean algebra, then $A$ is isomorphic to* $\mathrm{co}(\mathrm{S}(A))$.

Duality of homomorphisms and continuous maps of Stone spaces is summarized as follows (see [10, Theorem 8.2]).

**Proposition 1.33.** *If $A$ and $B$ are Boolean algebras, then $B$ is a homomorphic image of $A$ if and only if* $\mathrm{S}(B)$ *is homeomorphic to a subspace of* $\mathrm{S}(A)$.

## 1.6 Superatomic Boolean Algebras

Trough the text we will use several times the notion of *superatomic Boolean algebra*. This section is just a brief summary of the basic results of this class of Boolean algebras. So let us start by mentioning the following well known result (see [8, Exercise 7.22]).

**Proposition 1.34.** *Any two countable atomless Boolean algebras are isomorphic.*

We have already seen (Proposition 1.12) that whenever $X$ is a centered subset of a Boolean algebra $A$, $X$ generates a filter. In particular, if $c \neq 0$, then $\{c\}$ is a centered subset of $A$ and therefore $\{a \in A : c \leqslant a\}$ is a filter in $A$. More can be said when $c \in \mathrm{At}(A)$, as our following result shows.

**Lemma 1.35.** *Assume $A$ is a Boolean algebra, $c \in A$ and $F = \{a \in A : c \leqslant a\}$. Then, $F$ is an ultrafilter in $A$ if and only if $c \in \mathrm{At}(A)$.*

*Proof.* Let us assume that $F$ is an ultrafilter in $A$. Fix $x \in A^+$ such that $x \leqslant c$ and define $G := \{a \in A : x \leqslant a\}$. Since $G$ is a filter containing $F$, we infer that $G = F$ and so, $c \leqslant x$.

Now, we are going to suppose that $c \in \mathrm{At}(A)$ in order to prove that $F$ is an ultrafilter. Start by fixing $a \in A$. Since $c$ is an atom, we get that either $c \leqslant a$ or $c \leqslant -a$ and thus, $a \in F$ or $-a \in F$. $\square$

**Definition 1.36.** If $A$ is a Boolean algebra and $E \subseteq A$, we define

$$[E]_A := \left\{a \in A : \exists S \in [E]^{<\omega} \left(a \leqslant \bigvee S\right)\right\}.$$

Notice that if $E$ generates an ideal (in the sense of Proposition 1.10), then $[E]_A = \langle E \rangle^i_A$. If $E$ does not generate an a ideal, there is $S \in [E]^{<\omega}$ such that $\bigvee S = 1$ and in consequence, for every $a \in A$, $a \leqslant \bigvee S$. Hence, $A \subseteq [E]_A$ and so, $A = [E]_A$.

**Proposition 1.37.** *If $A$ is a Boolean algebra, then* $[\mathrm{At}(A)]_A = \{\bigvee S : S \in [\mathrm{At}(A)]^{<\omega}\}$.

*Proof.* By definition, $[\mathrm{At}\,(A)]_A \supseteq \{\bigvee S : S \in [\mathrm{At}\,(A)]^{<\omega}\}$. To prove the remaining inclusion, fix $x \in [\mathrm{At}(A)]_A$. Then, there is $S \in [\mathrm{At}(A)]^{<\omega}$ in such a way that $x \leqslant \bigvee S$. Define $G := \{a \in S : a \wedge x \neq 0\}$. Since $G \in [\mathrm{At}(A)]^{<\omega}$, we only need to show that $x = \bigvee G$. We will do this by proving two inequalities. First, for every $a \in G$ we get $a - x = a - (a \wedge x) < a$ and thus, $a - x = 0$, i.e., $a \leqslant x$. On the other hand, given that for any $E$, finite subset of $A$, we get $\bigvee E = \bigvee (E \setminus \{0\})$, we deduce that

$$x = x \wedge (\bigvee S) = \bigvee \{x \wedge a : a \in S\} = \bigvee \{x \wedge a : a \in G\} = x \wedge (\bigvee G);$$

which means that $x \leqslant \bigvee G$. $\square$

For the next definition recall that all our Boolean algebras are non-trivial.

**Definition 1.38.** Let $A$ be a Boolean algebra. If every homomorphic image of $A$ is atomic, we shall say that $A$ is *superatomic.*

It can be proved (see [10, Theorem 17.5]) that a Boolean algebra is superatomic if and only if every subalgebra of it has an atom.

Notice that every subalgebra of a superatomic Boolean algebra is also superatomic. Moreover, an easy recursive argument shows that every finite Boolean algebra is superatomic.

Superatomic Boolean algebras and their Stone spaces are related as follows (see [10, Proposition 17.8])

**Proposition 1.39.** *If $A$ is a Boolean algebra, then $A$ is superatomic if and only if* $\mathrm{S}(A)$ *is scattered.*

Now we present some relationships between superatomic Boolean algebras and independent subsets.

**Lemma 1.40.** *Let $A$ be a Boolean algebra. If $A$ is not superatomic, then $A$ contains an independent subset of size $\aleph_0$.*

*Proof.* Since $A$ is not superatomic, there is $B \leqslant A$ with $\mathrm{At}(B) = \emptyset$. In order to show that $A$ contains an independent set of size $\aleph_0$ we will argue that $\mathrm{Fr}(\omega)$ is isomorphic to some subalgebra of $B$ or, equivalently, that $\mathrm{S}(\mathrm{Fr}(\omega))$ is a continuos image of $\mathrm{S}(B)$ (see Proposition 1.33).

The fact $\mathrm{At}(B) = \emptyset$ implies that $\mathrm{S}(B)$ has no isolated points (see paragraph after Definition 1.29) and so, by Proposition 1.31, $2^\omega$ is a continuous image of $\mathrm{S}(B)$. Finally, according to [10, Proposition 9.7], $\mathrm{S}(\mathrm{Fr}(\omega))$ is homeomorphic to the topological product $2^\omega$.

□

Every superatomic Boolean algebra can be written as an increasing chain of special subsets of it. The process used to get this chain is known as *the Cantor-Bendixson derivation* and we will describe it below.

If $A$ is a Boolean algebra and $\alpha$ is an ordinal, define $I_\alpha \subseteq A$ recursively as follows (recall Definition 1.36): $I_0 := \{0\}$;

$$I_{\alpha+1} := \begin{cases} I_\alpha & \text{if the quotient algebra } A/I_\alpha \text{ is atomless} \\ \pi_\alpha^{-1}[[\mathrm{At}\,(A/I_\alpha)]_A] & \text{otherwise} \end{cases}$$

where $\pi_\alpha : A \to A/I_\alpha$ is the natural projection; and when $\alpha$ is limit, $I_\alpha := \bigcup_{\beta<\alpha} I_\beta$.

For every $\alpha$, $I_\alpha$ will be known as the *$\alpha$th Cantor-Bendixson subset of $A$*. Furthermore, an inductive argument shows that if $\alpha$ and $\beta$ are ordinals with $\alpha < \beta$, then $I_\alpha \subseteq I_\beta$.

The proof of the following result can be found in [10, Proposition 17.8] and shows that superatomic Boolean algebras can be characterized by the Cantor-Bendixson derivation process.

**Proposition 1.41.** *A Boolean algebra $A$ is superatomic if and only if $A = I_\alpha$ for some ordinal $\alpha$.*

## 1.7 The Cardinal $\mathfrak{h}$

In this section we shall define the cardinal $\mathfrak{h}$ and prove that it is a characteristic of the continuum, i.e., that $\omega_1 \leqslant \mathfrak{h} \leqslant \mathfrak{c}$. In section 3.5 $\mathfrak{h}$ will play a central role.

Given $a, b \in \mathcal{P}(\omega)$, we say that *$a$ is almost contained in $b$* (in symbols, $a \subseteq^* b$) whenever $a \setminus b \in [\omega]^{<\omega}$. On the other hand, the phrase *$a$ is compatible with $b$* means that $a \cap b \in [\omega]^\omega$; we will use the symbol $a \mid b$ to denote that relationship. Conversely, $a \perp b$ denotes that *$a$ is incompatible with $b$*, i.e., $a \cap b$ is finite. Finally, $a \subset^* b$ will be used whenever $a \setminus b \in [\omega]^{<\omega}$ and $b \setminus a \in [\omega]^\omega$.

The proof of our following result is straightforward and so, we omit it.

**Lemma 1.42.** *Given $a, b \in [\omega]^\omega$, $a \mid b$ if and only if there is $c \in [\omega]^\omega$ with $c \subseteq^* a$ and $c \subseteq^* b$.*

Consider $\mathcal{F} \subseteq [\omega]^\omega$. We will say that $X \in [\omega]^\omega$ is a *pseudointersection* of $\mathcal{F}$ whenever $X \subseteq^* F$, for every $F \in \mathcal{F}$.

The proof of our next result appears in [2, Proposition 6.4].

**Lemma 1.43.** *If $\{T_n : n \in \omega\} \subseteq [\omega]^\omega$ is such that $T_n \subseteq^* T_m$ whenever $m < n < \omega$, then $\{T_n : n \in \omega\}$ has a pseudointersection.*

For each $x \in [\omega]^\omega$, the phrase *$x_0$ and $x_1$ form a partition of $x$* means that $x_0$ and $x_1$ are infinite, disjoint and $x_0 \cup x_1 = x$.

**Definition 1.44.** Let $D \subseteq [\omega]^\omega$. We say that $D$ is *open* if for every $d \in D$ the collection $\{x \in [\omega]^\omega : x \subseteq^* d\}$ is contained in $D$. Moreover, $D$ is *dense* if for every $x \in [\omega]^\omega$ there is $d \in D$ such that $d \subseteq x$. Finally, we define the *distributivity number* as

$$\mathfrak{h} := \min\{|\mathcal{D}| : \forall\, D \in \mathcal{D}\, (D \text{ is dense and open}) \ \& \ \bigcap \mathcal{D} = \emptyset\}.$$

So far it is not clear why $\mathfrak{h}$ is well defined, so let us argue that there is a family of dense open sets with empty intersection. We will use the following notation: for each $\mathcal{A} \subseteq [\omega]^\omega$, define $\overline{\mathcal{A}} := \{x \in [\omega]^\omega : \exists\, a \in \mathcal{A}\, (x \subseteq^* a)\}$.

An infinite subset of $[\omega]^\omega$ will be called *almost disjoint* if any two different members of if are incompatible. Also, an almost disjoint family will be called *maximal* if it is not properly contained in any other almost disjoint family.

The phrase *maximal almost disjoint family* will be abbreviated by MADF.

**Lemma 1.45.** *If $D$ is a dense open family, then there is a MADF $\mathcal{A}$ such that $\overline{\mathcal{A}} \subseteq D$.*

*Proof.* Let us apply Zorn's Lemma to $\langle \mathbb{P}, \subseteq \rangle$, where $\mathbb{P} = \{\mathcal{A} \subseteq D : \mathcal{A} \text{ is almost disjoint}\}$. We claim that $\mathbb{P}$ is non empty. Indeed, let $d \in D$ and fix $f : \omega \times \omega \to d$, a bijection. Define $A := \{f\,[\{n\} \times \omega] : n \in \omega\}$. If $a \in A$, then $a \subseteq d$, and thus, $a \in D$ due to $D$ is open. So, $A$ is an almost disjoint family contained in $D$. Now, let $\mathcal{C}$ be a nonempty chain in $\mathbb{P}$. We shall verify that $\bigcup \mathcal{C} \in \mathbb{P}$. First, notice that $\bigcup \mathcal{C}$ is an infinite subset of $D$. Now, fix $x, y \in \bigcup \mathcal{C}$ and get $C_0, C_1 \in \mathcal{C}$ in such a way that $x \in C_0$ and $y \in C_1$. Since $\mathcal{C}$ is a chain, we can assume that $x, y \in C_1$ and so, $x \perp y$. Therefore, $\bigcup \mathcal{C}$ is almost disjoint. Since $\bigcup \mathcal{C}$ is an upper bound of $\mathcal{C}$, by Zorn's Lemma, there is $\mathcal{A}$ a maximal element of $\mathbb{P}$.

We shall verify that $\mathcal{A}$ is a MADF. Let $x \in [\omega]^\omega$. Use the fact that $D$ is dense to obtain $d \in D$ in such a way that $d \subseteq x$. We claim that there exists $z \in \mathcal{A}$ such that $d \mid z$. Otherwise, $\{d\} \cup \mathcal{A}$ would be a member of $\mathbb{P}$, contradicting that $\mathcal{A}$ is maximal in $\mathbb{P}$. Hence, $x \mid z$, which implies that $\mathcal{A}$ is a MADF. Finally, the inclusion $\overline{\mathcal{A}} \subseteq D$ comes as a result of $D$ being open. $\square$

**Lemma 1.46.** *If $\mathcal{A}$ is a maximal almost disjoint family, then $\overline{\mathcal{A}}$ is dense and open.*

*Proof.* Let $x \in \overline{\mathcal{A}}$ and $y \subseteq^* x$ be arbitrary and infinite. We know that there is $a \in \mathcal{A}$ such that $x \subseteq^* a$. Immediately, $y \subseteq^* a$ and we have just proved that $\overline{\mathcal{A}}$ is open.

To check that $\overline{\mathcal{A}}$ is dense, fix $x \in [\omega]^\omega \setminus \mathcal{A}$, and notice that $\mathcal{A} \cup \{x\}$ can not be almost disjoint. So, there is $a \in \mathcal{A}$ such that $a \mid x$. Therefore, $a \cap x \in \overline{\mathcal{A}}$ showing that $\overline{\mathcal{A}}$ is dense. $\square$

Let $x$ be an infinite subset of $\omega$ and enumerate it as $\{k_i : i < \omega\}$, where $k_i < k_{i+1}$ for each $i \in \omega$. Define $x_0 := \{k_{2i} : i < \omega\}$, $x_1 := x \setminus x_0$, and fix $\mathcal{A}_x$, a MADF in such a way that $x_0 \in \mathcal{A}_x$. We claim that $x \notin \overline{\mathcal{A}}_x$. Otherwise, there is $a \in \mathcal{A}_x$ such that $x \subseteq^* a$. It should be clear that $a \mid x_0$ and then $a = x_0$ because both belong to a MADF. Therefore, $x \subseteq^* x_0$ and in consequence, $x_1$ is finite; a contradiction. Thus $\left\{\overline{\mathcal{A}}_x : x \in [\omega]^\omega\right\}$ is a family of dense open sets with empty intersection. Hence, $\mathfrak{h}$ is well defined.

**Proposition 1.47.** $\aleph_1 \leqslant \mathfrak{h} \leqslant \mathfrak{c}$.

*Proof.* We have seen previously that $\mathcal{D} := \left\{\overline{\mathcal{A}}_x : x \in [\omega]^\omega\right\}$ is a family of dense open sets with empty intersection. Thus, $\mathfrak{h} \leqslant |\mathcal{D}| \leqslant \mathfrak{c}$.

Now, consider $\mathcal{E} := \{E_n : n \in \omega\}$, a family of dense open sets. Let us show that $\bigcap \mathcal{E} \neq \emptyset$ by building a sequence $\{T_n : n \in \omega\}$ such that:

(1) for every $n \in \omega$, $T_n \in E_n$, and

(2) $T_n \subseteq T_m$ whenever $m < n$.

Proceed by finite recursion. Fix $T_0 \in E_0$ and suppose that we have already defined $T_n \in E_n$. Since $E_{n+1}$ is dense, there is $T_{n+1} \in E_{n+1}$ in such a way that $T_{n+1} \subseteq T_n$. This completes the recursion.

According to Lemma 1.43, there is $X$, a pseudointersection of $\{T_n : n \in \omega\}$. We claim that $X \in \bigcap \mathcal{E}$. Certainly, fix $n \in \omega$ and note that $X \subseteq^* T_n \in E_n$. Since $E_n$ is open, $X \in E_n$.

In conclusion, $\aleph_1 \leqslant \mathfrak{h}$. $\square$

**Lemma 1.48.** *Let $\alpha < \mathfrak{h}$ be arbitrary and suppose that for every $\xi < \alpha$, $D_\xi$ is dense and open. Then, $\bigcap_{\xi<\alpha} D_\xi$ is dense and open.*

*Proof.* Let us start by showing that $\bigcap_{\xi<\alpha} D_\xi$ is open. Let $x \in \bigcap_{\xi<\alpha} D_\xi$ and $y \subseteq^* x$ be arbitrary and infinite. Fix $\eta < \alpha$ and notice that $x \in D_\eta$. Now, given that $D_\eta$ is open, $y \in D_\eta$. Hence, $y \in \bigcap_{\xi<\alpha} D_\xi$.

Now let us check that $\bigcap_{\xi<\alpha} D_\xi$ is dense. Fix $x \in [\omega]^\omega$ and define for every $\xi < \alpha$ the collection $D'_\xi := \{y \in D_\xi : y \subseteq x\}$ . Continue by fixing a bijection $f : x \to \omega$, in order to set $E_\xi := \left\{ f``y : y \in D'_\xi \right\}$ for every $\xi < \alpha$.

We claim that $\{E_\xi : \xi < \alpha\}$ is a family of dense open sets. Indeed, given $\xi < \alpha$, if $y \in D'_\xi$ and $z \in [\omega]^\omega$ are such that $z \subseteq^* f``y$, then $f^{-1}[z] \subseteq^* y \subseteq x$. Thus, $f^{-1}[z] \in D'_\xi$ because $D_\xi$ is open. The equality $f``f^{-1}[z] = z$ guarantees that $z \in E_\xi$. On the other hand, to verify that $D_\xi$ is dense suppose that $z \in [\omega]^\omega$ is arbitrary. Immediately, $f^{-1}[z] \in [\omega]^\omega$, and so, there is $y \in D_\xi$ such that $y \subseteq f^{-1}[z]$. However, $f^{-1}[z] \subseteq x$ and in consequence, $y \in D'_\xi$. Therefore, $f``y \in E_\xi$, and hence, $E_\xi$ is dense because $f``y \subseteq z$.

Since $\alpha < \mathfrak{h}$, there is $y \in \bigcap_{\xi<\alpha} E_\xi$. Then, $f^{-1}[y] \in \bigcap_{\xi<\alpha} D'_\xi \subseteq \bigcap_{\xi<\alpha} D_\xi$. Moreover, $f^{-1}[y] \subseteq x$ and so, we have just finished the proof. $\square$

Consider $\{D_\alpha : \alpha < \mathfrak{h}\}$, a family of dense open sets with empty intersection. We are going to modify this family in order to get a decreasing sequence of dense open sets of length $\mathfrak{h}$. Define for every $\alpha < \mathfrak{h}$ the collection $E_\alpha := \bigcap_{\xi\leqslant\alpha} D_\xi$. Immediately, $E_\alpha \subseteq E_\beta$ whenever $\beta < \alpha < \mathfrak{h}$. According to Lemma 1.48, $\{E_\alpha : \alpha < \mathfrak{h}\}$ is a family of dense open sets. Besides, observe that $\bigcap_{\alpha<\mathfrak{h}} E_\alpha \subseteq \bigcap_{\alpha<\mathfrak{h}} D_\alpha = \emptyset$.

**Proposition 1.49.** $\mathfrak{h}$ *is a regular cardinal.*

*Proof.* Suppose that $\gamma < \mathfrak{h}$ and let us argue that $\gamma \neq \mathrm{cf}(\mathfrak{h})$. Start by considering $\{E_\alpha : \alpha < \mathfrak{h}\}$, a decreasing sequence of dense open sets with empty intersection. Assume that $f : \gamma \to \mathfrak{h}$ is cofinal in $\mathfrak{h}$. Notice that $\bigcap_{\xi<\gamma} E_{f(\xi)} \neq \emptyset$.

Fix $x \in \bigcap_{\xi<\gamma} E_{f(\xi)}$ and let $\alpha < \mathfrak{h}$ be arbitrary. Since $f$ is cofinal, there is $\xi < \gamma$ such that $\alpha \leqslant f(\xi)$. Therefore, $x \in E_{f(\xi)} \subseteq E_\alpha$. Thus, we conclude that $x \in \bigcap_{\alpha<\mathfrak{h}} E_\alpha$; the contradiction we were looking for. $\square$

CHAPTER 2: MINIMALLY GENERATED BOOLEAN ALGEBRAS

This chapter presents the class of minimally generated Boolean algebras. Informally speaking, a Boolean algebra is minimally generated if one can construct it by small, indivisible steps. However, before we present the formal definition of a minimally generated algebra we are going to develop in the first section of this chapter one of the tools we will be using constantly.

## 2.1 The Ideal $J_A(x)$

Given a Boolean algebra $B$, along with a proper subalgebra $A$, for every $x \in B \setminus A$ there is an ideal in $B$ which describes the relationship between $x$ and the Boolean algebra $A$. We consider this ideal now.

Keeping the notation from the previous paragraph: we claim that $X := (A \upharpoonright x) \cup (A \upharpoonright (-x))$ (see Definition 1.8) satisfies the hypothesis of Proposition 1.10. If there is $E \in [(A \upharpoonright x) \cup (A \upharpoonright (-x))]^{<\omega}$ in such a way that $\bigvee E = 1$, we can assume that $E = E_0 \cup E_1$ with $E_0 \subseteq A \upharpoonright x$ and $E_1 \subseteq A \upharpoonright (-x)$. We assert that $E_0 \neq \emptyset \neq E_1$. Indeed, if $E = E_0$, then $\bigvee E_0 = 1$, but $x$ is an upper bound of $E_0$, so $x = 1 \in A$, which is a contradiction. We would do similar deductions if $E = E_1$. Now, $\bigvee E_0 < x$ and $\bigvee E_1 < -x$ imply that $x - \bigvee E_0 > 0$ and $x < -\bigvee E_1$. Hence, $x - \bigvee E_0 \leqslant (-\bigvee E_1) \wedge (-\bigvee E_0) = -\bigvee E = -1 = 0$, which is the contradiction we were looking for.

**Definition 2.1.** Let $A$ be a subalgebra of a Boolean algebra $B$. For each $x \in B \setminus A$, we define

$$J_A(x) := \langle (A \upharpoonright x) \cup (A \upharpoonright (-x)) \rangle^i_A.$$

Sometimes we will write $J_x$ instead of $J_A(x)$ to simplify our notation.

Now, let us argue that $y \in J_x$ implies the existence of $b \in A \upharpoonright x$ and $c \in A \upharpoonright (-x)$ such that $y \leqslant b \vee c$. Indeed, if $y \in J_x$, there exists $E \in [(A \upharpoonright x) \cup (A \upharpoonright (-x))]^{<\omega}$ such that $y \leqslant \bigvee E$. We can assume that $E = E_0 \cup E_1$, where $E_0 \subseteq A \upharpoonright x$ and $E_1 \subseteq A \upharpoonright (-x)$. Then, for every $z \in E_0$, we get $z \leqslant x$ and consequently, $\bigvee E_0 \in A \upharpoonright x$. By similar reasons, $\bigvee E_1 \in A \upharpoonright (-x)$.

**Lemma 2.2.** *Let $A$ be a subalgebra of some Boolean algebra $B$, $a \in A$ and $x \in B \setminus A$. Then the following statements are equivalent.*

1. $a \in J_x$.
2. $a \wedge x \in A$.
3. $a - x \in A$.
4. $\{y \in A(x) : y \leqslant a\} \subseteq A$.

*Proof.* (1) $\to$ (2) : Let $a \in J_x$. There are $b, c \in A$ such that $b \leqslant x, c \leqslant -x$, and $a \leqslant b \vee c$. We will be done if we prove that $a \wedge x = a \wedge b$. Actually,

$$\begin{aligned}
a &= a \wedge (b \vee c) = [(a \wedge x) \vee (a - x)] \wedge (b \vee c) \\
&= [(a \wedge x) \wedge (b \vee c)] \vee [(a - x) \wedge (b \vee c)] \\
&= [(a \wedge x) \wedge b] \vee [(a \wedge x) \wedge c] \vee [(a - x) \wedge b] \vee [(a - x) \wedge c] \\
&= [(a \wedge x) \wedge b] \vee [(a - x) \wedge c] = (a \wedge b) \vee [(a - x) \wedge c]
\end{aligned}$$

and therefore

$$a \wedge x = ((a \wedge b) \vee [(a - x) \wedge c]) \wedge x = a \wedge b.$$

(2) $\to$ (3) : Given that $a - (a \wedge x) = a - x$, we conclude that (3) is a consequence of (2).

(3) $\to$ (4) : Start by noticing that the assumption $a - x \in A$ implies $a \wedge x \in A$ because $a - (a - x) = a \wedge x$.

Let $y \in A\,(x)$ be such that $y \leqslant a$. We know from Proposition 1.6 that there are $b, c \in A$ such that $y = (b \wedge x) \vee (c - x)$. Since $y \leqslant a$, we obtain that $y = y \wedge a$. In consequence,

$$y = [(b \wedge x) \vee (c - x)] \wedge a = [(a \wedge x) \wedge b] \vee [(a - x) \wedge c].$$

This finishes the implication because $a \wedge x$ and $a - x$ are elements of $A$.

$(4) \rightarrow (1)$ : Notice that $a - x$ and $a \wedge x$ are members of $A$ because both are below $a$. In particular, $a - x \in A \upharpoonright x$ and $a \wedge x \in A \upharpoonright (-x)$. To finish we only have to observe that $a \leqslant (a - x) \vee (a \wedge x)$. □

As we noticed earlier, $J_x$ is an ideal in $A$, but the last lemma implies that $J_x$ is, actually, an ideal in $A\,(x)$. Indeed, the only non-trivial property of Definition 1.9 is (2). Let $b \in J_x$ and $c \in A\,(x)$ be in such a way that $c \leqslant b$; hence, $c \in A$ and therefore $c$ is also an element of $J_x$.

## 2.2 Minimal Extensions

Now we are interested in Boolean algebras that are minimal over some of their subalgebras in the sense of our next definition.

**Definition 2.3.** Let $B$ be a Boolean algebra and $A \leqslant B$. We will say that $B$ is *minimal over* $A$ if there is no subalgebra of $B$ lying properly between $A$ and $B$.

The symbol $A \leqslant_m B$ is used to denote that $B$ is minimal over $A$. As always, when $A$ is a proper subset of $B$, we shall write $A <_m B$.

Now, if $A <_m B$ and $x \in B \setminus A$, then $A < A(x) \leqslant B$ and so, $B = A\,(x)$. In other words, every minimal extension of a Boolean algebra turns out to be a simple extension, but the reverse implication is false as showed by our next example.

**Example 2.3.1.** *Consider* $X = \left\{ t + \dfrac{1}{2^n} : t \in \{0,1\} \ \ \& \ \ n \in \omega \right\} \cup \{0\}$ *as a topological subspace of the real line. Let us show that if* $A = \mathrm{co}(X)$*, then there are* $x, y \in \mathcal{P}\,(X)$ *such that* $A < A\,(y) < A\,(x)$.

*Assume that $t \in \{0,1\}$, and notice that $\{t + 2^{-n} : n \in \omega\}$ converges to $t$ an so, for every $a \in A$ such that $t \in a$, we get that $|a| = \omega$. Therefore, if we define $x := \{0,1\}$ and $y := \{0\}$, it happens that $\{x, y\} \cap A = \emptyset$.*

*Obviously $A \leqslant A(y)$. Moreover, we already saw that $y \notin A$ and in consequence, $A < A(y)$. Let us show that $A(y) \leqslant A(x)$, i.e., that $y \in A(x)$. Set $a := \{1 + 2^{-n} : n \in \omega\} \cup \{1\} \in A$ and notice that $X \cap (1/2, \infty) = a = X \cap [3/4, \infty)$. Now, being $y = x - a$, it follows that $y \in A(x)$.*

*It remains to show that $x \notin A(y)$. Proceed by an indirect argument, i.e., suppose that $x = (b \wedge y) \vee (c - y)$ for some $b, c \in A$. Since $1 \notin y$, $1 \in c - y$. In particular, $1 \in c$ implies that $\omega = |c| = |c - y| \leqslant |x|$, which is clearly a contradiction.*

**Lemma 2.4.** *Assume that $A \leqslant B$ and $x \in B \setminus A$. If for every $y \in A(x)$ we have that $y \in A$ or $x \in A(y)$, then $A <_m A(x)$.*

*Proof.* We have to confirm that if $C$ satisfies $A < C \leqslant A(x)$, then $C = A(x)$, so fix $a \in C \setminus A$. Hence, $a \in A(x)$ and by our hypothesis, $x \in A(a)$. But $A(a) \subseteq C$, so $x \in C$. Therefore, $A(x) \subseteq C$. □

**Corollary 2.5.** *Let $A$ be a proper subalgebra of $B$. If for every $x, y \in B \setminus A$ we have that $y \in A(x)$, then $A <_m B$.*

*Proof.* Start by fixing $x \in B \setminus A$. Notice that $B \supseteq A(x)$. Let us show that this sets are equal. Given $y \in B$, if $y \in A$, we obtain $y \in A(x)$. When $y \in B \setminus A$, our hypothesis implies that $y \in A(x)$. Therefore, $B = A(x)$.

Finally, we apply the previous proposition to get the result. □

The following proposition provides us with some equivalences for a simple extension to be minimal over the Boolean algebra that was extended from.

**Proposition 2.6.** *Let $A < B$ and $x \in B \setminus A$. The following are equivalent.*

1. *$A <_m A(x)$.*

*2. $J_x$ is a maximal ideal of $A$.*

*3. Exactly one ultrafilter of $A$ can be extended to more than one ultrafilter in $A(x)$.*

*4. $A = \langle\{y \in A : y$ is comparable with $x\}\rangle_A$.*

*5. There is $G \subseteq A$ such that $\langle G\rangle_A = A$ and for every $y \in G$, $y$ is comparable with $x$.*

*Proof.* (1) $\to$ (2) : By contrapositive, suppose that $J_x$ it is not a maximal ideal in $A$. Then, there is $a \in A$ such that neither $a$ nor $-a$ are elements of $J_x$. We will prove that $A < A(a \wedge x) < A(x)$.

Due to Proposition 2.2, $a \notin J_x$ implies $a \wedge x \notin A$. Then, $A < A(a \wedge x)$. Now we will prove that $x \notin A(a \wedge x)$. Let us proceed by contradiction, if $x \in A(a \wedge x)$, there are $b, c \in A$ satisfying

$$x = [b \wedge (a \wedge x)] \vee [c - (a \wedge x)] = [a \wedge (b \wedge x)] \vee (c - a) \vee (c - x).$$

Since $c - x \leqslant -x$, we get $c - x = 0$. Then $x = [a \wedge (b \wedge x)] \vee (c - a)$ and hence, $x - a = c - a \in A$. Thus, by Proposition 2.2 we conclude that $-a \in J_x$, contradicting our assumption.

(2) $\to$ (3) : Our objective will be to prove that $J_x^*$ is the only ultrafilter that can be extended to two different ultrafilters in $A(x)$.

Firstly, we assert that $J_x^* \cup \{x\}$ is centered. Indeed, let $E \in [J_x^* \cup \{x\}]^{<\omega}$. As we know, we have to show that $\bigwedge E \neq 0$. This assertion follows immediately when $x \notin E$ or $E = \{x\}$ (recall that $x \neq 0$).

Now, if $x \in E$ and $E \neq \{x\}$, then $E \backslash \{x\} \in \left([J_x^*]^{<\omega}\right) \backslash \{\emptyset\}$. Hence, $y := \bigwedge (E \setminus \{x\}) \in J_x^*$ (i.e., $-y \in J_x$) and $x \wedge y = \bigwedge E$. If $x \wedge y = 0$, then $x \leqslant -y$ and therefore, $x \in J_x \subseteq A$, contradicting our assumption on $x$. By similar arguments one can show that $J_x^* \cup \{-x\}$ is centered.

Therefore, there are $F_0, F_1$, ultrafilters in $A(x)$, such that $x \in F_0$, $-x \in F_1$, and $J_x^* \subseteq F_0 \cap F_1$.

To conclude, we have to verify that for every ultrafilter $U$ in $A$, different from $J_x^*$, $G := \langle U \rangle_{A(x)}^f$ is an ultrafilter in $A(x)$. Firstly, $U \setminus J_x^* \neq \emptyset$ due to Proposition 1.15, so fix $a \in U \setminus J_x^*$. Notice that $a \in J_x$ and by Lemma 2.2 we get that $a \wedge x$ and $a - x$ belong to $A$.

Now we are going to use Proposition 1.15 to prove that $G$ is an ultrafilter. Let $V$ be an ultrafilter in $A(x)$ with $G \subseteq V$. If $z \in V$, we know that there are $b, c \in A$ such that $z = (b \wedge x) \vee (c - x)$. Thus, $z \wedge a = [b \wedge (a \wedge x)] \vee [c \wedge (a - x)]$. We conclude that $z \wedge a \in A$. On the other hand, the fact $a, z \in V$ implies that $z \wedge a \in V$. Hence, $z \wedge a \in V \cap A$. As one easily checks, $V \cap A$ is a filter in $A$ containing $U$ and so, $V \cap A = U$. Therefore, $z \wedge a \in U$ and since $z \wedge a \leqslant z$, we deduce that $z \in G$. In conclusion, $V \subseteq G$.

(3) $\rightarrow$ (2) : Seeking a contradiction, suppose that $J_x$ is not a maximal ideal in $A$. So, there is $a \in A$ with $J_x \cap \{a, -a\} = \emptyset$.

The argument used for the implication (2) $\rightarrow$ (3) can be modified to show that for every $i < 2$, $J_x^* \cup \left\{a^i\right\}$ is a centered subset of $A$.

Let $U_0$ and $U_1$ be two ultrafilters in $A$ such that for every $i < 2$, $J_x^* \cup \left\{a^i\right\} \subseteq U_i$. We claim that for every $i, j < 2$, $U_j \cup \left\{x^i\right\}$ is a centered subset of $A(x)$. Observe that this claim guarantees that there are at least two filters in $A$ which can be extended to more than one ultrafilter in $A(x)$.

Following our previous arguments, we only we have to check that for every $z \in U_j$, $z \wedge x^i \neq 0$. By contrapositive, assume that $z \in A$ satisfies $z \wedge x^i = 0$. Then, $-x^i \geqslant z$ and, as a consequence, $z \in J_x$. Therefore, $-z \in J_x^* \subseteq U_j$ and so, $z \notin U_j$.

(2) $\rightarrow$ (4) : Considering Proposition 1.18 we get that $J_x$ generates $A$ and therefore it will be enough to prove that $\langle J_x \rangle_A = \langle E \rangle_A$, where $E := \{a \in A : a \text{ is comparable with } x\}$

First, let us check that $E \subseteq \langle J_x \rangle_A$. Fix $a \in E$, we have that $a \leqslant x$ or $x \leqslant a$ and therefore, $a \in (A \upharpoonright x)$ or $-a \in (A \upharpoonright (-x))$; in both cases it follows that $a \in \langle J_x \rangle_A$. Now, let $C \leqslant B$ such that $E \subseteq C$. When $a \in J_x$, we get $a \wedge x \in C$ (see Proposition 2.2) and since $a - x \leqslant -x$, we also get $-(a - x) \in C$. Thus, $a = (a \wedge x) \vee (a - x) \in C$; hence $\langle J_x \rangle_A \subseteq C$.

$(4) \rightarrow (5)$ : It is straightforward.

$(5) \rightarrow (1)$ : Denote by $E$ the set of all members of $A$ which are comparable with $x$. Since we are assuming (5), $E$ generates $A$. On the other hand, we already proved in $(2) \rightarrow (4)$ that $J_x$ and $E$ generate the same subalgebra of $A$. Therefore, by Proposition 1.18, $J_x$ is a maximal ideal.

By Lemma 2.4, to prove (1), we only need to show that for every $y \in A\,(x)$, $y \in A$ or $x \in A\,(y)$. According to Proposition 1.6, $y = (a \wedge x) \vee (b - x) \vee c$, for some $a, b, c \in A$ which are pairwise disjoint.

If it happens that $a$ and $b$ are elements of $J_x$, Lemma 2.2 gives $y \in A$. In case that $a \notin J_x$, as a result of $J_x$ being maximal, $-a \in J_x$. Now, $a \wedge (y - x) = (a \wedge y) - x = (a \wedge x) - x = 0$. This implies that $y - x \leqslant -a$. In a pretty similar way we can deduce that $x - y \leqslant -a$. Therefore, $x \bigtriangleup y \leqslant -a$. By Lemma 2.2.(4) we get that $x \bigtriangleup y \in A$. Finally, if we let $d := x \bigtriangleup y$, then $d, -d \in A$ and $x = d \bigtriangleup y = (y \wedge (-d)) \vee (d - y)$. Thus, $x \in A\,(y)$ by Proposition 1.6.

On the other hand, if $b \notin J_x$ proceed as in previous paragraph to get $(-x) \bigtriangleup y \in A$ and therefore,$-x \in A\,(y)$. $\square$

**Corollary 2.7.** *Assume that $E$ is a subset of the Boolean algebra $B$. Set $A = \langle E \rangle$ and fix $x \in B \setminus A$. Then, the following statements are equivalent.*

1. *$A <_m A\,(x)$*
2. *For each $e \in E$, $A \cap \{e \wedge x, x - e\} \neq \emptyset$.*

*Proof.* Proposition 2.6 ensures that (1) holds if and only if $J_x$ is maximal ideal in $A$, i.e., $J_x^*$ is an ultrafilter in $A$. Now, this last statement is, according to Lemma 1.14, equivalent to: for any $e \in E$, $e \in J_x^*$ or $-e \in J_x^*$, in other words, $J_x \cap \{e, -e\} \neq \emptyset$ for all $e \in E$. The final step in our argument is to recall Lemma 2.2 in order to obtain that $J_x \cap \{e, -e\} \neq \emptyset$ is equivalent to $A \cap \{x \wedge e, x - e\} \neq \emptyset$ for all $e \in E$. $\square$

Although the next result is fairly trivial, it illustrates an application of Proposition 2.6 and it will be used several times trough the text.

**Corollary 2.8.** *Let $B$ be a Boolean algebra. If $A < B$ and $x \in \mathrm{At}(B) \setminus A$, then $A <_m A(x)$.*

*Proof.* Let $G := \{y \in A : x \leqslant y\}$. Straightforward arguments show that $G$ is a filter in $A$. Moreover, the fact $x \in \mathrm{At}(B)$ implies that for every for every $a \in A$, $x \leqslant a$ or $x \leqslant -a$ and in consequence, $a \in G$ or $-a \in G$. So, $G$ is an ultrafilter in $A$ and all its elements are comparable with $x$.

Now, a routine modification to the argument used to prove Proposition 1.18 shows that every ultrafilter in $A$ generates $A$. In particular, $\langle G \rangle_A = A$. So, by Proposition 2.6, $A <_m A(x)$. □

## 2.3 The Forcing of Koszmider

The material we present in this section follows the basic definitions and notation of the classic textbook [13].

If we consider a Boolean algebra $A$ in the ground model $M$, our main goal is to find a generic extension $M[G]$ such that $M[G] \models A <_m A(g)$, for some $g$ that depends on the generic filter $G$.

**Definition 2.9.** Assume $X$ is a set and let $A$ be a subalgebra of $\mathcal{P}(X)$. If $F$ is a filter in $A$, define *the forcing of Koszmider* as the collection:

$$\mathbb{P}(A, F) := \left\{ (p_0, p_1) \in A^2 : p_0 \cap p_1 = \emptyset \;\;\&\;\; p_0 \cup p_1 \notin F \right\},$$

and for every $(p_0, p_1), (q_0, q_1) \in A^2$, $(p_0, p_1) \leqslant (q_0, q_1)$ if and only if $q_0 \subseteq p_0$ and $q_1 \subseteq p_1$.

It is routine to check that whenever $A$ is a Boolean algebra and $F \subseteq A$ is a filter, $\langle \mathbb{P}(A, F), \leqslant \rangle$ is partial order in the sense of [13, Definition 2.1, p. 52].

Through this section, $X, A$ and $F$ will be like in the previous definition.

**Lemma 2.10.** *Let $G$ be a $\mathbb{P}(A, F)$-generic filter, and define*

$$g := \bigcup \{p_0 : \exists p_1 \in A\, (p_0, p_1) \in G\}.$$

*Then, for every $(p_0, p_1) \in G$ we get the inclusions $p_0 \subseteq g \subseteq -p_1$.*

*Proof.* Let $(p_0, p_1) \in G$. The inclusion $p_0 \subseteq g$ follows by definition. To show that $g \subseteq -p_1$, suppose that there is $x \in g \cap p_1$. Since $x \in g$, there exists $(q_0, q_1) \in G$ such that $x \in q_0$. On the other hand, since $G$ is a filter, there is $(r_0, r_1) \in G$ in such a way that $(r_0, r_1) \leqslant (p_0, p_1), (q_0, q_1)$. Immediately, $q_0 \subseteq r_0$ and $p_1 \subseteq r_1$, and this implies that $x \in r_0 \cap r_1$, which is impossible. □

From now on, consider $g$ as in the previous result. For our following result, familiarity with the material presented in Section 2.1 is assumed.

**Proposition 2.11.** *Let $M$ be a countable transitive model of ZFC and $F$ be a non-principal (i.e., $F \cap \mathrm{At}(A) = \emptyset$) ultrafilter in $A$. If $G$ is a $\mathbb{P}(A, F)$-generic filter over $M$, then*

1. *$g \notin A$ and*
2. *$M[G] \models J_g = F^*$, i.e., the equality $J_g = F^*$ holds in $M[G]$.*

*Proof.* $(1)$ : For every $a \in F$ define

$$E_a := \{(p_0, p_1) \in \mathbb{P}(A, F) : \forall i \in 2\, (a \cap p_i \neq \emptyset)\}.$$

We shall show that $E_a$ is a dense subset of $\mathbb{P}(F, A)$. Let $(p_0, p_1) \in \mathbb{P}(A, F)$ be arbitrary. Immediately, $a - (p_0 \cup p_1) \in F$ (recall that $F$ is an ultrafilter). Since $F$ contains no atoms, there are $c_0, c_1 \in A^+$ such that $c_0 \cap c_1 = \emptyset$ and $c_0 \cup c_1 = a - (p_0 \cup p_1)$. From the fact that $F$ is a centered family we get that $c_0 \notin F$ or $c_1 \notin F$. Let us suppose that $c_0 \notin F$ (i.e., $-c_0 \in F$). We claim that $c_1 \in F$. Otherwise, $-(c_0 \cup c_1) = (-c_1) \cap (-c_1) \in F$, which contradicts the fact $c_0 \cup c_1 \in F$.

Apply once again that $F$ is disjoint from $\mathrm{At}(A)$ to get $d_0, d_1 \in A^+$ in such a way that $d_0 \cap d_1 = \emptyset$ and $d_0 \cup d_1 = c_1$. Proceeding as in the previous paragraph, we can assume that $d_0 \notin F$ and $d_1 \in F$. By letting $r_0 := p_0 \cup c_0$ and $r_1 := p_1 \cup d_0$ we obtain $r_0 \cap r_1 = \emptyset$ and, moreover, since $F$ is an ultrafilter, $r_0 \cup r_1 \notin F$. Thus, $(r_0, r_1)$ is a member of $\mathbb{P}(F, A)$ which extends $(p_0, p_1)$. Also, $c_0 \subseteq r_0 \cap a$ and $d_0 \subseteq r_1 \cap a$, i.e., $(r_0, r_1) \in E_a$.

Employ an indirect argument to show that $g \notin A$, i.e., suppose that $g \in A$. Thus, $g \in F$ or $-g \in F$. In case that $g \in F$, there is $(p_0, p_1) \in G \cap E_g$. By Lemma 2.10, $p_0 \subseteq g \subseteq -p_1$. Nonetheless, $(p_0, p_1) \in E_g$ implies that $p_1 \cap g \neq \emptyset$, against the previous result. When $-g \in F$ similar arguments apply.

(2) : Since $F$ is an ultrafilter, we have that $F^* = A \setminus F$, so we will argue that $J_g = A \setminus F$. Start by taking $a \in A \setminus F$. With the aim of verifying that $a \cap g \in A$, define

$$D_a := \{(p_0, p_1) \in \mathbb{P}(A, F) : a \subseteq p_0 \cup p_1\}.$$

We claim that $D_a$ is dense in $\mathbb{P}(A, F)$. Indeed, fix $(p_0, p_1) \in \mathbb{P}(A, F)$. Note that the inclusion $a - p_1 \subseteq a$ implies that $a - p_1 \notin F$. Hence, $p_0 \cup (a - p_1) \notin F$, and therefore, $q := (p_0 \cup (a - p_1), p_1) \in \mathbb{P}(A, F)$. It should be clear that $q \leqslant (p_0, p_1)$ and $q \in D_a$.

Being $G$ a $\mathbb{P}(A, F)$-generic filter over $M$, there is $(p_0, p_1) \in G \cap D_a$. Forthwith, $(p_0, p_1) \in G$ implies that $p_0 \subseteq g$, and we get that $a \cap p_0 \subseteq a \cap g$. Additionally, the fact $(p_0, p_1) \in D_a$ gives $a \cap g \subseteq (p_0 \cup p_1) \cap g$. Nonetheless, $(p_0 \cap g) \cup (p_1 \cap g) = p_0$ as a result of Lemma 2.10. Therefore, $a \cap g = a \cap p_0 \in A$.

Given that $F$ is an ultrafilter, $F^*$ is a maximal ideal in $A$, but we have just proved that $F^* \subseteq J_g$. Therefore, $F^* = J_g$. $\square$

If $H$ is an ultrafilter in a Boolean algebra $A$ and $M[G]$ is a generic extension of the ground model $M$, with $A \in M$, we claim that, in $M[G]$, $H$ is an ultrafilter in $A$. First note that being a filter is an upper absolute property, i.e., $M[G] \models$"$H$ is a filter in $A$". Now, $H$ is maximal given that for every $a \in A$ it follows that $a \in H \cap M[G]$ or $-a \in H \cap M[G]$.

We know that $A(x)$ is minimal over $A$, whenever $J_x$ is a maximal ideal of $A$ and $x \notin A$ (Proposition 2.6). Therefore, as long as we have $G$, a $\mathbb{P}(A, F)$-generic filter, the generic extension $M[G]$ must satisfy that $A\left(\bigcup\{p_0 : \exists\, p_1 \in A\, (p_0, p_1) \in G\}\right)$ is a minimal extension of $A$. This proves our last result of the section.

**Corollary 2.12.** *With the notation of Proposition 2.11, $M[G] \models A <_m A(g)$.*

## 2.4 Minimally Generated Boolean Algebras

In a naïve approach, a minimally generated Boolean algebra is constructed by an increasing chain of subalgebras indexed by an ordinal, and in such a way that forms a sequence of minimal extensions.

**Definition 2.13.** If $A$ is a Boolean algebra and $\alpha$ is an ordinal, we will say that $\{A_\xi : \xi < \alpha\}$ is a *continuous representation of* $A$ if the following statements hold.

1. If $\xi < \eta < \alpha$, then $A_\xi \leqslant A_\eta$,
2. if $\gamma < \alpha$ is a limit ordinal, then $A_\gamma = \bigcup_{\beta<\gamma} A_\beta$, and
3. $A = \bigcup_{\xi<\alpha} A_\xi$.

**Lemma 2.14.** *Let $\delta$ be a limit ordinal. If $A$ is a Boolean algebra such that $\langle\{x_\xi : \xi < \delta\}\rangle_A = A$, then $\left\{\langle x_\xi : \xi < \alpha\rangle_A : \alpha < \delta\right\}$ is a continuous representation of $A$.*

*Proof.* For every $\alpha < \delta$ define $A_\alpha := \langle x_\xi : \xi < \alpha\rangle_A$. We are going to prove that $\{A_\xi : \xi < \delta\}$ is a continuous representation of $A$. Condition (1) of Definition 2.13 follows easily from the fact that $\{x_\beta : \beta < \xi\} \subseteq \{x_\beta : \beta < \eta\}$, whenever $\xi < \eta < \delta$.

Assume that $\gamma < \delta$ is a limit ordinal. Then, $A_\beta \subseteq A_\gamma$ for each $\beta < \gamma$ and so $\bigcup_{\xi<\gamma} A_\xi \subseteq A_\gamma$. To prove the reverse inclusion fix $x \in A$. Thus, there are $n \in \omega$ and $\{E_i : i < n\} \subseteq \left[\left\{x_\xi^k : \xi < \delta \;\&\; k < 2\right\}\right]^{<\omega}$ in such a way that $x = \bigvee_{i<n} \bigwedge E_i$. Therefore, there is a finite set $S \subseteq \delta$ in such a way that $\bigcup_{i<n} E_i \subseteq \left\{x_\xi^k : k \in 2 \;\&\; \xi \in S\right\}$. By letting $\beta = \max S + 1$, we obtain $\beta < \delta$ and $x \in A_\beta$. Finally, a reasoning similar to the one we just exposed can be used to show that condition (3) from Definition 2.13 holds. $\square$

It is about time to present the central definition of the thesis.

**Definition 2.15.** A Boolean algebra $B$ is *minimally generated* if there is an ordinal $\alpha$ and a family $\{B_\xi : \xi < \alpha\}$ of subalgebras of $B$ such that the following statements hold.

1. $B_0 = \{0, 1\}$,
2. if $\xi + 1 < \alpha$, then $B_\xi \leqslant_m B_{\xi+1}$, and

3. $\{B_\xi : \xi < \alpha\}$ is a continuous representation of $B$.

The least ordinal $\alpha$ for which there is a family like the one described above will be called the *length of* $B$ and will be denoted by $\ell(B)$.

We will say that $B$ is *minimally generated over* $A$, and shall use the symbol $A \leqslant_{mg} B$, if there is a family of subalgebras of $B$, $\{B_\xi : \xi < \alpha\}$, satisfying properties (2) and (3) from Definition 2.15 and the equality $B_0 = A$. Likewise, $\ell(B/A)$ denotes the least ordinal $\alpha$ for which such a family exists and we call it the *length of* $B$ *over* $A$.

The symbol $A <_{mg} B$ will be used whenever $A \leqslant_{mg} B$ and $A \neq B$.

The phrase $\{B_\xi : \xi < \alpha\}$ *witnesses* $A \leqslant_{mg} B$ means that $\{B_\xi : \xi < \alpha\}$ satisfies conditions (2) and (3) of Definition 2.15 and $B_0 = A$. Similarly for the sentence $\{B_\xi : \xi < \alpha\}$ *witnesses that* $A$ *is minimally generated.*

**Lemma 2.16.** *Let* $A < B$*. Then* $A <_{mg} B$ *if and only if there is an ordinal* $\alpha$ *and a family* $\{B_\xi : \xi < \alpha\}$ *of subalgebras of* $B$ *such that the following statements hold.*

1. $B_0 = A$,

2. *if* $\xi + 1 < \alpha$*, then* $B_\xi <_m B_{\xi+1}$*, and*

3. $\{B_\xi : \xi < \alpha\}$ *is a continuous representation of* $B$*.*

*Proof.* We will argue the sufficiency only. Assume that $\{C_\xi : \xi < \gamma\}$ witnesses $A \leqslant_{mg} B$ and recursively define $f(0) = 0$,

$$f(\beta + 1) = \min\left\{\xi < \gamma : C_{f(\beta)} < C_\xi\right\},$$

as long as the set on the right is not empty, and $f(\beta) = \sup f``\beta$, whenever $\beta$ is limit. This produces a function $f$ whose domain is an ordinal, let us say, $\alpha$ and satisfies the following for each $\beta < \alpha$,

(i) $C_{f(0)} = C_0 = A$;

(ii) if $\beta + 1 < \alpha$, then $C_{f(\beta)} <_m C_{f(\beta+1)}$;

(iii) $C_{f(\beta)} = \bigcup_{\xi<\beta} C_{f(\xi)}$, when $\beta$ is limit, and

(iv) $\bigcup_{\xi<\alpha} C_{f(\xi)} = \bigcup_{\xi<\gamma} C_\gamma = B$.

Therefore, by letting $B_\xi := C_{f(\xi)}$, for each $\xi < \alpha$, we obtain that $\{B_\xi : \xi < \alpha\}$ is the required family. $\square$

**Corollary 2.17.** *If $A \leqslant_{mg} B$ and $\ell(A/B) = \alpha$, then there is a continuous representation of $B$ of length $\alpha$ which satisfies condition* (2) *of Lemma 2.16.*

The following proposition shows how minimally generated Boolean algebras behave under homomorphisms.

For every homomorphism $\pi$ define the *kernel of* $\pi$ as the set $\ker(\pi) := \pi^{-1}[\{0\}]$.

**Proposition 2.18.** *Assume $A, B$, and $Q$ are Boolean algebras. If $A \leqslant B$, $\pi : B \to Q$ is a surjective homomorphism, and $P := \pi[A]$, then the following hold.*

1. *If $A \leqslant_m B$, then $P \leqslant_m Q$.*

2. *When $\ker(\pi) \subseteq A$, $A \leqslant_m B$ if and only if $P \leqslant_m Q$.*

3. *If $A \leqslant_{mg} B$, then $P \leqslant_{mg} Q$ and $\ell(Q/P) \leqslant \ell(B/A)$.*

4. *When $\ker(\pi) \subseteq A$, $A \leqslant_{mg} B$ if and only if $P \leqslant_{mg} Q$.*

5. *Homomorphic images of minimally generated Boolean algebras are also minimally generated.*

*Proof.* (1) : By contraposition, if there is $T$ such that $P < T < Q$, a straightforward calculation shows that $A < \pi^{-1}[T] < B$.

(2) : Notice that we only need to prove the reverse implication. We will proceed by a contrapositive argumentation. Assume that there is $S$ such that $A < S < B$. Hence, $P \leqslant \pi[S] \leqslant Q$. We will show that $P < \pi[S] < Q$.

Let $y \in S \setminus A$. We claim that $\pi(y) \in \pi[S] \setminus P$. Certainly, if $\pi(y) \in P$, there is $z \in A$ such that $\pi(z) = \pi(y)$; hence $\pi(y - z) = 0$ and in consequence, $y - z \in A$. In a similar way we get $z - y \in A$. Besides, $y \wedge z \in A$ due to the following equalities:

$$z \triangle (z-y) = [z-(z-y)] \vee [(z-y)-z] = z \wedge y$$

So, $y = (y \wedge z) \vee (y - z)$ belongs to $A$, which is a contradiction. Therefore, $\pi(y) \in \pi[S] \setminus P$ and this implies that $P < \pi[S]$.

By a similar argument $\pi[S] < \pi[B] = Q$.

(3) : Assume that $\{B_\xi : \xi < \alpha\}$ witnesses that $A \leqslant_{mg} B$. So,

$$Q = \pi[B] = \pi\left[\bigcup_{\xi<\alpha} B_\alpha\right] = \bigcup_{\xi<\alpha} \pi[B_\xi].$$

Besides, if $\xi + 1 < \alpha$, then $\pi[B_\xi] \leqslant_m \pi[B_{\xi+1}]$ by (1). In case that $\gamma < \alpha$ is limit, then $\pi[B_\gamma] = \bigcup_{\beta<\gamma} \pi[B_\beta]$. Therefore, the sequence $\{\pi[B_\xi] : \xi < \alpha\}$ witness that $P \leqslant_{mg} Q$. The inequality $\ell(Q/P) \leqslant \ell(B/A)$ follows immediately.

(4) : Let $\{Q_\xi : \xi < \alpha\}$ be a sequence witnessing $P \leqslant_{mg} Q$. Naturally, $B = \bigcup_{\alpha<\xi} \pi^{-1}[Q_\alpha]$ and, according to (2), whenever $\xi + 1 < \alpha$, $\pi^{-1}[Q_\xi] \leqslant_m \pi^{-1}[Q_{\xi+1}]$.

(5) : Let $A$ be the two element algebra and apply (3). □

Now let us analyze the behavior of the class of minimally generated Boolean algebras under the operation of taking subalgebras.

**Proposition 2.19.** *Assume $M$ is a Boolean algebra, $A \leqslant B \leqslant M$, and $D \leqslant M$. If $P := A \cap D$ and $Q := B \cap D$, then the following hold.*

1. *If $A \leqslant_m B$, then $P \leqslant_m Q$.*

2. *Whenever $A \leqslant_{mg} B$, it follows that $P \leqslant_{mg} Q$ and $\ell(Q/P) \leqslant \ell(B/A)$.*

3. *If $A \leqslant_{mg} B$ and $A \leqslant C \leqslant B$, then $A \leqslant_{mg} C$.*

4. *Every subalgebra of a minimally generated Boolean algebra is also minimally generated.*

*Proof.* (1) : Clearly, $P \leqslant Q$. If $P$ and $Q$ are equal there is nothing of interest to prove, so let us assume that $P < Q$.

We are going to use Corollary 2.5 to verify that $P \leqslant_m Q$. Fix $x, y \in Q \setminus P \subseteq B \setminus A$ and observe that $B = A(x)$ because $A <_m B$.

The arguments presented in the proof of implication (5) $\to$ (1) of Proposition 2.6 can be used to show that $x \triangle y \in A$. Considering $Q \leqslant D$, we deduce that $x \triangle y \in D$. Hence, $x \triangle y \in P$ and in consequence, $y = x \triangle (x \triangle y) \in P(x)$.

(2) : Using (1), one easily verifies that if $\{B_\xi : \xi < \alpha\}$ witnesses $A \leqslant_{mg} B$, then $\{D \cap B_\xi : \xi < \alpha\}$ witnesses $P \leqslant_{mg} Q$.

To prove (3) and (4) take $D = C = \{0, 1\}$, respectively, and use (2). □

If a product $\prod_{i \in I} B_i$ of Boolean algebras is minimally generated, then for every $i \in I$, $B_i$ is minimally generated as a result of being a homomorphic image of the product (see Proposition 2.18). Later, we will see in Corollary 3.14 that the converse fails. Nonetheless, a product of finitely many minimally generated Boolean algebras is minimally generated, as the following result shows.

**Proposition 2.20.** *Let $A$ and $B$ be minimally generated Boolean algebras. Then $A \times B$ is minimally generated.*

*Proof.* Let $\{A_\xi : \xi < \alpha\}$ and $\{B_\xi : \xi < \beta\}$ be sequences which witness that $A$ and $B$ are minimally generated, respectively. Moreover, let us assume that whenever $\xi + 1 < \alpha$, we have that $A_\xi <_m A_{\xi+1}$ and similarly for $\{B_\xi : \xi < \beta\}$ (see Lemma 2.16). Additionally, from this point forward, $2_B := \{0_B, 1_B\}$.

Define the sequence $\{C_\xi : \xi < \alpha + \beta\}$ by $C_0 = \{(0_A, 0_B), (1_A, 1_B)\}$ and

$$C_\xi := \begin{cases} A_\xi \times 2_B, \text{ if } 0 < \xi < \alpha \\ \\ A \times B_\eta, \text{ if } \xi = \alpha + \eta \end{cases}$$

We claim that $A \times B = \bigcup_{\xi < \alpha + \beta} C_\xi$. Certainly, if $(a, b) \in A \times B$, then there is $\eta < \beta$ such that $b \in B_\eta$ and so, $(a, b) \in C_{\alpha + \eta}$. The reverse inclusion is trivial.

We shall continue by proving the remaining properties which prove that $\{C_\xi : \xi < \alpha + \beta\}$ witnesses that $A \times B$ is minimally generated.

Let $\gamma < \alpha$ be a limit ordinal. It follows that

$$C_\gamma = A_\gamma \times 2_B = \left(\bigcup_{\xi<\gamma} A_\xi\right) \times 2_B = \bigcup_{\xi<\gamma} (A_\xi \times 2_B) = \bigcup_{\xi<\gamma} C_\xi.$$

In a similar way it can be proved that for every limit ordinal $\gamma < \beta$, $\bigcup_{\xi<\alpha+\gamma} C_\xi = C_{\alpha+\gamma}$.

Finally, we have to show that whenever $\xi + 1 < \alpha + \beta$, $C_\xi <_m C_{\xi+1}$. Let us start with the case $\xi < \alpha$, i.e., we will argue that $A_\xi \times 2_B <_m A_{\xi+1} \times 2_B$.

Fix $x \in A_{\xi+1} \setminus A_\xi$. We know that $A_\xi (x) = A_{\xi+1}$ and due to Proposition 2.6.(4), if $G := \{y \in A_\xi : y \text{ is comparable with } x\}$, then $\langle G \rangle = A_\xi$. Notice that $0_A$ and $1_A$ belong to $G$.

Let $H := \{(y, 1_B) : y \in G\}$ and fix $(y, 1_B) \in H$. Immediately, $y$ is comparable with $x$, i.e., $y \leqslant x$ or $x \leqslant y$. So, $(y, 1_B) \leqslant (x, 1_B)$ or $(x, 1_B) \leqslant (y, 1_B)$. Therefore, every element of $H$ is comparable with $(x, 1_B)$.

We claim that $H$ generates $C_\xi$. Indeed, let $(z, k) \in C_\xi$. By Proposition 1.4 there exist $n \in \omega$ and $\{E_i : i < n\} \subseteq [G \cup G^*]^{<\omega}$ such that $z = \bigvee_{i<n} \bigwedge E_i$.

Now, if $k = 0_B$, for every $i < n$ we define

$$F_i := [(E_i \cap G) \times \{1_B\}] \cup [(E_i \cap G^*) \times \{0_B\}] \cup \{(1_A, 0_B)\} \subseteq H \cup H^*.$$

Then, $\bigvee_{i<n} \bigwedge F_i = \left(\bigvee_{i<n} \bigwedge E_i, 0_B\right) = (z, 0_B)$.

On the other hand, if $k = 1_B$, for every $i < n$ we set

$$F_i := [(E_i \cap G) \times \{1_B\}] \cup [(E_i \cap G^*) \times \{0_B\}]$$

and $F_n := \{(0_A, 1_B)\}$. Therefore, $\bigvee_{i<n+1} \bigwedge F_i = \left(\bigvee_{i<n} \bigwedge E_i, 1_B\right) = (z, 1_B)$.

In conclusion, $C_\xi <_m C_\xi (x, 1_B)$ due to Proposition 2.6.(5). Thus, we have to show that $C_\xi (x, 1_B) = C_{\xi+1}$. Equivalently, will verify that $A_\xi (x) \times 2_B = (A_\xi \times 2_B) (x, 1_B)$. Let $(y, i) \in A_\xi (x) \times 2_B$. There are $a, b \in A$ such that $y = (a \wedge x) \vee (b - x)$. Then,

$$(y, i) = ((a \wedge x) \vee (b - x), i) = (a \wedge x, i) \vee (b - x, 0) = [(a, i) \wedge (x, i)] \vee [(b, 0_B) - (x, 1_B)].$$

The remaining inclusion is proved in a similar way.

In case that $\xi = \alpha + \eta$, let $x \in B_{\eta+1} \setminus B_\eta$. It can be proved, using arguments similar to the ones employed above, that $(A \times B_\eta)(1_A, x) = A \times (B_\eta(x))$, so it is enough to show that $C_{\alpha+\eta} <_m C_{\alpha+\eta}(1_A, x)$.

Recall that the set $G := \{y \in B_\eta : y \text{ is comparable with } x\}$ generates $B_\eta$. We shall prove that $G$ and $G^*$ are closed under $\wedge$. Fix $a, b \in G$. If $a, b < x$ or $a, b > x$, we get that $a \wedge b < x$ or $a \wedge b > x$, respectively, and so, $a \wedge b \in G$. To verify the remaining cases suppose that $a < x < b$ and get $a \wedge b < x$.

If $a, b \in G^*$, immediately $-a, -b \in G$. In case that $-a, -b < x$ or $-a, -b > x$, it follows that $(-a) \vee (-b) < x$ or $(-a) \vee (-b) > x$, respectively; either way, $a \wedge b \in G^*$. Finally, if $-a < x < -b$, we deduce that $-(a \wedge b) = (-a) \vee (-b) = -b$, which shows that $a \wedge b \in G^*$.

Define

$$H := \{(1_A, y) : y \in G \ \ \& \ \ y > x\} \cup \{(a, y) : y \in G \ \ \& \ \ a \in A \ \ \& \ \ y < x\}.$$

Notice that every element of $H$ is comparable with $(1_A, x)$. We claim that $\langle H \rangle = C_{\alpha+\eta}$. Indeed, let $(a, z) \in C_{\alpha+\eta}$. We know that there are $n \in \omega$ and $\{E_i : i < n\} \subseteq [G \cup G^*]^{<\omega}$ in such a way that $z = \bigvee_{i<n} \bigwedge E_i$. Now,

$$(a, z) = \big(a, \textstyle\bigvee_{i<n} \bigwedge E_i\big) = \big(a, \bigvee_{i<n} \bigwedge [(E_i \cap G) \cup (E_i \cap G^*)]\big) =$$
$$\big(a, \textstyle\bigvee_{i<n} [(\bigwedge (E_i \cap G)) \wedge (\bigwedge (E_i \cap G^*))]\big).$$

Moreover, for every $i < n$, $\bigwedge (E_i \cap G) \in G$ and $\bigwedge (E_i \cap G^*) \in G^*$ because $G$ and $G^*$ are closed under $\wedge$.

Then, $(a, z) = \big(a, \bigvee_{i<n} (u_i \wedge w_i)\big) = \bigvee_{i<n} (a, u_i \wedge w_i)$ where for every $i < n$, $u_i$ and $w_i$ belong to $G$ and $G^*$, respectively. Observe that, given these conditions, we only have to prove that for every $u \in G$, $w \in G^*$ and $a \in A$, we get that $(a, u \wedge w) \in \langle H \rangle$.

Let $a \in A$, $u \in G$ and $w \in G^*$. Consider the following cases.

(i) $u, -w < x$. Immediately $(a, u)$ and $(1, -w)$ belong to $H$. Then, $(a, u) \wedge (0, w) \in \langle H \rangle$.

(ii) $u, -w > x$. It follows that $(1, u), (1, -w) \in H$ and since $(a, u \wedge w) = (a, 0) \vee [(1, u) - (1, -w)]$, we get the result.

(iii) $u > x$ and $-w < x$. We have $(0, w), (0, -u) \in \langle H \rangle$, and so we apply the following equality $(a, u \wedge w) = [(0, w) - (0, -u)] \vee (a, 0)$.

(iv) $u < x$ and $-w > x$. In this case $(0, w) \in \langle H \rangle$ and $(0, u) \in H$. To finish we use the following equation $(a, u \wedge w) = [(0, u) \wedge (0, w)] \vee (a, 0)$. □

To complete this section we shall present a criterion in order to determine whether a Boolean algebra is not minimally generated over some of its subalgebras. This result uses the notion of independence introduced in Definition 1.23.

**Lemma 2.21.** *Assume $A < B$. If $A$ is atomless and there is $x \in B$ which is independent of $A$, then $B$ is not minimally generated over $A$.*

*Proof.* Seeking a contradiction, assume that $A <_{mg} B$. Then, $A <_{mg} A(x)$ due to $A < A(x) \leqslant B$ and Proposition 2.19.(3).

Fix $u \in A(x) \setminus A$ in such a way that $A <_m A(u)$. Hence, $J_u := J_A(u)$ is a maximal ideal. According to Lemma 1.25, there are $a, b \in A$ in such a way that $A \restriction a = A \restriction u$ and $A \restriction b = A \restriction (-u)$. Immediately, $a \vee b \in J_u$ and so, $a \vee b \neq 1$. Besides, for every $z \in J_u$, $z \leqslant a \vee b$. We can conclude that $J_u \subseteq \langle \{a \vee b\} \rangle^i_A$, but this is actually an equality considering that $J_u$ is maximal in $A$ (see Proposition 2.6).

Set $c := -(a \vee b)$. To get the desired contradiction, we will check that $c$ is an atom of $A$. In order to prove this, notice that $c > 0$ and fix $d \in A$ such that $d < c$. Let us argue that $-d \notin J_u$. Indeed, if $-d \in J_u$, we get that $-d \leqslant a \vee b$, and in consequence, $d \geqslant c$, which can not happen because we are assuming that $d < c$. Therefore, $d \in J_u$ since $J_u$ is a maximal ideal in $A$. It follows that $d \leqslant a \vee b = -c$; enough to conclude that $d = 0$. □

Whenever $A$ and $B$ are Boolean algebras, $\mathrm{S}(A \oplus B)$ is homeomorphic to the topological product $\mathrm{S}(A) \times \mathrm{S}(B)$, where $A \oplus B$ is *the free product of $A$ and $B$* (see [10, Section 11]). Since $\mathrm{S}(A)$ and $\mathrm{S}(B)$ embed as subspaces of $\mathrm{S}(A) \times \mathrm{S}(B)$, by Proposition 1.33, both, $A$ and $B$, are homomorphic images of $A \oplus B$. Therefore, if $A \oplus B$ is minimally generated, then $A$ and $B$ have to be minimally generated. We also mention that there are minimally generated Boolean algebras whose free product fails to be minimally generated (see [9, Example 1]).

# CHAPTER 3: POSITIVE AND NEGATIVE EXAMPLES

The present chapter has two goals: to prove that some very well-known classes of Boolean algebras are subclasses of the class of minimally generated Boolean algebras and to exhibit several Boolean algebras which fail to be minimally generated.

## 3.1 Interval Algebras

This brief section presents the interval algebras as our first example of Boolean algebras that are minimally generated.

Consider $\langle L, \leqslant \rangle$, a linear order, and for every $x \in L$ define $(\leftarrow, x) := \{y \in L : y < x\}$.

**Definition 3.1.** $A$ is an *interval algebra* if there is $L$, an infinite linear order, such that

$$A = \langle \{(\leftarrow, x) : x \in L\} \rangle_{\mathcal{P}(L)}.$$

**Proposition 3.2.** *Every interval algebra is minimally generated.*

*Proof.* Let $A$ be an interval algebra generated by the lineal order $L$. Fix $\alpha = |L|$ and $C := \{(\leftarrow, x) : x \in L\}$. We get $\{i_\xi : \xi < \alpha\}$, an enumeration of $C$ without repetitions.

Define $C_\beta := \{i_\xi : \xi < \beta\}$ and $B_\beta := \langle C_\beta \rangle$ for each $\beta < \alpha$. We will argue that $\{B_\xi : \xi < \alpha\}$ witnesses that $A$ is minimally generated.

(i) $B_0 = \langle \emptyset \rangle = \{\emptyset, L\}$.

(ii) $B_\xi \leqslant_m B_{\xi+1}$, for every $\xi < \alpha$: since $L$ is a linear ordering, every element of $C_\xi$ is comparable with $i_\xi$. Keeping in mind Proposition 2.6, we only need to prove that $B_{\xi+1} = B_\xi\,(i_\xi)$. Start by noticing that $i_\xi \in B_{\xi+1}$ and in consequence, $B_{\xi+1} \supseteq B_\xi\,(i_\xi)$. For the reverse inclusion observe that $C_\xi \subseteq B_\xi$, so $B_{\xi+1} = \langle C_{\xi+1} \rangle = \langle C_\xi \cup \{i_\xi\} \rangle \subseteq \langle B_\xi \cup \{i_\xi\} \rangle = B_\xi\,(i_\xi)$.

(iii) $\{B_\xi : \xi < \alpha\}$ is a continuous representation of $A$: being $A = \langle \{i_\xi : \xi < \delta\} \rangle_A$, this statement is a straight consequence of Lemma 2.14. $\square$

## 3.2 Superatomic Boolean Algebras

In this section we will prove that all superatomic Boolean algebras are minimally generated. Also, we are going to show some other relationships between minimal generation and superatomicity.

**Lemma 3.3.** *Assume $B$ is a Boolean algebra, $A \leqslant B$, and $I$ is an ideal in $B$ with $I \subseteq A$. If $\pi : B \to B/I$ is the natural projection, then*

1. *$I$ is an ideal in $A$,*

2. *$\pi$ “$A = A/I$, and*

3. *$\pi^{-1}[\pi$ “$A] = A$.*

*Proof.* The argument for (1) is routine, so we omit it. Now, to prove (2) we only need to show that

$$\pi(a) = \{x \in A : a \triangle x \in I\},$$

for each $a \in A$. If $x \in \pi(a)$, then $x \in B$ and $a \triangle x \in I \subseteq A$; hence, $x = a \triangle (a \triangle x) \in A$. The reverse inclusion is clear.

For (3), start by noting that $A \subseteq \pi^{-1}[\pi$ “$A]$. On the other hand, $x \in \pi^{-1}[\pi$ “$A]$ implies $\pi(x) \in A/I$ and so, $x \in \pi(x) \subseteq A$. $\square$

**Lemma 3.4.** *Let $B$ be a superatomic Boolean algebra and $A < B$. Then, there is $x \in B \setminus A$ such that $A <_m A(x)$.*

*Proof.* Since $B$ is superatomic, by Proposition 1.41 there is an ordinal $\gamma$ such that $I_\gamma = B$, where $I_\gamma$ is the $\gamma$th Cantor-Bendixson subset of $B$ (see the paragraphs following Proposition 1.40). Hence, $I_\gamma \nsubseteq A$. Now, let $\alpha$ be the least ordinal satisfying $I_\alpha \nsubseteq A$. If $\alpha$ were limit, then $\bigcup_{\xi<\alpha} I_\xi \nsubseteq A$ which is a contradiction to our choice of $\alpha$. Thus, $\alpha = \beta + 1$, for some ordinal $\beta$. In particular, $I_\beta \subseteq A$ and so, $I_\alpha = I_{\beta+1} \neq I_\beta$. Then,

$$I_\alpha = \pi^{-1}[[\mathrm{At}(B/I_\beta)]_{B/I_\beta}],$$

where $\pi : B \to B/I_\beta$ is the corresponding quotient homomorphism.

Fix $x \in I_\alpha \setminus A$. Then $\pi(x) \in [\mathrm{At}(B/I_\beta)]_{B/I_\beta}$ and therefore, by Proposition 1.37, $\pi(x) = \bigvee S$, for some finite set $S \subseteq \mathrm{At}(B/I_\beta)$. Observe that if $\pi^{-1}[S] \subseteq A$, then $S \subseteq \pi$“A, i.e., $x \in \pi^{-1}[\pi“A] = A$ (see Lemma 3.3.(3)); a contradiction which shows that there exists $a \in \pi^{-1}[S] \setminus A$. Hence, $\pi(a) \in \mathrm{At}(B/I_\beta)$ and since $a \notin A$, we get $\pi(a) \notin \pi“A = A/I_\beta$ (again, Lemma 3.3).

Since $\pi(a) \in \mathrm{At}(B/I_\beta) \setminus (A/I_\beta)$, $A/I_\beta <_m (A/I_\beta)(\pi(a))$ (see Corollary 2.8). Then, by the previous lemma, $\pi“A <_m (\pi“A)(\pi(a))$. Moreover, $(\pi“A)(\pi(a)) = \pi[A(a)]$ as a result of Proposition 1.7. Now, let $h := \pi \restriction A(a)$; we claim that $\ker(h) \subseteq A$. Indeed, $\ker(h) = A(a) \cap \ker(\pi) = A(a) \cap I_\beta = I_\beta \subseteq A$. Therefore, according to Proposition 2.18.(2), $A <_m A(a)$. □

**Proposition 3.5.** *Every superatomic Boolean algebra is minimally generated.*

*Proof.* Let $A$ be a superatomic Boolean algebra. By recursion on $\alpha$ define define $C_\alpha \leqslant A$ as follows: $C_0 = \{0_A, 1_A\}$; if $A \setminus C_\alpha \neq \emptyset$, we fix $x_\alpha \in A \setminus C_\alpha$ in such a way that $C_\alpha <_m C_\alpha(x_\alpha)$ (recall Lemma 3.4) and set $C_{\alpha+1} := C_\alpha(x_\alpha)$; and when $\alpha$ is limit, $C_\alpha = \bigcup_{\beta<\alpha} C_\beta$.

Let $\delta$ be the least ordinal such that $A = C_\delta$. If $\delta = \beta+1$ we claim that $\{C_\xi : \xi < \delta + 1\}$ witnesses that $A$ is minimally generated. Indeed, notice that for every $\xi < \eta \leqslant \delta$, $C_\xi \subseteq C_\eta$ and thus, $\bigcup_{\xi<\delta+1} C_\xi = \bigcup_{\xi\leqslant\delta} C_\xi = C_\delta = A$. One easily verifies the remaining properties of Definition 2.15.

On the other hand, in case that $\delta$ is limit, it can be easily verified that $\{C_\xi : \xi < \delta\}$ witnesses that $A$ is minimally generated. □

Our next result shows that one can use minimally generated Boolean algebras to characterize superatomicity.

**Proposition 3.6.** *If $B$ is a Boolean algebra, then $B$ is superatomic if and only if for every $A \leqslant B$, $B$ is minimally generated over $A$.*

*Proof.* Let $B$ be a superatomic Boolean algebra and $A \leqslant B$. For every ordinal $\alpha$ we define $A_\alpha \leqslant B$ as follows: $A_0 := A$; whenever $B \setminus A_\alpha \neq \emptyset$, $A_{\alpha+1} := A_\alpha\,(a_\alpha)$, where $a_\alpha \in B \setminus A_\alpha$ is such that $A_\alpha \leqslant_m A_\alpha\,(a_\alpha)$ (see Lemma 3.4); finally, if $\alpha$ is a limit ordinal, $A := \bigcup_{\beta<\alpha} A_\beta$.

Clearly, for the construction described in the previous paragraph there is a least ordinal $\alpha$ such that $B = A_\alpha$. We have the following claims:

(i) If $\alpha = \beta + 1$, then $\{A_\xi : \xi < \alpha + 1\}$ witnesses that $A \leqslant_{mg} B$. Indeed, just note that $B = A_\alpha \subseteq \bigcup_{\xi<\alpha+1} A_\xi$ and so $B = \bigcup_{\xi<\alpha+1} A_\xi$.

(ii) In case that $\alpha$ is a limit ordinal, then $\{A_\xi : \xi < \alpha\}$ witnesses that $A \leqslant_{mg} B$. By construction, $B = \bigcup_{\xi<\alpha} A_\xi$ and the rest of the properties are easily verified.

For the reverse implication let us proceed by a contrapositive argument, i.e., we are going to assume that $B$ is not superatomic and we shall prove that there is $A \leqslant B$ in such a way that $B$ is not minimally generated over $A$.

According to Proposition 1.40, there is $X$, a countable independent subset of $B$. Fix $x \in X$, and recall Lemma 1.24 to conclude that $x$ is independent of $A := \langle X \setminus \{x\}\rangle_B$. Since $A$ is countable and generated by an independent set, $A$ is atomless (see Lemma 1.21). Hence, as a result of Proposition 2.21, we are done. □

We saw in Proposition 2.6 the relationship between $A\,(x)$ being a minimal extension of $A$ and the ideal $J_x$. Now let us study the impact of the assumption $A \leqslant_{mg} A(x)$ on the ideal $J_x$. We begin by introducing some topological tools.

**Definition 3.7.** If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, define *the disjoint union space* $(X \oplus Y, \mathcal{T}_{X\oplus Y})$ as follows: $X \oplus Y := (X \times \{0\}) \cup (Y \times \{1\})$ and

$$\mathcal{T}_{X\oplus Y} := \{(U \times \{0\}) \cup (V \times \{1\}) : U \in \mathcal{T}_X \ \ \& \ \ V \in \mathcal{T}_Y\}.$$

**Lemma 3.8.** *If $A$ is a subalgebra of $B$ and $x \in B \setminus A$, then $A\,(x)$ is superatomic whenever $A$ is superatomic.*

*Proof.* The plan is to use Proposition 1.39. Since $\mathrm{S}(A)$ is scattered, $\mathrm{S}(A) \oplus \mathrm{S}(A)$ is scattered. Hence, by showing that $\mathrm{S}(A(x))$ is homeomorphic to a subspace of $\mathrm{S}(A) \oplus \mathrm{S}(A)$, we will prove that $\mathrm{S}(A(x))$ is scattered.

Let $U \in \mathrm{S}(A(x))$ be arbitrary and for each $i < 2$, if $x^i \in U$, define

$$h(U) := \{a \in A : a \wedge x^i \in U\}.$$

Note that for each $U \in \mathrm{S}(A(x))$ there is exactly one $i < 2$ with $x^i \in U$.

We claim that $h(U) \in \mathrm{S}(A)$. Assume $x^i \in U$, for some $i < 2$. Certainly, $0 \wedge x^i = 0 \notin U$ and so, $0 \notin h(U)$. Since $1 \wedge x^i = x^i \in U$, $1 \in h(U)$. Now, consider $a, b \in h(U)$. Hence, $a \wedge x^i, b \wedge x^i \in U$. Since $U$ is a filter, $(a \wedge b) \wedge x^i = (a \wedge x^i) \wedge (b \wedge x^i) \in U$, and thus, $a \wedge b \in h(U)$. We continue by considering $a \in h(U)$ and $b \in A$ in such a way that $a \leqslant b$. It follows that $b \wedge x^i \geqslant a \wedge x^i \in U$, and given that $U$ is a filter, $b \in h(U)$. Therefore $h(U)$ is a filter. In order to check that $U$ is an ultrafilter, observe that for any $a \in A$ we get $a \in U$ or $-a \in U$ and deduce that $a \in h(U)$ or $-a \in h(U)$ as a result of $x^i$ being an element of $U$.

Define $f : \mathrm{S}(A(x)) \to \mathrm{S}(A) \oplus \mathrm{S}(A)$ as follows: for every $U \in \mathrm{S}(A(x))$ and $i < 2$, $f(U) = (h(U), i)$ whenever $x^i \in U$. Let us prove that $f$ is one-to-one. Fix $U, V \in \mathrm{S}(A)$ such that $U \neq V$. We consider two cases. If for some $i < 2$ we obtain $x^i \in U$ and $x^{1-i} \in V$, immediately $f(U) \neq f(V)$. When $x^k \in U \cap V$ for some $k < 2$, let $a \in U \setminus V$. Consider $a_0, a_1 \in A$ in such a way that $a = (a_0 \wedge x) \vee (a_1 - x)$. As a result of the observation following Definition 1.13, there is $j < 2$ such that $a_j \wedge x^j \in U$. Notice that the assumption $k = 1 - j$ implies that $0 = (a_j \wedge x^j) \wedge x^{1-j} \in U$, which is impossible. Thus, $j = k$ and so, $a_k \in h(U)$. On the other hand, from $a_k \wedge x^k \leqslant a$ and $a \in A(x) \setminus V$ we deduce that $a_k \wedge x^k \notin V$, i.e., $a_k \notin h(V)$. Therefore, $a_k \in h(U) \setminus h(V)$.

Let us argue that f is continuous. Fix $a \in A$, recall Definition 1.27 and note that the following inequalities hold for any $i < 2$.

$$\begin{aligned} f^{-1}[a^- \times \{i\}] &= \{U \in \mathrm{S}(A) : f(U) \in a^- \times \{i\}\} \\ &= \{U \in \mathrm{S}(A) : (h(U), i) \in a^- \times \{i\}\} \end{aligned}$$

$$\begin{aligned}
&= \left\{U \in \mathrm{S}(A) : h(U) \in a^{-} \ \ \& \ \ x^{i} \in U\right\} \\
&= \left\{U \in \mathrm{S}(A) : a \in h(U) \ \ \& \ \ x^{i} \in U\right\} \\
&= \left\{U \in \mathrm{S}(A) : a \wedge x^{i} \in U\right\} \\
&= \left\{U \in \mathrm{S}(A) : U \in \left(a \wedge x^{i}\right)^{-}\right\} = \left(a \wedge x^{i}\right)^{-}
\end{aligned}$$

Now, considering that $f$ is a continuous injection from a compact space into a a Hausdorff space, $f$ is an embedding; in other words, $\mathrm{S}(A(x))$ is homeomorphic to a subspace of $\mathrm{S}(A) \oplus \mathrm{S}(A)$. □

**Proposition 3.9.** *Let $B$ be a Boolean algebra, $A \leqslant B$ and $x \in B \setminus A$. If the factor algebra $A/J_x$ is superatomic, then $A \leqslant_{mg} A(x)$.*

*Proof.* Assume that $A/J_x$ is superatomic and recall that $J_x$ is an ideal in $A(x)$ (paragraph following Lemma 2.2). Denote by $\pi : A(x) \to A(x)/J_x$ the natural projection. We already saw in Proposition 1.7 that $A(x)/J_x = \pi[A(x)] = \pi[A](\pi(x))$, and by Lemma 3.3.(2) we obtain the equality $A(x)/J_x = (A/J_x)(\pi(x))$. Therefore, $A(x)/J_x$ is a simple extension of $A/J_x$ and so, $A(x)/J_x$ is superatomic as a result of Lemma 3.8.

Given that $A/J_x \leqslant A(x)/J_x$, $A(x)/J_x$ is minimally generated over $A/J_x$ (see Proposition 3.6). Besides, $\ker(\pi) = J_x \subseteq A$ and therefore, according to Proposition 2.18.(4), $A \leqslant_{mg} A(x)$. □

## 3.3 Free Boolean Algebras

Now we are going to concentrate on proving that the only free Boolean algebra with an infinite number of generators is $\mathrm{Fr}(\omega)$.

**Proposition 3.10.** $\mathrm{Fr}(\omega)$ *is minimally generated.*

*Proof.* Let us start by fixing $\{a_n : n \in \omega\}$, an independent family that generates $\mathrm{Fr}(\omega)$. Define for every $n \in \omega$ the Boolean algebra $A_n = \langle \{a_i : i < n\} \rangle_{\mathrm{Fr}(\omega)}$. We obtain that

$\{A_n : n \in \omega\}$ is a continuous representation of $\mathrm{Fr}(\omega)$ (see Lemma 2.14); in particular, $\mathrm{Fr}(\omega) = \bigcup_{n\in\omega} A_n$.

Fix $n \in \omega$ and notice that $A_{n+1}$ is superatomic because $|A_{n+1}| < \omega$. Use Proposition 3.6 to deduce that $A_n <_{mg} A_{n+1}$. Assume that $\left\{A^n_\xi : \xi < \alpha_n\right\}$ witnesses $A_n \leqslant_{mg} A_{n+1}$ and satisfies $A^n_\xi <_m A^n_{\xi+1}$ whenever $\xi + 1 < \alpha_n$ (we are using Lemma 2.16). Hence, if $\alpha_n$ were infinite, $A_{n+1}$ would be infinite as well. Therefore, $\alpha_n < \omega$.

From the previous paragraph we deduce that the set $I := \bigcup_{n\in\omega} (\{n\} \times \alpha_n)$, endowed with the lexicographic order is isomorphic to the ordinal $\omega$. Let $f : \omega \to I$ be an isomorphism and define for every $n \in \omega$ and $\xi \in \alpha_n$, $B_{f^{-1}(n,\xi)} := A^n_\xi$. A routine verification confirms that $\{B_i : i < \omega\}$ witnesses that $\mathrm{Fr}(\omega)$ is minimally generated. $\square$

For the rest of the section we will focus on showing that no uncountable free Boolean algebra is minimally generated.

**Lemma 3.11.** *Let $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ be continuous representations of $\mathrm{Fr}(\omega_1)$ such that for every $\alpha < \omega_1$, $A_\alpha$ and $B_\alpha$ are countable. Then, there exists a limit ordinal $\gamma < \omega_1$ in such a way that $A_\gamma = B_\gamma$.*

*Proof.* For every $\alpha < \omega_1$ there are $\beta_0, \beta_1 < \omega_1$ such that $A_\alpha \subseteq B_{\beta_0}$ and $B_\alpha \subseteq A_{\beta_1}$ because both, $A_\alpha$ and $B_\alpha$, are countable. Using this property we can construct two sequences $\{\alpha_n : n \in \omega\}$ and $\{\beta_n : n \in \omega\}$ in such a way that for every $n \in \omega$, $\omega \leqslant \alpha_n < \beta_n < \alpha_{n+1} < \omega_1$ and $A_{\alpha_n} \subseteq B_{\beta_n} \subseteq A_{\alpha_{n+1}}$. Notice that $\bigcup_{n<\omega} A_{\alpha_n} = \bigcup_{n<\omega} B_{\beta_n}$.

Define $\gamma := \sup\{\alpha_n : n \in \omega\} = \{\beta_n : n \in \omega\}$ and note that $\gamma$ is limit. Hence, $A_\gamma = \bigcup_{\xi<\gamma} A_\xi$ and $B_\gamma = \bigcup_{\xi<\gamma} B_\xi$. Thus, we shall conclude our proof by showing that $A_\gamma = \bigcup_{n<\omega} A_{\alpha_n}$ and $B_\gamma = \bigcup_{n<\omega} B_{\beta_n}$. The proofs of this equalities are similar, so we omit the second one.

By definition of $\gamma$, $\alpha_n < \gamma$ for every $n \in \omega$ and therefore, $A_{\alpha_n} \subseteq A_\gamma$. To prove the other inclusion fix $\xi < \gamma$. Hence, there is $m \in \omega$ such that $\alpha_m > \xi$. Thus, $A_{\alpha_m} \supseteq A_\xi$. $\square$

**Proposition 3.12.** $\mathrm{Fr}(\omega_1)$ *is not minimally generated.*

*Proof.* Let us suppose, in order to get a contradiction, that $\{B_\xi : \xi < \sigma\}$ witness that $\mathrm{Fr}(\omega_1)$ is minimally generated. Moreover, we will assume, following Corollary 2.17, that $\ell(\mathrm{Fr}(\omega_1)) = \sigma$ and that for every $\xi + 1 < \sigma$, $B_\xi <_m B_{\xi+1}$.

Assume that $\sigma < \omega_1$. We claim that for every $\xi < \sigma$, $|B_\xi| \leqslant \omega$. Certainly, let us proceeded by induction. Obviously, $|B_0| \leqslant \omega$. Now, if $\xi + 1 < \sigma$ and $|B_\xi| \leqslant \omega$, fix $x \in B_{\xi+1} \setminus B_\xi$. We get that $B_\xi(x) = B_\xi$ and therefore, $|B_{\xi+1}| = |\langle B_\xi \cup \{x\}\rangle| \leqslant \omega$. Finally, if $\gamma < \sigma$ is limit and $|B_\xi| \leqslant \omega$ for each $\xi < \gamma$, then $|B_\gamma| = \left|\bigcup_{\xi<\gamma} B_\xi\right| \leqslant \omega$. As a result of the claim, $|\mathrm{Fr}(\omega_1)| = \left|\bigcup_{\xi<\sigma} B_\xi\right| \leqslant \omega$; which is impossible because $\mathrm{Fr}(\omega_1)$ has a subset of size $\aleph_1$. Thus, $\omega_1 \leqslant \sigma$.

Now, let us prove that $\omega_1 = \sigma$. Assume that $\omega_1 < \sigma$ and notice that $B_{\omega_1} \in [\mathrm{Fr}(\omega_1)]^{\omega_1}$. Hence, by Proposition 1.22, there is $S \in [B_{\omega_1}]^{\omega_1}$ such that $S$ is independent. Since $\langle B_\xi : \xi < \omega_1\rangle$ witnesses that $B_{\omega_1}$ is minimally generated, $\ell(\langle S\rangle_B) \leqslant \ell(B_{\omega_1}) \leqslant \omega_1 < \sigma$ (see Proposition 2.19.(2)) and this is contradiction because $\langle S\rangle_B$ and $\mathrm{Fr}(\omega_1)$ are isomorphic.

Let $\{x_\xi : \xi < \omega_1\}$ be an independent set that generates $\mathrm{Fr}(\omega_1)$ and define for each $\alpha < \omega_1$, $A_\alpha := \langle x_\xi : \xi < \alpha\rangle_{\mathrm{Fr}(\omega_1)}$. Then, Lemma 2.14 applies and therefore $\{A_\alpha : \alpha < \omega_1\}$ is a continuous representation of $\mathrm{Fr}(\omega_1)$. According to Lemma 3.11, there is a limit ordinal $\gamma < \omega_1$ in such a way that $A_\gamma = B_\gamma$. Then, by Proposition 1.21, $A_\gamma$ is atomless. Besides, $x_{\gamma+1} \in \mathrm{Fr}(\omega_1) \setminus A_\gamma$ is independent over $A_\gamma$ (see Lemma 1.24). Following Lemma 2.21, $\mathrm{Fr}(\omega_1)$ is not minimally generated over $A_\gamma$; this contradicts that $B_\gamma \leqslant_{mg} \mathrm{Fr}(\omega_1)$ (again, Proposition 2.19.(2)). □

**Corollary 3.13.** $\mathrm{Fr}(\kappa)$ *is not minimally generated whenever* $\kappa \geqslant \aleph_1$.

*Proof.* Given that subsets of independent sets are independent, $\mathrm{Fr}(\kappa)$ has an independent set of size $\aleph_1$. Hence, $\mathrm{Fr}(\omega_1) \leqslant \mathrm{Fr}(\kappa)$ and the rest follows as consequence of our previous result and Proposition 2.19.(3). □

**Corollary 3.14.** *Assume that for each* $n < \omega$, $U_n$ *is an infinite independent subset of a Boolean algebra* $A_n$. *Then, the product* $\prod_{n<\omega} A_n$ *is not minimally generated. In particular,* $\mathrm{Fr}(\omega)$ *is minimally generated (see Proposition 3.10), but the power* $(\mathrm{Fr}(\omega))^\omega$ *is not.*

*Proof.* It can be proved that $\prod_{n<\omega} A_n$ has an independent subset of size $\left|\prod_{n<\omega} U_n\right| \geqslant \omega^\omega = \mathfrak{c} \geqslant \omega_1$ (check [10, Theorem 13.10]). Therefore, $\mathrm{Fr}(\omega_1)$ embeds into $\prod_{n<\omega} A_n$. $\square$

The Theorem of Balcar-Franěk (see [10, Theorem 13.6]) provides us with more examples of Boolean algebras that are not minimally generated. This result asserts that every infinite complete Boolean algebra $A$ has an independent subset of size $|A|$. Therefore, Proposition 3.12 along with Balcar-Franěk's theorem ensures that every complete Boolean algebra with at least $\aleph_1$ elements is not minimally generated.

## 3.4 Dense Trees

Following the same approach as S. Koppelberg did in [11], in this section we are going to consider trees inside Boolean algebras. In particular, we shall see that every minimally generated Boolean algebra contains a dense tree.

**Definition 3.15.** Let $A$ be a subalgebra of $B$. We say that $A$ *is dense in* $B$ if for each $b \in B^+$ there is $a \in A^+$ such that $a \leqslant b$.

Let us begin by studying under which circumstances $A$ is dense in $A(x)$.

**Definition 3.16.** Assume that $A$ is a Boolean algebra and $I$ is an ideal in $A$. If there exist $a \in A$ such that $I = A \upharpoonright a$, then $I$ is a *principal ideal.*

**Lemma 3.17.** *Let $A <_m B$ and $x \in B \setminus A$. If $A \upharpoonright x$ and $A \upharpoonright (-x)$ are not principal ideals in $A$ and $A <_m A(x)$, then $A$ is dense in $A(x)$.*

*Proof.* Given $c \in A(x)^+$ there are $a, b \in A$ with $c = (a \wedge x) \vee (b - x)$. Consider first the case $a \wedge x \neq 0$. If $a \in J_x$, then $a \wedge x \in A$ and $a \wedge x \leqslant c$. On the other hand, when $a \notin J_x$ we get $-a \in J_x$ (see Proposition 2.6) and thus, $x - a \in A$. Hence, $x - a \in A \upharpoonright x$, i.e., $A \upharpoonright (x-a) \subseteq A \upharpoonright x$ and so there is $d \in A \upharpoonright x \setminus A \upharpoonright (x-a)$ ($A \upharpoonright x$ is not principal). Since, $d \leqslant x$ and $d \nleqslant x - a$, we deduce that $d \wedge a \neq 0$. Therefore, $d \wedge a \in A^+$ and $d \wedge a \leqslant x \wedge a \leqslant c$. The second case, when $b - x > 0$, has a similar proof. $\square$

**Lemma 3.18.** *Let $A <_m B$ and $x \in B \setminus A$. If either $A \upharpoonright x$ or $A \upharpoonright (-x)$ are principal ideals in $A$, then there is $y \in \mathrm{At}(A(x)) \setminus A$.*

*Proof.* Assume that $d \in A$ is such that $A \upharpoonright x = A \upharpoonright d$. Define $y := x - d$ and let us prove that $y \in A(x) \setminus A$. Clearly, $y \in A(x)$. In order to get a contradiction, let us suppose that $y \in A$. By Lemma 2.2, $-d \in J_x$. Therefore, there are $a, b \in A$ in such a way that $a \leqslant x$, $b \leqslant -x$ and $-d \leqslant a \vee b$ (check the paragraph immediately after Definition 2.1). However, $A \upharpoonright x = A \upharpoonright d$ and thus, $-d \leqslant d \vee b$. Immediately, $-d \leqslant b$ and in consequence, $d \geqslant x$. On the other hand, $d \in A \upharpoonright x$. Therefore, $d = x$, contradicting the assumption $x \notin A$. Hence, $y \notin A$.

Now, apply Proposition 2.6 to deduce that $J_x^*$ is an ultrafilter in $A$. We claim that $G := \{c \in A : y \leqslant c\} = J_x^*$. Indeed, fix $c \in J_x^*$, equivalently, $-c \in J_x$. Then, there are $e_0, e_1 \in A$ such that $e_0 \leqslant x$, $e_1 \leqslant -x$ and $-c \leqslant e_0 \vee e_1$. Notice that $e_0 \leqslant d$. Hence, $-c \leqslant d \vee (-x)$ and this implies that $c \geqslant (-d) \wedge x = y$. Therefore, $G$ is a filter in $A$ containing $J_x^*$, i.e., they are equal as claimed.

We will show that $y \in \mathrm{At}(A(x))$. Let $z \in A(x)$ be such that $z \leqslant y$. We are going to prove that either $z = y$ or $z = 0$. Let $a, b \in A$ be such that $z = (a \wedge x) \vee (b - x)$. Considering that $z \leqslant y$, we have that $b - x = 0$ and in consequence, $z = a \wedge x$. Moreover, $a \wedge x \leqslant -d$. On the other hand, since $J_x^*$ is an ultrafilter in $A$, $a \in G$ or $-a \in J_x^*$. In the first case, $a \geqslant x - d$, and thus, $z \geqslant x - d = y$. Therefore, $z = y$. If $-a \in J_x^*$, then $a \in J_x^*$. Proceeding as we did in previous paragraphs, $a \leqslant d \vee (-x)$. Hence, $a \wedge x \leqslant d \wedge x \leqslant d$ and so, $z = 0$.

Now, if $A \upharpoonright (-x)$ is a principal ideal in $A$, let $d \in A$ be such that $A \upharpoonright (-x) = A \upharpoonright d$. Define $y := d - x$ and proceed as in the previous paragraphs. □

**Lemma 3.19.** *Let $A < B$ and $x \in B \setminus A$ be such that $A <_m A(x)$. If $D \subseteq A$ satisfies that $\langle D \rangle_A$ is dense in $A$ and $y \in \mathrm{At}(A(x)) \setminus A$, then $\langle D \cup \{y\} \rangle_{A(x)}$ is dense in $A(x)$.*

*Proof.* Given that $A <_m A(x)$, $A(x) = A(y)$. Fix $z \in (A(y))^+$ and get $a, b \in A$ such that $z = (a \wedge y) \vee (b - y)$. Notice that either $a \wedge y \neq 0$ or $b - y \neq 0$. When $a \wedge y \neq 0$, we obtain that $0 < a \wedge y \leqslant y$ and since $y$ is an atom, $y = a \wedge y$; thus, $y$ is a member of $\langle D \cup \{y\} \rangle_{A(x)}$ satisfying $y \leqslant z$.

In case that $b - y \neq 0$, fix $d \in \langle D \rangle_A^+$ in such a way that $d \leqslant b$. Observe that $d - y \leqslant z$ and $d - y \in \langle D \cup \{y\} \rangle_{A(x)}$. Now note that the equality $d - y = 0$ implies that $0 < d \leqslant y$ and so, $y = d \in A$. Hence, $d - y \neq 0$. $\square$

Consider $\langle T, \leqslant \rangle$ a partial order. Recall that $\langle T, \leqslant \rangle$ is a *tree* if for each $x \in T$, the collection $\{y \in T : y < x\}$ is well-ordered by $\leqslant$.

Now, let us introduce the notion of a tree inside a Boolean algebra.

**Definition 3.20.** Let $A$ be a Boolean algebra and $T \subseteq A^+$. We say that $T$ *is a tree in* $A$ if

1. for every $x, y \in T$, we have that $x \leqslant y$, $y \leqslant x$ or $x \wedge y = 0$, where $\leqslant$ is the natural order of $A$, and

2. $\langle T, \leqslant_T \rangle$ is a tree, where for each $x, y \in T$, $x \leqslant_T y$ if and only if $y \leqslant x$.

If $T$ is a tree in $A$ and $\langle T \rangle_A$ is dense in $A$, we shall say that $T$ *is a dense tree in* $A$.

**Proposition 3.21.** *If $B$ is minimally generated Boolean algebra, then there is $T$, a dense tree in $B$, such that $\langle T \rangle_B \leqslant_{mg} B$.*

*Proof.* Let $\{B_\alpha : \alpha < \sigma\}$ be a witness to the fact that $B$ is minimally generated. Moreover (see Lemma 2.16), assume that $B_\alpha <_m B_{\alpha+1}$ whenever $\alpha + 1 < \sigma$.

Denote by $E_0$ the set of all ordinals $\alpha$ such that $\alpha + 1 < \sigma$ and $B_\alpha$ is not dense in $B_{\alpha+1}$. Also, $E_1 := \sigma \setminus E_0$.

Now, for each $\alpha \in E_0$ use Lemmas 3.17 and 3.18 to obtain $x_\alpha \in \mathrm{At}(B_{\alpha+1}) \setminus B_\alpha$ and if $\alpha \in E_1$, fix an arbitrary point $x_\alpha \in B_{\alpha+1} \setminus B_\alpha$. Proceeding as in the proof of Lemma 2.14 we deduce that $B_\alpha = \langle \{x_\xi : \xi < \alpha\} \rangle$ for each $\alpha < \sigma$.

Set $T := \{x_\alpha : \alpha \in E_0\}$ and define $\leqslant_T$ as follows: $a \leqslant_T b$ if and only if $b \leqslant a$, for any $a, b \in T$. Let us argue that $\langle T, \leqslant_T \rangle$ is a tree in $B$.

Given $\alpha, \beta \in E_0$ with $\alpha < \beta$, we get $x_\alpha, x_\beta \in B_{\beta+1}$ and so, the condition $x_\beta \in \mathrm{At}(B_{\beta+1})$ implies that either $x_\beta \leqslant x_\alpha$ or $x_\alpha \wedge x_\beta = 0$. On the other hand, if $\alpha \in E_0$ and we set $S := \{y \in T : y <_T x_\alpha\}$ we claim that $S$ is well-ordered by $\leqslant_T$. Indeed, start by letting $E_2 := \{\xi \in \alpha \cap E_0 : x_\alpha < x_\xi\}$ and define $g : E_2 \to S$ by $g(\xi) = x_\xi$, for each $\xi \in E_2$. We will show that $g$ is an isomorphism between $\langle E_2, \in\rangle$ and $\langle S, <_T\rangle$. Assume $\xi,\eta \in E_2$ are such that $\xi < \eta$. Then, as we saw before, $x_\xi \leqslant x_\eta$ or $x_\xi \wedge x_\eta = 0$, but the fact $0 < x_\alpha \leqslant x_\xi \wedge x_\eta$ implies that $x_\eta < x_\xi$, i.e., $g(\xi) <_T g(\eta)$. Since $\langle E_2, \in\rangle$ is a linear ordering, we only need to argue that $g$ is onto. Suppose that $\beta \in E_0$ satisfies $x_\beta \in S$, i.e, $0 < x_\alpha < x_\beta$ and notice that if $\alpha \leqslant \beta$, then we would have that $x_\alpha \in B_{\beta+1}$ and $x_\beta \in \mathrm{At}(B_{\beta+1})$; a contradiction to $0 < x_\alpha < x_\beta$. Hence, $\beta < \alpha$ and therefore, $\beta \in E_2$. In conclusion, $g$ is an isomorphism, which implies that $\leqslant_T$ well-orders $S$.

Now that we know that $\langle T, \leqslant_T\rangle$ is a tree in $B$, we will focus on showing that $\langle T\rangle$ is dense in $B$. To do so, define, for each $\alpha < \sigma$, $C_\alpha := \langle\{x_\xi : \xi \in \alpha \cap E_0\}\rangle$. We claim that $C_\alpha$ is dense in $B_\alpha$ whenever $\alpha < \sigma$. Our proof will be by transfinite induction. Clearly, $C_0 = B_0$. Now assume that for some $\alpha < \sigma$ we have that $C_\alpha$ is dense in $B_\alpha$ and $\alpha + 1 < \sigma$. When $\alpha \in E_0$, $C_{\alpha+1} = C_\alpha(x_\alpha)$ and given that $x_\alpha \in \mathrm{At}(B_{\alpha+1}) \setminus B_\alpha$, Lemma 3.19 implies that $C_\alpha$ is dense in $C_{\alpha+1}$. When $\alpha \in E_1$, $C_\alpha = C_{\alpha+1}$ and $B_\alpha$ is dense in $B_{\alpha+1}$; thus, $C_{\alpha+1}$ is dense in $B_\alpha$, as needed. Finally, assume that $\alpha < \sigma$ is limit and has the property that $C_\xi$ is dense in $B_\xi$ for all $\xi < \alpha$. Hence, given $x \in B_\alpha^+$, there is $\beta < \alpha$ with $x \in B_\beta$ and by our inductive hypothesis, there exists $y \in C_\beta^+ \subseteq C_\alpha^+$ in such a way that $y \leqslant x$, i.e., $C_\alpha$ is dense in $B_\alpha$.

As a consequence of the previous paragraph, $C := \bigcup_{\alpha<\sigma} C_\alpha$ is a dense subalgebra of $\bigcup_{\alpha<\sigma} B_\alpha = B$, which together with the fact $C \leqslant \langle T\rangle$ implies that $\langle T\rangle$ is, indeed, a dense subalgebra of $B$.

Set $A := \langle T\rangle$. To finish our proof, let us argue that $A \leqslant_{mg} B$. Start by noticing that when $E_1 = \emptyset$, $A = B$ an so, $A \leqslant_{mg} B$, trivially. Hence, assume that $E_1 \neq \emptyset$ and denote by $\eta_0$ and $\eta_1$ the order types of $\langle E_0, \in\rangle$ and $\langle E_1, \in\rangle$, respectively. Set $\eta = \eta_0 + \eta_1$, and fix $f : \eta \to E_0 \cup E_1$ in such a way that both, $f \upharpoonright \eta_0 : \eta_0 \to E_0$ and $f \upharpoonright (\eta \setminus \eta_0) : \eta \setminus \eta_0 \to E_1$, are isomorphisms.

For each $\alpha < \eta$, let $D_\alpha := \left\langle\left\{x_{f(\xi)} : \xi < \alpha\right\}\right\rangle$. Since $D_{\eta_0} = A$, we only need to show that $\{D_\alpha : \alpha < \eta\}$ witnesses that $B$ is minimally generated.

Given that $B$ is generated by $\{x_\alpha : \alpha < \sigma\}$, as we mentioned at the beginning of our proof, Lemma 2.14 guarantees that $\{D_\alpha : \alpha < \eta\}$ is a continuous representation of $B$. Now consider $\alpha < \eta$ in such a way that $\alpha + 1 < \eta$. To prove that $D_\alpha \leqslant_m D_{\alpha+1}$ we consider two cases.

First, when $\alpha < \eta_0$, our assumption on $f \restriction \eta_0$ guarantees that $\left\{x_{f(\xi)} : \xi < \alpha\right\} \subseteq \{x_\xi : \xi < f(\alpha)\}$ and therefore, $D_\alpha \subseteq B_{f(\alpha)}$. Also, the fact $f(\alpha) \in E_0$ implies that $x_{f(\alpha)} \in \operatorname{At}\left(B_{f(\alpha)+1}\right) \setminus B_{f(\alpha)} \subseteq \operatorname{At}\left(B_{f(\alpha)}\right) \setminus D_\alpha$ and according to Corollary 2.8, $D_\alpha <_m D_\alpha\left(x_{f(\alpha)}\right) = D_{\alpha+1}$.

Before we embark on the case $\eta_0 \leqslant \alpha$, let us prove that if $\eta_0 \leqslant \gamma < \eta$, then $B_{f(\gamma)} \leqslant D_\gamma$. Assume that $\xi < f(\gamma)$ is arbitrary. Since $\sigma = E_0 \cup E_1$, we consider two possibilities. When $\xi \in E_0$, we obtain $f^{-1}(\xi) < \eta_0 \leqslant \gamma$ and thus, $x_\xi = x_{f(f^{-1}(\xi))} \in D_\gamma$. On the other hand, from $\xi \in E_1$ we deduce that $\xi < f(\gamma) \in E_1$ and so, $f^{-1}(\xi) < \gamma$ which, once again, gives $x_\xi = x_{f(f^{-1}(\xi))} \in D_\gamma$. As a consequence, $\{x_\xi : \xi < f(\gamma)\} \subseteq D_\gamma$ and therefore, $B_{f(\gamma)} \leqslant D_\gamma$.

We are ready to finish the proof of our proposition. Suppose that $\eta_0 \leqslant \alpha < \alpha + 1 < \eta$ and fix $\xi < \alpha$. If we show that $D_\alpha \cap \left\{x_{f(\alpha)} \wedge x_{f(\xi)}, x_{f(\alpha)} - x_{f(\xi)}\right\} \neq \emptyset$, then by Corollary 2.7 would give us $D_\alpha \leqslant_m D_{\alpha+1}$. To simplify notation, set $y := x_{f(\xi)}$ and $z := x_{f(\alpha)}$. There are two cases. If $f(\xi) < f(\alpha)$, then $y \in B_{f(\alpha)} \leqslant_m B_{f(\alpha)+1} = B_{f(\alpha)}(z)$ and according to Corollary 2.7 we obtain $\emptyset \neq B_{f(\alpha)} \cap \{z \wedge y, z - y\} \subseteq D_\alpha \cap \{z \wedge y, z - y\}$. Now, when $f(\alpha) < f(\xi)$, the assumption $\eta_0 \leqslant \alpha < \eta$ gives $f(\alpha) \in E_1$. Notice that if $f(\xi) \in E_1$, the fact that $f \restriction (\eta \setminus \eta_0)$ is an isomorphism would produce $\alpha < \xi$. Thus, $f(\xi) \notin E_1$, i.e., $f(\xi) \in E_0$. As a consequence, $y \in \operatorname{At}\left(B_{f(\xi)+1}\right)$ and since $z \in B_{f(\xi)} \subseteq B_{f(\xi)+1}$, we get that either $y \leqslant z$ or $y \wedge z = 0$. In other words, $y \wedge z \in \{y, 0\} \subseteq D_\alpha$. $\square$

## 3.5 The Factor Algebra $\mathcal{P}(\omega)/\operatorname{fin}$

The main goal of this section is to present the factor algebra $\mathcal{P}(\omega)/\operatorname{fin}$ as a negative example of minimal generation. Reading this section requires from the reader knowledge of definitions and results presented in Section 1.7.

It can be verified that that $\operatorname{fin} := [\omega]^{<\omega}$ is an ideal in $\mathcal{P}(\omega)$, in the sense of Definition 1.9. From now on, $\mathcal{P}(\omega)/\operatorname{fin}$ will be considered as a Boolean algebra.

For every $a \in \mathcal{P}(\omega)$ define $[a]_{\mathrm{fin}}$ as the equivalence class of $a$ modulo the ideal fin.

Let us mention some immediate properties of $\mathcal{P}(\omega)/\operatorname{fin}$: for every $a, b, \in \mathcal{P}(\omega)$ we get $[a]_{\mathrm{fin}} \leqslant [b]_{\mathrm{fin}}$ if and only if $a \subseteq^* b$; $[a]_{\mathrm{fin}} > 0$ as long as $a$ is infinite, and when $a$ is finite, it follows that $[a]_{\mathrm{fin}} = 0$. On the other hand, given that $a =^* b$ if and only if $a \triangle b$ is finite, it follows that $[a]_{\mathrm{fin}} < [b]_{\mathrm{fin}}$ if and only if $a \subset^* b$.

**Proposition 3.22.** $\mathcal{P}(\omega)/\operatorname{fin}$ *is an atomless Boolean algebra.*

*Proof.* Let $x \in \mathcal{P}(\omega)/\operatorname{fin}$ be a positive element. Fix $a \in x$ and notice that $a \in [\omega]^{\omega}$. So, we can find $b, c \in [a]^{\omega}$ in such a way that $b \cap c = \emptyset$. Since $b$ is infinite, $[b]_{\mathrm{fin}} > 0$ and moreover, $[b]_{\mathrm{fin}} \leqslant x$ because $b \subseteq a$. We claim that $x \neq b$. Certainly, $a \triangle b \in [\omega]^{\omega}$ because $c \subseteq a \setminus b \subseteq a \triangle b$. Therefore, $0 < [b]_{\mathrm{fin}} < x$. □

It is well known that there is at least one almost disjoint family of size $\mathfrak{c}$ (the proof appears in [13, Theorem 1.3, Section 2.1]), and this fact can be used to show that $\mathcal{P}(\omega)/\operatorname{fin}$ has size $\mathfrak{c}$. Now let us show that $\operatorname{Fr}(\mathfrak{c})$ embeds as a subalgebra of $\mathcal{P}(\omega)/\operatorname{fin}$.

**Proposition 3.23.** $\mathcal{P}(\omega)/\operatorname{fin}$ *has an independent family of size* $\mathfrak{c}$.

*Proof.* As usual, $\mathbb{R}$ and $\mathbb{Q}$ will denote the sets of all real numbers and of all rational numbers, respectively. Recall that if $a, b \in \mathbb{R}$, then $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. Define

$$\mathcal{S} := \{(a, b) : a, b \in \mathbb{Q} \ \& \ a < b\} \text{ and } \mathcal{B} := \left\{\bigcup \mathcal{E} : \mathcal{E} \in [\mathcal{S}]^{<\omega}\right\}.$$

Clearly, $|\mathcal{B}| \leqslant \left|[\mathcal{S}]^{<\omega}\right| = \omega$. Moreover, $\{(0, n) : n \in \omega\} \subseteq \mathcal{B}$, and thus, $|\mathcal{B}| = \omega$.

Now, for every $r \in \mathbb{R}$ set $F_r := \{B \in \mathcal{B} : r \in B\}$. We claim that $\{F_r : r \in \mathbb{R}\}$ has size $\mathfrak{c}$. Indeed, let $r_0, r_1 \in \mathbb{R}$ be such that $r_0 < r_1$ and find $q \in \mathbb{Q}$ in such a way that $q \in (r_0, r_1)$. Observe that $r_0 \in (q_0, q)$ for some $q_0 \in \mathbb{Q}$; this guarantees that $(q_0, q) \in F_{r_0} \setminus F_{r_1}$. Therefore, $\mathfrak{c} \leqslant |\{F_r : r \in \mathbb{R}\}|$. The other inequality is fairly trivial.

We are going to prove that $\{F_r : r \in \mathbb{R}\}$ satisfies the following: for every $G, H \in [\mathbb{R}]^{<\omega} \setminus \{\emptyset\}$ such that $G \cap H = \emptyset$, we obtain that $\left|\bigcap_{r \in G} F_r \setminus \bigcup_{s \in H} F_s\right| = \omega$. In order to do that, use the density of $\mathbb{Q}$ in $\mathbb{R}$ to get $\{I_r : r \in G\} \subseteq \mathcal{S}$ such that:

1. $r \in I_r$, for every $r \in G$,

2. for every $r, s \in G$, if $r \neq s$ then $I_r \cap I_s = \emptyset$, and
3. $H \cap \bigcup \{I_r : r \in G\} = \emptyset$.

Fix $t \in G$ and assume that $I_t = (a, b)$. For every $q \in S := (t, b) \cap \mathbb{Q}$ define $B_q := (a, q) \cup \bigcup \{I_r : r \in G \setminus \{t\}\}$. Immediately, $B_q \in \bigcap_{r \in G} F_r$ for every $q \in S$. Moreover, property (3) implies that $B_q \notin F_s$, for every $s \in H$. Hence, $\{B_q : q \in S\} \subseteq \bigcap_{r \in G} F_r \setminus \bigcup_{s \in H} F_s$. Finally, if $q, q' \in S$ and $q \neq q'$, then $B_q \neq B_{q'}$; this makes certain that $\bigcap_{r \in G} F_r \setminus \bigcup_{s \in H} F_s$ has cardinality $\omega$.

Since $\mathcal{B}$ is countable infinite, the sets $\mathcal{P}(\omega)$ and $\mathcal{P}(\mathcal{B})$ are isomorphic and, as a consequence, there is $\{a_\alpha : \alpha \in \mathfrak{c}\} \subseteq \mathcal{P}(\omega)$ with the property that for every $G, H \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$ such that $G \cap H = \emptyset$, we have that $\left|\bigcap_{\beta \in G} a_\beta \setminus \bigcup_{\gamma \in H} a_\gamma\right| = \omega$.

Let us prove that $\{[a_\alpha]_{\text{fin}} : \alpha \in \mathfrak{c}\}$ is an independent family in $\mathcal{P}(\omega)/\operatorname{fin}$. Fix $G, H \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$ such that $G \cap H = \emptyset$. Then,

$$\left(\bigwedge_{\beta \in G} [a_\beta]_{\text{fin}}\right) - \left(\bigvee_{\gamma \in H} [a_\gamma]_{\text{fin}}\right) = \left[\bigcap_{\beta \in G} a_\beta \setminus \bigcup_{\gamma \in H} a_\gamma\right]_{\text{fin}} > 0.$$

This inequality completes the proof. □

By the previous proposition, $\operatorname{Fr}(\mathfrak{c})$ embeds into $\mathcal{P}(\omega)/\operatorname{fin}$ and the next results follows (see Corollary 3.13 and Proposition 2.19.(3)).

**Corollary 3.24.** $\mathcal{P}(\omega)/\operatorname{fin}$ *is not minimally generated.*

Denote by $\beta\omega$ the Čech-Stone compactification of the integers. It is a well known fact that if $F$ is an infinite closed set of $\beta\omega$, then $|F| = 2^{\mathfrak{c}}$ (see [6, Theorem 5.4, Chapter 3]). On the other hand, given $X$, an infinite Hausdorff space, and $\{x_n : n \in \omega\} \subseteq X$, a converging sequence to $x$, we can easily prove that $\{x_n : n \in \omega\} \cup \{x\}$ is compact and in consequence, it is also a closed set of $X$. Therefore, $\beta\omega$ has no infinite convergent sequences (equivalently, it contains no copy of the linearly ordered topological space $\omega + 1$).

By definition, an *Efimov space* is an infinite compact Hausdorff space which contains neither a copy of $\beta\omega$ nor a copy of $\omega+1$. Thus, Efimov's Problem, which was posed originally in [4], asks if there are Efimov spaces. This question has been answered consistently several times (for example, Fedorčuk constructed in [5] an Efimov space using $\diamondsuit$), but it is an open problem to get an Efimov space in ZFC (or show that, consistently, there are no Efimov spaces).

Now, since the natural projection $\pi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)/\operatorname{fin}$ is a surjective homomorphism, Proposition 2.18.(3) and Corollary 3.24 guarantee that $\mathcal{P}(\omega)$ is not minimally generated. Given that the Stone space of $\mathcal{P}(\omega)$ is $\beta\omega$ (check [6, Theorem 5.1, Chapter 3]), we can use a duality argument to conclude that if $A$ is minimally generated, then $\mathrm{S}(A)$ contains no copy of $\beta\omega$.

By previous paragraph, to solve Efimov's problem we only need to construct a minimally generated Boolean algebra such that its Stone space does not contain an infinite convergent sequence. This was done in [3] via Koszmider's forcing.

Previously, we have seen in Proposition 3.21 that every minimally generated Boolean algebra possesses a dense tree. However, we shall verify that this property does not characterize them. To be precise, we will prove next that $\mathcal{P}(\omega)/\operatorname{fin}$ has a dense tree (this result was originally proved by Balcar, Simon and Pelant in [1]).

**Lemma 3.25.** *If $\{D_\alpha : \alpha < \mathfrak{h}\}$ is a decreasing family of dense open sets in $\omega$ such that $\bigcap_{\alpha<\mathfrak{h}} D_\alpha = \emptyset$, then there is a family $\{T_\alpha : \alpha < \mathfrak{h}\}$ satisfying the following for each $\alpha < \mathfrak{h}$.*

1. *$T_0 = \{\omega\}$,*
2. *$T_\alpha$ is a MADF whenever $\alpha > 0$,*
3. *if $\beta < \alpha$ and $s \in T_\alpha$, there is $t \in T_\beta$ such that $s \subset^* t$,*
4. *$T_{2\alpha+1} \subseteq D_\alpha$, and*
5. *for each $a \in [\omega]^\omega$, if $|\{s \in T_{2\alpha+1} : a \mid s\}| = \mathfrak{c}$, then there exists $t \in T_{2\alpha+2}$ such that $t \subseteq a$.*

*Proof.* Let us start by defining $T_0 := \{\omega\}$ and $T_1$ as an arbitrary MADF. The next step is to fix $\lambda < \mathfrak{h}$, a limit ordinal, and to assume that $\{T_\alpha : \alpha < \lambda\}$ satisfies conditions (1)-(5) ((4) and (5) only apply when $2\alpha+1 < \lambda$). By Lemma 1.46, $\overline{T_\alpha}$ is dense and open, for every $\alpha < \lambda$. Then, $\bigcap_{\alpha<\lambda} \overline{T_\alpha}$ is dense and open (recall Lemma 1.48). Apply Lemma 1.45 to get

$\mathcal{A}$, a MADF, in such a way that $\overline{\mathcal{A}} \subseteq \bigcap_{\alpha<\lambda} \overline{T_\alpha}$, and define $T_\lambda := \mathcal{A}$. Routine arguments show that $\{T_\xi : \xi \leqslant \lambda\}$ satisfies (1), (2), (4) and (5). Now, for (3): if $s \in T_\lambda$ and $\beta < \lambda$ are arbitrary, then $T_\lambda \subseteq \overline{T_\lambda} \subseteq \overline{T_{\beta+1}}$ and so, there is $u \in T_{\beta+1}$ with $s \subseteq^* u$. According to our inductive hypothesis, there exists $t \in T_\beta$ such that $u \subset^* t$ and hence, $s \subset^* t$, as needed.

To complete the recursion, suppose that $\gamma < \mathfrak{h}$ is a successor ordinal for which the family $\{T_\xi : \xi < \gamma\}$ has been constructed according to conditions (1)-(5). We face two cases: either $\gamma = 2\alpha + 1$ or $\gamma = 2\alpha + 2$, for some ordinal $\alpha$. Firstly, assume that $\gamma = 2\alpha + 1$ and for every $y \in T_{2\alpha}$, fix $y^+$ and $y^-$, two infinite sets, in such a way that they form a partition of $y$. Next, define $T'_{2\alpha} = \{y^- : y \in T_{2\alpha}\} \cup \{y^+ : y \in T_{2\alpha}\}$. A straightforward argument shows that $T'_{2\alpha}$ is a MADF and in consequence, there is $\mathcal{A}$, a MADF, such that $\overline{\mathcal{A}} \subseteq \overline{T'_{2\alpha}} \cap D_\alpha$. We shall define $T_{2\alpha+1} := \mathcal{A}$. Clearly, (4) is satisfied. Now, regarding (3), let $s \in T_{2\alpha+1}$ and $\beta \leqslant 2\alpha$ be arbitrary. There are $y \in T_{2\alpha}$ and $z \in \{y^-, y^+\}$ with $s \subseteq^* z \subseteq^* y$. Thus, when $\beta = 2\alpha$, we set $t := y$ to obtain $t \in T_\beta$ and $s \subset^* t$. On the other hand, for $\beta < 2\alpha$ we use our inductive hypothesis to get $t \in T_\beta$ such that $y \subset^* t$ and so, $s \subset^* t$. In conclusion, $\{T_\xi : \xi \leqslant \gamma\}$ satisfies (1)-(5).

For the case $\gamma = 2\alpha + 2$ start by setting, for every $a \in [\omega]^\omega$, the collection $E_a := \{s \in T_{2\alpha+1} : a \mid s\}$. Also, define $X := \{a \in [\omega]^\omega : |E_a| = \mathfrak{c}\}$. Now consider $f : X \to |X|$ a bijection, and $\varphi : |X| \to T_{2\alpha+1}$ in such a way that for every $\beta < |X|$, $\varphi(\beta)$ is an arbitrary member of $E_{f^{-1}(\beta)} \setminus \varphi[\beta]$ (note that $|X| \leqslant \mathfrak{c}$). Finally, let us define $\psi := \varphi \circ f$. In this way, $a \mid \psi(a)$ for every $a \in X$ and $\psi(b) \neq \psi(c)$, whenever $b, c \in X$ are different.

Fix $b \in T_{2\alpha+1}$. If $b \notin \operatorname{ran}(\psi)$, partition $b$ into two infinite sets, $b^-$ and $b^+$. For the case that there is $a \in X$, with $\psi(a) = b$, use that $a$ and $b$ are compatible to get $b_0$ and $b_1$, a pair of infinite sets, in such a way that they form a partition of $a \cap b$, and define $b^- := b_0$ and $b^+ := b_1 \cup (b \setminus a)$. Next, establish $T_{2\alpha+2}$ as the collection $\{b^- : b \in T_{2\alpha+1}\} \cup \{b^+ : b \in T_{2\alpha+1}\}$. Let us verify condition (5): assume that $a \in X$ is arbitrary and set $b := \psi(a)$. Thus, $b^- \in T_{2\alpha+2}$ and $b^- \subset^* a$.

To finish our argument, let us check that condition (3) holds. To do so, fix $\beta \leqslant 2\alpha + 1$ and $s \in T_{2\alpha+2}$. There is $b \in T_{2\alpha+1}$ with $s \subset^* b$. When $\beta = 2\alpha + 1$, set $t := b$ to get $s \subset^* t$

and if $\beta < 2\alpha + 1$, apply the inductive hypothesis to get $t \in T_\beta$ with $b \subset^* t$ and therefore, $s \subset^* t$, as wanted. $\square$

Given a tree $S$ and an ordinal $\alpha$, we will denote by $S(\alpha)$ the $\alpha$th level of $S$.

**Proposition 3.26.** *There is $T \subseteq [\omega]^\omega$ in such a way that the following conditions hold.*

1. *$\langle T, \supseteq^* \rangle$ is a tree (in the traditional sense) of height $\mathfrak{h}$,*

2. *$T(0) = \{\omega\}$,*

3. *for every $\alpha < \mathfrak{h}$, $T(\alpha)$ is a MADF, and*

4. *for each $a \in [\omega]^\omega$ there is $t \in T$ such that $t \subseteq a$.*

*Proof.* Suppose that $\{T_\alpha : \alpha < \mathfrak{h}\}$ and $\{D_\alpha : \alpha < \mathfrak{h}\}$ as described in Lemma 3.25, and define $T := \bigcup_{\alpha<\mathfrak{h}} T_\alpha$. Let us begin by proving that $\langle T, \supseteq^* \rangle$ is a partial order. Reflexivity and transitivity of $\supseteq^*$ are straightforward, so will only show that it is antisymmetric. If $s, t \in T$ are such that $t \supseteq^* s$ and $s \supseteq^* t$, then there are $\beta, \alpha < \mathfrak{h}$ in such a way that $s \in T_\alpha$ and $t \in T_\beta$. We claim that $\alpha = \beta$. Otherwise, assume that $\beta < \alpha$ and apply the previous lemma to get $r \in T_\beta$ satisfying $r \supset^* s$. Hence, $r \mid t$ as a result of Lemma 1.42. Given that $r$ and $t$ are compatible elements of $T_\beta$, an almost disjoint family, $r = t$. We deduce that $t \supset^* s$, which contradicts the assumption $t \subseteq^* s$. Therefore, $s, t \in T_\beta$. Moreover, $s \mid t$ and so, $s = t$.

Fix $t \in T$ and $\alpha < \mathfrak{h}$ such that $t \in T_\alpha$. We shall show that $S := \{s \in T : s \supset^* t\}$ is order isomorphic to $\alpha$. Let us start by proving that for each $\beta < \alpha$, $|S \cap T_\beta| = 1$. By Lemma 3.25, $S \cap T_\beta \neq \emptyset$. Let $s, s' \in S \cap T_\beta$. Then, $s \supset^* t$ and $s' \supset^* t$, this guarantees that $s \mid s'$. Since $T_\beta$ is almost disjoint, $s = s'$. For every $\beta < \alpha$ denote by $f(\beta)$ the only member of $S \cap T_\beta$. This gives a function $f : \alpha \to S$. In order to prove that $f$ is order-preserving, fix $\gamma < \beta < \alpha$ and get $r \in T_\gamma$ in such a way that $r \supset^* f(\beta) \supseteq^* t$ (recall condition (3) in Lemma 3.25). Since $f(\gamma) \in S$, we get $f(\gamma) \supseteq^* t$ and so, $f(\gamma) = r$ because $T_\gamma$ is almost disjoint. Thus, $f(\gamma) \supseteq^* f(\beta)$. This argument also shows that $f$ is one to one.

Now, to verify that $f$ is onto, fix $s \in S$ and obtain $\beta < \mathfrak{h}$ in such a way that $s \in T_\beta$. We claim that $\beta < \alpha$. Certainly, if $\beta = \alpha$, then $s, t \in T_\alpha$ and in consequence, $s = t$. When $\alpha < \beta$, there exists $r \in T_\alpha$ with $r \supset^* s$. Straightaway, $r \supset^* t$ and this implies that $r = t$. Therefore, $t \supset^* s$ and $s \supset^* t$ which is impossible. Therefore, we have just proved that for each $\alpha < \mathfrak{h}$, $T_\alpha \subseteq T(\alpha)$. For the reverse inclusion note that if $u \in T(\alpha)$, then, for some $\beta < \mathfrak{h}$, $u \in T_\beta \subseteq T(\beta)$ and so, $\alpha = \beta$.

Finally, let $a \in [\omega]^\omega$. In order to find $t \in T$, with $t \subseteq a$, we need to prove that $|\{s \in T_{2\gamma+1} : a \mid s\}| = \mathfrak{c}$, for some $\gamma < \mathfrak{h}$ (see Lemma 3.25). We will recursively construct a strictly increasing sequence $\{\alpha_n : n \in \omega\} \subseteq \mathfrak{h}$ and $\{s_n : n \in \omega\}$ in such a way that, for each $n \in \omega$, $s_n$ is a subset of $T_{\alpha_n}$ of size $2^n$ and for each $s \in s_n$ we have that $s \mid a$ and there are $s', s'' \in s_{n+1}$ such that $s \supseteq s'$, $s \supseteq s''$, and $s' \neq s''$.

Firstly, set $\alpha_0 = 0$ and $s_0 = \{\omega\}$. Next, assume that $s_n$ is already constructed and fix $s \in s_n$. Since $\bigcap_{\alpha<\mathfrak{h}} D_\alpha = \emptyset$, there is $\eta < \mathfrak{h}$ such that $s \cap a \notin D_\eta$. Fix $\xi(s) < \mathfrak{h}$ satisfying $\xi(s) > \max\{\alpha_n, \eta\}$.

We claim that for every $\beta \geqslant \xi(s)$, there are $t_0, t_1 \in T_{2\beta+1}$ in such a way that $t_0 \neq t_1$, $t_0 \mid (s \cap a)$, $t_1 \mid (s \cap a)$ and $t_0 \cup t_1 \subseteq^* s$. Certainly, there is $t_0 \in T_{2\beta+1}$, with $t_0 \mid (s \cap a)$, because $T_{2\beta+1}$ is a MADF. Let us assume that $t_0$ is the only member of $T_{2\beta+1}$ that is compatible with $s \cap a$. Then, $s \cap a \subseteq^* t_0$. On the other hand, $T_{2\beta+1} \subseteq D_\beta$. So, use that $D_\beta$ is open to get that $s \cap a \in D_\beta \subseteq D_\eta$, contradicting our choice for $\eta$.

Now, fix $i \in 2$ and let us prove that $t_i \subseteq^* s$. Start by using that $\alpha_n < 2\beta + 1$, along with Lemma 3.25, to get $r \in T_{\alpha_n}$ such that $t_i \subseteq^* r$. Since $s \cap r \supseteq^* s \cap t_i \supseteq (s \cap a) \cap t_i$, we deduce that $s \mid r$. However, $s$ and $r$ belong to $T_{\alpha_n}$, which is an almost disjoint family; therefore, $s = r$.

Define $\alpha_{n+1} := 2\sup\{\xi(s) : s \in s_n\} + 1$. By propositions 1.47 and 1.49, $\alpha_{n+1} < \mathfrak{h}$. For each $s \in s_n$ use the two previous paragraphs to obtain $s', s'' \in T_{\alpha_{n+1}}$ satisfying $s' \neq s''$, $s' \mid a$, $s'' \mid a$ and $s' \cup s'' \subseteq^* s$. We set $s_{n+1}$ as the collection $\{s' : s \in s_n\} \cup \{s'' : s \in s_n\}$. Let us verify that whenever $r$ and $s$ are different elements of $s_n$, $\{r', r''\} \cap \{s', s''\} = \emptyset$. Indeed, $s \neq r$ entails that $s \perp r$, which together with the assumption $r' \cup r'' \subseteq^* r$, implies that

$s \perp r'$ and $s \perp r''$. Given that $s' \cup s'' \subseteq^* s$, this concludes the recursion.

Now we are able to get a subset of $T$, order isomorphic to the Cantor tree. Indeed, define $S := \bigcup_{n\in\omega} s_n$ and notice that there is $h : {}^{<\omega}2 \to S$ such that the following hold.

(1) $h$ is a bijection,

(2) for each $n \in \omega$, $h\,[{}^{n}2] = s_n$, and

(3) for every $f \in {}^{\omega}2$ , if $m < n < \omega$, then $h\,(f \restriction n) \subset^* h\,(f \restriction m)$.

Notice that for every $f \in {}^{\omega}2$ and $n \in \omega$, $h\,(f \restriction n) \in s_n$; thus, $a \mid h\,(f \restriction n)$. Therefore, we can apply Lemma 1.43 to get $a_f$, a pseudointersection of $\{a \cap h\,(f \restriction n) : n \in \omega\}$. Clearly, $a_f \subseteq^* a$.

Define $\gamma := \sup\{\alpha_n : n \in \omega\}$. Since $\mathfrak{h}$ is a regular uncountable ordinal (see propositions 1.47 and 1.49), $\gamma + 1 < \mathfrak{h}$. Moreover, use that $\gamma$ is limit to get the equality $2\gamma = \gamma$. Finally, we shall prove that $|\{s \in T_{2\gamma+1} : a \mid s\}| = \mathfrak{c}$. To verify this fact, use that $T_{2\gamma+1}$ is a MADF and get, for each $f \in {}^{\omega}2$, $y_f \in T_{2\gamma+1}$ such that $a_f \mid y_f$. Clearly, $a \mid y_f$. We only need to check that for every $f, g \in {}^{\omega}2$, $y_f \neq y_g$, as long as $f \neq g$. To do this we need an auxiliary result.

We claim that $y_f \subseteq^* h\,(f \restriction n)$, for every $n \in \omega$. If $n \in \omega$, then $a_f \subseteq^* h\,(f \restriction n)$. Immediately, $y_f \mid h\,(f \restriction n)$. Use that $\alpha_n < 2\gamma + 1$ to get $u \in T_{\alpha_n}$ with $y_f \subset^* u$. Then, $u$ and $h\,(f \restriction n)$ are compatible, and thus, $u = h\,(f \restriction n)$ as a result of both being elements of the MADF $T_{\alpha_n}$.

We are ready to finish our proof: assume that $f, g \in {}^{\omega}2$ satisfy $f \neq g$. Choose $n \in \omega$ in such a way that $f\,(n) \neq g\,(n)$. Then, $h\,(f \restriction (n+1)) \neq h\,(g \restriction (n+1))$, which implies that $h\,(f \restriction (n+1)) \perp h\,(g \restriction (n+1))$, as a result of being elements of $T_{\alpha_{n+1}}$. Therefore, $y_f \neq y_g$ because $y_f \subseteq^* h\,(f \restriction (n+1))$ and $y_g \subseteq^* h\,(g \restriction (n+1))$. □

**Corollary 3.27.** $\mathcal{P}(\omega)/\operatorname{fin}$ *has a dense tree.*

*Proof.* Denote by $T$ the tree given in Proposition 3.26 and let $\pi : \mathcal{P}(\omega) \to \mathcal{P}(\omega)/\operatorname{fin}$ be the natural projection. Let us prove that $\langle \pi\,[T]\,, \geqslant \rangle$ is a dense tree in $\mathcal{P}(\omega)/\operatorname{fin}$.

We claim that $\pi \restriction T : T \to \pi\,[T]$ is an isomorphism. If $s, t \in T$ and $\pi\,(s) = \pi\,(t)$, then $(s \setminus t) \cup (t \setminus s)$ is finite an so, $s \supseteq^* t$ and $t \supseteq^* s$. Considering $\langle T, \supseteq^* \rangle$ is a tree, it follows that $s = t$. Immediately, $\pi \restriction T$ is a bijection. Now, recall that for every $s, t \in \mathcal{P}(\omega)$, $s \subset^* t$ as long as $\pi(s) < \pi(t)$ (see the paragraph before Proposition 3.22) Therefore, $\langle \pi[T], \geqslant\rangle$ is a tree of height $\mathfrak{h}$.

Let us continue by proving that $\langle \pi[T], \geqslant\rangle$ is a tree in $\mathcal{P}(\omega)/\operatorname{fin}$, i.e., we shall show that for every $s, t \in T$, either $\pi(s)$ and $\pi(t)$ are $\leqslant$-comparable or $\pi(t) \wedge \pi(s) = 0$. Equivalently, let us verify that for each $s, t \in T$, either $s$ and $t$ are $\subseteq^*$-comparable or $s \perp t$. Fix $s, t \in T$ and get $\alpha, \beta < \mathfrak{h}$ in such a way that $s \in T(\alpha)$ and $t \in T(\beta)$. If $\alpha = \beta$, then $s = t$ or $s \perp t$ because $T(\alpha)$ is a MADF. On the other hand, if $\alpha < \beta$, the fact that $T$ is a tree gives us a node $r \in T(\alpha)$ such that $t \subset^* r$. Since $s, r \in T(\alpha)$, $r = s$ or $r \perp s$ and in consequence, $t \subset^* s$ or $t \perp s$. A similar reasoning works for the case $\beta < \alpha$.

Finally, we will prove that $\langle \pi[T]\rangle_{\mathcal{P}(\omega)/\operatorname{fin}}$ is dense in $\mathcal{P}(\omega)/\operatorname{fin}$. Let $a \in (\mathcal{P}(\omega)/\operatorname{fin})^+$. Then, there is $s \in [\omega]^\omega$ such that $a = \pi(s)$. Use Proposition 3.26 to get $t \in T$ satisfying $t \subseteq s$. Notice that $\pi(t) \in \pi[T]$ and $\pi(t) \leqslant a$. $\square$

Given a topological space $X$, a collection $\mathcal{U}$ of non-empty open subsets of $X$ is called a *$\pi$-base* for $X$ if any non-empty open subset of $X$ contains a member of $\mathcal{U}$. Thus, when $A$ is a Boolean algebra and $D$ is a dense subset of $A$, we deduce (following the notation from Definition 1.27) that $\{d^- : d \in D\}$ is a $\pi$-base for $\mathrm{S}(A)$. Hence, our proof of Corollary 3.27 implies that the remainder of the Čech-Stone compactification of the integers, $\mathrm{S}(\mathcal{P}(\omega)/\mathrm{fin})$, has a *tree $\pi$-base*, i.e., if $\mathcal{T} := \left\{(\pi(t))^- : t \in T\right\}$, then $\langle \mathcal{T}, \supseteq\rangle$ is a set-theoretic tree (in general, when $A$ is a Boolean algebra and $a, b \in A$ it happens that $a^- \subseteq b^-$ if and only if $a \leqslant b$).

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