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MULTIPLICITY OF SOLUTIONS TO YAMABE TYPE EQUATIONS

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PRESENTA:
M. EN C. JUAN CARLOS FERNÁNDEZ MORELOS

DIRECTOR DE TESIS
DRA. MÓNICA ALICIA CLAPP JIMÉNEZ-LABORA
(INSTITUTO DE MATEMÁTICAS CU, UNAM)

MIEMBROS DEL COMITÉ TUTOR
DR. CARLOS PRIETO DE CASTO
(INSTITUTO DE MATEMÁTICAS CU, UNAM)
DRA. MARÍA ASUNCIÓN BEGOÑA FERNÁNDEZ FERNÁNDEZ
(FACULTAD DE CIENCIAS, UNAM)

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A mis padres,
Samuel Fernández
y Yolanda Morelos Olivares

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ABSTRACT

We study two elliptic problems with critical Sobolev exponent. In the first one, we show that the problem

$$-\Delta u = |u|^{\frac{4}{N-2}} u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

has at least

$$\max\{\operatorname{cat}(\Theta, \Theta \smallsetminus B_r M), \operatorname{cupl}(\Theta, \Theta \smallsetminus B_r M) + 1\} \geq 2$$

pairs of nontrivial solutions in every domain Ω obtained by deleting from a given bounded smooth domain $\Theta \subset \mathbb{R}^N$ a thin enough tubular neighborhood $B_r M$ of a closed smooth submanifold M of Θ of dimension $\leq N-2$, where "cat" is the Lusternik-Schnirelmann category and "cupl" is the cup-length of the pair.

For the second one, we consider a compact Riemannian manifold (M, g) without boundary of dimension $m \geq 3$ and under some symmetry assumptions, we establish existence and multiplicity of positive and sign changing solutions to the following Yamabe type equation

$$-\operatorname{div}_g(a\nabla u) + bu = c|u|^{2^*-2}u \quad \text{ on } M$$

where div_g denotes the divergence operator on (M, g), a, b and c are smooth functions with a and c positive, and $2^* = \frac{2m}{m-2}$ denotes the critical Sobolev exponent. In particular, if R_g denotes the scalar curvature, we give some examples where the Yamabe equation

$$-\frac{4(m-1)}{m-2}\Delta_g u + R_g u = \kappa u^{2^*-2} \quad \text{ on M.}$$

admits a prescribed number of sign changing solutions. We also study the lack of compactness of these problems in a symmetric setting and how the symmetries restore it at some energy levels. This allows us to use a suitable variational principle to show the existence and multiplicity of such solutions.

INTRODUCCIÓN (SPANISH VERSION)

En el presente trabajo consideramos el problema anisotrópico con exponente crítico

$$-\operatorname{div}_g(a\nabla_g u) + bu = c|u|^{2^*-2}u, \qquad \text{en } M \tag{1}$$

donde (M^m, g) es una variedad Riemanniana, div_g denota el gradiente respecto a la métrica g, $a, b, c \in \mathcal{C}^\infty(M)$ y $2^* = 2m/(m-2)$ es el exponente crítico de Sobolev del encaje $H_g^1(M) \hookrightarrow L_g^p(M)$. Este problema es una generalización del famoso problema de Yamabe, que consiste en encontrar métricas que sean conformes con g y para las cuales la curvatura escalar sea constante. Este problema resulta ser equivalente a resolver un problema de exponente crítico de la forma arriba mencionada.

Derivados de esta formulación, distinguimos dos casos. El primero se obtiene al considerar $M = \Omega$ como un abierto acotado de $\mathbb{R}^m$ con frontera suave, donde a $\mathbb{R}^m$ lo dotamos de la metrica plana estándard. El problema con condiciones de frontera

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{en } \Omega, \\ \quad u = 0 & \text{sobre } \partial\Omega, \end{cases}$$

se conoce como el problema de *Bahri-Coron*. En el caso en que (M, g) sea una variedad riemanniana compacta y sin frontera, el problema

$$\Delta_g u + \frac{m-2}{4(m-1)} R_g u = \kappa\, |u|^{2^*-2} u, \qquad \text{en } M,$$

donde $\Delta_g = -\operatorname{div}_g \nabla_g$ es el operador de Laplace-Beltrami, R_g denota la curvatura escalar y κ es una constante, es el problema de Yamabe clásico.

La principal dificultad de este tipo de problemas se debe a la falta de compacidad. En efecto, ya sea que M se trate de una variedad Riemanniana compacta o un dominio acotado y suave de $\mathbb{R}^m$, el encaje de Sobolev $H_g^1(M) \hookrightarrow L_g^{2^*}(M)$ es continuo pero no es compacto. Esto último se debe a la *invariancia bajo dilataciones* propia de este tipo de problemas. En ambos casos nos interesa obtener resultados de multiplicidad para soluciones positivas y que cambian de signo. Aunque la multiplicidad de soluciones positivas para este problema es bien conocido (ver la Sección 1.1), se sabe muy poco sobre soluciones nodales al problema de Yamabe. Obtendremos no sólo múltiples soluciones nodales para este problema, sino también para el problema más general (1) bajo restricciones adecuadas en los coeficientes a, b y c, y bajo condiciones de simetría para la variedad (M, g). Los resultados principales de esta tesis, aquí ampliados, fueron escritos junto con mi tutora de doctorado y están incluídos en los artículos [26, 27].

Esta tesis está organizada como sigue. En el capítulo 1 damos una breve introducción histórica a ambos problemas y establecemos los resultados principales. El primer resultado es sobre multiplicidad de soluciones positivas al problema de Bahri-Coron, considerando dominios suaves y acotados a los que se les ha removido una vecindad tubular suficientemente delgada de una subvariedad encajada en dicho dominio. Este resultado mejora al dado en [29] en la ausencia de simetrías en algunos casos interesantes. En la subsecuente sección del mismo capítulo establecemos dos resultados de multiplicidad de soluciones nodales para el problema general (1) en una variedad Riemanniana compacta y sin frontera, obteniendo como corolarios la existencia de una infinidad de soluciones nodales para la ecuación de Yamabe en el caso que la variedad tenga muchas simetrías, y, además, un resultado en el que establecemos la existencia de un número prescrito de soluciones de este tipo en problemas menos simétricos. Incluímos además una sección con una observación sobre la no existencia de soluciones de energía mínima para algunos problemas de tipo Brezis-Nirenberg [19], que son, asimismo, derivados de la ecuación general (1). En la última sección de este capítulo proponemos algunos problemas abiertos que se desprenden de la presente investigación y que nos gustaría seguir estudiando. En el capítulo 2 damos el planteamiento variacional al problema de Bahri-Coron y probamos el resultado de multiplicidad de soluciones establecido anteriormente en el capítulo 1. En el tercer capítulo damos el correspondiente planteamiento variacional al problema general (1) y establecemos dos resultados clave para la obtención de múltiples soluciones nodales: Un teorema de compacidad tipo Struwe [94] para el problema anisotrópico con simetrías, el cual nos permitirá restaurar bajo ciertas condiciones la compacidad, y un principio variacional para soluciones que cambian de signo, similar al dado por Clapp y Pacella en [31] (ver los teoremas 3.1.1 y 3.1.2 más adelante). Pospondremos la prueba de estos resultados para el capítulo 4 y el apéndice C respectivamente. En la siguiente sección, probamos los resultados principales del capítulo usando el teorema de compacidad y el principio variacional anteriormente mencionados, mientras que en la última sección del mismo probamos el resultado de no existencia para soluciones de enrgía mínima. El capítulo 4 está enteramente orientado a la prueba del teorema de compacidad 3.1.1. Agregamos tres apéndices: en el primero, damos una breve introducción a la teoría de integración sobre variedades riemannianas y a los espacios y encajes de Sobolev asociados. En la primera sección del apéndice B recordamos la definición de la categoría de Lusternick-Schnirelmann y su relación con el cup-length, dando asimismo las herramientas necesarias para probar el resultado de existencia de múltiples soluciones al problema de Bahri-Coron; en la segunda sección damos la construcción del transfer de punto fijo adecuada a nuestras necesidades y enunciamos sus principales propiedades. Finalmente, en el tercer y último apéndice probamos el principio variacional para soluciones nodales establecido en el capítulo 3.

PREFACE (ENGLISH VERSION)

In this work we consider the general anisotropic critical Sobolev exponent problem

$$- \operatorname{div}_g(a\nabla_g u) + bu = c|u|^{2^*-2}u, \qquad \text{on } M \tag{2}$$

where (M^m, g) is a Riemannian manifold, div_g denotes the gradient with respect to the metric g, $a, b, c \in \mathcal{C}^\infty(M)$ and $2^* = 2m/(m-2)$ is the critical Sobolev exponent of the imbedding $H_g^1(M) \hookrightarrow L_g^p(M)$. This problem is a generalization of the so called Yamabe problem consisting in finding metrics conformal to g such that the scalar curvature is constant. It turns out that solving this problem is equivalent to solving a critical exponent elliptic PDE in the form (2).

Derived from the above formulation, we distinguish two cases. In the first one, taking $M = \Omega$ as an smooth and bounded domain in $\mathbb{R}^m$ with the standard Euclidean metric, the boundary problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is known as the *Bahri-Coron problem*. In case (M, g) is a compact Riemannian manifold without boundary, the problem

$$\Delta_g u + \frac{m-2}{4(m-1)} R_g u = \kappa\, |u|^{2^*-2}\, u, \qquad \text{on } M,$$

where $\Delta_g = -\operatorname{div}_g \nabla_g$ is the Laplace-Beltrami operator, R_g denotes the scalar curvature and κ is a constant, is just the classical Yamabe problem.

The main difficulty in this kind of problems is the lack of compactness. Indeed, whether M is a compact Riemannian manifolds or a bounded smooth domain of $\mathbb{R}^m$, the imbedding $H_g^1(M) \hookrightarrow L_g^{2^*}(M)$ is continuous but not compact, due to the *invariance under dilations*. In both cases we are interested in obtaining multiplicity of positive and sign-changing solutions. Though multiplicity of positive solutions to this problem is well understood (see Section 1.1), almost nothing is known about nodal solutions to the Yamabe Problem. In this thesis we will also obtain multiplicity of sign-changing solutions to the more general problem (2) under suitable restrictions on the coefficients a, b and c, and under some symmetry assumptions on the manifold (M, g). The main results of this thesis, here revisited and expanded, were written jointly with my PhD advisor are contained in the papers [26, 27].

This thesis is organized as follows. In Chapter 1 we give a brief historical background of both problems and state the main results. The first one is a multiplicity of positive solutions to the Bahri-Coron problem on domains obtained by removing a thin enough tubular neighborhood of a submanifold imbedded in a smooth bounded domain in the Euclidean space. This result improves the one given in [29] in the absence of symmetries in some interesting situations. In the following section we state two results about multiplicity of sign-changing solution to the more general problem (2) on a compact Riemannian manifold without boundary, obtaining as corollaries the existence of an infinite number of solutions to the Yamabe problem in case we have a lot of symmetries, and the existence of a prescribed number of sign changing solutions in less symmetric cases. We also include a section with a remark about the nonexistence of ground state solutions to some Brezis-Nirenberg type problems [19] derived from equation (2). In the final section of this chapter we propose some open problems derived from this work, which are interesting for further research. In Chapter 2 we set the variational framework to prove the multiplicity result to the Bahri-Coron problem stated in Chapter 1. In Chapter 3 we begin with the variational setting of the problem and state two key ingredients to prove our multiplicity results: a Struwe compactness like theorem [94] for the anisotropic problem with symmetries on Riemannian manifolds, which allow us to restore compactness under suitable conditions, and a variational principle for nodal solutions similar to the one given by Clapp and Pacella in [31] (see Theorems 3.1.1 and 3.1.2 below). We postpone the proof of this results to Chapter 4 and Appendix C respectively. In the following section of this same chapter we prove the main results using these theorems and in the last section we prove the nonexistence result. Chapter 4 is entirely devoted to the proof of Theorem 3.1.1. There are three Appendixes: in the firs one we give a brief review of integration on manifolds and Sobolev spaces and Sobolev imbeddings for Riemannian manifolds. In the first section of Appendix B we recall the definition of the Lusternick-Schnirelmann category and its relation with the cup-length, giving the main tools to handle the proof of the multiplicity result for the Bahri-Coron problem, while in the second section we give the construction of the fixed point transfer we need in our situation, enouncing its general properties. Finally, in Appendix C we prove the variational principle for sign changing solutions.

CONTENTS

CHAPTER 1

INTRODUCTION TO THE PROBLEM

In this work we consider the general anisotropic critical Sobolev exponent problem

$$-\operatorname{div}_g(a\nabla_g u) + bu = c|u|^{2^*-2}u, \qquad \text{on } M \qquad (\wp_{g,a,b,c})$$

where (M^m, g) is a Riemannian manifold, div_g denotes the divergence and ∇_g the gradient with respect to the metric g, $a, b, c \in \mathcal{C}^\infty(M)$ and $2^* = 2m/(m-2)$ is the critical Sobolev exponent of the imbedding $H^1_g(M) \hookrightarrow L^p_g(M)$. In case $M = \Omega$ is an smooth and bounded open subset of $\mathbb{R}^m$ with the standard Euclidean metric, the boundary problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is known as the *Bahri-Coron problem*. In case (M, g) is a compact Riemannian manifold without boundary, the problem

$$\Delta_g u + \frac{m-2}{4(m-1)} R_g u = \kappa \, |u|^{2^*-2} \, u, \qquad \text{on } M,$$

where $\Delta_g = -\operatorname{div}_g \nabla_g$ is the Laplace-Beltrami operator, R_g denotes the scalar curvature and κ is a constant, is the so-called *Yamabe problem.*

The main difficulty in this kind of problems is the lack of compactness. Indeed, whether M is a compact Riemannian manifolds or a bounded smooth subset of $\mathbb{R}^m$, the imbedding $H^1_g(M) \hookrightarrow L^{2^*}_g(M)$ is continuous but not compact, due to the *invariance under dilations.* In both cases we are interested in obtaining multiplicity of positive and sign-changing solutions. We will also obtain multiplicity of sign-changing solutions to the more general problem ($\wp_{g,a,b,c}$) with suitable restrictions on the coefficients a, b and c. The main results of this thesis, here revisited and expanded, were written jointly with my PhD advisor are contained in the papers [26, 27].

1.1 The Yamabe and the Bahri-Coron Problems: Historical background

1.1.1 The Yamabe Problem

Given a closed Riemannian manifold (M, g) of dimension $m \geq 3$, the Yamabe problem consists in finding a metric conformal to g with constant scalar curvature. In 1960, H. Yamabe [103] attempted to solve this problem using calculus of variations techniques, but his proof had a gap, discovered by N. Trudinger [99] in 1968. Trudinger was able to repair the proof, but only with very restrictive hypothesis on the Riemannian manifold (M, g). In 1976, T. Aubin discovered the difficulties of the problem and extended Trudinger's result for a large class of closed Riemannian manifolds. The remaining cases were studied by R. Schoen [96] and his theorem completed the solution of the Yamabe problem.

Writing the conformal metric as $\hat{g} = u^{4/(m-2)}g$ with $u \in \mathcal{C}^\infty(M)$, $u > 0$, the scalar curvature $R_{\hat{g}}$ satisfies the equation

$$R_{\hat{g}} u^{2^*} = R_g u - \frac{4(m-1)}{m-2} \Delta_g u. \tag{1.1}$$

So, the Yamabe Problem is equivalent to solving (1.1) with constant $R_{\hat{g}} \equiv \kappa$, where the solution u must be positive and smooth. This leads to proving the existence of a positive solution of the PDE

$$\Delta_g u + \frac{m-2}{4(m-1)} R_g u = \kappa \left|u\right|^{2^*-2} u, \qquad u \in \mathcal{C}^\infty(M), \tag{1.2}$$

with $\kappa \in \mathbb{R}$.

Problem (1.2) has a natural variational formulation in terms of the Yamabe functional

$$Y_g(u) := \frac{\int_M \mathfrak{c}_m^{-1} |\nabla u|^2 + R_g u^2 dV_g}{\left(\int_M |u|^{2^*} dV_g\right)^{2/2^*}},$$

where $\mathfrak{c}_m = \frac{m-2}{4(m-1)}$. The proof of the existence of at least one positive solution to problem (1.2) consist in showing that the infimum

$$Y(M, g) := \inf_{h \in [g]} \frac{\int_M R_h dV_h}{Vol(M, h)^{(m-2)/m}} = \inf_{u \in C^\infty(M)} Y_g(u)$$

is always attained by Y_g [5, 96, 99, 103], where $[g]$ denotes the conformal class of g. This constant is called the Yamabe invariant or the Yamabe quotient and it is conformally invariant, that is, if g_1 and g_2 are in the same conformal class $[g]$, then $Y(M, g_1) = Y(M, g_2)$. Aubin [5] proved the following result.

Theorem 1.1.1 (Aubin, 1976) *If $(\mathbb{S}^m, g_0)$ denotes the unit round sphere, then $Y(M, g) \leq Y(\mathbb{S}^m, g_0)$. Moreover, if we have the strict inequality, there is a positive and smooth solution u to problem (1.2) with $R_{\hat{g}} = Y(M, g)$.*

To decide whether a Riemannian manifold satisfies the strict inequality or not, lies in the heart of the problem. Aubin was able to construct "local" test functions $\varphi \in \mathcal{C}^\infty(M)$ satisfying $Y_g(\varphi) < Y(\mathbb{S}^m, g_0)$, in case $m \geq 6$ and M is not locally conformally flat. As we have already mentioned, Schoen give a proof of this inequality in the remainder cases and his theorem reads as follows.

Theorem 1.1.2 (Schoen, 1984) *If M has dimension 3,4 or 5, or if M is locally conformally flat, then $Y(M,g) < Y(\mathbb{S}^m, g_0)$, unless it is conformally equivalent to the round sphere $(\mathbb{S}^m, g_0)$.*

The proof of this theorem consisted in constructing a "global" test function such that $Y_g(\varphi) < Y(\mathbb{S}^m, g_0)$. To this end, Schoen introduced two important new ideas. First, he recognized the key role of the Green function for the operator $\Delta_g + R_g$; in fact, his test function was simply the Green function with its singularity smoothed out. Second, he discovered the unexpected relevance of the positive mass theorem of general relativity, which was proved in dimensions 3 and 4 by Schoen and S.-T. Tau [97, 98]. A curious feature of Schoen's proof is that it works only in the cases not covered by Aubin's theorem. For more about the Yamabe Problem and its solution, we refer the interested reader to the paper [69] or to the book [6] and the references therein.

Multiplicity and uniqueness results are also very interesting. For instance, if the Yamabe constant is negative, the Yamabe problem has a unique positive solution, while for a zero Yamabe constant, the solution is unique up to a constant factor. Another uniqueness result was obtained by Obata [87] in 1971, who showed that if g is an Einstein metric, there exists a unique metric of constant scalar curvature and unit volume. However, a richer set of solutions is obtained in the case of positive Yamabe constant. For example, the set of solutions of the Yamabe Problem for the standard sphere $(\mathbb{S}^m, g_0)$ is not compact. Ambrossetti and Malchiodi [2] showed that the sphere $(\mathbb{S}^m, \hat{g})$ has at least two solutions, where the metric $\hat{g}$ is a perturbation of the standard metric in $\mathbb{S}^m$. This result was improved later by Berti and Malchiodi in [13]. A very striking result was given by Pollack, who showed in [86] the existence of a prescribed number of positive solutions to the Yamabe problem with constant positive scalar curvature, for arbitrarily close metrics to any given metric in the C^0 topology. Khuri, Marques and Schoen [67] proved in 2009 that compactness of the set of positive solutions in the C^2-topology for (M,g) not conformally equivalent to the round sphere holds true up to dimension 24. In contrast, Brendle [15] and Brendle and Marques [17] showed that compactness fails in every dimensions grater than or equal to 25. For more about the compactness problem, we recommend the article [67], the survey articles [78], [18] and [16] and the references therein.

Concerning the qualitative behavior of the solutions, Hebey and Vaugon [59] showed that, for any closed subgroup Γ of the group of isometries of (M,g) there is a minimal Γ-invariant solution to the Yamabe problem. We recommend the survey paper [58] and the references therein to learn more about the symmetric Yamabe problem.

However, the question of multiplicity of nodal solutions to the Yamabe problem is not yet well understood. Ammann and Humbert [4], based on the fact that the operator $L_g := \Delta_g + \mathfrak{c}_m R_g$ has a discrete sprectrum

$$Spec(L_g) = \{\lambda_1(g), \lambda_2(g), \ldots\},$$

where the eigenvalues satisfy

$$\lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \lambda_k(g) \cdots \to +\infty$$

and inspired in the variational equality for the Yamabe constant

$$Y(M,g) = \inf_{\hat{g}\in[g]} \lambda_1(\hat{g}) Vol(M,\hat{g})^{2/m},$$

they defined the kth Yamabe invariant, $k \geq 2$, as

$$Y_k(M,g) = \inf_{\hat{g}\in[g]} \lambda_k(\hat{g}) Vol(M,\hat{g})^{2/m}$$

which is also a conformal invariant. The problem of finding nodal solutions is closely related to the second Yamabe invariant. Indeed, it was proved by Ammann and Humbert in [4] that the second invariant is never achieved by a Riemannian metric. Nevertheless, allowing "generalized" metrics in the conformal class, they showed that this infimum is achieved if the Yamabe invariant is nonnegative, M not locally conformally flat and $m \geq 11$. An extension of this result was given later by El Sayed in [51]. In fact, a nodal solution to the Yamabe problems gives rise to a generalized metric of the form $\hat{g} = |u|^{2^*-2} g$. Notice $\hat{g}$ is not a metric since it is no longer smooth and it vanishes on the zero set of u. There are examples of closed manifolds for which the second Yamabe invariant is never attained, even by a generalized metric, as it is showed by the case of the sphere with the standard metric. Using the approach of the second Yamabe invariant, Petean [83] and Henry [60] showed the existence of nodal solutions on some product manifolds. Other examples of existence results on nodal solutions to Yamabe-type equations were given by Holcman [62, 63], Jourdain [65], Djadli-Jourdain [43] and Vétois [100].

1.1.2 The Bahri-Coron Problem

The Yamabe problem on bounded smooth domains in $\mathbb{R}^m$ with Dirichlet boundary condition is particularly interesting and it depends on the shape of the domain. One of the first results is due to Pohozaev [85], who showed in 1965 that the problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has no nontrivial solution if Ω is strictly starshaped and $p \geq 2^*$. On the other hand, the situation is completely different when Ω is an annulus

$$\Omega := \{x \in \mathbb{R}^m \ : \ 0 < a < |x| < b\}.$$

In this case, Kazdan and Warner [66] proved that the problem (1.3) has an infinite number of radial solutions for every $p > 2$. The more general result on the existence of at least one positive solution to this problem was given by Bahri and Coron [7]. They showed that if the domain has non trivial topology, then there exist a positive solution to problem (1.3) for $p = 2^*$. Concretely, they proved the following.

Theorem 1.1.3 (Bahri-Coron, 1988) *If the reduced homology groups with coefficients in $\mathbb{Z}/2$ of the domain satisfy $\tilde{H}_*(\Omega, \mathbb{Z}/2) \neq 0$, then, problem (1.3) with $p = 2^*$ has at least one positive solution.*

After this result, the Yamabe problem in smooth bounded domains in $\mathbb{R}^m$ is known as the Bahri-Coron problem.

Nevertheless, nontrivial topology of the domain is not necessary for a solution to exist. Examples of contractible domains for which problem the Bahri-Coron problem has at least one nontrivial solution were given, for example, by Ding [42] and Passaseo [82]. Concerning the multiplicity of positive and nodal solutions to the Bahri-Coron problem, many result have been established in the presence of symmetries (see for example [23, 24, 29, 35, 77, 82]). Multiplicity of positive and nodal solutions under general conditions, as the one given by Bahri and Coron in Theorem 1.1.3 is widely open.

1.2 Main results: Multiplicity of solutions to the Bahri-Coron problem

Let Θ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and let M be a compact smooth submanifold of $\mathbb{R}^N$, without boundary, contained in Θ. Consider the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Theta_r, \\ u = 0 & \text{on } \partial\Theta_r, \end{cases} \tag{1.4}$$

where $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent and

$$\Theta_r := \{x \in \Theta : \operatorname{dist}(x, M) > r\}, \qquad r > 0.$$

Our aim is to establish multiplicity of solutions for r small.

If M is a point and r is small enough, Coron showed in [38] that this problem has at least one positive solution. The existence of at least two solutions was established by Clapp and Weth in [37]. More recently, Ge, Musso and Pistoia [53] proved that the number of sign changing solutions becomes arbitrarily large as r goes to zero. Their solutions are bubble-towers, i.e. they look like superpositions of standard bubbles with alternating signs concentrating at the point M. Under additional assumptions, positive and sign changing solutions which look like a sum of standard bubbles one of which concentrates at the point M and the others at some points in $\Theta \smallsetminus M$ were constructed in [30]. There are also various results on the existence and shape of solutions to this problem when M is a finite set of points and r is small enough, see e.g. [72, 73, 79, 89].

In contrast to this, if M has positive dimension only few results are known. Hirano and Shioji established the existence of two solutions in an annular domain with a thin straight tunnel in [61]. Some multiplicity results were recently obtained by Clapp, Grossi and Pistoia in [29] when both Θ and M are invariant under the action of some group of symmetries. They also showed that, without any symmetry assumption, this problem has at least $\operatorname{cat}(\Theta, \Theta_r)$ positive solutions for small enough r, where $\operatorname{cat}(\Theta, \Theta_r)$ is the Lusternik-Schnirelmann category of the pair (Θ, Θ_r).

Here we show that for some domains there is an additional solution. We write $\operatorname{cupl}(\Theta, \Theta_r)$ for the cup-length of the pair (Θ, Θ_r). The definitions of category and cup-length are given in appendix B.1. We prove the following result.

Theorem 1.2.1 *Assume that* $\dim M \leq N-2$. *Then there exists* $r_0 > 0$ *such that, if* Ω *is a bounded smooth domain in* $\mathbb{R}^N$ *which satisfies*

$$M \cap \overline{\Omega} = \emptyset \qquad \text{and} \qquad \Theta_r \subset \Omega \subset \Theta,$$

for some $r \in (0, r_0)$, *then problem*

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

has at least

$$\max\{\operatorname{cat}(\Theta, \Theta_r),\ \operatorname{cupl}(\Theta, \Theta_r) + 1\} \geq 2$$

pairs of nontrivial solutions.

It is well known that $\operatorname{cat}(\Theta, \Theta_r) \geq \operatorname{cupl}(\Theta, \Theta_r)$ (see Lemma B.1.1). So Theorem 1.2.1 improves Corollary 1.2 in [29] when $\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r)$. There are some interesting situations in which this occurs. For example, the following ones.

Example 1.2.2 *If* M *is contractible in* Θ, *then* $\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r) = 1$ *for* r *small enough.*

Example 1.2.3 *If* Θ *is a tubular neighborhood of* M *and* $\operatorname{cat}(M) = \operatorname{cupl}(M)$, *then* $\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r) = \operatorname{cupl}(M)$ *for* r *small enough.*

Examples of manifolds M such that $\operatorname{cat}(M) = \operatorname{cupl}(M)$ are those having the homotopy type of a sphere $\mathbb{S}^k$, of a real $\mathbb{R}P^k$, a complex $\mathbb{C}P^k$ or a quaternionic $\mathbb{H}P^k$ projective space, or of a product of such spaces.

Note that both $\operatorname{cat}(\Theta, \Theta_r)$ and $\operatorname{cupl}(\Theta, \Theta_r)$ depend on the embedding of M into Θ. For example, if M is the circle $C := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$ and Θ is the torus $\{x \in \mathbb{R}^3 : \operatorname{dist}(x, C) < \frac{1}{2}\}$, then Θ is a tubular neighborhood of M and Example 1.2.3 gives

$$\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r) = \operatorname{cupl}(\mathbb{S}^1) = 2$$

for $r \in (0, \frac{1}{2})$. On the other hand, if M is the circle $\{(x_1, 0, x_3) \in \mathbb{R}^3 : (x_1 - 1)^2 + x_3^2 = \frac{1}{4}\}$, then Example 1.2.2 gives $\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r) = 1$ for $r \in (0, \frac{1}{4})$. Theorem 1.2.1 asserts the existence of three solutions in the first case, and two solutions in the second one.

As we shall show in Proposition 2.3.1, at least $\operatorname{cat}(\Theta, \Theta_r) \geq 1$ solutions are positive. Our methods do not allow us to conclude whether the additional solution is sign changing or not.

We wish to stress the fact that multiplicity results for problem (1.5) are only available for some particular types of domains. One expects to have multiple solutions in every domain satisfying the hypothesis of the Bahri-Coron's theorem, but, as it was already mentioned in the previous section, the proof of this fact remains open. Classical variational methods cannot be applied to establish multiplicity due to the lack of compactness of the associated energy functional. Under suitable symmetry assumptions compactness is restored: if Ω is invariant under the action of a group Γ of linear isometries of $\mathbb{R}^N$ and every Γ-orbit in Ω is infinite, problem (1.5) is known to have infinitely many Γ-invariant solutions [21]. Recently, Clapp and Faya [23] considered domains having finite Γ-orbits and gave conditions for the existence of a prescribed number of solutions.

In a non-symmetric setting, the Lyapunov-Schmidt reduction method has been successfully applied to obtain multiplicity results for problem (1.4) when M is a point or a finite set (see [53] and the references therein), but this method becomes very hard to apply when M has positive dimension.

1.3 Main results: Multiplicity of nodal solutions to the Yamabe problem

Let (M, g) be a closed Riemannian manifold of dimension $m \geq 3$ and Γ be a closed subgroup of the group of isometries $\mathrm{Isom}_g(M)$ of (M, g). We denote by $\Gamma p := \{\gamma p : \gamma \in \Gamma\}$ the Γ-orbit of a point $p \in M$ and by $\#\Gamma p$ its cardinality. A subset X of M is said to be Γ-invariant if $\Gamma x \subset X$ for every $x \in X$, and a function $f : X \to \mathbb{R}$ is Γ-invariant if it is constant on each orbit Γx of X.

We shall study the following Yamabe type equation

$$\begin{cases} -\operatorname{div}_g(a\nabla_g u) + bu = c|u|^{2^*-2}u, \\ u \in H_g^1(M)^\Gamma, \end{cases} \tag{1.6}$$

where ∇_g denotes the gradient and div_g the divergence operator on (M, g), $a, b, c \in \mathcal{C}^\infty(M)$ are Γ-invariant functions, a and c are positive on M, and

$$H_g^1(M)^\Gamma := \{u \in H_g^1(M) : u \text{ is } \Gamma\text{-invariant}\}.$$

For the definition and elementary properties of this the Sobolev spaces $H_g^1(M)$ and $H_g^1(M)^\Gamma$, see Appendix A.

Set

$$\mu_{a,b}^\Gamma(M, g) := \inf_{v \in H_g^1(M)^\Gamma \setminus \{0\}} \frac{\int_M [a|\nabla_g u|^2 + b|u|^2]\, dV_g}{\int_M [|\nabla_g u|^2 + |u|^2]\, dV_g}.$$

Observe that $\mu_{a,b}^\Gamma(M, g) > 0$ if and only if the operator $-\operatorname{div}_g(a\nabla_g) + b$ is coercive in $H_g^1(M)^\Gamma$; this means the existence of a positive constant $C > 0$ such that $\int_M a|\nabla u|_g^2 + b|u|^2 dV_g \geq C\|u\|_{H_g^1(M)}^2$ for all $u \in H_g^1(M)^\Gamma$. If $a \equiv \mathfrak{c}_m^{-1}$ and $b = R_g$ then coercivity of $-\operatorname{div}_g(a\nabla_g) + b$ on $H_g^1(M)^\Gamma$ implies the Yamabe invariant of (M, g) is also positive (see [4])

We will prove the following result.

Theorem 1.3.1 *If the operator $-div_g(a\nabla_g) + b$ is coercive in $H_g^1(M)^\Gamma$ and if $1 \leq \dim(\Gamma p) < m$ for every $p \in M$, then problem (1.6) has at least one positive solution and infinitely many sign-changing Γ-invariant solutions.*

A special case is the following multiplicity result for the Yamabe problem (1.2). Hereafter, we assume $\kappa > 0$.

Corollary 1.3.2 *If the operator* $\Delta_g + \mathfrak{c}_m R_g$ *is coercive in* $H_g^1(M)^\Gamma$ *and* $1 \leq \dim(\Gamma p) < m$ *for all* $p \in M$*, then the Yamabe equation* (1.2) *has infinitely many* Γ*-invariant sign-changing solutions in* $H_g^1(M)$*.*

The following examples illustrate this result.

Example 1.3.3 $\mathbb{S}^1$ *acts freely and isometrically on the round sphere* $\mathbb{S}^{2k+1} = (\mathbb{S}^{2k+1}, g_0)$*. So for* $k \geq 1$ *the Yamabe equation* (1.2) *has infinitely many sign-changing* $\mathbb{S}^1$*-invariantsolutions on* $\mathbb{S}^{2k+1}$.

Example 1.3.4 *Let* (N, h) *a Riemannian manifold of dimension* n *and* $f \in \mathcal{C}^\infty(N)$ *be a positive function. Then* $\mathbb{S}^1$ *acts freely and isometrically on the warped product* $N \times_f \mathbb{S}^{2k+1} = (N \times \mathbb{S}^{2k+1}, g)$ *by multiplication on the second factor, where* $g = h + f^2 g_0$*. So, if* $k \geq 0$, $2k+n \geq 2$ *and* $\Delta_g + \mathfrak{c}_m R_g$ *is coercive in* $H_g^1(N \times_f \mathbb{S}^{2k+1})^{\mathbb{S}^1}$*, the Yamabe equation* (1.2) *has infinitely many* $\mathbb{S}^1$*-invariant sign-changing solutions on* $N \times_f \mathbb{S}^{2k+1}$.

It was shown in [44] that the scalar curvatures R_h *of* N *and* $R_{h+f^2 g_0}$ *of* $N \times_f \mathbb{S}^{2k+1}$ *are related by the equation*

$$\frac{2(2k+1)}{k+1}\Delta_h \phi + R_h \phi + 2k(2k+1)\phi^{(k-1)/(k+1)} = R_{h+f^2 g_0}\phi,$$

where $\phi := f^{k+1}$.

As a special case we have that, if $R_h > -2k(2k+1)$*, the Yamabe equation* (1.2) *has infinitely many* $\mathbb{S}^1$*-invariant sign-changing solutions on the product* $N \times \mathbb{S}^{2k+1}$*. This generalizes Theorem 1.2 in [83], which asserts the existence of one sign-changing solution on* $N \times \mathbb{S}^1$.

Theorem 1.3.1 requires that the Γ-orbit of every point in M is infinite. Next, we study a case in which M is allowed to have finite Γ-orbits. We consider the following setting:

Let M be a closed smooth m-dimensional manifold and $a, b, c \in \mathcal{C}^\infty(M)$ be such that a and c are positive on M. We fix an open subset Ω of M, a Riemannian metric h on Ω and a compact subgroup Λ of $\mathrm{Isom}_h(\Omega)$ such that $\dim(\Lambda p) < m$ for all $p \in \Omega$, the restrictions of a, b, c to Ω are Λ-invariant and the operator $-\mathrm{div}_h(a\nabla_h) + b$ is coercive on the space $\mathcal{C}_c^\infty(\Omega)^\Gamma$ of smooth Γ-invariant functions with compact support in Ω. Under these assumptions, we will prove the following multiplicity result.

Theorem 1.3.5 *There exists an increasing sequence* (ℓ_k) *of positive real numbers, depending only on* (Ω, h), a, b, c *and* Λ*, with the following property: For any Riemanniann metric* g *on* M *and any closed subgroup* Γ *of* $\mathrm{Isom}_g(M)$ *which satisfy*

1. $g = h$ *in* Ω*;*

2. Γ *is a subgroup of* Λ *and* a, b, c *are* Γ*-invariant;*

3. $-div_g(a\nabla_g) + b$ *is coercive on* $H_g^1(M)^\Gamma$ *;*

4. $\min_{p\in M} \frac{a(p)^{m/2}\#\Gamma p}{c(p)^{\frac{m-2}{2}}} > \ell_k$;

problem (1.6) *has at least k pairs of Γ-invariant solutions $\pm u_1, \ldots, \pm u_k$ such that u_1 is positive, $u_2, \ldots, u_k$ change sign, and*

$$\int_M c\,|u_j|^{2^*}\,dV_g \leq \ell_j S^{m/2} \qquad \text{for every } j = 1, \ldots, k, \tag{1.7}$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m)$.

If $m \geq 4$, $a = c \equiv 1$ and $\Delta_g + b$ is coercive on $H^1_g(M)$, Vétois showed that problem (1.6) has at least $\frac{n+2}{2}$ solutions under the Brezis-Nirenberg-type hypothesis that $b(p_0) < \mathfrak{c}_m R_g(p_0)$ at some point $p_0 \in M$ [100], but nothing is said about the sign of the solutions, except for the cases where the positive solution is known to be unique. It is important to remark that the hypothesis considered by Vétois are not satisfied by the Yamabe equation.

Theorem 1.3.5 provides sign-changing solutions for more general data, but symmetries are required. In fact, property (4) requires that the group Λ has large enough subgroups. The group $\mathbb{S}^1$ has this property. Set $\Gamma_n := \{e^{2\pi i j/n} : j = 0, ..., n-1\}$. The next application illustrates the role of the symmetries.

We derive the following multiplicity result for the Yamabe problem (1.2).

Corollary 1.3.6 *Let (M, h) be a closed Riemannian manifold on which $\mathbb{S}^1$ acts freely and isometrically, such that $\Delta_h + \mathfrak{c}_m R_h$ is coercive on $H^1_h(M)$. Fix an open $\mathbb{S}^1$-invariant subset Ω of M such that $R_h > 0$ on $M \smallsetminus \Omega$. Then there exist a sequence (ℓ_k) in $(0, \infty)$ and an open neighborhood $\mathcal{O}$ of h in the space of Riemannian metrics on M with the $\mathcal{C}^0$-topology, with the following property: for every $g \in \mathcal{O}$ such that $g = h$ in Ω and $\Gamma_n \subset \mathrm{Isom}_g(M)$ for some $n > \kappa^{(m-2)/2}\ell_k$, the Yamabe problem (1.2) has at least k pairs of Γ-invariant solutions $\pm u_1, \ldots, \pm u_k$ such that u_1 is positive, $u_2, \ldots, u_k$ change sign, and*

$$\int_M |u_j|^{2^*}\,dV_g \leq \kappa^{-1}\ell_j S^{m/2} \qquad \text{for every } j = 1, \ldots, k.$$

For instance, we may take Ω to be the complement of a closed tubular neighborhood of an $\mathbb{S}^1$-orbit in (M, h) on which $R_h > 0$. Then $M \smallsetminus \Omega$ is $\mathbb{S}^1$-diffeomorphic to $\mathbb{S}^1 \times \mathbb{B}^{m-1}$, where $\mathbb{B}^{m-1}$ is the closed unit ball in $\mathbb{R}^{m-1}$. We choose $n > \kappa^{(m-2)/2}\ell_k$. Then, if we modify the metric in the interior of the piece of $M \smallsetminus \Omega$ which corresponds to $\{e^{2\pi i\vartheta/n} : 0 \leq \vartheta \leq 1\} \times \mathbb{B}^{m-1}$ and translate this modification to each of the pieces corresponding to $\{e^{2\pi i\vartheta/n} : j-1 \leq \vartheta \leq j\} \times \mathbb{B}^{m-1}$, $j = 2, ..., n$, we obtain a metric g on M such that $g = h$ in Ω and $\Gamma_n \subset \mathrm{Isom}_g(M)$. If g is chosen to be close enough to h, then the previous corollary asserts the existence of k pairs of solutions to the Yamabe problem (1.2). This way we obtain many examples of Riemannian manifolds with finite symmetries which admit a prescribed number of solutions to the Yamabe problem. As far as we know, this is the first result of multiplicity of nodal solutions to the Yamabe equation.

The main ingredients to prove these theorems are a compactness result and a variational principle for sign changing critical points. The first one is Theorem

3.1.1, stated in Chapter 3 and proved in Chapter 4. The variational principle we use is an slight modification of the Clapp-Pacella's variational principle for sign changing solutions [31, Theroem 3.7]. In Appendix C we give the proof of the fundamental lemma needed to follow the lines of the proof given in section 3 of [31]

An interesting phenomenon about the existence of ground state solutions occurs when the function b in equation (1.6) is taken strictly above or below $\mathfrak{c}_m R_g$. We have the following.

Theorem 1.3.7 *For the round sphere* $(\mathbb{S}^m, g_0)$ *with* $m \geq 3$, *equation*

$$\Delta_g u + bu = |u|^{2^*-2}u \qquad \text{on } (\mathbb{S}^m, g_0), \tag{1.8}$$

admits no ground state solutions for any function $b \in \mathcal{C}^\infty(\mathbb{S}^m)$ *such that* $b \geq \mathfrak{c}_m R_{g_0} = \frac{m(m-2)}{4}$ *and* $b \not\equiv \mathfrak{c}_m R_{g_0}$, *i.e.,*

$$\inf_{\substack{u \in \mathcal{C}^\infty(\mathbb{S}^m)\\ u \neq 0}} \frac{\int_{\mathbb{S}^m} \left[|\nabla_{g_0} u|^2_{g_0} + bu^2 \right] dV_{g_0}}{\left(\int_{\mathbb{S}^m} |u|^{2^*} dV_{g_0} \right)^{2/2^*}}$$

is not attained.

Following Aubin's ideas in [5] in case b is taken strictly below $\mathfrak{c}_m R_{g_0}$, one can show that equation (1.8) always admits a ground state solution. From this it is evident that the Yamabe equation is a double limiting problem, first because of the presence of the critical Sobolev exponent and, second, for the presence of the function $\mathfrak{c}_m R_g$.

1.4 Open problems and further research

In this subsection we indicate some of the open problems which are motivated by this work and which we plan to continue investigating.

1. It would be interesting to know the sign of the extra solution to the Bahri-Coron problem obtained in Theorem 1.2.1. Our methods were not enough to do so. To prove the existence of nodal solutions is a very difficult problem due to the lack of compactness and little progress has been made in this direction, for example, when the domain has many symmetries [23, 24, 29].

2. The supercritical Yamabe problem is difficult to handle due to the lack of a Sobolev imbedding. The method of reducing a supercritical exponent problem to a critical or subcritical one via harmonic morphisms, has proved to be a powerful tool to handle supercritical nonlinearities on bounded and smooth domains (Cf. [90, 25] and more recently the survey paper [32]). In case of supercritical problems on Riemannian manifolds, recent progress has been made, for example, in the case of perturbations of the Yamabe problem [28, 52, 84]. The important feature of this technique is that one reduces an equation of the form

$$\Delta_g u + u = |u|^{p-2}u, \qquad \text{on } (M^m, g)$$

with $p > 2m/(m-2)$, $m = \dim M$, to an anisotropic problem of the type

$$-\operatorname{div}_h(a\nabla_h v) + b\, v = c\, |u|^{p-2}u \qquad \text{on } (N^n, h) \tag{1.9}$$

with the same exponent satisfying now that $p \leq 2n/(n-2)$, where our techniques could apply. For example, in [28] the case when M is a warped product and the reduction is performed by the projection onto the first factor, leads to an anisotropic problem of the form (1.9) with a and c positive. In this case, the map $M \to N$ is not a harmonic morphism, for it does not preserve the Laplace-Beltrami operator [9]. We would like to study more general maps between Riemannian manifolds conserving a divergence type operator, giving a relatively easy condition to decide whether a map is of this kind or not, like the Baird- Eells formula [8], and apply then Theorem 1.3.5 to prove the existence of nodal solutions as it was done in [25, 34]. Some of this type of maps were studied by Loubeau [74, 75] in case of p-Laplacian preserving maps and more recently by Ou [88] for maps taking a divergence type operator into a Laplace-Beltrami operator.

3. Theorem 1.3.7 suggests the problem of nonexistence of ground state solutions to the equation
$$\Delta u + bu = |u|^{2^*-2}u, \qquad \text{on } M,$$
where $b \in \mathcal{C}^\infty(M)$ is such that $b > \mathfrak{c}_m R_g$. It would be interesting to know if the Sobolev type constant
$$S_b := \inf_{\substack{u\in\mathcal{C}^\infty(M)\\ u\neq 0}} \frac{\int_M \left[|\nabla_g u|_g^2 + bu^2\right] dV_g}{\left(\int_M |u|^{2^*} dV_g\right)^{2/2^*}}$$
is equal to $S_{\mathfrak{c}_m R_g}$ or if it stabilizes above some critical function. The main difficulty here is to find suitable tests functions to prove or disprove the equality. We believe this is closely related to the solution of the Yamabe problem given by Aubin and Schoen.

4. It could be interesting to obtain a global compactness result on closed Riemannian manifolds in the presence of symmetries like the one obtained in [24]. In the non symmetric case a compactness result has been obtained in [49, Chapter 3]. In this same direction, we would like to study the Yamabe problem for non complete manifolds, in which the injectivity radius is positive and the scalar and Ricci curvatures are bounded. An example of this situation is provided by considering smooth open subsets of a closed Riemannian manifold. This could be useful to get multiplicity results to the Dirichlet problem
$$\begin{cases} \Delta_g u = |u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$
where Ω is an open subset of a closed manifold M with nonempty smooth boundary and $p > 2$.

This thesis is organized as follows. In Chapter 2 we set the variational framework to prove Theorem 1.2.1. At the end of this chapter we prove examples 1.2.2

and 1.2.3. In Chapter 3 we begin with the variational setting of the problem and state two key ingredients to prove Theorems 1.3.1 and 1.3.5: a compactness theorem for the anisotropic problem with symmetries on Riemannian manifolds, and a variational principle for nodal solutions similar to the one given by Clapp and Pacella in [31] (see Theorems 3.1.1 and 3.1.2 below). We postpone the proof of this results to Chapter 4 and Appendix C. In section 3.2 we prove Theorems 1.3.1 and 1.3.5, and Corollary 1.3.6. In the following section we prove the nonexistence result Theorem 1.3.7 translating the problem on the sphere to a critical exponent problem on $\mathbb{R}^m$, via the stereographic projection. Chapter 4 is entirely devoted to the proof of Theorem 3.1.1. In the first section of this chapter, we develop the main tools we need to prove it and, among them, we state and prove a comparison result between the L^p-norms in the space $H_g^1(M)$ and the usual L^p-norms on $L^p(\mathbb{R}^m)$. After that, we present a reduction argument consisting in reducing the compactness problem for the functional associated to problem 1.6 to a simpler functional with $b = 0$. In the final section of this chapter we give a complete proof of the compactness theorem 3.1.1. There are three Appendices: in the firs one we give a brief review of integration on manifolds and Sobolev spaces and Sobolev imbeddings for Riemannian manifolds. In the first section of Appendix B we recall the definition of the Lusternik-Schnirelmann category and its relation with the cup-length, giving the main tools to handle the proof of Theorem 1.2.1, while in the second section we give the construction of the fixed point transfer we need in our situation, and state its general properties. Finally, in Appendix C we prove the variational principle Theorem 3.1.2.

CHAPTER 2

MULTIPLE SOLUTIONS TO THE BAHRI-CORON PROBLEM

This chapter is devoted to the proof of Theorem 1.2.1. Our proof of this theorem uses variational methods and some tools from algebraic topology which include the fixed point transfer introduced by Dold in [46]. Key elements of our variational approach are a refinement of the deformation lemma which was proved in [37] and a lower bound for the energy of sign changing solutions to the limit problem in $\mathbb{R}^N$ obtained by Weth in [101]. These results are stated in section 2.1. Section 2.2 is devoted to the construction of two auxiliary maps which play an important role in the proof of Theorem 1.2.1. The proof of this theorem and of Examples 1.2.2 and 1.2.3 are given in section 2.3. In Appendix B we recall the definition and properties of the Lusternik-Schnirelmann category, the cup-length and the fixed point transfer. In this chapter, m will denote the dimension of a submanifold in $\mathbb{R}^N$.

2.1 Variational setting

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We consider the Sobolev space $H_0^1(\Omega)$ with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2\right)^{1/2}.$$

We write $|u|_p$ for the L^p-norm of u, $1 \leq p \leq \infty$.

The solutions to problem (1.5) are the critical points of the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ given by

$$J(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}.$$

The nontrivial solutions are the critical points of the restriction of J to the Nehari manifold

$$\mathcal{N} := \{u \in H_0^1(\Omega) : u \neq 0,\ \|u\|^2 = |u|_{2^*}^{2^*}\},$$

which is a $\mathcal{C}^2$-manifold, radially diffeomorphic to the unit sphere in $H_0^1(\Omega)$.

Recall that J is said to satisfy the Palais-Smale condition $(PS)_c$ on $\mathcal{N}$ at the level $c \in \mathbb{R}$ if every sequence (u_k) in $\mathcal{N}$ such that $J(u_k) \to c$ and $\nabla_{\mathcal{N}} J(u_k) \to 0$ contains a convergent subsequence. Here $\nabla_{\mathcal{N}} J$ denotes the gradient of the restriction of J to $\mathcal{N}$, i.e. $\nabla_{\mathcal{N}} J(u)$ is the orthogonal projection of $\nabla J(u)$ onto the tangent space to $\mathcal{N}$ at u.

We write $\mathcal{C}_0^1\left(\overline{\Omega}\right)$ for the Banach space of $\mathcal{C}^1$-functions on $\overline{\Omega}$ which vanish on $\partial\Omega$, endowed with the norm

$$\|u\|_{\mathcal{C}^1} := |u|_\infty + |\nabla u|_\infty .$$

For $d \in \mathbb{R}$ we write $\mathcal{N}^d := \{u \in \mathcal{N} : J(u) \leq d\}$. The following refinement of the deformation lemma was proved in [37, Lemma 1].

Lemma 2.1.1 *Assume that J has no critical values in the interval $[b,d]$ and that it satisfies $(PS)_c$ for every $b \leq c \leq d$. Then there exists a continuous map*

$$\eta : [0,1] \times \mathcal{N}^d \to \mathcal{N}^d$$

with the following properties:

(a) $\eta(0,u) = u$ and $\eta(1,u) \in \mathcal{N}^b$ for every $u \in \mathcal{N}^d$, and $\eta(t,v) = v$ for every $v \in \mathcal{N}^b$, $t \in [0,1]$.

(b) If $u \in \mathcal{N}^d \cap \mathcal{C}_0^1(\overline{\Omega})$, then $\eta(t,u) \in \mathcal{C}_0^1(\overline{\Omega})$ for every $t \in [0,1]$.

(c) If $B \subset \mathcal{N}^d \cap \mathcal{C}_0^1(\overline{\Omega})$ is bounded in $\mathcal{C}_0^1(\overline{\Omega})$, then $\widehat{B} := \{\eta(t,u) : u \in B,\ t \in [0,1]\}$ is bounded in $\mathcal{C}_0^1(\overline{\Omega})$.

(d) If $u \in \mathcal{N}^d \cap \mathcal{C}_0^1(\overline{\Omega})$ and $u \geq 0$, then $\eta(t,u) \geq 0$ for every $t \geq 0$.

Next, we consider the limit problem

$$-\Delta u = |u|^{2^*-2}u, \qquad u \in D^{1,2}\left(\mathbb{R}^N\right), \tag{2.1}$$

where the space $D^{1,2}(\mathbb{R}^N)$ is the completion of the space $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2}$. The energy functional $J_\infty : D^{1,2}\left(\mathbb{R}^N\right) \to \mathbb{R}$ associated to (2.1) is given by

$$J_\infty(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*},$$

and the Nehari manifold is

$$\mathcal{N}_\infty := \{u \in D^{1,2}\left(\mathbb{R}^N\right) : u \neq 0,\ \ \|u\|^2 = |u|_{2^*}^{2^*}\}.$$

As usual we consider $H_0^1(\Omega)$ as a Hilbert subspace of $D^{1,2}\left(\mathbb{R}^N\right)$ via trivial extensions. Then J is the restriction of J_∞ to $H_0^1(\Omega)$ and $\mathcal{N} = \mathcal{N}_\infty \cap H_0^1(\Omega)$. The radial projection $\rho : D^{1,2}\left(\mathbb{R}^N\right) \setminus \{0\} \to \mathcal{N}_\infty$ onto the Nehari manifold is given by

$$\rho(u) = \left(\frac{\|u\|^2}{|u|_{2^*}^{2^*}}\right)^{\frac{N-2}{4}} u.$$

Set

$$c_\infty := \inf_{\mathcal{N}_\infty} J_\infty.$$

The standard bubbles

$$U_{\lambda,y}(x) = [N(N-2)]^{\frac{N-2}{4}} \frac{\lambda^{\frac{N-2}{2}}}{(\lambda^2 + |x-y|^2)^{\frac{N-2}{2}}} \qquad \lambda \in (0,\infty), \quad y \in \mathbb{R}^N,$$

are the only positive solutions to problem (2.1). They satisfy $J(U_{\lambda,y}) = c_\infty$. It is a well known fact that

$$\inf_{\mathcal{N}} J = \inf_{\mathcal{N}_\infty} J_\infty = c_\infty,$$

independently of Ω, and that c_∞ is not attained by J on $\mathcal{N}$ if Ω is bounded, see e.g. [95, 102].

We consider the barycenter map $\beta : H_0^1(\Omega) \setminus \{0\} \to \mathbb{R}^N$, given by

$$\beta(u) := \frac{\int_{\mathbb{R}^N} x|u(x)|^{2^*} dx}{\int_{\mathbb{R}^N} |u(x)|^{2^*} dx}.$$

The following fact will be used below.

Lemma 2.1.2 *Let X be a closed subset of $\mathbb{R}^N$ such that $\overline{\Omega} \cap X = \emptyset$. Then*

$$c_X := \inf\{J(u) : u \in \mathcal{N},\ \beta(u) \in X\} > c_\infty.$$

Proof. Arguing by contradiction, assume there exist $u_k \in \mathcal{N}$ with $\beta(u_k) \in X$ and $J(u_k) \to c_\infty$. Using Ekeland's variational principle [50, 102], we may assume that (u_k) is a $(PS)_{c_\infty}$ sequence. Then, by Struwe's global compactness theorem [94, 102], there exist $y_k \in \Omega$ and $\lambda_k > 0$ such that, after passing to a subsequence,

$$\|u_k - U_{\lambda_k,y_k}\| \to 0 \qquad \text{as} \quad k \to \infty.$$

It follows that $|\beta(u_k) - y_k| \to 0$ as $k \to \infty$ and, hence, that $\operatorname{dist}(\beta(u_k), \Omega) \to 0$ as $k \to \infty$. This is a contradiction. ∎

We shall also use the following result, which was proved by Weth in [101].

Theorem 2.1.3 *There exists an $\varepsilon_0 > 0$ such that*

$$J_\infty(u) > 2c_\infty + 3\varepsilon_0$$

for every sign changing solution u of problem (2.1).

2.2 Two auxiliary maps

Let Θ be a bounded smooth domain in $\mathbb{R}^N$ and let M be a compact smooth submanifold of Θ, without boundary, such that $\dim M \le N-2$. We write

$$\mathrm{d}(x) := \operatorname{dist}(x, M).$$

For $a > 0$ we set

$$B_aM := \{x \in \mathbb{R}^N : \mathrm{d}(x) < a\},$$

and write $\overline{B}_aM$ for its closure and S_aM for its boundary in $\mathbb{R}^N$.

We fix $R > 0$ such that $\overline{B}_RM$ is a tubular neighborhood of M in $\mathbb{R}^N$ and $\overline{B}_RM \subset \Theta$. Then, for any $x \in \overline{B}_RM$, there is a unique point $q(x) \in M$ satisfying

$$d(x) = |x - q(x)|$$

and the map $q : \overline{B}_RM \to M$ is well defined and smooth (see [55]). For any $a < R$ and a monotone decreasing sequence of positive numbers $(r_k) \subset (0, \infty)$ such that $r_k \to 0$ as $i \to \infty$, $2r_{k+1} < r_k$ and $2r_1 < a$, the the following statement holds true.

Lemma 2.2.1 *There exists a sequence of functions $(\psi_k) \subset C^\infty(\mathbb{R}^N)$ such that*

$$\psi_k = 1 \text{ in } \overline{B}r_kM, \qquad \text{and} \qquad \mathrm{supp}\psi_k \subset \overline{B}_{2a}M, \quad \text{for all } k \in \mathbb{N}.$$

Moreover $\|\psi_k\|, |\psi_k|_2 \to 0$ as $k \to \infty$.

Proof. For each $i \in \mathbb{N}$, define the function $g_k : \mathbb{R}^N \to \mathbb{R}$ by

$$g_k(x) := \begin{cases} 1 & \text{if } d(x) \le \frac{4r_k}{3} \\ 5 - \frac{3d(x)}{r_k} & \text{if } \frac{4r_k}{3} \le d(x) \le \frac{5r_k}{3} \\ 0 & \text{if } d(x) \ge \frac{5r_k}{3} \end{cases}$$

Observe $g_k \in C_c^0\left(\mathbb{R}^N\right)$ and note it is smooth in $\mathbb{R}^N \smallsetminus (S_{r_k}M \cup S_{2r_k}M)$. This functions clearly satisfies

$$0 \le g_k \le 1, \qquad \mathrm{supp}(g_k) = \overline{B}_{\frac{5r_k}{3}}M, \qquad \text{and} \qquad g_k = 1 \text{ in } \overline{B}_{\frac{4r_k}{3}}M.$$

We claim the existence of a constant $C > 0$ depending only on M, a, N, and m, such that

$$\int_{\mathbb{R}^N} |\nabla g_k|^2 \le C, \qquad \text{for all } \; i \in \mathbb{N}.$$

In fact, since M is a compact submanifold of $\mathbb{R}^N$, it suffices to show the claim when M has the form

$$M = \{(x, F(x)) \in \mathbb{R}^N \; : \; x \in \mathbb{B}_1^m\}$$

where $\mathbb{B}_r^m := \{x \in \mathbb{R}^m \; : \; |x| \le r\}$ for $r > 0$, and $F : \mathbb{B}_1^m \to \mathbb{R}^{N-m}$ is smooth.

Consider a diffeomorphism $\varphi : \mathbb{B}_1^m \times \mathbb{B}_R^{N-m} \to B_RM$ such that $\varphi(\mathbb{B}_1^m) = M$ and that it sends $\{x\} \times \mathbb{B}_R^{N-m}$ isometrically onto the fiber $q(\varphi(x))^{-1}$. Hence $|v| = d(\varphi(x), v)$, for all $(x, v) \in \mathbb{B}_1^m \times \mathbb{B}_R^{N-m}$. Since $r_k \le R$ and $N - m \ge 2$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla g_k(y)|^2 \, dy &= \int_{B_{\frac{5r_k}{3}} M \setminus \overline{B}_{\frac{4r_k}{3}}} 9/r_k^2 \, |\nabla d(y)|^2 \, dy \\
&\leq C_1 r_k^{-2} \int_{B_{\frac{5r_k}{3}} M \setminus \overline{B}_{\frac{4r_k}{3}}} dy \\
&\leq C_2 r_k^2 \int_{\mathbb{B}_1^m \times \left(\mathbb{B}^{N-m}_{\frac{5r_k}{3}} \setminus \overline{\mathbb{B}}^{N-m}_{\frac{4r_k}{3}}\right)} dx dv \\
&\leq C_3 r_k^{-2} \int_{\frac{4r_k}{3}}^{\frac{5r_k}{3}} t^{N-m-1} dt \\
&= C_4 r_k^{N-m-2} \\
&\leq C_5 := C_4 R^{N-m-2}
\end{aligned}
$$

where constant C_1 depends on M and R, and the constant C_j depends on N, m, R and M for $j = 2, 3, 4$ and 5, and the claim is proved.

Let $\eta \in C_c^\infty\left(\mathbb{R}^N\right)$ be the standard mollifier and, for all $\varepsilon > 0$ define

$$
\eta_\varepsilon := \varepsilon^{-N} \eta\left(\frac{x}{\varepsilon}\right).
$$

Then $\eta_\varepsilon \in C_c^\infty\left(\mathbb{R}^N\right)$ satisfies that $\operatorname{supp}(\eta_\varepsilon) \subset B_\varepsilon(0)$ and $\int_{\mathbb{R}^N} \eta_\varepsilon dx = 1$.

Next, for each $k \in \mathbb{N}$, take $\varepsilon_k < \frac{r_k}{6}$ and consider the convolution

$$
f_k(y) := (\eta_{\varepsilon_k} * g_k)(y) = \int_{\mathbb{R}^N} \eta_{\varepsilon_k}(y - x) g_k(x) dx.
$$

Using the properties of the convolution, the sequence $(f_k) \subset C_c^\infty\left(\mathbb{R}^N\right)$ have the following properties:

1. $0 \leq f_k \leq 1, \quad \operatorname{supp} f_k \subset \overline{B}_{2r_k} M, \quad \text{and} \quad f_k = 1 \text{ in } \overline{B}_{r_k} M;$
2. $\operatorname{supp}(f_k) \subset \operatorname{Int}\{f_k = 1\};$
3. There exists $C > 0$ independent of r_k such that

$$
\int_{\mathbb{R}^N} |\nabla f_k|^2 \leq C \quad \text{for all} \quad k \in \mathbb{N}.
$$

The first two are readily checked. Let us prove the third assertion:

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla f_k|^2 &= |\nabla(\eta_{\varepsilon_k} * g_k)|_2^2 \\
&\leq |\eta_{\varepsilon_k} * (\nabla g_k)|_2^2 \\
&\leq |\eta_{\varepsilon_k}|_1^2 \, |\nabla g_k|_2^2 \leq C_5,
\end{aligned}
$$

where C_5 was the constant independent of r_k obtained just above.

Finally, set $S_k = \sum_{i=1}^{k} \frac{1}{i}$ and define $\psi_k : \mathbb{R}^N \to \mathbb{R}$ by

$$\psi_k(y) = \frac{1}{S_k} \sum_{i=1}^{k} \frac{f_i(y)}{i}.$$

The first two properties of ψ_k follow directly from the properties of f_k. To prove the third one, note that $\mathrm{supp}|\nabla f_k| \subset \overline{B}_{2r_k}M \smallsetminus B_{2r_{k+1}}M$, so

$$\begin{aligned}
\|\psi_k\|^2 &= \int_{\mathbb{R}^N} \left| \frac{1}{S_k} \sum_{i=1}^{k} \frac{\nabla f_i(y)}{i} \right|^2 dy \\
&= \left(\frac{1}{S_k}\right)^2 \sum_{i=1}^{k} \frac{1}{i^2} \int_{\mathbb{R}^N} |\nabla f_i(y)|^2 \, dy \\
&\leq C \left(\frac{1}{S_k}\right)^2 \sum_{i=1}^{k} \frac{1}{i^2} \\
&\to 0
\end{aligned}$$

as $k \to \infty$ as we wished. ■

Fix $u \in C_c^\infty(\mathbb{R}^N) \cap \mathcal{N}_\infty$ and set $u_y(x) := u(x-y)$. If $(1-\psi_k)u_y \neq 0$ we define

$$u_{k,y} := \rho([1-\psi_k]\, u_y).$$

where ρ is the radial projection onto $\mathcal{N}_\infty$.

Lemma 2.2.2 *Given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$(1-\psi_k)u_y \neq 0, \qquad |J_\infty(u_{k,y}) - J_\infty(u_y)| < \varepsilon \quad \text{and} \quad |\beta(u_{k,y}) - \beta(u_y)| < \varepsilon$$

for all natural numbers $k \geq k_0$ and all $y \in \mathbb{R}^N$.

Proof. As $u \neq 0$, $|u| > 0$ in some ball of radius γ. Fix $i \in \mathbb{N}$ such that $B_\gamma(y) \smallsetminus \overline{B}_{Rr_i}M \neq \emptyset$ for every $y \in \mathbb{R}^N$. Then, $(1-\psi_k)u_y \neq 0$ for every $k \geq i$ and $y \in \mathbb{R}^N$. Moreover, since u has compact support, there exists $L > 0$ such that $u_{k,y} = u_y$ for every $k \geq i$ and $|y| \geq L$.

By Lemma 2.2.1,

$$\begin{aligned}
\|u_{k,y} - u_y\|^2 = \|\psi_k u_y\|^2 &= \int_{\mathbb{R}^N} |u_y \nabla\psi_k - \psi_k \nabla u_y|^2 \\
&\leq 4 \int_{\mathbb{R}^N} |u_y|^2 |\nabla\psi_k|^2 + |\psi_k|^2 |\nabla u_y|^2 \\
&\leq C \left(\|\psi_k\|^2 + |\psi_k|_2^2 \right) \to 0 \qquad \text{as } k \to \infty.
\end{aligned}$$

The functions $J_\infty \circ \rho$ and $\beta \circ \rho$ are both uniformly continuous on the compact set

$$\mathcal{K} = \{u_y : |y| \leq M\} \cup \{u_{k,y} : |y| \leq M,\ k \geq i\} \subset D^{1,2}(\mathbb{R}^N),$$

implying the existence of a natural number $k_0 \geq i$ such that

$$|J_\infty(u_{k,y}) - J_\infty(u_y)| < \varepsilon \quad \text{and} \quad |\beta(u_{k,y}) - \beta(u_y)| < \varepsilon$$

for all $i \geq k_0$ and all $y \in \mathbb{R}^N$, as claimed. ■

We can now construct a useful map form $\mathbb{R}^N$ to $\mathcal{N}_\infty$

Lemma 2.2.3 *For any given $a \in (0, R)$ and $\varepsilon > 0$ there exists $a_0 \in (0, a)$ such that, for every $\delta > 0$, there is a continuous function $h_\delta : \mathbb{R}^N \to D^{1,2}(\mathbb{R}^N)$ with the following properties:*

(a) $h_\delta(y) \in \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ *and* $h_\delta(y) \geq 0$ *for all* $y \in \mathbb{R}^N$,

(b) The $\mathcal{C}^1$*-norm of* $h_\delta(y)$ *is uniformly bounded on* $\mathbb{R}^N$, *i.e.*

$$\sup\{|h_\delta(y)|_\infty + |\nabla(h_\delta(y))|_\infty : y \in \mathbb{R}^N\} < \infty.$$

(c) $J_\infty(h_\delta(y)) \leq c_\infty + \varepsilon$ *for all* $y \in \mathbb{R}^N$,

(d) $J_\infty(h_\delta(y)) \leq c_\infty + \delta$ *for all* $y \in \mathbb{R}^N \smallsetminus B_a M$.

(e) $\operatorname{supp}(h_\delta(y)) \subset \overline{B}_{a+\delta}M \smallsetminus B_{a_0}M$ *for all* $y \in \overline{B}_a M$,

(f) $\operatorname{supp}(h_\delta(y)) \subset B_\delta(y)$ *for all* $y \in \mathbb{R}^N \smallsetminus B_a M$,

(g) $\beta(h_\delta(y)) = y$ *for all* $y \in \mathbb{R}^N \smallsetminus B_a M$,

(h) $\beta(h_\delta(y)) \in \overline{B}_a M$ *for all* $y \in \overline{B}_a M$, *and there is a continuous map* $\vartheta : [0,1] \times \overline{B}_a M \to \overline{B}_a M$ *such that* $\vartheta(0, y) = y$, $\vartheta(1, y) = \beta(h_\delta(y))$ *for all* $y \in \overline{B}_a M$, *and* $\vartheta(t, z) = z$ *for all* $z \in S_a M$.

Proof. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be a radial function such that $\chi(x) = 1$ if $|x| \leq \frac{a}{8}$, $\chi(x) \in (0, 1]$ if $|x| < \frac{a}{4}$ and $\chi(x) = 0$ if $|x| \geq \frac{a}{4}$. Fix $\mu > 0$ so that the function

$$w := \rho(\chi U_{\mu,0})$$

satisfies $J_\infty(w) \leq c_\infty + \min\{\delta, \frac{\varepsilon}{2}\}$, were $U_{\mu,0}$ is the standard bubble. For $\lambda > 0$ and $y \in \mathbb{R}^N$ we define

$$w_{\lambda,y}(x) := \lambda^{\frac{2-N}{2}} w\left(\frac{x-y}{\lambda}\right).$$

Then $w_{\lambda,y} \in \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$, $J_\infty(w_{\lambda,y}) = J_\infty(w) \leq c_\infty + \min\{\delta, \frac{\varepsilon}{2}\}$, $\operatorname{supp}(w_{\lambda,y}) \subset B_{\frac{\lambda a}{4}}(y)$ and $\beta(w_{\lambda,y}) = y$.

Set $\gamma := \min\{\delta, \frac{a}{4}\} > 0$ and choose a nonincreasing function $\Lambda \in \mathcal{C}^\infty[0, \infty)$ such that $\Lambda(t) = 1$ if $t \leq \frac{a}{2}$, $\Lambda(t) = \frac{4\gamma}{a}$ if $t \geq a$ and $\Lambda(t) \leq \frac{4}{a}(a + \gamma - t)$ for all $t \leq a$. Define $\widetilde{h} : \mathbb{R}^N \to \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ as

$$\widetilde{h}(y) := w_{\Lambda(\mathrm{d}(y)),y}.$$

Note that $\widetilde{h}(y) = w_{1,y}$ if $\mathrm{d}(y) \leq \frac{a}{2}$, $\operatorname{supp}(\widetilde{h}(y)) \subset B_\gamma(y)$ if $\mathrm{d}(y) \geq a$ and $\operatorname{supp}(\widetilde{h}(y)) \subset B_{a-\mathrm{d}(y)+\gamma}(y) \subset \overline{B}_{a+\delta}M$ for all $y \in \overline{B}_a M$.

Fix $a_1 \in (0, \frac{a}{4})$. Since $\dim M \leq N - 2$, for $k > \frac{1}{a_1}$, Lemma 2.2.1 gives functions $\psi_k \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\psi_k(x) = 1$ if $\mathrm{d}(x) \leq \frac{1}{k}$, $\psi_k(x) = 0$ if $\mathrm{d}(x) \geq a_1$, and $\|\psi_k\| \to 0$ as $k \to \infty$. Then, $[1 - \psi_k]\,\widetilde{h}(y) = \widetilde{h}(y) \neq 0$ if $\mathrm{d}(y) \geq \frac{a}{2}$ and $[1 - \psi_k]\,\widetilde{h}(y) = [1 - \psi_k]\, w_{1,y} \neq 0$ if $\mathrm{d}(y) \leq \frac{a}{2}$, because $w_{1,y} > 0$ in $B_{\frac{a}{4}}(y)$ and $B_{\frac{a}{4}}(y) \smallsetminus \overline{B}_{a_1}M \neq \emptyset$. Therefore, the function $\widetilde{h}_k : \mathbb{R}^N \to \mathcal{N}_\infty \cap \mathcal{C}_c^\infty(\mathbb{R}^N)$ given by

$$\widetilde{h}_k(y) := \rho([1 - \psi_k]\,\widetilde{h}(y))$$

is well defined. It satisfies that $\mathrm{supp}(\widetilde{h}_k(y)) \subset \overline{B}_{a+\delta}M \smallsetminus B_{\frac{1}{k}}M$ for all $y \in \overline{B}_a M$. Moreover, by Lemma 2.2.2 there exists $k_0 > \frac{1}{a_1}$ such that

$$\left| J_\infty(\widetilde{h}_k(y)) - J_\infty(\widetilde{h}(y)) \right| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \left| \beta(\widetilde{h}_k(y)) - y \right| < \frac{a}{2}$$

for all $k \geq k_0$ and all $y \in \mathbb{R}^N$. Set

$$h_\delta(y) := \widetilde{h}_{k_0}(y) \qquad \text{and} \qquad a_0 := \frac{1}{k_0}.$$

Then, $\mathrm{supp}(\widetilde{h}_\delta(y)) \subset \overline{B}_{a+\delta}M \smallsetminus B_{a_0}M$ for all $y \in \overline{B}_a M$ and, since $J_\infty(\widetilde{h}(y)) \leq c_\infty + \frac{\varepsilon}{2}$, we have that

$$J_\infty(h_\delta(y)) \leq c_\infty + \varepsilon \qquad \text{for all } y \in \mathbb{R}^N,$$

Clearly, $h_\delta(y)$ satisfies (a) and (b) and, since $h_\delta(y) = \widetilde{h}(y)$ if $\mathrm{d}(y) \geq \frac{a}{2}$, it also satisfies properties (d), (f) and (g). The map $\vartheta(t, y) := (1-t)y + t\beta(h_\delta(y))$ is well defined and has the properties stated in (h). ■

It is convenient sometimes to write the elements of $\overline{B}_R M$ as

$$[\zeta, t] := q(\zeta) + \frac{t}{R}(\zeta - q(\zeta)) \qquad \text{with } \zeta \in S_R M \text{ and } t \in [0, R]. \tag{2.2}$$

Define

$$\mathcal{E} := \{u \in \mathcal{N} : u^+, u^- \in \mathcal{N}\},$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. We prove the following statement.

Lemma 2.2.4 *Given $\varepsilon \in (0, c_\infty)$ there exist $b_0, b_1, b_2 \in (0, R)$ such that, if Ω is a bounded smooth domain in $\mathbb{R}^N$ which satisfies*

$$M \cap \overline{\Omega} = \emptyset \qquad \textit{and} \qquad (\Theta \smallsetminus B_r M) \subset \Omega \subset \Theta,$$

for some $r \in (0, b_0)$, and $c_\infty < c_0 < c_1 < c_\infty + \varepsilon$ are such that J has no critical values in $(c_1, c_\infty + \varepsilon]$, then there exists a continuous map $G : \overline{B}_{b_1}M \times \overline{B}_{b_2}M \to \mathcal{E}$ with the following properties:

(i) $J(G(x, y)) \leq 2c_\infty + 2\varepsilon$ *for all* $(x, y) \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M$,

(ii) $J(G(x, y)) \leq c_0 + c_1$ *for all* $(x, y) \in \left(S_{b_1}M \times \overline{B}_{b_2}M\right) \cup \left(\overline{B}_{b_1}M \times S_{b_2}M\right)$,

(iii) $J(G(x, y)^+) \leq c_0$ *and* $\beta(G(x, y)^+) = x$ *for all* $(x, y) \in \left(S_{b_1}M \times \overline{B}_{b_2}M\right)$,

(iv) $J(G(x, y)^-) \leq c_0$ *and* $\beta(G(x, y)^-) = y$ *for all* $(x, y) \in \left(\overline{B}_{b_1}M \times S_{b_2}M\right)$.

Proof. Fix $a_1 \in (0, R)$. For $a := a_1$ and the given ε, let $a_{1,0} \in (0, a_1)$ be as in Lemma 2.2.3. Now fix $a_2 \in (0, a_{1,0})$ and, for $a := a_2$ and the same ε, let $a_{2,0} \in (0, a_2)$ be as in Lemma 2.2.3. Set $b_0 := a_{2,0}$ and $r \in (0, b_0)$. Let

$$\delta := \frac{1}{4} \min\{R - a_1,\ a_{1,0} - a_2, a_2 - b_0,\ b_0 - r,\ 2\varepsilon,\ c_0 - c_\infty,\ c_\infty + \varepsilon - c_1\}$$

and choose a function $h_\delta^i : \mathbb{R}^N \to D^{1,2}(\mathbb{R}^N)$ with the properties stated in Lemma 2.2.3 for a_i, ε and $a_{i,0}$.

Since

$$\operatorname{supp}(h_\delta^i(x)) \subset \left(\overline{B}_{a_i+\delta}M \smallsetminus B_{a_{i,0}}M\right) \subset \left(\overline{B}_R M \smallsetminus B_{a_{i,0}}M\right) \qquad \text{for all } x \in \overline{B}_{a_i}M,$$

we have that

$$h_\delta^i\left(\overline{B}_{a_i}M\right) \subset \mathcal{N} \cap \mathcal{C}_0^1(\overline{\Omega}) \qquad \text{if } \ r \in (0, b_0), \quad i = 1, 2,$$

In addition, $h_\delta^i(x) \geq 0$ and $J(h_\delta^i(x)) \leq c_\infty + \varepsilon$ for all $x \in \overline{B}_{a_i}M$; $J(h_\delta^i(y)) \leq c_\infty + \delta$, $\operatorname{supp}(h_\delta^i(y)) \subset B_\delta(y)$ and $\beta(h_\delta^i(y)) = y$ for all $y \in S_{a_i}M$; and the set $\{h_\delta^i(x) : x \in \overline{B}_{a_i}M\}$ is bounded in $\mathcal{C}_0^1(\overline{\Omega})$.

Since J satisfies $(PS)_c$ at every $c \in (c_\infty, 2c_\infty)$ and J has no critical values in $(c_1, c_\infty + \varepsilon]$, Lemma 2.1.1 yields a deformation

$$\eta : [0,1] \times \mathcal{N}^{c_\infty+\varepsilon} \to \mathcal{N}^{c_\infty+\varepsilon}$$

such that $\eta(0,u) = u$ and $\eta(1,u) \in \mathcal{N}^{c_1+\delta}$ for every $u \in \mathcal{N}^{c_\infty+\varepsilon}$; $\eta(t,v) = v$ for every $v \in \mathcal{N}^{c_1+\delta}$, $t \in [0,1]$; and $\eta(t, h_\delta^i(x)) \geq 0$ for every $x \in \overline{B}_{a_i}M$. Moreover, the sets

$$\mathcal{K}_i := \{\eta(t, h_\delta^i(x)) : x \in \overline{B}_{a_i}M,\ t \in [0,1]\}, \qquad i = 1, 2,$$

are compact in $H_0^1(\Omega)$ and, by statement (c) in Lemma 2.1.1, they are bounded in the $\mathcal{C}^1$-norm.

Fix a radial function $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $\phi(x) = 1$ if $|x| \geq 2$ and $\phi(x) = 0$ if $|x| \leq 1$. Then, there is a $\gamma \in (0, \frac{\delta}{2})$ such that the function $\phi_x(y) := \phi(\frac{y-x}{\gamma})$ satisfies

$$\phi_x u \neq 0 \quad \text{and} \quad |J(\rho(\phi_x u)) - J(u)| < \delta, \qquad \text{for all } (x,u) \in \mathbb{R}^N \times (\mathcal{K}_1 \cup \mathcal{K}_2),$$

see [37, Lemma 2]. Note that $\phi_x u \equiv 0$ in $\overline{B}_\gamma(x)$ for every $u \in \mathcal{K}_1 \cup \mathcal{K}_2$, $x \in \mathbb{R}^N$.

Next, we choose $\lambda_i \in \mathcal{C}^\infty[0,\infty)$ nonincreasing and such that $\lambda_i(t) = 1$ if $t \leq a_i$ and $\lambda_i(t) = \frac{\gamma}{\delta}$ if $t \geq a_i + \delta$. Using the notation introduced in (2.2), we define $G_i : \overline{B}_{a_i+2\delta}M \to \mathcal{N}$ as follows:

$$G_i([\zeta,t]) := \begin{cases} h_\delta^i([\zeta,t]) & \text{if } t \in [0,a_i], \quad \zeta \in S_R M, \\ (h_\delta^i([\zeta,a_i]))_{\lambda(t),[\zeta,t]-\lambda(t)[\zeta,a_i]} & \text{if } t \in [a_i, a_i+2\delta], \quad \zeta \in S_R M, \end{cases}$$

where $u_{\lambda,x}(y) := \lambda^{\frac{2-N}{2}} u(\frac{y-x}{\lambda})$. Then, $J(G_i(y)) \leq c_\infty + \delta$ and $\beta(G_i(y)) = y$ if $a_i \leq \mathrm{d}(y) \leq a_i + 2\delta$, and $\operatorname{supp}(G_i(y)) \subset \overline{B}_\gamma(y)$ if $a_i + \delta \leq \mathrm{d}(y) \leq a_i + 2\delta$.

We define $G : \overline{B}_{a_1+2\delta}M \times \overline{B}_{a_2+2\delta}M \to \mathcal{E}$ in the following way: for $x \in \overline{B}_{a_1+2\delta}$, $y \in \overline{B}_{a_2+2\delta}M$, let

$$G(x,y) := \begin{cases} G_1(x) - G_2(y) & \text{if } \mathrm{d}(x) \leq a_1+\delta, \ \ \mathrm{d}(y) \leq a_2+\delta, \\ \rho\left[\phi_y \eta\left(\frac{\mathrm{d}(y)-a_2-\delta}{\delta}, G_1(x)\right)\right] - G_2(y) & \text{if } \mathrm{d}(x) \leq a_1+\delta, \ \ a_2+\delta \leq \mathrm{d}(y), \\ G_1(x) - \rho\left[\phi_x \eta\left(\frac{\mathrm{d}(x)-a_1-\delta}{\delta}, G_2(y)\right)\right] & \text{if } a_1+\delta \leq \mathrm{d}(x), \ \ \mathrm{d}(y) \leq a_2+\delta, \\ G_1(x) - G_2(y) & \text{if } a_1+\delta \leq \mathrm{d}(x), \ \ a_2+\delta \leq \mathrm{d}(y). \end{cases}$$

Note that in all four cases, the first summand and the second summand in the definition of $G(x,y)$ have disjoint supports. Therefore, the first summand is $G(x,y)^+$

and the second one is $G(x,y)^-$. Since both summands belong to $\mathcal{N}$ we conclude that $G(x,y) \in \mathcal{E}$. Moreover,

$$J(G(x,y)) = J(G(x,y)^+) + J(G(x,y)^-).$$

Setting $b_i := a_i + 2\delta$, one can easily check that G has the desired properties. ∎

2.3 Proof of Theorem 1.2.1

As before, we fix R small enough so that $\overline{B}_R M$ is a tubular neighborhood of M contained in Θ. Fix $\varrho \in (0, \mathrm{dist}(\overline{B}_R M, \partial\Theta))$ small enough so that $\overline{B}_\varrho(\partial\Theta)$ is a tubular neighborhood of $\partial\Theta$, and set $\Theta^- := \Theta \smallsetminus B_\varrho(\partial\Theta)$ and $\Theta^+ := \Theta \cup B_\varrho(\partial\Theta)$. Define

$$d^* := \inf\{J_\infty(u) : u \in \mathcal{N}_\infty \cap H_0^1(\Theta), \quad \beta(u) \notin \Theta^+\}.$$

By Lemma 2.1.2 we have that $d^* > c_\infty$.

Choose $\varepsilon_0 \in (0, \frac{c_\infty}{3})$ as in Theorem 2.1.3 and such that $c_\infty + \varepsilon_0 < d^*$. For $\varepsilon := \frac{3}{2}\varepsilon_0$ fix $b_0, b_1, b_2 \in (0, R)$ as in Lemma 2.2.4, and for $a := b_1$ and $\varepsilon := \varepsilon_0$ fix $a_0 \in (0, a)$ as in Lemma 2.2.3.

Set $r_0 := \min\{a_0, b_0\}$ and let Ω be a bounded smooth domain in $\mathbb{R}^N$ which satisfies

$$M \cap \overline{\Omega} = \emptyset \qquad \text{and} \qquad \left(\Theta \smallsetminus \overline{B}_r M\right) \subset \Omega \subset \Theta$$

for some $r \in (0, r_0)$. Set $r_1 := \frac{1}{2}\mathrm{dist}(\overline{\Omega}, M)$ and define

$$c^* := \inf\{J(u) : u \in \mathcal{N}, \ \beta(u) \in \overline{B}_{r_1} M\}.$$

By Lemma 2.1.2 we have that $c^* > c_\infty$. Now fix a regular value c_0 of J such that

$$c_\infty < c_0 < \min\{c^*, c_\infty + \varepsilon_0\}.$$

Note that

$$\begin{aligned} &\beta(u) \in \Theta^+ \quad \text{ for all } u \in \mathcal{N}^c \text{ with } c \in (0, d^*), \\ &\beta(u) \in \Theta^+ \smallsetminus \overline{B}_{r_1} M \quad \text{ for all } u \in \mathcal{N}^{c_0}. \end{aligned}$$

Let $\mathcal{H}^*$ be Čech cohomology with $\mathbb{Z}/2$-coefficients and define

$$c_1 := \inf\{c \in [c_0, d^*) : \beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M) \to \mathcal{H}^*(\mathcal{N}^c, \mathcal{N}^{c_0}) \text{ is a monomorphism}\}$$

For these data all statements below hold true.

Proposition 2.3.1 *$c_0 < c_1 \le c_\infty + \varepsilon_0$, and problem (1.5) has at least*

$$\mathrm{cat}(\Theta, \Theta \smallsetminus B_r M) \ge 1$$

positive solutions with energy in $[c_0, c_\infty + \varepsilon_0]$, and at least

$$\mathrm{cupl}(\Theta, \Theta \smallsetminus B_r M) \ge 1$$

positive solutions with energy in $[c_0, c_1]$. Moreover, c_1 is a critical value of J.

Proof. Since $c_0 \in (c_\infty, 2c_\infty)$ and $(PS)_c$ holds true for every $c \in (c_\infty, 2c_\infty)$, there exist $\alpha > 0$ and a deformation of $\mathcal{N}^{c_0+\alpha}$ into $\mathcal{N}^{c_0}$ which keeps $\mathcal{N}^{c_0}$ fixed. Hence $\mathcal{H}^*(\mathcal{N}^{c_0+\alpha}, \mathcal{N}^{c_0}) = 0$. On the other hand, the inclusion $i : (\overline{B}_{r_1}M, S_{r_1}M) \hookrightarrow (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$ induces an isomorphism in cohomology

$$i^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \cong \mathcal{H}^*(\overline{B}_{r_1}M, S_{r_1}M)$$

by excision. $\mathcal{H}^{N-m}(\overline{B}_{r_1}M, S_{r_1}M)$ contains a nontrivial element: the Thom class of the disk bundle $q : B_{r_1}M \to M$, where $m := \dim M$. Therefore, $\mathcal{H}^{N-m}(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \neq 0$. This implies that $c_1 \geq c_0 + \alpha > c_0$. Note that it also implies that

$$\operatorname{cupl}(\Theta, \Theta \smallsetminus B_r M) = \operatorname{cupl}(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \geq 1. \tag{2.3}$$

Set $\delta := \min\{c_0 - c_\infty, \varrho\}$. Then, Lemma 2.2.3 yields a map $h_\delta : \mathbb{R}^N \to D^{1,2}(\mathbb{R}^N)$ with $\operatorname{supp}(h_\delta(x)) \subset \Theta \smallsetminus B_{a_0}M \subset \Omega$ for all $x \in \Theta^-$, which restricts to a map of pairs

$$h_\delta : (\Theta^-, \Theta^- \smallsetminus B_a M) \to (\mathcal{N}^{c_\infty+\varepsilon_0}, \mathcal{N}^{c_0}) \tag{2.4}$$

such that the composition

$$(\Theta^-, \Theta^- \smallsetminus B_a M) \xrightarrow{h_\delta} (\mathcal{N}^{c_\infty+\varepsilon_0}, \mathcal{N}^{c_0}) \xrightarrow{\beta} (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$$

is homotopic to the inclusion $\iota : (\Theta^-, \Theta^- \smallsetminus B_a M) \hookrightarrow (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$. Since

$$\iota^* = h_\delta^* \circ \beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*(\Theta^-, \Theta^- \smallsetminus B_r M)$$

is an isomorphism, we have that

$$\beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^{c_\infty+\varepsilon_0}, \mathcal{N}^{c_0})$$

is a monomorphism. Hence, $c_1 \leq c_\infty + \varepsilon_0$.

If $c \in (c_0, 2c_\infty)$, the number of pairs $\pm u$ of critical points of J on $\mathcal{N}$ with critical values in $[c_0, c]$ is at least $\operatorname{cat}(\widetilde{\mathcal{N}}^c, \widetilde{\mathcal{N}}^{c_0})$, where $\widetilde{\mathcal{N}}^c$ is the quotient space of $\mathcal{N}^c$ obtained by identifying u with $-u$ (see [10, 33]). Note that $\beta(u) = \beta(-u)$. Hence, there is a map $\widetilde{\beta} : (\widetilde{\mathcal{N}}^c, \widetilde{\mathcal{N}}^{c_0}) \to (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$ such that $\widetilde{\beta} \circ \kappa = \beta$, where $\kappa : (\mathcal{N}^c, \mathcal{N}^{c_0}) \to (\widetilde{\mathcal{N}}^c, \widetilde{\mathcal{N}}^{c_0})$ is the quotient map.

Set $c := c_\infty + \varepsilon_0$ and let h_δ be the map given in (2.4). Since $\widetilde{\beta} \circ \kappa \circ h_\delta = \beta \circ h_\delta$ is homotopic to ι, and each of the inclusions $(\Theta^-, \Theta^- \smallsetminus B_a M) \hookrightarrow (\Theta, \Theta \smallsetminus B_r M) \hookrightarrow (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$ is a homotopy equivalence of pairs, using Lemma B.1.2 we conclude that

$$\operatorname{cat}(\widetilde{\mathcal{N}}^{c_\infty+\varepsilon_0}, \widetilde{\mathcal{N}}^{c_0}) \geq \operatorname{cat}(\Theta, \Theta \smallsetminus B_r M).$$

Hence, problem (1.5) has at least $\operatorname{cat}(\Theta, \Theta \smallsetminus B_r M)$ pairs of solutions $\pm u$ with $J(u) \in [c_0, c_\infty + \varepsilon_0]$. Moreover, Lemma B.1.1 and inequality (2.3) allow us to conclude that

$$\operatorname{cat}(\Theta, \Theta \smallsetminus B_r M) \geq \operatorname{cupl}(\Theta, \Theta \smallsetminus B_r M) \geq 1.$$

Now set $c := c_1$. Since $\beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^{c_1}, \mathcal{N}^{c_0})$ is a monomorphism, $\widetilde{\beta}^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*(\widetilde{\mathcal{N}}^{c_1}, \widetilde{\mathcal{N}}^{c_0})$ is also a monomorphism. Lemmas B.1.1 to B.1.3 imply that

$$\operatorname{cat}(\widetilde{\mathcal{N}}^{c_1}, \widetilde{\mathcal{N}}^{c_0}) \geq \operatorname{cupl}(\widetilde{\mathcal{N}}^{c_1}, \widetilde{\mathcal{N}}^{c_0}) \geq \operatorname{cupl}(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) = \operatorname{cupl}(\Theta, \Theta \smallsetminus B_r M).$$

Hence, problem (1.5) has at least $\operatorname{cupl}(\Theta, \Theta \smallsetminus B_r M) \geq 1$ pairs of solutions $\pm u$ with $J(u) \in [c_0, c_1]$.

Note that c_1 must be a critical value. Otherwise, for $\alpha > 0$ small enough, we would be able to deform $\mathcal{N}^{c_1+\alpha}$ into $\mathcal{N}^{c_1-\alpha}$ keeping $\mathcal{N}^{c_0}$ fixed. Since $\beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M) \to \mathcal{H}^*(\mathcal{N}^{c_1+\alpha}, \mathcal{N}^{c_0})$ is a monomorphism, this would imply that $\beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M) \to \mathcal{H}^*(\mathcal{N}^{c_1-\alpha}, \mathcal{N}^{c_0})$ is also a monomorphism, contradicting the definition of c_1.

Finally, recall that every critical point u of J with $J(u) \in (c_\infty, 2c_\infty)$ does not change sign. Otherwise, we would have that $u^+ \neq 0$ and $u^- \neq 0$ and, hence, that $u^\pm \in \mathcal{N}$, because $\|u^\pm\|^2 - |u^\pm|_{2^*}^{2^*} = J'(u)u^\pm = 0$. But then $J(u) = J(u^+) + J(u^-) \geq 2c_\infty$, which is a contradiction. ∎

To conclude the proof of Theorem 1.2.1 we shall show next that J has a critical value in $(c_1, 2c_\infty + 3\varepsilon_0]$. We need the following two lemmas.

Lemma 2.3.2 *The connecting homomorphism $\delta^* : \widetilde{\mathcal{H}}^{*-1}(\Theta^+ \smallsetminus B_{r_1} M) \to \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M)$ of the reduced cohomology exact sequence of the pair $(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M)$ is an epimorphism.*

Proof. Since the sequence

$$\widetilde{\mathcal{H}}^{*-1}(\Theta^+ \smallsetminus B_{r_1} M) \xrightarrow{\delta^*} \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M) \xrightarrow{j^*} \widetilde{\mathcal{H}}^*(\Theta^+)$$

is exact, we need only to show that the homomorphism j^*, induced by the inclusion, is trivial. The diagram

$$\begin{array}{ccc} \mathcal{H}^*(\mathbb{R}^N, \mathbb{R}^N \smallsetminus B_{r_1} M) & \longrightarrow & \widetilde{\mathcal{H}}^*(\mathbb{R}^N) \\ \cong \downarrow & & \downarrow \\ \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1} M) & \xrightarrow{j^*} & \widetilde{\mathcal{H}}^*(\Theta^+) \end{array}$$

induced by inclusions, commutes. The left vertical arrow is an isomorphism by excision. Since $\widetilde{\mathcal{H}}^*(\mathbb{R}^N) = 0$, we conclude that $j^* = 0$. ∎

The next lemma is a consequence of Struwe´s global compactness theorem [94, 102] and Theorem 2.1.3.

Lemma 2.3.3 *If J does not have a critical value in $(c_1, 2c_\infty)$, then J satisfies $(PS)_c$ for every $c \in (c_\infty + c_1, 2c_\infty + 3\varepsilon_0]$.*

Proof. See [37, Lemma 6]. ∎

Proposition 2.3.4 *J has a critical value in $(c_1, 2c_\infty + 3\varepsilon_0]$.*

Proof. Arguing by contradiction, assume that J does not have a critical value in $(c_1, 2c_\infty + 3\varepsilon_0]$. By Lemma 2.2.4 there is a continuous map $G : \overline{B}_{b_1} M \times \overline{B}_{b_2} M \to \mathcal{E}$ such that

$$\begin{aligned} J(G(x,y)) \leq 2c_\infty + 3\varepsilon_0 \quad & \text{for all } (x,y) \in \overline{B}_{b_1} M \times \overline{B}_{b_2} M, \\ J(G(x,y)) \leq c_0 + c_1 \quad & \text{for all } (x,y) \in \left(S_{b_1} M \times \overline{B}_{b_2} M\right) \cup \left(\overline{B}_{b_1} M \times S_{b_2} M\right), \\ J(G(x,y)^+) \leq c_0 \ \text{ and } \ \beta(G(x,y)^+) = x \quad & \text{for all } (x,y) \in \left(S_{b_1} M \times \overline{B}_{b_2} M\right), \\ J(G(x,y)^-) \leq c_0 \ \text{ and } \ \beta(G(x,y)^-) = y \quad & \text{for all } (x,y) \in \left(\overline{B}_{b_1} M \times S_{b_2} M\right), \end{aligned}$$

where b_1, b_2 were chosen at the beginning of this section. By Lemma 2.3.3 there is a continuous map

$$\eta : [0,1] \times \mathcal{N}^{2c_\infty + 3\varepsilon_0} \to \mathcal{N}^{2c_\infty + 3\varepsilon_0}$$

such that $\eta(0,u) = u$ and $\eta(1,u) \in \mathcal{N}^{c_0+c_1}$ for every $u \in \mathcal{N}^{2c_\infty + 3\varepsilon_0}$, and $\eta(t,v) = v$ for every $v \in \mathcal{N}^{c_0+c_1}$, $t \in [0,1]$.

For $t \in [0,1]$ we define $g_t : \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] \to \mathcal{N}^{2c_\infty + 3\varepsilon_0}$ by

$$g_t(x,y,\lambda) := \eta(t, \rho\left((1+\lambda)G(x,y)^+ + (1-\lambda)G(x,y)^-\right)),$$

where ρ is the radial projection onto $\mathcal{N}$. Then,

$$\begin{aligned} g_t(x,y,\lambda) = \rho\left((1+\lambda)G(x,y)^+ + (1-\lambda)G(x,y)^-\right) \\ \text{for all } (x,y) \in \partial\left(\overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1]\right). \end{aligned}$$

Consider the sets

$$\begin{aligned} \mathcal{E}^* &:= \{u \in \mathcal{E} : \beta(u^-) \in M\}, \\ K &:= \{\mathbf{z} \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] : g_1(\mathbf{z}) \in \mathcal{E}^*\}. \end{aligned}$$

Since K is compact and

$$c_0 + c_1 \geq J(g_1(\mathbf{z})) = J(g_1(\mathbf{z})^+) + J(g_1(\mathbf{z})^-) > J(g_1(\mathbf{z})^+) + c_0 \qquad \text{for all } \mathbf{z} \in K,$$

we have that

$$d := \max_{\mathbf{z} \in K} J(g_1(\mathbf{z})^+) < c_1.$$

We claim that $\beta : (\mathcal{N}^d, \mathcal{N}^{c_0}) \to (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$ induces a monomorphism

$$\beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*(\mathcal{N}^d, \mathcal{N}^{c_0}). \tag{2.5}$$

This contradicts the definition of c_1, and proves the proposition by contradiction.

The rest of the argument is devoted to the proof of this claim. Let $\gamma_0 : H_0^1(\Omega) \to \mathbb{R}$ be given by

$$\gamma_0(u) := \begin{cases} \frac{|u|_{2^*}^{2^*}}{\|u\|^2} - 1 & \text{if } u \neq 0, \\ -1 & \text{if } u = 0. \end{cases}$$

Then γ_0 is continuous, and $\gamma_0(u) = 0$ iff $u \in \mathcal{N}$. Define $\gamma : \mathcal{N} \to \mathbb{R}$ as

$$\gamma(u) := \gamma_0(u^+) - \gamma_0(u^-).$$

Note that

$$\gamma(u) = -1 \quad \text{iff} \quad u \leq 0, \qquad \gamma(u) = 1 \quad \text{iff} \quad u \geq 0, \qquad \gamma(u) = 0 \quad \text{iff} \quad u \in \mathcal{E}.$$

Denote by $\mathbf{z} := (x,y,\lambda) \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1]$ and, for each $t \in [0,1]$, define $\widetilde{\beta}_t : \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] \to \mathbb{R}^N$ by

$$\widetilde{\beta}_t(\mathbf{z}) := \begin{cases} [1 - \gamma(g_t(\mathbf{z}))]\beta(g_t(\mathbf{z})^-) + \gamma(g_t(\mathbf{z}))y & \text{if } g_t(\mathbf{z})^- \neq 0, \\ y & \text{if } g_t(\mathbf{z})^- = 0. \end{cases}$$

This function is continuous and depends continuously on t.

Next, consider the map $\theta_t : \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] \to \mathbb{R}^N \times \mathbb{R}$ defined by

$$\theta_t(\mathbf{z}) := \begin{cases} (\widetilde{\beta}_t(\mathbf{z}),\ \gamma(g_t(\mathbf{z}))) & \text{if } t \in [0,1], \\ -t(y,\lambda) + (1+t)\theta_0(\mathbf{z}) & \text{if } t \in [-1,0]. \end{cases}$$

We write $\theta_t(\mathbf{z}) = (\theta_{t,1}(\mathbf{z}), \theta_{t,2}(\mathbf{z})) \in \mathbb{R}^N \times \mathbb{R}$. It is easy to check that θ_t has the following properties (cf. [37, Lemma 7]):

(a) If $\theta_t(\mathbf{z}) \in M \times \{0\}$ then $g_t(\mathbf{z}) \in \mathcal{E}^*$ for all $t \in [0,1]$.

(b) If $\lambda \in \{-1,1\}$ then $\theta_{t,2}(\mathbf{z}) = \lambda$ for all $t \in [-1,1]$.

(c) If $y \in S_{b_2}M$ then $\theta_{t,1}(\mathbf{z}) = y$ for all $t \in [-1,1]$.

(d) If $(y,\lambda) \in \partial\left(\overline{B}_{b_2}M \times [-1,1]\right)$ then $\theta_t(\mathbf{z}) \notin M \times \{0\}$ for all $t \in [-1,1]$.

Performing a translation, if necessary, we may assume that $0 \in M$. Now, for each $t \in [-1,1]$, we define the map $f_t : \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] \to \mathbb{R}^N \times \mathbb{R}$ by

$$f_t(x,y,\lambda) := (x, (y,\lambda) - \theta_t(x,y,\lambda)).$$

This is a map over $\overline{B}_{b_1}M$, i.e. $p \circ f_t = p$ where $p : \overline{B}_{b_1}M \times \mathbb{R}^N \times \mathbb{R} \to \overline{B}_{b_1}M$ is the projection. Its set of fixed points,

$$\mathrm{Fix}(f_t) := \{(x,y,\lambda) \in \overline{B}_{b_1}M \times \overline{B}_{b_2}M \times [-1,1] : f_t(x,y,\lambda) = (x,y,\lambda)\},$$

is the set of zeroes of θ_t. Thus, by property (d), $\mathrm{Fix}(f_t) \subset \overline{B}_{b_1}M \times B_{b_2}M \times (-1,1)$ and, since $\mathrm{Fix}(f_t)$ is compact, the restriction $p : \mathrm{Fix}(f_t) \to \overline{B}_{b_1}M$ of the projection is a proper map. Hence, f_t is compactly fixed in the sense of Dold [46], and there exist transfer homomorphisms

$$\begin{aligned} \mathrm{t}_{f_t} &: \mathcal{H}^*\left(\mathrm{Fix}(f_t), \mathrm{Fix}(f_t) \cap p^{-1}(S_{b_1}M)\right) \to \mathcal{H}^*\left(\overline{B}_{b_1}M, S_{b_1}M\right), \\ \mathrm{t}_{f_t} &: \mathcal{H}^*\left(\mathrm{Fix}(f_t) \cap p^{-1}(S_{b_1}M)\right) \to \mathcal{H}^*\left(S_{b_1}M\right), \end{aligned}$$

for each $t \in [-1,1]$. The definition and properties of the fixed point transfer were introduced in [46]. In section B.2 of the Appendix we give the definition and state the elementary properties we need.

Observe that the map $g_1^+ : (\mathrm{Fix}(f_1), \mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)) \to \left(\mathcal{N}^d, \mathcal{N}^{c_0}\right)$ is well defined, and consider the diagram

$$\begin{array}{ccc} \mathcal{H}^{*-1}(\Theta^+ \smallsetminus B_{r_1}M) & \xrightarrow{\delta^*} & \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \\ \downarrow \beta^* & & \downarrow \beta^* \\ \mathcal{H}^{*-1}\left(\mathcal{N}^{c_0}\right) & \xrightarrow{\delta^*} & \mathcal{H}^*\left(\mathcal{N}^d, \mathcal{N}^{c_0}\right) \\ \downarrow (g_1^+)^* & & \downarrow (g_1^+)^* \\ \mathcal{H}^{*-1}\left(\mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)\right) & \xrightarrow{\delta^*} & \mathcal{H}^*\left(\mathrm{Fix}(f_1), \mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M)\right) \\ \downarrow \mathrm{t}_{f_1} & & \downarrow \mathrm{t}_{f_1} \\ \mathcal{H}^{*-1}\left(S_{b_1}M\right) & \xrightarrow{\delta^*} & \mathcal{H}^*\left(\overline{B}_{b_1}M, S_{b_1}M\right) \end{array} \tag{2.6}$$

Due to the naturality property of the transfer, this diagram commutes.

Note that $f_{-1} = s \circ p$, where $s : \overline{S}_{b_1}M \to S_{b_1}M \times \overline{B}_{b_2}M \times [-1,1]$ is the zero section $s(x) := (x, 0, 0)$. So the units property of the transfer gives

$$\mathrm{t}_{f_{-1}} = s^* : \mathcal{H}^*(s(S_{b_1}M)) \to \mathcal{H}^*\left(S_{b_1}M\right),$$

and the homotopy property yields

$$\mathrm{t}_{f_1} \circ p^* = \mathrm{t}_{f_{-1}} \circ p^* = s^* \circ p^* = id : \mathcal{H}^*(S_{b_1}M) \to \mathcal{H}^*\left(S_{b_1}M\right).$$

Note also that

$$\beta(g_1(\mathbf{z})^+) = \beta\left(G(x,y)^+\right) = x = p(\mathbf{z}) \qquad \text{for all } \mathbf{z} = (x,y,\lambda) \in \mathrm{Fix}(f_1) \cap p^{-1}(S_{b_1}M).$$

Therefore,

$$i^* = \mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* : \mathcal{H}^*(\Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*\left(S_{b_1}M\right), \tag{2.7}$$

where $i : (B_{b_1}M, S_{b_1}M) \hookrightarrow (\Theta^+, \Theta^+ \smallsetminus B_{r_1}M)$ is the inclusion. The commutativity of the diagram (2.6) and equality (2.7) yield

$$\mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* \circ \delta^* = \delta^* \circ \mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* = \delta^* \circ i^* = i^* \circ \delta^*.$$

Since δ^* is an epimorphism (see Lemma 2.3.2), we conclude that

$$\mathrm{t}_{f_1} \circ (g_1^+)^* \circ \beta^* = i^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*\left(\overline{B}_{b_1}M, S_{b_1}M\right).$$

But $i^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*\left(\overline{B}_{b_1}M, S_{b_1}M\right)$ is an isomorphism. Hence, $\beta^* : \mathcal{H}^*(\Theta^+, \Theta^+ \smallsetminus B_{r_1}M) \to \mathcal{H}^*\left(\mathcal{N}^d, \mathcal{N}^{c_0}\right)$ is a monomorphism, and claim (2.5) is proved. ∎

Proof of Theorem 1.2.1. It follows immediately from Propositions 2.3.1 and 2.3.4. ∎

Proof of Example 1.2.2. Let $R > 0$ be such that $\overline{B}_R M$ is a tubular neighborhood of M contained in Θ. Since M is contractible in Θ, so is $B_R M$. For every $r \in (0, R)$ we have that $\Theta = \Theta_r \cup B_R M$. Thus $\mathrm{cat}(\Theta, \Theta_r) \le 1$. Lemma B.1.1 and Proposition 2.3.1 yield $1 \le \mathrm{cupl}(\Theta, \Theta_r) \le \mathrm{cat}(\Theta, \Theta_r) \le 1$. ∎

Proof of Example 1.2.3. If $\Theta = B_R M$ is a tubular neighborhood of M and $r \in (0, R)$ then, for $s \in (r, R)$, the inclusion $(\overline{B}_s M, S_s M) \hookrightarrow (\Theta, \Theta_r)$ is a homotopy equivalence of pairs. The Thom isomorphism theorem asserts that

$$\Phi : \mathcal{H}^*(M) \to \mathcal{H}^{N-m+*}(\overline{B}_s M, S_s M), \qquad \Phi(\omega) = \tau \smile q^*(\omega),$$

is an isomorphism, where $\tau \in \mathcal{H}^{N-m}(\overline{B}_s M, S_s M)$ is the Thom class of the disk bundle $q : \overline{B}_s M \to M$, see e.g. [93]. Hence, $\mathrm{cupl}(M) = \mathrm{cupl}(\overline{B}_s M, S_s M)$. Clearly, $\mathrm{cat}(\overline{B}_s M, S_s M) \le \mathrm{cat}(\overline{B}_s M) = \mathrm{cat}(M)$. Since we are assuming that $\mathrm{cat}(M) = \mathrm{cupl}(M)$, using Lemmas B.1.1 and B.1.2 we obtain

$$\begin{aligned} \mathrm{cat}(\Theta, \Theta_r) &= \mathrm{cat}(\overline{B}_s M, S_s M) = \mathrm{cat}(M) = \mathrm{cupl}(M) \\ &= \mathrm{cupl}(\overline{B}_s M, S_s M) = \mathrm{cupl}(\Theta, \Theta_r) \le \mathrm{cat}(\Theta, \Theta_r), \end{aligned}$$

which proves our claim. ∎

CHAPTER 3

MULTIPLICITY OF SOLUTIONS FOR YAMABE TYPE EQUATIONS

In this chapter we will prove Theorems 1.3.1, 1.3.5 and 1.3.7. In the first section we give the variational setting of the problem and state the compactness theorem and the variational principle (Theorems 3.1.1 and 3.1.2 below) we need. Section 3.2 is devoted to prove the first two main theorems and Corollary 1.3.6. In the third and final section we prove the nonexistence of ground states established in Theorem 1.3.7.

3.1 Variational setting of the problem

Let (M,g) be a closed Riemannian manifold of dimension $m \geq 3$, Γ be a closed subgroup of $\mathrm{Isom}_g(M)$, and $a,b,c \in \mathcal{C}^\infty(M)$ be Γ-invariant functions. Define

$$\|u\|_{g,a,b} := \left(\int_M \left[a|\nabla_g u|_g^2 + bu^2\right]\right)^{1/2} dV_g$$

We will assume throughout that $a>0$, $c>0$. Notice the coercivity of $-\mathrm{div}_g(a\nabla_g)+b$ on the space $H_g^1(M)^\Gamma$ implies that

$$\langle u,v\rangle_{g,a,b} := \int_M \left[a\langle \nabla_g u, \nabla_g v\rangle_g + buv\right] dV_g$$

is an interior product in $H_g^1(M)^\Gamma := \{u \in H_g^1(M) : u \text{ is } \Gamma\text{-invariant}\}$ and the induced norm $\|\cdot\|_{g,a,b}$, is equivalent to the standard norm $\|\cdot\|_g$ in $H_g^1(M)^\Gamma$. For any $p \in [1,\infty)$,

$$|u|_{g,p} := \left(\int_M |u|^p \, dV_g\right)^{1/p}$$

will denote the standard norm in the space $L_g^p(M)$ and, for $p=\infty$, $|u|_\infty$ will denote the norm in $L_g^\infty(M)$. As $c>0$,

$$|u|_{g,c,2^*} := \left(\int_M c|u|^{2^*} dV_g\right)^{1/2^*}$$

defines a norm in $L_g^{2^*}(M)$ which is equivalent to the standard norm $|\cdot|_{g,2^*}$. In this section we will assume that $-\mathrm{div}_g(a\nabla_g)+b$ is coercive on $H_g^1(M)$.

By the principle of symmetric criticality [81], the solutions to problem (1.6) are the critical points of the energy functional

$$\begin{aligned} J_g(u) &= \frac{1}{2}\int_M \left[a|\nabla_g u|_g^2 + bu^2\right] dV_g - \frac{1}{2^*}\int_M c|u|^{2^*} dV_g \\ &= \frac{1}{2}\|u\|_{g,a,b}^2 - \frac{1}{2^*}|u|_{g,c,2^*}^{2^*} \end{aligned}$$

defined on the space $H_g^1(M)^\Gamma$. The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}_g^\Gamma := \{u \in H_g^1(M)^\Gamma : u \neq 0, \quad \|u\|_{g,a,b}^2 = |u|_{g,c,2^*}^{2^*}\} \tag{3.1}$$

which is of class $\mathcal{C}^2$, radially diffeomorphic to the unit sphere in $H_g^1(M)^\Gamma$, and a natural constraint for J_g. Moreover, for every $u \in H_g^1(M)^\Gamma$, $u \neq 0$,

$$u \in \mathcal{N}_g^\Gamma \iff J_g(u) = \max_{t\geq 0} J_g(tu). \tag{3.2}$$

In fact, if $u \neq 0$, then $t_u u \in \mathcal{N}_g^\Gamma$ where

$$t_u := \left(\frac{\|u\|_{g,a,b}^2}{|u|_{g,c,2^*}^{2^*}}\right)^{\frac{m-2}{4}}. \tag{3.3}$$

For these and more properties of the Nehari manifold, we refer the reader to [3], [22] and [102].

Set

$$\tau_g^\Gamma := \inf_{\mathcal{N}_g^\Gamma} J_g. \tag{3.4}$$

The continuity of the Sobolev embedding $H_g^1(M) \hookrightarrow L_g^{2^*}(M)$ (see Theorem A.2.3) implies that $\tau_g^\Gamma > 0$.

The proofs of Theorems 1.3.1 and 1.3.5 follow the scheme introduced in [23, 24, 33]. They are based on a compactness result and a variational principle for sign-changing solutions, which we state next.

Definition 1 *A Γ-invariant Palais-Smale sequence for the functional J_g at the level τ is a sequence (u_k) such that,*

$$u_k \in H_g^1(M)^\Gamma, \qquad J_g(u_n) \to \tau, \qquad J_g'(u_k) \to 0 \text{ in } \left(H_g^1(M)\right)'.$$

We shall say that J_g satisfies condition $(PS)_\tau^\Gamma$ in $H_g^1(M)$ if every Γ-invariant Palais-Smale sequence for J_g at the level τ contains a subsequence which converges strongly in $H_g^1(M)$.

The presence of symmetries allows to increase the lowest level at which this condition fails. The following result will be proved in Chapter 4.

Theorem 3.1.1 (Compactness) *The functional J_g satisfies condition $(PS)_\tau^\Gamma$ in $H_g^1(M)$ for every*

$$\tau < \left(\min_{q \in M} \frac{a(q)^{m/2} \, \#\Gamma q}{c(q)^{(m-2)/2}} \right) \frac{1}{m} S^{m/2},$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m)$.

With the hypothesis on a, b and c stated in the beginning of this section, the variational principle that we will use is the following one.

Theorem 3.1.2 (Sign-changing critical points) *Let W be a nontrivial finite dimensional subspace of $H_g^1(M)^\Gamma$. If J_g satisfies $(PS)_\tau^\Gamma$ in $H_g^1(M)$ for every $\tau \leq \sup_W J_g$, then J_g has at least one positive critical point u_1 and $\dim W - 1$ pairs of sign-changing critical points $\pm u_2, ..., \pm u_k$ in $H_g^1(M)^\Gamma$ such that $J_g(u_1) = \tau_g^\Gamma$ and $J_g(u_i) \leq \sup_W J_g$ for $i = 1, ..., k$.*

The proof, up to minor modifications, is the same as in Theorem 3.7 in [33]. For the reader convenience, we give a proof of this theorem in Appendix C.

Our main results follow from these two theorems.

3.2 Proof of the main theorems 1.3.1 and 1.3.5

Proof of Theorem 1.3.1. Take $k \in \mathbb{N}$. M contains an open dense subset Ω such that the Γ-orbit of each point $p \in \Omega$ is Γ-diffeomorphic to Γ/H for some fixed closed subgroup H of Γ. Moreover, Γp has Γ-invariant neighborhood Ω_p contained in Ω which is Γ-diffeomorphic to $\mathbb{B} \times \Gamma/H$, where $\mathbb{B}$ is the euclidean unit ball of dimension $m - \dim(\Gamma p)$; see Theorem I.5.14 in [39]. Since we are assuming that $\dim(\Gamma p) < m$, for any given $k \in \mathbb{N}$ we may choose k different Γ-orbits $\Gamma p_1, \ldots, \Gamma p_k \subset \Omega$ and Γ-invariant neighborhoods Ω_{p_i} as before, with $\Omega_{p_i} \cap \Omega_{p_j} = \emptyset$ if $i \neq j$. Then, we can choose a Γ-invariant function $\omega_i \in \mathcal{C}_c^\infty(\Omega_{p_i})$ for each $i = 1, \ldots, k$.

Let $W := \operatorname{span}\{\omega_1, \ldots, \omega_k\}$ be the linear subspace of $H_g^1(M)^\Gamma$ spanned by $\{\omega_1, \ldots, \omega_k\}$. As ω_i and ω_j have disjoint supports for $i \neq j$, the set $\{\omega_1, \ldots, \omega_k\}$ is orthogonal in $H_g^1(M)^\Gamma$. Hence, $\dim W = k$. On the other hand, as $\dim(\Gamma p) \geq 1$, we have that $\#\Gamma p = \infty$ for every $p \in M$. So, by Theorem 3.1.1, J_g satisfies $(PS)_\tau^\Gamma$ in $H_g^1(M)$ for every $\tau \in \mathbb{R}$. Therefore, Theorem 3.1.2 yields at least one positive and $k - 1$ sign-changing Γ-invariant solutions to problem (1.6). As $k \in \mathbb{N}$ is arbitrary, we conclude that there are infinitely many sign-changing solutions. ■

Proof of Corollary 1.3.2. Just recall the curvature tensor is invariant under isometries and, thus, the scalar curvature R_g is Γ-invariant for every subgroup Γ of the isometry group $\operatorname{Isom}_g(M)$. ■

An easy modification of the proof of Theorem 1.2 in [24] yields Theorem 1.3.5. We include the proof for the reader convenience.

Proof of Theorem 1.3.5. We divide the proof into four steps.

Step 1. *We define the sequence (ℓ_k) and show it is strictly increasing and consists of positive real numbers.*

After replacing Ω by a Λ-invariant open subset of it, if necessary, we may assume that Λp is Λ-diffeomorphic to Λ/H for every $p \in \Omega$ and some fixed subgroup H of Λ; see Theorem I.5.14 in [39]. Let $\mathcal{P}_1(\Omega)$ be the family of all nonempty Λ-invariant open subsets of Ω and, for each $\tilde{\Omega} \in \mathcal{P}_1(\Omega)$, set

$$\mathcal{D}(\tilde{\Omega}) := \{\varphi \in \mathcal{C}_c^\infty(\tilde{\Omega}) : \varphi \text{ is } \Lambda\text{-invariant}, \ \varphi \neq 0, \ \|\varphi\|_{h,a,b}^2 = |\varphi|_{h,c,2^*}^{2^*}\}.$$

For each $k \in \mathbb{N}$ let

$$\mathcal{P}_k(\Omega) := \{(\Omega_1, \dots, \Omega_k) : \Omega_i \in \mathcal{P}_1(\Omega), \quad \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j\}.$$

Arguing as in the proof of Theorem 1.3.1 we see that $\mathcal{P}_k(\Omega) \neq \emptyset$ and $\mathcal{D}(\tilde{\Omega}) \neq \emptyset$. Set

$$\tau_k := \inf \left\{ \sum_{i=1}^k \frac{1}{m} \|\varphi_i\|_{h,a,b}^2 : \varphi_i \in \mathcal{D}(\Omega_i), \quad (\Omega_1, \dots, \Omega_k) \in \mathcal{P}_k(\Omega) \right\},$$

and define

$$\ell_k := (\frac{1}{m} S^{m/2})^{-1} \tau_k.$$

Next, we show that the sequence (ℓ_k) has the desired properties.

Fix $k \in \mathbb{N}$, and let (M, g) be a Riemanniann manifold and Γ be a closed subgroup of $\mathrm{Isom}_g(M)$ which satisfy (1)-(4). As $g = h$ in Ω and Γ is a subgroup of Λ, extending $\varphi \in C_c^\infty(\tilde{\Omega})$ by zero outside $\tilde{\Omega}$, we have that $\mathcal{D}(\tilde{\Omega}) \subset \mathcal{N}_g^\Gamma$ for every $\tilde{\Omega} \in \mathcal{P}_1(\Omega)$, $J_g(\varphi) = \frac{1}{m} \|\varphi\|_{h,a,b}^2$ for every $\varphi \in \mathcal{D}(\tilde{\Omega})$ and $\tau_1 \geq \tau_g^\Gamma > 0$. For arbitrary $(\Omega_1, \dots, \Omega_k) \in \mathcal{P}_k(\Omega)$ and $\varphi_i \in \mathcal{D}(\Omega_i)$ we have

$$\sum_{i=1}^k \frac{1}{m} \|\varphi_i\|_{h,a,b}^2 = \sum_{i=1}^{k-1} \frac{1}{m} \|\varphi_i\|_{h,a,b}^2 + \frac{1}{m} \|\varphi_k\|_{h,a,b}^2 \geq \tau_{k-1} + \tau_1 > \tau_{k-1}.$$

Hence $\tau_k \geq \tau_{k-1} + \tau_1 > \tau_{k-1}$ and the sequence (ℓ_k) is strictly increasing and consists of positive numbers.

Step 2. *For each $\varepsilon \in (0, \tau_1)$, J_g has k pairs of critical points $\pm u_1, \dots, \pm u_k \in H_g^1(M)^\Gamma$ such that u_1 is positive, $u_2, \dots, u_k$ change sign and*

$$J_g(u_j) \leq \tau_k, \ j = 1, \dots, k-1, \quad \textit{and} \quad J_g(u_k) \leq \tau_k + \varepsilon.$$

Since we are assuming that

$$\ell_{a,c}^\Gamma := \min_{p \in M} \frac{a(p)^{m/2} \, \#\Gamma p}{c(p)^{(m-2)/2}} > \ell_k,$$

we may choose $\varepsilon \in (0, \tau_1)$ such that $\tau_k + \varepsilon < \ell_{a,c}^\Gamma(\frac{1}{m} S^{m/2})$. Since $g = h$ in Ω, by definition of τ_k, there exist $(\Omega_1, \dots, \Omega_k) \in \mathcal{P}_k(\Omega)$ and $\omega_i \in \mathcal{D}(\Omega_i)$, such that

$$\tau_k \leq \sum_{i=1}^k J_g(\omega_i) < \tau_k + \varepsilon.$$

For each $n = 1, \dots, k$ set $W_n := \operatorname{span}\{\omega_1 \dots, \omega_n\}$. As ω_i and ω_j have disjoint supports for $i \neq j$, the set $\{\omega_1, \dots, \omega_k\}$ is orthogonal in $H^1_g(M)^\Gamma$. Hence, $\dim W_n = n$. Moreover, if $u \in W_n$, $u = \sum_{i=1}^n t_i\omega_i$, then

$$\Big\| \sum_{i=1}^n t_i\omega_i \Big\|^2_{g,a,b} = \sum_{i=1}^n \|t_i\omega_i\|^2_{g,a,b} \quad \text{and} \quad \Big| \sum_{i=1}^n t_i\omega_i \Big|^{2^*}_{g,c,2^*} = \sum_{i=1}^n |t_i\omega_i|^{2^*}_{g,c,2^*}.$$

Consequently, (3.2) yields

$$J_g(u) = \sum_{i=1}^n J_g(t_i\omega_i) \leq \sum_{i=1}^n J_g(\omega_i) < \tau_k + \varepsilon.$$

Therefore,

$$\sigma_n := \sup_{W_n} J_g \leq \tau_k + \varepsilon < \ell^\Gamma_{a,c}(\frac{1}{m} S^{m/2}).$$

So Theorems 3.1.1 and 3.1.2 yield a positive critical point u_1 and $n-1$ pairs of sign changing critical points $\pm u_{n,2}, \dots, \pm u_{n,n}$ of J_g in $H^1_g(M)^\Gamma$ such that $J_g(u_1) = \tau^\Gamma_g$ and

$$J_g(u_{n,j}) \leq \sigma_n \qquad \text{for all } \; j = 2, \dots, n.$$

Now, for each $2 \leq n \leq k$, we inductively choose $u_n \in \{u_{n,2}, \dots, u_{n,n}\}$ such that $u_n \neq u_j$ for all $1 \leq j < n$. Observe that $\tau_1 \leq J_g(\omega_i)$ for every $i = 1, \dots, k$. Consequently, for each $2 \leq n \leq k$ we obtain

$$\begin{aligned} \sigma_n + (k-n)\tau_1 &\leq \sum_{i=1}^n J_g(\omega_i) + \underbrace{(\tau_1 + \cdots \tau_1)}_{k-n \text{ times}} \leq \sum_{i=1}^n J_g(\omega_i) + \sum_{i=n+1}^k J_g(\omega_i) \\ &= \sum_{i=1}^k J_g(\omega_i) < \tau_k + \varepsilon. \end{aligned}$$

As $\varepsilon \in (0, \tau_1)$, for $k - n \geq 1$ this inequality actually gives

$$\sigma_n + (k-n)\tau_1 < \tau_k + \varepsilon < \tau_k + \tau_1 < \tau_k + (k-n)\tau_1.$$

Subtracting $(k-n)\tau_1$ from both sides we conclude

$$J_g(u_n) \leq \sigma_n < \tau_k \;\text{ if } n < k \qquad \text{and} \qquad J_g(u_k) \leq \sigma_k < \tau_k + \varepsilon,$$

and Step 2 follows.

Step 3. *J_g has k pairs of critical points $\pm u_1, \dots, \pm u_k \in H^1_g(M)^\Gamma$ such that u_1 is positive, $u_2, \dots, u_k$ are sign-changing and*

$$J_g(u_n) \leq \tau_k, \qquad \textit{for every } n = 1, \dots, k$$

Let $(\varepsilon_l) \subset (0, \tau_1)$ be such that $\varepsilon_{l+1} < \varepsilon_l$ and $\varepsilon_l \to 0$ as $l \to \infty$. For each $l \in \mathbb{N}$, Step 2 gives k pairs of critical points $\pm w_{l,1}, \dots, \pm w_{l,k} \in H^1_h(M)^\Gamma$ for J_g so that $w_{l,1}$ is positive, $w_{l,2}, \dots, w_{l,k}$ are sign-changing, $J_g(w_{l,j}) < \tau_k$ and $J_g(w_{l,k}) \leq \tau_k + \varepsilon_l$. If

$J_g(w_{l_0,k}) \leq \tau_k$ for some $l_0 \in \mathbb{N}$, then $u_1 := w_{l_0,1}, \ldots, u_k := w_{l_0,k}$ satisfy what we wanted. If not, then $\tau_k < J_g(w_{l,k}) \leq \tau_k + \varepsilon$ for all $l \in \mathbb{N}$ and $J_g(w_{l,k}) \to \tau_k$ as $l \to \infty$. Moreover, as $w_{l,k}$ is a critical point for J_g, $J_g'(w_{l,k}) = 0$ for all $l \in \mathbb{N}$. This implies that $(w_{l,k})$ is a Palais-Smale sequence for J_g at the level τ_k. As we showed in Step 2, $\tau_k < \ell^\Gamma_{a,b}\frac{1}{m}S^{m/2}$ and, therefore, J_g satisfies condition $(PS)^\Gamma_{\tau_k}$, so, in a suitable subsequence, $w_{l,k} \to u_k$ strongly in $H^1_g(M)^\Gamma$ as $l \to \infty$, with $u_k \neq 0$ and $u_k \in \mathcal{N}^\Gamma_g$. By continuity of J_g and J_g',

$$J_g(u_k) = \lim_{l\to\infty} J_g(w_{l,k}) = \tau_k, \quad \text{and} \quad J_g'(u_k) = \lim_{l\to\infty} J_g(w_{l,k}) = 0,$$

and, hence, u_k is a nontrivial critical point for J_g. Notice that if $k = 1$, then $w_{l,1}$ lies in the convex cone of positive functions in $H^1_h(M)^\Gamma$. This set is closed in $H^1_g(M)^\Gamma$ and, hence, u_1 must be positive. If $k \geq 2$, observe for each $l \in \mathbb{N}$, $w_{l,k} \in \mathcal{E}^\Gamma_g := \{v \in \mathcal{N}^\Gamma_g \ : \ v^+, v^- \in \mathcal{N}^\Gamma_g\}$ the equator of the Nehari manifold; again, this set is closed in $H^1_h(M)^\Gamma$ and u_k changes sign. As $J_g(w_{1,j}) < \tau_k$ for each $j = 1, \ldots, k-1$, then $u_k \neq w_{1,j}$ for each $j = 1, \ldots, k-1$. Taking $u_j := w_{1,j}$ for each $j = 1, \ldots, k-1$ and u_k, we conclude the proof of Step 3.

Step 4. *J_g has k pairs of critical points $\pm u_1, \ldots, \pm u_k \in H^1_g(M)^\Gamma$ such that u_1 is positive, $u_2, \ldots, u_k$ are sign-changing and*

$$J_g(u_n) \leq \tau_n, \qquad \textit{for every } n = 1, \ldots, k$$

As $\ell^\Gamma_{a,b} > \ell_k$ and as the sequence (ℓ_k) is strictly increasing, then $\ell^\Gamma_{a,b} > \ell_n$ for each $n = 1, \ldots, k$. For each $n = 1, \ldots, k$, using Step 3 we obtain n pairs of critical points $\pm v_{n,1}, \ldots, \pm v_{n,n} \in H^1_g(M)^\Gamma$ such that $J_g(v_{n,j}) \leq \tau_n$, $j = 1, \ldots, n$, where $v_{n,1}$ is positive and the rest are sign-changing. As in Step 2, let $u_1 = v_{1,1}$ and for $2 \leq n \leq k$ choose inductively, $u_n \in \{v_{n,1}, \ldots, v_{n,n}\}$ such that $u_n \neq u_j$ if $1 \leq j < n$. Thus, $J_g(u_n) \leq \tau_n$ by construction.

Finally, notice that $|u_n|_{g,c,2^*} = \|u_n\|_{g,a,b}$ because $u_n \in \mathcal{N}^\Gamma_g$. Therefore

$$\int_M c|u_n|^{2^*} = mJ_g(u_n) \leq m\tau_n = \ell_n S^{m/2},$$

and this concludes the proof or Theorem 1.3.5. ∎

Proof of Corollary 1.3.6. Let $\mathfrak{M}$ be the space of Riemannian metrics on M with the distance induced by the $\mathcal{C}^0$-norm in the space of covariant 2-tensor fields τ on M, taken with respect to the fixed metric h, i.e.

$$\|\tau\|_{\mathcal{C}^0} := \max_{p\in M} \max_{X,Y\in T_pM\smallsetminus\{0\}} \frac{|\tau(X,Y)|}{|X|_h\,|Y|_h}.$$

As the functions $\mathfrak{M} \to \mathcal{C}^0(M)$ given by $g \mapsto R_g$ and $g \mapsto \sqrt{|g|}$ are continuous, where $|g| := \det(g)$, the sets

$$\mathcal{O}_1 := \left\{g \in \mathfrak{M} : \frac{1}{2}R_h(p) < R_g(p) < 2R_h(p) \ \ \forall p \in M \smallsetminus \Omega\right\},$$
$$\mathcal{O}_2 := \left\{g \in \mathfrak{M} : \frac{1}{2}\sqrt{|h|\,(p)} < \sqrt{|g|\,(p)} < 2\sqrt{|h|\,(p)} \ \ \forall p \in M \smallsetminus \Omega\right\},$$

are open neighborhoods of h in $\mathfrak{M}$, where $\mathfrak{c}_m = \frac{m-2}{4(m-1)}$. Moreover, as

$$|\nabla_g u(p)|_g = \max_{X \in T_pM \smallsetminus \{0\}} \frac{|duX|}{|X|_g},$$

for every $u \in \mathcal{C}^\infty(M)$ we have that

$$\frac{1}{2}|\nabla_h u|_h^2 \leq |\nabla_g u|_g^2 \leq 2|\nabla_h u|_h^2 \qquad \text{if } \|g-h\|_{\mathcal{C}^0} < \frac{1}{2}.$$

Set $\mathcal{O} := \{g \in \mathfrak{M} : \|g-h\|_{\mathcal{C}^0} < \frac{1}{2}\} \cap \mathcal{O}_1 \cap \mathcal{O}_2$. Then there are positive constants $C_1 \leq 1$ and $C_2 \geq 1$ such that, for every $g \in \mathcal{O}$ and $u \in \mathcal{C}^\infty(M)$,

$$\int_{M \smallsetminus \Omega} \left[|\nabla_g u|^2 + \frac{m-2}{4(m-1)} R_g |u|^2\right] dV_g \geq C_1 \int_{M \smallsetminus \Omega} \left[|\nabla_h u|^2 + \frac{m-2}{4(m-1)} R_h |u|^2\right] dV_h,$$
$$\int_{M \smallsetminus \Omega} \left[|\nabla_g u|^2 + |u|^2\right] dV_g \leq C_2 \int_{M \smallsetminus \Omega} \left[|\nabla_h u|^2 + |u|^2\right] dV_h.$$

Therefore, if $g \in \mathcal{O}$ and $g = h$ in Ω, we have that

$$\frac{\int_M \left[|\nabla_g u|^2 + \frac{m-2}{4(m-1)} R_g |u|^2\right] dV_g}{\int_M \left[|\nabla_g u|^2 + |u|^2\right] dV_g} \geq \frac{C_1 \int_M \left[|\nabla_h u|^2 + \frac{m-2}{4(m-1)} R_h |u|^2\right] dV_h}{C_2 \int_M \left[|\nabla_h u|^2 + |u|^2\right] dV_h}$$

for every $u \in \mathcal{C}^\infty(M)$. So, if $\Gamma_n \subset \mathrm{Isom}_g(M)$, setting $b_h := \frac{m-2}{4(m-1)} R_h$ and $b := \frac{m-2}{4(m-1)} R_g$, we obtain

$$\mu_{1,b}^{\Gamma_n}(M,g) \geq \mu_{1,b_h}^{\Gamma_n}(M,h) \geq \mu_{1,b_h}(M,h) > 0,$$

and $\Delta_g + \mathfrak{c}_m R_g$ is coercive on $H_h^1(M)$.

Set (Ω, h) as given, $\Lambda = \mathbb{S}^1$, $a \equiv 1$, $b = \mathfrak{c}_m R_g$ and $c \equiv \kappa$. Then, if $g \in \mathcal{O}$ is such that $g = h$ in Ω and $\Gamma_n \subset Isom_g(M)$ for some $n > \kappa^{(m-2)/2}\ell_k$, these data satisfy assumptions (1)-(4) in Theorem 1.3.5, and the conclusion follows. ∎

3.3 Nonexistence of ground state solutions

In this section we prove Theorem 1.3.7.Let $(\mathbb{S}^m, g_0)$ be the round sphere and $b \in \mathcal{C}^\infty(\mathbb{S}^m)$ be such that $b \geq \mathfrak{c}_m R_{g_0} = \frac{m(m-2)}{4}$ and $b \not\equiv \mathfrak{c}_m R_{g_0}$. Fix $N := (0, \ldots, 0, 1) \in \mathbb{S}^m \subset \mathbb{R}^{m+1}$ and let $\sigma : \mathbb{S}^m \smallsetminus \{N\} \to \mathbb{R}^m$ be the stereographic projection, which is given by

$$\sigma(x,t) := \frac{x}{1-t},$$

and its inverse is

$$\varphi(x); = \sigma^{-1}(x) = \left(\frac{2x}{1+|x|^2}, 1 - \frac{2}{1+|x|^2}\right).$$

Notice

$$\frac{\partial \varphi^k}{\partial x_i} = \begin{cases} \frac{2}{(|x|^2+1)^2}\left[\left(|x|^2+1\right)\delta_{ik} - 2x^k x^i\right] & \text{if } k = 1, \ldots, m \\ \frac{2}{(|x|^2+1)^2}(2x^i) & \text{if } k = m+1. \end{cases}$$

Hence, the components of the metric in the coordinate $(\mathbb{R}^m, \varphi)$ are given by

$$\begin{aligned} g_{0,ii} &= \left\langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_i} \right\rangle \\ &= \sum_{k=1}^{m+1} \left(\frac{\partial \varphi^k}{\partial x_i} \right)^2 \\ &= \frac{4}{(|x|^2+1)^4} \left[\sum_{k=1}^{m} 4(x^k)^2(x^i)^2 - 4(x^i)^2 \left(|x|^2+1\right) + \left(|x|^2+1\right)^2 + 4(x^i)^2 \right] \\ &= \left(\frac{2}{|x|^2+1} \right)^2, \end{aligned}$$

while for $i \neq j$,

$$\begin{aligned} g_{0,ij} &= \left\langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right\rangle = \sum_{k=1}^{m+1} \frac{\partial \varphi^k}{\partial x_i} \frac{\partial \varphi^k}{\partial x_i} \\ &= \sum_{k=1}^{m} 4(x^k)^2 x^i x^j - 4x^i x^j \left(|x|^2+1\right) + 4x^i x^j = 0. \end{aligned}$$

It follows that

$$g_0^{ij} = \begin{cases} \left[\frac{2}{|x|^2+1}\right]^{-2} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

and

$$\sqrt{|g_0|} = \sqrt{g_{0,11} \cdots g_{0,mm}} = \sqrt{\prod_{i=1}^{m} \left[\frac{2}{|x|^2+1} \right]^2} = \left[\frac{2}{|x|^2+1} \right]^m$$

Denote $z(x) = 2/(|x|^2+1)$ and let ξ be the standard Euclidean metric in $\mathbb{R}^m$. Then $\sigma^{-1} : (\mathbb{R}^m, \xi) \to (\mathbb{S}^m \setminus \{N\}, g_0)$ is a conformal equivalence such that

$$(\sigma^{-1})^* g_0 = z^2 \xi = \psi^{\frac{4}{m-2}} \xi,$$

where the function $\psi := z^{\frac{m-2}{2}} = \left[\frac{2}{1+|x|^2}\right]^{\frac{m-2}{2}}$ satisfies the scalar curvature equation

$$\Delta_\xi \psi + \frac{m-2}{4(m-1)} R_\xi \psi - \frac{m-2}{4(m-1)} R_{g_0} \psi^{2^*-1} = 0,$$

(see [41]). Since $\Delta_\xi = -\Delta$, $R_{g_0} = m(m-1)$ and $R_\xi = 0$, this equation yields that

$$-\Delta \psi = \frac{m(m-2)}{4} \psi^{2^*-1}. \tag{3.5}$$

Lemma 3.3.1 *A function $u \in \mathcal{C}^\infty(M)$ solves*

$$\Delta_{g_0} u + bu = |u|^{2^*-2} u, \tag{3.6}$$

iff and only if

$$v(x) := \left(\frac{2}{1+|x|^2} \right)^{\frac{m-2}{2}} u(\sigma^{-1}(x))$$

is a solution of

$$-\Delta v+\widetilde{b}v=|v|^{2^*-2}v \qquad in \;\; \mathbb{R}^m, \tag{3.7}$$

where $\Delta = div\nabla$ *is the usual Laplace operator on* $\mathbb{R}^m$ *(without a sign) and*

$$\widetilde{b}(x):=\left(b(\sigma^{-1}(x))-\frac{m(m-2)}{4}\right)\left(\frac{2}{1+|x|^2}\right)^2.$$

Moreover, u *is a ground state for problem (3.6) iff* v *is a minimizer for*

$$S_b:=\inf_{\substack{v\in D^{1,2}(\mathbb{R}^m)\\ v\neq 0}}\frac{\int_{\mathbb{R}^m}\left(|\nabla v|^2+\widetilde{b}v^2\right)dx}{\left(\int_{\mathbb{R}^m}|v|^{2^*}dx\right)^{2/2^*}}.$$

Proof. For simplicity, we adopt the notation $\widehat{u}:=u\circ\sigma^{-1}$, $\widehat{b}:=b\circ\sigma^{-1}$. The Laplace-Beltrami operator in coordinates given by the inverse of the stereographic projection is

$$\begin{aligned}\Delta_{g_0}u&=-\frac{1}{\sqrt{|g_0|}}\sum_{j=1}^{m}\frac{\partial}{\partial x^j}\left(\sqrt{|g_0|}g_0^{jj}\frac{\partial\widehat{u}}{\partial x^j}\right)\\&=-(m-2)z^{-3}\nabla z\cdot\nabla\widehat{u}-z^{-2}\Delta\widehat{u}.\end{aligned}$$

Also observe that

$$\begin{aligned}\Delta(z^{\frac{m-2}{2}})&=\frac{(m-2)(m-4)}{4}z^{\frac{m-6}{2}}|\nabla z|^2+\frac{m-2}{2}z^{(\frac{m-3}{2})}\Delta z\\&=\frac{(m)(m-2)}{4}z^{\frac{m+2}{2}}|x|^2-\frac{m(m-2)}{2}z^{m/2},\end{aligned}$$

and that

$$|v|^{2^*-2}v=z^{\frac{m+2}{2}}|\widehat{u}|^{2^*-1}\widehat{u}.$$

As a consequence of these identities we obtain

$$\begin{aligned}&-\Delta v+(\widehat{b}-\frac{m(m-2)}{4})z^2v=-\Delta(z^{\frac{m-2}{2}}\widehat{u})+\left(\widehat{b}-\frac{m(m-2)}{4}\right)z^{\frac{m+2}{2}}\widehat{u}\\&=-z^{\frac{m-2}{2}}\Delta\widehat{u}-2\nabla\widehat{u}\cdot\nabla(z^{\frac{m-2}{2}})-\widehat{u}\Delta(z^{\frac{m-2}{2}})+\left(\widehat{b}-\frac{m(m-2)}{4}\right)z^{\frac{m+2}{2}}\widehat{u}\\&=z^{\frac{m+2}{2}}\left(-z^{-2}\Delta\widehat{u}-(m-2)z^3\nabla\widehat{u}\cdot\nabla(z)\right)-\widehat{u}\Delta(z^{\frac{m-2}{2}})+\left(\widehat{b}-\frac{m(m-2)}{4}\right)z^{\frac{m+2}{2}}\widehat{u}\\&=z^{\frac{m+2}{2}}\left[\Delta_{g_0}u-\widehat{u}z^{-\frac{m+2}{2}}\Delta(z^{\frac{m-2}{2}})\right]+\left(\widehat{b}-\frac{m(m-2)}{4}\right)\widehat{u}\\&=z^{\frac{m+2}{2}}\left[\Delta_{g_0}u+\widehat{u}\widehat{b}-\widehat{u}\left(\frac{(m)(m-2)}{4}|x|^2-\frac{m(m-2)}{2}z^{-1}\right)+\frac{m(m-2)}{4}\right]\\&=z^{\frac{m+2}{2}}\left[\Delta_{g_0}u+\widehat{u}\widehat{b}\right].\end{aligned}$$

Then v satisfies equation 3.7 if and only if

$$\begin{aligned}z^{\frac{m+2}{2}}\left[\Delta_{g_0}u+\widehat{u}\widehat{b}\right]&=-\Delta v+\widetilde{b}v=|v|^{2^*-2}v\\&=z^{\frac{m+2}{2}}|\widehat{u}|^{2^*-1}v=z^{\frac{m+2}{2}}|\widehat{u}|^{2^*-1}\widehat{u}.\end{aligned}$$

As $z(x) > 0$ for every $x \in \mathbb{R}^m$, this happens if an only if u is a solution of equation 3.7.

The second part of the lemma follows immediately from the following identity

$$\frac{\int_{\mathbb{S}^m}\left(|\nabla_{g_0}u|_{g_0}^2 + bu^2\right)dV_{g_0}}{\left(\int_{\mathbb{S}^m}|u|^{2^*}dV_{g_0}\right)^{2/2^*}} = \frac{\int_{\mathbb{R}^m}\left(|\nabla v|^2 + \widetilde{b}v^2\right)dx}{\left(\int_{\mathbb{R}^m}|v|^{2^*}dx\right)^{2/2^*}}, \tag{3.8}$$

Let us then show (3.8). Notice on the one hand that in the orthogonal coordinates given by the stereographic projection the gradient is written as

$$\nabla_{g_0}u = \sum_{i=1}^{m} g_0^{ii}\frac{\partial\widehat{u}}{\partial x^i}\frac{\partial}{\partial x^i},$$

so,

$$|\nabla_{g_0}u|_{g_0}^2 = \sum_{i=1}^{m}(g_0^{ii})^2\left(\frac{\partial\widehat{u}}{\partial x^i}\right)g_{0,ij} = z^{m-2}|\nabla\widehat{u}|^2. \tag{3.9}$$

On the other hand, multiplying equation (3.5) by $\psi\widehat{u}^2$, integrating and using Green's formula we obtain

$$\int_{\mathbb{R}^m}\nabla\psi\nabla(\psi\widehat{u}^2)dx = \int_{\mathbb{R}^m}\frac{m(m-2)}{4}\psi^{2^*}\widehat{u}^2dx. \tag{3.10}$$

Hence, equalities (3.9) and (3.10) yield

$$\begin{aligned}
\int_{\mathbb{R}^m}|\nabla v|^2dx &= \int_{\mathbb{R}^m}|\nabla(\psi\widehat{u})|^2dx\\
&= \int_{\mathbb{R}^m}\widehat{u}^2|\nabla\psi|^2 + 2\widehat{u}\psi\nabla\psi\nabla\widehat{u} + \psi^2|\nabla\widehat{u}|^2dx\\
&= \int_{\mathbb{R}^m}z^m z^{-2}|\nabla\widehat{u}|^2dx + \int_{\mathbb{R}^m}\widehat{u}^2|\nabla\psi|^2 + 2\widehat{u}\psi\nabla\widehat{u}\nabla\psi dx\\
&= \int_{\mathbb{S}\setminus\{N\}}|\nabla_{g_0}u|_{g_0}^2dV_{g_0} + \int_{\mathbb{R}^m}\nabla\psi\nabla(\psi\widehat{u}^2)dx\\
&= \int_{\mathbb{S}^m}|\nabla_{g_0}u|_{g_0}^2dV_{g_0} + \int_{\mathbb{R}^m}\frac{m(m-2)}{4}\psi^{2^*}\widehat{u}^2dx\\
&= \int_{\mathbb{S}^m}|\nabla_{g_0}u|_{g_0}^2dV_{g_0} + \int_{\mathbb{R}^m}z^m\frac{m(m-2)}{4}\widehat{u}^2dx\\
&= \int_{\mathbb{S}^m}|\nabla_{g_0}u|_{g_0}^2dV_{g_0} + \int_{\mathbb{S}^m\setminus\{N\}}\frac{m(m-2)}{4}u^2dV_{g_0}\\
&= \int_{\mathbb{S}}|\nabla_{g_0}u|_{g_0}^2 + \frac{m(m-2)}{4}u^2dV_{g_0}.
\end{aligned}$$

From this we get

$$
\begin{aligned}
&\int_{\mathbb{S}^m} \left[|\nabla_{g_0} u^2|_{g_0}^2 + bu^2 \right] dV_{g_0} \\
&= \int_{\mathbb{S}^m} \left(|\nabla_{g_0} u|_{g_0}^2 + \frac{m(m-2)}{4} u \right) dV_{g_0} + \int_{\mathbb{S}^m} \left[b - \frac{m(m-2)}{4} \right] u^2 dV_{g_0} \\
&= \int_{\mathbb{R}^m} |\nabla v|^2 \, dx + \int_{\mathbb{S}^m \smallsetminus \{N\}} \left[b - \frac{m(m-2)}{4} \right] u^2 dV_{g_0} \\
&= \int_{\mathbb{R}^m} |\nabla v|^2 \, dx + \int_{\mathbb{R}^m} z^m \left[\widehat{b} - \frac{m(m-2)}{4} \right] \widehat{u}^2 dx \\
&= \int_{\mathbb{R}^m} |\nabla v|^2 \, dx + \int_{\mathbb{R}^m} \left[\widehat{b} - \frac{m(m-2)}{4} \right] z^2 v^2 dx \\
&= \int_{\mathbb{R}^m} |\nabla v|^2 + \widetilde{b} v^2 dx.
\end{aligned}
$$

We conclude by noticing that

$$
\int_{\mathbb{R}^m} |v|^{2^*} \, dx = \int_{\mathbb{R}^m} z^m \, |\widehat{u}|^{2^*} \, dx = \int_{\mathbb{S}^m \smallsetminus \{N\}} |u|^{2^*} \, dV_{g_0} = \int_{\mathbb{S}^m} |u|^{2^*} \, dV_{g_0}.
$$

■

If $b \equiv \frac{m(m-2)}{4}$ then $\widetilde{b} \equiv 0$ and $S_{\frac{m(m-2)}{4}} =: S$ is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m)$. This constant is attained at the so-called standard bubble

$$
U(x) = [m(m-2)]^{\frac{m-2}{4}} \left(\frac{1}{1 + |x|^2} \right)^{\frac{m-2}{2}}
$$

and at any dilation $U_\varepsilon(x) := \varepsilon^{\frac{2-m}{2}} U\left(\frac{x}{\varepsilon}\right)$ of it, with $\varepsilon > 0$.

Lemma 3.3.2 *If $b \geq \frac{m(m-2)}{4}$ then $S_b = S$.*

Proof. Clearly, $S_b \geq S$. Fix $\alpha \in (1/2, 1)$. Then, for all $\varepsilon \in (0,1)$,

$$
\widetilde{b}(x) U_\varepsilon^2(x) \leq C \left(\frac{1}{1+|x|^2} \right)^2 \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^{m-2} \leq C \varepsilon^{m-2} \left(\frac{1}{\varepsilon^2 + |x|^2} \right)^{m-2+\alpha}.
$$

Hence, we have that

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^m} \widetilde{b}(x) U_\varepsilon^2(x) dx = \int_{|x| \leq \varepsilon} \widetilde{b}(x) U_\varepsilon^2(x) dx + \int_{|x| \geq \varepsilon} \widetilde{b}(x) U_\varepsilon^2(x) dx \\
&\leq C \varepsilon^2 \int_{|y| \leq 1} U^2(y) dy + C \varepsilon^{m-2} \int_{|x| \geq \varepsilon} |x|^{-2m+4-2\alpha} dx \\
&= C \varepsilon^2 + C \varepsilon^{2(1-\alpha)} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0.
\end{aligned}
$$

Therefore,

$$
\lim_{\varepsilon \to 0} \frac{\int_{\mathbb{R}^m} \left(|\nabla U_\varepsilon|^2 + \widetilde{b} U_\varepsilon^2 \right) dx}{\left(\int_{\mathbb{R}^m} |U_\varepsilon|^{2^*} dx \right)^{2/2^*}} = \frac{\int_{\mathbb{R}^m} |\nabla U_\varepsilon|^2 \, dx}{\left(\int_{\mathbb{R}^m} |U_\varepsilon|^{2^*} dx \right)^{2/2^*}} = S.
$$

This shows that $S \geq S_b$. ■

Proof of Theorem 1.3.7. If S_b were attained at some $\widetilde{v} \in D^{1,2}(\mathbb{R}^m)$ then, as $\widetilde{b} \geq 0$ and $\widetilde{b} \not\equiv 0$, we would have that

$$S = S_b = \frac{\int_{\mathbb{R}^m} \left(|\nabla v|^2 + \widetilde{b} v^2 \right) dx}{\left(\int_{\mathbb{R}^m} |v|^{2^*} dx \right)^{2/2^*}} > \frac{\int_{\mathbb{R}^m} |\nabla v|^2 \, dx}{\left(\int_{\mathbb{R}^m} |v|^{2^*} dx \right)^{2/2^*}} \geq S,$$

a contradiction. ■

CHAPTER 4

COMPACTNESS

This chapter is devoted to the proof of Theorem 3.1.1. The first section deals with all the preliminary results we need to achieve this, among them, we prove some useful comparison lemmas between the L^p-norms in $\mathbb{R}^m$ and the corresponding L^p norms in M. After that, in the following section, we give some general properties of the functional J_g and the Palais-Smale sequences. In the third one, we give reduce the problem to a simpler energy functional. Then we state and prove a Struwe-like compactness theorem in another section concluding with the proof of Theorem 3.1.1.

4.1 Main Tools

First we begin with the following Brezis-Lieb type lemma. Let (X, σ, μ) where σ is a σ-algebra of subsets of X and μ is a measure in (X, σ). If $1 \leq p \leq \infty$, denote by $L^p_\mu(X) := L^p(X, \sigma, \mu)$ the L^p-spaces associated to this data and let $|\cdot|_{p,\mu}$ be their norm. The following statements holds true.

Lemma 4.1.1 *Let $p \in [1, \infty)$, $(c_k), (\varphi_k)$ be two bounded sequences in $L^\infty_\mu(X)$ and (u_k) a bounded sequence in $L^p_\mu(X)$ satisfying $c_k(x) \to \bar{c}(x)$, $\varphi_k(x) \to 1$ and $u_k(x) \to u(x)$ μ-a.e. in X respectively. Then $u \in L^p_\mu(X)$ and*

$$\lim_{k\to\infty} \int_X (c_k|u_k|^p - c_k|u_k - \varphi_k u|^p)\, d\mu = \int_X \bar{c}|u|^p d\mu. \tag{4.1}$$

Proof. Let $p \in [1, \infty)$ and $\varepsilon > 0$. Here $C_{p,\varepsilon}$ denotes a positive constant depending only on p and ε, not necessarily the same one. With this, just note that, as (φ_k) is bounded in $L^\infty_\mu(X)$,

$$\begin{aligned}\Big||u_k(x)|^p - |u_k(x) - \varphi_k(x)u(x)|^p\Big| - \varepsilon|u_k(x) - \varphi_k(x)u(x)|^p \\ \leq C_{p,\varepsilon}|\varphi_k(x)|^p|u(x)|^p \leq C_{p,\varepsilon}|\varphi_k|_{L^\infty_\mu(X)}|u(x)|^p \leq C_{p,\varepsilon}|u(x)|^p.\end{aligned}$$

This inequality and the fact $u_k(x) - \varphi_k(x)u(x) \to 0$ μ-a.e. in X, allow us to replicate the proof of Lemma 3.4 in [24], to obtain a similar statement in our situation. ∎

Let now (M, g) be a compact Riemannian manifold without boundary with dimension $m \geq 3$. Denote by d_g the geodesic distance in M induced by the metric g. Then (M, d_g) is a complete metric space and by the Hopf-Rinow Theorem, the exponential map is defined on the whole tangent bundle $\exp : TM \to M$, and, so, $\exp_p : \mathbb{R}^m \equiv T_pM \to M$ is globally defined for all $p \in M$ (see, for example, [45, 71]). If i_g denotes the injectivity radius of the manifold, compactness of M implies $i_g > 0$. Fix $\delta > 0$ such that $3\delta < i_g$.

Lemma 4.1.2 *Then there positive constants A_1, A_2 such that for every $q \in M$ and every $u \in H_g^1(M)$,*

$$A_1 \sum_{i=1}^{m} |\partial_i \tilde{u}|^2 \leq |\nabla_g u|_g^2 \circ \exp_q \leq A_2 \sum_{i=1}^{m} |\partial_i \tilde{u}|^2, \tag{4.2}$$

where $\tilde{u} = u \circ \exp_q$ is written in normal coordinates around q and $|\cdot|$ is the standard Euclidean metric.

Proof. Let $q \in M$ and write g_{ij} in normal coordinates. As the matrix $G := (g_{ij})$ is symmetric and positive definite, it is diagonalizable and all its eigenvalues $\lambda_1, \ldots, \lambda_m$ are positive real numbers [76, Theorem 1.4 and 1.8]. By Schurs' Decomposition Theorem [76, Theorem 1.12 and 1.13], there exists an orthogonal matrix Q such that $Q^tGQ = D$, where D is the diagonal matrix whose diagonal entries are the eigenvalues of G. Now, if we take $x \in \mathbb{R}^m$ and set $y := Q^tx$, then

$$x^tGx = (QQ^tx)^tG(QQ^tx) = (Q^tx)^tQ^tGQ(Q^tx) = y^tDy = \sum_{i=1}^{m} \lambda_i y_i^2, \tag{4.3}$$

where $y = (y_1, \ldots, y_m)$. Since $\lambda_i > 0$ for all $i = 1, \ldots, m$, we obtain

$$\begin{aligned}
\min\{\lambda_1, \ldots, \lambda_m\} \left(y^tI_my\right) &= \min\{\lambda_1, \ldots, \lambda_m\} \left(\sum_{i=1}^{m} y_i^2\right) \\
&\leq \sum_{i=1}^{m} \lambda_i y_i^2 \leq \max\{\lambda_1, \ldots, \lambda_m\} \left(\sum_{i=1}^{m} y_i^2\right) \\
&= \max\{\lambda_1, \ldots, \lambda_m\} \left(y^tI_my\right).
\end{aligned}$$

Further, observe that

$$y^tI_my = (Q^tx)^tI_m(Q^tx) = x^tQI_mQ^tx = x^tQQ^tx = x^tI_mx. \tag{4.4}$$

Hence,

$$0 \leq \min\{\lambda_1, \ldots, \lambda_m\} \left(x^tI_mx\right) \leq x^tGx \leq \max\{\lambda_1, \ldots, \lambda_m\} \left(x^tI_mx\right),$$

As the eigenvalues of a matrix do not depend on the basis, the eigenvalues of the metric g are independent of the chart in which we write the components g_{ij}. Then, we have continuous functions $\lambda_i : M \to \mathbb{R}, i = 1, \ldots, m$, which achieve its maximum and its minumum on the compact manifold M. Define

$$0 < A_1 := \min_{p \in M} \min\{\lambda_1(p), \ldots, \lambda_m(p)\} \leq \max_{p \in M} \max\{\lambda_1(p), \ldots, \lambda_m(p)\} =: A_2.$$

Now, if $u \in H^1_g(M)$, $(\partial_1 u(p), \ldots, \partial_m u(p)) \in \mathbb{R}^m$ for each $p \in M$ and

$$A_1 \sum_{i=1}^{m} |\partial_i \tilde{u}|^2 \leq \sum_{i,j=1}^{m} g^{ij}(\partial_i u \partial_j u) \circ \exp_q = |\nabla_g u|_g^2 \leq A_2 \sum_{i=1}^{m} |\partial_i \tilde{u}|^2,$$

and A_1 and A_2 do not depend on u. ■

As a consequence, we have the following integral comparison lemma.

Lemma 4.1.3 *With the same notation as in the previous lemma, there is a constant $C_1 > 1$ such that, for every $q \in M$, $\varrho \in (0, 3\delta]$, $u \in H^1_g(M)$ and $s \in [1, 2^*]$,*

$$C_1^{-1} \int_{B(0,\varrho)} |\tilde{u}|^s \, dx \leq \int_{B_g(q,\varrho)} |u|^s \, dV_g \leq C_1 \int_{B(0,\varrho)} |\tilde{u}|^s \, dx, \tag{4.5}$$

$$C_1^{-1} \int_{B(0,\varrho)} |\nabla \tilde{u}|^2 \, dx \leq \int_{B_g(q,\varrho)} |\nabla_g u|_g^2 \, dV_g \leq C_1 \int_{B(0,\varrho)} |\nabla \tilde{u}|^2 \, dx, \tag{4.6}$$

where $B_g(q, r)$ denotes the ball in (M, g) with center q and radius r.

Proof. Given a set $A \subset \mathbb{R}^m$, we denote its closure respect to the Euclidean metric as $\overline{A}$. As M is compact, the set

$$K := \bigcup_{q \in M} \overline{B}_{T_q M}(0, 3\delta) \subset TM.$$

is also compact. Since the exponential map $\exp : TM \to M$ and the metric g are continuous, writing g in normal coordinates around any $q \in M$ with fixed radius 3δ, we obtain that the function $\sqrt{|g|} : K \to \mathbb{R}^m$ is continuous and positive. Therefore, there exist two positive numbers B_1 and B_2, depending only on M, g and δ, such that

$$0 < B_1 \leq (\sqrt{|g_q|}(x) \leq B_2 \tag{4.7}$$

for all $q \in M$ and all $x \in \overline{B}(0, \varrho)$.

Take $u \in H^1_g(M)$, $\varrho \in (0, 3\delta]$, $q \in M$ and $s \in [1, 2^*]$. Hereafter, we identify $B(0, \varrho) \equiv \overline{B}_{T_q M}(0, 3\delta)$. Then, the function $\tilde{u} u \circ \exp_q$ written in normal coordinates around q is well defined in $B(0, \varrho)$, $\tilde{u} \in L^s(B(0, \varrho))$ and $|\nabla \tilde{u}|^2 \in L^2(B(0 \varrho)$. Moreover, by Lemma (4.1.2)

$$A_1 |\nabla \tilde{u}|^2 \leq |\nabla_g u|_g^2 \circ \exp_q \leq A_2 |\nabla \tilde{u}|^2.$$

independently of $u \in H^1_g(M)$, $q \in M$ and $x \in \overline{B}(0, \varrho)$.

Thus, on the one hand we have that

$$
\begin{aligned}
A_1B_1\int_{B(0,\varrho)}|\nabla\tilde{u}|^2\,dx &= A_1B_1\int_{B(0,\varrho)}\sqrt{|g|}^{-1}|\nabla\tilde{u}|^2\sqrt{|g|}\,dx\\
&\le A_1\int_{B(0,\varrho)}|\nabla\tilde{u}|^2\sqrt{|g|}\,dx\\
&\le \int_{B(0,\varrho)}|\nabla_g u|^2\circ\exp_q\sqrt{|g|}\,dx\\
&= \int_{B_g(q,\varrho)}|\nabla_g u|^2\,dV_g\\
&\le A_2\int_{B(0,\varrho)}|\nabla_g u|^2\circ\exp_q\,dx\\
&\le A_2B_2\int_{B(0,\varrho)}|\nabla u|^2\,dx.
\end{aligned}
$$

And on the other hand

$$
\begin{aligned}
B_1\int_{B(0,\varrho)}|\tilde{u}|^s\,dx &= B_1\int_{B(0,\varrho)}\sqrt{|g|}^{-1}|\tilde{u}|^s\sqrt{|g|}\,dx \le \int_{B(0,\varrho)}|\tilde{u}|^s\sqrt{|g|}\,dx\\
&= \int_{B_g(q,\varrho)}|u|^s\,dV_g \le B_2\int_{B(0,\varrho)}|\tilde{u}|^s\,dx.
\end{aligned}
$$

independently of $s\in[1,2^*]$. So, $C_1 := \max\{B_1^{-1}, A_1^{-1}B_1^{-1}, A_2B_2, B_2\}$ is the constant we where looking for. ∎

Remark 4.1.4 *From the proof of the previous lemma we can derive the following inequality*

$$
C_1^{-1}\int_{B(z,\varrho)}|\widetilde{u}|^s\,dx \le \int_{\exp_q(B(z,\varrho))}|u|^s\,dV_g \le C_1\int_{B(z,\varrho)}|\widetilde{u}|^s\,dx, \tag{4.8}
$$

for every $q\in M$, every $\varrho>0$ and every $z\in\mathbb{R}^m$ satisfying $|z|+\varrho<i_g$, where the constant C_1 is the same as before.

Lemma 4.1.5 *For every $0<a<i_g$, there exists a constant $C_2\ge 1$ depending only on (M,g) and a, such that, for all $q\in M$*

$$
C_2^{-1}\,|y-z| \le d_g(\exp_q(y),\exp_q(z)) \le C_2\,|y-z| \qquad \forall y,z\in B(0,a) \tag{4.9}
$$

Proof. As $0<a<i_g$, the metric $h_q := \exp_q^* g$ on $\overline{B}(0,a)\subset\mathbb{R}^m$ is well defined for every $q\in M$. Denote by $|\cdot|_{h_q(x)}$ the induced norm and by $|\cdot|$ the usual Euclidean metric in $\mathbb{R}^m$.

Define the set

$$
\begin{aligned}
\mathcal{K} &:= \{(q,x,v)\in M\times TB(0,i_g)\ :\ x\in\overline{B}(0,a), v\in T_xB(0,i_g), |v|=1\}\\
&\approx M\times\overline{B}(0,a)\times\mathbb{S}^{m-1},
\end{aligned}
$$

and the function

$$G : \mathcal{K} \to \mathbb{R}, \qquad G(q,x,v) = |v|_{h_q(x)}$$

As M is compact, $\mathcal{K}$ is also compact and continuity of G gives a constant $C_2 > 0$ depending only on (M,g) and a, such

$$0 < C_2^{-1} \leq G(q,x,v) \leq C_2,$$

for all $(q,x,v) \in M \times \overline{B}(0,a) \times \mathbb{S}^{m-1}$.

For $w \neq 0 \in \mathbb{R}^m$, applying the previous inequality we obtain

$$0 < C_2^{-1} \leq G(q,x,w/|w|) = \left|\frac{w}{|w|}\right|_{h_q(x)} = \frac{1}{|w|}|w|_{h_q(x)} \leq C_2.$$

Hence,

$$C_2^{-1}|w| \leq |w|_{h_q(x)} \leq C_2|w|, \tag{4.10}$$

for every $q \in M, x \in \overline{B}(0,a)$ and $w \in \mathbb{R}^m$.

Letting $\gamma : [0,1] \to \overline{B}(0,a)$ be any admissible curve, this last inequality implies

$$C_2^{-1}|\dot\gamma(t)| \leq |\dot\gamma(t)|_{h_q(\gamma(t))} \leq C_2|\dot\gamma(t)|, \qquad \text{for almost all } t \in [0,1].$$

Integrating this inequality we obtain

$$C_2^{-1}\mathcal{L}_e(\gamma) = C_2^{-1}\int_0^1 |\dot\gamma(t)|dt \leq \int_0^1 |\dot\gamma(t)|_{h_q}dt = \mathcal{L}_{h_q}(\gamma) \leq C_2\int_0^1 |\dot\gamma(t)|dt = C_2\mathcal{L}_e(\gamma),$$

where $\mathcal{L}_e$ denotes the euclidian length induced by $|\cdot|$ and $\mathcal{L}_{h_q}$ the corresponding length induced by h_q (see chapter 6 in [71]). Recall the length of admissible curves is preserved under isometries. Since the distance between two points $y,z \in \mathbb{R}^m$ with the Euclidean metric is just $|y-z|$ and since $\exp_q : (\overline{B}(0,a), h_q) \to (\overline{B}_g(q,a), g)$ is an isometry for every $q \in M$, inequality (4.34) follows from the definition of d_g as an infimum of the length of admissible curves in M. ∎

4.2 Properties of the energy functional and Palais-Smale sequences

Let (M,g) be a Riemannian manifold with dimension $m \geq 3$, Γ be a closed subgroup of $\mathrm{Isom}_g(M)$, $a,b,c \in \mathcal{C}^\infty(M)$. Hereafter we use the notation introduced in Chapter 3, considering a and c positive, but not assuming the coercivity of $-\mathrm{div}_g(a\nabla_g) + b$ on the space $H_g^1(M)\Gamma$, unless this is explicitly stated.

We shall need the following lemmas for functionals in a Hilbert space.

Lemma 4.2.1 *Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a Hilbert space and $J : H \to \mathbb{R}^m$ be a functional of class $\mathcal{C}^1$. If (v_k) and (φ_k) are sequences in H such that $J'(v_k) \to 0$ in H' and (φ_k) is bounded in H, then*

$$J'(v_k)\varphi_k \to 0 \text{ as } k \to \infty.$$

Proof. Since (φ_k) is bounded in H, there exists a constant $C > 0$ such that $\|\varphi\| \leq C$ for all k. Now, $J'(v_k) \to 0$ in H' implies that $\|J'(v_k)\|_{H'} \to 0$ as $k \to \infty$ and , hence,

$$|J'(v_k)\varphi_k| \leq \|J'(v_k)\|_{H'}\|\varphi_k\| \leq C\|J'(v,k)\|_{H'} \to 0$$

as $k \to \infty$ ∎

Lemma 4.2.2 *Let $(H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ be a Hilbert space, V be a dense subspace of H, (u_n) a bounded subsequence in H and $u \in H$. If $\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle$, for all $\varphi \in V$, then*

$$\langle u_n, v \rangle \to \langle u, v \rangle, \qquad \textit{for all } \; v \in H.$$

Proof. Take $v \in H$, $\varepsilon > 0$ and let $d > 0$ be any bound for the sequence $(\|u_n\|) \subset \mathbb{R}$. By density of V in H, there exists $\varphi \in V$ such that

$$\|v - \varphi\| < \frac{\varepsilon}{2(d + \|u\|)}$$

Now, by hypothesis, there exists $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$,

$$|\langle u_n - u, \varphi \rangle| < \frac{\varepsilon}{2}.$$

Then, for any $n \geq N_0$ we have by Cauchy-Schwartz inequality,

$$\begin{aligned} |\langle u_n, v \rangle - \langle u, v \rangle| &\leq |\langle u_n - u, v - \varphi \rangle| + |\langle u_n - u, \varphi \rangle| \\ &\leq \|u_n - u\|\|v - \varphi\| + |\langle u_n - u, \varphi \rangle| \\ &\leq (\|u_n\| + \|u\|)\|v - \varphi\| + |\langle u_n - u, \varphi \rangle| \\ &< (d + \|u\|)\frac{\varepsilon}{2(d + \|u\|)} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and the result follows. ∎

The Γ-invariant Palais-Smale sequences for the functional J_g are bounded in $H_g^1(M)^\Gamma$, as we show in the next result.

Lemma 4.2.3 *Let (u_k) be a Γ-invariant Palais-Smale sequence for the functional J_g at the level τ. Then (u_k) is bounded in $H_g^1(M)$.*

Proof. Hereafter, C will denote a positive constant, not necessarily the same one. Let (u_k) be a sequence in $H_g^1(M)$ such that $J_g(u_n) \to \tau$ and $J_g'(u_k) \to 0$ in $\left(H_g^1(M)\right)'$. Then, $J_g'(u_k)u_k = o(\|u_k\|_g)$ and

$$|u_k|_{g,2^*}^{2^*} \leq C\left(\frac{1}{m}|u_k|_{g,c,2^*}^{2^*}\right) = C\left(J_g(u_k) - \frac{1}{2}J_g'(u_k)u_k\right) \leq C + o(\|u_k\|_g).$$

Hence,

$$\int_M a|\nabla_g u|^2 + b|u|^2 dV_g = 2\left(J_g(u_k) + \frac{1}{2^*}|u_k|_{g,c,2^*}^{2^*}\right) \leq C + o(\|u_k\|_g). \qquad (4.11)$$

Moreover, as $(J_g(u_k))$ is bounded in $\mathbb{R}$ and M is compact, Hölder's inequality implies that

$$\begin{aligned}|u_k|_{g,2}^2 &\le C|u_k|_{g,2^*}^2 \le C\left(|u_k|_{g,c,2^*}^{2^*}\right)^{2/2^*}\\ &= C\left(mJ_g(u_k) - \frac{m}{2}J_g'(u_k)u_k\right)^{2/2^*}\\ &\le C + o(\|u_k\|_g^{2/2^*}).\end{aligned} \tag{4.12}$$

As b is bounded, inequalities (4.11) and (4.12) yield

$$\begin{aligned}a_0\|u_k\|_g^2 &\le \int_M a|\nabla_g u|^2 + b|u|^2 dV_g + \int_M (-b+a_0)\, u_k^2\, dV_g\\ &\le \int_M a|\nabla_g u|^2 + b|u|^2 dV_g + C|u_k|_{g,2}^2\\ &\le C + o(\|u_k\|_g) + o(\|u_k\|_g^{2/2^*}),\end{aligned}$$

where $a_0 := \min_M a$. This implies that (u_k) is bounded in $H_g^1(M)$. ∎

Corollary 4.2.4 *If (u_k) is a Γ-invariant Palais-Smale sequence for J_g at the level τ and $u_k \to 0$ in $L_g^{2^*}(M)$, then $u_k \to 0$ strongly in $H_g^1(M)$.*

Proof. The norm

$$\|u\|_{g,a,1} := \int_M a|\nabla u_k|^2 + |u_k|^2 dV_g$$

is well defined in $H_g^1(M)$ and it is equivalent to the standard one. As (u_k) is bounded in $H_g^1(M)$, Lemma 4.2.1 yields

$$\begin{aligned}\|u_k\|_{a,1}^2 - |u_k|_{g,c,2^*}^{2^*} dV_g &= J_g'(u_k)u_k + \int_M (1-b)|u_k|^2 dV_g\\ &\le J_g'(u_k)u_k + C|u_k|_{g,2}^2 = o(1) + C|u_k|_{g,c,2^*}^2.\end{aligned} \tag{4.13}$$

where C denotes a positive constant, not necessarily the same one. Since $|u_k|_{g,c,2^*} \to 0$, we conclude that $\|u_k\|_{a,1} = o(1)$. Therefore, $u_k \to 0$ strongly in $H_g^1(M)$ ∎

4.3 Reduction argument

In this section we will give a well known reduction argument to reduce Palais-Smale sequences for the functional J_g to Palais-Smale sequences for a simpler functional not depending on b (Cf. [24, 102]). Consider the functional J_g with $b = 0$ and denote it by $J_{g,0}$, i.e.,

$$J_{g,0} := \frac{1}{2}\int_M a|\nabla u|_g dV_g - \frac{1}{2^*}\int_M c|u|^{2^*} dV_g$$

Lemma 4.3.1 *Let $a \in L_g^\infty(M)$. If $u_k \rightharpoonup u$ weakly in $H_g^1(M)$, then, up to a subsequence*

$$\int_M a\langle \nabla u_k, \nabla \varphi\rangle_g dV_g \to \int_M a\langle \nabla u, \nabla\varphi\rangle_g dV_g \ \text{as } k \to \infty,$$

for all $\varphi \in H_g^1(M)$.

Proof. Take $\varphi \in H_g^1(M)$ and consider the linear functional $T_\varphi : H_g^1(M) \to \mathbb{R}$ given by

$$T_\varphi(v) := \int_M a\langle \nabla v, \nabla \varphi\rangle_g dV_g.$$

This functional is linear and continuous, for Cauchy-Schwartz and Hölder's inequalities yield

$$\begin{aligned} |T_\varphi(v)| &\le |a|_\infty \int_M |\nabla v|_g |\nabla \varphi|_g dV_g \le |a|_\infty \left(\int_M |\nabla v|_g^2 dV_g\right)^{1/2} \left(\int_M |\nabla \varphi|_g^2 dV_g\right)^{1/2} \\ &\le |a|_\infty \|\varphi\|_g \|v\|_g. \end{aligned}$$

Since $u_k \rightharpoonup u$ weakly in $H_g^1(M)$, then $T(u_k) \to T(u)$ and the limit follows. ∎

Lemma 4.3.2 *Let $c_k, \bar{c} \in L_g^\infty(M)$ and $u_k, u \in H_g^1(M)$ such that (c_k) is bounded in $L_g^\infty(M)$, $c_k \to \bar{c}$ almost everywhere in M and $u_k \rightharpoonup u$ weakly in $H_g^1(M)$. Then, up to a subsequence,*

$$\int_M c_k |u_k|^{2^*-2} u_k v \, dV_g \to \int_M \bar{c}|u|^{2^*-2} uv \, dV_g \qquad \text{for all } \ v \in H_g^1(M).$$

Proof. As u_k converges weakly to u in $H_g^1(M)$, (u_k) is bounded in $H_g^1(M)$. Up to a subsequence, compactness of the imbedding $H_g^1(M) \hookrightarrow L_g^{2^*-1}(M)$ (Theorem A.2.3) yields

$$u_k \to u \text{ in } L_g^{2^*-1}(M), \qquad u_k \to u \quad \text{a.e in } M$$

and the existence of a function $h \in L_g^{2^*-1}(M)$ such that

$$|u_k| \le h \quad \text{a.e. in } M \ \text{ for all } \ k$$

Since $\sup_k\{|c_k|_\infty\} < \infty$ and $\sup_k\{\|u_k\|_g\} < \infty$, the above limits allow us to use Lebesgue's Dominated Convergence Theorem to obtain

$$\int_M c_k |u_k|^{2^*-1} u_k \varphi dV_g \to \int_M \bar{c}|u|^{2^*-1} u\varphi dV_g \qquad \text{as } k \to \infty.$$

for any $\varphi \in C^\infty(M)$. To conclude this for all $\varphi \in H_g^1(M)$, define for each k the functionals $\phi_k, \bar{\phi} : H_g^1(M) \to \mathbb{R}$,

$$\begin{aligned} \phi_k(v) &:= \frac{1}{2^*}\int_M c_k |v|^{2^*} dV_g \\ \bar{\phi}(v) &:= \frac{1}{2^*}\int_M \bar{c}|v|^{2^*} dV_g \end{aligned}$$

By Frechet-Riez Representation Theorem and the above limit we obtain

$$\langle \nabla\phi_k(u_k), \varphi\rangle_{H_g^1(M)} = \phi_k'(u_k)\varphi = \int_M c_k|u_k|^{2^*-1}u_k\varphi dV_g$$
$$\to \int_M \bar{c}|u|^{2^*-1}u\varphi dV_g = \bar{\phi}'(u)v = \langle \nabla\bar{\phi}(u), \varphi\rangle_{H_g^1(M)}$$

for all $\varphi \in \mathcal{C}^\infty(M)$. Now we prove the sequence $(\nabla\phi_k(u_k))$ is bounded in $H_g^1(M)$: Hereafter, C will denote a positive constant, not necessarily the same one. By Hölder's and Sobolev's inequalities we obtain that

$$\left|\langle \nabla\phi_k(u_k), v\rangle\right| = |\phi_k'(u_k)v| \le C\int_M |u_k|^{2^*-1}|v|dV_g$$
$$\le C\left(\int_M |u_k|^{2^*}\right)^{\frac{2^*-1}{2^*}} |v|_{g,2^*} \le C\|u_k\|_g^{2^*-1}\|v\|_g \le C\|v\|_g$$

for all $v \in H_g^1(M)$, where the constant C is independent of k. Hence

$$\|\nabla\phi_k(u_k)\|_g \le C,$$

and the sequence $(\nabla\phi_k(u_k))$ is bounded in $H_g^1(M)$. Lemma 4.2.2 yields

$$\int_M c_k|u_k|^{2^*-2}u_kv\; dV_g = \phi_k'(u_k)v = \langle \nabla\phi_k(u_k), v\rangle_{H^1(M)}$$
$$\to \langle \nabla\bar{\phi}(u), v\rangle_{H_g^1(M)} = \bar{\phi}'(u)v = \int_M \bar{c}|u|^{2^*-2}uv\; dV_g,$$

for every $v \in H_g^1(M)$ as we wanted to prove. ∎

The following two lemmas are an adaptation of Lemmas 3.5 and 3.6 in [24].

Lemma 4.3.3 *Let $a_k, \bar{a} \in L_g^\infty(M)$ and $u_k, u \in H_g^1(M)$ be such that (a_k) is bounded in L_g^∞, $a_k \to \bar{a}$ in $L_g^\infty(M)$, and $u_k \rightharpoonup u$ weakly in $H_g^1(M)$. Then, in a subsequence, the following limit holds true*

$$\lim_{k\to\infty}\left(\int_M a_k|\nabla u_k|_g^2 - a_k|\nabla(u_k-u)|_g^2 dV_g\right) = \int_M \bar{a}|\nabla u|_g^2 dV_g.$$

Proof. Write

$$a_k|\nabla u_k|_g^2 - a_k|\nabla(u_k-u)|_g^2 - \bar{a}|\nabla u|_g^2 = a_k\langle 2\nabla u_k - \nabla u, \nabla u\rangle_g - \bar{a}|\nabla u|_g^2$$
$$= (a_k-\bar{a})\langle \nabla(2u_k-u), \nabla u\rangle_g + 2\bar{a}\langle \nabla(u_k-u), \nabla u\rangle_g \tag{4.14}$$

As $u_k \rightharpoonup u$ weakly in $H_g^1(M)$, (u_k) is bounded in $H_g^1(M)$. Let $C := \sup_{n\in\mathbb{N}}\|u_k\|_g$. On the one hand, Cauchy-Schwartz, Hölder's and Sobolev's inequalities give

$$\left|\int_M (a_k-\bar{a})\langle \nabla(2u_k-u), \nabla u\rangle_g dV_g\right| \le |a_k-\bar{a}|_\infty \int_M |\nabla(2u_k-u)|_g|\nabla u|_g dV_g$$
$$\le |a_k-\bar{a}|_\infty \int_M 2|\nabla u_k|_g|\nabla u|_g + |\nabla u|_g^2 dV_g$$
$$\le |a_k-\bar{a}|_\infty\left\{2\left(\int_M |\nabla u_k|_g^2 dV_g\right)^{1/2}\left(\int_M |\nabla u|_g^2\right)^{1/2} + \int_M |\nabla u|_g^2\right\}$$
$$\le |a_k-\bar{a}|_\infty 2\left\{C\|u\|_g + \|u\|_g^2\right\} \to 0, \tag{4.15}$$

On the other hand, since $u_k \rightharpoonup u$ weakly in $H_g^1(M)$, passing, if necessary to a subsequence, Lemma 4.3.1 implies

$$\lim_{k\to\infty} \int_M \bar{a}\langle \nabla(u_k - u), \nabla u\rangle_g dV_g = 0. \tag{4.16}$$

Therefore identity (4.14) together with limits (4.15) and (4.16) give

$$\begin{aligned}\lim\nolimits_{k\to\infty} &\int_M \left(a_k|\nabla u_k|_g^2 - a_k|\nabla(u_k-u)|_g^2 - \bar{a}|\nabla u|_g^2\right) dV_g \\ &= \lim_{k\to\infty} \left[\int_M (a_k - \bar{a})\langle\nabla(2u_k - u), \nabla u\rangle_g dV_g + 2\int_M \bar{a}\langle\nabla(u_k-u), \nabla u\rangle_g dV_g \right] = 0,\end{aligned}$$

as claimed. ■

Lemma 4.3.4 *Let (c_k) be a bounded sequence in $L_g^\infty(M)$ and $\bar{c} \in L_g^\infty(M)$ such that $c_k \to \bar{c}$ in $L_g^\infty(M)$. Let (u_k) be a sequence in $H_g^1(M)$ such that $u_k \rightharpoonup u$ weakly in $H_g^1(M)$ with $u \in L_g^\infty(M)$. Then, up to a subsequence,*

$$c_k|u_k|^{2^*-2}u_k - c_k|u_k-u|^{2^*-2}(u_k-u) \to \bar{c}|u|^{2^*-2}u \qquad \textit{in } \left(H_g^1(M)\right)'$$

Proof. First notice the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(t) := |t|^{2^*-2}t$$

satisfies

$$|f(t+s) - f(t)| \leq (2^*-1)(|t|+|s|)^{2^*-2}|s|, \qquad \text{for all } \ s,t \in \mathbb{R} \tag{4.17}$$

Define $\varphi_k : H_g^1(M) \to \mathbb{R}$ by

$$\varphi_k(v) := \int_M c_k f(u_k)v - c_k f(u_k-u)v - \bar{c}f(u)v dV_g.$$

We need to show that.

$$\|\varphi_k\|_{(H_g^1(M))'} = \sup_{v\neq 0} \frac{|\varphi_k(v)|}{\|v\|_g} \to 0 \qquad \text{as } k \to \infty.$$

First notice that for any $p \leq 2^*$, the inclusions

$$H_g^1(M) \hookrightarrow L_g^{2^*}(M) \hookrightarrow L_g^p(M) \tag{4.18}$$

are continuous, the first one been the Sobolev imbedding and the second one because M is compact. Then, there exists a positive number $\tilde{C}_p > 0$ such that $|v|_p \leq \tilde{C}_p\|v\|_g$ for any $v \in H_g^1(M)$

Define $r := \frac{2m}{5} > 1$ and $1 < p := \frac{2m}{2m-5} \leq 2^*$ and observe $1/p + 1/r = 1$. If we prove that

$$c_k f(u_k) - c_k f(u_k - u) - \bar{c}f(u) \in L_g^r(M), \qquad \text{for all } k \in \mathbb{N}, \tag{4.19}$$

and that, up to a subsequence,

$$c_k f(u_k) - c_k f(u_k - u) \to \bar{c} f(u) \text{ in } L^r_g(M), \tag{4.20}$$

then Hölder's inequality will allow us to conclude that

$$\begin{aligned}
|\varphi_k(v)| &\leq \int_M |c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)|\, |v| dV_g \\
&\leq \Big(\int_M |c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)|^r dV_g \Big)^{1/r} |v|_{g,p} \\
&\leq \tilde{C}_p \|v\|_g \left(\int_M |c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)|^r dV_g \right)^{1/r} \\
&\longrightarrow 0
\end{aligned} \tag{4.21}$$

for every $v \in H^1_g(M)$, from which we will conclude the proof.

For the rest of the proof, C will denote a positive constant, not necessarily the same one. We show the first assertion. As $u, \bar{c} \in L^\infty_g(M)$ and M is compact, $\bar{c} f(u) \in L^r_g(M)$. By (4.17), since $u \in L^\infty_g(M)$ and (c_k) is bounded in $L^\infty_g(M)$, we obtain

$$\begin{aligned}
|c_k f(u_k) - c_k f(u_k - u)| &\leq C \left(|u_k| + |u|\right)^{2^*-2} |u| \\
&\leq C|u|_\infty |u_k|^{2^*-2} + C|u|_\infty^{2^*-1} \leq C \left(|u_k|^{2^*-2} + 1\right),
\end{aligned} \tag{4.22}$$

almost everywhere in M. Thus,

$$|c_k f(u_k) - c_k f(u_k - u)|^r \leq C \left(|u_k|^{(2^*-2)r} + 1\right).$$

for every $k \in \mathbb{N}$. Inclusions (4.18) allow to conclude that $|u_k|^q \in L^1_g(M)$ for $q := (2^* - 2)r < 2^*$. As constant functions lie in $L^r_g(M)$ when M is compact, $c_k f(u_k) - c_k f(u_k - u) \in L^r_g(M)$, and therefore $c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u) \in L^r_g(M)$ as we claimed.

Next, we proceed to prove that, up to a subsequence, limit (4.20) holds true.

Consider again $q = (2^* - 2)r < 2^*$. Since (u_k) converges weakly in $H^1_g(M)$, it is bounded in this space and compactness of the imbedding $H^1_g(M) \hookrightarrow L^q_g(M)$ gives us the existence of a subsequence, which we denote the same, such that

$$u_k \to u \quad \text{in} \quad L^q_g(M).$$

Passing again, if necessary, to a subsequence, we can suppose that

$$u_k \to u \quad \text{a.e. in } M$$

and also the existence of a function $h \in L^q_g(M)$ such that

$$|u_k| \leq h, \quad \text{a.e. in } M.$$

Then

$$|c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)| \leq C \left(|u_k|^{2^*-2} + 1\right)$$

and the inequality holds a.e. in M, C is independent of k. Thus

$$|c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)|^r \leq C\left(h^q + 1\right).$$

a.e. in M. As $c_k \to \bar{c}$ in $L_g^\infty(M)$, $c_k \to \bar{c}$ a.e. in M and

$$|c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)|^r \to 0 \quad \text{a.e. in } M.$$

Therefore, Lebesgue's Dominated Convergence Theorem yields

$$|c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u)|_r^r \to 0,$$

and $c_k f(u_k) - c_k f(u_k - u) - \bar{c} f(u) \to 0$ in $L_g^r(M)$ as we wanted. ∎

Proposition 4.3.5 *Suppose $a, b, c \in C^\infty(M)$ are Γ-invariant with a and c positive and that $-div_g(a\nabla_g) + b$ is coercive on $H_g^1(M)^\Gamma$. Let (u_k) be a Γ-invariant Palais-Smale sequence for J_g at the level τ and let $u \in H_g^1(M)$ be such that*

$$\begin{array}{lll} u_k & \rightharpoonup & u \quad \text{weakly in } H_g^1(M), \\ u_k & \to & u \quad \text{strongly in } L_g^2(M), \text{ and} \\ u_k & \to & u \quad \text{a.e. in } M. \end{array}$$

Then $u \in H_g^1(M)^\Gamma$, $J_g'(u) = 0$ and the sequence $w_k := u_k - u \in H_g^1(M)^\Gamma$ satisfies, in a subsequence, that

$$\|w_k\|_g^2 = \|u_k\|_g^2 - \|u\|_{H_g^1(M)}^2 + o(1) \tag{4.23}$$

$$J_{g,0}(w_k) \to \tau - J_g(u) \text{ in } \mathbb{R} \tag{4.24}$$

$$J_{g,0}'(w_k) \to 0 \text{ in } \left(H_g^1(M)\right)' \tag{4.25}$$

where $o(1) \to 0$ as $k \to \infty$. In other words, (w_k) is a Γ-invariant Palais-Smale sequence for $J_{g,0}$ at the level $J_g(u) - \tau$.

Proof. As $H_g^1(M)^\Gamma$ is a closed subspace of $H_g^1(M)$, it is weakly closed. In this way, weak convergence of the sequence (u_k) implies $u \in H_g^1(M)^\Gamma$ and $w_k = u_k - u \in H_g^1(M)^\Gamma$. To prove assertion (4.23), just note that $u_k \rightharpoonup u$ implies $\langle u_k, u\rangle_{H_g^1(M)} = \langle u, u\rangle_{H_g^1(M)} + o(1)$ and, therefore,

$$\begin{aligned} \|w_k\|_{H_g^1(M)}^2 &= \|u_k\|_{H_g^1(M)}^2 - 2\langle u_k, u\rangle_{H_g^1(M)} + \|u\|_g^2 \\ &= \|u_k\|_{H_g^1(M)}^2 - 2\langle u, u\rangle_{H_g^1(M)} + \|u\|_g^2 + o(1) \\ &= \|u_k\|_{H_g^1(M)}^2 - \|u\|_{H_g^1(M)}^2 + o(1). \end{aligned}$$

Next, we prove that $J_g'(u) = 0$. Since $u_k \rightharpoonup u$ weakly in $H_g^1(M)$, Lemma 4.3.2 gives, up to a subsequence, that

$$\int_M c|u_k|^{2^*-2} u_k \varphi dV_g = \int_M c|u|^{2^*-2} u\varphi dV_g + o(1).$$

As (u_k) is a Palais-Smale sequence for the functional J_g and $-\mathrm{div}_g(a\nabla_g)+b$ is coercive on $H_g^1(M)^\Gamma$ this identity gives

$$\begin{aligned} o(1)=J_g'(u_k)\varphi &= \langle u_k,\varphi\rangle_{g,a,b}-\int_M c|u_k|^{2^*-2}u_k\varphi dV_g \\ &= \langle u,\varphi\rangle_{g,a,b}-\int_M c|u|^{2^*-2}u\varphi dV_g+o(1) \\ &= J_g'(u)\varphi+o(1) \end{aligned}$$

for every $\varphi\in H_g^1(M)$. Hence $J_g'(u)=0$.

To prove (4.24), on the other hand, as u_k converges weakly in $H_g^1(M)$, (u_k) is bounded in $H_g^1(M)$, in $L_g^2(M)$ and also in $L_g^{2^*}(M)$. By hypothesis $u_k\to u$ a.e. in M, so Lemma (4.1.1) and Lemma 4.3.3 yield

$$J_g(w_k)=J_g(u_k)-J_g(u)+o(1).$$

On the one hand we have that

$$J_g(w_k)-J_{g,0}(w_k)=\frac{1}{2}\int_M b|w_k|^2dV_g=o(1).$$

because $w_k\to 0$ strongly in $L_g^2(M)$. So,

$$J_{g,0}(w_k)=J_g(w_k)+o(1)=J_g(u_k)-J_g(u)+o(1)=\tau-J_g(u)+o(1).$$

Consequently, $J_{g,0}(w_k)\to\tau-J_g(u)$ as we claimed.

Finally we show limit (4.25) holds true. Hereafter C will denote a positive constant, not necessarily the same one, not depending on k. Notice that

$$J_g'(w_k)-J_{g,0}'(w_k)\to 0 \qquad \text{in} \quad \left(H_g^1(M)\right)'. \tag{4.26}$$

Indeed, take $\varphi\in H_g^1(M)$, then Hölder's inequality gives

$$\left|J_g'(w_k)\varphi-J_0'(w_k)\varphi\right|\le C\int_M |w_k|\ |\varphi|dV_g\le |w_k|_{g,2}\|\varphi\|_g$$

Hence, for every non zero element $\varphi\in H_g^1(M)$ we have that

$$\frac{\left|J_g'(w_k)\varphi-J_0'(w_k)\varphi\right|}{\|\varphi\|_g}\le C|w_k|_{g,2}\to 0$$

and limit (4.26) follows.

Now, we shall prove that

$$J_g'(w_k)\to 0 \qquad \text{in} \ \left(H_g^1(M)\right)'. \tag{4.27}$$

As u is a critical point of J_g, regularity theory gives $u\in L_g^\infty(M)$ (Cf. [48]). Then 4.3.4 yields

$$\begin{aligned}
&\left|J_g'(w_k)\varphi\right| \\
&= \left|J_g'(u_k)\varphi - J_g'(u)\varphi + \left(\int_M cf(u_k)\varphi - cf(u_k-u)\varphi - cf(u)\varphi dV_g\right)\right| \\
&\le \|J_g'(u_k)\|_{(H_g^1(M))'}\|\varphi\|_g + o(\|\varphi\|_g) = o(\|\varphi\|_g)
\end{aligned}$$

because $J_g'(u_k) \to 0$ in $\left(H_g^1(M)\right)'$ for (u_k) is a Γ-invariant Palais-Smale sequence. Therefore limit (4.27) holds true.

From limits (4.26) and (4.27) we conclude that $J_{g,0}'(w_k) \to 0$ in $\left(H_g^1(M)\right)'$. ■

4.4 Proof of Theorem 3.1.1

This section is devoted to the proof of Theorem 3.1.1. We use the notation introduced in the previous sections. As in Chapter 2, we consider the Euclidean limit problem

$$\begin{cases} -\Delta v = |v|^{2^*-2}v, \\ v \in D^{1,2}(\mathbb{R}^m), \end{cases} \tag{4.28}$$

and its associated energy functional

$$J_\infty(v) := \frac{1}{2}\int_{\mathbb{R}^m} |\nabla v|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^m} |v|^{2^*} dx, \qquad v \in D^{1,2}(\mathbb{R}^m).$$

The following inequality will prove to be useful later on.

Lemma 4.4.1 *If $u \in D^{1,2}(\mathbb{R}^m)$ and $v \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, then*

$$\int_{\mathbb{R}^m} v^2|u|^{2^*} dx \le S^{-1}\left(\int_{\operatorname{supp}(v)} |u|^{2^*} dx\right)^{2/m}\left(\int_{\mathbb{R}^m} |\nabla(vu)| dx\right)^{2/2^*} \tag{4.29}$$

Proof. For the proof, see Lemma 8.12 in [102]. ■

The proof of Theorem 3.1.1 will follow easily from Proposition 4.3.5 and the following one.

Proposition 4.4.2 *Assume that $b \equiv 0$. Let (u_k) be a Γ-invariant Palais-Smale sequence for J_0 at the level $\tau > 0$ such that $u_k \rightharpoonup 0$ weakly in $H_g^1(M)$. Then, after passing to a subsequence, there exist a point $p \in M$ and a nontrivial solution $\widehat{v}$ to problem (4.28) such that $\#\Gamma p < \infty$ and*

$$\tau \ge \left(\frac{a(p)^{m/2}\,\#\Gamma p}{c(p)^{(m-2)/2}}\right) J_\infty(\widehat{v}) \ge \left(\min_{q\in M}\frac{a(q)^{m/2}\,\#\Gamma q}{c(q)^{(m-2)/2}}\right)\frac{1}{m}S^{m/2}. \tag{4.30}$$

Proof. Fix δ such that $3\delta \in (0, i_g)$, where i_g is the injectivity radius of M. Let $C_1 > 0$ be as in Lemma (4.1.3).

By Lemmas 4.2.3 and 4.2.1 we have that

$$|u_k|_{g,c,2^*}^{2^*} = m\left(J_0(u_k) - \frac{1}{2}J_0'(u_k)u_k\right) \to m\tau =: \beta > 0.$$

So, as M is compact, after passing to a subsequence, there exist $q_0 \in M$ and $\lambda_0 \in (0, \beta)$ such that

$$\int_{B_g(q_0,\delta)} c|u_k|^{2^*} dV_g \geq \lambda_0 \qquad \forall k \in \mathbb{N}.$$

where, recall, $B_g(q,r)$ denotes the ball in (M,g) with center q and radius r. For each k, the Levy concentration function $Q_k : [0,\infty) \to [0,\infty)$ given by

$$Q_k(r) := \max_{q \in M} \int_{B_g(q,r)} c|u_k|^{2^*} dV_g$$

is continuous, nondecreasing, and satisfies $Q_k(0) = 0$ and $Q_k(\delta) \geq \lambda_0$. We fix $\lambda \in (0, \lambda_0)$ such that

$$\lambda < C_1^{-m-1}(\min_M c)\left[2^{-1}S(\min_M a)(\max_M c)^{-1}\right]^{m/2}. \tag{4.31}$$

where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^m) \hookrightarrow L^{2^*}(\mathbb{R}^m)$. Then, for each $k \in \mathbb{N}$, continuity of Q_k and compactness of M give the existence of $p_k \in M$ and $r_k \in (0, \delta]$ such that

$$Q_k(r_k) = \int_{B_g(p_k,r_k)} c|u_k|^{2^*} dV_g = \lambda \tag{4.32}$$

and, after passing to a subsequence, $p_k \to p$ in M.

Fix a cut-off function $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ such that $0 \leq \zeta \leq 1$, $\zeta(y) = 1$ if $|y| \leq 2\delta$ and $\zeta(y) = 0$ if $|y| \geq 3\delta$. For each k, define

$$v_k(x) := r_k^{(m-2)/2}(u_k \circ \exp_{p_k})(r_k x), \qquad \zeta_k(x) := \zeta(r_k x),$$

$$a_k(x) := (a \circ \exp_{p_k})(r_k x) \qquad \text{and} \qquad c_k(x) := (c \circ \exp_{p_k})(r_k x).$$

Then, $\operatorname{supp}(\zeta_k v_k) \subset \overline{B(0, 3\delta r_k^{-1})}$ and, extending $\zeta_k v_k$ by 0 outside $B(0, 3\delta r_k^{-1})$, we have that $\zeta_k v_k \in D^{1,2}(\mathbb{R}^m)$. As $\zeta \equiv 1$ in $B(0, r_k)$, using (4.32) and (4.5) and performing the change of variable $y = r_k x$ we obtain

$$0 < \lambda = \int_{B_g(p_k,r_k)} c|u_k|^{2^*} dV_g \leq C_1 \int_{B(0,r_k)} (c \circ \exp_{p_k})|\zeta(u_k \circ \exp_{p_k})|^{2^*} dy \tag{4.33}$$
$$= C_1 \int_{B(0,1)} c_k|\zeta_k v_k|^{2^*} dx \leq C \int_{B(0,1)} |\zeta_k v_k|^{2^*} dx.$$

Here and hereafter C stands for a positive constant, not necessarily the same one. Moreover, inequalities (4.5) and (4.6) yield

$$\begin{aligned}\int_{B(0,3\delta r_k^{-1})} |\nabla(\zeta_k v_k)|^2\, dx &= \int_{B(0,3\delta)} \left|\nabla(\zeta(u_k \circ \exp_{p_k}))\right|^2 dy \\ &\leq C \int_{B(0,3\delta)} \left[\zeta^2 \left|\nabla(u_k \circ \exp_{p_k})\right|^2 + |\nabla\zeta|^2 (u_k \circ \exp_{p_k})^2\right] dy \\ &\leq C \int_{B(0,3\delta)} \left[\left|\nabla(u_k \circ \exp_{p_k})\right|^2 + (u_k \circ \exp_{p_k})^2\right] dy \\ &\leq C \int_{B_g(p_k,3\delta)} \left[|\nabla_g u_k|_g^2 + u_k^2\right] dV_g \\ &\leq C\|u_k\|_g,\end{aligned}$$

so Lemma 4.2.3 implies that $(\zeta_k v_k)$ is bounded in $D^{1,2}(\mathbb{R}^m)$. Therefore, after passing to a subsequence, we have that $\zeta_k v_k \rightharpoonup v$ weakly in $D^{1,2}(\mathbb{R}^m)$, $\zeta_k v_k \to v$ in $L^2_{loc}(\mathbb{R}^m)$ and $\zeta_k v_k \to v$ a.e. in $\mathbb{R}^m$. The proof of the proposition will follow from the next three claims.

Claim 1. $v \neq 0$.

To prove this claim first note that, take C_2 as in Lemma 4.1.5 so that for every $q \in M$,

$$C_2^{-1}|y-z| \leq d_g(\exp_q(y), \exp_q(z)) \leq C_2|y-z| \qquad \forall y,z \in B(0,2\delta). \tag{4.34}$$

Set $\varrho := C_2^{-1}$. Notice that if $z \in \mathbb{R}^m$ satisfies that $|z| < 1$ and if $q = \exp_{p_k}(x) \in \exp_{p_k} B(r_k z, r_k\varrho)$, then $|r_k z| \leq 2\delta$, $|x| \leq 2\delta$ and

$$d_g(q, \exp_{p_k}(r_k z)) \leq C_2|x - r_k z| \leq r_k.$$

Hence, for every $z \in \overline{B(0,1)}$ we have that

$$\exp_{p_k} B(r_k z, r_k\varrho) \subset B_g(\exp_{p_k}(r_k z), r_k). \tag{4.35}$$

Now, arguing by contradiction, assume that $v = 0$. Let $\vartheta \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ be such that $\operatorname{supp}(\vartheta) \subset B(z,\varrho)$ for some $z \in \overline{B(0,1)}$. Then, $\operatorname{supp}(\vartheta) \subset B(0,2)$. Set $\hat{\vartheta}_k(q) := \vartheta(r_k^{-1}\exp_{p_k}^{-1}(q))$. As $\zeta_k \equiv 1$ in $B(0,2) \subset B(0,2\delta r_k^{-1})$, $\zeta_k v_k \to 0$ in $L^2_{loc}(\mathbb{R}^m)$, $J_0'(u_k) \to 0$ in $\left(H_g^1(M)\right)'$ and $(\hat{\vartheta}_k^2 u_k)$ is bounded in $H_g^1(M)$, using inequalities (4.5), (4.6) and 4.29 and Sobolev's inequalities, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m} |\nabla(\vartheta\zeta_k v_k)|^2\,dx = \int_{B(0,2)} |\nabla(\vartheta v_k)|^2\,dx = \int_{B(0,2r_k)} \left|\nabla\left((\hat{\vartheta}_k u_k)\circ \exp_{p_k}\right)\right|^2 dy\\
&\leq C_3 \int_{B_g(p_k,2r_k)} a\left|\nabla_g(\hat{\vartheta}_k u_k)\right|_g^2 dV_g\\
&= C_3 \int_{B_g(p_k,2r_k)} a\left[\hat{\vartheta}_k^2 |\nabla_g(u_k)|_g^2 + 2\hat{\vartheta}_k u_k \left\langle \nabla_g u_k, \nabla_g \hat{\vartheta}_k\right\rangle_g + \left|\nabla_g\hat{\vartheta}_k\right|^2 u_k^2\right] dV_g\\
&= C_3 \int_{B_g(p_k,2r_k)} a\left\langle \nabla_g u_k, \nabla_g(\hat{\vartheta}_k^2 u_k)\right\rangle_g dV_g + C\int_{B(p_k,2r_k)} a|u_k|^2|\nabla\hat{\vartheta}_k|^2 dV_g\\
&\leq C_3 \int_{B_g(p_k,2r_k)} a\left\langle \nabla_g u_k, \nabla_g(\hat{\vartheta}_k^2 u_k)\right\rangle_g dV_g + C\int_{B(0,2)} |\zeta_k v_k|^2 dx\\
&= C_3 \int_{B_g(p_k,2r_k)} a\left\langle \nabla_g u_k, \nabla_g(\hat{\vartheta}_k^2 u_k)\right\rangle_g dV_g + o(1)\\
&= C J'_{g,0}(u_k)(\hat{\vartheta}_k^2 u_k) - \int_{B_g(p_k,2r_k)} b\, u_k^2 \hat{\vartheta}_k^2 dV_g + C_3\int_{B_g(p_k,2r_k)} c\,|u_k|^{2^*-2}(\hat{\vartheta}_k u_k)^2\,dV_g + o(1)\\
&= C_3 \int_{B_g(p_k,2r_k)} c\,|u_k|^{2^*-2}(\hat{\vartheta}_k u_k)^2\,dV_g + o(1)\\
&\leq C_4 \int_{B(0,2)\cap B(z,\rho)} |v_k|^{2^*-2}(\vartheta v_k)^2\,dx + o(1)\\
&\leq C_4 \left(\int_{B(z,\rho)} |v_k|^{2^*} dx\right)^{2/m} \left(\int_{B(0,2)} |\vartheta\zeta_k v_k|^{2^*} dx\right)^{2/2^*} + o(1)\\
&\leq C_4 S^{-1} \left(\int_{B(z,\rho)} |v_k|^{2^*} dx\right)^{2/m} \int_{\mathbb{R}^m} |\nabla(\vartheta\zeta_k v_k)|^2\,dx + o(1),
\end{aligned}$$

where C stands for constants, not necessarily the same ones, $C_3 := C_1(\min_M a)^{-1}$ and $C_4 := C_1(\max_M c)C_3$. On the other hand, from (4.8), (4.35) and (4.32) we derive

$$\begin{aligned}
\int_{B(z,\rho)} |v_k|^{2^*} dx &= \int_{B(r_k z, r_k\rho)} \left|u_k \circ \exp_{p_k}(y)\right|^{2^*} dy\\
&\leq C_1 \int_{\exp_{p_k}(B(r_k z, r_k\rho))} |u_k|^{2^*} dV_g\\
&\leq C_1 (\min_M c)^{-1} \int_{B_g(\exp_{p_k}(r_k z), r_k)} c\,|u_k|^{2^*} dV_g\\
&\leq C_1 (\min_M c)^{-1}\lambda.
\end{aligned}$$

It follows from (4.31) that $(C_1(\min_M c)^{-1}\lambda)^{2/m} < 2^{-1}C_4^{-1}S$. Therefore,

$$\lim_{k\to\infty} \int_{\mathbb{R}^m} |\nabla(\vartheta\zeta_k v_k)|^2\,dx = 0$$

and Sobolev's inequality yields

$$\lim_{k\to\infty} \int_{\mathbb{R}^m} |\vartheta\zeta_k v_k|^{2^*} dx = 0$$

for every $\vartheta \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ such that $\operatorname{supp}(\vartheta) \subset B(z,\varrho)$ for some $z \in \overline{B(0,1)}$. As $B(0,1)$ can be covered by a finite number of balls $B(z_j,\varrho)$ with $z_j \in \overline{B(0,1)}$, choosing a partition of unity $\{\vartheta_j^{2^*}\}$ subordinated to this covering, we conclude that

$$\int_{B(0,1)} |\zeta_k v_k|^{2^*} dx \le \int_{B(0,1)} \left|\left(\sum_j \vartheta_j\right)\zeta_k v_k\right|^{2^*} dx \le \sum_j \int_{\mathbb{R}^m} |\vartheta_j \zeta_k v_k|^{2^*} dx \longrightarrow 0,$$

contradicting (4.33). This finishes the proof of Claim 1.

Claim 2. $\widehat{v} := \left(\frac{c(p)}{a(p)}\right)^{(m-2)/4} v$ is a nontrivial solution to problem (4.28).

First we show that, after passing to a subsequence, $r_k \to 0$. Arguing by contradiction, assume that $r_k > \theta > 0$ for all k large enough. Then, as $\zeta_k v_k \to v$ a.e. in $\mathbb{R}^m$ and $\operatorname{supp}(\zeta_k v_k) \subset \overline{B(0,3\delta r_k^{-1})} \subset \overline{B(0,3\delta\theta^{-1})}$, $v \neq 0$ in $\overline{B(0,3\delta\theta)}$. Hence, since $\zeta_k v_k \to v$ in $L^2_{loc}(\mathbb{R}^m)$, and $0 \le \zeta \le 1$, using inequality (4.5) and performing again the change of variables $y = r_k x$ we obtain

$$\begin{aligned}
0 \neq \int_{B(0,3\delta\theta^{-1})} |v|^2\, dx &= \int_{B(0,3\delta\theta^{-1})} |\zeta_k v_k|^2\, dx + o(1) \\
&= \int_{B(0,3\delta r_k^{-1})} |\zeta_k v_k|^2\, dx + o(1) \\
&= r_k^{-2} \int_{B(0,3\delta)} \left|\zeta(u_k \circ \exp_{p_k})\right|^2 dy + o(1) \\
&\le \theta^{-2} \int_{B(0,3\delta)} \left|u_k \circ \exp_{p_k}\right|^2 dy + o(1) \\
&\le C_1\theta^{-2} \int_{B_g(p_k,3\delta)} |u_k|^2\, dV_g + o(1) \\
&\le C_1\theta^{-2} \int_M |u_k|^2 dV_g.
\end{aligned}$$

This yields a contradiction because, as we are assuming that $u_k \rightharpoonup 0$ weakly in $H^1_g(M)$, we have that $u_k \to 0$ strongly in $L^2_g(M)$ by the compactness of the imbedding $H^1_g(M) \hookrightarrow L^2_g(M)$.

Claim 2 is equivalent to showing that v satisfies

$$-a(p)\Delta v = c(p)|v|^{2^*-2}v, \qquad v \in D^{1,2}(\mathbb{R}^m),$$

i.e. we need to show that

$$\int_{\mathbb{R}^m} a(p)\left\langle \nabla v, \nabla\varphi\right\rangle\, dx = \int_{\mathbb{R}^m} c(p)|v|^{2^*-2} v\varphi\, dx \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^m). \tag{4.36}$$

To this end, take $\varphi \in C_c^\infty(\mathbb{R}^m)$ and let $R > 0$ be such that $\operatorname{supp}(\varphi) \subset B(0,R)$. For k such that $Rr_k < 2\delta$ define $\hat{\varphi}_k \in H^1_g(M)$ by

$$\hat{\varphi}_k(q) := r_k^{\frac{2-m}{2}} \varphi(r_k^{-1} \exp_{p_k}^{-1}(q)).$$

Note first that, as the exponential map $\exp : TM \to M$ is continuous, then $\exp_{p_k}(r_k x) \to \exp_p(0) = p$ for every $x \in \mathbb{R}^m$. Consequently $a_k \to a(p)$ and $c_k \to c(p)$ in $L^\infty_{loc}(\mathbb{R}^m)$ and since $\zeta_k v_k \rightharpoonup v$ weakly in $D^{1,2}(\mathbb{R}^m)$ we have that

$$\int_{\mathbb{R}^m} a_k \left\langle \nabla\left(\zeta_k v_k\right), \nabla\varphi\right\rangle\, dx = \int_{\mathbb{R}^m} a(p)\left\langle \nabla v, \nabla\varphi\right\rangle\, dx + o(1).$$

By Lemma 3.6 in [24] we also have that

$$\int_{\mathbb{R}^m} c_k|\zeta_k v_k|^{2^*-2}(\zeta_k v_k)\,\varphi\,dx = \int_{\mathbb{R}^m} c(p)|v|^{2^*-2}v\varphi\,dx + o(1).$$

Next observe that, if (g_{ij}^k) is the metric g written in normal coordinates around p_k, (g_k^{ji}) is its inverse, $\left|g^k\right| := \det(g_{ij}^k)$ and (∂^{ji}) is the identity matrix then, for every $i, j = 1, ..., m$,

$$\lim_{|y|\to 0} g_k^{ji}(y) = \partial^{ji} \quad \text{and} \quad \lim_{|y|\to 0} \left|g^k\right|^{1/2}(y) = 1, \tag{4.37}$$

uniformly in k. Therefore, as $\operatorname{supp}(\hat{\varphi}_k \circ \exp_{p_k}) \subset B(0, Rr_k) \subset B(0, 2\delta)$, $r_k \to 0$, and $(u_k \circ \exp_{p_k})$ and $(\hat{\varphi}_k \circ \exp_{p_k})$ are bounded in $D^{1,2}(\mathbb{R}^m)$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^m} (a \circ \exp_{p_k}) \left\langle \nabla(u_k \circ \exp_{p_k}), \nabla(\hat{\varphi}_k \circ \exp_{p_k}) \right\rangle dy - \int_M a \left\langle \nabla_g u_k, \nabla_g \hat{\varphi}_k \right\rangle_g dV_g \\
&= \sum_{i,j} \int_{B(0,Rr_k)} (a \circ \exp_{p_k})(\partial^{ji} - \left|g^k\right|^{1/2} g_k^{ji})\, \partial_i(u_k \circ \exp_{p_k})\, \partial_j(\hat{\varphi}_k \circ \exp_{p_k})\, dy \\
&= o(1),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^m} (c \circ \exp_{p_k})|u_k \circ \exp_{p_k}|^{2^*-2}(u_k \circ \exp_{p_k})(\hat{\varphi}_k \circ \exp_{p_k})dy - \int_M c \left|u_k\right|^{2^*-2} u_k \hat{\varphi}_k dV_g \\
&= \int_{B(0,Rr_k)} (c \circ \exp_{p_k})|u_k \circ \exp_{p_k}|^{2^*-2}(u_k \circ \exp_{p_k})(\hat{\varphi}_k \circ \exp_{p_k})(1 - \left|g^k\right|^{1/2})\, dy \\
&= o(1).
\end{aligned}$$

Finally, as $J_0'(u_k) \to 0$ in $\left(H_g^1(M)\right)'$ and $(\hat{\varphi}_k)$ is bounded in $H_g^1(M)$ we conclude from Lemma 4.2.1 that, for k large enough,

$$\begin{aligned}
&\int_{\mathbb{R}^m} a(p) \left\langle \nabla v, \nabla \varphi \right\rangle\, dx \\
&\quad = \int_{\mathbb{R}^m} a_k \left\langle \nabla\left(\zeta_k v_k\right), \nabla \varphi \right\rangle\, dx + o(1) \\
&\quad = \int_{\mathbb{R}^m} (a \circ \exp_{p_k}) \left\langle \nabla(u_k \circ \exp_{p_k}), \nabla(\hat{\varphi}_k \circ \exp_{p_k}) \right\rangle\, dy + o(1) \\
&\quad = \int_M a \left\langle \nabla_g u_k, \nabla_g \hat{\varphi}_k \right\rangle_g dV_g + o(1) \\
&\quad = J_{g,0}'(u_k)(\hat{\varphi}_k) + \int_M c \left|u_k\right|^{2^*-2} u_k \hat{\varphi}_k\, dV_g + o(1) \\
&\quad = \int_M c \left|u_k\right|^{2^*-2} u_k \hat{\varphi}_k\, dV_g + o(1) \\
&\quad = \int_{\mathbb{R}^m} (c \circ \exp_{p_k})|u_k \circ \exp_{p_k}|^{2^*-2}(u_k \circ \exp_{p_k})(\hat{\varphi}_k \circ \exp_{p_k})\, dy + o(1) \\
&\quad = \int_{\mathbb{R}^m} c_k|\zeta_k v_k|^{2^*-2}(\zeta_k v_k)\,\varphi\, dx + o(1) \\
&\quad = \int_{\mathbb{R}^m} c(p)|v|^{2^*-2}v\varphi\, dx + o(1).
\end{aligned}$$

This proves (4.36).

Claim 3. $\#\Gamma p < \infty$ and $\tau \geq \left(\frac{a(p)^{m/2}\,\#\Gamma p}{c(p)^{(m-2)/2}}\right) J_\infty(\widehat{v})$.

Let $\gamma_1 p, ..., \gamma_n p$ be n distinct points in the Γ-orbit Γp of p, and fix $\eta \in (0, \delta]$ such that $d_g(\gamma_i p, \gamma_j p) \geq 4\eta$ if $i \neq j$. For k sufficiently large, $d_g(p_k, p) < \eta$ so, as γ_i is an isometry, $d_g(\gamma_i p_k, \gamma_j p_k) > 2\eta$ for all $k \in \mathbb{N}$ and $i \neq j$, for, if not,

$$\begin{aligned} d_g(\gamma_i p, \gamma_j p) &\leq d(\gamma_i p, \gamma_i p_k) + d_g(\gamma_i p_k, \gamma_j p_k) + d_g(\gamma_j p_k, \gamma_j p) \\ &= d(p, p_k) + d_g(\gamma_i p_k, \gamma_j p_k) + d_g(p_k, p) < 4\eta, \end{aligned}$$

a contradiction. Since c and u_k are Γ-invariant, for each $\rho \in (0, \eta]$ we obtain that

$$n \int_{B_g(p_k,\rho)} c\,|u_k|^{2^*}\,dV_g = \sum_{i=1}^{n} \int_{B_g(\gamma_i p_k,\rho)} c\,|u_k|^{2^*}\,dV_g \leq \int_M c\,|u_k|^{2^*}\,dV_g. \tag{4.38}$$

Let $\varepsilon > 0$. By (4.37) there exists $\rho \in (0, \eta]$ such that $(1+\varepsilon)^{-1} < \left|g^k\right|^{1/2} < (1+\varepsilon)$ in $B(0, \rho)$ for k large enough. As $1_{B(0,\rho r_k^{-1})} c_k \to c(p)$ and $\zeta_k v_k \to v$ a.e. in $\mathbb{R}^m$, Fatou's lemma, inequality (4.38) and identity (A.1) yield

$$\begin{aligned} \frac{n}{m}\int_{\mathbb{R}^m} c(p)\,|v|^{2^*}\,dx &\leq \liminf_{k\to\infty} \frac{n}{m}\int_{B(0,\rho r_k^{-1})} c_k\,|\zeta_k v_k|^{2^*}\,dx \\ &\leq \liminf_{k\to\infty} \frac{n}{m}\int_{B(0,\rho)} (c\circ \exp_{p_k})\left|u_k\circ \exp_{p_k}\right|^{2^*} dy \\ &\leq \liminf_{k\to\infty} \frac{n}{m}\int_{B(0,\rho)} (1+\varepsilon)\left|g^k\right|^{1/2} (c\circ \exp_{p_k})\left|u_k\circ \exp_{p_k}\right|^{2^*} dy \\ &\leq (1+\varepsilon)\liminf_{k\to\infty} \frac{n}{m}\int_{B_g(p_k,\rho)} c\,|u_k|^{2^*}\,dV_g \\ &\leq (1+\varepsilon)\lim_{k\to\infty} \frac{1}{m}\int_M c\,|u_k|^{2^*}\,dV_g = (1+\varepsilon)\tau. \end{aligned}$$

This implies that n is bounded and, therefore, $\#\Gamma p < \infty$. Moreover, as ε is arbitrary, taking $n = \#\Gamma p$, we conclude that

$$\begin{aligned} \left(\frac{a(p)^{m/2}\,\#\Gamma p}{c(p)^{(m-2)/2}}\right) J_\infty(\widehat{v}) &= \left(\frac{a(p)^{m/2}\,\#\Gamma p}{c(p)^{(m-2)/2}}\right)\frac{1}{m}\int_{\mathbb{R}^m} |\widehat{v}|^{2^*}\,dx \\ &= \frac{\#\Gamma p}{m}\int_{\mathbb{R}^m} c(p)\,|v|^{2^*}\,dx \leq \tau, \end{aligned}$$

as claimed.

This finishes the proof of the proposition. ∎

Proof of Theorem 3.1.1. Let (u_k) be a sequence in $H_g^1(M)^\Gamma$ such that $J_g(u_k) \to \tau < (\min_{q\in M} \frac{a(q)^{m/2}\,\#\Gamma q}{c(q)^{(m-2)/2}})\frac{1}{m}S^{m/2}$ and $J_g'(u_k) \to 0$ in $\left(H_g^1(M)\right)'$. By Lemma 4.2.3, (u_k) is bounded in $H_g^1(M)$ so, after passing to a subsequence, $u_k \rightharpoonup u$ weakly in $H_g^1(M)$, $u_k \to u$ strongly in $L_g^2(M)$ and $u_k \to u$ almost everywhere in M. By Proposition 4.3.5, $u \in H_g^1(M)^\Gamma$, $J_g'(u) = 0$. Moreover the sequence $\widetilde{u}_k := u_k - u$ is a

Γ-invariant Palais-Smale sequence for the functional $J_{g,0}$ at the level $\widetilde{\tau} := \tau - J_g(u)$ satisfying $\widetilde{u}_k \rightharpoonup 0$ weakly in $H_g^1(M)$. If $\widetilde{\tau} \leq 0$, then

$$0 \geq \widetilde{\tau} = \lim_{k\to\infty} J_{g,0}(\widetilde{u}_k) = \lim_{k\to\infty} \frac{1}{2} J_{g,0}(\widetilde{u}_k)\widetilde{u}_k + \lim_{k\to\infty} \frac{1}{m} |\widetilde{u}_k|_{g,c,2^*}^{2^*} = \lim_{k\to\infty} \frac{1}{m} |\widetilde{u}_k|_{g,c,2^*}^{2^*} \geq 0,$$

and Lemma 4.2.4 implies $u_k \to u$ strongly in $H_g^1(M)$ in a subsequence. If $\widetilde{\tau} < (\min_{q\in M} \frac{a(q)^{m/2}\,\#\Gamma q}{c(q)^{(m-2)/2}}) \frac{1}{m} S^{m/2}$, Proposition 4.4.2 implies that $\widetilde{\tau} = 0$ and, by the above remark, $u_k \to u$ strongly in $H_g^1(M)$ in a subsequence. ■

APPENDIX A

SOBOLEV SPACES ON MANIFOLDS

A.1 Integration on manifolds

We begin with the definition of the integral of continuous functions with compact support.

Definition 2 *Let (M^m, g) be a Riemannian manifold with $\dim M = m$, $\{(\Omega_i, \varphi_i)\}_{i\in I}$ a countable atlas for M and $\{\gamma_i\}_{i\in I}$ a partition of unity subordinated to the atlas. Then, for all functions $f : M \to \mathbb{R}$ which are continuous and compactly supported on M, we define*

$$\int_M f dV_g := \sum_{i\in I} \int_{\varphi(\Omega_i)} (\gamma_i \sqrt{|g|} f) \circ \varphi_i^{-1} dx,$$

where $|g| = \det(g_{ij})$ in the corresponding coordinates.

It can be readily checked that the definition does not depend on the atlas and the subordinated partition of unity (Cf [56, Chapitre 4]).

Remark A.1.1 *We will frequently encounter the following situation: If (Ω, φ) is a chart for M and $f : M \to \mathbb{R}$ is continuous with supp$(f) \subset \Omega$, then we can simply define the integral as follows*

$$\int_M f dV_g := \int_\Omega f dV_g := \int_{\varphi(\Omega)} (\sqrt{|g|} f) \circ \varphi^{-1} dx, \tag{A.1}$$

and, as before, this definition does not depend on the chart containing supp(f).

Since we will deal with more general functions on manifolds, it is desirable to extend the definition of the integral. To do this, we use de Lebesgue measure on $\mathbb{R}^m$ and define the *Lebesgue volume measure on M*, $\lambda_{M,g}$. A detailed exposition of what follows can be found, for instance, in [1, Chapter XII].

First, we say that a subset A of M is *Lebesgue measurable on* M if for every $p \in A$ there exists a chart (Ω, φ) such that $\varphi(\Omega \cap A)$ is a Lebesgue measurable subset of $\mathbb{R}^m$. We define

$$\mathcal{L}(M) := \{A \subset M \ : \ A \text{ is Lebesgue measurable on } M\}.$$

This set is a σ-algebra over M and it contains the Borel σ-algebra of M.

Next, we proceed to define a volume for these sets. To do so, let $\{(\Omega_i, \varphi_i)\}_{i\in I}$ an atlas for M and $A \in \mathcal{L}(M)$. Then, the volume of A is

$$Vol_g(A) := \sum_{i\in I} \int_{\varphi_i(A\cap\Omega_i)} \sqrt{|g|} \circ \varphi^{-1} dx \in [0, \infty].$$

It is not hard to see that this definition does not depend on the chosen atlas. We define the *Lebesgue measure on* M,

$$\lambda_{M,g} : \mathcal{L}(M) \to \mathbb{R}, \qquad \lambda_{M,g}(A) := vol_g(A).$$

In this manner, we have defined a measure space $(M, \mathcal{L}(M), \lambda_{M,g})$ and we can define the integral in a measure theoretic way. For every subset Ω of M, we define the set of measurable functions on Ω

$$\mathcal{L}_0(\Omega, \lambda_{M,g}) := \{f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \ : \ f \text{ is } \lambda_{M,g} \text{ measurable } \},$$

and the set of integrable functions on Ω,

$$\mathcal{L}_1(\Omega, \lambda_{M,g}) := \{f \in \mathcal{L}_0(\Omega, \lambda_{M,g}) \ : \ \int_\Omega |f| d\lambda_{M,g} < \infty\}.$$

We have the following characterization of $\lambda_{M,g}-$ measurable and integrable functions.

Proposition A.1.2 *1. $f \in \mathcal{L}(M, \lambda_{M,g})$ if and only if $f \circ \varphi^{-1} : \varphi(\Omega) \to \mathbb{R} \cap \{\pm\infty\}$ is Lebesgue measurable for all charts (Ω, φ) on M.*

2. If (Ω, φ) is a chart for M, and $f \in \mathcal{L}_0(M, \lambda_{M,g})$, then $f \in \mathcal{L}_1(\Omega, \lambda_{M,g})$ if and only if $(f\sqrt{|g|}) \circ \varphi^{-1} \in \mathcal{L}_1(\varphi(\Omega))$, and, in this case

$$\int_\Omega f d\lambda_{M,g} = \int_{\varphi(\Omega)} (f\sqrt{|g|}) \circ \varphi^{-1} dx$$

As a corollary we have the following result.

Corollary A.1.3 *Let $\{(\Omega_i, \varphi_i)\}_{i\in I}$ be an atlas for M, $\{\gamma_i\}_{i\in I}$ a subordinated partition of unity and $f \in \mathcal{L}_0(M, \lambda_{M,g})$. Then $f \in \mathcal{L}_1(M, \lambda_{M,g})$ if and only if $\gamma_i f \in \mathcal{L}_1(\Omega_i, \lambda_{M,g})$ for all $i \in I$ and*

$$\sum_{i=1}^{\infty} \int_{\Omega_i} \gamma_i |f| d\lambda_{M,g} < \infty.$$

In this case

$$\begin{aligned}\int_M f d\lambda_{M,g} &= \sum_{i\in I}\int_{\Omega_i}\gamma_i f d\lambda_{M,g} \\ &= \sum_{i\in I}\int_{\varphi(\Omega_i)}(\gamma_i f\sqrt{|g|})\circ\varphi^{-1}dx.\end{aligned} \tag{A.2}$$

In particular, if $f : M \to \mathbb{R}$ is continuous and compactly supported on M, we recover the former definition of the integral, that is

$$\int_M f d\lambda_{M,g} = \int_M f dV_g = \sum_{i\in I}\int_{\varphi(\Omega_i)}(\gamma_i f\sqrt{|g|})\circ\varphi^{-1}dx$$

for every atlas $\{(\Omega_i, \varphi_i)\}_{i\in I}$ and every partition of unity $\{\gamma_i\}_{i\in I}$ subordinated to the atlas.

We define the Lebesgue spaces by $L^p_g(M) := L^p(M, \mathcal{L}(M), \lambda_{M,g})$, for any $p \in [1,\infty]$. In what follows, we will use only the notation dV_g instead of the measure theoretic one, $d\lambda_{M,g}$ and we will denote the L^p_g norm of $f \in L^p_g(M)$ simply by

$$|f|_p := |f|_{L^p_g(M)} = \left(\int_M |f|^p dV_g\right)^{1/p}$$

if there is no risk of confusion. Instead of saying that a proposition happens $\lambda_{M,g}$−a.e. in M, we will simply say that the proposition happens almost everywhere (a.e.) in M.

A.2 The space $H^1_g(M)$

Here and hereafter, we shall consider only a compact manifolds without boundary. For a treatment of the Sobolev spaces of non compact complete Riemannian manifolds with or without boundary, we refer the reader to the books [6, 56, 57].

Let (M^m, g) a compact Riemannian manifold without boundary with $\dim M = m$ and $u : M \to \mathbb{R}$ smooth. If we denote by ∇u the first covariant derivative of u, its Hilbert-Schmidt norm is defined in a local chart by

$$|\nabla u|^2_g := g^{ij}\partial_i u \partial_j u$$

where we have used the Einstein summation convention. For $u \in \mathcal{C}^\infty(M)$, the integrals $\int_M |u|^2 dV_g$ and $\int_M |\nabla u|^2_g dV_g$ are both finite and we can define the norm

$$\|u\|_g := \left(\int_M |u|^2 dV_g + \int_M |\nabla u|^2_g dV_g\right)^{1/2}.$$

Definition 3 *The Sobolev space $H^1_g(M)$ is the completion of $\mathcal{C}^\infty(M)$ with respect to the norm $\|\cdot\|_g$.*

Remark A.2.1 *As (M,g) is compact and without boundary, it is a complete space and the Sobolev space can be also defined as the closure of $C_c^\infty(M)$ with respect to the norm $\|\cdot\|_g$. If the manifold is not complete, this is true in general.*

Notice a sequence (u_k) in $H_g^1(M)$ converges strongly in this space if and only if the sequences (u_k) and $(\partial_i u_k)$ converge strongly in $L_g^2(M)$ for all $i = 1,\dots,m$. In fact, we have the following.

Lemma A.2.2 *Let (u_k) be a sequence in $\mathcal{C}^\infty(M)$ and $u, v_1 \dots, v_m \in L_g^2(M)$ such that $u_k \to u$ in $L_g^2(M)$ and $\partial_i u_k \to v_i$ in $L_g^2(M)$ for all $i = 1,\dots,m$. Then $u \in H_g^1(M)$, $\partial_i u = v_i$ and $u_k \to u$ in $H_g^1(M)$.*

Proof. First we show (u_k) is a Cauchy sequence in $H_g^1(M)$. Let $\varepsilon > 0$, then the convergence in $L_g^2(M)$ of the sequences (u_k) and $(\partial_i u_k)$ imply they are Cauchy sequences in $L_g^2(M)$ and there exist $N_0, N_1, \dots, N_m \in \mathbb{N}$ such that $|u_j - u_l|_2^2 \leq \varepsilon/(m+1)$ for every $j, l \geq N_0$ and $|\partial_i u_j - \partial_i u_l|_2^2 \leq \varepsilon/(m+1)$ if $j, l \geq N_i$ for each $i = 1,\dots,m$. Then, for every $j, l \geq N = \max\{N_0, N_1, \dots, N_m\}$ we have that

$$\|u_j - u_l\|_g^2 = |u_j - u_l|_2^2 + \sum_{i=1}^{N} |u_j - u_l|_2^2 \leq \varepsilon,$$

concluding that (u_k) is a Cauchy sequence in $H_g^1(M)$. Therefore, there exists $w \in H_g^1(M)$ such that $u_k \to w$ in $H_g^(M)$. Since $|u_k - w|_2 \leq \|u_k - w\|_g$ and $|\partial_i u_k - \partial_i w|_2 \leq \|\partial_i u_k - \partial_i w\|_g$ for all $i = 1,\dots,m$, then also $u_k \to w$ in $L_g^2(M)$ and $\partial_i u_k \to \partial_i w$ in $L_g^2(M)$ for all $i = 1,\dots,m$. We conclude that $u = w$ and $v_i = \partial_i w$ for all $i = 1\dots,m$. So, $\partial_i u = \partial_i w = v_i$ by definition, $u \in H_g^1(M)$ and $u_k \to u$ in $H_g^1(M)$. ■

Because of this lemma, given a function $u \in H_g^1(M) \smallsetminus \mathcal{C}^\infty(M)$, we denote by $\partial_i u$ to the limit in $L_g^2(M)$ of the converging sequence $(\partial_i u_k)$.

For any $u \in H_g^1(M)$, the positive part and the negative part of u, $u^+ := \max\{0, u\}$ and $u^- := \min\{0, u\}$, also lie in this space. The proof of this fact is entirely the same as in the Euclidean case, Cf. Corolary 20.12 in [64].

As in the Euclidean case, we have the following Sobolev's imbedding Theorem for compact manifolds

Theorem A.2.3 *If (M,g) is a compact Riemannian manifold without boundary, $\dim M \geq 3$, then the imbedding $H_g^1(M) \hookrightarrow L_g^p(M)$ is continuous for every $1 \leq p \leq 2^*$ and is compact for any $1 \leq p < 2^*$.*

Proof. See Chapter 2 in [6] or the same chapter in [57]. ■

A.3 Sobolev spaces with symmetries

To end this Appendix, we give a few words about Sobolev spaces in the presence of symmetries.

Given a compact Riemannian manifold (M, g), we denote by $\mathrm{Isom}_g(M) = \{\gamma : (M, g) \to (M, g) \ : \ \gamma \text{ is an isometry}\}$ its group of isometries. It is well known that this is a Lie group acting smoothly on M and that it is compact whenever M is compact (Cf. [68, Chapter II, Theorem 1.2]). Thus, every closed subgroup Γ of $\mathrm{Isom}_g(M)$ is also a compact Lie group by the Closed Subgroup Theorem [70, Theorem 20.12]. For any closed subgroup Γ of $\mathrm{isom}_g(M)$, we say a function $f : M \to \mathbb{R}$ is Γ-*invariant* if $f \circ \gamma = f$ for each $\gamma \in \Gamma$. Denote by

$$H^1_g(M)^\Gamma := \{u \in H^1_g(M) \ : \ u \text{ is } \Gamma - \text{invariant}\}.$$

the subspace of $H^1_g(M)$ consisting of Γ-invariant functions. This is a closed subspace of $H^1_g(M)$ and it is a Hilbert space endowed with the metric $\| \cdot \|$.

The following approximation result is useful.

Proposition A.3.1 *The subspace*

$$\mathcal{C}^\infty(M)^\Gamma := \{u \in \mathcal{C}^\infty(M) \ : \ u \textit{ is } \Gamma - \textit{invariant}\}$$

is dense in $H^1_g(M)^\Gamma$.

Proof. See Theorem VI.4.2 in [14]. ∎

APPENDIX B

TOPOLOGICAL METHODS

B.1 Category and cup-length

A pair consisting of a topological space X and a subset A of X is denoted by (X, A). A map of pairs $f : (X, A) \to (Y, B)$ is a continuous function $f : X \to Y$ such that $f(a) \in B$ for every $a \in A$. Two maps of pairs $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic if there exists a map of pairs $F : ([0, 1] \times X, [0, 1] \times A) \to (Y, B)$ such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$ for every $x \in X$.

Definition 4 *The Lusternik-Schnirelmann category of the pair (X, A) is the smallest number $k =: \mathrm{cat}(X, A)$ such that there exists an open cover $U_0, U_1, ..., U_k$ of X with the following properties:*

(LS_1) $A \subset U_0$ and there exists a homotopy $F : ([0, 1] \times U_0, [0, 1] \times A) \to (X, A)$ such that $F(0, x) = x$ and $F(1, x) \in A$ for every $x \in U_0$,

(LS_2) U_j is contractible in X for every $j = 1, ..., k$.

If no such cover exists we set $\mathrm{cat}(X, A) := \infty$.

If $A = \emptyset$ we write $\mathrm{cat}(X)$ instead of $\mathrm{cat}(X, \emptyset)$.

Let $\mathcal{H}^*$ be Čech cohomology with $\mathbb{Z}/2$-coefficients. We write $\widetilde{\mathcal{H}}^*$ for reduced Čech cohomology. The cup-product endows $\mathcal{H}^*(X, A)$ with a (graded right) $\mathcal{H}^*(X)$-module structure

$$\smile : \mathcal{H}^i(X, A) \times \mathcal{H}^j(X) \to \mathcal{H}^{i+j}(X, A),$$

see e.g. [47].

Definition 5 *The cup-length of (X, A) is the smallest number $k \in \mathbb{N} \cup \{0\}$ such that*

$$\xi_0 \smile \zeta_1 \smile \cdots \smile \zeta_k = 0 \qquad \text{for all } \xi_0 \in \mathcal{H}^*(X, A), \quad \text{for all } \zeta_1, \ldots, \zeta_k \in \widetilde{\mathcal{H}}^*(X).$$

We denote it by $\mathrm{cupl}(X, A)$. If no such number exists we define $\mathrm{cupl}(X, A) := \infty$.

We write $\mathrm{cupl}(X)$ instead of $\mathrm{cupl}(X, \emptyset)$. Note that $\mathrm{cupl}(X, A) \geq 1$ iff $\mathcal{H}^*(X, A) \neq 0$.

The category and the cup-length are related as follows.

Lemma B.1.1 $\mathrm{cat}(X, A) \geq \mathrm{cupl}(X, A)$.

Proof. See [33, Proposition 4.3]. ∎

Lemma B.1.2 *If $f : (X, A) \to (Y, B)$ and $h : (Y, B) \to (X, A)$ are maps of pairs whose composition $h \circ f : (X, A) \to (X, A)$ is homotopic to the identity of the pair (X, A) then*

$$\mathrm{cat}(X, A) \leq \mathrm{cat}(Y, B) \qquad \textit{and} \qquad \mathrm{cupl}(X, A) \leq \mathrm{cupl}(Y, B).$$

Proof. The proof is straightforward. ∎

Lemma B.1.3 *If the map of pairs $f : (X, A) \to (Y, B)$ induces a monomorphism $f^* : \mathcal{H}^*(Y, B) \to \mathcal{H}^*(X, A)$, then*

$$\mathrm{cupl}(Y, B) \leq \mathrm{cupl}(X, A).$$

Proof. Let $\xi_0 \in \mathcal{H}^*(Y, B), \zeta_1, \ldots, \zeta_r \in \widetilde{\mathcal{H}}^*(Y)$ be such that $\xi_0 \smile \zeta_1 \smile \cdots \smile \zeta_r \neq 0$. Then, since $f^* : \mathcal{H}^*(Y, B) \to \mathcal{H}^*(X, A)$ is a monomorphism, we have that

$$0 \neq f^*(\xi_0 \smile \zeta_1 \smile \cdots \smile \zeta_r) = f^*(\xi_0) \smile f^*(\zeta_1) \smile \cdots \smile f^*(\zeta_r).$$

This proves our claim. ∎

If Θ is a bounded smooth domain in $\mathbb{R}^N$, M is an m-dimensional compact smooth manifold without boundary, and $\overline{B}_r M$ is a tubular neighborhood of M contained in Θ, then the inclusion $i : (\overline{B}_r M, S_r M) \hookrightarrow (\Theta, \Theta \smallsetminus B_r M)$ induces an isomorphism in cohomology

$$i^* : \mathcal{H}^*(\Theta, \Theta \smallsetminus B_r M) \cong \mathcal{H}^*(\overline{B}_r M, S_r M)$$

by excision. Let $\tau \in \mathcal{H}^{N-m}(\overline{B}_r M, S_r M)$ be the Thom class of the disk bundle $q : B_r M \to M$ and let $\widetilde{\tau} \in \mathcal{H}^{N-m}(\Theta, \Theta \smallsetminus B_r M)$ be such that $i^*(\widetilde{\tau}) = \tau$. The cup-lenght of $(\Theta, \Theta \smallsetminus B_r M)$ can be computed in terms of $\widetilde{\tau}$, as follows.

Proposition B.1.4 $\mathrm{cupl}(\Theta, \Theta \smallsetminus B_r M)$ *is the smallest number $k \in \mathbb{N}$ such that*

$$\widetilde{\tau} \smile \zeta_1 \smile \cdots \smile \zeta_k = 0 \qquad \textit{for all } \zeta_1, \ldots, \zeta_k \in \widetilde{\mathcal{H}}^*(\Theta). \tag{B.1}$$

Proof. Let $k \in \mathbb{N}$ be such that (B.1) holds true, and let $\xi_0 \in \mathcal{H}^*(\Theta, \Theta \smallsetminus B_r M)$ and $\zeta_1, \ldots \zeta_k \in \widetilde{\mathcal{H}}^*(\Theta)$. By the Thom isomorphism theorem, $i^*(\xi_0) = \tau \smile q^*(\omega) = i^*(\widetilde{\tau}) \smile q^*(\omega)$ for some $\omega \in \mathcal{H}^*(M)$. Since $\widetilde{\tau} \smile \zeta_1 \smile \cdots \smile \zeta_k = 0$ we obtain that

$$\begin{aligned} i^*(\xi_0 \smile \zeta_1 \smile \cdots \smile \zeta_k) &= \tau \smile q^*(\omega) \smile i^*(\zeta_1 \smile \cdots \smile \zeta_k) \\ &= q^*(\omega) \smile i^*(\widetilde{\tau}) \smile i^*(\zeta_1 \smile \cdots \smile \zeta_k) \\ &= q^*(\omega) \smile i^*(\widetilde{\tau} \smile \zeta_1 \smile \cdots \smile \zeta_k) = 0 \end{aligned}$$

and, since $i^* : \mathcal{H}^*(\Theta, \Theta \smallsetminus B_r M) \to \mathcal{H}^*(\overline{B}_r M, S_r M)$ is an isomorphism, we conclude that $\xi_0 \smile \zeta_1 \smile \cdots \smile \zeta_n = 0$. Hence $\mathrm{cupl}(\Theta, \Theta \smallsetminus B_r M) \leq k$. The opposite inequality is trivial. ∎

B.2 Fixed Point transfer

For convenience to the reader, we include the definition and properties of the fixed point transfer used in this paper. The details can be found in [46].

Let B be a metric space, let U be an open subset of $B \to \mathbb{R}^n$, and let $P : B \times \mathbb{R}^n \to B$ and $\pi : B \times \mathbb{R}^n \to \mathbb{R}^n$ be the projections. A map $F : \overline{U} \to B \times \mathbb{R}^n$ is said to be *compactly fixed over* B if $P(F(z)) = p(z)$ for every $z \in \overline{U}$, $\mathrm{Fix}(F) := \{z \in \overline{U} F(z) = z\} \subset U$, and there exists a continuous function $\varrho : B \to (0, \infty)$ such that

$$\mathrm{Fix}(F) \subset T_\varrho := \{(b, x) \in B \times \mathbb{R}^n \ : \ |x| \leq \varrho(b)\}.$$

Let A be a closed subset of B. and let $Y \subset X$ be open subsets of U such that

$$\mathrm{Fix}(F) \subset X, \quad \mathrm{Fix}(F) \cap (A \times \mathbb{R}^n) \subset Y, \quad \mathrm{Fix}(F) \cap (P(Y) \times \mathbb{R}^n)\,. \tag{A.1}$$

Set $B' := B \setminus P(\mathrm{Fix}(F))$ and consider the following sequence of maps.

$$\begin{array}{rcl}
(X, Y) \times (\mathbb{R}^n, \mathbb{R}^{\setminus}\{0\}) & \overset{(id, i-F)}{\longleftarrow} & (X, (X \setminus \mathrm{Fix}(F)) \cup Y) \\
& \overset{i_1}{\hookrightarrow} & (B \times \mathbb{R}^n, (B, \mathbb{R}^{\setminus}\mathrm{Fix}(F)) \cup (P(Y) \times \mathbb{R}^n)) \\
& \overset{i_2}{\hookleftarrow} & (B \times \mathbb{R}^n, (B \times \mathbb{R}^n \setminus T_\varrho) \cup ((P(Y) \cup B') \times \mathbb{R}^n)) \\
& \overset{i_3}{\hookrightarrow} & (B, P(Y) \cup B') \times (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \\
& \overset{i_4}{\hookleftarrow} & (B, a) \times (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})
\end{array}$$

where $(id, i-F)(b, x) := (b, x, x - \pi(F(b, x)))$ and all other maps are inclusions. Here we write, $(B, A) \times (D, C) := (B \times D, B \times C \cup A \times D)$. Let H^* the Alexander-Spanier cohomology or any other continuous cohomology theory. The inclusion i_1 is an excision, so it induces an isomorphism in cohomology. The excision, homotopy and exactness properties of cohomology ensures that the inclusion i_3 induces an isomorphism too. Hence, applying cohomology to this sequence of maps, and composing both ends with the suspension isomorphism $H^{i+n}((B, A) \times (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) \cong H^i(B.A)$ we obtain a homomorphism

$$t_F^{X,Y} : H^*(X, Y) \to H^*(B, A).$$

The set of all open subsets $Y \subset X$ of U which satisfies (A.1) is a cofinal subset of $\mathcal{U} := \{(X, Y) \ : \ X, Y \text{ open in } B \times \mathbb{R}^n, X \supset \mathrm{Fix}(F), Y \supset \mathrm{Fix}(F) \cap (A \times \mathbb{R}^n)\}$. So passing to the direct limit

$$H^*(\mathrm{Fix}(F), \mathrm{Fix}(F) \cap (A \times \mathbb{R}^n)) \cong \lim_{\leftarrow}\{H^*(X, Y) \ : \ (X, Y) \in \mathcal{U}\}$$

we obtain a homomorphism

$$\tau_F : H^*(\mathrm{Fix}(F), \mathrm{Fix}(F) \cap (A \times \mathbb{R}^n)) \to H^*(B, A)$$

called the *(relative) fixed point transfer of* F. We state the properties we need for our purposes.

Naturality: *It commutes with connecting homomorphisms, that is, the diagram*

$$\begin{array}{ccc} H^i(Fix(F)\cap(A\times\mathbb{R}^n)) & \xrightarrow{\delta^*} & H^{i+1}(Fix(F), Fix(F)\cap(A\times\mathbb{R}^n)) \\ \downarrow{\scriptstyle\tau_F} & & \downarrow{\scriptstyle\tau_F} \\ H^i(A) & \xrightarrow{\delta^*} & H^{i+1}(B,A) \end{array}$$

commutes.

Units: *If $S : B \to B\times\mathbb{R}^n$ is a section of P, (i.e, $P\circ S = id$) and $F = S\circ P : B\times\mathbb{R}^n \to B\times\mathbb{R}^n$, then $Fix(F) = S(B)$ and $\tau_F = S^* : H^*(S(B)) \to H^*(B)$.*

Homotopy: *Let $h : \overline{W} \to B\times[0,1]\times\mathbb{R}^n$ be a compactly fixed map over $B\times[0,1]$. For each $t\in[0,1]$ set $W_t := \{(b,x)\in B\times\mathbb{R}^n \ :\ (b,t,x)\in W\}$ and let $h_t : W_t \to B\times\mathbb{R}^n$ be given by $h_t(b,x) := q(h(b,t,x))$ where $q : B\times[0,1]\times\mathbb{R}^n \to B\times\mathbb{R}^n$ is the projection. Then*

$$\tau_{h_0}\circ P^* = \tau_{h_1}\circ P^*.$$

APPENDIX C

A VARIATIONAL PRINCIPLE FOR SIGN-CHANGING SOLUTIONS

Let (M, g) be a closed Riemannian manifold with dimension $m \geq 3$, Γ a closed subgroup of $\mathrm{Isom}_g(M)$, $a, b, c \in \mathcal{C}^\infty(M)^\Gamma$. Since we are assuming that $a > 0$ and the operator $-\mathrm{div}_g(a\nabla_g) + b$ is coercive on $H^1(M)^\Gamma$, there exists $\mu > 0$ such that

$$\int_M \left[a|\nabla_g u|^2 + b|u|^2\right] dV_g \geq \mu \int_M \left[a|\nabla_g u|^2 + |u|^2\right] dV_g \qquad \forall u \in H_g^1(M)^\Gamma.$$

Fix $\theta > \max\{1, \mu, |b|_\infty\}$ and consider the scalar product

$$\langle u, v\rangle_{g,a,\theta} := \int_M \left[a\langle\nabla_g u, \nabla_g v\rangle_g + \theta uv\right] dV_g \tag{C.1}$$

in $H_g^1(M)^\Gamma$. As $\theta > 0$, this interior product is well defined in this space and the induced norm $\|\cdot\|_{g,a,\theta}$ is equivalent to the standard one, $\|\cdot\|_g$. For any closed subset $\mathcal{D}$ of $H_g^1(M)^\Gamma$ and any $\alpha > 0$ we define

$$B_\alpha(\mathcal{D}) := \{u \in H_g^1(M)^\Gamma \ : \ \mathrm{dist}_\theta(u, \mathcal{D}) \leq \alpha\}.$$

where $\mathrm{dist}_\theta(u, \mathcal{D}) := \inf_{v\in\mathcal{D}} \|u - v\|_{g,a,\theta}$. Continuity of the function $u \mapsto \mathrm{dist}_\theta(u, \mathcal{D})$ imply $B_\alpha(\mathcal{D})$ is also closed in $H_g^1(M)^\Gamma$.

The gradient of the functional $J_g : H_g^1(M)^\Gamma \to \mathbb{R}$ at $u \in H_g^1(M)^\Gamma$, with respect to the scalar product (C.1), is the vector $\nabla J_g(u)$ which satisfies

$$\begin{aligned}\langle\nabla J_g(u), v\rangle_{g,a,\theta} = J_g'(u)v &= \langle u, v\rangle_{g,a,\theta} - \int_M (\theta - b)uv\, dV_g - \int_M c\, |u|^{2^*-2}\, uv\, dV_g \\ &= \langle u, v\rangle_{g,a,\theta} - \langle L(u), v\rangle_{g,a,\theta,} - \langle G(u), v\rangle_{g,a,\theta,} \qquad \forall v \in H_g^1(M)^\Gamma,\end{aligned}$$

i.e., $\nabla J_g(u) = u - Lu - Gu$ where Lu, $Gu \in H_g^1(M)^\Gamma$ are the unique solutions to

$$-\mathrm{div}_g(a\nabla_g(Lu)) + \theta\,(Lu) = (\theta - b)u, \tag{C.2}$$

$$-\mathrm{div}_g(a\nabla_g(Gu)) + \theta\,(Gu) = c\,|u|^{2^*-2}\, u. \tag{C.3}$$

Observe these functions are uniquely determined by the relations

$$\langle L(u), v\rangle_{g,a\theta} = \int_M (\theta - b)uv\, dV_g, \quad \text{and} \quad \langle G(u), v\rangle_{g,a\theta} = \int_M c|u|^{2^*-2}uv\, dV_g \quad (C.4)$$

for all $v \in H_g^1(M)^\Gamma$.

The proof of the following statement was suggested to us by Jérôme Vétois. It fills in a small gap in the proof of Lemma 2.1 in [100].

Lemma C.0.1 *For every $u \in H_g^1(M)^\Gamma$,*

$$\|Lu\|_{g,a,\theta} \leq \frac{\theta - \mu}{\theta + \mu} \|u\|_{g,a,\theta}.$$

Proof. Note first that, for every $u \in H_g^1(M)^\Gamma$, coercivity of $-\mathrm{div}_g(\nabla_g) + b$ and our election of θ imply

$$\begin{aligned}
\int_M (\theta - b)u^2 dV_g &= \int_M \theta u^2 dV_g - bu^2 dV_g \\
&\leq \int_M \theta u^2 dV_g - \mu \int_M \left[a|\nabla_g u|^2 + |u|^2\right] dV_g + \int_M a\,|\nabla_g u|^2\, dV_g \\
&= (\theta - \mu) \int_M u^2 dV_g + (1 - \mu) \int_M a|\nabla_g u|^2 dV_g \\
&= (\frac{\theta - \mu}{\theta}) \int_M \theta u^2 dV_g + (\frac{\theta - \theta\mu}{\theta}) \int_M a|\nabla_g u|^2 dV_g \\
&\leq (\frac{\theta - \mu}{\theta}) \int_M \theta u^2 dV_g + (\frac{\theta - \mu}{\theta}) \int_M a|\nabla_g u|^2 dV_g \\
&= (\frac{\theta - \mu}{\theta})\|u\|^2_{g,a,\theta}
\end{aligned}$$

Hence, from (C.4) we obtain

$$\begin{aligned}
\|Lu\|^2_{g,a,\theta} = \langle L(u), L(u)\rangle_{g,a,\theta} = \int_M (\theta - b)u(Lu)\, dV_g &\leq \frac{1}{2} \int_M (\theta - b)\left[u^2 + (Lu)^2\right] dV_g \\
&\leq \frac{\theta - \mu}{2\theta} \left(\|u\|^2_{g,a,\theta} + \|Lu\|^2_{g,a,\theta}\right).
\end{aligned}$$

It follows that

$$\frac{\theta + \mu}{2\theta} \|Lu\|^2_{g,a,\theta} \leq \frac{\theta - \mu}{2\theta} \|u\|^2_{g,a,\theta},$$

as claimed. ■

As J_g is of class C^2, $-\nabla J_g : H_g^1(M)^\Gamma \to H_g^1(M)^\Gamma$ is a C^1 vector field in the whole space $H_g^1(M)^\Gamma$. Therefore, $-\nabla J_g$ is Lipschitz continuous and for any $u \in H_g^1(M)^\Gamma$ the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\psi(t, u) &= -\nabla J_g(\psi(u)) \\ \psi(0, u) &= u \end{cases},$$

has a unique solution defined for all $0 \leq t < T(u)$, where $T(u) \in (0, \infty)$ is the maximal existence time for this solution. Define $\mathcal{G} := \{(t, u) : u \in H_g^1(M)^\Gamma, 0 \leq$

$t < T(u)\}$. The negative gradient flow of J_g is the function $\psi : \mathcal{G} \to H^1_g(M)^\Gamma$ defined by the above Cauchy problem.

A subset $\mathcal{D}$ of $H^1_g(M)^\Gamma$ is called *strictly positively invariant under* ψ if

$$\psi(t, u) \in \mathrm{int}\mathcal{D} \text{ for any } u \in \mathcal{D} \text{ and any } t \in (0, T(u))$$

where $\mathrm{int}\mathcal{D}$ denotes the interior of $\mathcal{D}$ in $H^1_g(M)^\Gamma$.

Let $\mathcal{P}^\Gamma := \{u \in H^1_g(M)^\Gamma : u \geq 0\}$ be the convex cone of nonnegative functions, and

$$\mathcal{E}^\Gamma_g := \{u \in \mathcal{N}^\Gamma_g : u^+, u^- \in \mathcal{N}^\Gamma_g\},$$

where $u^+ := \max\{0, u\}$, $u^- := \min\{0, u\}$ and $\mathcal{N}^\Gamma_g$ is the Nehari manifold defined in (3.1). Note that the sign-changing solutions to problem (1.6) lie in $\mathcal{E}^\Gamma_g$.

The proof of the following lemma is similar to that of Lemma 2 in [36]. We give a sketch for the reader´s convenience.

Lemma C.0.2 *There exists $\alpha_0 > 0$ such that, for every $\alpha \in (0, \alpha_0)$,*

(a) $\left[B_\alpha(\mathcal{P}^\Gamma) \cup B_\alpha(-\mathcal{P}^\Gamma)\right] \cap \mathcal{E}^\Gamma_g = \emptyset$,

(b) $B_\alpha(\mathcal{P}^\Gamma)$ *and* $B_\alpha(-\mathcal{P}^\Gamma)$ *are strictly positively invariant.*

Proof. By symmetry considerations, it is enough to prove this only for $B_\alpha(\mathcal{P}^\Gamma)$.

(a): Note that $|u^-(p)| \leq |u(p) - v(p)|$ for every $u, v : M \to \mathbb{R}$ with $v \geq 0$, $p \in M$. Sobolev's inequality yields a positive constant C such that

$$\left|u^-\right|_{g,c,2^*} = \min_{v\in\mathcal{P}^\Gamma} |u - v|_{g,c,2^*} \leq C \min_{v\in\mathcal{P}^\Gamma} \|u - v\|_{g,a,\theta} = C\, \mathrm{dist}_\theta(u, \mathcal{P}^\Gamma) \qquad \text{(C.5)}$$

for every $u \in H^1_g(M)^\Gamma$. If $u \in \mathcal{E}^\Gamma_g$, then $u^- \in \mathcal{N}^\Gamma_g$ and, therefore, $|u^-|^{2^*}_{g,c,2^*} = mJ_g(u^-) \geq m\tau^\Gamma_g > 0$. This proves that $\mathrm{dist}_\theta(u, \mathcal{P}^\Gamma) \geq \alpha_1 > 0$ for all $u \in \mathcal{E}^\Gamma_g$.

(b): As the operator $-\mathrm{div}_g(\nabla_g) + (\theta - b)$ is strictly elliptic and θ and $\theta - b$ are nonngetaive, for any $v \in \mathcal{P}^\Gamma$, equations (C.2) and (C.3) give that $-\mathrm{div}_g(a\nabla_g(Lv)) + \theta\,(Lv) \geq 0$ and $-\mathrm{div}_g(a\nabla_g(Gv)) + \theta\,(Gv) \geq 0$. Hence, the weak maximum principle [54, Chapter 8] yields that $Lv \in \mathcal{P}^\Gamma$ and $Gv \in \mathcal{P}^\Gamma$ if $v \in \mathcal{P}^\Gamma$. As $\mathcal{P}^\Gamma$ is closed and convex in $H^1_g(M)^\Gamma$, for $u \in H^1_g(M)^\Gamma$, the orthogonal projection of u onto $\mathcal{P}^\Gamma$ $v \in \mathcal{P}^\Gamma$, is such that $\mathrm{dist}_\theta(u, \mathcal{P}^\Gamma) = \|u - v\|_{g,a,\theta}$. Then, linearity of the function L and Lemma C.0.1 yields

$$\mathrm{dist}_\theta(Lu, \mathcal{P}^\Gamma) \leq \|Lu - Lv\|_{g,a,\theta} \leq \frac{\theta - \mu}{\theta + \mu} \|u - v\|_{g,a,\theta} = \frac{\theta - \mu}{\theta + \mu} \mathrm{dist}_\theta(u, \mathcal{P}^\Gamma). \quad \text{(C.6)}$$

On the other hand, from (C.4), Hölder and Sobolev's inequality and (C.5) we get that

$$\begin{aligned}
\mathrm{dist}_\theta(Gu, \mathcal{P}^\Gamma) \left\|G(u)^-\right\|_{g,a,\theta} &\leq \left\|G(u)^-\right\|^2_{g,a,\theta} = \left\langle G(u), G(u)^-\right\rangle_{g,a,\theta} \\
&= \int_M c\,|u|^{2^*-2} uG(u)^-\, dV_g \leq \int_M c\left|u^-\right|^{2^*-2} u^- G(u)^-\, dV_g \\
&\leq \left|u^-\right|^{2^*-1}_{g,c,2^*} \left|G(u)^-\right|_{g,c,2^*} \leq C^{2^*} \mathrm{dist}_\theta(u, \mathcal{P}^\Gamma)^{2^*-1} \left\|G(u)^-\right\|_{g,a,\theta}.
\end{aligned}$$

Hence,
$$\operatorname{dist}_\theta(Gu, \mathcal{P}^\Gamma) \le C^{2^*} \operatorname{dist}_\theta(u, \mathcal{P}^\Gamma)^{2^*-1} \qquad \forall u \in H_g^1(M)^\Gamma. \tag{C.7}$$

Fix $\nu \in \left(\frac{\theta-\mu}{\theta+\mu}, 1\right)$ and let $\alpha_2 > 0$ be such that $C^{2^*}\alpha_2^{2^*-2} \le \nu - \frac{\theta-\mu}{\theta+\mu}$. Then, from inequalities (C.6) and (C.7) we obtain that

$$\operatorname{dist}_\theta(Lu + Gu, \mathcal{P}^\Gamma) \le \nu \operatorname{dist}_\theta(u, \mathcal{P}^\Gamma) \qquad \forall u \in B_\alpha(\mathcal{P}^\Gamma),$$

if $\alpha \in (0, \alpha_2)$. Therefore, $Lu + Gu \in \operatorname{int} B_\alpha(\mathcal{P}^\Gamma)$ if $u \in B_\alpha(\mathcal{P}^\Gamma)$. Since $B_\alpha(\mathcal{P}^\Gamma)$ is closed and convex, Theorem 5.2 in [40] yields that

$$\psi(t, u) \in B_\alpha(\mathcal{P}^\Gamma) \;\text{ for all }\; t \in (0, T(u)) \;\text{ if } u \in B_\alpha(\mathcal{P}^\Gamma).$$

Now we can argue as in the proof of Lemma 2 in [36] to shows that $B_\alpha(\mathcal{P}^\Gamma)$ is strictly positively invariant. ∎

Proof of Theorem 3.1.2. As the operator $-\operatorname{div}_g(\nabla_g) + b$ is coercive in $H_g^1(M)^\Gamma$ and $\dim W \ge 1$, the existence of a critical point follows from the classical mountain-pass theorem, and a well-known argument (see [12]) shows that it does not change sign. Once we have established Lemma C.0.2, the rest of the proof is completely analogous to that of Theorem 3.7 in [33]. ∎

BIBLIOGRAPHY

[1] H. Amann, J. Escher. Analysis III. *Birkhäuser Verlag AG Basel-Boston-Berlin*, 2009.

[2] A. Ambrosetti, A. Malchiodi. A multiplicity result for the Yamabe Problem on S^n. *J. Funct. Anal.* **168** (1999), 529-561.

[3] A. Ambrosetti, A. Malchiodi. Nonlinear analysis and semilinear elliptic problems. Cambridge studies in advance mathematics **104**. *Cambridge University Press*, 2007.

[4] B. Ammann, E. Humbert. The second Yamabe invariant. *J. Funct. Anal.* **235** (2006), 377-412.

[5] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl.* **55** (1976), 269-296.

[6] T. Aubin. Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. *Springer-Verlag Berlin-Heidelberg*, 1998.

[7] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain. *Comm. Pure Appl. Math.* **41** (1988), 253-294.

[8] P.Baird, J.Eells. A conservation law for harmonic maps. Geometry Symposium Utrecht 1980. Lecture Notes in Mathematics **894**, *Springer-Verlag Berlin-Heildelberg* (1981), 1-25

[9] P. Baird, J.C. Wood, Harmonic morphisms between Riemannian manifolds. London Mathematical Society Monographs. New Series,**29**.*The Clarendon Press, Oxford University Press, Oxford*, 2003.

[10] T. Bartsch, Topological Methods for Variational Problems with Symmetries. Lecture Notes in Math. **1560**, *Springer-Verlag, Berlin-Heidelberg,* 1993.

[11] T. Bartsch, M. Clapp, T. Weth. Configuration spaces, transfer, and 2-nodal solutions of a semiclassical nonlinear Schrödinger equation. *Math. Ann.* **338**, 2007, 147-185.

[12] V. Benci, G. Cerami. The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems. *Arch. Ration. Mech. Anal.* **114** (1991), 79-93

[13] M. Berti, A. Malchiodi. Non-compactness and multiplicity results for the Yamabe problem on S^n. *J. Funct. Anal.* **180** (2001), 210-241.

[14] G. E. Bredon, Introduction to compact transformation groups. Pure and Applied Mathematics **46**, *Academic Press, New York - London*, 1972

[15] S. Brendle. Blow-up phenomena for the Yamabe equation. *J. Amer. Math. Soc.* **21** (2008), 951-979.

[16] S. Brendle. On the conformal scalar curvature equation and related problems. *Surveys in Differential Geometry* **12** (2008), 1-20.

[17] S. Brendle, F. C. Marques. Blow-up phenomena for the Yamabe equation. II. *J. Differential Geom.* **81** (2009), 225-250.

[18] S. Brendle, F. C. Marques. Recent progress on the Yamabe Problem. Preprint arXiv:1010.4960 [math.DG], 2010. http://arxiv.org/abs/1010.4960

[19] H. Brezis, L. Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.*Comm. Pure Appl. Math*, **36** (1983), 437477

[20] H. Brezis, L. Yanyan. Some nonlinear elliptic equations have only constant solutions.*J. Partial Diff. Eqs.*, **19** (2006), 208-217.

[21] M. Clapp, A global compactness result for elliptic problems with critical nonlinearity on symmetric domains, in Nonlinear Equations: Methods, Models and Applications, 117-126, *Progr. Nonlinear Differential Equations Appl.* **54** Birkhauser, Boston, 2003.

[22] M. Clapp. Métodos variacionales en ecuaciones diferenciales parciales. Lecture Notes, 2016

[23] M. Clapp, J. Faya. Multiple solutions to the Bahri-Coron problem in some domains with nontrivial topology. *Proc. Amer. Math. Soc.* **141** (2013), 4339-4344.

[24] M. Clapp, J. Faya. Multiple solutions to anisotropic critical and supercritical problems in symmetric dommains, in Contributions to Nonlinear Elliptic Equations and Systems, 99-120, *Progr. Nonlinear Differential Equations Appl.* **86** Birkhäuser 2015.

[25] M. Clapp, J. Faya, A. Pistoia Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents. *Calc. Var. Partial Differ. Equ.* **48** (2013), 611623

[26] M. Clapp, J.C. Fernndez. Multiple solutions to the Bahri-Coron problem in a bounded domains without a thin neighborhood of a manifold. *Topol. Methods Nonlinear Anal.* **46** (2015), 11191137

[27] M. Clapp, J.C. Fernndez. Multiplicity of nodal solution to the Yamabe problem. Preprint.

[28] M. Clapp, M. Ghimenti, A. M. Micheletti. Solutions to a singularly perturbed supercritical elliptic equation on a Riemannian manifold concentrating at a submanifold. *J. Math.Anal.Appl.* **420 (1)** (2014), 314-333.

[29] M. Clapp, M. Grossi, A. Pistoia, Multiple solutions to the Bahri-Coron problem in domains with a shrinking hole of positive dimension. *Complex Var. Elliptic Equ.* **57** (2012), 1147–1162.

[30] M. Clapp, M. Musso, A. Pistoia, Multipeak solutions to the Bahri-Coron problem in domains with a shrinking hole. *J. Funct. Anal.* **256** (2009), 275-306.

[31] M. Clapp, F. Paccela. Multiple solutions to the pure critical exponent problem in domains with a hole of arbitrary size. *Mat. Z.* **259** (2008), 575-589.

[32] M. Clapp, A. Pistoia. Symmetries, Hopf fibrations and supercritical elliptic problems. Preprint arXiv:1501.03349v1 [math.AP]. (2015) https://arxiv.org/abs/1501.03349.

[33] M. Clapp, D. Puppe, Critical point theory with symmetries. *J. reine angew. Math.* **418** (1991), 1-29.

[34] M. Clapp, S. Tiwari Multiple solutions to a pure supercritical problem for the p-Laplacian. *Calc. Var. Partial Differ. Equ.* **55:7** (2016), 2-23.

[35] M. Clapp, T.Weth. Minimal nodal solutions of the pure critical exponent problem on a symmetric domain. *Calc. Var.* **21** (2004), 1-14.

[36] M. Clapp, T. Weth. Multiple solutions for the Brezis-Nirenberg Problem. *Adv. Differential Equations* **10** (2005), 463-480.

[37] M. Clapp, T. Weth, Two solutions of the Bahri-Coron problem in punctured domains via the fixed point transfer. *Commun. Contemp. Math.* **10** (2008), 81-101.

[38] J.M. Coron, Topologie et cas limite des injections de Sobolev. *C.R. Acad. Sc. Paris* **299**, Ser. I (1984), 209-212.

[39] T. tom Dieck, Transformation groups. De Gruyter Studies in Mathematics **8**, *Walter de Gruyter, Berlin - New York*, 1987

[40] K. Deimling. Ordinary differential equations in Banach spaces. Lecture Notes in Mathematics **596**. *Springer-Verlag, Berlin-New York* (1977).

[41] W. Ding. On a conformally invariant elliptic equation on $\mathbb{R}^n$. *Commun. Math. Phys.* **107** (1986), 331-335.

[42] W. Ding. Positive solutions of $\Delta u + u^{2^*-1} = 0$ on contractible domains. *J. Partial Diff. Eq.* **2** (1989), 83-88.

[43] Z. Djadli, A. Jourdain, Nodal solutions for scalar curvature type equations with perturbation terms on compact Riemannian manifolds. *Boll. Unione Mat. Ital. Serie 8* **5-B** (2002), 205226.

[44] F. Dobarro, E. Lami Dozo. Scalar curvature and warped products of Riemann manifolds. Trans. Amer. Math. Soc. 303 (1987), no. 1, 161–168.

[45] M. Do Carmo, Riemannian Geometry. Mathematics: Theory and Applications. *Birkhuser Boston-Basel-Berlin* 1992.

[46] A. Dold, The fixed point transfer of fibre-preserving maps. *Math. Z.* **148** (1976), 215-244.

[47] A. Dold, Lectures on algebraic topology. Second edition. Grundlehren der Mathematischen Wissenschaften **200**. *Springer-Verlag, Berlin-New York*, 1980.

[48] O. Druet Generalized scalar curvature type equations on compact Riemannian manifolds. *Proc. Roy Soc. Edinburgh,* **130A** (2000), 767-788.

[49] O. Druet, E. Hebey, F. Robert. Blow-up theory for elliptic PDE's in Riemannian geometry. Mathematical Notes. *Princeton University Press* 2004

[50] I. Ekeland, On the variational principle. *J. Math. Anal. Appl.* **47** (1974), 324–353.

[51] S. El Sayed. Second eigenvalue of the Yamabe operator and applications. *Calc. Var. Partial Differential Equations.* **50** (2014), 665-692.

[52] P. Esposito, A. Pistoia, J. Vétois. Blow-up solutions for linear perturbations of the Yamabe equation. Concentration Analysis and Applications to PDE: ICTS Workshop, Bangalore, January 2012. Trends in Mathematics. *Birkhäuser* 2013, 29-47

[53] Y. Ge, M. Musso, A. Pistoia, Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains. *Comm. Partial Differential Equations* **35** (2010), 1419-1457.

[54] D. Gilbarg, N. S. Trudinger Elliptic partial differential equations of second order. *Classics in Mathematics.* Springer-Verlag Berlin Heidelberg New York, 2001.

[55] V. Guillemin, A. Pollack. Differential Topology. AMS Chelsea Publishing. *American mathematical Society* 1974.

[56] E. Hebey, Introduction à l'ànlyse non linéaire sur les variétés. *Diderot Multimedia* 1997.

[57] E. Hebey, Nonlinear analysis on manifolds: Sobolev spaces and inequalities. Courant Lecture Notes **5**, *American mathematical Society* 1999.

[58] E. Hebey. From the Yamabe problem to the equivariant Yamabe problem. *Séminaires et Congrès SMF* **1** (1996), 377-402.

[59] E. Hebey, M. Vaugon. Le problème de Yamabe équivariant. *Bull. Sc. math.* **117** (1993), 241-286.

[60] G. Henry. Second Yamabe constant on Riemannian products. Preprint arXiv:1505.00981v1 [math.DG] (2015). http://arxiv.org/abs/1505.00981

[61] N. Hirano, N. Shioji, Existence of two solutions for the Bahri-Coron problem in an annular domain with a thin hole. *J. Funct. Anal.* **261** (2011), 3612–3632.

[62] D. Holcman. Solutions nodales sur les variétés Riemanniennes. *J. Funct. Anal.* **161** (1999), 219-245.

[63] D. Holcman. Nonlinear PDE with vector fields. *J. Anal. Math.* **81** (2000), 111-137.

[64] J. Jost. Postmodern analysis. Universitext. *Springer-Verlag Berlin Heidelberg* (2005)

[65] A. Jourdain, Solutions nodales pour des équations dy type courbure scalaire sur la sphère. *Bull. Sci. math.*, **123** (1999), 299-327.

[66] J. Kazdan, F. Warner. Remarks on some quasilinear elliptic equations. *Comm. Pure Appl. Math.* **38** (1975), 557-569.

[67] M. A. Khuri, F. C. Marques, R. M. Schoen. A compactness theorem for the Yamabe problem. *J. Differential Geom.* **81** (2009), 143-196.

[68] S. Kobayashi. Transformation Groups in Differential Geometry. Classics in Mathematics, *Springer-Verlag Berlin Heidelberg* 1995

[69] J. Lee, T. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)* **17** (1987), 37-91.

[70] J. Le, Introduction to Smooth Manifolds. Graduate Texts in Mathematics **218**, *Springer Science+Bussiness Media* 2013.

[71] J. Le, Riemannian Manifolds: an introduction to curvature. Graduate Texts in Mathematics **176**,*Springer-Verlag, New York* 1997.

[72] R. Lewandowski, Little holes and convergence of solutions of $-\Delta u = u^{\frac{N+2}{N-2}}$. *Nonlinear Anal.* **14** (1990), 873-888.

[73] G. Li, S. Yan, J. Yang, An elliptic problem with critical growth in domains with shrinking holes. *J. Diff Eqns.* **198** (2004), 275-300.

[74] E. Loubeau. On p-harmonic morphisms. *Differential Geom. Appl.* **12**, (2000) 219229

[75] E. Loubeau, J. M. Burel. p-harmonic morphisms: the $1 < p < 2$ case and some nontrivial exmaples. *Contemp. Math.* 308 (2002), 21-37.

[76] J. R. Magnus, H. Neudecker, Matrix Differential Calculus with Applications to Statistiscs and Econometrics. *John Wyley & Sons Ltd.* 2007.

[77] M.V. Marchi, F. Pacella. On the existence of nodal solutions of the equation $-\Delta u = |u|^{2^*-2}u$ with Dirichlet boundary conditions. *Diff. Int. Eq.* **6** (1993), 849- 862.

[78] F. C. Marques Recent developments on the Yamabe problem. *Matemtica Contempornea,* **35** (2008), 115-130.

[79] M. Musso, A. Pistoia, Sign changing solutions to a Bahri-Coron problem in pierced domains. *Discr. Contin. Dyn. Syst.* **21** (2008), 295-306.

[80] F. Morabito, A. Pistoia, G. Vaira. Towering phenomena for the Yamabe equation on symmetric manifolds. Preprint arXiv:1603.01538 [math.AP] (2016). http://arxiv.org/abs/1603.01538

[81] R.S. Palais, The principle of symmetric criticality. *Comm. Math. Phys.* **69** (1979), 19-30

[82] D. Passaseo. Multiplicity of positive solutions of nonlinear elliptic equations with critical Sobolev exponent in some contractible domains. *Manuscripta Math.* **65** (1989), 147-165.

[83] J. Petean. On nodal solutions of the Yamabe equation on products. *J. Geom. Phys.* **59** (2009), 1395-1401.

[84] A. Pistoia, G. Vaira. Clustering phenomena for linear perturbation of the Yamabe equation. Preprint arXiv:1511.07028 [math.AP] (2015). https://arxiv.org/abs/1511.07028

[85] S.I. Pohozhaev. Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. *Soviet Math. Dokl.* **6** (1965), 1408-1411.

[86] D. Pollack. Nonuniqueness and high energy solutions for a conformally invariant scalar equation. *Comm. Anal. Geom.* **1** (1993), 347-414.

[87] M. Obata. The conjectures on conformal transformations of Riemannian manifolds. *J. Differential Geom.* **6** (1971), 265-270

[88] Y. Ou. f-harmonic horphisms between Riemannian manifolds. *Chin. Ann. Math.* **35B(2)** (2014), 225236

[89] O. REY, Sur un probléme variationnel non compact: L'effect de petits trous dans le domain,*C.R. Acad. Sci. Paris* **308** (1986), 349-352.

[90] B. RUF, P.N. SRIKANTH. Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit. *J. Eur. Math. Soc. (JEMS)* **12**, (2010) 413427.

[91] F. ROBERT, J. VÉTOIS. Sign-changing solutions to elliptic second order equations: glueing a peak to a degenerate critical manifold. *Calc. Var. Partial Differential Equations.* **54** (2015), 693-716

[92] N. SAINTIER. Blow-up theory for symmetric critical equations involving the p-Laplacian. *Nonlinear Differ. Equ. Appl.* **15** (2008), 227245

[93] E. SPANIER, Algebraic Topology. *Springer-Verlag New York, Inc.,* 1966.

[94] M. STRUWE, A global compactness result for elliptic boundary value problems involving limiting nonlinearities. *Math. Z.* **187** (1984), 511–517.

[95] M. STRUWE,Variational methods. Ergebnisse der Mathematik und ihrer Grenzgebiete **34**,*Springer Verlag Berlin-Heidelberg*, 2008.

[96] R. SCHOEN. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.* **20** (1984), 479-495.

[97] R. SCHOEN, S. T. YAU. On the proof of the positive mass conjecture in general relativity. *Commun. Math. Phys.* **65** (1979), 46-76.

[98] R. SCHOEN, S. T. YAU. Proof of the positive mass Theorem. II. *Commun. Math. Phys.* **79** (1981), 231-260.

[99] N. TRUDINGER. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 22 (1968), 265-274.

[100] J. VÉTOIS. Multiple solutions for nonlinear elliptic equations on compact manifolds.*Int. J. Mat.* **18** (2007), 10711111.

[101] T. WETH, Energy bounds for entire nodal solutions of autonomous superlinear equations. *Calc. Var. Partial Differential Equations* **27** (2006), 421-437.

[102] M. WILLEM, Minimax theorems. *Progress in Nonlinear Differential Equations and their Applications***24**. Birkhäuser Boston Inc., Boston MA, 1996.

[103] H. YAMABE. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.* 12 (1960), 21-37.