Universidad Nacional Autónoma de México
Posgrado en Ciencias de la Tierra
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Centro de Ciencias de la Atmósfera

A Simple Earth System Model to Simulate Past Glacial Cycles

A Dynamical Systems Approach

TESIS
QUE PARA OPTAR POR EL GRADO DE
MAESTRO EN CIENCIAS
(Física de la Atmósfera)

presenta:
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DISSERTATION
to aim for the degree of
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Preface

The intriguing problem of Earth's glacial cycles was analysed in this work with a dynamical systems approach. This work puts forward a conceptual box model of Earth System — that encompasses Carbon and Energy Subsystems — in which the importance of internal components in long timescales is revealed by stressing that simulated Carbon Subsystem is enough to drive glacial cycles and its internal flux variations could explain transitional events such as Mid-Pleistocene Transition.

This work — represented by this written report — explains thoroughly and in a clear manner — the author wants to believe — every aspect of the model. Therefore, one can expect a basic structure in the chapters and sections that follow: there will be some introductory ideas. Only then, ideas will be developed in deep, for having no logical jumps or mischievous assumptions, since the approach taken in this research is not quite standard in the field. Therefore, maybe reader would have some questions once he/she has read introductory ideas, this questions will be answered duly when we delve deep into those.

Ideas and explanations — that have their origin on phenomenological and mathematical reasoning — shall be rewritten in hypotheses, axioms, theorems, propositions and the like; in order to have a clear and readable summary once the reasoning is sound in the mind of the reader.

It is worth to remark the difference between axioms and hypotheses. Hypotheses are suppositions concerning phenomenological models and axioms are assumptions made about the mathematical model. That is, while hypotheses talk about nature itself, axioms are translation rules of natural principles into mathematical objects. This is a point that has not been emphazised enough in physical sciences, not only in an epistemological way but in a practical and interpretative one.

Once given an overview on the form in which information is presented, it is also necessary to give a prospect of the themes that this monography will cover. First, it is presented the particular problem accompanied by a short overview on the literature about it and a proper discussion by the author. In addition, it shall be shown information about the original problem that encouraged the development of the model: Vostok Ice Core time series. This panorama is closed by a statement in which lie the guide behind the construction of the model.

Following chapters describe thoroughly the models of chosen subsystems. The presen-

tation leave no doubts about their construction. Building of Carbon Subsystem Model is not standard, in the sense that it centers on the relationships and phenomenology between reservoirs, in opposition to the complexity shown in some recent models: complexity with its origin in a process-oriented analysis. This difference in focus leads us to another way to learn about the system, that shows clearly the reasons about why past and present models (and future ones, if paradigm does not change) could not capture the workings of Earth System in all its glory: they have minimized the effects that the Earth can have over herself, e.g. life (profoundly intertwined in Carbon and Water Cycles), and have detailed descriptions of physical processes, in spite of evidence that long-term evolution of Earth's climate is modulated by the Subsystems on its own right, as is shown in the first chapter of this thesis. Finally, the model in that way constructed is applied to the situation that gave rise to it: Vostok Ice Core. Results show that idealized reconstruction generate some of the observed periodicities (e.g. 100ka) and also show the observed assymetries: fast heating towards interglacial period and slow cooling en route to glacial period. It is concluded that reconstruction is good despite the model simplicity.

In this way, present work pretends to show that explanations for glacial cycles —and any type of periodicities and transitions between them— on the palæoclimatic record must be seen as consequence to structural changes in the way in which planet works. This changes lead it to states consistent with the main external forcing. Maybe some of this changes are made by this forcing, nonetheless transitional states and final state are determined by the Earth System itself. With this perspective and the methods presented to study the dynamical system just modeled, author believes that is of primary interest to use extensively this type of analysis for simple models in order to widen our understanding in terms of relationships between processes: we already know a large part of the biology, geology, physics and chemistry behind Earth System workings, but we do not know how processes interact. Is it not worth to unravel this relationships which give rise to new unexpected phenomena? Is this not the key to study the Earth as a whole, instead of a ensemble of processes without relationship? Would this not better our understanding in order to gain more knowledge of our planet and to think in realistic ways to mitigate anthropogenic contribution to climate change?

From the author's point of view, this work has not only a value for the palæoclimatology, but an episthemological and methodological value for Earth Sciences in general —and Atmospheric Sciences in particular— where, given the desirable multidisciplinary and interdisciplinary approach, there are fundamental problems in the application of mathematical methods, either by the informality in the way they are applied or because they are not adequate. But above all, there is a lack of awareness of certain methods that could help to delve deep into the intrinsic complexity of our studies (through modeling) of a complex system like, without doubt, a planet.

Prefacio

El interesante problema de los ciclos glaciales de la Tierra ha sido analizado en este trabajo desde una perspectiva de sistemas dinámicos. Este trabajo propone un modelo del Sistema Tierra —que abarca los Subsistemas Carbono y Energía— en el que la importancia de los componentes internos en escalas de tiempo largas se revela al hacer incapié en que el Subsistema Carbono es suficiente para controlar los ciclos glaciales y que las variaciones internas en sus flujos pueden explicar los eventos de transición tales como la Transición del Pleistoceno Medio.

Este trabajo —cuyo producto es este reporte escrito— explica a cabalidad y en una forma clara —según piensa el autor— cada aspecto del modelo. Por tanto, se puede esperar una estructura básica en los capítulos y secciones, a saber: se presentarán ideas introductorias. Una vez hecho eso, las ideas se desarrollarán en profundidad, para impedir saltos lógicos o supuestos erróneos, dado que la perspectiva usada en esta investigación no es la estándar en el campo de estudio. Así, si el lector tuviera preguntas una vez ha leído las ideas introductorias, esas preguntas serán respondidas satisfactoriamente cuando indaguemos profundamente en ellas.

Las ideas y explicaciones —que tengan su origen en razonamientos fenomenológicos y matemáticos— serán reescritas en hipótesis, axiomas, teoremas, proposiciones y similares; de manera que se tenga un resumen claro y legible una vez los razonamientos hayan cristalizado en la mente del lector.

Es importante destacar la diferencia entre axioma e hipótesis. Las hipótesis son suposiciones que le conciernen a los modelos fenomenológicos y los axiomas son supuestos hechos en el contexto del modelo matemático. Mientras las hipótesis hablan de la naturaleza como tal, los axiomas son reglas de traducción de éstos principios naturales en objetos matemáticos. Éste es un punto en el que no se ha hecho suficiente énfasis en las ciencias físicas, no sólo en un sentido espistemológico sino en los sentidos práctico e interpretativo.

Una vez dada someramente la forma en que se presenta el material, también daremos un panorama de los temas que se cubrirán en el texto. Por principio, se presenta el problema en particular acompañado de una revisión de la literatura sobre el tema y una discusión de ésta por parte del autor. Además, se mostrará el problema original que motivó el desarrollo del modelo: las series de tiempo del Barreno Glacial de Vostok. Este panorama se cierra

con una declaración de intenciones que se siguió para el diseño del modelo.

En los siguientes capítulos se describen exhaustivamente los modelos de los subsistemas modelados, no dejando dudas sobre su construcción. La construcción del modelo del Subsistema Carbono no es estándar en el sentido de que se centra en las relaciones y la fenomenología entre los reservorios, a diferencia del enfoque centrado en el complejísimo entramado de relaciones que se usa en muchos de los modelos de la actualidad. Esta diferencia en enfoque nos permite conocer de otra forma el mismo sistema, pero así presentado nos permite ver claramente las razones por las que los modelos pasados y actuales (y también los futuros, si es que no se cambia de paradigma) no han podido capturar del todo el funcionamiento del Sistema Tierra: han minimizado los efectos que la propia maquinaria de la Tierra puede tener sobre sí misma, en particular la vida (muy asociada al Ciclo de Carbono y al Ciclo Hidrológico), detallando más que nada los aspectos físicos; aún cuando hay evidencias claras que nos indican que la evolución a largo plazo del clima de la Tierra se da con la participación fundamental de los propios Subsistemas, como se menciona en el capítulo primero. Finalmente el modelo construido se aplica a la situación que lo motivó, el Barreno Glacial de Vostok, para reconstruir a primer orden las series de tiempo de Carbono y Temperatura. Los resultados muestran que la reconstrucción, aunque simple, genera algunas de las periodicidades observadas (en particular, la principal de 100ka) —además de mostrar las asimetrías observadas en el registro: calentamiento rápido desde la glaciación y enfriamiento lento desde el período interglacial. Se concluyó que tal reconstrucción es bastante buena para la simplicidad del modelo.

Es así como, con este trabajo, se pretende mostrar que las explicaciones para los periodos glaciales y en realidad para cualquier tipo de periodicidades —y transiciones entre periodicidades— en el registro paleoclimático deben de ser vistas como consecuencia de cambios estructurales en la forma en que el planeta funciona, que lo llevan a estados consistentes con el forzador externo principal. Tal vez algunos de esos cambios estructurales son iniciados por el forzador externo, mas los estados transicionales y el estado final están determinados por el Sistema Tierra. Con esta perspectiva y los métodos presentados para estudiar el sistema dinámico modelado, el autor cree que es de capital importancia comenzar a usar de manera extensiva este tipo de análisis para modelos simples de forma que se amplie nuestro entendimiento de las relaciones entre procesos: ya conocemos la biología, la geología, la física y la química detrás de los procesos en el sistema Tierra, pero no sabemos como es que esos procesos interactúan ¿No acaso valdría la pena desentrañar esas relaciones que dan origen a nuevos fenómenos inesperados? ¿No es esa la clave para comenzar a estudiar la Tierra como un todo y no como un conjunto de procesos por separado? ¿No mejoraría ésto nuestra visión para poder tener un mayor conocimiento del planeta y poder pensar en verdaderas formas de mitigación del cambio climático antropogénico?

Así, el trabajo —desde el punto de vista del autor— no sólo tiene un valor como tal para el campo de la paleoclimatología, sino un valor epistemológico y metodológico para las Ciencias de la Tierra en general —y las Ciencias Atmosféricas, en particular— donde, dado el carácter teórico de ser multidisciplinarias e interdisciplinarias, hay problemas fundamentales de aplicación de métodos matemáticos, ya sea por la informalidad con la que se aplican o porque no son adecuados. Pero sobre todo, hay un desconocimiento de ciertos métodos que podrían ayudar a desentrañar —como dijimos— la complejidad inherente en el estudio (a través de la modelación) de un sistema complejo —como lo es, sin duda, un planeta.

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Part I

Prelude

1 State-of-the-Art and principles.

> Die Wahrheit triumphiert nie, sondern ihre Gegner sterben nur aus.
>
> *(Rephrased quote of Max Planck (1858-1947))*

1.1 Introduction.

Throughout Earth's history there are examples of climate change, sometimes severe. Earliest registered glaciation — Huronian glaciation in the Siderian and Rhyacian periods of Palæoproterozoic era, circa 2400Ma before present — take place in the context of the changing young Earth – in which biogeochemical cycles were changing rapidly due to processes such mantle plumes and weathering – the relative-to-present faint young Sun and the rise of aerobic organisms. These factors made the onset of a glacial age a matter of reorganization of the biogeochemical cycles (Melezhik, 2006): CO_2 was sequestered by silicate weathering of basalts coming from massive floods caused by mantle plumes, which in turn reinforced carbon burial due to flourishment of aerobic organisms and CH_4 was nearly eliminated by aerobic organisms' byproduct, namely O_2. What brought about Huronian glaciation was an unusual combination of factors that next glaciations — Sturtian and Marinoan glaciations — occur until similar conditions were met in the Cryogenian period of Neoproterozoic era, circa 700Ma before present (Melezhik, 2006). Huronian, Sturtian and Marinoan glaciations are theorized to have been so severe that possibly Earth was entirely covered by snow and ice — Snowball Earth theory — or it was covered aside from a equatorial channel — a Slushball Earth.

Following these ancient events, three glaciations occur in Phanerozoic eon — the present eon — which are the Andean-Saharan — circa 450Ma before present — Karoo and Pleistocene glaciations. Karoo glaciation — circa 300Ma before present — has a trigger in common with Huronian glaciation: an oxigenation event (Berner, 1999).

More recently — in the late Pliocene epoch of the Neogene period and Pleistocene epoch of the Quaternary period, both in the Cenozoic era, 3Ma before present — the last glacial age was established. This glaciation is the most studied of all known glaciations, since it is the most recent and there are proxies and evidence, which are still well-preserved. From this

evidence, it is possible to say that climate in Pleistocene glaciation have evolved through intervals with extensive ice-covered surface — the so-called glacial periods — and others — which are called interglacial periods — with narrower ice cover and milder temperatures; glacial and interglacial periods form glacial cycles. In fact, present climate constitutes the last interglacial period of the last glaciation. Glacial cycles have been also identified in Karoo glaciation (Soreghan and Giles, 1999); therefore, they are not a peculiarity of Pleistocene glacial age.

An idea that has been popular as a driver of glacial cycles was proposed since XIX century: variations of Earth's orbital parameters could be responsible for glacial cycles. Earlier proponents were french mathematician Joseph Adhémar and scottish scientist James Croll (as quoted by Paillard, 2001). This idea was not formalized until the seminal works by Milankovic on the orbital theory of climate (Milankovic, 1920, 1930, 1941). This theory takes into account that variations in orbital parameters — mainly eccentricity (e), obliquity (ϵ) and precession (measured by the sine of the longitude of the periapsis ϖ) — make Earth's received solar radiation a function of time, in particular the insolation in the Northern Hemisphere. Thus, insolation forcing triggers a feedback based on ice-albedo mechanism, which is implemented in classical models like those by Budyko, (1969) and Sellers, (1969).

However, this theory was not accepted thoroughly until high resolution proxies came — like deep-ocean cores and ice cores — and the work by Hays et al., (1976) revealed the fingerprints of the variations of $e, \epsilon, \sin\varpi$ in deep-ocean proxies: periodicities of glacial cycles seem to be in accordance with those of orbital parameters (including those from Karoo glaciation — that show 100ka periodicity associated with e — as shown by Belt et al., 2015).

Nevertheless, it is not necessarily true that, because fingerprints are present in proxies, orbital forcing is the driver of glacial cycles. In fact, a common problem that is elusive for orbital theory of climate is the existence of transitional events, in which the dominant periodicity of glacial cycles change. One of such transitional events is the *mid-Pleistocene Transition* (Ashkenazy and Tziperman, 2004; Huybers, 2007; Tziperman and Gildor, 2003) — which ocurred in-between the Pleistocene glaciation, circa 1Ma ago — characterized by a change from 41ka periodicity (associated with ϵ) to 100ka cycles (related to e), an asymmetrization of glacial cycles — which were essentially symmetric until that time — and with greater amplitude. Oscillation of orbital parameters has been always present and, therefore, it is deeply imprinted in Earth System. Thus, it is not strange that we find this orbital periodicities in palæoclimate proxies.

Then, the question is about what selects dominant periodicities and if orbital forcing is only a trigger instead of driver. The obvious candidates to tackle these points come from Earth System's internal factors: some of them are ice-sheet dynamics, Ocean circulation,

greenhouse gases (GHG) and inherent nonlinear responses (Huybers, 2007; Ikeda and Tajika, 1999; Paillard, 2001; Tajika, 1998). Efforts in order to incorporate these factors into a model have been put forward in classical works by Källén et al., (1979) and Saltzman and Maasch, (1988) or the comprehensive model shown recently by Fowler et al., (2013). Another candidate could be an external factor: the intrinsic variation of solar activity across history, such as in the Huronian glaciation.

The role of GHG over climate has been acknowledged since XIX century — for example, by Fourier, (1824),Tyndall, (1861) and Arrhenius, (1896) — until the most recent IPCC assessment reports. However, their impact over glacial cycles remains obscure.

We propose a simple model of Earth System that could account for the 100ka periodicity and assymetries depicted in the data from Vostok proxy (Petit et al., 1999) and for transitional events without relying upon astronomical and astrophysical factors. This model is based on the systems theory approach due to the International Geosphere and Biosphere Programme (IGBP) (Falkowski et al., 2000; Steffen, 2000). The aim is to show that Earth System's internal mechanisms — in particular, those related to GHG — could drive glacial cycles and not only serve as an amplifying mechanism of orbital forcing.

Some efforts in this way have been put forward by Hogg, (2008), who proposed a model of atmospheric carbon that depends on surface temperature (T). However, orbital parameters already drive Carbon Subsystem.

1.2 The Vostok proxy

We give a brief description of the Vostok ice core (Petit et al., 1999). This proxy has three time series: one for CO_2, another for CH_4 and the third for the anomaly of T (ΔT). We shall focus in time series of atmospheric CO_2 — which is measured by concentration ($[CO_2]$) in ppmv — and ΔT — measured in $^\circ$C, with respect to the normal in Vostok Station, Antarctica.

$[CO_2]$ time series was obtained from air trapped inside bubbles within ice as a result of glacier formation, while ΔT series relies on the fact that H_2O enthalpy of evaporation varies with its isotopic composition, therefore one can measure ratios of $\delta^{18}O$ to $\delta^{16}O$ — or δD to δH — to obtain an empirical function of T which can be inverted to obtain T as a function of these ratios, as described by Petit et al., (1999). Depth-age correlation and data were downloaded from NOAA-NGDC (Petit et al., 1999).

In figure 1, we present both time series with time labeled from zero — the deepest data and, therefore, the oldest one — to the most recent data.

Time series show the following features (Falkowski et al., 2000; McGehee and Lehman, 2012; Petit et al., 1999; Steffen, 2000):

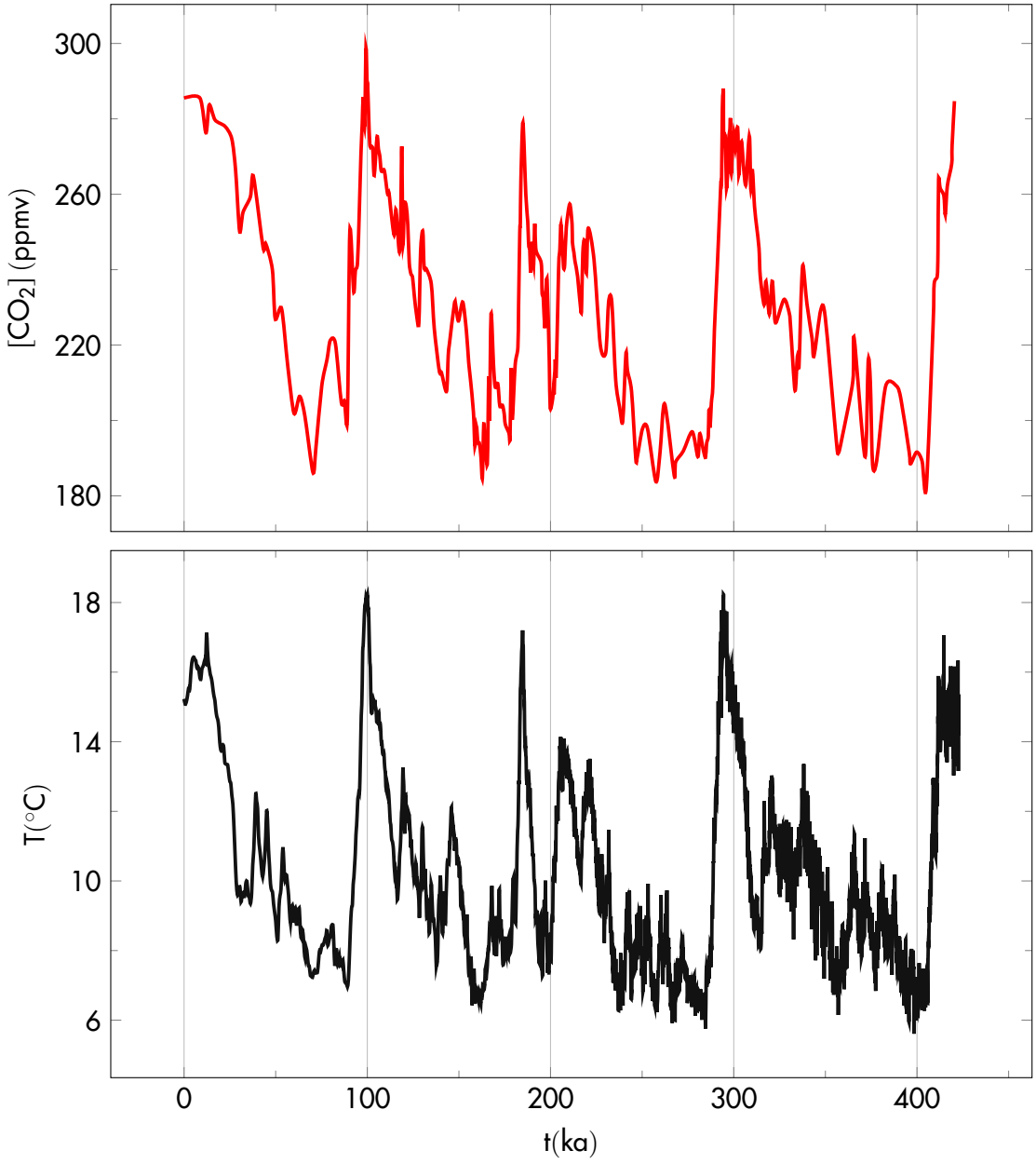

Figure 1: Time series from Vostok proxy, Petit et al., (1999). $T = \overline{T} + \Delta T$ — where $\overline{T} = 15°C$ is global annual average temperature — and $[CO_2]$. Time is set in *kiloannum*.

Periodicity. There are four cycles with five maxima and four minima, which are evenly distributed in time.

Almost in phase. As Garduño et al., (2005) remark, extrema are matched over time. This hints a correlation between ΔT and $[CO_2]$ across a millenial time scale.

Boundedness. This feature, together with periodicity, means that Earth System oscillates between two states in this lapse (Petit et al., 1999; Steffen, 2000).

Assymetries. Evolution from a glacial to an interglacial period is faster than passage in the converse direction: heating ocurrs very fast, while cooling is slow. Also, it is possible to observe that glacial periods are wider in time than integlacials.

Qualitative explanation of these features has been put forward by Falkowski et al., (2000) and Steffen, (2000). This explanation may be summarized in the following terms: starting in an interglacial period, T and $[CO_2]$ are at their highest. Increased precipitation causes increased biological activity. Then, carbon is transferred to Continent (emerged lands) from Atmosphere (troposphere) until a saturation level is reached. At that point — triggered by runoff and airborne dust — the Ocean (saline waters that surround Continent) absorbs CO_2, while its biological activity gradually recovers — because of formerly described sowing by Continent — and makes more space for absorbing carbon. T and CO_2 drop until the Ocean reaches its saturation level and the process is now reversed. This "control switching" between Ocean and Continent — mediated by Atmosphere — is the main driver of glacial cycles.

Part II

Earth System Model

2 Preliminaries.

> Oft macht man eine Bemerkung und sieht erst später, wie wahr sie ist.
>
> *(Ludwig Wittgenstein (1889-1951))*

The modeling idea behind present work is to divide Earth in subsystems, which will interact to produce climate.

Actual modeled subsystems described here are Energy and Carbon. Energy is represented by an energy balance and Carbon is depicted by a model of global carbon cycle based on qualitative ideas by Steffen, (2000).

Carbon Subsystem has been selected, because it is portrayed in Vostok proxy. Thus, this selection has not been done for we consider Carbon Subsystem be more important than Water Subsystem: it is well known this is not the case, since water vapor is the most important GHG and also water has a decisive role through contributions of cloud cover, snow and ice to albedo.

However, it is clear enough that Carbon and Water Subsystems have key factors that are connected through Clausius-Clapeyron relation (which links phase changes to state changes): when there is some increment on Atmospheric Carbon, Energy Subsystem is modified in such a way that there is a T increment due to a rise in internal energy. This allows for an increment on evaporation, since more liquid water molecules overcome intermolecular forces due to this increment in internal energy — and which makes latent heat dependent on T. Freed water molecules add up to water vapor amount in the Atmosphere, which further modify Energy Subsystem by acting as a feedback on carbon-dependent initial boost on T. But, the new T increment rises Ocean temperature and makes that Oceanic water has lesser dissolving capacity for gases — because Henry's law solubility constant depends on temperature — which causes outgassing that increments Atmospheric Carbon. This closes the feedback of Carbon Subsystem into Water Subsystem, as well as the converse feedback.

Energy Subsystem is key to understand the dynamics of weather and climate, since dynamics is closely interwoven with thermodynamics. That is, circulation — in short- and long-term — relies heavily on balances and imbalances due to thermal energy gradients and these gradients are established because of solar radiation — and its absortion by

Earth's surface — is not equally distributed.

However, as thermodynamics is subsidiary to dynamics in short timescales (meteorology), it is also true that thermodynamics dominate over dynamics in long timescales (climate). That is thermodynamics is commonly parametrized in short time scales and dynamics is subject to constitutive relations in large time scales. An example of this modeling idea is pivotal in the model by Adem, (1962).

In the present case, we consider that — in the millenial timescales of Vostok proxy — ΔT is representative of a global anomaly of T — because time step is wide enough to think that Atmosphere is well-mixed. Therefore, we make use of global energy budget in terms of the radiation that is captured from the Sun.

3 Modeling Carbon Subsystem.

> Wahrlich ist es nicht das Wissen, sondern das Lernen, nicht das Besitzen, sondern das Erwerben, nicht das Da-Seyn, sondern das Hinkommen, was den größten Genuss gewährt.
>
> *(Carl Friedrich Gauß (1777-1855))*

Global carbon cycle is complex, because of the many different processes that transform carbon between several chemical compounds — and not to mention its key relationship to life — most of which relies heavily upon carbon-based biochemistry. In fact, complexity is more evident when one looks deeply into Oceanic and Continental parts of Carbon Subsystem, which resemble a box full of even more little boxes connected with a complex mesh of tubes, representing every kind of transformation between different carbon chemical compounds that takes place within Ocean or Continent: due to phenomena of physical or chemical nature or a combination of both (mainly biological phenomena).

None of this complex internal dynamics is the purpose of proposed model — because precise knowledge of this adds up nothing, but would obscure what we want to show — and will be simplified in the following manner. As one can fix the limits of a thermodynamic system — selecting the actual system under study in an arbitrary way and guiding this freedom by the purpose of our scientific inquiries — we will define precisely which part of Carbon Subsystem will be modeled. If we were to take the complete subsystem, we must take into account necessarily every single process and introduce full complexity. Thus, we formulate the following

Hypothesis 1 *We constrain the modeled Carbon Subsystem to Atmospheric reservoir and Continental and Oceanic subreservoirs that interact directly with its surroundings: that is, those Continental and Oceanic subreservoirs having carbon stock readily available to exchange with the outside of its parent reservoir (Continental or Oceanic).*

Graphical representation — for the sake of clarity — is shown in the figure 2.

This hypothesis has an important consequence: this model of Carbon Subsystem shall not be conservative with respect to total carbon stock as becomes apparent from construction

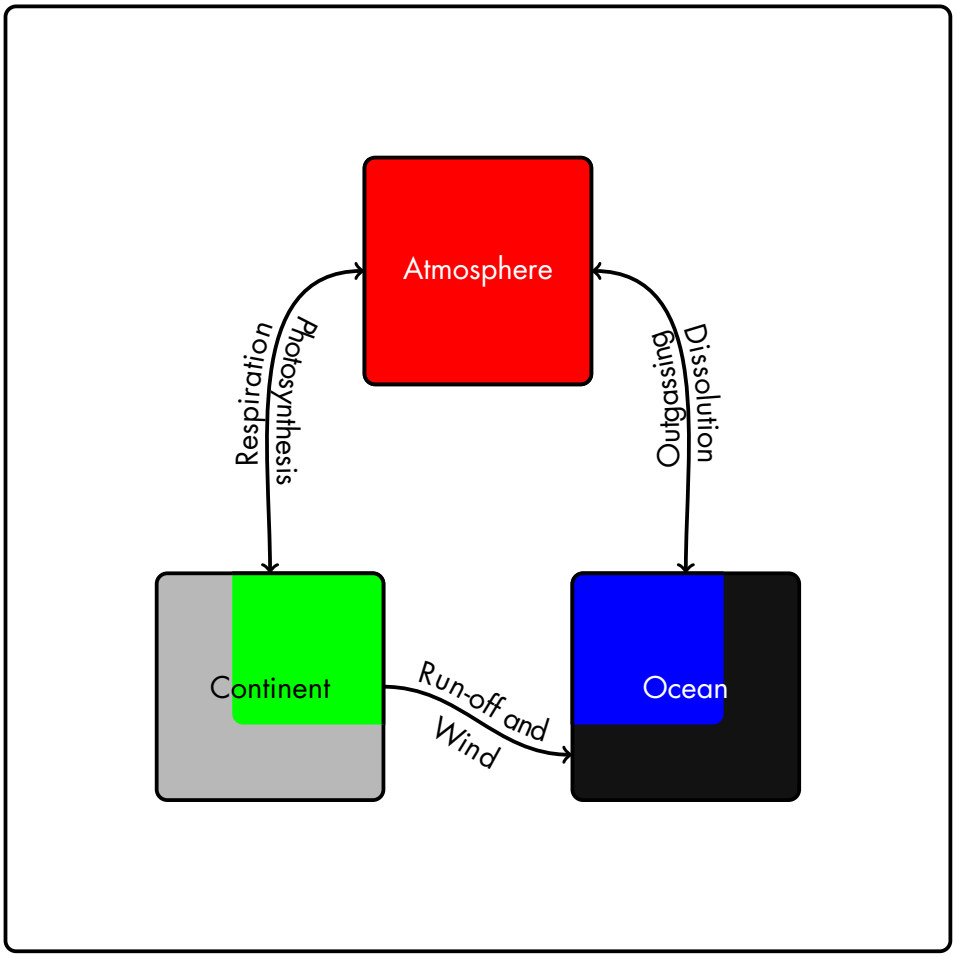

Figure 2: Carbon Subsystem. Actual modeled subsystem in color.

of the hypothesis and the related figure.

In spite of the former consideration, we must make another remark to ease modeling tasks. Since we want not to deal with various carbon compounds — which arise in Continental case — then we do not distinguish between compounds or their phases. That means, we put forward a generic carbon stock.

Hypothesis 2 *Generic carbon is the carbon which each reservoir — as defined in hypothesis 1 — have as stock, independently of chemical compounds it forms.*

This is completely valid, because we will see in a while that the processes linking reservoirs and interchanging stocks are clearly not intertwined with respect to carbon compounds.

The former hypotheses 1 and 2 make the description of Carbon Subsystem more elegant and susceptible of extensions as we will see next, when we proceed to build mathematical model from phenomenological grounds.

We have three reservoirs, whose stocks are going to flow between each other. This flows are determined by the nature of the processes that drive them. The first thing to consider is the following rule

Axiom 1 *Every single flow between reservoirs can be of one of three sorts, with respect to controlling stock: donor-controlled flow, recipient-controlled flow, donor- and recipient-controlled flow (or interaction flow).*

Then, we must analyze phenomena in order to choose a class for each flow in the modeled subsystem. Let us begin with Continental stock and we look at the relationships to Atmospheric and Oceanic stocks. Relationship with Atmosphere comes through processes that rely on weathering processes and biological activity, since basis of Continental ecosystems and food chains are plants which need light and CO_2 to produce more complex carbon compounds, in order to nurture higher levels of food chain. However, the relationship is also on the side of Continent acting as a donor for the decomposition of organic matter and respiration of organisms and other physical and chemical processes such as fires and volcanic eruptions (in a lesser extent). Thus, Atmospheric-Continental flow of carbon depends on both Atmospheric and Continental stocks: it is an interaction flow. Relationship with Ocean is special and we will tackle it from two perspectives. For now, we will take the perspective of Continental stock. How does carbon flow from Continent to Ocean? Organic matter due to Continental biological productivity and inorganic carbon reach Ocean through run-off mechanism mainly and secondarily through dust blown by winds. Then, Ocean has no direct way to intervene this processes. Thus, we conclude that Oceanic-Continental flow — as seen from Continental perspective — relies only on Continental stock: it is a donor-controlled flow.

Next, we seek for an analogous analysis from Ocean's perspective. Connection to At-

mosphere is done in a physical way, since Oceanic reservoir — as chosen — spans only water and has nothing to do with biology, in opposition to Continental case. The link is the gases dissolved into Oceanic water, then flow depends on the quasistatic processes of dissolving and outgassing, which depend on the concentrations of gases dissolved into Oceanic water and in the Atmospheric gaseous phase. Thus, Atmospheric-Oceanic flow relies on both Atmospheric and Oceanic stocks: it is an interaction flow as Atmospheric-Continental flow was. Now, we tackle Continental-Oceanic flow from the perpective of Ocean. One can become aware of the reason why this flow is special, when one reasons that carbon flow from Continent does not enter directly to our Oceanic reservoir, but to the Oceanic food chain and Ocean water acid-alkaline balance — things that are out of the system by our hypotheses. Thus, from a perspective of the ocean, Continental-Oceanic flow only depends on Oceanic stock, which could be taken as an indicator of food chain activity and acid-alkaline balance in an implicit way. Then, Continental-Oceanic flow — as seen from Oceanic perspective — is a recipient-controlled flow.

We have left Atmosphere's perspective at the end, because it is somewhat trivial. Both, Atmospheric-Continental and Atmospheric-Oceanic flows, are interaction flows. Then, the analyses are symmetric when done from Atmospheric perspective. That is: Continental-Atmospheric and Oceanic-Atmospheric flows are the same as the former ones.

We make a summary of the results deduced before

Proposition 1 *For Carbon subsystem the following are the flows and their nature as regards controlling stock.*

- *Interaction flows: Atmospheric-Oceanic and Atmospheric-Continental flows.*
- *Recipient-controlled flows: Continental-Oceanic flow (Oceanic perspective).*
- *Donor-controlled flows: Continental-Oceanic flow (Continental perspective).*

Former phenomenological model of the flows is enough to set a mathematical model, in this case a system of ordinary differential equations (SODE) that model stocks evolution quantitatively. SODE is intended to express the temporal rate of change of stocks — then it will be a first-order SODE — in terms of the flows. The rule, that we propose to translate phenomenological model to mathematical one, is the following set of notation and rule.

Notation 1 Continental carbon stock is represented by the symbol C_1, while Atmospheric carbon stock is designed by the symbol C_2 and Oceanic carbon stock is given the symbol C_3

Axiom 2 *Let Φ be a given flow between reservoirs i, j, where i is the donor and j the recipient. If Φ is an interaction flow, it will be represented by a term containing the product*

of stocks $C_i C_j$ and if it is a flow that is controlled by either side but only one of them, it will appear as a term containing only the donor or the recipient stock, that is C_i or C_j.

Remark 1 The flow will depend on the stock — the quantity of carbon that the reservoir keeps — and, then, it is impossible to transfer carbon if controlling reservoir is empty. Interaction flows are controlled by both, and a product is as simple and as valid to describe the same fact.

With this rule in our hands, we simply write the system from phenomenological model summarized in proposition 1.

Proposition 2 *For Continental stock we write down*

$$\dot{C}_1 = -\alpha C_1 + \beta C_1 C_2$$

and for Atmospheric stock we say

$$\dot{C}_2 = -\beta C_1 C_2 + \varepsilon C_2 C_3$$

and, finally, Oceanic stock ODE reads

$$\dot{C}_3 = \zeta C_3 - \varepsilon C_2 C_3$$

However, this is not quite complete. As one can see, there are some coefficients multiplying representation of flows. These are the missing part in flow modeling. While flows are controlled by stock of carbon available, the rate could be modified by the phenomena that drive flows: that is exactly what coefficients mean. Then — in a full description of the driving phenomena — they are complex parametrizations of these phenomena, other models in its own right: ε must be a function of solubility (through Henry's law of dissolution), of T and of Ocean dynamics and thermodynamics — i.e. mixing layer depth, thermohaline circulation and Ocean temperature —, α depends on Continental biological productivity, rainfall and winds in the same manner as ζ relies on Oceanic biological productivity or acid-alkaline balance and β must be a function of Atmosphere dynamics and thermodynamics and Continental biological productivity. However, we also want our parametrizations be correct but not a burden.

As a first approximation to the processes described before, we will take constant coefficients. This is not unrealistic, since — in a millennial timescale and not taking into account transitional events — processes that drive non-interaction flows could be assummed to be constant because fluctuations are small. Interaction flows may reverse in sign — because of the control switching aforementioned in section 1.2. Yet that could be a little difficult to model without a further analysis, that is in the course of another research goal and out of the scope of this work.

Remark 2 For that reason, we will introduce an auxiliary flow in Atmospheric ODE in order to take into account any excess or deficit of carbon due to constant parametrizations of interaction flows: that is, a corrective flow. It will only depend on Atmospheric stock.

Therefore, because of the former remark, our SODE reads

Theorem 3 *According to the rules we established*

$$\begin{cases} \dot{C}_1 = -\alpha C_1 + \beta C_1 C_2 \\ \dot{C}_2 = \quad \gamma C_2 - \beta C_1 C_2 + \varepsilon C_2 C_3 \\ \dot{C}_3 = \quad \zeta C_3 \qquad\qquad - \varepsilon C_2 C_3 \end{cases} \tag{1}$$

where all coefficients are constant and positive (with the possible exception of γ).

Remark 3 Note that interaction terms — due to their nature — appear in a skew-symmetric fashion.

The next thing to do is an analysis of SODE 1. We will do it in a dynamic-systemic fashion.

3.1 Very short overview on techniques for dynamical systems.

3.1.1 Definition of dynamical system and interpretations.

SODEs, and in general differential equations, are expressions of dynamical systems. Formally, we can write the following

Definition 1 [Dynamical System] *A dynamical system on an open set $X \subset \mathbb{R}^n$ is a C^1- mapping (differentiable mapping with continuous derivative)*

$$\phi : \mathbb{R} \times X \longrightarrow X$$

such that, when one defines the family of maps $\phi_t := \phi(t, x)$, satisfies

(i) $\phi_0(x) = x$, $\forall x \in X$

(ii) $(\phi_t \circ \phi_s)(x) = \phi_{t+s}(x)$, $\forall s, t \in \mathbb{R}$, $x \in X$

Definition 2 [States and phase space] *The points of the open set X are known as the states of the dynamical system ϕ. X is known as the phase space of the dynamical system.*

Because the scalar variable t represents time in applications, ϕ can be interpreted as a mapping that, if $x = x_0 \in X$ is fixed, it gives the new state of the system $x \in X$ at time t. More clearly, let us fix an initial $x_0 \in X$, then $x(t) = \phi(t, x_0)$ is the function that describes the evolution of the system (the states that dynamical system visit), when we begin at x_0. The

set of states X can be interpreted as the possible combinations that state variables, which characterize the system, can take. Thus, in the applications ϕ represents all possible paths of evolution for the system, e.g. the possible trajectories a pendulum can take.

On the other hand, for a fixed t_f, ϕ represents the new state x_f to which $x \in X$ will evolve, when we reach time t_f. That is the same as having all possible initial states of the system and evolving them in time. This is the meaning of the ϕ_t mappings.

Therefore, $\phi(t, x)$ has all the information we need about evolution of dynamical system.

Remark 4 In this context, definition 1 says that at initial time $t = 0$, ϕ_t is the identity map on X (since to get the system at initial time is to do nothing), and that we can do translations in time by composing these mappings (and therefore we can redefine initial time).

Let us make it more explicit to the reader. We fix a $t_0, t_1 \in \mathbb{R}$ and select some $x \in X$, then

$$\begin{aligned}\phi(0, x) &= x\\ \phi(t_0, x) &= x_0\\ \phi(t_1, x) &= x_1\end{aligned}$$

i.e. the system at the state x at time $t = 0$, is at the state x_0 when $t = t_0$ and at the state x_1 when $t = t_1$. However, it is also true that if the system is at state x_0 at time $t = 0$ (since $\phi(0, x_0) = x_0$), then

$$\begin{aligned}\phi(-t_0, x_0) &= \phi(-t_0, \phi(t_0, x))\\ &= (\phi_{-t_0} \circ \phi_{t_0})(x)\\ &= \phi_{-t_0+t_0}(x) = \phi_0(x)\\ &= x\end{aligned}$$

and it follows that

$$\begin{aligned}x_1 &= \phi(t_1, x)\\ &= \phi(t_1, \phi(-t_0, x_0))\\ &= (\phi_{t_1} \circ \phi_{-t_0})(x_0)\\ &= \phi_{t_1-t_0}(x_0)\\ &= \phi(t_1 - t_0, x_0)\end{aligned}$$

Thus,

$$\begin{aligned}\phi(-t_0, x_0) &= x\\ \phi(0, x_0) &= x_0\\ \phi(t_1 - t_0, x_0) &= x_1\end{aligned}$$

when we take as initial state (we select the time origin) as x_0.

When we choose x_1 as the initial state, for the sake of completeness, then

$$\begin{aligned}
\phi(-t_1, x_1) &= \phi(-t_1, \phi(t_1, x)) \\
&= (\phi_{-t_1} \circ \phi_{t_1})(x) \\
&= \phi_{-t_1+t_1}(x) = \phi_0(x) \\
&= x \\
\phi(t_0 - t_1, x_1) &= \phi(t_0 - t_1, \phi(t_1, x)) \\
&= (\phi_{t_0-t_1} \circ \phi_{t_1})(x) \\
&= \phi_{t_0-t_1+t_1}(x) \\
&= \phi_{t_0}(x) \\
&= \phi(t_0, x) \\
&= x_0
\end{aligned}$$

which leads us to

$$\begin{aligned}
\phi(-t_1, x_1) &= x \\
\phi(t_0 - t_1, x_1) &= x_0 \\
\phi(0, x_1) &= x_1
\end{aligned}$$

Then we always can redefine the origin of time.

It is almost trivial to acknowledge the following (we have used it before in a sense)

Proposition 4 *Let ϕ be a dynamical system on X. Then $\forall t \in \mathbb{R}$, ϕ_t is a diffeomorphism (a C^1-mapping with C^1-inverse).*

Proof. Since ϕ is a C^1-mapping, then its restrictions ϕ_t are also a one-parameter family of C^1-mappings (a mapping for each t) in the form $\phi_t : X \longrightarrow X$. Using the definition of dynamical system, this family satisfies that $\forall x \in X$

$$\begin{aligned}
(\phi_t \circ \phi_{-t})(x) &= \phi_{t-t}(x) \\
&= \phi_0(x) = x \\
&= \mathsf{id}(x)
\end{aligned}$$

therefore, ϕ_{-t}, which is C^1-mapping by hypothesis, is the inverse mapping of ϕ_t. We conclude that ϕ_t is a diffeomorphism for every $t \in \mathbb{R}$, as we wanted to prove. □

Remark 5 In the context of time, former proposition says that dynamical systems accept time-reversal transformation. Thus, dynamical systems are deterministic (as we only need

to know initial state of the system to know every other future state) and time-reversible (we also know, given initial state, every other past state — consistent with dynamics — using negative time).

Let us delve deeper into the meaning of a dynamical system to emerge in the differential equations realm. Let $\phi(t, x)$ be a dynamical system over X, then the function

$$f(x) := \left.\frac{d\phi}{dt}\right|_{t=0}$$

defines a C^1-vector field. Why we evaluate at $t = 0$ and not at $t = t_0$? We use limit definition of derivative

$$\left.\frac{d\phi}{dt}\right|_{t=0} = \lim_{h\to 0} \frac{\phi(h, x) - \phi(0, x)}{h}$$
$$\left.\frac{d\phi}{dt}\right|_{t=t_0} = \lim_{h\to 0} \frac{\phi(t_0 + h, x) - \phi(t_0, x)}{h}$$

both expressions are valid for any $x \in X$. Then we use definition 1 in the second expression

$$\begin{aligned}\left.\frac{d\phi}{dt}\right|_{t=t_0} &= \lim_{h\to 0} \frac{\phi(t_0 + h, x) - \phi(t_0, x)}{h} \\ &= \lim_{h\to 0} \frac{\phi(h + t_0, x) - \phi(0 + t_0, x)}{h} \\ &= \lim_{h\to 0} \frac{\phi(h, \phi(t_0, x)) - \phi(0, \phi(t_0, x))}{h} \\ &= \lim_{h\to 0} \frac{\phi(h, x_0) - \phi(0, x_0)}{h}\end{aligned}$$

but $x_0 \in X$. Therefore,

$$\left.\frac{d\phi}{dt}\right|_{t=0} = \left.\frac{d\phi}{dt}\right|_{t=t_0}, \forall t_0 \in \mathbb{R}$$

which means that $f \neq f(t, x)$ and validates our selection of $t = 0$ to evaluate derivative.

Now, reader must note that $f(x)$ is tangent to $\phi(t, x)$, $\forall x \in X$. Thus, $\phi(t, x_0)$ is the solution (integral curve or quadrature) of the following SODE with its corresponding initial value problem (IVP)

$$\begin{cases}\dot{x} = f(x) \\ x(0) = x_0\end{cases}$$

The converse — that any

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

for which f is a C^1-vector field, defines a dynamical system ϕ — is true by virtue of the global existence theorem (see Perko, 2001, ch. 3; in particular section 3.1, theorem 1). Therefore, the dynamical system is the general solution of former SODE.

From former revelation, one can think that dynamical systems only arise from autonomous SODEs — those SODEs for which $f \neq f(t, x)$ — because it is clear that for each time f changes and therefore we can not make time translations. However, what happens if we grow the dimension of phase space, by the definition of the following?

$$\begin{aligned} X &:= (t, x) \ \therefore \ X_0 = (t_0, x_0) \\ \frac{dt}{dT} &= 1 \ \therefore \ t = T + cst. \\ F(X) &:= (1, f(X)) \end{aligned}$$

then the non-autonomous SODE

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

can be rewritten as the following SODE

$$\begin{cases} \frac{dX}{dT} = F(X) \\ X(0) = X_0 \end{cases}$$

which is autonomous, and defines a dynamical system $\phi(T, X)$. What we have done, by the addition of t as another dimension of phase space, is to glue together the one-parameter family of C^1-vector fields $f_t(x) := f(t, x)$ to form a complete vector field F, and saying that the new t state variable is not coupled to the others.

Another interpretation of a dynamical system is the reason behind another name for ϕ: the flow of the SODE. ϕ determines how the system evolves in a lagrangian fluid-dynamical point of view (where the particles we follow are specific realizations of the dynamical system). The equivalent eulerian point of view — how the phase space (or a subset of it) flows — will be the evolution of every point in that subset: that means, the evolution of different realizations of the system.

3.1.2 Invariant sets of a dynamical system.

Former section leads us to formulate new questions. Is there any state in phase space that is fixed? That is, if we start in that state, do we continue in that state? A third form to express this is: Are there states that do not flow at all? Also, we could think about subsets of phase space that do not flow or that their states flow inside them: the flow is constrained to those subsets. These sets are known as invariant sets and, in the case of states, they are called fixed points or — in other fields such as Physics — they are also known as equilibria. From former questions, we can formulate definitions of invariant sets.

Definition 3 [Invariant set of a dynamical system] *Let* $\phi : \mathbb{R} \times X \longrightarrow X$ *be a dynamical system and let* $Y \subset X$ *satisfy that*

$$\phi(t, Y) \subseteq Y, \ \forall t \in \mathbb{R}$$

that is, for every time t*, all points in* Y *flow to other points in* Y*. Then* Y *is an invariant subset of* ϕ*.*

Remark 6 In a more informal way, dynamical system is constrained to evolve inside this subset of states, if it begins in one of these states.

Definition 4 [Fixed point of a dynamical system] *Let* $\phi : \mathbb{R} \times X \longrightarrow X$ *be a dynamical system and let* $x_0 \in X$ *satisfy that*

$$\phi(t, x_0) = x_0, \ \forall t \in \mathbb{R}$$

then x_0 *is a fixed point of* ϕ*.*

Remark 7 In relation to former definition, it is clear that a fixed point is an invariant set that consists of only one state.

From former definitions, one would think that the search for invariant sets is doomed to failure, since it seems that knowledge of invariant sets pass through finding an explicit form for the flow ϕ. That is impossible in a general case — since not every SODE has a closed form solution. In spite of this, one must query SODE in other ways to obtain this information, without obtaining a numerical (approximate) or closed solution. This is the goal of dynamical systems theory.

SODE associated to ϕ is written as $\dot{x} = f(x)$. This expression determines the tangent vector to the evolution curve in phase space corresponding to every initial state (in a lagrangian point of view) or it determines the flow velocity field in the phase space and, therefore, any possible realization of the system follow curves that have this field (constrained to them, in a eulerian point of view), as we have explained before. Thus, if a point x_0 is a fixed point, then $\dot{x} = 0$ since $x(t) = x_0$. Thus, we have shown that

Proposition 5 *Let $\phi : \mathbb{R} \times X \longrightarrow X$ be a dynamical system and $\dot{x} = f(x)$ the corresponding SODE. Then, every fixed point $x_0 \in X$ satisfies that $\dot{x} = 0$, $\forall t \in \mathbb{R}$ or, equivalently*

$$f(x_0) = 0$$

Conversely, every zero of f is a fixed point.

Then, to look for fixed points, simply we seek the root of the vector field that defines the flow.

For other invariant sets, criteria are more complicated. The following is for invariant surfaces: we will not need more for our purposes.

Proposition 6 *Let $\phi : \mathbb{R} \times X \longrightarrow X$ be a dynamical system and $\dot{x} = f(x)$ the corresponding SODE. Then an invariant surface $S \subseteq X$ satisfies that, if $y \in S$, $f(y) \in \mathcal{T}_y S$, where $\mathcal{T}_y S$ is the tangent space of S at point y.*

Proof. Take any point $y \in S$ as an initial state. Then the solution $x(t)$, defined in terms of dynamical system as the flow of y or $x(t) = \phi(t, y)$, is a curve in phase space such that its tangents at any time are determined by $(f \circ x)(t)$. By hypothesis, $x(t) \in S$ for any $t \in \mathbb{R}$.

Suppose that $(f \circ x)(t_b) \notin \mathcal{T}_{x(t_b)} S$ for some $t_b \in \mathbb{R}$ but $x(t_b) \in S$. Such supposition tells us that, since f is C^1-vector field, the point $x(t_b)$ would have a neighborhood, where the curve were not contained inside S. That is, we would conclude that S is not invariant: we obtain a contradiction because of supposition that tangent vectors to the solution are not contained in tangent space of the surface S.

Thus, $\forall t$, $(f \circ x)(t) \in \mathcal{T}_{x(t)} S$ and — since $x(t)$ is defined for all $y \in S$ — then, for every $y \in S$, it follows that $f(y) \in \mathcal{T}_y S$; which is what we were trying to show. □

Remark 8 The former proof could be summarized in the following manner: If any surface is invariant, then all the trajectories beginning inside the surface will remain inside it. Thus, tangent vectors of trajectories must be tangent to the surface.

An equivalent criteria can be developed for surfaces, when we are given its normal space (e.g. for planar surface it is common to give the normal vector). This is a direct consequence of former proposition.

Corolary 7 *Let $\phi : \mathbb{R} \times X \longrightarrow X$ be a dynamical system and $\dot{x} = f(x)$ the corresponding SODE. Then an invariant surface $S \subseteq X$ satisfies that $\forall y \in S$, $f(x) \perp \mathcal{N}_y S$, where $\mathcal{N}_y S$ is the normal space of S at point y.*

Former criteria show that the vector field $f(x)$ contains the same information as ϕ. However, we must make some workarounds to overcome the general impossibility to obtain a closed expression for ϕ.

Remark 9 Why we do not speak of invariant curves? As we can see, from the definition of invariant surface, vector field defined by dynamical system must be inside the tangent space at any point of the surface. If we apply that idea to a curve, then the vector field shall be tangent to the curve at every point. Thus, an invariant curve is nothing else but an integral curve of the vector field, which we call a solution. Then, for us, there are no invariant curves. However, as we shall see in next subsection, there can be manifolds (in particular curves) made of fixed points.

3.1.3 Classification of fixed points. Bifurcations.

Once we identified invariant sets for a given dynamical system, we want to know how the system behaves in a neighborhood near them. In general, that task will involve a careful evaluation using linearization or second-order approximation (termed Normal Form Theory) around fixed points and, in some cases, Lyapunov functions. For surfaces or other invariant sets, there is a more complex set of techniques, which will not be summarized.

The flow around equilibria is important, since it can give us a general portrait of the evolution of the system without solving it (neither closed nor approximate solution): if a fixed point is stable, the system will evolve towards it (trajectories will flow towards it or the flow of phase space will sink into it); while if it is unstable, the system will evolve away (trajectories will flow away or the flow of phase space have a source on it).

There are many types of equilibrium, depending on dimensionality of phase space. In a $2-$dimensional framework, they are classified entirely and we present this classification.

First subdivision is between hyperbolic and non-hyperbolic fixed points. A hyperbolic equilibrium is one with a linearization that has non-zero real eigenvalues. In opposition, a non-hyperbolic fixed point has complex eigenvalues with non-vanishing imaginary part.

We look first into hyperbolic type. They are called nodes. There are three posibilities for the eigenvalues of the linearization: both eigenvalues are positive, both are negative or we have eigenvalues of different sign. Positive eigenvalues of linearization are associated with fundamental solutions of the form $e^{\lambda t} \mid \lambda > 0$, which increase arbitrarily when time increases; negative ones are related to fundamental solutions of the form $e^{-\lambda t} \mid \lambda > 0$, which tend to vanish when time increases. Thus, positive eigenvalues define (in the linearization) eigenspaces in which the dynamical system tends to go away from node (in non-linearized system there will be a submanifold in which the flow tends to go away of the node and such that the eigenspace found for the linearization is tangent to the submanifold at the node). In opposition, negative eigenvalues define eigenspaces, and the corresponding submanifolds, in which flow goes towards node. From former discussion, we classify nodes

Definition 5 [Hyperbolic fixed points] *Let $x \in X$ be a node for a dynamical system, for which X is a $2-$manifold. Then it is a*

1. *Stable node iff all eigenvalues of linearization are negative.*
2. *Unstable node iff all eigenvalues of linearization are positive.*
3. *Saddle node iff one eigenvalue is positive and the other is negative.*

If eigenvalues are equal, for a stable or an unstable node, then the node is termed improper node or degenerate node (if related eigenspace has dimension 1 *and therefore system is non-diagonalizable) or star (if the entire plane is eigenspace): e.g. there can be improper stable nodes (degenerate stable nodes) or stable stars.*

Remark 10 A glimpse to the topology of the flow around these equilibria is given in figure 3.

Now, we turn to non-hyperbolic fixed points. There is no general name for them. Since their eigenvalues are complex with non-vanishing imaginary part, then, if λ is an eigenvalue, λ^* is the remaining eigenvalue (by virtue of the Fundamental Theorem of Algebra). Thus, there are three posibilities for the eigenvalues: both eigenvalues are purely imaginary, both eigenvalues have positive real part or both have negative real part. There are no degenerate cases as in the case of hyperbolic ones: the diagonalization is always possible. As before, the pair of purely imaginary eigenvalues have fundamental solutions of the form $e^{\pm i\lambda t}$ (or can be written in terms of cosine and sine functions). Therefore, the flow does not tend towards or away from equilibrium. Trajectories are closed and in the linearization look like ellipses. The other two options, since the fundamental solutions are of the form $e^{\mathrm{Re}\,\lambda t}e^{\pm i\,\mathrm{Im}\,\lambda t}$, have flows that spiral down (negative real part) or away (positive real part) from fixed point. This discussion leads to

Definition 6 [Non-hyperbolic fixed points] *Let $x \in X$ be a non-hyperbolic fixed point for a dynamical system, for which X is a $2-$manifold. Then it is a*

1. *Center iff eigenvalues of linearization are purely imaginary.*
2. *Stable focus iff eigenvalues of linearization have positive real part.*
3. *Unstable focus iff eigenvalues of linearization have negative real part.*

Remark 11 Centers are somewhat weak, because if one detects a center in a linearization, it is possible that the non-linear (true) flow has not a center but a focus: that means that the transformation of linearization may not preserve the topology of the flow between the linearized flow and the true flow. In that case, to confirm the character of a non-hyperbolic fixed point, it is necessary to use Normal Form Theory (a second order approximation at the equilibrium) or Lyapunov functions. A glimpse to the topology of the flow around this equilibria is given in figure 4.

Until now, we have not seen equilibria with null eigenvalues in the linearization. These ones will lead to manifolds of fixed points associated to the eigenspace of the null eigenvalue, since the fundamental solutions for the null eigenvalue are constant. This are called non-isolated fixed points. For defining isolation without relying on linearization, we define it with the following: a fixed point x is isolated if there exists an $\epsilon > 0$ such that there is no other fixed point inside the ball of radius ϵ centered in the fixed point x. Negation of this is our definition of non-isolated fixed point.

Definition 7 [Non-isolated fixed point] *Let x be a fixed point. If for every $\epsilon > 0$, there is at least one fixed point inside the ball, other than x, then x is non-isolated.*

Remark 12 A glimpse to the topology of the flow around non-isolated fixed points is given in figure 5.

In an $n-$dimensional setting (being X an $n-$manifold) classification is more complicated, since there could be a manifold that pases through fixed point, in which it is a stable focus; but another manifold passes through, in which it is unstable node; and also another manifold in which the behavior is inconclusive from only linearization. In general, for each equilibria we will have three submanifolds: The stable manifold (in which equilibrium is stable), the unstable manifold (in which equilibrium is unstable), and the center manifold (in which linearization is not enough to deduce stability). The sum of dimensionalities of these submanifolds must be the whole space, unless there is some degeneration. In the context of $2-$dimensional systems, hyperbolic equilibria have no center manifold and centers have only center manifold. Something to note is the following: while stable and unstable manifolds are unique, center manifold may not be unique. This means that we can find at least one center manifold, but maybe there is another manifold passing through the fixed point, that also serves as center manifold and is different from the one we have found.

However, there are another questions, and maybe the most important ones. When a system has parameters, how the flow of the phase space is modified? Do fixed points or invariant sets change their characteristics because of them? Do invariant sets vanish with parameter modification? Since $f(x)$ depends on parameters, it is clear that existence, characterization, rise or vanishing of invariant sets is closely related with parameters. The character changes on fixed points are termed bifurcations and can be seen as phase changes relying on parameters. These are important, because bifurcations make structural changes in the flow of phase space, which can be related to changes in the evolution of physical phenomena described by the dynamical system under inquiry.

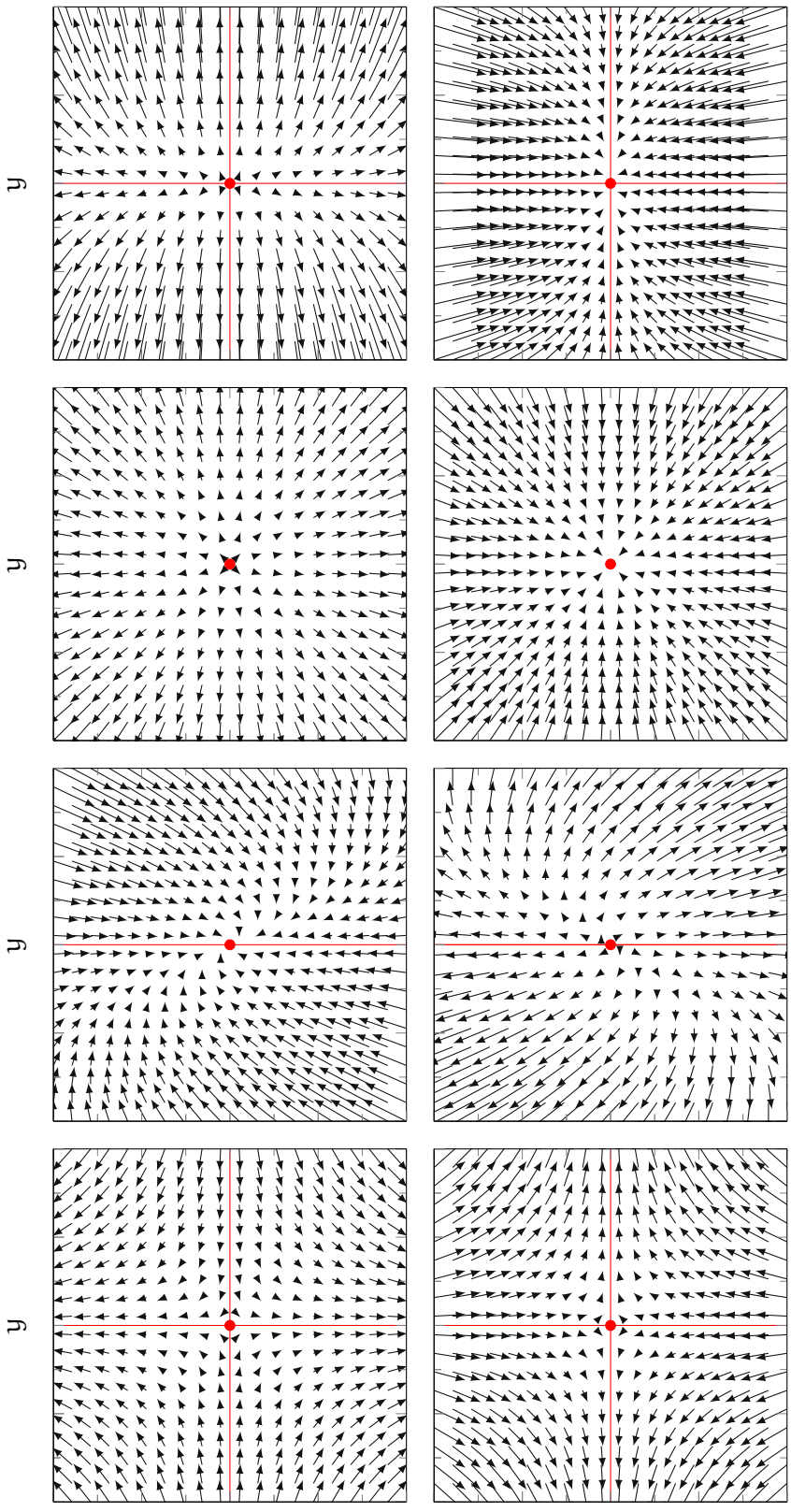

Figure 3: Phase space around fixed points (red points). First column: unstable cases. Second column: stable cases. In rows: nodes, stars, improper nodes and saddle nodes. Red lines: eigenspaces.

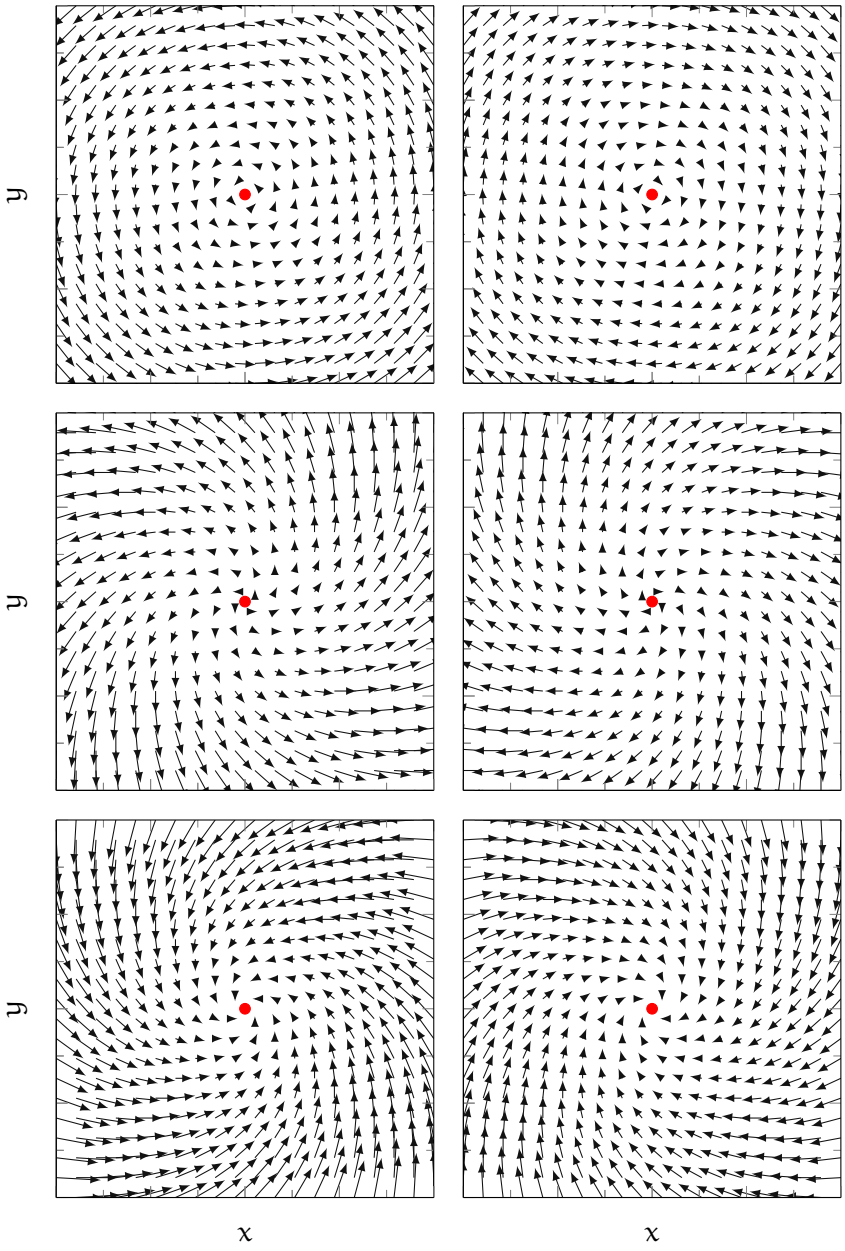

Figure 4: Phase space around fixed points (red points). First column: counterclockwise cases. Second column: clockwise cases. In rows: centers, unstable foci and stable foci.

Figure 5: Phase space around fixed points (red points). Examples of flow around non-isolated fixed points (a fixed point with a vanishing eigenvalue). Red line: line of fixed points. Note that it acts as a separatrix of flow regimes.

3.1.4 2D Example: Analysis of a non-linear pendulum.

We will illustrate these techniques, and show bifurcations, with a physicist-friendly non-linear mechanical system: the pendulum. Pendulum, that we will analyze, has two terms: the usual gravity term and a dampening term. The state of the system is given by θ the angle formed by the pendulum and the local plumb line. Equation reads

$$\ddot{\theta} = -\mu\dot{\theta} - \frac{g}{L}\sin\theta$$

where μ is a parameter controlling dampening or forcing[1] and L is the longitude of the pendulum.

Every $n-$order ODE can be transformed into an $n-$dimensional SODE of first order. In this case we introduce a new state variable $\omega := \dot{\theta}$: this variable is angular velocity. Thus, $\ddot{\theta} = \dot{\omega}$ and former equation can be recasted as the following SODE

$$\begin{cases} \dot{\theta} = & \omega \\ \dot{\omega} = -\frac{g}{L}\sin\theta - \mu\omega \end{cases}$$

First, we search for fixed points

$$f(\theta, \omega) = (\omega, -\frac{g}{L}\sin\theta - \mu\omega) = 0$$

First component of former expression establishes that $\omega \equiv 0$. Second component establishes that

$$\omega = -\frac{g}{\mu L}\sin\theta$$

in order to vanish. Both conditions must be fulfilled at once, thus

$$\begin{aligned} -\frac{g}{\mu L}\sin\theta &= 0 \\ \sin\theta &= 0 \therefore \\ \theta &= k\pi,\ k \in \mathbb{Z} \end{aligned}$$

if we suppose $\mu \neq 0$. If we allow $\mu = 0$, at the end we obtain the same condition. Then there is a family of countable isolated fixed points given by

$$(\theta, \omega) = (k\pi, 0)$$

[1] This type of forcing is somewhat artificial, but for the purpose of showing dynamical systems techniques is perfect.

We linearize the pendulum SODE using Taylor series of f at each equilibrium. At first order of approximation:

$$\begin{aligned} f &\sim f(\theta_0, \omega_0) + (\theta - \theta_0, \omega - \omega_0)\nabla f|_{(\theta_0,\omega_0)} \\ &\sim 0 + (\theta - k\pi, \omega) \left(\begin{array}{cc} 0 & -\frac{g}{L}\cos\theta \\ 1 & -\mu \end{array} \right)\Bigg|_{(\theta_0,\omega_0)} \\ &\sim (\theta - k\pi, \omega) \left(\begin{array}{cc} 0 & -\frac{g}{L}\cos k\pi \\ 1 & -\mu \end{array} \right) \end{aligned}$$

$$\begin{aligned} f &\sim \left(\omega, -\frac{g}{L}(\cos k\pi)(\theta - k\pi) - \omega\mu\right) \\ &\sim \left(\omega, (-1)^{k+1}\frac{g}{L}(\theta - k\pi) - \omega\mu\right) \\ &\sim \begin{cases} \left(\omega, -\frac{g}{L}(\theta - k\pi) - \omega\mu\right) & k = 2\ell,\ \ell \in \mathbb{Z} \\ \left(\omega, \frac{g}{L}(\theta - k\pi) - \omega\mu\right) & k = 2\ell + 1,\ \ell \in \mathbb{Z} \end{cases} \end{aligned}$$

then linearized system is

$$\begin{cases} \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{L}(\theta - k\pi) - \mu\omega \end{cases} & , k = 2\ell,\ \ell \in \mathbb{Z} \\ \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \frac{g}{L}(\theta - k\pi) - \mu\omega \end{cases} & , k = 2\ell + 1,\ \ell \in \mathbb{Z} \end{cases}$$

and, for the sake of clarity, we make an invariant translation on θ: $\theta_n := \theta - k\pi$ and $\dot{\theta}_n = \dot{\theta}$. Thus,

$$\begin{cases} \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{L}\theta - \mu\omega \end{cases} & , k = 2\ell,\ \ell \in \mathbb{Z} \\ \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \frac{g}{L}\theta - \mu\omega \end{cases} & , k = 2\ell + 1,\ \ell \in \mathbb{Z} \end{cases}$$

or in matrix form

$$\begin{aligned} (\dot{\theta}, \dot{\omega}) &= (\theta, \omega) \left(\begin{array}{cc} 0 & -\frac{g}{L} \\ 1 & -\mu \end{array} \right),\ k = 2\ell,\ \ell \in \mathbb{Z} \\ (\dot{\theta}, \dot{\omega}) &= (\theta, \omega) \left(\begin{array}{cc} 0 & \frac{g}{L} \\ 1 & -\mu \end{array} \right),\ k = 2\ell + 1,\ \ell \in \mathbb{Z} \end{aligned}$$

Let us obtain the eigenvalues of system matrix:

$$\begin{aligned}\det\begin{pmatrix} -\lambda & \mp\frac{g}{L} \\ 1 & -(\mu+\lambda) \end{pmatrix} &= 0 \\ (\mu+\lambda)\lambda \mp \frac{g}{L} &= 0 \\ \lambda^2 + \mu\lambda \mp \frac{g}{L} &= 0\end{aligned}$$

or, for $\mu = 0$

$$\lambda^2 \mp \frac{g}{L} = 0$$

Thus, the two solutions are the following

$$\begin{aligned}\lambda_\pm &= -\frac{1}{2}\mu \pm \frac{1}{2}\sqrt{\mu^2 \pm 4\frac{g}{L}} \\ &= -\frac{1}{2}\mu \pm \frac{1}{2}\mu\sqrt{1 \pm \frac{4g}{\mu^2 L}} \\ &= \frac{1}{2}\mu\left(-1 \pm \sqrt{1 \pm \frac{4g}{\mu^2 L}}\right)\end{aligned}$$

or for $\mu = 0$

$$\lambda_\pm = \pm\sqrt{\pm\frac{g}{L}}$$

where the switch in signs inside square root comes from the parity of k.

All the characteristics of eigenvalues rely upon the expression inside square root (a discriminant). Let us look closer into it

$$D_\mp = 1 \pm \frac{4g}{\mu^2 L}$$

or for $\mu = 0$

$$D_\mp = \pm\frac{g}{L}$$

If $D_\mp < 0$, then we have complex eigenvalues with non-vanishing imaginary part (non-hyperbolic fixed points). Else ($D_\mp \geqslant 0$) eigenvalues are real (nodes). We divide this task by cases.

1. $\mu \neq 0$ (forced — $\mu < 0$ — or dampened — $\mu > 0$ — pendulum)

a) $D_- = 1 + \frac{4g}{\mu^2 L}$

i. $D_- < 0$: impossible, since $g, L > 0$, $\mu \in \mathbb{R}$.

ii. $D_- = 0$: impossible, since $g, L > 0$, $\mu \in \mathbb{R}$.

iii. $D_- > 0$: unique available behavior.

A. $\frac{1}{\mu}\lambda_+ > 0$: always true, since $D_- > 1$. If $\mu > 0$, then equilibrium is a saddle node. If $\mu < 0$, it is a stable node.

B. $\frac{1}{\mu}\lambda_+ = 0$: impossible, since $D_- > 1$.

C. $\frac{1}{\mu}\lambda_+ < 0$: impossible, since $D_- > 1$.

b) $D_+ = 1 - \frac{4g}{\mu^2 L}$

i. $D_+ < 0$: true, if $\mu^2 L < 4g$. Complex conjugate eigenvalues

$$\lambda = \frac{1}{2}\mu\left(-1 \pm i\sqrt{\frac{4g}{\mu^2 L} - 1}\right)$$

then this equilibrium is a focus. It is stable if $\mu > 0$ and unstable if $\mu < 0$.

ii. $D_+ = 0$: true, if $\mu^2 L = 4g$. Eigenvalues with multiplicity. Both eigenvalues are

$$\lambda = -\frac{1}{2}\mu$$

and means that equilibrium is unstable if $\mu < 0$ and stable if $\mu > 0$. Possibly an improper node.

iii. $D_+ > 0$: true, if $\mu^2 L > 4g$.

A. $\frac{1}{\mu}\lambda_+ > 0$: impossible, since $D_+ < 1$.

B. $\frac{1}{\mu}\lambda_+ = 0$: impossible, since $D_+ < 1$.

C. $\frac{1}{\mu}\lambda_+ < 0$: always true, since $D_+ < 1$. If $\mu > 0$, then equilibrium is a stable node. If $\mu < 0$, it is a saddle node.

2. $\mu = 0$ (Classical simple pendulum)

a) $D_- = \frac{g}{L}$: symmetric real eigenvalues

$$\lambda_\pm = \pm\sqrt{\frac{g}{L}}$$

which means a saddle node.

b) $D_+ = -\frac{g}{L}$: complex conjugate eigenvalues, pure imaginary

$$\lambda_\pm = \pm i\sqrt{\frac{g}{L}}$$

which means a center.

In the case of fixed points with odd k, these pass from stable nodes to saddle nodes with increasing μ. Fixed points with even k have more rich behaviour: for $\mu < 0$ they pass from unstable foci, through improper unstable nodes, to saddle nodes. For $\mu = 0$ they are centers. For $\mu > 0$ they pass from stable foci, through improper stable nodes, to stable nodes. These situations can be seen in figures 6, 7 and 8

The linearization centers are centers in the non-linear flow. This is confirmed by the existence of a Hamiltonian: a conserved quantity, that in this case is related to the sum of kinetic and potential energy. Hamiltonian function is a Lyapunov function in this case.

3.2 Analysis of Carbon Subsystem Model.

We will analyze SODE in equation 1 with the theory and ideas portrayed formerly.

3.2.1 Fixed points.

First, we identify our $f(x)$ or, precisely, our $f(C)$.

$$f(C) = (-\alpha C_1 + \beta C_1 C_2, \gamma C_2 - \beta C_1 C_2 + \varepsilon C_2 C_3, \zeta C_3 - \varepsilon C_2 C_3) \tag{2}$$

and we find zeros of this vector field and the conditions, if there are any, for this to happen. These may reveal fixed points according to proposition 5. From the first component of 2

$$-\alpha C_1 + \beta C_1 C_2 = 0$$
$$(-\alpha + \beta C_2) C_1 = 0$$

out of which we have two conditions: $\forall t\, C_1(t) \equiv 0$ or $-\alpha + \beta C_2$ vanishes or, equivalently, $\forall t\, C_2(t) = \alpha/\beta$.

We can do an analogous analysis of the third component of 2

$$\zeta C_3 - \varepsilon C_2 C_3 = 0$$
$$(\zeta - \varepsilon C_2) C_3 = 0$$

from which we have two conditions $\forall t\, C_3(t) \equiv 0$ or $\zeta - \varepsilon C_2$ is equal to zero or, equivalently, $\forall t\, C_2(t) = \zeta/\varepsilon$.

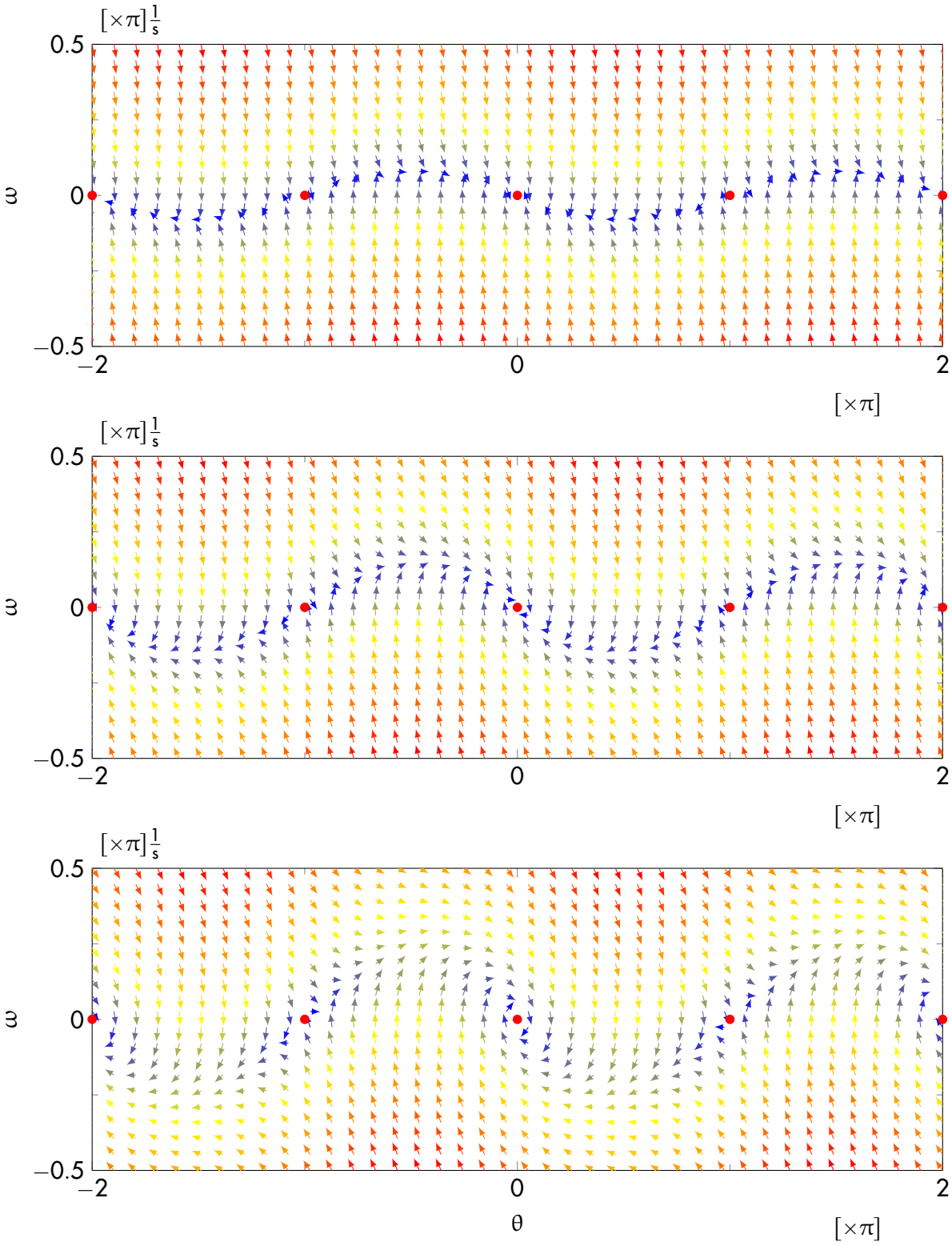

Figure 6: Phase space of dampened pendulum with $\frac{g}{L} = \frac{1}{s^2}$. In panels, from top to bottom: overdampened $\mu = 4\frac{1}{s}$, critically dampened $\mu = 2\frac{1}{s}$ and dampened $\mu = 1\frac{1}{s}$. Red points are fixed points.

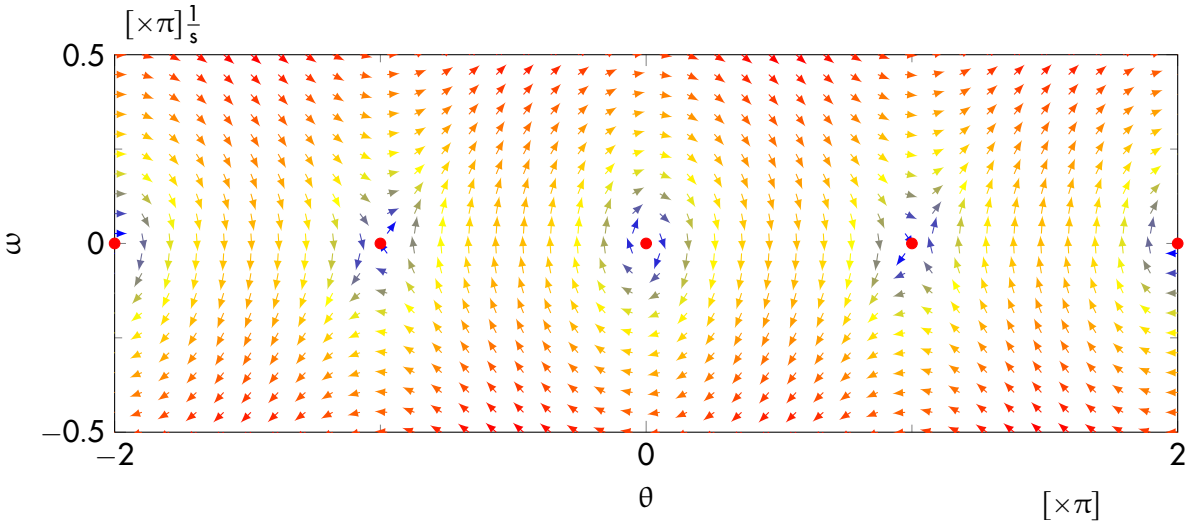

Figure 7: Phase space of classical simple pendulum with $\frac{g}{L} = 1\frac{1}{s^2}$.

Second component of 2 is more involved

$$\gamma C_2 - \beta C_1 C_2 + \varepsilon C_2 C_3 = 0$$
$$(\gamma - \beta C_1 + \varepsilon C_3) C_2 = 0$$

and this leads us to say that $\forall t\, C_2(t) \equiv 0$ or $\gamma - \beta C_1 + \varepsilon C_3$ becomes zero or, equivalently, $\forall t\, C_3(t) = (\beta/\varepsilon) C_1(t) - (\gamma/\varepsilon)$.

At first glance, it is obvious that certain conditions are incompatible. Let us experiment with all combinations for the sake of completeness.

Firstly, we take $C_1(t) \equiv 0$. Since C_1 does not appear in third component, we can have both conditions from this component:

- $C_1(t), C_3(t) \equiv 0$
- $C_1(t) \equiv 0 \wedge C_2(t) \equiv \zeta/\varepsilon$

then we proceed to compare this conditions with those from second component. It is clear that, unless $\gamma/\varepsilon = 0$ (that is $\gamma = 0$, which is not possible because our construction), second condition on second component clashes with the first of former conditions. In the same manner, unless $\zeta = 0$ (which is also not possible), first condition on second component conflicts with the second of former conditions. This results in the following valid conditions

- $C_1(t), C_2(t), C_3(t) \equiv 0$
- $C_1(t) \equiv 0 \wedge C_2(t) \equiv \zeta/\varepsilon \wedge C_3(t) \equiv -\gamma/\varepsilon$

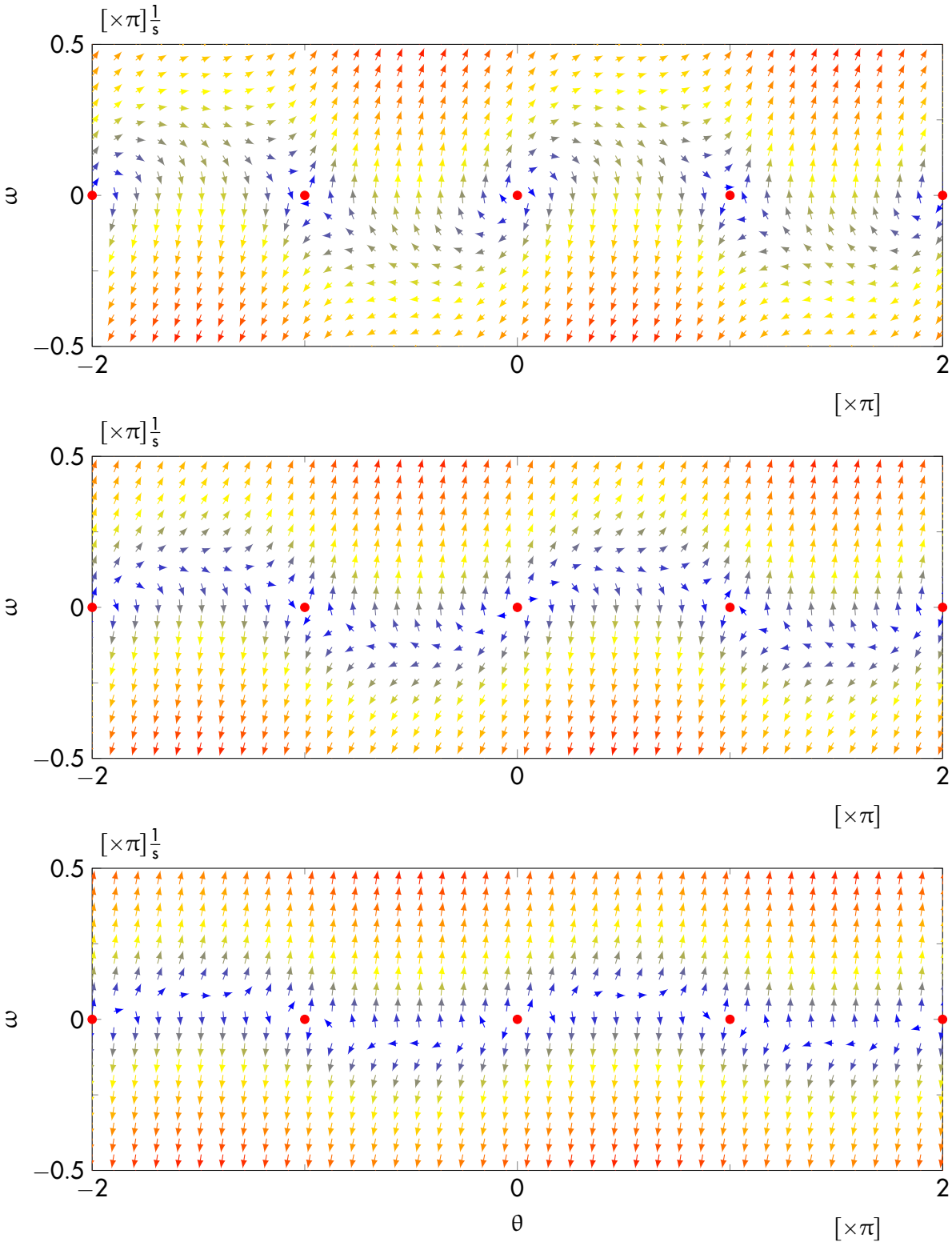

Figure 8: Phase space of forced pendulum with $\frac{g}{L} = 1\frac{1}{s^2}$. In panels, from top to bottom: forced $\mu = -1\frac{1}{s}$, critically forced $\mu = -2\frac{1}{s}$ and overforced $\mu = -4\frac{1}{s}$.

Now, we make tests with second condition of the first component: $C_2(t) \equiv \alpha/\beta$. Third component first option is compatible at a glance. Second option is not as clear as before. The compliance of both of them means that

$$\frac{\alpha}{\beta} = \frac{\zeta}{\varepsilon}$$

which is a strong, but plausible, condition — it lowers dimensionality to parameter space in one degree of freedom. Then valid conditions are, until now, the following

- $C_2(t) \equiv \alpha/\beta \wedge C_3(t) \equiv 0$
- $C_2(t) \equiv \alpha/\beta \equiv \zeta/\varepsilon$

Former conditions clash with first condition on the second component, unless a parameter or two vanishes, but they are fully compatible with the second condition. Valid options are

- $C_1(t) \equiv \gamma/\beta \wedge C_2(t) \equiv \alpha/\beta \wedge C_3(t) \equiv 0$
- $\zeta = \alpha\varepsilon/\beta \wedge C_2(t) \equiv \alpha/\beta \wedge C_3(t) = (\beta/\varepsilon)C_1(t) - (\gamma/\varepsilon)$

Finally, we summarize the valid conditions that we have obtained

1. $C_1(t), C_2(t), C_3(t) \equiv 0$
2. $C_1(t) \equiv 0 \wedge C_2(t) \equiv \zeta/\varepsilon \wedge C_3(t) \equiv -\gamma/\varepsilon$
3. $C_1(t) \equiv \gamma/\beta \wedge C_2(t) \equiv \alpha/\beta \wedge C_3(t) \equiv 0$
4. $\zeta = \alpha\varepsilon/\beta \wedge C_2(t) \equiv \alpha/\beta \wedge C_3(t) = (\beta/\varepsilon)C_1(t) - (\gamma/\varepsilon)$

First valid condition says that origin is a fixed point, second condition reveals that always there is a fixed point in $C_2 - C_3$ plane. In the same manner, third condition shows that there is always a fixed point in $C_1 - C_2$ plane. Nevertheless, both coordinate plane equilibria can vary depending on parameters. This could be an evidence for invariant surfaces. Fourth condition unveils that the only other fixed points that Carbon Subsystem model has, are those on a line Γ (given in parametric form) subjected to the existence of a relationship between flow parameters. It is worth to note that these fixed points are non-isolated.

Once we have interpreted results from our analysis we condense them in the following theorem.

Theorem 8 *SODE 1 has the following equilibria*

1. *The origin.*
2. $(0, \zeta/\varepsilon, -\gamma/\varepsilon)$

3. $(\gamma/\beta, \alpha/\beta, 0)$

4. $(\tau, \alpha/\beta, (\beta/\varepsilon)\tau - (\gamma/\varepsilon)) \longleftrightarrow \zeta = \alpha\varepsilon/\beta,\ \tau \in \mathbb{R}$

Remark 13 From this theorem, it is possible to say something about bifurcations in Carbon Subsystem. If Carbon Subsystem flows do not satisfy condition $\beta\zeta = \alpha\varepsilon$, then the line of non-isolated fixed points vanishes. Thus, former condition establishes a strong structural change (a bifurcation and a structural instability of the system) in the flow of the dynamical system. This can be an indication of the existence of invariant manifolds, that cut this line of fixed points (or not — if the flow is parallel to this line). Also, it is worth to note that second and third equilibria are contained inside the line of non-isolated fixed points, when the condition for its existence is fulfilled.

3.2.2 Invariant surfaces.

Two things make us wonder if $C_1 - C_2$ and $C_2 - C_3$ planes (and the remaining one, too) are invariant: always there are two fixed points inside them (origin and the other equilibrium) and it is somewhat strange to have negative concentrations that pass to positive values or the other way round. Thus, we test their invariance with help of proposition 6 or its corolary. Normal vectors for the three coordinate planes are e_3, e_1, e_2 respectively: the canonical coordinate vectors. Let us check that they are perpendicular to $f(C)$ vector field at each point of the planes[2]. We make dot product

$$\begin{aligned} f(C) \cdot e_3 &= (\zeta - \varepsilon C_2)C_3 \\ f(C) \cdot e_1 &= (\alpha + \beta C_2)C_1 \\ f(C) \cdot e_2 &= (\gamma - \beta C_1 + \varepsilon C_3)C_2 \end{aligned}$$

and then we evaluate over corresponding planes

$$\begin{aligned} [f(C) \cdot e_3](C_1, C_2, 0) &= (\zeta - \varepsilon C_2)0 = 0 \\ [f(C) \cdot e_1](0, C_2, C_3) &= (\alpha + \beta C_2)0 = 0 \\ [f(C) \cdot e_2](C_1, 0, C_3) &= (\gamma - \beta C_1 + \varepsilon C_3)0 = 0 \end{aligned}$$

Thus, we have proven that $f(C)$ is tangent to the coordinate planes. Therefore, these are invariant and contain the flow of the dynamical system in each octant: that is, if we begin in a given octant we will stay on that octant or at the boundary of it (coordinate planes). Thus, if we have positive concentrations of carbon, we will have non-negative concentrations in

[2] In a three-dimensional setting, curves have one-dimensional tangent space and two-dimensional normal space and surfaces have two-dimensional tangent space and one-dimensional normal space.

all the evolution of the system. Also, if one reservoir gets depleted (a catastrophic event in reality), it will never recover.

We summarize, as we have done with equilibria, former results in the following theorem

Theorem 9 *For SODE 1, coordinate planes are invariant. Thus, flow of the dynamical system is constrained to the octants: if the system begins in an octant, it stays in that octant or its boundary (coordinate planes).*

Before we begin to seek other invariant surfaces of this system in a non-educated manner, we can use remarks 3 and 13 to guide our prospects: if we sum SODE 1 equations, we get rid of non-linear terms and obtain the following

$$\frac{d}{dt}(C_1 + C_2 + C_3) = -\alpha C_1 + \gamma C_2 + \zeta C_3 \tag{3}$$

this means that Carbon content is not a conserved quantity, as we proposed when we constructed the model. Now, from former study, we can restate original system in the following manner, considering that $C_i \neq 0$,

$$\begin{cases} \frac{1}{C_1}\dot{C}_1 = -\alpha + \beta C_2 \\ \frac{1}{C_2}\dot{C}_2 = \quad \gamma - \beta C_1 + \varepsilon C_3 \\ \frac{1}{C_3}\dot{C}_3 = \quad \zeta \qquad\quad - \varepsilon C_2 \end{cases}$$

in which left-hand sides can be rewritten as logarithmic derivatives

$$\begin{cases} \frac{d}{dt}\ln C_1 = -\alpha + \beta C_2 \\ \frac{d}{dt}\ln C_2 = \quad \gamma - \beta C_1 + \varepsilon C_3 \\ \frac{d}{dt}\ln C_3 = \quad \zeta \qquad\quad - \varepsilon C_2 \end{cases} \tag{4}$$

Let us try to write the right-hand side of the time derivative of the sum of the carbon stocks with former expressions. From a quick and shallow examination, it is clear that second equation and one of the remaining are enough to write it.

Let us try the first and the second equations of system 4: we need a γC_2, then we multiply first equation by γ/β. We need a $-\alpha C_1$, then we multiply second equation by α/β. We obtain

$$\begin{cases} \frac{\gamma}{\beta}\frac{d}{dt}\ln C_1 = -\alpha\frac{\gamma}{\beta} + \beta\frac{\gamma}{\beta}C_2 \\ \frac{\alpha}{\beta}\frac{d}{dt}\ln C_2 = \quad \gamma\frac{\alpha}{\beta} - \beta\frac{\alpha}{\beta}C_1 + \varepsilon\frac{\alpha}{\beta}C_3 \end{cases}$$

and a simplification gives

$$\begin{cases} \frac{d}{dt}\ln C_1^{\gamma/\beta} = -\frac{\alpha\gamma}{\beta} + \gamma C_2 \\ \frac{d}{dt}\ln C_2^{\alpha/\beta} = \quad \frac{\alpha\gamma}{\beta} - \alpha C_1 + \frac{\alpha\varepsilon}{\beta}C_3 \end{cases}$$

which upon summation gives

$$\frac{d}{dt}\left[\ln\left(C_1^{\gamma/\beta}C_2^{\alpha/\beta}\right)\right] = -\alpha C_1 + \gamma C_2 + \frac{\alpha\varepsilon}{\beta}C_3$$

Substraction of former expression from that of the sum of stocks tells us that

$$\frac{d}{dt}\left[C_1 + C_2 + C_3 - \ln\left(C_1^{\gamma/\beta}C_2^{\alpha/\beta}\right)\right] = \left(\zeta - \frac{\alpha\varepsilon}{\beta}\right)C_3$$

If we expect expression under time derivative to be conserved, right-hand side shall vanish. But $C_3 \neq 0$, therefore $\zeta = \alpha\varepsilon/\beta$. Nevertheless, this is the condition for having the line of fixed points! If condition is fullfilled, former expression means that

$$C_1 + C_2 + C_3 - \ln\left(C_1^{\gamma/\beta}C_2^{\alpha/\beta}\right) = H_1 = \text{cst.}$$

but this expression defines a one-parameter family of surfaces given by

$$g_1(C_1, C_2) = H_1 - (C_1 + C_2) + \ln\left(C_1^{\gamma/\beta}C_2^{\alpha/\beta}\right)$$

where H_1 is the parameter. Thus, we can question ourselves about the possibility that this surfaces are invariant. It is very likely. Let us prove it. We obtain the normal vector from the expression of $g_1(C_1, C_2)$.

$$\begin{aligned} n &= (\partial_1 g_1, \partial_2 g_1, -1) \\ &= \left(-1 + \frac{\gamma}{\beta C_1}, -1 + \frac{\alpha}{\beta C_2}, -1\right) \\ &= -(1,1,1) + \left(\frac{\gamma}{\beta C_1}, \frac{\alpha}{\beta C_2}, 0\right) \end{aligned}$$

and if we make dot product with the vector field 2

$$f(C)\cdot n = -f(C)\cdot(1,1,1) + f(C)\cdot\left(\frac{\gamma}{\beta C_1}, \frac{\alpha}{\beta C_2}, 0\right)$$

but the first term is the equivalent to sum the three original equations of the SODE and the second term is what we have done with logarithmic equations:

$$\begin{aligned} f(C)\cdot n &= \alpha C_1 - \gamma C_2 - \zeta C_3 - \frac{\alpha\gamma}{\beta} + \gamma C_2 + \frac{\alpha\gamma}{\beta} - \alpha C_1 + \frac{\alpha\varepsilon}{\beta}C_3 \\ &= -\left(\zeta - \frac{\alpha\varepsilon}{\beta}\right)C_3 \end{aligned}$$

thus, if line of equilibria exists, then these surfaces are invariant! We summarize these results in the following

Theorem 10 *For SODE 1 where $\zeta = \alpha\varepsilon/\beta$ the quantity*

$$H_1(C) = C_1 + C_2 + C_3 - \ln\left(C_1^{\gamma/\beta} C_2^{\alpha/\beta}\right) \tag{5}$$

is a constant of movement or conserved quantity and it defines invariant surfaces

$$g_1(C_1, C_2) = H_1 - (C_1 + C_2) + \ln\left(C_1^{\gamma/\beta} C_2^{\alpha/\beta}\right)$$

In a similar way, let us construct a similar function using the remaining logarithmic equation (that for $\ln C_3$) — third equation of 4 — and the second one of the system 4. We need a γC_2, then we multiply third equation by $-\gamma/\varepsilon$. Since we need ζC_3, we multiply second equation by ζ/ε. This leads to the following pair of equations

$$\begin{cases} \frac{d}{dt}\ln C_2^{\zeta/\varepsilon} = & \frac{\gamma\zeta}{\varepsilon} - \frac{\beta\zeta}{\varepsilon} C_1 + \zeta C_3 \\ \frac{d}{dt}\ln C_3^{-\gamma/\varepsilon} = -\frac{\gamma\zeta}{\varepsilon} & + \gamma C_2 \end{cases}$$

and, when we sum both equations, we obtain

$$\frac{d}{dt}\left[\ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right)\right] = -\frac{\beta\zeta}{\varepsilon} C_1 + \gamma C_2 + \zeta C_3$$

and substracting this last expression from the sum of reservoirs 3

$$\frac{d}{dt}\left[C_1 + C_2 + C_3 - \ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right)\right] = \left(-\alpha + \frac{\beta\zeta}{\varepsilon}\right) C_1$$

and if we hope that this quantity is conserved, as in the first case, we need that the factor inside parenthesis at the right-hand side vanishes — since $C_3 \neq 0$. But, analogously as before, the vanishing of that factor means that we have the line of fixed points! This makes us clear that this line of fixed points is somewhat special: it gives birth to conserved quantities! As before, conservation means

$$C_1 + C_2 + C_3 - \ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right) = H_2 = cst.$$

and this defines a one-parameter family of surfaces given by

$$g_2(C_2, C_3) = H_2 - (C_2 + C_3) + \ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right)$$

where H_2 is the parameter. Also, let us check it is invariant. The normal vector is

$$\begin{aligned} n &= (-1, \partial_2 g_2, \partial_3 g_2) \\ &= \left(-1, -1 + \frac{\zeta}{\varepsilon C_2}, -1 - \frac{\gamma}{\varepsilon C_3}\right) \\ &= -(1,1,1) + \left(0, \frac{\zeta}{\varepsilon C_2}, -\frac{\gamma}{\varepsilon C_3}\right) \end{aligned}$$

and if we make dot product with $f(C)$

$$f(C) \cdot n = -f(C) \cdot (1,1,1) + f(C) \left(0, \frac{\zeta}{\varepsilon C_2}, -\frac{\gamma}{\varepsilon C_3}\right)$$

but the first term is the equivalent to sum the three original equations of the SODE and the second term is what we have done with logarithmic equations:

$$\begin{aligned} f(C) \cdot n &= \alpha C_1 - \gamma C_2 - \zeta C_3 - \frac{\beta\zeta}{\varepsilon} C_1 + \gamma C_2 + \zeta C_3 \\ &= \left(\alpha - \frac{\beta\zeta}{\varepsilon}\right) C_1 \end{aligned}$$

then these surfaces are also invariant! We summarize these results in the following

Theorem 11 *For SODE 1 where $\zeta = \alpha\varepsilon/\beta$ the quantity*

$$H_2(C) = C_1 + C_2 + C_3 - \ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right) \tag{6}$$

is a constant of movement or conserved quantity and it defines invariant surfaces

$$g_2(C_2, C_3) = H_2 - (C_2 + C_3) + \ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right)$$

Remark 14 The existence of a second family g_2 makes us wonder, if members of both families have points in common. This is important, since if one point is common to both invariant surfaces, then it must remain in both surfaces. The intersection of surfaces is a curve and, therefore, this curve is invariant. But invariant curves, as we have remarked, are solutions to the dynamical system.

Remark 15 Since each of g_1-surfaces is invariant, then flow of SODE is constrained to this surfaces. We also can do a one-one correspondence of this surfaces (and therefore the numerical value of the constant H_1) with points on the line of fixed points: that means that the dynamics near the fixed points are only determined by the dynamics inside each invariant surface. We have reduced a $3-$dimensional problem to a $2-$dimensional situation in the particular case of $\zeta = \alpha\varepsilon/\beta$. That is, since there is a line of fixed points it means that these points have one null eigenvalue and the eigendirection is the vector that determines the line. Since we have found an invariant $2-$manifold associated to each equilibrium in that line, then flow is not parallel to the line and will depend on the character of the two remaining eigenvalues.

Note the form of the conserved quantities 5 and 6. It consists of the sum of the reservoirs minus an expression under a logarithm. The invariant surfaces that they generate and their intersections (which must be solutions) could be difficult to describe. However, at a glance,

if we substract conserved quantity 6 from 5 we obtain another conserved quantity, say H_3, which is

$$\begin{aligned} H_3 = H_2 - H_1 &= -\ln\left(C_2^{\zeta/\varepsilon} C_3^{-\gamma/\varepsilon}\right) + \ln\left(C_1^{\gamma/\beta} C_2^{\alpha/\beta}\right) \\ &= \ln\left(C_1^{\gamma/\beta} C_2^{(\alpha/\beta)-(\zeta/\varepsilon)} C_3^{\gamma/\varepsilon}\right) \end{aligned}$$

but the exponent of C_2 vanishes since we are in the case $\zeta = \alpha\varepsilon/\beta$. Thus, we write

$$\begin{aligned} H_3 &= \ln\left(C_1^{\gamma/\beta} C_3^{\gamma/\varepsilon}\right) \\ H_3\frac{\varepsilon}{\gamma} &= \frac{\varepsilon}{\gamma}\ln\left(C_1^{\gamma/\beta} C_3^{\gamma/\varepsilon}\right) \\ &= \ln\left(C_1^{\varepsilon/\beta} C_3\right) \\ e^{H_3\varepsilon/\gamma} &= C_1^{\varepsilon/\beta} C_3 \therefore \\ C_3 &= e^{H_3\varepsilon/\gamma} C_1^{-\varepsilon/\beta} \end{aligned}$$

Therefore, the invariant surface related to conserved quantity H_3 is a cylinder. That means, there are solutions of the system that lie in those surfaces. Last results just proved are summarized in the following

Theorem 12 *For SODE 1 where $\zeta = \alpha\varepsilon/\beta$, the quantity*

$$H_3(C) = \ln\left(C_1^{\gamma/\beta} C_3^{\gamma/\varepsilon}\right) \tag{7}$$

is a constant of movement or conserved quantity and it defines invariant surfaces

$$g_3(C_1, C_2) = e^{H_3\varepsilon/\gamma} C_1^{-\varepsilon/\beta}$$

Remark 16 From former three invariant surfaces — induced by the constants of movement 5, 6 and 7 —, we acknowledge that, in the case that there exist a line of equilibria, the structure of the solutions is determined by two of the invariant surfaces: solutions are given by their intersections. One of the families is dependent on the other two families.

Now, we shall analyze the stability of equilibria in theorem 8 in order to put former results in context.

3.2.3 Local stability around equilibria.

First we make first order approximation of the SODE around equilibria

$$\begin{aligned} f &\sim f(C_0) + (C - C_0)\nabla f|_{C_0} \\ &\sim 0 + (C - C_0) \begin{pmatrix} -\alpha + \beta C_2 & -\beta C_2 & 0 \\ \beta C_1 & \gamma - \beta C_1 + \varepsilon C_3 & -\varepsilon C_3 \\ 0 & \varepsilon C_2 & \zeta - \varepsilon C_2 \end{pmatrix}\Bigg|_{C_0} \\ &\sim (C - C_0) \begin{pmatrix} -\alpha + \beta C_2 & -\beta C_2 & 0 \\ \beta C_1 & \gamma - \beta C_1 + \varepsilon C_3 & -\varepsilon C_3 \\ 0 & \varepsilon C_2 & \zeta - \varepsilon C_2 \end{pmatrix}\Bigg|_{C_0} \end{aligned}$$

Thus, for each equilibrium in theorem 8, former expression reads

$$(C - C_0) \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \zeta \end{pmatrix},\ C_0 = (0,0,0)$$

$$(C - C_0) \begin{pmatrix} -\alpha + \frac{\beta\zeta}{\varepsilon} & -\frac{\beta\zeta}{\varepsilon} & 0 \\ 0 & 0 & \gamma \\ 0 & \zeta & 0 \end{pmatrix},\ C_0 = (0, \zeta/\varepsilon, -\gamma/\varepsilon)$$

$$(C - C_0) \begin{pmatrix} 0 & -\alpha & 0 \\ \gamma & 0 & 0 \\ 0 & \frac{\alpha\varepsilon}{\beta} & \zeta - \frac{\alpha\varepsilon}{\beta} \end{pmatrix},\ C_0 = (\gamma/\beta, \alpha/\beta, 0)$$

$$(C - C_0) \begin{pmatrix} 0 & -\alpha & 0 \\ \beta\tau & 0 & -\beta\tau + \gamma \\ 0 & \frac{\alpha\varepsilon}{\beta} & 0 \end{pmatrix},\ C_0(\tau) = (\tau, \alpha/\beta, (\beta/\varepsilon)\tau - (\gamma/\varepsilon))$$

the C_0 term is not important, since we can do a translation to make new origin coincide with our fixed point, as we have done in the pendulum case. Thus, we will obtain the eigenvalues of former matrices to analyze local flow.

However, for the sake of clarity, we make analysis in batch. Characteristic polynomials

of former linearized system matrices are

$$\det\begin{pmatrix} -\alpha-\lambda & 0 & 0 \\ 0 & \gamma-\lambda & 0 \\ 0 & 0 & \zeta-\lambda \end{pmatrix} = (-\alpha-\lambda)(\gamma-\lambda)(\zeta-\lambda)$$

$$\det\begin{pmatrix} -\alpha+\frac{\beta\zeta}{\varepsilon}-\lambda & -\frac{\beta\zeta}{\varepsilon} & 0 \\ 0 & -\lambda & \gamma \\ 0 & \zeta & -\lambda \end{pmatrix} = (\lambda-\sqrt{\gamma\zeta})(\lambda+\sqrt{\gamma\zeta})\left(-\alpha+\frac{\beta\zeta}{\varepsilon}-\lambda\right)$$

$$\det\begin{pmatrix} -\lambda & -\alpha & 0 \\ \gamma & -\lambda & 0 \\ 0 & \frac{\alpha\varepsilon}{\beta} & \zeta-\frac{\alpha\varepsilon}{\beta}-\lambda \end{pmatrix} = (\lambda-i\sqrt{\alpha\gamma})(\lambda+i\sqrt{\alpha\gamma})\left(\zeta-\frac{\alpha\varepsilon}{\beta}-\lambda\right)$$

$$\begin{aligned}\det\begin{pmatrix} -\lambda & -\alpha & 0 \\ \beta\tau & -\lambda & -\beta\tau+\gamma \\ 0 & \frac{\alpha\varepsilon}{\beta} & -\lambda \end{pmatrix} &= -\lambda\left[\lambda^2-\frac{\alpha\varepsilon}{\beta}(-\beta\tau+\gamma)\right]-\alpha\beta\tau\lambda \\ &= -\lambda\left[\lambda^2+\alpha\beta\tau+\frac{\alpha\varepsilon}{\beta}(\beta\tau-\gamma)\right]\end{aligned}$$

and, therefore, eigenvalues are

$$\begin{cases} \lambda_1=-\alpha,\ \lambda_2=\gamma,\ \lambda_3=\zeta & C_0=(0,0,0) \\ \lambda_1=-\alpha+\frac{\beta\zeta}{\varepsilon},\ \lambda_{2,3}=\pm\sqrt{\gamma\zeta} & C_0=(0,\zeta/\varepsilon,-\gamma/\varepsilon) \\ \lambda_1=\zeta-\frac{\alpha\varepsilon}{\beta},\ \lambda_{2,3}=\pm i\sqrt{\alpha\gamma} & C_0=(\gamma/\beta,\alpha/\beta,0) \\ \lambda_1=0,\ \lambda_{2,3}=\pm i\sqrt{\alpha\beta\tau+\frac{\alpha\varepsilon}{\beta}(\beta\tau-\gamma)} & C_0(\tau)=(\tau,\alpha/\beta,(\beta/\varepsilon)\tau-(\gamma/\varepsilon)) \end{cases}$$

At the first fixed point, the origin, there is a negative eigenvalue and one positive and another that depends on the sign of γ. Therefore, there is an stable $1-$eigenspace, one unstable $1-$eigenspaces and the other depending on γ (if $\gamma > 0$, there is another unstable $1-$eigenspace; if $\gamma < 0$, there is another stable $1-$eigenspace; or one unstable $2-$eigenspace if $\gamma = \zeta$). The second equilibrium has three different eigenvalues: at least there is one $1-$eigenspace that is unstable, if $\beta\zeta > \alpha\varepsilon$; and stable, if $\beta\zeta < \alpha\varepsilon$. The other eigenvalues are associated with two $1-$eigenspaces: if $\gamma > 0$, they have a saddle-like behaviour; if $\gamma < 0$, they have center-like behavior. The third fixed point introduces a $1-$eigenspace that is unstable or stable relying on the same criteria as for the former fixed point. The two remaining induce a center-like behavior, if $\gamma > 0$; and a saddle-like character, if $\gamma < 0$.

The line of equilibria deserves a separate analysis. We analyse

$$\begin{aligned} D &= \alpha\beta\tau + \frac{\alpha\varepsilon}{\beta}(\beta\tau - \gamma) \\ &= \alpha(\beta + \varepsilon)\tau - \frac{\alpha\varepsilon}{\beta}\gamma \end{aligned}$$

if $D < 0$, eigenvalues $\lambda_{2,3}$ become real, but remain pure imaginary if $D > 0$:

$$\begin{aligned} D &< 0 \\ \alpha(\beta + \varepsilon)\tau - \frac{\alpha\varepsilon}{\beta}\gamma &< 0 \\ (\beta + \varepsilon)\tau - \frac{\varepsilon}{\beta}\gamma &< 0 \\ (\beta + \varepsilon)\tau &< \frac{\varepsilon}{\beta}\gamma \\ \tau &< \frac{\varepsilon}{\beta(\beta + \varepsilon)}\gamma \end{aligned}$$

therefore, for equilibria on the line that have τ satisfiying latter condition, they have eigenspaces with saddle-like behavior related to $\lambda_{2,3}$. If not, they have center-like behavior. If $\gamma > 0$ there is saddle-like behavior at first octant, if $\gamma < 0$, first octant has purely center-like behavior.

We summarize former findings into the next two theorems.

Theorem 13 *For SODE 1 and its equilibria in theorem 8, the linearizations in a neighborhood of each equilibrium give the following system matrices*

1. $(0, 0, 0)$*:*

$$\begin{pmatrix} -\alpha & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \zeta \end{pmatrix}$$

2. $(0, \zeta/\varepsilon, -\gamma/\varepsilon)$*:*

$$\begin{pmatrix} -\alpha + \frac{\beta\zeta}{\varepsilon} & -\frac{\beta\zeta}{\varepsilon} & 0 \\ 0 & 0 & \gamma \\ 0 & \zeta & 0 \end{pmatrix}$$

3. $(\gamma/\beta, \alpha/\beta, 0)$*:*

$$\begin{pmatrix} 0 & -\alpha & 0 \\ \gamma & 0 & 0 \\ 0 & \frac{\alpha\varepsilon}{\beta} & \zeta - \frac{\alpha\varepsilon}{\beta} \end{pmatrix}$$

4. $(\tau, \alpha/\beta, (\beta/\varepsilon)\tau - (\gamma/\varepsilon))$:

$$\begin{pmatrix} 0 & -\alpha & 0 \\ \beta\tau & 0 & -\beta\tau+\gamma \\ 0 & \frac{\alpha\varepsilon}{\beta} & 0 \end{pmatrix}$$

Theorem 14 *For SODE 1 and its equilibria in theorem 8, the linearizations in a neighborhood of each equilibrium give the following eigenvalues and linear stability results*

1. $(0,0,0)$:

$$\lambda_1 = -\alpha \textbf{ (stable)}$$
$$\lambda_2 = \gamma \textbf{ (unstable if } \gamma > 0\textbf{, stable if } \gamma < 0\textbf{)}$$
$$\lambda_3 = \zeta \textbf{ (unstable)}$$

2. $(0, \zeta/\varepsilon, -\gamma/\varepsilon)$:

$$\lambda_1 = -\alpha + \frac{\beta\zeta}{\varepsilon} \textbf{ (unstable if } \beta\zeta > \alpha\varepsilon\textbf{, stable if } \beta\zeta < \alpha\varepsilon\textbf{)}$$
$$\lambda_{2,3} = \pm\sqrt{\gamma\zeta} \textbf{ (saddle-like if } \gamma > 0\textbf{, center-like if } \gamma < 0\textbf{)}$$

3. $(\gamma/\beta, \alpha/\beta, 0)$:

$$\lambda_1 = \zeta - \frac{\alpha\varepsilon}{\beta} \textbf{ (unstable if } \beta\zeta > \alpha\varepsilon\textbf{, stable if } \beta\zeta < \alpha\varepsilon\textbf{)}$$
$$\lambda_{2,3} = \pm i\sqrt{\alpha\gamma} \textbf{ (center-like if } \gamma > 0\textbf{, saddle-like if } \gamma < 0\textbf{)}$$

4. $(\tau, \alpha/\beta, (\beta/\varepsilon)\tau - (\gamma/\varepsilon))$:

$$\lambda_1 = 0 \textbf{ (non-isolated fixed points)}$$
$$\lambda_{2,3} = \pm i\sqrt{\alpha\beta\tau + \frac{\alpha\varepsilon}{\beta}(\beta\tau - \gamma)}$$

which are saddle-like if $\tau < \varepsilon\gamma/(\beta(\beta+\varepsilon))$ *and center-like for the other option; where, if* $\gamma < 0$*, all fixed points in first octant have center-like behavior.*

3.2.4 Stable, unstable and center-like subspaces.

Once we have made the analysis of the stability with a linear approximation, let us look for the eigenspaces of which we have declared their existence in former analysis. For that end, we shall obtain eigenvectors associated with eigenvalues. Thus, we need to find the vectors u that make the following: $\Lambda(u) = \lambda u$, where Λ is the linear function that represents

linearization of the SODE. Former expression can be written as $\Lambda(u) - \lambda\,\mathrm{id}(u) = 0$ or $S(u) = 0$ where $S = \Lambda - \lambda\,\mathrm{id}$. That is, we seek the kernel of the linear function S or the kernel of any matrix that represents S in a particular basis, e.g. $A - \lambda I$. Let us rewrite linearized system matrices in terms of their eigenvalues to ease this task.

Theorem 15 *For SODE 1, and its equilibria in theorem 8, the linearizations in a neighborhood of each equilibrium give the following system matrices in terms of eigenvalues.*

1. $(0,0,0)$*:*

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

2. $(0, \zeta/\varepsilon, -\gamma/\varepsilon)$*:*

$$\begin{pmatrix} \lambda_1 & -(\alpha+\lambda_1) & 0 \\ 0 & 0 & \frac{\lambda_{2,3}^2}{\zeta} \\ 0 & \frac{\lambda_{2,3}^2}{\gamma} & 0 \end{pmatrix}$$

3. $(\gamma/\beta, \alpha/\beta, 0)$*:*

$$\begin{pmatrix} 0 & \frac{\lambda_{2,3}^2}{\gamma} & 0 \\ -\frac{\lambda_{2,3}^2}{\alpha} & 0 & 0 \\ 0 & -(\lambda_1 - \zeta) & \lambda_1 \end{pmatrix}$$

4. $(\tau, \alpha/\beta, (\beta/\varepsilon)\tau - (\gamma/\varepsilon))$*:*

$$\begin{pmatrix} 0 & \frac{\lambda_{2,3}^2}{\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)} & 0 \\ \gamma - \frac{\beta}{\alpha\varepsilon}(\lambda_{2,3}^2 + \alpha\beta\tau) & 0 & \frac{\beta}{\alpha\varepsilon}(\lambda_{2,3}^2 + \alpha\beta\tau) \\ 0 & -\frac{\varepsilon}{\beta}\frac{\lambda_{2,3}^2}{\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)} & 0 \end{pmatrix}$$

and we analyze the kernels of each $A - \lambda_i I$ matrix. We begin with the origin

$$A - \lambda_i I = \begin{pmatrix} \lambda_1 - \lambda_i & 0 & 0 \\ 0 & \lambda_2 - \lambda_i & 0 \\ 0 & 0 & \lambda_3 - \lambda_i \end{pmatrix}$$

then, it follows that

$$((\lambda_1 - \lambda_i)C_1, (\lambda_2 - \lambda_i)C_2, (\lambda_3 - \lambda_i)C_3) = (0,0,0) \tag{8}$$

and if $i = 1$

$$(0, (\lambda_2 - \lambda_i)C_2, (\lambda_3 - \lambda_i)C_3) = (0,0,0) \longleftrightarrow C_2 = C_3 \equiv 0$$

then eigenvectors are of the form $(C_1, 0, 0)$ or we say that eigenspace is $U_1 = \langle\{(1,0,0)\}\rangle$. In an entirely analogous manner, it follows from 8 that $U_2 = \langle\{(0,1,0)\}\rangle$ and $U_3 = \langle\{(0,0,1)\}\rangle$.

We come to the second fixed point and we get

$$A - \lambda_i I = \begin{pmatrix} \lambda_1 - \lambda_i & -(\alpha + \lambda_1) & 0 \\ 0 & -\lambda_i & \frac{\lambda_{2,3}^2}{\zeta} \\ 0 & \frac{\lambda_{2,3}^2}{\gamma} & -\lambda_i \end{pmatrix}$$

and we write down transformed vector

$$\left((\lambda_1 - \lambda_i)C_1, -(\alpha + \lambda_1)C_1 - \lambda_i C_2 + \frac{\lambda_{2,3}^2}{\gamma}C_3, \frac{\lambda_{2,3}^2}{\zeta}C_2 - \lambda_i C_3\right) = (0,0,0) \qquad (9)$$

and if $i = 1$ former expression reduces to

$$\left(0, -(\alpha + \lambda_1)C_1 - \lambda_i C_2 + \frac{\lambda_{2,3}^2}{\gamma}C_3, \frac{\lambda_{2,3}^2}{\zeta}C_2 - \lambda_i C_3\right) = (0,0,0)$$

From third component of reduced 9, we can put C_3 in terms of C_2

$$\begin{aligned} \frac{\lambda_{2,3}^2}{\zeta}C_2 - \lambda_i C_3 &= 0 \\ \lambda_i C_3 &= \frac{\lambda_{2,3}^2}{\zeta}C_2 \\ C_3 &= \frac{\lambda_{2,3}^2}{\zeta\lambda_i}C_2 \end{aligned}$$

and from second component of reduced 9, we put C_2 in terms of C_1 eliminating C_3 from

former expression

$$
\begin{aligned}
-(\alpha+\lambda_1)C_1-\lambda_i C_2+\frac{\lambda_{2,3}^2}{\gamma}C_3&=0\\
-(\alpha+\lambda_1)C_1-\lambda_i C_2+\frac{\lambda_{2,3}^4}{\gamma\zeta\lambda_i}C_2&=0\\
-(\alpha+\lambda_1)C_1-\lambda_i C_2+\frac{\lambda_{2,3}^2}{\lambda_i}C_2&=0\\
-(\alpha+\lambda_1)C_1+\frac{\lambda_{2,3}^2-\lambda_i^2}{\lambda_i}C_2&=0\\
\frac{\lambda_{2,3}^2-\lambda_i^2}{\lambda_i}C_2&=(\alpha+\lambda_1)C_1\\
C_2&=\frac{\lambda_i(\alpha+\lambda_1)}{\lambda_{2,3}^2-\lambda_i^2}C_1
\end{aligned}
$$

which leads to

$$C_3=\frac{\gamma(\alpha+\lambda_1)}{\lambda_{2,3}^2-\lambda_i^2}C_1$$

and we conclude that eigenvectors have the form

$$C_1\left(1,\frac{\lambda_i(\alpha+\lambda_1)}{\lambda_{2,3}^2-\lambda_i^2},\frac{\gamma(\alpha+\lambda_1)}{\lambda_{2,3}^2-\lambda_i^2}\right)$$

or

$$C_1\frac{\alpha+\lambda_1}{\lambda_{2,3}^2-\lambda_i^2}\left(\frac{\lambda_{2,3}^2-\lambda_i^2}{\alpha+\lambda_1},\lambda_i,\gamma\right)$$

and, therefore, the related eigenspace is

$$U_1=\left\langle\left\{\left(\frac{\lambda_{2,3}^2-\lambda_1^2}{\alpha+\lambda_1},\lambda_1,\gamma\right)\right\}\right\rangle$$

Now, we do the same for the other eigenvalues at once. First component of 9, since $\lambda_1\neq\lambda_{2,3}$, gives $C_1\equiv 0$. The expression for C_3, that we have derived for the $i=1$ case, is already valid and it reads

$$C_3=\frac{\lambda_{2,3}}{\zeta}C_2$$

Also, expression for C_2 is valid, but previous manipulations:

$$\frac{\lambda_{2,3}^2 - \lambda_i^2}{\lambda_i} C_2 = 0$$

expression that leads to saying that C_2 is arbitrary. Therefore, eigenvectors for $i = 2, 3$ are of the form

$$C_2 \left(0, 1, \frac{\lambda_{2,3}}{\zeta}\right)$$

and the related eigenspaces are

$$U_{2,3} = \left\langle \left\{ \left(0, 1, \frac{\lambda_{2,3}}{\zeta}\right) \right\} \right\rangle$$

Once we are seasoned in the mechanics of this operations, we delve into the next fixed point, which has matrix

$$A - \lambda_i I = \begin{pmatrix} -\lambda_i & \frac{\lambda_{2,3}^2}{\gamma} & 0 \\ -\frac{\lambda_{2,3}^2}{\alpha} & -\lambda_i & 0 \\ 0 & -(\lambda_1 - \zeta) & \lambda_1 - \lambda_i \end{pmatrix}$$

and the related transformed vector is

$$\left(-\lambda_i C_1 - \frac{\lambda_{2,3}^2}{\alpha} C_2, \frac{\lambda_{2,3}^2}{\gamma} C_1 - \lambda_i C_2 - (\lambda_1 - \zeta) C_3, (\lambda_1 - \lambda_i) C_3\right) = (0, 0, 0) \tag{10}$$

However, in this chance, we begin with the first component and get C_1 in terms of C_2

$$\begin{aligned} -\lambda_i C_1 - \frac{\lambda_{2,3}^2}{\alpha} C_2 &= 0 \\ \lambda_i C_1 &= -\frac{\lambda_{2,3}^2}{\alpha} C_2 \\ C_1 &= -\frac{\lambda_{2,3}^2}{\alpha \lambda_i} C_2 \end{aligned}$$

Readily, we plug former result in the second component of 10 and reduce

$$\begin{aligned}
\frac{\lambda_{2,3}^2}{\gamma}C_1 - \lambda_i C_2 - (\lambda_1 - \zeta)C_3 &= 0 \\
-\frac{\lambda_{2,3}^4}{\alpha\gamma\lambda_i}C_2 - \lambda_i C_2 - (\lambda_1 - \zeta)C_3 &= 0 \\
\frac{\lambda_{2,3}^2}{\lambda_i}C_2 - \lambda_i C_2 - (\lambda_1 - \zeta)C_3 &= 0 \\
\frac{\lambda_{2,3}^2 - \lambda_i^2}{\lambda_i}C_2 - (\lambda_1 - \zeta)C_3 &= 0 \\
\frac{\lambda_{2,3}^2 - \lambda_i^2}{\lambda_i}C_2 &= (\lambda_1 - \zeta)C_3 \\
C_2 &= \frac{\lambda_i(\lambda_1 - \zeta)}{\lambda_{2,3}^2 - \lambda_i^2}C_3
\end{aligned}$$

and C_1 is in terms of C_3

$$C_1 = \frac{\gamma(\lambda_1 - \zeta)}{\lambda_{2,3}^2 - \lambda_i^2}C_3$$

Thus, if $i = 1$, third component of 10 says that C_3 is arbitrary and eigenvectors are of the form

$$C_3\left(\frac{\gamma(\lambda_1 - \zeta)}{\lambda_{2,3}^2 - \lambda_i^2}, \frac{\lambda_i(\lambda_1 - \zeta)}{\lambda_{2,3}^2 - \lambda_i^2}, 1\right)$$

or

$$C_3\frac{\lambda_1 - \zeta}{\lambda_{2,3}^2 - \lambda_i^2}\left(\gamma, \lambda_i, \frac{\lambda_{2,3}^2 - \lambda_i^2}{\lambda_1 - \zeta}\right)$$

and the related eigenspace results

$$U_1 = \left\langle \left\{ \left(\gamma, \lambda_1, \frac{\lambda_{2,3}^2 - \lambda_1^2}{\lambda_1 - \zeta}\right) \right\} \right\rangle$$

If we make an analogy with the former fixed point; if $i = 2, 3$, we obtain that third component of 10 says that $C_3 \equiv 0$ since $\lambda_1 \neq \lambda_{2,3}$. Expression for C_1 is already valid and it reads

$$C_1 = -\frac{\lambda_{2,3}}{\alpha}C_2$$

and we obtain that C_2 is arbitrary from second component (before reduction). Therefore, eigenvectors take the form

$$C_2\left(-\frac{\lambda_{2,3}}{\alpha},1,0\right)$$

Therefore, the corresponding eigenspaces are

$$U_{2,3}=\left\langle\left\{\left(-\frac{\lambda_{2,3}}{\alpha},1,0\right)\right\}\right\rangle$$

Finally, we will do the same operations for the case of the line of equilibria. Matrix is written as

$$A-\lambda_i I=\begin{pmatrix} -\lambda_i & \frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)} & 0 \\ \gamma-\frac{\beta}{\alpha\varepsilon}(\lambda_{2,3}^2+\alpha\beta\tau) & -\lambda_i & \frac{\beta}{\alpha\varepsilon}(\lambda_{2,3}^2+\alpha\beta\tau) \\ 0 & -\frac{\varepsilon}{\beta}\frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)} & -\lambda_i \end{pmatrix}$$

expression that leads to the following three equations

$$-\lambda_i C_1+\left(\gamma-\frac{\beta}{\alpha\varepsilon}(\lambda_{2,3}^2+\alpha\beta\tau)\right)C_2=0 \tag{11}$$

$$\frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)}C_1-\lambda_i C_2-\frac{\varepsilon}{\beta}\frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)}C_3=0 \tag{12}$$

$$\frac{\beta}{\alpha\varepsilon}(\lambda_{2,3}^2+\alpha\beta\tau)C_2-\lambda_i C_3=0 \tag{13}$$

and from equation 13, we find that

$$C_3=\frac{\beta}{\alpha\varepsilon\lambda_i}(\lambda_{2,3}^2+\alpha\beta\tau)C_2$$

Similarly, from equation 11

$$\begin{aligned} C_1&=\left(\frac{\gamma}{\lambda_i}-\frac{\beta}{\alpha\varepsilon\lambda_i}(\lambda_{2,3}^2+\alpha\beta\tau)\right)C_2 \\ &=\frac{\gamma}{\lambda_i}C_2-C_3 \end{aligned}$$

If we plug former expression into equation 12

$$\frac{\gamma}{\lambda_i}\frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)}C_2-\frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)}C_3-$$
$$\lambda_i C_2-\frac{\varepsilon}{\beta}\frac{\lambda_{2,3}^2}{\beta\tau+\frac{\varepsilon}{\beta}(\beta\tau-\gamma)}C_3=0$$

collecting terms

$$\frac{\gamma\lambda_{2,3}^2 - \left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\lambda_i^2}{\lambda_i\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]}C_2 - \left(1 + \frac{\varepsilon}{\beta}\right)\frac{\lambda_{2,3}^2}{\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)}C_3 = 0$$

and we conclude that

$$\left(1 + \frac{\varepsilon}{\beta}\right)\frac{\lambda_{2,3}^2}{\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)}C_3 = \frac{\gamma\lambda_{2,3}^2 - \left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\lambda_i^2}{\lambda_i\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]}C_2$$

$$\left(1 + \frac{\varepsilon}{\beta}\right)\lambda_{2,3}^2 C_3 = \frac{\gamma\lambda_{2,3}^2 - \left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\lambda_i^2}{\lambda_i}C_2$$

$$C_3 = \frac{\beta}{(\beta + \varepsilon)\lambda_{2,3}^2}\frac{\gamma\lambda_{2,3}^2 - \left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\lambda_i^2}{\lambda_i}C_2$$

$$= \frac{\gamma}{\lambda_i}\frac{\beta\lambda_{2,3}^2 - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\lambda_i^2}{(\beta + \varepsilon)\lambda_{2,3}^2}C_2$$

and

$$C_1 = \frac{\gamma}{\lambda_i}\frac{\varepsilon\lambda_{2,3}^2 - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\lambda_i^2}{(\beta + \varepsilon)\lambda_{2,3}^2}C_2$$

If $i = 1$, former expressions are not valid since $\lambda_1 = 0$. However, in that case equations 11 and 13 lead to the vanishing of C_2, and equation 12 reduces to $C_3 = (\beta/\varepsilon)C_1$, then eigenvectors are of the form $C_1(1, 0, \beta/\varepsilon)$ and $U_1 = \langle\{(1, 0, \beta/\varepsilon)\}\rangle$. Note that this is the direction vector of the line of fixed points.

If $i = 2, 3$, then expressions we derived are valid and become

$$C_1 = \frac{\gamma}{\lambda_{2,3}}\frac{\varepsilon - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]}{\beta + \varepsilon}C_2$$

$$C_3 = \frac{\gamma}{\lambda_{2,3}}\frac{\beta - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]}{\beta + \varepsilon}C_2$$

and eigenvectors have the form

$$\left(\varepsilon - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right], \frac{\beta + \varepsilon}{\gamma}\lambda_{2,3}, \beta - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right]\right)$$

Therefore, eigenspaces become

$$U_{2,3} = \left\langle \left\{ \left(\varepsilon - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right], \frac{\beta+\varepsilon}{\gamma}\lambda_{2,3}, \beta - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right] \right) \right\} \right\rangle$$

We summarize proven results in the following

Theorem 16 *For SODE 1 and its equilibria in theorem 8 the linearizations in a neighborhood of each equilibria give the following eigenspaces, related to the corresponding eigenvalues*

1. $(0,0,0)$*:*

$$U_1 = \langle\{e_1\}\rangle$$
$$U_2 = \langle\{e_2\}\rangle$$
$$U_3 = \langle\{e_3\}\rangle$$

2. $(0, \zeta/\varepsilon, -\gamma/\varepsilon)$*:*

$$U_1 = \left\langle \left\{ \left(\frac{\lambda_{2,3}^2 - \lambda_1^2}{\alpha + \lambda_1}, \lambda_1, \gamma \right) \right\} \right\rangle$$
$$U_{2,3} = \left\langle \left\{ \left(0, 1, \frac{\lambda_{2,3}}{\zeta} \right) \right\} \right\rangle$$

3. $(\gamma/\beta, \alpha/\beta, 0)$*:*

$$U_1 = \left\langle \left\{ \left(\gamma, \lambda_1, \frac{\lambda_{2,3}^2 - \lambda_1^2}{\lambda_1 - \zeta} \right) \right\} \right\rangle$$
$$U_{2,3} = \left\langle \left\{ \left(-\frac{\lambda_{2,3}}{\alpha}, 1, 0 \right) \right\} \right\rangle$$

4. $(\tau, \alpha/\beta, (\beta/\varepsilon)\tau - (\gamma/\varepsilon))$*:*

$$U_1 = \langle\{(1, 0, \beta/\varepsilon)\}\rangle$$
$$U_{2,3} = \left\langle \left\{ \left(\varepsilon - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right], \frac{\beta+\varepsilon}{\gamma}\lambda_{2,3}, \beta - \frac{\beta}{\gamma}\left[\beta\tau + \frac{\varepsilon}{\beta}(\beta\tau - \gamma)\right] \right) \right\} \right\rangle$$

Let us now identify manifolds, if possible, and discuss the dynamics.

3.2.5 Stable, unstable and center manifolds.

We have said in section 3.1.3 that manifolds related to stable and unstable cases are unique and manifolds related to center-like behavior may not be unique. Taking this into account, let us compare what we know from invariant manifolds and our previous work regarding stability in the linear approximation.

We begin with the origin. Eigenspaces are generated by the canonical vectors (theorem 16). And we have found that invariant surfaces through the origin are the coordinate planes, their intersections being the axes, which are lines whose direction vectors are the canonical vectors. Therefore, axes are solutions of the dynamical system. This can be seen in figures 9-12.

For the second and third fixed points, invariant manifolds coincide with manifolds in which center or saddle-like behavior takes place, depending on the sign of γ. However, only one of such fixed points is present at the first octant not matter if γ is positive or negative. The fixed point at the first octant has always a center-like behavior at the linear approximation. Therefore, at least for these points, $C_1 - C_2$ or $C_2 - C_3$ planes are center manifolds. In that cases, it remains to know, if in non-linear regime, those fixed points at the first octant are centers or foci. The other manifold will determine if the flux will be towards that plane or away from it depending on stability. However, with the information retrieved in former sections, there is no hint of this.

Until now, the most striking feature of Carbon Subsystem Model is the existence of a line of equilibria, subjected to the condition that $\zeta = \alpha\varepsilon/\beta$. In this case, the system has three families of invariant surfaces induced by three non-independent constants of motion. Constant given by equation 7 sheds light into the structure of the field: it is organized in those hyperbolic-like cylinders, g_3 surfaces, given by the invariant surfaces induced by that conserved quantity. Figures 9-12, show the differences between the phase space with $\zeta = \alpha\varepsilon/\beta$ and the cases where former condition is not fulfilled. It is evident the aforementioned organization of the vector field in the cases where the condition is present. In all the aformentioned figures, $\zeta > \alpha\varepsilon/\beta$. Therefore for $\zeta < \alpha\varepsilon/\beta$ flow will go towards coordinate planes, in opposition to the depicted behavior.

Now, each equilibrium point on the line — at the interior of the first octant — lies in one and only one g_3. Therefore, in each g_3 lies all the evolution around corresponding equilibrium points. Thus, we can think that these surfaces are center manifolds consistent with the previous linear local analysis. We want to know if there exist periodic solutions (or closed orbits).

Every single point of the first octant has a well-defined value of each H_1, H_2 and H_3. Therefore, the families of surfaces induced by H_i, that is g_i, do not intersect each other within the same family: since they are the level surfaces of H_i. From former reasoning,

each equilibrium has its own triplet of surfaces g_1, g_2, g_3 that contain it. We say that at each equilibrium, the corresponding g_i-triplet — and the corresponding pairs — intersect in only one point: the fixed point itself. If it were not true, then there would be a intersection that must be a solution of the system, which renders the fixed point to be a non-fixed point: that is a contradiction.

Let c be a fixed point on the line, $g_{3,c}$ be the surface that passes through it. Therefore, any point on that surface will have a different value of H_1 than the value of $H_{1,c}$ assigned to the fixed point. Take the surface $g_{1,c}$ of all fixed points such that H_1: $g_{1,c} = H_{1,c} - H_1(C_1, C_2, 0)$. For any other point C, in particular on $g_{3,c}$, those which have a certain $H_1 = h_1 = cst.$ lie on the surface $g_1 = h_1 - H_1(C_1, C_2, 0)$. Thus, $g_1 - g_{1,c} = h_1 - H_{1,c}$. We analyze cases: if $h_1 = H_{1,c}$, both surfaces are the same; if $h_1 < H_{1,c}$, g_1 is below $g_{1,c}$, which means that $g_1 \cap g_{3,c} = \varnothing$ since $g_1 \cap g_{1,c} = \varnothing$, because they are level surfaces of H_1; finally, if $h_1 > H_{1,c}$, g_1 is above $g_{1,c}$ and there must be a non-empty $g_1 \cap g_{3,c}$, in which $c \notin g_1 \cap g_{3,c}$.

Former paragraph is a proof that the function $\hat{H}_1 = H_1 - H_{1,c}$ is positive on $g_{3,c}\backslash\{c\}$ and satisfies that $\hat{H}_1(c) = H_1(c) - H_{1,c} = H_{1,c} - H_{1,c} = 0$. Direct consequence of the definition of $\hat{H}_1$ is that, it is another constant of motion, because we are only substracting a constant $H_{1,c}$ to H_1 (which is already a conserved quantity).

In order to see the purpose of the construction of $\hat{H}_1$, we state — without proof — a theorem, due to Aleksandr Lyapunov (Perko, 2001), and an auxiliary definition.

Definition 8 [Lyapunov functions] *Let $X \subseteq \mathbb{R}^n$ be an open subset that contains a point x_0 and let $\dot{x} = f \in C^1(X)$ such that $f(x_0) = 0$ (x_0 is a fixed point of f). A function $L \in C^1(X)$ such that $L(x_0) = 0$ and $L(x) > 0$ for all $x \in X\backslash\{x_0\}$ is called a Lyapunov function.*

Theorem 17 [Lyapunov stability criteria] *Let $X \subseteq \mathbb{R}^n$ be an open subset that contains a point x_0, let $\dot{x} = f \in C^1(X)$ such that $f(x_0) = 0$ and L be a Lyapunov function in X. If $\dot{L} \leqslant 0$, $\forall x \in X$, then x_0 is stable; if $\dot{L} < 0$ for $x \in X\backslash\{x_0\}$, then x_0 is asymptotically stable and if $\dot{L} > 0$ for $x \in X\backslash\{x_0\}$, then x_0 is unstable.*

Therefore, we have constructed a Lyapunov function — namely $\hat{H}$ — over the surface g_3, which can be taken as the space X since flow is constrained to this surface. But this function has a vanishing time derivative as we have proven previous to the establishment of Lyapunov stability criteria. In consequence,

Corolary 18 [Center-like dynamics of Carbon Subsystem] *Any equilibrium on the line of fixed points for the Carbon Subsystem Model, that has center-like linearization, behaves as a center (in the non-linear flow) on the family of surfaces g_3. Then these surfaces are filled with a continuum of closed orbits.*

This is a surprising result! Former corolary has a graphical representation in the figures

9 and 11, where it is possible to infer, from vector field, closed orbits surrounding line of equilibria in each case.

3.2.6 Physical consequences of former mathematical analysis.

Former analysis can be extended in order to obtain periods and amplitudes of the periodic solutions $g_1 \cap g_{3,c}$ or, at least approximations, in which we would see how flows of Carbon between Continent, Atmosphere and Ocean generate different periodicities. These will be in terms of the H_i, which are heavily relying on parameters (flows of carbon). This could be a future line of research.

However, the main results of this chapter — and maybe of this whole work — are the consequences of the corolary 18: for certain equivalence of ratios of the flows — the ratio of the productivity of the Ocean to its interaction with Atmosphere be equal to the ratio of the productivity of Continent to its interaction with Atmosphere, $\zeta/\varepsilon = \alpha/\beta$ — yields that Carbon Subsystem has a periodic evolution. If equivalence of ratios is not fulfilled, Carbon Subsystem renders aperiodic and spirals down or goes away. Then, there are two phases: one in which organization leads to periodicity (periodic phase) and one where there are no constants of motion and leads to decay (aperiodic phase). The model is structurally unstable. But this structural instability was discussed in chapter 1, where we presented the Mid-Pleistocene Transition. Structural instability exists in reality, we are only modeling it: in the context of the presented model Mid-Pleistocen Transition and similar events have an immediate explanation as phase transitions due to structural instability.

Transition itself is a lapse when the equivalence of ratios is not fulfilled, then the system migrates between invariant surfaces — which in aperiodic phase are no longer invariant. Internal mechanisms tend to restore the equivalence of ratios and system again gets in the periodic phase, but with different periodicity and amplitude. Some lapses that are in periodic phase are the glacial ages. These glacial ages contain several cycles of the periodic phase, which are the glacial cycles.

Short-term evolution depends on other internal processes. These processes alter flows of Carbon as perturbations. If the perturbations are sufficiently large, then a phase change to aperiodic phase is induced. If perturbations are relatively small, they give the in-cycle variability seen in Vostok proxy. Therefore, presented model is equivalent to the idea of the model of a large central object, e.g. the Sun, and the point-like object, e.g. the Earth, in which the orbit of the Earth is an ellipse. When we put in action the gravitational influences of the other planets and the Moon, orbit of the Earth is no longer an ellipse nor is closed. In spite of this, we can say that, at any instant, Earth go through an "instantaneous" ellipse: the whole orbit is made of bits of ellipses. In the context of our model, Carbon Subsystem follows trajectories that have "noise" because there are perturbations. However,

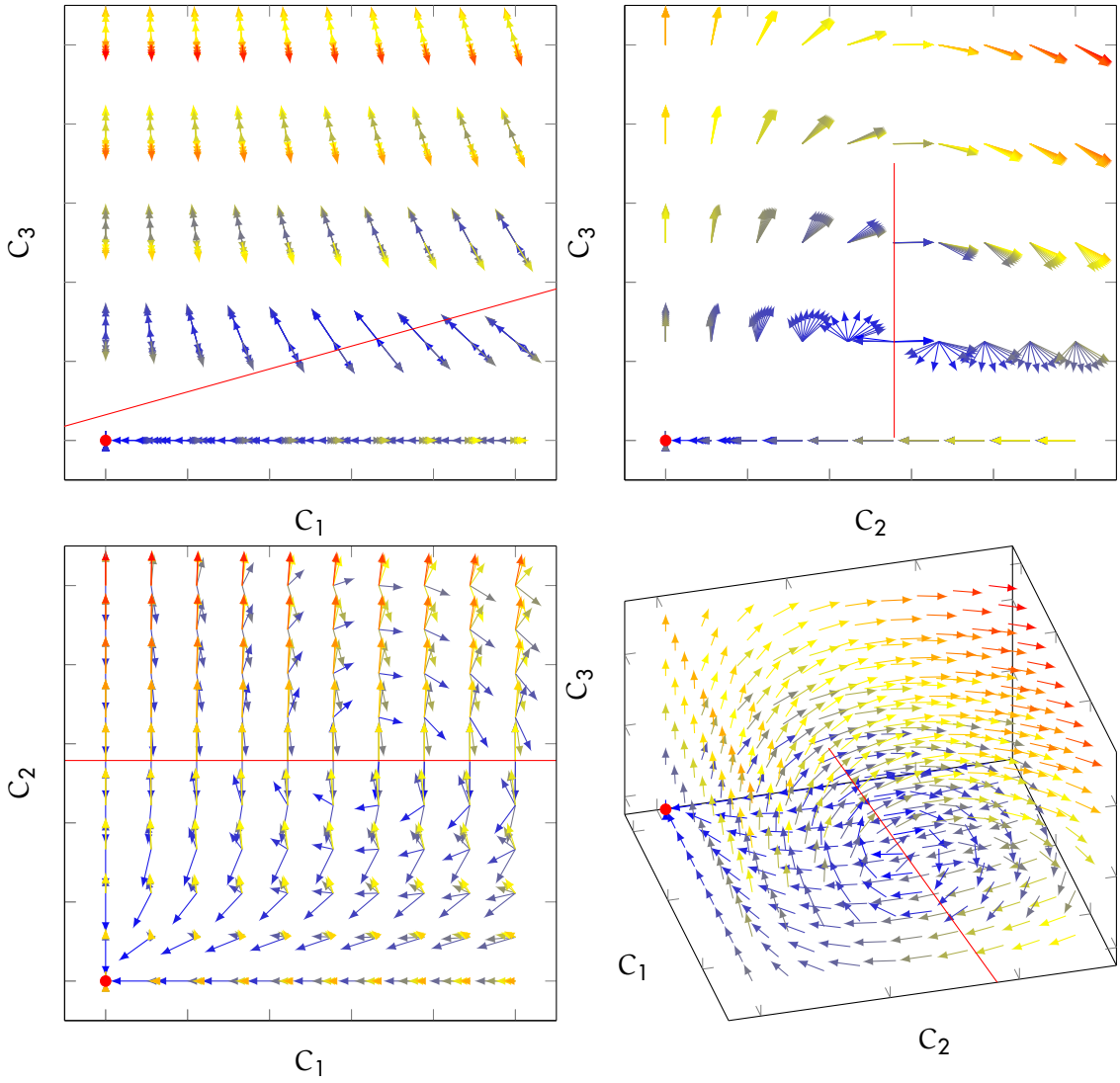

Figure 9: Phase space of Carbon Subsystem with $\zeta = \alpha\varepsilon/\beta$ and $\gamma < 0$

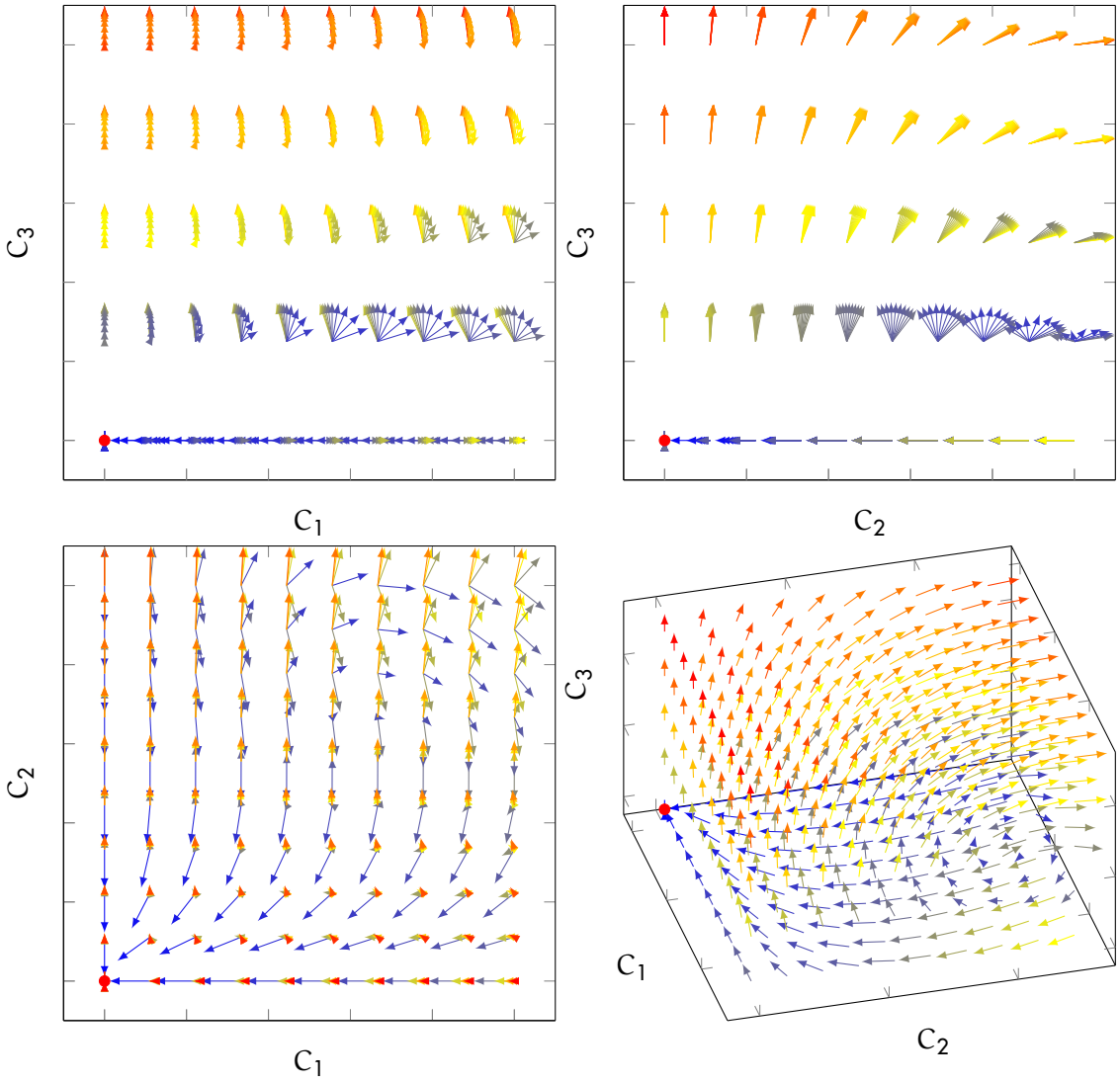

Figure 10: Phase space of Carbon Subsystem with $\zeta \neq \alpha\varepsilon/\beta$ and $\gamma < 0$

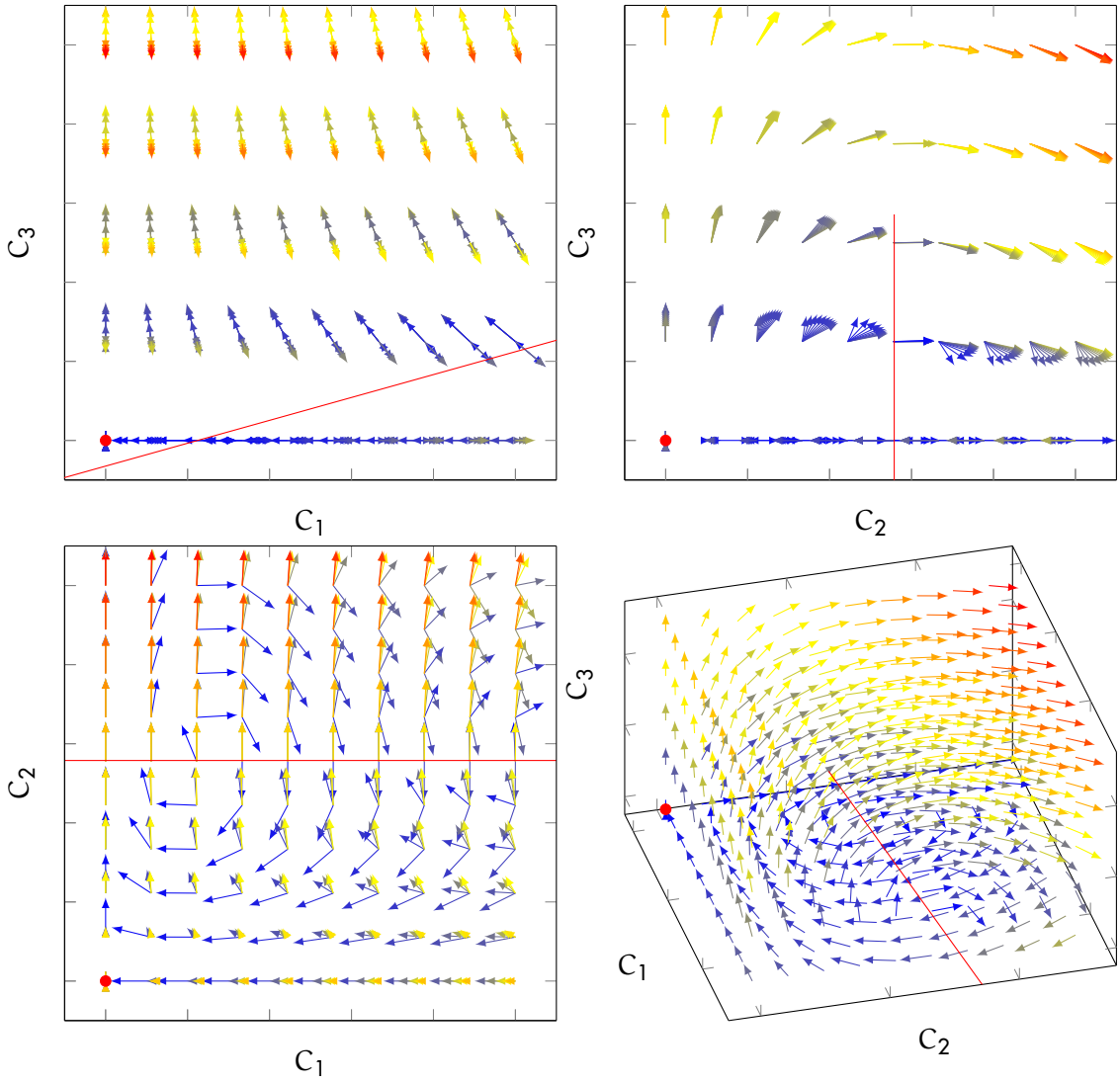

Figure 11: Phase space of Carbon Subsystem with $\zeta = \alpha\varepsilon/\beta$ and $\gamma > 0$

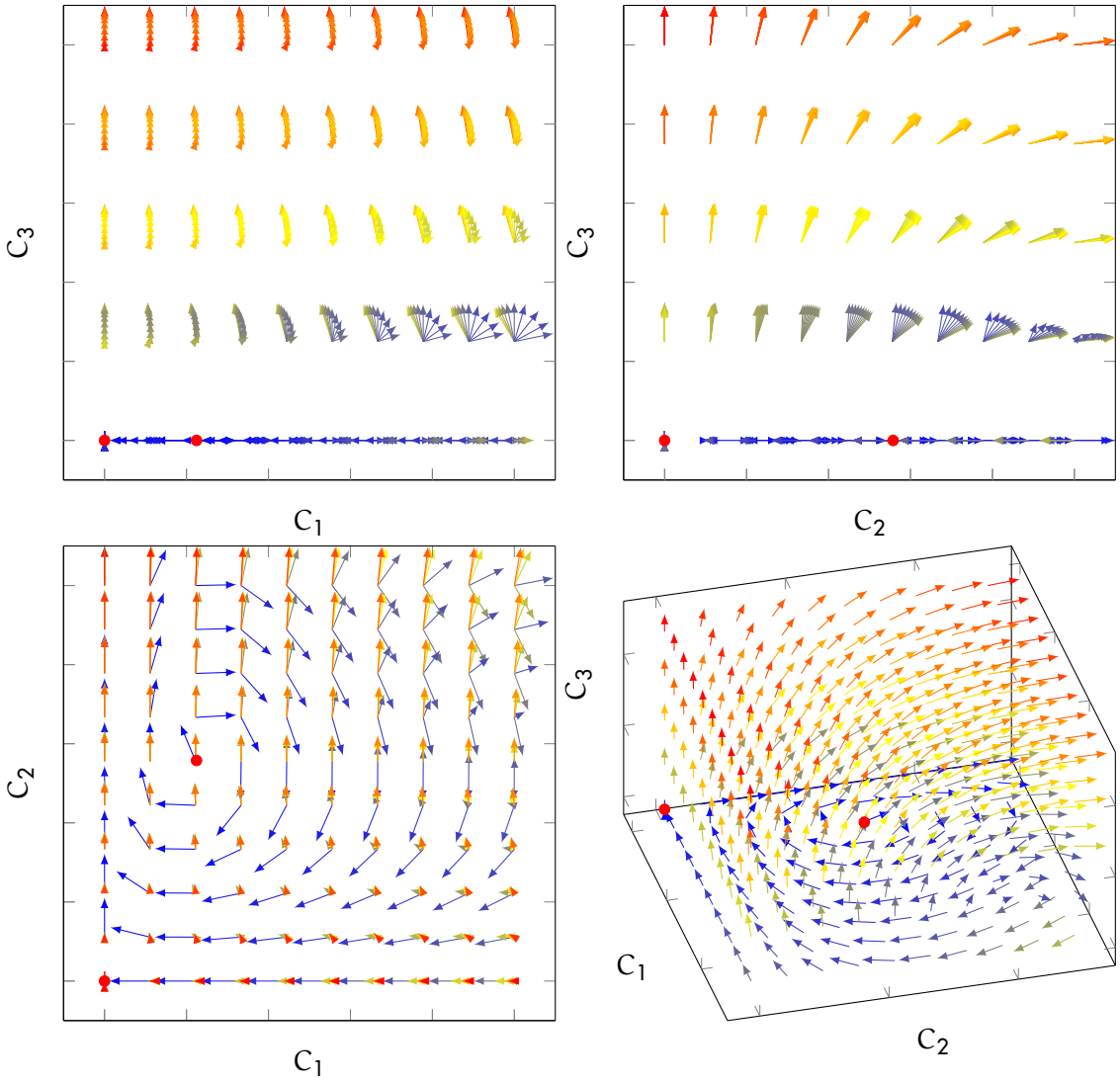

Figure 12: Phase space of Carbon Subsystem with $\zeta \neq \alpha\varepsilon/\beta$ and $\gamma > 0$

the model with constant coefficients we have presented and analyzed shows some sort of mean behavior of the Subsystem.

4 Modeling Energy Subsystem.

> Wir müssen wissen — wir werden wissen!
>
> *(David Hilbert (1862-1943))*

Energy is the key to the Earth System as we know it. It starts and keeps running Atmosphere dynamics and Ocean circulation and most of biological activity. In particular, it determines Surface (both Continent and Ocean) temperature — amongst other factors. The main energy source — on which Earth System relies — is, unsurprisingly, the Sun. Stars emit electromagnetic radiation — due to thermonuclear reactions inside them, mantained by equilibria of diverse nature, e.g. equilibrium between internal thermal pressure and gravitational collapse — in a characteristic spectrum, which is more or less related to surface temperature and stellar atmosphere composition.

In the case of the Sun, it is a main sequence G2 spectral type star: that means it has an effective temperature ($T_{\text{eff},\odot}$) about 5778K. Its emission spectrum can be approximated by that of a blackbody with this temperature. Shifts on this basic behavior are due to absortion by stellar atmosphere and the fact that blackbody is an idealization. However, we will use it.

Hypothesis 3 *Sun radiates as a spherical blackbody at $T_{\text{eff},\odot}$.*

With this in mind, we begin a series of calculations.

Lemma 19 *Sun luminosity — total energetic output — is*

$$L_\odot = 4\pi r_\odot^2 \sigma T_{\text{eff},\odot}^4 \tag{14}$$

where $r_\odot$ is Sun's radius and σ is Stefan-Boltzmann constant.

Proof. By hypothesis 3, solar output per unit area is

$$\sigma T_{\text{eff},\odot}^4$$

and we multiply this by Sun's area

$$4\pi r_\odot^2$$

we obtain

$$4\pi r_{\odot}^2 \sigma T_{\text{eff},\odot}^4$$

which turns out is the total energetic output, as we wanted to show. □

Lemma 20 *Solar irradiance — energetic output distributed at a sphere of radius* r *— is*

$$I_{\odot} = \left(\frac{r_{\odot}}{r}\right)^2 \sigma T_{\text{eff},\odot}^4 \tag{15}$$

Proof. From former lemma, we know the total output. Since by hypothesis 3 the output is assumed homogeneous and isotropic, if we divide luminosity by the area of the sphere of radius r we obtain the irradiance.

$$\begin{aligned} I_{\odot} &= \frac{L_{\odot}}{4\pi r^2} \\ &= \left(\frac{r_{\odot}}{r}\right)^2 \sigma T_{\text{eff},\odot}^4 \end{aligned}$$

as we required. □

Corolary 21 *At* $r = r_{\odot-\oplus} \approx 1\text{AU}$

$$I_{\odot} \approx (1370 \pm 10)\frac{W}{m^2} \tag{16}$$

where $r_{\odot-\oplus}$ *is the mean distance between Sun and Earth.*

Proof. At $r = r_{\odot-\oplus} \approx 1\text{AU}$, irradiance calculation gives

$$\begin{aligned} I_{\odot} &\approx \left(\frac{6.96342 \times 10^8 m}{1.49598023 \times 10^{11} m}\right)^2 5.6736713 \times 10^{-8} \frac{W}{m^2 K^4} (5778K)^4 \\ &\approx (1370 \pm 10)\frac{W}{m^2} \end{aligned}$$

as we meant. □

Remark 17 As one can see, equation 15 with $r = r_{\odot-\oplus}$ summarizes the factors through which $I_{\odot} = I_{\odot}(t)$. Astrophysical factors are given by variations of $T_{\text{eff},\odot}$ and $r_{\odot}$, while part of the orbital forcing enters through $r_{\odot-\oplus}$. Uncertainties were calculated with radius, distance and effective temperature.

Now, we will figure out the amount of energy Earth intercepts. For that, we must multiply irradiance by the area actually covered by Earth. Accordingly, we make the following assumptions.

Hypothesis 4 *We will ignore the fact that there exists a variation of distance due to non-flat Earth. That is, let Earth gather solar energy as a disk.*

Hypothesis 5 *Let Earth be spherical, when one distributes radiation across Surface.*

Lemma 22 *Intercepted solar energy by Earth is given by*

$$F_{\odot} = \pi \left(\frac{r_{\odot} r_{\oplus}}{r_{\odot-\oplus}} \right)^2 \sigma T_{\text{eff},\odot}^4 \tag{17}$$

where $r_{\oplus}$ is the mean Earth's radius.

Proof. We must multiply solar irradiance at $r = r_{\odot-\oplus}$ by the area Earth covers in accordance with hypothesis 4, that is

$$\begin{aligned} F_{\odot} &= \pi r_{\oplus}^2 I_{\odot} \\ &= \pi \left(\frac{r_{\odot} r_{\oplus}}{r_{\odot-\oplus}} \right)^2 \sigma T_{\text{eff},\odot}^4 \end{aligned}$$

which is the result we sought. □

Finally, one can calculate total energy flux integrated over the whole Earth's spherical area.

Proposition 23 *Solar energy flux distributed over Earth is*

$$\Omega_{\odot} = \frac{1}{4} I_{\odot} \tag{18}$$

Proof. Lemma 22 gives us $F_{\odot}$. Dividing this by Earth's area according to hypothesis 5 we obtain

$$\begin{aligned} \Omega_{\odot} &= \frac{F_{\odot}}{4\pi r_{\oplus}^2} \\ &= \frac{1}{4} \left(\frac{r_{\odot}}{r_{\odot-\oplus}} \right)^2 \sigma T_{\text{eff},\odot}^4 \end{aligned}$$

but the latter expression, in accordance with lemma 20, can be rewritten as

$$\Omega_{\odot} = \frac{1}{4} I_{\odot}$$

which is the result we wanted to prove. □

Corolary 24 *At $r = r_{\odot-\oplus}$*

$$\Omega_{\odot} \approx (343 \pm 2) \frac{W}{m^2} \tag{19}$$

Proof. By corolary 21, the result follows trivially. □

As we assumed constant quantities when we worked out former numerical values, we shall do that in the following, therefore

Hypothesis 6 *Radii and effective temperature are constant.*

Remark 18 This assumption is necessary, if we want to show that astrophysical and celestial-mechanical factors could be only triggers. Then, received solar input will be constant.

It is worth to note that values we used in former calculations — which are present values — are those that, possibly, were taken by variables during Vostok proxy timeline.

Once we had a glimpse on energy source, we must figure out how Earth System collects this energy. This could be complicated, if we choose to do it in a full-physics approach, since radiative transfer is a physical process in which many physical concepts — from electromagnetism through thermodynamics and quantum mechanics — are involved and mathematical formulation would be as complex as one wish.

Before Earth absorb any incoming solar radiation, part of it goes again out of Earth System towards space: due to reflection of this radiation by atmospheric aerosols, clouds, diverse kinds of surface — from water bodies to different kinds of soils — snow and ice and scattering by atmospheric gases. The ratio of the integrated value of these contributions to the total is the planetary albedo.

Obviously, in the meanwhile, solar radiation may interact — in a stronger way than scattering —with atmospheric gases, e.g. absortion of UV radiation by O_3 at the upper atmosphere and, of course, absortion by clouds. However, this attenuation of incoming solar radiation is small — since most atmospheric gases have quantum-mechanical properties, which make them non-reactive to incoming solar radiation, and absorbed parts of incoming radiation sprectrum are minimum in energetic content, because Sun emission is mostly at visible spectrum. Also, the absortion by lower clouds is quite low, since most of solar radiation inicident on them is reflected — they have high albedo — and higher clouds are not optically thick. Therefore, we shall not include this attenuation in our balance.

The remaining flux — total minus planetary albedo contribution — is absorbed by Earth. That is, it heightens Surface internal energy, which in turn rises its temperature. A body that has a temperature over 0K emits electromagnetic radiation. Thus, heated Surface radiates. We will do the following hypothesis, as we done it for Sun's emission

Hypothesis 7 *Surface radiates as a spherical blackbody at* $T_{\text{eff},\oplus}$.

A direct consequence of this hypothesis is the following proposition.

Proposition 25 *Surface emission is*

$$\Omega_{\oplus} = \sigma T_{\text{eff},\oplus}^4 \tag{20}$$

It is straightforward to see that typically $T_{\text{eff},\oplus} \ll T_{\text{eff},\odot}$, which in consequence means that their blackbody spectra have almost no overlapping — which means that the overlap occurs for wavelength ranges, where at least one of them has nearly no emission. While most of solar emission is within visible range, terrestrial radiation is within IR range: that makes reasonable the common terminology of incoming shortwave radiation (ISR) for incoming solar radiation and outgoing longwave radiation (OLR) for terrestrial radiation. Coincidentally, while Atmosphere is almost transparent to ISR, many atmospheric gases interact strongly with OLR — as their molecular levels become excited by the energy of these photons. These gases are the GHG, water vapor the most important of them, as we have explained in section 2.

When GHG molecules return to base state, they radiate again in the IR range. Depending on layers of GHG, there are three possibilities for a reemited photon: it is absorbed again by another GHG molecule, it leaves Earth System into space or it reaches Surface — where it is absorbed again — and heats it: GHG interaction with OLR and this back radiation constitute together the so-called Greenhouse effect. Models of layers of GHG require using integro-differential equations which are well beyond the complexity level we want to achieve.

OLR is not entirely shielded by GHG absortion bands: there are wavelength ranges — customarily called windows — through which OLR flux departs. Broadening or narrowing of windows relies on concentration of GHG (due to probability of interaction increases if the density of GHG molecules increases). This fact makes necessary to consider Atmosphere as a "grey" body regarding OLR.

Finally, some of Surface's internal energy is spent in other thermodynamical phenomena which are not radiative in nature, such as convection — that leads to Ocean circulation and Atmosphere dynamics — sensible heat flux to Atmosphere — that heats it — or in phase change processes. However, latent heat flux — due to Water Subsystem phase changes — is balanced in a global annual average, since global water vapor content in Atmosphere does not rises nor decreases.

Former discussion enables us to construct a model of Energy Subsystem analogously to the setting up of Carbon Subsystem model. The rules here are easier to understand, but we present them in summary as we have done with Carbon, and we give figure 13 in order to settle down ideas.

From this scheme we can formulate the following axiom and notation.

Axiom 3 *Energy transformations will be represented by non-dimensional and normalized fractions — that is, they take real values in* $[0,1]$.

Therefore, energy transformations divide fluxes of energy. In figure 13 these are represented by orange flux divisors.

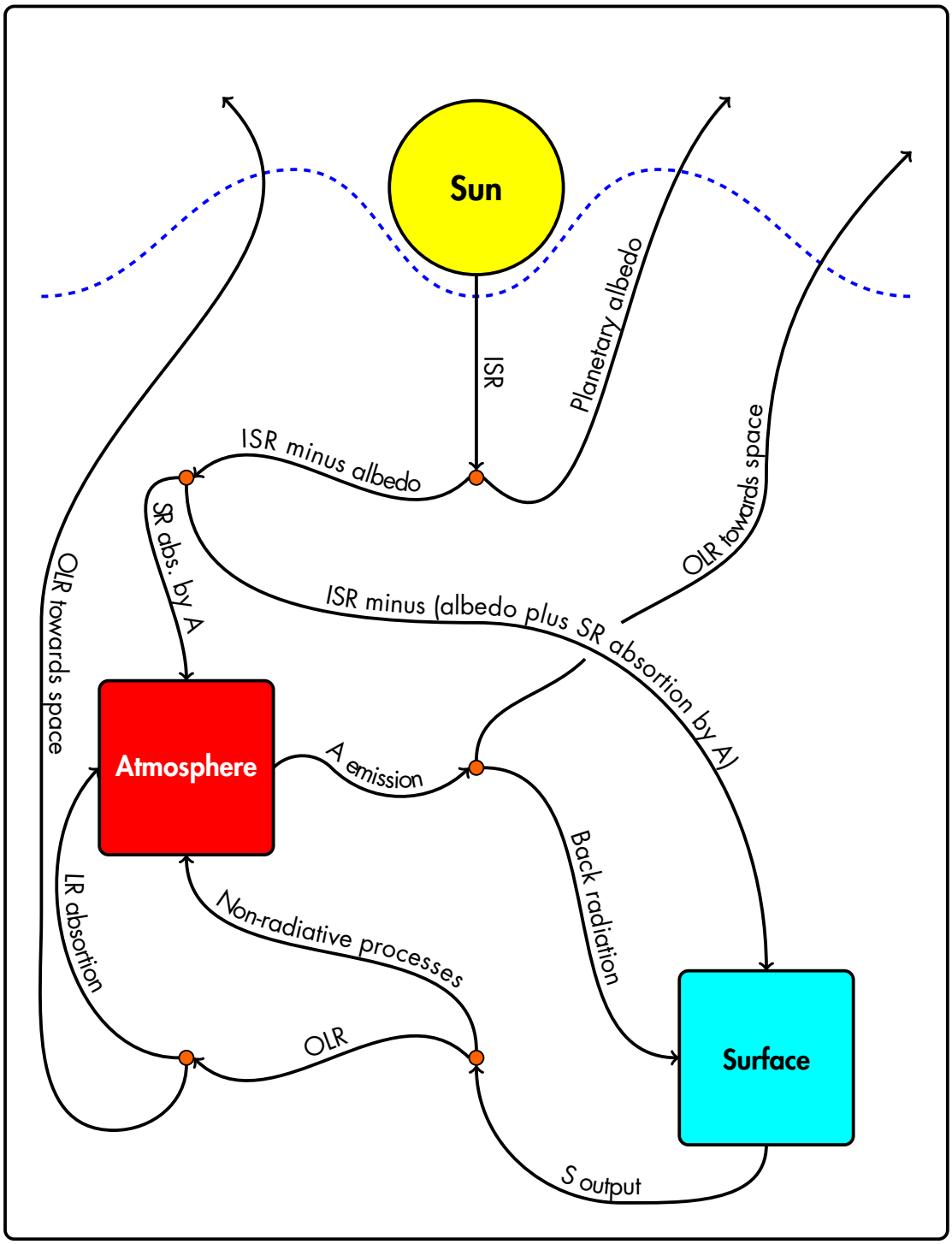

Figure 13: Energy Subsystem.

Notation 2 Non-dimensional fractions associated with Energy transformations are represented by the symbols α_i, where $i \in \mathbb{N}$.

We have proven the following proposition from the description of Energy Subsystem.

Proposition 26 *Earth System energy transformations are the following*

- ***Planetary albedo.*** *Fraction of ISR that is reflected towards space by Earth. Represented by non-dimensional fraction* α_1.
- ***Shortwave absortion.*** *Fraction of ISR minus albedo fraction that is absorbed by Atmosphere. Represented by* α_2. *Nonetheless,* $\alpha_2 \to 0$.
- ***Non-radiative.*** *Fraction of internal energy of Surface that is employed in convection or lost in sensible heat flow. Represented by* α_3.
- ***Longwave absortion.*** *Fraction of OLR absorbed by Atmosphere's GHGs. Represented by* α_4
- ***Back radiation.*** *Emitted Atmospheric LR towards Surface. Represented by* α_5.

Now, we formulate another axiom — concerning energy reservoirs.

Axiom 4 *Energy reservoirs are Surface and Atmosphere. Energy stock is measured in* J/m^2.

Notation 3 Surface reservoir is represented by S and Atmosphere is written as A.

Remark 19 Energy stock comprehends roughly internal energy per unit surface of Surface and Atmosphere.

We will also axiomatize the concept of Energy flux.

Axiom 5 *Energy flux is the exchange of energy between reservoirs.*

Available energy fluxes are the outputs Atmosphere and Surface would have as their stocks are emitted as radiation only.

With this in mind and the following notation, we begin to write down fluxes.

Notation 4 Energy fluxes will be represented by the following symbol: ϕ. Available energy fluxes will be represented by φ. All have a subindex according to their nature and are measured in W/m^2

Proposition 27 *The following are the fluxes taken into account in the model of Energy Subsystem:*

- $\phi_\odot = \Omega_\odot$

- $\phi_{alb} = a_1\Omega_\odot$
- $\phi_{abs,sr} = a_2(1-a_1)\Omega_\odot$
- $\phi_{abs,sfc} = (1-a_2)(1-a_1)\Omega_\odot$
- $\varphi_\oplus = \Omega_\oplus$
- $\phi_{abs,nr} = a_3\Omega_\oplus$
- $\phi_{OLR} = (1-a_3)\Omega_\oplus$
- $\phi_{abs,lr} = a_4(1-a_3)\Omega_\oplus$
- $\phi_{spc,1} = (1-a_4)(1-a_3)\Omega_\oplus$
- $\varphi_A = \Omega_{Atm}$
- $\phi_{bck,lr} = a_5\Omega_{Atm}$
- $\phi_{spc,2} = (1-a_5)\Omega_{Atm}$

Once we have done the former axioms and notations, we can sum inputs and outputs in order to obtain the rate of change of reservoirs at the following proposition.

Proposition 28 *For the Surface reservoir we have*

$$\dot{S} = (1-a_1)\Omega_\odot + a_5\Omega_{Atm} - \Omega_\oplus$$

and for Atmosphere reservoir we write down

$$\dot{A} = (a_3 + (1-a_3)a_4)\Omega_\oplus - \Omega_{Atm}$$

Proof. In the case of S we sum input fluxes and substract output fluxes

$$\begin{aligned}\dot{S} &= \phi_{abs,sfc} + \phi_{bck,lr} - \varphi_\oplus \\ &= (1-a_1)\Omega_\odot + a_5\Omega_{Atm} - \Omega_\oplus\end{aligned}$$

Similarly for Atmosphere

$$\begin{aligned}\dot{A} &= \phi_{abs,nr} + \phi_{abs,lr} - \varphi_A \\ &= a_3\Omega_\oplus + a_4(1-a_3)\Omega_\oplus - \Omega_{Atm} \\ &= (a_3 + (1-a_3)a_4)\Omega_\oplus - \Omega_{Atm}\end{aligned}$$

Both results are the ones we sought. □

Nevertheless, we need to couple Carbon and Energy Subsystems. What we have being saying from the very beginning makes us to link both SODEs through the effect of Carbon Cycle in the way Earth captures radiation. There is no other place to do that but in energy transformations: energy transformations will be carbon-dependent. But, how?

4.1 Parametrizations. Coupling with Carbon Subsystem.

Let us analyze energy transformations one by one to figure out how Carbon participates in each. We will reason deeply about factors, but from former sections it remains clear that C_2 is the parameter.

Planetary albedo. The way elements of Earth System reflect ISR rely upon biology and Water Subsystem.

Clouds are the most important contribution to albedo since they cover Earth's surface in an average of 50 percent — albeit they also support OLR absortion. Their contribution to albedo and OLR absortion is a function of their type and distribution: it is determined by the area they cover, their thickness, water content, droplet and/or ice cristal geometry and condensation nuclei chemical composition, besides other factors. Because of the discussion at the beginning of the chapter about coupling of Carbon Subsystem and Water Subsystem, we can conclude that — in interglacial periods — Atmosphere has more water vapor content and, since surface is warmer, there is a potential to have more convection and low cloud formation. Convection will raise rainfall that nurtures Continental food chain and soil types gradually change to ones with lesser albedo, for they tend to be covered by vegetation. That is, though clouds formation is favored, their role as reflectors does not overwhelm the Surface Energy stock to maintain interglacial warmer temperatures — because of OLR absortion role of increased water vapor content in the Atmosphere and their vanishing half life compared to kiloannum timescales. In glacial periods — even though low cloud activity lessens — Atmosphere's thermodynamical state is enough for precipitation being in the form of ice and snow. Gradually, cloud cover type that increases albedo diminishes but ice and snow cover grows. Mid-latitudes — due to the developing of the ice and snow cover — and tropical — due to decrease in rainfall — vegetation diminish and let that desertification augments further planetary albedo. Therefore, it is hinted that we can parametrize $a_1 = a_1(C_2)$ and this function is a decreasing function.

Non-radiative. As we described cloud role in the former reasoning, and since convection increases as a response to non-homogeneous thermodynamical state of the Atmosphere, subject to the boundary conditions of surface and higher atmosphere, it is clear that we can put forward that $a_3 = a_3(C_2)$ and it is an increasing function.

Longwave absortion. This trasformation is the clearest of all of them that relies upon GHG heavily, as we said when we drafted quantum-mechanical aspects of absortion of OLR by GHGs molecules. Therefore, it is clear that $a_4 = a_4(C_2)$ and it is an increasing function.

Back radiation. This is the other end of OLR absortion in the latter discussion. Therefore it is also straightforward to suggest that $a_5 = a_5(C_2)$ and that it is an increasing function.

We have found that all energy transformations could be written down as monotonic functions of C_2. But we need to figure out closed expressions — or at least a model — for these functions. In order to advance throughout this problem, we can discuss about characteristics of these functions in relation to natural phenomena described.

Since we are dealing with $a_i \in [0,1]$, the function values are bounded. Then, we must seek a model which is bounded. Concerning natural phenomena, it is clear that energy transformations never reach maximum or minimum values of the codomain. Extrema are in no way possible: matter can not be a perfect reflector or a perfect absorber of ISR; GHG can not absorb all OLR, although we make concentrations become overwhelmingly high and we raise temperatures and pressures — and in the process, we obliterate Earth System — in such a level that broadening of spectral absortion lines becomes larger and larger; emission of GHG can not be directed completely towards Surface and all energy stock can not be used by non-radiative processes, since them depend upon an imbalance that is corrected continuously by these very processes. Thus, there is always some saturation limit — for both extrema — in energy transformations.

From the former argumentation, it becomes apparent: saturation implies that functions are asymptotic to the boundaries of the codomain. This — in addition to monotonicity — means that: $a_i \to 0$ when $C_2 \to -\infty$ and $a_i \to 1$ when $C_2 \to \infty$ for $i = 3,4,5$ and the converse for $i = 1$. This lower the number of candidates for parametrizations. However, it would be also nice if our parametrization is continuous.

Instead of determining in a completely arbitrary way this function, we can let selection be done through the construction of an ODE that governs a_i: the arbitrariness shall only be that we restrict governing ODE to be a first-order ODE. Thus, we must think about the rate of change of a_i with C_2.

Natural first option — for the case of monotonical increasing function — would be to write down

$$\frac{da_i}{dC_2} = r_i a_i$$

that is, an exponential growth. Note that an steady-state solution is the null function, then asymptotical properties are fulfilled for the lower limit since solutions — by virtue of the ODE existence and uniqueness theorem — do not intersect each other. Nonetheless, this model fails utterly to capture the other limit: we must restrict exponential growth.

The following model can be the solution

$$\frac{da_i}{dC_2} = r_i a_i \left(1 - \ell_i(a_i)\right)$$

where ℓ_i is a known function of a_i. Extra term will stop exponential growth depending on the function ℓ_i. For making a selection of ℓ_i, we must consider the meaning of slope. Exponential growth ODE says us that slope of a_i grows arbitrarily since a_i is a monotonic increasing function and determines its own slope. Thus, second model must cut off this growth and, at the same time, keep slope positive — because we want a monotonic increasing function. Then, ℓ_i is such a function that makes rate of change of a_i non-negative but with at least one concavity. Function a_i will increase but there is a point which marks a decreasing rate of rise.

For simplicity, let us pick ℓ_i as follows

$$\ell_i(a_i) = \frac{1}{n_i} a_i$$

therefore, the rate of change reads

$$\begin{aligned} \frac{da_i}{dC_2} &= r_i a_i \left(1 - \frac{1}{n_i} a_i\right) \\ &= r_i a_i - \frac{r_i}{n_i} a_i^2 \end{aligned}$$

and we conclude that slope profile has a parabolic form. By the sign of quadratic term, we know this parabola opens downwards. First line gives us steady-state solutions which are $a_i = 0$ and $a_i = n_i$, and therefore (by symmetry of parabola) $a_i = \frac{n_i}{2}$ is where the apex of parabola is located — where maximum growth ocurrs amidst both steady-state solutions. Since we have two steady-state solutions of the governing ODE, every solution — other than steady-state — will be asymptotic to one or both. If we take as an IVP the condition $a_i(C_{2,0}) = a_{i,0} \in (0, n_i)$, former discussion lets us see that $a_i(C_2)$ will be monotonic increasing and asymptotic to both steady-state solutions — again, owing to existence and uniqueness of solutions. For IVP away from this band between 0 and n_i, the behavior shall be of growth towards 0 or decay to n_i — because of the sign of the field determined by parabola and in this context 0 is stable and n_i is unstable, while in the band 0 is unstable and n_i is stable — but the values that the function takes are quite unphysical.

Former dynamical-systemic analysis leads to the following interpretation of parameters of this governing ODE: r_i are intrinsic growth rates associated with exponential growth, which are bounded by the existence of a cap for the increase, n_i. For our purpose $n_i = 1$, $\forall i$. Former analysis is only reversed in certain details for the case where $r_i < 0$ and gives the corresponding model for albedo. As a summary, we propose the following

Theorem 29 *Let $a_i : \mathbb{R} \longrightarrow (0,1) \subset [0,1]$ the energy transformation functions. Then, a_i satisfy the following non-linear first-order ODE*

$$\frac{da_i}{dC_2} = r_i a_i (1 - a_i) \tag{21}$$

Remark 20 The governing ODE — which we have constructed before — is the so-called logistic equation, proposed by the first time by Verhulst, (1845) in connection with population dynamics.

We want an expression for the relevant solutions of ODE 21. Fortunately in this case, non-linearity is not an issue to get a closed form. We propose the IVP $a_i(C_{2,0}) = a_{i,0}$ and integrate using the method of partial fractions to recast ODE in a convenient form

$$\begin{aligned} a_i' &= r_i a_i(1 - a_i) \\ \frac{1}{a_i(1 - a_i)} a_i' &= r_i \\ \left(\frac{B}{a_i} + \frac{D}{1 - a_i}\right) a_i' &= r_i \\ \frac{B(1 - a_i) + Da_i}{a_i(1 - a_i)} a_i' &= r_i \end{aligned}$$

therefore

$$\begin{aligned} B(1 - a_i) + Da_i &= 1 \\ B + (D - B)a_i &= 1 \therefore \\ B = 1,\ D &= 1 \end{aligned}$$

which are the coefficients of the partial fraction expansion of the left hand side. By introducing this information into the ODE, we have recasted it into the following form

$$\left(\frac{1}{a_i} + \frac{1}{1 - a_i}\right) a_i' = r_i$$

which can be integrated readily using the fundamental theorem of calculus, as both terms on the left hand side are the derivatives of logarithms

$$\begin{aligned} \int \frac{1}{a_i}\, da_i + \int \frac{1}{1 - a_i}\, da_i &= \int r_i\, dC_2 \\ \int \frac{1}{a_i}\, da_i - \int -\frac{1}{1 - a_i}\, da_i &= \int r_i\, dC_2 \\ \ln a_i - \ln(1 - a_i) &= r_i C_2 + \ln c \\ \ln \frac{a_i}{1 - a_i} &= r_i C_2 + \ln c \\ \frac{a_i}{1 - a_i} &= c e^{r_i C_2} \end{aligned}$$

We shall do some algebra gymnastics to obtain a closed form as we promised

$$\begin{aligned} a_i &= ce^{r_i C_2} - ce^{r_i C_2} a_i \\ \left(1 + ce^{r_i C_2}\right) a_i &= ce^{r_i C_2} \\ a_i(C_2) &= \frac{ce^{r_i C_2}}{1 + ce^{r_i C_2}} \\ &= \frac{\frac{1}{c} e^{-r_i C_2}}{\frac{1}{c} e^{-r_i C_2}} \frac{ce^{r_i C_2}}{1 + ce^{r_i C_2}} \\ &= \frac{1}{1 + \frac{1}{c} e^{-r_i C_2}} \end{aligned}$$

and substitute IVP for determining c

$$\begin{aligned} a_i(C_{2,0}) = \frac{1}{1 + \frac{1}{c} e^{-r_i C_{2,0}}} = a_{i,0} &\therefore \\ a_{i,0} &= \frac{1}{1 + \frac{1}{c} e^{-r_i C_{2,0}}} \\ \frac{1}{a_{i,0}} &= 1 + \frac{1}{c} e^{-r_i C_{2,0}} \\ \frac{1}{a_{i,0}} - 1 &= \frac{1}{c} e^{-r_i C_{2,0}} \\ \frac{1}{c} &= \left(\frac{1}{a_{i,0}} - 1\right) e^{r_i C_{2,0}} \end{aligned}$$

and we introduce this in the explicit quadrature

$$a_i(C_2) = \frac{1}{1 + \left(\frac{1}{a_{i,0}} - 1\right) e^{-r_i (C_2 - C_{2,0})}}$$

and we have proven the following

Theorem 30 *Explicit formula for* $a_i : \mathbb{R} \longrightarrow (0,1) \subset [0,1]$ *is the following*

$$a_i(C_2) = \frac{1}{1 + \left(\frac{1}{a_{i,0}} - 1\right) e^{-r_i (C_2 - C_{2,0})}} \tag{22}$$

where $a_{i,0}$ *is the value* a_i *takes at the reference* $C_{2,0}$ *and* $r_i < 0$ *for* $i = 1$ *and* $r_i > 0$, $i = 3, 4, 5$.

In figure 14 we show some plots to exhibit the behavior of the parametrizations.

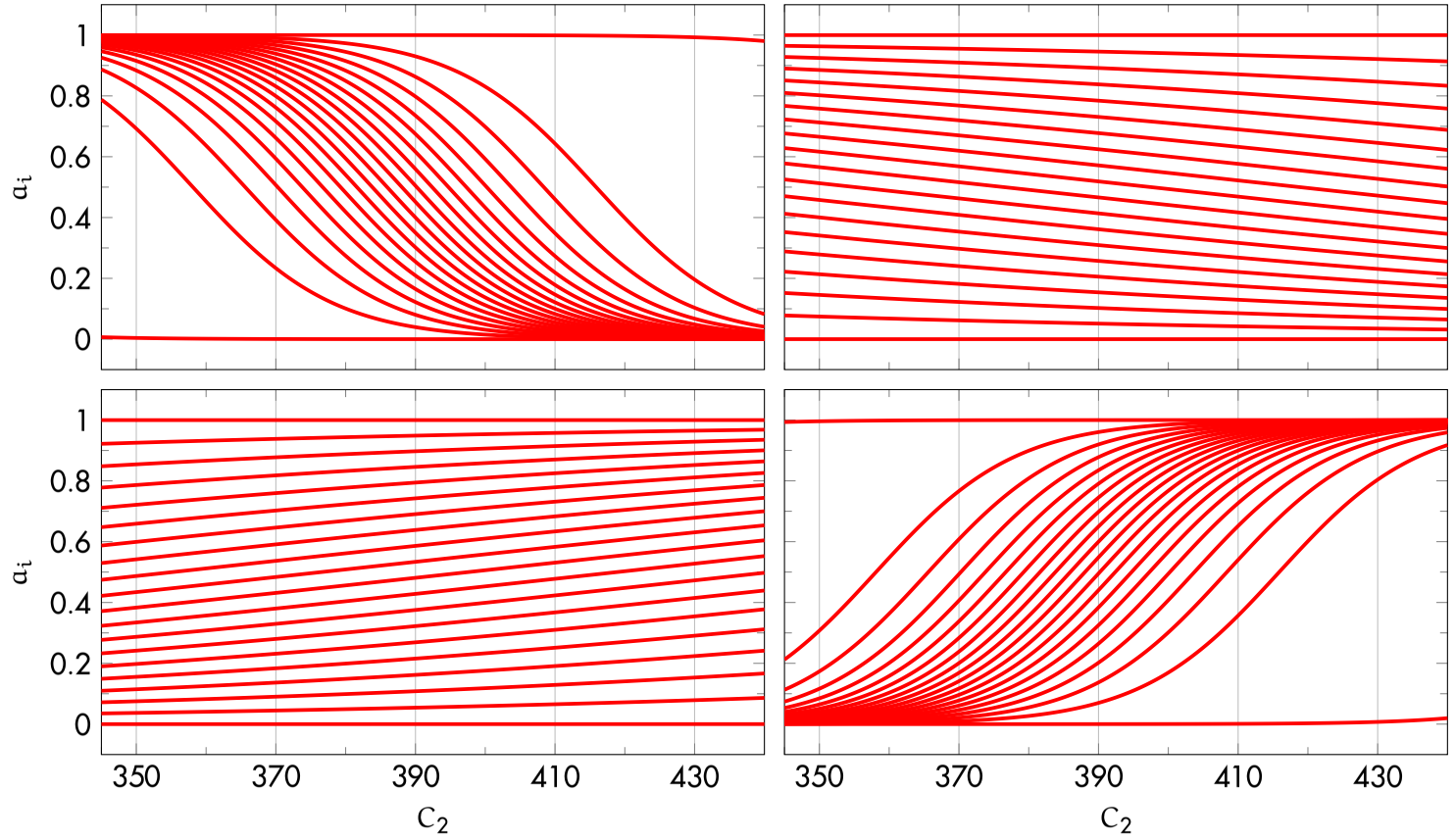

Figure 14: Some examples of solutions of logistic equation. Upper left: $r_i = -0.1$. Upper right: $r_i = -0.01$. Lower left: $r_i = 0.01$. Lower right: $r_i = 0.1$. Plots in each panel correspond to different IVPs.

4.2 Model set up.

From the axiom 5, we can recast equations in proposition 28 in a form that $\Omega_{\oplus} \mapsto S$ and $\Omega_{Atm} \mapsto A$. This mapping of variables is accomplished by multiplying by $1/s$ both stocks. Formally, that is equivalent to say that $\Omega_{\oplus}$ and Ω_{Atm} are proportional to S and A — respectively — through proportionality factors $k_{\oplus}, k_{Atm}$, which have units of $1/s$ and values of 1. This will establish a first-order SODE for Energy Subsystem.

We will also establish arbitrarily $\Omega_{\odot} \neq \Omega_{\odot}(t)$ as normalizing factor by dividing SODE by it. Therefore, we map $S/\Omega_{\odot} \mapsto s$ and $A/\Omega_{\odot} \mapsto b$

Theorem 31 *Energy Subsystem SODE is given by*

$$\begin{cases} \dot{s} = (1 - a_1) + a_5 b - s \\ \dot{b} = - b + (a_3 + (1 - a_3)a_4)s \end{cases} \tag{23}$$

in a non-dimensionalized way and in a normal form

$$\begin{cases} \dot{S} = (1 - a_1)\Omega_{\odot} + a_5 A - S \\ \dot{A} = - A + (a_3 + (1 - a_3)a_4)S \end{cases} \tag{24}$$

where all coefficients are given by parametrizations of theorem 30.

Remark 21 It is worth to note that Energy Subsystem SODE is linear, but with variable coefficients.

From the solutions of this SODE, one can promptly calculate T through Stefan-Boltzmann equation slightly modified for the effect of non-radiative processes: we use ϕ_{OLR} in the calculation instead of $\varphi_{\oplus}$, therefore — accounting for non-dimensionalization —

$$(1 - a_3)s\Omega_{\odot} = \sigma T^4 \therefore$$
$$T = \sqrt[4]{\frac{(1 - a_3)s}{\sigma}\Omega_{\odot}}$$

and we have proven that

Theorem 32 *From the solutions of the SODE 23, Surface temperature* T *(in Kelvin) is given by*

$$T = \sqrt[4]{\frac{(1 - a_3)s}{\sigma}\Omega_{\odot}} \tag{25}$$

or from SODE 24

$$T = \sqrt[4]{\frac{(1 - a_3)}{\sigma}S} \tag{26}$$

4.3 Standard energy equation.

We can do some algebra to mix equations in SODE 24 and obtain an equation for temporal evolution of T. First, we take equation 26, raise to fourth power, and get its time derivative

$$\sigma T^4 = (1-\alpha_3)S$$
$$4\sigma T^3\dot{T} = (1-\alpha_3)\dot{S} - \dot{\alpha}_3 S$$

and since $\alpha_3 = \alpha_3(C_2)$, we can use chain rule to calculate $\dot{\alpha}_3$

$$4\sigma T^3\dot{T} = (1-\alpha_3)\dot{S} - \frac{d\alpha_3}{dC_2}\dot{C}_2 S$$

and we remember that α_3 satisfies logistic equation by theorem 29, so we substitute this in the latter expression and obtain

$$4\sigma T^3\dot{T} = (1-\alpha_3)\dot{S} - r_3\alpha_3(1-\alpha_3)\dot{C}_2 S$$

and we do the same with $\dot{S}$ using corresponding equation of the SODE 24

$$4\sigma T^3\dot{T} = (1-\alpha_3)\left[(1-\alpha_1)\Omega_\odot + \alpha_5 A - S\right] - r_3\alpha_3(1-\alpha_3)\dot{C}_2 S$$

At the left hand side we will try to complete an S, since we have a σT^3 and on the right hand side we have a $(1-\alpha_3)$. In order to do that, we multiply former equation by T, divide it by $(1-\alpha_3)T$ and, subsequently, using relationship between T and S we get

$$4\frac{\sigma T^4}{1-\alpha_3}\frac{1}{T}\dot{T} = (1-\alpha_1)\Omega_\odot + \alpha_5 A - S - r_3\alpha_3\dot{C}_2 S$$
$$4\frac{S}{T}\dot{T} = (1-\alpha_1)\Omega_\odot + \left(\alpha_5 A - r_3\alpha_3\dot{C}_2 S\right) - S$$
$$= (1-\alpha_1)\Omega_\odot + \left(\alpha_5 A - \frac{r_3\alpha_3\dot{C}_2}{1-\alpha_3}\sigma T^4\right) - \frac{1}{1-\alpha_3}\sigma T^4$$

From proposition 27 we see that $\phi_{OLR} = (1-\alpha_3)\Omega_\oplus$ and from proposition 25 we can write down $\phi_{OLR} = (1-\alpha_3)\sigma T_{eff,\oplus}^4$. Therefore, we can conclude that $T^4 = (1-\alpha_3)T_{eff,\oplus}^4$ and we can recast former equation in terms of effective temperature

$$4\frac{S}{T}\dot{T} = (1-\alpha_1)\Omega_\odot + \left(\alpha_5 A - r_3\alpha_3\dot{C}_2\sigma T_{eff,\oplus}^4\right) - \sigma T_{eff,\oplus}^4 \tag{27}$$

Former equation can be compared with the one presented by Hogg, (2008) and that is an standard in the literature in its generic form:

$$R\dot{T}_{std} = S_{std} + G_{std} - \sigma T_{std}^4$$

where R is the surface heat capacity, S_{std} is insolation and G_{std} is greenhouse effect term. In our equation

$$R \mapsto 4\frac{S}{T},\ S_{std} \mapsto (1 - a_1)\Omega_{\odot},\ G_{std} \mapsto a_5 A - r_3 a_3 \dot{C}_2 \sigma T_{eff,\oplus}^4$$

Part III

Numerical Results and Conclusions

5 Simulation and results

> Rationem vero harum gravitatis proprietatum ex phænomenis nondum potui deducere, et hypotheses non fingo.
>
> *(Isaac Newton (1643-1727))*

Once we have constructed and made an analysis of our Earth System Model, it is our task to use it. Equations from SODEs 1 and 23 are numerically solved using information of previous chapters concerning the dynamical structure of the Carbon Subsystem mainly: the fundamental corolary 18. The objective is to simulate Vostok time series for atmospheric CO_2 and T: then we seek for closed orbits in Carbon Subsystem.

5.1 Workflow.

We made an algorithm to begin numerical explorations of the model. The summarized version is the following: We searched for reasonable initial conditions and chosen parameter values. This was made in an iterative process at the end of which we obtained the best fitting parameters to Vostok data. The approximation algorithm was a manual process, that is, we did not use an automatized optimization algorithm. However, we used MSE between observations and simulations as a guide to judge best fitting.

We used formulas to link model and observation spaces, since we have made use of non-dimensionalized SODEs 1 and 23. In the case of Carbon Subsystem, we used a concentration translation and a homothetic transformation on the observed data to obtain an initial approximation for the amplitude and base level in the model space. This was also an iterative process.

The results from this iterative process yielded the following expression:

$$C_{2,\mathrm{model}} = 5.202 \times 10^{-3}\,(C_{2,\mathrm{obs}}) - 0.9 \tag{28}$$

Similarly, for ΔT, we assumed an annual global climatic normal of $15\,^{\circ}C$, that is,

$$T_{\mathrm{model,celsius}} = \Delta T_{\mathrm{obs}} + 15\,^{\circ}C \tag{29}$$

All iterations in both processes were carried out in *Python programming language* using two approaches: using *Numpy* numerical libraries (based on *Fortran*-written numerical

methods) — in particular the solver `odeint` based on `lsoda` from *ODEPACK* and `ode` based on `vode` from *NETLIB* — and the autor-written library of numerical methods — in particular an implementation of Adams-Bashforth-Moulton predictor-corrector method with one iteration of an explicit fourth order Runge-Kutta to start the method.

The final output was processed, taking it to observation space. We have done a linear interpolation of both proxy data and numerical solutions to obtain regularly spaced samples, and then we have run DFT-based spectral analysis on both sources in order to compare spectral composition of observations with that of simulation. For that end, we have used periodogram naïve spectral estimation, since we thought that a more complex technique is not necessary in this case (such as multitaper or wavelet analysis). We have used native *Numpy* signal processing methods to obtain periodogram. Previously, signals were linearly detrended and standarized in order to comparison of power spectra be feasible.

5.2 Initialization and results.

The initial conditions for the Energy Subsystem were estimated taking into account that Atmospheric and Surface fluxes must be greater than $\Omega_\odot$. Additionally, considering the recent value of $C_{2_{ref}} = 387\text{ppmv}$, we also fix $a_{i,0}$: $a_{1,0} = 0.313, a_{2,0} = 0.207, a_{3,0} = 0.897, a_{4,0} = 0.624$, from considerations of actual albedo and outgoing radiation and estimation through simulations of non-radiative processes contribution.

Unfortunately, initial conditions for the Carbon Subsystem were more difficult to establish. Although there are proxies for Continental and Oceanic stocks, they are not in the form we need: generic interchangeable carbon. Given our modeling assumptions — dating since chapter 1 and construction of Carbon Subsystem Model in chapter 3 — and the fact that we begin at an interglacial period with an active Continent and a dormant Ocean — we may infer the following ordering for the initial values of the carbon stocks: $C_{3,0} \leqslant C_{1,0} \leqslant C_{2,0}$.

Results from the iterative process are presented in table 1.

In table 1 we do not summarize the values for $C_{3,0}$, γ and ζ. The following is done to obtain them. Initial conditions fulfill $C_{1,0} + C_{2,0} + C_{3,0} = 1$. From this, we set γ as

$$\gamma = \alpha \frac{C_{1,0}}{C_{2,0}} - \frac{\alpha\varepsilon}{\beta} \frac{C_{3,0}}{C_{2,0}} \tag{30}$$

in order to the sum of Carbon Subsystem equations be zero at the starting time. ζ becomes determined by the condition $\zeta = \alpha\varepsilon/\beta$ since we consider Vostok proxy pictures a periodic phase of the system. The final values are: $C_{3,0} \approx 0.1400$, $\gamma \approx -0.0118$ and $\zeta \approx 0.1012$.

Initial conditions				Parameters			
Carbon		Energy		Carbon		Energy	
Symb.	Stock	Symb.	Stock	Symb.	Val.	Symb.	Val. $\times 10^{-3}$
C_1	0.2305	s	1.3879	α	0.0293	$-r_1$	4.122
C_2	0.6295	b	1.2591	β	0.1050	r_2	4.135
				ε	0.3630	r_3	0.476
						r_4	0.074

Table 1: Initial conditions and best fitting parameters for Carbon and Energy Subsystems.

5.3 Analysis and discussion of the results.

Time series product of simulation for Carbon Subsystem are shown in figure 15, where we can note the phase shift between Continental and Oceanic stocks (and also Atmospheric one). This phase shift makes clear that closed orbits in phase space are not circles nor ellipses and reveal the change in control, from Continental to Oceanic and the converse: Our model captures assymetries we have identified in Vostok proxy, not only in Atmospheric stock but in the other stocks. Variability of Atmosphere across glacial cycles is large, since it acts as a conduit, while Continent and Ocean have smaller amplitude. It is worth to note that Ocean gets its highest concentrations, when we are arriving to the interglacial period, while Continent peaks amidst the journey to glacial period. This makes sense with the explanation we presented in chapter 1: that is, our results are in accordance with proxies and the qualitative explanations already given.

The computed time series for Atmospheric CO_2 and ΔT depicted in figure 16 show the assymetries with faster heating and slow cooling, and interglacials periods of shorter duration than glacial ones.

There are some interesting things, when we analyze the (frequency) power spectrum, p_{sp}. In figure 17, we mark the frequencies of our simulation and also mark the nearest or coincident periodicities on proxy (labeled in period and not in frequency). It can be seen that proxy power spectra has more spectral components than simulation one, then power decays more rapidly on simulation than in observation. In spite of the shifts, periodicities of simulation are remarkably similar to those of proxy in both spectra. Periodicities, that do not appear in simulation, point at processes and perturbations that we are not taking into account in our model. However, our simple model capture the main points of observed behavior. This is reinforced by the figure 18, in which is presented, side by side, Vostok

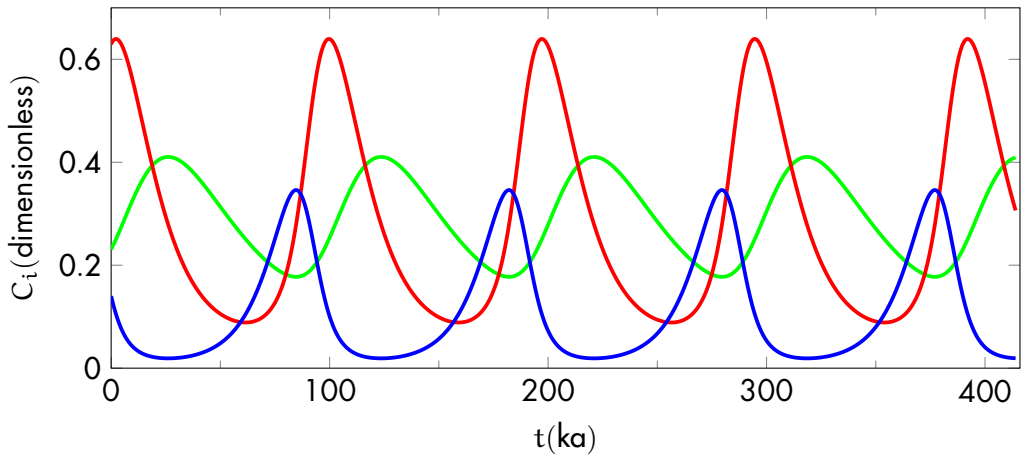

Figure 15: Carbon stocks from simulation. Green C_1, red C_2 and blue C_3.

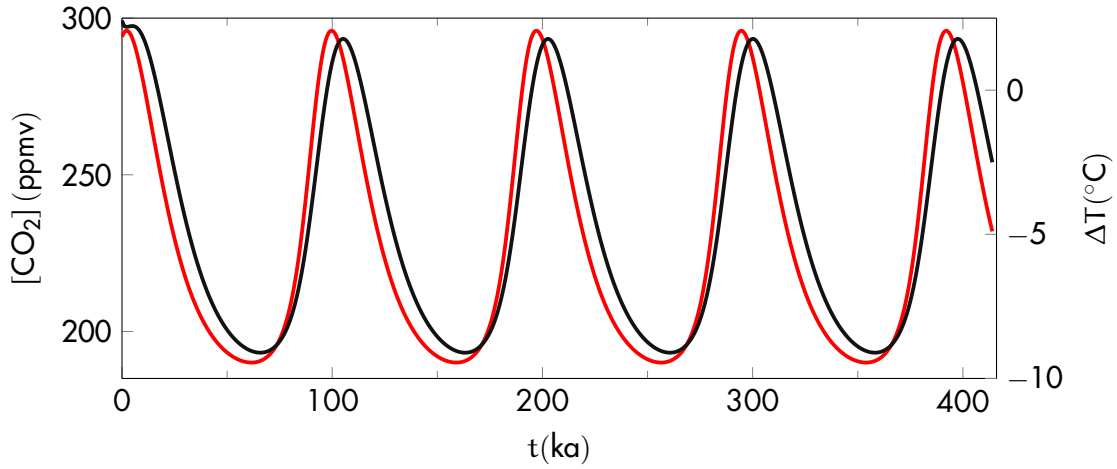

Figure 16: Simulated Atmospheric carbon (red) and ΔT (black).

time series and the corresponding simulated series.

In figure 18, it can be seen clearly what we have said at section 3.2.6: Parameters in reality are functions of time, even in periodic phases. That is the reason behind, e.g. the variation of period seen at the end of the series, where the glacial period is extended for a while.

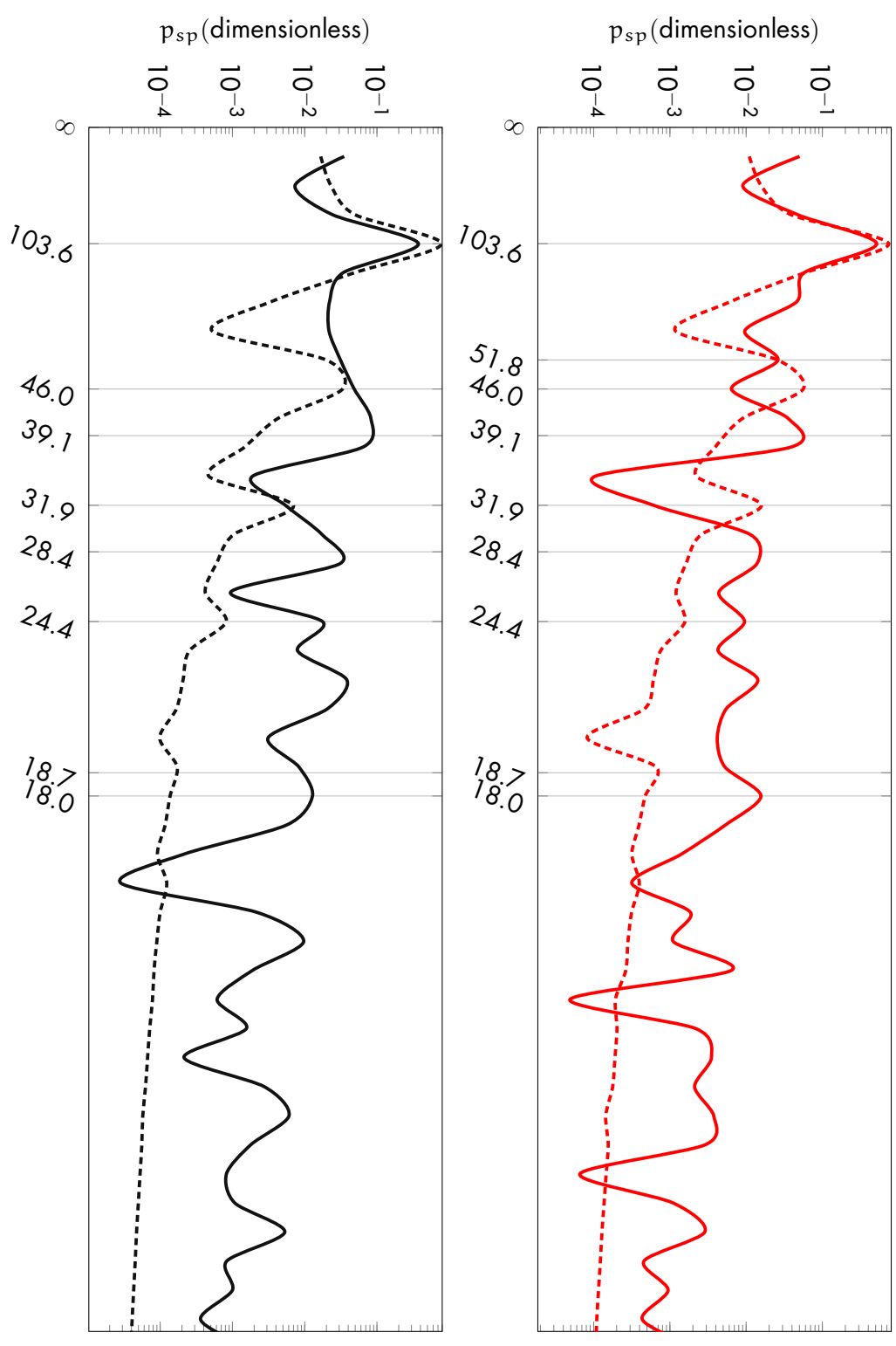

Figure 17: Periodogram estimation. Upper panel: Atmospheric Carbon. Lower panel: Temperature. Proxy (solid), model (dashed).

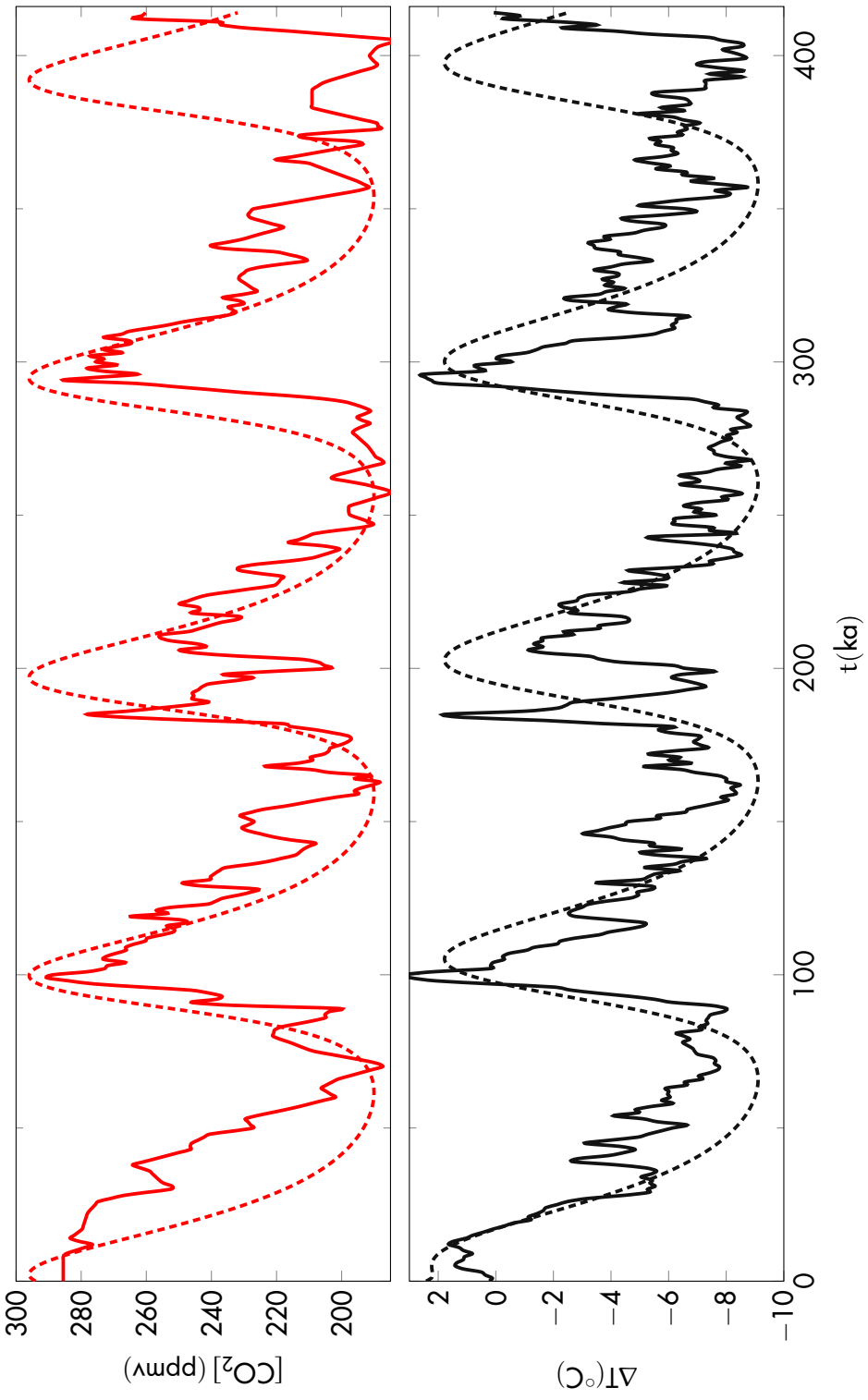

Figure 18: Comparison between Vostok time series and model numerical solutions.

6 Conclusions.

Autoritätsdusel ist der größte
Feind der Wahrheit.

(Albert Einstein (1879-1955))

As we have seen along all the sections of this work, celestial mechanical considerations — characteristic of orbital theory of climate — are not explicit. This is what distinguishes the model just presented from other models that community has been proposed.

The analysis of Carbon Subsystem, through the model we propose, has shown that periodic behavior is possible without taking into account orbital parameters. In spite of that fact, we say that orbital parameters and solar intrinsic variability enter as the background in which Earth System is immersed and, therefore, orbital parameters force Earth System to be suceptible to certain periodicities: transitional events end only when the System is at a new state compatible with equilibrium of the flows and with orbital forcing. That means, orbital forcing makes that some parts of phase space be more probable — for the system to evolve there — than others. However, the control upon the evolution of the System is internally driven.

Perturbations and shorter timescale phenomena are responsible of variations in fluxes in Earth System that modify periodicity, as said in section 3.2.6 and seen in section 5.3.

Although results presented hightens the trust in our model, another work would be comparing the results of Carbon Subsystem for Ocean and Continent with reality. For that task, we must process proxies for the Carbon stock in Ocean and Continent to get an estimate of the interchangeable Carbon, as we defined it at chapter 3.

Dynamical-systemic results make us wonder if we can obtain the period for a given closed orbit in terms of parameters. This will help us determine readily possible states of the system for certain principal periodicity and identify readily periodicities for other glacial ages.

Also we can ask, how much perturbation is needed to break a periodic phase into an aperiodic one and what is the decaying speed (a measure of how it evolves away of the periodic solution). This could help us estimate, if antropogenic action over the system is sufficiently strong to induce a relatively long transition event and estimate the general characteristics of the new periodic phase that will be established.

Part IV

Epilogue

7 Remarks on modeling: an epistemological perspective.

> Die Grenzen meiner Sprache
> bedeuten die Grenzen meiner
> Welt...
> Wovon man nicht sprechen kann,
> darüber muss man schweigen.
>
> *(Ludwig Wittgenstein*
> *(1889-1951))*

This chapter, first in this part of the present monography, contain some thoughts of my own that I want to share with the readers of the scientific community of Atmospheric Sciences and other disciplines.

As I have suggested at the preface to this thesis, that the greatest problem on modeling (and, in science, in general) is that a lot of scientists do not know what they really do.

When I talk about modeling I mean that kind of models that arise from phenomenology (physics, biology, chemistry or the like, including sociology), which is a result of a precise observation of nature.

First, models are collection of propositions that describe the phenomena that observations and experimentation had isolated. They do not need mathematization —a thoroughly translation of the principles into axioms to manipulate some mathematical objects— unless scientists make use of some mathematical concepts to draw simple relations between variables; e.g. you can talk about physics without a heavy mathematical language. Therefore, this model is a phenomenological model.

For several causes, scientists want to have a construct to work with, in which other relationships between phenomena arise, once they realize that certain rules are followed by phenomena. Here mathematization comes in. However, mathematization is not a unique process for several reasons. First, there is no unique rule that assign one and only one mathematical object to each object in the phenomenological model. Second, there can be many ways to describe the same objects, thereof there is not a one-one correspondence between certain phenomenological framework and mathematical models of it. Third, not every phenomena in nature is described by the phenomenological model, which results in mathematical constructs that lack certain characteristics that are found later in nature.

Fourth, we can not expect that any conclusion, which is a logical consequence of the mathematical model, has an expression in nature, inasmuch as we could not grasp the whole nature.

Then we can not expect to get a perfect representation of reality. In fact, such thing is useless, because it would mean that we know already everything and we can build a new universe from scratch: model becomes real. However, the information we took for giving rise to a certain model is far from complete, since it becomes clear that we only sense one part of the reality, a concept that is obscured by this very crippled perception.

Thus, the objective of the modeling is, for starters, to approximate perceived reality and not to reproduce it. Notwithstanding our lack of information, modern research science wants to have monstruous models that consume high quantities of energy and computational power but with poor understanding of the complexity that arises when phenomena of diverse nature interact. They only pull the lever and do not question why they pull it. They try to reduce nature to a bunch of phenomena that is barely connected with the help of parameterizations, that are small models heavily idealized. In addition, actual model is a numerical model.

To this moment we have three levels: phenomenological model, mathematical model and numerical model. Three steps that are not the same. Maybe phenomenological and mathematical models are, given a certain mathematization and a buch of luck, isomorphically related, but numerical model is an approximation to the mathematical model. Therefore it is a second order approximation to phenomena.

In Atmospheric Sciences, numerical models are monsters that approximate solutions to Navier-Stokes equations and are embroidered with parameterizations. We are running detailed Navier-Stokes and thermodynamical solvers with poor parameterizations of the Earth Subsystems that are not physical in nature, which is equivalent to say that Earth is only a great rotating and spherical tank of moist air with certain small complexities due to its relief, land use, convection, biology, chemistry and the like. This loss of respect for the other subsystems of Earth, comparable to the mechanistic way of thinking, pervades modeling community. We have lost curiosity. We are becoming engineers into the search of new algorithms that make more efficient our calculations. But we do not see that a review of the basis is what we really need. We do not see that we are using tools not appropiate to the task, we just use them the way others have used them, no matter if they were wrong.

Meteorology and climatology where raised by mathematicians. In spite of that, it amazes me the lack of critical thinking within the community. I know it is not only this scientific community that have this kind of problems. But as a physicist and a mathematician this decline or stagnation of sciences is not nice. Mathematics is not a minor tool of natural sciences. Mathematics is not only numbers. Mathematics is thinking. Its etymology comes from the greek *mathema*, which means knowledge. I give my own definition of Mathematics as a

tool: Mathematics is the art of analyzing unsolvable problems and obtaining conclusions in an elegant and near-effortless way. Contrary to this definition, most of the community conceive Mathematics as a toolbox that can be used without care: like speaking a language in the worst way possible.

Most scientists do not realise that a mathematical way of thinking could help us to make insight in the complex problems that we study this days. We are no more concerned with individual phenomena, but with the interaction of phenomena. To unravel this interactions, to see through non-linearity is the use that we could make of Mathematics, and in this way obtain simplifications to the monstruous models that are timescale- or spacescale-aware in a smart way: simplifications that are relevant to our purpouses and not monsters that want to describe everything (as I said at the beginning).

And you must never forget: *Models are gimmicks to study nature. Models are never nature itself.*

8 Acknowledgements and final words.

> In re mathematica ars
> proponendi quæstionem pluris
> facienda est quam solvendi.
>
> *(Georg Cantor (1845-1918))*

I know this is not included inside the tradition of thesis writing. In fact, that tradition states that acknowledgements and dedications be short and come at the beginning of the monograph. I think this kind of rules, that come from traditional rituals, are quite outdated: Science is not a task of extremely earnest people only, but one of human beings that have emotions and enjoy their work. Creativity and intuition forms an integral part of sciences and, when it is balanced by formality, becomes the true science.

Examples of this attitude come to my mind immediately, but the most important for me is the case of Johannes Kepler, who shared with their readers the surprise and disappointment that his findings — and the process to get them — made in his spirit.

I think that the text of the former monograph about a model of the Earth System is well written in that point, since conclusions are only the ending of the journey to them. And the journey itself is more important than the goal: since the goal could be disappointing or not what we wanted.

In the case of the present work it is really reassuring that the conclusions are in accordance with my expectations, that were a fruit of a rigurous mathematical thinking, previous to the developing of more formal work. In spite of that, the main idea is one of my own, the simple preeliminar work and ideas are not only due to me. There were scattered out there in the conversations with my peers, which are my closest friends in a lot of cases.

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