UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

POSGRADO EN CIENCIAS FÍSICAS

Study of Modifications of Gravity in Three Dimensions

TESIS

QUE PARA OPTAR POR EL GRADO DE
DOCTORA EN CIENCIAS (FÍSICA)

PRESENTA

LORENA PARRA RODRÍGUEZ

TUTOR PRINCIPAL

DR. JOSÉ DAVID VERGARA OLIVER
INSTITUTO DE CIENCIAS NUCLEARES

MIEMBROS DEL COMITÉ TUTOR

DRA. ROCÍO JAÚREGUI RENAUD
INSTITUTO DE FÍSICA

DR. JOSÉ ANTONIO GARCÍA ZENTENO
INSTITUTO DE CIENCIAS NUCLEARES

CIUDAD UNIVERSITARIA, CD.MX., 2016

The work described in this thesis was performed on a double degree program between the Van Swinderen Institute for Particle Physics and Gravity of the University of Groningen under the supervision of Prof. Eric A. Bergshoeff and the Instituto de Ciencias Nucleares of the Universidad Nacional Autónoma de México under the supervision of Prof. J. David Vergara. It was supported by the University of Groningen, the Universidad Nacional Autónoma de México, and the Consejo Nacional de Ciencia y Tecnología (CONACyT).

Chapters about the Supersymmetric Massive Extension, the Non Relativistic and the Carroll limits, are published under the title "Extensions and Limits of Gravity in Three Dimensions" with:
ISBN: 978-90-367-8124-4 (printed version)
ISBN: 978-90-367-8123-7 (electronic version)

Contents

1 Introduction

General Relativity (GR) has resisted the test of time even a hundred years after its conception. From an experimental point of view it has been tested to incredible accuracy by solar system experiments and pulsar timing measurements, and it has predicted with precision, such phenomena as the deflection of light, the perihelion advance of Mercury, and the gravitational frequency shift of light. GR also predicts that over time a binary system's orbital energy will be converted to gravitational radiation and the measured rate of orbital period decay turns out to be almost precisely as predicted. From a theoretical point of view GR provides a comprehensive, consistent and elegant description of gravity, spacetime and matter on the macroscopic level.

However, GR loses precision and predictability in strong gravity regimes and it fails to explain some phenomena in weak gravity regimes such as the galaxy rotation problem and the missing mass problem in clusters of galaxies. Furthermore, there is not a satisfactory model that explains dark energy and the accelerated expansion of the universe. The theory also has some shortcomings such as its failure to unify gravity with strong and electroweak forces, its prediction of spacetime singularities, and its incompatibility with quantum mechanics.

Due to GR's experimental and theoretical importance, many efforts have been made to study its possible deformations. Extending GR is technically very challenging because any new theory would have to resolve the above-mentioned large scale problems while taking into account the behavior that arises in quantum theory. In other words, we are searching for a mathematically consistent theory (free of instabilities and ghosts), that can reproduce GR at cosmological scales, provide an explanation of today's acceleration of the universe and at the same time, capable of giving a description of gravity following the principles of quantum mechanics.

In view of the complexity of this task, this work aims to take a first step towards dissecting GR and studying a number of different extensions and limits of the theory in order to have a better understanding of it. Modifying a known theory is one of the best ways to discover new structures, which may have unforeseen applications. In this thesis we therefore address the following question: What are the possible ways of modifying and studying different limits of gravity? Table 1.1 lists the extensions and limits studied in this thesis.

Extensions	a)	Add new symmetries: Supersymmetry.
	b)	Add new parameters: Mass parameter.
	c)	Modify symmetries: Lorentz violation.
	d)	Modify quantization scheme: Polymer quantization.
Limits	e)	Study the non-relativistic limit.
	f)	Study the ultra-relativistic limit.

Table 1.1
Extensions and Limits of Gravity as described in this thesis.

In order to explain the usefulness of supersymmetry, case (a) in Table 1.1, we should keep in mind the current status of the Standard Model of particle physics. The Standard Model was developed in the early 1970s and has since then successfully explained observations from particle colliders. The Standard Model has the symmetry group

$$SU(3)_C \times SU(2)_L \times U(1)_Y \,. \tag{1.1}$$

Roughly speaking, the three factors give rise to the fundamental interactions. The first $SU(3)_C$ part is related to the chromodynamics sector. The gauge group $SU(2)_L \times U(1)_Y$ is related to the electroweak theory which describes two different forces below the breaking scale: the weak force and the electromagnetic force.

Despite the huge success of the Standard Model in describing high energy phenomena, it does leave some features unexplained. For instance, the Standard Model Higgs sector is not natural, the theory does not provide a dark matter candidate, fails to explain the origin of the three gauge interactions and cannot provide an explanation for the inflation period in the early universe [1]. There are several possible approaches to address these issues, such as adding more particles or interactions to the model, including more general internal symmetries (in other words, the symmetries of the Standard Model are obtained from the symmetry breaking of a larger symmetry group) or adding more general spacetime symmetries (such as extra spacetime dimensions or supersymmetry).

With the development of quantum mechanics, symmetry principles and conservation laws came to play a fundamental role in physics. Particle physics symmetries include the Poincaré invariance (translations and rotations), ‘internal’ symmetries which can be further divided into global (related to conserved quantum numbers such as the electric charge) and local symmetries (which form the basis for gauge theories and are therefore related to forces) and discrete symmetries (such as charge conjugation, parity transformation and time reversal invariance).

In 1967 Coleman and Mandula [2] proved that, given certain assumptions, these are the only possible symmetries for a physical system. The Coleman-Mandula theorem states that for quantum field theories, spacetime and internal symmetries can only be combined in a direct product symmetry group (in other words, Poincaré and internal symmetries do not mix) [3]. This theorem is based on certain assumptions: that the Poincaré, internal and discrete symmetries are the only symmetries of the S matrix, that there is a mass gap and that internal symmetries form a so called Lie group.

It is possible to bypass this theorem by relaxing some of these assumptions. For example, Poincaré and internal symmetries do in fact mix in the case of spontaneous broken symmetries. Another option is to consider conformal field theories, in which there is no mass gap. If the theory only presupposes massless particles, the Poincaré algebra is extended to the conformal algebra. Yet another alternative is to relax the Lie group structure of the Poincaré and internal symmetries and to unify them into a superalgebra. In particular, the theorem assumes that the symmetry generators of the algebra only involve commutators. Weakening this assumption to allow both anticommuting and commuting generators opens the possibility of supersymmetry, in other words, an algebra that includes fermionic and bosonic symmetry generators.

Performing a supersymmetry transformation on a field results in an interchange of bosonic and fermionic fields. Exact invariance under supersymmetry implies that every particle in nature should have a partner with the same mass and same quantum numbers but with a spin differing by one-half. Performing two successive supersymmetry transformations on a field leads to the same field but with a different dependence of spacetime coordinates. It turns out that supersymmetry is the only possible extension of the known spacetime symmetries of particle physics, so it is natural to include it in the study of gravity which is the gauge field of the Poincaré spacetime symmetries.

Supergravity is an extension of Einstein's general relativity that treats supersymmetry as a gauge symmetry. In other words, supergravity theories have local supersymmetry and corresponding gauge fields, the gravitini. The gravitini are the fermionic partners of the graviton. Including supersymmetry in a theory of gravity turns out to be a valuable and useful technical tool. Moreover, supergravity theories arise naturally from string theory which is a major candidate for a theory of quantum gravity. Supergravity is also an essential ingredient for supersymmetric phenomenology and for obtaining precise predictions, often relating strong- to weak-coupling regimes leading to the AdS/CFT correspondence.

In addition to adding new symmetries, such as supersymmetry, one could also add new parameters to gravity. For instance, adding a mass parameter to GR leads to a theory of massive gravitons, corresponding to case (b) in Table 1.1. Recent years have witnessed a growing interest in massive gravity as a result of the cosmological constant problem and the discovery of the acceleration of the universe [4, 5], which could be better explained in terms of an infrared modification of GR [6, 7] that gives the graviton a mass than by using a dark energy component. Further motivation arises from the conjecture that some theories involving massive gravitons could be the low energy limit of a non-critical string-theory underlying quantum chromodynamics [8].

There are number of ways to acquire a theory of massive gravitons, one of which is to include explicit mass terms in the Einstein-Hilbert action as in the so-called Fierz–Pauli theory [9]. A second alternative is to consider higher-dimensional scenarios, as in the Dvali-Gabadadze-Porrati model [10]. Other theories, referred to as bi-metric theories, consider two dynamical spin-2 fields [11–13]. Finally, one could introduce higher-derivative terms in the action as in the New Massive Gravity (NMG) theory [14].

Massive gravity theories pose two main problems. First of all, they usually make it necessary to work with non-linear theories because linear theories would not yield the predictions of GR when taking the massless limit. The second problem that we must take into account is that most massive gravity theories are plagued with ghost instabilities (fields with negative kinetic energy). In an attempt to resolve these problems, researchers have tried to construct massive gravity theories in three dimensions.

Although we live in a four dimensional spacetime it is often convenient to study gravity in different dimensions. Dimensionality is one of the main features used to describe physical systems. For example, it is possible to build theoretical models in dimensions higher than four, or to study systems that are in lower spatial dimensions, such as one dimensional wires, or two dimensional interfaces.

Recently, the study of GR extensions in more than four dimensions has gained a lot of attention, in particular thanks to the black hole solutions this offers [15]. Another motivation for studying gravity in higher spacetime dimensions is that the AdS/CFT correspondence relates the properties of a D-dimensional black hole with those of a quantum field theory in $D-1$ dimensions [16], thus relating an AdS black hole in the gravitational theory with a strongly coupled CFT theory at finite temperature [17]. Additionally, many studies have focused on this approach because string theory is formulated in higher dimensions and it includes gravity [18].

The quantum theory of gravitons is non-renormalizable in four dimensions. Therefore, theorists have considered GR and its variants in three dimensions because one expects less severe short-distance behavior in lower dimensions. This makes it interesting to use three-dimensional gravity models to study problems in quantum gravity.

Three dimensions are special because a massless graviton in three dimensions has no propagating degrees of freedom (we discuss in detail the counting of degrees of freedom in section (3.1)). In three dimensional gravity, the Weyl tensor vanishes identically and the Riemann tensor is therefore completely determined by the Ricci tensor. Thus, all the components of the Riemann tensor are fixed by Einstein's equations and there are no free degrees of freedom. Despite this, three dimensional gravity allows for solutions with non-trivial topology. A massive graviton in three dimensions has two degrees of freedom which is the same number as a massless graviton in four dimensions. Therefore, in three dimensions it should be possible to construct a theory of massive gravity that is invariant under diffeomorphisms.

New Massive Gravity (NMG) is a higher-derivative extension of three-dimensional (3D) Einstein–Hilbert gravity, with a particular set of terms quadratic in the 3D Ricci tensor and Ricci scalar [14]. The NMG model is interesting because although the theory contains higher derivatives, it nevertheless describes, unitarily, two massive degrees of freedom of helicity $+2$ and -2. Furthermore, it has been shown that even at the non-linear level ghosts are absent [19]. The 3D NMG model is an interesting laboratory to study the validity of the AdS/CFT correspondence in the presence of higher derivatives. A supersymmetric version of NMG was constructed in [20]. In addition to the fourth-order-derivative terms of the metric tensor, this model also contains third-order-derivative terms involving the gravitino.

For many purposes, it is convenient to work with a formulation of this model without higher derivatives, see, e.g. [21]. In the linearized NMG model, this can be achieved by introducing an auxiliary symmetric tensor that couples to (the

Einstein tensor of) the 3D metric tensor and has an explicit mass term [14].

The first GR modification that we consider in this work combines extensions (a) and (b) from Table 1.1, we modify GR by adding an additional symmetry (supersymmetry) and an additional parameter (a mass parameter). We work with a supersymmetric reformulation of the NMG model (SNMG) *without* higher derivatives. This requires us to introduce both an auxiliary symmetric tensor and further auxiliary fermionic fields that effectively lower the number of derivatives of the gravitino kinetic terms.

As previously mentioned, the rules of quantum mechanics have not been applied to gravity in a satisfactory manner up to now, although recently, many theories have emerged as possible avenues towards the quantization of GR such as string theory, supergravity, loop quantum gravity, Hořava-Lifshitz gravity... Despite these efforts, the candidate models still need to overcome major formal and conceptual problems besides, with our current technology development on particle physics and cosmological observations, there is no way to validate their predictions through experimental tests.

Even though there are no experimental accessible situations where one can check the quantum gravity framework, there are many physical scenarios that one can imagine that require a quantum theory of gravity for their description where we can study their properties and outcomes with our current knowledge.

Recently, Hořava proposed a new theory of gravity [22–24] based on an anisotropic scaling of the spatial and time coordinates, which leads to the breaking of the relativistic invariance at short distances, with GR being recovered for low-energy limits. The main ingredient of this theory is the anomalous scaling relation between the Minkowskian space and time coordinates, breaking the spacetime diffeomorphism invariance.

The second modification that we study in this thesis (case (c) on Table 1.1) was inspired by the Lorentz violation in Hořava's theory. We use Noether's theorem to study the symmetries of anisotropic actions for arbitrary fields which generally depend on higher order spatial derivatives, and find the corresponding current densities and charges. In particular, we consider scale invariance on higher derivative four dimensional extensions of the scalar field and electrodynamics, and a three dimensional Chern-Simons theory.

On the other hand, in the mid 1980s, Ashtekar pointed out that it is possible to rewrite the equations of gravity in terms of variables that made the theory resemble the theories of particle physics in such a way that techniques from particle physics could be imported to the quantization of gravity. The resulting approach is called loop quantum gravity (see for example [25] for a brief introduction).

Loop Quantum Gravity considers GR's insight that spacetime is a dynamical field and is therefore a quantum object so there should be no background metric, implying that geometry and matter should both be 'born quantum mechanically' [25]. A second consideration in loop quantum gravity is that the quantum discreteness that determines the particle-like behavior of other field theories also affects the structure of space. One of the main results of loop quantum gravity is the derivation of a granular structure of space at the Planck length.

Polymer Quantum Mechanics is a quantization that replicates some features of the quantization in Loop Quantum Gravity. It was introduced by Ashtekar in [26] and later used in [27–30] as a novel quantization for scalar field theories, to construct a quantization scheme not equivalent to Schrödinger Quantum Mechanics. This scheme has been successfully applied to the quantization of several cosmologies, avoiding the big bang singularity.

In order to gain a better insight on this quantization, in our third study case (case (d) on Table 1.1) we analyze the polymer quantization via the path integral procedure of some toy models such as the non-relativistic free particle, the relativistic free particle, the harmonic oscillator and the simplest Friedman-Robertson-Walker model. This scheme provides us with an effective action that we can use to understand more deeply the non-perturbative aspects of the dynamical system at classical and quantum levels.

Apart from these gravity extensions, another approach to study gravity is to test its properties in limiting cases [31–34]. Here we consider the supersymmetric non-relativistic and ultra-relativistic limits of GR: the limits when the speed of light goes to infinity (we combine extension (a) and limit (c) of Table 1.1) and to zero (combining cases (a) and (d)) respectively. Geometrically, the non-relativistic transition can be viewed as the opening of the light cones (the cones become space-like hypersurfaces) while the opposite ultra-relativistic transition can be understood as the shrinking of the light cones (the cones become time-like hypersurfaces). While the non-relativistic limit is governed by the Galilei algebra, the corresponding algebra in the ultra-relativistic limit is the Carroll algebra. Both invariance groups can be obtained from adequate contractions of the Poincaré group. When starting from the AdS group, different contractions will lead to the non-relativistic Newton-Hooke and the ultra-relativistic AdS-Carroll groups.

There are many non-relativistic conformal field theories which describe physical systems within condensed matter physics [35], atomic physics [36] and nuclear physics [37]. Non-relativistic versions of the AdS/CFT correspondence have recently been investigated because they open the way for possible applications of gauge-gravity duality to a variety of real-life strongly interacting systems. Non-relativistic symmetry groups, such as the Schrödinger or the Galilean conformal symmetry groups, are relevant for the study of cold atoms, which have a gravity dual possessing these symmetries [38, 39]. Furthermore, in the case of strings, non-relativistic limits may have applications in the context of non-relativistic versions of AdS/CFT [40].

In addition, the so-called Carroll symmetries that arise in the ultra-relativistic limit, have played an important role in recent investigations [41], for example in studies of tachyon condensation [42]. More recently, they have also appeared in the study of warped conformal field theories [43].

There are essentially two procedures for constructing non-relativistic and ultra-relativistic gravity/supergravity theories. The first involves using the limits from vielbein formulations of relativistic gravity/supergravity. As shown in [44], such a limit can be defined and implemented in a consistent manner in the non-relativistic case. The second procedure consists of gauging the algebras with non-relativistic or ultra-relativistic symmetries (the gauging procedure in the non-relativistic case was studied in [45–47]). In other words, the first procedure considers the non-relativistic and ultra-relativistic limits of Einstein gravity while the second procedure involves performing a gauging of the limit versions of the Poincaré algebra. It turns out that the relevant non-relativistic and ultra-relativistic versions of the Poincaré algebra represent a particular contraction of this algebra.

Using the non relativistic limit of GR leads to the well-known non-relativistic Galilean gravity (valid in frames with constant acceleration), Curved Galilean gravity (valid in frames with time-dependent acceleration) and, Newton–Cartan gravity (valid in arbitrary frames). So far, no satisfactory ultra-relativistic limit of GR has been found.

1.1. Outline of the thesis

This thesis is organized as follows. The first part of Chapter 2 contains a review of supersymmetry and supergravity, as these topics play an important role in this thesis. Since we discuss different types of limits and contractions of the AdS groups, we dedicate a section of this chapter to reviewing some features of the different kinematical groups. We also explain the non-linear realizations method which allows us to construct actions invariant under the symmetry group under consideration.

In Chapter 3 we make the case for constructing a supersymmetric model of New Massive Gravity without higher derivatives. We show how to explicitly construct the linearized, massive, off-shell, spin 1 three dimensional $\mathcal{N} = 1$ supermultiplet and we comment about its massless limit as a warming-up example before moving on to the spin-2 case. We then discuss how to obtain a linearized supersymmetric new massive gravity theory without higher derivatives and we offer some comments on the non-linear case.

At the linearized level, the NMG model decomposes into the sum of a massless spin-2 Einstein–Hilbert theory and a massive spin-2 Fierz–Pauli (FP) model [14]. In the supersymmetric case we therefore need a 3D massless and a 3D massive spin-2 supermultiplet.

Chapters 4 and 5 are devoted to the symmetry analysis of higher order spatial derivatives systems with the anisotropic scaling between the space and time coordinates introduced by Hořava and the construction of the path integral for some physical systems in the framework of polymer quantization respectively. In Chapter 5 we also study the measure of the integral in detail in order to obtain the form of the exact propagators. Afterwards we analyze the effective action of several systems and their dynamics.

Chapter 6 addresses the problem of how to describe a non-relativistic superparticle in a curved background. Describing a non-relativistic superparticle in a curved background requires first a supersymmetric extension of the gravity backgrounds. Since non-relativistic supergravity multiplets have to our knowledge only been explicitly constructed in three dimensions, we will only consider superparticles in a three-dimensional (3D) background. A supersymmetric version of the 3D Galilean background was recently constructed by gauging the Galilei, or more accurately the Bargmann, superalgebra [48]. Our aim is to investigate the action of a 3D superparticle first in a flat background and then in a Galilean supergravity background without a cosmological constant.

In Chapter 7 we investigate particles whose dynamics are invariant under the Carroll superalgebra. We investigate the geometry of flat and curved (AdS) Carroll space and the symmetries of a particle moving in such a space both in the bosonic as well as in the supersymmetric case. In the bosonic case we find that the Carroll particle possesses an infinite-dimensional symmetry which only includes dilatations in the flat case. The duality between the Bargmann and Carroll algebra that is relevant for the flat case does not extend to the curved case.

In the supersymmetric case we study the dynamics of the $\mathcal{N}=1$ AdS Carroll superparticle. Only in the flat limit do we find that the action is invariant under an infinite-dimensional symmetry that includes a supersymmetric extension of the Lifshitz Carroll algebra with dynamical exponent $z=0$. We also discuss the extension to $\mathcal{N}=2$ supersymmetry in the flat case and show that the flat $\mathcal{N}=2$ superparticle is equivalent to the (non-moving) $\mathcal{N}=1$ superparticle making it non-BPS, unlike its Galilei counterpart. This is due to the fact that in this case the so-called kappa-symmetry eliminates the linearized supersymmetry.

In the last chapter, Chapter 8, we conclude and offer some possible directions for future research.

2
Preliminaries

In the first part of this chapter we review some general theoretical aspects that we will use through this thesis. We start with supersymmetry and supergravity and we discuss the construction of massless and massive representations of supersymmetry algebras in four and three dimensions respectively in order to have a better understanding of the supermultiplets involved in the New Massive Supergravity section.

We also make a review of the kinematical groups obtained from the different contractions of the AdS group that we use in the non-relativistic and in the Carroll sections. We give the generalities of the non-linear realizations procedure to obtain particle actions and we discuss how to study the symmetries of Hamiltonian systems.

2.1. Supersymmetry

The Standard Model has a huge and continued success in providing experimental predictions, however it does leave some unexplained phenomena like the hierarchy problem or the cosmological constant problem. Besides, it describes three of the four fundamental interactions at the quantum level, and we need a quantum theory of gravity because at energies higher than the Planck scale, gravity becomes comparable with other forces and cannot be neglected. At this scales quantum effects of gravity have to be included but Einstein's theory is non-renormalizable and therefore it can not provide proper answers to observables beyond this scale.

Supersymmetry solves the naturalness issue of the hierarchy problem. Combined with string theory it solves the quantum gravity issue. It also provides the best example for dark matter candidates and it also allows one to evade the Coleman-Mandula theorem which asserted the impossibility of putting together spacetime symmetries and internal symmetries in a non-trivial way. Moreover, Haag, Lopuszanski and Sohnius showed that supersymmetry is the only possible option of non-trivial interactions between internal symmetries and spacetime symmetries. There is a lot of literature devoted to these topics, throughout this thesis we will mainly follow the notation from [49]. Other useful references are [50–55].

Supersymmetry is a spacetime symmetry mapping bosonic states (integer spin) into fermionic states (half-integer spin) and vice versa. The operator Q that generates such transformation must be a spinor with

$$Q|\text{boson}\rangle = |\text{fermion}\rangle \qquad \text{and} \qquad Q|\text{fermion}\rangle = |\text{boson}\rangle \,. \tag{2.1}$$

Supersymmetry is not an internal symmetry but a spacetime symmetry because it changes the spin of a particle and the behaviour of bosons and fermions is different under rotations. This can be seen trough the commutation relations between the supersymmetry generators and the other ones. The operator Q commutes with time and spatial translations and also with internal quantum numbers (like gauge and global symmetries). However, it does not commute with Lorentz generators

$$[Q, P_\mu] = 0 \,, \qquad [Q, M_{\mu\nu}] \neq 0 \,. \tag{2.2}$$

We should remark here that the full supersymmetry algebra contains the Poincaré algebra as a subalgebra, and since each irreducible representation of the Poincaré algebra corresponds to a particle, an irreducible representation of the supersymmetry algebra in general corresponds to several particles (because any representation of the full supersymmetry algebra also gives in general a reducible representation of the Poincaré algebra). The resulting states are related between them by the Q generators and have spins differing by units of one half. They form a supermultiplet. A supermultiplet always contains an equal number of bosonic and fermionic degrees of freedom (here it is necessary to make a distinction between on- and off-shell multiplets. In the off-shell case, it is necessary to add auxiliary fields to the theory in order to close the algebra off-shell and then we will have equality of bosonic and fermionic degrees of freedom). In other words, in all supersymmetric models each one-particle state has at least one superpartner, therefore instead of having single particle states we have supermultiplets of particle states. From the commutation relations we can also see that the particles belonging to the same supermultiplet have different spin but same mass and same quantum numbers.

It is possible to write down theories with any number $\mathcal{N}$ of supersymmetry generators but this number cannot be arbitrarily large, because as we will show later on, $\mathcal{N}$ is related with particle states of a given spin, so by increasing $\mathcal{N}$ the spin will also increase and it is very difficult to built higher spin consistent theories. For example in 4 dimensions to describe local and interacting massless theories with global supersymmetry, $\mathcal{N}$ can be at most as large as 4 for theories with maximal spin 1 (gauge theories) and as large as 8 for theories with maximal spin 2 (supergravity theories). For $\mathcal{N} \geq 9$ the representations contain some particles with spin $s > 2$. It turns out, that there is a one-to-one correspondence between the massless on shell supermultiplets of $\mathcal{N}$-extended four dimensional supersymmetry and the massive on shell-supermultiplets of $\mathcal{N}$ extended three-dimensional supersymmetry [56], we discuss both cases in detail in the following sections.

2.1.1. Massless multiplets

The maximum allowed number of supersymmetry generators $\mathcal{N}$, depends on the dimension of the spinor representation of the Lorentz group which at the same time depends on the spacetime dimensions where one is working (for example, in ten dimensions, which is the dimension where superstring theory lives, only two supersymmetry generators are allowed). In this section we will focus in the case of four dimensions.

Lets consider superalgebras containing $\mathcal{N} \geq 1$ Majorana spinor charges $Q_{i\alpha}$, where α is the spinor index and i is the index that labels the supercharges $i = 1, \ldots \mathcal{N}$. We define the supercharges $Q_{i\alpha}$ as the left-handed chiral components and use the Weyl representation, then $Q_{i\alpha} = (Q_{i1}, Q_{i2}, 0, 0)$, where the i index stands up for their hermitian conjugates $Q^{\dagger i\alpha} = ((Q_{i1})^*, (Q_{i2})^*, 0, 0)$ so we can use the index range $\alpha = 1, 2$. This gives the algebra

$$[P_\mu, Q_{i\alpha}] = 0\,, \quad [M_{\mu\nu}, Q_{i\alpha}] = -\frac{1}{2}[\gamma_{\mu\nu}]_\alpha^\beta Q_{i\beta}\,, \quad \{Q_{i\alpha}, Q^{\dagger j\beta}\} = \frac{1}{2}\delta_i^j [\gamma_\mu \gamma^0]_\alpha^\beta P^\mu\,. \tag{2.3}$$

Since the momentum operators P_μ commute with the supercharges, we may consider the states at arbitrary but fixed momentum P_μ which for massless representations, satisfies $P^2 = 0$, so it will be enough to consider the action of the supersymmetry generators on a set of particle states $|p^\mu, s, \lambda\rangle$ where s is the spin and λ is the helicity of the particle (the component of angular momentum in the direction of motion of the particle). Going to the rest frame $P^\mu = (E, 0, 0, E)$ and

$$\{Q_{i\alpha}, Q^{\dagger j\beta}\} = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \delta_i^j\,, \tag{2.4}$$

hence half of the spinors must vanish on physical states while the other half will generate a Clifford algebra. From the non-trivial generators we can choose Q_{i2} as the annihilation and $Q^{\dagger i2}$ as the creation operators. In the present frame

$$[J^3, Q^{\dagger i2}] = -\frac{1}{2} Q^{\dagger i2} , \tag{2.5}$$

which means that $Q^{\dagger i2}$ lowers the helicity of a state by 1/2 and Q_{i2} arises it by the same amount. Now, it is possible to choose a vacuum state (a state annihilated by all the annihilation operators). This vacuum state will carry some irreducible representation of the Poincaré algebra. We denote this state as $|\lambda_0\rangle$. We can construct the supermultiplet by applying the creation operator over $|\lambda_0\rangle$ creating a tower of helicity states of maximum helicity λ_0 and minimum helicity $\lambda_0 - \frac{1}{2}\mathcal{N}$ as

$$|\lambda_0\rangle \, , |\lambda_0 - \tfrac{1}{2}\rangle_i \, , |\lambda_0 - 1\rangle_{[ij]} \, \ldots |\lambda_0 - \tfrac{\mathcal{N}}{2}\rangle_{[i\ldots\mathcal{N}]} \, , \tag{2.6}$$

where $i = 1, \ldots \mathcal{N}$. Note that the particle spin $s = |\lambda|$ is a redundant label and p_μ is fixed so we omitted them when writing the states. Since the action of different creation operators is antisymmetric, the interchange of the indices $i, j, \ldots$ is antisymmetric too and states with helicity $\lambda = \lambda_0 - \frac{k}{2}$ have multiplicity $\binom{N}{k}$ with $k = 0, 1, \ldots \mathcal{N}$, see Table 2.1. The sequence stops at the multiplicity 1 state of lowest helicity $\lambda_0 - \frac{1}{2}\mathcal{N}$ and it is easy to see that in every multiplet $\lambda_{\max} - \lambda_{\min} = \frac{\mathcal{N}}{2}$. Summing the binomial coefficients gives a total of $2^{\mathcal{N}}$ states

State	Number of states
$\|\lambda_0\rangle$	$1 = \binom{\mathcal{N}}{0}$
$\|\lambda_0 - \frac{1}{2}\rangle$	$\mathcal{N} = \binom{\mathcal{N}}{1}$
$\|\lambda_0 - 1\rangle$	$\frac{1}{2!}\mathcal{N}(\mathcal{N}-1) = \binom{\mathcal{N}}{2}$
$\vdots$	$\vdots$
$\|\lambda_0 - \frac{\mathcal{N}}{2}\rangle$	$1 = \binom{\mathcal{N}}{\mathcal{N}}$

TABLE 2.1
This table gives the number of states that we will get in 4 dimensions depending on the number of supersymmetry generators $\mathcal{N}$ that we are considering.

with $2^{\mathcal{N}-1}$ having integer helicity (bosons) and $2^{\mathcal{N}-1}$ having half-integer helicity (fermions).

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}} = (2^{\mathcal{N}-1})_{bosons} + (2^{\mathcal{N}-1})_{fermions}\,, \tag{2.7}$$

For unextended supersymmetry $\mathcal{N} = 1$ each massless supermultiplet only contains two states $|\lambda_0\rangle$ and $|\lambda_0 - \frac{1}{2}\rangle$. We denote these multiplets by $(\lambda_0, \lambda_0 - \frac{1}{2})$. To satisfy CPT invariance and since they can never be CPT self-conjugate and one needs to double these multiplets by adding their CPT conjugate with opposite helicities and opposite quantum numbers. Thus one arrives at the following massless $\mathcal{N} = 1$ multiplets.

- The chiral multiplet ($\lambda_0 = 0$): contains $(0, -\frac{1}{2})$ and its CPT conjugate $(+\frac{1}{2}, 0)$. The degrees of freedom correspond to one Weyl fermion and one complex scalar. This is the representation to describe a matter multiplet.
- The vector multiplet ($\lambda_0 = -\frac{1}{2}$): consists of $(-\frac{1}{2}, -1)$ plus $(+1, +\frac{1}{2})$, corresponding to a gauge boson (massless vector) and one Weyl fermion (gaugino). This representation describes gauge fields in a supersymmetric theory.
- The gravitino multiplet ($\lambda_0 = -1$): contains $(-1, -\frac{3}{2})$ and $(+\frac{3}{2}, +1)$, which describe a gravitino and a gauge boson. Notice that there is a corresponding free field theory for a massless spin 3/2 fermion but no interacting field theory is know for this multiplet without supergravity.
- The graviton multiplet ($\lambda_0 = -\frac{3}{2}$): describes a $(-\frac{3}{2}, -2)$ and $(+2, +\frac{3}{2})$ supermultiplets corresponding to the graviton and the gravitino. This is the supergravity multiplet.

For $\mathcal{N} = 2$ the supermultiplets will contain $2^2 = 4$ states and again it is necessary to add their CPT conjugates so we will get

- The matter multiplet or hypermultiplet ($\lambda_0 = \frac{1}{2}$): contains $2\times(\frac{1}{2}, 0, 0, -\frac{1}{2})$ corresponding to two Weyl fermions and two complex scalars. Note that although this multiplet is self-conjugate it requires to be doubled because of hermiticity. This can be decomposed in terms of two $\mathcal{N} = 1$ chiral multiplets.
- The gauge multiplet ($\lambda_0 = 0$): consisting of $(0, -\frac{1}{2}, -\frac{1}{2}, -1)$ and $(+1, +\frac{1}{2}, +\frac{1}{2}, 0)$, so the degrees of freedom are those of one vector, two Weyl fermions and one complex scalar. This supermultiplet can be decomposed in terms of one $\mathcal{N} = 1$ vector and one $\mathcal{N} = 1$ chiral multiplets.

- The gravitino multiplet ($\lambda_0 = \frac{3}{2}$): describing a $(+\frac{3}{2}, +1, +1, +\frac{1}{2})$ and $(-\frac{1}{2}, -1, -1, -\frac{3}{2})$ corresponding to a spin 3/2 fermion, two vectors and one Weyl fermion.

- The graviton multiplet ($\lambda_0 = 2$): containing $(+2, +\frac{3}{2}, +\frac{3}{2}, +1)$ and $(-1, -\frac{3}{2}, -\frac{3}{2}, -2)$ a graviton, two gravitini and a vector.

Consider now the $\mathcal{N} = 4$ case

- The vector multiplet ($\lambda_0 = +1$): consisting of $(1 \times (+1), 4 \times (+\frac{1}{2}), 6 \times (0), 4 \times (-\frac{1}{2}), 1 \times (-1))$. This is the only multiplet in $\mathcal{N} = 4$ with states with helicity $\lambda < 2$. It consist of three $\mathcal{N} = 1$ chiral multiplets plus their CPT conjugates and one $\mathcal{N} = 1$ vector. Or one $\mathcal{N} = 2$ vector multiplet and two $\mathcal{N} = 2$ hypermultiplets plus their CPT conjugates.

- The gravitino multiplet ($\lambda_0 = +\frac{3}{2}$): contains $(1 \times (+\frac{3}{2}), 4 \times (+1), 6 \times (+\frac{1}{2}), 4 \times (0), 1 \times (-\frac{1}{2}))$ plus its CPT conjugate.

- The graviton multiplet ($\lambda_0 = +2$): consisting of $(1 \times (+2), 4 \times (+\frac{3}{2}), 6 \times (+1), 4 \times (+\frac{1}{2}), 1 \times (0))$ and its CPT conjugate.

Finally, for $\mathcal{N} = 8$ generators we obtain

- The maximum multiplet: $(1 \times (\pm 2), 8 \times (\pm\frac{3}{2}), 28 \times (\pm 1), 56 \times (\pm\frac{1}{2}), 70 \times (0))$.

It is important to note that renormalizable theories have spin $s \leq 1$ implying $\mathcal{N} \leq 4$. Therefore $\mathcal{N} = 4$ is the largest supersymmetry for renormalizable theories with global supersymmetry. Besides this, the maximum number of supersymmetries is $\mathcal{N} = 8$ otherwise we will have particles with spin higher than two and theories with more than one graviton. Supergravity theories will be those involving one spin-2 graviton, $\mathcal{N}$ spin-3/2 gravitini, and, for $\mathcal{N} \geq 2$ lower spin particles.

2.1.2. Massive supermultiplets

Having discussed the algebra and representations of extended massless supersymmetry, we will turn now to the case of massive supersymmetry $\mathcal{N} \geq 1$ in three dimensions. In this section we follow the discussion given in [56]. The algebra in this case is

$$[P_\mu, Q_{i\alpha}] = 0\,, \quad [M_{\mu\nu}, Q_{i\alpha}] = -\frac{1}{2}[\gamma_{\mu\nu}]_\alpha^\beta Q_{i\beta}\,, \quad \{Q_{i\alpha}, Q^{\dagger j\beta}\} = \frac{1}{2}\delta_i^j[\gamma_\mu\gamma^0]_\alpha^\beta P^\mu\,,$$
$$\{Q_{i\alpha}, Q_{j\beta}\} = \epsilon_{ij}\mathcal{Z}_{\alpha\beta}\,, \quad \{Q^{\dagger i\alpha}, Q^{\dagger j\beta}\} = \epsilon^{ij}\bar{\mathcal{Z}}^{\alpha\beta}\,, \tag{2.8}$$

these are superalgebras with central charges. The generator $\mathcal{Z}$ and its complex conjugate $\bar{\mathcal{Z}}$ commute with all generators of the full algebra. Because of the presence of ϵ_{ij}, there is no possibility of central charges for the simplest algebra $\mathcal{N}=1$.

In general, massive multiplets are bigger than massless ones because the number of non-trivial generators is not reduced, unlike for the massless case where one-half of the generators vanishes. However, it is possible to have a shortening of the massive representation in the presence of special values of the central charges. The shortened supermultiplets are known as BPS multiplets.

In the rest frame where $P_\mu = (m, 0, 0)$ and $P^2 = m^2 > 0$, the supersymmetry algebra becomes

$$\{Q_{i\alpha}, Q^{\dagger j\beta}\} = m\delta_i^j \delta_\alpha^\beta\,, \quad \{Q_{i\alpha}, Q_{j\beta}\} = \epsilon_{ij}\mathcal{Z}_{\alpha\beta}\,, \quad \{Q^{\dagger i\alpha}, Q^{\dagger j\beta}\} = \epsilon^{ij}\bar{\mathcal{Z}}^{\alpha\beta}\,. \quad (2.9)$$

Since the central charges can or can not be non-vanishing, we will just focus in the vanishing central charges $\mathcal{Z}=0$ case.

There are $2\mathcal{N}$ creation and annihilation operators $Q^{\dagger iA}$ and Q_{iA} respectively (with $A=1,2$), in this case $Q^{\dagger iA}$ lowers the helicity by $\frac{1}{2}$ and Q_{iA} rises it by the same amount. In 3D the helicity λ is equal to the Pauli-Lubanski pseudo scalar divided by the mass. We can define a vacuum state with mass m and spin s as $|\Omega\rangle$ which is annihilated by both Q_{iA} and act with the creation operators to construct the corresponding massive representation

$$|\Omega\rangle, Q^{\dagger iA}|\Omega\rangle, Q^{\dagger jB}Q^{\dagger iA}|\Omega\rangle, Q^{\dagger kC}Q^{\dagger jB}Q^{\dagger iA}|\Omega\rangle, \ldots, Q^{\dagger \mathcal{N}B}\ldots Q^{\dagger iA}|\Omega\rangle, \qquad (2.10)$$

leading to $2^{\mathcal{N}}$ states with helicities ranging from λ to $\lambda + \mathcal{N}/2$. It turns out that formally these are the same massless particle multiplets in four dimensions for $\mathcal{N}$-extended supersymmetry.

Helicity	+2	+3/2	+1	+1/2	0	-1/2	-1	-3/2	-2
$\mathcal{N}=1$	1	1							
$\mathcal{N}=2$	1	2	1						
$\mathcal{N}=4$	1	4	6	4	1				
$\mathcal{N}=8$	1	8	28	56	70	56	28	8	1

Table 2.2
This table gives the multiplets containing the +2 helicities for $\mathcal{N}=1,2,4$ and 8 supersymmetries.

One important difference between the 4D and 3D theories is that the last one has half the total number of supersymmetries. The correspondence between 4D and 3D theories is for supermultiplets, and not for free field theories as it is explained in [56].

2.2. Supergravity

Supergravity is a field theory that combines general relativity and supersymmetry, therefore, supersymmetry holds locally in a supergravity theory (in other words, supergravity is the supersymmetric theory of gravity, or equivalently, a theory of local supersymmetry). The theory will be invariant under supersymmetry transformations in which the spinor parameters are arbitrary functions of the spacetime coordinates. The gauge field of supersymmetry is a spin 3/2 field ψ_μ^α called gravitino and it is the supersymmetric partner of the graviton.

The supersymmetry algebra (see (2.8)) will then involve local translation parameters which must be viewed as diffeomorphisms. Take the anticommutation relation

$$\{Q_{i\alpha}, Q^{\dagger j\beta}\} = \tfrac{1}{2}\delta_i^j[\gamma_\mu\gamma^0]_\alpha^{\ \beta} P^\mu \,, \tag{2.11}$$

this indicates that having general coordinate transformations is equivalent to have local supersymmetry. Thus local supersymmetry requires gravity. The converse is also true. In any supersymmetric theory which includes gravity, supersymmetry has to be realized locally. The reason is that a constant spinor is not compatible with the symmetries required in a theory of gravity with fermions. One must extend the constant parameter to a local one.

Since Einstein's gravity is a gauge theory of local symmetry, then the generators of rotations M_{ab} and the translation generators P_a have corresponding gauge fields ω_μ^{ab}, the spin connection, and e_μ^a, the vierbein (where $a, b = 0, \ldots, D-1$ are Lorentz gauge group indices and $\mu, \nu = 0, \ldots, D-1$ are spacetime indices). Thus, the vierbein and the spin connection transform under general coordinate transformations as collections of vectors. A vierbein formulation of gravity is necessary since supersymmetry will require the presence of spinor fields. The gauge fields have corresponding field strengths $R_{\mu\nu}^{ab}$, the Riemann curvature tensor, and $C_{\mu\nu}^a$ the torsion. Setting the torsion to zero allows us to solve for the spin connection in terms of the vierbein, which leaves a theory with two degrees of freedom: the graviton.

A supergravity theory is an interacting field theory that contains the gravity multiplet plus other matter multiplets of the underlying global supersymmetry algebra. The gauge multiplet consists of the vierbein $e_\mu^a(x)$ describing the graviton plus the supersymmetric partners of it, a specific number $\mathcal{N}$ of vector–spinor fields $\psi^i_\mu(x)$, $i = 1, ..., N$, the gravitini. In the basic case of $\mathcal{N} = 1$ supergravity in four spacetime dimensions, the gauge multiplet consists entirely of the graviton and one Majorana spinor gravitino.

2.3. Kinematical Groups

Spacetime symmetries have played a central role in the understanding of various physical theories such as Newtonian Gravity, Maxwell's Electromagnetism, Special Relativity, General Relativity, Strings and Supergravity. Most of these models are based on relativistic symmetries. An example of a model with non-relativistic symmetries is Newtonian Gravity which is based on the Galilei symmetries. Such non-relativistic symmetries arise when the velocity of light is sent to infinity.

A less well known example of a non-relativistic symmetry are the Carroll symmetries which arise when the velocity of light is sent to zero [32]. In this sense the Carroll symmetries are the opposite to the Galilei symmetries. This can also be seen by looking at the light cone which in the Carroll case, at each point of spacetime, collapses to the time axis whereas in the Galilei case it coincides with the space axis, see Table 2.3.

Table 2.3
The diagram at the left shows the transition Einstein→Newton that occurs in the limit $c \to \infty$, geometrically it can be viewed as the opening of the light cones. The diagram at the right shows the transition Einstein→Carroll that occurs in the limit $c \to 0$ which can be interpreted as the shrinking of the light cones.

A systematic investigation of the possible relativity groups[1] was initiated by Bacry and Lévy-Leblond [31]. They showed that all these groups can be obtained by a contraction of the anti-de Sitter (AdS) and de Sitter (dS) groups. There are eleven kinematical Lie algebras:

[1] A relativity group is an invariance group of a physical theory that contains the generators of special relativity: time translations, spatial translations, boosts and spatial rotations.

- Two simple Lie algebras: the dS and AdS algebras.
- Five solvable Lie algebras: the Poincaré (P) algebra, two Newton-Hook ($NH_\pm$) and two AdS-Carroll ($AC_\pm$) algebras.
- Three nilpotent Lie algebras: the Galilei (G), the Carroll (C) and two Para-Galilei (PG) algebras.

$$\begin{tikzcd}
 & & AdS \arrow[dll, "c\to 0"'] \arrow[d, "R\to\infty"] \arrow[drr, "c\to\infty"] & & \\
AC \arrow[dr, "R\to\infty"'] & & P \arrow[dl, "c\to 0"'] \arrow[dr, "c\to\infty"] & & NH \arrow[dl, "R\to\infty"] \\
 & C & & G &
\end{tikzcd}$$

TABLE 2.4
The figure displays the different contractions of the AdS group. The different abbreviations are explained in the text.

As Table 2.4 shows there are three different types of contractions[2]: the non-relativistic limit $c \to \infty$ of the AdS group leads to the Newton-Hooke (NH) group. The flat limit $R \to \infty$ leads to the Poincaré (P) group and the ultra-relativistic limit $c \to 0$ leads to the AdS-Carroll (AC) group [32][3]. In a second stage, the flat limit of the AdS-Carroll group and the ultra-relativistic limit of the Poincaré group leads to the Carroll (C) group while the non-relativistic limit of the Poincaré group and the flat limit of the Newton-Hooke group leads to the Galilean (G) group.

All the algebras corresponding to the kinematical groups given in Table 2.4 contain the same commutators involving spatial rotations. These commutators are given by

$$[M_{ab}, M_{cd}] = 2\eta_{a[c}M_{d]b} - 2\eta_{b[c}M_{d]a}\,, \tag{2.12}$$

$$[M_{ab}, P_c] = 2\delta_{c[b}P_{a]}\,, \qquad [M_{ab}, K_c] = 2\delta_{c[b}K_{a]}\,, \tag{2.13}$$

[2]In this thesis we will only consider the AdS case.
[3]Bacry and Lévy-Leblond [31] call this algebra the para-Poincaré algebra.

where $a = 1, \ldots, D-1$, for a D-dimensional space-time. The Galilean algebra can be extended with a central charge generator Z to the so-called Bargmann algebra [57]. It has been recently shown that there is a duality between this Bargmann algebra and the Carroll algebra by the exchange of Z and the generator of time translations H [33]. Note that this duality does not extend to a duality between the Newton-Hooke and AdS-Carroll algebras. This is due to the expression for the commutator $[P_a, P_b]$, see Table 2.5[4].

	$[P_a, K_b]$	$[H, K_a]$	$[H, P_a]$	$[P_a, P_b]$	$[K_a, K_b]$
AdS	$\delta_{ab}H$	P_a	$-\frac{1}{R^2}K_a$	$\frac{1}{R^2}M_{ab}$	M_{ab}
Poincaré	$\delta_{ab}H$	P_a	0	0	M_{ab}
Newton-Hooke	$\delta_{ab}Z$	P_a	$-\frac{1}{R^2}K_a$	0	0
AdS-Carroll	$\delta_{ab}H$	0	$-\frac{1}{R^2}K_a$	$\frac{1}{R^2}M_{ab}$	0
Galilei	$\delta_{ab}Z$	P_a	0	0	0
Carroll	$\delta_{ab}H$	0	0	0	0

TABLE 2.5
This table gives an overview of the algebras of the relativity groups that we consider.

The next thing to do is to consider a supersymmetric extension of the bosonic relativity groups given in 2.4. We construct the $\mathcal{N} = 1$ superalgebras in any dimension (see Tables 2.5 and 2.6, where Q stands for the generator of supersymmetry).
All these algebras contain the same commutator between spatial rotations and supersymmetry generators

$$[M_{ab}, Q] = -\frac{1}{2}\gamma_{ab}Q\,. \tag{2.14}$$

Note that an adequate Newton-Hooke superalgebra should contain commutator relations that yield $\{Q, Q\} \sim H$ *and* $\{Q, Q\} \sim \not{P}$, that is, the commutator of two supersymmetries gives time- *and* space-translations. For $\mathcal{N} = 1$ this cannot be achieved, in order to obtain a true supersymmetric extension of the Bargmann algebra in which the anti-commutator of two supersymmetry generators gives

[4] Note that the AdS Carroll algebra is formally isomorphic to the Poincaré algebra by replacing the AC generators of time/spatial translations and boosts according to $H \to \frac{1}{R}H$, $P_a \to \frac{1}{R}K_a$ and $K_a \to P_a$. We thank Sergey Krivonos for pointing this out to us.

$\mathcal{N}=1$	$[K_a,Q]$	$[H,Q]$	$[P_a,Q]$	$\{Q_\alpha,Q_\beta\}$
AdS	$-\frac{\gamma_{a0}}{2}Q$	$\frac{\gamma_0}{2R}Q$	$\frac{\gamma_a}{2R}Q$	$2[\gamma^A C^{-1}]_{\alpha\beta}P_A+\frac{1}{R}[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}$
Poincaré	$-\frac{\gamma_{a0}}{2}Q$	0	0	$2[\gamma^A C^{-1}]_{\alpha\beta}P_A$
Newton-Hooke	0	0	0	$2\delta_{\alpha\beta}Z$
AdS-Carroll	0	0	$\frac{1}{2R}\gamma_a Q$	$[\gamma^0 C^{-1}]_{\alpha\beta}H+\frac{2}{R}[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a$
Galilei	0	0	0	$2\delta_{\alpha\beta}Z$
Carroll	0	0	0	$[\gamma^0 C^{-1}]_{\alpha\beta}H$

Table 2.6
In this table we give the (anti-)commutators of the $\mathcal{N}=1$ superalgebras that involve the generators Q of supersymmetry.

both a time and a space translation we need at least two supersymmetries. On the other hand the AdS Carroll superalgebra has a conventional supersymmetry algebra, where the energy and boost generators appear in the anti-commutator of the supersymmetries.

One way to obtain $\mathcal{N}=2$ locally supersymmetric theories in three spacetime dimensions is through dimensional reduction from four dimensional $\mathcal{N}=1$ supersymmetric systems, however, there are theories which cannot be obtained from such procedure. Such theories contain Chern-Simons couplings, in [58] is showed that in three dimensions $\mathcal{N}$-extended AdS supergravity exists in several incarnations [59,60]. These were called the (p,q) AdS supergravity theories where the non-negative integers $p\geq q$ are such that $\mathcal{N}=p+q$. The $\mathcal{N}=(p,q)$ notation refers to the associated 3D AdS supergroups $\mathrm{OSp}(2,p)\oplus\mathrm{OSp}(2,q)$ with R-symmetry group $\mathrm{SO}(p)\times\mathrm{SO}(q)$.

There are two different independent versions of the $\mathcal{N}=2$ supergravities in three dimensions, the so-called $\mathcal{N}=(1,1)$ and $\mathcal{N}=(2,0)$ representations.

The basic commutators of the 3D $\mathcal{N}=(2,0)$ AdS algebra are given by those on Table 2.5 plus

$$\Big[M_{AB},Q^i\Big]=-\frac{1}{2}\gamma_{AB}Q^i\,,\qquad [P_A,Q^i]=x\gamma_A Q^i\,,\qquad [\mathcal{R},Q^i]=2x\epsilon^{ij}Q^j\,,$$
$$\{Q^i_\alpha,Q^j_\beta\}=2[\gamma^A C^{-1}]_{\alpha\beta}P_A\delta^{ij}+2x[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}\delta^{ij}+2[C^{-1}]_{\alpha\beta}\epsilon^{ij}\mathcal{R}\,, \tag{2.15}$$

where $(A=0,1,2;i=1,2)$. Here $P_A, M_{AB}, \mathcal{R}$ and Q^i_α are the generators of spacetime translations, Lorentz rotations, SO(2) R-symmetry transformations and supersymmetry transformations, respectively. The bosonic generators P_A, M_{AB} and $\mathcal{R}$ are anti-hermitian while the fermionic generators Q^i_α are hermitian. The parameter $x = 1/(2R)$, with R being the AdS radius. Note that the generator of the SO(2) R-symmetry becomes the central element of the Poincaré algebra in the flat limit $x \to 0$.

To show that the above algebra corresponds to the $\mathcal{N} = (2, 0)$ AdS algebra it is convenient to define the new generators

$$M_C = \epsilon_{CAB} M^{AB} \,, \qquad\qquad J^{\pm}_A = P_A \pm x M_A \,. \tag{2.16}$$

In terms of these new generators we obtain the following (anti-)commutation relations:

$$[J^+_A, Q^i] = 2x\gamma_A Q^i \,, \qquad\qquad \{Q^i_\alpha, Q^j_\beta\} = 2[\gamma^A C^{-1}]_{\alpha\beta} J^+_A \delta^{ij} \,, \tag{2.17}$$

while the charges Q^i do not transform under J^-_A. This identifies the algebra as the $\mathcal{N} = (2, 0)$ AdS algebra.

We now consider the $3D$ $\mathcal{N} = (1, 1)$ anti-de Sitter algebra which is given by the commutators on Table 2.5 and the following (anti-)commutators:

$$\begin{aligned} [M_{AB}, Q^\pm] &= -\frac{1}{2}\gamma_{AB} Q^\pm \,, \qquad [P_A, Q^\pm] = \pm x \gamma_A Q^\pm \,, \\ \{Q^\pm_\alpha, Q^\pm_\beta\} &= 4[\gamma^A C^{-1}]_{\alpha\beta} P_A \pm 4x[\gamma^{AB} C^{-1}]_{\alpha\beta} M_{AB} \,, \end{aligned} \tag{2.18}$$

Here P_A, M_{AB} and $Q^\pm_\alpha$ are the generators of space-time translations, Lorentz rotations and supersymmetry transformations, respectively. The bosonic generators P_A and M_{AB} are anti-hermitian while the fermionic generators $Q^\pm_\alpha$ are hermitian. The parameter $x = 1/(2R)$ is a contraction parameter.

If we consider the AdS superalgebra (2.18) it is easy to show that this algebra corresponds to the $\mathcal{N} = (1, 1)$ representation by redefining

$$M_C = \epsilon_{CAB} M^{AB} \,, \qquad J_\pm = P_A \pm x M_A \,. \tag{2.19}$$

This gives the (anti-)commutation relations

$$[J_+, Q^+] = 2x\gamma_A Q^+ \,, \quad [J_-, Q^-] = -2x\gamma_A Q^- \,, \quad \{Q^\pm_\alpha, Q^\pm_\beta\} = 4[\gamma^A C^{-1}]_{\alpha\beta} J_\pm \,, \tag{2.20}$$

where the $\mathcal{N} = (1, 1)$ character is manifest.

Together with the bosonic commutators in Table 2.5 the algebras of the kinematical groups that we study in this thesis are given in Tables 2.7 and 2.8 for

$\mathcal{N}=(2,0)$	$[K_a,Q^+]$	$[K_a,Q^-]$	$[H,Q^+]$	$[H,Q^-]$	$[P_a,Q^+]$	$[P_a,Q^-]$	$[Z,Q^+]$	$[Z,Q^-]$
AdS	$-\frac{\gamma_{a0}}{2}Q^-$	$-\frac{\gamma_{a0}}{2}Q^+$	$-\frac{\gamma_0}{2R}Q^+$	$\frac{3\gamma_0}{2R}Q^-$	$\frac{\gamma_a}{2R}Q^-$	$\frac{\gamma_a}{2R}Q^+$	$-\frac{3\gamma_0}{4R}Q^+$	$-\frac{\gamma_0}{4R}Q^-$
P	$-\frac{\gamma_{ab}}{2}Q^-$	$-\frac{\gamma_{ab}}{2}Q^+$	0	0	0	0	0	0
NH	$-\frac{\gamma_{a0}}{2}Q^-$	0	$-\frac{\gamma_0}{2R}Q^+$	$\frac{3\gamma_0}{2R}Q^-$	$\frac{\gamma_a}{2R}Q^-$	0	0	0
AC	0	0	0	0	$\frac{\gamma_a}{2R}Q^-$	$\frac{\gamma_a}{2R}Q^+$	0	0
G	$-\frac{\gamma_{a0}}{2}Q^-$	0	0	0	0	0	0	0
C	0	0	0	0	0	0	0	0

TABLE 2.7
This table gives an overview of the commutators of the $\mathcal{N}=(2,0)$ superalgebras that involve the supersymmetry generators Q_+ and Q_-

$\mathcal{N}=(2,0)$	$\{Q^+_\alpha,Q^+_\beta\}$	$\{Q^-_\alpha,Q^-_\beta\}$	$\{Q^+_\alpha,Q^-_\beta\}$
AdS	$[\gamma^0C^{-1}]_{\alpha\beta}H + \frac{1}{2R}[\gamma^{ab}C^{-1}]_{\alpha\beta}M_{ab}$	$2[\gamma^0C^{-1}]_{\alpha\beta}Z$	$[\gamma^aC^{-1}]_{\alpha\beta}P_a + \frac{1}{2R}[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}$
Poincaré	$[\gamma^0C^{-1}]_{\alpha\beta}H$	$2[\gamma^0C^{-1}]_{\alpha\beta}Z$	$[\gamma^aC^{-1}]_{\alpha\beta}P_a$
Newton-Hooke	$[\gamma^0C^{-1}]_{\alpha\beta}H + \frac{1}{2R}[\gamma^{ab}C^{-1}]_{\alpha\beta}M_{ab}$	$2[\gamma^0C^{-1}]_{\alpha\beta}Z$	$[\gamma^aC^{-1}]_{\alpha\beta}P_a + \frac{1}{2R}[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a$
AdS-Carroll	$[\gamma^0C^{-1}]_{\alpha\beta}(H+2Z)$	$[\gamma^0C^{-1}]_{\alpha\beta}(H-2Z)$	$\frac{1}{2R}[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a$
Galilei	$[\gamma^0C^{-1}]_{\alpha\beta}H$	$2[\gamma^0C^{-1}]_{\alpha\beta}Z$	$[\gamma^aC^{-1}]_{\alpha\beta}P_a$
Carroll	$[\gamma^0C^{-1}]_{\alpha\beta}(H+2Z)$	$[\gamma^0C^{-1}]_{\alpha\beta}(H-2Z)$	0

TABLE 2.8
This table gives the anticommutators of the $\mathcal{N}=(2,0)$ superalgebras that we studied.

the $\mathcal{N}=(2,0)$ case and in Tables 2.9 and 2.10 for the $\mathcal{N}=(1,1)$ case. In order to compare the (anti-)commutators of all the kinematical groups we rewrote the generators of the AdS algebras P_0, M_{a0} and $Q^{1,2}$ in terms of the generators H, K_a and $Q^\pm$.

The Newton–Hooke superalgebra that we will use is obtained by contracting the $\mathcal{N}=(2,0)$ AdS superalgebra. The reason why we do not use the $\mathcal{N}=(1,1)$ AdS algebra for the contraction in this case is essentially the same reason as why we are interested in $\mathcal{N}=2$ rather than $\mathcal{N}=1$ algebras. Taking the non-relativistic contraction thereof amounts to taking the simultaneous contractions of two independent $\mathcal{N} = 1$ algebras. However, we already argued that this cannot lead to a superalgebra of the desired form. On the other hand, for the ultra-relativistic limit, the $\mathcal{N} = (2,0)$ and $\mathcal{N} = (1,1)$ AdS Carroll algebras are not isomorphic. We will see in Chapter 7 that the associated particle actions are rather different.

$\mathcal{N} = (1,1)$	$[K_a, Q^-]$	$[H, Q^+]$	$[H, Q^-]$	$[P_a, Q^+]$	$[P_a, Q^-]$
AdS	0	$\frac{\gamma_0}{2R}Q^+$	$-\frac{\gamma_0}{2R}Q^-$	$\frac{\gamma_a}{2R}Q^+$	$-\frac{\gamma_a}{2R}Q^-$
P	0	0	0	0	0
AC	0	0	0	$\frac{\gamma_a}{2R}Q^+$	$-\frac{\gamma_a}{2R}Q^-$
C	0	0	0	0	0

Table 2.9
This table gives the commutators of the $\mathcal{N} = (1,1)$ superalgebras that involve the supersymmetry generators Q_+ and Q_-.

$\mathcal{N} = (1,1)$	$\{Q^+_\alpha, Q^+_\beta\}$	$\{Q^-_\alpha, Q^-_\beta\}$
AdS	$[\gamma^A C^{-1}]_{\alpha\beta} P_A + \frac{1}{2R}[\gamma^{AB} C^{-1}]_{\alpha\beta} M_{AB}$	$[\gamma^A C^{-1}]_{\alpha\beta} P_A - \frac{1}{2R}[\gamma^{AB} C^{-1}]_{\alpha\beta} M_{AB}$
Poincaré	$[\gamma^A C^{-1}]_{\alpha\beta} P_A$	$[\gamma^A C^{-1}]_{\alpha\beta} P_A$
AdS-Carroll	$2[\gamma^0 C^{-1}]_{\alpha\beta} H + \frac{4}{R}[\gamma^{a0} C^{-1}]_{\alpha\beta} K_a$	$2[\gamma^0 C^{-1}]_{\alpha\beta} H - \frac{4}{R}[\gamma^{a0} C^{-1}]_{\alpha\beta} K_a$
Carroll	$2[\gamma^0 C^{-1}]_{\alpha\beta} H$	$2[\gamma^0 C^{-1}]_{\alpha\beta} H$

Table 2.10
This table gives the anticommutators of the $\mathcal{N} = (1,1)$ superalgebras of the kinematical groups that we are considering.

2.4. Non-Linear Realizations

Non-linear realizations have played a very important role in our understanding of symmetries in quantum field theories and in particle physics in general.

Given a quantum field theory with a rigid symmetry group G which is spontaneously broken to a subgroup H, in many circumstances, its effective theory is described by the non-linear realization of G with local subgroup H. Having deduced the non-linear realization, one has discovered the symmetry G of the underlying theory and possessing the symmetry one may then use it to try to understand the underlying dynamics that it has. In this section we review this procedure and later on we will use it when we discuss the non relativistic and the Carroll superparticle sections.

In the original approach [61,62] Callan, Coleman, Wess and Zumino follow a method in terms of coset representatives[5]. We use a reformulation of the theory of non-linear realizations in terms of the group elements of G themselves rather than those of the coset G/H [65,66]. Let us consider a Lie supergroup G, a subgroup H, and the coset G/H. We will use the following notation for the Lie generators:

$$G_A \in G\,, \quad H_i \in H\,, \quad G_I \in G/H\,. \tag{2.21}$$

The generators of the supergroup satisfy the graded commutation relations

$$[G_A, G_B\} = f_{AB}^{C} G_C\,. \tag{2.22}$$

The bracket structure $\{\,,\,]$ signifies either commutator or anticommutator, according to the even or odd character of the generators. The structure constants are graded anti-symmetric

$$f_{AB}^{C} = -(-1)^{AB} f_{BA}^{C}\,, \quad \text{where} \quad \begin{array}{c} (-)^{AB} = -1 \text{ when } A \text{ and } B \text{ refer to odd} \\ \text{generators}\,, \\ (-)^{AB} = +1 \text{ otherwise}\,. \end{array} \tag{2.23}$$

We will use the conventions corresponding to independent parity for forms and Grassmann variables. For instance, the rule for a 0-form as G_A and for a 1-form as L^B would be

$$L^A G_B = (-)^{AB} G_B L^A\,, \tag{2.24}$$

whereas for two 1-forms

$$L^A L^B = -(-)^{AB} L^B L^A\,. \tag{2.25}$$

We consider group elements $g(z)$ of G and take the symmetries of the non-linearly realized theory given by

$$g(z) \to g_0 g(z) \text{ and } g(z) \to g(z)h(z)\,, \tag{2.26}$$

[5]For an early application of this method in a different situation than the one considered in this work, namely to the construction of worldline actions of conformal and superconformal particles, see [63,64].

where $h \in H$ is a local (that is, spacetime dependent) transformation that belongs to H and $g_0 \in G$ is a rigid transformation (that is, does not depend on spacetime) that belongs to G. We note that the rigid and local transformations may be carried out completely independently. This defines what the non-linear realization is and one only has to find the dynamics that is invariant under this symmetry.

This theory contains gauge degrees of freedom that can be fixed using the local H transformations. This formulation is a non-linear realization of a group G with local subgroup H. The global G transformations of the coordinates z's are determined up to the local H transformations and are in general non-linear. This general approach can be used for spacetime symmetries as well as internal symmetries of fields. One can recover the original approach using from the beginning these local H transformations to set to zero some fields (fixing the gauge freedom) and so work only with coset representatives, it would also require the H-compensating transformations for the global g_0 transformations. In this section we will review this more general way of constructing non-linear realizations and in the non-relativistic and Carroll sections will choose specific examples of the coset.

Let us now consider the structure equations for the forms L^A. The usual method is to consider the Maurer-Cartan (MC) one form which belongs to the Lie algebra and so can be written in the form

$$\Omega = g^{-1}dg = G_A L^A \,. \tag{2.27}$$

This equation defines the set of 1-forms L^A and satisfies the MC equation

$$d\Omega = G_A dL^A = d(g^{-1}dg) = dg^{-1}dg = -g^{-1}dgg^{-1}dg = -\Omega \wedge \Omega \,, \tag{2.28}$$

and

$$\Omega\wedge\Omega = (-)^{AB}G_A G_B L^A\wedge L^B = \frac{1}{2}(-)^{AB}[G_A, G_B\} L^A\wedge L^B = -\frac{1}{2}f_{AB}^{C} G_C L^B\wedge L^A \,, \tag{2.29}$$

therefore we get the structure equations for the L^A's

$$dL^C - \frac{1}{2}f_{AB}^{C} L^B \wedge L^A = 0 \,. \tag{2.30}$$

When the forms L^A are considered as depending on the parameters z^A, we have $L^A = dz^B L_B^A$. In order to evaluate the properties of transformations of the MC form with respect to an infinitesimal variation of the group element g, it is very useful to introduce the following quantity

$$\Omega_\delta = g^{-1}\delta g \,. \tag{2.31}$$

Notice that formally $\Omega_\delta \to \Omega$ when $\delta g \to dg$. We may now evaluate the variation of Ω to get

$$\delta\Omega = d\Omega_\delta - [\Omega_\delta, \Omega]\,, \tag{2.32}$$

which does not depend on the grading. Let us now consider a generic variation of the parameters z^A and define

$$\Omega_\delta = G_A[\delta z^A] = (-)^A[\delta z^A]G_A, \qquad [\delta z^A] = \delta z^B L_B^A\,, \tag{2.33}$$

we get

$$\delta(G_A L^A) = d(G_A[\delta z^A]) - [G_B[\delta z^B], G_C L^C]\,. \tag{2.34}$$

Using the identity

$$[O_1 c_1, O_2 c_2] = [O_1, O_2\} c_2 c_1\,, \tag{2.35}$$

valid for graded c-numbers c_i's and graded operators O_i's, $i = 1, 2$, with $\deg[c_i] = \deg[O_i]$, we get

$$[G_B[\delta z^B], G_C L^C] = f_{BC}^A G_A L^C [\delta z^B]\,, \tag{2.36}$$

and therefore

$$\delta L^A = d([\delta z^A]) - f_{BC}^A L^C [\delta z^B]\,. \tag{2.37}$$

Now, the central problem in the theory of non-linear realizations is to construct an action. Let us consider $\{B_i\}$ as the set of generators of broken (super)translations (it can also include central charge extensions), $\{T_i\}$ the unbroken translation generators, $\{H_i\}$ the generators of the stability group H and $\{K_i\}$ the remaining broken transformations. We now consider the coset element $g = g_0\, U$, where g_0 is the coset representing the group (super)space of broken and unbroken translations and U is in terms of the generators $\{K_a\}$. We may write g_0 and U in terms of an exponential parametrization as

$$g_0 = e^{T_i z_T^i(x)} e^{B_i z_B^i(x)}\,, \tag{2.38}$$

where the z_i are the Goldstone bosons of the symmetry generators T_i and B_i, and $U = e^{K_i z_K^i(x)}$, where U is parametrized by the Goldstone bosons of the G_i symmetry transformations.

As a warming up example let us consider the basic commutators of the AdS algebra given by $(A = 0, 1, \ldots, D-1)$

$$\begin{gathered}
[M_{AB}, M_{CD}] = 2\eta_{A[C}M_{D]B} - 2\eta_{B[C}M_{D]A}\,, \\
[M_{AB}, P_C] = -2\eta_{C[A}P_{B]}\,, \qquad [P_A, P_B] = \frac{1}{R^2}M_{AB}\,.
\end{gathered} \tag{2.39}$$

Here P_A, and M_{AB}, are the generators of spacetime translations, and rotations respectively and both of them are anti-hermitian.

To make the non-relativistic and the ultrarelativistic contractions in the following sections, we rescale the generators with parameters ω, α and β as follows ($a = 1, \dots, D-1$):

$$P_0 = \alpha\, H\,, \qquad M_{a0} = \omega\, K_a\,, \qquad R = \beta \tilde{R}\,, \tag{2.40}$$

and divide the commutators in the following way

$$\begin{aligned} &[M_{ab}, M_{cd}] = 2\delta_{a[c}M_{d]b} - 2\delta_{b[c}M_{d]a}\,, \qquad [J_{ab}, (P/K)_c] = -2\,\delta_{c[a}(P/K)_{b]}\,,\\ &[K_a, K_b] = \frac{1}{\omega^2}M_{ab}\,, \qquad [P_a, P_b] = \frac{1}{\beta^2 R^2}M_{ab}\,, \qquad [P_a, K_b] = \frac{\alpha}{\omega}\,\delta_{ab}H\,,\\ &[H, K_a] = \frac{1}{\alpha\omega}P_a\,, \qquad [H, P_a] = -\frac{\omega}{\alpha\beta^2 R^2}\,K_a\,. \end{aligned} \tag{2.41}$$

We consider the coset

$$\frac{G}{H} = \frac{\text{AdS}}{\text{SO}(D-1)}\,, \tag{2.42}$$

and the coset element $g = g_0\, U$, where $g_0 = e^{Ht}e^{P_a x^a}$ is the coset representing the AdS space and $U = e^{K_a v^a}$ is a general boost. Here, the complete set of generators is given by $\{G_i\} = \{H, P_a, K_a, M_{ab}\}$, where

$\{T_i\} = H$	is the unbroken time translation,
$\{B_i\} = P_a$	are the broken translation generators,
$\{K_i\} = K_a$	are the broken AdS boost generators,
$\{H_i\} = M_{ab}$	are the generators of the stability group in this case the spatial rotations.

The x^a ($a = 1, ..., D-1$) are the Goldstone bosons of broken translations, t is the Goldstone boson of the unbroken time translation[6] and U is parametrized by the v_a Goldstone bosons of the broken AdS boost transformations.

The reason to consider the coset element in terms of g_0 and U is because in this way we have that for a general symmetric spacetime g_0 is the coset element representing the 'empty' (super-)spacetime, while U represents the broken symmetries that are due to the presence of a dynamical object, in our case a particle,

[6]The unbroken translation P_0 generates via a right action [67, 68] a transformation which is equivalent to the world-line diffeomorphisms.

in the 'empty' (super-)spacetime. For the case of a particle U is given by the general rotation that mixes the 'longitudinal' time direction with the 'transverse' space directions. If we would like to consider as a dynamical object a p-brane, we should consider as U the general rotations that mix the longitudinal and tranverse directions [68].

First we write out the Maurer-Cartan form Ω_0 associated to the AdS space

$$\Omega_0 = g_0^{-1} dg_0 = He^0 + P_a e^a + K_a \omega^{a0} + M_{ab}\omega^{ab} \,, \tag{2.43}$$

where (e^0, e^a) and $(\omega^{a0}, \omega^{ab})$ are the space and time components of the Vielbein and spin connection 1-forms of the AdS space, respectively. If we parametrize the AdS space as $e^{Ht}e^{P_a x^a}$, the Vielbein and spin-connection 1-forms corresponding to the AdS space are given by

$$\begin{aligned} e^0 &= dt \cosh \frac{x}{\beta R} \,, \\ e^a &= \frac{\beta R}{x} dx^a \sinh \frac{x}{\beta R} + \frac{1}{x^2} x^a x^b dx_b \Big(1 - \frac{\beta R}{x} \sinh \frac{x}{\beta R}\Big) \,, \\ \omega^{a0} &= -\frac{\omega}{\alpha \beta x R} dt x^a \sinh \frac{x}{R} \,, \\ \omega^{ab} &= \frac{1}{x^2} x^{[b} dx^{a]} \Big(\cosh \frac{x}{\beta R} - 1 \Big) \,. \end{aligned} \tag{2.44}$$

We now insert a particle in the empty AdS space and consider the Maurer-Cartan form of the combined system:

$$\Omega = g^{-1} dg = U^{-1}\Omega_0 U + U^{-1} dU = G^A L_A \,. \tag{2.45}$$

Using these formulae we find that the Maurer-Cartan form Ω is given by

$$\begin{aligned} L_H &= e^0 \cosh \frac{v}{\omega} + \frac{\alpha}{v} \sinh \frac{v}{\omega} v_a e^a \,, \\ L_P{}^a &= e^a + \frac{\alpha}{v} \sinh \frac{v}{\omega} e^0 v^a + \frac{\beta R}{v^2} \Big(\cosh \frac{v}{\omega} - 1 \Big) v^a v^b e^b \,, \\ L_K{}^a &= \frac{\omega}{v} \sinh \frac{v}{\omega} dv^a + \frac{1}{v^2} \Big(1 - \frac{\omega}{v} \sinh \frac{v}{\omega} \Big) v^a v^b dv^b + \omega^{a0} \cosh \frac{v}{\omega} \\ &\quad + \frac{1}{v^2} \Big(1 - \cosh \frac{v}{\omega} \Big) \omega^{b0} v_b v^a + \frac{2\omega}{v} \sinh \frac{v}{\omega} v_b \, \omega^{ab} \,, \\ L_M{}^{ab} &= \omega^{ab} + \frac{1}{\omega v} \sinh \frac{v}{\omega} v^{[b}\omega^{a]0} + \frac{1}{v^2} \Big(\cosh \frac{v}{\omega} - 1 \Big) v^{[b} dv^{a]} \\ &\quad - \frac{2}{v^2} \Big(\cosh \frac{v}{\omega} - 1 \Big) v_c v^{[b}\omega^{a]c} \,. \end{aligned} \tag{2.46}$$

Finally, the construction of the most general action with the lowest number of derivatives is obtained by taking the pull-back of all the L's invariant under the subgroup H of the representations that arise in the Cartan form. The result is unique up to a few possible constants.

2.4.1. Limits vs Non Linear Realizations

There are two ways of obtaining the (super)particle action starting from the AdS algebra (see Table 2.11).

$$\begin{tikzcd}
& \begin{array}{c}\text{AdS}\\\text{Algebra}\end{array} \arrow[rrr, "NLR"] \arrow[dl, "c\to 0"'] \arrow[dr, "c\to\infty"] & & & \begin{array}{c}\text{AdS}\\\text{Action}\end{array} \arrow[dl, "c\to 0"'] \arrow[dr, "c\to\infty"] & \\
\begin{array}{c}\text{AC}\\\text{Algebra}\end{array} & & \begin{array}{c}\text{NH}\\\text{Algebra}\end{array} \arrow[r, "NLR"] & \begin{array}{c}\text{AC}\\\text{Action}\end{array} & & \begin{array}{c}\text{NH}\\\text{Action}\end{array}
\end{tikzcd}$$

TABLE 2.11
Contruction of the AC and the NH action.

The first method consist in using the non-linear realizations method (NLR) over the AdS algebra (2.41) written in terms of the contraction parameters and taking the AC or the NH limits from the Maurer-Cartan one-forms (2.46) as follows

- Newton–Hooke limit: define the contraction parameters as

$$\alpha = \tfrac{1}{\omega}\,, \quad \beta = \omega\,, \quad \omega \to \infty\,. \tag{2.47}$$

- Carroll limit: in this case, the parametes are given by

$$\alpha = \frac{\omega}{2}\,, \quad \beta = 1\,, \qquad \omega \to \infty\,. \tag{2.48}$$

After taking these limits the results are the same as those given in sections 6 and 7 respectively. Even though this is a general procedure from where we can obtain the results of both limits, a drawback arises when going to the supersymmetric case where the calculations become tedious.

The second way, explained in detail in sections 6 and 7 for the non-relativistic and Carroll limits respectively, consists in taking the NH and AC limits over the AdS Algebra, leading to the corresponding non-relativistic and ultra-relativistic algebras, then taking the non-linear realization in each limit to obtain the action and transformation rules.

2.5. Group averaging method

The group averaging method (see for example [69–71]) is a formalism to derive the physical Hilbert space for a system with constraints.

Let us assume that there is only an unique constraint $\hat{\mathcal{G}}$. For a given arbitrary state $|\psi_{kin}\rangle$ in the kinematical space $\mathcal{H}_{kin}$ (where all the constraints are well defined) we can obtain an averaged state $|\psi_{phys}\rangle$ (solution of the constraint equation, i.e. $\hat{\mathcal{G}}|\psi_{phys}\rangle = 0$) by

$$|\psi_{phys}\rangle = \int_{-\infty}^{\infty} d\lambda\, e^{\frac{i}{\hbar}\lambda\hat{\mathcal{G}}}|\psi_{kin}\rangle\,, \tag{2.49}$$

where we introduced an $\hbar$ factor for convenience. The Kernel for this transformation is given by

$$G(x_f, t_f;\, x_i, t_i) := \int_{-\infty}^{\infty} d\lambda\, \langle x_f, t_f\, | e^{\frac{i}{\hbar}\lambda\hat{\mathcal{G}}} | x_i, t_i\rangle\,. \tag{2.50}$$

If we decompose the evolution in N intervals of length $\epsilon = 1/N$

$$e^{\frac{i}{\hbar}\lambda\hat{\mathcal{G}}} = \prod_{n=1}^{N} e^{\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}_n}, \tag{2.51}$$

and inserting a set of a complete base of the form $\mathbf{1} = \int dt_n\, dx_n |x_n, t_n\rangle\,\langle x_n, t_n|$, we can rewrite the Kernel as

$$G(x_f, t_f;\, x_i, t_i) = \int_{-\infty}^{\infty} d\lambda \prod_{n=1}^{N-1}\left[\int_{-\infty}^{\infty} dt_n \int_{-\infty}^{\infty} dx_n\right] \prod_{n=1}^{N} \langle x_n, t_n\, | e^{\frac{i}{\hbar}\epsilon\,\lambda\,\hat{\mathcal{G}}_n} | x_{n-1}, t_{n-1}\rangle\,, \tag{2.52}$$

where the final and initial state are $\langle x_n, t_n\,| \equiv \langle x_f, t_f\,|$ and $|x_0, t_0\,\rangle \equiv |x_i, t_i\,\rangle$ respectively.

On the other hand, we can write the physical state as

$$\psi_{phys}(x,t) = \langle x, t|\psi_{phys}\rangle = \int dx' \int dt'\, G(x, t;\, x', t')\psi_{kin}(x', t')\,, \tag{2.53}$$

which means that we can use the Kernel $G(x_f, t_f;\, x_i, t_i)$ to extract a physical state from a kinematical state.

Now, if $|\phi_{kin}\rangle$ and $|\psi_{kin}\rangle$ are kinematical states mapped to physical states ($|\phi_{phys}\rangle$ and $|\psi_{phys}\rangle$ respectively) according with the definition (2.49) the physical internal product is defined as

$$\begin{aligned}\langle\phi_{phys}|\psi_{phys}\rangle :=& \langle\phi_{kin}|\int d\lambda\, e^{\frac{i}{\hbar}\lambda\hat{\mathcal{G}}}|\psi_{kin}\rangle \\ =& \int dx\, dx'\, dt\, dt'\, \phi_{kin}(x,t)\, G(x,t;\, x',t')\, \psi_{kin}(x',t')\,,\end{aligned} \tag{2.54}$$

and all the information of the quantum dynamics is codified in the extraction amplitude $G(x_f, t_f;\, x_i, t_i)$, which is the transition amplitude from x_i at the time t_i to x_f at the time t_f.

To exemplify the procedure, in the next section we will show how to use the averaging method to calculate the Kernel for the relativistic particle to obtain the usual result. This will be useful for the polymer path integral which we will discuss in chapter 5.

2.5.1. The relativistic free particle

The action of a relativistic free particle of mass m is

$$S = \frac{1}{2}c\int_0^1 d\tau\left(\frac{1}{\lambda}\dot{x}_\mu\dot{x}^\mu - m^2\lambda\right), \tag{2.55}$$

and can be written in an equivalent way as

$$S = \int_0^1 d\tau\left[p_\mu\dot{x}^\mu + \frac{1}{2}\lambda\left(\frac{1}{c}p_\mu p^\mu + m^2 c\right)\right], \tag{2.56}$$

where $x^\mu(\tau)$ are the coordinates of the particle in a D dimensional Minkowski space in function of the arbitrary time parameter τ. The parameter λ acts as the Lagrange multiplier for the constraint

$$\hat{\mathcal{G}} = \frac{1}{2}\left(\frac{1}{c}\hat{p}_\mu\hat{p}^\mu + m^2 c\right). \tag{2.57}$$

Using that

$$\begin{aligned}&\langle x_n|e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}}|x_{n-1}\rangle = \\ &= \left(\frac{1}{2\pi\hbar}\right)^D\int_{-\infty}^{\infty} d^D p_n \exp\left\{\frac{i}{\hbar}\left[-\epsilon\lambda\frac{1}{2}\left(\frac{1}{c}p_{\mu n}p^\mu_n + m^2 c\right) + p_{\mu n}\left(x^\mu_n - x^\mu_{n-1}\right)\right]\right\}.\end{aligned} \tag{2.58}$$

We can use the group averaging method to calculate the Kernel (2.50) where the physical internal product is

$$
\begin{aligned}
\langle x_f | e^{-\frac{i}{\hbar}\lambda\hat{\mathcal{G}}} | x_i \rangle &= \prod_{n=1}^{N-1} \left[\int_{-\infty}^{\infty} d^D x_n \right] \prod_{n=1}^{N} \langle x_n | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}} | x_{n-1} \rangle \\
&= \left(\frac{1}{2\pi\hbar} \right)^D \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} d^D p_n \right] \prod_{n=1}^{N-1} \left[\prod_{\mu=0}^{D-1} \left[\delta\left(p_{\mu_n} - p_{\mu_{n+1}} \right) \right] \right] \\
&\quad \exp\left\{ \frac{i}{\hbar} \left[p_{\mu_N} \left(x^\mu_{_N} - x^\mu_{_0} \right) - \sum_{n=1}^{N} \epsilon\lambda \frac{1}{2} \left(\frac{1}{c} p_{\mu_n} p^\mu_{_n} + m^2 c \right) \right] \right\} \\
&= \left(\frac{1}{2\pi\hbar} \right)^D \int_{-\infty}^{\infty} d^D p_N \exp\left\{ -\frac{i}{\hbar} \left[\epsilon N \lambda \frac{1}{2} \left(\frac{1}{c} p_{\mu_N} p^\mu_{_N} + m^2 c \right) - p_{\mu_N} \left(x^\mu_{_N} - x^\mu_{_0} \right) \right] \right\} .
\end{aligned}
\tag{2.59}
$$

The term $p_{\mu_n}\left(x^\mu_{_n} - x^\mu_{_{n-1}}\right)$ inside the exponential (2.58) combined with the integral over x_n in the product (2.59) would lead to a Dirac delta over the momentum p_{μ_n} for each one of the D coordinates. We will only have $N-1$ of these Dirac deltas but N integrals over p_n so an integration over the p_N momentum would remain.

Integrating over λ we get the Kernel

$$
\begin{aligned}
G(x_f; x_i) = \int d\lambda \left(\frac{1}{2\pi\hbar} \right)^D \int_{-\infty}^{\infty} d^D p_N \\
\exp\left\{ -\frac{i}{\hbar} \left[\epsilon N \lambda \frac{1}{2} \left(\frac{1}{c} p_{\mu_N} p^\mu_{_N} + m^2 c \right) - p_{\mu_N} \left(x^\mu_{_N} - x^\mu_{_0} \right) \right] \right\} .
\end{aligned}
\tag{2.60}
$$

We can simplify this expression by making the integration over the momentum or over the Lagrange multiplier. In the first case, integrating over p_N we obtain

$$
G(x_f; x_i) = \left(\frac{c}{2\pi i\hbar\lambda} \right)^{\frac{D}{2}} \int d\lambda \exp\left\{ \frac{ic}{2\hbar} \left(\frac{X_\mu X^\mu}{\lambda} - \lambda m^2 \right) \right\} , \tag{2.61}
$$

where we used $X^\mu = x^\mu_N - x^\mu_0$ and $\epsilon N = 1$. This is the result obtained in [72].

In the second case, integrating over the Lagrange multiplier, we get

$$G(x_f;x_i) = \left(\frac{1}{2\pi\hbar}\right)^D \int_{-\infty}^{\infty} d^D p_N \left(\frac{4\pi\hbar c}{\tau}\right) \delta(p_{\mu N}p^\mu_N + m^2c^2) e^{\frac{i}{\hbar}p_{\mu N}(x^\mu_N - x^\mu_0)}$$
$$= \left(\frac{1}{2\pi\hbar}\right)^{D-1} \left(\frac{2c}{\tau}\right) \int_{-\infty}^{\infty} d^{D-1}p_N \frac{e^{\frac{i}{\hbar}\vec{p}_N(\vec{x}_N-\vec{x}_0)}}{2\sqrt{\vec{p}^{\,2}+m^2c^2}} \left[e^{\frac{i}{\hbar}(x^0_N-x^0_0)\sqrt{\vec{p}^{\,2}+m^2c^2}} + e^{-\frac{i}{\hbar}(x^0_N-x^0_0)\sqrt{\vec{p}^{\,2}+m^2c^2}}\right]. \tag{2.62}$$

In the last step we integrated over p_0 using the Dirac Delta and we used the metric $(-,+,++\ldots)$. This propagator coincides with the usual result (for example given in [73]).

2.6. Killing Equations

Killing vectors ξ^μ are vector fields that generate a symmetry of the metric $g_{\mu\nu}$ (the symmetries of a metric are also called isometries of spacetime). Given a metric, Killing vectors should satisfy the Killing equations. Solutions, if they exist, represent symmetry transformations of the spacetime. Killing vectors form a Lie algebra and this resulting algebra of these Killing vectors is the Lie algebra of the isometry group of the metric.

The Killing equations are applied to two kinds of problems:

1. Determining the metric of a spacetime whose symmetries are assumed.

2. Finding the symmetries of a spacetime.

In this section we focus on the second problem. Given an action we show the Lagrangian and Hamiltonian procedure to find the Killing equations, whose solutions will give the symmetries of the system. Both formalisms coincide. As warm up examples we apply these formalisms to the massive relativistic particle in the bosonic and in the supersymmetric cases.

2.6.1. Massive Relativistic Particle

The first example that we consider is the bosonic free relativistic particle of mass M, here $\mu = 0, 1, \ldots D-1$.

Lagrangian Formalism

We consider the action of the free massive relativistic particle

$$S = M \int d\tau \sqrt{-\dot{x}_\mu \dot{x}^\mu} \, . \tag{2.63}$$

Let us consider that the transformation of the coordinates is given by

$$\delta x^\mu = \xi^\mu(x) \, , \tag{2.64}$$

the invariance of the variation of the action under the symmetry transformations $\delta S = 0$ lead to the following Killing equations

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0 \, . \tag{2.65}$$

The solutions of (2.65) are simply $\xi^\mu(x) = a^\mu + \lambda^{\mu\nu} x_\nu$, where a^μ and $\lambda^{\mu\nu}$ are constant parameters. These are precisely the infinitesimal translations and Lorentz transformations of the Poincaré algebra.

Hamiltonian Formalism

The canonical action of the free massive relativistic particle given by

$$S = \int d\tau (p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 + M^2)) \, . \tag{2.66}$$

The basic Poisson brackets of the canonical variables occurring in this action are

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu}, \qquad \{e, \pi_e\} = 1 \, , \tag{2.67}$$

Dirac's Hamiltonian will be

$$H_D = \frac{e}{2}(p^2 + M^2) + \lambda \pi_e \, . \tag{2.68}$$

The equations of motion are

$$\dot{x}^\mu = e p^\mu \, , \qquad \dot{p}^\mu = 0 \, , \qquad \dot{e} = \lambda \, , \qquad \dot{\pi}_e = -\frac{1}{2}(p^2 + M^2) \, . \tag{2.69}$$

We will take as the generator of canonical transformations

$$G = p_\mu \, \xi^\mu + \gamma \pi_e \, , \tag{2.70}$$

with parameters $\xi^\mu = \xi^\mu(x)$ and $\gamma = \gamma(x)$. Here $\lambda = \lambda(\tau)$ is an arbitrary function and π_e is constrained $\dot{\pi}_e = 0$. This leads to the following restrictions on the parameters:

$$\dot{G} = p_\mu \dot{x}^\nu \partial_\nu \xi^\mu - \frac{1}{2}\gamma(p^2 + M^2) = e p_\mu p^\nu \partial_\nu \xi^\mu - \frac{1}{2}\gamma(p^2 + M^2)\,, \tag{2.71}$$

and the Killing equations that we obtain are

$$\gamma = 0\,, \quad \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0\,. \tag{2.72}$$

As we see, both formalisms lead to the same Killing equations (and thus the same symmetries) (2.65) and (2.72).

2.6.2. Massive Relativistic Superparticle

It is time to extend the analysis of symmetries to the supersymmetric case. The superparticle theory has been studied because of its simplicity and its analogous properties with superstring theory, it has also been considered that the classical superparticle describes the dynamics of the classical spinning particle or spin-1/2 particle [74, 75].

Lagrangian Formalism

Now let us consider the supersymmetric case. The action of the free massive relativistic superparticle is given by

$$S = M \int d\tau \sqrt{-(\dot{x}^\mu + i\bar{\theta}\gamma^\mu\dot{\theta})^2}\,. \tag{2.73}$$

Let us consider that the transformation of the coordinates is given by

$$\delta x^\mu = \xi^\mu(x,\theta)\,, \qquad \delta\theta = \chi(x,\theta)\,. \tag{2.74}$$

In order to make the action invariant under the transformation rules, $\delta S = 0$, we find the following conditions

$$\begin{aligned} \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + i\bar{\theta}\gamma_\mu \partial_\nu \chi + i\bar{\theta}\gamma_\nu \partial_\mu \chi = 0\,, \\ i\partial_\theta \xi_\mu \dot{\theta}\bar{\theta}\gamma^\mu\dot{\theta} - \bar{\chi}\gamma_\mu \dot{\theta}\bar{\theta}\gamma^\mu\dot{\theta} - \bar{\theta}\gamma^\mu\dot{\theta}\bar{\theta}\gamma_\mu\partial_\theta\chi\dot{\theta} = 0\,, \\ \partial_\theta\xi_\mu + i\bar{\chi}\gamma_\mu + i\bar{\theta}\gamma_\mu\partial_\theta\chi + i\partial_\mu\xi_\nu\bar{\theta}\gamma^\nu - \bar{\theta}\gamma_\nu\partial_\mu\chi\bar{\theta}\gamma^\nu = 0\,. \end{aligned} \tag{2.75}$$

Making the redefinition

$$h_\mu = \xi_\mu + i\bar{\theta}\gamma_\mu\chi\,, \tag{2.76}$$

we can rewrite the Killing equations in a more compact form as

$$\begin{aligned} \partial_\mu h_\nu + \partial_\nu h_\mu = 0\,, \\ \partial_\theta h_\mu \dot\theta \bar\theta \gamma^\mu \dot\theta = 0\,. \end{aligned} \tag{2.77}$$

The third equation $\partial_\theta h_\mu + i\partial_\mu h_\nu \bar\theta\gamma^\nu = 0$, is a direct consequence of the other two. The solutions of (2.77) are the transformations of Poincaré symmetry and supersymmetry:

$$\xi^\mu(x,\theta) = a^\mu + \lambda^{\mu\nu}x_\nu + \bar\epsilon\gamma^\mu\theta\,, \qquad \chi = \frac{1}{4}\lambda^{\mu\nu}\gamma_{\mu\nu}\theta + i\epsilon\,. \tag{2.78}$$

Hamiltonian Formalism

The canonical action of the free massive relativistic superparticle will be

$$S = \int d\tau (p_\mu \dot x^\mu + P_\theta\dot\theta - \frac{e}{2}(p^2+M^2) - (\bar P_\theta + ip_\mu\bar\theta\gamma^\mu)\rho) \tag{2.79}$$

The basic Poisson brackets of the canonical variables occurring in this action are

$$\{x_\mu, p_\nu\} = \delta_{\mu\nu}, \qquad \{e,\pi_e\} = 1\,, \qquad \{P_\theta^\alpha, \theta_\beta\} = -\delta^\alpha_\beta\,, \qquad \{\Pi_\rho^\alpha, \rho_\beta\} = -\delta^\alpha_\beta\,. \tag{2.80}$$

and the corresponding Dirac's Hamiltonian of this action is given by

$$H_D = \frac{e}{2}(p^2+M^2) + (\bar P_\theta + ip_\mu\bar\theta\gamma^\mu)\rho + \lambda\pi_e + \bar\pi_\rho\Lambda\,. \tag{2.81}$$

As it occurs in the main text, π_e and Π_ρ are the primary constraints and $\lambda = \lambda(\tau)$ and $\Lambda = \Lambda(\tau)$ are arbitrary functions. The primary hamiltonian equations of motion are given by

$$\begin{aligned} \dot x^\mu = ep^\mu + i\bar\theta\gamma^\mu\rho\,, \qquad \dot p^\mu = 0\,, \qquad \dot{\bar P}_\theta = -ip_\mu\bar\rho\gamma^\mu\,, \qquad \dot\theta = \rho\,, \\ \dot e = \lambda\,, \qquad \dot\pi_e = -\frac{1}{2}(p^2+M^2)\,, \qquad \dot\rho = \lambda\,, \qquad \dot{\bar\Pi}_\rho = \bar P_\theta + iP_\mu\bar\theta\gamma^\mu\,. \end{aligned} \tag{2.82}$$

The stability of primary constraints give secondary bosonic and fermionic constraints, given by $p^2 + M^2 = 0$ and $\bar P_\theta + iP_\mu\bar\theta\gamma^\mu = 0$ respectively. By requiring the stability of these secondary constraints one gets $\rho = 0$, leading to the same equations of motion as in the Lagrangian formalism.

We will take as the generator of canonical transformations

$$G = p_\mu\,\xi^\mu(x,\theta) + \bar P_\theta\chi(x,\theta) + \gamma(x,\theta)\pi_e + \bar\Pi_\rho\,\Gamma(x,\theta)\,, \tag{2.83}$$

with parameters $\xi^\mu = \xi^\mu(x,\theta)$, $\chi = \chi(x,\theta)$, $\gamma = \gamma(x,\theta)$ and $\Gamma = \Gamma(x,\theta)$ that have the following restrictions on the parameters:

$$\begin{aligned}\dot{G} &= p_\mu(\dot{x}^\nu\partial_\nu\xi^\mu + \partial_\theta\xi^\mu\dot{\theta}) - \frac{1}{2}\gamma(p^2+M^2) - ip_\mu\bar{\rho}\gamma^\mu\chi + P_\theta(\partial_\mu\chi\dot{x}^\mu + \partial_\theta\chi\dot{\theta}) \\ &\quad + (\bar{P}_\theta + ip_\mu\bar{\theta}\gamma)\Gamma\,, \\ &= ep_\mu p^\nu\partial_\nu\xi^\mu + ip_\mu\partial_\nu\xi^\mu\bar{\theta}\gamma^\mu\rho + p_\mu\partial_\theta\xi^\mu\rho - \frac{1}{2}\gamma(p^2+M^2) \\ &\quad - ip_\mu\bar{\rho}\gamma^\mu\chi + ep^\mu P_\theta\partial_\mu\chi + i\bar{\theta}\gamma^\mu\rho\bar{P}_\theta\partial_\mu\chi + \bar{P}_\theta\partial_\theta\chi\rho + \bar{P}_\theta\Gamma + ip_\mu\bar{\theta}\gamma^\mu\Gamma\end{aligned} \tag{2.84}$$

and the Killing equations are given by

$$\begin{aligned}\gamma = 0\,, \quad \partial_\mu\chi = 0\,, \quad \Gamma = \partial_\theta\chi\rho\,, \\ \partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 0\,, \\ \partial_\theta\xi^\mu + +i\bar{\chi}\gamma^\mu + i\partial_\nu\xi^\mu\bar{\theta}\gamma^\nu + i\bar{\theta}\gamma^\mu\partial_\theta\chi = 0\,.\end{aligned} \tag{2.85}$$

By making the redefinition (2.76) we obtain again the Killing equations (2.77).

3
Supersymmetric Massive Extension

3.1. Introduction

The first modification from general relativity that we will consider is to search for a theory with a spin-2 massive particle. Consistent theories for both massless and massive particles with spin less than two are well known and their quantum versions play an important role in theoretical particle physics. In fact, the Standard Model is a consistent quantum theory of massive and massless fields with spin 0, 1/2 and 1, but compared with the lower spin cases, massive spin-2 particles turn out to be more difficult to handle.

One motivation to search for a theory of massive gravity is related with the cosmological constant problem. It is possible to introduce the cosmological constant as a free parameter in the classical action of a gravitational field as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R-2\Lambda) + S_{\text{matter}}[g_{\mu\nu}, X] \tag{3.1}$$

where $\kappa \equiv \frac{8\pi G}{c^4} \equiv \frac{8\pi}{m_{\text{Pl}}^2} \equiv \frac{1}{M_{Pl}^2}$ with m_{Pl} and M_{Pl} being Planck's mass and the reduced Planck's mass respectively; the first term stands for the Einstein-Hilbert action, the second term is the cosmological constant term (which can be assigned units of 1 over distance squared) and the third term describes the matter part where X represents a generic matter field. The variation of equation (3.1) will lead to the Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu} \tag{3.2}$$

here, $T_{\mu\nu}$ is the energy-momentum tensor, this term gathers the ordinary matter and radiation contributions to the total energy (and momentum) of the universe.

A much less obvious source of energy on the structure of the universe is the vacuum energy density and is proportional to the cosmological constant.

A nonzero vacuum energy density would have a visible effect on the geometry of space-time. A large negative(positive) cosmological constant would produce a space with negative(positive) constant curvature so our Euclidean geometry would not be valid any longer. If the cosmological constant is nonzero but quite small we would have to look over large distances to see its effects on space-time structure. It would also affect the expansion rate of the universe, a negative cosmological constant would slow the expansion of the galaxies at a constant rate and a positive cosmological constant would accelerate galaxies away from each other and increase the expansion rate of the universe.

Now, theoretically, the vacuum energy density is naively expected to be of the order $\Lambda \sim \mathrm{km}^{-2}$ and the acceleration rate of the universe H is computed to be $H \geq 10^{-3}\mathrm{eV}$ but these values will cause tremendous distortions in the space-time structure. According to observational data the value of Λ if nonzero has to be of the order of $\Lambda \sim 10^{-52}\mathrm{m}^{-2}$ or similarly the expansion rate has to be less than or equal to the present-day value of the Hubble parameter $H \leq 10^{-33}\mathrm{eV}$.

Since the discrepancy between theory and experiment is so big we may be missing an essential part of the puzzle. Some authors argue that giving mass to the graviton may solve this unnaturalness problem because long range interactions would be damped exponentially and this would narrow the gap between the expected and the observed value of the cosmological constant.

There are also purely theoretical motivations of working with a massive theory of gravity. The main theoretical aim is to better understand our current theory, to test it and to try to get consistent modifications and to obtain new insights and points of view.

The idea of adding a mass to a spin-1 and a spin-2 particle is old. Between 1936 and 1941 Proca developed the theory of the massive vector boson fields governing the weak interaction and the motion of the spin-1 mesons. In 1939 the free theory for massive spin-2 particles was constructed by Fierz and Pauli [9]. A massless spin-1 particle enjoys a $U(1)$ symmetry, and in the same way a massless spin-2 particle has a diffeomorphism symmetry, however a main difference arises between both cases when coupling with external matter fields. In the spin-2 theory this coupling makes general relativity a fully non-linear theory with non-linear diffeomorphism invariance, this feature is inherited in the massive theory (although the symmetry is broken) making it very difficult to obtain.

There are some concerns when dealing with a massive graviton, one of them is that when taking the massless limit of a massive spin-2 field, it propagates too many degrees of freedom, and it would seem that we can never get general relativity from it. This phenomena is called the vDVZ discontinuity and it can be solved by considering non-linear extensions of the theory. A second problem to overcome is that non-linearities introduce ghosts into the physical spectrum. Ghosts correspond to fields with negative kinetic energy that lead to instabilities at the classical level and to non-unitarity at the quantum level.

In the past decade a lot of effort has been put into constructing models that can cure the vDVZ discontinuity and that are ghost free, for a review see for example [76]. Here we will concentrate on the three-dimensional theory of massive gravity called New Massive Gravity. New Massive Gravity (NMG) is a higher-derivative extension of three-dimensional (3D) Einstein–Hilbert gravity with a particular set of terms quadratic in the 3D Ricci tensor and Ricci scalar [14]. The interest in the NMG model lies in the fact that, although the theory contains higher derivatives, it nevertheless describes, unitarily, two massive degrees of freedom of helicity $+2$ and -2. Furthermore, it has been shown that even at the non-linear level ghosts are absent [19]. The 3D NMG model is an interesting laboratory to study the validity of the AdS/CFT correspondence in the presence of higher derivatives. Its extension to 4D remains an open issue and has only been established so far at the linearized level [77].

A supersymmetric version of NMG was constructed in [20]. Besides the fourth-order-derivative terms of the metric tensor this model also contains third-order-derivative terms involving the gravitino. The purpose of this section is to show how to construct a reformulation of the supersymmetric NMG model (SNMG) without higher derivatives.

This chapter is organized as follows, first we will show how to construct the massive actions, then we will give a general review of New Massive Gravity, after that we will use the spin-1 case as a warm up exercise to study the spin-2 case to explicitly construct the linearized, massive, off-shell supermultiplet performing a Kaluza-Klein reduction over a circle and projecting onto the first massive Kaluza Klein sector. The final form of the 3-dimensional off-shell massive supermultiplet is then obtained after a truncation and gauge-fixing a few Stückelberg symmetries. We will look in detail at the massless limit along the construction. Then we will construct the linearized version of SNMG.

3.2. Constructing Massive Actions

3.2.1. Proca Action

Spin-1 theories share several features with the spin-2 analogues (for example, unlike the spin-0 or spin-1/2 cases gauge invariance is needed in the massless case to obtain the correct number of degrees of freedom), that is why it is instructive to begin with the simpler spin-1 case.

First we want to construct a relativistic Lagrangian for a massive spin-1 field. In particle physics, particles with spin-1 are the mediators of interactions and they are described in field theory as quanta of vector fields. The most important examples are the gauge fields of the electromagnetic and strong interactions (massless photons and gluons respectively) and the ones of weak interactions (massive $W^{\pm}$ and Z^0 bosons). Thus, we start with a vector field A_μ.

The vector field A_μ has D degrees of freedom which is more than is required to describe a spin-1 particle (see Table 3.1). Here a distinction has to be made that distinguishes massless and massive particles. In the massless off-shell case we can use the gauge transformation to fix one component of A_μ therefore we will have $D-1$ degrees of freedom; in the on-shell case the Lorentz subsidiary condition ($\partial^\mu A_\mu = 0$) imposes a constraint on the polarization vectors reducing again the degrees of freedom by one, so we end with $D-2$ degrees of freedom. For example, in electrodynamics, the photon has only two polarization states, both of which are transverse. For the massive off-shell case the gauge invariance is lost, because the field A_μ transforms inhomogeneously and thus the mass term in the Lagrangian is not invariant, so only after imposing the Lorentz condition there will be left $D-1$ degrees of freedom. That is why a massive vector field allows an additional longitudinal polarization state in addition to the two transverse ones.

A_μ	Off-Shell	$4D$	$3D$	On-Shell	$4D$	$3D$
Massless	$D-1$	3	2	$D-2$	2	1
Massive	D	4	3	$D-1$	3	2

Table 3.1
This table gives the counting of degrees of freedom for a vector field A_μ that describes a spin-1 particle.

Here we will consider the massive classical spin-1 field also known as the Proca field (we will assume that A_μ is a real-valued neutral field, for a charged field A_μ is complex). That the vector field carries mass m means it satisfies the field equation

$$(\Box - m^2)A_\mu = 0\,, \tag{3.3}$$

that has $D-1$ degrees of freedom. The sign of the mass term depends on the metric choice that we made, in this case we are using the mostly plus signature convention $(-,+,\ldots,+)$. Thus we must impose a constraint to cut down the number of degrees of freedom by one. The only Lorentz covariant possibility (linear in A_μ) is

$$\partial^\mu A_\mu = 0\,, \tag{3.4}$$

Now we make an ansatz for a possible action that contains both (3.3) and (3.4). The first part of this ansatz will contain the leading Klein–Gordon term, the second part consist on all possible quadratic terms with the order of derivatives not higher than the leading term. For this case, the only possibility (for first-order derivatives) is a $(\partial^\mu A_\mu)^2$ term

$$S_{\text{Proca}} = \int d^D x \{\frac{1}{2} A^\mu (\Box - m^2) A_\mu + \frac{a}{2} (\partial^\mu A_\mu)^2\}\,, \tag{3.5}$$

where a is a free parameter that we have to tune in such a way we can obtain both (3.3) and (3.4).

From the variation $\frac{\delta S}{\delta A_\mu} = 0$ we obtain an equation of motion of rank-1

$$(\Box - m^2) A_\mu - a \partial_\mu (\partial^\nu A_\nu) = 0\,, \tag{3.6}$$

by taking the divergence of the previous equation we obtain an equation of rank-0

$$(\Box - a\Box - m^2) \partial^\mu A_\mu = 0\,, \tag{3.7}$$

setting $a = 1$ we obtain the divergenceless condition (3.4) and substituting these into (3.6) the Klein–Gordon equation is also derived.

3.2.2. Fierz-Pauli action

Now let us move on with the graviton case. In this case, particles with spin-2 are described by a second-rank symmetric tensor $g_{\mu\nu}$. Let us count the degrees of freedom (see Table 3.2), since $g_{\mu\nu}$ is a symmetric matrix it has $D(D+1)/2$ independent components. In the massless off-shell case, gauge invariance (general coordinate transformations) fix D components; in the on shell-case the subsidiary conditions will constrain the polarization tensor reducing the degrees of freedom by D so we will have at the end $D(D-3)/2$ degrees of freedom. These last constraints are the reason why the polarization states of the massless graviton are only transverse. For the massive scenario the gauge invariance is broken and only on-shell we can reduce the degrees of freedom due to the $D+1$ constraints that we obtain from the divergenceless and traceless conditions.

$g_{\mu\nu}$	Off-Shell	4D	3D	On-Shell	4D	3D
Massless	$D(D-1)/2$	6	3	$D(D-3)/2$	2	0
Massive	$D(D+1)/2$	10	6	$D(D-1)/2-1$	5	2

Table 3.2
This table gives the counting of degrees of freedom for a tensor field $g_{\mu\nu}$ that describes a spin-2 particle.

The next step is to build an action for the massive graviton. We start from the field equation, the divergenceless equation and the tracelessness equation:

$$(\Box - m^2)h_{\mu\nu} = 0\,, \qquad \partial^\mu h_{\mu\nu} = 0\,, \qquad \eta^{\mu\nu}h_{\mu\nu} = 0\,, \tag{3.8}$$

we need to make an ansatz for a possible action from where we can derive (3.8), that contains the Klein–Gordon term as the leading term and all quadratic terms with no more than second-order derivatives

$$S_{\text{FP}} = \int d^Dx\{\tfrac{1}{2}h^{\mu\nu}(\Box - m^2)h_{\mu\nu} + \tfrac{1}{2}h(a\Box - bm^2)h + \tfrac{1}{2}ch^{\mu\nu}\partial_\mu\partial^\rho h_{\rho\nu} + dh^{\mu\nu}\partial_\mu\partial_\nu h\}\,, \tag{3.9}$$

where $h_{\mu\nu}$ is a symmetric tensor, $h \equiv \eta^{\mu\nu}h_{\mu\nu}$, and a, b, c and d are parameters to be tuned.

We obtain an equation of motion of rank-2 from $\frac{\delta S}{\delta h_{\mu\nu}} = 0$

$$(\Box - m^2)h_{\mu\nu} + \eta_{\mu\nu}(a\Box - bm^2)h + c\partial_{(\mu}\partial^\rho h_{\nu)\rho} + d\partial_\mu\partial_\nu h + d\eta_{\mu\nu}\partial^\rho\partial^\sigma h_{\rho\sigma} = 0\,, \tag{3.10}$$

From the divergence of (3.10), i.e. $\partial^\mu(\frac{\delta S}{\delta h_{\mu\nu}}) = 0$, we get a rank-1 equation

$$(\Box - m^2)\partial^\mu h_{\mu\nu} + (a\Box - bm^2)\partial_\nu h + \tfrac{1}{2}c\Box\partial^\rho h_{\nu\rho} + \tfrac{1}{2}c\partial_\nu\partial^\mu\partial^\rho h_{\mu\rho} + d\Box\partial_\nu h + d\partial_\nu\partial^\rho\partial^\sigma h_{\rho\sigma} = 0\,, \tag{3.11}$$

from the double divergence $\partial^\nu\partial^\mu(\frac{\delta S}{\delta h_{\mu\nu}}) = 0$ the rank-0 equation

$$(\Box - m^2)\partial^\mu\partial^\nu h_{\mu\nu} + (a\Box^2 - bm^2\Box)h + \tfrac{1}{2}c\Box\partial^\mu\partial^\nu h_{\mu\nu} + \tfrac{1}{2}c\Box\partial^\mu\partial^\nu h_{\mu\nu} + d\Box^2 h + d\Box\partial^\mu\partial^\nu h_{\mu\nu} = 0\,, \tag{3.12}$$

and from the traceless equation $\eta^{\mu\nu}(\frac{\delta S}{\delta h_{\mu\nu}}) = 0$, another rank-0 equation

$$(\Box - m^2)h + D(a\Box - bm^2)h + c\partial^\mu\partial^\nu h_{\mu\nu} + d\Box h + dD\partial^\mu\partial^\nu h_{\mu\nu} = 0\,, \tag{3.13}$$

It is possible to tune the parameters

$$a = -1\,, \qquad b = -1\,, \qquad c = -2 \quad \text{and} \quad d = 1\,, \tag{3.14}$$

where the choice of d should satisfy $-2 + dD \neq 0$ (here we only consider cases $D \geq 3$). Thus the action becomes

$$S_{\text{FP}} = \int d^Dx\{\frac{1}{2}h^{\mu\nu}(\Box - m^2)h_{\mu\nu} - \frac{1}{2}h(\Box - m^2)h - h^{\mu\nu}\partial_\mu\partial^\rho h_{\rho\nu} + h^{\mu\nu}\partial_\mu\partial_\nu h\}\,, \tag{3.15}$$

There is another way of understanding the tuning, let us focus on the mass terms and let us change the coefficients by $-\frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} + (1-\alpha)h^2)$, a proper Lagrangian should be ghost and tachyon free (free fields should have positive energy and should not propagate faster than the speed of light) but this change will give an action describing an additional scalar ghost satisfying the equation

$$\left(\Box - \frac{1 + (\alpha - 1)D}{(2 - D)\alpha}m^2\right)h = 0\,, \tag{3.16}$$

for $\alpha \neq 0$. Taking the limit $\alpha \to 0$ the mass of the scalar ghost goes to infinity so it becomes non-dynamical. As a last step we rearrange the terms in (3.15) using the linearized Einstein tensor

$$G^{\text{lin}}_{\mu\nu}(h) = \Box h_{\mu\nu} - 2\partial_{(\mu}\partial^\rho h_{\nu)\rho} + 2\partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h\,, \tag{3.17}$$

so the action reads

$$S_{FP} = \frac{1}{2}\int d^Dx\{h^{\mu\nu}G^{\text{lin}}_{\mu\nu}(h) - m^2(h^{\mu\nu}h_{\mu\nu} - h^2)\}\,. \tag{3.18}$$

The kinetic term of the Fierz-Pauli action (3.18) comes from the Einstein term and is invariant under diffeomorphisms $\delta h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$ but the mass term in the Lagrangian breaks the gauge invariance. Stückelberg's trick consists in introducing additional gauge fields to restore the gauge symmetry which had been broken by the mass term.

Now, it is natural to think that due to the precision of the predictions of general relativity, massive gravity theories should be deformations of it. However, when we take the limit of the mass going to zero in the Fierz-Pauli theory not all of the experimental outcomes of the theory approach to those of general relativity, in particular, the light-bending angle obtained from the massive theory has a disagreement of 25 percent with respect to the one predicted by general relativity. Besides, taking the massless limit of the Fierz-Pauli system coupled to a conserved energy-momentum tensor does not lead to massless gravity, but rather to linearized Einstein gravity plus extra degrees of freedom. This discontinuity in the physical predictions of general relativity and the massless limit of Fierz-Pauli theory is known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity. This vDVZ discontinuity can be verified introducing the Stückelberg fields mentioned above and coupling the massive field $h_{\mu\nu}$ to matter via a conserved energy-momentum tensor $T_{\mu\nu}$, in the massless limit one observes a non-vanishing coupling of the trace h to the trace of $T_{\mu\nu}$, therefore, the FP theory gives linearized general relativity plus a scalar mode that does not decouple.

Consider the FP Lagrangian (3.18) of a massive graviton coupled to the energy-momentum tensor $T_{\mu\nu}$

$$\mathcal{L} = \mathcal{L}_{\text{FP}} + h_{\mu\nu}T^{\mu\nu} = \frac{1}{2}\{h^{\mu\nu}G^{\text{lin}}_{\mu\nu}(h) - m^2(h^{\mu\nu}h_{\mu\nu} - h^2)\} + h_{\mu\nu}T^{\mu\nu}\,, \qquad (3.19)$$

now we introduce Stückelberg fields to preserve the gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu\,, \qquad V_\mu \to V_\mu + \partial_\mu\phi\,. \qquad (3.20)$$

Inserting (3.20) into (3.19), together with the requirement that the energy-momentum tensor is conserved ($\partial^\mu T_{\mu\nu} = 0$), considering the scalings $V_\mu \to \frac{1}{m}V_\mu$, $\phi \to \frac{1}{m}\phi$ and taking the limit $m \to 0$ we obtain the action

$$\mathcal{L}_{\text{FP}} = \frac{1}{2}h^{\mu\nu}G^{\text{lin}}_{\mu\nu}(h) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2(h_{\mu\nu}\partial^\mu\partial^\nu\phi - h\partial^2\phi) + h_{\mu\nu}T^{\mu\nu} \qquad (3.21)$$

Here we see that the vector V_μ decouples, but we get off-diagonal terms mixing the scalar ϕ and the tensor $h_{\mu\nu}$. This action, however, can be diagonalized by shifting $h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{D-2}\eta_{\mu\nu}\phi$, to obtain

$$\mathcal{L}_{\text{FP}} = \frac{1}{2}h^{\mu\nu}G^{\text{lin}}_{\mu\nu}(h) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2\frac{D-1}{D-2}\partial_\mu\phi\partial^\mu\phi + h_{\mu\nu}T^{\mu\nu} + \frac{2}{D-2}\phi T,\,, \quad (3.22)$$

where the gauge transformations are simply $\delta h_{\mu\nu} = \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu$ and $\delta V_\mu = \partial_\mu\Lambda$. Here we see explicitly the coupling between the scalar and the trace of the energy-momentum tensor. This is the origin of the vDVZ discontinuity. We will illustrate how a supersymmetric version of this discontinuity arises in section 3.5.2 (see also [78] for an earlier discussion).

Lets do the counting of degrees of freedom, we have $D(D-3)/2$ from the canonical massless graviton, $D-2$ from the canonical massless vector, and one from the canonical massless scalar, giving in total the $D(D-1)/2-1$ degrees of freedom from the massive graviton.

After the discovery of the vDVZ discontinuity, Vainshtein realized that the discontinuity can be cured in a fully nonlinear Fierz-Pauli theory, but shortly afterwards Boulware and Deser claimed that any nonlinear completion of Fierz-Pauli theory leads to a ghost [79].

Here is where it becomes important the construction of a three dimensional theory of massive gravity. For a massless graviton in three dimensions there are no propagating degrees of freedom (see Table 3.1), so any potentially ghost-like feature connected to them would be harmless, since it will not constitute any physical degree of freedom. On the other hand, for a massive graviton in three dimensions there are two degrees of freedom, therefore, it is possible to write down an action with "healthy" massive spin-2 mode and a massless spin-2 ghost mode which is pure gauge, such a theory is called New Massive Gravity (NMG) and we will briefly discuss about it in the next section.

3.3. New Massive Gravity

It is possible to solve the differential constraint of Fierz Pauli action (the divergenceless equation, see (3.8)) by increasing the number of derivatives, to do that we rewrite $h_{\mu\nu}$ in terms of a new symmetric spin-two field $h'_{\mu\nu}$

$$h_{\mu\nu} = G^{(\mathrm{lin})}_{\mu\nu}(h')\,, \tag{3.23}$$

so the remaining equation of motion and tracelessness equation (3.8) can be written as

$$(\Box - m^2)G^{(\mathrm{lin})}_{\mu\nu}(h') = 0\,, \qquad R^{(\mathrm{lin})}(h') = 0\,, \tag{3.24}$$

where $G^{(\mathrm{lin})}(h')$ is the trace of the linearized Einstein tensor which in three dimensions is proportional to the linearized Ricci scalar, therefore we can write together both equations of motion in a single equation of the form

$$G^{(\mathrm{lin})}_{\mu\nu} - \frac{1}{m^2}\left[\Box G^{(\mathrm{lin})}_{\mu\nu} - \frac{1}{4}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)R^{(\mathrm{lin})}\right] = 0\,. \tag{3.25}$$

An action that reproduces equation (3.25) at the linear level is

$$S_{\mathrm{NMG}} = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\left(R + \frac{1}{m^2}K\right)\,, \qquad K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\,, \tag{3.26}$$

where $R_{\mu\nu}$ is the Ricci tensor, R its trace and κ has mass dimension $[\kappa] = -1/2$ in fundamental units, κ will be the 3D analog of the square root of Newton's constant, while m is a relative mass parameter. Note that the Einstein-Hilbert term has a "wrong" sign but this term together with the curvature-squared invariant constructed from K avoid ghosts in the theory. This is the action of New Massive Gravity (NMG), the spectrum in three dimensions consists of a propagating massive spin-two mode and a non-propagating massless spin-two mode. The tuning $-\frac{3}{8}$ of the second term of K suppress a massive spin-0 mode.

In short, at the linearized level the NMG model decomposes into the sum of a massless spin-2 Einstein–Hilbert theory and a massive graviton with two propagating degrees of freedom (spin-2 modes of helicity +2 and -2) [14].

For many purposes, it is convenient to work with a formulation of the model without higher derivatives, see, e.g. [21]. This can be achieved by introducing an auxiliary symmetric tensor that couples to (the Einstein tensor of) the 3D metric tensor and has an explicit mass term [14]. Using auxiliary fields, we can make manifest the connection of NMG with the FP theory, at the level of the Lagrangian. Using a symmetric auxiliary field $f_{\mu\nu}$ with trace $f = g^{\mu\nu} f_{\mu\nu}$ we can write the action (3.26) as

$$S_{\text{aux-NMG}} = \frac{1}{\kappa^2} \int d^3x \sqrt{-g} \left[R + f^{\mu\nu} G_{\mu\nu} - \frac{m^2}{4} \left(f^{\mu\nu} f_{\mu\nu} - f^2 \right) \right], \tag{3.27}$$

expanding about a Minkowski background at the linearized level, the Lagrangian will take the form

$$\mathcal{L}_{\text{lin-aux-NMG}} = \left(\hat{f}^{\mu\nu} - \frac{1}{2} h^{\mu\nu} \right) G^{\text{lin}}_{\mu\nu}(h) - \frac{m^2}{4} \left(\hat{f}^{\mu\nu} \hat{f}_{\mu\nu} - \hat{f}^2 \right), \tag{3.28}$$

where $\hat{f}_{\mu\nu}$ denotes the perturbation of $f_{\mu\nu}$, rearranging the terms in last equation and defining $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \hat{f}_{\mu\nu}$ we obtain

$$\mathcal{L}_{\text{FP-NMG}} = -\frac{1}{2} \tilde{h}^{\mu\nu} G^{\text{lin}}_{\mu\nu}(\tilde{h}) + \frac{1}{2} \left[\hat{f}^{\mu\nu} G^{\text{lin}}_{\mu\nu}(\hat{f}) - \frac{m^2}{2} \left(\hat{f}^{\mu\nu} \hat{f}_{\mu\nu} - \hat{f}^2 \right) \right], \tag{3.29}$$

From here we can clearly identify the non-propagating Einstein mode $\tilde{h}_{\mu\nu}$ which is a ghost, and the decoupled, ghost free, massive spin-2 mode $\hat{f}_{\mu\nu}$ given by a FP Lagrangian.

We can consider now a supersymmetric generalization of New Massive Gravity (SNMG). There are two ways of doing that. It can be done using the metric and a higher-derivative action like in [20], or one may consider the auxiliary field version of NMG, this requires that besides an auxiliary symmetric tensor, we introduce further auxiliary fermionic fields that effectively lower the number of derivatives of the gravitino kinetic terms. In the supersymmetric case we need a 3D massless and a 3D massive spin-2 supermultiplet. The massless multiplet is already known [14] and here we will show how to construct the linearized, massive, off-shell spin-2 supermultiplet.

For the SNMG we only consider the case of simple $\mathcal{N} = 1$ supersymmetry. Having constructed the off-shell massive spin-2 supermultiplet, it is rather straightforward to construct a linearized version of SNMG without higher derivatives, by appropriately combining a massless and a massive spin-2 multiplet.

We will show the procedure explicitly for the easier spin-1 case in section 3.4. In section 3.5 we will show the results for the Fierz–Pauli case. Along the construction of the multiplets we will look in detail at the massless limit.

As we saw in chapter 2, three dimensional spin-2 supermultiplets contain gravitini, so for the sake of completeness we show now the counting of degrees of freedom for a 3/2 field ψ_μ: ψ_μ is a complex spinor with $2^{[D/2]}D$ components (where $2^{[D/2]} = 2^{D/2}$ for D even and $2^{[D/2]} = 2^{(D-1)/2}$ for D odd) for spacetime dimension D, since we can use supersymmetry to fix $2^{[D/2]}$ this means that we have $(D-1)2^{[D/2]}$ independent degrees of freedom. These are the off-shell degrees of freedom.

Now, the action of a free 3/2 field (also called free Rarita-Schwinger field) is

$$S = -\int d^D x \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho \tag{3.30}$$

and the equation of motion obtained from (3.30) reads

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho = 0\,, \tag{3.31}$$

with the gauge condition

$$\gamma^i\psi_i = 0\,, \tag{3.32}$$

where i stands only for the spatial indices. The spatial components ψ_i satisfy the Dirac equation

$$\gamma\cdot\partial\psi_i = 0\,, \tag{3.33}$$

however, there is an additional constraint from contracting this last equation with ∂^i which give

$$\partial^i\psi_i = 0\,, \tag{3.34}$$

which led us to find $3\times 2^{[D/2]}$ independent constraints. These constrains imply that there are only $2^{[D/2]}(D-3)$ classical degrees of freedom. The on-shell degrees of freedom are half this number (see Table 3.3).

ψ_μ	Off-Shell	4D	3D	On-Shell	4D	3D
Massless	$(D-1)2^{[D/2]}$	12	4	$\frac{1}{2}(D-3)2^{[D/2]}$	2	0

Table 3.3
This table gives the counting of degrees of freedom for gravitino field ψ_μ that describes a spin-3/2 particle.

3.4. Supersymmetric Proca

In this section we show how to obtain the 3D supersymmetric Proca theory from the KK reduction of an off-shell 4D $\mathcal{N}=1$ supersymmetric Maxwell theory and a subsequent truncation to the first massive KK sector. This is a warming-up exercise for the spin-2 case which will be discussed in the next section.

3.4.1. Kaluza–Klein reduction

Our starting point is the 4D $\mathcal{N}=1$ supersymmetric Maxwell multiplet which consists of a vector $\hat{V}_{\hat{\mu}}$, a 4-component Majorana spinor $\hat{\psi}$ and a real auxiliary scalar $\hat{F}$. We indicate fields depending on the 4D coordinates and 4D indices with a hat. We do not indicate spinor indices. The supersymmetry rules, with a constant 4-component Majorana spinor parameter ϵ, and gauge transformation, with local parameter $\hat{\Lambda}$, of these fields are given by

$$\delta\hat{V}_{\hat{\mu}} = -\bar{\epsilon}\Gamma_{\hat{\mu}}\hat{\psi} + \partial_{\hat{\mu}}\hat{\Lambda} \ , \qquad \delta\hat{\psi} = \frac{1}{8}\Gamma^{\hat{\mu}\hat{\nu}}\hat{F}_{\hat{\mu}\hat{\nu}}\epsilon + \frac{1}{4}i\Gamma_5\hat{F}\epsilon \ , \qquad \delta\hat{F} = i\bar{\epsilon}\Gamma_5\Gamma^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\psi} \ , \tag{3.35}$$

where $\hat{F}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{V}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{V}_{\hat{\mu}}$.

In the following, we will split the 4D coordinates as $x^{\hat{\mu}} = (x^\mu, x^3)$, where x^3 denotes the compactified circle coordinate. Since all fields are periodic in x^3, we can write them as a Fourier series. For example:

$$\hat{V}_{\hat{\mu}}(x^{\hat{\mu}}) = \sum_n V_{\hat{\mu},n}(x^\mu)e^{inmx^3} \ , \qquad n \in \mathbb{Z} \ , \tag{3.36}$$

where $m \neq 0$ has mass dimensions and corresponds to the inverse circle radius. The Fourier coefficients $V_{\hat{\mu},n}(x^\mu)$ correspond to three-dimensional (un-hatted) fields. We first consider the bosonic fields. The reality condition on the 4D vector and scalar implies that only the 3D ($n=0$) zero modes are real. All other modes are complex but only the positive ($n \geq 1$) modes are independent, since

$$V_{\hat{\mu},-n} = V^\star_{\hat{\mu},n} \ , \qquad F_{-n} = F^\star_n \ , \qquad n \neq 0 \ . \tag{3.37}$$

In the following we will be mainly interested in the $n = 1$ modes whose real and imaginary parts we indicate by

$$
\begin{aligned}
V_\mu^{(1)} &\equiv \frac{1}{2}(V_{\mu,1} + V_{\mu,1}^\star) \ , \qquad & V_\mu^{(2)} &\equiv \frac{1}{2i}(V_{\mu,1} - V_{\mu,1}^\star) \ , \\
\phi^{(1)} &\equiv \frac{1}{2}(V_{3,1} + V_{3,1}^\star) \ , \qquad & \phi^{(2)} &\equiv \frac{1}{2i}(V_{3,1} - V_{3,1}^\star) \ , \\
F^{(1)} &\equiv \frac{1}{2}(F_1 + F_1^\star) \ , \qquad & F^{(2)} &\equiv \frac{1}{2i}(F_1 - F_1^\star) \ .
\end{aligned}
\tag{3.38}
$$

Similarly, the Majorana condition of the 4D spinor $\hat{\psi}$ implies that the $n = 0$ mode is Majorana but that the independent positive ($n \geq 1$) modes are Dirac. This is equivalent to two (4-component, 3D reducible) Majorana spinors which we indicate by

$$
\psi^{(1)} = \frac{1}{2}(\psi_1 + B^{-1}\psi_1^\star) \ , \qquad \psi^{(2)} = \frac{1}{2i}(\psi_1 - B^{-1}\psi_1^\star) \ . \tag{3.39}
$$

Here B is the 4×4 matrix $B = iC\Gamma_0$, where C is the 4×4 charge conjugation matrix.

Substituting the harmonic expansion (3.36) of the fields and a similar expansion of the gauge parameter $\hat{\Lambda}$ into the transformation rules (3.35), we find the following transformation rules for the first ($n = 1$) KK modes:

$$
\begin{aligned}
\delta\phi^{(1)} &= -\bar{\epsilon}\Gamma_3\psi^{(1)} - m\Lambda^{(2)} - m\xi\phi^{(2)} \ , \qquad \delta\phi^{(2)} = -\bar{\epsilon}\Gamma_3\psi^{(2)} + m\Lambda^{(1)} + m\xi\phi^{(1)} \ , \\
\delta V_\mu^{(1)} &= -\bar{\epsilon}\Gamma_\mu\psi^{(1)} + \partial_\mu\Lambda^{(1)} - m\xi V_\mu^{(2)} \ , \quad \delta V_\mu^{(2)} = -\bar{\epsilon}\Gamma_\mu\psi^{(2)} + \partial_\mu\Lambda^{(2)} + m\xi V_\mu^{(1)} \ , \\
\delta F^{(1)} &= i\bar{\epsilon}\Gamma_5\Gamma^\mu\partial_\mu\psi^{(1)} - im\bar{\epsilon}\Gamma_5\Gamma_3\psi^{(2)} - m\xi F^{(2)} \ , \\
\delta F^{(2)} &= i\bar{\epsilon}\Gamma_5\Gamma^\mu\partial_\mu\psi^{(2)} + im\bar{\epsilon}\Gamma_5\Gamma_3\psi^{(1)} + m\xi F^{(1)} \ , \\
\delta\psi^{(1)} &= \frac{1}{8}\Gamma^{\mu\nu}F_{\mu\nu}^{(1)}\epsilon + \frac{1}{4}\Gamma^\mu\Gamma_3\partial_\mu\phi^{(1)}\epsilon + \frac{i}{4}\Gamma_5 F^{(1)}\epsilon + \frac{m}{4}\Gamma^\mu\Gamma_3 V_\mu^{(2)}\epsilon - m\xi\psi^{(2)} \ , \\
\delta\psi^{(2)} &= \frac{1}{8}\Gamma^{\mu\nu}F_{\mu\nu}^{(2)}\epsilon + \frac{1}{4}\Gamma^\mu\Gamma_3\partial_\mu\phi^{(2)}\epsilon + \frac{i}{4}\Gamma_5 F^{(2)}\epsilon - \frac{m}{4}\Gamma^\mu\Gamma_3 V_\mu^{(1)}\epsilon + m\xi\psi^{(1)} \ ,
\end{aligned}
$$

where we have defined

$$
\Lambda^{(1)} = \frac{1}{2}(\Lambda_1 + \Lambda_1^\star) \ , \qquad \Lambda^{(2)} = \frac{1}{2i}(\Lambda_1 - \Lambda_1^\star) \ . \tag{3.40}
$$

Apart from global supersymmetry transformations with parameter ϵ and gauge transformations with parameters $\Lambda^{(1)}$, $\Lambda^{(2)}$, the transformations (3.40) also contain a global SO(2) transformation with parameter ξ, that rotates the real and imaginary parts of the 3D fields. This SO(2) transformation corresponds to a central charge transformation and is a remnant of the translation in the compact circle direction.[1]

In order to write the 3D 4-component Majorana spinors in terms of two irreducible 2-component Majorana spinors it is convenient to choose the following representation of the Γ-matrices in terms of 2×2 block matrices:

$$\Gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.41}$$

The 3D 2×2 matrices γ_μ satisfy the standard relations $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ and can be chosen explicitly in terms of the Pauli matrices by

$$\gamma_\mu = (i\sigma_1, \sigma_2, \sigma_3)\,. \tag{3.42}$$

In this representation the 4D charge conjugation matrix C is given by

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \quad \text{with} \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.43}$$

where ε is the 3D charge conjugation matrix.

Using the above representation the 4-component Majorana spinors decompose into two 3D irreducible Majorana spinors as follows:

$$\psi^{(1)} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \qquad \psi^{(2)} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}. \tag{3.44}$$

In terms of these 2-component spinors the transformation rules (3.40) read

$$\begin{aligned}
\delta\phi^{(1)} &= -\bar\epsilon_1\chi_2 + \bar\epsilon_2\chi_1 - m\Lambda^{(2)} - m\xi\phi^{(2)}\,, \\
\delta\phi^{(2)} &= -\bar\epsilon_1\psi_2 + \bar\epsilon_2\psi_1 + m\Lambda^{(1)} + m\xi\phi^{(1)}\,, \\
\delta V_\mu^{(1)} &= -\bar\epsilon_1\gamma_\mu\chi_1 - \bar\epsilon_2\gamma_\mu\chi_2 + \partial_\mu\Lambda^{(1)} - m\xi V_\mu^{(2)}\,, \\
\delta V_\mu^{(2)} &= -\bar\epsilon_1\gamma_\mu\psi_1 - \bar\epsilon_2\gamma_\mu\psi_2 + \partial_\mu\Lambda^{(2)} + m\xi V_\mu^{(1)}\,, \\
\delta F^{(1)} &= -\bar\epsilon_1\gamma^\mu\partial_\mu\chi_2 + \bar\epsilon_2\gamma^\mu\partial_\mu\chi_1 - m(\bar\epsilon_1\psi_1 + \bar\epsilon_2\psi_2) - m\xi F^{(2)}\,, \\
\delta F^{(2)} &= -\bar\epsilon_1\gamma^\mu\partial_\mu\psi_2 + \bar\epsilon_2\gamma^\mu\partial_\mu\psi_1 + m(\bar\epsilon_1\chi_1 + \bar\epsilon_2\chi_2) + m\xi F^{(1)}\,,
\end{aligned} \tag{3.45}$$

[1]This is a conventional central charge transformation. Three-dimensional supergravity also allows for non-central charges from extensions by non-central R-symmetry generators [80], recently discussed in [81].

$$\begin{aligned}
\delta\chi_1 &= \frac{1}{8}\gamma^{\mu\nu}F^{(1)}_{\mu\nu}\epsilon_1 + \frac{1}{4}(\gamma^\mu\partial_\mu\phi^{(1)} + F^{(1)} + m\gamma^\mu V^{(2)}_\mu)\epsilon_2 - m\xi\psi_1 \ , \\
\delta\chi_2 &= \frac{1}{8}\gamma^{\mu\nu}F^{(1)}_{\mu\nu}\epsilon_2 - \frac{1}{4}(\gamma^\mu\partial_\mu\phi^{(1)} + F^{(1)} + m\gamma^\mu V^{(2)}_\mu)\epsilon_1 - m\xi\psi_2 \ , \\
\delta\psi_1 &= \frac{1}{8}\gamma^{\mu\nu}F^{(2)}_{\mu\nu}\epsilon_1 + \frac{1}{4}(\gamma^\mu\partial_\mu\phi^{(2)} + F^{(2)} - m\gamma^\mu V^{(1)}_\mu)\epsilon_2 + m\xi\chi_1 \ , \\
\delta\psi_2 &= \frac{1}{8}\gamma^{\mu\nu}F^{(2)}_{\mu\nu}\epsilon_2 - \frac{1}{4}(\gamma^\mu\partial_\mu\phi^{(2)} + F^{(2)} - m\gamma^\mu V^{(1)}_\mu)\epsilon_1 + m\xi\chi_2 \ .
\end{aligned} \tag{3.46}$$

If we take $m \to 0$ in the above multiplet we obtain two decoupled multiplets, ($\phi^{(1)}$, $V^{(1)}_\mu$, $F^{(1)}$, χ_1, χ_2) and ($\phi^{(2)}, V^{(2)}_\mu, F^{(2)}, \psi_1, \psi_2$). Either one of them constitutes a massless $\mathcal{N} = 2$ vector multiplet. This massless limit has to be distinguished from the massless limits discussed in subsections 3.4.3 and 3.5.2, which refer to limits taken after truncating to $\mathcal{N} = 1$ supersymmetry.

3.4.2. Truncation

In the process of KK reduction, the number of supercharges stays the same. The 3D multiplet (3.46) we found in the previous subsection thus exhibits four supercharges and hence corresponds to an $\mathcal{N} = 2$ multiplet, containing two vectors and a central charge transformation. One can, however, truncate it to an $\mathcal{N} = 1$ multiplet, not subjected to a central charge transformation and containing only one vector. This truncated multiplet will be the starting point to obtain an $\mathcal{N} = 1$ supersymmetric version of the Proca theory. The $\mathcal{N} = 1$ truncation is given by:

$$\phi^{(2)} = V^{(1)}_\mu = F^{(2)} = \chi_2 = \psi_1 = 0 \ , \tag{3.47}$$

provided that at the same time we truncate the following symmetries:

$$\epsilon_1 = \Lambda^{(1)} = \xi = 0 \ . \tag{3.48}$$

Substituting this truncation into the transformation rules (3.46), we find the following $\mathcal{N} = 1$ massive vector supermultiplet: [2]

$$\begin{aligned}
\delta\phi^{(1)} &= \bar\epsilon_2\chi_1 - m\Lambda^{(2)} \ , &\qquad \delta V^{(2)}_\mu &= -\bar\epsilon_2\gamma_\mu\psi_2 + \partial_\mu\Lambda^{(2)} \ , \\
\delta\psi_2 &= \frac{1}{8}\gamma^{\mu\nu}F^{(2)}_{\mu\nu}\epsilon_2 \ , &\qquad \delta\chi_1 &= \frac{1}{4}(\gamma^\mu\partial_\mu\phi^{(1)} + F^{(1)} + m\gamma^\mu V^{(2)}_\mu)\epsilon_2 \ , \\
\delta F^{(1)} &= \bar\epsilon_2\gamma^\mu\partial_\mu\chi_1 - m\bar\epsilon_2\psi_2 \ .
\end{aligned} \tag{3.49}$$

[2] Note that the field content given in (3.49) is that of massless $\mathcal{N} = 2$. In the massive case, however, the scalar field ϕ will disappear after gauge-fixing the Stückelberg symmetry.

Redefining $\epsilon_2 \to \epsilon\,, \Lambda^{(2)} \to \Lambda$ and

$$\phi^{(1)} \to 4\phi\,, \quad V_\mu^{(2)} \to V_\mu\ , \quad F^{(1)} \to -F\,, \quad \psi_2 \to \psi\,, \quad \chi_1 \to \chi \quad \text{and} \quad m \to 4m\ , \tag{3.50}$$

we obtain

$$\begin{aligned} &\delta\phi = \frac{1}{4}\bar{\epsilon}\chi - m\Lambda\ , \qquad \delta V_\mu = -\bar{\epsilon}\gamma_\mu\psi + \partial_\mu\Lambda\ , \qquad \delta F = -\bar{\epsilon}\gamma^\mu\partial_\mu\chi + 4m\bar{\epsilon}\psi\ , \\ &\delta\psi = \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}\epsilon\ , \qquad \delta\chi = \gamma^\mu D_\mu\phi\,\epsilon - \frac{1}{4}F\epsilon\ , \end{aligned} \tag{3.51}$$

where the covariant derivative D_μ is defined as

$$D_\mu\phi = \partial_\mu\phi + mV_\mu\ . \tag{3.52}$$

The transformation rules (3.51) leave the following action invariant:

$$I_1 = \int d^3x \left(-\frac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{2}\,D_\mu\phi\,D^\mu\phi - 2\bar{\psi}\,\partial\!\!\!/\,\psi - \frac{1}{8}\bar{\chi}\,\partial\!\!\!/\,\chi + \,m\,\bar{\psi}\chi + \frac{1}{32}F^2\right)\ . \tag{3.53}$$

The gauge transformation with parameter Λ is a Stückelberg symmetry, that can be fixed by imposing the gauge condition

$$\phi = \text{const}\ . \tag{3.54}$$

Taking the resulting compensating gauge transformation

$$\Lambda = \frac{1}{4m}\,\bar{\epsilon}\chi \tag{3.55}$$

into account, we obtain the final form of the supersymmetry transformation rules of the $\mathcal{N}=1$ supersymmetric Proca theory:

$$\begin{aligned} \delta V_\mu &= -\bar{\epsilon}\gamma_\mu\psi + \frac{1}{4m}\bar{\epsilon}\partial_\mu\chi\ , \qquad & \delta\psi &= \frac{1}{8}\gamma^{\mu\nu}F_{\mu\nu}\epsilon\ , \\ \delta F &= -\bar{\epsilon}\gamma^\mu\partial_\mu\chi + 4m\bar{\epsilon}\psi\ , \qquad & \delta\chi &= m\gamma^\mu\epsilon V_\mu - \frac{1}{4}F\epsilon\ . \end{aligned} \tag{3.56}$$

The supersymmetric Proca action is then given by

$$I_{\text{Proca}} = \int d^3x \left(-\tfrac{1}{4}F_{\mu\nu}\,F^{\mu\nu} - \tfrac{1}{2}\,m^2V_\mu\,V^\mu - 2\bar{\psi}\,\partial\!\!\!/\,\psi - \tfrac{1}{8}\bar{\chi}\,\partial\!\!\!/\,\chi + \,m\,\bar{\psi}\chi + \tfrac{1}{32}F^2\right). \tag{3.57}$$

The supersymmetric Proca theory describes 2+2 on-shell and 4+4 off-shell degrees of freedom.

This finishes our description of how to obtain the 3D off-shell massive $\mathcal{N}=1$ vector multiplet from a KK reduction and subsequent truncation onto the first massive KK sector of the 4D off-shell massless $\mathcal{N}=1$ vector multiplet.

3.4.3. Massless limit

We end this section with some comments on the massless limit ($m \to 0$). Taking the massless limit in (3.51), we see that the Proca multiplet splits into a massless vector multiplet and a massless scalar multiplet. Note that a massless vector multiplet can be coupled to a current supermultiplet. This is a feature that we would like to incorporate, in view of the upcoming spin-2 discussion. We will do so by coupling the above supersymmetric Proca system to a conjugate multiplet $(J_\mu, \mathcal{J}_\psi, \mathcal{J}_\chi, J_F)$, where J_μ is a vector, $\mathcal{J}_\psi$ and $\mathcal{J}_\chi$ are spinors and J_F is a scalar. Our starting point is then the action

$$I = I_{\text{Proca}} + I_{\text{int}} \ , \tag{3.58}$$

where the interaction part I_{int} describes the coupling between the Proca multiplet and the conjugate multiplet:

$$I_{\text{int}} = V^\mu J_\mu + \bar{\psi}\mathcal{J}_\psi + \bar{\chi}\mathcal{J}_\chi + F J_F \ . \tag{3.59}$$

Requiring that I_{int} is separately invariant under supersymmetry, determines the transformation rules of the conjugate multiplet:

$$\begin{aligned} \delta J_\mu &= \frac{1}{4}\,\bar{\epsilon}\,\gamma_{\mu\nu}\partial^\nu \mathcal{J}_\psi + m\bar{\epsilon}\,\gamma_\mu \mathcal{J}_\chi \ , & \delta\mathcal{J}_\psi &= -\gamma^\mu \epsilon\, J_\mu - 4m\epsilon\, J_F \ , \\ \delta J_F &= \frac{1}{4}\,\bar{\epsilon}\mathcal{J}_\chi \ , & \delta\mathcal{J}_\chi &= \frac{1}{4m}\,\epsilon\,\partial^\mu J_\mu + \gamma^\mu \epsilon\,\partial_\mu J_F \ . \end{aligned} \tag{3.60}$$

Taking the massless limit in the action (3.58) and transformation rules (3.56), (3.60) is non-trivial, due to the factors of $1/m$ that appear in the transformation rules. In order to be able to take the limit in a well-defined fashion, we will work in the formulation where the Stückelberg symmetry is not yet fixed. Note that this formulation can be easily retrieved from the gauge fixed version, by making the following redefinition in the action (3.57) and transformation rules (3.56):

$$V_\mu = \tilde{V}_\mu + \frac{1}{m}\partial_\mu \phi \ . \tag{3.61}$$

Applying this redefinition to (3.57) and (3.56) indeed brings one back to the action (3.53) and to the transformation rules (3.51), whose massless limit is well-defined. The massless limit of the interaction part I_{int} (after performing the above substitution) and of the transformation rules (3.60) of the conjugate multiplet, is however not well-defined. In order to remedy this, we will impose the constraint that J_μ corresponds to a conserved current, i.e. that

$$\partial^\mu J_\mu = 0 \ . \tag{3.62}$$

In order to preserve supersymmetry, we will also take $\mathcal{J}_\chi = 0$ and $J_F = 0$.[3] The conjugate multiplet then reduces to a spin-1 current supermultiplet.

The massless limit is now everywhere well-defined. The transformation rules (3.51) reduce to the transformation rules of a massless vector $(\tilde{V}_\mu, \psi)$ and scalar (ϕ, χ, F) multiplet, see eqs. (3.128) and (3.123), respectively. Performing the above outlined procedure and taking the limit $m \to 0$ leads to the action

$$I = \int d^3x \Big[\Big(-\tfrac{1}{4}\tilde{F}_{\mu\nu}\,\tilde{F}^{\mu\nu} - 2\bar{\psi}\,\partial\!\!\!/\,\psi + \tilde{V}^\mu J_\mu + \bar{\psi}\mathcal{J}_\psi \Big) - \tfrac{1}{2}\Big(\partial_\mu\phi\,\partial^\mu\phi + \tfrac{1}{4}\bar{\chi}\,\partial\!\!\!/\,\chi - \tfrac{1}{16}F^2 \Big) \Big] \,, \tag{3.63}$$

which is the sum of the supersymmetric massless vector and scalar multiplet actions, see eqs. (3.129) and (3.124), respectively. The vector multiplet action is coupled to a spin-1 current multiplet. Note that there is no coupling left between the current multiplet and the scalar multiplet. This will be different in the spin-2 case, as we will see later.

3.5. Supersymmetric Fierz–Pauli

In this section we extend the discussion of the previous section to the spin-2 case, skipping some of the details we explained in the spin-1 case.

3.5.1. Kaluza–Klein reduction and truncation

Our starting point is the off-shell 4D $\mathcal{N} = 1$ massless spin-2 multiplet which consists of a symmetric tensor $\hat{h}_{\hat{\mu}\hat{\nu}}$, a gravitino $\hat{\psi}_{\hat{\mu}}$, an auxiliary vector $\hat{A}_{\hat{\mu}}$ and two auxiliary scalars $\hat{M}$ and $\hat{N}$. This corresponds to the linearized version of the 'old minimal supergravity' multiplet. The supersymmetry rules, with constant spinor parameter ϵ, and gauge transformations of these fields, with local vector parameter $\hat{\Lambda}_{\hat{\mu}}$ and local spinor parameter $\hat{\eta}$, are given by [82, 83]:

$$\begin{aligned}
&\delta\hat{h}_{\hat{\mu}\hat{\nu}} = \bar{\epsilon}\,\Gamma_{(\hat{\mu}}\,\hat{\psi}_{\hat{\nu})} + \partial_{(\hat{\mu}}\,\hat{\Lambda}_{\hat{\nu})}\;, \quad \delta\hat{M} = -\bar{\epsilon}\,\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{\psi}_{\hat{\lambda}}\;, \quad \delta\hat{N} = -i\,\bar{\epsilon}\,\Gamma_5\,\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{\psi}_{\hat{\lambda}}\;,\\
&\delta\hat{\psi}_{\hat{\mu}} = -\tfrac{1}{4}\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{h}_{\hat{\lambda}\hat{\mu}}\epsilon - \tfrac{1}{12}\Gamma_{\hat{\mu}}(\hat{M} + i\,\Gamma_5\hat{N})\,\epsilon + \tfrac{1}{4}i\,\hat{A}_{\hat{\mu}}\Gamma_5\epsilon - \tfrac{1}{12}i\,\Gamma_{\hat{\mu}}\Gamma^{\hat{\rho}}\hat{A}_{\hat{\rho}}\Gamma_5\epsilon + \partial_{\hat{\mu}}\hat{\eta}\;,\\
&\delta\hat{A}_{\hat{\mu}} = \frac{3}{2}i\,\bar{\epsilon}\,\Gamma_5\,\Gamma_{\hat{\mu}}{}^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\,\hat{\psi}_{\hat{\lambda}} - i\,\bar{\epsilon}\,\Gamma_5\,\Gamma_{\hat{\mu}}\Gamma^{\hat{\rho}\hat{\lambda}}\,\partial_{\hat{\rho}}\hat{\psi}_{\hat{\lambda}}\;.
\end{aligned} \tag{3.64}$$

[3]Strictly speaking, preservation of the constraint $\partial^\mu J_\mu = 0$ under supersymmetry leads to the constraint $\partial\!\!\!/\mathcal{J}_\chi = 0$ and preservation of this new constraint leads to the constraint $\Box J_F = 0$. We are however interested in the massless limit, in which the conserved currents $(J_\mu, \mathcal{J}_\psi)$ and the fields $(\mathcal{J}_\chi, J_F)$ form two separate multiplets, that couple to a massless vector and scalar multiplet respectively. Since we are mostly interested in the coupling of the supercurrent multiplet $(J_\mu, \mathcal{J}_\psi)$ to the vector multiplet, we will simply set the fields $(\mathcal{J}_\chi, J_F)$ equal to zero.

Like in the spin-1 case we first perform a harmonic expansion of all fields and local parameters and substitute these into the transformation rules (3.64). Projecting onto the lowest KK massive sector we then obtain all the transformation rules of the real and imaginary parts of the $n = 1$ modes, like in eq. (3.40) for the spin-1 case. We indicate the real and imaginary parts of the bosonic modes by:

$$
\begin{aligned}
h^{(1)}_{\mu\nu} &\equiv \frac{1}{2}\left(h_{\mu\nu,1} + h^{\star}_{\mu\nu,1}\right) , & h^{(2)}_{\mu\nu} &\equiv \frac{1}{2i}\left(h_{\mu\nu,1} - h^{\star}_{\mu\nu,1}\right) ,\\
V^{(1)}_{\mu} &\equiv \frac{1}{2}\left(h_{\mu 3,1} + h^{\star}_{\mu 3,1}\right) , & V^{(2)}_{\mu} &\equiv \frac{1}{2i}\left(h_{\mu 3,1} - h^{\star}_{\mu 3,1}\right) ,\\
\phi^{(1)} &\equiv \frac{1}{2}\left(h_{33,1} + h^{\star}_{33,1}\right) , & \phi^{(2)} &\equiv \frac{1}{2i}\left(h_{33,1} - h^{\star}_{33,1}\right) ,\\
P^{(1)} &\equiv \frac{1}{2}\left(A_{3,1} + A^{\star}_{3,1}\right) , & P^{(2)} &\equiv \frac{1}{2i}\left(A_{3,1} - A^{\star}_{3,1}\right) ,\\
M^{(1)} &\equiv \frac{1}{2}\left(M_1 + M^{\star}_1\right) , & M^{(2)} &\equiv \frac{1}{2i}\left(M_1 - M^{\star}_1\right) ,\\
N^{(1)} &\equiv \frac{1}{2}\left(N_1 + N^{\star}_1\right) , & N^{(2)} &\equiv \frac{1}{2i}\left(N_1 - N^{\star}_1\right) ,
\end{aligned}
\tag{3.65}
$$

while the fermionic modes decompose into two Majorana modes:

$$
\begin{aligned}
\psi^{(1)}_{\mu} &\equiv \frac{1}{2}\left(\psi_{\mu,1} + B^{-1}\psi^{\star}_{\mu,1}\right) , & \psi^{(2)}_{\mu} &\equiv \frac{1}{2i}\left(\psi_{\mu,1} - B^{-1}\psi^{\star}_{\mu,1}\right) ,\\
\psi^{(1)}_{3} &\equiv \frac{1}{2}\left(\psi_{3,1} + B^{-1}\psi^{\star}_{3,1}\right) , & \psi^{(2)}_{3} &\equiv \frac{1}{2i}\left(\psi_{3,1} - B^{-1}\psi^{\star}_{3,1}\right) .
\end{aligned}
\tag{3.66}
$$

We next use the representation (3.41) of the Γ-matrices and decompose the 4-component spinors into two 2-component spinors as follows:

$$
\begin{gathered}
\psi^{(1)}_{\mu} = \begin{pmatrix} \psi_{\mu 1} \\ \psi_{\mu 2} \end{pmatrix} , \quad \psi^{(1)}_{3} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} , \quad \psi^{(2)}_{\mu} = \begin{pmatrix} \chi_{\mu 1} \\ \chi_{\mu 2} \end{pmatrix} , \quad \psi^{(2)}_{3} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} ,\\
\eta^{(1)} = \begin{pmatrix} \eta^{(1)}_1 \\ \eta^{(1)}_2 \end{pmatrix} , \qquad \eta^{(2)} = \begin{pmatrix} \eta^{(2)}_1 \\ \eta^{(2)}_2 \end{pmatrix} , \qquad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} .
\end{gathered}
\tag{3.67}
$$

Furthermore, we perform the following consistent truncation of the fields [4]

$$
\phi^{(2)} = V^{(1)}_{\mu} = h^{(2)}_{\mu\nu} = M^{(2)} = N^{(1)} = P^{(2)} = A^{(1)}_{\mu} = \chi_2 = \psi_1 = \psi_{\mu 1} = \chi_{\mu 2} = 0
\tag{3.68}
$$

[4] If we take the massless limit before the mentioned truncation we find two copies of a $\mathcal{N} = 2$ massless spin-2 multiplet plus two copies of a $\mathcal{N} = 2$ massless spin-1 multiplet, see also text after (3.46).

and of the parameters $\Lambda_\mu^{(2)} = \Lambda_3^{(1)} = \epsilon_1 = \eta_1^{(1)} = \eta_2^{(2)} = \xi = 0$.

For simplicity, from now on we drop all numerical upper indices, e.g. $\phi^{(1)} = \phi$, and all numerical lower indices, e.g. $\psi_{\mu 1} = \psi_\mu$ of the remaining non-zero fields (but not of the parameters). We find that the transformation rules of these fields under supersymmetry, with constant 2-component spinor parameter ϵ, and Stückelberg symmetries, with local scalar and vector parameters $\Lambda_3\,, \Lambda_\mu\,,$ and 2-component spinor parameters η_1 and η_2, are given by [5]

$$\begin{aligned}
\delta h_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\psi_{\nu)} + \partial_{(\mu}\Lambda_{\nu)}\ , \\
\delta\phi &= -\bar\epsilon\,\chi - m\Lambda_3\ , \qquad \delta V_\mu = \frac{1}{2}\bar\epsilon\gamma_\mu\psi - \frac{1}{2}\bar\epsilon\chi_\mu + \frac{1}{2}\partial_\mu\Lambda_3 + \frac{1}{2}m\Lambda_\mu\ , \\
\delta\psi_\mu &= -\frac{1}{4}\gamma^{\rho\lambda}\partial_\rho h_{\lambda\mu}\epsilon + \frac{1}{12}\gamma_\mu M\epsilon + \frac{1}{12}\gamma_\mu P\epsilon + \partial_\mu\eta_2\ , \\
\delta\psi &= -\frac{1}{4}\gamma^{\rho\lambda}\partial_\rho V_\lambda\epsilon - \frac{1}{12}N\epsilon - \frac{1}{12}\gamma^\rho A_\rho\epsilon + m\eta_2\ , \\
\delta\chi_\mu &= -\frac{1}{4}\gamma^\rho\partial_\rho V_\mu\epsilon + \frac{1}{4}m\gamma^\rho h_{\rho\mu}\epsilon - \frac{1}{12}\gamma_\mu N\epsilon + \frac{1}{4}A_\mu\epsilon - \frac{1}{12}\gamma_\mu\gamma^\rho A_\rho\epsilon + \partial_\mu\eta_1\ , \\
\delta\chi &= -\frac{1}{4}\gamma^\rho\partial_\rho\phi\epsilon - \frac{1}{12}M\epsilon + \frac{1}{6}P\epsilon - \frac{1}{4}m\gamma^\rho V_\rho\epsilon - m\eta_1\ , \\
\delta M &= -\bar\epsilon\gamma^\rho\partial_\rho\chi + \bar\epsilon\gamma^{\rho\lambda}\partial_\rho\psi_\lambda - m\bar\epsilon\gamma^\rho\chi_\rho\ , \\
\delta N &= -\bar\epsilon\gamma^\rho\partial_\rho\psi - \bar\epsilon\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + m\bar\epsilon\gamma^\rho\psi_\rho\ , \\
\delta P &= \bar\epsilon\gamma^\rho\partial_\rho\chi + \frac{1}{2}\bar\epsilon\gamma^{\rho\lambda}\partial_\rho\psi_\lambda + m\bar\epsilon\gamma^\rho\chi_\rho\ , \\
\delta A_\mu &= \frac{3}{2}\bar\epsilon\gamma_\mu{}^{\rho\lambda}\partial_\rho\chi_\lambda - \bar\epsilon\gamma_\mu\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + \frac{1}{2}\bar\epsilon\gamma_\mu{}^\rho\partial_\rho\psi - \bar\epsilon\partial_\mu\psi - \frac{1}{2}m\bar\epsilon\gamma_\mu{}^\rho\psi_\rho + m\bar\epsilon\psi_\mu\ .
\end{aligned} \tag{3.69}$$

The action invariant under the transformations (3.69) is given by

$$\begin{aligned}
I_m = \int d^3x\Big\{ & h^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) - m^2\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) \\
&+2h^{\mu\nu}\partial_\mu\partial_\nu\phi - 2h\partial^\alpha\partial_\alpha\phi - F^{\mu\nu}F_{\mu\nu} + 4mh^{\mu\nu}\partial_{(\mu}V_{\nu)} - 4mh\partial^\mu V_\mu \\
&-4\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar\chi_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8\bar\psi\gamma^{\mu\nu}\partial_\mu\chi_\nu + 8\bar\psi_\mu\gamma^{\mu\nu}\partial_\nu\chi + 8m\bar\psi_\mu\gamma^{\mu\nu}\chi_\nu \\
&-\tfrac{2}{3}M^2 - \tfrac{2}{3}N^2 + \tfrac{2}{3}P^2 + \tfrac{2}{3}A_\mu A^\mu\Big\}\ ,
\end{aligned} \tag{3.70}$$

[5]The 4D analogue of this multiplet, in superfield language, can be found in [84].

where $h = \eta^{\mu\nu} h_{\mu\nu}$ and $G^{\rm lin}_{\mu\nu}(h)$ is the linearized Einstein tensor. We observe that the action is non-diagonal in the bosonic fields $(h_{\mu\nu}\,, V_\mu\,, \phi)$ and the fermionic fields $(\psi_\mu\,, \chi)$ and $(\chi_\mu\,, \psi)$.

Finally, we fix all Stückelberg symmetries by imposing the gauge conditions

$$\phi = \text{const}\ , \qquad V_\mu = 0\ , \qquad \psi = 0\ , \qquad \chi = 0\ . \tag{3.71}$$

Taking into account the compensating gauge transformations

$$\begin{aligned} \Lambda_3 &= 0\ , & \Lambda_\mu &= \frac{1}{m}\bar\epsilon\chi_\mu\ , \\ \eta_1 &= -\frac{1}{12m}\,(M-2P)\,\epsilon\ , & \eta_2 &= \frac{1}{12m}\,(N+\gamma^\rho A_\rho)\,\epsilon\ , \end{aligned} \tag{3.72}$$

we obtain the final form of the supersymmetry rules of the 3D $\mathcal{N}=1$ off-shell massive spin-2 multiplet:

$$\begin{aligned} \delta h_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar\epsilon\partial_{(\mu}\chi_{\nu)}\ , \\ \delta\psi_\mu &= -\frac14\gamma^{\rho\lambda}\partial_\rho h_{\lambda\mu}\epsilon + \frac{1}{12}\gamma_\mu(M+P)\epsilon + \frac{1}{12m}\partial_\mu(N+\gamma^\rho A_\rho)\epsilon\ , \\ \delta\chi_\mu &= \frac14 m\gamma^\rho h_{\rho\mu}\epsilon + \frac14 A_\mu\epsilon - \frac{1}{12}\gamma_\mu(N+\gamma^\rho A_\rho)\epsilon - \frac{1}{12m}\partial_\mu(M-2P)\epsilon\ , \\ \delta M &= \bar\epsilon\gamma^{\rho\lambda}\partial_\rho\psi_\lambda - m\bar\epsilon\gamma^\rho\chi_\rho\ , \\ \delta N &= -\bar\epsilon\gamma^{\rho\lambda}\partial_\rho\chi_\lambda + m\bar\epsilon\gamma^\rho\psi_\rho\ , \\ \delta P &= \frac12\bar\epsilon\gamma^{\rho\lambda}\partial_\rho\psi_\lambda + m\bar\epsilon\gamma^\rho\chi_\rho\ , \\ \delta A_\mu &= \frac32\bar\epsilon\gamma_\mu{}^{\rho\lambda}\partial_\rho\chi_\lambda - \bar\epsilon\gamma_\mu\gamma^{\rho\lambda}\partial_\rho\chi_\lambda - \frac12 m\bar\epsilon\gamma_\mu{}^\rho\psi_\rho + m\bar\epsilon\psi_\mu\ . \end{aligned} \tag{3.73}$$

These transformation rules leave the following action invariant:

$$\begin{aligned} I_{m\neq 0} = \int d^3x\,\Big\{ & h^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) - m^2(h^{\mu\nu}h_{\mu\nu} - h^2) \\ & - 4\bar\psi_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar\chi_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar\psi_\mu\gamma^{\mu\nu}\chi_\nu \\ & - \frac23 M^2 - \frac23 N^2 + \frac23 P^2 + \frac23 A_\mu A^\mu \Big\}\ . \end{aligned} \tag{3.74}$$

This action describes 2+2 on-shell and 12+12 off-shell degrees of freedom. The first line is the standard Fierz–Pauli action. The fermionic off-diagonal mass term can easily be diagonalized by going to a basis in terms of the sum and difference of the two vector-spinors.[6]

The above action shows that the three scalars M, N, P and the vector A_μ are auxiliary fields which are set to zero by their equations of motion. We thus obtain the on-shell massive spin-2 multiplet with the following supersymmetry transformations:

$$\delta h_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)} \ , \qquad \delta\psi_\mu = -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon \ , \qquad \delta\chi_\mu = \frac{m}{4}\gamma^\nu h_{\mu\nu}\epsilon \ . \tag{3.75}$$

It is instructive to consider the closure of the supersymmetry algebra for the above supersymmetry rules given the fact that, unlike in the massless case, the symmetric tensor $h_{\mu\nu}$ does not transform under the gauge transformations $\delta h_{\mu\nu} = \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu$ and the only symmetries left to close the algebra are the global translations. We find that the commutator of two supersymmetries on $h_{\mu\nu}$ indeed gives a translation,

$$[\delta_1, \delta_2]h_{\mu\nu} = \xi^\rho\partial_\rho h_{\mu\nu} \ , \tag{3.76}$$

with parameter

$$\xi^\mu = \frac{1}{2}\bar{\epsilon}_2\gamma^\mu\epsilon_1 \ . \tag{3.77}$$

To close the commutator on the two gravitini requires the use of the equations of motion for these fields. From the action (3.74) we obtain the following equations:

$$\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho = m\gamma^{\mu\nu}\psi_\nu \ , \tag{3.78}$$

and a similar equation for ψ_μ. These equations of motion imply the standard spin-3/2 Fierz–Pauli equations

$$\mathcal{R}^{(1)}_\mu \equiv \partial\!\!\!/\chi_\mu + m\psi_\mu = 0 \ , \qquad \partial^\mu\chi_\mu = 0 \ , \qquad \gamma^\mu\chi_\mu = 0 \ , \tag{3.79}$$

and similar equations for ψ_μ. A useful alternative way of writing the equations of motion (3.78) is

$$\mathcal{R}^{(2)}_{\mu\nu} \equiv \partial_{[\mu}\chi_{\nu]} + m\gamma_{[\mu}\psi_{\nu]} = 0 \ . \tag{3.80}$$

Using these two ways of writing the equations of motion as well as the FP conditions that follow from them we find that the commutator on the two gravitini

[6] The $+3/2$ and $-3/2$ helicity states are described by the sum and difference of the two vector-spinors. See also appendix 3.C.

gives the same translations (3.77) up to equations of motion. More specifically, we find the following commutators

$$\begin{aligned}[\delta_1,\delta_2]\,\psi_\mu &= \xi^\nu\partial_\nu\psi_\mu - \frac{1}{4m}\xi^\alpha\partial_\mu\mathcal{R}^{(1)}_\alpha - \frac{1}{8m}\xi^\alpha\gamma_\alpha\partial_\mu(\gamma^{\rho\sigma}\partial_\rho\chi_\sigma)\\ &\quad + \frac{1}{4m}\xi^\alpha\partial_\mu\partial_\alpha(\gamma^\sigma\chi_\sigma) - \frac{1}{8}\xi^\alpha\gamma_\mu\gamma_\alpha(\gamma^{\rho\sigma}\partial_\rho\psi_\sigma)\ ,\\ [\delta_1,\delta_2]\,\chi_\mu &= \xi^\nu\partial_\nu\chi_\mu + \frac{1}{2}\xi^\nu\mathcal{R}^{(2)}_{\mu\nu} - \frac{1}{8}\xi^\rho\gamma_\rho\mathcal{R}^{(1)}_\mu\\ &\quad - \frac{1}{8}\xi^\rho\gamma_\rho\partial_\mu(\gamma^\nu\chi_\nu) + \frac{m}{8}\xi^\rho\gamma_\rho\gamma_\mu(\gamma^\nu\psi_\nu)\ .\end{aligned} \tag{3.81}$$

Hence, the algebra closes on-shell.

3.5.2. Massless limit

Finally, we discuss the massless limit $m \to 0$ of the supersymmetric FP theory. This is particularly interesting in view of the fact that the massless limit of the ordinary spin-2 FP system, coupled to a conserved energy-momentum tensor does not lead to linearized Einstein gravity. Instead, it leads to linearized Einstein gravity plus an extra force, mediated by a scalar that couples to the trace of the energy-momentum tensor with gravitational strength. This phenomenon is known as the van Dam–Veltman–Zakharov discontinuity. In the following, we will pay particular attention to this discontinuity in the supersymmetric case.

In order to discuss the massless limit, it turns out to be advantageous to trade the scalar fields M and P for scalars S and F, defined by

$$S = \frac{1}{6}(M+P)\ , \qquad F = \frac{4}{3}(M-2P)\ . \tag{3.82}$$

This field redefinition will make the multiplet structure of the resulting massless theory more manifest. In order to discuss the vDVZ discontinuity, we will include a coupling to a conjugate multiplet $(T_{\mu\nu}, \mathcal{J}^\psi_\mu, \mathcal{J}^\chi_\mu, T_S, T_N, T_F, T^A_\mu)$, as we did in the Proca case. Here $T_{\mu\nu}$ is a symmetric two-tensor, $\mathcal{J}^\psi_\mu$, $\mathcal{J}^\chi_\mu$ are vector-spinors, T^A_μ is a vector and T_F, T_S, T_N are scalars. We will thus start from the action

$$I = I_{\mathrm{FP}} + I_{\mathrm{int}}\ , \tag{3.83}$$

where I_{FP} is the supersymmetric FP action (3.74) and the interaction part I_{int} is given by

$$I_{\mathrm{int}} = h_{\mu\nu}T^{\mu\nu} + \bar{\psi}_\mu\mathcal{J}^\mu_\psi + \bar{\chi}_\mu\mathcal{J}^\mu_\chi + S\,T_S + F\,T_F + N\,T_N + A_\mu\,T^\mu_A\ . \tag{3.84}$$

Requiring that I_{int} is separately invariant under supersymmetry determines the transformation rules of the conjugate multiplet:

$$
\begin{aligned}
\delta T_{\mu\nu} &= \frac{1}{4}\bar{\epsilon}\,\gamma_{\alpha(\mu}\partial^{\alpha}\mathcal{J}^{\psi}_{\nu)} + \frac{m}{4}\bar{\epsilon}\,\gamma_{(\mu}\mathcal{J}^{\chi}_{\nu)} \ , \\
\delta \mathcal{J}^{\psi}_{\mu} &= \gamma^{\alpha}\epsilon\, T_{\alpha\mu} + \frac{1}{4}\gamma_{\mu\alpha}\epsilon\,\partial^{\alpha}T_S + m\gamma_{\mu}\epsilon\, T_N + \frac{m}{2}\gamma_{\mu\alpha}\epsilon\, T^{\alpha}_{A} - m\epsilon\, T^{A}_{\mu} \ , \\
\delta \mathcal{J}^{\chi}_{\mu} &= \frac{1}{m}\epsilon\,\partial^{\alpha}T_{\mu\alpha} - \gamma_{\mu\alpha}\epsilon\,\partial^{\alpha}T_N - 4m\gamma_{\mu}\epsilon\, T_F - \frac{3}{2}\gamma_{\mu\alpha\beta}\epsilon\,\partial^{\alpha}T^{\beta}_{A} + \gamma_{\mu\alpha}\gamma_{\beta}\epsilon\,\partial^{\alpha}T^{\beta}_{A} \ , \\
\delta T_S &= \frac{1}{2}\bar{\epsilon}\,\gamma^{\mu}\mathcal{J}^{\psi}_{\mu} \ , \\
\delta T_N &= \frac{1}{12m}\bar{\epsilon}\,\partial^{\mu}\mathcal{J}^{\psi}_{\mu} - \frac{1}{12}\bar{\epsilon}\,\gamma^{\mu}\mathcal{J}^{\chi}_{\mu} \ , \\
\delta T_F &= -\frac{1}{16m}\bar{\epsilon}\,\partial^{\mu}\mathcal{J}^{\chi}_{\mu} \ , \\
\delta T^{A}_{\mu} &= -\frac{1}{4}\bar{\epsilon}\mathcal{J}^{\chi}_{\mu} + \frac{1}{12}\bar{\epsilon}\,\gamma_{\mu}\gamma^{\rho}\mathcal{J}^{\chi}_{\rho} - \frac{1}{12m}\bar{\epsilon}\,\gamma_{\mu}\partial^{\rho}\mathcal{J}^{\psi}_{\rho} \ .
\end{aligned}
\tag{3.85}
$$

As in the Proca case, one should go back to a formulation that is still invariant under the Stückelberg symmetries, in order to take the massless limit in a well-defined way. This may be achieved by making the following field redefinitions in the final transformation rules (3.73) and action (3.74) thereby re-introducing the fields $(V_\mu\,, \phi'\,, \chi'\,, \psi)$ that were eliminated by the gauge-fixing conditions (3.71):

$$
\begin{aligned}
h_{\mu\nu} &= \tilde{h}_{\mu\nu} - \frac{1}{m}(\partial_\mu V_\nu + \partial_\nu V_\mu) + \frac{1}{m^2}\partial_\mu\partial_\nu\phi' \ , \\
\psi_\mu &= \tilde{\psi}_\mu - \frac{1}{m}\partial_\mu\psi \ , \qquad \chi_\mu = \tilde{\chi}_\mu + \frac{1}{4m}\partial_\mu\chi' \ .
\end{aligned}
$$

Applying this field redefinition in (3.73) then leads to transformation rules[7], whose massless limit is well-defined. In order to make the massless limit of the interaction part I_{int} and of the transformation rules (3.85) well-defined, we impose that $T_{\mu\nu}$ and $\mathcal{J}^{\psi}_{\mu}$ are conserved

$$
\partial^{\nu}T_{\mu\nu} = 0 \ , \qquad \partial^{\mu}\mathcal{J}^{\psi}_{\mu} = 0 \ , \tag{3.86}
$$

and we put $\mathcal{J}^{\chi}_{\mu}$, T_F, T_N and T^{A}_{μ} to zero in order to preserve supersymmetry and to obtain an irreducible multiplet in the massless limit. The conjugate multiplet

[7] These resulting transformation rules are given by the transformation rules (3.69), provided one makes the following substitution: $h_{\mu\nu} \to \tilde{h}_{\mu\nu}$, $\psi_\mu \to \tilde{\psi}_\mu$, $\chi_\mu \to \tilde{\chi}_\mu$, $\phi \to -\phi'$ and $\chi \to \chi'/4$.

(3.85) then reduces to a spin-2 supercurrent multiplet $(T_{\mu\nu}, \mathcal{J}^{\psi}_{\mu}, T_S)$ that contains the energy-momentum tensor $T_{\mu\nu}$ and supersymmetry current $\mathcal{J}^{\psi}_{\mu}$.

As in the Proca case, the massless limit is now well-defined. Performing the above outlined steps on the action (3.83) and taking the massless limit leads, however, to an action that is in off-diagonal form. This action can be diagonalized by making the following field redefinitions:

$$\tilde{h}_{\mu\nu} = h'_{\mu\nu} + \eta_{\mu\nu}\phi' \ , \qquad \tilde{\psi}_{\mu} = \psi'_{\mu} + \frac{1}{4}\gamma_{\mu}\chi' \ , \qquad S = S' - \frac{1}{8}F \ , \qquad \tilde{\chi}_{\mu} = \chi'_{\mu} - \gamma_{\mu}\psi \ . \tag{3.87}$$

The resulting action is given by

$$\begin{aligned} I = \int d^3x \Big\{ & h'^{\mu\nu} G^{\text{lin}}_{\mu\nu}(h') - 4\bar{\psi}'_{\mu}\gamma^{\mu\nu\rho}\partial_{\nu}\psi'_{\rho} - 8S'^2 + h'_{\mu\nu}T^{\mu\nu} + \bar{\psi}'^{\mu}\mathcal{J}^{\psi}_{\mu} + S'T_S \\ & - F^{\mu\nu}F_{\mu\nu} - \frac{2}{3}N^2 + \frac{2}{3}A^{\mu}A_{\mu} - 4\bar{\chi}'_{\mu}\gamma^{\mu\nu\rho}\partial_{\nu}\chi'_{\rho} - 8\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \\ & + 2\big[- \partial_{\mu}\phi'\partial^{\mu}\phi' - \frac{1}{4}\bar{\chi}'\gamma^{\mu}\partial_{\mu}\chi' + \frac{1}{16}F^2\big] \\ & + \phi'\eta^{\mu\nu}T_{\mu\nu} - \frac{1}{4}\bar{\chi}'\,\gamma^{\mu}\mathcal{J}^{\psi}_{\mu} - \frac{1}{8}FT_S \Big\} \ . \end{aligned} \tag{3.88}$$

This is an action for three massless multiplets : a spin two multiplet $(h'_{\mu\nu}, \psi'_{\mu}, S')$, a mixed gravitino-vector multiplet[8] $(V_{\mu}, \chi'_{\mu}, \psi, N, A_{\mu})$ and a scalar multiplet (ϕ', χ', F). These multiplets and their transformation rules are collected in appendix 3.B.[9] The spin-2 multiplet couples to the supercurrent multiplet in the usual fashion. Unlike the Proca case however, the supercurrent multiplet does not only couple to the spin-2 multiplet, but there is also a coupling to the scalar multiplet, given in the last line of (3.88). Indeed, defining

$$T_{\phi} = \eta^{\mu\nu}T_{\mu\nu} \ , \qquad \mathcal{J} = -\frac{1}{4}\gamma^{\mu}\mathcal{J}^{\psi}_{\mu} \ , \qquad T_F = -\frac{1}{8}T_S \ , \tag{3.89}$$

one finds that the fields $(T_{\phi}, \mathcal{J}, T_F)$ form a conjugate scalar multiplet with transformation rules

$$\delta T_{\phi} = -\bar{\epsilon}\,\gamma^{\mu}\partial_{\mu}\mathcal{J} \ , \quad \delta\mathcal{J} = -\frac{1}{4}\epsilon\,T_{\phi} + \gamma^{\mu}\epsilon\,\partial_{\mu}T_F \ , \quad \delta T_F = \frac{1}{4}\bar{\epsilon}\mathcal{J} \ , \tag{3.90}$$

[8] An on-shell version of this multiplet was introduced in [85].

[9] The transformation rules of the different multiplets can also be found by starting from the transformation rules of the massive FP multiplet and carefully following all redefinitions as outlined in the main text, provided one performs compensating gauge transformations.

such that the last line of (3.88) is invariant under supersymmetry.

We have thus obtained a 3D supersymmetric version of the 4D vDVZ discontinuity. The above discussion shows that the massless limit of the supersymmetric FP theory coupled to a supercurrent multiplet, leads to linearized $\mathcal{N}=1$ supergravity, plus an extra scalar multiplet that couples to a multiplet that includes the trace of the energy-momentum tensor and the gamma-trace of the supercurrent.

3.6. Linearized SNMG without Higher Derivatives

Using the results of the previous section we will now construct linearized New Massive Supergravity without higher derivatives but with auxiliary fields. Furthermore, we will show how, by eliminating the different "non-trivial" bosonic and fermionic auxiliary fields, one re-obtains the higher-derivative kinetic terms for both the bosonic and fermionic fields. We remind that by a "non-trivial" auxiliary field we mean an auxiliary field whose elimination leads to higher-derivative terms in the action.

Consider first the bosonic case. The linearized version of lower-derivative ("lower") NMG is described by the following action [14]:

$$I^{\rm lin}_{\rm NMG}(\text{lower}) = \int d^3x \left\{ -h^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) + 2q^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) - m^2(q^{\mu\nu}q_{\mu\nu} - q^2)\right\}\,, \tag{3.91}$$

where $h_{\mu\nu}$ and $q_{\mu\nu}$ are two symmetric tensors and $q=\eta^{\mu\nu}q_{\mu\nu}$. The above action can be diagonalized by making the redefinitions

$$h_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu}\ , \qquad q_{\mu\nu} = B_{\mu\nu}\ , \tag{3.92}$$

after which we obtain

$$I^{\rm lin}_{\rm NMG}\left[A,B\right] = \int d^3x \left\{ -A^{\mu\nu}G^{\rm lin}_{\mu\nu}\left(A\right) + B^{\mu\nu}G^{\rm lin}_{\mu\nu}\left(B\right) - m^2\left(B^{\mu\nu}B_{\mu\nu} - B^2\right)\right\}\,. \tag{3.93}$$

Using this diagonal basis it is clear that we can supersymmetrize the action in terms of a massless multiplet $(A_{\mu\nu}, \lambda_\mu, S)$ and a massive multiplet $(B_{\mu\nu}, \psi_\mu, \chi_\mu, M, N, P, A_\mu)$. Transforming this result back in terms of $h_{\mu\nu}$ and $q_{\mu\nu}$ and making the redefinition

$$\lambda_\mu = \rho_\mu - \psi_\mu \tag{3.94}$$

we find the following linearized lower-derivative supersymmetric NMG action

$$\begin{aligned} I^{\rm lin}_{\rm SNMG}(\text{lower}) = \int d^3x \Big\{ & -h^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) + 2q^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) - m^2(q^{\mu\nu}q_{\mu\nu} - q^2) + 8S^2 \\ & - \frac{2}{3}M^2 - \frac{2}{3}N^2 + \frac{2}{3}P^2 + \frac{2}{3}A_\mu A^\mu \\ & + 4\bar{\rho}_\mu\gamma^{\mu\nu\rho}\partial_\nu\rho_\rho - 8\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\rho_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \Big\}. \end{aligned} \tag{3.95}$$

This action describes 2+2 on-shell and 16+16 off-shell degrees of freedom. It is invariant under the following transformation rules

$$\begin{aligned} \delta h_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\rho_{\nu)}\ , \qquad \delta S = \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\rho_{\mu\nu} - \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\psi_{\mu\nu}\ , \\ \delta\rho_\mu &= -\frac{1}{4}\gamma^{\rho\sigma}(\partial_\rho h_{\mu\sigma})\epsilon + \frac{1}{2}S\gamma_\mu\epsilon + \frac{1}{12}\gamma_\mu(M+P)\epsilon\ , \end{aligned} \tag{3.96}$$

where

$$\rho_{\mu\nu} = \frac{1}{2}(\partial_\mu\rho_\nu - \partial_\nu\rho_\mu)\ , \qquad \psi_{\mu\nu} = \frac{1}{2}(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu)\ , \tag{3.97}$$

plus the transformation rules for the massive multiplet ($q_{\mu\nu}\,, \psi_\mu\,, \chi_\mu\,,\ M,\ N,\ P,\ A_\mu$) which can be found in eq. (3.73), with $h_{\mu\nu}$ replaced by $q_{\mu\nu}$. We have deleted $1/m$ terms in the transformation of $h_{\mu\nu}$ and ρ_μ since they take the form of a gauge transformation. Note also that the auxiliary field S transforms to the gamma trace of the equation of motion for ρ_μ.

The action (3.95) contains the trivial auxiliary fields ($S,\ M,\ N,\ P,\ A_\mu$) and the non-trivial auxiliary fields ($q_{\mu\nu}\,, \psi_\mu\,, \chi_\mu$). The elimination of the trivial auxiliary fields does not lead to anything new. These fields can simply be set equal to zero and disappear from the action. Instead, as we will show now, the elimination of the non-trivial auxiliary fields leads to higher-derivative terms in the action. To start with, the equation of motion for $q_{\mu\nu}$ can be used to solve for $q_{\mu\nu}$ as follows:

$$q_{\mu\nu} = \frac{1}{m^2}G^{\rm lin}_{\mu\nu}(h) - \frac{1}{2m^2}\eta_{\mu\nu}G^{\rm lin}_{\rm tr}(h)\ , \tag{3.98}$$

where $G^{\rm lin}_{\rm tr}(h) = \eta^{\mu\nu}G^{\rm lin}_{\mu\nu}(h)$. One of the vector-spinors, ψ_μ, occurs as a Lagrange multiplier. Its equation of motion enables one to solve for χ_μ:

$$\chi_\mu = -\frac{1}{2m}\gamma^{\rho\sigma}\gamma_\mu\,\rho_{\rho\sigma}\ . \tag{3.99}$$

The equation of motion of the other vector-spinor, χ_μ, can be used to solve for ψ_μ in terms of χ_μ:

$$\psi_\mu = -\frac{1}{2m}\gamma^{\rho\sigma}\gamma_\mu\,\chi_{\rho\sigma}\ , \tag{3.100}$$

and hence, via eq. (3.99), in terms of ρ_μ. One can show that the solution of ψ_μ in terms of (two derivatives of) ρ_μ is such that it solves the constraint

$$\gamma^{\mu\nu}\psi_{\mu\nu} = 0\ . \tag{3.101}$$

We now substitute the solutions (3.98) for $q_{\mu\nu}$ and (3.99) for χ_μ back into the action and make use of the identity

$$-4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho = \frac{8}{m^2}\bar{\rho}^{\mu\nu}\not{\partial}\rho_{\mu\nu} - \frac{2}{m^2}\bar{\rho}_{\mu\nu}\gamma^{\mu\nu}\not{\partial}\gamma^{\sigma\rho}\rho_{\sigma\rho}\ , \tag{3.102}$$

where we ignore a total derivative term. One thus obtains the following linearized higher-derivative ("higher") supersymmetric action of NMG [20]:

$$\begin{aligned} I^{\rm lin}_{\rm SNMG}(\text{higher}) = \int d^3x \Big\{ & - h^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) + 4\bar{\rho}_\mu\gamma^{\mu\nu\rho}\partial_\nu\rho_\rho + 8S^2 \\ & + \frac{4}{m^2}\big(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\big)^{\rm lin} + \frac{8}{m^2}\bar{\rho}_{ab}\not{\partial}\rho_{ab} - \frac{2}{m^2}\bar{\rho}_{ab}\gamma^{ab}\not{\partial}\gamma^{cd}\rho_{cd}\Big\}\ . \end{aligned} \tag{3.103}$$

The action (3.103) is invariant under the supersymmetry rules

$$\delta h_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\rho_{\nu)}\ , \qquad \delta\rho_\mu = -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon + \frac{1}{2}S\gamma_\mu\epsilon\ , \qquad \delta S = \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\rho_{\mu\nu}\ , \tag{3.104}$$

where we made use of the constraint (3.101) to simplify the transformation rule of S. Under supersymmetry the auxiliary field S transforms to the gamma-trace of the equation of motion for ρ_μ, since the higher-derivative terms in this equation of motion are gamma-traceless and therefore drop out.

Alternatively, the higher-derivative kinetic terms for ρ_μ can be obtained by boosting up the derivatives in the massive spin-3/2 FP equations in the same way as that has been done for the spin-2 FP equations in the construction of New Massive Gravity [14], except for one subtlety, see appendix 3.C.

This finishes our construction of linearized SNMG. In the next section we will discuss to which extent this result can be extended to the non-linear case.

3.7. The non-linear case

Supersymmetric NMG *without* "non-trivial" auxiliary fields, i.e. with higher derivatives, has already been constructed some time ago [20]. This action only contains the auxiliary field S of the massless multiplet. A characteristic feature is that there is no kinetic term for S and in the bosonic terms S occurs as a torsion contribution to the spin-connection. However, due to its coupling to the fermions it cannot be eliminated from the action. Thus, in the non-linear case we cannot anymore identify S as a "trivial" auxiliary field.

We recall that, apart from the auxiliary field S, in the linearized analysis of section 3.5 and 3.6 we distinguish between the trivial auxiliary fields (M, N, P, A_μ) and the non-trivial ones ($q_{\mu\nu}\,, \psi_\mu\,, \chi_\mu$). Only the elimination of the latter ones leads to higher derivatives in the Lagrangian. In the formulation of [20] only the auxiliary field S occurs. One could now search either for a formulation in which all other auxiliary fields occur or for an alternative formulation in which only the non-trivial auxiliary fields ($q_{\mu\nu}\,, \psi_\mu\,, \chi_\mu$) are present. In this work we will not consider the inclusion of all auxiliary fields any further. It is not clear to us whether such a formulation exists. This is based on the fact that our construction of the linearized massive multiplet makes use of the existence of a consistent truncation to the first massive KK level. Such a truncation can only be made consistently at the linearized level.

Before discussing the inclusion of the non-trivial auxiliary fields ($q_{\mu\nu}\,, \psi_\mu\,, \chi_\mu$) it is instructive to first consider the linearized case and see how, starting from the (linearized) formulation of [20] these three non-trivial auxiliary fields can be included and a formulation with lower derivatives can be obtained. Our starting point is the higher-derivative action (3.103) and corresponding transformation rules (3.104). We first consider the bosonic part of the action (3.103), i.e.

$$I^{\rm lin}_{\rm bos}(\text{higher}) = \int d^3x \,\Big\{ - h^{\mu\nu} G^{\rm lin}_{\mu\nu}(h) + 8S^2 + \frac{4}{m^2}\big(R^{\mu\nu}R_{\mu\nu} - \frac{3}{8}R^2\big)^{\rm lin} \Big\}\,. \quad (3.105)$$

We already know from the construction of the bosonic theory that the derivatives can be lowered by introducing a symmetric auxiliary field $q_{\mu\nu}$ and writing the equivalent bosonic action

$$I^{\rm lin}_{\rm bos}(\text{lower}) = \int d^3x \,\Big\{ - h^{\mu\nu} G^{\rm lin}_{\mu\nu}(h) + 8S^2 + 2q^{\mu\nu}G^{\rm lin}_{\mu\nu}(h) - m^2\left(q^{\mu\nu}q_{\mu\nu} - q^2\right) \Big\}\,. \quad (3.106)$$

The field equation of $q_{\mu\nu}$ is given by eq. (3.98) and substituting this solution back into the lower-derivative bosonic action (3.106) we re-obtain the higher-derivative bosonic action (3.105).

We next consider the fermionic part of the higher-derivative action (3.103),

$$I^{\rm lin}_{\rm ferm}(\text{higher}) = \int d^3x \,\Big\{4\bar\rho_\mu\gamma^{\mu\nu\rho}\partial_\nu\rho_\rho + \frac{8}{m^2}\bar\rho_{ab}\partial\!\!\!/\rho_{ab} - \frac{2}{m^2}\bar\rho_{ab}\gamma^{ab}\partial\!\!\!/\gamma^{cd}\rho_{cd}\Big\}\,. \quad (3.107)$$

To lower the number of derivatives we first replace the terms that are quadratic in $\rho_{\mu\nu}$ by the kinetic term of an auxiliary field χ_μ, while adding another term with a Lagrange multiplier ψ_μ to fix the relation between $\rho_{\mu\nu}$ and χ_μ:

$$I^{\rm lin}_{\rm ferm}({\rm lower}) = \int d^3x \left\{4\bar{\rho}_\mu\gamma^{\mu\nu\rho}\partial_\nu\rho_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho - 8\bar{\psi}_\mu\left(\gamma^{\mu\nu\rho}\rho_{\nu\rho} - m\gamma^{\mu\nu}\chi_\nu\right)\right\} . \tag{3.108}$$

The equation of motion for ψ_μ enables us to express χ_μ in terms of $\rho_{\mu\nu}$. The result is given in eq. (3.99). Substituting this solution for χ_μ back into the action, the terms linear in the Lagrange multiplier ψ_μ drop out and we re-obtain the higher-derivative fermionic action given in eq. (3.107).

Adding up the lower-derivative bosonic action (3.106) and the lower-derivative fermionic action (3.108) we obtain the lower-derivative supersymmetric action (3.95), albeit without the bosonic auxiliary fields (M, N, P, A_μ). We only consider a formulation in which these auxiliary fields are absent.

Having introduced the new auxiliary fields $(q_{\mu\nu}\,, \psi_\mu\,, \chi_\mu)$ we should derive their supersymmetry rules. They can be derived by starting from the solutions (3.98), (3.99) and (3.100) of these auxiliary fields in terms of $h_{\mu\nu}$ and ρ_μ and applying the supersymmetry rules of $h_{\mu\nu}$ and ρ_μ given in eq. (3.104). This leads to supersymmetry rules that do not contain the auxiliary fields. These can be introduced by adding to the supersymmetry rules a number of (field-dependent) equation of motion symmetries. We thus find the intermediate result:

$$\begin{aligned}
&\delta h_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\rho_{\nu)}\ , \qquad \delta\rho_\mu = -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon + \frac{1}{2}S\gamma_\mu\epsilon\ , \qquad \delta\psi_\mu = -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho q_{\mu\sigma}\epsilon\ ,\\
&\delta q_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)} + \frac{1}{m}\bar{\epsilon}\partial_{(\mu}\chi_{\nu)}\ , \quad \delta S = \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\rho_{\mu\nu}\,, \quad \delta\chi_\mu = \frac{m}{4}\gamma^\nu q_{\mu\nu}\epsilon + \frac{1}{2m}\epsilon\partial_\mu S\ .
\end{aligned} \tag{3.109}$$

These transformation rules are not yet quite the same as the ones given in eq. (3.96). In particular, the transformation rules of S and χ_μ are different. The difference is yet another “on-shell symmetry” of the action eq. (3.95), with spinor parameter η, given by

$$\delta S = -\frac{1}{4}\bar{\eta}\gamma^{\mu\nu}\psi_{\mu\nu}\ , \qquad \delta\chi_\mu = -\frac{1}{2m}\eta\partial_\mu S\ . \tag{3.110}$$

The transformation rules in eqs. (3.96) and (3.109) are therefore equivalent up to an on-shell symmetry with parameter $\eta = \epsilon$:

$$\delta_{\rm susy}(\text{eq. (3.96)}) = \delta_{\rm susy}(\text{eq. (3.109)}) + \delta_{\text{on-shell}}(\eta = \epsilon)\ . \tag{3.111}$$

We now wish to discuss in which sense the previous analysis can be extended to the non-linear case. For simplicity, we take the approximation in which one considers only the terms in the action that are independent of the fermions and the terms that are bilinear in the fermions. Furthermore, we ignore in the supersymmetry variation of the action terms that depend on the auxiliary scalar S. Since terms linear in S only occur in terms bilinear in fermions this effectively implies that we may set $S = 0$ in the action. In this approximation the higher-derivative action of SNMG is given by [20]

$$\begin{aligned} I_{\text{SNMG}}^{\text{nonlin}}\,(\text{higher}) = \int d^3x\; e\Big\{ &-4R\,(\hat{\omega}) + \frac{1}{m^2}R^{\mu\nu ab}\,(\hat{\omega})\,R_{\mu\nu ab}\,(\hat{\omega}) - \frac{1}{2m^2}R^2\,(\hat{\omega}) \\ &+ 4\bar{\rho}_\mu\gamma^{\mu\nu\rho}D_\nu\,(\hat{\omega})\,\rho_\rho + \frac{8}{m^2}\bar{\rho}_{ab}\,(\hat{\omega})\,\not{D}\,(\hat{\omega})\,\rho^{ab}\,(\hat{\omega}) - \frac{2}{m^2}\bar{\rho}_{\mu\nu}(\hat{\omega})\gamma^{\mu\nu}\not{D}(\hat{\omega})\gamma^{\rho\sigma}\rho_{\rho\sigma}(\hat{\omega}) \\ &- \frac{2}{m^2}R_{\mu\nu ab}\,(\hat{\omega})\,\bar{\rho}_\rho\gamma^{\mu\nu}\gamma^\rho\rho^{ab}\,(\hat{\omega}) - \frac{2}{m^2}R\,(\hat{\omega})\,\bar{\rho}^\mu\gamma^\nu\rho_{\mu\nu}\,(\hat{\omega}) \\ &+ \text{ higher-order fermions and S-dependent terms}\Big\}\,. \end{aligned} \tag{3.112}$$

Note that we have replaced the symmetric tensor $h_{\mu\nu}$ by a Dreibein field ${e_\mu}^a$. Keeping the same approximation discussed above the action (3.112) is invariant under the supersymmetry rules

$$\delta {e_\mu}^a = \frac{1}{2}\bar{\epsilon}\gamma^a\rho_\mu\;, \qquad \delta\rho_\mu = D_\mu\,(\hat{\omega})\,\epsilon\;. \tag{3.113}$$

We first consider the lowering of the number of derivatives in the bosonic part of the action. Since the Ricci tensor now depends on a torsion-full spin connection we need a *non-symmetric* auxiliary tensor $q_{\mu,\nu}$. The action (3.112) can then be converted into the following equivalent action:

$$\begin{aligned} I_{\text{SNMG}}^{\text{nonlin}}\,(\text{higher}) = \int d^3x\; e\Big\{ &-4R\,(\hat{\omega}) - m^2\left(q^{\mu,\nu}q_{\mu,\nu} - q^2\right) + 2q^{\mu,\nu}G_{\mu,\nu}\,(\hat{\omega}) \\ &+ 4\bar{\rho}_\mu\gamma^{\mu\nu\rho}D_\nu\,(\hat{\omega})\,\rho_\rho + \frac{8}{m^2}\bar{\rho}_{ab}\,(\hat{\omega})\,\not{D}\,(\hat{\omega})\,\rho^{ab}\,(\hat{\omega}) - \frac{2}{m^2}\bar{\rho}_{\mu\nu}(\hat{\omega})\gamma^{\mu\nu}\not{D}(\hat{\omega})\gamma^{\rho\sigma}\rho_{\rho\sigma}(\hat{\omega}) \\ &- \frac{2}{m^2}R_{\mu\nu ab}(\hat{\omega})\bar{\rho}_\rho\gamma^{\mu\nu}\gamma^\rho\rho^{ab}(\hat{\omega}) - \frac{2}{m^2}R(\hat{\omega})\bar{\rho}^\mu\gamma^\nu\rho_{\mu\nu}\,(\hat{\omega}) \\ &+ \text{ higher-order fermions and S-dependent terms}\Big\}\,. \end{aligned}$$

The equivalence with the previous action can be seen by solving the equation of motion for $q_{\mu,\nu}$:

$$q_{\mu,\nu} = \frac{1}{m^2}G_{\mu,\nu}\,(\hat{\omega}) - \frac{1}{2m^2}g_{\mu\nu}G^{\text{tr}}\,(\hat{\omega}) \tag{3.114}$$

and substituting this solution back into the action. Note that the solution for $q_{\mu,\nu}$ is not super-covariant.

We next consider the lowering of the number of derivatives in the fermionic terms in the action. Following the linearized case we define an auxiliary vector-spinor χ_μ as

$$\chi_\mu = -\frac{1}{2m}\gamma^{\rho\sigma}\gamma_\mu\rho_{\rho\sigma}\left(\hat{\omega}\right) \ , \tag{3.115}$$

or equivalently

$$\rho_{\mu\nu}\left(\hat{\omega}\right) = -m\gamma_{[\mu}\chi_{\nu]} \ . \tag{3.116}$$

The first equation is the non-linear generalization of eq. (3.99). Using this definition one can show the following identity

$$\begin{aligned}\frac{8}{m^2}e\bar{\rho}_{ab}(\hat{\omega})\not{D}(\hat{\omega})\rho^{ab}(\hat{\omega}) - \frac{2}{m^2}e\bar{\rho}_{\mu\nu}(\hat{\omega})\gamma^{\mu\nu}\not{D}(\hat{\omega})\left[\gamma^{\rho\sigma}\rho_{\rho\sigma}(\hat{\omega})\right] =& \\ = -4e\bar{\chi}_\mu\gamma^{\mu\nu\rho}D_\nu(\hat{\omega})\chi_\rho - \frac{1}{m}eR_{\mu\nu ab}(\hat{\omega})\bar{\rho}_\rho\gamma^{\mu\nu\rho}\gamma^{ab}\gamma^\sigma\chi_\sigma& \\ + \text{ higher-order fermions and total derivative terms} \ ,&\end{aligned} \tag{3.117}$$

which is the non-linear generalization of the identity (3.102). This identity can be used to replace the higher-derivative kinetic terms of the fermions by lower-derivative ones. At the same time we may use eq. (3.116) to replace $\rho_{\mu\nu}$ by χ_μ. This can be done by introducing a Lagrange multiplier ψ_μ whose equation of motion allows us to use eq. (3.115). This leads to the following action:

$$\begin{aligned}I^{\text{nonlin}}_{\text{SNMG}}\left(\text{lower}\right) = \int d^3x \ e\Big\{& -4R\left(\hat{\omega}\right) + 2q^{\mu,\nu}G_{\mu,\nu}\left(\hat{\omega}\right) - m^2\left(q^{\mu,\nu}q_{\mu,\nu} - q^2\right) \\ &+ 4\bar{\rho}_\mu\gamma^{\mu\nu\rho}D_\nu\left(\hat{\omega}\right)\rho_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}D_\nu\left(\hat{\omega}\right)\chi_\rho - 8\bar{\psi}_\mu\gamma^{\mu\nu\rho}\rho_{\nu\rho}\left(\hat{\omega}\right) + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \\ &- \frac{1}{m}R_{\mu\nu ab}\left(\hat{\omega}\right)\bar{\rho}_\rho\gamma^{\mu\nu\rho}\gamma^{ab}\gamma^\sigma\chi_\sigma + \frac{2}{m}R_{\mu\nu ab}\left(\hat{\omega}\right)\bar{\rho}_\rho\gamma^{\mu\nu}\gamma^\rho\gamma^a\chi^b \\ &- \frac{1}{m}R\left(\hat{\omega}\right)\bar{\rho}_\mu\gamma^{\mu\nu}\chi_\nu - \frac{2}{m}R\left(\hat{\omega}\right)\bar{\rho}^\mu\chi_\mu \\ &+ \text{ higher-order fermions and S-dependent terms}\Big\} \ .\end{aligned}$$

Our next task is to derive the supersymmetry rules of the auxiliary fields $q_{\mu,\nu}$, ψ_μ and χ_μ. Using the solutions of the auxiliary fields in terms of $e_\mu{}^a$ and ρ_μ we derived these supersymmetry rules. In this way one obtains supersymmetry rules that do not contain any of the auxiliary fields and, consequently, do not reduce to the supersymmetry rules (3.96) upon linearization. To achieve this, we must add to these transformation rules a number of field-dependent equation of motion symmetries, like we did in the linearized case. Since the results we obtained are not illuminating we refrain from giving the explicit expressions here.

A disadvantage of the present approach is that, although in principle possible in the approximation we considered, one cannot maintain the interpretation of S as a torsion contribution to the spin-connection. This makes the result rather cumbersome. Without further insight the lower-derivative formulation of SNMG, if it exists at all at the full non-linear level, does not take the same elegant form as the higher-derivative formulation presented in [20].

3.8. Discussion

In this work we considered the $\mathcal{N}=1$ supersymmetrization of New Massive Gravity in the presence of auxiliary fields. All auxiliary fields are needed to close the supersymmetry algebra off-shell. At the linearized level, we distinguished between two types of auxiliary fields: the "non-trivial" ones whose elimination leads to higher derivatives in the Lagrangian (these are the fields $q_{\mu\nu}, \psi_\mu$ and χ_μ) and the "trivial" ones whose elimination (if possible at all at the full non-linear level) does not lead to higher derivatives (these are the fields S, M, N, P and A_μ). We found that at the linearized level all auxiliary fields could be included leading to a linearized SNMG theory without higher derivatives. At the non-linear level we gave a partial answer for the case that only the trivial auxiliary S and the non-trivial auxiliaries $q_{\mu\nu}, \psi_\mu$ and χ_μ were included. To obtain the full non-linear answer one should perhaps make use of superspace techniques. The answer without the non-trivial auxiliaries and with higher derivatives can be found in [20].

We discussed a 3D supersymmetric analog of the 4D vDVZ discontinuity by taking the massless limit of the supersymmetric FP model coupled to a supercurrent multiplet. We showed that in the massless limit there is a non-trivial coupling of a scalar multiplet (containing the scalar mode ϕ of the metric) to a current multiplet (containing the trace of the energy-momentum tensor). This is the natural supersymmetric extension of what happens in the bosonic case and supports the analysis of [78].

As a by-product we found a way to "boost up" the derivatives in the spin-3/2 FP equation, see appendix 3.C. The trick is based upon the observation that, before boosting up the derivatives like in the construction of the NMG model, one should first combine the equations of motion describing the helicity $+3/2$ and $-3/2$ states into a single parity-even equation with one additional derivative.

3.A. General Multiplets and Degrees of Freedom

In any realization of a supersymmetry algebra of the form $\{Q, Q\} \sim P$ the number of bosonic degrees of freedom should match the number of fermionic degrees of freedom. On-shell, this always holds, but off-shell bosonic and fermionic degrees of freedom coincide when the theory has auxiliary fields which "close the algebra off-shell".

Field		Off-Shell	$4D$	$3D$	On-Shell	$4D$	$3D$
ϕ	Massless Massive	1	1	1	1	1	1
ψ	Massless	$2^{[D/2]}$	4	2	$\frac{1}{2}2^{[D/2]}$	2	1
A_μ	Massless	$D-1$	3	2	$D-2$	2	1
	Massive	D	4	3	$D-1$	3	2
ψ_μ	Massless	$(D-1)2^{[D/2]}$	12	4	$\frac{1}{2}(D-3)2^{[D/2]}$	2	0
$g_{\mu\nu}$	Massless	$D(D-1)/2$	6	3	$D(D-3)/2$	2	0
	Massive	$D(D+1)/2$	10	6	$D(D-1)/2-1$	5	2

TABLE 3.4
This table contains the results for the degrees of freedom for a scalar field ϕ, a Majorana fermion ψ, a gauge field A_μ, a Majorana gravitino ψ_μ and a graviton field $g_{\mu\nu}$.

To illustrate the off-shell and on-shell degrees of freedom, consider for example the 3D massless "mixed gravitino-vector" multiplet with field content $\{V_\mu, A_\mu, N, \chi_\mu, \psi\}$, where A_μ and N are auxiliary fields. The off-shell counting of degrees of freedom is established in the following way: the gauge field V_μ describes two off-shell degrees of freedom, while the auxiliary field A_μ describes three, since there is no gauge invariance for this field, and the auxiliary field N describes one degree of freedom (a propagating and an auxiliary scalar fields always have one degree of freedom). In addition, the gravitino χ_μ contains four degrees of freedom, while the spinor ψ describes two. Therefore, the off-shell counting of bosonic and fermionic degrees of freedom match $6+6$ giving a total of twelve. On the other hand, the on-shell degrees of freedom will be: one for the gauge field V_μ, zero for the auxiliary fields and for the gravitino, and one for the spinor ψ. Once more the number of bosonic and fermionic degrees of freedom match $1+1$ leading to a total of two degrees of freedom.

3.B. Off-shell $\mathcal{N} = 1$ Massless Multiplets

In this section we collect the off-shell formulations of the different 3D massless multiplets with $\mathcal{N} = 1$ supersymmetry. A useful reference where more properties about 3D supersymmetry can be found is [86]. The field content of the different multiplets can be found in Table 3.5.

multiplet	fields	off-shell	on-shell
$s = 2$	$h_{\mu\nu}\,,\psi_\mu\,,S$	4+4	0+0
$s = 1$	$V_\mu\,,N\,,A_\mu\,,\chi_\mu\,,\psi$	6+6	1+1
$s = 0$	$\phi\,,\chi\,,F$	2+2	1+1
gravitino multiplet	$\chi_\mu\,,A_\mu\,,D$	4+4	0+0
vector multiplet	$V_\mu\,,\psi$	2+2	1+1

Table 3.5
This Table indicates the field content and off-shell/on-shell degrees of freedom of the different massless multiplets. Only the massless multiplets above the double horizontal line occur in the massless limit of the FP model.

s=2 The off-shell version of the 3D massless spin-2 multiplet is well-known. The multiplet is extended with an auxiliary real scalar field S. The off-shell supersymmetry rules are given by

$$\delta h_{\mu\nu} = \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}\ , \qquad \delta\psi_\mu = -\frac{1}{4}\gamma^{\rho\sigma}\partial_\rho h_{\mu\sigma}\epsilon + \frac{1}{2}S\gamma_\mu\epsilon\ , \qquad \delta S = \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\psi_{\mu\nu}\ , \tag{3.118}$$

where

$$\psi_{\mu\nu} = \frac{1}{2}(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu)\ . \tag{3.119}$$

These transformation rules leave the following action invariant:

$$I_{s=2} = \int d^3x\left\{h^{\mu\nu}G^{\text{lin}}_{\mu\nu}(h) - 4\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 8S^2\right\}\ . \tag{3.120}$$

s=1 The off-shell "mixed gravitino-vector" multiplet consists of a propagating vector V_μ, an auxiliary vector A_μ, an auxiliary scalar N, a vector spinor χ_μ and a spinor ψ. An on-shell version of this multiplet, called "vector-spinor" multiplet, has been considered in [85]. The off-shell supersymmetry rules are given by

$$\begin{aligned}
\delta V_\mu &= \bar\epsilon\gamma_\mu\psi - \tfrac{1}{2}\bar\epsilon\chi_\mu \ , \\
\delta\psi &= -\tfrac{1}{8}\gamma^{\rho\lambda}F_{\rho\lambda}\epsilon - \tfrac{1}{12}N\epsilon - \tfrac{1}{12}\gamma^\alpha A_\alpha\epsilon \ , \\
\delta\chi_\mu &= -\tfrac{1}{4}\gamma^\alpha F_{\alpha\mu}\epsilon - \tfrac{1}{8}\gamma_\mu\gamma^{\rho\lambda}F_{\rho\lambda}\epsilon - \tfrac{1}{6}\gamma_\mu N\epsilon + \tfrac{1}{4}A_\mu\epsilon - \tfrac{1}{6}\gamma_\mu\gamma^\alpha A_\alpha\epsilon \ , \\
\delta N &= \bar\epsilon\gamma^\alpha\partial_\alpha\psi - \bar\epsilon\gamma^{\alpha\beta}\partial_\alpha\chi_\beta \ , \\
\delta A_\mu &= \tfrac{3}{2}\bar\epsilon\gamma_\mu{}^{\alpha\beta}\partial_\alpha\chi_\beta - \bar\epsilon\gamma_\mu\gamma^{\alpha\beta}\partial_\alpha\chi_\beta + \bar\epsilon\gamma_\mu{}^\alpha\partial_\alpha\psi + \bar\epsilon\partial_\mu\psi \ .
\end{aligned} \tag{3.121}$$

Note that this multiplet is irreducible. It cannot be written as the sum of a gravitino and vector multiplet. These multiplets are given below. The supersymmetric action for this multiplet is given by

$$I_{s=1} = \int d^3x \Big\{ - F^{\mu\nu}F_{\mu\nu} - \tfrac{2}{3}N^2 + \tfrac{2}{3}A_\mu A^\mu - 4\bar\chi_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho - 8\bar\psi\gamma^\mu\partial_\mu\psi \Big\} \ , \tag{3.122}$$

with $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$.

s=0 The off-shell scalar multiplet consists of a scalar ϕ, a spinor χ and an auxiliary scalar F. The off-shell supersymmetry rules are given by

$$\delta\phi = \tfrac{1}{4}\bar\epsilon\chi \ , \qquad \delta\chi = \gamma^\mu\epsilon\,(\partial_\mu\phi) - \tfrac{1}{4}F\epsilon \ , \qquad \delta F = -\bar\epsilon\gamma^\mu\partial_\mu\chi \ . \tag{3.123}$$

The supersymmetric action for a scalar multiplet is given by

$$I_{s=0} = \int d^3x \Big\{ - \partial^\mu\phi\partial_\mu\phi - \tfrac{1}{4}\bar\chi\gamma^\mu\partial_\mu\chi + \tfrac{1}{16}F^2 \Big\} \ . \tag{3.124}$$

Besides the massless multiplets discussed so-far there is a separate gravitino and vector multiplet. The vector multiplet arises in section 3.4 in the massless limit of the Proca theory. For completeness we give these two multiplets below.

gravitino multiplet The off-shell gravitino multiplet consists of a gravitino χ_μ, an auxiliary vector A_μ and an auxiliary scalar D. The off-shell supersymmetry rules are given by

$$\delta\chi_\mu = \frac{1}{4}\gamma^\lambda\gamma_\mu\epsilon A_\lambda + \frac{1}{2}\gamma_\mu\epsilon D \ , \qquad \delta A_\mu = \frac{1}{2}\bar\epsilon\gamma^{\rho\sigma}\gamma_\mu\chi_{\rho\sigma} \ , \qquad \delta D = \frac{1}{4}\bar\epsilon\gamma^{\rho\sigma}\chi_{\rho\sigma} \ , \tag{3.125}$$

where

$$\chi_{\mu\nu} = \frac{1}{2}(\partial_\mu\chi_\nu - \partial_\mu\chi_\mu) \ . \tag{3.126}$$

These transformation rules leave the following action invariant:

$$I_{s=3/2} = \int d^3x \Big\{ -4\bar{\chi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \chi_\rho - \frac{1}{2}A^\mu A_\mu + 2D^2 \Big\} \ . \tag{3.127}$$

vector multiplet The off-shell vector multiplet consists of a vector V_μ and a spinor ψ. The off-shell supersymmetry rules are given by

$$\delta V_\mu = -\bar{\epsilon}\gamma_\mu \psi \ , \qquad \delta\psi = \frac{1}{8}\gamma^{\mu\nu}\epsilon\, F_{\mu\nu} \ , \tag{3.128}$$

with $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. The supersymmetric action for a vector multiplet is given by

$$I_{s=1} = \int d^3x \Big\{ -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - 2\bar{\psi}\gamma^\mu \partial_\mu \psi \Big\} \ . \tag{3.129}$$

This finishes our discussion of the massless multiplets in three dimensions.

3.C. Boosting up the Derivatives in Spin-3/2 FP

In this appendix we show how the higher-derivative kinetic terms for the gravitino ρ_μ can be obtained by boosting up the derivatives in the massive spin-3/2 FP equations in the same way as that has been done for the spin-2 FP equations in the construction of New Massive Gravity [14] except for one subtlety.

Our starting point is the following fermionic action with two massive gravitini, ψ_μ and χ_μ, each of which carries only one physical degree of freedom in 3D,

$$I\,[\psi,\chi] = \int d^3x \Big\{ -4\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - 4\bar{\chi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\chi_\rho + 8m\bar{\psi}_\mu\gamma^{\mu\nu}\chi_\nu \Big\} \ . \tag{3.130}$$

The equations of motion following from this action are given by

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho - m\gamma^{\mu\nu}\chi_\nu = 0 \ , \qquad \gamma^{\mu\nu\rho}\partial_\nu\chi_\rho - m\gamma^{\mu\nu}\psi_\nu = 0 \ . \tag{3.131}$$

Note that each one of the equations (3.131) can be used to solve for one gravitino in terms of the other one. However, this solution does not solve the other equation. Therefore, one cannot substitute only one solution back into (3.130) because one would lose information about the differential constraint encoded in the other equation. After diagonalization

$$\zeta^1_\mu = \psi_\mu + \chi_\mu \ , \qquad \zeta^2_\mu = \psi_\mu - \chi_\mu \ . \tag{3.132}$$

we obtain the massive FP equations for a helicity +3/2 and -3/2 state:

$$\left(\partial\!\!\!/ + m\right)\zeta^1_\mu = 0\ ,\ \gamma^\mu\zeta^1_\mu = 0\ ,\ \partial^\mu\zeta^1_\mu = 0\ , \tag{3.133}$$

$$\left(\partial\!\!\!/ - m\right)\zeta^2_\mu = 0\ ,\ \gamma^\mu\zeta^2_\mu = 0\ ,\ \partial^\mu\zeta^2_\mu = 0\ . \tag{3.134}$$

To boost up the derivatives in these equations we may proceed in two ways. One option is to boost up the derivatives in each equation separately by solving the corresponding differential constraint. In a second step one should then combine the two higher-derivative equations by a single equation in terms of ρ_μ by a so-called "soldering" technique which has also been applied to construct New Massive Gravity out of two different Topologically Massive Gravities [87]. Alternatively, it is more convenient to first combine the two equations into the following equivalent second-order equation which is manifestly parity-invariant:

$$\left(\Box - m^2\right)\zeta_\mu = \left(\partial\!\!\!/ \mp m\right)\left(\partial\!\!\!/ \pm m\right)\zeta_\mu = 0\ , \qquad \gamma^\mu\zeta_\mu = 0\ ,\ \partial^\mu\zeta_\mu = 0\ . \tag{3.135}$$

Note that the action corresponding to these equations of motion cannot be used in a supersymmetric action since the fermionic kinetic term would have the same number of derivatives as the standard bosonic kinetic term describing a spin-2 state.

We are now ready to perform the procedure of "boosting up the derivatives" in the same way as in the bosonic theory where it leads to the higher-derivative NMG theory. To be specific, we solve the divergenceless condition $\partial^\mu\zeta_\mu = 0$ in terms of a new vector-spinor ρ_μ as follows:

$$\zeta_\mu = \mathcal{R}_\mu(\rho) \equiv \varepsilon_\mu{}^{\nu\rho}\partial_\nu\rho_\rho\ . \tag{3.136}$$

Substituting this solution back into the other two equations in (3.135) leads to the higher-derivative equations

$$\left(\Box - m^2\right)\mathcal{R}_\mu(\rho) = 0\ , \qquad \gamma^\mu\mathcal{R}_\mu(\rho) = 0\ . \tag{3.137}$$

These equations of motion are invariant under the gauge symmetry

$$\delta\rho_\mu = \partial_\mu\eta\ . \tag{3.138}$$

Furthermore, they can be integrated to the following action:

$$I[\rho] = \int d^3x\left\{\bar{\rho}^\mu\mathcal{R}_\mu(\rho) - \frac{1}{2m^2}\bar{\rho}^\mu\partial\!\!\!/\left[\partial\!\!\!/\mathcal{R}_\mu(\rho) + \varepsilon_\mu{}^{\sigma\tau}\partial_\sigma\mathcal{R}_\tau(\rho)\right]\right\}\ . \tag{3.139}$$

One can show that the equations of motion following from this action implies the algebraic constraint given in (3.137). The action (3.139) is precisely the fermionic part of the action (3.103) of linearized SNMG.

4

Symmetry Analysis for Anisotropic Field Theories

4.1. Introduction

The second modification from general relativity that we will consider implies to give up on diffeomorphism invariance. Since at the quantum level, general relativity is non-renormalizable, it should be viewed as an effective theory that breaks down at some scale. Lorentz symmetry breaking in Hořava-Lifshitz Gravity theory [22–24] can lead to a modification of the graviton propagator in the UV, thus rendering the theory power-counting renormalizable.

Abandoning Lorentz invariance by introducing higher order spatial derivatives and forbidding the introduction of any higher order time derivatives, improves the UV behaviour of the propagator and at the same time, guarantees the absence of ghosts [88, 89].

It is important to point out that there is nothing wrong with using Lorentz violations as there is no reason why a theory should necessarily exhibit Lorentz symmetry in the far UV. Our theory will be consistent as long as it allows Lorentz invariance to be restored as an emergent symmetry in the IR.

Systems without Lorentz invariance are quite common, for example, multifractal spacetimes [90], modified gravities described in [91, 92], aethereal gravity [93], they all deal with Lorentz violations. Furthermore, a Lifshitz scalar field theory has been studied in the context of condensed matter physics as a description of tricritical phenomena involving spatially modulated phases [94, 95] and low energy Lorentz breaking can be originated by high energy physics such as string theory [39, 96].

On the other hand, theories that are invariant under scale and conformal transformations have been studied for over a hundred years. Current understanding, at least on the classical level, of several nonlinear physical systems and critical phenomena is based on the invariance under global scaling transformations $x_\mu \to b x_\mu$ (e.g. [97, 98]).

Hořava's papers devoted to gravity models are characterized by an specific anisotropic scaling between space and time:

$$t \to b^z\, t\,, \quad \vec{x} \to b\,\vec{x}\,; \qquad b > 0\,, \ \ z \in \mathrm{Re}\,. \tag{4.1}$$

Note that spatial isotropy is still assumed to be kept; the degree of anisotropy between space and time is measured by z, which plays the role of a dynamical critical exponent.

Noether's theorem [99] states that every differentiable symmetry of the action of a physical system has a corresponding conservation law, it associates to a Lagrangian symmetry the conserved current whose total differential vanishes on-shell. In this chapter, we use Noether's theorem for general classical fields and for general symmetry transformations in actions that depend on higher order spatial derivatives to find the correspondent current densities that lead to a conservation law.

Then, we analyze several important applications: the scalar field, an electrodynamics compatible with the transformations (4.1) for $z = 2$ as in [100, 101], and higher derivative Chern-Simons extensions (see [102]). For these cases we obtain the canonical energy-momentum tensor and the conserved current associated with scale invariance. Conclusions are presented in the last section.

4.2. Noether's Theorem

Let us consider the action $S[\Phi]$, for a collection of arbitrary fields $\Phi = \{\phi_a\}$, in D dimensions, where a represents space-time or internal indices:

$$S[\Phi] = \int d^D x\, \mathcal{L}\left(\Phi,\, \partial_\mu\Phi,\, \partial_{i_{1,2}}\Phi,\, \partial_{i_{1,3}}\Phi,\, \ldots\, ,\partial_{i_{1,n}}\Phi\right)\,, \tag{4.2}$$

where $\frac{\partial^n \Phi(x)}{\partial x_{i_1} \ldots \partial x_{i_n}} \equiv \partial_{i_{1,n}}\Phi\,.$ Throughout this paper $n = 0, 1, 2, \ldots$ indicates the derivative order, $i_n = 1, 2, \ldots, D-1$ stand for spatial indices, while $\mu = 0,\ 1,\ 2,$ $\ldots,\ D-1$ denotes space-time indices.

Under arbitrary infinitesimal stationary variations $\delta\Phi$, the equations of motion are

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} + \partial_{i_{1,2}} \frac{\partial\mathcal{L}}{\partial(\partial_{i_{1,2}}\Phi)} - \cdots + (-1)^n \partial_{i_{1,n}} \frac{\partial\mathcal{L}}{\partial(\partial_{i_{1,n}}\Phi)} = 0\,, \tag{4.3}$$

note that they contain the usual first two terms, plus *higher order spatial derivatives*.

Now, the addition of a fourth divergence

$$\partial_\mu \Lambda^\mu = \partial_\mu \, \Theta^\mu(\Phi) + \partial_m \, \Xi^m \left(\Phi,\, \partial_{i_1}\Phi,\, \partial_{i_1\, i_2}\Phi,\, \ldots,\, \partial_{i_1 \ldots i_{n-1}}\Phi\right) \tag{4.4}$$

to the Lagrangian, does not modify the Euler-Lagrange equations.

For convenience we introduce the notation

$$\delta^\mu_{i_1} \mathcal{D}^{i_1} = \delta^\mu_{i_1} \left(D^{i_1}_2 + \cdots + D^{i_1}_n \right) , \quad \delta^\mu_{i_1} \mathcal{W}^{i_1} = \delta^\mu_{i_1} \left(W^{i_1}_2 + \cdots + W^{i_1}_n \right) , \tag{4.5}$$

where

$$\delta^\mu_{i_1} D^{i_1}_n \equiv \delta^\mu_{(i_1} \left\{ \sum_{k=0}^{(n-1)} (-1)^k \binom{n}{k} \partial_{n-1-k)} \left[\left(\partial_{k)} \frac{\partial \mathcal{L}}{\partial(\partial_{i_{1,n}}\Phi)} \right) \delta\Phi \right] \right\} ,$$

$$\delta^\mu_{i_1} W^{i_1}_n \equiv \delta^\mu_{(i_1} \left\{ \sum_{k=0}^{(n-1)} (-1)^k \binom{n}{k} \partial_{n-1-k)} \left[\left(\partial_{k)} \frac{\partial \mathcal{L}}{\partial(\partial_{i_{1,n}}\Phi)} \right) \partial_\nu \Phi \, \delta x^\nu \right] \right\} ,$$

here the lower index on $D^{i_1}_n$ and $W^{i_1}_n$ indicates the order of the term, the upper index i_1 should be contracted with the derivative index and on the right side of the expression all the indices should be contracted. The parenthesis on the subindexes as usual, denotes that these should be symmetric.

On the other hand, if the action integral (4.2) remains invariant up to a fourth divergence $\delta S = \int d^D x \, \partial_\mu \delta\Omega^\mu$ [103] when the coordinates are subject to an infinitesimal transformation of the kind $x'_\mu = x_\mu + \delta x_\mu$, $\Phi'(x') = \Phi(x) + \delta\Phi(x)$, then the conserved Noether currents ($\partial_\mu \mathcal{J}^\mu = 0$) are given by

$$\mathcal{J}^\mu = \left(\delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi \right) \delta x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi - \delta\Omega^\mu - \delta^\mu_{i_1} \left(\mathcal{W}^{i_1} - \mathcal{D}^{i_1} \right) , \tag{4.6}$$

and the corresponding conserved charge $\mathcal{Q}$ is

$$\mathcal{Q} = \int_{V_t} d^3x \, \mathcal{J}^0 = \int_{V_t} d^3x \left[\left(\delta^0_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} \partial_\nu \Phi \right) \delta x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} \delta\Phi - \delta\Omega^0 \right] , \tag{4.7}$$

Expresion (4.7) is exactly the same conserved charge as in the case of a Lagrangian depending only on the first space-time derivative

$$\mathcal{L}(x) = \mathcal{L}\left(\Phi,\, \partial_\mu \Phi\right) . \tag{4.8}$$

If the theory is invariant under translations ($\delta x^\nu = a^\mu$), the canonical energy-momentum tensor is given by

$$\mathcal{T}^\mu{}_\nu = \delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta^\mu_{(i} \frac{\partial \mathcal{L}}{\partial(\partial_i \partial_j \Phi)} \partial_{j)} \partial_\nu \Phi + \delta^\mu_{(i} \partial_{j)} \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \partial_j \Phi)} \right) \partial_\nu \Phi + \dots \tag{4.9}$$

4.3. Examples

4.3.1. Scalar Field

For a massless scalar field ϕ in D dimensions, we will consider the time-related term:

$$S_t[\phi] = \int d^D x \left(\partial_0 \phi \, \partial^0 \phi \right) . \tag{4.10}$$

We want to find the conditions in order to make 4.10 invariant under a scale transformation of the form

$$t' = b^z t \,, \qquad x'_i = b x_i \,, \qquad \phi' = b^{-\Delta} \phi \,, \tag{4.11}$$

where $b > 0$, $z, \Delta \in \mathrm{Re}$ and $i = 1, \dots, D-1$. We find that

$$S'_t[\phi] = b^{z+D-1} \int d^D x \, b^{-2(z+\Delta)} \left(\partial_0 \phi \partial^0 \phi \right) , \tag{4.12}$$

is scale invariant if

$$\Delta = \frac{D-1-z}{2} . \tag{4.13}$$

We can also add an interaction term ϕ^n invariant under the scaling symmetry 4.11 when

$$n = \frac{D-1+z}{\Delta} = 2 \left(\frac{D-1+z}{D-1-z} \right) . \tag{4.14}$$

If we wish to include a term with second derivatives of the field $\left(\partial_i \partial^i \phi\right)^m$, invariance under the scale rules leads to

$$m = \frac{D-1+z}{2+\Delta} . \tag{4.15}$$

Let us consider the four dimensional case (see Table 4.1). The usual isotropic scaling is given when $z = 1$. In the case when the dynamical critical exponent $z = 2$ the only interaction term that will preserve the scaling symmetry will be

ϕ^{10} and a quadratic term if we add second order spatial derivatives. When $z = 3$ we cannot add an interaction term since in this case $n \to \infty$, but we can add to the action a cubic term of second order derivatives that remains invariant under the scaling.

z	**Δ**		**n**		**m**	
	3D	4D	3D	4D	3D	4D
1	$\frac{1}{2}$	1	6	4	$\frac{6}{5}$	$\frac{4}{3}$
2	0	$\frac{1}{2}$	-	10	2	2
3	$-\frac{1}{2}$	0	-10	-	$\frac{10}{3}$	3

TABLE 4.1
Anisotropic scaling terms for a three-dimensional and four-dimensional scalar field with different dynamical critical exponents z.

Now we will find the conserved Noether currents and charges in the four dimensional case with $z = 2$

$$\mathcal{L} = \partial_0\phi\, \partial^0\phi + \partial_{ij}\phi\, \partial^{ij}\phi\,, \quad i,j = 1,2,3 \quad \Rightarrow \quad -\partial_0\partial^0\phi + \partial_{ij}\partial^{ij}\phi = 0\,. \quad (4.16)$$

The scalar field theory obtained when $i = j$ gives an action that is a sum of a *kinetic term* involving time derivatives, and a *potential term* which is of a special form: It can be derived from a variational principle

$$\frac{1}{4}\left(\Delta\phi(x)\right)^2 = \left(\frac{1}{2}\frac{\delta W}{\delta\phi(x)}\right) \quad (4.17)$$

where $W = \frac{1}{2}\int d^{D-1}x\,(\partial_i\phi\partial_i\phi)$. This resulting action is a prototype of a class of models introduced and studied in the context of tri-critical phenomena in condensed matter physics by Lifshitz [94] in 1941, and is consequently referred to in the literature as the "Lifshitz scalar" field theory [95, 104]. In the context originally studied by Lifshitz, t is Wick rotated to become one of the spatial dimensions, and the Lifshitz scalar then describes the tricritical point connecting the phases with a zero, homogeneous or spatially modulated condensate of ϕ. In [105] Hořava introduced a new class of nonrelativistic gravity models, characterized by anisotropic scaling between space and time with a nontrivial value of the dynamical critical exponent $z = 2$. This anisotropy leads to a change of the critical dimension of the system to $2 + 1$, and makes the theory suitable for the worldvolume of a membrane where it can be coupled to quantum critical matter with $z = 2$, described in the simplest case by Lifshitz scalars with dynamical critical exponent $z = 2$.

Using Noether's theorem as shown before, we obtain the conserved currents and charges of these kind of field theories. The canonical energy-momentum tensor (4.9) is (using $\nabla^2 \equiv \partial_k \partial^k$)

$$\mathcal{T}^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}\mathcal{L} - 2\,\delta^{\mu}_{0}\,\partial^0\phi\,\partial_\nu\phi + 2\,\delta^{\mu}_{(i}\left[-\partial^{i\,j}\phi\,\partial_{j)}\partial_\nu\phi + (\partial^{i}\nabla^2\phi)\,\partial_\nu\phi\right]. \tag{4.18}$$

This Lagrangian is also invariant under scale transformations

$$\delta t = 2b\,t\,, \quad \delta\vec{x} = b\,\vec{x}\,, \quad \delta\varphi = -\frac{1}{2}b\,\varphi\,, \quad b > 0\,, \tag{4.19}$$

and its corresponding current can be written as [106]

$$\mathcal{J}^\mu = 2\,x_0\,\mathcal{T}^{\mu 0} + x_s\,\mathcal{T}^{\mu s} + \partial_\lambda\,\sigma^{\mu\lambda}\,, \tag{4.20}$$

where

$$\sigma^{0\lambda} = -\frac{1}{2}g^{0\lambda}\varphi^2\,, \qquad \sigma^{i\lambda} = g^{i\lambda}\left(2\varphi\nabla\varphi - \nabla\varphi^2\right) + g^{im}g^{\lambda n}\varphi\,\partial_{mn}\varphi\,. \tag{4.21}$$

Defining a new tensor

$$\begin{aligned}\Theta^{\mu\nu} = \mathcal{T}^{\mu\nu} + \frac{1}{2}\Bigg[\frac{1}{2}\left(g^{\mu 0}g^{\nu 0}\,\partial_{\lambda\rho}\,X^{\lambda\rho}{}_{00} + g^{\mu i}g^{\nu 0}\,\partial_{\lambda\rho}\,X^{\lambda\rho}{}_{i0}\right)& \\ + \left(g^{\mu i}g^{\nu j}\,\partial_{\lambda\rho}\,X^{\lambda\rho}{}_{ij} + g^{\mu 0}g^{\nu j}\,\partial_{\lambda\rho}\,X^{\lambda\rho}{}_{0j}\right)\Bigg]&\,, \\ X^{\lambda\rho\mu\nu} = g^{\lambda\rho}\,\sigma^{\mu\nu} - g^{\lambda\mu}\,\sigma^{\rho\nu} - g^{\lambda\nu}\,\sigma^{\mu\rho} + g^{\mu\nu}\,\sigma^{\lambda\rho} + \frac{1}{3}\left(g^{\lambda\mu}g^{\rho\nu} - g^{\lambda\rho}g^{\mu\nu}\right)\sigma^{\alpha}_{\ \alpha}&\,,\end{aligned} \tag{4.22}$$

which has the following properties

$$\partial_\mu\Theta^{\mu\nu} = 0\,, \qquad 2\,\Theta^{0}{}_{0} + \Theta^{m}{}_{m} = 2\,\mathcal{T}^{0}_{\ 0} + \mathcal{T}^{m}_{\ m} + \frac{1}{2}\,\partial_{\lambda\rho}\,X^{\lambda\rho\mu}{}_{\mu}\,, \tag{4.23}$$

the conserved currents take a simpler form

$$\mathcal{J}^\mu = 2\,x_0\,\Theta^{\mu 0} + x_s\,\Theta^{\mu s}\,. \tag{4.24}$$

4.3.2. Electrodynamics

Let us consider the four dimensional electrodynamics Lagrangian with $z = 2$ [101]:

$$\mathcal{L} = \alpha\, F_{0i}F^{0i} + \beta\,(\partial_k F_{ij})\,(\partial^k F^{ij})\,, \quad \Rightarrow \quad \begin{array}{c} \partial_i F^{0i} = 0 \\ \alpha\,\partial_0 F^{0i} - 2\,\beta\nabla^2\partial_j F^{ji} = 0 \end{array}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,;\ \ \alpha,\beta \in \mathrm{Re}\,;\ \ i,j,k = 1,2,3\,;$ and $\mu,\nu = 0,1,2,3$, is invariant under translations ($\delta x^\mu = a^\mu$, $\delta A_\mu = 0$). The canonical tensor is:

$$\mathcal{T}^0{}_\nu = \delta^0_\nu\mathcal{L} - 2\,\alpha\,F^{0m}\partial_\nu A_m\,, \quad \mathcal{T}^k_{\ \nu} = \delta^k_\nu\mathcal{L} - 2\,\alpha\,F^{k0}\partial_\nu A_0 - \mathcal{W}^k_\nu\,, \tag{4.25}$$

where $\mathcal{W}^k_\nu = 4\,\beta\,\left(\partial^k F^{mn}\,\partial_{m\nu}A_n - \partial_\nu A_n\,\partial^k_m F^{mn}\right)$.

For the transformation of the fields

$$A'_i = b^{-1/2}\,A_i\,, \quad A'_0 = b^{-3/2}\,A_0\,, \tag{4.26}$$

the Lagrangian remains invariant and the corresponding conserved currents are

$$\mathcal{J}^0 = 2\,x^0\,\mathcal{T}^0{}_0 + x^s\,\mathcal{T}^0{}_s - \alpha\,F^{0i}A_i\,, \tag{4.27}$$

$$\mathcal{J}^p = 2\,x^0\,\mathcal{T}^p{}_0 + x^s\,\mathcal{T}^p{}_s - 3\,\alpha\,F^{p0}A_0 - 2\,\beta\,\delta^p_{(m}\left(3\,\partial^{(m}F^{n)l}\,\partial_{n)}A_l - \partial_{n)}\partial^{(m}F^{n)l}\,A_l\right). \tag{4.28}$$

4.3.3. Higher Derivative Chern-Simons Extensions

In this example we will consider three dimensions and an arbitrary dynamical critical exponent z. The action [102] will take the form

$$S_{ECS} = (2m)^{-1}\int d^3x\;\epsilon^{\alpha\beta\gamma}\,\nabla^2 A_\alpha\,\partial_\beta A_\gamma\,, \quad \Rightarrow \quad \epsilon^{\alpha\beta\gamma}\,\nabla^2 F_{\beta\gamma} = 0\,, \tag{4.29}$$

where $F_{\beta\gamma} = \partial_\beta A_\gamma - \partial_\gamma A_\beta$. The theory is invariant under scaling transformations when

$$A'_0 = b^{1-z}A_0\,, \quad A'_i = A_i\,. \tag{4.30}$$

Moreover, it is also gauge invariant $(\delta x^\nu = 0\,,\ \delta A_\alpha = \partial_\alpha\Lambda)\,$, up to a 4th divergence

$$\mathcal{J}^\mu = (2m)^{-1}\left[-\epsilon^{\mu\alpha\nu}\,\nabla^2\,A_\alpha\partial_\nu\Lambda + \delta^\mu_i\epsilon^{\beta\alpha\nu}\left(\partial_\beta A_\alpha\,\partial^i\partial_\nu\Lambda - \partial^i\partial_\beta A_\alpha\,\partial_\nu\Lambda\right)\right] - \delta\Omega^\mu$$

where $\delta\Omega^\mu = (2m)^{-1}\,\epsilon^{\mu\alpha\nu}\,A_\alpha\,\nabla^2\,\partial_\nu\Lambda\,.$

4.4. Discussion

The principal goal of this chapter was to study an anisotropic theory which can be applied to several systems where the Lorentz symmetry is broken. Noether's theorem for actions with an anisotropic dependence on time and space was applied. The general form of the conserved currents and charges for arbitrary symmetry transformations was obtained. In particular, for the case of translations, the canonical energy-momentum tensor was presented.

An interesting result was that the usual Noether's charge (when the action depends only on the first space-time derivatives) is not modified even if the Lagrangian depends on additional higher order spatial derivatives.

We made the symmetry analysis for three anisotropic systems, for the free scalar field and electrodynamics we studied translation and scaling invariance and for the Chern-Simons theory scaling and gauge transformations. With the method presented here, it is possible to give the explicit form of the conserved currents.

5

Path Integrals and Polymer Quantum Mechanics

5.1. Introduction

Loop quantum gravity is a theory that attempts to merge quantum mechanics and general relativity. The theory is based on quantum geometry, the essential discreteness of which permeates all constructions and results. The fundamental excitations are 1-dimensional and polymer-like.

Polymer quantum mechanics is an alternative formalism of quantum mechanics that supports the idea of the existence of a minimal length scale μ, known as the polymer length, which enters into the Hamiltonian of the system to deform its functional form into a new one called the polymer Hamiltonian. Since no discreteness of space has been detected, this fundamental length must be much smaller than the natural length parameters associated to a mechanical system. In contrast to the standard Schrödinger picture, this representation gives an ultraviolet cutoff due to the existence of the polymer length as a possible minimum length scale for the system under consideration. [26, 107–110]

Polymer quantization was re-introduced by Ashtekar, et. al. [26] in the context of Loop quantum cosmology to construct a scheme not equivalent to Schrödinger's representation. This formalism has been successfully applied to the quantization of several cosmologies, avoiding the big bang singularity, but also for quantum theories of one degree of freedom like the free particle, a particle in a box [111], the harmonic oscillator [112], as well as some other potentials. For such quantization the first step is to build a representation of the Weyl algebra on a polymer Hilbert space. The next step is to implement a Hamiltonian constraint on this space which can not be done directly because of the discreetness of space, so the procedure introduces a length scale (apart from Planck's constant) into the quantum theory.

It is interesting to study the polymer version of different simple toy systems in order to learn about some of the properties of the loop quantization such as the quantum constraints, the structure of the physical Hilbert space, the path integral, the quantum solutions, the effective action and properties of the observables.

This chapter is organized as follows. In section 5.2 we summarily review the polymer Quantum Mechanics scheme introducing the Weyl–Heisenberg algebra.

Section 5.3 contains the construction of the path integral for the polymer quantization. This quantization procedure has the advantage of being quite easy generalized to the polymer case and furthermore this scheme provides us with an effective action that we can use to understand more deeply the non-perturbative aspects of a dynamical system at classical and quantum levels. Besides, it is possible to analyze carefully the measure of the integral, and we found the interesting result that the measure for the spatial momenta is modified with a final bounded integral.

We study the polymer version of several systems: the non-relativistic free particle, the relativistic particle, the harmonic oscillator and cosmology. We found some interesting properties in the phase space of this systems such as non trivial restrictions over the energy and the phase space variables.

5.2. Polymer Quantum Mechanics

In this section we are going to describe the Weyl-Heisenberg algebra which we are going to use to construct a new representation known as polymer representation (which is not unitarily equivalent to Schödinger's representation) [26, 111].

In Schrödinger's representation the Hilbert space $\mathcal{H}_{Sch}$ is the space of the quadratically integrable functions ($\mathcal{H}_{Sch} = L^2(\mathrm{Re}, \mathrm{d}x)$) and the action of this operators is given by

$$\hat{U}_\lambda\, \psi(x) = e^{i\lambda\, x}\psi(x) \qquad \text{and} \qquad \hat{V}_\mu\, \psi(x) = \psi(x+\mu)\,, \tag{5.1}$$

for all $\psi \in \mathcal{H}_{Sch}$. The unitary groups $\hat{U}_\lambda$ and $\hat{V}_\mu$ (which depend only in one parameter) are weakly continuous in λ and μ. This property assures that the autoadjoint operators $\hat{x}$ and $\hat{p}$ exist in $\mathcal{H}_{Sch}$ such that

$$\hat{U}_\lambda := e^{i\lambda\, \hat{x}} \qquad \text{and} \qquad \hat{V}_\mu := e^{i\frac{\mu}{\hbar}\hat{p}}\,. \tag{5.2}$$

Now, the Stone–von Neumann theorem stablished that any irreducible representation with commutation relations $\hat{U}_\lambda\, \hat{V}_\mu = e^{i\lambda\mu}\, \hat{V}_\mu\, \hat{U}_\lambda$ is unitarily equivalent to

Schrödinger's representation (5.1), so all the measures dx that can be obtain from the action of the operators $\hat{U}$ and $\hat{V}$ are equivalent [113].

The polymer representation of the Weyl–Heisenberg algebra **is not** equivalent to Schrödinger's representation because it violates one of the assumptions of the Stone–von Neumann theorem, in the new representation, the operator $\hat{V}_\mu$ is not weakly continuos in μ due to the discrete structure of the space, therefore the autoadjoint operator $\hat{p}$ which satisfies $\hat{V}_\mu := e^{\frac{i}{\hbar}\mu\hat{p}}$ does not exist (in other words, the operator $\hat{V}_\mu$ is not related with an Hermitian operator as an infinitesimal generator). This is what we expected from the absence of an spatial continuous, because if the Riemannian geometry is discrete we do not expect that the operator $\hat{p} = -i\hbar\frac{d}{dx}$ exists in a fundamental level.

The central difference between Schrödinger and polymer quantization is the choice of a non-separable Hilbert space $\mathcal{H}_{Poly}$. A Hilbert space with an uncountable orthonormal basis characterized by kets $|x_j\rangle$ labelled by real numbers x_j. In this representation $\{|x_j\rangle\}$ is an orthonormal base, and it's internal product is defined as

$$\langle x_i|x_j\rangle = \frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp_i\, e^{\frac{i}{\hbar}\,p_i\,(x_i - x_j)} = \delta_{x_i,x_j}\,, \tag{5.3}$$

note that in the right side we have a Kronecker delta instead of a Dirac delta. The action of the basic operators $\hat{U}$ and $\hat{V}$ is given by

$$\hat{U}_\lambda\,|x_j\rangle = e^{i\lambda\,x_j}|x_j\rangle \qquad \text{and} \qquad \hat{V}_\mu\,|x_j\rangle = |x_j - \mu\rangle\,, \tag{5.4}$$

in this case $\hat{U}_\lambda$ is weakly continuous in λ and therefore the autoadjoint operator $\hat{x}$ exists in $\mathcal{H}_{Poly}$ with $\hat{U}_\lambda = e^{i\lambda\,\hat{x}}$. This action can be written as $\hat{x}|x_j\rangle = x_j|x_j\rangle$.

However there is an important difference with respect with Schrödinger's representation, now the eigenkets of $\hat{x}$ are *normalizable* and also the elements of the Hilbert space. In this sense the eigenvalues are 'discrete'.

On the other hand, although the family $\hat{V}_\mu$ has an unitary group $\mathcal{H}_{Poly}$, the operator is not weakly continuous in μ. This is because no matter how small is μ, $|x_j\rangle$ and $\hat{V}_\mu\,|x_j\rangle$ are always orthonormal

$$\lim_{\mu\mapsto 0}\langle x_j|\hat{V}_\mu\,|x_j\rangle = 0\,. \tag{5.5}$$

The commutation relation between these operators is

$$\left[\hat{x}, \hat{V}_\mu\right] = -\mu\hat{V}_\mu\,. \tag{5.6}$$

To define the analogue of the Schrödinger momentum operator we expand $e^{-\frac{i}{\hbar}\mu p}$ (since $\frac{\mu}{\hbar}p$ is small)

$$p = \frac{\hbar}{\mu}\frac{e^{\frac{i}{\hbar}\mu p} - e^{-\frac{i}{\hbar}\mu p}}{2i} + \mathcal{O}\left(\frac{\mu^2}{\hbar^2}p^2\right) , \tag{5.7}$$

so we define the operator momentum in a similar way choosing μ small and as a fix scale and we define the operator in $\mathcal{H}_{Poly}$ as

$$\hat{p} \approx \frac{\hbar}{\mu}\frac{\hat{V}_\mu - \hat{V}_\mu^\dagger}{2i} . \tag{5.8}$$

Therefore, the square of the momentum operator will be simply

$$\hat{p}_x^2 \approx -\frac{\hbar^2}{4\mu^2}\left(\hat{V}_\mu^2 + \hat{V}_\mu^{\dagger\,2} - \hat{V}_\mu\,\hat{V}_\mu^\dagger - \hat{V}_\mu^\dagger\,\hat{V}_\mu\right) , \tag{5.9}$$

and since the action of the operator $\hat{V}_\mu$ over a spatial state $|x_n\rangle$ produces a shift on it as shown in eq. (5.4), the action of $\hat{p}_x^2$ will be

$$\hat{p}_x^2\,|x_n\rangle = \frac{\hbar^2}{4\mu^2}\left(2|x_n\rangle - |x_n - 2\mu\rangle - |x_n + 2\mu\rangle\right) . \tag{5.10}$$

5.3. Path Integrals for Polymer Quantization

In this section we will imitate Feynman's formalism using the evolution operator $e^{\frac{i}{\hbar}\lambda\hat{\mathcal{G}}}$ and integrating over λ in the context of the polymer quantization in order to find the matrix elements of the evolution operator.

Let us remember that in the polymer construction the solutions of the constraint equation are decomposed in sectors in which the wave functions are only defined in grids of the spatial variable x (in other words x has only discrete values), so we need to impose that $x = n\mu$ where n is an integer over the grid.

Note that in the usual systems the variables x and p are symmetric (because both of them are quadratic) so one can think alternatively on x or p as the configuration variable. Here, we will consider x as the momentum variable of the particle over a circle, while p is in the interval $(0, \pi\frac{\hbar}{\mu}))$.

As in section 2.5, we decompose the evolution in N intervals of length $\epsilon = 1/N$

$$e^{\frac{i}{\hbar}\lambda\hat{\mathcal{G}}} = \prod_{n=1}^{N} e^{\frac{i}{\hbar}\epsilon\lambda\,\hat{\mathcal{G}}_n}, \tag{5.11}$$

but this time, the complete base contains a discrete part due to the spatial variable x

$$\mathbf{1} = \int_{-\infty}^{\infty} dt_n \sum_{x_n=-\infty}^{\infty} |x_n, t_n\rangle \langle x_n, t_n| \,, \tag{5.12}$$

and the Kernel will have the form

$$G(x_f, t_f; x_i, t_i) = \int_{-\infty}^{\infty} d\lambda \prod_{n=1}^{N-1} \left[\int_{-\infty}^{\infty} dt_n\right] \prod_{n=1}^{N-1} \left[\sum_{x_n=-\infty}^{\infty}\right] \prod_{n=1}^{N} \langle x_n, t_n | e^{\frac{i}{\hbar}\epsilon\,\lambda\,\hat{\mathcal{G}}_n} | x_{n-1}, t_{n-1}\rangle \,. \tag{5.13}$$

In the next sections we will study the polymer cases of the non-relativistic and relativistic particle, the harmonic oscillator and some cosmologies.

5.3.1. Polymer non-relativistic free particle

The action for a non-relativistic free particle of mass m is

$$\mathcal{S} = \int d\tau \left[p_t\, \dot{t} + p_x\, \dot{x} - \lambda \left(p_t + \frac{1}{2m} p_x^2\right)\right] \,, \tag{5.14}$$

with the constraint

$$\hat{\mathcal{G}} = \hat{p}_t + \frac{1}{2m}\hat{p}_x^2 \,. \tag{5.15}$$

It is possible to treat independently the spatial and the temporal coordinates. For the momentum $\hat{p}_t$ the propagator will be

$$\langle t_n | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{p}_t} | t_{n-1}\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp_{t_n} \exp\left[-\frac{i}{\hbar}\epsilon\lambda p_{t_n} + \frac{i}{\hbar} p_{t_n}(t_n - t_{n-1})\right] \,. \tag{5.16}$$

To compute the spatial propagator we need the definition of the delta function (5.3) and to apply the action of the square of the spatial momentum operator over a spatial state (5.10). Using these results we obtain

$$\langle x_n | e^{-\frac{i}{\hbar}\frac{\epsilon\lambda}{2m}\hat{p}_x^2} | x_{n-1}\rangle = \frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp_{x_n} \exp\left[\frac{i}{\hbar} p_{x_n}(x_n - x_{n-1}) - \frac{i}{\hbar}\frac{\epsilon\,\lambda\hbar^2}{2m\mu^2}\sin^2\frac{\mu\, p_{x_n}}{\hbar}\right] \,, \tag{5.17}$$

therefore the complete propagator will be

$$\langle x_f,\, t_f | e^{-\frac{i}{\hbar}\epsilon\,\lambda\,\hat{\mathcal{G}}} | x_i,\, t_i\rangle = \delta\left(\lambda - (t_N - t_0)\right) \prod_{n=1}^{N} \left[\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp_{x_n}\right] \prod_{n=1}^{N-1}\left[\sum_{x_n=-\infty}^{\infty}\right] \exp\left[\frac{i}{\hbar}\sum_{n=1}^{N-1} x_n(p_{x_n} - p_{x_{n+1}}) + \mathcal{H}\right] \,. \tag{5.18}$$

Since the temporal part is the same as in the usual case, we obtained a Dirac delta directly from it of the form $\delta\left(\lambda-(t_N-t_0)\right)$. Besides, we defined

$$\mathcal{H}=\frac{i}{\hbar}\left[-x_0\,p_{x_1}+x_N p_{x_N}-\frac{\epsilon\lambda\hbar^2}{2m\mu^2}\sum_{n=1}^{N}\sin^2\frac{\mu\,p_{x_n}}{\hbar}\right]. \tag{5.19}$$

Applying Poisson's summation formula, we can transform the coordinate sum to a proper path integral adding an infinite sum with guarantees the cyclic invariance on the momentum (see [114])

$$\sum_{x_n=-\infty}^{\infty}e^{\frac{i}{\hbar}x_n(p_{x_n}-p_{x_{n+1}})}=\sum_{l_n=-\infty}^{\infty}\int_{-\infty}^{\infty}dx_n e^{\frac{i}{\hbar}\left[x_n(p_{x_n}-p_{x_{n+1}})+2\pi x_n l_n\right]}, \tag{5.20}$$

so the propagator becomes

$$\langle x_f,\,t_f|e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}}|x_i,\,t_i\rangle=\delta\left(\lambda-(t_N-t_0)\right)\prod_{n=1}^{N}\left[\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}}dp_{x_n}\right]$$
$$\prod_{n=1}^{N-1}\left[\sum_{l_n=-\infty}^{\infty}\int_{-\infty}^{\infty}dx_n\right]exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N-1}\left[x_n(p_{x_n}-p_{x_{n+1}})+2\pi x_n l_n\right]+\mathcal{H}\right\}. \tag{5.21}$$

In the last expression the sums over l_n can be absorbed into the variables p_n by extending their range of integration from $(-\pi\hbar/\mu,\pi\hbar/\mu)$ to $(-\infty,\infty)$. Only in the last integral $\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}}dp_{x_N}$ this is impossible and we arrive at

$$\langle x_f,\,t_f|e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}}|x_i,\,t_i\rangle=\delta\left(\lambda-(t_N-t_0)\right)\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}}dp_{x_N}\prod_{n=1}^{N-1}\left[\frac{\mu}{2\pi\hbar}\int_{-\infty}^{\infty}dp_{x_n}\right]$$
$$\prod_{n=1}^{N-1}\left[\int_{-\infty}^{\infty}dx_n\right]\exp\left\{\frac{i}{\hbar}\sum_{n=1}^{N-1}x_n(p_{x_n}-p_{x_{n+1}})+\mathcal{H}\right\}. \tag{5.22}$$

We can simplify the last expression by obtaining $N-1$ Dirac deltas on the momentum to obtain

$$\langle x_f,\,t_f|e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}}|x_i,\,t_i\rangle=\delta\left(\lambda-(t_N-t_0)\right)$$
$$\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}}dp_{x_N}\exp\left\{-\frac{i}{\hbar}\left[\frac{\epsilon N\lambda\hbar^2}{4m\mu^2}\left(1-\cos\frac{2\mu\,p_{x_n}}{\hbar}\right)-p_{x_N}(x_N-x_0)\right]\right\}, \tag{5.23}$$

and by making the following definitions

$$z = \frac{\epsilon N \lambda \hbar^2}{4m\mu^2}, \qquad \zeta = \frac{2\mu\, p_{x_n}}{\hbar}, \qquad x_N - x_0 = 2\Delta\mu\,, \tag{5.24}$$

we obtain the integral form of a Bessel function $J_\Delta(z)$

$$\begin{aligned} \langle x_f,\, t_f | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}} | x_i,\, t_i\rangle &= \delta\left(\lambda - (t_N - t_0)\right) \frac{1}{2\pi} e^{-iz} \int_0^{2\pi} d\zeta e^{i(z\cos\zeta + \Delta\zeta)} \\ &= \delta\left(\lambda - (t_N - t_0)\right) i^\Delta J_\Delta(z) e^{-iz}\,. \end{aligned} \tag{5.25}$$

To obtain the Kernel of the non-relativistic free particle in the polymer case, we just have to integrate over λ

$$G(x_f,\, t_f;\, x_i,\, t_i) = i^{\frac{1}{2\mu}(x_N - x_0)} J_{\frac{1}{2\mu}(x_N - x_0)}\left(\frac{\hbar^2}{4m\mu^2}(t_N - t_0)\right) e^{-\frac{\hbar^2}{4m\mu^2}(t_n - t_0)}\,, \tag{5.26}$$

where we used $\epsilon N = 1$.

This is the result obtained by the method of images on [115]. It is necessary to stress that the residual bounded integral in the measure for the spatial momenta in (5.23) is needed in order to obtain the right result.

Equations of motion

The last step in the construction of the path integral consists in taking the limit $N \to \infty$. Generally in this limit we have to make the following substitutions $\epsilon \sum_{n=0}^N \to \int d\tau$, $(t_n - t_{n-1})/\epsilon \to dt/d\tau$ and $(x_n - x_{n-1})/\epsilon \to dx/d\tau$ to obtain a continuos action (over τ). Therefore, the Kernel for the free particle in the polymer case is

$$\begin{aligned} G(x_f,\, t_f;\, x_i,\, t_i) &:= \int d\lambda\, \langle x_f,\, t_f | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}} | x_i,\, t_i\rangle \\ &= \int d\lambda \int [\mathcal{D}x(\tau)]\, [\mathcal{D}t(\tau)]\, [\mathcal{D}p_t(\tau)]\, [\mathcal{D}p_x(\tau)]\, e^{\frac{i}{\hbar}\mathcal{S}}\,, \end{aligned} \tag{5.27}$$

where $\mathcal{S}$ represents the effective polymer action, in this case is given by

$$\mathcal{S} = \int_0^1 d\tau \left[p_t \dot{t} + p_x\, \dot{x} - \lambda\left(p_t + \frac{\hbar^2}{2m\mu^2}\sin^2\frac{\mu\, p_x}{\hbar}\right)\right]. \tag{5.28}$$

The equations of motion of the polymer action for the free particle given in (5.28) are

$$\dot{p}_t = 0, \qquad \dot{p}_x = 0\,, \tag{5.29a}$$

$$\dot{t} = \lambda, \qquad p_x = \frac{\hbar}{2\mu}\arcsin\left(\frac{2m\mu\dot{x}}{\lambda\hbar}\right)\,. \tag{5.29b}$$

Now we can use this equations of motion to rewrite the effective action (using the redefinition $x' = \dot{x}/\lambda$)

$$\mathcal{S} = \int_{t_1}^{t_2} dt\left[\frac{\hbar}{2\mu}x'\arcsin\left(\frac{2m\mu}{\hbar}x'\right) - \frac{\hbar^2}{4m\mu^2}\left(1-\sqrt{1-\frac{4\,m^2\mu^2}{\hbar^2}x'^2}\right)\right]\,, \tag{5.30}$$

and the Hamiltonian

$$H = \frac{\partial\mathcal{L}}{\partial x'}x' - \mathcal{L} = \frac{\hbar^2}{4m\mu^2}\left(1-\sqrt{1-\frac{4\,m^2\mu^2}{\hbar^2}x'^2}\right) = \frac{\hbar^2}{2m\mu^2}\sin^2\left(\frac{\mu p_x}{\hbar}\right)\,, \tag{5.31}$$

in the last step we used that $p = \frac{\partial\mathcal{L}}{\partial x'} = \frac{\hbar}{2\mu}\arcsin(\frac{2m\mu}{\hbar}x')$. Since the energy H is constant, we write the velocity x' in terms of it

$$x'^2 = \frac{2H}{m}\left(1-\frac{2m\mu^2}{\hbar^2}H\right)\,. \tag{5.32}$$

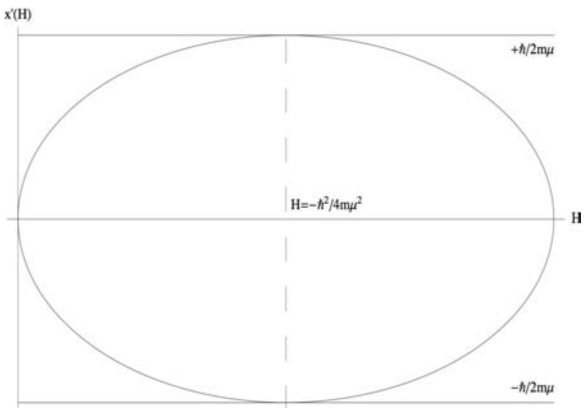

FIGURE 5.1
Velocity graph. Note that both the velocity and the energy are bounded.

In order that the velocity stay well defined, lets note that the values of the energy are restricted by

$$0 < H < \frac{\hbar^2}{2m\mu^2} \,, \tag{5.33}$$

in other words, the velocity of the free particle has a limit

$$-\frac{\hbar}{2m\mu} < x'(H) < \frac{\hbar}{2m\mu} \,, \tag{5.34}$$

when the energy takes the values $H = 0$ or $H = \frac{\hbar^2}{2m\mu^2}$ the velocity of the particle is zero; on the other hand when the energy is equal to half of it's maximum value $H = \frac{\hbar^2}{4m\mu^2}$ the velocity is maximum as we can see in the fig. 5.1.

We can integrate the velocity eq. (5.32) to obtain the solutions for the system

$$t_{poly}(x) = \pm \frac{1}{\sqrt{\frac{2H}{m}(1 - \frac{2m\mu^2}{\hbar^2}H)}}\, x + t_{0_{poly}} \,, \tag{5.35}$$

where $t_{0_{poly}}$ is a constant.

In the continuous limit, when $\mu \to 0$, the energy has to be positive $H > 0$ and the velocity is unbounded, in other words we recover the case of the classical non-relativistic free particle. Furthermore, in this limit, the polymer trajectories $t_{poly\pm}$ would be the classical solutions $t_{clas}(x) = \pm\sqrt{\frac{m}{2H}}\, x + t_0$.

5.3.2. Polymer relativistic particle

The constraint in the case of the relativistic particle is

$$\hat{\mathcal{G}} = \frac{1}{2}\left(\frac{1}{c}\hat{p}_\mu \hat{p}^\mu + m^2 c\right) \,, \tag{5.36}$$

let us assume that only the spatial variables are discrete, while the temporal variable remains continuous, we are only going to consider a space-time in $D = (1,1)$ dimensions with metric $(-,+)$. Then, we can compute the propagator $\langle {x^1}_f, {x^0}_f | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}} | x^1_i, x^0_i \rangle$ (where the first index denote the space-time component and the second index the evolution partition) as follows

$$
\begin{aligned}
&\langle x^1{}_f, x^0{}_f | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}} | x^1{}_i, x^0{}_i \rangle = \\
&= \prod_{n=1}^{N-1} \left[\int_{-\infty}^{\infty} dx^0{}_n \right] \prod_{n=1}^{N-1} \left[\sum_{x^1{}_n=-\infty}^{\infty} \right] \prod_{n=1}^{N} \langle x^1{}_n, x^0{}_n | e^{-\frac{i}{\hbar}\epsilon\lambda\hat{\mathcal{G}}_n} | x^1{}_{n-1}, x^0{}_{n-1} \rangle \\
&= \prod_{n=1}^{N-1} \left[\int_{-\infty}^{\infty} dx^0{}_n \right] \prod_{n=1}^{N} \left[\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp^0{}_n \right] \prod_{n=1}^{N-1} \left[\sum_{x^1{}_n=-\infty}^{\infty} \right] \prod_{n=1}^{N} \left[\frac{\mu}{2\pi\hbar} \int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp^1{}_n \right] \\
&\quad \exp\left\{ \frac{i}{\hbar} \sum_{n=1}^{N} \left[-\epsilon\lambda \frac{1}{2} \left(\frac{1}{c} p_{0,n} p^0{}_n + \frac{\hbar^2}{c\mu^2} \sin^2 \frac{\mu p^1{}_n}{\hbar} + m^2 c \right) \right.\right. \\
&\qquad\qquad \left.\left. + p_{0,n} \left(x^0{}_n - x^0{}_{n-1} \right) + p_{1,n} \left(x^1{}_n - x^1{}_{n-1} \right) \right] \right\} .
\end{aligned}
\tag{5.37}
$$

We will use here all the tricks that we learned from the previous example. For the temporal coordinate it is possible to perform all the integrals except for the last one over the momentum $p^0{}_N$. For the spatial coordinate first we convert all the summations into integrals over $x^1{}_n$ using Poisson's summation formula given on eq. (5.20) which will add an extra infinite sum. We will use these additional sums to extend the range of integration on the spatial momentum variables $p^1{}_n$. We can do this except for $p^1{}_N$ so this integral will remain bounded. Therefore the Kernel will have the form

$$
\begin{aligned}
G(x^1{}_f, x^0{}_f; x^1{}_i, x^0{}_i) &= \int_{-\infty}^{\infty} d\lambda\, \mu \left(\frac{1}{2\pi\hbar} \right)^2 \int_{-\infty}^{\infty} dp^0{}_N \int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp^1{}_N \\
&\quad e^{-\frac{i}{\hbar}\left[\frac{\epsilon N\lambda}{2c} \left(p_{0,N} p^0{}_N + \frac{\hbar^2}{\mu^2} \sin^2 \frac{\mu p^1{}_N}{\hbar} + m^2 c^2 \right) - p_{0,N}(x^0{}_N - x^0{}_0) - p_{1,N}(x^1{}_N - x^1{}_0) \right]} .
\end{aligned}
\tag{5.38}
$$

Now we can integrate over λ to obtain a Dirac delta

$$
\begin{aligned}
G(x^1{}_f, x^0{}_f; x^1{}_i, x^0{}_i) &= \mu \left(\frac{1}{2\pi\hbar} \right)^2 \left(\frac{4\pi\hbar c}{\epsilon N} \right) \int_{-\infty}^{\infty} dp^0{}_N \int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp^1{}_N \\
&\quad \delta\left(p_{0,N} p^0{}_N + \frac{\hbar^2}{\mu^2} \sin^2 \frac{\mu p^1{}_N}{\hbar} + m^2 c^2 \right) e^{\frac{i}{\hbar}\left[p_{0,N}(x^0{}_N - x^0{}_0) + p_{1,N}(x^1{}_N - x^1{}_0) \right]} .
\end{aligned}
\tag{5.39}
$$

Finally, we integrate over ${p^0}_N$ and take $\epsilon N = 1$ to obtain

$$G({x^1}_f, {x^0}_f; {x^1}_i, {x^0}_i) = \frac{2\mu c}{\pi\hbar} \int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} d{p^1}_N \frac{e^{\frac{i}{\hbar} p_{1,N}({x^1}_N - {x^1}_0)}}{2\sqrt{\frac{\hbar^2}{\mu^2}\sin^2\frac{\mu {p^1}_N}{\hbar} + m^2c^2}} \left[e^{\frac{i}{\hbar}({x^0}_N - {x^0}_0)\sqrt{\frac{\hbar^2}{\mu^2}\sin^2\frac{\mu {p^1}_N}{\hbar} + m^2c^2}} + e^{-\frac{i}{\hbar}({x^0}_N - {x^0}_0)\sqrt{\frac{\hbar^2}{\mu^2}\sin^2\frac{\mu {p^1}_N}{\hbar} + m^2c^2}} \right]. \tag{5.40}$$

Equations of motion

Taking the limit $N \to \infty$ and substituting $\epsilon \sum_{n=0}^{N} \to \int d\tau$, $\left(x^0_{n+1} - x^0_n\right)/\epsilon \to dx^0/d\tau$ and $\left(x^1_{n+1} - x^1_n\right)/\epsilon \to dx^1/d\tau$ we get

$$\begin{aligned} G(x^1_f,\, x^0_f;\, x^1_i,\, x^0_i) &:= \int d\lambda\, \langle x_f,\, t_f | e^{-\frac{i}{\hbar}\epsilon\,\lambda\,\mathcal{G}} | x^1_i,\, x^0_i \rangle \\ &= \int d\lambda \int \left[\mathcal{D}x^0(\tau)\right] \left[\mathcal{D}x^1(\tau)\right] \left[\mathcal{D}p^0(\tau)\right] \left[\mathcal{D}p^1(\tau)\right] e^{\frac{i}{\hbar}\mathcal{S}}\,, \end{aligned} \tag{5.41}$$

where the effective action is given by

$$\mathcal{S} = \int_0^1 d\tau \left[p_0 \dot{x}^0 + p_1\, \dot{x}^1 - \frac{\lambda}{2}\left(\frac{1}{c} p_0 p^0 + \frac{\hbar^2}{c\mu^2}\sin^2\frac{\mu p^1}{\hbar} + m^2 c \right) \right]. \tag{5.42}$$

The equations of motion corresponding to the effective action of the relativistic particle (5.42) would be

$$\dot{p}_0 = 0, \qquad \dot{p}_1 = 0\,, \tag{5.43a}$$

$$p_0 = \frac{c}{\lambda}\dot{x}^0, \qquad p_1 = \frac{\hbar}{2\mu}\arcsin\left(\frac{2c\mu\dot{x}}{\lambda\hbar}\right), \tag{5.43b}$$

substituting the equations of motion (5.43a) in the action (5.42), using the metric $(-,+)$ and choosing the coordinates $(x_0, x_1) = (c\,t, x)$, this action has the form

$$\mathcal{S} = \int d\tau \left[-\frac{c^3}{2\lambda}\dot{t}^2 + \frac{\hbar}{2\mu}\dot{x}\arcsin\left(\frac{2c\mu}{\lambda\hbar}\dot{x}\right) - \frac{\lambda\hbar^2}{4c\mu^2}\left(1 - \sqrt{1 - \frac{4c^2\mu^2}{\lambda^2\hbar^2}\dot{x}^2}\right) - \frac{1}{2}\lambda m^2 c \right], \tag{5.44}$$

now we are going to eliminate the Lagrange multiplier λ with it's equation of motion

$$-\frac{cm^2}{2}+\frac{c^3}{2\lambda^2}\dot{t}^2-\frac{\hbar^2}{4c\mu^2}\left(1-\sqrt{1-\frac{4c^2\mu^2}{\lambda^2\hbar^2}\dot{x}^2}\right)=0\,, \tag{5.45}$$

this is a fourth order equation in λ so there are four solutions for the Lagrange multiplier, fixing $\dot{t}=\tau$ we get

$$\lambda=\pm\sqrt{\frac{c^2+\frac{2c^4m^2\mu^2}{\hbar^2}-\dot{x}^2\pm\sqrt{c^4-2c^2\dot{x}^2+\dot{x}^4-\frac{4c^4m^2\mu^2}{\hbar^2}\dot{x}^2}}{2(m^2+c^2m^4\mu^2/\hbar^2)}}\,, \tag{5.46}$$

once more the velocity values are bounded, we can find that the velocity must satisfy that

$$\dot{x}^2\leq c^2+\frac{2c^4m^2\mu^2}{\hbar^2}-2\sqrt{\frac{c^6m^2\mu^2}{\hbar^2}+\frac{c^8m^4\mu^4}{\hbar^4}}\,, \tag{5.47}$$

now we can get μ in terms of the maximum velocity $\dot{x}_{max}=\nu$ (remember that μ is the length of the grid and it has to be positive $\mu>0$)

$$\mu=\frac{\hbar(c^2-\nu^2)}{2c^2m\nu}\,. \tag{5.48}$$

For a proton accelerated in the LHC at 7 TeV and substituting the values of its mass, the speed of light and Planck's constant

$$\begin{aligned}c&=299\ 792\ 458\ m/s\,,\\ \hbar&=1.054\ 571\ 726\times10^{-34}\ J\cdot s\,,\\ m&=1.672\ 621\ 777\times10^{-27}\ kg\,,\end{aligned} \tag{5.49}$$

we obtain for the length of the grid

$$\mu=1.889\times10^{-24}\ m\,, \tag{5.50}$$

this is $10^{11}\,l_P$ where l_P is Plank's length ($l_P=1.616\ 199\times10^{-35}\ m$).

5.3.3. Polymer Harmonic Oscillator

We will now show how to quantize the polymer harmonic oscillator using the path integral, the action for the parameterized system is

$$\mathcal{S}=\int d\tau\left[p_t\,\dot{t}+p_x\,\dot{x}-\lambda\left(p_t+\frac{1}{2m}p_x^2+\frac{k}{2}x^2\right)\right]\,, \tag{5.51}$$

with the constraint given by

$$\mathcal{G} = p_t + \frac{1}{2m}p_x^2 + \frac{k}{2}x^2 \approx 0\,. \tag{5.52}$$

In the polymer bra-ket space we can evaluate the harmonic potential, we thus get

$$\begin{aligned}\langle x_n \Big| \frac{k}{2}x^2 \Big| x_{n-1}\rangle &= \frac{k}{2}x_n x_{n-1}\delta_{x_n,x_{n-1}} \\ &= \frac{k}{2}x_n\, x_{n-1}\,\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp_{x_n}\, e^{\frac{i}{\hbar}\,p_{x_n}\,(x_n - x_{n-1})}\,,\end{aligned} \tag{5.53}$$

in consequence for the spatial part of the constraint we obtain

$$\begin{aligned}&\langle x_n| e^{-\frac{i}{\hbar}\epsilon\lambda\left(\frac{1}{2m}p_x^2+\frac{k}{2}x^2\right)}\,|x_{n-1}\rangle = \\ &= \langle x_n|1 - \frac{i}{\hbar}\epsilon\,\lambda\left(\frac{\hbar^2}{2m\mu^2}\sin^2\frac{\mu\,p_x}{\hbar} + \frac{k}{2}x^2\right)|x_{n-1}\,\rangle \\ &= \frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp_{x_n}\, e^{\frac{i}{\hbar}\,p_{x_n}(x_n - x_{n-1}) - \frac{i}{\hbar}\epsilon\,\lambda\left(\frac{\hbar^2}{2m\mu^2}\sin^2\frac{\mu\,p_{x_n}}{\hbar} + \frac{k}{2}x_n\,x_{n-1}\right)}\,.\end{aligned} \tag{5.54}$$

Therefore, using the expressions (5.54) and (5.16) we get for the path integral

$$\begin{aligned}&\langle x_f,\, t_f| e^{-\frac{i}{\hbar}\epsilon\,\lambda\,\mathcal{G}}|x_i,\, t_i\rangle = \\ &= \prod_{n=1}^{N-1}\left[\int_{-\infty}^{\infty} dt_n\right]\prod_{n=1}^{N-1}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dp_{t_n}\right] e^{\frac{i}{\hbar}\epsilon\sum_{n=1}^{N}\left(-\lambda\,p_{t_n} + p_{t_n}\frac{t_n - t_{n-1}}{\epsilon}\right)}\prod_{n=1}^{N-1}\left[\sum_{x_n=-\infty}^{\infty}\right] \\ &\prod_{n=1}^{N-1}\left[\frac{\mu}{2\pi\hbar}\int_{-\frac{\pi\hbar}{\mu}}^{\frac{\pi\hbar}{\mu}} dp_{x_n}\right] e^{\frac{i}{\hbar}\epsilon\sum_{n=1}^{N}\left(p_{x_n}\frac{x_n - x_{n-1}}{\epsilon} - \frac{\lambda\hbar^2}{2m\mu^2}\sin^2\frac{\mu\,p_{x_n}}{\hbar} - \frac{k\lambda}{2}x_n x_{n-1}\right)}\,.\end{aligned} \tag{5.55}$$

Taking the limit $N \to \infty$ and using $\epsilon\sum_{n=0}^{N} \to \int d\tau$, $(t_n - t_{n-1})/\epsilon \to dt/d\tau$, $(x_n - x_{n-1})/\epsilon \to, dx/d\tau$. Then, we conclude

$$\begin{aligned}G(x_f,\, t_f;\, x_i,\, t_i) &= \int d\lambda \langle x_f,\, t_f|e^{-\frac{i}{\hbar}\epsilon\,\lambda\,\mathcal{G}}|x_i,\, t_i\rangle \\ &= \int d\lambda \int [\mathcal{D}x(\tau)]\,[\mathcal{D}t(\tau)]\,[\mathcal{D}p_t(\tau)]\,[\mathcal{D}p_x(\tau)]\, e^{\frac{i}{\hbar}\mathcal{S}}\,,\end{aligned} \tag{5.56}$$

where

$$\mathcal{S} = \int_0^1 d\tau\left[p_t\dot{t} + p_x\,\dot{x} - \lambda\left(p_t + \frac{\hbar^2}{2m\mu^2}\sin^2\frac{\mu\,p_x}{\hbar} + \frac{k}{2}x^2\right)\right]. \tag{5.57}$$

In this form, we get the Kernel of the polymer harmonic oscillator. To deduce the corresponding Schrödinger equation, we use the relations $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, $k = m\omega^2$ and taking into account that the system is quadratic in the potential we use the p_x representation. Furthermore, we assume that from the effective action (5.57), we can read the effective constraint, then the time independent Schrödinger equation is

$$E\,\psi = \frac{\hbar^2}{4m\mu^2}\left(1-\cos\frac{2\,\mu\,p_x}{\hbar}\right)\psi - \frac{m\omega^2\hbar^2}{2}\frac{\partial^2\psi}{\partial p_x^2}\,. \tag{5.58}$$

Note that if we would want to define de quantum amplitude we should take into account in 5.56 that the momenta are fixed, while the coordinates are not, as it was done in [116]. Next, let us define

$$u = \frac{\mu p_x}{\hbar} + \pi/2, \qquad g = m\omega\mu, \qquad \alpha = \frac{2\mu E}{g\omega\hbar^2} - \frac{1}{2g^2}\,. \tag{5.59}$$

Then, the eq. (5.58) is transformed in Mathieu's differential equation. This equation has been studied in several works, see for example [30,117], and reference therein. In this work, they obtained periodic solutions for special values of α

$$\begin{aligned} \psi_{2n}(u) &= \pi^{-1/2}\mathrm{ce}_n(1/4g^2,u), \qquad && \alpha = A_n(g)\,,\\ \psi_{2n+1}(u) &= \pi^{-1/2}\mathrm{se}_{n+1}(1/4g^2,u), \qquad && \alpha = B_n(g)\,. \end{aligned} \tag{5.60}$$

We denote by ce_n and se_n $(n = 0, 1, \ldots)$ the cosine and sine elliptic functions respectively. For a pendulum with a single potential well m=1 [117]. Note that only even order ce and se, functions occur. A_n and B_n correspond to Mathieu characteristic value functions. for n even ce_n and se_n are π-periodic, but for n odd are 2π-antiperiodic. Explicitly, the eigenvalues corresponding to the periodic eigenfunctions (5.60) are:

$$\frac{E_{2n}}{\omega} = \frac{2g^2A_n(g)+1}{4g}, \qquad \frac{E_{2n+1}}{\omega} = \frac{2g^2B_{n+1}(g)+1}{4g}\,. \tag{5.61}$$

Dynamical analysis

In this subsection we want to study the full dynamics of the system in order to try to find non- pertubative solutions to the action (5.57). The equations of motion are

$$\dot p_t = 0, \qquad \dot p_x = -\lambda\,k\,x\,, \tag{5.62a}$$

$$\dot t = \lambda\,, \qquad p_x = \frac{\hbar}{2\mu}\arcsin\left(\frac{2m\mu\dot x}{\lambda\hbar}\right)\,. \tag{5.62b}$$

Using the eq. (5.62a) and (5.57) and defining the new time $x' = \dot{x}/\lambda$ to de-parametrize the action we get

$$\mathcal{S} = \int dt \left[\frac{\hbar}{2\mu} x' \arcsin\left(\frac{2m\mu x'}{\hbar}\right) - \frac{\hbar^2}{4m\mu^2}\left(1 - \sqrt{1 - \frac{4\,m^2\mu^2}{\hbar^2}x'^2}\right) - \frac{k}{2}x^2 \right] . \tag{5.63}$$

From this expression we derive the effective Hamiltonian

$$H = \frac{\partial \mathcal{L}}{\partial x'}x' - \mathcal{L} = \frac{\hbar^2}{4m\mu^2}\left(1 - \sqrt{1 - \frac{4\,m^2\mu^2}{\hbar^2}x'^2}\right) - \frac{k}{2}x^2 . \tag{5.64}$$

It is clear that the values of the velocity $x'$are bounded by the same limit as in the free particle case

$$-\frac{\hbar}{2m\mu} < x' < \frac{\hbar}{2m\mu} . \tag{5.65}$$

Now, as H is a constant of motion, we can solve for x'

$$x'^2 = \frac{1}{m}\left[2H\left(1 - \frac{2m\mu^2}{\hbar^2}H\right) + k\left(\frac{4m\mu^2}{\hbar^2}H - 1\right) - \frac{m\mu^2k^2}{\hbar^2}x^4\right] , \tag{5.66}$$

in consequence we obtain the complete evolution of the system

$$t = \pm\sqrt{\frac{m}{k(1 - \frac{2m\mu^2}{\hbar^2}H)}}\,\text{EllipticF}\left[\arcsin\left(\sqrt{\frac{k}{2H}}\,x\right) , \frac{\frac{2m\mu^2}{\hbar^2}H}{\frac{2m\mu^2}{\hbar^2}H - 1}\right] + t_0 . \tag{5.67}$$

To simplify the dynamical analysis we write the system in the phase space. In terms of the momentum $p = \frac{\partial \mathcal{L}}{\partial x'} = \frac{\hbar}{2\mu}\arcsin(\frac{2m\mu}{\hbar}x')$ the Hamiltonian will be

$$H = \frac{\hbar^2}{2m\mu^2}\sin^2\frac{\mu p}{\hbar} + \frac{k}{2}x^2, \tag{5.68}$$

solving for p from (5.68) results

$$p = \pm\frac{\hbar}{\mu}\arcsin\sqrt{\frac{2m\mu^2}{\hbar^2}\left(H - \frac{k}{2}x^2\right)} . \tag{5.69}$$

Now, using as configuration variable the momentum p, the system evolves under a potential $V(p) = \frac{\hbar^2}{2m\mu^2}\sin^2\frac{\mu p}{\hbar}$. Solving eq. (5.69) for the “momentum variable" x we get

$$x = \pm\sqrt{\frac{1}{km}\left(2mH - \frac{\hbar^2}{\mu^2}\sin^2\frac{p\mu}{\hbar}\right)} . \tag{5.70}$$

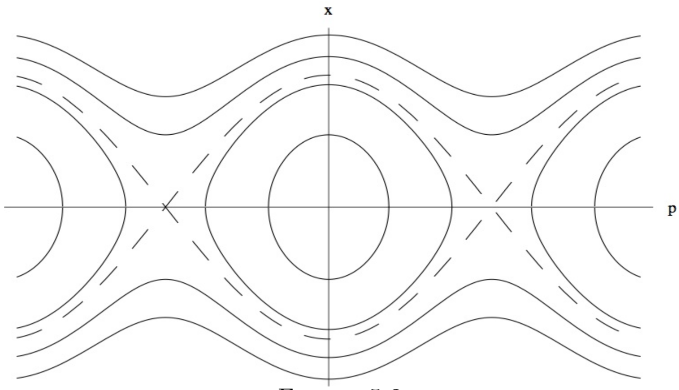

FIGURE 5.2
Graphic in the phase space. The dashed line corresponds to the separatrix of the movement given by $H = \frac{\hbar^2}{2m\mu^2}$.

So we see that the potential is periodic in p (with period $\frac{\pi\hbar}{\mu}$). Furthermore, it is always positive and corresponds to a pendulum. Then, the motion in p, and the separatrix divide the phase space in two regions with phase space curves of different characteristics, in one region we get libration and in the other rotation (Figure 5.2). In the case of libration the system approaches to a standard harmonic oscillator this limit corresponds to $g = 0$, i.e. $\mu = 0$, in this limit the quantum states are equally spaced. The case $g \gg 0 \sim \mu \gg 0$ corresponds to a rigid rotor limit and here we have quadratically spaced energy levels.

On the other hand, the fixed points are found in $p = n\frac{\hbar}{2\mu}\arcsin(1)$, where $n \in Z$; for n even correspond to minima and n odd correspond the maxima of the potential. In the region of libration, x is found between $x^2 < \frac{2H}{k}$, whereas p is bounded between $p^2 < \frac{\hbar^2}{\mu^2}\arcsin^2\sqrt{\frac{2m\mu^2}{\hbar^2}H}$.

In the rotation region, if n is even, $x = +\sqrt{\frac{2H}{k}}$ or $x = -\sqrt{\frac{2H}{k}}$. For n odd we get that $x = +\sqrt{\frac{2H}{k} - \frac{\hbar^2}{mk\mu^2}}$ or $x = -\sqrt{\frac{2H}{k} - \frac{\hbar^2}{mk\mu^2}}$.

Using the Hamiltonian equation, with Hamiltonian (5.68) we get

$$\dot{p} = \frac{\partial H}{\partial x} = kx = \pm\sqrt{\frac{k}{m}\left(2mH - \frac{\hbar^2}{\mu^2}\sin^2\frac{\mu p}{\hbar}\right)}. \tag{5.71}$$

In the case of libration, with initial conditions given by

$$t_{poly_L}\left(p_0 = \pm\frac{\hbar}{\mu}\arcsin\sqrt{\frac{2m\mu^2}{\hbar^2}H}\right) = 0\,, \tag{5.72}$$

we obtain

$$t_{poly_L} = \frac{\hbar}{\mu}\sqrt{\frac{1}{2kH}}\left\{\pm \text{EllipticF}\left[\frac{p\mu}{\hbar},\ \frac{\hbar^2}{2Hm\mu^2}\right]\right.$$
$$\left.\pm(4n\mp 1)\text{EllipticF}\left[\arcsin\sqrt{\frac{2m\mu^2}{\hbar^2}H},\frac{\hbar^2}{2m\mu^2 H}\right]\right\}. \tag{5.73}$$

In the case of rotations, for the initial condition

$$t_{poly_R}\,(p=0) = 0\,, \tag{5.74}$$

results

$$t_{poly_R} = \frac{\hbar}{\mu}\sqrt{\frac{1}{2kH}}\left\{\pm \text{EllipticF}\left[\frac{p\mu}{\hbar},\frac{\hbar^2}{2Hm\mu^2}\right]\right\} + t_{0_R}\,. \tag{5.75}$$

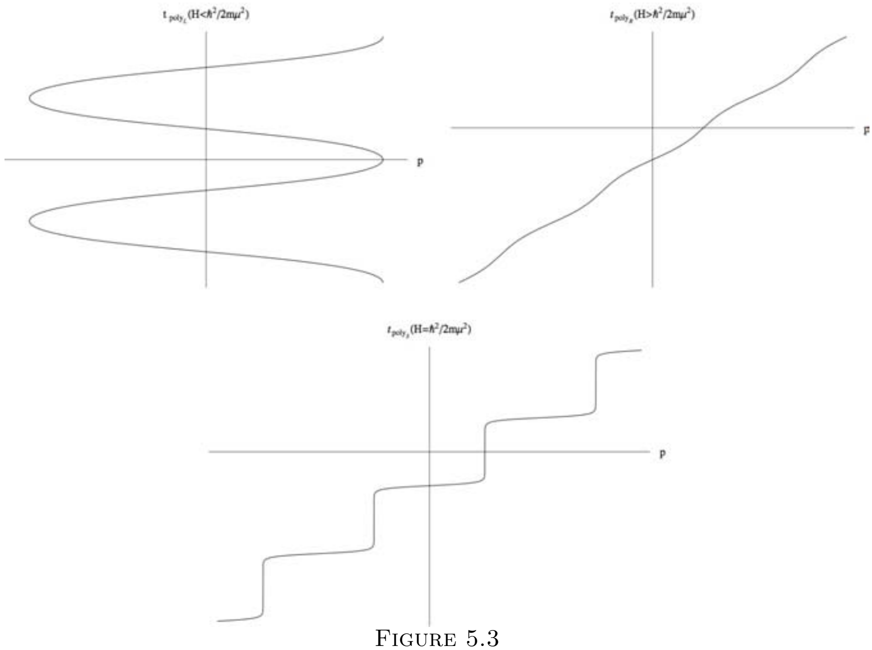

FIGURE 5.3
Solutions for the p variable. The upstairs left plot corresponds to the solutions inside the libration region, the upstairs right plot corresponds to the rotation region and the downstairs plot correspond to the separatrix solution

.

From figure 5.3 we observe that in the case of libration the movement in p is bounded and oscillatory. Whereas, in the case of rotation the movement is not bounded but is still periodic (5.3.3). The case $H = \frac{\hbar^2}{2m\mu^2}$ corresponds to the separatrix, in the quantum case this solution corresponds to the quantum tunneling and we observe from the (5.3.2) that this solution has the form of a kink, but we must remember that this solution corresponds to the p variable and the must be interpreted carefully It will be quite interesting to see if for the polymer quantum mechanics this kind of solutions can be interpreted as a map between a coherent state from the oscillatory side to a squeezed pendular state in the rotor side. This is the case in the quantum pendulum [117]. Furthermore it will be interesting to analyze the consequence of this kind of solutions in the case of quantum cosmology.

5.3.4. Polymer Cosmology

Loop quantum cosmology is the application of loop quantum gravity to cosmological models. In loop quantum cosmology the geometric observables display a fundamental discreteness causing that the big bang is replaced by a quantum bounce (the quantum Hamiltonian constraint of the theory does not break down when the scale factor vanishes and the classical singularity occurs replacing the big bang by a quantum bounce).

Almost all phenomenological work in cosmology is based on the flat, homogeneous and isotropic Friedmann Robertson Walker (FRW) space-times and perturbations thereof. To construct a Hamiltonian formulation of the FRW models is necessary to introduce a finite elementary cell $\mathcal{V}$ [118] in the space with topology Re^3 and restrict all integrations to it. It is easiest to fix a fiducial flat metric $\mathring{q}_{ab}dx^a dx^b = dx_1^2 + dx_2^2 + dx_3^2$ (where x^a are Cartesian coordinates), that is related with the physical 3-metric by $q_{ab} = a^2 \mathring{q}_{ab}$ and a is the scale factor. The volume of the cell $\mathcal{V}$ with respect to the metric $\mathring{q}_{ab}$ will be denoted by V_0 and the physical volume will be just $V = a^3 V_0$.

The standard canonical pairs are (a, p_a) for geometry and (ϕ, p) for the scalar field. To compare with LQC results [69] it is convenient to replace the scalar factor a by a variable ν which is proportional to the volume of the cell $\mathcal{V}$ and the conjugate momentum p_a by a new variable b defined as

$$\nu = \frac{a^3 V_0}{2\pi G}, \qquad b = -\frac{4\pi G}{3V_0}\frac{p_a}{a^2}. \tag{5.76}$$

With this replacement ν has the dimension of length and it ranges over $(-\infty, \infty)$ and b has dimensions of inverse length. Their Poisson bracket is then given by $\{\nu, b\} = -2$ while for the matter phase space $\{\phi, p\} = 1$.

We will discuss in some detail the simplest cosmological space-time, the spatially flat FLRW model with cosmological constant zero ($k = 0$, $\Lambda = 0$). The effective action for this model obtained after applying the path integral formulation that we studied in the previous sections, is given in [119] and it takes the form

$$\mathcal{S} = \int_0^1 d\tau \left\{ p\,\dot{\phi} - \frac{1}{2} b\,\dot{\nu} - \lambda \left[p^2 - \frac{3\pi G\hbar^2}{\mu^2}\nu^2 \sin^2 \frac{\mu\, b}{\hbar} \right] \right\}, \tag{5.77}$$

where γ is the Barbero-Immirzi parameter of LQC, λ is a Lagrange multiplier and μ is the length of the grid as in the previous examples. By making the redefinition $\eta = 3\pi G$, the effective action will take the form

$$\mathcal{S} = \int_0^1 d\tau \left\{ p\,\dot{\phi} - \frac{1}{2} b\,\dot{\nu} - \lambda \left[p^2 - \frac{\eta\hbar^2}{\mu^2}\nu^2 \sin^2 \frac{\mu\, b}{\hbar} \right] \right\}. \tag{5.78}$$

In here, we will study the dynamics of the system by analyzing the phase space, so first we find that the equations of motion will be

$$\begin{aligned} \dot{p} &= 0\,, & \dot{b} &= -\frac{4\lambda\,\nu\eta\hbar^2}{\mu^2}\sin^2\frac{\mu\, b}{\hbar}\,, \\ \dot{\phi} &= 2\lambda\, p\,, & b &= \frac{\hbar}{2\mu}\arcsin\left(\frac{\mu\,\dot{\nu}}{2\lambda\eta\hbar\,\nu^2}\right). \end{aligned} \tag{5.79}$$

Now we substitute these equations in the action (5.78), we make the identification $\lambda = \dot{\phi}$ and use that $\nu' = \dot{\nu}/\lambda$ to obtain

$$\mathcal{S} = \int d\phi \left\{ \frac{1}{4} - \frac{\hbar\,\nu'}{2\mu}\arcsin\left(\frac{\mu\,\nu'}{2\eta\hbar\,\nu^2}\right) + \nu^2\frac{\eta\hbar}{2\mu^2}\left(1 - \sqrt{1 - \frac{\mu^2\,\nu'^2}{4\eta^2\hbar^2\,\nu^4}}\right) \right\}. \tag{5.80}$$

Writing the Hamiltonian in terms of the momentum $b = \frac{\partial\mathcal{L}}{\partial\nu'} = -\frac{\hbar}{4\mu}\arcsin\left(\frac{\mu\nu'}{2\eta\hbar\,\nu^2}\right)$ gives $H = -\nu^2\frac{\eta\hbar^2}{\mu^2}\sin^2\frac{2\mu\, b}{\hbar}$. If we consider that the energy is constant from (5.3.4) we get

$$\nu(b) = \pm\sqrt{\frac{-H}{\frac{\eta\hbar^2}{\mu^2}\sin^2\frac{2\mu\, b}{\hbar}}}\,. \tag{5.81}$$

From here we see that there are different restrictions over the energy H and the values of ν are bounded.

When the universe is flat, there are two types of classical trajectories: one in which the universe begins with a big bang and expands (eternally expanding universe) and another one when it contracts into a big crunch (eternally contracting universe). However, for the LQC evolution the quantum solution follows the classical one at low densities and curvatures (LQC has good infrared behavior) but instead of falling into the singularity undergoes a quantum bounce and joins on to the classical trajectory that was expanding to the future.

The expansion volume will be given then by

$$\nu = \pm\sqrt{-\frac{\mu^2 H}{\eta\hbar^2}\frac{1}{\sin^2\frac{2\mu b}{\hbar}}}\,. \tag{5.82}$$

From here we can see the periodicity of the velocity of expansion b and that the values of H and ν are restricted by $H < 0$, and $\nu \notin \left[-\sqrt{-\frac{\mu^2 H}{\eta\hbar^2}}, \sqrt{-\frac{\mu^2 H}{\eta\hbar^2}}\right]$. In the classical this result will be

$$\nu_{clas} = \pm\sqrt{-\frac{H}{\eta\, b_{clas}^2}}\,, \tag{5.83}$$

therefore if we go back in time when the expansion velocity increases, then the volume of the universe goes to zero into the singularity. However, in the polymer case, when the expansion velocity increases, the volume bounce intend of going into the singularity and reaches the minimal value $\sqrt{-\frac{\mu^2 H}{\eta\hbar^2}}$.

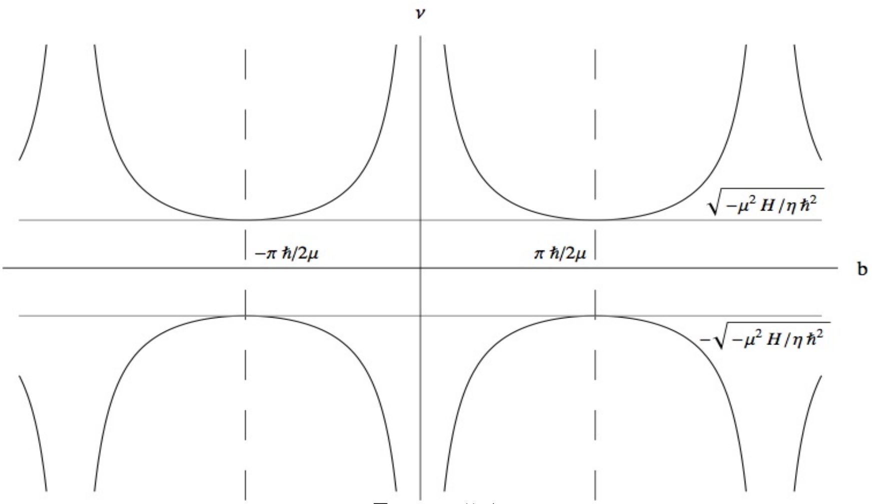

FIGURE 5.4
Phase space diagram in the case $k = 0$.

5.4. Discussion

In the frame of the polymer representation, we obtained the propagators of the non-relativistic and relativistic particle, the harmonic oscillator and Friedmann Robertson Walker cosmologies in the cases of flat and positive curvatures.

One of the most important characteristics of the group averaging procedure that is not commonly address in the literature and that we stress in this work, is that it gives us the correct measure of the path integral, as we saw there is always a remaining integration over the momentum even without taking into account the polymer representation. This result is not relevant when one analyzes the effective actions of the systems, but is crucial to take it into account in order to compute the exact propagator.

Since in quantum field theory the effective action is an important tool for the investigation of properties of the theories such as non-perturbative effects it was interesting to take it as our starting point in the analysis of the dynamics of our systems. The first interesting feature that we found, is that the velocity of the particle in the systems is bounded. Furthermore, in the relativistic case we found that this velocity is always smaller than the speed of light.

6

Non Relativistic Limit

6.1. Introduction

Non-relativistic theories have been interesting tools in theoretical physics in recent years. The main motivation is to widen the applications of the conjectured anti-de Sitter/conformal field theory (AdS/CFT) correspondence and to test if it also holds away from its original relativistic setting. AdS/CFT correspondence can be used to understand strongly interacting field theories by mapping them to classical gravity. In the study of collective phenomena in condensed matter physics it is quite common to observe this strong-weak coupling duality since a strongly coupled system reorganizes itself in such a way that new weakly coupled degrees of freedom emerge dynamically and the system can be better described in terms of fields representing the emergent excitations. There are reasons to expect that non-relativistic holography has applications in effective descriptions of strongly correlated condensed matter systems [39, 120–123]; for a review, see e.g. [124]. Usually, one only considers a non-relativistic setting at the boundary but one might also consider non-relativistic models both in the bulk and at the boundary [40]. Originally, non-relativistic superstring theories and superbranes were studied as special points in the parameter space of M-theory with non-relativistic symmetries [125, 126]. Non-relativistic strings were also thought as a possible soluble sector within string theory or M-theory [127, 128]. This latter expectation was based on a similar experience with the pp-wave/BMN limit [129].

Furthermore, a formulation of non-relativistic gravity that is invariant under diffeomorphisms was introduced by Cartan [130], see also [131–135]. This so-called Newton-Cartan gravity can be reformulated as a gauge theory of the Bargmann algebra [45, 136]. The interest in Galilean-invariant theories with diffeomorphism invariance has increased recently due to their relation with condensed matter systems [38, 137, 138], see also [139, 140] and references therein. Galilean-invariant theories have also appeared recently in studies of Lifshitz holography [141, 142].

In this chapter we make a first step in improving our knowledge on non-relativistic gravity and particle/string/brane theories that underlie a holography with a non-relativistic setting at both the boundary and the bulk. In this work we will consider supersymmetric theories and we will restrict to particles, or more precisely superparticles [143, 144], only.

Non-relativistic $\mathcal{N} = 2$ massive superparticles in ten dimensions in a flat background have already been studied in [127]. In this chapter we wish to extend this analysis and consider superparticles in a *curved* non-relativistic supergravity background. It should be stressed that, independent of the relation with non-relativistic holography, there is not much literature on non-relativistic supersymmetry; however, see e.g. [145–151]. This in itself provides ample reason to investigate this topic.

Part of this chapter consists of a review of known results on non-relativistic superparticles in a *flat* background, we will use the non linear realizations method to study the dynamics and the symmetries of these particles. To the best of our knowledge all non-relativistic superparticle actions with background fields are new.

In the construction of massive (super-)particle actions an important role is played by symmetries, both global and local ones. In particular, it is well-known that relativistic massive superparticles have an infinitely-reducible gauge symmetry, called κ-symmetry [152, 153], that eliminates half of the fermions. In the non-relativistic setting this κ-symmetry corresponds to a fermionic gauge shift symmetry, i.e. a Stückelberg symmetry [127].

When discussing the symmetries of (super-)particles in a curved background it is important to distinguish between 'proper' and 'sigma model' symmetries [154, 155]. In the case of proper symmetries the background (super)gravity fields only transform through their dependence on the embedding coordinates of the (super)particle. On the other hand, in the case of sigma model symmetries the background fields have their own transformation rules. We will clarify how these sigma model transformations are related to the transformations of the (super)gravity fields when viewed as the components of a (super)gravity multiplet defined in the target space.

To explain our construction of non-relativistic superparticles in a curved background, we will first consider the bosonic case and take as our starting point a single massive non-relativistic particle in a flat background. The action of such a particle is invariant under the (global) Galilei symmetries, hence the name 'Galilean' particle. We next partially gauge the spatial target space translations of the Galilean particle such that the constant parameter of a spatial translation is promoted to an arbitrary function of time. The resulting extended symmetries are sometimes called the 'acceleration-extended' Galilean symmetries. In order to achieve this partial gauging the particle must move in a curved Galilean background that is characterized by the Newton potential Φ [156].[1] We will call this particle a 'Curved Galilean' particle. Finally, we perform a full gauging of the Galilean symmetries such that the parameter of spatial translations becomes an arbitrary function of the spacetime coordinates and the full symmetries of the particle action are the general coordinate transformations. Actually, to perform this gauging it is necessary to extend the Galilei symmetries with an additional central charge transformation[2] [45,136,159]. The background is now promoted to a Newton–Cartan gravity background that is characterized by a spacelike Vielbein ${e_\mu}^a$, a timelike Vielbein τ_μ and a central charge gauge field m_μ. We will call the corresponding particle a 'Newton–Cartan' (NC) particle. Figure 6.1 indicates the different backgrounds that we will consider and outlines the how to move between backgrounds, either by a gauging of symmetries or by gauge fixing some of the symmetries.

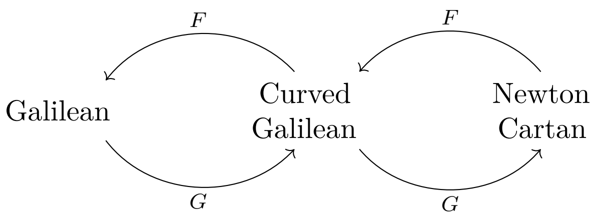

Figure 6.1

This figure displays the different backgrounds used in this chapter. For each background, section 6.4 discusses the bosonic case while section 6.5 treats the supersymmetric case. The upper arrows marked with an F indicate the direction of gauge-fixing, while the lower arrows marked with a G indicate the gauging direction.

Next, we will consider the superparticle. This requires a supersymmetric extension of the gravity backgrounds in the first place. Since non-relativistic su-

[1] There is also a different way of gauging the spatial translations where the background is given by a vector field rather than a scalar potential [136, 157].

[2] In section 6.3, where we discuss supersymmetric particle actions, we will restrict ourselves to $d = 3$. In that case, it is known that the Bargmann algebra admits a second central charge, see e.g. [158]. We will however not consider it in this thesis.

pergravity multiplets to our knowledge have only been explicitly constructed in three dimensions, we will only consider superparticles in a three-dimensional (3D) background. A supersymmetric version of the 3D Galilean and NC backgrounds was recently constructed by gauging the Galilei, or better Bargmann, superalgebra [48]. We will make full use of the construction of [48] which, in particular, explains how to switch between different backgrounds, with different symmetries, by partial gauging or partial gauge-fixing. Our aim will be to investigate the action of a 3D superparticle first in a flat background and, next, in a Galilean and NC supergravity background with and without a cosmological constant. To indicate the different cases we will use the same nomenclature as in the bosonic case but with the word particle replaced by superparticle.

This work is organized as follows. In section 6.2 we go through the known descriptions of the bosonic Newton-Hooke particle to introduce our notation and the non linear realizations method to find particle actions. Moreover, we use the Lagrangian formalism to find the Killing equations of this system to study the symmetries that is contains. We end this section with comments on the flat limit. In section 6.4 we discuss the different gaugings and gauge-fixings in a simple setting. Sections 6.3 and 6.5 are devoted to the supersymmetrization of the theory discussed in section 6.2.

6.2. The Free Newton-Hooke Particle

This section will serve as a useful warm-up exercise to show the bosonic results before discussing the superparticle case. We consider a negative cosmological constant $\Lambda < 0$ with the AdS radius given by $R^2 = -1/\Lambda$ [3]. In global coordinates the metric of an AdS spacetime is given by

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\varphi^2\,, \qquad\qquad f(r) = 1 - \Lambda r^2\,. \tag{6.1}$$

By taking the non-relativistic limit of a relativistic particle in such an AdS background we arrive at the action

$$S = \int d\tau\, \frac{m}{2}\left[\frac{\dot{x}^i \dot{x}^i}{\dot{t}} - \frac{\dot{t}\, x^i x^i}{R^2}\right]\,, \tag{6.2}$$

once a total derivative term is eliminated. Physically, this system is equivalent to the non-relativistic harmonic oscillator.

[3] In the following we will express everything in terms of the radius R.

The NH algebra with a central charge extension can be derived as a contraction of the AdS algebra (2.39) extended to the direct sum of the AdS algebra and a commutative subalgebra spanned by Z. To make the non-relativistic contraction we rescale the generators with a parameter ω as follows ($a = 1, \ldots, D-1$):

$$P_0 = \omega\, Z + \frac{1}{2\omega}\, H\,, \qquad M_{a0} = \omega\, K_a\,, \qquad R = \omega \tilde{R}\,. \tag{6.3}$$

Taking the limit $\omega \to \infty$ and dropping the tilde on the R we get the following Newton–Hooke algebra:

$$\begin{gathered} [M_{ab}, M_{cd}] = 2\eta_{a[c}M_{d]b} - 2\eta_{b[c}M_{d]a}\,, \qquad [J_{ab}, (P/K)_c] = -2\,\delta_{c[a}(P/K)_{b]}\,, \\ [H, K_a] = P_a\,, \qquad [H, P_a] = -\frac{1}{R^2}\, K_a\,, \qquad [P_a, K_b] = \delta_{ab}\, Z\,. \end{gathered} \tag{6.4}$$

The generators $\{H\,, J_{ab}\,, P_a\,, K_a\}$ generate time translations, spatial rotations, space translations and boost transformations, respectively. The central extension Z that leads to the Bargmann version of the NH algebra occurs only in the last commutator. Physically, the occurrence of the central charge transformations is related to the fact that at the non-relativistic level the mass of a particle is conserved.

6.2.1. Non-linear Realizations

In this section we obtain the action and transformation rules for the Newton–Hooke particles by using the method of non-linear realizations. We will derive the action and transformation rules of the NH superparticle and afterwards we will take the flat limit to obtain the results for the Galilean particle, see e.g. [68, 160].

The starting point is the AdS algebra with a central extension given in section 2.3. We derive the transformation rules for the coordinates (t, x^i, s, v^i) using the coset $NH/SO(D-1)$ with the coset element $g = g_0 U$, where $g_0 = e^{Ht} e^{P_i x^i} e^{Zs}$ is the coset representing the 'empty' NH space with a central charge extension and $U = e^{K_i v^i}$ is a general NH boost. This leads to the Maurer–Cartan form Ω_0 associated to the NH space

$$\Omega_0 = g_0^{-1} dg_0 = H\, e^0 + P_i e^i + Z e_z + K_i \omega^{i0} + M_{ij}\omega^{ij}\,, \tag{6.5}$$

where (e^0, e^i, e_z) and $(\omega^{i0}, \omega^{ij})$ are the space, time and central charge components of the Vielbein and spin-connection 1-forms corresponding to the NH space are given by

$$e^0 = dt\,, \qquad e^i = dx^i\,, \qquad e_z = ds + \frac{x^2}{2R^2}\, dt\,, \qquad \omega^{i0} = -\frac{dt}{R^2}\, x^i\,, \tag{6.6}$$

where $x^2 = x^i x^i$. By inserting a particle in the empty NH space we obtain the Maurer-Cartan form of the combined system

$$\Omega = g^{-1}dg = U^{-1}\Omega_0 U + U^{-1}dU = H\, L_H + P_i L_P^i + Z L_Z + K_i L_K^i + M_{ij}L^{ij}\,, \quad (6.7)$$

where the explicit 1-forms are given by

$$L_H = e^0\,, \qquad L_P^i = e^i + v^i\, e^0\,, \qquad L_Z = e_z + v^i\, e^i + \frac{1}{2}v^2\, e^0\,, \qquad L_K^i = \omega^{i0} + dv^i\,. \quad (6.8)$$

Note that we can obtain the Maurer–Cartan forms of the NH space by a matrix representation of the NH boost.

$$(\ L_H,\, L_P{}^i,\, L_Z\ ) = (\ e^0,\, e^i,\, e_z\ ) \begin{pmatrix} 1 & v^i & \frac{v^2}{2} \\ 0 & 1 & v^i \\ 0 & 0 & 1 \end{pmatrix}\,. \quad (6.9)$$

The action of the NH particle is given by the pull-back of all L's that are invariant under rotations, hence

$$\begin{aligned} S &= a\int (L_H)^* + b\int (L_Z)^* = a\int\, (e^0\,)^* + b\int \left(e_z + v^i\, e^i + \frac{1}{2}v^2\, e^0\right)^* \\ &= b\int d\tau\, \left[\frac{x^2}{2R^2}\,\dot{t} + v^i \dot{x}^i + \frac{v^2}{2}\dot{t} + \dot{s}\right]\,. \end{aligned} \quad (6.10)$$

Note that neglecting total derivative terms, by choosing $b = -m$ and using the equation of motion of $v^i = -\frac{\dot{x}}{\dot{t}}$, we recover the action given in eq. (6.2).

Using the Maurer–Cartan form one can derive the transformation rules of the Goldstone fields that realize the NH algebra (6.4):

$$\begin{aligned} \delta t = -\zeta\,, \qquad & \delta x^i = \lambda^i{}_k x^k - a^i \cos\frac{t}{R} + \lambda^i R \sin\frac{t}{R}\,, \\ \delta v^i = \lambda^i{}_k v^k - \frac{a^i}{R}\sin\frac{t}{R} - \lambda^i \cos\frac{t}{R}\,, \qquad & \delta s = \frac{a^i x^i}{R}\sin\frac{t}{R} + \lambda^i x^i \cos\frac{t}{R}\,. \end{aligned} \quad (6.11)$$

6.2.2. The Killing Equations of the Newton-Hooke Particle

In this section we find the Killing symmetries of the NH particle using the Lagrangian formalism. We start from the action (6.10) and consider that the coordinates transform as:

$$\delta x^i = \xi^i(t,x)\,, \qquad \delta t = \xi^0(t,x)\,, \qquad \delta v^i = \zeta^i(t,x,v)\,, \qquad \delta s = \eta(t,x)\,. \quad (6.12)$$

After imposing the invariance of the variation of the action ($\delta S = 0$) under the symmetry transformations, we find that the Killing equations are given by

$$\begin{aligned}\Big(\frac{x^2}{2R^2}+\frac{v^2}{2}\Big)\partial_t\xi^0 + v^i(\partial_t\xi^i+\zeta^i) + \frac{1}{R^2}x^i\xi^i + \partial_t\eta = 0\,,\\ \Big(\frac{x^2}{2R^2}+\frac{v^2}{2}\Big)\partial_i\xi^0 + v^j\partial_i\xi^j + \zeta^i + \partial_i\eta = 0\,.\end{aligned} \tag{6.13}$$

It is easy to prove that the solutions of the Killing equations (6.13) correspond to the symmetry transformations that we found in (6.11).

6.2.3. The Flat Limit

Taking the flat limit $R \to \infty$ of the Newton–Hooke algebra (6.4) we obtain the Galilean algebra

$$\begin{aligned}[M_{ab}, M_{cd}] = 2\eta_{a[c}M_{d]b} - 2\eta_{b[c}M_{d]a}\,, \qquad [J_{ab}, (P/K)_c] = -2\,\delta_{c[a}(P/K)_{b]}\,,\\ [H, K_a] = P_a\,, \qquad [P_a, K_b] = \delta_{ab}\,Z\,.\end{aligned} \tag{6.14}$$

In this case, the time, space and central charge components of the Vielbein simplify to

$$e^0 = dt\,, \qquad e^i = dx^i\,, \qquad e_z = ds\,, \tag{6.15}$$

and since we are studying the flat case, all the components of the spin-connection vanish and the action of the Galilean particle is just

$$S = b\int\Big(e_z + v^i\,e^i + \frac{1}{2}v^2\,e^0\Big)^* = \frac{m}{2}\int d\tau\frac{\dot{x}^i\dot{x}^i}{\dot{t}}\,, \tag{6.16}$$

where we already substituted the equation of motion for v^i. This action is invariant under the transformation rules

$$\delta t = -\zeta\,, \qquad \delta x^i = \lambda^i{}_k x^k - a^i + v^i\,, \qquad \delta v^i = \lambda^i{}_k v^k - v^i\,, \qquad \delta s = v^i x^i\,. \tag{6.17}$$

6.3. The Newton–Hooke Superalgebra

In the same way that the non-relativistic bosonic particle is based upon the Galilei algebra, or its centrally extended version, the Bargmann algebra, the action and transformation rules of the non-relativistic 3D $\mathcal{N} = 2$ superparticle are based upon the supersymmetric extension of the Galilei or Bargmann algebra. It turns out that we need *two* supersymmetries since one of the supersymmetries is, like the time translations in the bosonic case, a Stückelberg symmetry.

The NH superalgebra can be derived as a contraction of the AdS superalgebra. In the case of two supersymmetries there are two independent versions of the latter one, the so-called $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ algebras. As we explained in Section 2, in the presence of a cosmological constant the relativistic AdS superalgebra in three dimensions is not unique. Instead, in the case of $\mathcal{N}$ supersymmetries, one always finds $\mathcal{N}$ different versions, often referred to as (p,q) AdS superalgebras [58]. In Section 2.3 we give both the $(1,1)$ and $(2,0)$ $\mathcal{N} = 2$ AdS superalgebras. As explained in Section 2.3, the Newton–Hooke superalgebra that we will use below is obtained by contracting the $\mathcal{N} = (2,0)$ AdS superalgebra.

We proceed by discussing the contraction of the 3D $\mathcal{N} = (2,0)$ AdS algebra. The basic commutators are given by $(A = 0,1,2)$

$$\begin{aligned}
[M_{AB}, M_{CD}] &= 2\eta_{A[C}M_{D]B} - 2\eta_{B[C}M_{D]A}\,, & [M_{AB}, Q^i] &= -\frac{1}{2}\gamma_{AB}Q^i\,,\\
[M_{AB}, P_C] &= -2\eta_{C[A}P_{B]}\,, & [P_A, Q^i] &= x\gamma_A Q^i\,,\\
[P_A, P_B] &= 4x^2 M_{AB}\,, & [\mathcal{R}, Q^i] &= 2x\,\varepsilon^{ij}Q^j\,,\\
\{Q^i_\alpha, Q^j_\beta\} &= 2[\gamma^A C^{-1}]_{\alpha\beta}P_A\,\delta^{ij} + 2x[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}\,\delta^{ij} + 2C^{-1}_{\alpha\beta}\,\varepsilon^{ij}\mathcal{R}\,.
\end{aligned} \tag{6.18}$$

Here $P_A, M_{AB}\,, \mathcal{R}$ and Q^i_α are the generators of spacetime translations, Lorentz rotations, SO(2) R-symmetry transformations and supersymmetry transformations, respectively. The bosonic generators P_A, M_{AB} and $\mathcal{R}$ are anti-hermitian while the fermionic generators Q^i_α are hermitian. The parameter x is a contraction parameter. Note that the generator of the SO(2) R-symmetry becomes the central element of the Poincaré algebra in the flat limit $x \to 0$.

To make the non-relativistic contraction first we define new supersymmetry charges by

$$Q^\pm_\alpha = \frac{1}{2}\left(Q^1_\alpha \pm \gamma_0 Q^2_\alpha\right), \tag{6.19}$$

and rescale the generators with a parameter ω as follows:

$$\begin{aligned}
&P_0 = \omega\, Z + \frac{1}{2\omega}\, H\,, \qquad \mathcal{R} = -\omega\, Z + \frac{1}{2\omega}\, H\,, \qquad M_{a0} = \omega\, K_a\,,\\
&Q^+ = \frac{1}{\sqrt{\omega}}\,\tilde{Q}^+\,, \qquad Q^- = \sqrt{\omega}\,\tilde{Q}^-\,.
\end{aligned} \tag{6.20}$$

We also set $x = 1/(2\omega R)$. Taking the limit $\omega \to \infty$ and dropping the tildes on the $Q^\pm$ we get the following 3D $\mathcal{N}=(2,0)$ Newton–Hooke superalgebra:

$$\begin{aligned}
[J_{ab},(P/K)_c] &= -2\,\delta_{c[a}(P/K)_{b]}\,, & [H,K_a] &= P_a\,,\\
[H,P_a] &= -\frac{1}{R^2}\,K_a\,, & [H,Q^-] &= \frac{3}{2R}\,\gamma_0 Q^-\\
[J_{ab},Q^\pm] &= -\frac{1}{2}\,\gamma_{ab}Q^\pm\,, & [H,Q^+] &= -\frac{1}{2R}\,\gamma_0 Q^+\,,\\
[K_a,Q^+] &= -\frac{1}{2}\,\gamma_{a0}Q^-\,, & [P_a,Q^+] &= \frac{1}{2R}\,\gamma_a Q^-\,,\\
\{Q^+_\alpha,Q^+_\beta\} &= [\gamma^0C^{-1}]_{\alpha\beta}\,H+\frac{1}{2R}\,[\gamma^{ab}C^{-1}]_{\alpha\beta}J_{ab}\,,\\
\{Q^+_\alpha,Q^-_\beta\} &= [\gamma^aC^{-1}]_{\alpha\beta}\,P_a+\frac{1}{R}\,[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a\,,\\
[P_a,K_b] &= \delta_{ab}\,Z\,, & \{Q^-_\alpha,Q^-_\beta\} &= 2[\gamma^0C^{-1}]_{\alpha\beta}\,Z\,.
\end{aligned} \tag{6.21}$$

The central extension Z that leads to the Bargmann version of the NH superalgebra occurs in the last two (anti-)commutation relations.

6.3.1. Non-linear Realizations

In this section we obtain the action and transformation rules for the flat Galilean and Newton–Hooke superparticles by using the method of non-linear realizations [61,62].[4] We will derive in the first subsection the κ-symmetric action and transformation rules of the Galilean superparticle, see e.g. [68,160]. In the second subsection we will do the same for the NH superparticle. The normalizations of the (to-be) embedding coordinates that occur in this section differ from those in the main text.

The starting point is the $\mathcal{N}=2$ Bargmann superalgebra given in section 6.18. We derive the transformation rules for the coordinates $(t,x^i,s,\theta^\alpha_-,\theta^\alpha_+,v^i)$ using the coset $G/H=\mathcal{N}=2$ super Newton–Hooke$/SO(D-1)$. The coset element is given by $g=g_0U$, where $g_0=e^{Ht}\,e^{P_ix^i}\,e^{Zs}\,e^{Q^-_\alpha\theta^\alpha_-}\,e^{Q^+_\alpha\theta^\alpha_+}$ is the coset representing the ‘empty’ $\mathcal{N}=2$ NH superspace with a central charge extension and $U=e^{K_iv^i}$ is a general NH boost representing the insertion of the particle.

The Maurer–Cartan form associated to the super NH space is given by

$$\Omega_0=g_0^{-1}dg_0=H\,E^0+P_iE^i+ZE_Z+K_i\omega^{i0}+M_{ij}\omega^{ij}-\bar{Q}^-E_- -\bar{Q}^+E_+\,, \tag{6.22}$$

[4]For an early application of this method in a different situation than the one considered in this work, namely to the construction of worldline actions of conformal and superconformal particles, see [63,64].

where (E^0, E^i, E_Z, $E_{-\alpha}$, $E_{+\alpha}$) and (ω^{i0}, ω^{ij}) are the time and space supervielbein and spin-connection components of the NH superspace given explicitly by

$$\begin{aligned}
E^0 &= dt\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2}\,\bar\theta_+\gamma^0 d\theta_+\,,\\
E^i &= dx^i\,(1+\frac{1}{4R}\,\bar\theta_+\theta_+) + \frac{dt}{2R}\,(3\bar\theta_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\,\bar\theta_+\theta_+) - \bar\theta_+\gamma^i d\theta_-\,,\\
E_Z &= ds + \frac{dt}{2R}\,(\frac{x^ix^i}{R} + 3\,\bar\theta_-\theta_-) - \bar\theta_-\gamma^0 d\theta_-\,,\\
\omega^{i0} &= \frac{dt}{R}\,(-\frac{x^i}{R} - \frac{x^i}{4R^2}\,\bar\theta_+\theta_+ + \frac{3}{2R}\,\bar\theta_+\gamma^i\theta_-) - \frac{dx^k\epsilon_{ki}}{4R^2}\,\bar\theta_+\theta_+ - \frac{1}{R}\,\bar\theta_+\gamma^{i0}d\theta_-\,,\\
\omega^{ij} &= -\frac{dt\,\epsilon_{ab}}{8R^2}\,\bar\theta_+\theta_+ - \frac{1}{4R}\,\bar\theta_+\gamma^{ab}d\theta_+\,,
\end{aligned}$$

$$\begin{aligned}
E_- &= d\theta_-\,(1-\frac{1}{2R}\bar\theta_+\theta_+) - \frac{dt}{2R}\,(3\gamma_0\theta_- - \frac{3}{2R}\gamma_0\theta_-\bar\theta_+\theta_+ - \frac{x^i}{R}\,\gamma^{i0}\theta_+) - \frac{dx^i}{2R}\,\gamma^i\theta_+\,,\\
E_+ &= d\theta_+ + \frac{dt}{2R}\,\gamma_0\theta_+\,.
\end{aligned} \tag{6.23}$$

In terms of the supervielbein and the spin-connection the Maurer–Cartan form of the $\mathcal{N}=(2,0)$ NH superparticle is given by with the L's given by

$$\begin{aligned}
&L_H = E^0\,, && L_P^i = E^i + v^i\,E^0\,, && L_Z = E_Z + v^i\,E^i + \frac{v^iv^i}{2}\,E^0\,,\\
&L_K^i = \omega^{i0} - 2v^j\,\omega^{ji} + dv^i\,, && && L_J^{ij} = \omega^{ij}\,,\\
&L_- = E_- - \frac{v^i}{2}\,\gamma_{i0}\,E_+ && && L_+ = E_+\,.
\end{aligned} \tag{6.24}$$

Note that we can writhe the space-time super-translations in matrix form in terms of the vielbein of NH superspace as

$$(\ L_H, L_P{}^a,\ L_{-\alpha},\ L_{+\alpha},\ L_Z\ ) = (\ E^0, E^a,\ E_{-\alpha},\ E_{+\alpha},\ E_Z\ )\begin{pmatrix} 1 & v^i & 0 & 0 & \frac{v^2}{2}\\ 0 & 1 & 0 & 0 & v^i\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & \frac{v^i\gamma_{i0}}{2} & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix} \tag{6.25}$$

allowing to obtain the Maurer–Cartan forms of the NH superspace by a matrix representation of the NH boost.

Using the Maurer–Cartan form one can derive the transformation rules of the Goldstone fields that realize the NH superalgebra (6.21). We find that the bosonic transformation rules are given by

$$\begin{aligned}
\delta t &= -\zeta\,,\\
\delta x^i &= \lambda^i{}_k x^k - a^i\cos\frac{t}{R} + \frac{a^k\varepsilon_{ki}}{4R}\sin\frac{t}{R}\bar\theta_+\theta_+ + v^i R\sin\frac{t}{R} + \frac{v^k\varepsilon_{ki}}{4}\cos\frac{t}{R}\bar\theta_+\theta_+\,,\\
\delta k^i &= \lambda^i{}_k k^k - \frac{a^i}{R}\sin\frac{t}{R}\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - v^i\cos\frac{t}{R}\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+)\,,\\
\delta s &= \frac{a^i x^i}{R}\sin\frac{t}{R} + \frac{a^i}{2R}\sin\frac{t}{R}\bar\theta_-\gamma^i\theta_+ + v^i x^i\cos\frac{t}{R} + \frac{v^i}{2}\cos\frac{t}{R}\bar\theta_-\gamma^i\theta_+\,,\\
\delta\theta_+ &= \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_+\,,\\
\delta\theta_- &= \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_- + \frac{a^i}{2R}\sin\frac{t}{R}\,\gamma^{i0}\theta_+ + \frac{v^i}{2}\cos\frac{t}{R}\,\gamma^{i0}\theta_+\,.
\end{aligned} \tag{6.26}$$

The transformation rules under the ϵ_--supersymmetry transformations are given by

$$\delta\theta_- = \epsilon_-(t) = \exp\Big(\frac{3t}{2R}\,\gamma_0\Big)\epsilon_-\,, \qquad \delta s = \bar\epsilon_-(t)\gamma^0\theta_-\,, \tag{6.27}$$

while all others fields are invariant. For the ϵ_+-transformations we find the following rules

$$\begin{aligned}
\delta t &= \frac{1}{2}\,\bar\epsilon_+(t)\gamma^0\theta_+\,, \qquad\qquad \delta x^i = \bar\epsilon_+(t)\gamma^i\theta_-\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+)\,,\\
\delta k^i &= \frac{k^k}{2R}\,\bar\theta_+\gamma_{ki}\epsilon_+(t) + \frac{x^i}{2R^2}\,\bar\epsilon_+(t)\gamma^0\theta_+ - \frac{1}{R}\,\bar\theta_-\gamma^{i0}\epsilon_+(t)\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+)\,,\\
\delta s &= -\frac{x^i}{2R}\,\bar\epsilon_+(t)\gamma^{i0}\theta_-\,(1-\frac{1}{4R}\bar\theta_+\theta_+) - \frac{x^i x^i}{4R^2}\,\bar\epsilon_+(t)\gamma^0\theta_+ - \frac{1}{2R}\,\bar\epsilon_+(t)\gamma^0\theta_+\,\bar\theta_-\theta_-\,,\\
\delta\theta_+ &= \epsilon_+(t)\,(1-\frac{1}{8R}\,\bar\theta_+\theta_+)\,,\\
\delta\theta_- &= \frac{x^i}{2R}\,\gamma_i\epsilon_+(t)\,(1+\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2R}\,\gamma_i\theta_+\bar\epsilon_+(t)\gamma^i\theta_- + \frac{3}{4R}\,\gamma_0\theta_-\bar\epsilon_+(t)\gamma^0\theta_+\,,
\end{aligned} \tag{6.28}$$

with

$$\epsilon_+(t) = \exp\Big(\frac{-t}{2R}\,\gamma_0\Big)\epsilon_+\,. \tag{6.29}$$

6.3.2. The Killing Equations of the Newton-Hooke Superparticle

The action of the $\mathcal{N}=(2,0)$ NH superparticle is given by the pull-back of all the L's that are invariant under rotations:

$$
\begin{aligned}
S &= a\int (L_H)^* + b\int (L_Z)^* \\
&= a\int d\tau\Big(\dot t\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2}\,\bar\theta_+\gamma^0\dot\theta_+\Big) \\
&\quad + b\int d\tau\Big[ds + \frac{dt}{2R}\,(\frac{x^i x^i}{R} + 3\,\bar\theta_-\theta_-) - \bar\theta_-\gamma^0 d\theta_- \\
&\qquad + v^i\Big(dx^i\,(1+\frac{1}{4R}\,\bar\theta_+\theta_+) + \frac{dt}{2R}\,(3\bar\theta_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\,\bar\theta_+\theta_+) - \bar\theta_+\gamma^i d\theta_-\Big) \\
&\qquad + \frac{v^2}{2}\Big(\dot t\,(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2}\,\bar\theta_+\gamma^0\dot\theta_+\Big)\Big]\,.
\end{aligned}
\tag{6.30}
$$

We will consider that the coordinates transform in the following way:

$$
\begin{gathered}
\delta x^i = \xi^i(t,x,\theta_\pm)\,,\qquad \delta t = \xi^0(t,x,\theta_\pm)\,,\qquad \delta\theta_\pm = \chi_\pm(t,x,\theta_\pm)\,,\\
\delta v^i = \zeta^i(t,x,v,\theta_\pm)\,,\qquad \delta s = \eta(t,x,\theta_\pm)\,.
\end{gathered}
\tag{6.31}
$$

Since a and b are independent, the invariance of the variation of the action $\delta S=0$ lead to the following sets of Killing equations

$$
\begin{aligned}
0 &= \partial_t\xi^0(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2R}\bar\theta_+\chi_+ - \frac{1}{2}\bar\theta_+\gamma^0\partial_t\chi_+\,,\\
0 &= \partial_j\xi^0(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2}\bar\theta_+\gamma^0\partial_t\chi_+\,,\\
0 &= \partial_{\theta_-}\xi^0(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{2}\bar\theta_+\gamma^0\partial_{\theta_-}\chi_+\,,\\
0 &= \partial_{\theta_+}\xi^0(1-\frac{1}{4R}\,\bar\theta_+\theta_+) + \frac{1}{2}\bar\chi_+\gamma^0 - \frac{1}{2}\bar\theta_+\gamma^0\partial_{\theta_+}\chi_+\,,
\end{aligned}
\tag{6.32}
$$

and for the second part of the action

$$
\begin{aligned}
0 &= \frac{1}{2R}\partial_t\xi^0\Big(\frac{x^2}{R} + 3\bar\theta_-\theta_- + 3\bar\theta_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\bar\theta_+\theta_+\Big) + v^i\partial_t\xi^i(1+\frac{1}{4R}\,\bar\theta_+\theta_+)\\
&\quad + v^i\zeta^i(1-\frac{1}{4R}\,\bar\theta_+\theta_+) - \frac{1}{4R^2}v^i\xi^k\epsilon_{ki}\bar\theta_+\theta_+ + \partial_t\eta + \frac{1}{R^2}x^i\xi^i\\
&\quad + \frac{1}{2R}\zeta_i\Big(3\bar\theta_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\bar\theta_+\theta_+\Big) + \frac{3}{R}\bar\theta_-\chi_- - \bar\theta_-\gamma^0\partial_t\chi_-\\
&\quad + \frac{3v^i}{2R}\bar\theta_+\gamma^{0i}\chi_+\gamma^{0i}\chi_- - v^i\bar\theta_+\gamma^i\partial_t\chi_- - \frac{3}{2R}v^i\bar\theta_-\gamma^{0i}\chi_+ - \frac{1}{2R^2}v^i x^k\epsilon_{ki}\bar\theta_+\chi_+\,,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{1}{2R}\partial_i\xi^0\Big(\frac{x^2}{R} + 3\bar{\theta}_-\theta_- + 3\bar{\theta}_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\bar{\theta}_+\theta_+\Big) + v^j\partial_i\xi^j(1 + \frac{1}{4R}\,\bar{\theta}_+\theta_+)\\
&\quad + \zeta^i(1 + \frac{1}{4R}\,\bar{\theta}_+\theta_+) + \partial_i\eta - \bar{\theta}_-\gamma^0\partial_i\chi_- - v^j\bar{\theta}_+\gamma^j\partial_i\chi_- + \frac{1}{2R}v^i\bar{\theta}_+\chi_+\,,\\
0 &= \partial_{\theta_-}\xi^0\Big(\frac{x^2}{R} + 3\bar{\theta}_-\theta_- + 3\bar{\theta}_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\bar{\theta}_+\theta_+\Big) + v^i\partial_{\theta_-}\xi^i(1 + \frac{1}{4R}\,\bar{\theta}_+\theta_+)\\
&\quad - \zeta^i\bar{\theta}_+\gamma^i + \partial_{\theta_-}\eta - \bar{\chi}_-\gamma^0 - \bar{\theta}_-\gamma^0\partial_{\theta_-}\chi_- + v^i\partial_{\theta_-}\chi_- - v^i\bar{\chi}_+\gamma^i\,,\\
0 &= \partial_{\theta_+}\xi^0\Big(\frac{x^2}{R} + 3\bar{\theta}_-\theta_- + 3\bar{\theta}_+\gamma^{0i}\theta_- - \frac{x^k\epsilon_{ki}}{2R}\bar{\theta}_+\theta_+\Big) + v^i\partial_{\theta_+}\xi^i(1 + \frac{1}{4R}\,\bar{\theta}_+\theta_+)\\
&\quad - \frac{1}{2}v^i\zeta^i\bar{\theta}_+\gamma^0 + \partial_{\theta_+}\eta - \bar{\theta}_-\gamma^0\partial_{\theta_+}\chi_- - v^i\bar{\theta}_+\gamma^i\partial_{\theta_+}\chi_-\,.
\end{aligned}
\tag{6.33}
$$

The transformations rules given in eqs. (6.26)-(6.28) are solutions of these sets of Killing equations.

6.3.3. The Kappa-symmetric Newton Hooke Superparticle

We are now ready to derive the action and κ-transformation rules. To derive an action that is invariant under κ-transformations we need to find a fermionic gauge-transformation that leaves L_H and/or L_Z invariant.[5]

It is convenient to define the line-elements

$$
\pi^0 = \dot{t}\,(1 + \frac{1}{8R}\,\bar{\theta}_+\theta_+) + \frac{1}{4}\,\bar{\theta}_+\gamma^0\dot{\theta}_+\,,
\tag{6.34}
$$

$$
\pi^i = \big(\dot{x}^i + \tfrac{1}{4}\,\bar{\theta}_+\gamma^i\dot{\theta}_- + \tfrac{1}{4}\,\bar{\theta}_-\gamma^i\dot{\theta}_+\big)\,\big(1 - \tfrac{1}{8R}\,\bar{\theta}_+\theta_+\big) - \tfrac{\dot{t}}{4R}\,\big(3\bar{\theta}_+\gamma^{0i}\theta_- - \tfrac{x^k\varepsilon_{ki}}{2R}\,\bar{\theta}_+\theta_+\big)\,.
\tag{6.35}
$$

which are related to the Maurer–Cartan form via the pull-backs

$$
(L_H)^* = \pi^0\,, \qquad (L_P)^* = \pi^i + k^i\pi^0\,.
\tag{6.36}
$$

Now, the variation of L_H and L_Z under gauge-transformations is given by

$$
\begin{aligned}
\delta L_H &= d[\delta z_H] - \bar{L}_+\gamma^0[\delta z_+]\,,\\
\delta L_Z &= d[\delta z_Z] - 2\bar{L}_-\gamma^0[\delta z_-] - \delta_{ab}\big(L_P^a\,[\delta z_G^b] - L_G^a\,[\delta z_P^b]\big)\,.
\end{aligned}
\tag{6.37}
$$

[5] The case of the $SU(1,1|2)$ superconformal particle is discussed in [161].

For κ-transformations we find, using the explicit expressions for L_+ and L_-,

$$\begin{aligned}
(\delta L_H)^* &= [\delta\bar{z}_+](\gamma^0\dot{\theta}_+ + \frac{\dot{t}}{2R}\theta_+)\,,\\
(\delta L_Z)^* &= 2[\delta\bar{z}_-]\gamma^0\Big(\dot{\theta}_-(1+\frac{1}{2R}\,\bar{\theta}_+\theta_+) - \frac{\dot{t}}{2R}\,(3\gamma_0\theta_- - \frac{3}{2R}\gamma_0\theta_-\bar{\theta}_+\theta_+ - \frac{x^i}{R}\,\gamma^{i0}\theta_+)\\
&\qquad - \frac{\dot{x}^i}{2R}\,\gamma^i\theta_+ - \frac{k^i}{2}\,\gamma_{i0}(\dot{\theta}_+ + \frac{\dot{t}}{2R}\gamma_0\theta_+\Big)\,.
\end{aligned} \tag{6.38}$$

It follows that to obtain a κ-symmetric action we need to take the pull-back of either L_H or L_Z, with $[\delta z_+]=0$ or $[\delta z_-]=0$, respectively. We focus here on the second case, i.e. we choose $a=0$ and we take

$$S = \int (L_Z)^*\,, \qquad [\delta z_+] = \kappa\,, \qquad [\delta z_-] = 0\,, \tag{6.39}$$

with κ an arbitrary (local) parameter. To compare to the action and transformations rules given in [162] one needs to make the following redefinitions

$$t\to -t\,, \qquad x^i \to -x^i + \frac{1}{2}\,\bar{\theta}_+\gamma^i\theta_-\,, \qquad \pi^0\to-\pi^0\,, \qquad \pi^i\to-\pi^i\,, \tag{6.40}$$

rescale all spinors by $1/\sqrt{2}$, together with $R\to -R$ this leads to the κ-symmetric Newton–Hooke superparticle action:

$$S = \int d\tau\,\tfrac{m}{2}\,\Big[\tfrac{\pi^i\pi^i}{\pi^0} - \bar{\theta}_-\gamma^0\dot{\theta}_- - \dot{t}\,\tfrac{x^ix^i}{R^2} + \tfrac{3\dot{t}}{2R}\,\bar{\theta}_-\theta_- + \tfrac{\dot{t}\,x^i}{2R^2}\,\bar{\theta}_+\gamma^i\theta_- - \tfrac{\dot{t}}{16R^2}\,\bar{\theta}_+\theta_+\,\bar{\theta}_-\theta_-\Big]\,, \tag{6.41}$$

The κ-transformations read as follows:

$$\begin{aligned}
\delta t &= \frac{1}{4}\,\bar{\kappa}\gamma^0\theta_+\,, & \delta x^i &= \frac{1}{4}\,\bar{\kappa}\gamma^i\theta_- - \frac{\pi^j}{8\pi^0}\,\bar{\kappa}\gamma_j\gamma_{i0}\theta_+\,,\\
\delta\theta_+ &= \kappa\,(1-\tfrac{1}{16R}\,\bar{\theta}_+\theta_+)\,, & \delta\theta_- &= -\tfrac{\pi^i}{2\pi^0}\,\gamma_{i0}\kappa - \tfrac{3}{8R}\,\gamma_0\theta_-\bar{\theta}_+\gamma^0\kappa + \tfrac{x^i}{16R^2}\,\gamma^i\kappa\,\bar{\theta}_+\theta_+\,.
\end{aligned} \tag{6.42}$$

The κ-symmetry can be gauge-fixed by imposing the gauge condition $\theta_+=0$. We find that the action of the NH superparticle takes the form

$$S = \int d\tau\,\frac{m}{2}\,\left[\frac{\dot{x}^i\dot{x}^i}{\dot{t}} - \bar{\theta}_-\gamma^0\dot{\theta}_- - \frac{\dot{t}\,x^ix^i}{R^2} + \frac{3\dot{t}}{2R}\,\bar{\theta}_-\theta_-\right]\,. \tag{6.43}$$

A realization of the Newton–Hooke superalgebra, whose explicit form is given in eq. (6.21), on the embedding coordinates is given by the following bosonic transformation rules

$$\delta t = -\zeta\,, \qquad \delta x^i = \lambda^i{}_k x^k - \xi^i(t)\,, \qquad \delta\theta_- = \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_-\,, \tag{6.44}$$

supplemented with the following fermionic transformations:

$$\delta t = 0\,, \qquad \delta x^i = -\frac{1}{2}\,\bar\epsilon_+(t)\gamma^i\theta_-\,, \qquad \delta\theta_- = \epsilon_-(t) - \frac{\dot x^i}{2\dot t}\,\gamma_{0i}\,\epsilon_+(t) + \frac{x^i}{2R}\,\gamma_i\,\epsilon_+(t)\,. \tag{6.45}$$

Here, the time-dependence of the parameters $\xi^i(t)$ and $\epsilon_\pm(t)$ is given by

$$\begin{aligned} &\xi^i(t) = v^i R\sin\frac{t}{R} + a^i\cos\frac{t}{R}\,, \\ &\epsilon_-(t) = \exp\Big(\frac{3t}{2R}\,\gamma_0\Big)\,\epsilon_-\,, \qquad \epsilon_+(t) = \exp\Big(-\frac{t}{2R}\,\gamma_0\Big)\,\epsilon_+\,. \end{aligned} \tag{6.46}$$

6.3.4. The Flat Limit

Since the NH superalgebra is a deformation of the Galilei algebra, by taking the flat limit $R\to\infty$, the 3D $\mathcal{N}=2$ Galilei superalgebra is given by the bosonic commutation relations

$$\begin{aligned} &[J_{ab}, P_c] = -2\,\delta_{c[a}P_{b]}\,, \qquad [K_a, H] = -P_a\,, \\ &[J_{ab}, K_c] = -2\,\delta_{c[a}K_{b]}\,, \qquad [J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}\,, \\ &[P_a, K_b] = \delta_{ab}\,Z\,, \end{aligned} \tag{6.47}$$

plus the additional relations [150, 151]

$$\begin{aligned} &[J_{ab}, Q^\pm] = -\frac{1}{2}\,\gamma_{ab}\,Q^\pm\,, \qquad [K_a, Q^+] = -\frac{1}{2}\,\gamma_{a0}\,Q^-\,, \\ &\{Q^+_\alpha, Q^+_\beta\} = 2\,[\gamma^0 C^{-1}]_{\alpha\beta}\,H\,, \qquad \{Q^+_\alpha, Q^-_\beta\} = [\gamma^a C^{-1}]_{\alpha\beta}\,P_a\,, \\ &\{Q^-_\alpha, Q^-_\beta\} = 2\,[\gamma^0 C^{-1}]_{\alpha\beta}\,Z\,. \end{aligned} \tag{6.48}$$

In the flat case the spin-connection vanishes and the time and spatial components of the supervielbein simplify to

$$\begin{aligned} &E^0 = dt - \frac{1}{2}\,\bar\theta_+\gamma^0 d\theta_+\,, \qquad E^i = dx^i - \bar\theta_+\gamma^i d\theta_-\,, \qquad E_Z = ds - \bar\theta_-\gamma^0 d\theta_-\,, \\ &E_- = d\theta_-\,, \qquad E_+ = d\theta_+\,. \end{aligned} \tag{6.49}$$

and the Maurer–Cartan 1-forms are given by

$$\begin{aligned} &L_H = E^0\,, \qquad L_P^i = E^i + v^i\,E^0\,, \qquad L_Z = E_Z + v^i\,E^i + \frac{v^i v^i}{2}\,E^0\,, \\ &L_K^i = dv^i\,, \qquad L_J^{ij} = 0\,, \qquad L_- = E_- - \frac{v^i}{2}\,\gamma_{i0}\,E_+ \qquad L_+ = E_+\,. \end{aligned} \tag{6.50}$$

The action of the Galilean superparticle is given by the pull-back of all L's that are invariant under rotations, hence

$$S = a\int (L_H)^* + \int (L_Z)^* = \int d\tau\,\left[-\frac{a}{2}\,\bar{\theta}_+\gamma^0\dot{\theta}_+ - \bar{\theta}_-\gamma^0\dot{\theta}_- - \frac{\pi^i\pi^i}{2\pi^0}\right]\,, \tag{6.51}$$

where the line-elements are

$$\pi^0 = \dot{t} - \frac{1}{2}\,\bar{\theta}_+\gamma^0\dot{\theta}_+\,, \qquad \pi^i = \dot{x}^i - \bar{\theta}_+\gamma^i\dot{\theta}_-\,, \tag{6.52}$$

Here we replaced the Goldstone field v^i by its equation of motion $v^i = -\pi^i/\pi^0$. This procedure is know as inverse Higgs mechanism [163], see also [164]. The bosonic transformations of the embedding coordinates are

$$\begin{aligned} &\delta t = -\zeta\,, \qquad \delta x^i = \lambda^i{}_k x^k - a^i + v^i t + \frac{v^k\varepsilon_{ki}}{4}\,\bar{\theta}_+\theta_+\,, \\ &\delta\theta_+ = \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_+\,, \qquad \delta\theta_- = \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_- + \frac{v^i}{2}\,\gamma^{i0}\theta_+\,, \end{aligned} \tag{6.53}$$

and the supersymmetry transformations are

$$\delta t = \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\theta_+\,, \qquad \delta x^i = \bar{\epsilon}_+\gamma^i\theta_-\,, \qquad \delta\theta_\pm = \epsilon_\pm\,. \tag{6.54}$$

These transformations leave the action (6.51) and all L's, in particular the line-elements (6.52), invariant. The κ-transformations of the coordinates are

$$\delta t = -\frac{1}{2}\,\bar{\kappa}\gamma^0\theta_+\,, \quad \delta x^i = -\frac{\pi^j}{2\pi^0}\,\bar{\theta}_+\gamma^i\gamma_{j0}\kappa\,, \quad \delta\theta_+ = \kappa\,, \quad \delta\theta_- = -\frac{\pi^i}{2\pi^0}\,\gamma_{i0}\kappa\,, \tag{6.55}$$

and the corresponding κ-symmetric action is given by

$$S = \int d\tau\,\frac{m}{2}\,\left[-2\bar{\theta}_-\gamma^0\dot{\theta}_- - \frac{\pi^i\pi^i}{\pi^0}\right]\,. \tag{6.56}$$

6.4. Gauging Procedure: Bosonic Case

6.4.1. The Galilean Particle

In this section we will briefly review the gauging procedure for the particle case as it was done in [165]. The action describing a particle of mass m moving in a D-dimensional Minkowski spacetime parametrized by the evolution parameter τ is

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,, \qquad \mu = 0, 1, \dots D-1\,, \tag{6.57}$$

This Lagrangian is invariant under worldline reparametrizations and under Poincaré transformations

$$\delta x^\mu = \lambda^\mu_\nu x^\nu + \zeta^\mu\,, \tag{6.58}$$

where λ^μ_ν and ζ^μ are the parameters for Lorentz transformations and translations respectively. Taking the non-relativistic limit by rescaling the longitudinal coordinate $x^0 \equiv t$ and the mass m with a parameter ω

$$t \to \omega t\,, \qquad m \to \omega m\,, \tag{6.59}$$

and taking the limit $\omega \to \infty$, this rescaling is such that the kinetic term remains finite. In this limit, action (6.57) can be written as

$$S = \frac{m}{2} \int \frac{\dot{x}^i \dot{x}^i}{\dot{t}} d\tau\,. \tag{6.60}$$

This non-relativistic action is invariant under worldline reparametrizations and under the Galilei symmetries

$$\begin{aligned} \delta t &= \rho(\tau)\dot{t}\,, \qquad & \delta x^i &= \rho(\tau)\dot{x}^i\,, \\ \delta t &= -\zeta\,, \qquad & \delta x^i &= \lambda^i{}_j x^j - v^i t - a^i\,, \end{aligned} \tag{6.61}$$

where $(\zeta, a^i, \lambda^i{}_j, v^i)$ parametrize a constant time and space translations, spatial rotation and a boost transformation and $\rho(\tau)$ is the diffeomorphisms parameter. However, this Lagrangian is invariant under boosts only up to a total τ-derivative, leading to a modified Nöether charge giving rise to a centrally extended Galilei algebra containing an extra central charge generator Z. This centrally extended Galilei algebra is called the Bargmann algebra.

This results apply to free falling frames without any gravitational interactions. These frames are connected to each other through the Galilei symmetries. We now wish to extend these results to include frames that are accelerated with respect to the free falling frames. There are two procedures to achieve this: the first one valid in frames with time-dependent acceleration, consists on gauging the spatial translations and it is described in section 6.4.2. In the second one, described in section 6.4.3, one gauges all the symmetries of the Bargmann algebra.

6.4.2. The Curved Galilean Particle

The Curved Galilean Particle is valid in frames with constant acceleration. The procedure to obtain it goes as follows: gauge the spatial translations by allowing arbitrary time-dependent boosts, with parameters $\xi^i(t)$. The complete bosonic and fermionic transformation rules now read. By doing so we obtain a gauged action containing the gravitational potential $\Phi(x)$

$$L = \frac{m}{2}\left(\frac{\dot{x}^i\dot{x}^i}{\dot{t}} - 2\dot{t}\Phi(x)\right) . \tag{6.62}$$

Now the Lagrangian is invariant under worldline reparametrizations and the acceleration extended symmetries

$$\delta t = -\zeta + \rho(\tau)\dot{t}\,, \qquad \delta x^i = \lambda^i{}_j x^j - \xi^i(t) + \rho(\tau)\dot{x}^i\,, \tag{6.63}$$

The acceleration-extended symmetries are not a proper symmetry of the action (6.62). Instead, the Newton potential should be viewed as a background field and the acceleration-extended symmetries as sigma model symmetries. In particular, the transformation rule of the background field, that we will denote by the symbol $\delta_{\rm bg}$, lacks the transport terms that are present in the transformation rule associated to a proper symmetry, denoted in this chapter by $\delta_{\rm pr}$: [6]

$$\delta_{\rm bg} = \delta_{\rm pr} + \delta x^\mu \partial_\mu\,. \tag{6.64}$$

Using this we find that the action (6.62) is invariant under the acceleration-extended symmetries (6.63) provided the Newton potential Φ transforms as follows:

$$\delta\Phi = \frac{1}{\dot{t}}\frac{d}{d\tau}\left(\frac{\dot{\xi}^i}{\dot{t}}\right)x^i + \rho(\tau)\dot{\Phi} + \sigma(t)\,. \tag{6.65}$$

The last term represents gives a boundary term in the action.

[6] Assuming that $x^\mu \to x^\mu + \delta x^\mu$ a transport term is given by $-\delta x^\mu\partial_\mu$ so that the second term in (6.64) cancels the transport term present in the proper transformation rule represented by the first term.

6.4.3. The Newton-Cartan Particle

We now wish to extend the Curved Galilean particle to a NC particle, i.e. a particle moving in a NC gravity background and invariant under general coordinate transformations, see [45] for the detailed procedure. The first step to obtain the corresponding action is to gauge all the symmetries of the Bargmann algebra: time and space translations, spatial rotations, boosts and central charge transformations. Associate a gauge field to each of the symmetries and promote the constant parameters describing the transformations to arbitrary functions of the spacetime coordinates x^μ.

Besides these transformations all gauge fields transform under general coordinate transformations with parameters $\xi^\mu(x^\mu)$. Therefore, for each generator we have associated gauge fields, gauge parameters and curvatures.

The next step in the gauging procedure is to impose a set of constraints on the curvatures of the gauge fields. With these constraints the time and space translations become equivalent to general coordinate transformations modulo the other symmetries of the algebra. This enables one to solve for the gauge fields of boosts and spatial rotations ${\omega_\mu}^a$ and ${\omega_\mu}^{ab}$ in terms of the other ones, so the independent gauge fields will be $(\tau_\mu, {e_\mu}^a, m_\mu)$. The gauge fields τ_μ and ${e_\mu}^a$ of time and spatial translations are identified as the temporal and spatial vielbeins.

The dynamics of the Newton-Cartan point particle is described by the following Lagrangian

$$L = \frac{m}{2}\left(\frac{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\tau_\rho\dot{x}^\rho} - 2m_\mu\dot{x}^\mu\right), \qquad h_{\mu\nu} = {e_\mu}^a {e_\nu}^b \delta_{ab}\,. \tag{6.66}$$

The theory is invariant under general coordinate transformations plus boosts, spatial rotations and central charge transformations, all with parameters that are arbitrary functions of the spacetime coordinates.

The embedding coordinates transform under these general coordinate transformations, with parameters $\xi^\mu(x^\nu)$, in the standard way:

$$\delta x^\mu = -\xi^\mu(x^\nu)\,. \tag{6.67}$$

The transformation rules of the background fields τ_μ, ${e_\mu}^a$ and m_μ follow from the known proper transformation rules by omitting the transport term, see eq. (6.64):

$$\begin{aligned}
\delta_{\rm bg}\tau_\mu &= \partial_\mu\xi^\rho\,\tau_\rho\,,\\
\delta_{\rm bg}{e_\mu}^a &= \partial_\mu\xi^\rho\,{e_\rho}^a + {\lambda^a}_b\,{e_\mu}^b + \lambda^a\,\tau_\mu\,,\\
\delta_{\rm bg}m_\mu &= \partial_\mu\xi^\rho\,m_\rho + \partial_\mu\sigma + \lambda_a {e_\mu}^a\,.
\end{aligned} \tag{6.68}$$

Here, $\lambda^a{}_b\,,\lambda^a$ and σ are the parameters of a local spatial rotation, boost transformation and central charge transformation, respectively.

The proper transformation rules of the background fields $\tau_\mu\,, e_\mu{}^a$ and m_μ can be obtained by gauging the Bargmann algebra, see e.g. [45]. Since the NC background is the most general background one must be able to obtain the transformations of the Curved Galilean and flat backgrounds discussed in the two previous subsections by gauge-fixing some of the general coordinate transformations. This is discussed in detail in [48].

For the convenience of the reader, we list in table 6.1 the gauge-fixing conditions that need to be imposed on the NC background fields, and the compensating gauge transformations that come along with it, that bring us to the Curved Galilean background in terms of the Newton potential Φ.

gauge condition	compensating transformations
$\tau_\mu(x^\nu) = \delta_\mu{}^\emptyset$	$\xi^\emptyset(x^\nu) = \xi^\emptyset$
$\omega_\mu{}^{ab} = 0$	$\lambda^{ab}(x^\nu) = \lambda^{ab}$
$e_i{}^a = \delta_i{}^a$	$\xi^i(x^\nu) = \xi^i(t) - \lambda^i{}_j x^j$
$\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu)$	
$m(x^\nu) = 0$	$\sigma(x^\nu) = \sigma(t) + \partial_t\xi^i(t)\, x^i$
$\tau_i(x^\nu) = 0$	$\lambda^i(x^\nu) = -\partial_t\xi^i(t)$
$m_\emptyset(x^\nu) = \Phi(x^\nu)$	$\omega_\emptyset{}^a = -\partial_a\Phi(x^\nu)$

TABLE 6.1

This table indicates the gauge-fixing conditions, chronologically ordered from top to bottom, and the corresponding compensating transformations, that lead from the NC particle, to the Curved Galilean particle. Note that $\tau_i(x^\nu) \equiv e_i{}^0(x^\nu)$. The restriction $\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu)$ follows from the gauge conditions made at that point.

6.5. Gauging Procedure: Supersymmetric Case

6.5.1. The Galilean Superparticle

The Galilean superparticle was already discussed in [127]. In terms of the bosonic and fermionic embedding coordinates $\{t, x^i, \theta_-\}$ the action is given by

$$S = \int d\tau\, \frac{m}{2}\left[\frac{\dot{x}^i\dot{x}^i}{\dot{t}} - \bar{\theta}_-\gamma^0\dot{\theta}_-\right]. \qquad (6.69)$$

This action is invariant under the following Galilei symmetries:

$$\delta t = -\zeta\,, \qquad \delta x^i = \lambda^i{}_j x^j - v^i t - a^i\,. \qquad \delta\theta_- = \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_-\,. \tag{6.70}$$

The same action is invariant under two supersymmetries with constant parameters ϵ_+ and ϵ_-:

$$\delta t = 0\,, \qquad \delta x^i = -\frac{1}{2}\bar{\epsilon}_+\gamma^i\theta_-\,, \qquad \delta\theta_- = \epsilon_- - \frac{\dot{x}^i}{2\dot{t}}\gamma_{0i}\epsilon_+\,. \tag{6.71}$$

One may verify that the above set of transformation rules (6.70) and (6.71) closes off-shell. Note that the transformation with parameter ϵ_+ is realized linearly. Instead, the one with parameter ϵ_- is realized non-linearly, i.e. it is a broken symmetry. This implies that the superparticle corresponds to a 1/2 BPS state.

6.5.2. The Curved Galilean Superparticle

We will now extend the Galilean superparticle to a Curved Galilean superparticle thereby replacing the flat background by a Galilean supergravity background. This corresponds to extending the bosonic particle in a Galilean gravity background, discussed in section 6.4.2, to the supersymmetric case.

Our starting point is the superparticle action in a flat background, see eq. (6.69). We will now partially gauge the transformations (6.70) and (6.71) to allow for arbitrary time-dependent boosts, with parameters $\xi^i(t)$, and arbitrary supersymmetry transformations, with parameters $\epsilon_-(t)$. The complete bosonic and fermionic transformation rules now read

$$\delta t = -\zeta\,, \qquad \delta x^i = \lambda^i{}_j x^j - \xi^i(t)\,, \qquad \delta\theta_- = \frac{1}{4}\lambda^{ab}\gamma_{ab}\theta_-\,, \tag{6.72}$$

and

$$\delta t = 0\,, \qquad \delta x^i = -\frac{1}{2}\bar{\epsilon}_+\gamma^i\theta_-\,, \qquad \delta\theta_- = \epsilon_-(t) - \frac{\dot{x}^i}{2\dot{t}}\gamma_{0i}\epsilon_+\,, \tag{6.73}$$

respectively.

The Galilean supergravity multiplet, that we will use to perform the partial gauging of the transformations (6.70) and (6.71), was introduced in [48]. Alongside the Newton potential $\Phi(x)$, it also contains a fermionic background field $\Psi(x)$. The equations of motion for these two background fields are:

$$\partial^i\partial_i\Phi = 0\,, \qquad \gamma^i\partial_i\Psi = 0\,. \tag{6.74}$$

There is a slight subtlety regarding this Galilean supergravity multiplet, stemming from the fact that $\Psi(x)$ is the superpartner of the Newton force $\Phi_i \equiv \partial_i\Phi(x)$ and not of the Newton potential itself. The transformation rules of the Newton force are, however, compatible with the integrability condition $\partial_{[i}\Phi_{j]} = 0$, so that they can be integrated to transformation rules of the Newton potential $\Phi(x)$. This is done via the introduction of a fermionic prepotential $\chi(x)$, that will be called the 'Newtino potential', defined via

$$\partial_i\,\chi = \gamma_i\Psi \quad (i = 1, 2)\,, \qquad\qquad \gamma^1\partial_1\chi = \gamma^2\partial_2\chi\,, \tag{6.75}$$

where the second equation represents a constraint obeyed by $\chi(x)$, as a consequence of its definition. This constraint can be interpreted (upon choosing a specific basis for the γ-matrices) as the Cauchy–Riemann equations, expressing holomorphicity of $\chi_1 + \mathrm{i}\,\chi_2$, where $\chi_{1,2}$ are the components of χ. Since the Newton potential Φ obeys the Laplace equation in two spatial dimensions, it can also be seen as the real part of a holomorphic function $\Phi + \mathrm{i}\,\Xi$. The imaginary part $\Xi(x)$ of this function was called the 'dual Newton potential' in [48] and is related to $\Phi(x)$ via the Cauchy–Riemann equations for $\Phi + \mathrm{i}\,\Xi$:

$$\partial_i\Phi = \varepsilon_{ij}\partial^j\Xi\,, \qquad\qquad \partial_i\Xi = -\varepsilon_{ij}\partial^j\Phi\,. \tag{6.76}$$

The dual Newton potential was introduced in [48] in order to write down the supersymmetry transformation rule for χ. Since this is a transformation rule for both real and imaginary parts of $\chi_1 + \mathrm{i}\,\chi_2$, it is natural to expect that it involves also both real and imaginary parts of $\Phi + \mathrm{i}\,\Xi$ and this is indeed the case.

We find that the action of the Curved Galilean superparticle in terms of the Galilean supergravity background fields Φ and Ψ is given by

$$S = \int d\tau\,\frac{m}{2}\left[\frac{\dot{x}^i\dot{x}^i}{\dot{t}} - \bar{\theta}_-\gamma^0\dot{\theta}_- - 2\dot{t}\,\Phi + 2\dot{t}\,\bar{\theta}_-\gamma^0\Psi\right]\,. \tag{6.77}$$

One may verify that the action (6.77) is invariant under the transformations (6.72) and (6.73) provided that the background fields transform under the bosonic symmetries as

$$\delta_{\rm bg}\Phi = \frac{1}{\dot{t}}\frac{d}{d\tau}\left(\frac{\dot{\xi}^i}{\dot{t}}\right)x^i + \sigma(t)\,, \qquad\qquad \delta_{\rm bg}\Psi = \frac{1}{4}\lambda^{ab}\gamma_{ab}\Psi\,, \tag{6.78}$$

and under the fermionic symmetries as

$$\delta_{\rm bg}\Phi = \bar{\epsilon}_-\gamma^0\Psi + \frac{1}{2}\,\bar{\epsilon}_+\partial_t\chi - \frac{1}{2}\,\bar{\epsilon}_+\gamma^i\theta_-\,\partial_i\Phi\,, \tag{6.79}$$

$$\delta_{\rm bg}\Psi = \frac{1}{\dot{t}}\,\dot{\epsilon}_- - \frac{1}{2}\,\partial_i\Phi\gamma_{i0}\epsilon_+ - \frac{1}{2}\,\bar{\epsilon}_+\gamma^i\theta_-\,\partial_i\Psi\,. \tag{6.80}$$

The only invariance that is non-trivial to show is the one under the linear ϵ_+-transformations. Varying the action (6.77) under ϵ_+-transformations one is left with the following terms:

$$\delta_+ S = \int d\tau \, \frac{m}{2} \Big[- \bar{\epsilon}_+ \, \dot{t} \, \partial_t \chi - \bar{\epsilon}_+ \, \dot{x}^i \, \partial_i \chi - \frac{\dot{t}}{2} \, \bar{\epsilon}_+ \gamma^k \theta_- \, \bar{\theta}_- \gamma^{0i} \partial_i \partial_k \chi \Big] \, . \tag{6.81}$$

The first two terms combine into a total τ-derivative, since χ is a function of x^i and t and therefore

$$\frac{d}{d\tau} \chi = (\dot{t} \, \partial_t + \dot{x}^i \, \partial_i) \chi \, . \tag{6.82}$$

The second term vanishes upon using the equation of motion for the background field χ.

To calculate the commutator algebra it is important to keep in mind that the background fields do not transform as fundamental fields but, instead, according to background fields, see eq. (6.64). This explains the 'wrong' sign transport term in the ϵ_+-transformation and the absence of transport terms for all other symmetries. It also has the consequence that partial derivatives do not commute with background variations:

$$\big[\delta_{\mathrm{bg}}, \partial_\mu \big] = -(\partial_\mu \delta x^\nu) \partial_\nu \, . \tag{6.83}$$

Another subtlety when calculating the commutator algebra is related to the fact that the parameters ξ^i and ϵ_- are functions of the time t but that t itself is a scalar function $t(\tau)$ of the world-line parameter τ. This implies that when we calculate commutators we have to vary the t inside the parameters. Keeping the above subtleties in mind we find that the commutation relations close off-shell on the embedding coordinates and the background fields.

Imposing the gauge-fixing conditions

$$\Phi = 1 \, , \qquad\qquad \chi = 0 \, , \tag{6.84}$$

we recover the Galilean superparticle with the flat spacetime transformation rules (6.70) and (6.71). Imposing the additional gauge-fixing condition

$$t = \tau \tag{6.85}$$

we find agreement with the algebra obtained in [48].

6.5.3. The Newton–Cartan Superparticle

We wish to extend the result of the previous subsection to arbitrary frames corresponding to a superparticle in a Newton–Cartan supergravity background. Due to the complexity of the calculations we only give the result up to quartic fermions in the action. We find that using this approximation the action is given by

$$S = \int d\tau \, \frac{m}{2} \left[\frac{\dot{x}^\mu e_\mu{}^a \, \dot{x}^\nu e_{\nu\, a}}{\dot{x}^\rho \tau_\rho} - 2m_\mu \dot{x}^\mu - \bar{\theta}_- \gamma^0 D_\tau \theta_- + 2\, \bar{\theta}_- \gamma^0 \psi_{\mu -} \dot{x}^\mu \right. \\ \left. - \frac{\dot{x}^\mu e_{\mu a}}{\dot{x}^\rho \tau_\rho} \, \bar{\theta}_- \gamma^a \psi_{\nu +} \dot{x}^\nu \right] , \tag{6.86}$$

where the Lorentz-covariant derivative D_τ is defined as

$$D_\tau \theta_- = \dot{\theta}_- - \frac{1}{4} \, \dot{x}^\mu \omega_\mu{}^{ab} \gamma_{ab} \theta_- \,. \tag{6.87}$$

To lowest order in the fermions the action (6.86) is invariant under the following bosonic and fermionic symmetries of the embedding coordinates:

$$\delta x^\mu = -\xi^\mu(x^\alpha) \,, \qquad \delta\theta_- = \frac{1}{4} \, \lambda^{ab}(x^\alpha) \gamma_{ab} \, \theta_- \,, \tag{6.88}$$

and

$$\delta x^\mu = -\frac{1}{2} \, \bar{\epsilon}_+(x^\alpha) \gamma^a \theta_- \, e^\mu{}_a \,, \qquad \delta\theta_- = \epsilon_-(x^\alpha) - \frac{\dot{x}^\mu e_\mu{}^a}{2\dot{x}^\rho \tau_\rho} \, \gamma_{0a} \, \epsilon_+(x^\alpha) \,. \tag{6.89}$$

In the following we refrain from explicitly denoting the local x^μ-dependence of the parameters.

The transformation rules of the background fields follow from the supergravity result given in [48] and application of the identity (6.64). We find that the bosonic transformation rules are given by

$$\begin{aligned} \delta_{\rm pr} \tau_\mu &= 0 \,, & \delta_{\rm pr} m_\mu &= \partial_\mu \sigma + \lambda_a e_\mu{}^a \,, \\ \delta_{\rm pr} e_\mu{}^a &= \lambda^a{}_b \, e_\mu{}^b + \lambda^a \, \tau_\mu \,, & \delta_{\rm pr} \psi_{\mu +} &= \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{\mu +} \,, \\ \delta_{\rm pr} \omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} \,, & \delta_{\rm pr} \psi_{\mu -} &= \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{\mu -} - \frac{1}{2} \lambda^a \gamma_{a0} \psi_{\mu +} \,, \\ \delta_{\rm pr} \omega_\mu{}^a &= \partial_\mu \lambda^a - \lambda_b \omega_\mu{}^{ab} + \lambda^{ab} \omega_{\mu b} \,. \end{aligned} \tag{6.90}$$

To keep the formulas simple we have given here as well as below only the proper transformation rules. The background transformations are obtained by supplementing each of these rules with an additional transformation under general coordinate transformations, see eq. (6.64). For the fermionic transformations we find the following expressions:

$$
\begin{aligned}
\delta_{\rm pr}\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,, & \delta_{\rm pr}m_\mu &= \bar\epsilon_-\,\gamma^0\psi_{\mu-}\,,\\
\delta_{\rm pr}e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,, & \delta_{\rm pr}\psi_{\mu+} &= D_\mu\epsilon_+\,,\\
\delta_{\rm pr}\omega_\mu{}^{ab} &= 0\,, & \delta_{\rm pr}\psi_{\mu-} &= D_\mu\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+\,.
\end{aligned}
\tag{6.91}
$$

The variation of $\omega_\mu{}^{ab}$ is only zero on-shell, i.e. upon using the equations of motion of the background fields. The explicit form of these equations of motion are given in [48]. In the same manner we can write the variation of $\omega_\mu{}^a$ as

$$
\delta_{\rm pr}\omega_\mu{}^a = \frac{1}{2}\,\bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- + \frac{1}{2}\,\tau_\mu\,\bar\epsilon_-\gamma^0\hat\psi_0{}^a{}_- + \frac{1}{4}\,e_\mu{}^b\,\bar\epsilon_+\gamma^b\hat\psi^a{}_{0-} + \frac{1}{4}\,\bar\epsilon_+\gamma^a\hat\psi_{\mu0-}\,, \tag{6.92}
$$

where $\hat\psi_{\mu\nu-}$ is the covariant curvature of $\psi_{\mu-}$, see [48]. One may check that, to lowest order in fermions, the action (6.86) is invariant under the transformations (6.89), (6.91) and (6.92), upon use of the equations of motion of the background fields.

As a consistency check we have verified that by imposing the gauge-fixing conditions of [48] the action and transformation rules of the NC superparticle reduce to those of the Curved Galilean superparticle.

6.6. Discussion

In this chapter we have constructed with the non-linear realizations method, the free Newton-Hooke (super)particle actions and we analyzed the dynamics and the symmetries of a particle moving in such spaces. We also studied the superparticle actions describing the dynamics of a supersymmetric particle in a 3D Curved Galilean and Newton–Cartan supergravity background. It is also possible to construct the actions for a superparticle moving in the cosmological extension of these backgrounds by including a cosmological constant, see [162]. Due to the computational complexity we gave the action in the Newton–Cartan case only up to terms quartic in the fermions. The Newton–Cartan background is characterized by more fields and corresponds to more symmetries than the Galilean background. One can switch between the two backgrounds either by a partial gauging of symmetries (from Galilean to Newton–Cartan) or by gauge-fixing some of the symmetries (from Newton–Cartan to Galilean). An important role in the construction is played by symmetries. At several occasions we stressed that, as far as the background fields are concerned, one should use the background transformations and not the proper transformations. The latter are used in the definition of the supergravity multiplet. The relation between the two kind of transformations is given in eq. (6.64).

A noteworthy feature is that the proof of invariance of the superparticle action requires that the background fields satisfy their equations of motion. This is reminiscent to what happens with the fermionic κ-symmetry in the relativistic case. We showed that the non-relativistic superparticle also allows a κ-symmetric formulation but that in the non-relativistic case the κ-symmetry is of a simple Stückelberg type [127]. Although being rather trivial, we expect that the formulation with κ-symmetry is indispensable for a reformulation of our results in terms of a non-relativistic superspace and superfields, see e.g. [166]. Such a superspace formulation would be useful to construct the superparticle actions in the Newton–Cartan background to all orders in the fermions.

"Well, in our country," said Alice, still panting a little, "you'd generally get to somewhere else if you run very fast for a long time, as we've been doing."
"A slow sort of country!" said the Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"

Lewis Carroll

7 Carroll Limit

7.1. Introduction

The Carroll limit is defined as the limit when the velocity of light goes to zero $c \to 0$. This can be interpreted as the shrinking of the light cones till they collapse into the time axis, in this sense, it is the opposite of the non-relativistic limit and therefore it is also called ultra-relativistic limit. In this limit, there are no interactions between spatially separated events and the equations of motion are trivial, therefore no true motion occurs, except for tachyonic motion.

As we mention in section 2.3, applying the Carroll limit over the Poincaré group leads to the Carroll group. An interesting feature of this group is that in the contracted Lie algebra, the generator of time translations H, appears as a central charge. In other words if P_a generate the spatial translations and K_a the boosts, the only non-vanishing commutator among these generators is[1]

$$[P_a, K_b] = H \,. \tag{7.1}$$

Recently, the Carroll group has attracted attention because it appears as the symmetry group of different systems. For example, some studies have shown that the Carroll group might have an essential role in the phenomenon of tachyon

[1] The remaining non-vanishing commutators are those involving spatial rotations, see eq. (2.12).

condensates in string theory [42]. Sen [167] had a remarkable insight into the nature of the open bosonic string tachyon, he observed that it should be thought of as ending on a space-filling D-brane. He pointed out that this D-brane is unstable in the bosonic theory, as it does not carry any conserved charge, and he suggested that the open bosonic string tachyon should be interpreted as the instability mode of the D-brane. This led him to conjecture that the open string field theory could be used to precisely determine a new vacuum for the open string, namely one in which the D-brane is annihilated through condensation of the tachyonic unstable mode [168]. The open string excitations are subject to a time-like half line collapsed lightcone which corresponds to the Carroll limit [169].

It has also been pointed out, that there is a duality between the Galilean and the Carroll limits. In [33] it is shown that the Carroll group can be viewed as a subgroup of the Poincaré group in $(D + 1, 1)$ dimensions and they study non-Einsteinian electrodynamics in the Carroll limit. They also give an example where both Galilean and Carrollian symmetries coexist in a Chaplygin gas, a non-relativistic system in D dimensions that carries a Carroll symmetry.

Additionally, Carroll symmetries allow to build a fully covariant formalism for warped conformal field theories (WCFTs) [43] (the simplest field theories without Lorentz invariance that can be described holographically) in curved spaces. In [43] a procedure to gauge the Carroll algebra is provided, which eventually can lead to the construction of a Carroll gravity.

Finally, in [170] it is shown that the Bondi-Metzner-Sachs (BMS) group is the conformal extension of the Carroll group and it is gaining interest with its applications to conformal field theory.

The aim of this chapter is to study the general structure of the Carroll symmetries along the same lines as this has been done for the Galilean symmetries. This will be done in two stages. As a first step we will study the geometry of the empty Carroll space considering the coset $G/H = \text{AC}/\text{Hom AC}$, where Hom AC is the homogeneous part of the AC algebra. In a second step we will put a particle in this Carroll space and construct an action describing its dynamics.

More specifically, in the first part of this chapter we consider the bosonic AC algebra. In particular, we will construct the action of a particle invariant under the symmetries corresponding to this algebra using the method of non-linear realizations [61, 62]. This so-called AC particle reduces, in the limit that the AdS radius goes to infinity, to the Carroll particle that we studied in [41].

A characteristic feature of the free Carroll particle is that it does not move [33, 41, 171][2]. As we will see the AC particle does not move, but unlike the

[2] If we consider two particles or a particle interacting with Carroll gauge fields the dynamics is non-trivial. The same phenomenon occurs in tachyon condensation when the tachyon interacts

Carroll particle the momenta are not a constant of motion as a consequence of the AdS-Carroll symmetry. Another difference with the Carroll particle is that the mass-shell constraint depends on the coordinates of the AC space, therefore the AC particle 'sees' the geometry. This is different from the Carroll case where the energy of the particle is equal to plus or minus the mass [33, 41]. We find that only in the massless limit the mass-shell constraint coincides with the flat Carroll case. Using the AC particle action we will construct the Killing equations for the AC space. We find that the solution of the Killing equations produces an infinite-dimensional algebra that contains the symmetries of the AC algebra. The Lifshitz dilatations are not included in these symmetries. Only in the flat case the dilatations with $z = 0$ are part of the infinite dimensional algebra.

In the second part of this chapter we consider the supersymmetric extension of the Carroll algebras[3]. We first construct the $\mathcal{N} = 1$ AC superalgebra in any dimension (see Tables 2.5 and 7.1, where Q stands for the generator of supersymmetry). The AC superalgebra in the flat limit contains the supersymmetric extension of the 'Lifshitz boost extended Carroll algebra' introduced in appendix B of [172]. We construct the AC superparticle action both as the non-relativistic limit of the relativistic massive superparticle [143, 144] as well as by applying the non-linear realization technique. As we will see the $\mathcal{N} = 1$ AC superparticle like in the Relativistic and Galilean case is non-BPS, i.e. the supersymmetries are non-linearly realized. We will study the super-Killing equations and we find in general an infinite-dimensional algebra of symmetries thereby extending the finite $\mathcal{N} = 1$ super AC transformations.

$\mathcal{N} = 1$	$[M_{ab}, Q]$	$[P_a, Q]$	$\{Q_\alpha, Q_\beta\}$
AdS-Carroll	$-\frac{1}{2}\gamma_{ab}Q$	$\frac{1}{2R}\gamma_a Q$	$[\gamma^0 C^{-1}]_{\alpha\beta} H + \frac{2}{R}[\gamma^{a0}C^{-1}]_{\alpha\beta} K_a$

Table 7.1
In this table we give the (anti-)commutators of the $\mathcal{N} = 1$ Newton-Hooke and AdS-Carroll superalgebras that involve the generators Q of supersymmetry. Note that here is no duality between the two algebras.

Inspired by the relativistic and Galilei case we will investigate whether the $\mathcal{N} = 2$ Carroll superparticle is BPS or not. For simplicity we restrict to the three-dimensional case. We first construct the $\mathcal{N} = 2$ Carroll superalgebra as a contraction of the $\mathcal{N} = 2$ Poincaré superalgebra. This leads to the result given

with a gauge field [42].

[3] A first attempt in this direction was done in the unpublished notes [171].

in Table 7.2. We see that, unlike in the bosonic case, there is no duality in the supersymmetric case. Next, we construct the action for the $\mathcal{N}=2$ Carroll superparticle. This action has two terms, one of them is a Wess-Zumino (WZ) term. If we properly choose the coefficients of the two terms we find a so-called kappa gauge symmetry [152, 153] that kills half of the fermions. This gauge symmetry has the form of a Stückelberg symmetry, similar to what we found in the Galilean case [127, 162]. We find that after fixing the kappa-symmetry the super-Carroll action reduces to the action we found in the $\mathcal{N}=1$ case. The linearly realized supersymmetry acts trivially on all the fields and therefore the $\mathcal{N}=2$ Carroll superparticle reduces to the $\mathcal{N}=1$ Carroll superparticle and hence is not BPS. This is rather different from the $\mathcal{N}=2$ super-Galilei case were BPS particles do exist. The main difference between the super-Carroll and super-Galilei cases comes from the kappa symmetry transformations, in the former case it eliminates the linearized supersymmetry and it the last case it does not.

$\mathcal{N}=2$	$[K_a, Q^+]$	$\{Q^+_\alpha, Q^+_\beta\}$	$\{Q^+_\alpha, Q^-_\beta\}$	$\{Q^-_\alpha, Q^-_\beta\}$
Galilei	$-\frac{1}{2}\gamma_{a0}Q^-$	$[\gamma^0 C^{-1}]_{\alpha\beta}H$	$[\gamma^a C^{-1}]_{\alpha\beta}P_a$	$2[\gamma^0 C^{-1}]_{\alpha\beta}Z$
Carroll	0	$\frac{1}{2}[\gamma^0 C^{-1}]_{\alpha\beta}(H+2Z)$	0	$\frac{1}{2}[\gamma^0 C^{-1}]_{\alpha\beta}(H-2Z)$

TABLE 7.2
In this table we give the (anti-)commutators of the $\mathcal{N}=2$ Galilei and Carroll supersymmetry algebras. Both algebras contain also the commutator $[M_{ab}, Q^\pm] = -\frac{1}{2}\gamma_{ab}Q^\pm$. Note that there is no duality between these two algebras.

In a separate Appendix we extend our investigations to the $\mathcal{N}=2$ curved case and consider the Carroll contraction of the so-called (p,q) AdS superalgebras [58] for the particular cases of $(p,q)=(2,0)$ and $(p,q)=(1,1)$. We find that the associated particle actions are rather different. While in the (2,0) case we have kappa-symmetry, we find that this is not the case in the (1,1) case. The two models have different degrees of freedom.

This chapter is organized as follows. In section 2 we discuss the bosonic free AC particle thereby extending our previous analysis [41] to the curved case. In particular, we construct the action and investigate the Killing equations. In section 3 we consider the $\mathcal{N}=1$ AC superparticle. At the end of this section we discuss the flat limit. Finally, in section 4 we investigate the $\mathcal{N}=2$ Super Carroll particle. Our conclusions are presented in section 5. Some technical details and the extension of the $\mathcal{N}=2$ Super Carroll particle to the curved case, for three dimensions only, are given in the Appendix.

7.2. The Free AdS Carroll Particle

Before discussing the supersymmetric case we will first study in this section different aspects of the free AdS Carroll (AC) particle.

7.2.1. The AdS Carroll Algebra

In order to write the commutators corresponding to the AC algebra, we will start with the contraction of the D-dimensional AdS algebra. The basic commutators are given by ($A = 0, 1, \ldots, D-1$)

$$[M_{AB}, M_{CD}] = 2\eta_{A[C}M_{D]B} - 2\eta_{B[C}M_{D]A}\,, \tag{7.2}$$

$$[M_{AB}, P_C] = 2\eta_{C[B}P_{A]}\,, \qquad [P_A, P_B] = \frac{1}{R^2}M_{AB}\,, \tag{7.3}$$

where R is the AdS radius. Here P_A and M_{AB} are the (anti-hermitian) generators of space-time translations and Lorentz rotations, respectively.

To make the Carroll contraction we rescale the generators with a parameter ω as follows [31,32]:

$$P_0 = \frac{\omega}{2}H\,, \qquad M_{a0} = \omega K_a\,. \tag{7.4}$$

Taking the limit $\omega \to \infty$ we find that the commutators corresponding to the D-dimensional AC algebra are given by ($a = 1, \ldots, D-1$):

$$[M_{ab}, M_{cd}] = 2\delta_{a[c}M_{d]b} - 2\delta_{b[c}M_{d]a}\,, \qquad [M_{ab}, K_c] = 2\delta_{c[b}K_{a]}\,, \tag{7.5}$$

$$[M_{ab}, P_c] = 2\delta_{c[b}P_{a]}\,, \qquad [P_a, K_b] = \frac{1}{2}\delta_{ab}H\,, \tag{7.6}$$

$$[P_a, P_b] = \frac{1}{R^2}M_{ab}\,, \qquad [P_a, H] = \frac{2}{R^2}K_a\,. \tag{7.7}$$

Notice that the commutation relations of space-time translation coincide with the same commutation relations of the AdS algebra. The difference between the AdS and AC algebra is in the different commutation relations that involve the boost generators. Note that this is not the case for the Newton-Hook algebras.

The AC algebra can be expressed in terms of the left invariant Maurer-Cartan 1-forms L^a, which satisfy the Maurer-Cartan equations $dL^C - \frac{1}{2}f^C{}_{AB}L^BL^A = 0$. Explicitly, these equations read

$$dL_H + \frac{1}{2}L_P^a\,L_K^a = 0\,, \qquad dL_P^a - 2L_P^b\,L_M^{ab} = 0\,, \tag{7.8}$$

$$dL_K^a - 2L_K^b\,L_M^{ab} = \frac{2}{R^2}L_H L_P{}^a\,, \qquad dL_M^{ab} - 2L_M^{ca}\,L_M^{cb} = \frac{1}{2R^2}L_P{}^b L_P{}^a\,. \tag{7.9}$$

7.2.2. Non-Linear Realizations

In this subsection we apply the method of non-linear realizations [61, 62] and use the algebra (7.5) to construct the action of the AC particle.

We consider the coset $G/H = \mathrm{AC/SO(D\text{-}1)}$ and the coset element $g = g_0\, U$, where $g_0 = e^{Ht} e^{P_a x^a}$ is the coset representing the AC space and $U = e^{K_a v^a}$ is a general Carroll boost. The $x^a\,(a = 1, ...D-1)$ are the Goldstone bosons of broken translations, t is the Goldstone boson of the unbroken time translation[4] and U is parametrized by the Goldstone bosons of the broken Carroll boost transformations.

The reason to consider the coset element in terms of g_0 and U is because in this way we have that for a general symmetric space-time g_0 is the coset element representing the 'empty' space-time, while U represents the broken symmetries that are due to the presence of a dynamical object, in our case a particle, in the 'empty' space-time. For the case of a particle U is given by the general rotation that mixes the 'longitudinal' time direction with the 'transverse' space directions, i.e. the Carroll boosts. If we would like to consider as a dynamical object a p-brane, we should consider as U the general rotations that mix the longitudinal and transverse directions [68].

Returning to the AC particle, it is interesting to write out the Maurer-Cartan form Ω_0 associated to the AC space

$$\Omega_0 = g_0^{-1} dg_0 = He^0 + P_a e^a + K_a \omega^{a0} + M_{ab}\omega^{ab}\,, \tag{7.10}$$

where (e^0, e^a) and $(\omega^{a0}, \omega^{ab})$ are the space and time components of the Vielbein and spin connection 1-forms of the AdS space, respectively. If we parametrize the AdS space as $e^{Ht}e^{P_a x^a}$, the Vielbein and spin-connection 1-forms corresponding to the AC space are given by

$$\begin{aligned} e^0 &= dt \cosh\frac{x}{R}\,,\\ e^a &= \frac{R}{x} dx^a \sinh\frac{x}{R} + \frac{1}{x^2} x^a x^b dx_b \Big(1 - \frac{R}{x}\sinh\frac{x}{R}\Big)\,,\\ \omega^{a0} &= -\frac{2}{xR} dt x^a \sinh\frac{x}{R}\,,\\ \omega^{ab} &= \frac{1}{2x^2}(x^b dx^a - x^a dx^b)\Big(\cosh\frac{x}{R} - 1\Big)\,. \end{aligned} \tag{7.11}$$

[4] The unbroken translation P_0 generates via a right action [67] [68] a transformation which is equivalent to the world-line diffeomorphisms.

These 1-forms satisfy the structure equations

$$de^0 + \frac{1}{2}e^{\,a}\,\omega^{\,a0} = 0\,, \qquad de^{\,a} - 2e^{\,b}\,\omega^{\,ab} = 0\,, \tag{7.12}$$

$$de^{\,a} - 2\omega^{\,b0}\,\omega^{\,ab} = \frac{2}{R^2}e^0 e^a\,, \qquad d\omega^{\,ab} - 2\omega^{\,ca}\,\omega^{\,cb} = \frac{1}{2R^2}e^b e^a\,. \tag{7.13}$$

We see that the Vielbein satisfies the torsionless condition and that the AC space, like the ancestor AdS space, has constant negative curvature.

We now insert a particle in the empty AC space and consider the Maurer-Cartan form of the combined system:

$$\Omega = g^{-1}dg = U^{-1}\Omega_0 U + U^{-1}dU\,. \tag{7.14}$$

In order to derive an expression for Ω we need to know how the space-time translation generators and the boost generators transform under a general Carroll boost:

$$\begin{aligned} U^{-1}\,H\,U &= H\,, & U^{-1}\,P_a\,U &= P_a + \frac{1}{2}v_a H\,, \\ U^{-1}\,K_a\,U &= K_a\,, & U^{-1}\,M_{ab}\,U &= M_{ab} + v_b K_a - v_a K_b\,. \end{aligned} \tag{7.15}$$

We have also $U^{-1}dU = dv^a K_a$. Using these formulae we find that the Maurer-Cartan form Ω is given by

$$\begin{aligned} L_H &= e^0 + \frac{1}{2}v_a e^a\,, & L_P{}^a &= e^a\,, \\ L_K{}^a &= \omega^{0a} + dv^a + 2v_b\,\omega^{ab}\,, & L_M{}^{ab} &= \omega^{ab}\,. \end{aligned} \tag{7.16}$$

We note that that the Maurer-Cartan forms of space-time translations can be written in matrix-form as follows:

$$\big(\ L_H, \quad L_P{}^a\ \big) = \big(\ e^0, \quad e^a\ \big)\begin{pmatrix} 1 & 0 \\ \frac{1}{2}v_a & 1 \end{pmatrix}\,. \tag{7.17}$$

The matrix appearing at the right-hand-side is the most general Carroll boost in the vector representation.

We now proceed with the construction of an action of the AC particle. An action with the lowest number of derivatives is obtained by taking the pull-back of all the L's that are invariant under rotations, see for example [68]. In this way we obtain the following action:

$$\begin{aligned} S &= M\int (L_H)^* = M\int \Big(e^0 + \frac{1}{2}v_a e^a\Big)^* \\ &= M\int d\tau\,\Big(\dot{t}\cosh\frac{x}{R} + \frac{R}{2x}v_a\dot{x}^a\sinh\frac{x}{R} + \frac{1}{2x^2}x^b v_b x_a \dot{x}^a\Big(1 - \frac{R}{x}\sinh\frac{x}{R}\Big)\Big)\,. \end{aligned} \tag{7.18}$$

This action is invariant under the following transformation rules with constant parameters $(\zeta, a^i, \lambda^i, \lambda^i_j)$ corresponding to time translations, spatial translations, boosts and spatial rotations, respectively:

$$\begin{aligned}
\delta t &= -\zeta + \frac{R}{2x}\lambda^k x_k \tanh\frac{x}{R} + \frac{t}{Rx}a^k x_k \tanh\frac{x}{R}\,,\\
\delta x^i &= -\frac{1}{x^2}\left(x^i a^k x_k - \frac{x}{R}\coth\frac{x}{R}(x^i a^k x_k - a^i x^2)\right) - 2\lambda^i_{\ k}\, x^k\,,\\
\delta v^i &= -\lambda^i - \frac{1}{x^2}\lambda^k x_k x^i \mathrm{sech}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right) - 2\lambda^i_{\ j}\, v^j - \frac{2t}{R^2}a^i\\
&\quad - \frac{2t}{R^2x^2}x^i a^k x_k\,\mathrm{sech}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right) + \frac{2}{Rx}v_b a^{[i}x^{b]}\mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right).
\end{aligned} \tag{7.19}$$

The equations of motion for t, x^a and v^a read

$$\begin{aligned}
0 &= \frac{1}{xR}x^a\dot{x}_a \sinh\frac{x}{R}\,,\\
0 &= -\frac{R}{2x}\dot{x}^a\sinh\frac{x}{R} - \frac{1}{2x^2}x^a x_b\dot{x}^b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\,,\\
0 &= \frac{R}{2x}\dot{v}_a\sinh\frac{x}{R} - \frac{1}{xR}\dot{t}x_a\sinh\frac{x}{R} + \frac{1}{2x^2}x_a x^b\dot{v}_b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\\
&\quad + \frac{\dot{x}^b}{2x^2}(v_a x^b - x_a v_b)\Big(\cosh\frac{x}{R}-1\Big)\,.
\end{aligned} \tag{7.20}$$

These equations imply that

$$\begin{aligned}
&\dot{x}^a = 0\,,\\
&\frac{1}{xR}\dot{t}x_a\sinh\frac{x}{R} = \frac{R}{2x}\dot{v}_a\sinh\frac{x}{R} + \frac{1}{2x^2}x_a x^b\dot{v}_b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\,.
\end{aligned} \tag{7.21}$$

Notice that the evolution of v^a is non-trivial. If we take the limit $R\to\infty$ we recover the flat bosonic equations of motion $\dot{x}_a = \dot{v}_a = 0$ and therefore a trivial dynamics for both x^a, v^a [41].

The energy and spatial momenta of the free AC particle are given by

$$\begin{aligned}
E &= -\frac{\partial\mathcal{L}}{\partial\dot{t}} = -M\cosh\frac{x}{R}\,,\\
p_a &= \frac{\partial\mathcal{L}}{\partial\dot{x}_a} = M\left[\frac{R}{2x}v_a\sinh\frac{x}{R} + \frac{1}{2x^2}x_a x^b v_b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\right]\,.
\end{aligned} \tag{7.22}$$

They satisfy the constraint

$$E^2 - M^2 \cosh^2 \frac{x}{R} = 0\,. \tag{7.23}$$

The canonical action of the AC particle is given by [5]

$$S = \int d\tau \left[-E\dot{t} + p_a \dot{x}^a - \frac{e}{2} \left(E^2 - M^2 \cosh^2 \frac{x}{R} \right) \right] . \tag{7.24}$$

Note that if we calculate $\dot{p}_a$ and impose both equations of motion (7.21) we obtain

$$\dot{p}_a = \frac{M}{Rx} \dot{t} x_a \sinh \frac{x}{R} = \frac{eM^2}{Rx} x_a \cosh \frac{x}{R} \sinh \frac{x}{R}\,. \tag{7.25}$$

In the last step we have used that $\dot{t} = -eE = eM \cosh \frac{x}{R}$, see eq. (7.27). This is the same result one finds using the Hamiltonian form given in eq. (7.42).

7.2.3. The Killing Equations of the AdS Carroll Particle

In order to find the Killing symmetries of the AC space, it is convenient to consider the symmetries of the canonical action (7.24). The basic Poisson brackets of the canonical variables occurring in the action (7.24) are given by

$$\{E, t\} = 1\,, \qquad \{e, \pi_e\} = 1\,, \qquad \{x_i, p_j\} = \delta_{ij}\,. \tag{7.26}$$

This leads to the following equations of motion:

$$\begin{gathered} \dot{t} = -eE\,, \qquad \dot{x}^i = 0\,, \qquad \dot{E} = 0\,, \qquad \dot{p}^i = \frac{eM^2}{2Rx} x^i \sinh \frac{2x}{R}\,, \\ \dot{\pi}_e = -\frac{1}{2} \left(E^2 - M^2 \cosh^2 \frac{x}{R} \right)\,, \qquad \dot{e} = \lambda\,. \end{gathered} \tag{7.27}$$

Here $\lambda = \lambda(\tau)$ is an arbitrary function and π_e is constrained by $\dot{\pi}_e = 0$.

We take as the generator of canonical transformations

$$G = -E\xi^0(t, \vec{x}, e) + p_i\, \xi^i(t, \vec{x}, e) + \gamma(t, \vec{x}, e)\pi_e\,, \tag{7.28}$$

where $\xi^0 = \xi^0(t, \vec{x}, e), \xi^i = \xi^i(t, \vec{x}, e)$ and $\gamma = \gamma(t, \vec{x}, e)$. The condition that this generator generates a Noether symmetry is that it is a constant of motion and it

[5] Alternatively, we can obtain this action by taking the Carroll limit of the canonical action of a massive particle in AdS, see appendix A.

leads to the following restrictions:

$$\begin{aligned}\dot{G} = 0 = &-E(\dot{t}\partial_t\xi^0 + \dot{e}\partial_e\xi^0) + \dot{p}_i\xi^i + p_i(\dot{t}\partial_t\xi^i + \dot{e}\partial_e\xi^i) + \gamma\dot{\pi}_e \\ = &\, eE^2\partial_t\xi^0 - \lambda E\partial_e\xi^0 + \frac{eM^2}{2Rx}x_i\xi^i\sinh\frac{2x}{R} \\ &- eEp_i\partial_t\xi^i + \lambda p_i\partial_e\xi^i - \frac{\gamma}{2}\left(E^2 - M^2\cosh^2\frac{x}{R}\right).\end{aligned} \tag{7.29}$$

From this equation we deduce the following equations describing the symmetries of the AC space:

$$\begin{gathered}\partial_e\xi^0 = 0\,,\qquad \partial_e\xi^i = 0\,,\qquad \partial_t\xi^i = 0\,,\\ \gamma = 2e\partial_t\xi^0\,,\qquad\qquad \frac{e}{xR}x_i\xi^i\sinh\frac{x}{R} + \frac{1}{2}\gamma\cosh\frac{x}{R} = 0\,.\end{gathered} \tag{7.30}$$

The last two equations can be combined into the single condition

$$\partial_t\xi^0 = -\frac{1}{xR}x_i\xi^i\tanh\frac{x}{R}\,. \tag{7.31}$$

The generator G is given by

$$G = -E\xi^0(t,\vec{x}) + p_i\,\xi^i(\vec{x}) + \gamma(t,\vec{x},e)\pi_e\,. \tag{7.32}$$

From the variation of the momenta we can obtain the transformation rules for v_i as follows. First, we use that

$$\begin{aligned}\delta p_i &= \{p_i, G\} = \{p_i, -E\xi^0(t,\vec{x}) + p_i\,\xi^i(\vec{x}) + 2e\partial_t\xi^0(t,\vec{x})\pi_e\} \\ &= E\partial_i\xi^0 - p_k\partial_i\xi^k - 2e\partial_t\partial_i\xi^0\pi_e\,.\end{aligned} \tag{7.33}$$

Next, using eq. (7.22) and $\pi_e = 0$ we obtain

$$\delta p_i = -M\cosh\frac{x}{R}\partial_i\xi^0 - M\left[\frac{R}{2x}v_i\sinh\frac{x}{R} + \frac{1}{2x^2}x_ix^bv_b\Big(1 - \frac{R}{x}\sinh\frac{x}{R}\Big)\right]\partial_i\xi^k\,. \tag{7.34}$$

Finally, using the expression for p_i given in eq. (7.22), we obtain the following transformations of the variables v_i:

$$
\begin{aligned}
\delta v_i =& -\frac{2x}{R}\partial_i\xi^0\coth\frac{x}{R} - v^a\partial_i\xi_a - \frac{1}{Rx}v_b x^b x_k \partial_i\xi^k\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\\
&+\frac{2}{Rx}\coth\frac{x}{R}\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)x_i x_a\partial^a\xi^0 + \frac{1}{Rx}\Big(\frac{R}{x}-\coth\frac{x}{R}\Big)v_i x_b\xi^b\\
&-\frac{1}{Rx}\Big(\frac{R}{x}-\frac{R^2}{x^2}\sinh\frac{x}{R}\Big)\Big(x_i x_b\xi^b - \frac{1}{x^2}x_i x_a\partial^a\xi_k v^k\Big)\\
&+\frac{1}{Rx^3}\mathrm{csch}\frac{x}{R}\Big(-\frac{R}{x}\sinh\frac{x}{R}-\frac{R^2}{x^2}\sinh^2\frac{x}{R}+1+\cosh\frac{x}{R}\Big)x_i x_b\xi^b x_k v^k\\
&+\frac{1}{Rx}\mathrm{csch}\frac{x}{R}\Big(-2\frac{R}{x}\sinh\frac{x}{R}-\frac{R^2}{x^2}\sinh^2\frac{x}{R}+1+\cosh\frac{x}{R}\Big)x_i x_b\xi^b x_k v^k\\
&+\frac{1}{Rx}\mathrm{csch}\frac{x}{R}\Big(\frac{R}{x}\sinh\frac{x}{R}-1\Big)\xi_i x_b v^b\,.
\end{aligned}
\tag{7.35}
$$

We see that the free Carroll particle in an AdS background has an infinite-dimensional symmetry. A possible solution to these equations is given by eq. (7.19) which are the symmetry transformations of the Carroll group. We do not find any Lifshitz dilatations in this case i.e., a transformation with parameters $\xi^i = x^i$, $\xi^0 = zt$.

The Massles Limit

Using the canonical action

$$S = \int d\tau\left[-E\dot{t} + p_a\dot{x}^a - \frac{e}{2}\left(E^2 - M^2\cosh^2\frac{x}{R}\right)\right], \tag{7.36}$$

it is straightforward to take the massless limit $M \to 0$ and obtain the action

$$S = \int d\tau\left(-E\dot{t} + p_a\dot{x}^a - \frac{e}{2}E^2\right). \tag{7.37}$$

We see that in the massless limit the R-dependence of the AC particle has disappeared. This means that the *massive* Carroll particles are affected by the geometry but the *massless* Carroll particles are not. Consequently, in the massless limit there is no difference between particles in an AdS or flat background. Furthermore, the isometries should be given by the most general conformal Carroll group as it was analyzed in [41]. In this case dilatations are included i.e., with parameters $\xi^i = x^i$, $\xi^0 = zt$.

7.2.4. The Carroll action as a Limit of the AdS action

In this section we show how to obtain the action of the D-dimensional free AdS Carroll particle starting from the massive particle moving in a D-dimensional AdS spacetime and to take the Carroll limit. The canonical form of the action before taking the limit is given by

$$S = \int d\tau [p^\mu \dot{x}_\mu - \frac{\tilde{e}}{2}(g_{\mu\nu}p^\mu p^\nu + m^2)] \,, \tag{7.38}$$

where τ is the evolution parameter, $g_{\mu\nu}$ is the metric of an AdS space and $\tilde{e}$ is a Lagrange multiplier. We use that the signature of the metric is $(-,+,+,\ldots)$ and that the AdS line element is given by

$$ds^2 = -\cosh^2\frac{x}{R}(dx^0)^2 + \frac{R^2}{x^2}\sinh^2\frac{x}{R}(dx^a)^2 - \left(\frac{R^2}{x^2}\sinh^2\frac{x}{R} - 1\right)(dx)^2 \,, \tag{7.39}$$

where $x = \sqrt{x_a x^a}$. To take the Carroll limit we first consider a re-scaling of the variables

$$x^0 = \frac{t}{\omega} \,, \quad p^0 = \omega E \,, \quad m = \omega M \,, \quad \tilde{e} = -\frac{e}{\omega^2} \,, \tag{7.40}$$

and next take the limit $\omega \to \infty$ to obtain

$$S = \int d\tau [-E\dot{t} + p^a \dot{x}_a - \frac{e}{2}(E^2 - M^2\cosh^2\frac{x}{R})] \,. \tag{7.41}$$

The equations of motion are given by

$$\begin{aligned} \dot{t} &= -eE \,, & \dot{E} &= 0 \,, \\ \dot{x}^a &= 0 \,, & \dot{p}^a &= \frac{eM^2}{Rx}x^a \cosh\frac{x}{R}\sinh\frac{x}{R} \,, \\ \dot{e} &= \lambda \,, & \pi_e &= -\frac{1}{2}(E^2 - M^2\cosh^2\frac{x}{R}) \,. \end{aligned} \tag{7.42}$$

Note that although the dynamics of x is trivial, i.e. $\dot{x}^a = 0$ (the particle is not changing its position), the momentum is changing over τ because $\dot{p}^a \neq 0$. In the flat limit (the limit when $R \to \infty$) the particle is at rest and does not move.

Finally, the mass-shell constraint reads

$$E^2 - M^2\cosh^2\frac{x}{R} = 0 \,. \tag{7.43}$$

7.3. The $\mathcal{N}=1$ AdS Carroll Superalgebra

We start by taking the contraction of the D-dimensional $\mathcal{N}=1$ AdS algebra. The basic commutators are given by $(A=0,1,\ldots,D-1)$

$$
\begin{aligned}
[M_{AB},M_{CD}] &= 2\eta_{A[C}M_{D]B}-2\eta_{B[C}M_{D]A}\,,\\
[M_{AB},P_C] &= 2\eta_{C[B}P_{A]}\,, & [P_A,P_B] &= 4x^2M_{AB}\,,\\
[M_{AB},Q] &= -\frac{1}{2}\gamma_{AB}Q\,, & [P_A,Q] &= \frac{1}{2R}\gamma_A Q\,,\\
\{Q_\alpha,Q_\beta\} &= 2[\gamma^A C^{-1}]_{\alpha\beta}P_A+\frac{1}{R}[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}\,,
\end{aligned}
\tag{7.44}
$$

where R is the AdS radius and P_A, M_{AB} and Q_α are the generators of spacetime translations, Lorentz rotations, and supersymmetry transformations, respectively. The bosonic generators P_A and M_{AB} are anti-hermitian while the fermionic generator Q_α is hermitian.

To make the Carroll contraction we rescale the generators with a parameter ω as follows:

$$
P_0=\frac{\omega}{2}H\,,\qquad M_{a0}=\omega K_a\,,\qquad Q=\sqrt{\omega}\tilde{Q}\,;\qquad a=1,2,\ldots,D-1
\tag{7.45}
$$

Taking the limit $\omega\to\infty$ and dropping the tildes on the Q we get the following $\mathcal{N}=1$ AdS Carroll superalgebra:

$$
\begin{aligned}
[M_{ab},P_c] &= 2\delta_{c[b}P_{a]}\,, & [M_{ab},K_c] &= 2\delta_{c[b}K_{a]}\,,\\
[P_a,P_b] &= \frac{1}{R^2}M_{ab}\,, & [P_a,K_b] &= \frac{1}{2}\delta_{ab}H\,, & [P_a,H] &= \frac{2}{R^2}K_a\,,\\
[P_a,Q] &= \frac{1}{2R}\gamma_a Q\,, & [M_{ab},Q] &= -\frac{1}{2}\gamma_{ab}Q\,,\\
\{Q_\alpha,Q_\beta\} &= [\gamma^0C^{-1}]_{\alpha\beta}H+\frac{2}{R}[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a\,.
\end{aligned}
\tag{7.46}
$$

The Maurer-Cartan equation $dL^C-\frac{1}{2}f^C{}_{AB}L^BL^A=0$ in components reads

$$
\begin{aligned}
dL_H &= -\tfrac{1}{2}L_P^{\,a}\,L_K^{\,a}-\tfrac{1}{2}\bar{L}_Q\gamma^0L_Q\,, & dL_P^{\,a} &= 2L_P^{\,b}\,L_M^{\,ab}\,,\\
dL_K^{\,a} &= 2L_K^{\,b}\,L_M^{\,ab}+\tfrac{2}{R^2}L_HL_P{}^a-\tfrac{1}{R}\bar{L}_Q\gamma^{a0}L_Q\,, & dL_M^{\,ab} &= 2L_M^{\,ca}\,L_M^{\,cb}+\tfrac{1}{2R^2}L_P{}^bL_P{}^a\,,\\
dL_Q &= \tfrac{1}{2}\gamma_{ab}L_Q\,L_M^{\,ab}-\tfrac{1}{2R}\gamma_aL_QL_P{}^a\,.
\end{aligned}
\tag{7.47}
$$

7.3.1. Superparticle Action

We now use the algebra (7.46) to construct the action of the AC superparticle with the coset

$$\frac{G}{H} = \frac{\mathcal{N} = 1 \text{ AdS Carroll}}{\text{SO(D-1)}} \tag{7.48}$$

that is locally parametrized as $g = g_0\, U$, where $g_0 = e^{Ht}e^{P_a x^a}e^{Q_\alpha \theta^\alpha}$ is the coset representing the 'empty' curved AC Carroll superspace and $U = e^{K_a v^a}$ is a general Carroll boost representing the particle inserted in the empty space. The Maurer-Cartan form Ω_0 associated to the empty AC superspace is given by

$$\Omega_0 = g_0^{-1} dg_0 = HE^0 + P_a E^a + K_a \omega^{a0} + M_{ab}\omega^{ab} - \bar{Q}E\,, \tag{7.49}$$

where (E^0, E^a, E_α) and $(\omega^{a0}, \omega^{ab})$ are the time and space components of the supervielbein and the spin connection of super-AdS if we parametrize the AdS superspace as $e^{Ht}e^{P_a x^a}e^{Q_\alpha \theta^\alpha}$. The explicit expressions for these components are given by

$$\begin{aligned}
E^0 &= dt \cosh\frac{x}{R} - \frac{1}{2}\bar{\theta}\gamma^0 d\theta - \frac{1}{2}\omega^{ab}\bar{\theta}\gamma_{ab}\gamma^0\theta\,,\\
E^a &= \frac{R}{x}dx^a \sinh\frac{x}{R} + \frac{1}{x^2}x^a x^b dx_b\Big(1 - \frac{R}{x}\sinh\frac{x}{R}\Big)\,,\\
\omega^{a0} &= -\frac{2}{xR}dt x^a \sinh\frac{x}{R} - \frac{1}{R}\bar{\theta}\gamma^{a0}d\theta - \frac{1}{2R^2}\bar{\theta}\gamma_{ab}\gamma^0\theta E^b\,,\\
\omega^{ab} &= \frac{1}{2x^2}(x^b dx^a - x^a dx^b)\Big(\cosh\frac{x}{R} - 1\Big)\,,\\
E_\alpha &= d\theta_\alpha - \frac{1}{2R}[\gamma_a\theta]_\alpha E^a + \frac{1}{2}\omega^{ab}[\gamma_{ab}\theta]_\alpha\,.
\end{aligned} \tag{7.50}$$

In this case we have torsion given by $T_0 = -\frac{1}{2}\bar{E}^\alpha\gamma^0 E_\alpha$ and a non-vanishing spin connection. The Maurer-Cartan form for the $\mathcal{N} = 1$ AC superparticle inserted in the AC superspace is given by

$$\Omega = g^{-1}dg = U^{-1}\Omega_0 U + U^{-1}dU\,, \tag{7.51}$$

where

$$\begin{aligned}
L_H &= E^0 + \frac{1}{2}v_a E^a\,, & L_P{}^a &= E^a\,,\\
L_K{}^a &= \omega^{a0} + dv^a + 2v_b\,\omega^{ab}\,, & L_M{}^{ab} &= \omega^{ab}\,,\\
L_{Q_\alpha} &= E_\alpha\,.
\end{aligned} \tag{7.52}$$

Note that the Maurer-Cartan forms of the spacetime supertranslations can be written in matrix form in terms of the Supervielbein components of the AC superspace as follows:

$$\begin{pmatrix} L_H, & L_P{}^a, & L_{Q_\alpha} \end{pmatrix} = \begin{pmatrix} E^0, & E^a, & E_\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}v_a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{7.53}$$

Like in the bosonic case the Maurer-Cartan forms of the supertranslations of the AC superparticle can be obtained from the Maurer-Cartan forms of the AC superspace by a matrix representation of the Carroll boost.

The action of the $\mathcal{N}=1$ AC superparticle is given by the pull-back of all the L's that are invariant under rotations:

$$\begin{aligned} S &= M\int (L_H)^* = M\int \left(E^0 + \frac{1}{2}v_a E^a\right)^* = \\ &= M\int d\tau \left(\dot{t}\cosh\frac{x}{R} + \frac{R}{2x}v_a\dot{x}^a\sinh\frac{x}{R} + \frac{1}{2x^2}x^b v_b x_a \dot{x}^a\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\right. \\ &\qquad\left. - \frac{1}{2}\bar{\theta}\gamma^0\dot{\theta} - \frac{1}{4x^2}x^b\dot{x}^a\bar{\theta}\gamma_{ab}\gamma^0\theta\left(\cosh\frac{x}{R}-1\right)\right) . \end{aligned} \tag{7.54}$$

The equations of motion corresponding to this action can be written as follows

$$\begin{aligned} &\dot{x}^i = 0\,, \qquad\qquad \dot{\theta} = 0\,, \\ &\frac{1}{xR}\dot{t}x_a\sinh\frac{x}{R} = \frac{R}{2x}\dot{v}_a\sinh\frac{x}{R} + \frac{1}{2x^2}x_a x^b\dot{v}_b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big) . \end{aligned} \tag{7.55}$$

We can write a Hamiltonian version of this action with the momenta given by

$$\begin{aligned} p_t &= M\cosh\frac{x}{R}\,, \\ p_a &= M\left[\tfrac{R}{2x}v_a\sinh\tfrac{x}{R} + \tfrac{1}{2x^2}x_a x^b v_b\Big(1-\tfrac{R}{x}\sinh\tfrac{x}{R}\Big) - \tfrac{1}{4x^2}x^b\bar{\theta}\gamma_{ab}\gamma^0\theta\Big(\cosh\tfrac{x}{R}-1\Big)\right] , \\ \bar{P}_\theta &= \frac{M}{2}\bar{\theta}\gamma^0 . \end{aligned} \tag{7.56}$$

Then, the canonical form of (7.54) is

$$S = \int d\tau\left[-\dot{t}E + \dot{x}_a p^a + \dot{\bar{\theta}}P_\theta - \tfrac{e}{2}\Big(E^2 - M^2\cosh^2\tfrac{x}{R}\Big) - \Big(\bar{P}_\theta\cosh\tfrac{x}{R} + \tfrac{1}{2}E\bar{\theta}\gamma^0\Big)\rho\right] . \tag{7.57}$$

The bosonic transformation rules for the coordinates with constant parameters (ζ, a^i, λ^i, λ^i_j) corresponding to time translations, spatial translations, boosts and rotations, respectively, are given by

$$
\begin{aligned}
\delta t &= -\zeta + \frac{R}{2x}\lambda^k x_k \tanh\frac{x}{R} + \frac{t}{Rx}a^k x_k \tanh\frac{x}{R}\,,\\
\delta x^i &= -\frac{1}{x^2}\left(x^i a^k x_k - \frac{x}{R}\coth\frac{x}{R}(x^i a^k x_k - a^i x^2)\right) - 2\lambda^i_{\ k}\, x^k\,,\\
\delta v^i &= -\lambda^i - \frac{1}{x^2}\lambda^k x_k x^i \mathrm{sech}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right) - 2\lambda^i_{\ j}\, v^j - \frac{2t}{R^2}a^i\\
&\quad - \frac{2t}{R^2x^2}x^i a^k x_k\, \mathrm{sech}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right) + \frac{2}{Rx}v_b a^{[i}x^{b]}\mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,,\\
\delta\theta &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta + \frac{1}{2Rx}a^k x^b\gamma_{kb}\theta\mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,.
\end{aligned}
$$

The fermionic transformation rules with constant parameter ϵ corresponding to the supersymmetry transformation are given by

$$
\begin{aligned}
\delta t &= \frac{1}{2}\bar\epsilon\gamma^0\theta\mathrm{sech}\frac{x}{R}\cosh\frac{x}{2R} - \frac{1}{2x}x^k\bar\epsilon\gamma^{k0}\theta\mathrm{sech}\frac{x}{R}\sinh\frac{x}{2R}\,,\\
\delta x^i &= 0\,,\\
\delta v^i &= \frac{1}{Rx}x^i\tanh\frac{x}{R}\Big(\bar\epsilon\gamma^0\theta\cosh\frac{x}{2R} - \frac{1}{x}x^k\bar\epsilon\gamma^{k0}\theta\sinh\frac{x}{2R}\Big)\\
&\quad - \frac{1}{Rx}x^i\bar\epsilon\gamma^0\theta\sinh\frac{x}{2R} + \frac{1}{R}\bar\epsilon\gamma^{i0}\theta\cosh\frac{x}{2R} + \frac{1}{Rx}x^k\bar\epsilon\gamma^{ik0}\theta\sinh\frac{x}{2R}\,,\\
\delta\theta &= \epsilon\cosh\frac{x}{2R} + \frac{1}{x}x^k\gamma_k\epsilon\sinh\frac{x}{2R}\,.
\end{aligned}
\tag{7.58}
$$

7.3.2. The Super Killing Equations

The basic Poisson brackets of the canonical variables are given by

$$
\begin{gathered}
\{E,t\} = 1\,,\qquad \{e,\pi_e\} = 1\,,\qquad \{x_i,p_j\} = \delta_{ij}\,,\\
\{P^\alpha_\theta,\theta_\beta\} = -\delta^\alpha_\beta\,,\qquad \{\Pi^\alpha_\rho,\rho_\beta\} = -\delta^\alpha_\beta\,,
\end{gathered}
\tag{7.59}
$$

and the corresponding Dirac Hamiltonian of the action (7.57) is given by

$$
H_D = \frac{e}{2}\Big(E^2 - M^2\cosh^2\frac{x}{R}\Big) + \lambda\pi_e + \Big(\bar P_\theta\cosh\frac{x}{R} + \frac{1}{2}E\bar\theta\gamma^0\Big)\rho + \bar\pi_\rho\Lambda\,, \tag{7.60}
$$

$\pi_e = 0$ and $\Pi_\rho = 0$ are the primary constraints, $\lambda = \lambda(\tau)$ and $\Lambda = \Lambda(\tau)$ are arbitrary functions. The corresponding primary Hamiltonian equations of motion are given by

$$\begin{aligned} \dot{t} &= -eE - \frac{1}{2}\bar{\theta}\gamma^0\rho\,, \qquad \dot{E} = 0\,, \\ \dot{x}^i = 0\,, \qquad \dot{p}^i &= \frac{eM^2}{xR}x^i\cosh\frac{x}{R}\sinh\frac{x}{R} - \frac{1}{xR}x^i\sinh\frac{x}{R}\bar{P}_\theta\rho\,, \end{aligned} \tag{7.61}$$

$$\begin{aligned} \dot{\pi}_e = -\frac{1}{2}\left(E^2 - M^2\cosh^2\frac{x}{R}\right)\,, \qquad \dot{e} = \lambda\,, \\ \dot{\theta} = -\rho\cosh\frac{x}{R}\,, \qquad \dot{\bar{P}}_\theta = -\frac{1}{2}E\bar{\rho}\gamma^0\,, \qquad \dot{\rho} = -\Lambda\,, \qquad \dot{\bar{\Pi}}_\rho = \bar{P}_\theta\cosh\frac{x}{R} + \frac{1}{2}E\bar{\theta}\gamma^0\,. \end{aligned} \tag{7.62}$$

The stability of primary constraints give as secondary constraint the mass-shell condition $E^2 - M^2\cosh^2\frac{x}{R} = 0$ and the fermionic constraint $\bar{P}_\theta\cosh\frac{x}{R} + \frac{1}{2}E\bar{\theta}\gamma^0 = 0$. If we require the stability of the secondary constraints we get $\rho = 0$. Substituting this into (7.62) and using the canonical momenta (7.56) we obtain equations (7.55).

The generator of canonical transformations has a bosonic and a fermionic part given by

$$G = -E\xi^0(t,\vec{x},\theta) + p_i\,\xi^i(t,\vec{x},\theta) + \gamma(t,\vec{x},\theta)\pi_e - \bar{P}_\theta\chi(t,\vec{x},\theta) + \bar{\Pi}_\rho\Gamma(t,\vec{x},\theta)\,, \tag{7.63}$$

the parameters $\xi^0 = \xi^0(t,\vec{x},\theta)$, $\xi^i = \xi^i(t,\vec{x},\theta)$, $\chi = \chi(t,\vec{x},\theta)$, $\gamma = \gamma(t,\vec{x},\theta)$ have the following restrictions

$$\begin{aligned} 0 &= \dot{G} \\ &= -E(\dot{t}\partial_t\xi^0 + \partial_\theta\xi^0\dot{\theta}) + \dot{p}_i\xi^i + p_i(\dot{t}\partial_t\xi^i + \partial_\theta\xi^i\dot{\theta}) + \gamma\dot{\pi}_e - \dot{\bar{P}}_\theta\chi - \bar{P}_\theta(\partial_t\chi\dot{t} + \partial_\theta\chi\dot{\theta}) + \dot{\bar{\Pi}}_\rho\Gamma \\ &= eE^2\partial_t\xi^0 + \frac{1}{2}E\partial_t\xi^0\bar{\theta}\gamma^0\rho + E\partial_\theta\xi^0\rho\cosh\frac{x}{R} + \frac{eM^2}{xR}x^i\xi_i\cosh\frac{x}{R}\sinh\frac{x}{R} \\ &\quad - \frac{1}{xR}x^i\xi_i\sinh\frac{x}{R}\bar{P}_\theta\rho - eEp_i\partial_t\xi^i - \frac{1}{2}p_i\partial_t\xi^i\bar{\theta}\gamma^0\rho - p_i\partial_\theta\xi^i\rho\cosh\frac{x}{R} \\ &\quad - \frac{1}{2}\gamma\left(E^2 - M^2\cosh^2\frac{x}{R}\right) + \frac{E}{2}\bar{\rho}\gamma^0\chi + eE\bar{P}_\theta\partial_t\chi + \frac{1}{2}\bar{P}_\theta\partial_t\chi\bar{\theta}\gamma^0\rho \\ &\quad + \bar{P}_\theta\partial_\theta\chi\rho\cosh\frac{x}{R} + \bar{P}_\theta\Gamma\cosh\frac{x}{R} + \frac{E}{2}\bar{\theta}\gamma^0\Gamma\,. \end{aligned} \tag{7.64}$$

From this equation we derive the super-Killing equations

$$\begin{gathered}\gamma = 2e\partial_t\xi^0\,, \qquad \Gamma = -\partial_\theta\chi\rho + \frac{1}{xR}x^i\xi_i \tanh\frac{x}{R}\rho\,,\\ \partial_t\xi^0 = -\frac{1}{xR}x^i\xi_i\tanh\frac{x}{R}\,, \qquad \partial_\theta\xi^0 = \frac{1}{2}\bar{\chi}\gamma^0\mathrm{sech}\frac{x}{R} + \frac{1}{2}\bar{\theta}\gamma^0\partial_\theta\chi\mathrm{sech}\frac{x}{R}\,,\\ \partial_t\xi^i = 0\,, \qquad \partial_\theta\xi^i = 0\,, \qquad \partial_t\chi = 0\,.\end{gathered} \tag{7.65}$$

The solution to this equations is given by eqs. (7.58) and (7.58) with the symmetry generator G given by

$$\begin{aligned}G = & -E\xi^0(\vec{x},\theta) + p_i\,\xi^i(t,\vec{x}) + 2e\partial_t\xi^0(\vec{x},\theta)\pi_e - \bar{P}_\theta\chi(\vec{x},\theta)\\ & + \bar{\Pi}_\rho\Big(-\partial_\theta\chi(\vec{x},\theta)\rho + \frac{1}{xR}x^i\xi_i(\vec{x},\theta)\tanh\frac{x}{R}\rho\Big)\,.\end{aligned} \tag{7.66}$$

Then, the $\mathcal{N}=1$ AC superparticle has an infinite dimensional algebra with the transformation rules given by (7.58) and (7.58).

7.3.3. The Flat Limit

We end this section with some comments on the flat limit ($R\to\infty$) which can be taken directly from the AC curved case in order to obtain the dynamics and symmetries of the $\mathcal{N}=1$ flat Carroll superparticle. In this case, the time and space components of the supervielbein simplify to

$$E^0 = dt - \frac{1}{2}\bar{\theta}\gamma^0 d\theta, \qquad E^a = dx^a\,, \qquad E_\alpha = d\theta_\alpha\,. \tag{7.67}$$

In the $R\to\infty$ limit, the torsion becomes $T_0 = -\frac{1}{2}d\bar{\theta}\gamma^0 d\theta$ and since we are studying the flat case, the spin connection vanishes. The supertranslations can be again written in terms of the supervielbein in matrix form as in (7.53) and the action is given by

$$S = M\int\Big(E^0 + \frac{1}{2}v_a E^a\Big)^* = M\int d\tau(\dot{t} - \frac{1}{2}\bar{\theta}\gamma^0\dot{\theta} + \frac{1}{2}v_a\dot{x}^a)\,. \tag{7.68}$$

The equations of motion that follow from this action are:

$$\dot{\vec{x}} = \dot{\vec{v}} = \dot{\theta} = 0\,. \tag{7.69}$$

Therefore, the superparticle does not move. The transformation rules of the different variables are given by

$$\begin{aligned}&\delta t = -\zeta + \frac{1}{2}\lambda^i x_i + \frac{1}{2}\bar{\epsilon}\gamma^0\theta\,, && \delta x^i = -a^i - 2\lambda^i_{\ j}\,x^j\,,\\ &\delta v^i = -\lambda^i - 2\lambda^i_{\ j}\,v^j\,, && \delta\theta = -\frac{1}{2}\lambda_{ij}\gamma^{ij}\theta + \epsilon\,.\end{aligned} \tag{7.70}$$

As we can see from the transformation of θ the $\mathcal{N}=1$ Carroll superparticle is not BPS like in the relativistic and Galilean case.

If we rewrite the action (7.68) in Hamiltonian form

$$S=\int d\tau\left[-\dot{t}E+\dot{x}_a p^a+\dot{\bar{\theta}}P_\theta-\frac{e}{2}(E^2-M^2)-\left(\bar{P}_\theta+\frac{1}{2}E\bar{\theta}\gamma^0\right)\rho\right], \tag{7.71}$$

it turns out that the super-Killing equations can be obtained as the flat limit of the equations (7.65)

$$\begin{gathered}\gamma=0\,,\quad \Gamma=-\partial_\theta\chi\rho\,,\quad \partial_t\xi^0=0\,,\quad \partial_t\xi^i=0\,,\quad \partial_\theta\xi^i=0\,,\quad \partial_t\chi=0\,,\\ \partial_\theta\xi^0=\frac{1}{2}\bar{\chi}\gamma^0+\frac{1}{2}\bar{\theta}\gamma^0\partial_\theta\chi\,,\end{gathered} \tag{7.72}$$

where the symmetry generator G is

$$G=-E\xi^0(\vec{x},\theta)+p_i\,\xi^i(\vec{x})-\bar{P}_\theta\chi(\vec{x},\theta)-\bar{\Pi}_\rho\partial_\theta\chi(\vec{x},\theta)\rho\,. \tag{7.73}$$

From the variation of the momenta

$$\delta p_i=\{p_i,G\}=E\partial_i\xi^0-p_k\partial_i\xi^k+\bar{P}_\theta\partial_i\chi \tag{7.74}$$

and using that the energy, the spatial momenta and the fermionic momenta are given by

$$E=-M\,,\qquad p_i=\frac{M}{2}v_i\,,\qquad \bar{P}_\theta=\frac{M}{2}\bar{\theta}\gamma^0\,, \tag{7.75}$$

we find that the transformation rule of v^i

$$\delta v_i=-2\partial_i\xi^0-v_k\partial_i\xi^k+\bar{\theta}\gamma^0\partial_i\chi\,. \tag{7.76}$$

Note that the above symmetries include the dilatations given by

$$\delta t=0,\qquad \delta x^a=x^a,\qquad \delta\theta=0,\qquad \delta v^a=-v^a\,. \tag{7.77}$$

These dilatations, together with the super-Carroll transformations, form a supersymmetric extension of the Lifshitz Carroll algebra [172] with dynamical exponent z=0. The Lifshitz Carroll algebra with z=0 has appeared in a recent study of warped conformal field theories [43].

7.3.4. The super-AdS Carroll action as a limit of the super-AdS action

In the supersymmetric case, we obtain the action of the free AdS Carroll superparticle starting from the massive superparticle moving in an AdS spacetime whose action is given by

$$S = \int d\tau [\dot{x}_\mu p^\mu + \dot{\bar{\phi}} P_\phi - \frac{\tilde{e}}{2}(g_{\mu\nu}p^\mu p^\nu + m^2) + (\bar{P}_\phi + g_{\mu\nu}p^\mu \bar{\phi}\gamma^\nu)\lambda] \,, \tag{7.78}$$

where $g_{\mu\nu}$ is the AdS metric with line element given by eq. (A.2). Rescaling the variables as

$$\begin{gathered} x^0 = \frac{t}{\omega} \,, \quad p^0 = \omega E \,, \quad m = \omega M \,, \quad \tilde{e} = -\frac{e}{\omega^2} \,, \\ \phi = \frac{1}{\sqrt{\omega}}\theta \,, \quad P_\phi = \sqrt{\omega}\, P_\theta \,, \quad \lambda = \frac{1}{\sqrt{\omega}}\rho \,, \end{gathered} \tag{7.79}$$

allows us to take the Carroll limit with $\omega \to \infty$ to obtain

$$S = \int d\tau [-\dot{t}E + \dot{x}_a p^a + \dot{\bar{\theta}} P_\theta - \frac{e}{2}\left(E^2 - M^2 \cosh^2 \frac{x}{R}\right) + (\bar{P}_\theta \cosh \frac{x}{R} + E\,\bar{\theta}\gamma^0)\rho] \,. \tag{7.80}$$

The primary equations of motion are

$$\begin{aligned} &\dot{t} = -eE - \bar{\theta}\gamma^0\rho \,, && \dot{E} = 0 \,, \\ &\dot{x}^a = 0 \,, && \dot{p}^a = \frac{eM^2}{Rx}x^a \cosh\frac{x}{R}\sinh\frac{x}{R} - \frac{1}{xR}x^a \sinh\frac{x}{R}\bar{P}_\theta\rho \,, \\ &\dot{e} = \lambda \,, && \pi_e = -\frac{1}{2}(E^2 - M^2\cosh^2\frac{x}{R}) \,, \\ &\dot{\theta} = -\cosh\frac{x}{R}\rho \,, && \dot{\bar{P}}_\theta = -E\,\bar{\rho}\gamma^0 \,, \\ &\dot{\rho} = -\Lambda \,, && \dot{\bar{\Pi}}_\rho = \bar{P}_\theta \cosh\frac{x}{R} + E\,\bar{\theta}\gamma^0 \,. \end{aligned} \tag{7.81}$$

After requiring the stability of all the constraints we obtain the equations of motion (7.55). Like in the bosonic case we find that the dynamics of x is trivial, $\dot{x}^a = 0$ (the particle is not changing its position), but that the momentum is changing over τ because $\dot{p}^a \neq 0$.

7.4. The $\mathcal{N} = 2$ Flat Carroll Superparticle

In this Section we extend our investigations to the $\mathcal{N} = 2$ supersymmetric case. The flat case is discussed in this Section while the curved case will be dealt with in Appendix C.

7.4.1. The $\mathcal{N}=2$ Carroll Superalgebra

Our starting point is the $\mathcal{N}=2$ super-Poincaré algebra. For simplicity, we consider 3D only. The basic commutators are $(A=0,1,2; i=1,2)$

$$\begin{aligned}
[M_{AB}, M_{CD}] &= 2\eta_{A[C}M_{D]B} - 2\eta_{B[C}M_{D]A}\,,\\
[M_{AB}, P_C] &= 2\eta_{C[B}P_{A]}\,,\\
[M_{AB}, Q^i] &= -\frac{1}{2}\gamma_{AB}Q^i\,,\\
\{Q^i_\alpha, Q^j_\beta\} &= 2[\gamma^A C^{-1}]_{\alpha\beta}P_A\delta^{ij} + 2[C^{-1}]_{\alpha\beta}\epsilon^{ij}Z\,.
\end{aligned} \tag{7.82}$$

To make the Carroll contraction we define new supersymmetry charges by

$$Q^\pm_\alpha = \frac{1}{2}(Q^1_\alpha \pm \gamma_0 Q^2_\alpha) \tag{7.83}$$

and rescale the different symmetry generators with a parameter ω as follows:

$$\begin{aligned}
&P_0 = \frac{\omega}{2}H\,, \qquad M_{a0} = \omega K_a\,, \qquad Z = \omega\tilde{Z}\,,\\
&\qquad Q^+ = \sqrt{\omega}\tilde{Q}^+\,, \qquad Q^- = \sqrt{\omega}\tilde{Q}^-\,.
\end{aligned} \tag{7.84}$$

Taking the limit $\omega \to \infty$ we obtain the following 3D Carroll algebra

$$\begin{aligned}
[M_{ab}, K_c] &= 2\delta_{c[b}K_{a]}\,, & [M_{ab}, P_c] &= 2\delta_{c[b}P_{a]}\,,\\
[K_a, P_b] &= -\frac{1}{2}\delta_{ab}H\,, & [M_{ab}, \tilde{Q}^\pm] &= -\frac{1}{2}\gamma_{ab}\tilde{Q}^\pm\\
\{\tilde{Q}^+_\alpha, \tilde{Q}^+_\beta\} &= [\gamma^0 C^{-1}]_{\alpha\beta}\Big(\frac{1}{2}H + \tilde{Z}\Big)\,, & \{\tilde{Q}^-_\alpha, \tilde{Q}^-_\beta\} &= [\gamma^0 C^{-1}]_{\alpha\beta}\Big(\frac{1}{2}H - \tilde{Z}\Big)\,.
\end{aligned} \tag{7.85}$$

The Maurer-Cartan equation $dL^C - \frac{1}{2}f_{AB}^{C}L^B \wedge L^A = 0$ in components reads:

$$\begin{aligned}
dL_H &= -\frac{1}{2}L_P^{\,a}\,L_K^{\,a} - \frac{1}{4}\bar{L}_-\gamma^0 L_- - \frac{1}{4}\bar{L}_+\gamma^0 L_+\,, & dL_P^{\,a} &= 2L_P^{\,b}\,L_M^{\,ab}\,,\\
dL_Z &= -\frac{1}{2}\bar{L}_+\gamma^0 L_+ + \frac{1}{2}\bar{L}_-\gamma^0 L_-\,, & dL_K^{\,a} &= 2L_K^{\,b}\,L_M^{\,ab}\,,\\
dL_- &= \frac{1}{2}\gamma_{ab}L_-\,L_M^{\,ab}\,, & dL_+ &= \frac{1}{2}\gamma_{ab}L_+\,L_M^{\,ab}\,,\\
dL_M^{\,ab} &= 2L_M^{\,ca}\,L_M^{\,cb}\,.
\end{aligned} \tag{7.86}$$

7.4.2. Superparticle Action and Kappa Symmetry

To construct the action of the $\mathcal{N}=2$ Carrollian superparticle we consider the following coset:

$$\frac{G}{H}=\frac{\mathcal{N}=2\text{ super Carroll}}{\text{SO(D-1)}}\,. \tag{7.87}$$

The coset element is given by $g=g_0\,U$, where $g_0=e^{Ht}e^{P_a x^a}e^{Q^-_\alpha\theta^\alpha_-}e^{Q^+_\alpha\theta^\alpha_+}e^{Zs}$ is the coset representing the 'empty' $\mathcal{N}=2$ Carroll superspace with a central charge extension and $U=e^{K_a v^a}$ is a general Carroll boost representing the insertion of the particle.

The Maurer-Cartan form associated to the super-Carroll space is given by

$$\Omega_0=(g_0)^{-1}dg_0=HE^0+P_aE^a-\bar{Q}^-E_- -\bar{Q}^+E_+ +ZE_Z\,, \tag{7.88}$$

where $(E^0,E^a,E_{-\alpha},E_{+\alpha},E_Z)$ are the supervielbein components of the Carroll superspace given explicitly by

$$\begin{aligned}
E^0&=dt-\frac{1}{4}\bar{\theta}_-\gamma^0 d\theta_- -\frac{1}{4}\bar{\theta}_+\gamma^0 d\theta_+\,, & E^a&=dx^a\,,\\
E_{-\alpha}&=d\theta_{-\alpha}\,, & E_{+\alpha}&=d\theta_{+\alpha}\,,\\
E_Z&=ds+\frac{1}{2}\bar{\theta}_-\gamma^0 d\theta_- -\frac{1}{2}\bar{\theta}_+\gamma^0 d\theta_+\,. & &
\end{aligned} \tag{7.89}$$

In terms of the supervielbein the Maurer-Cartan form of the $\mathcal{N}=2$ Carroll superparticle is given by

$$\begin{aligned}
L_H&=E^0+\frac{1}{2}v_aE^a\,, & L_P^a&=E^a\,,\\
L_Z&=E_Z\,, & L_K^a&=dv^a\,,\\
L_{-\alpha}&=E_{-\alpha}\,, & L_{+\alpha}&=E_{+\alpha}\,.
\end{aligned} \tag{7.90}$$

As before, we can write the space-time super-translations in matrix form in terms of the Vielbein of Carroll superspace as follows:

$$\left(\,L_H,\,{L_P}^a,\,L_{-\alpha},\,L_{+\alpha},\,L_Z\,\right)=\left(\,E^0,\,E^a,\,E_{-\alpha},\,E_{+\alpha},\,E_Z\,\right)\begin{pmatrix}1&0&0&0&0\\ \frac{1}{2}v_a&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\end{pmatrix}. \tag{7.91}$$

The action of the $\mathcal{N}=2$ Carrollian superparticle is given by the pull-back of all the L's that are invariant under rotations:

$$\begin{aligned} S &= a\int (L_H)^* + b\int (L_Z)^* \\ &= a\int d\tau\left(\dot{t} - \tfrac{1}{4}\bar{\theta}_-\gamma^0\dot{\theta}_- - \tfrac{1}{4}\bar{\theta}_+\gamma^0\dot{\theta}_+ + \tfrac{1}{2}v_a\dot{x}^a\right) + b\int d\tau\left(\dot{s} + \tfrac{1}{2}\bar{\theta}_-\gamma^0\dot{\theta}_- - \tfrac{1}{2}\bar{\theta}_+\gamma^0\dot{\theta}_+\right). \end{aligned} \tag{7.92}$$

The equations of motion corresponding to this action are given by

$$\dot{x}_a = 0\,,\quad \dot{v}_a = 0\,,\quad \dot{\theta}_- = 0\,,\quad \dot{\theta}_+ = 0\,. \tag{7.93}$$

The transformation rules for the coordinates with constant parameters (ζ, η, a^i, λ^i, λ^i_j, ϵ_+, ϵ_-) corresponding to time translations, Z transformations, spatial translations, boosts, rotations and supersymmetry transformations, respectively, are given by

$$\begin{aligned} \delta t &= -\zeta + \frac{1}{2}\lambda^i x_i + \frac{1}{4}\bar{\epsilon}_-\gamma^0\theta_- + \frac{1}{4}\bar{\epsilon}_+\gamma^0\theta_+\,, & \delta x^i &= -a^i - 2\lambda^i_j\, x^j\,, \\ \delta s &= -\eta - \frac{1}{2}\bar{\epsilon}_-\gamma^0\theta_- + \frac{1}{2}\bar{\epsilon}_+\gamma^0\theta_+\,, & \delta v^i &= -\lambda^i - 2\lambda^i_j\, v^j\,, \\ \delta\theta_+ &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_+ + \epsilon_+\,, & \delta\theta_- &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_- + \epsilon_-\,. \end{aligned} \tag{7.94}$$

To derive an action that is invariant under additional κ-transformations we need to find a fermionic gauge-transformation that leaves L_H and/or L_Z invariant. The variation of L_H and L_Z under gauge-transformations is given by

$$\begin{aligned} \delta L_H &= d([\delta z_H]) + \frac{1}{2}L^a_P[\delta z^a_K] + \frac{1}{2}L^a_K[\delta z^a_P] + \frac{1}{2}\bar{L}_-\gamma^0[\delta z_-] + \frac{1}{2}\bar{L}_+\gamma^0[\delta z_+]\,, \\ \delta L_Z &= d([\delta z_Z]) - \bar{L}_-\gamma^0[\delta z_-] + \bar{L}_+\gamma^0[\delta z_+]\,. \end{aligned} \tag{7.95}$$

where $[\delta z^a_K]$ is obtained from L_H by changing the 1-forms dt, $d\theta_+$, $d\theta_-$ with the transformations δt, $\delta\theta_+$, $\delta\theta_-$. In analogous way we can construct the other terms appearing in (7.95).

For κ-transformations, $[\delta z_H] = 0$, $[\delta z^a_K] = 0$, $[\delta z^a_P] = 0$,

$$\begin{aligned} 0 = \delta L_H &= \frac{1}{2}\delta\bar{\theta}_-\gamma^0[\delta z_-] + \frac{1}{2}\delta\bar{\theta}_+\gamma^0[\delta z_+]\,, \\ 0 = \delta L_Z &= -\delta\bar{\theta}_-\gamma^0[\delta z_-] + \delta\bar{\theta}_+\gamma^0[\delta z_+]\,. \end{aligned} \tag{7.96}$$

It follows that to obtain a κ-symmetric action we need to take $b = \pm\frac{1}{2}a$. We focus here on the case $b = -\frac{1}{2}a$. With this choice the action and κ-symmetry rules are given by

$$S = a\int (L_H - \frac{1}{2}L_Z)^*\,, \qquad [\delta z_+] = \kappa\,, \qquad [\delta z_-] = 0\,, \tag{7.97}$$

where $\kappa = \kappa(\tau)$ is an arbitrary local parameter. Using this we find the following κ-transformations of the coordinates

$$\begin{aligned} \delta t &= \frac{1}{4}\bar{\theta}_+\gamma^0\kappa\,, \qquad & \delta x^a &= 0\,, \qquad & \delta\theta_+ &= \kappa\,, \\ \delta s &= \frac{1}{2}\bar{\theta}_+\gamma^0\kappa\,, \qquad & \delta v_a &= 0\,, \qquad & \delta\theta_- &= 0\,. \end{aligned} \tag{7.98}$$

After fixing the κ-symmetry, by imposing the gauge condition $\theta_+ = 0$, the action reduces to

$$S = a\int d\tau\left(\dot{t} - \frac{1}{2}\dot{s} - \frac{1}{2}\bar{\theta}_-\gamma^0\dot{\theta}_- + \frac{1}{2}v_a\dot{x}^a\right)\,. \tag{7.99}$$

The residual transformations that leave this action invariant are given by

$$\begin{aligned} \delta t &= -\zeta + \frac{1}{2}\lambda^i x_i + \frac{1}{4}\bar{\epsilon}_-\gamma^0\theta_-\,, \qquad & \delta x^i &= -a^i - 2\lambda^i_{\ j}\, x^j\,, \\ \delta s &= -\eta - \frac{1}{2}\bar{\epsilon}_-\gamma^0\theta_-\,, \qquad & \delta v^i &= -\lambda^i - 2\lambda^i_{\ j}\, v^j\,, \\ \delta\theta_- &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_- + \epsilon_-\,. \end{aligned} \tag{7.100}$$

The linearly realized supersymmetry acts trivially on all the fields and therefore the $\mathcal{N} = 2$ Super Carroll particle reduces to the $\mathcal{N} = 1$ Super Carroll particle and hence is not BPS since the kappa-symmetry eliminates the linearized supersymmetry. This is different from the $\mathcal{N} = 2$ Super Galilei case were BPS particles do exist.

7.5. Discussion

In this chapter we have investigated the geometry of the flat and curved (AdS) Carroll space both in the bosonic as well as in the supersymmetric case. We furthermore have analyzed the symmetries of a particle moving in such a space. In the bosonic case we constructed the Vielbein and spin connection of the AdS Carroll (AC) space which shows that this space is torsionless with constant (negative) curvature. We constructed the action of a massive particle moving in this space thereby extending the flat case analysis of [41]. Like in the flat case, we found that the AC particle does not move. However, in the curved case the momenta are not conserved. Particles moving in a Carroll space, whether flat or curved, do not have a relation among their velocities and momenta.

Using the symmetries of the AC particle we have computed the Killing equations of the AC space. We found that these Killing equations allow an infinite-dimensional algebra of symmetries that, unlike in the flat case, does not include dilatations. Another difference with the flat case is that there is no duality between the Newton-Hooke and AdS Carroll algebras. Furthermore, in the curved case the mass-shell constraint depends on the coordinates of the AC space.

In the second part of this chapter we have extended our investigations to the supersymmetric case. Unlike the bosonic case, the $\mathcal{N} = 1$ AC superspace has torsion with constant curvature due to the presence of fermions. Like in the bosonic case, we found that the $\mathcal{N} = 1$ AC superparticle does not move and the momenta are conserved. We have constructed the super-Killing equations and showed that the symmetries form an infinite dimensional superalgebra. After taking the flat limit we found that among the symmetries of the $\mathcal{N} = 1$ Carroll superparticle we have a supersymmetric extension of the Lifshitz Carroll algebra [172] with dynamical exponent $z = 0$. The bosonic part of this algebra has appeared as a symmetry of warped conformal field theories [43].

We also showed that the $\mathcal{N} = 2$ Carroll superparticle has a fermionic kappa-symmetry such that, when this gauge symmetry is fixed, the $\mathcal{N} = 2$ Carroll superparticle reduces to the $\mathcal{N} = 1$ Carroll superparticle. Apparently, in flat Carroll superspace the number of supersymmetries is not physically relevant. This is due to the fact that the kappa gauge symmetry neutralizes the extra linear supersymmetries beyond $\mathcal{N} = 1$. Unlike the bosonic case, there is no duality between the $\mathcal{N} = 2$ Super Galilei and Super Carroll algebras.

In a separate appendix we investigated the $\mathcal{N} = 2$ AC superparticle [6]. We studied the so-called (2,0) and (1,1) super-Carroll spaces and the corresponding superparticles. Physically, the (2,0) and (1,1) cases are different, they have unequal degrees of freedom. For instance, only the (2,0) superparticle has a kappa-symmetry. Apparently, for the AC superparticle the type of supersymmetry one considers does make a difference.

7.A. The 3D $\mathcal{N} = 2$ AdS Carroll Superparticle

There are two independent versions of the 3D $\mathcal{N} = 2$ AdS algebra, the so-called $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ algebras. Correspondingly, there are two possible $\mathcal{N} = 2$ AdS Carroll superalgebras which we consider below.

[6] For simplicity we did only consider the 3D case.

7.A.1. The $\mathcal{N}=(2,0)$ AdS Carroll Superalgebra

We will start with the contraction of the 3D $\mathcal{N}=(2,0)$ AdS algebra. The basic commutators are given by $(A=0,1,2; i=1,2)$

$$
\begin{aligned}
&[M_{AB}, M_{CD}] = 2\eta_{A[C}M_{D]B} - 2\eta_{B[C}M_{D]A}\,, && [M_{AB}, Q^i] = -\frac{1}{2}\gamma_{AB}Q^i\,,\\
&[M_{AB}, P_C] = 2\eta_{C[B}P_{A]}\,, && [P_A, Q^i] = x\gamma_A Q^i\,,\\
&[P_A, P_B] = 4x^2 M_{AB}\,, && [\mathcal{R}, Q^i] = 2x\epsilon^{ij}Q^j\,,\\
&\{Q^i_\alpha, Q^j_\beta\} = 2[\gamma^A C^{-1}]_{\alpha\beta}P_A\delta^{ij} + 2x[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}\delta^{ij} + 2[C^{-1}]_{\alpha\beta}\epsilon^{ij}\mathcal{R}\,.
\end{aligned}
\tag{7.101}
$$

Here $P_A, M_{AB}, \mathcal{R}$ and Q^i_α are the generators of space-time translations, Lorentz rotations, SO(2) R-symmetry transformations and supersymmetry transformations, respectively. The bosonic generators P_A, M_{AB} and $\mathcal{R}$ are anti-hermitian while the fermionic generators Q^i_α are hermitian. The parameter $x = 1/(2R)$, with R being the AdS radius. Note that the generator of the SO(2) R-symmetry becomes the central element of the Poincaré algebra in the flat limit $x \to 0$.

To take the Carroll contraction we define new supersymmetry charges by

$$
Q^\pm_\alpha = \frac{1}{2}(Q^1_\alpha \pm \gamma_0 Q^2_\alpha) \tag{7.102}
$$

and rescale the generators with a parameter ω as follows:

$$
P_0 = \frac{\omega}{2}H\,, \qquad \mathcal{R} = \omega Z\,, \qquad M_{a0} = \omega K_a\,, \qquad Q^\pm = \sqrt{\omega}\tilde{Q}^\pm\,. \tag{7.103}
$$

Taking the limit $\omega \to \infty$ and dropping the tildes on the $Q^\pm$ we get the following 3D $\mathcal{N}=(2,0)$ Carroll superalgebra:

$$
\begin{aligned}
&[M_{ab}, P_c] = 2\delta_{c[b}P_{a]}\,, \qquad [M_{ab}, K_c] = 2\delta_{c[b}K_{a]}\,,\\
&[P_a, P_b] = \frac{1}{R^2}M_{ab}\,, \qquad [P_a, K_b] = \frac{1}{2}\delta_{ab}H\,, \qquad [P_a, H] = \frac{2}{R^2}K_a\\
&[P_a, Q^\pm] = \frac{1}{2R}\gamma_a Q^\mp\,, \qquad [M_{ab}, Q^\pm] = -\frac{1}{2}\gamma_{ab}Q^\pm\,,\\
&\{Q^+_\alpha, Q^+_\beta\} = \frac{1}{2}[\gamma^0 C^{-1}]_{\alpha\beta}\,(H+2Z)\,, \qquad \{Q^-_\alpha, Q^-_\beta\} = \frac{1}{2}[\gamma^0 C^{-1}]_{\alpha\beta}\,(H-2Z)\,,\\
&\{Q^+_\alpha, Q^-_\beta\} = \frac{1}{R}[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a\,.
\end{aligned}
\tag{7.104}
$$

In components the Maurer-Cartan equation $dL^C - \frac{1}{2} f^C{}_{AB} L^B L^A = 0$ reads as follows:

$$
\begin{aligned}
dL_H &= -\tfrac{1}{2} L_P^{\,a}\, L_K^{\,a} - \tfrac{1}{4}\bar{L}_- \gamma^0 L_- - \tfrac{1}{4}\bar{L}_+ \gamma^0 L_+\,, & dL_P^{\,a} &= 2L_P^{\,b}\, L_M^{\,ab}\,,\\
dL_K^{\,a} &= 2L_K^{\,b}\, L_M^{\,ab} + \tfrac{2}{R^2} L_H L_P{}^a - \tfrac{1}{R}\bar{L}_- \gamma^{a0} L_+\,, & dL_Z &= -\tfrac{1}{2}\bar{L}_+\gamma^0 L_+ + \tfrac{1}{2}\bar{L}_-\gamma^0 L_-\,,\\
dL_- &= \tfrac{1}{2}\gamma_{ab} L_-\, L_M^{\,ab} - \tfrac{1}{2R}\gamma_a L_+ L_P{}^a\,, & dL_+ &= \tfrac{1}{2}\gamma_{ab} L_+\, L_M^{\,ab} - \tfrac{1}{2R}\gamma_a L_- L_P{}^a\,,\\
dL_M^{\,ab} &= 2L_M^{\,ca}\, L_M^{\,cb} + \tfrac{1}{2R^2} L_P{}^b L_P{}^a\,.
\end{aligned}
\tag{7.105}
$$

7.A.2. Superparticle action

We use the algebra (7.104) to construct the action of the $\mathcal{N}=2$ Carrollian superparticle. The coset that we will consider is

$$
\frac{G}{H} = \frac{\mathcal{N}=(2,0)\ \text{AdS Carroll}}{\text{SO(D-1)}}\,, \tag{7.106}
$$

with the coset element $g = g_0\, U$, where $g_0 = e^{Ht} e^{P_a x^a} e^{Q_\alpha^- \theta_-^\alpha} e^{Q_\alpha^+ \theta_+^\alpha} e^{Zs}$ is the coset representing the $\mathcal{N}=(2,0)$ Carroll superspace with a central charge extension and $U = e^{K_a v^a}$ is a general Carroll boost that represents the superparticle.

The Maurer-Cartan form associated to the super-Carroll space is given by

$$
\Omega_0 = (g_0)^{-1} dg_0 = HE^0 + P_a E^a + K_a \omega^{a0} + M_{ab}\omega^{ab} - \bar{Q}^- E_- - \bar{Q}^+ E_+ + ZE_Z\,, \tag{7.107}
$$

where $(E^0, E^a, E_{-\alpha}, E_{+\alpha}, E_Z)$ and $(\omega^{a0}, \omega^{ab})$ are the supervielbein and the spin connection of the Carroll superspace which are given explicitly by

$$
\begin{aligned}
E^0 &= dt \cosh\frac{x}{R} - \frac{1}{4}(\bar\theta_-\gamma^0 d\theta_- + \bar\theta_+\gamma^0 d\theta_+) - \frac{1}{4}\omega^{ab}(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-)\\
&\quad + \frac{1}{4R}\bar\theta_-\gamma^{a0}\theta_+ E^a\,,\\
E^a &= \frac{R}{x} dx^a \sinh\frac{x}{R} + \frac{1}{x^2} x^a x^b dx_b \Big(1 - \frac{R}{x}\sinh\frac{x}{R}\Big)\,,\\
E_Z &= ds + \frac{1}{2}\bar\theta_-\gamma^0 d\theta_- - \frac{1}{2}\bar\theta_+\gamma^0 d\theta_+\\
&\quad - \frac{1}{2}\omega^{ab}(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-) + \frac{1}{2R}\bar\theta_-\gamma^{a0}\theta_+ E^a\,.\\
\omega^{ab} &= \frac{1}{2x^2}(x^b dx^a - x^a dx^b)\Big(\cosh\frac{x}{R} - 1\Big)\,,
\end{aligned}
$$

$$
\begin{aligned}
\omega^{a0} &= -\frac{2}{xR}dtx^a \sinh\frac{x}{R} - \frac{1}{R}\bar\theta_+\gamma^{a0}d\theta_- - \frac{1}{4R^2}(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-)E^b \\
&\quad - \frac{1}{R}\omega^{bc}\bar\theta_-\gamma_{bc}\gamma^{a0}\theta_+\,, \\
E_{-\alpha} &= [d\theta_-]_\alpha - \frac{1}{2R}[\gamma_a\theta_+]_\alpha E^a + \frac{1}{2}\omega^{ab}[\gamma_{ab}\theta_-]_\alpha\,, \\
E_{+\alpha} &= [d\theta_+]_\alpha - \frac{1}{2R}[\gamma_a\theta_-]_\alpha E^a + \frac{1}{2}\omega^{ab}[\gamma_{ab}\theta_+]_\alpha\,,
\end{aligned} \tag{7.108}
$$

We can use the supervielbein to write the Maurer-Cartan form of the $\mathcal{N}=(2,0)$ Carroll superparticle as follows:

$$
\begin{aligned}
L_H &= E^0 + \frac{1}{2}v_a E^a\,, & L_P^a &= E^a\,, \\
L_K^a &= \omega^{a0} + dv^a + 2v_b\,\omega^{ab}\,, & L_Z &= E_Z\,, \\
L_{-\alpha} &= E_{-\alpha}\,, & L_{+\alpha} &= E_{+\alpha}\,.
\end{aligned} \tag{7.109}
$$

7.A.3. Global Symmetries and Kappa symmetry

The action of the Carrollian superparticle is given by the pull-back of all L's that are invariant under rotations:

$$
\begin{aligned}
S &= a\int (L_H)^* + b\int (L_Z)^* \\
&= a\int d\tau\,\Big(\dot t\cosh\frac{x}{R} + \frac{R}{2x}v_a\dot x^a\sinh\frac{x}{R} + \frac{1}{2x^2}x^b v_b x_a\dot x^a\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big) \\
&\qquad - \frac{1}{4}\bar\theta_-\gamma^0\dot\theta_- - \frac{1}{4}\bar\theta_+\gamma^0\dot\theta_+ \\
&\qquad - \frac{1}{8x^2}x^b\dot x^a(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-)\Big(\cosh\frac{x}{R}-1\Big) \\
&\qquad + \frac{1}{4x}\bar\theta_-\gamma^{a0}\theta_+\Big[\dot x_a\sinh\frac{x}{R} + \frac{1}{Rx}x_a x_b\dot x^b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\Big]\Big) \\
&\quad + b\int d\tau\,\Big(\dot s + \tfrac{1}{2}\bar\theta_-\gamma^0\dot\theta_- - \tfrac{1}{2}\bar\theta_+\gamma^0\dot\theta_+ \\
&\qquad - \tfrac{1}{4x^2}x^b\dot x^a(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ - \bar\theta_-\gamma_{ab}\gamma^0\theta_-)\Big(\cosh\tfrac{x}{R}-1\Big) \\
&\qquad + \frac{1}{2x}\bar\theta_-\gamma^{a0}\theta_+\Big[\dot x_a\sinh\frac{x}{R} + \frac{1}{Rx}x_a x_b\dot x^b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\Big]\Big)\,,
\end{aligned} \tag{7.110}
$$

which is invariant under the following bosonic transformation rules for the coordinates with constant parameters (ζ, η, a^i, λ^i, λ^i_j) corresponding to time translations, Z transformations, spatial translations, boosts, rotations, respectively

$$\begin{aligned}
\delta t &= -\zeta + \frac{R}{2x}\lambda^k x_k \tanh\frac{x}{R} + \frac{t}{Rx}a^k x_k \tanh\frac{x}{R}\,,\\
\delta x^i &= -\frac{1}{x^2}\left(x^i a^k x_k - \frac{x}{R}\coth\frac{x}{R}(x^i a^k x_k - a^i x^2)\right) - 2\lambda^i_{\ k}\, x^k\,,\\
\delta s &= -\eta\,,\\
\delta v^i &= -\lambda^i - \frac{1}{x^2}\lambda^k x_k x^i \mathrm{sech}\frac{x}{R}\left(1-\cosh\tfrac{x}{R}\right) - 2\lambda^i_{\ j}\, v^j - \frac{2t}{R^2}a^i\\
&\quad - \tfrac{2t}{R^2x^2}x^i a^k x_k\,\mathrm{sech}\tfrac{x}{R}\left(1-\cosh\frac{x}{R}\right) + \tfrac{2}{Rx}v_b a^{[i}x^{b]}\mathrm{csch}\tfrac{x}{R}\left(1-\cosh\tfrac{x}{R}\right)\,,\\
\delta\theta_+ &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_+ + \frac{1}{2Rx}a^k x^b\gamma_{kb}\theta_-\mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,,\\
\delta\theta_- &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_- + \frac{1}{2Rx}a^k x^b\gamma_{kb}\theta_+\mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,.
\end{aligned} \tag{7.111}$$

The same action is invariant under fermionic transformation rules with constant parameters (ϵ_+,ϵ_-) corresponding to the supersymmetry transformations

$$\begin{aligned}
\delta t &= \frac{1}{4}\bar\epsilon_+\gamma^0\theta_+\mathrm{sech}\frac{x}{R}\cosh\frac{x}{2R} - \frac{1}{4x}x^k\bar\epsilon_+\gamma^{k0}\theta_-\mathrm{sech}\frac{x}{R}\sinh\frac{x}{2R}\\
&\quad + \frac{1}{4}\bar\epsilon_-\gamma^0\theta_-\mathrm{sech}\frac{x}{R}\cosh\frac{x}{2R} - \frac{1}{4x}x^k\bar\epsilon_-\gamma^{k0}\theta_+\mathrm{sech}\frac{x}{R}\sinh\frac{x}{2R}\,,\\
\delta x^i &= 0\,,\\
\delta v^i &= \frac{1}{Rx}x^i\bar\epsilon_+\gamma^0\theta_+\Big(\frac{1}{2}\tanh\frac{x}{R}\cosh\frac{x}{2R} - 2\sinh\frac{x}{2R}\Big) + \frac{1}{R}\bar\epsilon_+\gamma^{i0}\theta_-\cosh\frac{x}{2R}\\
&\quad - \frac{1}{2Rx^2}x^i x^k\bar\epsilon_+\gamma^{k0}\theta_-\tanh\frac{x}{R}\sinh\frac{x}{2R}\\
&\quad + \frac{1}{2Rx}x^i\bar\epsilon_-\gamma^0\theta_-\tanh\frac{x}{R}\cosh\frac{x}{2R} - \frac{1}{Rx}x^b\bar\epsilon_-\gamma_b\gamma^{i0}\theta_-\sinh\frac{x}{2R}\\
&\quad - \frac{1}{2Rx^2}x^i x^k\bar\epsilon_-\gamma^{k0}\theta_+\tanh\frac{x}{R}\sinh\frac{x}{2R}\,,\\
\delta s &= \frac{1}{2}\bar\epsilon_+\gamma^0\theta_+\cosh\frac{x}{2R} + \frac{1}{2x}x^k\bar\epsilon_+\gamma^{k0}\theta_-\sinh\frac{x}{2R}\\
&\quad - \frac{1}{2}\bar\epsilon_-\gamma^0\theta_-\cosh\frac{x}{2R} - \frac{1}{2x}x^k\bar\epsilon_-\gamma^{k0}\theta_+\sinh\frac{x}{2R}\,,
\end{aligned}$$

$$
\begin{aligned}
\delta\theta_+ &= \epsilon_+ \cosh\frac{x}{2R} + \frac{1}{x} x^k \gamma_k \epsilon_- \sinh\frac{x}{2R}\,, \\
\delta\theta_- &= \epsilon_- \cosh\frac{x}{2R} + \frac{1}{x} x^k \gamma_k \epsilon_+ \sinh\frac{x}{2R}\,.
\end{aligned}
\tag{7.112}
$$

To derive an action that is invariant under κ-transformations we need to find a fermionic gauge-transformation that leaves L_H and/or L_Z invariant. The variation of L_H and L_Z under gauge-transformations are given by

$$
\begin{aligned}
\delta L_H &= d([\delta z_H]) + \frac{1}{2} L_P^a [\delta z_K^a] + \frac{1}{2} L_K^a [\delta z_P^a] + \frac{1}{2} \bar{L}_- \gamma^0 [\delta z_-] + \frac{1}{2} \bar{L}_+ \gamma^0 [\delta z_+]\,, \\
\delta L_Z &= d([\delta z_Z]) - \bar{L}_- \gamma^0 [\delta z_-] + \bar{L}_+ \gamma^0 [\delta z_+]\,,
\end{aligned}
\tag{7.113}
$$

where, for example, $[\delta z_K^a]$ is obtained from L_H by changing the 1-forms dt, $d\theta_+$, $d\theta_-$ with the transformations δt, $\delta\theta_+$, $\delta\theta_-$. For κ-transformations we have $[\delta z_H] = 0$, $[\delta z_K^a] = 0$, $[\delta z_P^a] = 0$ and hence we find

$$
\begin{aligned}
\delta L_H &= \frac{1}{2} \delta\bar{\theta}_- \gamma^0 [\delta z_-] + \frac{1}{2} \delta\bar{\theta}_+ \gamma^0 [\delta z_+]\,, \\
\delta L_Z &= -\delta\bar{\theta}_- \gamma^0 [\delta z_-] + \delta\bar{\theta}_+ \gamma^0 [\delta z_+]\,.
\end{aligned}
\tag{7.114}
$$

It follows that to obtain a κ-symmetric action we need to take the pull-back of either L_H or L_Z, with $b = \pm\frac{1}{2}a$. We focus here on the case $b = -\frac{1}{2}a$. For this choice the action and κ-symmetry rules are given by

$$
S = a \int (L_H - \frac{1}{2} L_Z)^*\,, \qquad [\delta z_+] = \kappa\,, \qquad [\delta z_-] = 0\,,
\tag{7.115}
$$

where $\kappa = \kappa(\tau)$ is an arbitrary local parameter. Using this we find the following κ-transformations of the coordinates

$$
\begin{aligned}
\delta t &= \frac{1}{4} \mathrm{sech}\frac{x}{R} \bar{\theta}_+ \gamma^0 \kappa\,, & \delta x^a &= 0\,, & \delta\theta_+ &= \kappa\,, \\
\delta s &= \frac{1}{2} \bar{\theta}_+ \gamma^0 \kappa\,, & \delta v_a &= \frac{1}{2Rx} x^a \bar{\theta}_+ \gamma^0 \kappa \tanh\frac{x}{R}\,, & \delta\theta_- &= 0\,.
\end{aligned}
\tag{7.116}
$$

After κ-gauge fixing (setting $\theta_+ = 0$) the action reads

$$
\begin{aligned}
S = a \int d\tau \Big(&\dot{t} \cosh\frac{x}{R} - \frac{1}{2}\dot{s} + \frac{R}{2x} v_a \dot{x}^a \sinh\frac{x}{R} + \frac{1}{2x^2} x^b v_b x_a \dot{x}^a \Big(1 - \frac{R}{x}\sinh\frac{x}{R}\Big) \\
&- \frac{1}{2} \bar{\theta}_- \gamma^0 \dot{\theta}_- - \frac{1}{4x^2} x^b \dot{x}^a \bar{\theta}_- \gamma_{ab} \gamma^0 \theta_- \Big(\cosh\frac{x}{R} - 1\Big)\Big)\,.
\end{aligned}
\tag{7.117}
$$

This action is invariant under the following transformation rules

$$\begin{aligned}
\delta t &= -\frac{1}{4x}x^k\bar{\epsilon}_+\gamma^{k0}\theta_-\mathrm{sech}\frac{x}{R}\sinh\frac{x}{2R} + \frac{1}{4}\bar{\epsilon}_-\gamma^0\theta_-\mathrm{sech}\frac{x}{R}\cosh\frac{x}{2R}\,,\\
\delta x^i &= 0\,,\\
\delta v^i &= \frac{1}{R}\bar{\epsilon}_+\gamma^{i0}\theta_-\cosh\frac{x}{2R} - \frac{1}{2Rx^2}x^ix^k\bar{\epsilon}_+\gamma^{k0}\theta_-\tanh\frac{x}{R}\sinh\frac{x}{2R}\\
&\quad + \frac{1}{2Rx}x^i\bar{\epsilon}_-\gamma^0\theta_-\tanh\frac{x}{R}\cosh\frac{x}{2R} - \frac{1}{Rx}x^b\bar{\epsilon}_-\gamma_b\gamma^{i0}\theta_-\sinh\frac{x}{2R}\\
\delta s &= -\frac{1}{2x}x^k\bar{\epsilon}_+\gamma^{k0}\theta_-\sinh\frac{x}{2R} - \frac{1}{2}\bar{\epsilon}_-\gamma^0\theta_-\cosh\frac{x}{2R}\,,\\
\delta\theta_- &= \epsilon_-\cosh\frac{x}{2R} + \frac{1}{x}x^k\gamma_k\epsilon_+\sinh\frac{x}{2R}\,.
\end{aligned} \tag{7.118}$$

7.A.4. The $\mathcal{N}=(1,1)$ AdS Carroll Superalgebra

We now consider the $3D$ $\mathcal{N}=(1,1)$ anti-de Sitter algebra which is given by

$$\begin{aligned}
[M_{AB},M_{CD}] &= 2\eta_{A[C}M_{D]B} - 2\eta_{B[C}M_{D]A}\,, & [M_{AB},Q^\pm] &= -\frac{1}{2}\gamma_{AB}Q^\pm\,,\\
[M_{AB},P_C] &= 2\eta_{C[B}P_{A]}\,, & [P_A,Q^\pm] &= \pm x\gamma_A Q^\pm\,,\\
\{Q^\pm_\alpha,Q^\pm_\beta\} &= 4[\gamma^AC^{-1}]_{\alpha\beta}P_A \pm 4x[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}\,, & [P_A,P_B] &= 4x^2M_{AB}\,.
\end{aligned} \tag{7.119}$$

Here P_A, M_{AB} and $Q^\pm_\alpha$ are the generators of space-time translations, Lorentz rotations and supersymmetry transformations, respectively. The bosonic generators P_A and M_{AB} are anti-hermitian while the fermionic generators $Q^\pm_\alpha$ are hermitian. Like in the previous case, the parameter $x=1/(2R)$ is a contraction parameter.

To make the Carroll contraction we rescale the generators with a parameter ω as follows:

$$P_0 = \frac{\omega}{2}H\,, \qquad M_{a0} = \omega K_a\,, \qquad Q^\pm = \sqrt{\omega}\tilde{Q}^\pm\,. \tag{7.120}$$

Taking the limit $\omega\to\infty$ and dropping the tildes on the $Q^\pm$ we get the following 3D $\mathcal{N}=(1,1)$ Carroll superalgebra:

$$
\begin{aligned}
&[M_{ab}, P_c] = 2\delta_{c[b}P_{a]}\,, \qquad [M_{ab}, K_c] = 2\delta_{c[b}K_{a]}\,, \\
&[P_a, P_b] = \frac{1}{R^2}M_{ab}\,, \qquad [P_a, K_b] = \frac{1}{2}\delta_{ab}H\,, \qquad [P_a, H] = \frac{2}{R^2}K_a \\
&[P_a, Q^\pm] = \pm\frac{1}{2R}\gamma_a Q^\pm\,, \qquad [M_{ab}, Q^\pm] = -\frac{1}{2}\gamma_{ab}Q^\pm\,, \\
&\{Q^\pm_\alpha, Q^\pm_\beta\} = 2[\gamma^0 C^{-1}]H \pm \frac{4}{R}[\gamma^{a0}C^{-1}]_{\alpha\beta}K_a\,.
\end{aligned}
\tag{7.121}
$$

The corresponding components of the Maurer-Cartan equation are given by

$$
\begin{aligned}
dL_H &= -\frac{1}{2}L_P^{\,a}\,L_K^{\,a} - \bar{L}_+\gamma^0 L_+ - \bar{L}_-\gamma^0 L_-\,, \\
dL_P^{\,a} &= 2L_P^{\,b}\,L_M^{\,ab}\,, \\
dL_K^{\,a} &= 2L_K^{\,b}\,L_M^{\,ab} + \frac{2}{R^2}L_H L_P{}^a - \frac{2}{R}\bar{L}_+\gamma^{a0}L_+ + \frac{2}{R}\bar{L}_-\gamma^{a0}L_-\,, \\
dL_M^{\,ab} &= 2L_M^{\,ca}\,L_M^{\,cb} + \frac{1}{2R^2}L_P{}^b L_P{}^a\,, \\
dL_+ &= \frac{1}{2}\gamma_{ab}L_+\,L_M^{\,ab} - \frac{1}{2R}\gamma_a L_+ L_P{}^a\,, \\
dL_- &= \frac{1}{2}\gamma_{ab}L_-\,L_M^{\,ab} + \frac{1}{2R}\gamma_a L_- L_P{}^a\,.
\end{aligned}
\tag{7.122}
$$

7.A.5. Superparticle Action

Taking the algebra (7.121) we consider the following coset

$$
\frac{G}{H} = \frac{\mathcal{N} = (1,1)\ \text{AdS Carroll}}{\text{SO(D-1)}}\,. \tag{7.123}
$$

The coset element is $g = g_0\,U$, where $g_0 = e^{Ht}e^{P_a x^a}e^{Q^-_\alpha \theta^\alpha_-}e^{Q^+_\alpha \theta^\alpha_+}$ is the coset representing the $\mathcal{N} = (1,1)$ Carroll superspace and $U = e^{K_a v^a}$ is a general Carroll boost representing the insertion of the superparticle..

The Maurer-Cartan form associated to the super-Carroll space is given by

$$
\Omega_0 = (g_0)^{-1}dg_0 = HE^0 + P_a E^a + K_a\omega^{a0} + M_{ab}\omega^{ab} - \bar{Q}^- E_- - \bar{Q}^+ E_+\,, \tag{7.124}
$$

where $(E^0, E^a, E_{-\alpha}, E_{+\alpha})$ and $(\omega^{a0}, \omega^{ab})$ are the supervielbein and the spin connection of the Carroll superspace:

$$
\begin{aligned}
E^0 &= dt\cosh\frac{x}{R} - \bar\theta_-\gamma^0 d\theta_- - \bar\theta_+\gamma^0 d\theta_+ - \omega^{ab}(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-)\,,\\
E^a &= \frac{R}{x}dx^a\sinh\frac{x}{R} + \frac{1}{x^2}x^a x^b dx_b\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\,,\\
\omega^{a0} &= -\frac{2}{xR}dtx^a\sinh\frac{x}{R} - \frac{1}{R^2}(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-)E^b\\
&\quad -\frac{2}{R}(\bar\theta_+\gamma^{a0}d\theta_+ - \bar\theta_-\gamma^{a0}d\theta_-)\,,\\
\omega^{ab} &= \frac{1}{2x^2}(x^b dx^a - x^a dx^b)\Big(\cosh\frac{x}{R}-1\Big)\,,\\
E_{-\alpha} &= [d\theta_-]_\alpha + \frac{1}{2R}[\gamma_a\theta_-]_\alpha E^a + \frac{1}{2}\omega^{ab}[\gamma_{ab}\theta_-]_\alpha\,,\\
E_{+\alpha} &= [d\theta_+]_\alpha - \frac{1}{2R}[\gamma_a\theta_+]_\alpha E^a + \frac{1}{2}\omega^{ab}[\gamma_{ab}\theta_+]_\alpha\,.
\end{aligned}
\tag{7.125}
$$

We can use the supervielbein to write the Maurer-Cartan form of the $\mathcal{N}=(1,1)$ Carroll superparticle as follows:

$$
\begin{aligned}
L_H &= E^0 + \frac{1}{2}v_a E^a\,, & L_P^a &= E^a\,,\\
L_K^a &= \omega^{a0} + dv^a + 2v_b\,\omega^{ab}\,, & &\\
L_{-\alpha} &= E_{-\alpha}\,, & L_{+\alpha} &= E_{+\alpha}\,.
\end{aligned}
\tag{7.126}
$$

7.A.6. Global Symmetries

The action of the Carrollian superparticle is given by the pull-back of all L's that are invariant under rotations:

$$
\begin{aligned}
S &= M\int (L_H)^*\\
&= M\int d\tau\,\Big(\dot t\cosh\frac{x}{R} + \frac{R}{2x}v_a\dot x^a\sinh\frac{x}{R} + \frac{1}{2x^2}x^b v_b x_a\dot x^a\Big(1-\frac{R}{x}\sinh\frac{x}{R}\Big)\\
&\quad -\bar\theta_-\gamma^0\dot\theta_- - \bar\theta_+\gamma^0\dot\theta_+ - \frac{1}{2x^2}x^b\dot x^a(\bar\theta_+\gamma_{ab}\gamma^0\theta_+ + \bar\theta_-\gamma_{ab}\gamma^0\theta_-)\Big(\cosh\frac{x}{R}-1\Big)\Big)\,.
\end{aligned}
\tag{7.127}
$$

This action is invariant under the following bosonic transformation rules for the coordinates with constant parameters (ζ, a^i, λ^i, λ^i_j) corresponding to time translations, spatial translations, boosts and rotations, respectively

$$
\begin{aligned}
\delta t &= -\zeta + \frac{R}{2x}\lambda^k x_k \tanh\frac{x}{R} + \frac{t}{Rx}a^k x_k \tanh\frac{x}{R}\,, \\
\delta x^i &= -\frac{1}{x^2}\left(x^i a^k x_k - \frac{x}{R}\coth\frac{x}{R}(x^i a^k x_k - a^i x^2)\right) - 2\lambda^i_{\ k}\, x^k\,, \\
\delta v^i &= -\lambda^i - \frac{1}{x^2}\lambda^k x_k x^i \mathrm{sech}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right) - 2\lambda^i_{\ j}\, v^j \\
&\quad - \frac{2t}{R^2}a^i - \frac{2t}{R^2x^2}x^i a^k x_k\, \mathrm{sech}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right) \\
&\quad + \frac{2}{Rx}v_b a^{[i}x^{b]}\mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,, \\
\delta\theta_+ &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_+ + \frac{1}{2Rx}a^k x^b\gamma_{kb}\theta_+ \mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,, \\
\delta\theta_- &= -\frac{1}{2}\lambda^{ab}\gamma_{ab}\theta_- + \frac{1}{2Rx}a^k x^b\gamma_{kb}\theta_- \mathrm{csch}\frac{x}{R}\left(1-\cosh\frac{x}{R}\right)\,.
\end{aligned}
\tag{7.128}
$$

The same action is invariant under the following fermionic transformation rules with constant parameters (ϵ_+, ϵ_-) corresponding to supersymmetry transformations:

$$
\begin{aligned}
\delta t &= \bar\epsilon_+\gamma^0\theta_+ \mathrm{sech}\frac{x}{R}\cosh\frac{x}{2R} - \frac{1}{x}x^k\bar\epsilon_+\gamma^{k0}\theta_+\mathrm{sech}\frac{x}{R}\sinh\frac{x}{2R} \\
&\quad + \bar\epsilon_-\gamma^0\theta_- \mathrm{sech}\frac{x}{R}\cosh\frac{x}{2R} + \frac{1}{x}x^k\bar\epsilon_-\gamma^{k0}\theta_-\mathrm{sech}\frac{x}{R}\sinh\frac{x}{2R}\,, \\
\delta x^i &= 0\,, \\
\delta v^i &= \frac{2}{R}\bar\epsilon_+\gamma^{i0}\theta_+\cosh\frac{x}{2R} - \frac{2}{Rx}x^k\bar\epsilon_+\gamma_k\gamma^{i0}\theta_+\sinh\frac{x}{2R} \\
&\quad + \frac{2}{xR}x^i\tanh\frac{x}{R}\Big(\bar\epsilon_+\gamma^0\theta_+\cosh\frac{x}{2R} - \frac{1}{x}x^k\bar\epsilon_+\gamma^{k0}\theta_+\sinh\frac{x}{2R}\Big) \\
&\quad - \frac{2}{R}\bar\epsilon_-\gamma^{i0}\theta_-\cosh\frac{x}{2R} - \frac{2}{Rx}x^k\bar\epsilon_-\gamma_k\gamma^{i0}\theta_-\sinh\frac{x}{2R} \\
&\quad + \frac{2}{xR}x^i\tanh\frac{x}{R}\Big(\bar\epsilon_-\gamma^0\theta_-\cosh\frac{x}{2R} + \frac{1}{x}x^k\bar\epsilon_-\gamma^{k0}\theta_-\sinh\frac{x}{2R}\Big) \\
\delta\theta_+ &= \epsilon_+\cosh\frac{x}{2R} + \frac{1}{x}x^k\gamma_k\epsilon_+\sinh\frac{x}{2R}\,, \\
\delta\theta_- &= \epsilon_-\cosh\frac{x}{2R} - \frac{1}{x}x^k\gamma_k\epsilon_-\sinh\frac{x}{2R}\,.
\end{aligned}
\tag{7.129}
$$

One may not assume the validity of field equations at very high density of field and matter and one may not conclude that the beginning of the expansion should be a singularity in the mathematical sense.

Albert Einstein

8 Conclusions and Outlook

The discovery of general relativity 100 years ago is considered to be one of the greatest scientific and intellectual achievements of all time. Its importance and relevance have since then only increased as a result of the number and range of observations that can be made, and the applications that have since appeared [173, 174].

For example, the prediction of light-bending (which grew into gravitational lensing), the gravitational redshift and the anomaly in the perihelion advance of Mercury have made it possible to use satellite navigation systems (GPS), to study binary pulsars, to use microlensing to infer the distribution of mass within galaxies and the distribution of dark matter, to detect the presence of new exoplanets, and to develop radio astronomy, which has since led to the discovery of quasars and pulsars.

The corresponding mathematical and conceptual progress has also been outstanding: for example, Einstein's equations are one of the most interesting and important systems in the theory of partial differential equations and geometrical analysis. Moreover, the geometric concepts of connection and curvature have become fundamental to modern gauge theories.

In spite of its powerful scope, the theory has always had drawbacks in tests outside the solar system, including the understanding of compact stars such as white dwarfs, supernovae and neutron stars, in which the enormous nuclear density counterbalances the gravitational force, which in return requires the simultaneous use of quantum theory and GR. So far, there has not been a satisfactory physical meaning given to a singularity occurring in a given spacetime, and the modelling of properties of inflation, dark matter and dark energy remains incomplete.

Serious efforts to modify GR theory have taken a number of forms. Each modification adopts a different starting point and perspective, treating certain aspects of GR as more fundamental, either hoping that once the main difficulties are resolved, the remaining aspects can be handled successfully, or studying specific features of the theory to better understand it.

Many modifications of GR have been attempted over the last 50 years, such as supersymmetry, higher dimensions, extended objects, higher spin theories, infinite towers of particles and fields, massive gravitons, higher derivative theories, holography, etc. Many new theories have emerged, such as string theory, M-theory, twistor theory, Hořava-Lifshitz gravity, loop quantum gravity, etc. Some of these have attempted to modify Einstein gravity at short distances and thus focus on the quantum description, while others have concentrated their efforts on choosing adequate matter that would improve the ultra-violet behavior, or focus on infrared modifications to try to improve the dark matter and dark energy problems of cosmology. Others have studied inflationary scenarios or the unification of the gauge couplings. Each of these approaches has its own virtues and drawbacks but together they have instigated the study of a variety of challenging problems and led to unforeseen applications in diverse areas of mathematics, cosmology and physics.

In the context of this intricate enterprise, the aim of this thesis was to study a supersymmetric massive extension of GR, the symmetry properties of Hořava-Lifshitz anisotropic scaling and the features of some systems using the polymer representation. We also studied two different (and opposite) supersymmetric limits of the theory, the non-relativistic limit and the Carroll limit. Throughout this thesis, we have concentrated our attention on the simplest three-dimensional cases, not only because these are technically simpler, but also because three-dimensional gravity is interesting in its own right.

Supersymmetry has been widely studied in the literature because it can be used in many modern developments in mathematical and theoretical physics: it automatically appears as a symmetry of string theory, as well as in brane theories, the AdS/CFT correspondence, the Seiberg-Witten duality, etc.

Massive gravity is a different extension; it considers gravity to be propagated by a massive spin-2 particle. One of the main motivations for massive gravity is that adding a mass to the graviton leads to long distance modifications of GR that affect dark energy and reduce the cosmological constant problem.

Instead of adding new symmetries it is also possible to take a different approach by modifying the existing GR symmetries. The anomalous scaling in

Hořava-Lifshitz theory breaks the spacetime diffeomorphism invariance in such a way, that the gravitational action is no longer a topological invariant and the theory becomes dynamical. Nonetheless, this Lorentz symmetry violation does not represent a problem if the theory recovers this symmetry at energy scales where it has already been tested.

Another modification that we can study is to give up on the Schrödinger quantization scheme to try to handle the GR quantization. The motivation for studying polymer representations is that they provide the simplest analogues of the representation featuring in loop quantum gravity. It is interesting to study this approach because in some cosmology models [175–177] it leads to the avoidance of the classical singularity which in loop quantum cosmology is replaced by a quantum bounce, and is a microscopic basis for calculating the entropy of some black holes that agrees with the Bekenstein–Hawking's semiclassical formula [178].

In addition, studying the various limits of GR helps to understand it better because certain problems are easy to study in limiting cases in which these problems often become simpler and easier to analyze. Furthermore it is interesting to study these limiting cases because of the applications that may arise from them.

There are two ways to consider the limits: The first one, known in the literature as limiting procedure, see [44], obtains the limiting gravity/supergravity theories directly from the GR theory/supertheory. The second procedure consists of obtaining the gravity theory from the gauging of the contracted algebra. In this work, we have chosen the latter approach and investigated the non-relativistic and ultra-relativistic limits by choosing appropriate contractions of the Poincaré superalgebra and studying the particle actions that emerge from gauging the algebras.

The first GR modification that we studied is the supersymmetric-massive extension. We motivated and discussed the Kaluza Klein formulation to obtain the 3D, off-shell massive spin-2 supermultiplet. This multiplet, together with the previously known off-shell massless spin-2 multiplet, can be used to write a supersymmetric version of linearized NMG in the auxiliary field form.

The massless limit involves a non-trivial coupling of a scalar multiplet to a current multiplet. In the usual Fierz–Pauli case, it is possible to cure the vDVZ discontinuity by taking into account its non-linear version, but a non-linear version of the SNMG is not obvious since the construction of the massive spin-2 supermultiplet is based on the truncation of the Kaluza Klein expansion to the first massive level, which can only be performed at the linearized level. To obtain a non-linear SNMG theory one may have to use superspace techniques.

The second GR modification that we considered deals with the anisotropic spacetime scaling that breaks Lorentz symmetry. There are many open issues regarding Hořava-Lifshitz gravity. For example, renormalizability beyond power counting has not been explicitly shown, it has not been proved if the theory approaches GR in the infrared limit or not, the role of matter and its coupling to gravity have not been fully clarified yet and couplings between the matter and the scalar graviton could lead to violations of the equivalence principle [91].

Despite these drawbacks, it is interesting to apply the anisotropic scaling to several systems in order to try to learn more about their features and later on apply this lessons in the gravity framework. Our study takes a first step on this direction by the symmetry analysis of some higher derivative toy systems. After applying Noether's theorem for actions with an anisotropic dependence between time and space we obtained the explicit form of the conserved currents for a scalar field and for the electrodynamics Lagrangian, both systems with dynamical critical exponent $z = 2$ and additionally for a higher derivative Chern-Simons extension. We found that Noether's charge is unaffected because of the additional higher order spatial derivatives.

The third GR modification that we took under consideration is the polymer quantization scheme. In a similar way as in the previous case we studied some simpler systems in pursuance of gaining a deeper understanding about some features of the loop quantum gravity framework. We obtained the path integral for the polymer non-relativistic free particle, the relativistic particle and the harmonic oscillator. We calculated the propagators of the systems and obtained that the velocity of the particle in all the systems is bounded.

The fourth modification that we considered was the non-relativistic superparticle in a curved background. We first investigated the symmetries of a superparticle moving on a flat and curved (AdS) non-relativistic space. We then explained the gauging procedure for constructing the corresponding non-relativistic supergravity theory.

The fifth and last modification that we investigated was the supersymmetric-Carroll limit. Carroll symmetries have recently appeared in a number of contexts. For instance, an identification of a duality between the Galilean and the Carroll limits has led to the study of some non-Einsteinian systems, and these symmetries have also appeared in studies of tachyon condensation and warped conformal field theories.

We constructed the vielbein and spin connection of the AdS Carroll space, as well as the action of a massive particle moving in this space and its symmetries. We then extended the study of the supersymmetric case by focusing on the $\mathcal{N} = 1, 2$ AdS Carroll particle.

Our results can be extended in a number of ways. In the massive case, for instance, it would natural to extend the results of this work to the case of extended, i.e. $\mathcal{N} > 1$, supersymmetry, or to 'cosmological' massive gravity theories. Higher-derivative, linearized versions of NMG with extended supersymmetry, or anti-de Sitter vacua, are given in [56, 179]. Of special interest is the case of maximal supersymmetry since this corresponds to the KK reduction of the $\mathcal{N} = 8$ massless maximal supergravity multiplet which only exists in a formulation without (trivial) auxiliary fields. We expect that a formulation of this maximal SNMG theory without higher-derivatives will be useful in finding out whether this massive 3D supergravity model has the same miraculous ultraviolet properties as in the 4D massless case.

As for the polymer quantization it is possible to continue in different directions. It would be interesting to study some features of the solutions of the systems, for example, it was recently proved that the polymer harmonic oscillator possesses instanton solutions [116]. It would be also interesting to apply the formalism to models with curvature and a cosmological constant different to zero.

Regarding the non-relativistic particle, it would be useful to construct a four-dimensional analog of our results. In order to do so, one would first have to be able to construct the Galilean and Newton–Cartan supergravity multiplets in four spacetime dimensions. This has not yet been done.

Another generalization of our results might involve going from superparticles to superstrings or even super p-branes. This would require using a 'stringy' generalization of the non-relativistic limits we have considered here, see for example [165]. The case of a non-relativistic superstring in a flat background was already considered in [128]. In the case of a non-relativistic curved background one could apply holography and study the corresponding non-relativistic supersymmetric boundary theory.

In the case of a particle/superparticle propagating in three spacetime dimensions, the Galilei algebra contains a second central charge that might also be included. This leads to the notion of a *non-commutative* non-relativistic particle/superparticle in which the embedding coordinates are non-commutative with respect to the Dirac brackets [180, 181]. Finally, we have found that some of our superparticles are 1/2 BPS, thus corroborating recent results for relativistic superparticles [182]. It would be interesting to verify whether the statement that "all superparticles are BPS" applies to non-relativistic superparticles as well.

As a possible continuation of our ideas regarding the Carroll superparticle, it would be interesting to find the coupling of the AdS Carroll particle, and the corresponding superparticle, to the (super) AdS gauge fields. As in the flat Carroll case [41] we would expect the particle/superparticle to have a non-trivial dynamics.

It would be also interesting to study whether it is possible to construct a corresponding Carroll gravity/supergravity theory. This can be approached in one of two ways. One approach would be to gauge the (super)Carroll algebra and/or the Lifshitz Carroll algebra with $z = 0$ by extending the results in [183] in order to gauge the AdS-Carroll algebra and to extend this in the supersymmetric case. A second alternative approach would be to try to define an ultra-relativistic limit of relativistic gravity/supergravity similar to the non-relativistic limit [44].

Finally, in the supersymmetric-massive extension and in the limiting cases, it would be desirable to construct a four-dimensional analogue of our results, but so far, this has not yet been achieved.

List of Publications

- Lorena Parra, J. David Vergara, *Symmetry analysis for anisotropic field theories*, AIP Conf. Proc. 1473 (2011) 243-247.
- Eric A. Bergshoeff, Marija Kovacevic, Lorena Parra, Jan Rosseel, Yihao Yin and Thomas Zojer, *New Massive Supergravity and Auxiliary Fields*, Class. Quant. Grav. 30 (2013) 195004, [arXiv:1304.5445].
- Eric A. Bergshoeff, Marija Kovacevic, Lorena Parra and Thomas Zojer, *A new road to massive gravity?*, PoS, Corfu2012, (2013) 053.
- Lorena Parra, *Kaluza Klein reduction of supersymmetric Fierz-Pauli*, Phys. Part. Nucl. Lett. 11 (2014) 7, 984-986.
- Lorena Parra, J. David Vergara, *Polymer quantum mechanics some examples using path integrals*, AIP Conf. Proc. 1577 (2014), 269-280.
- Eric A. Bergshoeff, Joaquim Gomis, Marija Kovacevic, Lorena Parra, Jan Rosseel and Thomas Zojer, *The Non-Relativistic Superparticle in a Curved Background*, Phys. Rev. D 90 (2014) 6, 065006, [arXiv:1406.7286]
- Eric A. Bergshoeff, Joaquim Gomis and Lorena Parra, *The Symmetries of the Carroll Superparticle*, J. Phys. A 49 (2016) 18, 185402

Bibliography

[1] M. M. Nojiri, "Beyond the Standard Model," arXiv:1406.1410 [hep-ph].

[2] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," *Phys.Rev.* **159** (1967) 1251–1256.

[3] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality: Foundations and Applications*. Cambridge University Press, 2015. https://books.google.nl/books?id=JithBwAAQBAJ.

[4] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et al.*, "Measurements of Omega and Lambda from 42 high redshift supernovae," *Astrophys.J.* **517** (1999) 565–586, arXiv:astro-ph/9812133 [astro-ph].

[5] **Supernova Search Team** Collaboration, A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant," *Astron.J.* **116** (1998) 1009–1038, arXiv:astro-ph/9805201 [astro-ph].

[6] C. Deffayet, "Cosmology on a brane in Minkowski bulk," *Phys.Lett.* **B502** (2001) 199–208, arXiv:hep-th/0010186 [hep-th].

[7] C. Deffayet, G. Dvali, and G. Gabadadze, "Accelerated universe from gravity leaking to extra dimensions," *Phys.Rev.* **D65** (2002) 044023, arXiv:astro-ph/0105068 [astro-ph].

[8] G. 't Hooft, "Unitarity in the Brout-Englert-Higgs Mechanism for Gravity," arXiv:0708.3184 [hep-th].

[9] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," *Proc.Roy.Soc.Lond.* **A173** (1939) 211–232.

[10] G. Dvali, G. Gabadadze, and M. Porrati, "4-D gravity on a brane in 5-D Minkowski space," *Phys.Lett.* **B485** (2000) 208–214, arXiv:hep-th/0005016 [hep-th].

[11] S. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," *JHEP* **1202** (2012) 126, arXiv:1109.3515 [hep-th].

[12] S. Hassan and R. A. Rosen, "Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity," *JHEP* **1204** (2012) 123, arXiv:1111.2070 [hep-th].

[13] E. A. Bergshoeff, S. de Haan, O. Hohm, W. Merbis, and P. K. Townsend, "Zwei-Dreibein Gravity: A Two-Frame-Field Model of 3D Massive Gravity," *Phys.Rev.Lett.* **111** no. 11, (2013) 111102, arXiv:1307.2774.

[14] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, "Massive Gravity in Three Dimensions," *Phys.Rev.Lett.* **102** (2009) 201301, arXiv:0901.1766 [hep-th].

[15] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions," *Living Rev.Rel.* **11** (2008) 6, arXiv:0801.3471 [hep-th].

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys.Rept.* **323** (2000) 183–386, arXiv:hep-th/9905111 [hep-th].

[17] M. Natsuume, "AdS/CFT Duality User Guide," arXiv:1409.3575 [hep-th].

[18] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," *Phys.Lett.* **B379** (1996) 99–104, arXiv:hep-th/9601029 [hep-th].

[19] C. de Rham, G. Gabadadze, D. Pirtskhalava, A. J. Tolley, and I. Yavin, "Nonlinear Dynamics of 3D Massive Gravity," *JHEP* **1106** (2011) 028, arXiv:1103.1351 [hep-th].

[20] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin, *et al.*, "Massive 3D Supergravity," *Class.Quant.Grav.* **27** (2010) 025010, arXiv:0907.4658 [hep-th].

[21] O. Hohm and E. Tonni, "A boundary stress tensor for higher-derivative gravity in AdS and Lifshitz backgrounds," *JHEP* **1004** (2010) 093, arXiv:1001.3598 [hep-th].

[22] P. Horava, "Quantum Gravity at a Lifshitz Point," *Phys. Rev.* **D79** (2009) 084008, arXiv:0901.3775 [hep-th].

[23] P. Horava, "Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point," *Phys. Rev. Lett.* **102** (2009) 161301, arXiv:0902.3657 [hep-th].

[24] P. Horava and C. M. Melby-Thompson, "General Covariance in Quantum Gravity at a Lifshitz Point," *Phys. Rev.* **D82** (2010) 064027, arXiv:1007.2410 [hep-th].

[25] A. Ashtekar, "Introduction to Loop Quantum Gravity," *PoS* **QGQGS2011** (2011) 001, arXiv:1201.4598 [gr-qc].

[26] A. Ashtekar, S. Fairhurst, and J. L. Willis, "Quantum gravity, shadow states, and quantum mechanics," *Class.Quant.Grav.* **20** (2003) 1031–1062, arXiv:gr-qc/0207106 [gr-qc].

[27] T. Thiemann, *Modern canonical quantum general relativity*. Cambridge University Press, 2008. arXiv:gr-qc/0110034 [gr-qc]. http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521842631.

[28] A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A Status report," *Class. Quant. Grav.* **21** (2004) R53, arXiv:gr-qc/0404018 [gr-qc].

[29] L. Smolin, "An Invitation to loop quantum gravity," in *Proceedings, 3rd International Symposium on Quantum theory and symmetries (QTS3)*, pp. 655–682. 2004. arXiv:hep-th/0408048 [hep-th]. [Submitted to: Rev. Mod. Phys.(2004)].

[30] G. M. Hossain, V. Husain, and S. S. Seahra, "The Propagator in polymer quantum field theory," *Phys.Rev.* **D82** (2010) 124032, arXiv:1007.5500 [gr-qc].

[31] H. Bacry and J. Lévy-Leblond, "Possible kinematics," *J.Math.Phys.* **9** (1968) 1605–1614.

[32] J.-M. Lévy-Leblond, "Une nouvelle limite non-relativiste du groupe de poincaré," *Annales de l'institut Henri Poincaré (A) Physique théorique* **3** no. 1, (1965) 1–12. http://eudml.org/doc/75509.

[33] C. Duval, G. Gibbons, P. Horvathy, and P. Zhang, "Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time," *Class.Quant.Grav.* **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[34] X. Bekaert and K. Morand, "Connections and dynamical trajectories in generalised Newton-Cartan gravity II. An ambient perspective," arXiv:1505.03739 [hep-th].

[35] S. Sachdev, *Quantum Phase Transitions*. Cambridge University Press, 2001. http://books.google.de/books?id=Ih_E05N5TZQC.

[36] K. O'Hara, S. Hemmer, M. Gehm, S. Granade, and J. Thomas, "Observation of a Strongly Interacting Degenerate Fermi Gas of Atoms," *Science* **298** (2002) 2179–2182, arXiv:cond-mat/0212463 [cond-mat.supr-con].

[37] T. Mehen, I. W. Stewart, and M. B. Wise, "Conformal invariance for nonrelativistic field theory," *Phys.Lett.* **B474** (2000) 145–152, arXiv:hep-th/9910025 [hep-th].

[38] D. Son and M. Wingate, "General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas," *Annals Phys.* **321** (2006) 197–224, arXiv:cond-mat/0509786 [cond-mat].

[39] D. Son, "Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry," *Phys.Rev.* **D78** (2008) 046003, arXiv:0804.3972 [hep-th].

[40] A. Bagchi and R. Gopakumar, "Galilean Conformal Algebras and AdS/CFT," *JHEP* **0907** (2009) 037, arXiv:0902.1385 [hep-th].

[41] E. Bergshoeff, J. Gomis, and G. Longhi, "Dynamics of Carroll Particles," *Class.Quant.Grav.* **31** no. 20, (2014) 205009, arXiv:1405.2264 [hep-th].

[42] G. Gibbons, K. Hashimoto, and P. Yi, "Tachyon condensates, Carrollian contraction of Lorentz group, and fundamental strings," *JHEP* **0209** (2002) 061, arXiv:hep-th/0209034 [hep-th].

[43] D. M. Hofman and B. Rollier, "Warped Conformal Field Theory as Lower Spin Gravity," arXiv:1411.0672 [hep-th].

[44] E. Bergshoeff, J. Rosseel, and T. Zojer, "Newton-Cartan (super)gravity as a non-relativistic limit," arXiv:1505.02095 [hep-th].

[45] R. Andringa, E. Bergshoeff, S. Panda, and M. de Roo, "Newtonian Gravity and the Bargmann Algebra," *Class.Quant.Grav.* **28** (2011) 105011, arXiv:1011.1145 [hep-th].

[46] R. Andringa, E. A. Bergshoeff, J. Rosseel, and E. Sezgin, "3D Newton–Cartan supergravity," *Class.Quant.Grav.* **30** (2013) 205005.

[47] E. A. Bergshoeff, J. Hartong, and J. Rosseel, "Torsional Newton-Cartan Geometry and the Schrödinger Algebra," arXiv:1409.5555 [hep-th].

[48] R. Andringa, E. A. Bergshoeff, J. Rosseel, and E. Sezgin, "Newton-Cartan Supergravity," arXiv:1305.6737 [hep-th].

[49] D. Freedman and A. Van Proeyen, *Supergravity*. Cambridge University Press, 2012. https://books.google.nl/books?id=7ANrOHJLTjMC.

[50] M. Sohnius, "Introducing Supersymmetry," *Phys.Rept.* **128** (1985) 39–204.

[51] B. de Wit, "Supergravity," arXiv:hep-th/0212245 [hep-th].

[52] F. Quevedo, S. Krippendorf, and O. Schlotterer, "Cambridge Lectures on Supersymmetry and Extra Dimensions," arXiv:1011.1491 [hep-th].

[53] H. Nastase, "Introduction to Supergravity," arXiv:1112.3502 [hep-th].

[54] A. Bilal, "Introduction to supersymmetry," arXiv:hep-th/0101055 [hep-th].

[55] H. Baer and X. Tata, *Weak Scale Supersymmetry: From Superfields to Scattering Events*. Cambridge University Press, 2006. https://books.google.nl/books?id=QpwM_3PA2KAC.

[56] E. A. Bergshoeff, O. Hohm, J. Rosseel, and P. K. Townsend, "On Maximal Massive 3D Supergravity," *Class.Quant.Grav.* **27** (2010) 235012, arXiv:1007.4075 [hep-th].

[57] V. Bargmann, "On Unitary ray representations of continuous groups," *Annals Math.* **59** (1954) 1–46.

[58] A. Achucarro and P. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," *Phys.Lett.* **B180** (1986) 89.

[59] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, "Three-dimensional N=2 (AdS) supergravity and associated supercurrents," *JHEP* **1112** (2011) 052, `arXiv:1109.0496 [hep-th]`.

[60] S. M. Kuzenko, U. Lindstrom, M. Rocek, I. Sachs, and G. Tartaglino-Mazzucchelli, "Three-dimensional $\mathcal{N} = 2$ supergravity theories: From superspace to components," *Phys.Rev.* **D89** no. 8, (2014) 085028, `arXiv:1312.4267 [hep-th]`.

[61] S. R. Coleman, J. Wess, and B. Zumino, "Structure of phenomenological Lagrangians. 1.," *Phys.Rev.* **177** (1969) 2239–2247.

[62] J. Callan, Curtis G., S. R. Coleman, J. Wess, and B. Zumino, "Structure of phenomenological Lagrangians. 2.," *Phys.Rev.* **177** (1969) 2247–2250.

[63] E. Ivanov, S. Krivonos, and V. Leviant, "Geometry of Conformal Mechanics," *J.Phys.* **A22** (1989) 345.

[64] E. Ivanov, S. Krivonos, and V. Leviant, "Geometric Superfield Approach to Superconformal Mechanics," *J.Phys.* **A22** (1989) 4201.

[65] J. Gomis, K. Kamimura, and J. M. Pons, "Non-linear Realizations, Goldstone bosons of broken Lorentz rotations and effective actions for p-branes," *Nucl.Phys.* **B871** (2013) 420–451, `arXiv:1205.1385 [hep-th]`.

[66] M. Bando, T. Kugo, and K. Yamawaki, "Nonlinear Realization and Hidden Local Symmetries," *Phys.Rept.* **164** (1988) 217–314.

[67] I. N. McArthur, "Kappa symmetry of Green-Schwarz actions in coset superspaces," *Nucl.Phys.* **B573** (2000) 811–829, `arXiv:hep-th/9908045 [hep-th]`.

[68] J. Gomis, K. Kamimura, and P. C. West, "The Construction of brane and superbrane actions using non-linear realisations," *Class.Quant.Grav.* **23** (2006) 7369–7382, `arXiv:hep-th/0607057 [hep-th]`.

[69] A. Ashtekar, M. Campiglia, and A. Henderson, "Path Integrals and the WKB approximation in Loop Quantum Cosmology," *Phys.Rev.D* **82** (2010) 124043, `arXiv:1011.1024 [gr-qc]`.

[70] M. Bojowald, *Canonical Gravity and Applications*. Cambridge University Press, 2011.

[71] M. Bojowald, *Quantum Cosmology: A Fundamental Description of the Universe*. Lecture Notes in Physics. Springer, 2011.

[72] J. Henty, P. S. Howe, and P. Townsend, "Quantum mechanics of the relativistic spinning particle," *Class.Quant.Grav.* **5** (1988) 807.

[73] M. Henneaux and J. D. Vergara, "BRST Formalism and Gauge Invariant Operators: The Example of the Free Relativistic Particle," *Nova Science Publishers Inc* (1992) 111 – 123.

[74] H. Ikemori, "Superfield Formulation of Relativistic Superparticle," *Phys.Lett.* **B199** (1987) 239.

[75] F. Berezin and M. Marinov, "Particle Spin Dynamics as the Grassmann Variant of Classical Mechanics," *Annals Phys.* **104** (1977) 336.

[76] C. de Rham, "Massive Gravity," *Living Rev.Rel.* **17** (2014) 7, arXiv:1401.4173 [hep-th].

[77] E. A. Bergshoeff, J. Fernandez-Melgarejo, J. Rosseel, and P. K. Townsend, "On 'New Massive' 4D Gravity," *JHEP* **1204** (2012) 070, arXiv:1202.1501 [hep-th].

[78] S. Deser, J. Kay, and K. Stelle, "Hamiltonian Formulation of Supergravity," *Phys.Rev.* **D16** (1977) 2448.

[79] D. Boulware and S. Deser, "Can gravitation have a finite range?," *Phys.Rev.* **D6** (1972) 3368–3382.

[80] W. Nahm, "Supersymmetries and their Representations," *Nucl.Phys.* **B135** (1978) 149.

[81] E. A. Bergshoeff and O. Hohm, "A Topologically Massive Gauge Theory with 32 Supercharges," *Phys.Rev.* **D78** (2008) 125017, arXiv:0810.0377 [hep-th].

[82] K. Stelle and P. C. West, "Minimal Auxiliary Fields for Supergravity," *Phys.Lett.* **B74** (1978) 330.

[83] S. Ferrara and P. van Nieuwenhuizen, "The Auxiliary Fields of Supergravity," *Phys.Lett.* **B74** (1978) 333.

[84] I. Buchbinder, J. Gates, S. James, I. Linch, William Divine, and J. Phillips, "New 4-D, N=1 superfield theory: Model of free massive superspin 3/2 multiplet," *Phys.Lett.* **B535** (2002) 280–288, arXiv:hep-th/0201096 [hep-th].

[85] H. Nishino and S. Rajpoot, "Interacting N=1 vector-spinor multiplet in D-3," *Phys.Rev.* **D83** (2011) 127701.

[86] S. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry," *Front.Phys.* **58** (1983) 1–548, arXiv:hep-th/0108200 [hep-th].

[87] D. Dalmazi and E. L. Mendonca, "Generalized soldering of +-2 helicity states in D=2+1," *Phys.Rev.* **D80** (2009) 025017, arXiv:0906.4531 [hep-th].

[88] A. Padilla, "The good, the bad and the ugly .... of Horava gravity," *J. Phys. Conf. Ser.* **259** (2010) 012033, arXiv:1009.4074 [hep-th].

[89] T. P. Sotiriou, "Horava-Lifshitz gravity: a status report," *J. Phys. Conf. Ser.* **283** (2011) 012034, arXiv:1010.3218 [hep-th].

[90] G. Calcagni, "Lorentz Violations in Multifractal Spacetimes," arXiv:1603.03046 [gr-qc].

[91] D. Blas, "Lorentz violation in gravity," in *Proceedings, 50th Rencontres de Moriond Gravitation : 100 years after GR*, pp. 11–14. 2015. arXiv:1507.07687 [gr-qc]. https://inspirehep.net/record/1385303/files/arXiv:1507.07687.pdf.

[92] D. Blas, "Modified gravity and binary pulsars: the Lorentz violating case," in *14th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG14) Rome, Italy, July 12-18, 2015*. 2016. arXiv:1601.04653 [gr-qc]. https://inspirehep.net/record/1415749/files/arXiv:1601.04653.pdf.

[93] T. Jacobson, "Einstein-aether gravity: Theory and observational constraints," in *CPT and Lorentz symmetry. Proceedings, 4th Meeting, Bloomington, USA, Aug 8-11, 2007*, pp. 92–99. 2008. arXiv:0711.3822 [gr-qc]. https://inspirehep.net/record/768502/files/arXiv:0711.3822.pdf.

[94] E. M. Lifshitz, "On the theory of second order phase transitions i," *Zh. Eksp. Teor. Fiz.* **11** no. 269, (1941) 255.

[95] R. M. Hornreich, M. Luban, and S. Shtrikman, "Critical Behavior at the Onset of k –> -Space Instability on the lamda Line," *Phys. Rev. Lett.* **35** (1975) 1678–1681.

[96] G. Bertoldi, B. A. Burrington, and A. W. Peet, "Thermodynamics of black branes in asymptotically Lifshitz spacetimes," *Phys. Rev.* **D80** (2009) 126004, arXiv:0907.4755 [hep-th].

[97] P. C. Hohenberg and B. I. Halperin, "Theory of Dynamic Critical Phenomena," *Rev. Mod. Phys.* **49** (1977) 435–479.

[98] J. L. Cardy, *Scaling and renormalization in statistical physics*. 1996.

[99] E. Noether, "Invariante variationsprobleme," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918** (1918) 235–257. http://eudml.org/doc/59024.

[100] S. R. Das and G. Murthy, "Compact z=2 Electrodynamics in 2+1 dimensions: Confinement with gapless modes," *Phys. Rev. Lett.* **104** (2010) 181601, arXiv:0909.3064 [hep-th].

[101] J. M. Romero, J. A. Santiago, O. Gonzalez-Gaxiola, and A. Zamora, "Electrodynamics a la Horava," *Mod. Phys. Lett.* **A25** (2010) 3381–3393, arXiv:1006.0956 [hep-th].

[102] S. Deser and R. Jackiw, "Higher derivative Chern-Simons extensions," *Phys. Lett.* **B451** (1999) 73–76, arXiv:hep-th/9901125 [hep-th].

[103] E. L. Hill, "Hamilton's principle and the conservation theorems of mathematical physics," *Rev. Mod. Phys.* **23** (Jul, 1951) 253–260. http://link.aps.org/doi/10.1103/RevModPhys.23.253.

[104] P. Chaikin and T. Lubensky, *Principles of Condensed Matter Physics*. Cambridge University Press, 2000.
https://books.google.com.mx/books?id=P9YjNjzr9OIC.

[105] P. Horava, "Membranes at Quantum Criticality," *JHEP* **03** (2009) 020, arXiv:0812.4287 [hep-th].

[106] C. G. Callan, Jr., S. Coleman, and R. Jackiw, "A new improved energy-momentum tensor," *Annals of Physics* **59** (July, 1970) 42–73.

[107] K. Fredenhagen and F. Reszewski, "Polymer state approximations of Schrodinger wave functions," *Class. Quant. Grav.* **23** (2006) 6577–6584, arXiv:gr-qc/0606090 [gr-qc].

[108] A. Corichi, T. Vukasinac, and J. A. Zapata, "Polymer Quantum Mechanics and its Continuum Limit," *Phys. Rev.* **D76** (2007) 044016, arXiv:0704.0007 [gr-qc].

[109] A. Corichi, T. Vukasinac, and J. A. Zapata, "On a continuum limit for loop quantum cosmology," *AIP Conf. Proc.* **977** (2008) 64–71, arXiv:0711.0788 [gr-qc].

[110] M. Campiglia, "Polymer representations and geometric quantization," arXiv:1111.0638 [gr-qc].

[111] G. Chacon-Acosta, E. Manrique, L. Dagdug, and H. A. Morales-Tecotl, "Statistical thermodynamics of polymer quantum systems," *SIGMA* **7** (2011) 110, arXiv:1109.0803 [gr-qc].

[112] J. F. Barbero G., J. Prieto, and E. J. Villaseñor, "Band structure in the polymer quantization of the harmonic oscillator," *Class.Quant.Grav.* **30** (2013) 165011, arXiv:1305.5406 [gr-qc].

[113] J. Jauch, *Foundations of quantum mechanics*. Addison-Wesley Pub. Co., 1968.

[114] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*. World Scientific, 2004.

[115] E. Flores-Gonzalez, H. A. Morales-Tecotl, and J. D. Reyes, "Propagators in polymer quantum mechanics," *Annals of Physics* **336** (2013) 394 – 412.

[116] J. A. A. Olivares, A. Garcia-Chung, and J. D. Vergara, "Instanton solutions on the polymer harmonic oscillator," arXiv:1604.07288 [hep-th].

[117] M. Leibscher and B. Schmidt, "Quantum dynamics of a plane pendulum," *Phys. Rev. A* **80** (Jul, 2009) 012510.
http://link.aps.org/doi/10.1103/PhysRevA.80.012510.

[118] A. Ashtekar, M. Bojowald, and J. Lewandowski, "Mathematical structure of loop quantum cosmology," *Adv.Theor.Math.Phys.* **7** (2003) 233–268.

[119] A. Ashtekar and P. Singh, "Loop Quantum Cosmology: A Status Report," *Class.Quant.Grav.* **28** (2011) 213001, arXiv:1108.0893 [gr-qc].

[120] Y. Nishida and D. T. , "Nonrelativistic conformal field theories," *Phys.Rev.* **D76** (2007) 086004, arXiv:0706.3746 [hep-th].

[121] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," *Phys.Rev.Lett.* **101** (2008) 061601, arXiv:0804.4053 [hep-th].

[122] C. P. Herzog, M. Rangamani, and S. F. Ross, "Heating up Galilean holography," *JHEP* **0811** (2008) 080, arXiv:0807.1099 [hep-th].

[123] S. Kachru, X. Liu, and M. Mulligan, "Gravity duals of Lifshitz-like fixed points," *Phys.Rev.* **D78** (2008) 106005, arXiv:0808.1725 [hep-th].

[124] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Class.Quant.Grav.* **26** (2009) 224002, arXiv:0903.3246 [hep-th].

[125] J. Gomis and H. Ooguri, "Nonrelativistic closed string theory," *J.Math.Phys.* **42** (2001) 3127–3151, arXiv:hep-th/0009181 [hep-th].

[126] U. H. Danielsson, A. Guijosa, and M. Kruczenski, "IIA/B, wound and wrapped," *JHEP* **0010** (2000) 020, arXiv:hep-th/0009182 [hep-th].

[127] J. Gomis, K. Kamimura, and P. K. Townsend, "Non-relativistic superbranes," *JHEP* **0411** (2004) 051, arXiv:hep-th/0409219 [hep-th].

[128] J. Gomis, J. Gomis, and K. Kamimura, "Non-relativistic superstrings: A New soluble sector of AdS(5) x S**5," *JHEP* **0512** (2005) 024, arXiv:hep-th/0507036 [hep-th].

[129] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, "Strings in flat space and pp waves from N=4 superYang-Mills," *JHEP* **0204** (2002) 013, arXiv:hep-th/0202021 [hep-th].

[130] E. Cartan, "Sur les variétés à connexion affine et la théorie de la relativité généralisée. (première partie)," *Annales Sci.Ecole Norm.Sup.* **40** (1923) 325–412.

[131] P. Havas, "Four-Dimensional Formulations of Newtonian Mechanics and Their Relation to the Special and the General Theory of Relativity," *Rev.Mod.Phys.* **36** (1964) 938–965.

[132] J. Anderson, *Principles of relativity physics*. Academic Press, 1967. http://books.google.nl/books?id=mAVRAAAAMAAJ.

[133] A. Trautman, "Theories of Space, Time and Gravitation," in *Lectures on General Relativity,*, S. Deser and K. Ford, eds. Prentice-Hall, Englewood Cliffs, 1965.

[134] H. P. Künzle, "Galilei and lorentz structures on space-time : comparison of the corresponding geometry and physics," *Annales de l'institut Henri Poincaré (A) Physique théorique* **17** no. 4, (1972) 337–362. http://eudml.org/doc/75762.

[135] K. Kuchař, "Gravitation, geometry, and nonrelativistic quantum theory," *Phys. Rev. D* **22** (Sep, 1980) 1285–1299. http://link.aps.org/doi/10.1103/PhysRevD.22.1285.

[136] R. De Pietri, L. Lusanna, and M. Pauri, "Standard and generalized Newtonian gravities as 'gauge' theories of the extended Galilei group. I. The standard theory," *Class.Quant.Grav.* **12** (1995) 219–254, arXiv:gr-qc/9405046 [gr-qc].

[137] D. T. Son, "Newton-Cartan Geometry and the Quantum Hall Effect," arXiv:1306.0638 [cond-mat.mes-hall].

[138] M. Geracie, D. T. Son, C. Wu, and S.-F. Wu, "Spacetime Symmetries of the Quantum Hall Effect," arXiv:1407.1252 [cond-mat.mes-hall].

[139] K. Jensen and A. Karch, "Revisiting non-relativistic limits," arXiv:1412.2738 [hep-th].

[140] R. Banerjee, A. Mitra, and P. Mukherjee, "General algorithm for non-relativistic diffeomorphism invariance," arXiv:1501.05468 [gr-qc].

[141] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, "Boundary Stress-Energy Tensor and Newton-Cartan Geometry in Lifshitz Holography," *JHEP* **1401** (2014) 057, arXiv:1311.6471 [hep-th].

[142] J. Hartong, E. Kiritsis, and N. A. Obers, "Field Theory on Newton-Cartan Backgrounds and Symmetries of the Lifshitz Vacuum," arXiv:1502.00228 [hep-th].

[143] R. Casalbuoni, "The Classical Mechanics for Bose-Fermi Systems," *Nuovo Cim.* **A33** (1976) 389.

[144] L. Brink and J. Schwarz, "Quantum Superspace," *Phys.Lett.* **B100** (1981) 310–312.

[145] R. Puzalowski, "Galilean Supersymmetry," *Acta Phys.Austriaca* **50** (1978) 45.

[146] T. Clark and S. Love, "Non-Relativistic Supersymmetry," *Nucl.Phys.* **B231** (1984) 91.

[147] J. P. Gauntlett, J. Gomis, and P. Townsend, "Supersymmetry and the physical phase space formulation of spinning particles," *Phys.Lett.* **B248** (1990) 288–294.

[148] J. de Azcarraga and D. Ginestar, "Nonrelativistic limit of supersymmetric theories," *J.Math.Phys.* **32** (1991) 3500–3508.

[149] C. Duval and P. Horvathy, "On Schrodinger superalgebras," *J.Math.Phys.* **35** (1994) 2516–2538, arXiv:hep-th/0508079 [hep-th].

[150] O. Bergman and C. B. Thorn, "SuperGalilei invariant field theories in (2+1)-dimensions," *Phys.Rev.* **D52** (1995) 5997–6007, arXiv:hep-th/9507007 [hep-th].

[151] O. Bergman, "Galilean supersymmetry in (2+1)-dimensions," *Int.J.Mod.Phys.* **A12** (1997) 1173–1182, arXiv:hep-th/9609098 [hep-th].

[152] J. A. de Azcarraga and J. Lukierski, "Supersymmetric Particles with Internal Symmetries and Central Charges," *Phys.Lett.* **B113** (1982) 170.

[153] W. Siegel, "Hidden Local Supersymmetry in the Supersymmetric Particle Action," *Phys.Lett.* **B128** (1983) 397.

[154] G. W. Moore and P. C. Nelson, "Anomalies in Nonlinear σ Models," *Phys.Rev.Lett.* **53** (1984) 1519.

[155] C. Hull and A. Van Proeyen, "Pseudoduality," *Phys.Lett.* **B351** (1995) 188–193, arXiv:hep-th/9503022 [hep-th].

[156] A. Trautmann, "Foundations and Current Problems of General Relativity," in *Lectures on General Relativity,*, Prentice-Hall, ed. Brandeis Summer Institute, Waltham, Massachusets, 1965.

[157] R. Banerjee, A. Mitra, and P. Mukherjee, "A new formulation of non-relativistic diffeomorphism invariance," *Phys.Lett.* **B737** (2014) 369–373, arXiv:1404.4491 [gr-qc].

[158] J.-M. Lévy-Leblond, "Galilei group and Galilean invariance," in *Group Theory and Applications,*, II, p. 221. Acad. Press New York.

[159] J. Lukierski, P. Stichel, and W. Zakrzewski, "Acceleration-extended Galilean symmetries with central charges and their dynamical realizations," *Phys.Lett.* **B650** (2007) 203–207, arXiv:hep-th/0702179 [hep-th].

[160] J. Gomis, K. Kamimura, and P. C. West, "Diffeomorphism, kappa transformations and the theory of non-linear realisations," *JHEP* **0610** (2006) 015, arXiv:hep-th/0607104 [hep-th].

[161] A. Anabalon, J. Gomis, K. Kamimura, and J. Zanelli, "N=4 superconformal mechanics as a non linear realization," *JHEP* **0610** (2006) 068, arXiv:hep-th/0607124 [hep-th].

[162] E. Bergshoeff, J. Gomis, M. Kovacevic, L. Parra, J. Rosseel, *et al.*, "Nonrelativistic superparticle in a curved background," *Phys.Rev.* **D90** no. 6, (2014) 065006, arXiv:1406.7286 [hep-th].

[163] E. Ivanov and V. Ogievetsky, "The Inverse Higgs Phenomenon in Nonlinear Realizations," *Teor.Mat.Fiz.* **25** (1975) 164–177.

[164] I. McArthur, "Nonlinear realizations of symmetries and unphysical Goldstone bosons," *JHEP* **1011** (2010) 140, arXiv:1009.3696 [hep-th].

[165] R. Andringa, E. Bergshoeff, J. Gomis, and M. de Roo, "'Stringy' Newton-Cartan Gravity," *Class.Quant.Grav.* **29** (2012) 235020, arXiv:1206.5176 [hep-th].

[166] J. Lukierski, I. Prochnicka, P. Stichel, and W. Zakrzewski, "Galilean exotic planar supersymmetries and nonrelativistic supersymmetric wave equations," *Phys.Lett.* **B639** (2006) 389–396, arXiv:hep-th/0602198 [hep-th].

[167] A. Sen, "Universality of the tachyon potential," *JHEP* **9912** (1999) 027, arXiv:hep-th/9911116 [hep-th].

[168] W. Taylor, "Lectures on D-branes, tachyon condensation, and string field theory," arXiv:hep-th/0301094 [hep-th].

[169] K. Hashimoto and S. Terashima, "Brane decay and death of open strings," *JHEP* **0406** (2004) 048, arXiv:hep-th/0404237 [hep-th].

[170] C. Duval, G. Gibbons, and P. Horvathy, "Conformal Carroll groups and BMS symmetry," *Class.Quant.Grav.* **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[171] J. Gomis and F. Passerini, "unpublished notes," (2005) .

[172] G. Gibbons, J. Gomis, and C. Pope, "Deforming the Maxwell-Sim Algebra," *Phys.Rev.* **D82** (2010) 065002, arXiv:0910.3220 [hep-th].

[173] A. Eddington and P. S. of London, *Report on the Relativity Theory of Gravitation*. Fleetway Press, Limited, 1920. http://books.google.nl/books?id=cM8xAQAAMAAJ.

[174] A. Ashtekar, B. K. Berger, J. Isenberg, and M. A. H. MacCallum, "General Relativity and Gravitation: A Centennial Perspective," arXiv:1409.5823 [gr-qc].

[175] M. Bojowald, "Absence of singularity in loop quantum cosmology," *Phys. Rev. Lett.* **86** (2001) 5227–5230, arXiv:gr-qc/0102069 [gr-qc].

[176] M. Bojowald, "Quantum nature of cosmological bounces," *Gen. Rel. Grav.* **40** (2008) 2659–2683, arXiv:0801.4001 [gr-qc].

[177] A. Ashtekar, T. Pawlowski, and P. Singh, "Quantum Nature of the Big Bang: Improved dynamics," *Phys. Rev.* **D74** (2006) 084003, arXiv:gr-qc/0607039 [gr-qc].

[178] C. Rovelli, "Black hole entropy from loop quantum gravity," *Phys. Rev. Lett.* **77** (1996) 3288–3291, arXiv:gr-qc/9603063 [gr-qc].

[179] E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin, and P. K. Townsend, "More on Massive 3D Supergravity," *Class.Quant.Grav.* **28** (2011) 015002, arXiv:1005.3952 [hep-th].

[180] J. Lukierski, P. C. Stichel, and W. J. Zakrzewski, "Galilean invariant (2+1)-dimensional models with a Chern-Simons-like term and D = 2 noncommutative geometry," *Annals Phys.* **260** (1997) 224–249, arXiv:hep-th/9612017 [hep-th].

[181] C. Batlle, J. Gomis, and K. Kamimura, "Symmetries of the Free Schrödinger Equation in the Non-Commutative Plane," *SIGMA* **10** (2014) 011, `arXiv:1304.7293 [hep-th]`.

[182] L. Mezincescu, A. J. Routh, and P. K. Townsend, "All Superparticles are BPS," *J.Phys.* **A47** (2014) 175401, `arXiv:1401.5116 [hep-th]`.

[183] J. Hartong, "Gauging the Carroll Algebra and Ultra-Relativistic Gravity," *JHEP* **08** (2015) 069, `arXiv:1505.05011 [hep-th]`.

Resumen

La Relatividad General (RG), ha resistido la prueba del tiempo incluso después de cien años después de su concepción. Desde el punto de vista experimental, ha sido puesta a prueba con increíble exactitud por experimentos en el sistema solar y mediciones de *pulsar timing*, las cuales predicen con precisión, por ejemplo, la deflección de la luz, los adelantos en el perihelio de Mercurio y el cambio de frecuencia gravitacional de la luz. La RG también predice que a lo largo del tiempo la energía orbital de un sistema binario se convertirá en radiación gravitacional y la tasa de decaimiento del periodo orbital medido es casi precisamente el valor predicho por la teoría. Por otro lado desde el punto de vista teórico, la RG provee una descripción elegante, consistente y comprensiva del espacio-tiempo, la gravedad y la materia a nivel macroscópico.

Sin embargo, la RG pierde precisión y predictibilidad no sólo en regímenes gravitacionales fuertes sino que también existen fenómenos que aún no se han explicado en los regímenes débiles como el problema que surge de la rotación de las galaxias y el problema del déficit de masa en cúmulos de galaxias. Además, no existe un modelo satisfactorio que explique la energía oscura y la expansión acelerada del universo. También existen deficiencias de la teoría, por ejemplo, no ha sido posible unificar la gravedad con las fuerzas electrodébil y fuerte, la predicción de singularidades en el espacio-tiempo y su incompatibilidad con la mecánica cuántica.

Ha habido muchos esfuerzos que estudian posibles deformaciones de la RG debido a su importancia experimental y teórica. Es técnicamente muy desafiante tener una extensión de la RG, debido a que la nueva teoría debe hacer frente a los problemas a gran escala y al mismo tiempo tomar en cuenta el comportamiento que surgirá en la teoría cuántica. En otras palabras, estamos buscando una teoría matemáticamente consistente (libre de inestabilidades y fantasmas), capaz de reproducir la RG a escalas cosmológicas, que provea una explicación a la aceleración actual del universo, pero que al mismo tiempo, sea capaz de ofrecer una descripción de la gravedad siguiendo los principios de la mecánica cuántica.

Debido a que esta tarea es muy complicada, el objetivo de este trabajo es el de dar un primer paso en la disección de la RG y de estudiar diferentes extensiones y límites de la teoría para entenderla mejor. Modificar una teoría conocida es una de las mejores formas de descubrir nuevas estructuras que podrían tener aplicaciones no previstas. En esta tesis abordamos la siguiente pregunta: ¿Cuáles son las posibles formas de modificar y estudiar los diferentes límites de la gravedad? En la Tabla 1.1 hemos indicado las extensiones y los límites que hemos estudiado en esta tesis:

a) Añadir nuevas simetrías: supersimetría.

b) Añadir nuevos parámetros: parámetro de masa.

c) Modificar las simetrías existentes: abandonar la simetría de Lorentz.

d) Modificar el esquema de cuantización: cuantización polimérica.

e) Estudiar el límite no-relativista.

f) Estudiar el límite ultra-relativista.

Con el desarrollo de la mecánica cuántica, los principios de simetría y las leyes de conservación obtuvieron un papel fundamental en Física. Las simetrías conocidas en física de partículas son la invariancia de Poincaré (translaciones y rotaciones), las simetrías internas que pueden dividirse en globales (relacionadas con los números cuánticos conservados como la carga eléctrica) y las simetrías locales (que forman la base de las teorías de norma por lo que están relacionadas con las fuerzas de interacción) y las simetrías discretas (como la conjugación de carga, la transformación de paridad y la invariancia bajo inversión temporal).

En 1967 Coleman y Mandula demostraron que dadas ciertas suposiciones, estas son las únicas simetrías posibles para un sistema físico. El teorema de Coleman-Mandula establece que para teorías cuánticas de campos, las simetrías espacio-temporales y las simetrías internas sólo pueden combinarse en un producto directo del grupo de simetría (en otras palabras, las simetrías internas y de Poincaré no se mezclan). Hay algunas formas para sobrepasar este teorema relajando algunas de las suposiciones que requiere. En particular, el teorema supone que los generadores de simetría del álgebra involucran sólo conmutadores. Relajando esta suposición permitiendo tanto relaciones de conmutación así como de anticonmutación entre los generadores, uno obtiene la posibilidad de la supersimetría, en otras palabras se incluyen generadores de simetría bosónicos y fermiónicos.

La supersimetría ha sido estudiada ampliamente en la literatura debido a que se ha utilizado en muchos desarrollos modernos en física teórica y matemática, por ejemplo, emerge automáticamente como una simetría de teoría de cuerdas, de teorías de branas, en la correspondencia AdS/CFT, en la dualidad de Seiberg-Witten, etc.

Además de añadir supersimetría, también podemos añadir nuevos parámetros a la teoría de gravedad. Por ejemplo, un parámetro de masa. La gravedad masiva es una extensión diferente que parece natural debido a que sabemos que las partículas de interacción de las fuerzas electrodébiles deben de adquirir una masa a través del mecanismo de Higgs. Algunas de las principales motivaciones de la gravedad masiva provienen del hecho de que agregando una masa al gravitón se generarán modificaciones a gran distancia a la RG, afectando la energía oscura y disminuyendo el problema de la constante cosmológica. Existen algunas preocupaciones cuando se trata con un gravitón masivo, una de ellas surge cuando se toma el límite no-masivo de un campo de espín-2, debido a que propaga demasiados grados de libertad y parecería que nunca se podría obtener a la RG de este límite. Este fenómeno se conoce como la discontinuidad vDVZ y puede resolverse considerando extensiones no lineales de la teoría. Un segundo problema a superar es que las no-linearidades introducen fantasmas en el espectro físico. Los fantasmas corresponden a campos con energía cinética negativa que conducen a inestabilidades a nivel clásico y a la no-unitariedad a nivel cuántico.

En vista de que no ha sido posible aplicar las reglas de la mecánica cuántica a la teoría de la RG, recientemente han surgido diversos caminos hacia la cuantización. En este trabajo se han estudiado las características de diversos sistemas inspirados en dos modelos de cuantización: la gravedad de Hořava–Lifshitz y la gravedad cuántica de lazos. Cabe mencionar, que todos los posibles modelos que han surgido necesitan superar grandes problemas formales y conceptuales, además, con el desarrollo actual en física de partículas y en observaciones cosmológicas, no es posible validar sus predicciones mediante pruebas experimentales. A pesar de lo anterior, es posible probar los modelos dentro de escenarios físicos teóricos plausibles y estudiar sus propiedades y resultados.

La teoría que Hořava propuso está basada en un escalamiento anisotrópico entre las coordenadas espaciales y temporales que conduce al rompimiento de la invariancia bajo defeomorfismos en distancias cortas, dónde la RG se recupera en el límite de baja energía.

La segunda modificación a la RG que consideramos en esta tesis está inspirada en la violación de Lorentz de la gravedad de Hořava. Utilizamos el teorema de Noether para estudiar las simetrías de acciones anisotrópicas y encontramos las cargas y densidades de corriente respectivas.

Por otro lado, nos enfocamos en uno de los resultados de la teoría cuántica de lazos: la discretización cuántica del espacio. La mecánica cuántica polimérica es un esquema de cuantización (que no es equivalente al esquema de cuantización de Schrödinger) que replica alguna de las características de la teoría cuántica de lazos.

La siguiente modificación de la RG que estudiamos es precisamente este cambio de esquema de cuantización. Analizamos la cuantización polimérica a través del procedimiento de la integral de trayectoria, para analizar la dinámica de diversos sistemas. Con este esquema podemos obtener una acción efectiva que podemos utilizar para comprender mejor aspectos no-perturbativos de la dinámica de los sistemas a nivel clásico y cuántico.

El siguiente enfoque para estudiar a la RG además de modificar las simetrías y el esquema de cuantización es el de estudiar diferentes límites de RG para lograr un mejor entendimiento de la teoría, ya que usualmente podemos obtener claridad de algunos problemas estudiándolos en sus casos límite donde estos problemas se vuelven más simples y fáciles de analizar. Es interesante estudiar estos casos límite por las aplicaciones que pueden surgir de ellos.

En esta tesis consideramos los límites supersimétricos no-relativistas y ultra-relativistas de la RG: los límites cuando la velocidad de la luz tiende a infinito y a cero respectivamente. Geométricamente, la transición no-relativista puede verse como la apertura de los conos de luz (los conos se convierten en hipersuperficies tipo-espacio) mientras que de forma contraria, la transición ultra-relativista puede entenderse como la contracción de los conos de luz (los conos se convierten en hipersuperficies tipo tiempo). Mientras que el límite no-relativista está gobernado por el álgebra de Galileo, el álgebra que le corresponde al límite ultra-relativista es el álgebra de Carroll, ambos grupos de invariancia pueden obtenerse de contracciones adecuadas del grupo de Poincaré. Cuando se comienza del grupo AdS, diferentes contracciones conducirán a al grupo no-relativista de Newton-Hooke y al grupo ultra-relativista de AdS-Carroll.

Existen muchas teorías conformes de campos no-relativistas que describen sistemas físicos. Estos ejemplos surgen de la física de materia condensada, de la física atómica y de la física nuclear. Las versiones no relativistas de la correspondencia AdS-CFT se han investigado recientemente porque abren la puerta a posibles aplicaciones en la dualidad norma-gravedad a una variedad de sistemas con interacción fuerte. Los grupos de simetría no-relativistas, como los grupos de simetría conforme de Schrödinger y de Galileo son relevantes en el estudio de átomos fríos que tienen un dual que posee estas simetrías. Además, en el caso de las cuerdas, los límites no-relativistas pueden tener aplicaciones en el contexto de versiones no-relativistas de AdS-CFT.

Por otro lado, las llamadas simetrías de Carroll que surgen en el límite ultra-relativista han jugado un papel importante en investigaciones recientes. Por ejemplo, teorías con simetrías de Carroll surgen es estudios de condensación de taquiones. Recientemente también han aparecido en el estudio de teorías de campo conformes deformadas.

En esta tesis, consideramos la extensión masiva supersimétrica, el estudio de las simetrías de sistemas donde la simetría de Lorentz no se conserva, la cuantización polimérica y los límites supersimétricos no-relativista y ultra-relativista.

La primera modificación de la RG que estudiamos es la extensión masiva supersimétrica. Motivamos y discutimos la formulación de Kaluza Klein para obtener el supermultiplete espín-2 masivo off-shell tridimensional. Este multiplete junto con el multiplete espín-2 no massivo off-shell que ya era conocido, puede utilizarse para escribir una versión supersimétrica de la nueva gravedad masiva linearizada (linearized New Massive Gravity NMG) en la forma de campos auxiliares.

En el límite no-masivo hay un acoplamiento no-trivial del multiplete escalar con un multiplete de corriente. En el caso usual de Fierz–Pauli, es posible curar la discontinuidad de vDVZ tomando en cuenta la versión no-lineal de la teoría, pero una versión no lineal de la teoría SNMG no es obvia debido a que la construcción de un supermultiplete masivo de espín-2 se basa en truncar la expansión de Kaluza–Klein en el primer nivel masivo, lo cual sólo puede realizarse a nivel lineal. Para obtener la teoría no-lineal de la teoría SNMG, probablemente sea necesario utilizar técnicas de super-espacio.

La cuarta extensión que consideramos fue la de una superpartícula no-relativista en un fondo curvo. Primero, construimos con el método de realizaciones no-lineales, las acciones para la (super)partícula libre de Newton-Hooke, analizamos su dinámica y las simetrías de una partícula moviéndose en dichos espacios. También estudiamos las acciones de una superpartícula que describen la dinámica de una partícula supersimétrica en un fondo de supergravedad tridimensional de Galileo con curvatura y de Newton–Cartan. Debido a la complejidad computacional mostramos la acción en el caso de Newton–Cartan sólo hasta términos cuadráticos en los fermiones. El fondo de Newton–Cartan está caracterizado por más campos y corresponde a más simetrías que en el fondo de Galileo. Es posible cambiar de un fondo a otro ya sea normalizando parcialmente las simetrías (del fondo de Galileo al de Newton–Cartan) o fijando la norma de algunas simetrías (de Newton–Cartan a Galileo).

La quinta y última modificación que consideramos fue el límite de Carroll supersimétrico. Estudiamos partículas cuya dinámica sea invariante bajo la superálgebra de Carroll. En el caso bosónico encontramos que la partícula de Carroll posee una simetría infinita-dimensional que sólo en el caso plano incluye dilataciones. La dualidad entre el álgebra de Carroll y el álgebra de Bargmann, importante en el caso plano, no se extiende en el caso curvo. Sólo en el límite plano encontramos que la acción es invariante bajo una simetría infinito-dimensional que incluye a la extensión supersimétrica del álgebra de Lifshitz–Carroll.

A lo largo de esta tesis, nos centramos en los casos tridimensionales más simples, no sólo porque eran técnicamente más sencillos sino también porque la gravedad tridimensional es interesante por sí misma. En todos los casos sería deseable construir los resultados análogos en cuatro dimensiones, pero hasta el momento, no ha sido posible lograrlo.

Agradecimientos

Era muy difícil imaginarse que un doctorado se convertiría en un viaje lleno de conocimiento, retos, recuerdos y satisfacciones. Me gustaría aprovechar esta oportunidad para agradecer a las personas que enriquecieron mi camino durante estos cinco años.

First of all, I owe to my two supervisors Eric Bergshoeff and David Vergara a debt of gratitude for all the support they gave me during my PhD. Eric, thank you for giving me the opportunity to join your group, you widened my vision of physics through many interesting projects and encouraged me to attend international conferences, schools and workshops. I cannot truly express my appreciation for your permanent leadership, guidance, advices and enthusiasm, and for teaching me how to become a better and mature researcher. David, I am very thankful for the solid formation you gave me during my first two years, your experience and insights on many research topics inspired me to continue reading and learning. I wish to thank you for encouraging me to travel to the Netherlands, this PhD would not have been possible without the initial and final momentum you gave me.

I also wish to express my gratitude to the members of my PhD committee, Rocío Jaúregui and Antonio García for your help and advice trough all the PhD process. To my dutch reading committee Ana Achúcarro, Silvia Penati and Mees de Roo, and to my mexican reading committee, Myriam Mondragón, Hugo Morales and Mariano Chernicoff, I would like to thank you for carefully reading the manuscript and for the valuable comments you gave me to improve it. I am also very grateful to Quim Gomis, for the interesting discussions and clear explanations during the research projects we had together and for hosting me for a week in Barcelona in order to finish the Carroll project.

The van Swinderen Institute is a place full of incredible talented people starting from its faculty members, Elisabetta Pallante, Kyriakos Papadodimas and Diederik Roest, thank you for the interesting conversations on many aspects of life and physics. I would also like to thank my collaborators Marija Kovačević, Jan Rossell, Yihao Yin and Thomas Zojer, I learned a ton of things thanks to all of you.

Jan Willem, thank you for your kindness, friendship and for giving me help whenever I needed it. I already miss the coffee/tea/chocolate breaks. To my favorite latin people, thank you Víctor, Lucy and Julián, for your friendship, for all the conversations and the laughs we had. Lucy, thank you for your sweetness and all your support.

Outside the university, there were many people whose friendship I treasure. I would like to express my heartfelt appreciation towards Suruchi, Kshama, and

José for all the nice moments we spend together.

In the course of this journey, I would never had imagine that I will find two sisters in the Netherlands, I am very lucky and fortunate because of that. Thank you Ana Huerta and Ana Cunha. Anita, hermanita, thank you for building a home with me, for sharing all those laughs and tears, for encouraging me to become the best and joyful version of myself and for all the support and love you gave me. Anita, my dreamer, thank you for taking care of me, for those tasty dinners, for agreeing to disagree with me in all those interesting conversations we had and for the kindness and love you gave me. I love both of you and I am sure that it does not matter in which corner of the world we are, we will always stay friends.

A mis queridos amigos del MT, Ángeles, Angélica, Itzel, Kenji, Lucero, Mariana, Marianel, Paola y Víctor, gracias por todas esas risas y ese cariño que atravesaba el mundo. Alberto, muchas gracias por tus cartas, me alegrabas la tarde cada vez que encontraba una de ellas esperándome en casa. Lilian, gracias por tu apoyo y tu amistad.

A mi hermano Gabriel (a.k.a. Gabo), por tu cariño y aliento, sobretodo en los momentos más oscuros y difíciles. Gracias por siempre respaldarme y gracias por llevarme a Emporio contigo, hiciste que mi regreso a México se hiciera más colorido.

Quiero agradecer sobre todo a mi mamá Josefina por todo el cariño que me da, por llenar mi vida de música, libros y ciencia, por impulsarme a seguir mis sueños, por enseñarme a nunca rendirme y por alentarme a volar alto y sin límites. Mi querida mamá, gracias por toda tu ternura, por cuidarme y por enseñarme a ser valiente.

Edward, mi travesía habría sido mucho más difícil sin tu mano sosteniendo la mía, gracias por creer en mí, por alegrarme los días, por ser parte de esa fuerza que me impulsa a seguir adelante y por enseñarme que cuando el amor es mucho, cualquier distancia se hace corta.