UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

ESTADOS FUNDAMENTALES PARA UNA ECUACIÓN DE SCHRÖDINGER CON VARIOS PROBLEMAS LÍMITES

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

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MÉXICO, D. F. 21 DE ABRIL DE 2016

Agradecimientos

Durante la realización de esta tesis se recibió apoyo de los proyectos CONACYT 237661 y UNAM-DGAPA-PAPIIT IN104315.

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1 Introducción

1.1 Ecuación Estacionaria de Schrödinger

En este trabajo demostramos la existencia de un tipo particular de soluciones, llamadas estados fundamentales, para la ecuación dada de la siguiente manera:

$$-\Delta u + V(x)u = a(x)\,|u|^{p-2}\,u, \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

donde $H^1(\mathbb{R}^N)$ es el espacio de Sobolev usual, y suponemos $2 < p < 2^*$ ($2^* = 2N/(N-2)$ para $N \geq 3$, $2^* = \infty$ para $N = 1, 2$) y $a, V \in L^\infty(\mathbb{R}^N)$. Por ahora entenderemos los estados fundamentales como soluciones de mínima energía, más adelante precisaremos el significado de esto. En nuestra búsqueda de estados fundamentales haremos uso de métodos variacionales. Nos interesan los casos en los cuales a y V no tienen un límite global al infinito, es decir, $\lim_{|x|\to\infty} a(x)$ y $\lim_{|x|\to\infty} V(x)$ no existen.
El tipo de ecuaciones que trabajamos son un subconjunto de las llamadas ecuaciones estacionarias de Schrödinger no lineales, cuya forma general es la siguiente:

$$-\Delta u + V(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N). \tag{1.2}$$

Usualmente nos referimos a f como el término no lineal de la ecuación.
En el capítulo 2 de esta tesis daremos algunas definiciones y conceptos propios de las técnicas variacionales que utilizaremos, e introduciremos buena parte de la notación que utilizaremos a lo largo de este trabajo. En el capítulo 3 expondremos el resultado de Concentración Compacta que hemos obtenido para el caso donde los límites globales de a y V no existen e $\inf V > 0$. En el capítulo 4 utilizaremos estos resultados para obtener la existencia de estados fundamentales en tres situaciones distintas. El capítulo 5 está dedicado a exponer aplicaciones del resultado obtenido, dando una idea de que tanto este resultado expande la geometría que a y V pueden tener. El capítulo 6 trata sobre las preguntas que quedan abiertas, y resultados que siguen la dirección de lo que hemos hecho.

1.2 Antecedentes

Expondremos ahora un poco acerca de la ecuación de Schrödinger y de los resultados presentes en la literatura sobre el tema. Consideremos la ecuación de Schrödinger no lineal

$$i\frac{\partial\Psi}{\partial t} - \Delta\Psi + (V(x) + \omega)\Psi - a(x)\,|\Psi|^{p-2}\,\Psi = 0, \tag{1.3}$$

donde $i = \sqrt{-1}$. Un tipo particular de soluciones para esta ecuación son las llamadas ondas estacionarias, las cuales son soluciones de la forma

$$\Psi(x,t) = u(x)e^{iwt}. \tag{1.4}$$

Puede verificarse que una onda estacionaria es solución de (1.3) si y sólo si u es solución de la ecuación (1.1). De manera similar, considerando la ecuación no lineal de Klein-Gordon

$$\frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi + (V(x) + \omega^2)\Psi - a(x)\,|\Psi|^{p-2}\,\Psi = 0,$$

podemos ver que la búsqueda de ondas estacionarias nos lleva de nuevo a la ecuación (1.1). Esta relación con las ecuaciones de Schrödinger y Klein-Gordon hace que la ecuación que estudiamos, además de generar interés matemático por las técnicas que la búsqueda de sus soluciones lleva a desarrollar, presente también un interés de aspecto físico.

La principal dificultad al buscar soluciones de la ecuación (1.1) proviene de la no compacidad del encaje $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, lo cual impide el uso de condiciones de Palais-Smale y de Cerami. En el caso en el cual a y V son funciones constantes es de mucha importancia el artículo [5], en cuya bibliografía pueden encontrarse otros artículos que también han estudiado esta situación.

También el caso no autónomo ha sido abordado, bajo diversas hipótesis. En los artículos [20, 47] se trabaja con funciones V, a radialmente simétricas, lo cual permite recuperar compacidad al considerar el subespacio radialmente simétrico $H^1_r(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$. Otro acercamiento a este problema ha sido imponer hipótesis sobre V y a bajo las cuales se tiene de nuevo la compacidad del encaje $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, en ocasiones sustituyendo $H^1(\mathbb{R}^N)$ por algún espacio con peso. A grandes rasgos, estas hipótesis son que $\lim_{|x|\to\infty} V(x) = \infty$, $\limsup_{|x|\to\infty} a(x) \leq 0$ ó que $a^+ \in L^q(\mathbb{R}^N)$ para alguna $q > 0$ apropiada. Este método lo podemos encontrar en [2, 16, 17, 19, 22–24, 26, 28, 34, 39, 42, 43, 45] Otra técnica que ha sido utilizada es reemplazar el lado derecho de (1.1) por una función $f(x, u)$ linealmente asintótica en u [35, 52].

Aparte de las técnicas mencionadas arriba, otro método frecuentemente utilizado es el de asociar a la ecuación original un problema límite, pidiendo como hipótesis que el límite puntual de a y V exista cuando $|x| \to \infty$, para aplicar argumentos de Concentración Compacta [3, 4, 6–8, 10–14, 18, 20, 24, 25, 29, 30, 32, 36, 47, 49, 50]. Una variante de este método es suponer periodicidad, a veces en un sentido asintótico, de V y a con respecto a x [21, 28, 30, 31, 38, 44, 46, 51].

Antes de comenzar a explicar qué distingue a los resultados expuestos en esta tesis de los encontrados en la literatura ya existente hace falta mencionar otro artículo. Como se puede ver, los métodos anteriores buscan restaurar compacidad mediante la elección de hipótesis adecuadas. En el caso no periódico el único resultado que conocemos donde no se asume la existencia de límites para V y a cuando $|x| \to \infty$ es [9], en el cual se demuestra la existencia de un estado fundamental cuando $a \equiv 1$ y $\operatorname{ess\,inf} V > 0$. Los resultados que nosotros probamos aquí se suman a los de dicho artículo al tampoco suponer la existencia de límites para V y a, la mayor diferencia siendo que las hipótesis impuestas a V en [9] no son lo suficientemente explícitas para ser verificadas directamente, salvo en [9, Teorema 1.3], donde la solución encontrada no es un estado fundamental.

La otra característica importante de nuestros resultados tiene que ver con el signo de a. Convencionalmente, la ecuación (1.1) es llamada linealmente indefinida si V cambia de signo, y superlinealmente indefinida si a cambia de signo. Los resultados de existencia para el problema superlinealmente indefinido, ya sea linealmente indefinido o no, son vastos [14, 16–19, 22–24, 26, 29, 34, 42, 45], pero hasta ahora todos ellos requieren que, en algún sentido, a no cambie de signo cerca de infinito, por ejemplo pidiendo que $\limsup_{|x|\to\infty} a(x) \leq 0$ ó $\liminf_{|x|\to\infty} a(x) \geq 0$. De una manera más general, muchos autores piden que a^+ ó a^- pertenezcan a un espacio L^q adecuado. Nosotros hemos removido estas hipótesis, permitiendo que tanto V como a puedan cambiar de signo cerca de infinito.

Para finalizar esta introducción queremos comentar que una de las razones por las cuales el caso en el cual a puede cambiar de signo ha recibido más atención recientemente tiene que ver con el desarrollo en diseño de materiales. En el artículo [1] podemos encontrar una situación en la cual a no sólo puede cambiar de signo, sino que es bastante flexible en la forma que puede tomar. Este ejemplo trata, a grandes rasgos, de la propagación de luz monocromática a través de fibras ópticas. En dicha situación la función a representa una propiedad del medio de propagación, en el caso de la fibra óptica el índice de refracción. Dado el potencial actual de construir materiales dentro de un gran rango de propiedades prescritas, tenemos considerable libertad en la elección de las hipótesis para a.

2 Variational Framework

As we have said before we have used variational methods in our research, so now we will start by presenting some definitions and concepts related to these methods. First, for all $a, V \in L^\infty$, with $V > 0$, define the functional

$$\begin{aligned} I_{V,a}(u) &:= \frac{1}{2}\int (|\nabla u|^2 + Vu^2)dx - \frac{1}{p}\int a(x)\,|u|^p\,dx. \\ &= \frac{1}{2}\,\|u\|_V^2 - \frac{1}{p}\int a(x)\,|u|^p\,dx. \end{aligned}$$

Usually, when the global limits of V and a exist, one omits the subindex in the previous definition, and refer to the functional related to the original equation only as I, while the functional corresponding to the limit problem is denoted by I_∞. Since in our case we don't have a unique limit problem we need a more flexible notation, therefore the necessity of the subindex making explicit the potencial and the function a of every limit functional.
We can adapt the arguments of [48, Proposición 1.12] to see that $I_{V,a} \in C^2(H^1(\mathbb{R}^N))$ and

$$I'_{V,a}(u)v = \int (\nabla u \nabla v + Vuv - a(x)\,|u|^{p-2}\,uv)dx, \tag{2.1}$$

$$I''_{V,a}(u)[v,w] = \int (\nabla v \nabla w + Vvw - (p-1)a(x)\,|u|^{p-2}\,vw)dx. \tag{2.2}$$

Notice that $I'_{V,a}(u)v$ can be obtained multiplying equation (1.1) by v and integrating over $\mathbb{R}^N$. Since this happens, we have that the critical points of $I_{V,a}$ are weak solutions of (1.1).
The functional $I_{V,a}$ is commonly called the energy functional. As we mentioned before, we are interested in finding least energy solutions. Formally speaking we want to prove that there are nontrivial weak solutions $\hat{u}_{V,a}$ of (1.1) such that $I_{V,a}(\hat{u}_{V,a}) = \hat{c}_{V,a}$, where

$$\hat{c}_{V,a} := \inf\{I_{V,a}(u) \mid I'_{V,a}(u) = 0, u \in H^1(\mathbb{R}^N) \setminus \{0\}\}.$$

These solutions are the so called ground states.
In order to do this we will make use of the Nehari Manifold, $\mathcal{N}_{V,a}$, which is defined as

$$\mathcal{N}_{V,a} = \left\{u \neq 0 \mid I'_{V,a}(u)u = 0\right\}.$$

We also define $c_{V,a} := \inf_{\mathcal{N}_{V,a}} I_{V,a}$. By convention, $c_{V,a} := \infty$ if $\mathcal{N}_{V,a} = \varnothing$.
Since every nonzero critical point belongs to $\mathcal{N}_{V,a}$, a minimum of $I_{V,a}$ on $\mathcal{N}_{V,a}$ is a ground state, as we will show below. This is the main reason for using the Nehari manifold. We

now give an outline of the method that we will follow. The first thing we want is for $\mathcal{N}_{V,a}$ to be a natural constraint:

2.1 Definition *We say that the manifold $\mathcal{N}_{V,a} \subset H^1(\mathbb{R}^N)$ is a natural constraint for $I_{V,a}$ if every critical point of $I_{V,a}|_{\mathcal{N}_{V,a}}$ is also a critical point of $I_{V,a}$.*

Usually this property is proved defining $\mathcal{N}_{V,a}$ as the inverse image of a suitable operator (refer to the proof of Lemma 2.6). If we also prove that $I_{V,a}|_{\mathcal{N}_{V,a}}$ has a minimum, then the existence of a ground state for (1.1) follows. A common way to achieve this involves the Palais-Smale condition:

2.2 Definition *A sequence $(u_n) \in H^1(\mathbb{R}^N)$ is called a Palais-Smale sequence for $I_{V,a}$ if $(I_{V,a}(u_n))$ is bounded and $I'_{V,a}(u_n) \to 0$. If also $I_{V,a}(u_n) \to c$, we say that (u_n) is a $(PS)_c$-sequence for $I_{V,a}$.*

2.3 Definition *We say that $I_{V,a}$ satisfies the Palais-Smale condition (or $(PS)_c$-condition), if every Palais Smale sequence for $I_{V,a}$ (or, respectively, every $(PS)_c$-sequence) has a convergent subsequence.*

We are going to construct a minimizing Palais-Smale sequence for $I_{V,a}|_{\mathcal{N}_{V,a}}$ using Ekeland's Variational Principle, which is valid in C^1-manifolds. Since in our particular situation $I_{V,a}$ is a C^2 functional, and observing the way in which the Nehari manifold is defined, Ekeland's Variational Principle is valid under our hypothesis. The sequence we obtain in this way is a Palais-Smale sequence for $I_{V,a}|_{\mathcal{N}_{V,a}}$, but it can be proved that is also a Palais-Smale sequence for the unrestricted functional $I_{V,a}$.
In a recent paper [37], Noris and Verzini studied the Nehari Manifold in a very general setting. From this paper we will use Theorem 2.8. The next is a particular version of that theorem, adapted to our situation.

2.4 Definition *Define $E_V := (H^1(\mathbb{R}^N), \|\cdot\|_V)$.*

By convention, if $\overline{\mathcal{N}_{V,a}} \setminus \mathcal{N}_{V,a} = \varnothing$ then $\inf_{\overline{\mathcal{N}_{V,a}} \setminus \mathcal{N}_{V,a}} I_{V,a} = \infty$.

2.5 Theorem *Assume that*

i) $c_{V,a} \in \mathbb{R}$, $\inf_{\overline{\mathcal{N}_{V,a}} \setminus \mathcal{N}_{V,a}} I_{V,a} > c_{V,a}$.

Also assume that for some $0 < \delta < \delta'$ we have, for all $u \in \mathcal{N}_{V,a}$,

ii) $\|u\|_V \geq \delta$,

iii) $-I''_{V,a}(u)[u,u] \geq \delta \|u\|_V^2$,

and every $u \in \mathcal{N}_{V,a}$ with $I_{V,a}(u) \leq c_{V,a} + 1$ satisfies *iv)* $\left|I'_{V,a}(u)v\right| \leq \delta' \left\|v\right\|_V$ *and* $\left|I''_{V,a}(u)[v,w]\right| \leq \delta' \left\|v\right\|_V \left\|w\right\|_V$ *for all* $v, w \in E_V$.

Then there exists a $(PS)_{c_{V,a}}$*-sequence for* $I_{V,a}$ *and* $\mathcal{N}_{V,a}$ *is a natural constraint for* $I_{V,a}$ *(and a* C^1*-manifold).*

Proof. This result follows from the proof of [37, Theorem 2.8], with $V^+ = \{0\}, \xi_1(u) = u$ and $A = H^1(\mathbb{R}^N) \setminus \{0\}$. One of the conditions of this Theorem is for the Palais-Smale condition to hold for $I_{V,a}$ at the level $c_{V,a}$, but the rest of the hypotheses and [37, Corollary 2.4] are enough to conclude our result.

□

Observe that if $I_{V,a}$ satisfies the conditions of Theorem 2.5, and $I_{V,a}|_{\mathcal{N}_{V,a}}$ achieves its minimum, then $c_{V,a} = \hat{c}_{V,a}$, since all critical points of $I_{V,a}$ are contained in $\mathcal{N}_{V,a}$.

We are going to check the hypothesis of this theorem for equation (1.1), assuming $a^+ \neq 0$, where $a^\pm := \max\{\pm a, 0\}$. Once this is done, we will be ready to search for a minimizer. One idea that has proved very useful in the case of unbounded domains is Lions' Concentration Compactness Principle. Since our hypotheses will be different than the ones usually assumed (in fact we are going to work with weak*-convergence instead of pointwise convergence) we are going to obtain a weaker version of the standard Splitting Lemma, but this will be sufficient to prove the existence of a ground state. The only article we have found that uses weak*-convergence in a similar way is [35], in which a nonresonant asymptotically linear problem is studied, but there concentration compactness is not used.

2.1 The Nehari Manifold

In the next Lemmas we will establish conditions that assure that $a, V \in L^\infty$ satisfy the hypotheses of Theorem 2.5.

2.6 Lemma *For all* $a, V \in L^\infty$ *we have that* $\mathcal{N}_{V,a}$ *is closed and that there exists* $\gamma > 0$ *that only depends on* p*, a positive lower bound for* $\operatorname{ess\,inf} V$ *and an upper bound for* $\operatorname{ess\,sup} a$ *such that*

$$\inf_{u \in \mathcal{N}_{V,a}} \left\|u\right\|_V > \gamma.$$

Proof. It can be verified that the operator $G : E_V \to \mathbb{R}$ defined by $G(u) = I'_{V,a}(u)u$ is of class C^1, and that $\mathcal{N}_{V,a} = G^{-1}(0) \setminus \{0\}$.
To simplify the notation write $\alpha := \operatorname{ess\,sup} a$, and $\beta := \operatorname{ess\,inf} V$. By hypothesis $\beta > 0$. We will also denote by β the constant function taking that value. Let

$$C_p := \sup_{u \in E_V \setminus \{0\}} \frac{|u|_p}{\left\|u\right\|_\beta},$$

which is well defined because of the continuity of the embedding $E_\beta \hookrightarrow L^p$.
Let $u \in \mathcal{N}_{V,a}$. Since $DI_{V,a}(u)u = 0$ we have

$$\|u\|_\beta^2 \leq \|u\|_V^2 = \int_{\mathbb{R}^N} a\,|u|^p \leq \int_{\mathbb{R}^N} \alpha\,|u|^p \leq C_p\alpha\,\|u\|_\beta^p\,,$$

and so we have

$$\left(\frac{1}{C_p\alpha}\right)^{\frac{1}{p-2}} \leq \|u\|_\beta \leq \|u\|_V\,,$$

from where we obtain the result.

□

2.7 Lemma $c_{V,a} > 0$ *for all* $a, V \in L^\infty$.

Proof. For all $u \in \mathcal{N}_{V,a}$ we have

$$I_{V,a}(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{p}\int a(x)\,|u|^p = \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_V^2 \geq \left(\frac{1}{2}-\frac{1}{p}\right)\inf_{u\in\mathcal{N}_{V,a}}\|u\|_V^2\,.$$

The last inequality and the Lemma 2.6 imply the result.

□

Denote by $|E|$ the Lebesgue measure of a measurable subset E of $\mathbb{R}^N$. Also, if X is a normed space, denote $S_1X := \{x \in X \mid \|x\| = 1\}$. The next characterization will be useful in various places:

2.8 Lemma *For any* $u \in L^\infty$ *it holds true that*

$$\operatorname{ess\,sup} u = \sup\left\{\int_{\mathbb{R}^N} u\varphi \mid \varphi \in S_1L^1, \varphi \geq 0\right\} \tag{2.3}$$

and

$$\operatorname{ess\,inf} u = \inf\left\{\int_{\mathbb{R}^N} u\varphi \mid \varphi \in S_1L^1, \varphi \geq 0\right\}. \tag{2.4}$$

Proof. Denote by c the right hand side of (2.3). If $\varphi \in S_1L^1$ and $\varphi \geq 0$ then

$$\int_{\mathbb{R}^N} u\varphi \leq \operatorname{ess\,sup} u \int_{\mathbb{R}^N} \varphi = \operatorname{ess\,sup} u.$$

Therefore, $c \leq \operatorname{ess\,sup} u$. To show the inverse inequality, first fix some $R > 0$ and $x \in \mathbb{R}^N$. For any Lebesgue-measurable subset $E \subset B_R(x)$ we find

$$\frac{1}{|E|}\int_E u = \int_{\mathbb{R}^N} u\frac{\chi_E}{|E|} \leq c,$$

where χ_E denotes the characteristic function of E. From [40, Theorem 1.40] it follows that $u(x) \leq c$ a.e. in $B_R(x)$. Covering $\mathbb{R}^N$ with countably many balls $B_R(x)$ we obtain $u(x) \leq c$ a.e. in $\mathbb{R}^N$, i.e., $\operatorname{ess\,sup} u \leq c$. This proves (2.3).
Since $\operatorname{ess\,inf} u = -\operatorname{ess\,sup} -u$ we only need to apply (2.3) to $-u$ to obtain (2.4).

□

Until now we have not discarded the possibility that $c_{V,a} = \infty$. In the next lemma we do this.

2.9 Lemma *For all $a \in L^\infty$ such that $a^+ \neq 0$ we have that $\mathcal{N}_{V,a}$ is non-empty.*

Proof. From $a^+ \neq 0$ we obtain $\operatorname{ess\,sup} a > 0$, and Lemma 2.8 yields $\psi \in S_1 L^1$ such that $\psi \geq 0$ and $\int a\psi > 0$. Approximating ψ suitably in L^1 we obtain $\varphi_1 \in C_c^\infty$ such that $\int a\varphi_1 > 0$ and $\varphi_1 \geq 0$, and from here we can find $\varphi \in H^1(\mathbb{R}^N)$ such that $\int a(x) |\varphi|^p > 0$. Define the auxiliary function $h : \mathbb{R}^+ \to \mathbb{R}$ given by $h(t) = I_{V,a}(t\varphi)$, so that $h'(t) = I'_{V,a}(t\varphi)\varphi$. Notice that

$$\begin{aligned} I'_{V,a}(t\varphi)\varphi &= \langle t\varphi, \varphi\rangle_V - \int a(x) \left|t\varphi\right|^{p-2} (t\varphi)\varphi \\ &= t \left\|\varphi\right\|_V^2 - t^{p-1} \int a(x) \left|\varphi\right|^p . \end{aligned}$$

We can verify that under our hypotheses h has a unique positive critical point t_0, and by the linearity of the derivative we also have $I'_{V,a}(t_0\varphi)(t_0\varphi) = 0$, hence $t_0\varphi \in \mathcal{N}_{V,a}$ and $\mathcal{N}_{V,a}$ is non-empty.

□

We are now in position to prove the main result of this chapter.

2.10 Lemma *If $a \in L^\infty$ satisfies $a^+ \neq 0$ in L^∞ then there exists a $(PS)_{c_{V,a}}$-sequence for $I_{V,a}$ and $\mathcal{N}_{V,a}$ is a natural constraint (and a C^1-manifold).*

Proof. To prove this proposition we will check that the conditions of Theorem 2.5 are satisfied.
(i) holds because of Lemmas 2.6, 2.7 and 2.9.
Using Lemma 2.6, we can choose δ_0 such that (ii) holds, with δ_0 in place of δ.
Define $0 < \delta = \min(\delta_0, p-2)$. To prove (iii) observe that for all $u \in \mathcal{N}_{V,a}$ we have

$$\begin{aligned} I''_{V,a}(u)[u,u] &= \left\|u\right\|_V^2 - (p-1) \int a(x) \left|u\right|^p \\ &= \left\|u\right\|_V^2 - (p-1) \left\|u\right\|_V^2 \\ &= -(p-2) \left\|u\right\|_V^2 . \end{aligned}$$

Now only (iv) remains to be verified.
Notice that $u \in \mathcal{N}_{V,a}$ and $I_{V,a}(u) \leq c_{a,V} + 1$ imply that $\left\|u\right\|_V$ is bounded:

$$\begin{aligned} I'_{V,a}(u)u = 0 &\implies \left\|u\right\|_V^2 = \int a(x) \left|u\right|^p \\ &\implies c_{V,a} + 1 \geq I_{V,a}(u) = \left(\frac{1}{2} - \frac{1}{p}\right) \left\|u\right\|_V^2 . \end{aligned}$$

Hence, using Sobolev embeddings,

$$\begin{aligned}
\left|I'_{V,a}(u)v\right| &= \left|\langle u,v\rangle_V - \int a(x)\,|u|^{p-2}\,uv\right| \\
&\le |\langle u,v\rangle|_V + \left|\int a(x)\,|u|^{p-2}\,uv\right| \\
&\le \|u\|_V\,\|v\|_V + \|a\|_\infty \int |u|^{p-1}\,|v| \\
&\le \|u\|_V\,\|v\|_V + \|a\|_\infty \left\||u|^{p-1}\right\|_{\frac{p}{p-1}} \|v\|_p \\
&= \|u\|_V\,\|v\|_V + \|a\|_\infty \|u\|_p^{p-1}\,\|v\|_p \\
&\le M_1\,\|v\|_V\,.
\end{aligned}$$

In a similar way we can find M_2 such that $I''_{V,a}(u)[v,w] \le M_2\,\|v\|_V\,\|w\|_V$. Choosing $\delta' = \max(2\delta, M_1, M_2)$ all the required inequalities hold.

$\square$

3 Concentration Compactness Principle

Throughout this chapter we will assume $a, V \in L^\infty$, $a^+ \neq 0$ in L^∞ and $\inf V > 0$.

3.1 The Limit Problem.

The Concentration Compactness Principle, in its usual form, involves a limit problem, which is an equation that behaves in a similar way to the original equation at infinity. More precisely, the limit problem of (1.1) when $a_\infty = \lim_{x\to\infty} a$ and $V_\infty = \lim_{x\to\infty} V$ are well defined is given by

$$-\Delta u + V_\infty u = a_\infty |u|^{p-2} u, \tag{3.1}$$

so its set of solutions is invariant under translations. In the cases we have studied, where at least one of the limits involved does not exist, one must be more careful at the moment of choosing the limit problem. In order to explain the next ideas more clearly we are going to focus on the first situation we studied when we started this research, namely the case of $N = 1$, and which we generalized in the first of the examples included in this work. We are not giving here the full list of hypotheses, but for this section's purposes it is enough to assume that $V = 1, \lim_{t\to-\infty} a = 1$ and $\lim_{t\to\infty} a = -1$.
When we make the choice of the limit problem we must keep in mind the idea behind the Concentration Compactness Principle. When we pass from bounded domains to unbounded domains we lose compacity, because of the invariance of the solutions mentioned before: the original problem and the limit problem are similar near infinity, so in a sequence whose elements can be seen as a solution of the limit problem being translated at infinity, these elements behave more and more like solutions of the original problem, but the sequence does not converge.
Assume that a sequence is minimizing for the energy functional restricted to $\mathcal{N}_{V,a}$. In our case, in which V is constant, it makes sense to expect that the translations forming this sequences take place in the direction in which a takes larger values. Taking this into account, we choose as the limit problem the equation

$$-\Delta u + u = |u|^{p-2} u, \tag{3.2}$$

with the corresponding functional

$$I_{-\infty}(u) = \frac{1}{2}\left(\int |\nabla u|^2 + |u|^2 \, dx\right) - \frac{1}{p}\int |u|^p ,$$

the Nehari manifold

$$\mathcal{N}_{-\infty} = \left\{u \in H^1(\mathbb{R}) \setminus \{0\} \mid |I'_{-\infty}(u)u = 0\right\},$$

and the energy level

$$c_{-\infty} := \inf_{u \in \mathcal{N}_{-\infty}} I_{-\infty}(u).$$

What we prove in this case was that $c_{V,a} < c_{-\infty}$, which in turn helped us to prove that the Concentration Compactness Principle, or a version of it, still holds in this situation, so our election of limit problem was an adequate one. After this example we started working in more complicated ones, until we established the more general splitting lemma that will be exposed in the next section.

3.2 Concentration Compactness Principle Theorem for the Strongly Definite Case

3.1 Definition *For $y \in \mathbb{R}^N$ we define the translation operator τ_y on spaces of functions on $\mathbb{R}^N$ by*

$$(\tau_y f)(x) := f(x - y).$$

3.2 Definition *Define the sets*

$$\mathcal{P} := \{u \in L^\infty \mid \forall v \in \overline{B}_1 L^1 : \left|\int_{\mathbb{R}^N} uv\right| \le |a|_\infty\} = \overline{B}_{|a|_\infty} L^\infty,$$

$$\mathcal{Q} := \{u \in L^\infty \mid \forall v \in \overline{B}_1 L^1 : \left|\int_{\mathbb{R}^N} uv\right| \le |V|_\infty\} = \overline{B}_{|V|_\infty} L^\infty.$$

We endow $\mathcal{P}$ and $\mathcal{Q}$ with the weak-Topology, where we identify L^∞ with the dual of L^1. We also define the sets*

$$\mathcal{A}_\mathcal{P} := \{\tau_y a \mid y \in \mathbb{R}^N\} \subset \mathcal{P},$$
$$\mathcal{A}_\mathcal{Q} := \{\tau_y V \mid y \in \mathbb{R}^N\} \subset \mathcal{Q},$$

considered with the topology induced by $\mathcal{P}$ and by $\mathcal{Q}$, respectively and

$$\mathcal{B}_\mathcal{P} := \overline{\mathcal{A}_\mathcal{P}} \setminus \mathcal{A}_\mathcal{P} \subset \mathcal{P},$$

$$\mathcal{B}_\mathcal{Q} := \overline{\mathcal{A}_\mathcal{Q}} \setminus \mathcal{A}_\mathcal{Q} \subset \mathcal{Q}.$$

Since $\overline{B}_1 L^\infty$ is the polar of $\overline{B_1} L^1$ in L^∞, using [41, Theorems 3.15 and 3.16] it can be proved that the topological spaces $\mathcal{P}$ and $\mathcal{Q}$ are compact metrizable spaces.

3.3 Definition *We define the positive constant functions $V_{\sup} = \operatorname{ess\,sup} V$ and $V_{\inf} = \operatorname{ess\,inf} V$.*

3.4 Lemma *If $(u_n) \subset E_V$ and $(y_n) \subset \mathbb{R}^N$ are such that $\tau_{-y_n} u_n \rightharpoonup u_0$ in E_V then*

$$I_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n) - I_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n - u_0) - I_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0) = o(1) \tag{3.3}$$

and

$$DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n) - DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n - u_0) - DI_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0) = o(1). \tag{3.4}$$

Proof. We have

$$\begin{aligned}
&I_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n) - I_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n - u_0) - I_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0)\\
&= I_{V,a}(u_n) - I_{V,a}(u_n - \tau_{y_n}u_0) - I_{V,a}(\tau_{y_n}u_0)\\
&= \frac{1}{2}\left(\|u_n\|_V^2 - \|u_n - \tau_{y_n}u_0\|_V^2 - \|\tau_{y_n}u_0\|_V^2\right)\\
&- \frac{1}{p}\left(\int a(x)\,|u_n|^p - \int a(x)\,|u_n - \tau_{y_n}u_0|^p - \int a(x)\,|\tau_{y_n}u_0|^p\right)\\
&= \frac{1}{2}\left(\int |\nabla u_n|^2 - \int |\nabla(u_n - \tau_{y_n}u_0)|^2 - \int |\nabla \tau_{y_n}u_0|^2\right)\\
&+ \frac{1}{2}\left(\int V(x)u_n^2 - \int V(x)(u_n - \tau_{y_n}u_0)^2 - \int V(x)\tau_{y_n}u_0^2\right)\\
&- \frac{1}{p}\left(\int a(x)\,|u_n|^p - \int a(x)\,|u_n - \tau_{y_n}u_0|^p - \int a(x)\,|\tau_{y_n}u_0|^p\right).
\end{aligned}$$

As we can see in the proof of [48, Lemma 1.32] (Brézis-Lieb Lemma) $\tau_{-y_n}u_n \rightharpoonup u_0$ implies, after passing to a subsequence

$$\begin{aligned}
&\left|\int a(x)\left(|u_n|^p - |u_n - \tau_{y_n}u_0|^p - |\tau_{y_n}u_0|^p\right)\right|\\
&\le \|a\|_\infty \int \left||u_n|^p - |u_n - \tau_{y_n}u_0|^p - |\tau_{y_n}u_0|^p\right|\\
&= \|a\|_\infty \int \left||\tau_{-y_n}u_n|^p - |\tau_{-y_n}u_n - u_0|^p - |u_0|^p\right|\\
&= o(1).
\end{aligned}$$

In a similar way we can see that

$$\left|\int V(x)\left(u_n^2 - (u_n - \tau_{y_n}u_0)^2 - (\tau_{y_n}u_0)^2\right)\right| = o(1).$$

Denote $((u,v)) := \int \nabla u \nabla v$ for $u, v \in H^1$. Then

$$\begin{aligned}
&((\tau_{-y_n}u_n, \tau_{-y_n}u_n)) - ((\tau_{-y_n}u_n - u_0, \tau_{-y_n}u_n - u_0)) - ((u_0, u_0))\\
=&2\left((\tau_{-y_n}u_n, u_0)\right) - 2\left((u_0, u_0)\right)\\
=&o(1).
\end{aligned}$$

Combining these facts we obtain (3.3).
Now, let $v \in E_V$. We have:

$$
\begin{aligned}
&\left|DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n)v - DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n - u_0)v - DI_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0)v\right| \\
&= |DI_{V,a}(u_n)\tau_{y_n}v - DI_{V,a}(u_n - \tau_{y_n}u_0)\tau_{y_n}v - DI_{V,a}(\tau_{y_n}u_0)\tau_{y_n}v| \\
&= \left|\int a(x)\,|u_n|^{p-2}\,u_n\tau_{y_n}v - \int a(x)\,|u_n - \tau_{y_n}u_0|^{p-2}\,(u_n - \tau_{y_n}u_0)\tau_{y_n}v \right. \\
&\left. - \int a(x)\,|\tau_{y_n}u_0|^{p-2}\,\tau_{y_n}u_0\tau_{y_n}v\right| \\
&\leq \int |a(x)|\,\left||u_n|^{p-2}\,u_n - |u_n - \tau_{y_n}u_0|^{p-2}\,(u_n - \tau_{y_n}u_0) - |\tau_{y_n}u_0|^{p-2}\,\tau_{y_n}u_0\right|\,|\tau_{y_n}v| \\
&\leq \|a\|_\infty \left\||u_n|^{p-2}\,u_n - |u_n - \tau_{y_n}u_0|^{p-2}\,(u_n - \tau_{y_n}u_0) - |\tau_{y_n}u_0|^{p-2}\,\tau_{y_n}u_0\right\|_{L^{\frac{p}{p-1}}}\|v\|_{L^p}.
\end{aligned}
$$

Using [48, Lemma 8.1] we deduce (3.4) from the above inequality.

□

3.5 Lemma *Let* $(y_n) \in \mathbb{R}^N$ *and* $V^* \in L^\infty$ *such that* $\tau_{-y_n}V \overset{w^*}{\to} V^*$. *Then, for all* $u \in E_V$

$$\|u\|^2_{\tau_{-y_n}V} = \|u\|^2_{V^*} + o(1).$$

Proof. First we will prove that $V^* > 0$. Let $\varphi \in S_1L^1, \varphi \geq 0$.

$$
\begin{aligned}
\int V^*\varphi &= \lim_{n\to\infty}\int(\tau_{-y_n}V)\varphi \\
&= \lim_{n\to\infty}\int V(\tau_{y_n}\varphi) \\
&\geq \operatorname{ess\,inf} V,
\end{aligned}
$$

where in the last step we used (2.4). Using the inequality obtained and (2.4) again we obtain $\operatorname{ess\,inf} V^* \geq \operatorname{ess\,inf} V$, and from here $V^* \geq \operatorname{ess\,inf} V > 0$. Now we have

$$
\begin{aligned}
\|u\|^2_{\tau_{-y_n}V} &= \int |\nabla u|^2 + \int \tau_{-y_n}Vu^2 \\
&= \int |\nabla u|^2 + \int V^*u^2 + o(1) \\
&= \|u\|^2_{V^*} + o(1).
\end{aligned}
$$

□

3.6 Definition *If* $(u_n) \subset E_V$ *then we say that* $(DI_{V,a}(u_n))$ *vanishes in* E_V^* *(the dual of* E_V*) if*

$$DI_{\tau_{x_n}V,\tau_{x_n}a}(\tau_{x_n}u_n) \overset{w^*}{\to} 0$$

for every sequence $(x_n) \subset \mathbb{R}^N$.

3.7 Lemma *Let* $(u_n) \subset E_V$*,* $(y_n) \subset \mathbb{R}^N$ *and* $a^*, V^* \in L^\infty$ *be such that* $(DI_{a,V}(u_n))$ *vanishes in* E_V^**,* $\tau_{-y_n} u_n \rightharpoonup u_0$ *in* E_V*,* $\tau_{-y_n} a \overset{\mathrm{w}*}{\to} a^*$ *and* $\tau_{-y_n} V \overset{\mathrm{w}*}{\to} V^*$*. Define* $v_n := u_n - \tau_{y_n} u_0$*. Then, after passing to a subsequence,*

$$I_{V,a}(v_n) = I_{V,a}(u_n) - I_{V^*,a^*}(u_0) + o(1), \tag{3.5}$$

$$\|u_n\|_V^2 - \|v_n\|_V^2 \to \|u_0\|_{V^*}^2\,, \tag{3.6}$$

$$DI_{V^*,a^*}(u_0) = 0 \tag{3.7}$$

and $(DI_{V,a}(v_n))$ *vanishes in* E_V^**.*

Proof. Since $\tau_{-y_n} a \overset{w^*}{\to} a^*$, $\tau_{-y_n} V \overset{\mathrm{w}*}{\to} V^*$ it holds true that

$$I_{V^*,a^*}(u_0) = I_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0) + o(1).$$

Combining this with (3.3) we obtain (3.5).
Observe that, as in Lemma 3.4

$$\begin{aligned}
\|v_n\|_V^2 &= \|u_n - \tau_{y_n} u_0\|_V^2 \\
&= \|u_n\|_V^2 - 2\,\langle u_n, \tau_{y_n} u_0\rangle_V + \|\tau_{y_n} u_0\|_V^2\,, \\
&= \|u_n\|_V^2 - \|\tau_{y_n} u_0\|_V^2 + o(1),
\end{aligned}$$

so Lemma 3.5 gives (3.6).
Now we will prove that u_0 is a critical point of I_{V^*,a^*}. To simplify the notation we write $f(u) = |u|^{p-2}\, u$. Let $v \in E_V$. By the definition of weak* convergence, $\int \tau_{-y_n} V u_0 v \to \int V^* u_0 v$ and $\int \tau_{-y_n} a f(u_0) v \to \int a^* f(u_0) v$. Also, since $\tau_{-y_n} u_n \to u_0$ in L^2_{loc} and L^p_{loc}, we have that $(\tau_{-y_n} u_n) v \to u_0 v$ and $f(\tau_{-y_n} u_n) v \to f(u_0) v$ in L^1, so we can use arguments similar to those of Lemma 3.4 to obtain

$$\begin{aligned}
DI_{V^*,a^*}(u_0)(v) &= \int \nabla u_0 \nabla v + \int (\tau_{-y_n} V) u_0 v - \int_{\mathbb{R}^N} (\tau_{-y_n} a) f(u_0) v + o(1) \\
&= \int \nabla(\tau_{-y_n} u_n) \nabla v + \int (\tau_{-y_n} V)(\tau_{-y_n} u_n) v - \int_{\mathbb{R}^N} (\tau_{-y_n} a) f(\tau_{-y_n} u) v + o(1) \\
&= DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n} u_n) v + o(1) \\
&= o(1),
\end{aligned}$$

where we have used that $(DI_{V,a}(u_n))$ vanishes in E_V^*. Since v was arbitrarily chosen in E_V, u_0 is a critical point of I_{V^*,a^*}.
To prove that $(DI_{V,a}(v_n))$ vanishes in E_V^*, suppose that $(x_n) \subset \mathbb{R}^N$ and $v \in E_V$. If $DI_{\tau_{x_n}V,\tau_{x_n}a}(\tau_{x_n} v_n) v \to 0$ were not true we could pass to a subsequence such that

$$\liminf_{n\to\infty} \left| DI_{\tau_{x_n}V,\tau_{x_n}a}(\tau_{x_n} v_n) v \right| > 0 \tag{3.8}$$

and such that either $|x_n + y_n| \to \infty$ or $x_n + y_n \to -\xi$ for some $\xi \in \mathbb{R}^N$. In the first case it would follow that $\tau_{-x_n-y_n} v \rightharpoonup 0$ in E_V, then $\tau_{-x_n-y_n} v \rightharpoonup 0$ in L^p and hence $f(u_0)\tau_{-x_n-y_n}v \to 0$ in L^1. It also can be proved (again as in Lemma 3.4) that $\langle \tau_{-x_n-y_n}v, u_0\rangle_{\tau_{-y_n}V} = o(1)$. Therefore,

$$DI_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0)\tau_{-x_n-y_n}v = o(1). \tag{3.9}$$

Using (3.4), (3.9), and the fact that $(DI_{V,a}(u_n))$ vanishes in E_V^* we would obtain

$$\begin{aligned}
DI_{\tau_{x_n}V,\tau_{x_n}a}(\tau_{x_n}v_n)v &= DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}v_n)\tau_{-x_n-y_n}v \\
&= DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n)\tau_{-x_n-y_n}v - DI_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0)\tau_{-x_n-y_n}v + o(1) \\
&= DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n)\tau_{-x_n-y_n}v + o(1) \\
&= DI_{\tau_{x_n}V,\tau_{x_n}a}(\tau_{x_n}u_n)v + o(1) \\
&= o(1),
\end{aligned}$$

in contradiction with (3.8). In the second case we would obtain, using that translation is continuous in E_V and in L^q for $q \subset [1,\infty)$, (3.4), the fact that $(DI_{v,a}(u_n))$ vanishes in E_V^*, and the fact that u_0 is a critical point of I_{V^*,a^*}, that

$$\begin{aligned}
DI_{\tau_{x_n}V,\tau_{x_n}a}(\tau_{x_n}v_n)v &= DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}v_n)\tau_\xi v + o(1) \\
&= DI_{\tau_{-y_n}V,\tau_{-y_n}a}(\tau_{-y_n}u_n)\tau_\xi v - DI_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0)\tau_\xi v + o(1) \\
&= -DI_{\tau_{-y_n}V,\tau_{-y_n}a}(u_0)\tau_\xi v + o(1) \\
&= -DI_{V^*,a^*}(u_0)\tau_\xi v + o(1) \\
&= o(1),
\end{aligned}$$

contradicting (3.8). We have therefore proved that $(DI_{V,a}(v_n))$ vanishes in E_V^*.

□

3.8 Definition *A sequence $(u_n) \in E_V$ is said to vanish if $\tau_{-y_n}u_n \rightharpoonup 0$ in E_V for every sequence $(y_n) \subset \mathbb{R}^N$.*

If (u_n) vanishes in the above sense then (u_n) is bounded in E_V and $\tau_{-y_n}u_n \to 0$ in L^2_{loc} for every sequence $(y_n) \subset \mathbb{R}^N$. Hence

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} |u_n|^2 = 0$$

is satisfied for every $R > 0$ and Lions' Vanishing Lemma [33, Lemma I.1] implies that $u_n \to 0$ in L^q for every $q \in (2, 2^*)$.

In the proof of Lemma 3.5 we have already seen that $V_{\inf} \le V^*$. In a similar way it can also be seen that $a^* \le \operatorname{ess\,sup} a$. Using this facts and Lemma 2.6 we derive the next corollary.

3.9 Corollary *There exists $C > 0$ such that, for all $u \in \mathcal{N}_{V^*,a^*} \setminus \{0\}$, where $\tau_{-y_n}a \xrightarrow{w^*} a^*$ and $\tau_{-y_n}V \xrightarrow{w^*} V^*$ in L^∞ for some sequence (y_n), we have $\|u\|_{V^*} \ge C$.*

Using Lemma 3.7 we can now prove the Splitting Lemma in the case when there is no global limit for V and a at infinity. In the statement one can observe the changes that equations with many limit problems imply, and how every one of those limit problems can be involved in the splitting of the sequence.

3.10 Lemma (Splitting Lemma.) *Let (u_n) be a Palais-Smale sequence for $I_{V,a}$ at the level $c \in \mathbb{R}$. Then either $u_n \to 0$ in E_V or, after passing to a subsequence, there are $k \in \mathbb{N}$, sequences $(y_n^i)_n \subset \mathbb{R}^N$, functions $V^i, a^i \in L^\infty$, and functions $u^i \in E_V \setminus \{0\}$ (i=1,...,k) such that each u^i is a critical point of I_{V^i,a^i}, and such that the following hold true:*

$$\left| u_n - \sum_{i=1}^{k} \tau_{y_n^i} u^i \right|_p \to 0, \tag{3.10}$$

$$c \geq \sum_{i=1}^{k} I_{V^i,a^i}(u^i), \tag{3.11}$$

$$\tau_{-y_n^i} a \overset{w^*}{\to} a^i, \tau_{-y_n^i} V \overset{w^*}{\to} V^i, \tag{3.12}$$

and

$$\left| y_n^i - y_n^j \right| \to \infty \quad (i \neq j, n \to \infty). \tag{3.13}$$

Proof. Since (u_n) is a (PS)-sequence for $I_{V,a}$, it is bounded in E_V and $(DI_a(u_n))$ vanishes in E_V^*.
If (u_n) vanishes then $|u_n|_p \to 0$. Since $DI_{V,a}(u_n)(u_n) = o(1)$, also $\|u_n\|_V \to 0$.
If (u_n) does not vanish then, by definition, there exists $u^1 \in E_V \setminus \{0\}$ and a sequence $(y_n^1) \in \mathbb{R}^N$ such that, after passing to a subsequence and writing $u_n^1 := u_n, \tau_{-y_n^1} u_n^1 \rightharpoonup u^1$. By the compactness of $\mathcal{Q}$ and $\mathcal{P}$ we may also assume that $\tau_{-y_n^1} V \overset{w^*}{\to} V^1$ and $\tau_{-y_n^1} a \overset{w^*}{\to} a^1$ in L^∞. Now we define $u_n^2 := u_n^1 - \tau_{y_n^1} u^1$, so $\tau_{-y_n^1} u_n^2 \rightharpoonup 0$. Lemma 3.7 assures that

$$I_{V,a}(u_n^1) - I_{V,a}(u_n^2) \to I_{V^1,a^1}(u^1),$$

$$\left\|u_n^1\right\|_V^2 - \left\|u_n^2\right\|_V^2 \to \left\|u^1\right\|_{V^1}^2,$$

$$DI_{V^1,a^1}(u^1) = 0$$

and $(DI_{V,a}(u_n^2))$ vanishes in E_V^*. If (u_n^2) vanishes, then $|u_n^2|_p \to 0$ and hence

$$\left|u_n^1 - \tau_{y_n^1} u^1\right|_p \to 0.$$

Otherwise, there exist $V^2, a^2 \in L^\infty, u^2 \in E_V \setminus \{0\}$, and a sequence $(y_n^2) \subset \mathbb{R}^N$ such that, after passing to a subsequence, $\tau_{-y_n^2} V \overset{w^*}{\to} V^2, \tau_{-y_n^2} a \overset{w^*}{\to} a^2$ and $\tau_{-y_n^2} u_n^2 \rightharpoonup u^2$. Since $\tau_{-y_n^1} u_n^2 \rightharpoonup 0, |y_n^1 - y_n^2| \to \infty$.
Proceeding in this way, inductively we obtain sequences (y_n^i), functions $V^i, a^i \in L^\infty$, and

functions $u^i \in E_V \setminus \{0\}$, for $i = 1, 2, 3, ...$ By Corollary 3.9 there is $C > 0$, independent of i, such that $\|u^i\|_{V^i} \geq C$. For every j we have

$$0 \leq \left\|u_n^{j+1}\right\|^2 = \|u_n\|^2 - \sum_{i=1}^{j} \left\|u^i\right\|_{V^i}^2 + o(1),$$

so by the lower positive bound for $\|u^i\|_{V^i}$ and since (u_n) is bounded in E_V the process must stop after a finite number of iterations. Therefore, there is $k \in \mathbb{N}$ such that (u_n^{k+1}) vanishes,

$$\left|u_n^{k+1}\right|_p \to 0, \tag{3.14}$$

and (3.10) holds true.
Similarly, we have

$$-\int_{\mathbb{R}^N} a \left|u_n^{k+1}\right|^p \leq I_{V,a}(u_n^{k+1}) = I_{V,a}(u_n) - \sum_{i=1}^{k} I_{V^i,a^i}(u^i) + o(1),$$

so (3.11) is a consequence of (3.14) and of $c = \lim_{n\to\infty} I_{V,a}(u_n)$.
The remaining properties (3.12) and (3.13) were already proved along the way.

$\square$

4 Existence Of Ground States In Three General Situations

In this chapter we will still assume $a, V \in L^\infty$, $a^+ \neq 0$ in L^∞ and $\inf V > 0$. We will give additional hypothesis in each of the sections dedicated to prove the existence of a ground state.

4.1 Technical Lemmas

Now that the Splitting Lemma has been established, we will use it to prove the existence of a ground state. As a previous step we will establish a inequality between some energy levels, following the ideas of [15]. The three cases we are going to study are: a does not have a global limit but V does, the opposite case, and the case when neither V nor a have a global limit at infinity. By the nature of the hypotheses we are going to require, these three cases need to be treated individually, but we have isolated some technical lemmas that will help us in all three general situations.

It is usual in the case when the global limits at infinity exist that, in order to simplify calculations, one assumes that these global limits are 1 in both cases. The general result follows from this through a rescaling of the problem. In our case, when in some sense there are different limits in different directions, we can not make use of this simplification. Instead, we will make the rescaling explicit to show how the value of the limit affects the energy levels of the limit problems.

From here on we identify any constant $b \in \mathbb{R}$ with the constant function $b \in L^\infty$.

4.1 Lemma *Let $C_1, C_2 > 0$. Then u is a ground state of $I_{1,1}$ if and only if $w(x) := (C_1/C_2)^{1/(p-2)} u\left(C_1^{\frac{1}{2}} x\right)$ is a ground state for I_{C_1,C_2}, and*

$$I_{C_1,C_2}(w) = \left(\frac{1}{C_2}\right)^{\frac{2}{(p-2)}} (C_1)^{\frac{2}{p-2}+\frac{2-N}{2}} I_{1,1}(u).$$

Proof. Let $u \in E$, w as defined above and $v_1(x) := v(x/C_1^{\frac{1}{2}})$ for every $v \in E$. We have

$$
\begin{aligned}
DI_{C_1,C_2}(w)(v) &= \int \nabla w \nabla v + \int C_1 wv - \int C_2 |w|^{p-2} wv \\
&= \int C_1^{\frac{1}{p-2}+\frac{1}{2}} \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \nabla u \left(C_1^{\frac{1}{2}} x\right) \nabla v(x) dx \\
&+ \int C_1 C_1^{\frac{1}{p-2}} \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} u \left(C_1^{\frac{1}{2}} x\right) v(x) dx \\
&- \int C_2 C_1^{\frac{p-1}{p-2}} \left(\frac{1}{C_2}\right)^{\frac{p-1}{p-2}} \left|u \left(C_1^{\frac{1}{2}} x\right)\right|^{p-2} u \left(C_1^{\frac{1}{2}} x\right) v(x) dx \\
&= \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \int C_1^{\frac{1}{p-2}+\frac{1}{2}} \nabla u \left(C_1^{\frac{1}{2}} x\right) \left(C_1^{\frac{1}{2}} \nabla v_1 \left(C_1^{\frac{1}{2}} x\right)\right) dx \\
&+ \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \int C_1 C_1^{\frac{1}{p-2}} u \left(C_1^{\frac{1}{2}} x\right) v_1 \left(C_1^{\frac{1}{2}} x\right) dx \\
&- \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \int C_1^{\frac{p-1}{p-2}} \left|u \left(C_1^{\frac{1}{2}} x\right)\right|^{p-2} u \left(C_1^{\frac{1}{2}} x\right) v_1 \left(C_1^{\frac{1}{2}} x\right) dx \\
&= \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \int C_1^{\frac{1}{p-2}-\frac{N}{2}+1} \nabla u(x) \nabla v_1(x) dx + \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \int C_1 C_1^{\frac{1}{p-2}-\frac{N}{2}} u(x) v_1(x) dx \\
&- \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \int C_1^{\frac{p-1}{p-2}-\frac{N}{2}} |u(x)|^{p-2} u(x) v_1(x) dx \\
&= C_1^{\frac{1}{p-2}-\frac{N}{2}+1} \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} \left(\int \nabla u(x) \nabla v_1(x) dx + \int u(x) v_1(x) dx - \int |u(x)|^{p-2} u(x) v_1(x) dx\right) \\
&= C_1^{\frac{1}{p-2}-\frac{N}{2}+1} \left(\frac{1}{C_2}\right)^{\frac{1}{p-2}} DI_{1,1}(u)(v_1).
\end{aligned}
$$

From here we deduce that u is a critical point of $I_{1,1}$ if and only if w is a critical point of I_{C_1,C_2}. The equation about the energy levels is proved in a similar fashion, and from there we conclude the lemma.

□

It is well known that there exists a ground state $u_{1,1}$, positive on all of $\mathbb{R}^N$ and radially symmetric, for the equation whose weak solutions are critical points of $I_{1,1}$ (this can be checked in [27] and [32]). Define, for all $C_1, C_2 > 0$, the function $u_{C_1,C_2} := (C_1/C_2)^{1/(p-2)} u_{1,1}\left(C_1^{\frac{1}{2}} x\right)$. By Lemma 4.1, we have that u_{C_1,C_2} is a ground state of the functional I_{C_1,C_2}. By Lemma 2.10 we have that $\mathcal{N}_{C_1,C_2}$ is a natural constraint. It follows that $c_{C_1,C_2} = \hat{c}_{C_1,C_2} = I_{C_1,C_2}(u_{C_1,C_2}) = \left(\frac{1}{C_2}\right)^{\frac{2}{(p-2)}} (C_1)^{\frac{2}{p-2}+\frac{2-N}{2}} I_{1,1}(u)$.

Now, for fixed $\varepsilon \in (0,1)$, define a cut-off function $\chi \in C^\infty(\mathbb{R}^N)$ with $0 \le \chi \le 1, \chi(x) =$

1 if $|x| \leq 1-\varepsilon$, and $\chi(x) = 0$ if $|x| \geq 1$. For $u \in E_V$ and $R > 0$ define $u^R \in E_V$ by

$$u^R(x) = \chi\left(\frac{x}{R}\right) u(x),$$

for all $x \in \mathbb{R}^N$. Observe that

$$u^R \to u \in E_V \tag{4.1}$$

as $R \to \infty$.

From ([15, Lemma 2]) we obtain:

4.2 Lemma *For $0 < \varepsilon < 1$ and $C_1, C_2 > 0$ we have, as $R \to \infty$,*

$$\int_{\mathbb{R}^N} \left| |\nabla u_{C_1,C_2}|^2 - \left|\nabla u^R_{C_1,C_2}\right|^2 \right| = O(e^{-2(1-\varepsilon)C_1^{\frac{1}{2}}R}),$$

and

$$\int_{|x|\geq R} |u_{C_1,C_2}|^s \, dx = O(e^{-sC_1^{\frac{1}{2}}R})$$

for all $s > 0$.

Proof. Observe that

$$\begin{aligned}
u^R_{C_1,C_2}(x) &= \chi\left(\frac{x}{R}\right) u_{C_1,C_2}(x) \\
&= \left(\frac{C_1}{C_2}\right)^{\frac{1}{(p-2)}} \chi\left(\frac{x}{R}\right) u_{1,1}\left(C_1^{\frac{1}{2}}x\right) \\
&= \left(\frac{C_1}{C_2}\right)^{\frac{1}{(p-2)}} \chi\left(\frac{C_1^{\frac{1}{2}}x}{C_1^{\frac{1}{2}}R}\right) u_{1,1}\left(C_1^{\frac{1}{2}}x\right) \\
&= \left(\frac{C_1}{C_2}\right)^{\frac{1}{(p-2)}} u_{1,1}^{C_1^{\frac{1}{2}}R}\left(C_1^{\frac{1}{2}}x\right).
\end{aligned}$$

Using this and [15, Lemma 2] we obtain the result.

□

In the next lemmas we ask a^* and V^* to be positive constants for u_{V^*,a^*} to be well defined.

4.3 Lemma *Let $g_n : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be a sequence of continuous functions such that, for all $t \in \mathbb{R}^+ \cup \{0\}$, we have $g_n(t) \to g_0(t)$ for some continuous function g_0. Assume that each g_n achieves a unique global maximum in some $t_n \in \mathbb{R}^+$, for all $n \in \mathbb{N} \cup \{0\}$. Moreover, assume that each g_n is strictly increasing in $[0, t_n]$ and strictly decreasing in $[t_n, \infty)$. Also assume that $g_0(0) = 0$, $\lim_{t\to\infty} g_0(t) = -\infty$ and $g_0(t_0) > 0$. Then there exists $K' \in \mathbb{N}$ and $r_1 > r_0 > 0$ such that*

$$g_n(t) \leq \frac{g_0(t_0)}{2}$$

for all $n \geq K', t \in [0, r_0] \cup [r_1, \infty)$.

Proof. First we will prove that $t_n \to t_0$. We do this by contradiction. If the convergence does not hold then we can find $\varepsilon > 0$ and a subsequence of $(t_n)_n$, denoted in the same way, such that either $t_n \leq t_0 - \varepsilon$ for all n or $t_n \geq t_0 + \varepsilon$ for all n. For now we assume the first case.
Since $t_n \leq t_0 - \varepsilon$, we have that each g_n is decreasing in the interval $[t_0 - \varepsilon, t_0]$. Using the pointwise convergence we can find N_δ, for every $\delta > 0$, such that $n > N_\delta$ implies $g_n(t_0) > g_0(t_0) - \delta$. These two facts, combined with the pointwise convergence for every $t \in [t_0 - \varepsilon, t_0]$, imply $g_0(t) \geq g_0(t_0) - \delta$ for every $t \in [t_0 - \varepsilon, t_0]$. Since this is valid for every $\delta > 0$ we have $g_0(t) \geq g_0(t_0)$ for $t \in [t_0 - \varepsilon, t_0]$, which is a contradiction. The second case can be treated in a similar manner.
We have proved $t_n \to t_0$. Using this we can find K_1' such that $n > K_1'$ implies $t_n \in \left(\frac{t_0}{2}, \frac{3t_0}{2}\right)$. In particular, this implies that all the functions g_n with $n > K_1'$ are strictly increasing in $[0, t_0/2]$ and strictly decreasing in $[3t_0/2, \infty)$.
Because of the continuity of g_0 we can find $t_0' > 0$ such that $g_0(t) < g_0(t_0)/4$ for all $t \in [0, t_0']$. Define $t_0'' = \min\{t_0/2, t_0'\}$. Using pointwise convergence we can find K_2' such that $n > K_2'$ implies $g_n(t_0'') < g_0(t_0)/2$. So, if we define $K_2'' = \max\{K_1', K_2'\}$ we have that $n > K_2''$ implies $g_n(t) < g_0(t_0)/2$ for all $t \in [0, t_0']$.
In a similar way we can find K_3'' and t_1' such that $n > K_3''$ implies $g_n(t) < g_0(t_0)/2$ for all $t \in [t_1', \infty)$, and from here we deduce the result.

□

4.4 Lemma *Let $(y_n) \subset \mathbb{R}^N$ and $\tau_{-y_n} V \overset{w^*}{\to} V^*$,$\tau_{-y_n} a \overset{w^*}{\to} a^*$, where V^* and a^* are positive constants. Then there are $K' \in \mathbb{N}$ and $t_1 > t_0 > 0$ such that*

$$I_{V,a}(t(\tau_{y_n} u_{V^*,a^*}^{R_n})) \leq \frac{c_{V^*,a^*}}{2} \tag{4.2}$$

for all $n \geq K', t \in [0, t_0] \cup [t_1, \infty)$.

Proof. Define $h_n(t) := I_{V,a}(t(\tau_{y_n} u_{V^*,a^*}^{R_n}))$ and $h_0(t) := I_{V^*,a^*}(t u_{V^*,a^*})$. We are going to prove that some subsequence of $(h_n)_{n\in\mathbb{N}}$ satisfies the hypotheses of Lemma 4.3. Using Lemma 4.2 and the w*-convergence we see that

$$\begin{aligned}
&\lim_{n\to\infty} I_{V,a}(t(\tau_{y_n} u_{V^*,a^*}^{R_n})) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} \int \left|\nabla t(\tau_{y_n} u_{V^*,a^*}^{R_n})\right|^2 + \frac{1}{2} \int V(x)(t(\tau_{y_n} u_{V^*,a^*}^{R_n}))^2 - \frac{1}{p} \int a(x) \left|t(\tau_{y_n} u_{V^*,a^*}^{R_n})\right|^p \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{2} \int \left|\nabla t u_{V^*,a^*}^{R_n}\right|^2 + \frac{1}{2} \int (\tau_{-y_n} V) \left(t u_{V^*,a^*}^{R_n}\right)^2 - \frac{1}{p} \int (\tau_{-y_n} a) \left|t u_{V^*,a^*}^{R_n}\right|^p \right) \\
&= \frac{1}{2} \int |\nabla t u_{V^*,a^*}|^2 + \frac{1}{2} \int V^*(x)(t u_{V^*,a^*})^2 - \frac{1}{p} \int a^* |t u_{V^*,a^*}|^p \\
&= I_{V^*,a^*}(t u_{V^*,a^*}),
\end{aligned}$$

so we have the pointwise convergence $h_n(t) \to h(t)$. Next we will prove that there is $K' \in \mathbb{N}$ such that for all $n > K'$ there exists $t_n > 0$ with the property that h_n is strictly increasing in $[0, t_n]$ and strictly decreasing in $[t_n, \infty)$. For this notice that $V > 0$ implies that $\int \left|\nabla(\tau_{y_n} u^{R_n}_{V^*,a^*})\right|^2 + \int V(x)((\tau_{y_n} u^{R_n}_{V^*,a^*}))^2 > 0$. On the other hand we have

$$\begin{aligned}
&\lim_{n\to\infty} \left(\int a(x) \left|(\tau_{y_n} u^{R_n}_{V^*,a^*})\right|^p \right) \\
&= \int a^* \left|u_{V^*,a^*}\right|^p \\
&>0,
\end{aligned}$$

so, for some K', we have that $n \geq K'$ implies $\int a(x) \left|(\tau_{y_n} u^{R_n}_{V^*})\right|^p > 0$, and we can define t_n as the unique positive critical point of h_n. Defining $g_n = h_{K'-1+n}$ for $n \in \mathbb{N}$ and $g_0 = h_0$ we can apply Lemma 4.3 to obtain the result.

□

The technique used in the following lemma is well known, see, e.g. [20]:

4.5 Lemma *Let $a_1, a_2 \in L^\infty$ be such that $a_1 \geq a_2$ a.e., and let $V_1, V_2 \in L^\infty$ be such that $0 < V_1 \leq V_2$ a.e. Then, for all nontrivial critical points u of I_{V_2,a_2}, it holds true that $I_{V_2,a_2}(u) \geq c_{V_1,a_1}$. If in addition $a_1 \neq a_2$ or $V_1 \neq V_2$ then the obtained inequality is strict.*

Proof. We have

$$\|u\|^2_{V_2} = \int a_2 \left|u\right|^p,$$

so it follows $\int a_1 \left|u\right|^p \geq \int a_2 \left|u\right|^p > 0$, and we can define

$$\begin{aligned}
r &:= \left(\frac{\int a_2 \left|u\right|^p}{\int a_1 \left|u\right|^p} \right)^{\frac{1}{p-2}} \leq 1, \\
t &:= \left(\frac{\int V_1 \left|u\right|^2}{\int V_2 \left|u\right|^2} \right)^{\frac{1}{p-2}}, \\
s &:= \left(\frac{\int V_1 \left|u\right|^2}{\int V_2 \left|u\right|^2} \right)^{\frac{1}{2}}
\end{aligned}$$

and $w(x) := rtu(sx)$. We have

$$\begin{aligned}
DI_{V_1,a_1}(w)(w) &= \int |\nabla w|^2 + \int V_1 w^2 - \int a_1 |w|^p \\
&= \frac{r^2 t^2}{s^{N-2}} \int |\nabla u|^2 + \frac{r^2 t^2}{s^N} \int V_1 u^2 - \frac{r^p t^p}{s^N} \int a_1 |u|^p \\
&= \frac{r^2 t^2}{s^{N-2}} \left(\int |\nabla u|^2 + \frac{1}{s^2} \int V_1 u^2 - r^{p-2} \int a_1 |u|^p \right) \\
&= \frac{r^2 t^2}{s^{N-2}} \left(\int |\nabla u|^2 + \int V_2 u^2 - \int a_2 |u|^p \right) \\
&= \frac{r^2 t^2}{s^{N-2}} DI_{V_2,a_2}(u)(u) \\
&= 0,
\end{aligned}$$

so $w \in \mathcal{N}_{V_1,a_1}$.
We can verify directly, using $2 < p < 2^*$, that $\frac{p}{p-2} - \frac{N}{2} > 0$. Using this and $0 < V_1 \leq V_2$ we obtain $\frac{t^p}{s^N} \leq 1$, and from here we deduce $\frac{r^2 t^2}{s^{N-2}} = \left(\frac{r^2 t^2}{s^{N-2}}\right)\left(\frac{t^{p-2}}{s^2}\right) = r^2 \frac{t^p}{s^N} \leq 1$. It follows that

$$\begin{aligned}
I_{V_1,a_1}(w) &= \frac{1}{2} \int |\nabla w|^2 + \frac{1}{2} \int V_1 w^2 - \frac{1}{p} \int a_1 |w|^p \\
&= \frac{r^2 t^2}{2 s^{N-2}} \int |\nabla u|^2 + \frac{r^2 t^2}{2 s^N} \int V_1 u^2 - \frac{r^p t^p}{p s^N} \int a_1 |u|^p \\
&= \frac{r^2 t^2}{s^{N-2}} \left(\frac{1}{2} \int |\nabla u|^2 + \frac{1}{2 s^2} \int V_1 u^2 - \frac{r^{p-2}}{p} \int a_1 |u|^p \right) \\
&= \frac{r^2 t^2}{s^{N-2}} \left(\frac{1}{2} \int |\nabla u|^2 + \frac{1}{2} \int V_2 u^2 - \frac{1}{p} \int a_2 |u|^p \right) \\
&= \frac{r^2 t^2}{s^{N-2}} I_{V_2,a_2}(u) \\
&\leq I_{V_2,a_2}(u),
\end{aligned}$$

which implies $c_{V_1,a_1} \leq I_{V_2,a_2}(u)$.
If we have $a_1 > a_2$ or $V_1 < V_2$ then $r < 1$ or $\frac{t^p}{s^N} < 1$, so we can conclude $c_{V_1,a_1} \leq I_{V_1,a_1}(w) < I_{V_2,a_2}(u)$.

□

4.6 Lemma *Let $(y_n) \in \mathbb{R}^N$ be such that* w*-lim $\tau_{-y_n} V = V^*$ *and* w*-lim $\tau_{-y_n} a = a^*$ *are positive constants. Then there is $M > 0$ such that $n > M$ implies that $t\left[\tau_{y_n} u^{R_n}_{V^*,a^*}\right] \in \mathcal{N}_{V,a}$ for some $t > 0$.*

Proof. Defining h_n as in Lemma 4.4 we have $h_n'(t) = DI_{V,a}(t\tau_{y_n} u^{R_n}_{V^*,a^*})(\tau_{y_n} u^{R_n}_{V^*,a^*})$. As we have seen in the proof of that lemma there exists $M \in \mathbb{N}$ such that, for every

$n > M$, there exists $t_n > 0$ such that $h_n'(t_n) = 0$, which implies $0 = t_n h_n'(t_n) = DI_{V,a}(t_n \tau_{y_n} u^{R_n}_{V^*,a^*})(t_n \tau_{y_n} u^{R_n}_{V^*,a^*})$, and we obtain $t_n \left[\tau_{y_n} u^{R_n}_{V^*,a^*}\right] \in \mathcal{N}_{V,a}$

□

We conclude this section with some definitions.

4.7 Definition *We define*

$$\overline{a} := \sup_{u \in \mathcal{B}_{\mathcal{P}}} \operatorname{ess\,sup} u.$$

and

$$\overline{V} := \inf_{u \in \mathcal{B}_{\mathcal{Q}}} \operatorname{ess\,inf} u.$$

By convention, $\overline{a} := -\infty$ *if* $\mathcal{B}_{\mathcal{P}} = \varnothing$ *and* $\overline{V} := \infty$ *if* $\mathcal{B}_{\mathcal{Q}} = \varnothing$.

4.2 First General Situation

In this situation assume $a = 1$, so $\overline{a} = -\infty$ (a could be any positive constant, but we choose 1 to simplify the notation). Denote $I_V := I_{V,1}$. Assume the next set of hypotheses:

(V1) Either (i) V is constant or (ii) $V \leq \overline{V}$ and $V \not\equiv \overline{V}$ or (iii) there exist $\gamma \in (0, 2\overline{V}^{\frac{1}{2}})$, sequences $(z_n) \subset \mathbb{R}^N$ and $(R_n) \subset \mathbb{R}$, and a non-negative measurable function κ on $\mathbb{R}^N$ with $\left|\{x \in \mathbb{R}^N \mid \kappa(x) > 0\}\right| > 0$ such that $R_n \to \infty$, w*-$\lim_{n\to\infty} \tau_{-z_n} V = \overline{V}$ and $V(x) \leq \overline{V} - \kappa(x - z_n)e^{-\gamma R_n}$ holds true for all n and $x \in B_{R_n}(z_n)$.

(V2) $\inf V > 0$.

Observe that **(V1)** implies $0 < 1 - \gamma/2\overline{V}^{\frac{1}{2}} < 1$. We will use this in the proof of Proposition 4.8. Our next results have the purpose of prove an inequality between some energy levels that, together with the Concentration Compactness Principle, will imply the existence of a ground state.

4.8 Proposition *In case (iii) of (V1) there is* $K \in \mathbb{N}$ *and* $D > 0$ *such that*

$$I_V(t(\tau_{z_n} u^{R_n}_{\overline{V}})) \leq c_{\overline{V}} - De^{-\gamma R_n}$$

for all $n \geq K, t \geq 0$.

Proof. First, notice that from Lemma 4.4 we have

$$\lim_{n\to\infty} I_V(t(\tau_{z_n} u^{R_n}_{\overline{V}})) = I_{\overline{V}}(tu_{\overline{V}})$$

for all $t \geq 0$. Using Lemma 4.4, we can choose $n_0 \in \mathbb{N}$ and $t_1 > t_0 > 0$ such that

$$I_V(t(\tau_{z_n} u^{R_n}_{\overline{V}})) \leq \frac{c_{\overline{V}}}{2} \tag{4.3}$$

for all $n \geq n_0, t \in [0, t_0] \cup [t_1, \infty)$.
Observe that

$$\begin{aligned}
\int_{\mathbb{R}^N} V \left|(\tau_{z_n} u_{\overline{V}}^{R_n})\right|^2 &\leq \int_{\mathbb{R}^N} V \left|(\tau_{z_n} u_{\overline{V}})\right|^2 \\
&= \int_{\mathbb{R}^N} \tau_{-z_n} V \left|u_{\overline{V}}\right|^2 \\
&= \int_{\|x\| \leq R_n} \tau_{-z_n} V \left|u_{\overline{V}}\right|^2 + \int_{\|x\| > R_n} \tau_{-z_n} V \left|u_{\overline{V}}\right|^2 .
\end{aligned}$$

Passing to a subsequence we may assume that (R_n) is an increasing sequence. Since $u_{\overline{V}}$ is positive we may take n_0 large enough such that

$$C := \int_{\|x\| \leq R_{n_0}} \kappa \left|u_{\overline{V}}\right|^2 > 0.$$

It follows for $n \geq n_0$ that

$$\begin{aligned}
\int_{\|x\| \leq R_n} (\tau_{-z_n} V) \left|u_{\overline{V}}\right|^2 &\leq \int_{\|x\| \leq R_n} (\overline{V} - \kappa e^{-\gamma R_n}) \left|u_{\overline{V}}\right|^2 \\
&\leq \int_{\mathbb{R}^N} \overline{V} \left|u_{\overline{V}}\right|^2 - e^{-\gamma R_n} \int_{\|x\| \leq R_{n_0}} \kappa \left|u_{\overline{V}}\right|^2 \\
&\leq \int_{\mathbb{R}^N} \overline{V} \left|u_{\overline{V}}\right|^2 - C e^{-\gamma R_n},
\end{aligned}$$

and Lemma 4.2 implies

$$\int_{\|x\| > R_n} (\tau_{-z_n} V) \left|u_{\overline{V}}\right|^2 \leq \|V\|_\infty \int_{\|x\| > R_n} \left|u_{\overline{V}}\right|^2 dx \leq O(e^{-2\overline{V}^{\frac{1}{2}} R_n}).$$

From the above inequalities we deduce

$$\int_{\mathbb{R}^N} V(x) \left|\tau_{z_n} u_{\overline{V}}^{R_n}\right|^2 \leq \int_{\mathbb{R}^N} \overline{V} \left|u_{\overline{V}}\right|^2 - C e^{-\gamma R_n} + O(e^{-2\overline{V}^{\frac{1}{2}} R_n}). \tag{4.4}$$

From (4.4), Lemma 4.2 and $t \in [t_0, t_1]$ it follows, for $n \geq n_0$,

$$
\begin{aligned}
& I_V(t\tau_{z_n} u_{\overline{V}}^{R_n}) \\
&= \int \left|\nabla(t\tau_{z_n} u_{\overline{V}}^{R_n})\right|^2 + \int V(x) t^2 (\tau_{z_n} u_{\overline{V}}^{R_n})^2 - \int \left|t(\tau_{z_n} u_{\overline{V}}^{R_n})\right|^p \\
&\leq t^2 \int |\nabla u_{\overline{V}}|^2 + t^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V}\, |u_{\overline{V}}|^2 - t^2 C e^{-\gamma R_n} + t^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - t^p \int \left|u_{\overline{V}}^{R_n}\right|^p \\
&= t^2 \int |\nabla u_{\overline{V}}|^2 + t^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V}\, |u_{\overline{V}}|^2 - t^2 C e^{-\gamma R_n} + t^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - t^p \int |u_{\overline{V}}|^p \\
&+ t^p \int |u_{\overline{V}}|^p - t^p \int \left|u_{\overline{V}}^{R_n}\right|^p \\
&= t^2 \int |\nabla u_{\overline{V}}|^2 + t^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V}\, |u_{\overline{V}}|^2 - t^2 C e^{-\gamma R_n} + t^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - t^p \int |u_{\overline{V}}|^p \\
&+ t^p \int_{\|x\|>(1-\varepsilon)R_n} \left(|u_{\overline{V}}|^p - \left|u_{\overline{V}}^{R_n}\right|^p\right) \\
&\leq t^2 \int |\nabla u_{\overline{V}}|^2 + t^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V}\, |u_{\overline{V}}|^2 - t^2 C e^{-\gamma R_n} + t^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - t^p \int |u_{\overline{V}}|^p \\
&+ t^p \int_{\|x\|>(1-\varepsilon)R_n} |u_{\overline{V}}|^p \\
&\leq t^2 \int |\nabla u_{\overline{V}}|^2 + t_1^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V}\, |u_{\overline{V}}|^2 - t_0^2 C e^{-\gamma R_n} + t_1^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - t^p \int |u_{\overline{V}}|^p \\
&+ t_1^p \int_{\|x\|>(1-\varepsilon)R_n} |u_{\overline{V}}|^p \\
&\leq t^2 \int |\nabla u_{\overline{V}}|^2 + t_1^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V}\, |u_{\overline{V}}|^2 - t_0^2 C e^{-\gamma R_n} + t_1^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - t^p \int |u_{\overline{V}}|^p \\
&+ t_1^p O(e^{-p(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) \\
&= I_{\overline{V}}(tu_{\overline{V}}) + t_1^2 O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) - t_0^2 C e^{-\gamma R_n} + t_1^2 O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) + t_1^p O(e^{-p(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) \\
&\leq c_{\overline{V}} + O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) - t_0^2 C e^{-\gamma R_n}.
\end{aligned}
\tag{4.5}
$$

Choosing $\varepsilon \in (0, 1 - \gamma/(2\overline{V}^{\frac{1}{2}}))$ we can find D such that the conclusion holds.

□

4.9 Corollary *In cases (ii) and (iii) of (V1) we have $c_V < c_{\overline{V}}$.*

Proof. If (ii) happens Lemma 4.5 implies that $c_V < c_{\overline{V}}$ because $I_{\overline{V}}$ has a ground state. If (iii) happens, the result follows from Lemma 4.6 and Proposition 4.8.

□

4.10 Lemma *Assume (ii) or (iii) of (V1). Let $V^* \in \mathcal{B}_{\mathcal{Q}}$. If u^* is a nontrivial critical point of I_{V^*} then $c_V < I_{V^*}(u^*)$ holds.*

Proof. By the definition of $\mathcal{B}_Q$ we have $\overline{V} \leq V^*$, so Lemma 4.5 implies $c_{\overline{V}} \leq I_{V^*}(u^*)$. To conclude we use Corollary 4.9.

□

4.11 Theorem *There exists a ground state $\hat{u}$ for I_V.*

Proof. In case (i) of (A1) we have that $u_{\overline{V}}$ is the ground state of I_V, and there is nothing else to do. Assume we are in the case (ii) or (iii). By Lemma 2.10 there exists a $(PS)_{c_v}$-sequence (u_n) for I_V and we are in a position to use Lemma 3.10. It cannot happen that $u_n \to 0$ since $I_V(u_n) \to c_V > 0$ (2.7). Therefore, there exist $k \in \mathbb{N}$, functions $V^i \in \overline{\mathcal{A}}_Q$ and nontrivial critical points u^i of I_{V^i} such that

$$\sum_{i=1}^{k} I_{V_i}(u^i) \leq c_V. \tag{4.6}$$

Since the functions u^i are nontrivial critical points of I_{V^i} we have

$$I_{V^i}(u^i) = I_{V^i}(u^i) - \frac{1}{p} DI_{V^i}(u^i)(u^i) = \left(\frac{1}{2} - \frac{1}{p}\right) \left\| u^i \right\| > 0.$$

By Lemma 4.10 each V^i belongs to $\mathcal{A}_Q$ and is a translate of V. Hence $I_{V^i}(u^i) \geq c_V$ for every i. This implies that $k = 1$ and that a translate of u^1 is a ground state of I_V.

□

4.3 Second General Situation

Now we assume $V = 1$, and denote $I_a := I_{1,a}$.

Denote by $a^{\pm} := \max\{0, \pm a\}$ the positive and negative parts of a. We introduce the following condition:

(A1) Either (i) $\overline{a} \leq 0$ or (ii) $a \geq \overline{a}$ or (iii) there exist $\gamma \in (0, 2)$, sequences $(z_n) \subset \mathbb{R}^N$ and $(R_n) \subset \mathbb{R}$, and a non-negative measurable function κ on $\mathbb{R}^N$ with $\left|\{x \in \mathbb{R}^N \mid \kappa(x) > 0\}\right| > 0$ such that $R_n \to \infty$, w*-$\lim_{n\to\infty} \tau_{-z_n} a = \overline{a}$ and $a(x) \geq \overline{a} + \kappa(x - z_n) e^{-\gamma R_n}$ holds true for all n and $x \in B_{R_n}(z_n)$.

(A2) $a^+ \neq 0$ as an element of L^∞.

We are going to assume that a satisfies the previous conditions.

The outline of this section is similar to that of the last one, but we need to use other steps to adjust for the new hypotheses.

4.12 Proposition *In case (iii) of (A) there is $K \in \mathbb{N}$ and $D > 0$ such that*

$$I_a(t(\tau_{z_n} u_{\overline{a}}^{R_n})) \leq c_{\overline{a}} - De^{-\gamma R_n}$$

for all $n \geq K, t \geq 0$.

Proof. Using Lemma 4.4, we can choose $K' \in \mathbb{N}$ and $t_1 > t_0 > 0$ such that

$$I_a(t(\tau_{z_n} u_{\overline{a}}^{R_n})) \leq \frac{c_{\overline{a}}}{2} \tag{4.7}$$

for all $n \geq K', t \in [0, t_0] \cup [t_1, \infty)$.
Passing to a subsequence we may assume that (R_n) in an increasing sequence. Using (A) and the positivity of $u_{\overline{a}}$ choose $n_1 \geq n_0$ such that

$$C := \int_{\|x\| \leq (1-\varepsilon) R_{n_1}} \kappa u_{\overline{a}}^p > 0.$$

From Lemma 4.2 we have, for $n \geq n_1$,

$$\begin{aligned}
\int_{\mathbb{R}^N} a(x) \left|(\tau_{z_n} u_{\overline{a}}^{R_n})\right|^p dx &\geq \int_{\|x\| \leq R_n} (\tau_{-z_n})\, a \left|u_{\overline{a}}^{R_n}\right|^p dx \\
&\geq \int_{\|x\| \leq R_n} \left(\overline{a} + \kappa e^{-\gamma R_n}\right) \left|u_{\overline{a}}^{R_n}\right|^p dx \\
&\geq \overline{a} \int_{\|x\| \leq (1-\varepsilon) R_n} |u_{\overline{a}}|^p \, dx \\
&\quad + e^{-\gamma R_n} \int_{\|x\| \leq (1-\varepsilon) R_{n_1}} \kappa \left|u_{\overline{a}}\right|^p dx \\
&\geq \overline{a} \int_{\mathbb{R}^N} |u_{\overline{a}}|^p \, dx + O(e^{-p(1-\varepsilon)R_n}) \\
&\quad + e^{-\gamma R_n} \int_{\|x\| \leq (1-\varepsilon) R_{n_1}} \kappa \left|u_{\overline{a}}\right|^p dx \\
&\geq \overline{a} \int_{\mathbb{R}^N} |u_{\overline{a}}|^p \, dx + O(e^{-p(1-\varepsilon)R_n}) \\
&\quad + C e^{-\gamma R_n}.
\end{aligned} \tag{4.8}$$

Using this and Lemma 4.2 again, this implies for $t \in [t_0, t_1]$ and $n \geq n_1$,

$$\begin{aligned}
&I_a(t(\tau_{z_n} u_{\overline{a}}^{R_n})) \\
&= t^2 \int \left|\nabla(\tau_{z_n} u_{\overline{a}}^{R_n})\right|^2 + t^2 \int (\tau_{z_n} u_{\overline{a}}^{R_n})^2 - t^p \int a(x) \left|(\tau_{z_n} u_{\overline{a}}^{R_n})\right|^p \\
&\leq t^2 \int |\nabla u_{\overline{a}}|^2 + t^2 O(e^{-2(1-\varepsilon)R_n}) + t^2 \int u_{\overline{a}}^2 - t^p \int a(x) \left|(\tau_{z_n} u_{\overline{a}}^{R_n})\right|^p \\
&\leq I_{\overline{a}}(t u_{\overline{a}}) + t_1^2 O(e^{-2(1-\varepsilon)R_n}) + t_1^p O(e^{-p(1-\varepsilon)R_n}) - t_0^p C e^{-\gamma R_n} \\
&\leq c_{\overline{a}} + O(e^{-2(1-\varepsilon)R_n}) - t_0^p C e^{-\gamma R_n}.
\end{aligned} \tag{4.9}$$

For our election of ε we can deduce the existence of D such that the conclusion holds. $\square$

4.13 Corollary *We have $c_a < c_{\overline{a}}$.*

Proof. We consider the three cases from condition (A) separately. If (i) $\overline{a} \leq 0$ then $c_{\overline{a}} = \infty$. Since $a^+ \neq 0$, $c_a < \infty$ and there is nothing to prove. Therefore we may assume for the remaining cases that $\overline{a} > 0$. If (ii) $a \geq \overline{a}$ then $a \neq \overline{a}$ since $\mathcal{B}_{\mathcal{P}} \neq \varnothing$. Moreover, Lemma 4.5 implies that $c_a < c_{\overline{a}}$ because $I_{\overline{a}}$ has a ground state. The remaining case follows from Lemma 4.6 and Proposition 4.12.

□

4.14 Lemma *Let $a^* \in \mathcal{B}_{\mathcal{P}}$. If u^* is a nontrivial critical point of I_{a^*} then $c_a < I_{a^*}(u^*)$ holds.*

Proof. By the definition of $\mathcal{B}_{\mathcal{P}}$ we have $\overline{a} \geq a^*$, so Lemma 4.5 implies $c_{\overline{a}} \leq I_{a^*}(u)$. To conclude we use Corollary 4.13.

□

4.15 Theorem *There exists a ground state $\hat{u}$ for I_a.*

Proof. By Lemma 2.10 there exists a $(PS)_{c_v}$-sequence (u_n) for I_a we are in a position to use Lemma 3.10. It cannot happen that $u_n \to 0$ since $I_a(u_n) \to c_a > 0$. Therefore, there exist $k \in \mathbb{N}$, functions $a^i \in \overline{\mathcal{A}}_{\mathcal{P}}$ and nontrivial critical points u^i of I_{a^i} such that

$$\sum_{k=1}^{k} I_{a_i}(u^i) \leq c_a. \tag{4.10}$$

Since the functions u^i are nontrivial critical points of I_{a^i} we have

$$I_{a^i}(u^i) = I_{a^i}(u^i) - \frac{1}{p} DI_{a^i}(u^i)(u^i) = \left(\frac{1}{2} - \frac{1}{p}\right) \|u^i\| > 0.$$

By Lemma 4.14 each a^i belongs to $\mathcal{A}_{\mathcal{P}}$ and is a translate of a. Hence $I_{a^i}(u^i) \geq c_a$ for every i. This implies that $k = 1$ and that a translate of u^1 is a ground state of I_a.

□

4.4 Third General Situation

The different cases of conditions $(V1)$ and $(A1)$ can be combined to obtain existence of ground states. In this section we concentrate in one of these combination of hypotheses. The rest of the combinations can be proved with ideas used in this section.
We assume the next hypotheses:

(AV1) There exist $\gamma \in (0, 2\overline{V}^{\frac{1}{2}})$, sequences $(z_n) \subset \mathbb{R}^N$ and $(R_n) \subset \mathbb{R}$, and a non-negative measurable function κ_1 on $\mathbb{R}^N$ with $\left|\{x \in \mathbb{R}^N \mid \kappa_1(x) > 0\}\right| > 0$ such that $R_n \to \infty$, w*-$\lim_{n\to\infty} \tau_{-z_n} V = \overline{V}$ and $V(x) \leq \overline{V} - \kappa_1(x - z_n)e^{-\gamma R_n}$ holds true for all n and $x \in B_{R_n}(z_n)$.

(AV2) $\inf V > 0$.

(AV3) There exist a non-negative measurable function κ_2 on $\mathbb{R}^N$ with $\left|\{x \in \mathbb{R}^N \mid \kappa_2(x) > 0\}\right| > 0$ such that w*-$\lim_{n\to\infty} \tau_{-z_n} a = \overline{a}$ and $a(x) \geq \overline{a} + \kappa_2(x - z_n)e^{-\gamma R_n}$ holds true for all n and $x \in B_{R_n}(z_n)$.

(AV4) $a^+ \neq 0$ as an element of L^∞.

(AV5) For every $(y_n)_n$ we either have, after passing to a subsequence, that (w*-$\lim_{n\to\infty} \tau_{y_n} V$, w*-$\lim_{n\to\infty} \tau_{\text{}}$ $\mathcal{B}_\mathcal{Q}\times\mathcal{B}_\mathcal{P}$ or there is $\xi \in \mathbb{R}^N$ such that w*-$\lim_{n\to\infty} \tau_{y_n} V = \tau_\xi V$ and w*-$\lim_{n\to\infty} \tau_{y_n} a = \tau_\xi a$.

4.16 Proposition *There is $K \in \mathbb{N}$ and $D > 0$ such that*

$$I_{V,a}(t(\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}})) \leq c_{\overline{V},\overline{a}} - De^{-\gamma R_n}$$

for all $n \geq K, t \geq 0$.

Proof. Just as in propositions 4.8 and 4.12 we have for $n \geq n_1$

$$\int_{\mathbb{R}^N} V(x) \left|\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}}\right|^2 dx \leq \int_{\mathbb{R}^N} \overline{V} \left|u_{\overline{V},\overline{a}}\right|^2 - C_1 e^{-\gamma R_n} + O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) \tag{4.11}$$

and

$$\int_{\mathbb{R}^N} a(x) \left|(\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}})\right|^p dx \geq \overline{a} \int_{\mathbb{R}^N} \left|u_{\overline{V},\overline{a}}\right|^p dx + O(e^{-p(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + C_2 e^{-\gamma R_n}. \tag{4.12}$$

From (4.11),(4.12) and Lemma 4.2 it follows, for $t \in [t_0, t_1]$ and $n \geq n_1$,

$$\begin{aligned}
&I_{V,a}(t(\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}})) \\
&= \int \left|\nabla t(\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}})\right|^2 + \int V(x) t^2 (\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}})^2 - \int a(x) \left|t(\tau_{z_n} u^{R_n}_{\overline{V},\overline{a}})\right|^p \\
&\leq t^2 \int \left|\nabla u_{\overline{V},\overline{a}}\right|^2 + O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) + t^2 \int_{\mathbb{R}^N} \overline{V} \left|u_{\overline{V},\overline{a}}\right|^2 - C_1 e^{-\gamma R_n} + O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) \\
&- t^p \overline{a} \int_{\mathbb{R}^N} \left|u_{\overline{V},\overline{a}}\right|^p dx - O(e^{-p(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) - C_2 e^{-\gamma R_n} \\
&= I_{\overline{V},\overline{a}}(t u_{\overline{V},\overline{a}}) + O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) - C_1 e^{-\gamma \overline{V}^{\frac{1}{2}} R_n} + O(e^{-2\overline{V}^{\frac{1}{2}} R_n}) - O(e^{-p(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) - C_2 e^{-\gamma R_n} \\
&= I_{\overline{V^*}}(t u_{\overline{V^*}}) + O(e^{-2(1-\varepsilon)\overline{V}^{\frac{1}{2}} R_n}) - (C_1 + C_2) e^{-\gamma R_n}
\end{aligned} \tag{4.13}$$

Choosing $\varepsilon \in (0, 1 - \gamma/2\overline{V}^{\frac{1}{2}})$ we can conclude the result.

□

4.17 Corollary *We have $c_{V,a} < c_{\overline{V},\overline{a}}$.*

Proof. This follows from Lemma 4.6 and Proposition 4.16.

□

4.18 Lemma *Let $a^* \in \mathcal{B}_\mathcal{P}$ and $V^* \in \mathcal{B}_\mathcal{Q}$. If u^* is a nontrivial critical point of I_{V^*,a^*} then $c_{V,a} < I_{V^*,a^*}(u^*)$ holds.*

Proof. By the definition of $\mathcal{B}_\mathcal{P}$ and $\mathcal{B}_\mathcal{Q}$ we have $\overline{a} \geq a^*$ and $\overline{V} \leq V^*$, so Lemma 4.5 implies $c_{\overline{V},\overline{a}} \leq I_{V^*,a^*}(u)$. To conclude we use Corollary 4.17.

□

4.19 Theorem *There exists a ground state $\hat{u}$ for $I_{V,a}$.*

Proof. Again, we use Lemma 2.10 followed by Lemma 3.10. It cannot happen that $u_n \to 0$ since $I_{V,a}(u_n) \to c_{V,a} > 0$. Therefore, there exist $k \in \mathbb{N}$, functions $a^i \in \overline{\mathcal{A}}_\mathcal{P}, V^i \in \overline{\mathcal{A}}_\mathcal{Q}$ and nontrivial critical points u^i of I_{V^i,a^i} such that

$$\sum_{i=1}^{k} I_{V^i,a^i}(u^i) \leq c_{V,a}. \tag{4.14}$$

Since the functions u^i are nontrivial critical points of I_{V^i,a^i} we have $I_{V^i,a^i}(u^i) \geq c_{V^i,a^i} > 0$. By Lemma 4.18 and hypothesis **(AV5)** for each i there is ξ_i such that $a^i = \tau_{\xi_i} a$ and $V^i = \tau_{\xi_i} V$. Hence $I_{V^i,a^i}(u^i) \geq c_{V,a}$ for every i. This implies that $k = 1$ and that a translate of u^1 is a ground state of $I_{V,a}$.

□

5 Examples

5.1 Tools for the Construction of Examples

For $a \in L^\infty$ denote
$$\hat{a} := \lim_{R\to\infty} \operatorname{ess\,sup} a|_{\mathbb{R}^N \setminus B_R}.$$

5.1 Lemma *If* ***(A1)*** *is satisfied with $\overline{a}$ replaced by $\hat{a}$ then (the original condition)* ***(A1)*** *is satisfied.*

Proof. If $\mathcal{B}_\mathcal{P} = \varnothing$ then $\overline{a} = -\infty$ and (A) is satisfied. Assume therefore that $\mathcal{B}_\mathcal{P} \neq \varnothing$. In particular, a is not constant. We claim that
$$\overline{a} \leq \hat{a}. \tag{5.1}$$
To see this, suppose that $b \in \mathcal{B}_\mathcal{P}$. There is $(x_n) \subset \mathbb{R}^N$ such that $\tau_{x_n} a \overset{w^*}{\to} b$. If (x_n) contained a bounded subsequence then, after passing to a subsequence, there would exist $\xi \in \mathbb{R}^N$ such that $x_n \to \xi$ and $\tau_{x_n} a \overset{w^*}{\to} \tau_\xi a \in \mathcal{A}_\mathcal{P}$. This follows from weak*-continuity of translation in L^∞, which in turn is a consequence of continuity of translation on L^1. On the other hand, since $\mathcal{P}$ is metrizable, $\tau_\xi a = b \notin \mathcal{A}_\mathcal{P}$, a contradiction. Therefore $|x_n| \to \infty$.
Given $\epsilon > 0$, by Lemma 2.8 there is $\varphi \in S_1 L^1$ such that $\varphi \geq 0$ and
$$\int_{\mathbb{R}^N} b\varphi \geq \operatorname{ess\,sup} b - \frac{\varepsilon}{2}.$$
Take $\tilde{\psi} \in C_c^\infty$ such that $\tilde{\psi} \geq 0$ and
$$\left|\varphi - \tilde{\psi}\right|_1 \leq \frac{\varepsilon}{4\,|b|_\infty}.$$
Set $\psi := \tilde{\psi} / \left|\tilde{\psi}\right|_1$. Observe that
$$\left|\tilde{\psi} - \psi\right|_1 = |\psi|_1 \left|\left|\tilde{\psi}\right|_1 - 1\right|_1 = \left|\left|\tilde{\psi}\right|_1 - 1\right|_1 = \left|\left|\tilde{\psi}\right|_1 - |\varphi|_1\right|_1 \leq \left|\varphi - \tilde{\psi}\right|_1,$$
so
$$|\varphi - \psi|_1 \leq \frac{\varepsilon}{2\,|b|_\infty}$$
and $\psi \in S_1 L^1 \cap C_c^\infty$ satisfies $\psi \geq 0$. We obtain
$$\int_{\mathbb{R}^N} b\psi = \int_{\mathbb{R}^N} b\varphi - \int_{\mathbb{R}^N} b(\varphi - \psi) \geq \int_{\mathbb{R}^N} b\varphi - |b|_\infty\, |\psi - \varphi|_1 \geq \operatorname{ess\,sup} b - \varepsilon.$$

Suppose that $\operatorname{supp}\psi \subset B_R$. Then

$$\begin{aligned}
\operatorname{ess\,sup} b - \varepsilon &\leq \int_{\mathbb{R}^N} b\psi \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} (\tau_{x_n} a)\psi \\
&\leq \lim_{n\to\infty} \operatorname{ess\,sup} \left(a|_{B_R(-x_n)}\right) \int_{B_R} \psi \\
&\leq \lim_{n\to\infty} \operatorname{ess\,sup} \left(a|_{\mathbb{R}^N \setminus B_{|x_n|-R}}\right) \\
&= \hat{a}.
\end{aligned}$$

Letting $\varepsilon \to 0$ this yields $\operatorname{ess\,sup} b \leq \hat{a}$ and hence (5.1).
We now consider the three subcases of (A1), under the assumption $\mathcal{B}_{\mathcal{P}} \neq \varnothing$. In case (i) we obtain from (5.1) that $\overline{a} \leq \hat{a} \leq 0$. In case (ii) (5.1) implies $\overline{a} \leq \hat{a} \leq a$. And in case (iii) there is a sequence $(z_n) \subset \mathbb{R}^N$ such that $\tau_{-z_n} a \overset{w^*}{\to} \hat{a}$, i.e., $\hat{a} \in \overline{\mathcal{A}}_{\mathcal{P}}$. Since a is not constant, $\hat{a} \in \mathcal{B}_{\mathcal{P}}$. Therefore $\hat{a} \leq \overline{a}$, which implies together with (5.1) that $\hat{a} = \overline{a}$. In conclusion, the original condition (A1) is satisfied, with $\overline{a}$ instead of $\hat{a}$.

□

5.2 Lemma *Suppose that $k, l \in \mathbb{N}$ and that (A1) is satisfied for $a \in L^\infty(\mathbb{R}^k)$. Define $a' : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}$ by $a'(x,y) := a(x)$. Then (A1) is satisfied for $a' \in L^\infty(\mathbb{R}^k \times \mathbb{R}^l)$.*

Proof. Define the linear operator $\Gamma : \mathbb{R}^{\mathbb{R}^k} \to \mathbb{R}^{\mathbb{R}^k \times \mathbb{R}^l}$ by $(\Gamma u)(x,y) := u(x)$. Suppose that $a \in L^\infty(\mathbb{R}^k)$ and set $a' := \Gamma a$. Moreover, define by $\mathcal{A}'_{\mathcal{P}}$ and $\mathcal{B}'_{\mathcal{P}}$ the corresponding sets for a', and define $\overline{a}'$ correspondingly.
We claim that

$$\Gamma \text{ restricts to a bijection } \mathcal{B}_{\mathcal{P}} \to \mathcal{B}'_{\mathcal{P}}. \tag{5.2}$$

Clearly Γ is injective. We show first that $b' := \Gamma b \in \mathcal{B}'_{\mathcal{P}}$ if $b \in \mathcal{B}_{\mathcal{P}}$. There is $(x_n) \subset \mathbb{R}^k$ such that $\tau_{x_n} a \overset{w^*}{\to} b$. Since $b \notin \mathcal{A}_{\mathcal{P}}, |x_n| \to \infty$ (see the proof of Lemma 5.1). For any $\varphi \in L^1(\mathbb{R}^{k+l})$ the function

$$x \mapsto \int_{\mathbb{R}^l} \varphi(x,y)dy$$

is in $L^1(\mathbb{R}^k)$, by Fubini's Theorem. We obtain

$$\begin{aligned}
\int_{\mathbb{R}^{k+l}} (\tau_{(x_n,0)} a')\varphi &= \int_{\mathbb{R}^{k+l}} (\tau_{x_n} a)(x)\varphi(x,y)d(x,y) \\
&= \int_{\mathbb{R}^k} (\tau_{x_n} a)(x) \int_{\mathbb{R}^l} \varphi(x,y)dydx \\
&\to \int_{\mathbb{R}^k} b(x) \int_{\mathbb{R}^l} \varphi(x,y)dydx \\
&= \int_{\mathbb{R}^{k+l}} b'\varphi.
\end{aligned} \tag{5.3}$$

Hence $\tau_{(x_n,0)}a' \overset{w^*}{\to} b'$ and $b' \in \overline{\mathcal{A}'_\mathcal{P}}$. If $b' \in \mathcal{A}'_\mathcal{P}$ were true, there would exist $(x_0, y_0) \in \mathbb{R}^{k+l}$ such that $\tau_{(x_0,y_0)}b' = a'$. For all $(x, y) \in \mathbb{R}^{k+l}$ this would imply

$$b(x - x_0) = b'(x - x_0, y - y_0) = a'(x, y) = a(x)$$

and therefore $\tau_{x_0}b = a$. But this would contradict $b \notin \mathcal{A}_\mathcal{P}$. Therefore $b' \in \mathcal{B}'_\mathcal{P}$. We have shown that $\Gamma(\mathcal{B}_\mathcal{P}) \subset \mathcal{B}'_\mathcal{P}$.
To show surjectivity, suppose that $b' \in \mathcal{B}'_\mathcal{P}$. There is a sequence $((x_n, y_n)) \subset \mathbb{R}^{k+l}$ such that $\tau_{(x_n,y_n)}a' \overset{w^*}{\to} b'$. As before, $|(x_n, y_n)| \to \infty$ since $b' \notin \mathcal{A}'_\mathcal{P}$. Moreover, since $a'(x, y) = a(x)$ for all (x, y), we can assume that $y_n = 0$ for all n, so $\tau_{(x_n,0)}a' \overset{w^*}{\to} b'$ and $|x_n| \to \infty$.
Suppose that $(x^*, y), (x^*, y') \in \mathbb{R}^{k+l}$ are Lebesgue points of b'. We have

$$\tau_{(0,y-y')}b' = \mathrm{w}^* - \lim_{n\to\infty} \tau_{(x_n,y-y')}a' = \mathrm{w}^* - \lim_{n\to\infty} \tau_{(x_n,0)}a' = b'.$$

Therefore

$$\begin{aligned} b'(x^*, y) &= \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r((x^*,y))} b' \\ &= \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r((x^*,y))} \tau_{(0,y-y')}b' \\ &= \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r((x^*,y'))} b' \\ &= b'(x^*, y'). \end{aligned}$$

Since the complement of the set of Lebesgue points of b' has zero measure, this shows that b' is independent of the second argument and that there is $b \in L^\infty$ such that $b' = \Gamma b$.
We need to show that $b \in \mathcal{B}_\mathcal{P}$. Suppose that $\varphi \in L^1(\mathbb{R}^k)$. Pick any $\psi \in L^1(\mathbb{R}^l)$ such that $\int \psi = 1$. Then the map ϑ given by $\vartheta(x, y) := \varphi(x)\psi(y)$ is in $L^1(\mathbb{R}^{k+l})$, by Tonelli's Theorem. Moreover

$$\begin{aligned} \int_{\mathbb{R}^k} (\tau_{x_n}a)\varphi &= \int_{\mathbb{R}^k} (\tau_{x_n}a)(x) \int_{\mathbb{R}^l} \varphi(x)\psi(y)dydx \\ &= \int_{\mathbb{R}^{k+l}} (\tau_{(x_n,0)}a')\vartheta \\ &\to \int_{\mathbb{R}^{k+l}} b'\vartheta \\ &= \int_{\mathbb{R}^k} b\varphi. \end{aligned}$$

Hence $b \in \overline{\mathcal{A}_\mathcal{P}}$. If b belonged to $\mathcal{A}_\mathcal{P}$ then clearly b' would belong to $\mathcal{A}'_\mathcal{P}$, a contradiction. Therefore $b \in \mathcal{B}_\mathcal{P}$ and we have proved (5.2).
We now consider a number of cases. If $\mathcal{B}_\mathcal{P} = \varnothing$ then $\mathcal{B}'_\mathcal{P} = \Gamma(B) = \varnothing$ and $\overline{a}' = -\infty \le 0$, that is, a' satisfies (A1). We therefore assume now that $\mathcal{B}_\mathcal{P} \neq \varnothing$. It follows that

$$\overline{a}' = \sup_{b'\in\mathcal{B}'_\mathcal{P}} \operatorname{ess\,sup} b' = \sup_{b\in\mathcal{B}_\mathcal{P}} \operatorname{ess\,sup} \Gamma b = \sup_{b\in\mathcal{B}_\mathcal{P}} \operatorname{ess\,sup} b = \overline{a}.$$

In case (A1)(i) we obtain $\overline{a}' = \overline{a} \le 0$. In case (A1)(ii) we have $\overline{a} \le a$. This implies $\overline{a}' = \overline{a} \le a'$. And in case (A1)(iii) there are sequences $(z_n) \subset \mathbb{R}^k$ and $R_n \to \infty$ such that $\tau_{-z_n} a \overset{w^*}{\to} \overline{a}$ and

$$\forall x \in B_{R_n}(z_n) : a(x) \ge \overline{a} + \kappa(x - z_n)e^{-\gamma R_n}.$$

Put $z_n' := (z_n, 0)$ and define $k' := \Gamma k$. As in (5.3) $\tau_{-z_n'} a' \overset{w^*}{\to} \overline{a}'$. If $(x, y) \in B^{k+l}_{R_n}(z_n')$ then $x \in B^k_{R_n}(z_n)$ and hence

$$a'(x, y) = a(x) \ge \overline{a} + \kappa(x - z_n)e^{-\gamma R_n} = \overline{a}' + k'((x, y) - z_n')e^{-\gamma R_n}.$$

In all cases a' satisfies (A1). $\square$

5.2 Example 1

For the first example we consider this situation:

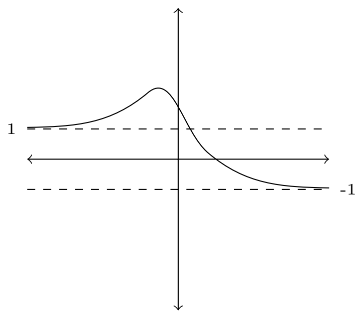
(B1) Let $A \in L^\infty$ be a function constant in the last $N-1$ coordinates, in other words, for all $y_1, y_2 \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$ we have $A(t, y_1) = A(t, y_2)$.

Define the function $a(\cdot) := A(\cdot, 0, \dots, 0)$, so we have $a(\cdot) = A(\cdot, y)$ for all $y \in \mathbb{R}^{N-1}$. Assume a has the next properties:

(B2) $\lim_{t \to -\infty} a(t) > 0, \lim_{t \to \infty} a(t) < 0$.

(B3) There are $\alpha \in (0, 2)$, $S_0 < 0$ and $M > 0$ such that $a(t) \ge (\lim_{s \to -\infty} a(s)) + Me^{-\alpha|t|}$ for all $t < S_0$.

The figure represents the behaviour of the function a in this example.



We have the next result:

5.3 Proposition *If A satisfies hypotheses **(B1)** - **(B3)** then I_A has a ground state.*

Proof. First, notice that $\lim_{R\to\infty} \operatorname{ess\,sup} a_{|\mathbb{R}^N\setminus B_R(0)} = \lim_{s\to-\infty} a(s)$. Define $z_n = S_0 - n$, $R_n = n$ and $\kappa(t) = Me^{\alpha(S_0+t)}$ for all $t \in \mathbb{R}$. For all n and $t \in B_{R_n}(z_n)$ we have $t < S_0$, so $\kappa(t-z_n)e^{-\alpha R_n} = Me^{\alpha(S_0+t-S_0+n)-\alpha n} = Me^{-\alpha|t|}$, so we can apply Lemma 5.1 to see that a satisfies **(A1)** and **(A2)**. Using now Lemma 5.2 and Theorem 4.15 we prove that I_A has a ground state.

□

5.3 Example 2

In this example most of the hypotheses need to be satisfied only in a cone. Denote by S^{N-1} the unit sphere in $\mathbb{R}^N$. Suppose that there exist $a_0 > 0, z_0 \in S^{N-1}, R_0, R_1 > 0,\alpha$ with $0 < \alpha < 2$ such that, setting

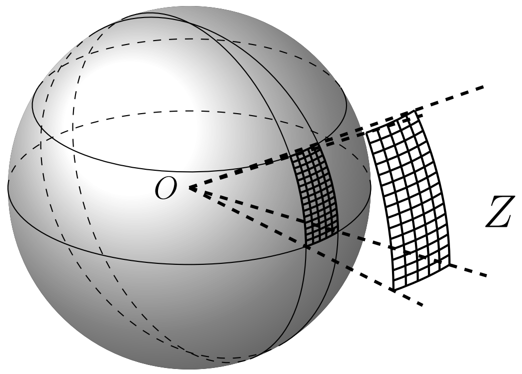
$$Z = \{tx \in \mathbb{R}^N \mid x \in B_{R_0}(z_0), t > 0\}$$

it holds true that $\lim_{x\to\infty, x\in Z} a = a_0$, and

$$\forall x \in Z \setminus \overline{B}_{R_1} : a(x) \geq a_0 + e^{-\alpha\|x\|}.$$

We also ask $R_0 > \frac{\alpha}{2}$ and $\operatorname{ess\,sup}(a|_{\mathbb{R}^N\setminus Z}) < a_0$.

The figure shows how the domain Z could look in $\mathbb{R}^3$:



5.4 Proposition *Under the hypotheses above I_a has a ground state.*

Proof. We have $a^+ \neq 0$ and $\lim_{R\to\infty} \operatorname{ess\,sup} a_{|\mathbb{R}^N\setminus B_R(0)} = a_0$. We are going to prove that a satisfies **(A1)**.
By the definition of Z we can construct sequences (z_n) and (R_n) such that $R_n \to \infty$, $\frac{R_n}{\|z_n\|} = R_0$ and $B_{R_n}(z_n) \subset Z \setminus \overline{B}_{R_1}$. Define $\kappa(v) = e^{-\alpha\|v\|}$ and $\gamma = \alpha/R_0 < 2$. We have, for all $n \in \mathbb{N}$ and $x \in B_{R_n}(z_n)$: $\kappa(x-z_n)e^{-\gamma R_n} = e^{-\alpha\|x-z_n\|}e^{-\alpha\|z_n\|} \leq e^{-\alpha\|x\|}$, and from here we obtain $a(x) \geq a_0 + \kappa(x-z_n)e^{-\gamma R_n}$. So Lemma 5.1 assures that **(A1)** is satisfied, and by Theorem 4.15 I_a has a ground state.

□

5.4 Example 3

In the next example a is radially symmetric, so there are already results about the existence of a radial ground state. However, the radial ground state is not necessarily a ground state, so Theorem 4.15 gives a better result.

Denote $Z_0 = \{x \in \mathbb{R}^N \mid 0 < \|x\| < 1\}$ and, for all $n \in \mathbb{N}$, the set $Z_i = \{x \in \mathbb{R}^N \mid 2^{i-1} < \|x\| < 2^i\}$. Define $a = -1$ in Z_i for i odd, and $a = 1 + e^{-2^{i-2}\gamma}$, with $0 < \gamma < 2$, for i even.

5.5 Proposition *I_a has a ground state.*

Proof. We define $z_n = (2^{2(n-1)} \cdot 3, 0, \ldots, 0)$ and $R_n = 2^{2(n-1)}$. It can be proved that $B_{R_n}(z_n) \subset Z_{2n}$, and then we have $a(x) = 1 + e^{-2^{2n-2}\gamma} = 1 + e^{-R_n\gamma}$, so the result follows from Lemma 5.1 and Theorem 4.15.

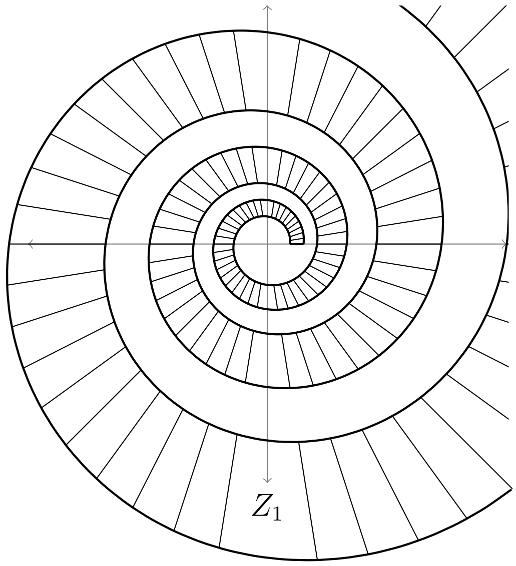
$\square$

5.5 Example 4

In this last example the function a is defined similar to the previous one, but this time instead of annular domains we consider logarithmic spirals, and a will not be a radial function. In order to make more clear the shape of the next subset of $\mathbb{R}^2$, we first define it using polar coordinates:

$$Z_1 := \left\{ \left(te^\theta + (1-t)\left(\frac{e^\theta + e^{\theta+2\pi}}{2}\right), \theta \right) \mid \theta > 0, 0 < t < 1 \right\}.$$

From the definition we can see that, in some sense, Z_1 is the open set delimited by two spirals, one given by $r = e^\theta$, and the other by $r = \frac{e^\theta + e^{\theta+2\pi}}{2}$. The elements of Z_1 are contained in line segments whose extremal points are elements of the spirals.



In cartesian coordinates we have

$$Z_1 = \left\{ \left(\cos\theta \left[te^{\theta} + (1-t)\left(\frac{e^{\theta}+e^{\theta+2\pi}}{2}\right)\right], \sin\theta \left[te^{\theta} + (1-t)\left(\frac{e^{\theta}+e^{\theta+2\pi}}{2}\right)\right]\right) \mid \theta > 0, 0 < t < 1 \right\}.$$

We will use the auxiliar function $f : \mathbb{R}^+ \times (0,1) \to f(\mathbb{R}^+ \times (0,1)) \subset \mathbb{R}^2$ defined as

$$f(t,\theta) = \left(\cos\theta \left[te^{\theta} + (1-t)\left(\frac{e^{\theta}+e^{\theta+2\pi}}{2}\right)\right], \sin\theta \left[te^{\theta} + (1-t)\left(\frac{e^{\theta}+e^{\theta+2\pi}}{2}\right)\right]\right).$$

Observe that $f(\mathbb{R}^+ \times (0,1)) = Z_1$. This function is a homeomorphism: the continuity and surjectivity are clear from the definitions. To prove the injectivity we will use polar coordinates. Assume that $f(\theta_1, t_1) = f(\theta_2, t_2)$. This implies that $\theta_1 = \theta_2 + 2m\pi$ for some $m \in \mathbb{Z}$. Without loss of generality we can assume $m \geq 0$. Now observe that, if $m > 0$, we will have

$$\begin{aligned}
\|f(\theta_2, t_2)\| &= t_2 e_2^{\theta} + (1-t_2)\left(\frac{e_2^{\theta}+e^{\theta_2+2\pi}}{2}\right) \\
&\leq \frac{e_2^{\theta}+e^{\theta_2+2\pi}}{2} \\
&< e^{\theta_2+2\pi} \\
&\leq e^{\theta_2+2m\pi} \\
&\leq \|f(\theta_1, t_1)\|,
\end{aligned}$$

which is a contradiction. We deduce $\theta_1 = \theta_2$, and from here follows $t_1 = t_2$, so f is injective. To prove that f is an open map consider the base of $\mathbb{R}^+ \times (0,1)$ whose elements are the sets of the form $(\theta_1, \theta_2) \times (t_1, t_2)$ with $(0 < \theta_1 < \theta_2, 0 < t_1 < t_2 < 1$. We use again polar coordinates (only to make clearer the shape of the set) to write

$$f((\theta_1, \theta_2) \times (t_1, t_2)) = \left\{ \left(te^{\theta} + (1-t)\left(\frac{e^{\theta} + e^{\theta+2\pi}}{2}\right), \theta \right) \mid \theta_1 < \theta < \theta_2, t_1 < t < t_2 \right\},$$

which is an open set. We conclude that f is a homeomorphism.
Now define

$$\begin{aligned}
z_n &:= \left(e^{(2n-1)\pi} + \left(\frac{e^{(2n+1)\pi} - e^{(2n-1)\pi}}{4} \right), \pi \right) \\
&= \left(\frac{1}{2} e^{(2n-1)\pi} + \frac{1}{2} \left(\frac{e^{(2n-1)\pi} + e^{(2n+1)\pi}}{2} \right), (2n-1)\pi \right) \\
&\in f\left(\left(\left(2n - \frac{3}{2}\right)\pi, \left(2n - \frac{1}{2}\right)\pi \right) \times (0,1) \right),
\end{aligned}$$

for $n \in \mathbb{N}$. The last set is open so we can find R_1 such that

$$B_{R_1}(z_1) \subset f\left(\left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \times (0,1) \right).$$

To simplify the exposition we are not going to find explicitly R_1.
Since f is invertible, we can define functions $g_1 : \mathbb{R}^2 \to \mathbb{R}^+$,$g_2 : \mathbb{R}^2 \to (0,1)$ such that $f^{-1}(z) = (g_1(z), g_2(z))$ for all $z \in \mathbb{R}^2$ (this functions will be the projection to the first and second coordinate, respectively, of $f^{-1}(z)$).
We are now able to define a. Set $a(z) := 1 + e^{-(e^{g_1(z)} R_1)}$ if $z \in Z_1$, and -1 otherwise. We will prove that I_a has a ground state.
Let $R > 0$, and $z \in Z_1 \bigcap (\mathbb{R}^2 \setminus B_R(0))$. We have, for $\theta = g_1(z), t = g_2(z)$

$$\begin{aligned}
R < \|z\| &= te^{\theta} + (1-t)\left(\frac{e^{\theta} + e^{\theta+2\pi}}{2}\right) \\
\implies R &< \frac{e^{\theta} + e^{\theta+2\pi}}{2} < e^{\theta+2\pi} \\
\implies \log(R) &- 2\pi < \theta,
\end{aligned}$$

so $1 + e^{-Re^{-2\pi}R_1} > a(z) > 1$. Taking limit when $R \to \infty$ we deduce $\hat{a} = 1$.
Set $R_n := e^{2(n-1)\pi} R_1$. We will prove that $B_{R_n}(z_n) \subset Z_1$ for all n. For this, let $x \in B_{R_n}(z_n)$. We have

$$\|x - z_n\| < R_n = e^{2(n-1)\pi} R_1 \implies \left\| \frac{x}{e^{2(n-1)\pi}} - \frac{z_n}{e^{2(n-1)\pi}} \right\| < R_1.$$

The polar coordinates of $z_n/e^{2(n-1)\pi}$ are

$$\begin{aligned}\frac{z_n}{e^{2(n-1)\pi}} &= \left(\frac{e^{(2n-1)\pi} + \left(\frac{e^{(2n+1)\pi} - e^{(2n-1)\pi}}{4}\right)}{e^{2(n-1)\pi}}, \pi\right) \\ &= \left(e^{\pi} + \left(\frac{e^{3\pi} - e^{\pi}}{4}\right), \pi\right) \\ &= z_1,\end{aligned}$$

so $x/e^{2(n-1)\pi} \in B_{R_1}(z_1) \subset Z_1$, and we can write

$$\frac{x}{e^{2(n-1)\pi}} = \left(te^{\theta} + (1-t)\left(\frac{e^{\theta} + e^{\theta+2\pi}}{2}\right), \theta\right)$$

for some $\theta \geq 0, 0 < t < 1$. It follows that

$$\begin{aligned} x &= \left(te^{2(n-1)\pi+\theta} + (1-t)\left(\frac{e^{2(n-1)\pi+\theta} + e^{2(n-1)\pi+\theta+2\pi}}{2}\right), \theta\right) \\ &= \left(te^{\theta_1} + (1-t)\left(\frac{e^{\theta_1} + e^{\theta_1+2\pi}}{2}\right), \theta_1\right),\end{aligned}$$

where $\theta_1 = 2(n-1)\pi + \theta > 0$. We deduce $x \in Z_1$, and from here $B_{R_n}(z_n) \in Z_1$. Still assuming $x \in B_{R_n}(z_n)$ we have

$$x/e^{2(n-1)\pi} \in B_{R_1}(z_1) \subset f\left(\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \times (0,1)\right).$$

Using the calculations made before we obtain

$$x \in f\left(\left(\left(2n - \frac{3}{2}\right)\pi, \left(2n - \frac{1}{2}\right)\pi\right) \times (0,1)\right),$$

and then

$$a(x) = 1 + e^{-(e^{g_1(x)}R_1)} \geq 1 + e^{-(e^{\left(2n-\frac{1}{2}\right)\pi}R_1)} \geq 1 + e^{-(e^{2n\pi}R_1)}.$$

Defining $\kappa \equiv e^{-R_1(e^{2n\pi} - e^{2(n-1)\pi})}$ we have $a(x) \geq 1 + \kappa(x - z_n)e^{-R_n}$, and the existence of a ground state follows from Lemma 5.1 and Theorem 4.15.

6 Future of this research

There are many problems in which the Concentration Compactness Principle of Lions has been used to prove existence of solutions, and we think that the variation of the Splitting Lemma we have presented here can be used to extend some of those results. Of special interest for us is the strongly indefinite case, in which a does not change sign, $0 \notin \sigma(-\Delta + V)$ and $\dim E((-\infty, 0)) = \infty$, where the subspace $E((-\infty, 0))$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the negative part of the spectrum. In this situation we have found difficulties trying to adapt the ideas of [15], given that Lemma 4.6 does not hold anymore. Obtaining this result will take us one step closer to prove the existence of ground states in the double indefinite case, where the hypoteses about the spectrum of $-\Delta + V$ are the same as in the strongly indefinite case but a can change sign.

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