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A GENERAL MIXTURE-DIFFUSION SDE AND CALIBRATION TO MARKET VOLATILITY SMILES

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1 Introduction

It is well known that Black-Scholes-Merton (1973) model is not the most appropriate way to price European options. For example, assumptions like constant volatility and continuous trajectories of the underlying asset price contribute that the option price of this model cannot be consistent with the option price given by the market in the real world.

Brigo and Mercurio (2000) proposed a new method based on the assumption that the density of the underlying asset price S of a financial instrument can be expressed as the weighted average of densities of some prices $S^1, ..., S^m$, where each S^i is driven by a known diffusion process under the risk-neutral measure Q. We will see later that this procedure can be considered as the first step to construct models that do the least departure from the lognormal world employed in the Black-Scholes-Merton model. Moreover, the mixture-diffusion SDE models that arise from this method are still analytically tractable to put in practice, and able to fit more general volatility models.

In this work we will follow the ideas given by Brigo (2002), and we will give some details of his results. In the first part of this work, given a mixture of probability densities, we will provide the necessary tools to define a candidate diffusion process whose marginal law follows the same evolution. In section 2, we shall see the particular case of mixture of Gaussian densities. In section 3, we will present diffusion processes whose marginal densities are mixtures of lognormal densities; possible solutions and algorithms to the volatility smile problem in mathematical finance; and differences between local volatility models and stochastic volatility models under terminal and instantaneous correlation criteria.

Throughout this work, unless otherwise stated, we shall work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ whose filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions (see Karatzas and Shreve, 2014), and we shall assume that our risky asset price process S under the risk-neutral measure Q is positive and continuous. Following the ideas of Brigo (2002), let us then start by assuming that the risky asset price S under the risk-neutral measure is driven by a stochastic differential equation (SDE)

$$dS_t = f_t(S_t)dt + \sigma_t(S_t)dW_t, \tag{1}$$

of diffusion type, with $S_0 = s_0$ deterministic initial condition; and $W = (W_t)_{t\geq 0}$ a standard Brownian motion. We also assume

(A1) The stochastic differential equation (1) with diffusion and drift coefficients f and σ respectively, and initial condition s_0, admits a unique strong solution whose support is assumed to be the interval (a, ∞) at all instants of time.

In order to describe the evolution of the distribution of probability of the price, we assume:

(A2) The unique solution S_t to the SDE (1) admits a density p_t which is absolutely continuous with respect to the Lebesgue measure and satisfies the Fokker-Planck equation:

$$\frac{\partial p_t}{\partial t} = -\frac{\partial (f_t p_t)}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma_t^2 p_t)}{\partial x^2},$$

where $p_t(x) := p(t,x)$, $f_t(x) := f(t,x)$, and $\sigma_t(x) := \sigma(t,x)$.

Now, let us assume that the information we have about each S^i is described by a basic parametric family of densities, say

$$\mathcal{D} = \{p(\cdot|\theta) : \theta \in \Theta\},$$

with $\Theta \subset \mathbb{R}^d$ open, d a positive integer; and all of its densities share the same support (a, ∞). Now, we are led to consider a particular collection of mixtures of elements of this family. So let us define for each fixed vector $\lambda = (\lambda_1, ...\lambda_m)$ of strictly positive weights with $\sum_{j=1}^m \lambda_j = 1$, the set

$$\mathcal{M}(\mathcal{D}, \lambda) := \{\lambda_1 p(\cdot|\theta_1) + \cdots + \lambda_m p(\cdot|\theta_m) : \theta_1, ...\theta_m \in \Theta\}.$$

Now, we are interested in finding conditions under which the density of S under the risk-neutral measure is the weighted average of known densities of some prices S^i. To do this we will introduce and solve the Problem 1.1 presented by Brigo (2002).

Problem 1.1 (Brigo, 2002)**.** *Let be given a drift $f_t(\cdot)$ and a mixture family $\mathcal{M}(\mathcal{D}, \lambda)$ of densities $p(\cdot|\theta)$ with support (a, ∞), and satisfying*

$$\text{(A3)} \qquad \lim_{s \downarrow a} f_t(s) p(s|\theta) = 0, \quad \text{for all} \quad t \geq 0, \quad \theta \in \Theta.$$

Let $\Sigma(f, s_0)$ denote the set of all diffusion coefficients σ^f such that the related SDE (1) satisfies the assumptions (A1) and (A2), and such that

$$\text{(A4)} \qquad \lim_{s \downarrow a} (\sigma_t^f(s))^2 p(s|\theta) = 0, \qquad \lim_{s \downarrow a} \frac{\partial \left((\sigma_t^f(s))^2 p(s|\theta) \right)}{\partial s} = 0,$$

for all $t \geq 0$, $\theta \in \Theta$.

Assume the set $\Sigma(f, s_0)$ to be non-empty. Then, given a curve $t \mapsto \sum_{i=1}^m \lambda_i p(\cdot|\theta_i(t))$ in $\mathcal{M}(\mathcal{D}, \lambda)$ (where $t \mapsto \theta_i(t)$ are C^1-curves in the parameter space Θ), find a diffusion coefficient in $\Sigma(f, s_0)$ whose related SDE (1) has solution with density $p_t(\cdot) = \sum_{i=1}^m \lambda_i p(\cdot|\theta_i(t))$.

Let us first give an introductory motivation to the Problem 1.1. Consider the scenario where we have partial information about the dynamics of S. For example, suppose that we know the drift $f_t(\cdot) = f(t, \cdot)$ and the weights λ_i of each one of the densities $p(\cdot|\theta_i(t))$ of the prices S^i, then the solution to Problem 1.1 allow us to find a diffusion coefficient σ^f such that the curve $t \mapsto p_t$ of densities of S matches with the given curve $t \mapsto \sum_{i=1}^m \lambda_i p(\cdot|\theta_i(t))$.

Proposition 1.2 (Brigo, 2002)**.** *Under assumptions and notation of Problem 1.1, consider the stochastic differential equation*

$$\begin{aligned} dS_t &= f_t(S_t) dt + \sigma_t^f(S_t) dW_t \qquad S_0 = s_0 \\ (\sigma_t^f(s))^2 &= \frac{2}{\sum_{i=1}^m \lambda_i p(s|\theta_i(t))} \left[\int_a^s \left(\int_a^x \sum_{i=1}^m \lambda_i \frac{\partial p(y|\theta_i(t))}{\partial t} dy \right) dx + \int_a^s f_t(x) \sum_{i=1}^m \lambda_i p(x|\theta_i(t)) dx \right]. \end{aligned} \tag{2}$$

If $\sigma^f \in \Sigma(f, s_0)$, then the solution to the SDE (2) solves the Problem 1.1, and $p_{S_t}(s) = \sum_{i=1}^m \lambda_i p(s|\theta_i(t))$ for $t \geq 0$.

Proof. As already mentioned by Brigo and Mercurio (2000), by assuming the hypothesis of Problem 1.1 this proof will be finished once we have proved that $q_t(s) := \sum_{i=1}^m \lambda_i p(\cdot|\theta_i(t))$ satisfies the Fokker-Plank equation with f and σ^f as in (2). We can do this by solving the Fokker-Planck equation with respect to σ. Thus, by using (A3), (A4), and integrating twice (with respect to x) the equation

$$\frac{\partial q_t}{\partial t} = -\frac{\partial (f_t q_t)}{\partial x} + \frac{1}{2} \frac{\partial^2 (\sigma_t^2 q_t)}{\partial x^2},$$

we get

$$\int_a^s \int_a^x \sum_{i=1}^m \lambda_i \frac{\partial p(y|\theta_i(t))}{\partial t} dydx = -\int_a^s f_t(x) \sum_{i=1}^m \lambda_i p(x|\theta_i(t))dx + \frac{1}{2}\sigma_t^2(s) \sum_{i=1}^m \lambda_i p(s|\theta_i(t)),$$

Therefore, the only candidate σ for which its associated SDE (1) satisfies the conditions (A1), (A2), (A3), and (A4) with density $q_t(\cdot)$ and drift f, must be equal to σ^f as in (2). □

We shall use a particular case of the above proposition.

Corollary 1.3 (Brigo, 2002)**.** *Under assumptions and notations of the previous proposition, if the basic densities $p(\cdot|\theta_i(t))$ evolving in $\mathcal{M}(\mathcal{D}, \lambda)$ are respectively the marginal densities of a family of (instrumental) SDEs*

$$dS_t^i = f_t^i(S_t^i)dt + \sigma_t^i dW_t, \quad S_0^i = s_0, \quad p_{S_t^i}(s) := p(s|\theta_i(t)), \quad i = 1, ..., m$$

all satisfying the assumptions (A1), (A2), (A3), and $\sigma^i \in \Sigma(f^i, s_0)$, then the solution to the Problem 1.1 takes the form

$$(\sigma_t^f(s))^2 = \sum_{i=1}^m \Lambda_t^i(s)(\sigma_t^i(s))^2 + \frac{2\sum_{i=1}^m \lambda_i \int_a^s (f_t(x) - f_t^i(x))p(x|\theta_i(t))dx}{\sum_{j=1}^m \lambda_i p(s|\theta_i(t))}, \qquad \Lambda_t^i(s) := \frac{\lambda_i p(s|\theta_i(t))}{\sum_{i=1}^m \lambda_i p(s|\theta_i(t))}. \quad (3)$$

If f satisfies

$$f_t(s) = \sum_{i=1}^m \Lambda_t^i(s) f_t^i(s) \quad (4)$$

then the second term in the right side of $(\sigma^f)^2$ in (3) vanishes, and then the squared diffusion and drift coefficients of our final SDE are "Λ-mixtures" of the squared diffusion and drift coefficients of the instrumental processes.

Proof. We substitute the Fokker-Planck instrumental equations

$$\frac{\partial p(s|\theta_i(t))}{\partial t} = -\frac{\partial (f_t^i(s)p(s|\theta_i(t))}{\partial s} + \frac{1}{2}\frac{\partial^2[(\sigma_t^i(s))^2 p(s|\theta_i(t))]}{\partial s^2}$$

in the equation (2) for $(\sigma_t^f)^2$. Thus, using (A3) and (A4), we get

$$(\sigma_t^f(s))^2 = \frac{2}{\sum_{i=1}^m \lambda_i p(s|\theta_i(t))} \left[\int_a^s \left(\sum_{i=1}^m -\lambda_i f_t^i(x) p(x|\theta_i(t)) \right) dx \right.$$
$$\left. + \sum_{j=1}^m \frac{\lambda_j (\sigma_t^j(s))^2 p(s|\theta_j(t))}{2} + \int_a^s f_t(x) \sum_{i=1}^m \lambda_i p(x|\theta_i(t)) dx \right].$$

By straightforward calculations, we obtain the result. □

The above problem could have been generalized by considering a generic density evolution $t \mapsto q_t$. In such case, and using analogous assumptions to (A3) and (A4), the diffusion coefficient would be

$$(\sigma_t^f(s))^2 = \frac{2}{q_t(s)} \left[\int_a^s \left(\int_a^x \frac{\partial q_t(y)}{\partial t} dy \right) dx + \int_a^s f_t(x) q_t(x) dx \right].$$

In the next sections, we shall see some implications of the above results related to the normal parametric family case and to mathematical finance. Although throughout this work we will make use of the assumptions made in this part, we will make explicit description of the cases where we can lighten some assumptions in order to get more useful results.

2 Diffusions whose densities follow mixtures of normal distributions

As already noticed by Brigo and Mercurio (2000), before late 90s no explicit rigorous attempts had been made to design time-continuous diffusion models whose densities were mixtures of normals or lognormals. Although there were some empirical works like (Ritchey, 1990) and (Guo, 1998) among others, the first theoretical point of view was due to Brigo and Mercurio.

This subject can be motivated in the sense that simplest generalization outside from the Gaussian world could begin by the use of mixtures of normal densities, at least in mathematical finance this subject can be relevant. In fact, as already mentioned by some authors, e.g. (Hull, 2012), financial instruments such as options have no volatility constant, nor do they have continuous price changes, then lognormal distributions are not the ideal way to price options. As a result, the use of other type of tools is needed to have a better explanation and prediction of option prices fluctuations.

We then begin by considering the case of the normal family

$$\mathcal{D} := \{p_{\mathcal{N}(m,v^2)} : m, v \in \mathbb{R}\}, \qquad a = -\infty,$$

and suppose we have some λ-mixture of normals given, $t \mapsto \sum_{i=1}^{m} \lambda_i p_{\mathcal{N}(m_i(t),v_i^2(t))}$. Now, we want to find a diffusion process such that its density matches the previous mixture. Under certain conditions this problem is tractable. Our aim is to connect each result to show a procedure of how to apply this theory in practice and providing an additional support to the Problem 1.1. Because of that, we shall show a particular tractable case. Consider the instrumental SDEs

$$dS_t^i = \mu_i(t)dt + \sigma_i(t)dW_t, \quad m_i(t) := \int_0^t \mu_i(l)dl, \quad v_i^2(t) := \int_0^t (\sigma_i(l))^2 dl,$$

for $i = 1, ..., m$, with null initial conditions. As a first step, we will verify that they satisfy the conditions (A1), (A2), and (A3). In fact, we know that each above instrumental SDE satisfies the Lipschitz condition (see Rogers and Williams, 2000, p. 128) and $S_t^i \sim \mathcal{N}(m_i(t), v_i^2(t))$ (see Appendix A), then the condition (A1) holds. Now, by straightforward calculations we can verify

$$\begin{aligned}
\frac{\partial p_t(s)}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\exp\left\{-\dfrac{(s-m(t))^2}{2v_i^2(t)}\right\}}{\sqrt{v_i^2(t)2\pi}} \right) \\
&= \frac{-\sigma_i^2(t)\exp\left\{-\dfrac{(s-m(t))^2}{2v_i^2(t)}\right\}}{2v_i^3(t)\sqrt{2\pi}} + \frac{\exp\left\{-\dfrac{(s-m(t))^2}{2v_i^2(t)}\right\}}{\sqrt{v_i^2(t)2\pi}}\left[\frac{2(s-m(t))\mu_i(t)}{2v_i^2(t)} + \frac{(s-m_i(t))^2\sigma_i^2(t)}{2v_i^4(t)}\right] \\
&= \exp\left\{-\frac{(s-m(t))^2}{2v_i^2(t)}\right\}\left[\frac{-\sigma_i^2(t)}{2v_i^3(t)\sqrt{2\pi}} + \frac{(s-m_i(t))\mu_i(t)}{\sqrt{2\pi}v_i^3(t)} + \frac{(s-m_i(t))^2\sigma_i^2(t)}{2v_i^5(t)\sqrt{2\pi}}\right],
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{\partial}{\partial s}\left(\mu_i(t)p_t(s)\right) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\left(\sigma_i^2(t)p_t(s)\right) \\
&\quad = \frac{\mu_i(t)(s-m_i(t))\exp\left\{-\dfrac{(s-m(t))^2}{2v_i^2(t)}\right\}}{v_i^3(t)\sqrt{2\pi}} \\
&\qquad + \frac{1}{2}\sigma_i^2(t)\left[\frac{-\exp\left\{-\dfrac{(s-m(t))^2}{2v_i^2(t)}\right\}}{v_i^3(t)\sqrt{2\pi}} + \frac{(s-m_i(t))^2\exp\left\{-\dfrac{(s-m(t))^2}{2v_i^2(t)}\right\}}{v_i^5(t)\sqrt{2\pi}}\right],
\end{aligned}$$

then the Fokker-Planck equation is satisfied and therefore (A2) holds. Finally, (A3) follows immediately from the definition $S_t^i \sim \mathcal{N}(m_i(t), v_i^2(t))$ when $s \to -\infty$. Now, by applying Corollary 1.3, we get that the drift and squared diffusion coefficient satisfies

$$f_t(s) = \sum_{i=1}^{m} \Lambda_t^i(s)\mu_i(t), \qquad (\sigma_t^f(s))^2 = \sum_{i=1}^{m} \Lambda_t^i(s)\sigma_i^2(t), \qquad \Lambda_t^i(s) = \frac{\lambda_i p_{\mathcal{N}(m_i(t), v_i^2(t))}(s)}{\sum_{j=1}^{m} \lambda_j p_{\mathcal{N}(m_j(t), v_j^2(t))}(s)} \tag{5}$$

Given (5), we will look for some conditions on μ_i's and σ_i's that allow us to get the assumptions (A1) and (A2) for our final SDE (1) associated to f and σ^f. One alternative to get strong solutions to the SDE (1) is via local Lipschitz property of its drift and diffusion coefficients. This can be consulted in the Theorem 12.1 in Section V.12 of (Rogers and Williams, 2000).

Theorem 2.1 (**SDE whose marginal law follows a given normal mixture; Brigo, 2002**). *Consider a SDE*

$$dS_t = f_t(S_t)dt + \sigma_t^f(Y_t)dW_t, \qquad S_0 = s_0 \tag{6}$$

with drift and diffusion coefficients f and σ^f as in (5), where the $\sigma_i's$ and $\mu_i's$ are at least C^1 times functions. Assume that $\mu_i(t) = \bar{\mu}$ and $\sigma_i(t) = \bar{\sigma}$ on an initial interval $[0, \epsilon)$ with $\epsilon > 0$; and each σ_i is bounded away from zero, that is, $\sigma_i(t) > K > 0$ for all i and t. Then the considered SDE admits a unique strong solution whose marginal density is a λ-normal mixture

$$p_{S_t} = \sum_{i=1}^{m} \lambda_i p_{\mathcal{N}(m_i(t), v_i^2(t))}.$$

Proof. Let us consider $t \mapsto \sigma_i(t)'s$ and $t \mapsto \mu_i(t)'s$ to be at least C^1. Since the initial condition is zero, we may have problems of regular behavior of f and σ^f when $(t, y) \to (0, 0)$. We avoid this problem by considering each σ_i to be bounded away from zero, and setting $\mu_i(t) = \bar{\mu}$ and $\sigma_i(t) = \bar{\sigma}$ for each $i = 1, ..., m$ and $t \in [0, \epsilon)$ with $\epsilon > 0$.

From the given assumptions, we get $\Lambda_t^i(s) = \lambda_i$ on $[0, \epsilon)$, and $0 \le \Lambda_t^i(s) \le 1$ for all t and s, and thus, f and σ^f are bounded on intervals of the form $[0, T]$. By the previously mentioned, we can easily verify that f and $(\sigma^f)^2$ are C^1 in both t and s, and moreover the σ_i's are also C^1 because they are bounded away from zero. Therefore, f and σ^f are locally Lipschitz, and now we can apply the Theorem 12.1 in Section V.12 of (Rogers and Williams, 2000) to conclude that our final SDE with drift and diffusion coefficients f and σ^f as in (5) admits a unique strong solution. $\square$

It is well known that the diffusion coefficient of a diffusion process that models the price of the underlying asset of an option is associated with its volatility. Therefore, it is natural to ask for types of relations between the process itself and its diffusion coefficient. Nowadays, some of the ways that are used in practice to measure relations between the process itself and its diffusion coefficient are the *terminal* correlation and the *instantaneous* correlation, which are also used as a tool to help to price financial instruments whose payoffs depend on the joint realizations of several prices or rates; for example, basket options, spread options, etc., (see Rebonato, 2005, Section 5.1).

There are some ways to introduce the concept of instantaneous correlation (see for instance Munk, 2015; Rebonato, 2005). Here we use the notation used by Brigo (2002) that gives some intuition to the reader unfamiliar with this concept. We recall the *quadratic variation* of a $(\mathcal{F}_t, Q)$-continuous semimartingale M as the increasing process $\langle M, M\rangle \equiv \langle M\rangle$ such that for every $t \ge 0$ and every sequence (Δ_n) of subdivisions of $[0, t]$ with $|\Delta_n| \to 0$,

$$\langle M, M\rangle_t = Q - \lim_{|\Delta_n| \to 0} T^{\Delta_n}, \qquad T_t^{\Delta_n} := \sum_i (M_{t_{i+1}} - M_{t_i})^2.$$

Other concept related to the previous one is the *covariation* of two $(\mathcal{F}_t, Q)$-continuous semimartingales M and N. This is the increasing process $\langle M, N\rangle$ such that for every $t \geq 0$ and every sequence (Δ_n) of subdivisions of $[0,t]$ with $|\Delta_n| \to 0$,

$$\langle M, N\rangle_t = Q - \lim_{|\Delta_n|\to 0} \tilde{T}^{\Delta_n}, \qquad \tilde{T}_t^{\Delta_n} := \sum_i (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}).$$

In this work we will use the notation Corr_t and Cov_t respectively for correlation and covariance, both conditional in the information available at time t. Given these concepts, a natural way to get information about changes on infinitesimal periods of time between $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ is via *instantaneous correlation*

$$\mathrm{Corr}_t(dM_t, dN_t) = \frac{d\langle M_., N_.\rangle_t}{\sqrt{d\langle M_.\rangle_t}\sqrt{d\langle N_.\rangle_t}},$$

where this concept has sense for well-behaving and strictly positive processes $\langle M\rangle$ and $\langle N\rangle$. In some sense, this criteria give us some information about linear behavior on instantaneous periods of time. For example, if $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ are well-behaving semimartingales with $N_t = \lambda M_t$ and $\lambda \neq 0$, we get $\mathrm{Corr}_t(dM_t, dN_t) = \mathrm{sgn}(\lambda)$. Moreover, if they are independent, their instantaneous correlation is zero.

Let us calculate the terminal correlation and instantaneous correlation of SDE (6). We recall $\sigma_t^f(S_t) = \sigma^f(t, S_t)$, then using Itô's formula with $\sigma := \sigma^f$ and the vector semimartingale $(I_t, S_t)_{\geq 0}$ we get

$$\sigma_t(S_t) = \int_0^t \left(D_1\sigma_l(S_l) + (D_2\sigma_l(S_l))f_l(S_l) + \frac{1}{2}(D_{22}\sigma_l(S_l))\sigma_l^2(S_l) \right) dl + \int_0^t (D_2\sigma_l(S_l))\sigma_l(S_l)dW_l,$$

where $I_t(w) \equiv t$, and D_i denotes the partial derivative with respect to the ith coordinate. Thus

$$\mathrm{Corr}_t(dS_t, d\sigma_t(S_t)) = \frac{d\langle S_., \sigma_.(S_.)\rangle_t}{\sqrt{d\langle S_.\rangle_t}\sqrt{d\langle \sigma_.(S_.)\rangle_t}} = \frac{(D_2\sigma_t(Y_t))\sigma_t^2(S_t)dt}{\sqrt{\sigma_t^2(S_t)dt}\sqrt{[(D_2\sigma_t(S_t))\sigma_t(S_t)]^2 dt}} = 1.$$

However, the results are different for the terminal correlation. By using (5) and $S_t \sim \sum_{i=1}^m \lambda_i p_{\mathcal{N}(m_i(t), v_i^2(t))}$, we get

$$\mathrm{Cov}(S_t, \sigma_t^2(S_t)) = E[S_t\sigma_t^2(S_t)] - E[S_t]E[\sigma_t^2(S_t)] = \sum_{i=1}^m \lambda_i m_i(t)\sigma_i^2(t) - \left(\sum_{i=1}^m \lambda_i m_i(t)\right)\left(\sum_{i=1}^m \lambda_i \sigma_i^2(t)\right)$$

Now consider the case $\mu_i(\cdot) = \mu(\cdot)$ for all i, and correspondingly $m_i(\cdot) = m(\cdot)$ for all i where $m(t) := \int_0^t \mu(l)dl$. Then, we get $f_t(s) = \mu(t)$ for all s, and the *terminal* correlation at time t is zero

$$\mathrm{Corr}(S_t, \sigma_t^2(S_t)) = \mathrm{Cov}(S_t, \sigma_t^2(S_t)) = 0$$

In this case we conclude that the instantaneous changes of the process itself and the instantaneous changes of its diffusion coefficient are perfectly positive correlated, whereas their terminal correlation at any time is zero, even when the time t is close to zero. Moreover, as we shall see later, the correlation between S_t and its average square diffusion $\int_0^t \sigma_l^2(S_l)dl$ is again zero

$$\mathrm{Corr}\left(S_t, \int_0^t \sigma_l^2(S_l)dl\right) = 0.$$

From the Theorem 2.1, we can get examples of mixtures of distributions of parametric families not necessary normal. For example, if we consider S the process given by Theorem 2.1, then applying Itô's formula to $Y_t := \exp(S_t)$ we get

$$dY_t = \left(Y_t f_t(\ln(Y_t)) + \frac{1}{2}Y_t\sigma_t^2(\ln(Y_t))\right) dt + Y_t\sigma_t(\ln(Y_t))dW_t, \qquad Y_0 = 1,$$

and by standard procedure, we conclude that $p_{Y_t}(y) = p_{S_t}(\ln(y))/y$. Thus, Y_t is distributed as a mixture of lognormals.

3 The smile phenomenon in option pricing

In this part, we will see some aspects of the smile problem in mathematical finance and explain a possible solution to them via the use of diffusions whose densities are λ-mixture lognormals.

3.1 The smile problem and implied distribution

The smile problem can be seen, especially for European options, as all the consequences that arise of the development and use of volatility smile curves (implied volatility vs strike price curves) to construct the (implied) densities of the underlying assets as a supporting tool to price options.

In order to introduce the smile problem in a more accurate way, let us first give some basic concepts of mathematical finance. In the common discrete-time financial scenario where we consider a market with two assets B_t and S_t, where B_t is the riskless asset(*numéraire*) satisfying $B_t = (1+r)B_{t-1}$, and S_t is the risky asset; we say that a contingent claim (*pay-off* of an option) C_T is *attainable*, if we can find a *self-financing* portfolio strategy $\xi_t := (\xi_t^0, \xi_t^1)$ (predictable process) such that the portfolio value $V_t := \xi_t \cdot (B_t, S_t)$ satisfies $V_T = C_T$, and the difference between its values at time $t+1$ and t equals

$$\begin{aligned} V_{t+1} - V_t &= \xi_{t+1} \cdot (B_{t+1}, S_{t+1}) - \xi_t \cdot (B_t, S_t) \\ &= \xi_{t+1} \cdot (B_{t+1}, S_{t+1}) - \xi_{t+1} \cdot (B_t, S_t) = \xi_{t+1}^0 (B_{t+1} - B_t) + \xi_{t+1}^1 (S_{t+1} - S_t). \end{aligned} \tag{7}$$

These portfolio value processes are useful to price financial instruments. In fact, under the assumption of a complete market without arbitrage opportunities, we can find a unique risk-neutral measure (martingale measure) Q that allow us to calculate the unique *arbitrage-free price* $E_Q[(1+r)^{-T} C_T]$ at time $t = 0$ of the contingent claim C_T with maturity T (see for instance Shreve, 2004).

As we might expect from (7), the continuous time scenario should be

$$dV_t = \xi_t^0 dB_t + \xi_t^1 dS_t$$

where $(B_t)_{t\in[0,T]}$ and $(S_t)_{t\in[0,T]}$ should follow dynamics of the type

$$dB_t = r(t) B_t dt, \qquad\qquad dS_t = r(t) S_t dt + v_t S_t dW_t,$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion; and the process $(\xi_t)_{t\in[0,T]}$ is adapted with respect to the filtration $\mathcal{F}_t = \sigma(S_l, l \le t) = \sigma(W_l, l \le t)$. Fortunately, the time-continuous scenario shares similar results from the discrete-time model. In fact, under complete markets, Harrison and Pliska (1981) established that the unique arbitrage-free price at time t for a contingent claim $C_T \in L^2(Q)$ is given by $\Pi_t(C_T) = B_t E_Q[C_T / B_T | \mathcal{F}_t]$ where Q is the risk-neutral measure such that $(S_t/B_t)_{t\in[0,T]}$ is $(\mathcal{F}_t, Q)$-martingale.

In the context of the Black-Scholes-Merton setup, we can deduce from the results given in Appendix B that the dynamics of S under the risk-neutral measure Q can be assumed to be of the form

$$dS_t = r(t) S_t dt + v(t) S_t dW_t, \qquad S_0 = s_0, \qquad t \in [0, T]; \tag{8}$$

where s_0 is a positive constant value; r and v are well-behaving and strictly positive functions (see also Elliott and Kopp, 2006, Section 7.5). By Itô's lemma applied to $\ln(S_t)$, we get

$$\ln(S_t) = \ln(S_0) + \int_0^t \left(r(s) - \frac{v^2(s)}{2} \right) ds + \int_0^t v(s) dW_s;$$

and using $\int_0^t v(s) dW_s \sim \mathcal{N}\left(0, \int_0^t v^2(s) ds\right)$, we conclude

$$\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left(R(0,t) - \frac{1}{2}V^2(t), V^2(t)\right), \qquad R(b,t) := \int_0^t r(s)ds, \qquad V^2(t) := \int_0^t v^2(s)ds. \tag{9}$$

When $a = 0$, we shall write $R(t)$ instead of $R(0,t)$. Now, we have the tools to compute the arbitrage-free price at time $t = 0$ (the initial investment) for a contingent claim $C_T \in L^2(Q)$ that is $\Pi_0(C_T) = B_0 E_Q[C_T/B_T|\mathcal{F}_0] = E_Q[C_T/B_T]$.

In the case of a European call option $C_t = f(S_T) = (S_T - K)^+$ where K is the *strike price*, and T is the *maturity* of the option, the discounted pay off (the initial investment) is calculated by using the formula of arbitrage-free price at time $t = 0$ with the lognormal distribution given by (9). This procedure drive us to the famous Black-Scholes-Merton (1973) call option formula, which we denote by "BSCall" and whose expression is given by

$$E_Q[(S_T - K)^+/B_T] = \mathrm{BSCall}(S_0, K, T, R(T), V(T)).$$

According to the formula, the (average) volatility $V(T)/\sqrt{T}$ is a quantity that does not depend on K. In fact, the volatility of the option is a characteristic of the stock S, and it has nothing to do with the contract itself.

Following the previously mentioned if we have two market prices of European call options denoted by $\mathrm{MKTCall}(S_0, K_1, T)$ and $\mathrm{MKTCall}(S_0, K_2, T)$ with the same maturity but with different strike prices K_1 and K_2, we should have a single volatility parameter $V(T)$ such that

$$\begin{aligned}\mathrm{MKTCall}(S_0, K_1, T) &= \mathrm{BSCal}(S_0, K_1, T, R(T), V(T))\\ \mathrm{MKTCall}(S_0, K_2, T) &= \mathrm{BSCal}(S_0, K_2, T, R(T), V(T))\end{aligned}$$

Unfortunately, this does not happen in the real world. What one really sees is that two different volatilities $V(K_1, T)$ and $V(K_2, T)$ are required to match the previous equations:

$$\begin{aligned}\mathrm{MKTCall}(S_0, K_1, T) &= \mathrm{BSCal}(S_0, K_1, T, R(T), V(T, K_1))\\ \mathrm{MKTCall}(S_0, K_2, T) &= \mathrm{BSCal}(S_0, K_2, T, R(T), V(T, K_2))\end{aligned}$$

This problem leads us to use different (average) implied volatilities $V(T,K)/\sqrt{T}$ for European options depending on the option strike K. If Black-Scholes-Merton model were consistent with respect to the strike prices, the *volatility smile* $K \mapsto \mathrm{V}^{\mathrm{MKT}}(T,K)/\sqrt{T}$ should be flat rather than the "smiley" ones that traders are used to.

Now, we shall see how volatility smiles are used to calculate implied distributions $P_{S_{T_j}}$ that are useful tools to study price fluctuations; and we will close this section showing some problems that arise of the use of interpolation to calculate the implied distribution, as often happens in practice.

Let us then consider (as in the real world happens) that the market only quote some strikes K_i's and maturities T_j's. One mathematical intuitive idea consists in doing interpolation in the points $(K, V(T_j, K))$ for each maturity T_j and thereby obtaining curves of the form $K \mapsto V(T_j, K)$ and

$$K \mapsto \mathrm{BSCall}(S_0, K, T_j, V(T_j, K)).$$

This procedure does not give any financial insight about the dynamics of S_t. However, in practice, it is accustomed to use the idea given by Breeden and Litzenberger (1978) assuming that the arbitrage-free price equation of Harrison and Pliska holds for each $\mathrm{MKTCall}(S_0, K_i, T_j)$, and thus

$$\begin{aligned}\mathrm{BSCall}(S_0, K_i, T_j, V(T_j, K_i)) = \mathrm{MKTCall}(S_0, K_i, T_j) &= e^{-\int_0^{T_j} r(l)dl} E_Q[(S_{T_j} - K_i)^+]\\ &= e^{-\int_0^{T_j} r(l)dl} \int_{K_i}^{\infty} (y - K_i) p_{S_{T_j}}(y)dy,\end{aligned} \tag{10}$$

where $p_{S_{T_j}}$ is the risk-neutral density of the stock price at maturity time T_j. Now, differentiating two times with respect to K (see also Hull, 2012) , we obtain

$$p_{S_{T_j}}(K) \approx e^{\int_0^{T_j} r(l)dl} \frac{\partial^2 \mathrm{BSCall}(S_0, K, T_j, V(T_j, K))}{\partial K^2}.$$

Although this method is easy to use and is in some way coherent, there are some problems. Here we mention some of them:

1. This method assumes that there are option prices for many strikes to make coherent the idea of using interpolation and extrapolation methods. Unfortunately this does not happen in practice.
2. Using the second order derivative approximation may increase even little errors of price options (see for instance Bedoui and Hamdi, 2010).

From these last problems, we might wonder what kind of dynamics that provides us mathematical and financial insight, and alternative to (8), do the densities $p_{S_{T_1}}, p_{S_{T_2}}, \ldots$ come from?

3.2 Local and stochastic volatility models

We will look for a partial solution to the volatility smile problem mentioned at the end of subsection 3.1. To do this, we will look for an alternative SDE whose density resembles the density of the SDE (8), and whose prices matches the prices quoted by the market for the different maturities and strikes K_i's and T_j's.

Let us then begin considering an alternative model

$$dS_t = r(t)S_t dt + \sigma(t, S_t)S_t dW_t \qquad S_0 = s_0 \tag{11}$$

where σ can be either a deterministic function of S_t, in which case the model will be called a *local volatility* model, or σ can have randomness by its own, i.e. driven by a SDE

$$d(\xi_t^2) = b(t, \xi_t^2)dt + \chi(t, \xi_t^2)dZ_t, \qquad \xi_t := \sigma(t, S_t),$$

in which case the model will be called a *stochastic volatility* model; and where Z is a standard Brownian motion satisfying

$$\mathrm{Cov}_t(dZ_t, dW_t) = d\langle W, Z\rangle_t = \rho dt.$$

Although the instantaneous correlation for stochastic volatility models is

$$\mathrm{Corr}_t(dS_t, d\sigma^2(t, S_t)) = \frac{\sigma(t, S_t)S_t\chi(t, \xi_t^2)\rho dt}{\sqrt{\sigma^2(t, S_t)S_t^2 dt}\sqrt{\chi^2(t, \xi_t^2)dt}} = \rho,$$

whereas for local volatility models is

$$\mathrm{Corr}_t(dS_t, \sigma^2(t, S_t)) = \frac{(D_2\sigma^2(t, S_t))(\sigma(t, S_t)S_t)^2 dt}{\sqrt{(\sigma(t, S_t)S_t)^2 dt}\sqrt{(D_2\sigma^2(t, S_t))^2\sigma^2(t, S_t)S_t^2 dt}} = 1,$$

this does not imply that stochastic volatility models are superior to local volatility models. In fact, this superiority does not appear when we consider terminal correlations, as we shall see later. There are some problems such as the computational complexity and the difficulty of fitting parameters to the current prices of vanilla options that arise in practice in stochastic volatility models (Gatheral and Lynch, 2002). Therefore, we shall focus on deterministic σ's flexible enough for practical purposes.

Now, given the specification that our alternative model is of local volatility, a problem we face is finding additional features that our alternative dynamics (11) should have in order to allow us to construct densities as similar as possible to the true densities $p_{S_{T_1}}, p_{S_{T_2}}, \ldots$ derived from the dynamics (8). This will be done by fitting the prices of our alternative model (11) to the prices of the market for the different maturities T_j's and strikes K_i's, and trying that the curves $K \mapsto V(T,K)/\sqrt{T}$ of our alternative model resemble the volatility smiles given by the market. The procedure is illustrated as follows:

1. Find the risk-neutral measure Q of the dynamics (11).
2. For each pair of (T_j, K_i) compute the prices given by the model (11)
$$\Pi(T_j, K_i) = e^{-\int_0^{T_j} r(l)dl} E_Q((S_{T_j} - K_i)^+),$$
and solve the equation
$$\mathrm{BSCall}(S_0, K_i, T_j, R(T_j), V(T_j, K_i)) = \Pi(T_j, K_i)$$
for $V(T_j, K_i)$.
3. Calculate the (average) implied volatilities $V(T_j, K_i)/\sqrt{T_j}$ for each pair (T_j, K_i).

Since the dynamics of our alternative model (11) are not lognormal, the curves $K \mapsto V(T_j, K)$ are not necessary flat. A possible solution to this problem is finding a diffusion σ in (11) flexible enough that allows the implied volatilities $V(T_j, K_i)/\sqrt{T_j}$ of the alternative model to be as close as possible to the market implied volatilities $V^{\mathrm{MKT}}(T_j, K_i)/\sqrt{T_j}$. As might be expected, the problem appears when we need to calibrate a large number of points. For example, if we consider the CEV model $\sigma(t, S_t) = \eta S_t^{\gamma}$ in (Cox, 1975), it has only the parameters γ and η, so its fitting capabilities are very limited. Therefore, λ-mixtures can be a useful tool instead of using methods that require interpolation and another techniques that may disturb data.

3.3 Local volatility lognormal mixture diffusion dynamics

Following the ideas suggested at the end of Subsection 3.2, we are interested in finding SDEs (11) whose densities follow mixtures of lognormal densities. One way to do this is via Corollary 1.3. Therefore, let us consider m instrumental processes of the Black-Scholes-Merton model

$$S_t^i = r(t)S_t^i dt + v_i(t)S_t^i dW_t, \qquad S_0^i = s_0 \tag{12}$$

associated to the SDE (11) with $v_i(t)$'s and $r(t)$ well-behaving and strictly positive functions, and drift coefficient $f(t,s) = r(t)s$. In order to use the Corollary 1.3, let us first verify that the assumptions (A1), (A2) and (A3) hold for each instrumental SDE (12). Since the drift and diffusion coefficients of (12) are locally Lipschitz, these SDEs have unique strong solutions, and moreover we deduce from (9) that their solutions are of the form

$$S_t^i = s_0 \exp\left\{\int_0^t \left(r(l) - \frac{v_i^2(l)}{2}\right) + \int_0^t v_i(l)dW_l\right\}, \qquad \ln(S_t^i/S_0^i) \sim \mathcal{N}\left(R(t) - \frac{V_i^2(t)}{2}, V_i^2\right).$$

Hence the condition (A1) holds. For (A2), we can verify after lengthy computations

$$\begin{aligned}\frac{\partial p_{S_t^i}(y)}{\partial t} = \exp\left\{\frac{-(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))^2}{2V_i^2(t)}\right\}\Bigg[&-\frac{v_i^2(t)}{2yV_i^3(t)\sqrt{2\pi}} \\ &+\frac{\left(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0)\right)(r(t) - v_i^2(t)/2)}{yV_i^3\sqrt{2\pi}} \\ &+\frac{v_i^2(t)(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))^2}{2yV_i^5(t)\sqrt{2\pi}}\Bigg],\end{aligned}$$

$$-\frac{\partial p_{S_t}(y) r(t) y}{\partial y} = \exp\left\{\frac{-(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))^2}{2V_i^2(t)}\right\} \times \left[\frac{r(t)(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))}{y V_i^3(t)\sqrt{2\pi}}\right],$$

and

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\frac{y v_i^2(t) \exp\left\{\frac{-(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))^2}{2V_i^2(t)}\right\}}{V_i(t)\sqrt{2\pi}}\right)$$
$$= \exp\left\{\frac{-(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))^2}{2V_i^2(t)}\right\}\left[-\frac{\left(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0)\right) v_i^2(t)}{2yV_i^3\sqrt{2\pi}} + \frac{v_i^2(t)(\ln(y) - R(t) + V_i^2(t)/2 - \ln(s_0))^2}{2yV_t^5(t)\sqrt{2\pi}} - \frac{v_i^2(t)}{2yV_i^3(t)\sqrt{2\pi}}\right].$$

As a result, for each $i = 1, ..., m$, the respective Fokker-Planck equation holds, and therefore (A2) is satisfied. Finally, we can see condition (A3) follows immediately by the definition of the lognormal density of S_i^t. Now, if we assume conditions of Corollary 1.3 hold for SDE (11) and (12), we get

$$\sigma^2_{\text{mix}}(t, s)s^2 := (\sigma_t^f(s))^2 = s^2 \sum_{i=1}^m \Lambda_i(t, s) v_i^2(t), \qquad \Lambda_i(t, s) = \frac{\lambda_i p_{\mathcal{N}(\ln s_0 + R(t) - V_i^2(t)/2, V_i^2(t))}(\ln s)}{\sum_{j=1}^m \lambda_j p_{\mathcal{N}(\ln s_0 + R(t) - V_j^2(t)/2, V_j^2(t))}(\ln s)}.$$

Brigo and Mercurio (2000) showed that under conditions similar to those mentioned in Theorem 2.1 for the time functions $t \mapsto \sigma_i(t)$'s, the final SDE

$$dS_t = r(t)S_t dt + \sigma_{\text{mix}}(t, S_t)S_t dW_t \tag{13}$$

has a unique strong solution whose marginal density is given by the mixture of lognormals densities of the instrumental SDEs (12). Therefore, we assume from now on that our SDE (13) with diffusion $\sigma_{\text{mix}}(t, s)s$, drift $f(r, s) = r(t)s$, and instrumental SDEs (12) satisfies the conditions of Corollary 1.3.

For our SDE (13), we confirm again the result $\text{Corr}_t(dS_t, d\sigma^2_{\text{mix}}(t, S_t)) = 1$. In fact, by using Itô's lemma we get

$$\sigma^2_{\text{mix}}(t, S_t) = \int_0^t \left(D_1\sigma^2_{\text{mix}}(l, S_l) + (D_2\sigma^2_{\text{mix}}(l, S_l))r(l)S_l + \frac{1}{2}(D_{22}\sigma^2_{\text{mix}}(l, S_l))\sigma^2_{\text{mix}}(l, S_l)S_l^2\right) dl + \int_0^t (D_2\sigma^2_{\text{mix}}(l, S_l))\sigma_{\text{mix}}(l, S_l)S_l dW_l,$$

thus

$$\text{Corr}_t(dS_t, d\sigma^2_{\text{mix}}(t, S_t)) = \frac{(D_2\sigma^2(t, S_t))\sigma^2_{\text{mix}}(t, S_t)S_t^2 dt}{\sqrt{\sigma^2_{\text{mix}}(t, S_t)S_t^2 dt}\sqrt{(D_2\sigma^2(t, S_t))^2\sigma^2_{\text{mix}}(t, S_t)S_t^2 dt}} = 1.$$

Although there is perfect positive instantaneous correlation, Brigo (2002) showed that under terminal correlation criteria the results are completely different.

Theorem 3.1 (Brigo, 2002)**.** *Consider the random variable*

$$\bar{v}(T) := \int_0^T \sigma^2_{\text{mix}}(t, S_t)dt,$$

with $\bar{v}(T)/T$ being the "average percentage variance" of the process $(S_t)_{t\in[0,T]}$. Then

$$\text{Corr}(\sigma^2_{\text{mix}}(T, S_T), S_T) = 0, \qquad \textit{and} \qquad \text{Corr}(\bar{v}(T), S_T) = 0. \tag{14}$$

Proof. We first notice that $\mathrm{Corr}(\sigma^2_{\mathrm{mix}}(T,S_T),S_T)=0$ is equivalent to

$$E[\sigma^2_{\mathrm{mix}}(T,S_T)S_T]-E[\sigma^2_{\mathrm{mix}}(T,S_T)]E[S_T]=0.$$

This last equation is verified by the results:

$$\begin{aligned}E[\sigma^2_{\mathrm{mix}}(T,S_T)S_T] &= \int_0^\infty \sum_{i=1}^m \Lambda_i(T,s)v_i^2(T)sp_{S_T}(s)ds = \int_0^\infty \sum_{i=1}^m \Lambda_i(T,s)v_i^2(T)s\left(\sum_{j=1}^m \lambda_j p_{S_T^j}(s)\right)ds \\ &= \int_0^\infty \sum_{i=1}^m s\lambda_i v_i^2(T)p_{S_T^i}(s)ds = s_0 e^{\int_0^T r(l)dl}\sum_{i=1}^m \lambda_i v_i^2(T),\end{aligned}$$

$E[S_T]=s_0e^{\int_0^T r(l)dl}$, and $E[\sigma^2_{\mathrm{mix}}(T,S_T)]=\sum_{i=1}^m \lambda_i v_i^2(T)$.

To show $\mathrm{Corr}(v(T),S_T)=0$, we begin by applying Itô's lemma to the product of semimartingales $\bar{v}(t)S_t$,

$$d(\bar{v}(t)S_t)=S_t\sigma^2_{\mathrm{mix}}(t,S_t)dt+r(t)S_t\bar{v}(t)dt+\bar{v}(t)\sigma_{\mathrm{mix}}(t,S_t)S_tdW_t. \tag{15}$$

Since

$$\begin{aligned}&E\left[\left\langle\int_0^\cdot \bar{v}(t)\sigma_{\mathrm{mix}}(t,S_t)S_tdW_t\right\rangle_T\right] \\ &= E\left[\int_0^T \left(\bar{v}(t)\sigma_{\mathrm{mix}}(t,S_t)S_t\right)^2dt\right] \\ &= \int_0^T E\left[\left(\bar{v}(t)\sigma_{\mathrm{mix}}(t,S_t)S_t\right)^2\right]dt \\ &\le \int_0^T E\left[\left(\sum_{i=1}^m\int_0^t v_i^2(l)dl\right)^2\left(\sum_{i=1}^m v_i^2(t)\right)S_t^2\right]dt \\ &= \int_0^T \left(\sum_{i=1}^m\int_0^t v_i^2(l)dl\right)^2\left(\sum_{i=1}^m v_i^2(t)\right)\sum_{j=1}^m \lambda_j E\left[\left(S_t^i\right)^2\right]dt \\ &= \int_0^T \sum_{j=1}^m\left(\sum_{i=1}^m\int_0^t v_i^2(l)dl\right)^2\left(\sum_{i=1}^m v_i^2(t)\right)\lambda_j e^{V_i^2(t)}s_0^2e^{2\int_0^t r(l)dl}dt<\infty,\end{aligned}$$

we conclude from the results given in Appendix A that $(\int_0^\cdot \bar{v}(l)\sigma_{\mathrm{mix}}(l,S_l)S_ldW_l)$ is a martingale in $[0,T]$. Thus, by taking expectations in (15) and using Fubini's theorem, we get

$$dE[\bar{v}(t)S_t]=E[S_t\sigma^2_{\mathrm{mix}}(t,S_t)]dt+r(t)E[\bar{v}(t)S_t]dt.$$

Let us now set $A_t:=E[S_t\sigma^2_{\mathrm{mix}}(t,S_t)]$ and $C_t:=E[\bar{v}(t)S_t]$. Thus, the above equation can be rewritten

$$\dot{C}_t=r(t)C_t+A_t,$$

whose solution is $C_t=e^{\int_0^t r(s)ds}\int_0^t e^{-\int_0^l r(u)du}A_ldl$. By Fubini's theorem, we conclude

$$E[\bar{v}(T)]=\int_0^T E[\sigma^2_{\mathrm{mix}}(l,S_l)]dl=\int_0^T\sum_{i=1}^m \lambda_i v_i^2(l)dl=\int_0^T s_0^{-1}e^{-\int_0^l r(u)du}A_ldl.$$

Now, by using $E[S_T]=s_0e^{\int_0^T r(u)du}$ and $\mathrm{Cov}(\bar{v}(T),S_T)=E[\bar{v}(T)S_T]-E[\bar{v}(T)]E[S_T]$, we conclude $\mathrm{Corr}(\bar{v}(T),S_T)=0$.

□

Although we know that these types of criteria are not a good a measure of dependence outside the Gaussian world, they are interesting results by themselves. In fact, we found examples of two processes for which their respective infinitesimal increments are perfect positive correlated, but at each time they have zero terminal correlation, even when the time T is close to zero.

Let us now set apart the results of correlation and return to understanding why λ-mixture dynamics may be needed in pricing options. As we mentioned at the beginning of this work, one of the reasons λ-mixture dynamics may be appealed in pricing options is when we can consider the price of an option as a linear combination of prices of underlying assets that are driven by known instrumental processes that are tractable. In the case of pricing a call option leaded by λ-mixture of lognormal dynamics, we get

$$\begin{aligned}\Pi(T,K) &= e^{-\int_0^T r(l)dl} E_Q[(S_T-K)^+] = e^{-\int_0^T r(l)dl} \int_K^\infty (s-K) \sum_{i=1}^m \lambda_i p_{S_T^i}(s) ds \\ &= \sum_{i=1}^m \lambda_i e^{-\int_0^T r(l)dl} \int_K^\infty (s-K) p_{S_T^i}(s) ds = \sum_{i=1}^m \lambda_i \mathrm{BSCall}(S_0, K, T, R(T), V_i(T))\end{aligned}$$

This is appreciated by traders, who usually prefer departures from the lognormal distribution and the corresponding Black-Scholes-Merton formula. Brigo (2002) observed that other equally important consequences of using λ-mixtures of lognormal dynamics is that the number m of instrumental processes chosen in order to get better capabilities to calibrate is arbitrary, and these dynamics can price options analytically. In fact, when one runs an optimization to find the V_i's and λ_i's that best reproduce the prices of the market, the target function of the optimization can be found in a closed form making not necessary the use of methods like Monte Carlo simulation, trees, etc.

3.4 Uncertain volatility geometric Brownian motion

This last part is dedicated to find some relations between stochastic volatility models and local volatility models when one works with λ-mixture diffusion processes.

It is known (Brigo, 2002; Gatheral and Lynch, 2002) that every stochastic volatility model has a local volatility version that features the same marginal distributions in time and thus the same initial prices for all *vanilla options* such as European options. We may wonder if λ-lognormal mixture dynamics (13) are the local version of some stochastic volatility model. Brigo (2002) showed that the answer is affirmative. In fact, by considering the uncertain volatility model

$$dS_t = r(t)S_t dt + \xi_t S_t dW_t, \qquad S_0 = s_0, \qquad \xi_t = \begin{cases} t \mapsto v_1(t), & \text{with probability } \lambda_1 \\ \vdots & \\ t \mapsto v_m(t), & \text{with probability } \lambda_m \end{cases}, \tag{16}$$

with well-behaving, strictly positive functions v_i's such that $v_i(t) = \bar{v}$ on $[0, \epsilon]$ for all i, and ξ independent of W, we get that the process S conditional on each value of ξ has the same behavior as the Black-Scholes-Merton model. In fact, we can verify by using Itô's lemma for $\ln(S_t)$ and then conclude that (16) has solution of the form

$$S_t = S_0 \exp\left\{ \int_0^t \left(r(l) - \frac{\xi_l^2}{2} \right) dl + \int_0^t \xi_l dW_l \right\}.$$

To make some remarks about the conditional transition density of the model (16), let us denote S^i the process that solves SDE (16) with $\xi \equiv v_i$, and set the function $f_A(X,Y) := 1_{\{XY \in A\}}$ where X and Y are random variables, and $A \in \mathcal{B}_\mathbb{R}$.

For the case $t > \epsilon > u$, we get

$$E[S_t \in A|S_u] = E\left[\sum_{i=1}^{m} 1_{\{\xi=v_i\}} 1_{\{S_t^i \in A\}}\middle| S_u\right] = E\left[\sum_{i=1}^{m} 1_{\{\xi=v_i\}} f_A(S_u^i, S_t^i/S_u^i)\middle| S_u\right]$$

$$= \sum_{i=1}^{m} Q(\xi = v_i) \int_A f(S_u^i, l) dp_{(S_t^i/S_u^i)}(l) = \sum_{i=1}^{m} \lambda_i \int_A \frac{\exp\left\{\frac{-\left(\ln(l) - \int_u^t \left(r(z) - \frac{v_i^2(z)}{2}\right) dz - \ln(S_u^i)\right)^2}{2\int_u^t v_i^2(z)dz}\right\}}{l\left(\int_u^t v_i^2(z)dz\right)^{1/2}\sqrt{2\pi}} dl,$$

therefore, we conclude

$$Q\left(S_t \in A|S_u = y\right) = \sum_{i=1}^{m} \lambda_i Q\left(S_t \in A|S_u = y, \xi = v_i\right), \qquad p_{S_t|S_u}(x|y) = \sum_{i=1}^{m} \lambda_i p_{S_t|S_u,\xi}(x|y, v_i).$$

In other words, the conditional transition density of the uncertain volatility model (16) is a mixture of lognormals transition densities. Now, if we condition on an instant $u > \epsilon$, including the information of the value v_i taken by ξ at the time $\epsilon < u$, information that is contained in the trajectory of S up to time $u > \epsilon$, we then get the conditional transition density between u and t is a lognormal density characterized by the relevant v_i.

For the particular case $u = 0$, we get that this model has the same marginal as the lognormal mixture diffusion seen in (13), but we get market incompleteness. Market incompleteness and the details of the construction of the model (16) are showed in the following result that is a consequence of the results presented by Fabio Mercurio in (Brigo et al., 2004).

Proposition 3.2. *Let $(\tilde{W}_t)_{t\in[0,T]}$ be a Brownian motion on a filtered probability space $(\Omega^{\tilde{W}}, \mathcal{F}_T^{\tilde{W}}, \mathcal{F}_t^{\tilde{W}}, P^{\tilde{W}})$; and let $(\Omega^{\tilde{\xi}}, \mathcal{F}^{\tilde{\xi}}, P^{\tilde{\xi}})$ be probability space with $\Omega^{\tilde{\xi}} = \{w_1, ..., w_m\}$ and probability measure satisfying $P^{\tilde{\xi}}(w_j) := \lambda_j > 0$ for $j = 1, ..., m$. Set the process $(\tilde{\xi}_t)_{t\in[0,T]}$ such that $\tilde{\xi}_t(w_j) = v_j(t)$ for each j and $t \geq 0$, where $v_1, ..., v_m$ are continuous, bounded away from zero, strictly positive functions such that $v_i(t) = \bar{v}$ on $t \in [0, \epsilon)$ for each $i = 1, ..., m$ with $\epsilon > 0$.*

Set $\Omega := \Omega^{\tilde{\xi}} \times \Omega^{\tilde{W}}$, $\mathcal{F}_t := \mathcal{F}^{\tilde{\xi}} \otimes \mathcal{F}_t^{\tilde{W}}$, $P := P^{\tilde{\xi}} \otimes P^{\tilde{W}}$; and let $\xi_t(w, y) := \tilde{\xi}_t(w)$ and $W_t(w, y) := \tilde{W}_t(y)$ be defined on $(\Omega, \mathcal{F}_T, \mathcal{F}_t, P)$. If we define a measure Q^ξ in $(\Omega, \mathcal{F}_T)$ by

$$\frac{dQ^\xi}{dP}(w_j, y) = \frac{p_j}{\lambda_j} \qquad \textit{for all} \quad y \in \Omega^{\tilde{W}},$$

for each $j = 1, ..., m$, where $p_j's$ are strictly positive with $\sum_{i=1}^m p_i = 1$; and the process $(S_t)_{t\in[0,T]}$ satisfies

$$dS_t = S_t \mu(t)dt + S_t \xi_t dW_t \qquad S_0 = s_0;$$

then there exist a risk-neutral measure Q in $(\Omega, \mathcal{F}_T)$ associated with $(S_t)_{t\in[0,T]}$ and the numéraire $B_t = e^{\int_0^t r(u)du}$ that satisfies

$$\frac{dQ}{dP} = \frac{dQ^\xi}{dP} \exp\left\{-\frac{1}{2}\int_0^T \left(\frac{\mu(t) - r(t)}{\xi_t}\right)^2 dt - \int_0^T \left(\frac{\mu(t) - r(t)}{\xi_t}\right) dW_t\right\}.$$

Proof. To prove this result, it is enough to show that $(S_t/B_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t, Q)$-martingale (see for instance Harrison and Pliska, 1981; Shreve, 2004; Privault, 2013). We observe that $(W_t)_{t\in[0,T]}$ and $(\xi_t)_{t\in[0,T]}$ are independent on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$. Moreover, $(W_t)_{t\in[0,T]}$ is a $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$-Brownian motion, and $(\xi_t)_{t\in[0,T]}$ has the same law as $(\tilde{\xi}_t)_{t\in[0,T]}$.

To prove that Q is a measure on $\mathcal{F}_T$, it is enough to verify $E_P[dQ/dP] = 1$. Let us then set

$$L_t := -\int_0^t \left(\frac{\mu(l) - r(l)}{\xi_l}\right) dW_l, \quad L_t^i := -\int_0^t \left(\frac{\mu(l) - r(l)}{v_i(l)}\right) dW_l, \quad Z_t^i := \exp\left\{L_t^i - \frac{1}{2}\langle L^i\rangle_t\right\}.$$

The process $(Z_t^i)_{t\in[0,T]}$ satisfies $Z_t^i = 1 + \int_0^t Z_s^i dL_s^i$. By using Fubini's theorem and Proposition A.3, we get $E_P\left[\langle Z^i\rangle_T\right] < \infty$; therefore $(Z_t^i)_{t\in[0,T]}$ is martingale (see Appendix A) and then $E[Z_l^i] = 1$ for all $l \in [0,T]$. Finally, we use linear combination of the indicator functions $1_{\{w_i\}\times\Omega^{\bar{W}}}$ and independence to conclude $E_P[dQ/dP] = 1$.

Now, by Bayes' formula (see appendix C), linear combination of the indicator functions $1_{\{w_i\}\times\Omega^{\bar{W}}}$, and independence, we conclude

$$
\begin{aligned}
& E_Q\left[S_T e^{-\int_0^T r(u)du}\middle|\mathcal{F}_t\right] \\
&= \frac{E_P\left[S_T e^{-\int_0^T r(u)du}\frac{dQ}{dP}\middle|\mathcal{F}_t\right]}{E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right]} \\
&= \frac{E_P\left[S_T e^{-\int_0^T r(u)du}\frac{dQ^\xi}{dP}e^{L_t-\frac{1}{2}\langle L\rangle_t}e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]}{E_P\left[\frac{dQ^\xi}{dP}e^{L_T-\frac{1}{2}\langle L\rangle_T}\middle|\mathcal{F}_t\right]} \\
&= \frac{S_t e^{-\int_0^t r(u)du}\frac{dQ^\xi}{dP}e^{L_t-\frac{1}{2}\langle L\rangle_t}E_P\left[\frac{S_T}{S_t}e^{-\int_t^T r(u)du}e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]}{\frac{dQ^\xi}{dP}e^{L_t-\frac{1}{2}\langle L\rangle_t}E_P\left[e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]} \\
&= \frac{S_t e^{-\int_0^t r(u)du}E_P\left[e^{\int_t^T\left(\mu(u)-\frac{\xi_u^2}{2}-r(u)\right)du+\int_t^T\xi_u dW_u}e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]}{E_P\left[e^{(L_T-L_t)-\frac{1}{2}\langle L\rangle_t^T}\middle|\mathcal{F}_t\right]} \\
&= S_t e^{-\int_0^t r(u)du}E_P\left[e^{\int_t^T\left(\frac{\xi_u^2-(\mu(u)-r(u))}{\xi_u}\right)dW_u-\frac{1}{2}\int_t^T\left(\frac{\xi_u^2-(\mu(u)-r(u))}{\xi_u}\right)^2du}\middle|\mathcal{F}_t\right] \\
&= S_t e^{-\int_0^t r(u)du}
\end{aligned}
$$

which ends the proof. □

From this result the incompleteness of the market follows immediately: for each vector of strictly positive weights $(p_1, ..., p_m)$ we can get a risk neutral measure which implies lack of uniqueness of the risk-neutral measure and therefore market incompleteness.

Now, we shall see a close relationship between the model (13) and the stochastic volatility mixture dynamics (16) given by Brigo (2002).

Proposition 3.3. *The lognormal mixture diffusion dynamics (13) is the local volatility version of the stochastic volatility mixture dynamics (16). The two models are linked by the relationship*

$$\sigma^2_{mix}(t,x) = E[\xi_t^2 | S_t = x]$$

Proof. The proof follows from a variant of the Bayes' formula:

$$\begin{aligned} E[\xi_t^2|S_t = x] &= E\left[\xi_t^2 \sum_{i=1}^m 1_{\{\xi=v_i\}} \middle| S_t = x\right] = \sum_{i=1}^m E\left[\xi_t^2 1_{\{\xi=v_i\}} \middle| S_t = x\right] \\ &= \sum_{i=1}^m E\left[\xi_t^2 \middle| S_t = x, \xi = v_i\right] Q[\xi = v_i | S_t = x] = \sum_{i=1}^m v_i^2(t) Q[\xi = v_i|S_t = x] = \sigma^2_{\text{mix}}(t,x) \end{aligned}$$

since, by Bayes' formula,

$$Q[\xi = v_i|S_t = x] = \frac{Q[S_t \in dx|\xi = v_i]Q[\xi = v_i]}{Q[S_t \in dx]} = \Lambda_i(t,x).$$

□

4 Conclusions

In this work, we were able to get conditions to find a candidate diffusion process whose density follows a given mixture of probability densities. We derived SDEs admitting strong solutions whose densities evolves as a mixture of normal densities. We introduced the concept of terminal and instantaneous correlation, and found some features and differences in the two types of volatility models. In the part of applications, we showed how the mixture diffusion SDE process can be used to model market smiles. Finally, we showed how local volatility mixture diffusion processes are able to fit more general more general volatility models.

Appendix A

Throughout this part, all the processes considered are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$ where $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions(see Karatzas and Shreve, 2014). The following results have been derived from (Meyer, 2000, Section I.9 and III.4) and (Shreve, 2004, Chapter 4).

Proposition A.1. *If M is a $(\mathcal{F}_t, Q)$-continuous local martingale with $M_0 \in L^2$ and $\langle M\rangle_T \in L^1$, then M is a L^2-martingale in $[0,T]$*

Proof. Let (T_n) be a sequence of increasing stopping times that simultaneously reduces $M^2 - \langle M\rangle$ and M. Since $M_0^2 - \langle M\rangle_0 = M_0^2$ is integrable, we get $M^2_{t\wedge T_n} - \langle M\rangle_{t\wedge T_n}$ is a $(\mathcal{F}_t, Q)$-martingale (for example see Meyer, 2000, Section I.8). Thus, $E[M^2_{t\wedge T_n}] \leq E[M_0^2] + E[\langle M\rangle_T] < \infty$ for each $n \in \mathbb{N}$ and as result $(M_{t\wedge T_n})_{n\in\mathbb{N}}$ is uniformly integrable.

Since $M_{t\wedge T_n} \xrightarrow[n\to\infty]{a.s} M_t$ and $(M_{t\wedge T_n})_{n\in\mathbb{N}}$ is uniformly integrable, we have $M_{t\wedge T_n} \xrightarrow[n\to\infty]{L^1} M_t$ and therefore

$$M_s = \lim_{n\to\infty} M_{s\wedge T_n} = \lim_{n\to\infty} E[M_{t\wedge T_n}|\mathcal{F}_s] = E[M_t|\mathcal{F}_s]$$

for all $0 \leq s \leq t \leq T$ □

Proposition A.2. *Let M be a $(\mathcal{F}_t, Q)$-continuous local martingale. If $E[\sup_{l\leq t}|M_l|] < \infty$ for each $t \geq 0$, then M is martingale.*

Proof. Since M satisfies $E[\sup_{l\leq t}|M_l|] < \infty$ for each $t \geq 0$, we have for each sequence of stopping times (T_n) that reduces M,

$$M_{t\wedge T_n} \xrightarrow[n\to\infty]{a.s} M_t \qquad \text{and} \qquad M_{t\wedge T_m} \leq \sup_{l\leq t}|M_l|$$

for each $m \in \mathbb{N}$. Therefore, $M_{t\wedge T_n} \xrightarrow[n\to\infty]{L^1} M_t$, and hence $M_s = \lim_{n\to\infty} M_{s\wedge T_n} = \lim_{n\to\infty} E[M_{t\wedge T_n}|\mathcal{F}_s] = E[M_t|\mathcal{F}_s]$ for each $0 \leq s \leq t$. □

Proposition A.3. *Let $W = (W_t)_{t\geq 0}$ be a $(\mathcal{F}_t, Q)$-Brownian motion and v a well-behaving, real valued function. If $M_t = \int_0^t v(s)dW_s$, then $M_t \sim \mathcal{N}\left(0, \int_0^t v^2(s)ds\right)$ for each $t \geq 0$.*

Proof. We define the Doleans exponential of a process X as

$$\mathcal{E}_t(X) := \exp\left\{X_t - \frac{1}{2}\langle X\rangle_t\right\}.$$

Let us now set $M_t := \int_0^t v(s)dW_s$ and $Y_t := \mathcal{E}_t(2M)$. We know from basic results for L^2_{Loc} processes (see Revuz and Yor, 1999, Section IV.2) and Itô's lemma that Y and $\mathcal{E}^2(M)$ are local martingales; therefore, we can get an increasing sequence of stopping times (T_n) that simultaneously reduces $\mathcal{E}^2(M)$ and Y. Now, by the equality $\mathcal{E}^2_t(M) = Y_t \exp(\langle M\rangle_t)$, we get $\mathcal{E}^2_{t\wedge T_n}(M) = Y_{t\wedge T_n}\exp(\langle M\rangle_{t\wedge T_n}) \leq Y_{t\wedge T_n}\exp(\langle M\rangle_t)$.

Since $\langle \int_0^{\cdot} v(s)dW_s\rangle_t = \int_0^t v^2(s)ds < \infty$, we get, by using L^2-Doob's maximal inequality, that

$$E\left[\sup_{s\in[0,t]} \mathcal{E}^2_{s\wedge T_n}(M)\right] \leq 4E[\mathcal{E}^2_{t\wedge T_n}(M)] \leq 4E[Y_{t\wedge T_n}]\exp(\langle M\rangle_t) = 4E[Y_0]\exp(\langle M\rangle_t) = 4\exp(\langle M\rangle_t) < \infty.$$

Because $\sup_{s\in[0,t]}\mathcal{E}^2_{s\wedge T_n}(M) \uparrow \sup_{s\in[0,t]}\mathcal{E}^2_s(M)$ as $n \to \infty$, we obtain $E\left[\sup_{s\in[0,t]}\mathcal{E}^2_s(M)\right] < 4\exp(\langle M\rangle_t) < \infty$ and therefore $E\left[\sup_{s\in[0,t]}\mathcal{E}_s(M)\right] < \infty$ for each $t \geq 0$. By Proposition A.2, we conclude $\mathcal{E}(M)$ is martingale, and then $E[\mathcal{E}_t(M)] = 1$ for all $t \geq 0$.

By similar arguments to Proposition A.1 and A.2, we get M is a martingale that, by Itô's isometry, has mean 0 and variance $\int_0^t v^2(s)ds$ for each $t \geq 0$. From these results we conclude

$$E\left[\exp\left\{\lambda M_t - \frac{\lambda^2}{2}\int_0^t v^2(s)ds\right\}\right] = 1,$$

for all λ, and therefore M_t has the same moment-generating function of a normal random variable with mean 0 and variance $\int_0^t v^2(s)ds$. □

Appendix B

Let $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$ be a filtered probability space where $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions, and consider the dynamics of the well-behaving, strictly positive risky and riskless assets respectively $(S_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}_T, (\mathcal{F}_t), P)$ being given by

$$dS_t = \gamma(t)S_t dt + v(t)S_t dW_t, \qquad dB_t = r(t)B_t dt,$$

where $W=(W_t)_{t\in[0,T]}$ is a $(\mathcal{F}_t,P)$-Brownian motion. If we set $\tilde{S}_t=S_t/B_t$, then, by applying Itô's lemma to the function $f(x,y)=xy$ evaluated on the vector semimartingale $(S_t,e^{-\int_0^t r(s)ds})$ we get

$$d\tilde{S}_t=-r(t)e^{-\int_0^t r(s)ds}S_t dt+e^{-\int_0^t r(s)ds}dS_t=(\gamma(t)-r(t))\tilde{S}_t dt+\tilde{S}_t v(t)dW_t. \tag{17}$$

Now, we will find the risk-neutral measure by using Girsanov's theorem. Let us then set

$$D_t:=\frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}=\exp\left\{-\int_0^t\lambda(s)dW_s-\frac{1}{2}\int_0^t\lambda^2(s)ds\right\},$$

where $\lambda(s)=\dfrac{\gamma(s)-r(s)}{v(s)}$ and $L_t=-\int_0^t\lambda(s)dW_s$. Since D_t satisfies $D_t=1+\int_0^t D_s dL_s$, we get that D is continuous local martingale. Moreover, by results showed in the proof Proposition A.3, we get $E[\langle D\rangle_T]<\infty$, and hence D is a martingale in $[0,T]$. Now, by Girsanov's theorem we have that

$$\tilde{W}_t=W_t-\langle W,L\rangle_t=W_t+\int_0^t\frac{\gamma(s)-r(s)}{v(s)}ds$$

is a $(\mathcal{F}_t,Q)$-Brownian motion. Then we get from (17):

$$d\tilde{S}_t=\tilde{S}_t v(t)d\tilde{W}_t,\qquad \tilde{S}_t=\tilde{S}_0\exp\left\{\int_0^t v(s)d\tilde{W}_s-\frac{1}{2}\int_0^t v^2(s)ds\right\},$$

and therefore $\tilde{S}$ is a $(\mathcal{F}_t,Q)$-martingale in $[0,T]$ (see proof of Proposition A.3). Finally, if S satisfies $dS_t=S_t\gamma(t)dt+v(t)S_t dW_t$ under the measure P, we get S also satisfies the $dS_t=S_t r(t)dt+v(t)S_t d\tilde{W}_t$ under the measure Q, and the reverse is also true. Moreover the arbitrage-free price does not depend on γ for the case of European options (see for instance Elliott and Kopp, 2006, Section 7.5). This shows that dynamics of our risky asset $(S_t)_{t\in[0,T]}$ under the risk-neutral measure Q can be assumed to be of the form (8) with numéraire $(B_t)_{t\in[0,T]}$ satisfying dynamics of the type $dB_t=r(t)B_t dt$.

Appendix C

Proposition C.1. *Let $(\Omega,\mathcal{F})$ a measurable space endowed with two probability measures P and Q such that $Q\ll P$. Then for any sub σ-algebra $\mathcal{G}$ and non-negative random variable X on $(\Omega,\mathcal{F})$ we get*

$$E_Q[X|\mathcal{G}]E_P\left[\frac{dQ}{dP}\middle|\mathcal{G}\right]=E_P\left[X\frac{dQ}{dP}\middle|\mathcal{G}\right].$$

Proof. By definition of conditional expectation, it is enough to prove

$$\int_A E_Q[X|\mathcal{G}]E_P\left[\frac{dQ}{dP}\middle|\mathcal{G}\right]dP=\int_A E_P\left[X\frac{dQ}{dP}\middle|\mathcal{G}\right]dP,$$

for each $A\in\mathcal{G}$. By using Radon-Nikodym's theorem and basic conditional expectation properties, we get

$$\begin{aligned}\int_A E_P\left[X\frac{dQ}{dP}\middle|\mathcal{G}\right]dP=\int_A XdQ=\int_A\frac{dQ}{dP}E_Q[X|\mathcal{G}]dP&=\int_A E_P\left[\frac{dQ}{dP}E_Q[X|\mathcal{G}]\middle|\mathcal{G}\right]dP\\&=\int_A E_Q[X|\mathcal{G}]E_P\left[\frac{dQ}{dP}\middle|\mathcal{G}\right]dP,\end{aligned}$$

which ends the proof. ∎

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