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ADELIC AHLFORS-BERS THEORY

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PRESENTA:
JUAN MANUEL BURGOS

TUTOR PRINCIPAL:
DR. ALBERTO VERJOVSKY SOLÁ
INSTITUTO DE MATEMÁTICAS UNAM-UNIDAD CUERNAVACA

MIEMBROS DEL COMITÉ TUTOR:
DR.WILSON ZUÑIGA
DEPARTAMENTO DE MATEMÁTICAS CINVESTAV-UNIDAD QUERÉTARO

DR.TIMOTHY GENDRON
INSTITUTO DE MATEMÁTICAS UNAM-UNIDAD CUERNAVACA

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To my parents.

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Contents

Introduction

The present work is a generalization of Ahlfors-Bers theory to the *algebraic solenoid* $\mathbb{C}^*_{\mathbb{Q}}$ defined as the following inverse limit: Consider the inverse system of coverings $\{\ p_{n,m} : \mathbb{C}^* \to \mathbb{C}^*\ \}_{n,m \geq 1, n|m}$ where $p_{n,m}(z) = z^{m/n}$ and $\mathbb{C}^*$ is the punctured plane. Then,

$$\mathbb{C}^*_{\mathbb{Q}} := \varprojlim_{p_n} \mathbb{C}^*$$

with canonical projections π_n. Along the same lines we define the *adelic Riemann sphere* $\hat{\mathbb{C}}_{\mathbb{Q}}$ as the inverse limit of branched coverings $p_{n,m} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ where $\hat{\mathbb{C}}$ is the Riemann sphere and the *adelic solenoid* $S^1_{\mathbb{Q}}$ as the inverse limit of coverings $p_{n,m} : S^1 \to S^1$. The functoriality of the construction implies the inclusion $S^1_{\mathbb{Q}} \subset \mathbb{C}^*_{\mathbb{Q}} \subset \hat{\mathbb{C}}_{\mathbb{Q}}$ such that $\hat{\mathbb{C}}_{\mathbb{Q}} = \mathbb{C}^*_{\mathbb{Q}} \cup \{\mathbf{0}, \infty\}$ where

$$\mathbf{0} = [(0, 0, 0 \ldots)]$$

$$\infty = [(\infty, \infty, \infty, \ldots)]$$

are cusps (cone over the adelic solenoid) and $\mathbb{C}^*_{\mathbb{Q}}$ is laminated by densely immersed leaves $\nu_a : \mathbb{C} \to \mathbb{C}^*$ where $a \in \hat{\mathbb{Z}}$, the adelic completion of integers. The algebraic solenoid can also be seen as a non trivial fiber bundle $\pi_1 : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*$ with fiber the adelic completion of integers $\hat{\mathbb{Z}}$ and also as a laminated Lie group with unit:

$$\mathbf{1} = [(1, 1, 1, \ldots)]$$

Definition 0.0.1. A homeomorphism h of $\hat{\mathbb{C}}_{\mathbb{Q}}$ is leaf preserving quasiconformal if it fixes $0, \infty$ and h_a is quasiconformal for every $a \in \hat{\mathbb{Z}}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{h_a} & \mathbb{C} \\
\downarrow{\scriptstyle \nu_a} & & \downarrow{\scriptstyle \nu_a} \\
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{h} & \mathbb{C}^*_{\mathbb{Q}}
\end{array}
$$

Definition 0.0.2. We say that $\mu \in L_\infty(\mathbb{C}^*_{\mathbb{Q}})$ is *horizontal* if $\nu_a^*(\mu) \in L_\infty(\mathbb{C})$ for every $a \in \hat{\mathbb{Z}}$. We denote the subspace horizontal elements by $L_{\infty,hor}(\mathbb{C}^*_{\mathbb{Q}})$.

Definition 0.0.3. Consider $\mu \in L_{\infty,hor}(\mathbb{C}^*_{\mathbb{Q}})_1$; i.e. $||\mu||_\infty < 1$. A leaf preserving quasiconformal map h of $\hat{\mathbb{C}}_{\mathbb{Q}}$ is a solution of the μ-Beltrami equation if h_a is a solution of the $(\nu_a)^*(\mu)$-Beltrami equation:

$$\partial_{\bar{z}} h_a = (\nu_a)^*(\mu)\ \partial_z h_a$$

in the distributional sense for every $a \in \hat{\mathbb{Z}}$ and the following diagram commutes for every $a \in \hat{\mathbb{Z}}$:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\ h_a\ } & \mathbb{C} \\
{\scriptstyle \nu_a}\downarrow & & \downarrow{\scriptstyle \nu_a} \\
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ h\ } & \mathbb{C}^*_{\mathbb{Q}}
\end{array}
$$

Remark 0.0.1. Consider $\mu \in L_{\infty,hor}(\mathbb{C}^*_{\mathbb{Q}})_1$ such that there is a leaf preserving quasiconformal solution to the μ-Beltrami equation. Because the map $a \mapsto h_a$ is continuous and the relation:

$$
(\nu_a)^*(\mu) = \frac{\partial_{\bar{z}} h_a}{\partial_z h_a}
$$

we have that the map $a \mapsto (\nu_a)^*(\mu)$ is continuous. This is a necessary condition for the existence of Beltrami solutions. Beltrami differentials satisfying this property will be called *vertical continuous* and the space of them will be denoted by $L^{vert}_\infty(\mathbb{C}^*_{\mathbb{Q}})$.

We look for vertical continuous Beltrami differentials $\mu \in L^{vert}_{\infty,hor}(\mathbb{C}^*_{\mathbb{Q}})_1$ such that there is a leaf preserving quasiconformal solution to the μ-Beltrami equation; i.e. Beltrami differentials such that the Ahlfors-Bers theorem holds. There is an obvious set of Beltrami differentials satisfying this property: Define the *periodic adelic Beltrami differentials*:

$$
Per = \bigcup_{n \in \mathbb{N}} Per_n
$$

such that $Per_n = \pi_n^*\big(L_\infty(\mathbb{C}^*)_1\big)$. See that

$$
Per_1 \subset Per_2 \subset Per_3 \ldots \subset Per
$$

If $\mu \in Per$ then there is a natural n and a Beltrami differential $\mu_n \in L_\infty(\mathbb{C}^*)_1$ such that $\mu = \pi_n^*(\mu_n)$. Consider the unique quasiconformal map f_n of the μ_n-Beltrami equation such that f_n fixes $0, 1, \infty$. There is a leaf preserving map f such that $f(1_{\mathbb{Q}}) = 1$ and the following diagram commutes:

$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \xrightarrow{\ f_n\ } & \hat{\mathbb{C}} \\
{\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} \\
\hat{\mathbb{C}}_{\mathbb{Q}} & \xrightarrow{\ f\ } & \hat{\mathbb{C}}_{\mathbb{Q}}
\end{array}
$$

Then, f is the leaf preserving quasiconformal solution to the μ-Beltrami equation such that f fixes $0, 1, \infty$.

Not every Beltrami differential $\mu \in L^{vert}_{\infty,hor}(\mathbb{C}^*_{\mathbb{Q}})_1$ admits a leaf preserving quasiconformal solution to the μ-Beltrami equation. In fact, there is a continuous complex function $\mu : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}$ such that $\|\mu\|_\infty < 1$ and:

$$
\nu_0^*(\mu)(z) = \frac{\frac{1}{2e} \sum_{n=1}^{+\infty} \left[\cos(x/n!) - \frac{2iy}{n!} \sin(x/n!)\right] \frac{e^{-\frac{y^2}{n!^2}}}{2n!}}{1 + \frac{1}{2e} \sum_{n=1}^{+\infty} \left[\cos(x/n!) + \frac{2iy}{n!} \sin(x/n!)\right] \frac{e^{-\frac{y^2}{n!^2}}}{2n!}} \frac{d\bar{z}}{dz}
$$

where $z = x + iy$. However, there is no leaf preserving quasiconformal solution to the μ-Beltrami equation.

It turns out that another condition is needed to assure the existence of Beltrami solutions. We introduce the renormalization maps $\mathcal{I}_n$:

Definition 0.0.4. Consider the normalized Haar measure η on $\hat{\mathbb{Z}}$ and the induced measure on the fibers $\pi_1^{-1}(x) \simeq \hat{\mathbb{Z}}$. We define the n-th renormalization map as the linear operator $\mathcal{I}_n : L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*) \to L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)$ such that:

$$\mathcal{I}_n(\mu)(x) = n \int_{\pi_n^{-1}(\pi_n(x))} d\eta \; \mu$$

Lema 0.0.1. *Consider $\mu \in L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)$. Then, the sequence $\big(\mathcal{I}_n(\mu)\big)_{n \in \mathbb{N}}$ converges uniformly to μ respect to the divisibility net.*

By definition, for every natural n there is $\mu_n \in L_\infty(\mathbb{C}^*)$ such that $\mathcal{I}_n(\mu) = \pi_n^*(\mu_n)$. In particular, we have that:
$$\text{Interior } \overline{Per}^\infty = L_{\infty,hor}^{vert}(\mathbb{C}_\mathbb{Q}^*)_1$$

Because not every Beltrami differential $\mu \in L_{\infty,hor}^{vert}(\mathbb{C}_\mathbb{Q}^*)_1$ admits a leaf preserving quasiconformal solution to the μ-Beltrami (example above), it is natural to ask for a finer topology $\mathcal{T}$ such that:

$$Bel(\mathbb{C}_\mathbb{Q}) = \text{Interior } \overline{Per}^\mathcal{T}$$

is a set of Beltrami differentials admitting Beltrami solutions.

Definition 0.0.5. Consider a cofinal totally ordered divisibility subsystem $\mathcal{S} = (n_j)_{j \in \mathbb{N}}$. We define the *renormalized metric*:

$$||\mu||_{Ren,\mathcal{S}} = n_1 \, ||\mathcal{I}_{n_1}(\mu)||_\infty \; + \; \sum_{j=1}^\infty n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \infty$$

Definition 0.0.6. We define the space of *adelic Beltrami differentials $Bel_\mathcal{S}$* as follows:

$$Bel_\mathcal{S} = \text{Interior } \overline{Per}^\mathcal{S} \subset L_{\infty,hor}^{vert}(\mathbb{C}_\mathbb{Q}^*)$$

where the closure is respect to metric topology induced by the metric $|| \cdot ||_{Ren,\mathcal{S}}$.

In particular, the adelic Beltrami differentials are the elements of $L_{\infty,hor}^{vert}(\mathbb{C}_\mathbb{Q}^*)$ with finite renormalized metric. The following is the adelic Ahlfors-Bers Theorem and it is the main result of the thesis:

Theorem 0.0.2. *Consider a cofinal totally ordered divisibility subsystem $\mathcal{S}$. For every adelic Beltrami differential $\mu \in Bel_\mathcal{S}$ there is a unique quasiconformal leaf preserving solution $f : \hat{\mathbb{C}}_\mathbb{Q} \to \hat{\mathbb{C}}_\mathbb{Q}$ to the μ-Beltrami equation such that f fixes $0, 1, \infty$.*

Consider the adelic hyperbolic space $H_{\mathbb{Q}} = \pi_1^{-1}(D(0,1))$ with boundary the adelic solenoid $S_{\mathbb{Q}}^1 = \pi_1^{-1}(S^1)$. Define the space of Beltrami differentials $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$ as those differentials $\mu \in Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}^*)$ such that:

$$\mu(z) = \mu(1/\bar{z}) \left(\frac{\bar{z}}{z}\right)^2$$

By the adelic Ahlfors-Bers Theorem, there is a unique leaf preserving quasiconformal solution f^μ to the μ-Beltrami equation such that f^μ fixes $0, 1, \infty$ and:

$$f^\mu(1/\bar{z}) = \frac{1}{\overline{f^\mu(z)}}$$

In particular, the solenoid $S_{\mathbb{Q}}^1$ and hence $H_{\mathbb{Q}}$ are invariant under f^μ. Define the following equivalence relation on $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$: $\mu \sim \eta$ if $f^\mu|_{S_{\mathbb{Q}}^1} = f^\eta|_{S_{\mathbb{Q}}^1}$. The universal Teichmüller space is defined as the quotient:

$$\mathcal{B} : Bel_{\mathcal{S}}(H_{\mathbb{Q}}) \to T_{\mathcal{S}}(1) = Bel_{\mathcal{S}}(H_{\mathbb{Q}})/ \sim$$

with the quotient topology induced by $(Bel_{\mathcal{S}}(H_{\mathbb{Q}}), \, ||\cdot||_{\mathcal{S}})$.

Definition 0.0.7. A leaf preserving homeomorphism $f : S_{\mathbb{Q}}^1 \to S_{\mathbb{Q}}^1$ is quasisymmetric if for every leaf ν_a the map $f_a = \nu_a^{-1} \circ f \circ \nu_a$ is quasisymmetric. Denote the space of these quasisymmetric maps by $QS(S_{\mathbb{Q}}^1)$ and by $QS(S_{\mathbb{Q}}^1)_1$ the subspace of those maps fixing 1.

We have an injective map $\Phi : T_{\mathcal{S}}(1) \to QS(S_{\mathbb{Q}}^1)_1$ such that $\mu \mapsto f^\mu|_{S^1}$. Denote by $QS_{\mathcal{S}}(S_{\mathbb{Q}}^1)_1$ the image of Φ. See that $Diff^m(S_{\mathbb{Q}}^1) \subset QS(S_{\mathbb{Q}}^1)$ for every $m = 1, 2, \ldots \infty$. Define $Diff_{\mathcal{S}}^m(S_{\mathbb{Q}}^1)_1 = QS_{\mathcal{S}}(S_{\mathbb{Q}}^1)_1 \cap Diff^m(S_{\mathbb{Q}}^1)$ for every $m = 1, 2, \ldots \infty$.

In particular we have the Nag-Verjovsky map:

$$\iota : Diff_{\mathcal{S}}^\infty(S_{\mathbb{Q}}^1)_1 \hookrightarrow T_{\mathcal{S}}(1)$$

There is a complex structure on $T_{\mathcal{S}}(1)$ induced by the complex structure on $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$: $\mu \mapsto i\mu$.

Consider the tangent space at the identity $T_{id}Diff^\infty(S_{\mathbb{Q}}^1)_1$ of C^∞-vector fields v such that:

$$\nu^*(v)(x) = \sum_{q \in \mathbb{Q}} a_q e^{iqx}$$

where $a_{-q} = \overline{a_q}$ and $a_0 = 0$. Define an (a priori) almost complex structure $\hat{J}$ such that:

$$\nu^*(\hat{J}v) = \sum_{q \in \mathbb{Q}} -i \; sg(q) a_q e^{iqx}$$

and translate it by conjugation with the adjoint map Ad to the whole tangent bundle $T \, Diff^\infty(S_{\mathbb{Q}}^1)_1$.

Consider a prime number p and define the inverse system $\mathcal{P} = \left(p^n\right)_{n \in \mathbb{N}}$. Consider the p-adic solenoid S_p^1. The following is the second major result of the thesis:

Theorem 0.0.3. *The p-adic Nag-Verjovsky map is differentiable analytic.*

The present work was born in the spirit of connecting Teichmüller theory with the adelic world. Besides the intrinsic beauty of this idea, this connection seems promising in future applications both in Mathematics and Theoretical Physics. In fact, S.Mochizuki [Mo] claims having proved the ABC conjecture in his formidable theory of p-adic and inter universal Teichmüller theory. The p-adic Teichmüller space proposed here is by far a more humble and down to earth version.

Concerning Dynamical systems theory, D.Sullivan [Su] studies the linking between universalities of Milnor-Thurston, Feigenbaum's (quantitative) and Ahlfors-Bers. As he points out, $S_2^1 \times S^1$ (his second example) is the basic solenoidal surface required in the dynamical theory of Feigenbaum's Universality [Fe]. Here S_2^1 is the 2-adic solenoid. We hope that the theory of adelic Beltrami differentials developed in this work shed some new light on these universalities.

In theoretical physics, one of the main problems of string theory is the lack of an exact theory. Actually, the theory is formulated as an asymptotic perturbative series from the very beginning [Na3] [Ne]. As an attempt to solution, Bowick and Rajeev [BR1], [BR2] propose a theory based on the Kähler geometry of the homogeneous space $Diff(S^1)/S^1$. This space was studied before by Kirillov [KY],[Ki]. At the end of that paper, they comment: "...It is of interest to know the relation between the universal Teichmüller space with which they work and $Diff(S^1)/S^1$". An important contribution in this direction was given by S.Nag and A.Verjovsky in [NV] where they find an analytic isometric map:

$$\iota : Diff^+(S^1)/M\ddot{o}b \hookrightarrow Homeo_{qs}^+(S^1)/M\ddot{o}b \simeq T(1)$$

where the left hand side is seen as a Virasoro coadjoint orbit. In accordance with R.Bowen and his result on the quasicircles Hausdorff dimension [Bo], at the end of [NV] it is shown that this map is transverse to any Teichmüller space of genus $g \geq 2$ Riemann surfaces: $\iota \pitchfork \mathcal{T}_g = T(G)$ where $\sum_g = \Delta/G$ is a Riemann surface of genus $g \geq 2$ and Δ is the hyperbolic disk.

In this sense, our work generalizes the Nag-Verjovsky map taking the solenoid S_∞^1 and the hyperbolic punctured disk Δ_∞^* (in Sullivan's notation, example four in [Su]) instead of their classical non laminated analogs. In fact, this was the guiding star of this work.

We briefly comment on the structure of this thesis, chapter description and results obtained: Chapter 1 introduces the adelic solenoid and the algebraic adelic solenoid. We will see them in three different ways:

- The algebraic solenoid is a topological group that fibers over the punctured complex plane with fiber the adelic completion of the integers $\hat{\mathbb{Z}}$. We have the following short exact sequence in the category of groups:

$$0 \longrightarrow \hat{\mathbb{Z}} \overset{\phi}{\longrightarrow} \mathbb{C}_\mathbb{Q}^* \overset{\pi_1}{\longrightarrow} \mathbb{C}^* \longrightarrow 1$$

This is Lemma 1.1.4.

- The algebraic solenoid is a topological space foliated by complex plane leaves. We have the following short exact sequence in the category of groups:

$$0 \longrightarrow \mathbb{Z} \overset{\iota}{\longrightarrow} \hat{\mathbb{Z}} \times \mathbb{C} \overset{exp}{\longrightarrow} \mathbb{C}_\mathbb{Q}^* \longrightarrow 1$$

  This is Lemma 1.1.7.

- The algebraic solenoid is the complex dynamical suspension of the shift $T : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ such that $T(x) = x + 1$.

As far as the author knows, the rest of the thesis is completely original. Given a continuous map on the algebraic solenoid $f : \mathbb{C}_\mathbb{Q}^* \to \mathbb{C}_\mathbb{Q}^*$ we develop the homotopic invariant $deg(f) \in \mathbb{Q}$. This invariant is the extension of the classical degree in the following sense: *For every continuous map $f : \mathbb{C}^* \to \mathbb{C}^*$ there is a unique continuous baseleaf preserving map $\hat{f}$ such that:*

$$\begin{array}{ccc} \mathbb{C}_\mathbb{Q}^* & \overset{\hat{f}}{\longrightarrow} & \mathbb{C}_\mathbb{Q}^* \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{C}^* & \overset{f}{\longrightarrow} & \mathbb{C}^* \end{array}$$

and $deg(f) = deg(\hat{f})$.

This is corollary 1.4.2. Following these ideas in the analytic category, we prove the Picard theorem for the adelic Riemann sphere $\hat{\mathbb{C}}_\mathbb{Q}$:

$$Pic(\hat{\mathbb{C}}_\mathbb{Q}) \simeq (\mathbb{Q}, +)$$

This is Proposition 1.4.5.

Chapter 2 develops the Renormalization technique to the adelic solenoid. This is the main tool we are going to use later in the construction of structures and objects on the solenoid. The first application of it gives Corollary 2.2.3: *For every C^∞ function $f : S_\mathbb{Q}^1 \to \mathbb{C}$ its Pontryagin series converges in the C^∞-topology.*

Chapter 3 is the heart of the thesis. It begins with an introduction, the definition of the adelic Riemann sphere and a series of preliminary technical Lemmas. Among these we have:

$$Hol(\hat{\mathbb{C}}_\mathbb{Q}) \simeq \mathbb{C}$$

The next section is a careful discussion of adelic Beltrami differentials mentioned before. The following section is devoted to the Ahlfors-Bers Theorem proof where the previous Renormalization technique is strongly used. The chapter ends with the infinitesimal counterpart.

Chapter 4 starts describing different equivalent Teichmüller models. This is a straightforward generalization of the classical models. The relation with Sullivan's work is described in Proposition 4.1.1: There is a canonical continuous injective map

$$T_\mathcal{S}(1) \hookrightarrow T_{Sullivan}(\Delta_\infty^*)$$

where $\mathcal{S}$ is a cofinal totally ordered divisibility sequence. After that, we immediately specialize in the p-adic case and prove the p-adic version of the Nag-Verjovsky map mentioned before.

Chapter 1

Adelic solenoid

1.1 Adelic solenoid

In what follows we will identify the group $U(1)$ with the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and the finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ with the group of nth roots of unity in S^1.

By covering space theory, for any integer $n \geq 1$, it is defined the unbranched covering space of degree n, $p_n : S^1 \to S^1$ given by $z \longmapsto z^n$. If $n, m \in \mathbb{Z}^+$ and n divides m, then there exists a covering map $p_{n,m} : S^1 \to S^1$ such that $p_n \circ p_{n,m} = p_m$ where $p_{n,m}(z) = z^{m/n}$. We also denote with the same letters the restriction of p_n and $p_{n,m}$ to the nth roots of unity. In particular we have the relation:

$$p_{n,m} \circ p_{m,l} = p_{n,l}$$

This defines an inverse system of covering spaces $\left(S^1, p_{n,m}\right)_{n,m \geq 1, n|m}$ whose inverse limit is the *universal one–dimensional solenoid* or *adelic solenoid*

$$S^1_{\mathbb{Q}} := \varprojlim_{p_n} S^1.$$

Thus $S^1_{\mathbb{Q}}$ consists of sequences $(z_n)_{n \in \mathbb{N}, \, z \in S^1}$ which are compatible with p_n i.e. $p_{n,m}(z_m) = z_n$ if n divides m.

The canonical projections of the inverse limit are the functions $S^1_{\mathbb{Q}} \overset{\pi_n}{\to} S^1$ defined by $\pi_n\left((z_j)_{j \in \mathbb{N}}\right) = z_n$. Each π_n is an epimorphism. In particular each π_n is a character which determines a locally trivial $\hat{\mathbb{Z}}$–bundle structure where the group

$$\hat{\mathbb{Z}} := \varprojlim_{p_n} \mathbb{Z}/m\mathbb{Z}$$

is the profinite completion of $\mathbb{Z}$, which is a compact, perfect and totally disconnected Abelian topological group homeomorphic to the Cantor set. Being $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$, it admits a canonical inclusion of $\mathbb{Z} \subset \hat{\mathbb{Z}}$ whose image is dense. We have an inclusion $\hat{\mathbb{Z}} \overset{\phi}{\to} S^1_{\mathbb{Q}}$ and a short exact sequence $0 \to \hat{\mathbb{Z}} \overset{\phi}{\to} S^1_{\mathbb{Q}} \overset{\pi_1}{\to} S^1 \to 1$.

The solenoid $S^1_{\mathbb{Q}}$ can also be realized as the orbit space of the $\mathbb{Q}$-bundle structure $\mathbb{Q} \hookrightarrow \mathbb{A} \to \mathbb{A}/\mathbb{Q}$, where $\mathbb{A}$ is the adèle group of the rational numbers which is a locally compact Abelian group, $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$ and $\mathbb{A}/\mathbb{Q} \cong S^1_{\mathbb{Q}}$ is a compact Abelian group (see [RV]). From this perspective, $\mathbb{A}/\mathbb{Q}$ can be seen as the inverse limit whose n–th component corresponds to the unique covering of degree $n \geq 1$ of $S^1_{\mathbb{Q}}$. The solenoid $S^1_{\mathbb{Q}}$ is also called the *algebraic universal covering space* of the circle S^1. The Grothendieck Galois group of the covering is $\hat{\mathbb{Z}}$, the *algebraic fundamental group* of $S^1_{\mathbb{Q}}$.

By considering the properly discontinuously free action of $\mathbb{Z}$ on $\hat{\mathbb{Z}} \times \mathbb{R}$ given by

$$n \cdot (x, t) = (x + n, t - n), \quad (n \in \mathbb{Z}, \ x \in \hat{\mathbb{Z}}, \ t \in \mathbb{R})$$

The solenoid $S^1_{\mathbb{Q}}$ is identified with the orbit space $\hat{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{R}$. Here, $\mathbb{Z}$ is acting on $\mathbb{R}$ by covering transformations and on $\hat{\mathbb{Z}}$ by translations. The path–connected component of the identity element $1 \in S^1_{\mathbb{Q}}$ is called the *baseleaf* [Od] and will be denoted by $\mathbb{R}_{BL}$. Clearly, $\mathbb{R}_{BL}$ is the image of $\{0\} \times \mathbb{R}$ under the canonical projection $exp : \hat{\mathbb{Z}} \times \mathbb{R} \to S^1_{\mathbb{Q}}$ defined below and it is a densely embedded copy of $\mathbb{R}$.

Hence $S^1_{\mathbb{Q}}$ is a compact, connected, Abelian topological group and also a one-dimensional lamination where each "leaf" is a simply connected one-dimensional manifold, homeomorphic to the universal covering space $\mathbb{R}$ of S^1, and a typical "transversal" is isomorphic to the Cantor group $\hat{\mathbb{Z}}$. The solenoid $S^1_{\mathbb{Q}}$ also has a leafwise C^∞ Riemannian metric (i.e., C^∞ along the leaves) which renders each leaf isometric to the real line with its standard metric dx. So, it makes sense to speak of a rigid translation along the leaves. The leaves also have a natural order equivalent to the order of the real line hence also an orientation.

Summarizing the above discussion we have the commutative diagram:

$$
\begin{array}{ccccccccc}
S^1_{\mathbb{Q}} = \lim S^1 & \longrightarrow & \dots S^1 & \xrightarrow{p_{m,n}} & S^1 & \longrightarrow & \dots S^1 \\
\phi \uparrow & & \uparrow{\scriptstyle l \mapsto e^{2\pi i l/n}} & & \uparrow{\scriptstyle l \mapsto e^{2\pi i l/m}} & & \uparrow{\scriptstyle 0 \mapsto 1} \\
\hat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \dots \mathbb{Z}/n\mathbb{Z} & \xrightarrow{p_{m,n}} & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \dots \{0\}
\end{array}
$$

where $\hat{\mathbb{Z}}$ is the adelic profinite completion of the integers and the image of the group monomorphism $\phi : (\hat{\mathbb{Z}}, +) \to (S^1_{\mathbb{Q}}, \cdot)$ is the *principal fiber*. We notice that $\pi_n(x) = \pi_n(y)$ implies $\pi_n(y^{-1}x) = 1$ and therefore $y^{-1}x = \phi(a)$ where $a \in n\hat{\mathbb{Z}}$ for some $n \in \mathbb{Z} \subset \hat{\mathbb{Z}}$.

Lema 1.1.1. *The following is a short exact sequence:*

$$0 \longrightarrow \hat{\mathbb{Z}} \xrightarrow{\phi} S^1_{\mathbb{Q}} \xrightarrow{\pi_1} S^1 \longrightarrow 1$$

and we have the commutative diagram:

$$0 \longrightarrow \hat{\mathbb{Z}} \xrightarrow{\phi} S_{\mathbb{Q}}^1 \xrightarrow{\pi_1} S^1 \longrightarrow 1$$

$$0 \longrightarrow n\hat{\mathbb{Z}} \xrightarrow{\phi} S_{\mathbb{Q}}^1 \xrightarrow{\pi_n} S^1 \longrightarrow 1$$

with vertical maps $=$ in the middle and p_n on the right.

Proof:

- By definition the following diagram commutes:

$$
\begin{array}{ccc}
S_{\mathbb{Q}}^1 & \xrightarrow{\pi_1} & S^1 \\
\phi \uparrow & & \uparrow 0 \mapsto 1 \\
\hat{\mathbb{Z}} & \longrightarrow & \{0\}
\end{array}
$$

In particular $\pi_1 \circ \phi = 1$ and $Im(\phi) \subset Ker(\pi_1)$. Suppose that $\pi_1(x) = 1$. Then

$$x = (\ldots, a_n, \ldots, a_m, \ldots 1) = (\ldots, e^{2\pi i b_n/n}, \ldots, e^{2\pi i b_m/m}, \ldots 1) = \phi(y)$$

such that $y = (\ldots, b_n, \ldots, b_m, \ldots 0)$. We have proved that $Ker(\pi_1) \subset Im(\phi)$. Because π_1 is an epimorphism and ϕ is a monomorphism we have the first item.

- For the second item, the second exact sequence follows exactly from the same arguments as the first. Because $\pi_1 = z^n \circ \pi_n$, we have the right commutative square. The left square is trivial (diagram chasing).

$\square$

We define the *principal baseleaf* as the image of the monomorphism $\nu : \mathbb{R} \to S_{\mathbb{Q}}^1$ defined as follows:

$$
\begin{array}{ccccccccc}
S_{\mathbb{Q}}^1 = \lim S^1 & \longrightarrow & \ldots S^1 & \xrightarrow{p_{m,n}} & S^1 & \longrightarrow & \ldots S^1 \\
\nu \uparrow & & \uparrow t \mapsto e^{it/n} & & \uparrow t \mapsto e^{it/m} & & \uparrow t \mapsto e^{it} \\
\mathbb{R} & \xrightarrow{=} & \ldots \mathbb{R} & \xrightarrow{=} & \mathbb{R} & \xrightarrow{=} & \ldots \mathbb{R}
\end{array}
$$

In particular, the inmersion ν is a group morphism such that $\nu(2\pi x) = \phi(x)$ for every integer x. Define:

$$exp : \ \hat{\mathbb{Z}} \times \mathbb{R} \to S_{\mathbb{Q}}^1$$

such that $exp(a, \theta) = \phi(a).\nu(\theta)$.

Lema 1.1.2. *We have the short exact sequence:*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \hat{\mathbb{Z}} \times \mathbb{R} \xrightarrow{exp} S_{\mathbb{Q}}^1 \longrightarrow 1$$

such that $\iota(a) = (a, -2\pi a)$.

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Proof: exp is epimorphism: Consider $x \in S^1_{\mathbb{Q}}$ and $a \in \mathbb{R}$ such that $e^{ia} = \pi_1(x)$. Because $\pi_1 \circ \nu = e^{i\theta}$ we have that $\pi_1(\nu(a)) = e^{ia} = \pi_1(x)$; i.e. $\pi_1(\nu(a)^{-1}x) = 1$. By Lemma 1.1.4 there is an adelic integer $b \in \hat{\mathbb{Z}}$ such that $\phi(b) = \nu(a)^{-1}x$; i.e. $x = \phi(b)\nu(a) = exp(b,a)$.

Ker(exp): Suppose that $exp(a, \theta) = \phi(a)\nu(\theta) = 1$. Then $\phi(a) = \nu(-\theta)$ and composing with π_1 we have $1 = e^{-i\theta}$ and $\theta = 2\pi k$ for some integer k. Then

$$1 = exp(a, 2\pi k) = \phi(a)\nu(2\pi k) = \phi(a)\phi(k) = \phi(a+k)$$

Because ϕ is monomorphism we have that $a + k = 0$. We conclude that a is an integer and $\theta = -2\pi a$ $\square$

Corollary 1.1.3.
- $\pi_1 : S^1_{\mathbb{Q}} \to S^1$ *is a fiber bundle with fiber isomorphic to* $\hat{\mathbb{Z}}$ *and monodromy the shift* $T(x) = x + 1$.

- *exp is a local homeomorphism.*

- *Restricted to a leaf, π_1 is a local homeomorphism.*

- $S^1_{\mathbb{Q}}$ *is the dynamical suspension of the shift* $T(x) = x + 1$.

- $S^1_{\mathbb{Q}}$ *is foliated by dense $\mathbb{R}$-leaves.*

Proof:

- If $diam(U) < 2\pi$ then U is a trivializing neighborhood of S^1.

- $\mathbb{Z}$ acts as translations by $\iota(\mathbb{Z})$ and because $\iota(\mathbb{Z})$ is discrete in $\hat{\mathbb{Z}} \times \mathbb{R}$ then $\mathbb{Z}$ acts proper and discontinuously. We conclude that exp is a local homeomorphism.

- By definition π_1 is an open continuous epimorphism. Restricted to a leaf and a trivializing neighborhood π_1 is one to one.

- $(x, 2\pi) + \iota(1) = (x + 1, 0)$ so $(x, 2\pi) \sim (x + 1, 0)$.

- The foliation $\hat{\mathbb{Z}} \times \mathbb{R}$ is invariant under translations by $\iota(a)$ for every integer a hence it induces a foliation in the solenoid. $\mathbb{Z}$ is dense in its profinite completion $\hat{\mathbb{Z}}$ and so is every coset of $\hat{\mathbb{Z}}/\mathbb{Z}$. By the preceding item, we have that every $\mathbb{R}$-leaf is dense in the solenoid.

$\square$

Geometrically, the structure of the fiber is the disjoint union:

$$\pi_1^{-1}(x) = \bigsqcup_{y^n = x} \pi_n^{-1}(y)$$

As an example, consider the subsystem $n_i = 2^i$ and the diadic solenoid S^1_2 with fiber $\mathbb{Z}_2$, the diadic profinite completion of the integers. The diadic solenoid is illustrated in Figure 1.1.

Figure 1.1: Diadic solenoid

Tensoring the adelic solenoid with the group $\mathbb{C}^*$ we get the algebraic solenoid $\mathbb{C}^*_{\mathbb{Q}}$:

$$\begin{array}{ccccccccc}
\mathbb{C}^*_{\mathbb{Q}} = \lim \mathbb{C}^* & \longrightarrow & \cdots \mathbb{C}^* & \xrightarrow{p_{m,n}} & \mathbb{C}^* & \longrightarrow & \cdots \mathbb{C}^* \\
\phi \uparrow & & \uparrow{\scriptstyle l \mapsto e^{2\pi i l/n}} & & \uparrow{\scriptstyle l \mapsto e^{2\pi i l/m}} & & \uparrow{\scriptstyle 0 \mapsto 1} \\
\hat{\mathbb{Z}} = \lim \mathbb{Z}/n\mathbb{Z} & \longrightarrow & \cdots \mathbb{Z}/n\mathbb{Z} & \xrightarrow{p_{m,n}} & \mathbb{Z}/m\mathbb{Z} & \longrightarrow & \cdots \{0\}
\end{array}$$

All the properties discussed before are shared by the algebraic solenoid with the natural extensions and the proofs are verbatim. For clarity purposes we mention them once again for the algebraic solenoid:

Lema 1.1.4. *The following is a short exact sequence:*

$$0 \longrightarrow \hat{\mathbb{Z}} \xrightarrow{\phi} \mathbb{C}^*_{\mathbb{Q}} \xrightarrow{\pi_1} \mathbb{C}^* \longrightarrow 1$$

and we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{\mathbb{Z}} & \xrightarrow{\phi} & \mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\pi_1} & \mathbb{C}^* & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle z^n} & & \\
0 & \longrightarrow & n\hat{\mathbb{Z}} & \xrightarrow{\phi} & \mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\pi_n} & \mathbb{C}^* & \longrightarrow & 1
\end{array}$$

We define the *principal baseleaf* $\nu : \mathbb{C} \to \mathbb{C}^*_{\mathbb{Q}}$ as follows:

$$\begin{array}{ccccccccc}
\mathbb{C}^*_{\mathbb{Q}} = \lim \mathbb{C}^* & \longrightarrow & \ldots \mathbb{C}^* & \xrightarrow{p_{m,n}} & \mathbb{C}^* & \longrightarrow & \ldots \mathbb{C}^* \\
\nu \uparrow & & \uparrow{\scriptstyle e^{iz/n}} & & \uparrow{\scriptstyle e^{iz/m}} & & \uparrow{\scriptstyle e^{iz}} \\
\mathbb{C} & \xrightarrow{=} & \ldots \mathbb{C} & \xrightarrow{=} & \mathbb{C} & \xrightarrow{=} & \ldots \mathbb{C}
\end{array}$$

In particular, the inmersion ν is a group morphism such that $\nu(2\pi x) = \phi(x)$ for every integer x. Define:

$$exp : \ \hat{\mathbb{Z}} \times \mathbb{C} \to \mathbb{C}^*_{\mathbb{Q}}$$

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such that $exp(a, z) = \phi(a).\nu(z)$.

Lema 1.1.5. *We have the short exact sequence:*

$$0 \longrightarrow \mathbb{Z} \overset{\iota}{\longrightarrow} \hat{\mathbb{Z}} \times \mathbb{C} \overset{exp}{\longrightarrow} \mathbb{C}_{\mathbb{Q}}^* \longrightarrow 1 \tag{1.1}$$

such that $\iota(a) = (a, -2\pi a)$.

Corollary 1.1.6.
- *$\pi_1 : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}^*$ is a fiber bundle with fiber isomorphic to $\hat{\mathbb{Z}}$ and monodromy the shift $T(x) = x + 1$.*

- *exp is a local homeomorphism.*

- *Restricted to a leaf, π_1 is a local homeomorphism.*

- *$\mathbb{C}_{\mathbb{Q}}^*$ is the complex dynamical suspension of the shift $T(x) = x + 1$.*

- *$\mathbb{C}_{\mathbb{Q}}^*$ is foliated by dense $\mathbb{C}$-leaves.*

Because $\nu(2\pi x) = \phi(x)$ for every integer x, we have the equivalent descriptions:

$$0 \longrightarrow n\mathbb{Z} \overset{\iota}{\longrightarrow} n\hat{\mathbb{Z}} \times \mathbb{C} \overset{exp}{\longrightarrow} \mathbb{C}_{\mathbb{Q}}^* \longrightarrow 1$$

for every natural n. These are the appropriate descriptions to lift the homeomorphisms $z^{p/q}$:

Lema 1.1.7. *We have the commutative diagram:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & q\mathbb{Z} & \overset{\iota}{\longrightarrow} & q\hat{\mathbb{Z}} \times \mathbb{C} & \overset{exp}{\longrightarrow} & \mathbb{C}_{\mathbb{Q}}^* & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle p/q} & & \downarrow{\scriptstyle p/q} & & \downarrow{\scriptstyle z^{p/q}} & & \\
0 & \longrightarrow & p\mathbb{Z} & \overset{\iota}{\longrightarrow} & p\hat{\mathbb{Z}} \times \mathbb{C} & \overset{exp}{\longrightarrow} & \mathbb{C}_{\mathbb{Q}}^* & \longrightarrow & 1
\end{array}
$$

such that $\iota(a) = (a, -2\pi a)$.

Remark 1.1.1. Because $\overline{z}^n = \overline{z^n}$ and the conjugation has a continuous extension to the Riemann sphere $\bar{z} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, there is a homeomorphism:

$$\bar{z} : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$$

such that $\pi_n(\bar{x}) = \overline{\pi_n(x)}$ for every $x \in \hat{\mathbb{C}}_{\mathbb{Q}}$. Because $\bar{z} = z^{-1}$ on S^1 this relation extends to the solenoid $S_{\mathbb{Q}}^1$ and by continuity we have that the composition:

$$1/\bar{z} : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$$

is a leaf preserving homeomorphism fixing $0, 1, \infty$.

As defined by D. Sullivan [Su]: A two dimensional solenoid is hyperbolic if every leaf is conformaly covered by the disk.

Corollary 1.1.8. *Consider the solenoid $H_{\mathbb{Q}} = \pi_1^{-1}(\Delta^*)$ where Δ^* is the open unit circle minus the origin. Then H_Q is a hyperbolic solenoid.*

Proof: By equation (1.1) we have the covering $exp : \hat{\mathbb{Z}} \times U \to H_q$ where U is the hyperbolic upper half plane. $\qquad\square$

1.2   Continuous maps and degree theory

The following lemmas and propositions tell us how continuity properties of solenoidal maps are related to limit periodic properties of their restriction on the baseleaf. For pedagogical reasons, we introduce the notion of limit periodic as a particular case of almost periodic functions.

Definition 1.2.1. A subset $A \subset \mathbb{R}$ is relatively dense if there is a real number $L > 0$ such that $[x, x + L] \cap A \neq \emptyset$ for every $x \in \mathbb{R}$.

The following definition is due to Harald Bohr in 1924 [Bo]:

Definition 1.2.2. A function $f : \mathbb{R} \to \mathbb{C}$ is almost periodic if for every $\epsilon > 0$ there is a relatively dense subset $A \subset \mathbb{R}$ such that:

$$|f(x + 2\pi t) - f(x)| < \epsilon$$

for every $x \in \mathbb{R}$ and $t \in A$.

There is a beautiful discussion of almost periodic functions in the context of constructive mathematics in [Br]. Restricting the reletively dense subsets to be of the form $N\mathbb{Z}$ for some natural N we have:

Definition 1.2.3. A function $f : \mathbb{R} \to \mathbb{C}$ is limit periodic if for every $\epsilon > 0$ there is a natural number N such that:

$$|f(x + 2\pi n) - f(x)| < \epsilon$$

for every $x \in \mathbb{R}$ and $n \in N\mathbb{Z}$.

An interesting discussion relating limit periodic functions, solenoids and adding machines can be found in [Be]. The following generalization is the appropriate one needed for our subsequent theory:

Definition 1.2.4. A function $f : \mathbb{C} \to \mathbb{C}$ is limit periodic respect to x if for every $\epsilon > 0$ and compact set $K \subset \mathbb{R}$ there is a natural number N such that:

$$|f(z + 2\pi n) - f(z)| < \epsilon$$

for every $z \in \mathbb{R} \times iK$ and $n \in N\mathbb{Z}$.

Lema 1.2.1. • *Consider a limit periodic function $f : \mathbb{R} \to \mathbb{C}$. Then the map $f \circ 2\pi_- :$ $\mathbb{Z} \to \mathbb{C}$ is uniformly continuous respect to the relative adelic topology on $\mathbb{Z}$. In particular, the map extends uniquely to a continuous map on $\hat{\mathbb{Z}}$.*

• *Consider a continuous limit periodic respect to x function $f : \mathbb{C} \to \mathbb{C}$. Then the map $h : \mathbb{Z} \times \mathbb{C} \to \mathbb{C}$ such that $h(n, z) = f(z + 2\pi n)$ extends uniquely to a continuous map on $\hat{\mathbb{Z}} \times \mathbb{C}$.*

Proof:

• Consider an $\epsilon > 0$. There is a natural N such that

$$|f(x + 2\pi n) - f(x)| < \epsilon$$

for every $x \in \mathbb{R}$ and $n \succ N$. In particular, if $n - m \in N\mathbb{Z}$ then

$$|f(2\pi n) - f(2\pi m)| = |f(2\pi m + 2\pi(n - m)) - f(2\pi m)| < \epsilon$$

• Consider a compact set $K \subset \mathbb{R}$ and the map $l : \mathbb{Z} \to C(\mathbb{R} \times K, \mathbb{C})$ such that $l(n)(z) = h(n, z)$. Consider an $\epsilon > 0$. There is a natural N such that

$$|f(z + 2\pi n) - f(z)| < \epsilon$$

for every $z \in \mathbb{R} \times K$ and $n \succ N$. In particular, if $n - m \in N\mathbb{Z}$ then

$$|f(z + 2\pi n) - f(z + 2\pi m)| = |f(z + 2\pi m + 2\pi(n - m)) - f(z + 2\pi m)| < \epsilon$$

for every $z \in \mathbb{R} \times K$; i.e. l is uniformly continuous

$$||l(n) - l(m)||_\infty < \epsilon$$

hence there is a unique continuous extension $\hat{l} : \hat{\mathbb{Z}} \to C(\mathbb{R} \times K, \mathbb{C})$. Finally, we have the unique continuous extension $\hat{h}$ such that $\hat{h}(a, z) = \hat{l}(a)(z)$. Because the real line is σ-compact and continuity is a local property, we have the result.

$\square$

The following Lemma justifies the name of limit periodic maps.

Lema 1.2.2. • *For every limit periodic map $f : \mathbb{R} \to \mathbb{C}$ there is a sequence $(f_n)_{n \in \mathbb{N}}$ such that f_n is $2\pi n$-periodic and (f_n) converges pointwise to f respect to the divisibility net.*

• *A map $f : \mathbb{R} \to \mathbb{C}$ is continuous limit periodic respect to x if and only if there is a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous maps such that f_n is $2\pi n$-periodic respect to x and (f_n) uniformly converges to f in bands $\mathbb{R} \times K$ where $K \subset \mathbb{R}$ is a compact set, respect to the divisibility net. Moreover, the sequence $(f_n)_{n \in \mathbb{N}}$ can be assumed to be equicontinuous.*

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Proof:

- Consider $F : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ such that $F(n, x) = f(x + 2\pi n)$. Because f is limit periodic, by Lemma 1.2.1 for every $x \in \mathbb{R}$ the function $F(\_, x)$ is uniformly continuous hence there is an extension $\hat{F} : \hat{\mathbb{Z}} \times \mathbb{R} \to \mathbb{C}$ such that for every $x \in \mathbb{R}$ the extension $\hat{F}(\_, x)$ is continuous on the compact $\hat{\mathbb{Z}}$. Consider the inverse limit morphisms $\pi_n : \hat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ and define

$$f_n(x) = n \int_{Ker(\pi_n)} da \ \hat{F}(a, x)$$

  where da denotes the normalized Haar measure on the compact abelian group $\hat{\mathbb{Z}}$. See that that for every $x \in \mathbb{R}$ the extension $\hat{F}(\_, x)$ is integrable for it is continuous.

  Consider the shift $T : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ such that $T(a) = a + 1$. Because the Haar measure is invariant under the shift, T^n is an automorphism of $Ker(\pi_n)$ and $\hat{F}(T(a), x) = \hat{F}(a + 1, x) = \hat{F}(a, x + 2\pi)$ we have that f_n is $2\pi n$-periodic:

$$f_n(x + 2\pi n) = n \int_{Ker(\pi_n)} da \ \hat{F}(a, x + 2\pi n) = n \int_{Ker(\pi_n)} da \ \hat{F}(T^n(a), x) = f_n(x)$$

  Finally, for every $\epsilon > 0$ and every $x \in \mathbb{R}$ there is a natural $N_{\epsilon,x}$ such that for every $n \succ N$ we have $\hat{F}(n\hat{\mathbb{Z}}, x) \subset U(\hat{F}(0, x), \epsilon)$. In particular,

$$|f(x) - f_n(x)| \leq n \int_{Ker(\pi_n)} da \ |\hat{F}(0, x) - \hat{F}(a, x)| < \epsilon$$

  for every $n \succ N$.

- Consider $F : \mathbb{Z} \times \mathbb{C} \to \mathbb{C}$ such that $F(n, z) = f(z + 2\pi n)$. Because f is limit periodic respect to x, by Lemma 1.2.1 there is a unique continuous extension $\hat{F} : \hat{\mathbb{Z}} \times \mathbb{C} \to \mathbb{C}$ of F. Consider the inverse limit morphisms $\pi_n : \hat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$ and define

$$f_n(z) = n \int_{Ker(\pi_n)} da \ \hat{F}(a, z)$$

  where da denotes the normalized Haar measure on the compact abelian group $\hat{\mathbb{Z}}$. Again, see that that for every $z \in \mathbb{C}$ the extension $\hat{F}(\_, z)$ is integrable for it is continuous.

  Because $F(n + 1, z) = F(n, z + 2\pi)$ and $\hat{F}$ is the continuous extension, we have the relation $\hat{F}(a + 1, x) = \hat{F}(a, x + 2\pi)$ hence there is a continuous function $\hat{f}$ such that:

Consider the annulus $D_{r,R}$ where r and R denote the inner and outer radius respectively. Because the solenoid $S^1_{\mathbb{Q}} \times [a,b] \simeq \pi_1^{-1}(D_{r,R})$ is compact, $\hat{f}$ is uniformly continuous there hence $\hat{F}$ is uniformly continuous. Then, for every $\epsilon > 0$ there is a $\delta_\epsilon > 0$ and a natural N_ϵ such that $\hat{F}(N\hat{\mathbb{Z}} \times U(z, \delta)) \subset U(\hat{F}(0,z), \epsilon)$ for every $z \in \mathbb{R} \times [a,b]$. In particular,

$$|f(z) - f_n(z)| \leq n \int_{Ker(\pi_n)} da \; |\hat{F}(0,z) - \hat{F}(a,z)| < \epsilon$$

for every $n \succ N$ and every $z \in \mathbb{R} \times [a,b]$; i.e. $(f_n)_{n \in \mathbb{N}}$ uniformly converges to f in bands $\mathbb{R} \times K$ where $K \subset \mathbb{R}$ is a compact set, respect to the divisibility net. By the same argument as before, f_n is $2\pi n$-periodic.

Let's see that the family on functions f_n is equicontinuous. Define the continuous function $g : \hat{\mathbb{Z}} \times \mathbb{C}^2 \to \mathbb{R}$ such that $g(a, z, w) = |\hat{F}(a, z) - \hat{F}(a, w)|$. Let $\epsilon > 0$ and $z \in \mathbb{R}$. Then $\hat{\mathbb{Z}} \times \Delta \subset g^{-1}(U(0; \epsilon))$ where $\Delta \subset \mathbb{C}^2$ is the diagonal and because $\hat{\mathbb{Z}}$ is compact, there is a $\delta > 0$ such that $\hat{\mathbb{Z}} \times U((z,z); \delta) \subset g^{-1}(U(0; \epsilon))$. In particular, for every $w \in \mathbb{C}$ such that $|z - w| < \delta$ we have $g(a, z, w) < \epsilon$ for every $a \in \hat{\mathbb{Z}}$. Then,

$$|f_n(z) - f_n(w)| \leq n \int_{Ker(\pi_n)} da \; |\hat{F}(a, z) - \hat{F}(a, w)| < \epsilon$$

if $|z - w| < \delta$ for every natural n.

Conversely, consider a compact set $K \subset \mathbb{R}$ and let $\epsilon > 0$. There is a natural N such that $n \succ N$ implies $||f - f_n||_\infty < \epsilon/2$ on the band $\mathbb{R} \times K$. Define $T : \mathbb{C} \to \mathbb{C}$ such that $T(z) = z + 2\pi$. Because $f_n = f_n \circ T^n$ we have:

$$
\begin{aligned}
||f - f \circ T^n||_\infty &= ||(f - f_n) - (f \circ T^n - f_n \circ T^n)||_\infty \\
&\leq ||f - f_n||_\infty + ||f \circ T^n - f_n \circ T^n||_\infty = 2||f - f_n||_\infty < \epsilon
\end{aligned}
$$

for every $n \succ N$; i.e. f is limit periodic respect to x. Because every f_n is continuous and the convergence is uniform on compact sets, we have that f is continuous.

$\square$

The first item of the above Lemma is surprising for a non-continuous limit periodic could be quite bizarre. However, it can always be approximated by periodic functions.

Definition 1.2.5. Define the baseleaf topology on $\mathbb{C}$ as the topology such that $\nu : \mathbb{C}_{BL} \to \mathbb{C}^*_{\mathbb{Q}}$ is an embedding (instead of just an inmersion) and denote the this new topological space as $\mathbb{C}_{BL}$. The baseleaf topology on $\mathbb{R}$ is defined analogously and will be denoted as $\mathbb{R}_{BL}$.

Because of the relation $\pi_m \circ \nu = e^{iz/n}$ and the fact that, by definition, the topology of $\mathbb{C}^*_{\mathbb{Q}}$ is the coarser topology such that every π_m is continuous, we have that the following sets

$$U = U' + 2\pi m\mathbb{Z}$$

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where $U' \subset \mathbb{C}$ is a usual open set and m is a natural number, constitute a basis for the baseleaf topology. In particular, we have the homeomorphism:

$$\mathbb{C}_{BL} \simeq \mathbb{R}_{BL} \times \mathbb{R} \tag{1.2}$$

Another form of the above homeomorphism is the following one:

$$\mathbb{C}_\mathbb{Q}^* \simeq S_\mathbb{Q}^1 \times \mathbb{R}$$

Remark 1.2.1. The space $\mathbb{C}_{BL}$ **is not** a topological vector space for the vector space action of $\mathbb{R}$ or $\mathbb{C}$ with the usual topologies is not continuous. However, $(\mathbb{C}_{BL}, +)$ is a topological group. Because $(\mathbb{C}^*, \cdot)$ is a complete topological group and the inverse limit of such groups is again a complete topological group, the algebraic solenoid $(\mathbb{C}_\mathbb{Q}^*, \cdot)$ is also a complete topological group. Because $\nu : \mathbb{C}_{BL} \hookrightarrow \mathbb{C}_\mathbb{Q}^*$ is a dense embedding we conclude that the topological completion of $(\mathbb{C}_{BL}, +)$ is the algebraic solenoid $(\mathbb{C}_\mathbb{Q}^*, \cdot)$:

$$\overline{(\mathbb{C}_{BL}, +)} \simeq (\mathbb{C}_\mathbb{Q}^*, \cdot)$$

as topological groups. A similar discussion holds for the solenoid and $\mathbb{R}_{BL}$. It is interesting to see that formulating the problem backwards is much more difficult:

Question: Given the topological group $(R_{BL}, +)$ with the explicit topology described before, what is its completion?

Answer: The adelic solenoid.

Lema 1.2.3. *Consider a continuous baseleaf preserving function $f : \mathbb{C}_\mathbb{Q}^* \to \mathbb{C}_\mathbb{Q}^*$. Then, there is a unique rational number q and a unique continuous limit periodic respect to x function h such that $f_0(z) = qz + h(z)$ where f_0 is defined by the commutative diagram:*

$$
\begin{array}{ccc}
\mathbb{C}_\mathbb{Q}^* & \xrightarrow{\ f\ } & \mathbb{C}_\mathbb{Q}^* \\
{\scriptstyle \nu} \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow {\scriptstyle \nu} \\
\mathbb{C} & \xrightarrow{\ f_0\ } & \mathbb{C}
\end{array}
$$

Proof: Endow $\mathbb{C}$ with the baseleaf topology. We have the commutative diagram:

$$
\begin{array}{ccc}
\mathbb{C}_\mathbb{Q}^* & \xrightarrow{\ f\ } & \mathbb{C}_\mathbb{Q}^* \\
{\scriptstyle \nu} \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow {\scriptstyle \nu} \\
\mathbb{C}_{BL} & \xrightarrow{\ f_0\ } & \mathbb{C}_{BL}
\end{array}
$$

Let's see that $f_0 : \mathbb{C}_{BL} \to \mathbb{C}_{BL}$ is continuous. Consider an open set $U \subset \mathbb{C}_{BL}$. There is an open set $U' \subset \mathbb{C}_\mathbb{Q}^*$ such that $U = \nu^{-1}(U')$. Because $f_0^{-1}(U) = \nu^{-1}(f^{-1}(U'))$ and ν is continuous, we have that $f_0^{-1}(U)$ is open.

Remark 1.2.2. Because the baseleaf topology is coarser than the usual one, every connected subset in the usual sense is also connected in the baseleaf sense.

Consider an annulus $D_{r,R}$ where r and R denote the inner and outer radius respectively. Because $S^1_{\mathbb{Q}} \times [a,b] \simeq \pi_1^{-1}(D_{r,R})$ is compact and f is continuous, the restrictions of f and therefore f_0 are uniformly continuous; i.e for every $\epsilon > 0$ and natural λ there is a real number $\delta_{\epsilon,\lambda} > 0$ and a natural number $N_{\epsilon,\lambda}$ such that

$$f_0(z + 2\pi N\mathbb{Z} + U(0,\delta)) \subset f_0(z) + 2\pi\lambda\mathbb{Z} + U(0,\epsilon) \tag{1.3}$$

for every $z \in \mathbb{R} \times [a,b]$.

Define g_m such that $g_m(z) = f_0(z + 2\pi Nm) - f_0(z)$ for every integer m. Consider $\epsilon < \pi/2$. We will prove that there is an integer $k_{\epsilon,\lambda}$ such that $g_m(\mathbb{R} \times [a,b]) \subset U(2\pi km, \epsilon)$ for every integer m. We will prove it in the following steps:

- *Base case:* Because g_1 is continuous and $\mathbb{R}_{BL} \times [a,b]$ is connected by remark 1.2.2, there is an integer k such that $g_1(\mathbb{R} \times [a,b]) \subset U(2\pi k, \epsilon)$.

- *Induction step:* Suppose that $g_m(\mathbb{R} \times [a,b]) \subset U(2\pi km, \epsilon)$ for every natural $m \leq M$. Because $g_{M+1}(z) = g_M(z + 2\pi N) + g_1(z)$ and the inductive hypothesis, we have that $g_{M+1}(\mathbb{R} \times [a,b]) \subset U(2\pi k(M+1), \pi)$. By equation (1.3) we have $g_m(\mathbb{R} \times [a,b]) \subset 2\pi\mathbb{Z} + U(0,\epsilon)$ for every integer m. Then,

$$g_{M+1}(\mathbb{R} \times [a,b]) \subset U(2\pi k(M+1), \pi) \cap (2\pi\mathbb{Z} + U(0,\epsilon)) = U(2\pi k(M+1), \epsilon)$$

- *Trivial case:* $g_0(\mathbb{R} \times [a,b]) = \{0\} \subset U(0,\epsilon)$.

- *Negative integers:* $g_{-m}(\mathbb{R} \times [a,b]) = -g_m(\mathbb{R} \times [a,b]) \subset -U(2\pi km, \epsilon) = U(2\pi km, \epsilon)$ for every natural m.

We have proved a stronger version of equation (1.3): For every $\epsilon > 0$ and natural number λ such that $\epsilon < \pi/2$ there is a real number $\delta_{\epsilon,\lambda} > 0$, a natural number $N_{\epsilon,\lambda}$ and an integer $k_{\epsilon,\lambda}$ such that

$$f_0(z + 2\pi Nm + U(0,\delta)) \subset f_0(z) + 2\pi km + U(0,\epsilon) \tag{1.4}$$

for every $z \in \mathbb{R} \times [a,b]$ an every integer m.

Let's see that the quotient $k_{\epsilon,\lambda}/N_{\epsilon,\lambda}$ is independent of the ϵ and λ chosen. Consider another $0 < \epsilon' < \pi/2$ and λ'. There is a real number $\delta'_{\epsilon',\lambda'} > 0$ such that $\delta' < \delta$, a natural number $N'_{\epsilon',\lambda'}$ and an integer $k'_{\epsilon',\lambda'}$ such that

$$f_0(z + 2\pi N'm' + U(0,\delta')) \subset f_0(z) + 2\pi k'm' + U(0,\epsilon') \tag{1.5}$$

for every $z \in \mathbb{R} \times [a,b]$ an every integer m'. Choose m and m' such that $N'm' = Nm$. Then,

$$\emptyset \neq f_0(2\pi N'm' + U(0,\delta')) \subset (f_0(0) + 2\pi k.m + U(0,\epsilon)) \cap (f_0(0) + 2\pi k'.m' + U(0,\epsilon'))$$

and because $\epsilon, \epsilon' < \pi/2$ we have that $k.m = k'.m'$ hence $k/N = k'/N'$. Denote this ϵ, λ-independent rational by q.

In particular, because the compact $[a, b]$ was arbitrary, we have proved that

$$f_0(z) = qz \ + \ h(z)$$

where h is continuous limit periodic respect to x: Because f_0 is continuous we have that h is continuous. It rest to show that it is limit periodic respect to x. Because we proved that the rational q was ϵ, λ-independent, equation (1.4) reads as follows: For every compact set $K \subset \mathbb{R}$ and real number $\epsilon > 0$ there is a real number $\delta_{K,\epsilon} > 0$ and a natural number $N_{K,\epsilon}$ such that:

$$h(z + 2\pi Nm) - h(z) = f_0(z + 2\pi Nm) - f_0(z) - 2\pi q \ Nm \in U(0, \epsilon)$$

for every $z \in \mathbb{R} \times [a, b]$ and every integer m. This proves the claim.

Moreover, this decomposition is unique for a linear limit periodic function must be zero. $\qquad \square$

Corollary 1.2.4. *For every uniformly continuous map $f : \mathbb{R}_{BL} \to \mathbb{C}_{BL}$ there is a unique rational number q and a unique continuous limit periodic function h such that $f(x) = qx + h(x)$. In particular, f is continuous respect to the usual topologies; i.e. $f : \mathbb{R} \to \mathbb{C}$ is continuous.*

Proof: Because f is uniformly continuous it extends continuously to the completions and by remark 1.2.1 we have the commutative diagram:

$$
\begin{array}{ccc}
S^1_{\mathbb{Q}} & \xrightarrow{\hat{f}} & \mathbb{C}^*_{\mathbb{Q}} \\
\big\uparrow{\scriptstyle \nu} & & \big\uparrow{\scriptstyle \nu} \\
\mathbb{R}_{BL} & \xrightarrow{f} & \mathbb{C}_{BL}
\end{array}
$$

By Lemma 1.2.3, we have the result. $\qquad \square$

Definition 1.2.6. The rational number q of the above lemma will be called the degree of f and will be denoted $deg(f)$.

A continuous map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$ can be assumed to be baseleaf preserving just by multiplying it by $f(1)^{-1}$. The following proposition gives the converse of Lemma 1.2.3.

Proposition 1.2.5. *There is a continuous (holomorphic) baseleaf preserving map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$ if and only if there is a continuous (holomorphic) limit periodic respect to x map g such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{f} & \mathbb{C}^*_{\mathbb{Q}} \\
\big\uparrow{\scriptstyle \nu} & & \big\uparrow{\scriptstyle \nu} \\
\mathbb{C} & \xrightarrow{deg(f)z+g(z)} & \mathbb{C}
\end{array}
$$

where $deg(f) \in \mathbb{Q}$ is the degree of f.

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Proof: By Lemma 1.2.3 there is such g. If f is holomorphic then it is holomorphic on every leaf. In particular it is holomorphic on the baseleaf and we have that g is holomorphic.

For the converse, suppose that $deg(f) = p/q$ such that p and q are coprime natural numbers. Define $F : q\mathbb{Z} \times \mathbb{C} \to p\mathbb{Z} \times \mathbb{C}$ such that $F(qn, z) = (pn, f_n(z))$ where

$$f_n(z) = f_0(z + 2\pi qn) - 2\pi pn = \frac{p}{q}z + g(z + 2\pi qn) \tag{1.6}$$

for every integer n. Because g is continuous limit periodic respect to x, by Lemma 1.2.1 function $h : q\mathbb{Z} \times \mathbb{C} \to \mathbb{C}$ such that $h(n, z) = g(z + 2\pi n)$ admits a unique continuous extension $\hat{h} : q\hat{\mathbb{Z}} \times \mathbb{C} \to \mathbb{C}$ such that $\hat{h}(a, z + 2\pi q) = \hat{h}(a + q, z)$. Then, there is a unique continuous extension $\hat{F} : q\hat{\mathbb{Z}} \times \mathbb{C} \to p\hat{\mathbb{Z}} \times \mathbb{C}$ of F such that

$$\hat{F}(qa, z) = (pa, \frac{p}{q}z + \hat{h}(qa, z))$$

and satisfies the same structural condition as F:

$$\hat{F}(qa, z + 2\pi q) = \hat{F}(q(a + 1), z) + (-p, 2\pi p)$$

By Lemma 1.1.7, there is a continuous map f such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\quad f \quad} & \mathbb{C}_{\mathbb{Q}}^* \\
\uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\
q\hat{\mathbb{Z}} \times \mathbb{C} & \xrightarrow{\quad \hat{F} \quad} & p\hat{\mathbb{Z}} \times \mathbb{C} \\
\downarrow & & \uparrow \\
\mathbb{C} & \xrightarrow{(p/q)z + g(z)} & \mathbb{C}
\end{array}
$$

Let $a \in q\hat{\mathbb{Z}}$ and consider a sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that $(q.n_i)$ converges to a. If f_0 is holomorphic then by equation (1.6) f_{n_i} is holomorphic for every natural i. By Lemma 1.2.1 the sequence of continuous maps (f_{n_i}) converges uniformly to $\hat{F}(a, \_)$ on compact sets hence $\hat{F}(a, \_)$ is holomorphic for every f_{n_i} is holomorphic. Then f is holomorphic on every leaf and by remark 1.3.1 we conclude that f is holomorphic.

$\square$

We have proved that for every uniformly continuous map $f : \mathbb{R}_{BL} \to \mathbb{R}_{BL}$ there is a unique rational number q and a continuous limit periodic map h such that $f(x) = qx + h(x)$. In particular, every uniformly continuous map $f : \mathbb{R}_{BL} \to \mathbb{R}$ is limit periodic. Because the baseleaf topology is coarser than the usual topology, we have the natural inclusion

$$C_{unif}(\mathbb{R}_{BL}, \mathbb{R}) \hookrightarrow C_{unif}(\mathbb{R}_{BL}, \mathbb{R}_{BL})$$

with cokernel the rational numbers. We have proved the following topological characterization of the rational numbers:

$$C_{unif}(\mathbb{R}_{BL}, \mathbb{R}_{BL})/C_{unif}(\mathbb{R}_{BL}, \mathbb{R}) \simeq \mathbb{Q}$$

Lema 1.2.6. *Consider a pair of continuous (holomorphic) baseleaf preserving maps $f, g : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}_{\mathbb{Q}}^*$. Then, f and g are homotopic (conformal isotopic) if and only if $\deg(f) = \deg(g)$.*

Proof: There is a continuous map $H : [0,1] \times \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}_{\mathbb{Q}}^*$ such that $H(0, \_) = f$ and $H(1, \_) = g$. By Lemma 1.2.5 there is a unique map $\hat{H} : [0,1] \times \mathbb{C}_{BL} \to \mathbb{C}_{BL}$ such that:

$$
\begin{array}{ccc}
[0,1] \times \mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\ \ H\ \ } & \mathbb{C}_{\mathbb{Q}}^* \\
\uparrow{\scriptstyle id \times \nu} & & \uparrow{\scriptstyle \nu} \\
[0,1] \times \mathbb{C}_{BL} & \xrightarrow{\ \ \hat{H}\ \ } & \mathbb{C}_{BL}
\end{array}
$$

where $\hat{H}(t, z) = q(t)z + h(t, z)$. Let's see that $\hat{H}$ is continuous. Consider an open set $U \subset \mathbb{C}_{BL}$. There is an open set $U' \subset \mathbb{C}_{\mathbb{Q}}^*$ such that $U = \nu^{-1}(U')$. Because $\hat{H}^{-1}(U) = (id \times \nu)^{-1}(H^{-1}(U')$ and $id \times \nu$ is continuous, we have that $\hat{H}^{-1}(U)$ is open. We conclude that $\hat{H}$ is continuous. In particular, the function $q : [0,1] \to \mathbb{Q}$ is continuous and because $\mathbb{Q}$ is totally disconnected we have that q is constant hence $q(0) = q(1)$; i.e. $\deg(f) = \deg(g)$.

Conversely, there is a rational $q = \deg(f) = \deg(g)$ such that:

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\ \ f\ \ } & \mathbb{C}_{\mathbb{Q}}^* \\
\uparrow{\scriptstyle \nu} & & \uparrow{\scriptstyle \nu} \\
\mathbb{C}_{BL} & \xrightarrow{\ qz+h(z)\ } & \mathbb{C}_{BL}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\ \ g\ \ } & \mathbb{C}_{\mathbb{Q}}^* \\
\uparrow{\scriptstyle \nu} & & \uparrow{\scriptstyle \nu} \\
\mathbb{C}_{BL} & \xrightarrow{\ qz+l(z)\ } & \mathbb{C}_{BL}
\end{array}
$$

where h and l are continuous (holomorphic) limit periodic respect to x. Because every linear combination of continuous (holomorphic) limit periodic functions respect to x is continuous (holomorphic) limit periodic respect to x, the map $\hat{H} : [0,1] \times \mathbb{C}_{BL} \to \mathbb{C}_{BL}$ such that $\hat{H}(t,z) = qz + (1-t)h(z) + tl(z)$ is continuous (such that $\hat{H}(t, \_)$ is holomorphic for every t). An almost verbatim construction to the one given in Proposition 1.2.5 gives a continuous map H such that:

$$
\begin{array}{ccc}
[0,1] \times \mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\ \ H\ \ } & \mathbb{C}_{\mathbb{Q}}^* \\
\uparrow{\scriptstyle id \times \nu} & & \uparrow{\scriptstyle \nu} \\
[0,1] \times \mathbb{C}_{BL} & \xrightarrow{\ \ \hat{H}\ \ } & \mathbb{C}_{BL}
\end{array}
$$

Then f and g are homotopic. If f and g are holomorphic, by Proposition 1.2.5 $H(t, \_)$ is holomorphic for every t hence H is a conformal isotopic. $\qquad\square$

Corollary 1.2.7. *For every baseleaf preserving continuous (holomorphic) map $f : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}_{\mathbb{Q}}^*$ there is a unique rational number q such that f is homotopic (conformal isotopic) to z^q. In particular, every character of the group $\mathbb{C}_{\mathbb{Q}}^*$ is of the form z^q for some rational number q.*

Proposition 1.2.8. *There is a continuous (holomorphic) map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}$ such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ f\ } & \mathbb{C} \\
{\scriptstyle \nu}\big\uparrow & \nearrow {\scriptstyle g} & \\
\mathbb{C} & &
\end{array}
$$

if and only if g is continuous (holomorphic) limit periodic respect to x.

Proof: Almost verbatim to 1.2.5 with $deg(f) = 0$. $\qquad\square$

The following Lemma shows that degree zero functions map all the solenoid to only one leaf.

Lema 1.2.9. *Consider a continuous (holomorphic) baseleaf preserving map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$. There is a continuous (holomorphic) map g such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ f\ } & \mathbb{C}^*_{\mathbb{Q}} \\
& \searrow {\scriptstyle g} & \big\uparrow {\scriptstyle \nu} \\
& & \mathbb{C}
\end{array}
$$

if and only if $deg(f) = 0$.

Proof: By proposition 1.2.8 there is a limit periodic respect to x continuous (holomorphic) map $h : \mathbb{C} \to \mathbb{C}$ such that $h = g \circ \nu$. Then $f \circ \nu = \nu \circ g \circ \nu = \nu \circ h$ and by Lemma 1.2.5 we have that $deg(f) = 0$.

Conversely, if $deg(f) = 0$ by proposition 1.2.5 there is a limit periodic respect to x continuous (holomorphic) map $h : \mathbb{C} \to \mathbb{C}$ such that $f \circ \nu = \nu \circ h$. By proposition 1.2.8 there is a continuous (holomorphic) map g such that $h = g \circ \nu$:

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ f\ } & \mathbb{C}^*_{\mathbb{Q}} \\
{\scriptstyle \nu}\big\uparrow & {\scriptstyle g}\searrow & \big\uparrow {\scriptstyle \nu} \\
\mathbb{C} & \xrightarrow{\ h\ } & \mathbb{C}
\end{array}
$$

Then, $f \circ \nu = \nu \circ h = \nu \circ g \circ \nu$. Because the maps are continuous and the image of ν is dense embedding, we have that $f = \nu \circ g$. $\qquad\square$

Corollary 1.2.10. *Consider a continuous (holomorphic) baseleaf preserving map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$. There is a continuous (holomorphic) map $g : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}$ such that $f = z^{deg(f)}.(\nu \circ g)$ where ν is the baseleaf.*

1.3 Differentiable structure and derivatives

Now we discuss the differentiable structure and derivatives.

Definition 1.3.1. Because restricted to a leaf π_1 is a local homeomorphism, we define the complex and differentiable structure of every leaf of the algebraic solenoid as the pullback of the respective structures of $\mathbb{C}^*$ by $\pi_1 : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*$.

Remark 1.3.1. Because π_1 is a group morphism, for every $a \in Ker(\pi_1) \simeq \hat{\mathbb{Z}}$ we have $\pi_1(a.\nu(z)) = e^{iz}$

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\pi_1} & \mathbb{C}^* \\
{\scriptstyle a.\nu}\uparrow & \nearrow{\scriptstyle e^{iz}} & \\
\mathbb{C} & &
\end{array}
$$

hence the complex and differential structures induced by π_1 and $a.\nu$ on the leaves coincide for e^{iz} is holomorphic. In particular, a function is holomorphic on $\mathbb{C}^*_{\mathbb{Q}}$ if and only if it is holomorphic on every leaf.

Thinking the leaves $a.\nu : \mathbb{C} \to \mathbb{C}^*_{\mathbb{Q}}$ as coordinate charts, we have the following definition:

Definition 1.3.2. Consider a continuous map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$. There exist the derivative $\partial_z^i \partial_{\bar{z}}^j f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}$ if it is continuous and

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{f} & \mathbb{C}^*_{\mathbb{Q}} \\
{\scriptstyle a.\nu}\uparrow & & \uparrow{\scriptstyle b.\nu} \\
\mathbb{C} & \xrightarrow{f_a} & \mathbb{C}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\partial_z^i \partial_{\bar{z}}^j f} & \mathbb{C} \\
{\scriptstyle a.\nu}\uparrow & \nearrow{\scriptstyle \partial_z^i \partial_{\bar{z}}^j f_a} & \\
\mathbb{C} & &
\end{array}
$$

for every $a \in Ker(\pi_1)$; i.e. for every leaf. We say that f is of class C^n if there exist $\partial_z^i \partial_{\bar{z}}^j f$ for every $i, j > 0$ such that $i + j \leq n$. We say that f is of class C^∞ if there exist $\partial_z^i \partial_{\bar{z}}^j f$ for every $i, j > 0$.

Proposition 1.3.1. *Consider a continuous baseleaf preserving map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$ such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{f} & \mathbb{C}^*_{\mathbb{Q}} \\
{\scriptstyle \nu}\uparrow & & \uparrow{\scriptstyle \nu} \\
\mathbb{C} & \xrightarrow{deg(f)z+g(z)} & \mathbb{C}
\end{array}
$$

Then, the continuous derivative $\partial_z^i \partial_{\bar{z}}^j f$ exists if and only if $\partial_z^i \partial_{\bar{z}}^j g$ exists and is continuous limit periodic respect to x. In particular, f is C^n if and only if $\partial_z^i \partial_{\bar{z}}^j g$ exists and is continuous limit periodic respect to x for every $i, j \geq 0$ such that $i + j \leq n$.

Proof: By definition, there are continuous maps $\partial_z^i \partial_{\bar{z}}^j f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}$ for every $i, j > 0$ and $i + j \leq n$ such that:

for every $a \in Ker(\pi_1)$; i.e. for every leaf. In particular, it is verified for the baseleaf $(a = 0)$ and by Lemma 1.2.8 the functions $\partial_z^i \partial_{\bar{z}}^j g : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}$ are continuous limit periodic respect to x for every $i, j > 0$ and $i + j \leq n$

Conversely, suppose that $\partial_z g$ is continuous limit periodic respect to x and $deg(f) = p/q$ such that p and q are coprime natural numbers. In the proof of proposition 1.2.5 we constructed the commutative diagram:

Define $F_z : q\mathbb{Z} \times \mathbb{C} \to \mathbb{C}$ such that

$$F_z(qn, z) = \partial_z f_0(z + 2\pi qn) = \frac{p}{q} + \partial_z g(z + 2\pi qn)$$

for every integer n. Because $\partial_z f_0$ is continuous limit periodic respect to x, by Lemma 1.2.1 there is a unique continuous extension $\hat{F}_z : q\hat{\mathbb{Z}} \times \mathbb{C} \to \mathbb{C}$ of F_z such that $\hat{F}_z(a, z + 2\pi q) = \hat{F}_z(a+q, z)$. By Lemma 1.1.7, there is a continuous map f_z such that the following diagram commutes:

Let $a \in q\hat{\mathbb{Z}}$ and consider a sequence of integers $(n_i)_{i \in \mathbb{N}}$ such that $(q.n_i)$ converges to a. Because $\hat{F}(n_i, \_)$ converges uniformly to $\hat{F}(a, \_)$ on compact sets and $\partial_z \hat{F}(n_i, \_) = (0, \hat{F}_z(n_i, \_))$ converges uniformly to $(0, \hat{F}_z(a, \_))$ on compact sets we conclude that

$$\partial_z \hat{F}(a, \_) = (0, \hat{F}_z(a, \_))$$

and it is continuous limit periodic respect to x. We have proved that there exist the partial derivative $\partial_z f = f_z$.

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In the case that f_0 is of class C^m such that $m = 1, 2, \ldots \infty$, an analogous inductive argument shows that there exist the all the other continuous partial derivatives of f; i.e. f is of class C^m. $\qquad\square$

We have a completely analogous proposition for functions with almost verbatim proof:

Proposition 1.3.2. *Consider a continuous function $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}$ such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ \ f\ \ } & \mathbb{C} \\
\nu \uparrow & \nearrow g & \\
\mathbb{C} & &
\end{array}
$$

Then, the continuous derivative $\partial_z^i \partial_{\bar{z}}^j f$ exists if and only if $\partial_z^i \partial_{\bar{z}}^j g$ exists and is continuous limit periodic respect to x. In particular, f is C^n if and only if $\partial_z^i \partial_{\bar{z}}^j g$ exists and is continuous limit periodic respect to x for every $i, j \geq 0$ such that $i + j \leq n$.

We have an improved version of Lemma 1.2.6:

Lema 1.3.3. *Consider a pair of C^n baseleaf preserving maps $f, g : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$. Then, f and g are C^n-isotopic if and only if $\deg(f) = \deg(g)$.*

Proof: Almost verbatim to the proof of Lemma 1.2.6. $\qquad\square$

1.4   Picard theorem

Proposition 1.4.1. *There is a continuous (holomorphic) map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*$ such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ \ f\ \ } & \mathbb{C}^* \\
\nu \uparrow & & \uparrow e^{iz} \\
\mathbb{C} & \xrightarrow{qz+g(z)} & \mathbb{C}
\end{array}
$$

if and only if g is continuous (holomorphic) limit periodic respect to x and q is a rational number.

Proof: Almost verbatim to the proof in Lemma 1.2.5. $\qquad\square$

Definition 1.4.1. We will call the above rational number the degree of f and denote it by $\deg(f)$. The following corollary justifies this notation:

The following corollary justifies this notation:

Corollary 1.4.2. *For every continuous (holomorphic) map $f : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*$ there is a unique continuous (holomorphic) baseleaf preserving map $\hat{f}$ such that:*

$$
\begin{array}{ccc}
 & & \mathbb{C}^*_{\mathbb{Q}} \\
 & \overset{\hat{f}}{\nearrow} & \downarrow{\pi_1} \\
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ f\ } & \mathbb{C}^*
\end{array}
$$

and $deg(f) = deg(\hat{f})$.

Proof: By proposition 1.4.1, there is a unique rational number $q = deg(f)$ and a continuous (holomorphic) limit periodic respect to x map g such that:

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ f\ } & \mathbb{C}^* \\
\big\uparrow{\nu} & & \big\downarrow{e^{iz}} \\
\mathbb{C} & \xrightarrow{qz+g(z)} & \mathbb{C}
\end{array}
$$

By proposition 1.2.5, there is a unique baseleaf preserving continuous (holomorphic) map $\hat{f}$ of degree q such that:

$$
\begin{array}{ccccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ \hat{f}\ } & \mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\pi_1} & \mathbb{C}^* \\
\big\uparrow{\nu} & & \big\uparrow{\nu} & \nearrow{e^{iz}} & \\
\mathbb{C} & \xrightarrow{qz+g(z)} & \mathbb{C} & &
\end{array}
$$

Then $\pi_1 \circ \hat{f} \circ \nu = f \circ \nu$ and because the image of ν is dense and the maps are continuous we have that $\pi_1 \circ \hat{f} = f$. $\qquad\square$

The following corollary shows the relation of the degree introduced here with the classical degree:

Corollary 1.4.3. *For every continuous (holomorphic) map $f : \mathbb{C}^* \to \mathbb{C}^*$ there is a unique continuous (holomorphic) baseleaf preserving map $\hat{f}$ such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{\ \hat{f}\ } & \mathbb{C}^*_{\mathbb{Q}} \\
\big\downarrow{\pi_1} & & \big\downarrow{\pi_1} \\
\mathbb{C} & \xrightarrow{\ f\ } & \mathbb{C}
\end{array}
$$

and $deg(f) = deg(\hat{f})$.

Proof: Because the map π_1 is holomorphic and the degree is multiplicative under composition we have that $f \circ \pi_1$ is a continuous (holomorphic) map with the same degree as f; i.e.

$$
\begin{array}{ccccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\pi_1} & \mathbb{C} & \xrightarrow{f} & \mathbb{C} \\
\big\uparrow{\scriptstyle\nu} & & \big\uparrow{\scriptstyle e^{iz}} & & \big\uparrow{\scriptstyle e^{iz}} \\
\mathbb{C} & \xrightarrow{z} & \mathbb{C} & \xrightarrow{deg(f)z+g(z)} & \mathbb{C}
\end{array}
$$

where $deg(f)$ is an integer and g is a continuous (holomorphic) map periodic respect to x. Then $deg(f) = deg(f \circ \pi_1)$. By the previous corollary there is a unique continuous (holomorphic) baseleaf preserving map $\hat{f}$ such that:

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\hat{f}} & \mathbb{C}_{\mathbb{Q}}^* \\
\big\downarrow{\scriptstyle\pi_1} & & \big\downarrow{\scriptstyle\pi_1} \\
\mathbb{C} & \xrightarrow{f} & \mathbb{C}
\end{array}
$$

and $deg(f) = deg(f \circ \pi_1) = deg(\hat{f})$. $\qquad\qquad\square$

Lema 1.4.4. *Consider a pair of continuous (holomorphic) maps $f, g : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}^*$. Then, f and g are homotopic (conformal isotopic) if and only if $deg(f) = deg(g)$.*

Proof: Almost verbatim to the proof of Lemma 1.2.6. $\qquad\qquad\square$

Proposition 1.4.5. $Pic(\hat{\mathbb{C}}_{\mathbb{Q}}) \simeq (\mathbb{Q}, +)$

Proof: Consider a complex holomorphic line bundle L over $\hat{\mathbb{C}}_{\mathbb{Q}}$. *Claim:* The open sets $U = \hat{\mathbb{C}}_{\mathbb{Q}} - \{0\}$ and $V = \hat{\mathbb{C}}_{\mathbb{Q}} - \{\infty\}$ constitute a trivializing cover: We only prove that $V = \mathbb{C}_{\mathbb{Q}}$ is trivializing for the other case is completely similar. Consider a holomorphic function $f : \mathbb{C}_{\mathbb{Q}} \to \mathbb{C}^*$. In particular, its restriction to $C_{\mathbb{Q}}^*$ is holomorphic and by Lemma 1.4.1 there is a map h such that:

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{f} & \mathbb{C}^* \\
\big\uparrow{\scriptstyle\nu} & & \big\uparrow{\scriptstyle e^{iz}} \\
\mathbb{C} & \xrightarrow{h(z)=deg(f)z+g(z)} & \mathbb{C}
\end{array}
$$

Restricted to the real line, h has the form:

$$
h(x) = deg(f)x + \sum_{q \in \mathbb{Q}} a_q e^{iqx}
$$

and because h is holomorphic we have:

$$
h(z) = deg(f)z + \sum_{q \in \mathbb{Q}} a_q e^{iqz}
$$

In particular, its imaginary part is the following:

$$
Im(h(z)) = deg(f)y + \sum_{q \in \mathbb{Q}} \left[Re(a_q)\sin(qx) + Im(a_q)\cos(qx) \right] e^{-qy}
$$

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Because f has a continuous extension at zero such that $f(0) \in \mathbb{C}^*$ and $|f(\nu(z))| = |e^{ih(z)}| = e^{-Im(h(z))}$, the limit of $Im(h)$ when y tends to $+\infty$ must be finite for every x. We conclude that $deg(f) = 0$ and $a_q = 0$ for every $q < 0$; i.e.

$$f(z) = e^{i \sum_{q \geq 0} a_q z^q}$$

Define the conformal isotopy:

$$f_t(z) = e^{it \sum_{q \geq 0} a_q z^q}$$

We have proved that every holomorphic function $f : \mathbb{C}_\mathbb{Q} \to \mathbb{C}^*$ is conformal isotopic to a constant function and we have the claim.

Then, the bundle L is determined by its holomorphic clutching function $f : \mathbb{C}_\mathbb{Q}^* \to \mathbb{C}^*$ and by Lemma 1.4.4 there is a unique rational number q such that f is conformal isotopic to z^q hence L is isomorphic to the complex holomorphic line bundle $\mathcal{O}(q)$ with clutching function the character z^q. Because $\mathcal{O}(p) \otimes \mathcal{O}(q) \simeq \mathcal{O}(p + q)$, the result follows. $\qquad\square$

Remark 1.4.1. It is tempting to argue just that U and V are contractible hence trvializing but this is true in the continuous category and we ere in the holomorphic one.

Proposition 1.4.6. *There is a natural group monomorphism* $\hat{\pi}_1^* : Pic(\hat{\mathbb{C}}) \hookrightarrow Pic(\hat{\mathbb{C}}_\mathbb{Q})$.

Proof: For every complex line bundle $\pi : L \to \hat{\mathbb{C}}$ we have its pullback:

$$\begin{array}{ccc} \hat{\pi}_1^*(L) & \longrightarrow & L \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \hat{\mathbb{C}}_\mathbb{Q} & \xrightarrow{\hat{\pi}_1} & \hat{\mathbb{C}} \end{array}$$

and because $\hat{\pi}_1$ is onto we have that $\hat{\pi}_1^*$ is a monomorphism: Take the trivializing cover $U = \hat{\mathbb{C}} - \{\infty\}$ and $V = \hat{\mathbb{C}} - \{0\}$ of the Riemann sphere $\hat{\mathbb{C}}$. For every clutching function $f : \mathbb{C}^* \to \mathbb{C}^*$ the clutching function of the pullback of its associated bundle respect to the trivializing cover $U' = \hat{\mathbb{C}}_\mathbb{Q} - \{\infty\}$ and $V' = \hat{\mathbb{C}}_\mathbb{Q} - \{0\}$ is $f \circ \pi_1$. Then the pullback $\hat{\pi}_1$ is injective for every pair of clutching functions such that $f \circ \pi_1 = g \circ \pi_1$ we have $f = g$. Because the tensor product of bundles with clutching functions f and g has the clutching function $f.g$, the pullback $\hat{\pi}_1^*$ is a group morphism for $(f.g) \circ \pi_1 = (f \circ \pi_1).(g \circ \pi_1)$; i.e.

$$\hat{\pi}_1^*(L \otimes L') \simeq \hat{\pi}_1^*(L) \otimes \hat{\pi}_1^*(L')$$

By general theory, if $L \simeq L'$ then $\hat{\pi}_1^*(L) \simeq \hat{\pi}_1^*(L')$. $\qquad\square$

Chapter 2

Renormalization

2.1 Renormalization

Definition 2.1.1. We say that $\mu \in L_{\infty}^{vert}(\mathbb{C}_{\mathbb{Q}}^{*})$ is vertical integrable if $\mu \in L_{\infty}(\mathbb{C}_{\mathbb{Q}}^{*})$ and $(F_x)^*(\mu) \in L_{\infty}(\hat{\mathbb{Z}})$ for almost every fiber $F_x : \mathbb{Z} \hookrightarrow \mathbb{C}_{\mathbb{Q}}$ (i.e. for almost every $x \in \mathbb{C}^*$).

Definition 2.1.2. Consider the normalized Haar measure η on $\hat{\mathbb{Z}}$ and the induced measure on the fibers $\pi_1^{-1}(x)$. We define the n-th renormalization map as the linear operator $\mathcal{I}_n : L_{\infty}^{vert}(\mathbb{C}_{\mathbb{Q}}^{*}) \to L_{\infty}^{vert}(\mathbb{C}_{\mathbb{Q}}^{*})$ such that:

$$\mathcal{I}_n(\mu)(x) = n \int_{\pi_n^{-1}(\pi_n(x))} d\eta \; \mu$$

The n-th renormalization map is the average respect the n-th level $\pi_n : \mathbb{C}_{\mathbb{Q}}^{*} \to \mathbb{C}^*$ of the algebraic solenoid renormalized such that its operator norm be one; i.e. $||\mathcal{I}_n||_{\infty} = 1$.

This is illustrated for the diadic solenoid in Figure 2.1.

Remark 2.1.1. See that, by definition, $\mathcal{I}_n(\mu)$ factors through π_n; i.e. there is a $\hat{\mu}_n \in Ł_{\infty}(\mathbb{C}^*)$ such that:

In particular, $\mathcal{I}_n(\mu)_0 = \mathcal{I}_n(\mu) \circ \nu = \hat{\mu}_n \circ e^{iz/n}$ is $2\pi n$-periodic respect to x.

Definition 2.1.3. Consider the left action $m : \hat{\mathbb{Z}} \to Aut(\mathbb{C}_{\mathbb{Q}}^{*})$ such that $m(a)(x) = \phi(a)x$. We say that $\mu \in L_{\infty}(\mathbb{C}_{\mathbb{Q}}^{*})$ is uniformly vertical L_{∞}-continuous if the map $\hat{\mathbb{Z}} \to L_{\infty}(\mathbb{C}_{\mathbb{Q}}^{*})$ such that $(a \to \mu \circ m(a))$ is continuous at zero.

Because $\pi_1 \circ \phi = 1$ we have $\pi_1 \circ m_a = id$ for every $a \in \hat{\mathbb{Z}}$. This way $m_a : \pi_1^{-1}(x) \to \pi_1^{-1}(x)$ for every $x \in \mathbb{C}^*$ and $a \in \hat{\mathbb{Z}}$; i.e. the fibers are invariant under the action m_a for
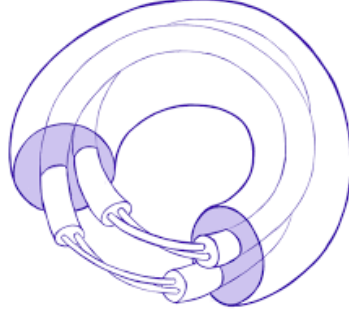
Figure 2.1: Van Dantzig-Vietoris solenoid

every $a \in \hat{\mathbb{Z}}$. To get a feel of this notion of continuity, see that every uniform continuous function is L_∞-continuous. For a less trivial example, see that the Dirichlet function ($\mathcal{D}(x) = 1$ if x is rational and $\mathcal{D}(x) = 0$ if x is irrational) is L_∞-continuous. On the contrary, the Heaviside or step function ($H(x) = 1$ if $x \geq 1$ and $H(x) = 0$ if $x < 0$) is not L_∞-continuous.

Lema 2.1.1. *If $\mu \in L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)$ is uniformly vertical L_∞-continuous then $\mathcal{I}_n(\mu)$ converges uniformly to μ respect to the divisibility net.*

Proof: Consider $\epsilon > 0$. There is a natural N such that $n \succ N$ implies $||\mu - \mu \circ m_a||_\infty < \epsilon$ for every $a \in n\hat{\mathbb{Z}}$. Then, for every x where μ is defined we have:

$$|\mu(x) - \mathcal{I}_n(\mu)(x)| \leq n \int_{\pi_n^{-1}(\pi_n(x))} dy \, |\mu(x) - \mu(y)| < \epsilon$$

because $y \in \pi_n^{-1}(\pi_n(x))$ implies $y^{-1}x = \phi(a)$ such that $a \in n\hat{\mathbb{Z}}$ and then $|\mu(x) - \mu(y)| < \epsilon$ for every $y \in \pi_n^{-1}(\pi_n(x))$. This implies $||\mu - \mathcal{I}_n(\mu)||_\infty < \epsilon$ for every $n \succ N$. □

Corollary 2.1.2. *Consider a uniformly vertical L_∞-continuous $\mu \in L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)$. Then, for almost every fiber $F_x : \hat{\mathbb{Z}} \to \mathbb{C}_\mathbb{Q}^*$ of the fiber bundle $\pi_1 : \mathbb{C}_\mathbb{Q}^* \to \mathbb{C}^*$ the pullback $F_x^*(\mu) : \hat{\mathbb{Z}} \to \mathbb{C}$ can be represented by a continuous function.*

Proof: For every natural n and almost every fiber $F_x : \hat{\mathbb{Z}} \to \mathbb{C}_\mathbb{Q}^*$ the map $F_x^*(\mathcal{I}_n(\mu))$ is locally constant (See remark 4.2.1). In particular they are continuous and by the previous Lemma they converge uniformly to $F_x^*(\mu)$ and we have the result. □

The above corollary can be written in the following way:

Definition 2.1.4. We say $\mu \in Per(\mathbb{C}_\mathbb{Q})$ if there is some natural n and $\mu_n \in L_\infty(\mathbb{C})$ such that $\mu = \pi_n^*(\mu_n)$.

Corollary 2.1.3. $\overline{Per}^\infty = L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)$

Lema 2.1.4.   • *If $f : \mathbb{C}_\mathbb{Q}^* \to \mathbb{C}$ is continuous then the family of functions $\mathcal{I}_n(f)$ is equicontinuous and the sequence $(\mathcal{I}_n(f))_{n \in \mathbb{N}}$ converges uniformly to f on compact sets.*

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- If $f : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}$ is C^m then $\mathcal{I}_n(f)$ is C^m and $(\mathcal{I}_n(f))_{n\in\mathbb{N}}$ converges to f in the C^m-topology on compact sets.

Proof:

- An analogous construction to the one given in the proof of Lemma 1.2.2 gives the commutative diagram:

such that F is continuous, $F(n,z) = f_0(z + 2\pi n)$ and $F(a+1,z) = F(a, z + 2\pi)$ for every integer n, $z \in \mathbb{C}$ and $a \in \hat{\mathbb{Z}}$. Define the function $\mathcal{I}_n(F)$ such that:

$$\mathcal{I}_n(F)(a,z) = n \int_{\pi_n^{-1}(\pi_n(a))} db \ F(b,z)$$

Because of the relation:

$$
\begin{aligned}
\mathcal{I}_n(F)(a, z + 2\pi) &= n \int_{\pi_n^{-1}(\pi_n(a))} d\eta \ F(b, z + 2\pi) = n \int_{\pi_n^{-1}(\pi_n(a))} d\eta \ F(b+1, z) \\
&= n \int_{\pi_n^{-1}(\pi_n(a))+1} d\eta \ F(b, z) = n \int_{\pi_n^{-1}(\pi_n(a+1))} d\eta \ F(b, z) \\
&= \mathcal{I}_n(F)(a+1, z)
\end{aligned}
$$

there is a function conjugated to F by the exp map. It is clear that this map is $\mathcal{I}_n(f)$ and we have the commutative diagram:

where $\mathcal{I}_n(f)_0 = \mathcal{I}_n(F)(0, \_)$:

$$\mathcal{I}_n(f)_0(z) = n \int_{Ker(\pi_n)} db \ F(b,z)$$

See that these maps coincide with the maps defined in the proof of Lemma 1.2.2 and by the same proof we have that they are periodic respect to x and equicontinuous. By Proposition 1.2.8, the family of functions $\mathcal{I}_n(f)$ is equicontinuous and the sequence $(\mathcal{I}_n(f))_{n\in\mathbb{N}}$ converges uniformly to f on compact sets.

- Suppose that there is a continuous derivative $\partial_z f$ such that:

Claim: $\partial_z(\mathcal{I}_n(f)_0) = \mathcal{I}_n(\partial_z f)_0$.

An analogous construction to the one given in the proof of Lemma 1.2.2 and Proposition 1.3.1 gives the commutative diagrams:

such that F and $\partial_z F$ are continuous, $F(n,z) = f_0(z+2\pi n)$, $F(a+1,z) = F(a,z+2\pi)$ and analogous relations for $\partial_z F$ for every $n \in \mathbb{Z}$, $z \in \mathbb{C}$ and $a \in \hat{\mathbb{Z}}$. In the same way as before, we have the commutative diagrams:

It only rest to show that $\partial_z \mathcal{I}_n(F) = \mathcal{I}_n(\partial_z F)$: Because $\partial_z F$ is continuous we can interchange the integral and the derivative:

$$\partial_z \mathcal{I}_n(F)(a,z) = n\,\partial_z \int_{\pi_n^{-1}(\pi_n(a))} d\eta\, F(b,z) = n \int_{\pi_n^{-1}(\pi_n(a))} d\eta\, \partial_z F(b,z) = \mathcal{I}_n(\partial_z F)(a,z)$$

and this proves the claim.

Because $\partial_z(\mathcal{I}_n(f)_0) = \mathcal{I}_n(\partial_z f)_0$ by the above item these functions are periodic respect to x and equicontinuous. By Proposition 1.3.2 and the above item, the equicontinuous derivatives $\partial_z \mathcal{I}_n(f)$ exists and

$$\partial_z \mathcal{I}_n(f) = \mathcal{I}_n(\partial_z f)$$

Finally, by the above item again and the last relation, the sequence $(\partial_z \mathcal{I}_n(f))_{n \in \mathbb{N}}$ converges uniformly to $\partial_z f$ on compact sets.

An inductive argument shows that the result holds for every derivative of order less than or equal to m and we have the result.

$\square$

2.2 Pontryagin series

To motivate the following discussion, recall the proof of uniform convergence of the Fourier series of a C^1 function: Consider the Fourier series

$$f(z) = \sum_{i \in \mathbb{Z}} a_i z^i$$

such that the series a priori converges in L_2. However, by the Cauchy-Schwarz inequality:

$$\sum_{i \in \mathbb{Z}} |a_i| \le |a_0| + \left(2 \sum_{i \in \mathbb{N}} \frac{1}{i^2}\right)^{1/2} \left(\sum_{i \in \mathbb{Z}} |i a_i|^2\right)^{1/2} = |a_0| + \frac{\pi}{\sqrt{3}} ||f'||_2 < \infty$$

and by the Weierstrass M-test we have that the Fourier series actually converges uniformly.

When we try to reproduce the above argument to a C^1 function on the solenoid it breaks down for:

$$\sum_{q \in \mathbb{Q}^+} \frac{1}{q^2} = \lim_n \sum_{q \in \frac{1}{n}\mathbb{N}} \frac{1}{q^2} = \lim_n n^2 \frac{\pi^2}{6} = \infty$$

Definition 2.2.1. If m divides n $(m|n)$, define the linear operator $R_{m,n} : C(S^1, \mathbb{C}) \to C(S^1, \mathbb{C})$ such that

$$R_{m,n}(f)(x) = m/n \sum_{y^{n/m}=x} f(y)$$

If $l|m|n$ then $R_{l,m} \circ R_{m,n} = R_{l,n}$ and R defines an inverse system of complex vector spaces over the divisibility net with inverse limit the complex vector space $\left(\varprojlim C(S^1, \mathbb{C}), \ p_n\right)$. Consider the inverse limit morphisms $\pi_n : S^1_{\mathbb{Q}} \to S^1$. By remark 2.1.1 the functions $\mathcal{I}_n(f)$ factor through π_n:

$$
\begin{array}{ccc}
S^1_{\mathbb{Q}} & \xrightarrow{\ \mathcal{I}_n(f)\ } & \mathbb{C} \\
\downarrow{\scriptstyle \pi_n} & \nearrow{\scriptstyle \hat{f}_n} & \\
S^1 & &
\end{array}
$$

such that:

$$\hat{f}_n(x) = n \int_{\pi_n^{-1}(x)} da f(a)$$

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If $m|n$, by definition $\pi_n^{n/m} = \pi_m$ hence:

$$\bigsqcup_{y^{n/m}=x} \pi_n^{-1}(y) = \pi_m^{-1}(x)$$

for every $x \in S^1$. Then,

$$R_{m,n}(\hat{f}_n)(x) = m/n \sum_{y^{n/m}=x} n \int_{\pi_n^{-1}(y)} da\ f(a) = m \int_{\pi_m^{-1}(x)} da\ f(a) = \hat{f}_m(x)$$

for every $x \in S^1$; i.e. $R_{m,n} \circ \hat{f}_n = \hat{f}_m$. We have a natural linear morphism $\mathcal{I} : C(S_{\mathbb{Q}}^1, \mathbb{C}) \to \varprojlim C(S^1, \mathbb{C})$ such that $\mathcal{I}(f) = (\hat{f}_n)_{n \in \mathbb{N}}$. Actually it a monomorphism:

Lema 2.2.1.
- *The linear morphism $\mathcal{I}$ is a monomorphism; i.e. $\mathcal{I} : C(S_{\mathbb{Q}}^1, \mathbb{C}) \hookrightarrow \varprojlim C(S^1, \mathbb{C})$.*

- *$(g_n) = \mathcal{I}(f)$ if and only if $(g_n \circ \pi_n)_{n \in \mathbb{N}}$ converges uniformly to f.*

Proof:

- Consider a pair of functions $f_1, f_2 \in C(S_{\mathbb{Q}}^1, \mathbb{C})$ such that $\mathcal{I}(f_1) = \mathcal{I}(f_2) = (g_n)$. By definition $\mathcal{I}_n(f_1) = \mathcal{I}_n(f_2) = g_n \circ \pi_n$ and because of Lemma 2.1.4 $f_1 = f_2$ for the sequence $(\mathcal{I}_n(f_i))_{n \in \mathbb{N}}$ uniformly converges to f_i and the limit is unique.

- By definition $\mathcal{I}_n(f) = g_n \circ \pi_n$ and because of Lemma 2.1.4 $(g_n \circ \pi_n)_{n \in \mathbb{N}}$ converges uniformly to f. For the converse, consider a natural n and let $\epsilon > 0$. There is a natural $N \geq n$ such that $||f - g_N \circ \pi_N||_\infty < \epsilon$. Because $\mathcal{I}_N(g_N \circ \pi_N) = g_N \circ \pi_N$ and $||\mathcal{I}_N||_\infty = 1$ we have $||\mathcal{I}_n(f) - g_N \circ \pi_N||_\infty < \epsilon$ hence $||\hat{f}_N - g_N||_\infty < \epsilon$. By the fact that $||R_{n,N}||_\infty = 1$ we have $||\hat{f}_n - g_n||_\infty < \epsilon$ and because $\epsilon > 0$ was arbitrary we conclude that $\hat{f}_n = g_n$.

$\square$

Proposition 2.2.2. *For every C^{m+1} function $f : S_{\mathbb{Q}}^1 \to \mathbb{C}$ such that $m \geq 0$ its Pontryagin series converges in the C^m-topology.*

Proof: Let's see how the operator $R_{m,n}$ acts on monomials:

$$R_{m,n}(z^{\lambda n/m})(x) = m/n \sum_{y^{n/m}=x} y^{\lambda n/m} = m/n \sum_{y^{n/m}=x} x^\lambda = x^\lambda$$

for every $x \in S^1$ hence $R_{m,n}(z^{\lambda n/m}) = z^\lambda$. Consider a natural r such that $1 \leq r \leq (n/m) - 1$. Choose a solution y' of the equation $y^{n/m} = x$. The set of points y/y' such that $y^{n/m} = x$ is the set of (n/m)-th roots of unity. If $r|(n/m)$ then the set of points $(y/y')^r$ such that $y^{n/m} = x$ is the set of (n/mr)-th roots of unity otherwise the set of

points is the set of (n/m)-th roots of unity as before. Either way, because the sum of all k-th roots of unity is zero for arbitrary k, we have that:

$$\sum_{y^{n/m}=x} y^r = y'^r \sum_{y^{n/m}=x} (y/y')^r = 0$$

Then, for every $x \in S^1$ we have:

$$R_{m,n}(z^{\lambda n/m+r})(x) = \frac{m}{n} \sum_{y^{n/m}=x} y^{\lambda n/m+r} = \frac{m}{n} x^\lambda \sum_{y^{n/m}=x} y^r = 0$$

hence $R_{m,n}(z^{\lambda n/m+r}) = 0$ for $r = 1, 2, \ldots (n/m) - 1$.

By remark 2.1.1 and Lemma 2.1.4, $\mathcal{I}_n(f)_0$ is C^{m+1} and $2\pi n$-periodic hence its Fourier series converges in the C^m-topology; i.e. we have that:

$$\mathcal{I}_n(f)(z) = \sum_{q \in \frac{1}{n}\mathbb{Z}} a_q^{(n)} z^q$$

and it converges in the C^m-topology for every natural n.

Claim: The coefficients $a_q^{(n)}$ are independent of n.

In particular we have that:

$$\hat{f}_n(z) = \sum_{i \in \mathbb{Z}} a_{i/n}^{(n)} z^i$$

and it converges in the C^m-topology for every natural n and because the linear operator $R_{m,n}$ is bounded (i.e. continuous, actually $||R_{m,n}|| = 1$) we have:

$$\hat{f}_m(z) = R_{m,n}(\hat{f}_n)(z) = R_{m,n}\left(\sum_{i \in \mathbb{Z}} a_{i/n}^{(n)} z^i\right) = \sum_{i \in \mathbb{Z}} a_{i/n}^{(n)} R_{m,n}(z^i) = \sum_{i \in \mathbb{Z}} a_{i/m}^{(n)} z^i$$

On the other hand:

$$\hat{f}_m(z) = \sum_{i \in \mathbb{Z}} a_{i/m}^{(m)} z^i$$

and because the Fourier series is unique we have the identity $a_{i/m}^{(m)} = a_{i/m}^{(n)}$ for every pair of naturals m, n such that $m|n$ and every integer i. We proved the claim.

Then, there are coefficients $a_q \in \mathbb{C}$ indexed on the rationals such that:

$$\mathcal{I}_n(f)(z) = \sum_{q \in \frac{1}{n}\mathbb{Z}} a_q z^q$$

and it converges in the C^m-topology for every natural n. By Lemma 2.1.4, the sequence $(\mathcal{I}_n(f))_{n \in \mathbb{N}}$ converges to f in the C^{m+1}-topology and we conclude that:

$$f(z) = \sum_{q \in \mathbb{Q}} a_q z^q$$

and the series converges in the C^m-topology. Because the solenoid is compact, in particular it also converges in L_2 and because the Potryagin series is unique, we have the result. $\square$

Corollary 2.2.3. *For every C^∞ function $f : S_\mathbb{Q}^1 \to \mathbb{C}$ its Pontryagin series converges in the C^∞-topology.*

Remark 2.2.1. Actually we have proved that the renormalization maps acts in the following way:

$$\mathcal{I}_n \left(\sum_{q \in \mathbb{Q}} a_q z^q \right) = \sum_{q \in \frac{1}{n}\mathbb{Z}} a_q z^q$$

where the series converges at least uniformly.

Corollary 2.2.4. *The linear operator $\mathcal{I}_n : L_p(S_\mathbb{Q}^1, \mathbb{C}) \to L_p(S_\mathbb{Q}^1, \mathbb{C})$ has operator norm $||\mathcal{I}_n||_p = 1$ for every $1 \leq p \leq \infty$.*

Proof: We already have the result for $p = \infty$. Because the operator $\mathcal{I}_n$ acts as a projection on modes, by Proposition 2.2.2 we have that, restricted to the C^1 functions, the linear operator $\mathcal{I}_n : C^1(S_\mathbb{Q}^1, \mathbb{C}) \to C^1(S_\mathbb{Q}^1, \mathbb{C})$ has operator norm $||\mathcal{I}_n||_p = 1$ for every $p > 1$. Because $C^1(S_\mathbb{Q}^1, \mathbb{C})$ is dense in $L_p(S_\mathbb{Q}^1, \mathbb{C})$ for every $p > 1$, there is a unique extension of $\mathcal{I}_n$ with the same norm. $\qquad\square$

Now, with these new tools at hand, we are able to tackle the problem we discuss at the beginning as a motivation.

Lema 2.2.5. *Every C^1 function on the solenoid has a L_1 Pontryagin transform.*

Proof: Consider a C^1 function f with its Pontryagin series:

$$f(z) = \sum_{q \in \mathbb{Q}} a_q z^q$$

and its derivative along the solenoid f':

$$f(z) = \sum_{q \in \mathbb{Q}} q\ a_q z^q$$

For every natural n consider the 2π-periodic function $\mathcal{I}_n(f) \circ \nu \circ (n_-)$ and its Fourier series:

$$\mathcal{I}_n(f)(\nu(nx)) = \sum_{j \in \mathbb{Z}} b_j e^{ijx}$$

and see that its derivative respect to x coincides with $\mathcal{I}_n(f') \circ \nu \circ (n_-)$:

$$\mathcal{I}_n(f')(\nu(nx)) = \sum_{j \in \mathbb{Z}} j\ b_j e^{ijx}$$

Because $a_{j/n} = b_j$ for every integer j, by Cauchy-Schwartz and Parseval identity we have:

$$\sum_{q \in \frac{1}{n}\mathbb{Z}} |a_q| = \sum_{j \in \mathbb{Z}} |b_j| \leq |b_0| + \left(2 \sum_{j \in \mathbb{N}} \frac{1}{j^2} \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} |j\ b_j|^2 \right)^{1/2}$$

$$\sum_{q \in \frac{1}{n}\mathbb{Z}} |a_q| = \sum_{j \in \mathbb{Z}} |b_j| \le |b_0| + \left(2\sum_{j \in \mathbb{N}} \frac{1}{j^2}\right)^{1/2} \left(\sum_{j \in \mathbb{Z}} |j\, b_j|^2\right)^{1/2}$$

$$= |a_0| + \frac{\pi}{\sqrt{3}}||\mathcal{I}_n(f') \circ \nu \circ (n_-)||_2$$

$$= \ldots$$

A simple direct calculation shows that $||\mathcal{I}_n(f') \circ \nu \circ (n_-)||_2 = ||\mathcal{I}_n(f')||_2$ and because $||\mathcal{I}_n||_2 = 1$ by the previous corollary we have:

$$\ldots = |a_0| + \frac{\pi}{\sqrt{3}}||\mathcal{I}_n(f')||_2$$

$$\le |a_0| + \frac{\pi}{\sqrt{3}}||f'||_2$$

Taking the limit on the left hand side we finally have:

$$\sum_{q \in \mathbb{Q}} |a_q| \le |a_0| + \frac{\pi}{\sqrt{3}}||f'||_2$$

$\square$

Remark 2.2.2. Because the solenoid has unit area by definition, the last useful identity can be written as:

$$||f||_\infty \le \sum_{q \in \mathbb{Q}} |a_q| \le |a_0| + \frac{\pi}{\sqrt{3}}||f'||_\infty$$

Chapter 3

Ahlfors-Bers theory

3.1 Introduction and Preliminaries

We define the adelic Riemann sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$ as the inverse limit of the ramified coverings:

$$
\begin{array}{ccccccccc}
\hat{\mathbb{C}}_{\mathbb{Q}} = \lim \hat{\mathbb{C}} & \longrightarrow & \dots \hat{\mathbb{C}} & \xrightarrow{p_{m,n}} & \hat{\mathbb{C}} & \longrightarrow & \dots \hat{\mathbb{C}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{C}^*_{\mathbb{Q}} = \lim \mathbb{C}^* & \longrightarrow & \dots \mathbb{C}^* & \xrightarrow{p_{m,n}} & \mathbb{C}^* & \longrightarrow & \dots \mathbb{C}^*
\end{array}
$$

with the natural inverse limit maps $\hat{\pi}_n : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}$ where $\hat{\mathbb{C}}$ is the Riemann sphere. We have the canonical inclusion $\mathbb{C}^*_{\mathbb{Q}} \hookrightarrow \hat{\mathbb{C}}_{\mathbb{Q}}$ and the new points:

$$\infty = [(\dots, \infty, \infty, \infty)]$$

$$0 = [(\dots, 0, 0, 0)]$$

Because these are the inverse limit of the ramification points, their topological nature is quite different from the other points. They are cusps. In particular, every homeomorphism of $\hat{\mathbb{C}}_{\mathbb{Q}}$ must fix these new points or permute them. In the following theory, this fixation will be a constraint of the theory and no longer a choice as in the classical theory.

Now we turn to the question of whether continuous maps and differentials on the algebraic solenoid $\mathbb{C}^*_{\mathbb{Q}}$ can be extended to the adelic sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$.

Lema 3.1.1. *Consider a continuous (holomorphic) map f and a continuous (holomorphic) function limit periodic respect to x function g such that:*

$$
\begin{array}{ccc}
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{f} & \mathbb{C} \\
{\scriptstyle \nu} \uparrow & \nearrow {\scriptstyle g} & \\
\mathbb{C} & &
\end{array}
$$

Then:

- *f can be continuously extended to $\mathbb{C}_{\mathbb{Q}}$ if and only if there is a complex number a such that:*

$$\lim_{y \to +\infty} ||a - g|_{Im(z) \geq y}||_{\infty} = 0$$

Moreover, the extension is $f(0) = a$.

- *f can be continuously extended to $\mathbb{C}_{\mathbb{Q}}^* \cup \{\infty\}$ if and only if there is a complex number b such that:*

$$\lim_{y \to -\infty} ||b - g|_{Im(z) \leq y}||_{\infty} = 0$$

Moreover, the extension is $f(\infty) = b$.

Proof: We prove the first item for the second one is completely analogous. It a simple calculus exercise to see that the extension $f(0) = a$ is continuous if and only if:

$$\lim_{r \to 0} ||a - f|_{|\pi_1(x)| \leq r}||_{\infty} = 0$$

Because f is continuous and the image of the baseleaf ν is dense, the above condition is equivalent to the one in the statement for $\pi_1 \circ \nu(z) = e^{iz}$ hence $|\pi_1(\nu(z))| \leq r$ if and only if $Im(z) \geq -ln(r)$ and we have the result. $\qquad\square$

Lema 3.1.2. *Consider a differential $\mu \in C(\mathbb{C}_{\mathbb{Q}}^*)d\overline{\pi_1} \otimes (d\pi_1)^{-1}$ and a differential $\eta \in C(\mathbb{C})d\bar{z} \otimes (dz)^{-1}$ such that $\eta = \nu^*(\mu)$ where ν is the baseleaf. Then, as a function μ has a continuous extension to the whole adelic sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$ if and only if there are constants a, b such that:*

$$\lim_{y \to +\infty} ||a.e^{2i\ Re(z)} + \eta|_{Im(z) \geq y}||_{\infty} = 0$$

$$\lim_{y \to -\infty} ||b.e^{2i\ Re(z)} + \eta|_{Im(z) \leq y}||_{\infty} = 0$$

Moreover, as a function the extension is $\mu(0) = a$ and $\mu(\infty) = b$.

Proof: If $\mu(x) = f(x)d\overline{\pi_1} \otimes (d\pi_1)^{-1}$ then $\eta(z) = \nu^*(\mu)(z) = f \circ \nu(z)(-e^{-2i\ Re(z)})d\bar{z} \otimes (dz)^{-1}$. Because taking the pullback of the differentials only adds a phase $-e^{-2i\ Re(z)}$ of unit norm, by Lemma 3.1.1 we have the result. $\qquad\square$

Lema 3.1.3. *Consider a continuous (holomorphic) baseleaf preserving map $f : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}_{\mathbb{Q}}^*$ such that $deg(f) \neq 0$ and a continuous (holomorphic) limit periodic respect to x map g such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{\ \ f\ \ } & \mathbb{C}_{\mathbb{Q}}^* \\
\Big\uparrow{\scriptstyle\nu} & & \Big\uparrow{\scriptstyle\nu} \\
\mathbb{C} & \xrightarrow{deg(f)z + g(z)} & \mathbb{C}
\end{array}
$$

If $||Im(g)||_{\infty} < \infty$ then f has a continuous extension fixing $0, \infty$ to the whole adelic sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$.

Proof: Define M and m such that $m \leq Im(g(z)) \leq M$ for every $z \in \mathbb{C}$. Because $Im(deg(f)z + g(z)) > y$ if $Im(z) > (y - m)/deg(f)$ we have that f is continuous at zero. Analogously, because $Im(deg(f)z + g(z)) < y$ if $Im(z) < (y - M)/deg(f)$ we have that f is continuous at ∞. $\square$

By Lemma 1.2.9, the degree zero case in the above Lemma is Lemma 3.1.1.

Lema 3.1.4. $Hol(\hat{\mathbb{C}}_{\mathbb{Q}}) \simeq \mathbb{C}$

Proof: Consider a holomorphic function $f : \hat{\mathbb{C}}_{\mathbb{Q}} \to \mathbb{C}$. Its restriction to the solenoid (equator) is:

$$f(\nu(x)) = \sum_{q \in \mathbb{Q}} a_q e^{iqx}$$

and because it is holomorphic we have:

$$f(\nu(z)) = \sum_{q \in \mathbb{Q}} a_q e^{iqz} = \sum_{q \in \mathbb{Q}} a_q e^{iqx} e^{-qy}$$

Because f is continuous on the adelic sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$, by Lemma 3.1.1 $a_q = 0$ for every non zero rational q; i.e. $f = a_0$. $\square$

Let's see how a homeomorphism permute leaves. Consider a homeomorphism $h : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}_{\mathbb{Q}}^*$ homotopic to $z^{p/q}$. Because exp is a local homeomorphism, there is a homeomorphism $\hat{h} : q\hat{\mathbb{Z}} \times \mathbb{C} \to p\hat{\mathbb{Z}} \times \mathbb{C}$ such that

$$
\begin{array}{ccc}
q\hat{\mathbb{Z}} \times \mathbb{C} & \xrightarrow{\hat{h}} & p\hat{\mathbb{Z}} \times \mathbb{C} \\
exp \downarrow & & \downarrow exp \\
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{h} & \mathbb{C}_{\mathbb{Q}}^*
\end{array}
$$

with the structural condition:

$$\hat{h}(a + q, z) = \hat{h}(a, z + 2\pi q) + (p, -2\pi p)$$

for every $a \in q\hat{\mathbb{Z}}$ and $z \in \mathbb{C}$.

Because $\hat{\mathbb{Z}}$ is totally disconnected, $\hat{h}$ maps leaves to leaves; i.e. there are homeomorphisms $s : \hat{\mathbb{Z}} \to \hat{\mathbb{Z}}$ and $f_a : \mathbb{C} \to \mathbb{C}$ such that $\hat{h}(a, z) = (s(a), f_a(z))$. The structural condition implies $s(a + q) = s(a) + p$ for every $a \in q\hat{\mathbb{Z}}$. In particular we have that $s(qn) = s(0) + pn$ for every integer n and because s is continuous we have:

$$s(qa) = s(0) + pa$$

for every $a \in \hat{\mathbb{Z}}$. We have proved the following lemma:

Lema 3.1.5. *Consider a homeomorphism $h : \mathbb{C}_{\mathbb{Q}}^* \to \mathbb{C}_{\mathbb{Q}}^*$ homotopic to $z^{p/q}$ and a homeomorphism $\hat{h} : q\hat{\mathbb{Z}} \times \mathbb{C} \to p\hat{\mathbb{Z}} \times \mathbb{C}$ such that*

$$\begin{array}{ccc}
q\hat{\mathbb{Z}} \times \mathbb{C} & \xrightarrow{\hat{h}} & p\hat{\mathbb{Z}} \times \mathbb{C} \\
{\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{h} & \mathbb{C}_{\mathbb{Q}}^*
\end{array}$$

where $\hat{h}(a, z) = (s(a), f_a(z))$. Then, there is $\lambda \in p\hat{\mathbb{Z}}$ such that

$$s(qa) = \lambda + pa$$

Corollary 3.1.6. *A homeomorphism is leaf preserving if and only if it is homotopic to the identity.*

Proof: Under the notation of the above Lemma, if h is leaf preserving then $s(a) = a + \lambda$ such that λ is now an integer. In particular, $deg(h) = 1$ and by Lemma 1.2.6 h is homotopic to z. For the converse, z is leaf preserving and because the space of leaves $\hat{\mathbb{Z}}/\mathbb{Z}$ is totally disconnected h is leaf preserving too. $\qquad\square$

Since we want to build a theory of continuous deformations of the identity, the above corollary shows that we only need leaf preserving homeomorphisms in our theory.

Definition 3.1.1. A leaf preserving homeomorphism h of $\hat{\mathbb{C}}_{\mathbb{Q}}$ is quasiconformal if it fixes $0, \infty$ and h_a is quasiconformal for every $a \in \hat{\mathbb{Z}}$ such that

$$\begin{array}{ccc}
\hat{\mathbb{Z}} \times \mathbb{C} & \xrightarrow{\hat{h}} & \hat{\mathbb{Z}} \times \mathbb{C} \\
{\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\
\mathbb{C}_{\mathbb{Q}}^* & \xrightarrow{h} & \mathbb{C}_{\mathbb{Q}}^*
\end{array}$$

where $\hat{h}(a, z) = (a, h_a(z))$; i.e. h restricted to every leaf is quasiconformal.

Definition 3.1.2. We say $\mu \in Per$ is a periodic adelic differential if there is some natural n and differential $\mu_n \in L_\infty(\mathbb{C}) \, d\bar{z} \otimes (dz)^{-1}$ such that $\mu = \pi_n^*(\mu_n)$. We say that μ is a periodic Beltrami adelic differential if $\mu \in Per$ and $||\mu||_\infty < 1$. We denote these differentials as Per_1.

The importance of the periodic adelic Beltami differentials is that they trivially have a quasiconformal solution to the respective Beltrami equation: Consider the periodic adelic Beltrami differential $\mu = \pi_n^*(\mu_n)$ and the quasiconformal solution f_n to the μ_n-Beltrami equation fixing $0, 1, \infty$. Define the leaf and orientation preserving homeomorphism f such that:

$$\begin{array}{ccc}
\hat{\mathbb{C}}_{\mathbb{Q}} & \xrightarrow{f} & \hat{\mathbb{C}}_{\mathbb{Q}} \\
{\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} \\
\hat{\mathbb{C}} & \xrightarrow{f_n} & \hat{\mathbb{C}}
\end{array}$$

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Then, f is the quasiconformal solution to the μ-Beltrami equation. At this point, it is natural to ask for a topology $\mathcal{T}$ such that the interior of the closure of these Beltrami differentials constitute new Beltrami differentials for which there exist quasiconformal solutions to their respective Beltrami equations; i.e.:

$$Bel(\mathbb{C}_\mathbb{Q}) = \text{Interior } \overline{Per_1}^{\mathcal{T}}$$

The first natural guess would be the metric topology $\mathcal{T}_\infty$ but this won't do for:

$$\text{Interior } \overline{Per_1}^\infty = L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)_1$$

and there are L_∞-vertical Beltrami differentials μ for which there is no solution to its Beltrami equation (See example 3.3.1).

The rest of the chapter is devoted to this problem and we will find a family of complete metric topologies $\mathcal{T}_{Ren,\mathcal{S}}$ solving it. However, the optimality of these solutions remains an open problem.

3.2   Adelic Beltrami differentials

In what follows, we will make the following abuse of notation:

Remark 3.2.1. Every leaf $\nu_a : \mathbb{C} \hookrightarrow \mathbb{C}_\mathbb{Q}^*$ is a translation surface modeled on π_1 and we will consider differentials in the space $L_\infty(\mathbb{C}_\mathbb{Q}^*) \, \overline{d\pi_1} \otimes (d\pi_1)^{-1}$. In particular, see that the space of Beltrami differentials $L_\infty(\mathbb{C}) \, \overline{dz} \otimes (dz)^{-1}$ embeds in this space via π_1^* for:

$$\pi_1^*(\mu \, \overline{dz} \otimes (dz)^{-1}) = \mu \circ \pi_1 \, \overline{\pi_1^*(dz)} \otimes (\pi_1^*(dz))^{-1} = \mu \circ \pi_1 \, \overline{d\pi_1} \otimes (d\pi_1)^{-1}$$

In pursuit to ease the notation we will make the following abuse of notation: Unless confusion, in what follows we will write a differential $\mu \, \overline{d\pi_1} \otimes (d\pi_1)^{-1}$ just as μ and identify $L_\infty(\mathbb{C}_\mathbb{Q}^*) \, \overline{d\pi_1} \otimes (d\pi_1)^{-1}$ with $L_\infty(\mathbb{C}_\mathbb{Q}^*)$; i.e. we will use the same notation to denote the differential and the function. Unless explicitly written, the context will make clear which one we are using.

Definition 3.2.1. An adelic differential is an element $\mu \in L_\infty(\mathbb{C}_\mathbb{Q})$ such that:

- μ is vertical and horizontal essentially bounded; i.e. $\nu_a^*(\mu) \in L_\infty(\mathbb{C})$ for every leaf ν_a and $F^*(\mu) \in L_\infty(\hat{\mathbb{Z}})$ for every fiber $F : \hat{\mathbb{Z}} \hookrightarrow \mathbb{C}_\mathbb{Q}$.

- μ is vertical L_∞-continuous; i.e. $\mu \in L_\infty^{vert}(\mathbb{C}_\mathbb{Q}^*)$

- There is a cofinal totally ordered divisibility subsystem $\mathcal{S} = (n_j)_{j \in \mathbb{N}}$ such that the following series converge:

$$\sum_{j=1}^\infty n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \infty$$

We will call the above series as an $\mathcal{S}$-renormalized average series. The set of adelic differentials with a convergent $\mathcal{S}$-renormalized average series will be denoted by $Ren_{\mathcal{S}}$. For every $\mu \in Ren_{\mathcal{S}}$ we define its renormalized norm as:

$$||\mu||_{Ren,\mathcal{S}} = n_1 \, ||\mathcal{I}_{n_1}(\mu)||_{\infty} \; + \; \sum_{j=1}^{\infty} n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_{\infty} < \infty$$

such that $\mathcal{S} = (n_j)_{j \in \mathbb{N}}$. An adelic Beltrami differential is an adelic differential μ such that $||\mu||_{\infty} < 1$ and the set of adelic Beltrami differentials with convergent $\mathcal{S}$-renormalized average series will be denoted by $Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$.

Remark 3.2.2. Is clear that $(Ren_{\mathcal{S}}, \; ||\cdot||_{Ren,\mathcal{S}})$ is a metric space.

Definition 3.2.2. Define the vector subspaces of adelic differentials:

$$Per_n = \pi_n^*(L_{\infty}(\mathbb{C}))$$

for every natural n. See that $Per_m \subset Per_n$ if $m|n$. Consider a cofinal totally ordered divisibility subsystem $\mathcal{S} = (n_j)_{j \in \mathbb{N}}$. Then,

$$Per_{\mathcal{S}} = \bigcup_j Per_{n_j} \subset Ren_{\mathcal{S}}$$

The space of periodic adelic Beltrami differentials $Per_{\mathcal{S},1}$ is the set of periodic adelic differentials $\mu \in Per_{\mathcal{S}}$ such that $||\mu||_{\infty} < 1$:

$$Per_{\mathcal{S},1} \subset Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$$

Lema 3.2.1.   • *The canonical inclusion $(Ren_{\mathcal{S}}, \; ||\cdot||_{Ren,\mathcal{S}}) \hookrightarrow L_{\infty}^{vert}(\mathbb{C}_{\mathbb{Q}}^*)$ is continuous. However, its inverse is not. In particular, $Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}) \subset Ren_{\mathcal{S}}$ is open.*

- $\overline{Per_{\mathcal{S}}}^{Ren,\mathcal{S}} = Ren_{\mathcal{S}}$. *In particular, $Per_{\mathcal{S},1} \subset Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$ is a dense subset.*

- $Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$ *is closed under multiplication by functions $\lambda \in L_{\infty}(\mathbb{C}_{\mathbb{Q}})$ such that $||\lambda||_{\infty} \leq 1$. In particular, $Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$ is star shaped respect to zero.*

Proof:

- By Lemma 2.1.1 we have:

$$\mu = \mathcal{I}_{n_1}(\mu) \; + \; \sum_{j=1}^{\infty} (\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu))$$

Then:

$$||\mu||_{\infty} \; \leq \; ||\mathcal{I}_{n_1}(\mu)||_{\infty} \; + \; \sum_{j=1}^{\infty} ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_{\infty}$$

$$\leq \; n_1 \, ||\mathcal{I}_{n_1}(\mu)||_{\infty} \; + \; \sum_{j=1}^{\infty} n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_{\infty} = ||\mu||_{Ren,\mathcal{S}}$$

and we have that the inclusion is continuous. Let's see that the inverse is not: Consider μ_n such that:

$$\nu^*(\mu_n)(z) = \frac{e^{i\frac{x}{n!}}}{n!}e^{-\frac{y^2}{n!^2}}\frac{d\bar{z}}{dz}$$

where ν is the baseleaf and $z = x + iy$. By Lemma 3.1.2 they are continuous in $\hat{\mathbb{C}}_{\mathbb{Q}}$ hence uniformly continuous in $\mathbb{C}_{\mathbb{Q}}^*$. In particular $\mu_n \in L_\infty^{vert}(\mathbb{C}_{\mathbb{Q}}^*)$ and because $||\mu_n||_{Ren,\mathcal{S}} = 1$ we have that $\mu_n \in Ren_\mathcal{S}$ for every n. However, $||\mu_n||_\infty = 1/n!$ tends to zero where $\mathcal{S} = (n!)_{n\in\mathbb{N}}$.

- For every $\epsilon > 0$ there is a natural I such that:

$$\sum_{j=I}^{\infty} n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \epsilon$$

Because of the fact:

$$\mathcal{I}_{n_j} \circ \mathcal{I}_{n_I}(\mu) = \begin{cases} \mathcal{I}_{n_j}(\mu) & j < I \\ \mathcal{I}_{n_I}(\mu) & j \geq I \end{cases}$$

we have:

$$(\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_{j+1}}(\mathcal{I}_{n_I}(\mu))) - (\mathcal{I}_{n_j}(\mu) - \mathcal{I}_{n_j}(\mathcal{I}_{n_I}(\mu))) = \begin{cases} \mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu) & j \geq I \\ 0 & j < I \end{cases}$$

then:

$$||\mu - \mathcal{I}_{n_I}(\mu)||_{Ren,\mathcal{S}} = \sum_{j=I}^{\infty} n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \epsilon$$

and we have that $\overline{Per(\mathbb{C}_{\mathbb{Q}})}^{Ren} = Bel_\mathcal{S}(\mathbb{C}_{\mathbb{Q}})$ for every $\mathcal{I}_{n_i}(\mu)$ is a periodic $\mathcal{S}$- adelic Beltrami differential.

- Is clear from the definition that $||\lambda.\mu||_{Ren,\mathcal{S}} \leq ||\mu||_{Ren,\mathcal{S}}$ for every $\lambda \in L_\infty(\mathbb{C}_{\mathbb{Q}})$ such that $||\lambda||_\infty \leq 1$ and $\mathcal{S}$-adelic Beltrami differential μ.

$\square$

Proposition 3.2.2. $Bel_\mathcal{S}(\mathbb{C}_{\mathbb{Q}})$ *is a Banach manifold modeled on* $Ren_\mathcal{S}$, $||\cdot||_{Ren,\mathcal{S}})$.

Proof: By the previous Lemma, $Bel_\mathcal{S}(\mathbb{C}_{\mathbb{Q}}) \subset Ren_\mathcal{S}$ is an open set and it only rest to show that $(Ren_\mathcal{S}, ||\cdot||_{Ren,\mathcal{S}})$ is complete. Consider a Cauchy sequence $(\mu_n)_{n\in\mathbb{N}}$ in $(Ren_\mathcal{S}, ||\cdot||_{Ren,\mathcal{S}})$. Again by the previous Lemma, the inclusion is continuous and $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(L_\infty^{vert}(\mathbb{C}_{\mathbb{Q}}^*), ||\cdot||_\infty)$. Because this space is complete, there is a unique $\mu \in L_\infty^{vert}(\mathbb{C}_{\mathbb{Q}}^*)$ such that the sequence converges to it respect to the $||\cdot||_\infty$ norm. Because the norm $||\cdot||_{Ren,\mathcal{S}}$ is a series of positive terms we can interchange the limit and

series and we have:

$$\lim_n \ ||\mu_n - \mu_m||_{Ren,\mathcal{S}}$$

$$= \ \lim_n \ \left(n_1 \ ||\mathcal{I}_{n_1}(\mu_n - \mu_m)||_\infty \ + \ \sum_{j=1}^{\infty} n_{j+1} \ ||\mathcal{I}_{n_{j+1}}(\mu_n - \mu_m) - \mathcal{I}_{n_j}(\mu_n - \mu_m)||_\infty \right)$$

$$= \ n_1 \ \lim_n \ ||\mathcal{I}_{n_1}(\mu_n - \mu_m)||_\infty \ + \ \sum_{j=1}^{\infty} n_{j+1} \ \lim_n \ ||\mathcal{I}_{n_{j+1}}(\mu_n - \mu_m) - \mathcal{I}_{n_j}(\mu_n - \mu_m)||_\infty$$

$$= \ n_1 \ ||\mathcal{I}_{n_1}(\mu - \mu_m)||_\infty \ + \ \sum_{j=1}^{\infty} n_{j+1} \ ||\mathcal{I}_{n_{j+1}}(\mu - \mu_m) - \mathcal{I}_{n_j}(\mu - \mu_m)||_\infty$$

$$= \ ||\mu - \mu_m||_{Ren,\mathcal{S}}$$

where we have used that $\mathcal{I}_n$ are bounded linear operators respect to the $||\cdot||_\infty$ norm. By the same argument and the fact that $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence we have:

$$\lim_m \ ||\mu - \mu_m||_{Ren,\mathcal{S}} = 0$$

and we conclude that $\mu \in Ren_{\mathcal{S}}$ and it is the limit of the Cauchy sequence. $\qquad\square$

3.3 Ahlfors-Bers theorem

Definition 3.3.1. Consider an adelic Beltrami differential μ. A quasiconformal map of $\hat{\mathbb{C}}_{\mathbb{Q}}$ is a solution of the Beltrami equation with coefficient μ (the μ-Beltrami equation) if h_a is a solution of the $(\nu_a)^*(\mu)$-Beltrami equation

$$\partial_{\bar{z}} h_a = (\nu_a)^*(\mu) \ \partial_z h_a$$

in the distributional sense for every $a \in \hat{\mathbb{Z}}$ where $\nu_a = exp(a, \_) : \mathbb{C} \hookrightarrow \hat{\mathbb{C}}_{\mathbb{Q}}$ is a leaf and

$$\begin{array}{ccc}
\hat{\mathbb{Z}} \times \mathbb{C} & \xrightarrow{\hat{h}} & \hat{\mathbb{Z}} \times \mathbb{C} \\
\downarrow{\scriptstyle exp} & & \downarrow{\scriptstyle exp} \\
\mathbb{C}^*_{\mathbb{Q}} & \xrightarrow{h} & \mathbb{C}^*_{\mathbb{Q}}
\end{array}$$

where $\hat{h}(a, z) = (a, h_a(z))$.

The following is the adelic version of Ahlfors-Bers theorem:

Theorem 3.3.1. *For every adelic Beltrami differential μ there is a unique quasiconformal leaf preserving solution $f : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ to the μ-Beltrami equation such that f fixes $0, 1, \infty$.*

Remark 3.3.1. If f is a solution to the μ-Beltrami equation then μ must be uniformly vertical continuous for $\mu = \partial_{\bar{z}} f / \partial_z f$. This is why we ask for the L_∞-vertical continuous condition in the adelic differential definition (See corollary 2.1.2).

Before presenting the proof of the adelic version of the Ahlfors-Bers theorem, it is important or at least pedagogical to describe some problems within and understand the capricious nature of the adelic Beltrami differential definition.

By Ahlfors-Bers theorem, there is a quasiconformal solution h_a for every leaf modulo postcompositions with affine transformations. In particular, the solutions can be chosen such that they verify the structural constraint:

$$h_{a+1}(z) + 2\pi = h_a(z + 2\pi)$$

for every $z \in \mathbb{C}$ and $a \in \hat{\mathbb{Z}}$ defining this way a leaf preserving map $h : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ fixing $0, \infty$. However, there is a priori no reason to expect that the resulting map would be continuous. Is clear that it will be continuous along the leaves but in general not across them. It's like drawing a picture separately in every piece of a puzzle and expect that the we get a clear picture after we put the pieces together. We have decomposed the foliated object in leaves and solved the problem for each leaf. To assure a continuous solution we need a global structural constraint.

The natural guess is that imposing some notion of vertical continuity (L_∞-vertical continuity) to the Beltrami differential would give the desired continuity of the solution across the leaves. Although this is a necessary condition, it is not enough. As the next example shows, even for a continuous Beltrami differential there is no need to continuity of the solution across the leaves.

Example 3.3.1. Consider the following function:

$$\mu(z) = \frac{\frac{1}{2e} \sum_{n=1}^{+\infty} \left[\cos(x/n!) - \frac{2iy}{n!} \sin(x/n!)\right] \frac{e^{-\frac{y^2}{n!^2}}}{2n!}}{1 + \frac{1}{2e} \sum_{n=1}^{+\infty} \left[\cos(x/n!) + \frac{2iy}{n!} \sin(x/n!)\right] \frac{e^{-\frac{y^2}{n!^2}}}{2n!}} \frac{d\bar{z}}{dz}$$

where $z = x + iy$. Let's see that it is a Beltrami differential; i.e. $||\mu||_\infty < 1$.

Define:

$$h_{\pm}(n)(z) = \frac{1}{2n!} e^{\pm ix/n!} \left(1 \mp 2y/n!\right) \frac{e^{-\frac{y^2}{n!^2}}}{2}$$

where $z = x + iy$. Because $||h_{\pm}(n)||_\infty < 1/2n!$ and the identity:

$$h_+(n) \pm h_-(n) = \left[\cos(x/n!) \mp \frac{2iy}{n!} \sin(x/n!)\right] \frac{e^{-\frac{y^2}{n!^2}}}{2n!}$$

we have that each term of the sum has supremum norm less than $1/n!$ hence the supremum norm of the sum is less than $e - 1$. We conclude that:

$$||\mu||_\infty < \frac{\frac{1}{2e}(e-1)}{1 - \frac{1}{2e}(e-1)} = \frac{e-1}{e+1} < 1/2$$

Because it is limit periodic respect to x and decays to zero when y tends to $\pm\infty$, by Lemmas 1.2.8 and 3.1.2 there is a continuous adelic Beltrami differential $\hat{\mu}$ on $\mathbb{C}_{\mathbb{Q}}^*$ with

a continuous extension to the whole adelic sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$ as a function such that $\mu = \nu^*(\hat{\mu})$ where ν is the baseleaf.

However, the quasiconformal solution of the μ-Beltrami equation:

$$w^\mu(z) = z + \frac{1}{2e} \sum_{n=1}^{+\infty} \sin(x/n!) e^{-\frac{y^2}{n!^2}}$$

is not of the type $z + g(z)$ with g limit periodic respect to x and by lemma 1.2.5 it is not the conjugation of any continuous map of the adelic sphere $\hat{\mathbb{C}}_{\mathbb{Q}}$ by the baseleaf ν; i.e. There is no continuous map $\hat{w}$ such that

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{w^\mu} & \mathbb{C} \\
\nu \downarrow & & \downarrow \nu \\
\hat{\mathbb{C}}_{\mathbb{Q}} & \xrightarrow{\hat{w}} & \hat{\mathbb{C}}_{\mathbb{Q}}
\end{array}
$$

$\square$

The above example shows that we still need some other global structural condition on the Beltrami differential to assure the continuity of its solution. This is precisely the convergence of the *renormalized average series*: *There is a cofinal totally ordered divisibility subsequence* $\mathcal{S} = (n_j)_{j\in\mathbb{N}}$ *such that the following series converge:*

$$\sum_{j=1}^{\infty} n_{j+1} \, ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \infty \tag{3.1}$$

The following definitions and Lemmas are the prelude to the Ahlfors-Bers theorem:

Theorem 3.3.2. *Consider k such that $0 \le k < 1$. Then, there exists a real number $p > 2$ only depending on k such that: For every Beltrami differential $\mu \in L_\infty(\mathbb{C})_1$ with $||\mu||_\infty \le k$ and compact support there is a unique quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ such that $f(0) = 0$ and $f_z - 1 \in L_p(\mathbb{C})$ (globally and not merely locally) verifying:*

$$f_{\bar{z}} = \mu f_z$$

on $\mathbb{C}$ in the sense of distributions.

A map verifying the conditions of the theorem is called a *normal quasiconformal solution*. The previous Theorem can be found in [Ah], [IT].

Lema 3.3.3. *If f is a normal solution of the μ-Beltrami equation such that μ has compact support, then there are constants A and $p > 2$ such that:*

$$|f(\zeta) - \zeta| \le A||\mu||_\infty |\zeta|^{1-2/p}$$

Moreover, the constant A is monotone respect to the area of the μ support and depends also on p.

Proof: By general theory [Ah], [IT], there is $2 < p < \infty$ such that:

$$f(z) = P(f_{\bar{z}})(z) + z$$

where P is the following linear operator on $L_p(\mathbb{C})$:

$$Ph(\zeta) = -\frac{1}{\pi} \int \int_{\mathbb{C}} h(z) \left(\frac{1}{z - \zeta} - \frac{1}{z} \right) \, dxdy$$

for every $h \in L_p(\mathbb{C})$ and $\zeta \in \mathbb{C}$. For any p such that $2 < p < \infty$ and every $h \in L_p(\mathbb{C})$, Ph is uniformly Hölder continuous with exponent $1 - 2/p$ and verifies $Ph(0) = 0$. Moreover, there is a constant K_p depending only on p such that:

$$|Ph(\zeta)| \leq K_p ||h||_p |\zeta|^{1-2/p}$$

for every $\zeta \in \mathbb{C}$. The map $\mu \mapsto f_{\bar{z}} \in L_p(\mathbb{C})$ from the space of Beltrami differentials with compact support is Lipschitz continuous; i.e. There is a constant C such that:

$$||f_{\bar{z}}||_p \leq C||\mu||_p \leq C \, Area(support(\mu))^{1/p} ||\mu||_\infty$$

Combining these relations, there is $2 < p < \infty$ such that:

$$
\begin{aligned}
|f(z) - z| &= |P(f_{\bar{z}})(z)| \leq K_p ||f_{\bar{z}}||_p |z|^{1-2/p} \\
&\leq \left(K_p C \, Area(support(\mu))^{1/p} \right) ||\mu||_\infty |z|^{1-2/p}
\end{aligned}
$$

and we have the Lemma. $\qquad\square$

Lema 3.3.4. *If f is a normal solution of the μ-Beltrami equation such that μ has compact support, then there are constants B and $p > 2$ such that:*

$$|\zeta| \leq B||\mu||_\infty |f(\zeta)|^{1-2/p} + |f(z)|$$

Moreover, the constant B is monotone respect to the area of the μ support and depends also on p.

Proof: Consider the inverse normal homeomorphism f^{-1} with Beltrami differential $\mu_{f^{-1}}$ such that:

$$\mu_{f^{-1}} \circ f = -\frac{f_z}{\bar{f_z}} \mu$$

Then, Hölder's inequality gives:

$$
\begin{aligned}
\int \int_{\mathbb{C}} |\mu_{f^{-1}}|^p \, dxdy &= \int \int_{\mathbb{C}} |\mu|^p (|f_z|^2 - |f_{\bar{z}}|^2) \, dxdy \\
&\leq \int \int_{\mathbb{C}} |\mu|^p |f_z|^2 \, dxdy = \int \int_{\mathbb{C}} |\mu|^{p-2} |f_{\bar{z}}|^2 \, dxdy \\
&\leq \left(\int \int_{\mathbb{C}} |\mu|^p \, dxdy \right)^{\frac{p-2}{p}} \left(\int \int_{\mathbb{C}} |f_{\bar{z}}|^p \, dxdy \right)^{\frac{2}{p}} \\
&= ||\mu||_p^{p-2} ||f_{\bar{z}}||_p^2 \leq ||\mu||_p^{p-2} (C||\mu||_p)^2 = C^2 ||\mu||_p^p
\end{aligned}
$$

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and we conclude that:
$$||\mu_{f^{-1}}||_p \ \leq \ C^{2/p}||\mu||_p$$

In the same way as in the previous proof, there is a constant C' (actually $C' = C$) such that:
$$||(f^{-1})_{\bar{z}}||_p \leq C'||\mu_{f^{-1}}||_p \ \leq \ C'C^{2/p} \ Area(support(\mu))^{1/p}||\mu||_\infty$$

and proceeding just as in the above Lemma, we have:
$$|f^{-1}(z) - z| \leq B||\mu||_\infty|z|^{1-2/p}$$

such that:
$$B = K_p C'C^{2/p} \ Area(support(\mu))^{1/p}$$

Substitution of $z = f(\zeta)$ and triangular inequality gives the desired result. $\qquad\square$

Consider $\mu \in L_\infty(\mathbb{C}^*_\mathbb{Q})_1$. For every natural n define the Beltrami differential $\mu_n \in L_\infty(\mathbb{C}^*)_1\overline{dz} \otimes (dz)^{-1}$ such that (Recall remark 3.2.1):
$$\mathcal{I}_n(\mu) \ \overline{d\pi_1} \otimes (d\pi_1)^{-1} = \pi_n^*(\mu_n)$$

Consider the quasiconformal normal solution $f_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the μ_n-Beltrami equation such that $f(0) = 0$ and $f_z - 1 \in L_p(\mathbb{C})$, $p > 2$. If $n|L$, define the maps $f_n^{\uparrow L}$ and $\hat{f}_n$ such that:

$$
\begin{array}{ccc}
\hat{\mathbb{C}}_\mathbb{Q} & \xrightarrow{\ \hat{f}_n\ } & \hat{\mathbb{C}}_\mathbb{Q} \\
{\scriptstyle \pi_L}\downarrow & & \downarrow{\scriptstyle \pi_L} \\
\hat{\mathbb{C}} & \xrightarrow{\ f_n^{\uparrow L}\ } & \hat{\mathbb{C}} \\
{\scriptstyle z^{L/n}}\downarrow & & \downarrow{\scriptstyle z^{L/n}} \\
\hat{\mathbb{C}} & \xrightarrow{\ f_n\ } & \hat{\mathbb{C}}
\end{array}
$$

Lema 3.3.5. • *The map $f_n^{\uparrow L}$ is a quasiconformal normal solution of the $\mu_n^{\uparrow L}$-Beltrami equation such that $\mu_n^{\uparrow L} = (z^{L/n})^*(\mu_n)$. Moreover, $(f_n^{\uparrow L})_z - 1 \in L_p(\mathbb{C})$ with the same $p > 2$ as the one we used for f_n.*

• *The composition of quasiconformal normal maps is a quasiconformal normal map.*

Proof:

• Define $n' = L/n$. First, let's see that $f_n^{\uparrow L}$ is quasiconformal. Indeed, it verifies Ahlfors *quasiconformal definition A* [Ah]:

    – *It is an orientation preserving homeomorphism:* Because $z^{n'}$ is a covering and f_n is an orientation preserving homeomorphism then $f_n^{\uparrow L}$ is so.

    – *It is ACL, absolutely continuous on lines:* Because f_n is absolutely continuous respect to any finite length rectifiable curve and the covering is C^1 we have that $f_n^{\uparrow L}$ is *ACL*.

– *It has bounded maximal dilatation:* Locally, the map $z^{1/n'}$ is defined outside zero and we have $f_n^{\uparrow L} = z^{1/n'} \circ f_n \circ z^{n'}$. Then,

$$\partial_z f_n^{\uparrow L}(z) = f_n(z^{n'})^{1/n'-1} \, \partial_z f_n(z^{n'}) \, z^{n'-1}$$

$$\partial_{\bar{z}} f_n^{\uparrow L}(z) = f_n(z^{n'})^{1/n'-1} \, \partial_{\bar{z}} f_n(z^{n'}) \, \bar{z}^{n'-1}$$

Because $\partial_{\bar{z}} f_n = \mu \, \partial_z f_n$ we have:

$$\partial_{\bar{z}} f_n^{\uparrow L} = \left(\mu \circ z^{n'} \frac{\bar{z}^{n'-1}}{z^{n'-1}} \right) \partial_z f_n^{\uparrow L} = \mu_n^{\uparrow L} \, \partial_z f_n^{\uparrow L}$$

hence $f_n^{\uparrow L}$ is a solution to the Beltrami equation with Beltrami differential $\mu_n^{\uparrow L}$. In particular, it has bounded maximal dilatation.

Now, let's see that it is normal. Consider the normal quasiconformal solution g to the $\mu_n^{\uparrow L}$-Beltrami equation. Because g and $f_n^{\uparrow L}$ are quasiconformal solutions of the same equation and both fix the origin, there is a non zero λ such that $g = \lambda f_n^{\uparrow L}$. Locally it means that:

$$g(z)^{n'} = \lambda f_n(z^{n'}) \tag{3.2}$$

for every $z \in \mathbb{C}$. Because $\mu_n^{\uparrow L}$ has compact support, they are both univalent outside a disk of sufficiently large radius R and we can write:

$$f_n(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots$$

and an analogous expression for g outside the disk. Substituting these expansions in equation (3.2) and comparing the leading term we get $\lambda = 1$. Because of the following relation:

$$||\mu_n^{\uparrow L}||_\infty = ||(z^{n'})^* \mu_n||_\infty = ||\mu_n||_\infty \le k$$

by Theorem 3.3.2 $(f_n^{\uparrow L})_z - 1 \in L_p(\mathbb{C})$ with the same $p > 2$ as the one we used for f_n and the claim is proved.

- Consider the quasiconformal normal maps f_1 and f_2 with respective Beltrami differentials μ_1 and μ_2. Their composition is a quasiconformal map fixing the origin with Beltrami differential μ with compact support. Consider the quasiconformal normal solution g to the μ-Beltrami equation. Again, because g and $f_1 \circ f_2$ are quasiconformal solutions of the same equation and both fix the origin, there is a non zero λ such that:

$$g = \lambda f_1 \circ f_2 \tag{3.3}$$

In the same way as before, because μ, μ_1 and μ_2 have compact support, they are univalent outside a disk of sufficiently large radius R and we can write:

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots$$

and an analogous expressions for f_1 and f_2 outside the disk. Substituting these expressions in equation (3.3) and comparing the leading terms we get $\lambda = 1$ just as before. This proves the claim.

$\square$

Remark 3.3.2. The above Lemma explains why we choose this normalization. The Douady-Hubbard normalization $f(z) - z \in O(1/|z|)$ is easy to work with but doesn't necessarily fix the origin and as we said before, this is no longer a choice but a constraint of the new theory. The normalization $f(0) = 1$ and $f(1) = 1$ is compatible with the maps $z \mapsto z^n$ of the inverse system. However, it is very difficult to control the growth of the maps in terms of their respective Beltrami differentials. The above Lemma shows that the chosen normalization is compatible with the inverse system with the advantage of having some control on the maps.

If $m|n|L$ define $f_{n,m}^{\uparrow L} = f_n^{\uparrow L} \circ (f_m^{\uparrow L})^{-1}$. See that $\hat{f}_{n,m} = \hat{f}_n \circ \hat{f}_m^{-1}$. The quasiconformal normal map $f_{n,m}^{\uparrow L}$ is the solution of the $\mu_{n,m}^{\uparrow L}$-Beltrami equation such that:

$$f_m^*(\mu_{n,m}^{\uparrow L}) = \frac{\mu_n^{\uparrow L} - \mu_m^{\uparrow L}}{1 - \overline{\mu_n^{\uparrow L}} \mu_m^{\uparrow L}} \; \overline{dz} \otimes (dz)^{-1}$$

where $\mu_n^{\uparrow L}$ and $\mu_m^{\uparrow L}$ on the right side denote the functions and not the differentials (recall remark 3.2.1). Because $||\mathcal{I}_n||_\infty = 1$ for every natural n, we have $||\mu_n^{\uparrow L}||_\infty = ||\mathcal{I}_n(\mu)||_\infty \leq ||\mu||_\infty$ for every natural L such that $n|L$. We also have:

$$||\mu_{n,m}^{\uparrow L}||_\infty \leq \frac{||\mu_n^{\uparrow L} - \mu_m^{\uparrow L}||_\infty}{1 - ||\mu_n^{\uparrow L}||_\infty ||\mu_m^{\uparrow L}||_\infty} \leq \frac{||\mathcal{I}_n(\mu) - \mathcal{I}_m(\mu)||_\infty}{1 - ||\mu||_\infty^2} \tag{3.4}$$

for every $m|n|L$, where the last step follows from the following calculus:

$$
\begin{aligned}
\| \mu_n^{\uparrow L} - \mu_m^{\uparrow L} \|_\infty &= \| \pi_L^*(\mu_n^{\uparrow L} - \mu_m^{\uparrow L}) \|_\infty \\
&= \| \pi_L^*((z^{L/n})^* \mu_n - (z^{L/m})^* \mu_m) \|_\infty \\
&= \| \pi_L^*((z^{L/n})^* \mu_n) - \pi_L^*((z^{L/m})^* \mu_m) \|_\infty \\
&= \| \pi_n^*(\mu_n) - \pi_m^*(\mu_m) \|_\infty \\
&= \| \mathcal{I}_n(\mu) - \mathcal{I}_m(\mu) \|_\infty
\end{aligned}
$$

See that the right hand side of relation (3.4) doesn't depend on L.

Lema 3.3.6. *Consider a vertical essentially bounded $\mu \in L_\infty(\mathbb{C}_\mathbb{Q})_1$ with compact support. Suppose there is a subsequence $(n_i)_{i \in \mathbb{N}}$ of the divisibility net such that:*

$$\lim_{i \to \infty} \| \mathcal{I}_{n_i}(\mu) - \mathcal{I}_{n_{i-1}}(\mu) \|_\infty = 0$$

Let $L = n_J$. There are constants A and A' such that:

- *If $i \leq J$ then*

$$|\pi_L \circ \hat{f}_{n_i}(x)| \leq (1 + A||\mathcal{I}_{n_i}(\mu)||_\infty) \; max\{ \; 1, \; |\pi_L(x)| \; \}$$

$$|\pi_L \circ \hat{f}_{n_i}(x)| \leq \left(1 + A'||\mathcal{I}_{n_i}(\mu) - \mathcal{I}_{n_{i-1}}(\mu)||_\infty\right) \; max\{ \; 1, \; |\pi_L \circ \hat{f}_{n_{i-1}}(x)| \; \}$$

- *If $i > J$ then*

$$|\pi_L \circ \hat{f}_{n_i}(x)| \leq (1 + A||\mathcal{I}_L||_\infty)e^{\frac{A'}{L}\sum_{j=J}^{i-1} n_{j+1} ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty} \, max\{ \, 1, \, |\pi_L(x)| \, \}$$

Proof: By hypothesis, equation (3.4) and the fact that:

$$||\mu_{n_i}^{\uparrow L}||_\infty = ||\mathcal{I}_{n_i}(\mu)||_\infty \leq ||\mu||_\infty < 1$$

for every natural i, we conclude that:

$$k = max\left\{ \, ||\mu||_\infty, \, ||\mu_{n_i}^{\uparrow L}||_\infty, \, ||\mu_{n_j,n_{j-1}}^{\uparrow L}||_\infty \quad such \ that \ \ i, j \in \mathbb{N} \, \right\} < 1$$

Hence, by Lemma 3.3.2 we can take the same value $p > 2$ for all the maps $f_{n_i}^{\uparrow L}$ and $f_{n_i,n_{i-1}}^{\uparrow L}$. Because the supports of all $\mu_{n_i}^{\uparrow L}$ and $\mu_{n_i,n_{i-1}}^{\uparrow L}$ are uniformly bounded:

$$supp\big(\mu_{n_i}^{\uparrow L}\big), \ supp\big(\mu_{n_i,n_{i-1}}^{\uparrow L}\big) \subset \pi_1\big(supp(\mu)\big) \cup \overline{D(0;1)}$$

by Lemma 3.3.3 we can also take the same constant A for all the maps $f_{n_i}^{\uparrow L}$ and $f_{n_i,n_{i-1}}^{\uparrow L}$. Then we have:

$$
\begin{aligned}
|\pi_L \circ \hat{f}_{n_i}(x)| &= |f_{n_i}^{\uparrow L}(\pi_L(x))| \\
&\leq A \, ||\mu_{n_i}^{\uparrow L}||_\infty \, |\pi_L(x)|^{1-2/p} + |\pi_L(x)| \\
&\leq A \, ||\mathcal{I}_{n_i}(\mu)||_\infty \, |\pi_L(x)|^{1-2/p} + |\pi_L(x)| \\
&\leq (1 + A||\mathcal{I}_{n_i}(\mu)||_\infty) \, max\{ \, 1, \, |\pi_L(x)| \, \}
\end{aligned}
$$

In particular,

$$|\pi_L \circ \hat{f}_L(x)| \leq (1 + A||\mathcal{I}_L||_\infty) \, max\{ \, 1, \, |\pi_L(x)| \, \} \tag{3.5}$$

For the second:

$$
\begin{aligned}
|\pi_L \circ \hat{f}_{n_i,n_{i-1}}(x)| &= |f_{n_i,n_{i-1}}^{\uparrow L}(\pi_L(x))| \\
&\leq A' \, ||\mathcal{I}_{n_i}(\mu) - \mathcal{I}_{n_{i-1}}(\mu)||_\infty \, |\pi_L(x)|^{1-2/p} + |\pi_L(x)| \\
&\leq (1 + A' \, ||\mathcal{I}_{n_i}(\mu) - \mathcal{I}_{n_{i-1}}(\mu)||_\infty) \, max\{ \, 1, \, |\pi_L(x)| \, \}
\end{aligned}
$$

where $A' = A/(1 - k^2)$. Because $\hat{f}_{n_i} = \hat{f}_{n_i,n_{i-1}} \circ \hat{f}_{n_{i-1}}$ the result follows.

Finally, for the third assertion we have:

$$
\begin{aligned}
|\pi_{n_j} \circ \hat{f}_{n_j,n_{j-1}}(x)| &= |f_{n_j,n_{j-1}}(\pi_{n_j}(x))| \\
&\leq A' \, ||\mathcal{I}_{n_j}(\mu) - \mathcal{I}_{n_{j-1}}(\mu)||_\infty \, |\pi_{n_j}(x)|^{1-2/p} + |\pi_{n_j}(x)| \\
&\leq \left(\frac{A'a_{n_j}}{n_j} + 1\right) \, max\{ \, 1, \, |\pi_{n_j}(x)| \, \}
\end{aligned}
$$

where $a_{n_j} = n_j \, ||\mathcal{I}_{n_j}(\mu) - \mathcal{I}_{n_{j-1}}(\mu)||_\infty$. Because $\pi_{n_j}^{n_j/L} = \pi_L$ we have:

$$
\begin{aligned}
|\pi_L \circ \hat{f}_{n_j,n_{j-1}}(x)| &= |\pi_{n_j} \circ \hat{f}_{n_j,n_{j-1}}(x)|^{n_j/L} \\
&\leq \left(\frac{A'a_{n_j}}{n_j} + 1\right)^{n_j/L} max\{ \, 1, \, |\pi_{n_j}(x)| \, \}^{n_j/L} \\
&\leq e^{\frac{A'}{L}a_{n_j}} \, max\{ \, 1, \, |\pi_L(x)| \, \}
\end{aligned}
$$

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In particular, because the right hand side of the above equation is greater than or equal to one, then:

$$max\{\ 1,\ |\pi_L \circ \hat{f}_{n_j, n_{j-1}}(x)|\ \} \leq e^{\frac{A'}{L}a_{n_j}}\ max\{\ 1,\ |\pi_L(x)|\ \} \tag{3.6}$$

and by the same argument, relation (3.5) implies:

$$max\{\ 1,\ |\pi_L \circ \hat{f}_L(x)|\ \} \leq (1 + A||\mathcal{I}_L||_\infty)\ max\{\ 1,\ |\pi_L(x)|\ \} \tag{3.7}$$

Because

$$\hat{f}_{n_i} = \hat{f}_{n_i, n_{i-1}} \circ \hat{f}_{n_{i-1}, n_{i-2}} \ldots \circ \hat{f}_{n_{J+1}, n_J} \circ \hat{f}_L$$

induction on relation (3.6) and relation (3.7) imply:

$$|\pi_L \circ \hat{f}_{n_i}(x)| \leq max\{\ 1,\ |\pi_L \circ \hat{f}_{n_i}(x)|\ \} \leq (1 + A||\mathcal{I}_L||_\infty)e^{\frac{A'}{L}\sum_{j=J}^{i-1}a_{n_{j+1}}}\ max\{\ 1,\ |\pi_L(x)|\ \}$$

and the result follows. $\square$

Corollary 3.3.7. *Consider a vertical essentially bounded $\mu \in L_\infty(\mathbb{C}_\mathbb{Q})_1$ with compact support. If there is a subsequence $(n_i)_{i\in\mathbb{N}}$ of the divisibility net such that the renormalized average series converge:*

$$\sum_{j=1}^{\infty} n_{j+1}\ ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \infty$$

Then, for every natural $i \geq J$:

$$|\pi_L \circ \hat{f}_{n_i}(x)| \leq (1 + A||\mathcal{I}_L||_\infty)e^{\frac{A'}{L}\sum_{j=J}^{\infty}n_{j+1}\ ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty}\ max\{\ 1,\ |\pi_L(x)|\ \}$$

where $L = n_J$ and A, A' are constants.

Proof: The convergence of the series implies the hypothesis of the previous Lemma 3.3.6. Taking the limit $i \to \infty$ on the right hand side of the relation gives the result. $\square$

Lema 3.3.8. *Under the same hypothesis of corollary 3.3.7 above we have:*

$$|\pi_L(x)| \leq (1 + B||\mathcal{I}_L||_\infty)e^{\frac{B'}{L}\sum_{j=J}^{\infty}n_{j+1}\ ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty}\ max\{\ 1,\ |\pi_L \circ \hat{f}_{n_i}(x)|\ \}$$

for every natural $i \geq J$ where $L = n_J$ and B, B' are constants.

Proof: The proof is almost verbatim to the proof of Lemma 3.3.6 with refernce to Lemma 3.3.4 instead of 3.3.3. $\square$

Lema 3.3.9. *Under the same hypothesis of corollary 3.3.7, for every natural $L = n_J$ there is a constant $M_L \geq 1$ such that:*

$$|\pi_L \circ \hat{f}_{n_{i+1}}(x) - \pi_L \circ \hat{f}_{n_i}(x)| \leq \frac{A'}{L}\ n_{i+1}\ ||\mathcal{I}_{n_{i+1}}(\mu) - \mathcal{I}_{n_i}(\mu)||_\infty\ M_L\ max\{\ 1,\ |\pi_L(x)|\ \}$$

for every $i \geq J$ where A' is a constant.

Proof: We take the same values of $k < 1$, $p > 2$ and constants A and $A' = A/(1 - k^2)$ as those in the proof of Lemma 3.3.6. Denote $n = n_i$ and $m = n_{i-1}$. By Lemma 3.3.3 and relation (3.4) we have:

$$|f_{n,m}(\pi_n(x)) - \pi_n(x)| \leq A' \parallel \mathcal{I}_n(\mu) - \mathcal{I}_m(\mu) \parallel_\infty | \pi_n(x) |^{1-2/p} \tag{3.8}$$

where $A' = A/(1 - k^2)$ and $\parallel \mu \parallel_\infty = k$. Define $n' = n/L$. Lagrange Theorem implies:

$$
\begin{aligned}
|\pi_L \circ \hat{f}_{n,m}(x) - \pi_L(x)| &= |f_{n,m}(\pi_n(x))^{n'} - \pi_L(x)| \\
&\leq n' \, |\xi|^{n'-1} \, |f_{n,m}(\pi_n(x)) - \pi_n(x)|
\end{aligned}
\tag{3.9}
$$

for some ξ in the interior of the segment joining $\pi_n \circ \hat{f}_{n,m}(x)$ and $\pi_n(x)$. In particular,

$$
\begin{aligned}
|\xi|^{n'-1} &\leq \; max\{ \, |\pi_n(x)|, \; |\pi_n \circ \hat{f}_{n,m}(x)| \, \}^{n'-1} \\
&= \; max\{ \, |\pi_L(x)|, \; |\pi_L \circ \hat{f}_{n,m}(x)| \, \}^{1-1/n'}
\end{aligned}
\tag{3.10}
$$

Equations 3.8, 3.9 and 3.10 imply:

$$
|\pi_L \circ \hat{f}_{n,m}(x) - \pi_L(x)| \leq \frac{A'}{L} \, n \, ||\mathcal{I}_n(\mu) - \mathcal{I}_m(\mu)||_\infty \ldots
$$
$$
\ldots max\{ \, |\pi_L(x)|, \; |\pi_L \circ \hat{f}_{n,m}(x)| \, \}^{1-1/n'} \, | \, \pi_L(x) \, |^{(1-2/p)1/n'} \tag{3.11}
$$

In particular, because $\hat{f}_n = \hat{f}_{n,m} \circ \hat{f}_m$ we have:

$$
|\pi_L \circ \hat{f}_n(x) - \pi_L \circ \hat{f}_m(x)| \leq \frac{A'}{L} \, n \, ||\mathcal{I}_n(\mu) - \mathcal{I}_m(\mu)||_\infty \ldots
$$
$$
\ldots max\{ \, |\pi_L \circ \hat{f}_m(x)|, \; |\pi_L \circ \hat{f}_n(x)| \, \}^{1-1/n'} \, | \, \pi_L \circ \hat{f}_m(x) \, |^{(1-2/p)1/n'} \tag{3.12}
$$

By the previous corollary 3.3.7 there is a constant M_L such that:

$$|\pi_L \circ \hat{f}_{n_i}(x)| \leq M_L \, max\{ \, 1, \; |\pi_L(x)| \, \} \tag{3.13}$$

for every $i \geq J$ where $L = n_J$. This bound implies:

$$
\begin{aligned}
&|\pi_L \circ \hat{f}_{n_{i+1}}(x) - \pi_L \circ \hat{f}_{n_i}(x)| \\
&\leq \frac{A'}{L} \, n_{i+1} \, ||\mathcal{I}_{n_{i+1}}(\mu) - \mathcal{I}_{n_i}(\mu)||_\infty \, (M_L \, max\{ \, 1, \; |\pi_L(x)| \, \})^{1-2/pn'} \\
&\leq \frac{A'}{L} \, n_{i+1} \, ||\mathcal{I}_{n_{i+1}}(\mu) - \mathcal{I}_{n_i}(\mu)||_\infty \, M_L \, max\{ \, 1, \; |\pi_L(x)| \, \}
\end{aligned}
\tag{3.14}
$$

where we have used that $M_L \, max\{ \, 1, \; |\pi_L(x)| \, \} \geq 1$ and the formula is proved. $\qquad \square$

Lema 3.3.10. *Under the same hypothesis of corollary 3.3.7 and previous definition of the maps $\hat{f}_{n_i}$, there is a continuous leaf preserving map $\hat{f} : \hat{\mathbb{C}}_\mathbb{Q} \to \hat{\mathbb{C}}_\mathbb{Q}$ such that $(\hat{f}_{n_i})_{i \in \mathbb{N}}$ converges pointwise to $\hat{f}$.*

Proof: For each $L = n_J$, Lemma 3.3.9 implies that $(\pi_L \circ \hat{f}_{n_i})_{i \in \mathbb{N}}$ is a uniform Cauchy sequence on compact sets so there is a continuous function $g_L : \mathbb{C}_{\mathbb{Q}} \to \mathbb{C}$ such that the sequence $(\pi_L \circ \hat{f}_{n_i})_{i \in \mathbb{N}}$ converges uniformly to g_L on compact sets. Consider another $L' = n_{J'}$ such that $J' > J$. Because $z^{L'/L} \circ \pi_{L'} \circ \hat{f}_{n_i} = \pi_L \circ \hat{f}_{n_i}$ for every $i \geq J'$ and the continuity of $z^{L'/L}$ we have that $z^{L'/L} \circ g_{L'} = g_L$. By the universal property of inverse limits there is a unique function $\hat{f} : \mathbb{C}_{\mathbb{Q}} \to \mathbb{C}_{\mathbb{Q}}$ such that $\pi_{n_i} \circ \hat{f} = g_{n_i}$ for every natural i. Because every g_{n_i} is continuous we have that $\hat{f}$ is continuous and verifies that $(\pi_L \circ \hat{f}_{n_i})_{i \geq J}$ converges uniformly to $\pi_L \circ \hat{f}$ on compact sets. In particular, $(\hat{f}_{n_i})_{i \in \mathbb{N}}$ converges pointwise to $\hat{f}$.

Let's see that $\hat{f}$ is proper: Consider a compact set $K \subset \hat{\mathbb{C}}_{\mathbb{Q}}$. The compact K is closed for every compact subset of a Hausdorff space is also compact and because $\hat{f}$ is continuous, $\hat{f}^{-1}(K)$ is closed. By Lemma 3.3.8 and the fact that $(\hat{f}_{n_i})_{i \in \mathbb{N}}$ converges pointwise to $\hat{f}$, we have that for every $L = n_J$ there is a constant M'_L such that:

$$|\pi_L(x)| \leq M'_L |\pi_L \circ \hat{f}(x)|$$

Choose some natural $L = n_J$. Define R such that $d(0, \pi_L(K)) = R < \infty$ for π_L is continuous; i.e. $\pi_L(K)$ is compact. By the above relation we have that

$$d(0, \pi_L(\hat{f}^{-1}(K))) \leq M'_L R$$

and because π_L is proper, the closed set $\hat{f}^{-1}(K)$ is contained in the compact $\pi_L^{-1}(D(0; M'_L R))$ hence $\hat{f}^{-1}(K)$ is compact and we have the claim.

In particular, the extension $\hat{f} : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ such that $\hat{f}(\infty) = \infty$ is continuous and because $\hat{f}_{n_i}(\infty) = \infty$ for every natural i, we have that $(\hat{f}_{n_i})_{i \in \mathbb{N}}$ converges pointwise to $\hat{f}$ on $\hat{\mathbb{C}}_{\mathbb{Q}}$. In particular, by definition every $\hat{f}_{n_i}$ is leaf preserving and so is $\hat{f}$. This finishes the proof. $\qquad\square$

Theorem 3.3.11. *For every adelic Beltrami differential μ there is a unique quasiconformal leaf preserving solution $f : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ to the μ-Beltrami equation such that f fixes $0, 1, \infty$.*

Proof: (Uniqueness) Suppose that f and g are quasiconformal solutions to the μ-Beltrami equation fixing $0, 1, \infty$. Then, $f \circ g^{-1}$ is leaf preserving 1-quasiconformal fixing $0, 1, \infty$. By Lemmas 1.2.5, 1.2.6 and Corollary 3.1.6 there is a holomorphic limit periodic respect to x function h such that $\nu^{-1} \circ f \circ g^{-1} \circ \nu(z) = z + h(z)$ where ν is the baseleaf. On the other hand, by Weyl's Lemma $\nu^{-1} \circ f \circ g^{-1} \circ \nu$ is a holomorphic homeomorphism of $\mathbb{C}$; i.e. an affine transformation. Because it fixes zero, we have that $\nu^{-1} \circ f \circ g^{-1} \circ \nu = id$ hence $f \circ g^{-1} = id$ and $f = g$.

(Existence) First suppose that μ has compact support in $\mathbb{C}_{\mathbb{Q}}$. Consider an arbitrary leaf $\nu : \mathbb{C} \to \mathbb{C}^*_{\mathbb{Q}} \subset \hat{\mathbb{C}}_{\mathbb{Q}}$. Under the same notation and definitions as before, by Lemma 3.3.10 there is a continuous baseleaf preserving map $\hat{f} : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ such that $(\hat{f}_{n_i})_{i \in \mathbb{N}}$ converges pointwise to $\hat{f}$. By Lemmas 3.3.7 and 3.3.8, actually we have the restriction $\hat{f} : \mathbb{C}^*_{\mathbb{Q}} \to \mathbb{C}^*_{\mathbb{Q}}$ and because $\hat{f}_{n_i}$ and $\hat{f}$ are leaf preserving, the compositions $\nu^{-1} \circ \hat{f}_{n_i} \circ \nu$ and $\nu^{-1} \circ \hat{f} \circ \nu$ are well defined. Moreover the sequence $(\nu^{-1} \circ \hat{f}_{n_i} \circ \nu)_{i \in \mathbb{N}}$ converges pointwise to $\nu^{-1} \circ \hat{f} \circ \nu$.

Claim: The maps $\nu^{-1} \circ \hat{f}_{n_i} \circ \nu$ are quasiconformal solutions of the respectives $\nu^*(\mathcal{I}_{n_i}(\mu))$-Beltrami equations. We have the following diagram:

Because the front, behind , left and right faces commute, the bottom face commutes. We already know that $\nu^{-1} \circ \hat{f}_{n_i} \circ \nu$ is continuous by Lemma 1.2.5 (or just because it is conjugated to a continuous function by a covering map). Locally the inverse of the map e^{iz/n_i} exists and we have

$$\partial_z \left(\nu^{-1} \circ \hat{f}_{n_i} \circ \nu \right) = \partial_z \left((e^{iz/n_i})^{-1} \circ f_{n_i} \circ e^{iz/n_i} \right)$$

Because $\partial_z f_{n_i} \in L_{2,loc}(\mathbb{C})$, by the above identity the same holds for $\partial_z \left(\nu^{-1} \circ \hat{f}_{n_i} \circ \nu \right)$. An analogous result holds for the other derivative. Finally, $\nu^{-1} \circ \hat{f}_{n_i} \circ \nu$ is the solution of the Beltrami equation with Beltrami differential

$$(e^{iz/n_i})^*(\mu_{n_i}) = (\pi_{n_i} \circ \nu)^*(\mu_{n_i}) = (\nu)^*(\pi_{n_i})^*(\mu_{n_i}) = (\nu)^*(\mathcal{I}_{n_i}(\mu))$$

This proves the claim.

Define the affine maps $A_i(z) = a_i z + b_i$ such that $A_i^{-1} \circ \nu^{-1} \circ \hat{f}_{n_i} \circ \nu$ is the quasiconformal solution of the $\nu^*(\mathcal{I}_{n_i}(\mu))$-Beltrami equation fixing $0, 1, \infty$ (See remark 3.3.3 below). Concretely:

$$a_i = \nu^{-1} \circ \hat{f}_{n_i} \circ \nu(1) - \nu^{-1} \circ \hat{f}_{n_i} \circ \nu(0)$$

$$b_i = \nu^{-1} \circ \hat{f}_{n_i} \circ \nu(0)$$

Because $(\hat{f}_{n_i})_{i \in \mathbb{N}}$ converges pointwise to $\hat{f}$, the sequence of affine maps $(A_i)_{i \in \mathbb{N}}$ converges locally uniformly to the map $A(z) = az + b$ such that:

$$a = \nu^{-1} \circ \hat{f} \circ \nu(1) - \nu^{-1} \circ \hat{f} \circ \nu(0)$$

$$b = \nu^{-1} \circ \hat{f} \circ \nu(0)$$

A priori a could be zero. Define the map g as the quasiconformal solution of the $\nu^*(\mu)$-Beltrami equation fixing $0, 1, \infty$. Because $\mathcal{I}_{n_i}(\mu)$ tends to μ in $L_\infty(\mathbb{C}_\mathbb{Q})$ we have that $\nu^*(\mathcal{I}_{n_i}(\mu))$ tends to $\nu^*(\mu)$ in $L_\infty(\mathbb{C})$ and by Lemma 3.3.12 we conclude that:

$$A_i^{-1} \circ \nu^{-1} \circ \hat{f}_{n_i} \circ \nu \to g$$

locally uniformly. Then:

$$\nu^{-1} \circ \hat{f}_{n_i} \circ \nu = A_i \circ A_i^{-1} \circ \nu^{-1} \circ \hat{f}_{n_i} \circ \nu \to A \circ g$$

and we conclude that:

$$\nu^{-1} \circ \hat{f} \circ \nu = A \circ g$$

Because $\hat{f}$ is continuous and fixes $0, \infty$ it cannot be constant. In particular $a \neq 0$ and we have that $\nu^{-1} \circ \hat{f} \circ \nu$ is a quasiconformal solution of the $\nu^*(\mu)$-Beltrami equation for every leaf ν. Finally, $\hat{f}$ is a homeomorphism for every continuous bijective map between compact sets is a homeomorphism. We have proved that $\hat{f}$ is quasiconformal. Multiplying by $\hat{f}(1)^{-1}$ we have the quasiconformal solution fixing $0, 1, \infty$.

Now we remove the hypothesis of the compact support of μ by the standard well known trick: Define $\mu_1 = \mu.\chi_{|\pi_1(z)| \geq 1}$ and consider the Möbius inversion $\gamma : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ such that $\gamma(z) = 1/z$. Because $\gamma^*(\mu_1)$ has compact support on $\mathbb{C}_{\mathbb{Q}}$, by the previous part there is a unique quasiconformal leaf preserving solution $g : \hat{\mathbb{C}}_{\mathbb{Q}} \to \hat{\mathbb{C}}_{\mathbb{Q}}$ to the $\gamma^*(\mu_1)$-Beltrami equation such that g fixes $0, 1, \infty$. Define f_1 such that the following diagram commutes:

$$\begin{array}{ccc}
\hat{\mathbb{C}}_{\mathbb{Q}} & \xrightarrow{\ g\ } & \hat{\mathbb{C}}_{\mathbb{Q}} \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\hat{\mathbb{C}}_{\mathbb{Q}} & \xrightarrow{\ f_1\ } & \hat{\mathbb{C}}_{\mathbb{Q}}
\end{array}$$

Claim: The map f_1 is the quasiconformal solution of the μ_1-Beltrami equation fixing $0, 1, \infty$: Because γ and g are homeomorphisms fixing $0, 1, \infty$ so is f_1. For every leaf ν_a we have the diagram:

Because every ν_a is injective and the left, right, top, bottom and front sides commute we have that the back face also commutes. By definition $\nu_{-a}^{-1} \circ g \circ \nu_{-a}$ is a quasiconformal solution of the $\nu_{-a}^* \circ \gamma^*(\mu_1)$-Beltrami equation so $\nu_a^{-1} \circ f_1 \circ \nu_a$ is a quasiconformal solution of the $(-z)^* \circ \nu_{-a}^* \circ \gamma^*(\mu_1)$-Beltrami equation. We have:

$$(-z)^* \circ \nu_{-a}^* \circ \gamma^*(\mu_1) = (\gamma \circ \nu_{-a} \circ (-z))^*(\mu_1) = \nu_a^*(\mu)$$

and this proves the claim.

Define the adelic differential μ_2 such that:

$$f_1^*(\mu_2) = \frac{\mu - \mu_1}{1 - \mu\overline{\mu_1}} \ \overline{d\pi_1} \otimes (d\pi_1)^{-1}$$

where μ and μ_1 on the right side denote the functions and not the differentials (recall remark 3.2.1). Under the same abuse of notation we have:

$$\begin{aligned}
\nu_a^*(\mu) &= (\mu \circ \nu_a)\, \overline{\nu_a^*(d\pi_1)} \otimes (\nu_a^*(d\pi_1))^{-1} \\
&= (\mu \circ \nu_a)\, \overline{d(\pi_1 \circ \nu_a)} \otimes (d(\pi_1 \circ \nu_a))^{-1} \\
&= (\mu \circ \nu_a)\, \overline{d\, e^{iz}} \otimes (d\, e^{iz})^{-1} \\
&= -e^{-i(z+\bar{z})}(\mu \circ \nu_a)\, \overline{dz} \otimes (dz)^{-1} = \mu_a\, \overline{dz} \otimes (dz)^{-1}
\end{aligned}$$

and a similar expression and definition for $\nu_a^*(\mu_1)$:

$$\nu_a^*(\mu_1) = -e^{-i(z+\bar{z})}(\mu_1 \circ \nu_a)\, \overline{dz} \otimes (dz)^{-1} = \mu_{1,a}\, \overline{dz} \otimes (dz)^{-1}$$

A similar calculation gives:

$$\begin{aligned}
(f_1 \circ \nu_a)^*(\mu_2) &= \nu_a^*(f_1^*(\mu_2)) = -e^{-i(z+\bar{z})}\left(\frac{\mu - \mu_1}{1 - \mu\overline{\mu_1}}\right) \circ \nu_a\, \overline{dz} \otimes (dz)^{-1} \\
&= \frac{\mu_a - \mu_{1,a}}{1 - \mu_a\overline{\mu_{1,a}}}\, \overline{dz} \otimes (dz)^{-1}
\end{aligned}$$

Because μ_2 has compact support on $\mathbb{C}_\mathbb{Q}$ there is a unique quasiconformal leaf preserving solution f_2 to the μ_2-Beltrami equation fixing $0, 1, \infty$. Define the map $f = f_2 \circ f_1$. It is clearly quasiconformal leaf preserving and fixes $0, 1, \infty$ for it is the composition of maps of the same kind. Because:

$$\nu_a^{-1} \circ f \circ \nu_a = \nu_a^{-1} \circ f_2 \circ f_1 \circ \nu_a = (\nu_a^{-1} \circ f_2 \circ \nu_a) \circ (\nu_a^{-1} \circ f_1 \circ \nu_a)$$

and the following fact:

$$(\nu_a^{-1} \circ f_1 \circ \nu_a)^*(\nu_a^*(\mu_2)) = (f_1 \circ \nu_a)^*(\mu_2) = \frac{\mu_a - \mu_{1,a}}{1 - \mu_a\overline{\mu_{1,a}}}\, \overline{dz} \otimes (dz)^{-1}$$

we conclude that $\nu_a^*(\mu) = \mu_a\, \overline{dz} \otimes (dz)^{-1}$ is the Beltrami differential of $\nu_a^{-1} \circ f \circ \nu_a$ for every leaf ν_a; i.e. f is the unique quasiconformal solution to the μ-Beltrami equation fixing $0, 1, \infty$. $\qquad\square$

Remark 3.3.3. At first sight it seems there is something terribly wrong in the above proof: While $\hat{f}$ fixes $0, \infty$ and has only one degree of freedom as a solution of the μ-Beltrami equation, its conjugated map $\nu^{-1} \circ \hat{f} \circ \nu$ has two degrees of freedom. Why the conjugated map has an extra degree of freedom? Let's see: The conjugated map has the same freedom as $\hat{f}$ plus the property of being uniformly limit periodic on horizontal bands. Once this last property is destroyed by an affine transformation, an extra degree of freedom comes out.

Remark 3.3.4. We have also proved that:

- The map $\hat{f}$ in Lemma 3.3.10 is in fact quasiconformal.

- A 1-quasiconformal map of $\hat{\mathbb{C}}_\mathbb{Q}$ fixing $0, 1, \infty$ is the identity.

The following Lemma is Proposition 4.36 of [IT]:

Proposition 3.3.12. *If μ_n tends to μ in $L_\infty(\mathbb{C})$ then f^{μ_n} tends to f^μ locally uniformly.*

Lema 3.3.13. *Consider adelic Beltrami differentials μ_n, μ and consider their respective quasiconformal solutions f_n, f of the Beltrami equation fixing $0, 1, \infty$. If $(\mu_n)_{n \in \mathbb{N}}$ converges to μ in $L_\infty(\mathbb{C}_\mathbb{Q})$, then $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f.*

Proof: Because $(\mu_n)_{n \in \mathbb{N}}$ converges to μ then $(\nu_a^*(\mu_n))_{n \in \mathbb{N}}$ converges to $\nu_a^*(\mu)$:

$$||\nu_a^*(\mu_n) - \nu_a^*(\mu)||_\infty \le ||\nu_a^*(\mu_n - \mu)||_\infty \le ||\mu_n - \mu||_\infty \to 0$$

for $||\nu_a||_\infty = 1$ for every leaf ν_a. By Proposition 3.3.12 the quasiconformal maps $\nu_a^{-1} \circ f_n \circ \nu_a$ converge locally uniformly to $\nu_a^{-1} \circ f \circ \nu_a$ for every leaf ν_a. We have the result. $\qquad\square$

Corollary 3.3.14. *Consider adelic Beltrami differentials μ_n, μ and consider their respective quasiconformal solutions f_n, f of the Beltrami equation fixing $0, 1, \infty$. If $(\mu_n)_{n \in \mathbb{N}}$ converges to μ in the Banach space $(Bel_{\mathcal{S}}(\mathbb{C}_\mathbb{Q}), ||\cdot||_{Ren,\mathcal{S}})$, then $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f.*

As a final remark, consider the morphism $\phi : \hat{\mathbb{Z}} \times \mathbb{C} \to S_\mathbb{Q}^1 \times S^1$ given by $\phi(a, x + iy) = (exp(a, x), e^{iy})$. Because $Ker(\phi) = Ker(exp)$ the morphism factors through exp and we have the commutative diagram:

$$\mathbb{C}_\mathbb{Q}^* \xrightarrow{\ \ p\ \ } S_\mathbb{Q}^1 \times S^1$$
$$exp \searrow \qquad \nearrow \phi$$
$$\hat{\mathbb{Z}} \times \mathbb{C}$$

The deck transformations group of the covering $p : \mathbb{C}_\mathbb{Q}^* \to S_\mathbb{Q}^1 \times S^1$ is generated by multiplication of $\nu(2\pi i)$ on the algebraic solenoid where ν is the baseleaf: $T(z) = \nu(2\pi i)z$ for every $z \in \mathbb{C}_\mathbb{Q}^*$.

Consider a group G acting on the algebraic solenoid $\mathbb{C}_\mathbb{Q}^*$. Define the space of G-invariant $\mathcal{S}$-adelic Beltrami differentials $Bel_{\mathcal{S}}(\mathbb{C}_\mathbb{Q}^*; G)$ as the set of differentials $\mu \in Bel_{\mathcal{S}}(\mathbb{C}_\mathbb{Q}^*)$ such that $g^*(\mu) = \mu$ for every $g \in G$. In particular, we have an Ahlfors-Bers theory on the torus $S_\mathbb{Q}^1 \times S^1$:

$$Bel_{\mathcal{S}}(S_\mathbb{Q}^1 \times S^1) = Bel_{\mathcal{S}}(\mathbb{C}_\mathbb{Q}^*; \langle T \rangle)$$

and a Teichmüller space of this adelic torus:

$$T_{\mathcal{S}}(S_\mathbb{Q}^1 \times S^1) = T_{\mathcal{S}}(\langle T \rangle)$$

Remark 3.3.5. The 2-adic case of the above construction, $S_2^1 \times S^1$, is the second example in [Su]. As it was commented there, this is the basic solenoidal surface required in the dynamical theory of Feigenbaum's Universality [Fe]. We hope that our theory of adelic Beltrami differentials could shed some new light on the link between this univeraslity and the Ahlfors-Bers one.

3.4 Infinitesimal deformations

Now we turn the discussion to infinitesimal deformations. We discuss it in the general case and then apply it to the p-adic case. We need an appropriate notion of $\dot{f}[\eta]$ where $\eta \in T_0 Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}) = Ren_{\mathcal{S}}$ (See remark 3.4.1).

Lema 3.4.1. *Consider* $\nu \in L_{\infty}(\mathbb{C})$. *Then,*

$$\dot{f}\left[(z^k)^* \nu\right] = (z^k)^* \left(\dot{f}[\nu]\right)$$

for every natural k.

    Proof: Consider the quasiconformal solutions $f^{\mu(t)}$ and $f^{(z^k)^* \mu(t)}$ of the $\mu(t)$ and $(z^k)^* \mu(t)$-Beltrami equations respectively fixing $0, 1, \infty$ such that $\mu(t) = t\nu + O(t^2)$. Unicity of the solutions imply the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\ f^{(z^k)^* \mu(t)}\ } & \mathbb{C} \\
{\scriptstyle z^k}\downarrow & & \downarrow{\scriptstyle z^k} \\
\mathbb{C} & \xrightarrow{\ f^{\mu(t)}\ } & \mathbb{C}
\end{array}
$$

Because $(z^k)^* \mu(t) = t\,(z^k)^* \nu + O(t^2)$, deriving respect to t gives:

$$\dot{f}\left[(z^k)^* \nu\right](\zeta) = \dot{f}[\nu](\zeta^k)\frac{1}{k\ \zeta^{k-1}} = (z^k)^* \left(\dot{f}[\nu]\right)(\zeta)$$

for every $\zeta \in \mathbb{C}$ where we have used on the right side that $\dot{f}[\nu]$ is actually a derivation and not a function. $\qquad\square$

The above Lemma motivates the following definition:

Definition 3.4.1. Consider $\nu \in T_0 Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}) = Ren_{\mathcal{S}}$. We define:

$$\dot{f}\left[\mathcal{I}_n(\nu)\right] = \pi_n^* \left(\dot{f}[\nu_n]\right) = (\dot{f}[\nu_n] \circ \pi_n).\,(n\ \pi_n^{n-1})\ \frac{d}{d\pi_1}$$

for $d\pi_1 = n\ \pi_n^{n-1}\ d\pi_n$.

See that $\dot{f}[\mathcal{I}_n(\nu)]$ is continuous on the whole adelic sphere. Now we define $\dot{f}[\nu]$ as the uniform limit of its periodic approximations just defined. The following Lemma is Theorem 4.37 in [IT]:

Lema 3.4.2. *Consider a family of Beltrami coefficients* $\{\mu(t)\}$ *depending on a real parameter* t *such that:*

$$\mu(t) = t\eta + o(t)$$

where $\eta \in L_{\infty}(\mathbb{C})$. *Then,*

$$\dot{f}[\eta](\zeta) = \lim_{t \to 0} \frac{f^{\mu(t)}(\zeta) - \zeta}{t}$$

exists for every $\zeta \in \mathbb{C}$ and the convergence is locally uniform on $\mathbb{C}$. Moreover,

$$\dot{f}[\eta](\zeta) = -\frac{1}{\pi} \int\int_{\mathbb{C}} \eta(z) \frac{\zeta(\zeta - 1)}{z(z-1)(z-\zeta)} dx dy$$

for every $\zeta \in \mathbb{C}$.

Lema 3.4.3. *Consider $\nu \in T_0 Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}) = Ren_{\mathcal{S}}$. There is a continuous derivation $\dot{f}[\nu]$ such that the sequence of continuous derivations $(\dot{f}[\mathcal{I}_{n_i}(\nu)])_{i \in \mathbb{N}}$ converges uniformly to it and $(n_i)_{i \in \mathbb{N}} = \mathcal{S}$.*

Proof: Consider $\nu \in T_0 Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}) = Ren_{\mathcal{S}}$, the tangent space of the $\mathcal{S}$-adelic Beltrami differentials at zero. Recall formula (3.4.2). For every $x \in S^1_{\mathbb{Q}}$ and naturals m, n such that $m|n$ we have:

$$\begin{aligned}
&|\dot{f}[\mathcal{I}_n(\nu)](x) - \dot{f}[\mathcal{I}_m(\nu)](x)| \\
&= |\dot{f}[\nu_n](\pi_n(x)) - \dot{f}[(z^{n/m})^*\nu_m](\pi_n(x))| \cdot |n \, \pi_n(x)^{n-1}| \\
&\leq n \, ||\nu_n - (z^{n/m})^*\nu_m||_\infty \, C \\
&= n \, ||\mathcal{I}_n(\nu) - \mathcal{I}_m(\nu)||_\infty \, C
\end{aligned}$$

where we used that $|\pi_n(x)| = 1$ for every $x \in S^1_{\mathbb{Q}}$ and the constant C:

$$C = \frac{1}{\pi} \max_{\zeta \in S^1} \, | \int\int_{\mathbb{C}} \frac{\zeta(\zeta - 1)}{z(z-1)(z-\zeta)} \, dx dy \, |$$

In particular, we have:

$$||\dot{f}[\mathcal{I}_{n_i}(\nu)] - \dot{f}[\mathcal{I}_{n_{i-1}}(\nu)]||_\infty \, \leq \, n_i \, ||\mathcal{I}_{n_i}(\nu) - \mathcal{I}_{n_{i-1}}(\nu)||_\infty \, C$$

Because the following renormalized average series converges:

$$\sum_{i=1}^{\infty} n_{i+1} \, ||\mathcal{I}_{n_{i+1}}(\mu) - \mathcal{I}_{n_i}(\mu)||_\infty < \infty \tag{3.15}$$

where $\mathcal{S} = (n_i)_{i \in \mathbb{N}}$ is cofinal totally ordered divisibility subsystem under consideration, the sequence $(\dot{f}[\mathcal{I}_{n_i}(\nu)])_{i \in \mathbb{N}}$ is a Cauchy sequence in the complete space of continuous derivations with the supremum norm. Then, there is a continuous derivation $\dot{f}[\nu]$ such that the sequence $(\dot{f}[\mathcal{I}_n(\nu)])_{n \in \mathbb{N}}$ converges uniformly to it respect to the divisibility net. $\square$

Remark 3.4.1. In particular we have the following property: There is a continuous function $g : \mathbb{C}_{\mathbb{Q}} \to \mathbb{C}$ such that $g(0) = 0$ and

$$\dot{f}[\eta](z) = g(z) \frac{d}{d\pi_1}$$

for every $z \in \mathbb{C}^*_{\mathbb{Q}}$. An analogous result holds for the whole adelic sphere. Although the expression in corollary 3.4.4 below would be the natural definition of $\dot{f}[\nu]$, this approach makes clear the continuity at zero of the map g previously defined that otherwise would be difficult to prove.

Corollary 3.4.4. *Consider $\eta \in T_0 Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}) = Ren_{\mathcal{S}}$. Then,*

$$\nu^*(\dot{f}[\eta]) = \frac{d}{dt} \left(\nu^{-1} \circ f^{t\eta} \circ \nu \right)$$

where ν is the baseleaf and the derivative is evaluated at zero.

Proof: Because $Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$ is star shaped and open, there is some $\epsilon > 0$ such that $f^{t\,\eta}$ is defined for $t \in [0, \epsilon)$. The rest of the proof is a calculation. Because everything converge uniformly on compacts we have:

$$
\begin{aligned}
\nu^*(\dot{f}[\eta]) &= \lim_n \nu^*(\dot{f}[\mathcal{I}_n(\eta)]) = \lim_n \nu^* \pi_n^*(\dot{f}[\eta_n]) \\
&= \lim_n \nu^* \pi_n^* \left(\frac{d}{dt} \, f^{t\eta_n} \right) = \lim_n \nu^* \left(\frac{d}{dt} \, f^{t\mathcal{I}_n(\eta)} \right) \\
&= \lim_n \frac{d}{dt} \left(\nu^{-1} \circ f^{t\mathcal{I}_n(\eta)} \circ \nu \right) = \frac{d}{dt} \lim_n \left(\nu^{-1} \circ f^{t\mathcal{I}_n(\eta)} \circ \nu \right) \\
&= \frac{d}{dt} \left(\nu^{-1} \circ f^{t\eta} \circ \nu \right)
\end{aligned}
$$

where the derivatives are evaluated at zero. $\qquad\qquad\square$

Chapter 4

Teichmüller space

4.1   Teichmüller models

Model A: Define the compact $\overline{H}_{\mathbb{Q}} = \pi_1^{-1}(D(0;1)) \subset \mathbb{C}_{\mathbb{Q}}$ and the set $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$ of $\mathcal{S}$-adelic Beltrami differentials with support in $\overline{H}_{\mathbb{Q}}$ where $\mathcal{S}$ is a cofinal totally ordered divisibility subsequence. See that the solenoid $S_{\mathbb{Q}}^1$ is the boundary of $H_{\mathbb{Q}}$. For every $\mu \in Bel_{\mathcal{S}}(H_{\mathbb{Q}})$ define the extension:

$$\breve{\mu}(z) = \mu(1/\bar{z}) \left(\frac{\bar{z}}{z}\right)^2$$

for every $z \in \mathbb{C}_{\mathbb{Q}}$ such that $\pi_1(z) > 1$. It is easy to see that $\breve{\mu} \in Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}})$ is actually an $\mathcal{S}$-adelic Beltrami differential. By theorem 3.3.11 there is a unique quasiconformal solution f^{μ} to the $\breve{\mu}$-Beltrami equation fixing $0, 1, \infty$. Because $\breve{\mu} = (1/\bar{z})^*(\breve{\mu})$ we have that

$$f^{\mu}(1/\bar{z}) = \frac{1}{\overline{f^{\mu}(z)}}$$

By remark 1.1.1, the solenoid $S_{\mathbb{Q}}^1$ and hence $H_{\mathbb{Q}}$ are invariant under f^{μ}. Define the following equivalence relation on $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$: $\mu \sim_A \eta$ if $f^{\mu}|_{S_{\mathbb{Q}}^1} = f^{\eta}|_{S_{\mathbb{Q}}^1}$. The universal Teichmüller space is defined as the quotient:

$$\mathcal{B} : Bel_{\mathcal{S}}(H_{\mathbb{Q}}) \to T_{\mathcal{S}}(1) = Bel_{\mathcal{S}}(H_{\mathbb{Q}})/\sim_A$$

with the quotient topology induced by $(Bel_{\mathcal{S}}(H_{\mathbb{Q}}), ||\cdot||_{\mathcal{S}})$. Although we have the usual group structure $\mu \star \nu = \eta$ if $f^{\mu} \circ f^{\nu} = f^{\eta}$, unfortunately the space $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$ is not closed under this product.

The following Lemma shows the relation between this Teichmüller space with the Sullivan's one [Su].

Proposition 4.1.1. *We have a continuous canonical injective map:*

$$T_{\mathcal{S}}(1) \hookrightarrow T_{Sullivan}(\Delta_{\infty}^*)$$

Proof: In the same way as in the classical case, by Weyl's Lemma every adelic Beltrami differential defines a complex structure on every leaf of $H_{\mathbb{Q}} = \Delta_{\infty}^*$ in Sullivan's notation

and we have a map $\vartheta : Bel_{\mathcal{S}}(H_{\mathbb{Q}}) \to T_{Sullivan}(\Delta_\infty^*)$. Because there is a bounded homotopy respect to the hyperbolic metric between $\vartheta(\mu)$ and $\vartheta(\eta)$ if and only if $f^\mu|_{S_{\mathbb{Q}}^1} = f^\eta|_{S_{\mathbb{Q}}^1}$, there is an injective map ϱ such that:

$$
\begin{array}{ccc}
Bel_{\mathcal{S}}(H_{\mathbb{Q}}) & & \\
{\scriptstyle \mathcal{B}}\downarrow & \searrow^{\vartheta} & \\
T_{\mathcal{S}}(1) & \xrightarrow{\hspace{1cm}\varrho\hspace{1cm}} & T_{Sullivan}(\Delta_\infty^*)
\end{array}
$$

By the same reason as before and Corollary 3.3.14, ϑ is continuous for pointwise convergence of continuous maps to a continuous map on a compact set (the solenoid $S_{\mathbb{Q}}^1$ in our case) is actually uniform. In particular, if $(\mu_n)_{n \in \mathbb{N}}$ converges to μ in the Banach space $(Bel_{\mathcal{S}}(\mathbb{C}_{\mathbb{Q}}), ||\cdot||_{Ren,\mathcal{S}})$, then $d(\vartheta(\mu_n), \vartheta(\mu))$ tends to zero where d is the Teichmüller metric defined in [Su]. Because the topology of $T_{\mathcal{S}}(1)$ is induced by the one in $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$, we have that ϱ is continuous. $\qquad\square$

Model B: This is the model of quasisolenoids fixing the unit . Define the compact $H_{\mathbb{Q}}^* = \pi_1^{-1}(\overline{D(0;1)}^c) \subset \mathbb{C}_{\mathbb{Q}}$. Now, define $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$ as the subspace of those Beltrami differentials $\mu \in Bel_{\mathcal{S}}(C_{\mathbb{Q}}^*)$ such that $\mu = 0$ on the complement of the adelic hyperbolic space $H_{\mathbb{Q}}^*$. Consider the quasiconformal solution f_μ to the μ-Beltrami equation fixing $0, 1, \infty$. See that $f_\mu|_{H_{\mathbb{Q}}^*}$ is univalent on every leaf. In fact, the application of the theory of univalent functions in Teichmüller theory is one of Bers great accomplishments.

Define the following equivalence relation on $Bel_{\mathcal{S}}(H_{\mathbb{Q}})$: $\mu \sim_B \eta$ if $f_\mu|_{H_{\mathbb{Q}}^*} = f_\eta|_{H_{\mathbb{Q}}^*}$. The universal Teichmüller space is defined as the quotient:

$$T_{\mathcal{S}}(1) = Bel_{\mathcal{S}}(H_{\mathbb{Q}}) / \sim_B$$

with the quotient topology induced by $(Bel_{\mathcal{S}}(H_{\mathbb{Q}}), ||\cdot||_{\mathcal{S}})$.

Lema 4.1.2. *Consider $\mu, \eta \in Bel_{\mathcal{S}}(H_{\mathbb{Q}})$. Then, $f^\mu|_{S_{\mathbb{Q}}^1} = f^\eta|_{S_{\mathbb{Q}}^1}$ if and only if $f_\mu|_{H_{\mathbb{Q}}^*} = f_\eta|_{H_{\mathbb{Q}}^*}$.*

Proof: (Modulo technicalities, the proof is almost verbatim as the one in [IT]). Define the map $g : \mathbb{C}_{\mathbb{Q}} \to \mathbb{C}_{\mathbb{Q}}$ such that:

$$
g(z) = \begin{cases} f^\mu \circ (f^\eta)^{-1}(z) & z \in H_{\mathbb{Q}} \\ z & z \in H_{\mathbb{Q}}^* \cup S_{\mathbb{Q}}^1 \end{cases}
$$

The map g is clearly leaf preserving continuous. For every leaf ν_a, the map $\nu_a^{-1} \circ g \circ \nu_a$ is explicitly given by:

$$
\nu_a^{-1} \circ g \circ \nu_a(z) = \begin{cases} \nu_a^{-1} \circ f^\mu \circ (f^\eta)^{-1} \circ \nu_a(z) & Im(z) > 0 \\ z & Im(z) \leq 0 \end{cases}
$$

and by definition A it is quasiconformal and we have that g is quasiconformal. The map $(f_\mu)^{-1} \circ g \circ f_\eta$ is 1-quasiconformal and fixes $0, 1, \infty$ hence by remark 3.3.4 it is the identity. Then, $f_\mu|_{H_{\mathbb{Q}}^*} = f_\eta|_{H_{\mathbb{Q}}^*}$.

For the converse, by continuity $f_\mu = f_\eta$ on $H_{\mathbb{Q}}^* \cup S_{\mathbb{Q}}^1$ and we have a 1-quasiconformal map $h = f^\mu \circ (f_\mu)^{-1} \circ f_\eta \circ (f^\eta)^{-1} : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$. By the same argument as before, $h = id$ and we have $f^\mu|_{S_{\mathbb{Q}}^1} = f^\eta|_{S_{\mathbb{Q}}^1}$. $\qquad\square$

Corollary 4.1.3. *Teichmüller space models A and B are homeomorphic.*

Model C: Here we present the quasisymmetric model.

Definition 4.1.1. A leaf preserving homeomorphism $f : S^1_{\mathbb{Q}} \to S^1_{\mathbb{Q}}$ is quasisymmetric if for every leaf ν_a the map $f_a = \nu_a^{-1} \circ f \circ \nu_a$ is quasisymmetric. Denote the space of these quasisymmetric maps by $QS(S^1_{\mathbb{Q}})$.

The following Lemma is the Beurling-Ahlfors Theorem [BA]:

Lema 4.1.4. *A real homeomorphism is quasisymmetric if and only if it admits a quasiconformal extension to the upper half plane.*

The next Lemma is the adelic analog.

Lema 4.1.5. *A map on the solenoid $S^1_{\mathbb{Q}}$ is quasisymmetric if and only if it admits a quasiconformal extension to $H_{\mathbb{Q}}$.*

Proof: Consider a quasisymmetric map $f : S^1_{\mathbb{Q}} \to S^1_{\mathbb{Q}}$. By almost verbatim Lemmas 1.2.5, 1.2.6 and Corollary 3.1.6, there are limit periodic functions $h_a : \mathbb{R} \to \mathbb{R}$ such that $f_a(x) = \nu_a^{-1} \circ f \circ \nu_a(x) = x + h_a(x)$ and $h_{a+1}(x) = h_a(x+1)$ where ν_a is a leaf. Define $w_a : U \to U$ such that:

$$w_a(x + iy) = \frac{1}{2} \int_0^1 dt \ [(1+i)f_a(x+ty) + (1-i)f_a(x-ty)] \tag{4.1}$$

Because every f_a is quasisymmetric the map w_a is quasiconformal for every leaf ν_a.

Claim: There is a homeomorphism $\hat{w}$ of $H_{\mathbb{Q}}$ such that $w_a = \nu_a^{-1} \circ \hat{w} \circ \nu_a$ for every leaf ν_a.

A direct calculation shows that there are continuous limit periodic maps g_a respect to x such that $w_a(z) = z + g_a(z)$ and $g_{a+1}(z) = g_a(z + 2\pi)$. By Lemma 1.2.1 there is a continuous map $w : \hat{\mathbb{Z}} \times U \to \hat{\mathbb{Z}} \times U$ such that $w(a, z) = (a, w_a(z))$ and $w(a + 1, z) = w(a, z + 2\pi) + (1, -2\pi)$. By Lemma 1.1.7 we have a continuous map $\hat{w}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
H_{\mathbb{Q}} - \{0\} & \xrightarrow{\hat{w}} & H_{\mathbb{Q}} - \{0\} \\
\uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\
\hat{\mathbb{Z}} \times U & \xrightarrow{w} & \hat{\mathbb{Z}} \times U \\
\uparrow & & \uparrow \\
U & \xrightarrow{w_a = id + g_a} & U
\end{array}
$$

It rest to show that $\hat{w}$ can be extended to a homeomorphism on $H_{\mathbb{Q}}$. Consider the continuous extension to the boundary $w : \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty) \to \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty)$ and define $w(+\infty) = +\infty$. Define the following neighborhood basis at $+\infty$:

$$\left\{ \{+\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{R} \times (y, +\infty) \quad such \ that \quad y \geq 0 \right\}$$

See that $\{+\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty)$ with the above basis at $+\infty$ is compact. Because every $w_0 : \mathbb{R} \times [0, +\infty) \to \mathbb{R} \times [0, +\infty)$ is a homeomorphism, for every $y \geq 0$ the preimage of the compact $\mathbb{R} \times [0, y]$ is a compact set hence there is some $y' \geq 0$ such that $w_0(\mathbb{R} \times (y', +\infty)) \subset \mathbb{R} \times (y, +\infty)$. Because $w_n(z) = w_0(z + 2\pi n) - 2\pi n$ we have that $w_n(\mathbb{R} \times (y', +\infty)) \subset \mathbb{R} \times (y, +\infty)$ for every integer and because the sequence $(w_n)n \in \mathbb{N}$ converges locally uniformly on horizontal bands respect to the divisibility net, we have that $w_a(\mathbb{R} \times (y', +\infty)) \subset \mathbb{R} \times (y, +\infty)$ for every $a \in \hat{\mathbb{Z}}$ taking a bigger y' if necessary; i.e. w is continuous at $+\infty$. Then w is a homeomorphism on $\{+\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty)$ for a continuous bijective map between compact sets is a homeomorphism. Define $\exp(+\infty) = 0$. Is clear that $\exp : \{+\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty) \to H_{\mathbb{Q}} \cup S_{\mathbb{Q}}^1$ is continuous hence a homeomorphism for the same reason as before. Because the following diagram commutes:

$$
\begin{array}{ccc}
H_{\mathbb{Q}} \cup S_{\mathbb{Q}}^1 & \xrightarrow{\quad \hat{w} \quad} & H_{\mathbb{Q}} \cup S_{\mathbb{Q}}^1 \\
{\scriptstyle exp} \uparrow & & {\scriptstyle exp} \uparrow \\
\{+\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty) & \xrightarrow{\quad w \quad} & \{+\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{R} \times [0, +\infty)
\end{array}
$$

the map $\hat{w}$ is a homeomorphism. This proves the claim.

A complete analogous construction to the one before gives the commutative diagram:

$$
\begin{array}{ccc}
\hat{\mathbb{C}}_{\mathbb{Q}} & \xrightarrow{\quad \hat{w} \quad} & \hat{\mathbb{C}}_{\mathbb{Q}} \\
{\scriptstyle exp} \uparrow & & {\scriptstyle exp} \uparrow \\
\{+\infty, -\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{C} & \xrightarrow{\quad w \quad} & \{+\infty, -\infty\} \cup \hat{\mathbb{Z}} \times \mathbb{C} \\
\uparrow & & \downarrow \\
\mathbb{C} & \xrightarrow{\quad w_a = id + g_a \quad} & \mathbb{C}
\end{array}
$$

where the maps on the top square are homeomorphisms and every w_a is quasiconformal. By equation (4.1) we have the relation $w_a(\bar{z}) = \overline{w_a(z)}$ hence $\hat{w}$ is a quasiconformal map fixing $0, \infty$ such that:

$$
\hat{w}(1/\bar{z}) = \frac{1}{\overline{\hat{w}(z)}}
$$

In particular, the map $\psi : T_{\mathcal{S}}(1) \to Bel_{\mathcal{S}}(H_{\mathbb{Q}})$ such that $\psi(f) = \mu_{\hat{w}}$ defines a section of the proyection $\mathcal{B}$: $\mathcal{B} \circ \psi = id$.

For the converse, consider a map $f : S_{\mathbb{Q}}^1 \to S_{\mathbb{Q}}^1$ such that there is a quasiconformal map $\hat{w} : H_{\mathbb{Q}} \to H_{\mathbb{Q}}$ with continuous extension f. Because $\hat{w}$ is a leaf preserving homeomorphism of $H_{\mathbb{Q}}$ then the continuous extension f is so and because $\nu_a^{-1} \circ \hat{w} \circ \nu_a$ is quasiconformal then by Lemma 4.1.4 $\nu_a^{-1} \circ f \circ \nu_a$ is quasisymmetric for every leaf ν_a. $\square$

Because the baseleaf ν is a morphism it defines leaf preserving left action $\rho : \mathbb{R}_{BL} \to S_{\mathbb{Q}}^1$ such that $\rho(a)(x) = \nu(a)x$ where $a \in \mathbb{R}$ and $x \in S_{\mathbb{Q}}^1$. This action is the translation along the leaves. By conjugation, it defines a left action on $QS(S_{\mathbb{Q}}^1)$ such that $a.f = \rho(a) \circ f \circ \rho(a)^{-1}$.

Consider a cofinal totally ordered divisibility subsequence $\mathcal{S}$ and the subspace $QS_{\mathcal{S}}(S_{\mathbb{Q}}^1)$ of restrictions of quasiconformal solutions of $\mathcal{S}$-adelic Beltrami differentials. Define the Teichmüller space $T_{\mathcal{S}}(1)$ as the quotient:

$$T_{\mathcal{S}}(1) = QS_{\mathcal{S}}(S_{\mathbb{Q}}^1)/R_{BL} \simeq QS_{\mathcal{S}}(S_{\mathbb{Q}}^1)_1$$

where $QS_{\mathcal{S}}(S_{\mathbb{Q}}^1)_1$ denote those maps fixing the unit. By the previous Lemma we have:

Corollary 4.1.6. *Teichmüller space models B and C are homeomorphic.*

4.2   p-adic case

In the same way we defined the adelic (algebraic) solenoid we define the p-adic one S_p^1 ($\mathbb{C}_p^*$) as the inverse limit of the inverse system $z^{p^n} : S^1 \to S^1$ ($z^{p^n} : \mathbb{C}^* \to \mathbb{C}^*$). The main difference is that the Pontryagin dual of the p-adic solenoid S_p^1 is now the set of characters z^q such that $q = m/p^n$ for some integer m and natural n. The fiber of S_p^1 as a fiber bundle $\pi_1 : S_p^1 \to S^1$ is now the group of p-adic integers $\mathbb{Z}_p$.

Remark 4.2.1. It is worth noting the following fact: Continuous real or complex valued functions on the p-adic or adelic integers **need not to be** locally constant. In fact:

- Consider the homeomorphism h between the 2-adic integers and the usual cantor set such that:
$$h(a_2 a_2 a_3 \ldots) = 2 \cdot 0.a_1 a_2 a_3 \ldots$$
where a_i denote the 2-adic numerical figures on the left side and the right side is the corresponding real number in basis three. Because the Cantor set is a subset of the real line, we have a continuous not locally constant function. In view of the Cantor set characterization, we have the same example on any perfect, compact and totally disconnected space.

- The p-adic norm $||\cdot||_p$ on the p-adic integers is continuous and not locally constant at zero.

- Actually, on any compact space X we have: A continuous function $f : X \to \mathbb{R}$ is locally constant if and only if its image is a finite set.

It is worth to note also that the directional derivative is the only one that makes sense on $\mathbb{Z}_p$ for every linear map $\phi : \mathbb{Z}_p \to \mathbb{C}$ is trivial hence every linear action $\rho : \phi : \mathbb{Z}_p \to Aut_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}$ is trivial too. In particular there can be no differential neither derivative, only directional derivative. Physically, the heuristic reason is that there is no **natural** extension of the total order relation of $\mathbb{Z}$ to $\mathbb{Z}_p$ and this can be seen in the following way: Geometrically, the (oriented) directions through the 0 point are the equivalence classes of monomorphisms $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$ such that two of them are equivalent if they are non trivial (positive) integer multiples. Because every linear map $m : \mathbb{Z} \hookrightarrow \mathbb{Z}_p$ is a monomorphism

if and only if it is of the form $m(x) = ax$ for some $a \in \mathbb{Z}_p^*$, we have that the space of directions are points of the projective space:

$$P(\mathbb{Z}_p) = \mathbb{Z}_p^* / \mathbb{Z}^*$$

where the quotient is respect to the multiplicative structures. Because this space is far from being trivial, we have multiple directions so the directional derivative is the one that makes sense.

Taking the analogy further, the space of oriented directions, the p-adic version of S^1, can be considered as the boundary of $\mathbb{Z}_p$:

$$\partial \overline{\mathbb{Z}_p} = \mathbb{Z}_p^* / \mathbb{Z}_+^*$$

and is the double covering of the projective space:

$$\partial \overline{\mathbb{Z}_p} \to P(\mathbb{Z}_p)$$

It seems that the p-adic solenoid S_p^1 has the boundary $\partial \overline{\mathbb{Z}_p} \times S^1$ or some dynamical suspension of $\partial \overline{\mathbb{Z}_p}$ with some monodromy map. We wonder whether these ideas connect to tropical geometry.

In the same way we defined a notion of vertical uniform continuity in L_∞ we define the notion of vertical uniform derivative:

Definition 4.2.1. Recall the left action $m : \mathbb{Z}_p \to Aut(\mathbb{C}_p^*)$ such that $m_a(x) = \phi(a)x$. We say that $\mu \in L_\infty(\mathbb{C}_p^*)$ has uniformly vertical L_∞-derivative $d\mu/d\mathbb{Z}_p$ if

$$\frac{d\mu}{d\mathbb{Z}_p} = \lim_{a \to 0} \frac{\mu \circ m_a - \mu}{||a||_p} \in L_\infty(\mathbb{C}_{\mathbb{Q}}^*)$$

As an example, consider the p-adic integers $\mathbb{Z}_p$ with its p-adic norm $||\cdot||_p$. By definition, the p-adic norm is continuous on this space. Let's see that it has a directional derivative at every point. If $x \neq 0$ then there is some natural number N such that $a \in N\mathbb{Z}_p$ imply $||x + a||_p = ||x||_p$ hence its derivative is zero at x. However, its directional derivative at zero is one for every direction. We have proved that the derivative exists and equals the delta Kronecker:

$$\frac{d\ ||x||_p}{dx} = \delta_x^0$$

Lema 4.2.1. *Consider a vertical L_∞-continuous $\mu \in L_\infty(\mathbb{C}_p^*)$ such that there is a converging $\mathcal{S}$-renormalized average series. Then and $d\mu/d\mathbb{Z}_p$ exists along the direction determined by $\mathcal{S}$ and it is zero; i.e:*

$$\frac{d\mu}{d\mathbb{Z}_p}\Big|_{\mathcal{S}} = 0$$

Proof: There is a subsequence $(n_j)_{j \in \mathbb{N}}$ of $(p^n)_{n \in \mathbb{N}}$ such that 3.1 holds. In particular, the non-renormalized average series converges:

$$\sum_{j=1}^\infty ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty < \infty$$

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and because of Lemma 2.1.1 we have:

$$\mu = \mathcal{I}_{n_1}(\mu) + \sum_{j=1}^{+\infty}(\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu))$$

Recall that $m_a(x) = \phi(a)x$ and $\phi(a) \in Ker(\pi_n)$ imply $\pi_n(\phi(a)x) = \pi_n(x)$ for π_n is a group morphism. Because $\phi(a) \in Ker(\pi_n)$ if and only if $a \in n\mathbb{Z}_p$, we have that $\pi_n^{-1}(\pi_n(x))$ is invariant under the action m_a such that $a \in n\mathbb{Z}_p$. In particular, because the measure is invariant under m_a, we have that $\mathcal{I}_n(\mu \circ m_a) = \mathcal{I}_n(\mu)$ for every $a \in n\mathbb{Z}_p$. Then,

$$\lim_J n_J ||\mu \circ m_{n_J} - \mu||_\infty$$

$$= \lim_J n_J || \sum_{j=J}^{+\infty}(\mathcal{I}_{n_{j+1}}(\mu \circ m_{n_J}) - \mathcal{I}_{n_j}(\mu \circ m_{n_J})) - \sum_{j=J}^{+\infty}(\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu))||_\infty$$

$$\leq 2 \lim_J n_J \sum_{j=J}^{+\infty} ||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty$$

$$\leq 2 \lim_J \sum_{j=J}^{+\infty} n_{j+1}||\mathcal{I}_{n_{j+1}}(\mu) - \mathcal{I}_{n_j}(\mu)||_\infty = 0$$

Finally we have:

$$\frac{d\mu}{d\mathbb{Z}_p}\Big|_{\mathcal{S}} = \lim_J \frac{\mu \circ m_{n_J} - \mu}{||n_J||_p} = \lim_J n_J(\mu \circ m_{n_J} - \mu) = 0$$

$\square$

Moreover, we conjecture the following:

Conjecture 1. *In the p-adic case, the renormalized average series condition in the adelic Beltrami differential definition can be substituted by the condition $d\mu/d\mathbb{Z}_p = 0$. We don't mean these conditions are equivalent, we mean that the whole theory that follows, in particular Ahlfors-Bers theorem, holds with this alternative condition.*

In what follows, we will take the sequence $\mathcal{P} = (p^n)_{n \in \mathbb{N}_0}$ and in pursuit of alleviating the notation, when there were no confusion, we will make the following abuses:

- Denote $|| \cdot ||_{\mathcal{P}} = || \cdot ||_{Ren,\mathcal{P}}$.

- Denote $\mathcal{I}_n = \mathcal{I}_{p^n}$.

- For every complex continuous function f from the solenoid, algebraic solenoid or the adelic sphere, denote $\mathcal{I}_n(f) = \mathcal{I}_n(f \circ \nu)$ and $||f||_{\mathcal{P}} = ||f \circ \nu||_{\mathcal{P}}$ where ν is the baseleaf.

- For degree zero maps f, by Lemma 1.2.9 there is a complex function g such that $f = \nu \circ g$. Denote $||f||_{\mathcal{P}} = ||g||_{\mathcal{P}}$.

- We will omit the notation making reference to orientation an leaf preserving p-adic solenoidal diffeomorphisms: $Diff_{lp}^+(S_p^1) = Diff(S_p^1)$. Instead, we will denote the orientation and leaf preserving p-adic solenoidal C^m-diffeomorphisms fixing the unit by $Diff^m(S_p^1)_1$. The same abuses hold for the p-adic solenoidal quasisymmetric maps $QS(S_p^1)$.

Lema 4.2.2. $(Ren_\mathcal{P}, ||\cdot||_\mathcal{P})$ *is a Banach algebra.*

Proof: By Proposition 3.2.2 $(Ren_\mathcal{P}, ||\cdot||_\mathcal{P})$ is a Banach space and it rest to proof that

$$||fg||_\mathcal{P} \le ||f||_\mathcal{P}||g||_\mathcal{P}$$

for every $f, g \in Ren_\mathcal{P}$. Denote $\mathcal{I}_{-1} = 0$. In the p-adic case we have the graded algebra:

$$Per_\mathcal{P} = \bigoplus_{j \in \mathbb{N}_0} A_j$$

such that $A_0 = Per_1$ and $A_j = Per_{p^j} - Per_{p^{j-1}}$. Moreover, for every $f \in Per_\mathcal{P}$ we have:

$$f = \sum_{n \in \mathbb{N}_0} (\mathcal{I}_n(f) - \mathcal{I}_{n-1}(f))$$

where any a finite amount of terms in the sum are zero and every one of them satisfy the property:

$$\mathcal{I}_n(f) - \mathcal{I}_{n-1}(f) \in A_n$$

for every natural $n \in \mathbb{N}_0$. Consider $f, g \in Per_\mathcal{P}$. Then $fg \in Per_\mathcal{P}$ and by the argument above we have:

$$\mathcal{I}_n(fg) - \mathcal{I}_{n-1}(fg) = \sum_{\substack{a+b=n \\ a,b \ge 0}} (\mathcal{I}_a(f) - \mathcal{I}_{a-1}(f))(\mathcal{I}_b(g) - \mathcal{I}_{b-1}(g))$$

Then,

$$\begin{aligned}
||fg||_\mathcal{P} &= \sum_{n \in \mathbb{N}_0} p^n ||\mathcal{I}_n(fg) - \mathcal{I}_{n-1}(fg)||_\infty \\
&\le \sum_{n \in \mathbb{N}_0} p^n \sum_{\substack{a+b=n \\ a,b \ge 0}} ||\mathcal{I}_a(f) - \mathcal{I}_{a-1}(f)||_\infty ||\mathcal{I}_b(g) - \mathcal{I}_{b-1}(g)||_\infty \\
&= \left(\sum_{n \in \mathbb{N}_0} p^n ||\mathcal{I}_n(f) - \mathcal{I}_{n-1}(f)||_\infty\right)\left(\sum_{m \in \mathbb{N}_0} p^m ||\mathcal{I}_m(g) - \mathcal{I}_{m-1}(g)||_\infty\right) \\
&= ||f||_\mathcal{P}||g||_\mathcal{P}
\end{aligned}$$

By Lemma 3.2.1 $Per_\mathcal{P}$ is dense in $(Ren_\mathcal{P}, ||\cdot||_\mathcal{P})$ and we have the result. $\qquad\square$

4.3 p-adic Nag-Verjovsky map

Consider p-adic solenoidal orientation and leaf preserving C^m-diffeomorphisms $f, g \in Diff^m(S_p^1)_1$ fixing the unit. Define:

$$d(f, g)_{C^m} = ||\frac{d^m}{dx^m}(\nu^{-1} \circ f \circ \nu - \nu^{-1} \circ g \circ \nu)||_{\mathcal{P}}$$

In general, these functions would be just pseudometrics but because of remark 2.2.2 and the fact that f and g fix the unit $(a_0 = 0)$, it is easy to see that:

$$d(f, g)_{C^i} \leq \left(\frac{\pi}{\sqrt{3}}\right)^{m-i} d(f, g)_{C^m}$$

for every $m \geq i \geq 0$. We conclude that the functions considered are actually metrics. In the case $m = \infty$, the C^∞-topology is defined as the one generated by the union of all the C^∞-topologies. In particular, the inclusions $Diff^n(S_p^1)_1 \subset Diff^m(S_p^1)_1$ are continuous for $m \geq n$ including $m = \infty$.

Lema 4.3.1. *The space $Diff^m(S_p^1)_1$ is complete for $m = 0, 1, 2, \ldots \infty$.*

Proof: Consider a C^m-Cauchy sequence $(f_n)_{n \in \mathbb{N}}$. By the above relation we have that the sequence of limit periodic respect to x functions $\left(\frac{d^i}{dx^i}(\nu^{-1} \circ f_n \circ \nu - id)\right)_{n \in \mathbb{N}}$ is a Cauchy sequence respect to $|| \cdot ||_{\mathcal{P}}$ for every $0 \leq i \leq m$. By Lemma 3.2.2, for every $0 \leq i \leq m$ there is a limit periodic respect to x function g_i such that the Cauchy sequence $\left(\frac{d^i}{dx^i}(\nu^{-1} \circ f_n \circ \nu - id)\right)_{n \in \mathbb{N}}$ converges to it respect to $|| \cdot ||_{\mathcal{P}}$. In particular, by the first item of Lemma 3.2.1, the convergence is uniform for every $0 \leq i \leq m$ and we have $g_i = \frac{d^i}{dx^i} g_0$ for such i. By Lemma 1.2.5 there is a continuous leaf preserving map f such that $\nu^{-1} \circ f \circ \nu = id + g_0$ and we conclude that the sequence $(f_n)_{n \in \mathbb{N}}$ converges to f in the C^m topology. $\square$

We define the set $Diff_{\mathcal{P}}^m(S_p^1)_1$ of p-adic solenoidal leaf and orientation preserving C^m diffeomorphisms f fixing the unit with the property that there is an adelic Beltrami differential $\mu \in Bel_{\mathcal{P}}(H_p)$ such that $f = f^\mu|_{S_p^1}$. Because every diffeomorphism y quasisymmetric, we have the Nag-Verjovsky map [NV]

$$\iota : Diff_{\mathcal{P}}^m(S_p^1)_1 \hookrightarrow QS_{\mathcal{P}}(S_p^1)_1 \simeq T_{\mathcal{P}}(1)$$

See that $Diff_{\mathcal{P}}^0(S_p^1)_1 = QS_{\mathcal{P}}(S_p^1)_1$.

Lema 4.3.2. *The Nag-Verjovsky map is differentiable for $m \geq 2$.*

Proof: ι is continuous: Consider diffeomorphisms $f, g \in Diff_{\mathcal{P}}^m(S_p^1)_1$ such that $f = f^\mu|_{S_p^1}$ and $g = f^\eta|_{S_p^1}$ where $\mu, \eta \in Bel_{\mathcal{P}}(H_p)$. Because $(Ren_{\mathcal{P}}, || \cdot ||_{\mathcal{P}})$ is a Banach algebra, for every $\eta \in Bel_{\mathcal{P}}(H_p)$ such that $||\eta||_{\mathcal{P}} < ||\mu||_{\mathcal{P}}^{-1}$ we have:

$$f \circ g^{-1} = f^{\mu \star \eta^{-1}}|_{S_p^1} \in Diff_{\mathcal{P}}^m(S_p^1)_1$$

for:

$$||\mu \star \eta^{-1}||_{\mathcal{P}} = ||\frac{\mu - \eta}{1 - \mu \, \overline{\eta}}||_{\mathcal{P}} \leq \frac{||\mu||_{\mathcal{P}} + ||\eta||_{\mathcal{P}}}{1 - ||\mu||_{\mathcal{P}}||\eta||_{\mathcal{P}}} < \infty$$

for every $m \geq 0$. Hence it is sufficient to prove the continuity at the identity.

Because the inclusions $Diff^m(S_p^1)_1 \subset Diff^n(S_p^1)_1$ are continuous for $m \leq n$, it is enough to prove the result for $m = 2$. Using the Ahlfors-Beurling extension formula (4.1), given the C^2 diffeomorphism f in $Diff_{\mathcal{P}}^2(S_p^1)_1$ such that:

$$\nu^{-1} \circ f \circ \nu(x) = x + \sum_{q \in \mathbb{Q}} a_q e^{iqx}$$

where $a_{-q} = \overline{a_q}$ and $a_0 = 0$, we have the quasiconformal extension w_f such that:

$$\nu^{-1} \circ w_f \circ \nu(z) = z + \sum_{q \in \mathbb{Q}} a_q e^{iqx} l(qy)$$

where $z = x + iy$ and l is the real function:

$$l(x) = \frac{\sin x}{x} - \frac{1 - \cos x}{x}$$

See that the above expression makes explicit the relation $w(\bar{z}) = \overline{w(z)}$. By Lemma 3.1.2 there is a continuous addelic Beltrami differential μ_f on the p-adic sphere $\hat{\mathbb{C}}_p$ such that:

$$\nu^*(\mu_f)(z) = \frac{\sum_{q \in \mathbb{Q}} iq \, a_q \, e^{iqx} \left(\frac{l(qy) + l'(qy)}{2} \right)}{1 + \sum_{q \in \mathbb{Q}} iq \, a_q \, e^{iqx} \left(\frac{l(qy) - l'(qy)}{2} \right)}$$

The above expression is well defined for $1/2|l(x) \pm l'(x)| < 1$ and because the solenoid is compact and f is a diffeomorphism, there is a constant $k > 0$ such that:

$$0 < k \leq 1 + \sum_{q \in \mathbb{Q}} iq \, a_q e^{iqx}$$

for every $x \in \mathbb{R}$. Now we have an explicit expression for the Nag-Verjorvsky map:

$$
\begin{array}{ccc}
 & Bel_{\mathcal{P}}(H_p) & \\
 & \overset{\varphi}{\nearrow} \quad \downarrow {\scriptstyle \mathcal{B}} & \\
Diff_{\mathcal{P}}^+(S_p^1)_1 & \overset{\iota}{\longrightarrow} & T_{\mathcal{P}}(1)
\end{array}
$$

where $\varphi(f) = \mu_f$ and $\iota(f) = [\varphi(f)] = [\mu_f]$. We claim that (recall the abuse of notation 4.2):

$$||\sum_{q \in \mathbb{Q}} iq \, a_q \, e^{iqx} \left(\frac{l(qy) \pm l'(qy)}{2} \right) ||_{\mathcal{P}} \leq \frac{\pi}{\sqrt{3}}||f||_{C^2}$$

This is just a calculation. Because:

$$||\sum_{q \in \mathbb{Q}} iq \, a_q \, e^{iqx} \left(\frac{l(qy) \pm l'(qy)}{2} \right) ||_{\mathcal{P}} = \sum_{n \in \mathbb{N}_0} p^n ||\mathcal{I}_n(\ldots) - \mathcal{I}_{n-1}(\ldots)||_\infty$$

and because of the fact $1/2|l(x) \pm l'(x)| < 1$ and remark 2.2.2 for each term we have:

$$||\mathcal{I}_n(\ldots) - \mathcal{I}_{n-1}(\ldots)||_\infty \leq \sum_{\substack{q \in p^{-n}\mathbb{Z} \\ q \notin p^{-n+1}\mathbb{Z}}} |q a_q| \leq \frac{\pi}{\sqrt{3}} ||\mathcal{I}_n(f'') - \mathcal{I}_{n-1}(f'')||$$

hence

$$|| \ldots ||_\mathcal{P} \leq \frac{\pi}{\sqrt{3}} ||f''||_\mathcal{P} = \frac{\pi}{\sqrt{3}} d(id, f)_{C^2}$$

and we have the claim.

In particular, φ is continuous at the identity for:

$$
\begin{aligned}
||\varphi(f)||_\mathcal{P} &\leq \frac{|| \sum_{q \in \mathbb{Q}} iq\, a_q\, e^{iqx} \left(\frac{l(qy)+l'(qy)}{2} \right) ||_\mathcal{P}}{1 - || \sum_{q \in \mathbb{Q}} iq\, a_q\, e^{iqx} \left(\frac{l(qy)-l'(qy)}{2} \right) ||_\mathcal{P}} \\
&\leq \frac{\frac{\pi}{\sqrt{3}} d(id, f)_{C^2}}{1 - \frac{\pi}{\sqrt{3}} d(id, f)_{C^2}}
\end{aligned}
$$

for f sufficiently close to the identity in the C^2 topology. In particular, ι is continuous.

ι is differentiable: Again, by the same argument as before, it is enough to prove it at the identity. For every degree zero solenoidal map h such that $id.h$ is a C^2 diffeomorphism in $Diff_\mathcal{P}^2(S_p^1)_1$ (h is a perturbation of the identity in the C^2 topology), define the linear map $d_{id}\varphi$ such that:

$$\nu^*(d_{id}\varphi\ (h))(z) = \sum_{q \in \mathbb{Q}} iq\, a_q\, e^{iqx} \left(\frac{l(qy) + l'(qy)}{2} \right) \frac{d\bar{z}}{dz}$$

where

$$\nu^{-1} \circ h \circ \nu(x) = \sum_{q \in \mathbb{Q}} a_q e^{iqx}$$

Actually, $d_{id}\varphi$ is the differential at the identity for:

$$\frac{||\varphi(id.h) - d_{id}\varphi(h)||_\mathcal{P}}{d(1, h)_{C^2}} \leq \left(\frac{\pi}{\sqrt{3}} \right)^2 \frac{d(1, h)_{C^2}}{1 - \frac{\pi}{\sqrt{3}} d(1, h)_{C^2}} = o(d(1, h)_{C^2})$$

This concludes the Lemma for $d_{id}\iota = [d_{id}\varphi]$. $\qquad\square$

Remark 4.3.1. In the above proof, several different quasiconformal extensions can be defined with appropriate functions l, for example:

$$l(x) = e^{-x^2}$$

This function has the property of decaying to zero when x tends to $\pm\infty$, $l(0) = 1$ and $1/2|l(x) \pm l'(x)| \leq 1$ for every $x \in \mathbb{R}$. Actually, this is the extension used in example 3.3.1.

From now on we will consider $m \geq 2$. We will define complex structures on both ends of the Nag-Verjovsky map and conclude that it is analytic respect to these structures.

Consider the tangent space at the identity $T_{id}Diff_{\mathcal{P}}^m(S_p^1)_1$ of C^∞-vector fields v such that:

$$\nu^*(v)(x) = \sum_{q \in \mathbb{Q}} a_q e^{iqx}$$

where $a_{-q} = \overline{a_q}$ and $a_0 = 0$. Because of the monomorphism:

$$d_{id}\varphi_{id}: \ T_{id}Diff_{\mathcal{P}}^m(S_p^1)_1 \hookrightarrow T_0 Bel_{\mathcal{P}}(H_p) = Ren_{\mathcal{P}}$$

these vector fields can be characterized as those for which there is some $\eta \in Ren_{\mathcal{P}}$ such that $v = \dot{f}[\eta]$.

Define an (a priori) almost complex structure $\hat{J}$ such that:

$$\nu^*(\hat{J}v) = \sum_{q \in \mathbb{Q}} -i \ sg(q) a_q e^{iqx}$$

and translate it by conjugation with the adjoint map Ad to the whole tangent bundle $T \ Diff_{\mathcal{P}}^m(S_p^1)_1$. Although $Diff_{\mathcal{P}}^m(S_p^1)_1$ is not a group, this translation of structure can be done.

On the other hand, the canonical linear complex structure of $L_\infty(H_p)$ induces a complex structure J on the Banach manifold of solenoidal quasisymmetric maps $QS_{\mathcal{P}}(S_p^1)_1$.

Theorem 4.3.3. *The Nag-Verjovsky map is analytic.*

Proof: Consider $\eta \in T_0 Bel_{\mathcal{P}}(H_{\mathbb{Q}})$ such that $\dot{f}[\eta]$ is a C^∞ derivation and its restriction to the solenoid is determined by:

$$\nu^*(\dot{f}[\eta])(x) = \sum_{q \in \mathbb{Q}} a_q e^{iqx} \ \frac{d}{dx}$$

where $a_{-q} = \overline{a_q}$ and $a_0 = 0$. Define the derivation F such that:

$$F(z) = \dot{f}[\eta] + i\dot{f}[i\eta]$$

See that F is continuous on the whole adelic sphere (see remark 3.4.1), in particular at zero. Because of the following fact:

$$\overline{\partial}\dot{f}[\eta] = \eta$$

in distributional sense on every leaf where $\overline{\partial} = d\overline{z} \ \partial_{\overline{z}}$, we have that $\partial_{\overline{z}}F = 0$ on every leaf of $H_{\mathbb{Q}}$ in distributional sense. By Weyl's lemma on every leaf, F is holomorphic on $H_{\mathbb{Q}}$ such that the real part of its continuous extension to the solenoid is the function defining $\dot{f}[\eta]$ respect to $d/d\pi_1$. The only derivation F satisfying this property is the continuous one such that:

$$\nu^*(F)(z) = 2\sum_{q>0} a_q e^{iqz} \ \frac{d}{dz}$$

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Because the imaginary part of its continuous extension to the solenoid is the function defining $\dot{f}[i\eta]$ respect to $d/d\pi_1$, we conclude that:

$$\nu^*(\dot{f}[i\eta])(x) = \left(-i \sum_{q>0} a_q e^{iqx} + i \sum_{q<0} a_q e^{iqx} \right) \frac{d}{dx}$$

and this proves the result for $\dot{f}[i\eta] = \hat{J}(\dot{f}[\eta])$. $\qquad\square$

Corollary 4.3.4. *The automorphism $\hat{J}$ defines a complex structure.*

Appendix A

Quasiconformal mappings

A.1 Minkowski inequality and Convolution

The following is the Minkowski inequality:

Theorem A.1.1. *Suppose $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. Consider $1 \leq p < \infty$ and a measurable complex function F respect to the product space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$. Then:*

$$\left\| \int_X F(x,y) d\mu(x) \right\|_{L_p(d\nu)} \leq \int_X \|F(x,y)\|_{L_p(d\nu)} \, d\mu(x)$$

Proof: Without loss of generality we may assume that F is positive real and that both sides of the inequality are finite taking simple functions approximation if necessary. The case $p = 1$ follows directly by Fubini's Theorem. Consider $1 < p < \infty$ and its conjugate $q = p/(p-1)$. Define:

$$G(y) = \left(\int_X F(x,y) d\mu(x) \right)^{p-1}$$

We have that $G \in L_q(Y, \nu)$ for:

$$
\begin{aligned}
\|G\|_{L_q(d\nu)} &= \left\| \left(\int_X F(x,y) d\mu(x) \right)^{p-1} \right\|_{L_p(d\nu)} \\
&= \left(\int_Y \left(\int_X F(x,y) d\mu(x) \right)^p d\nu(y) \right)^{1/q} \\
&= \left\| \int_X F(x,y) d\mu(x) \right\|_{L_p(d\nu)}^{p-1}
\end{aligned}
$$

By Fubini's and Hölder's Theorems we have:

$$
\begin{aligned}
\left\| \int_X F(x,y) d\mu(x) \right\|^p_{L_p(d\nu)} &= \int_Y G(y) \int_X F(x,y)\ d\mu(x)\ d\nu(y) \\
&= \int_X \int_Y G(y) F(x,y)\ d\nu(y)\ d\mu(x) \\
&\leq \int_X \|G(y)\|_{L_q(d\nu)} \|F(x,y)\|_{L_p(d\nu)}\ d\mu(x) \\
&= \|G\|_{L_q(d\nu)} \int_X \|F(x,y)\|_{L_p(d\nu)}\ d\mu(x)
\end{aligned}
$$

Dividing both sides by $\|G\|_{L_q(d\nu)}$ we have the result. $\qquad\square$

Remark A.1.1. Taking the countable measure on the set $X = \{1,2\}$ we have the classical Minkowski inequality:

$$
\|f+g\|_p \leq \|f\|_p + \|g\|_p
$$

Moreover, taking the countable measure on $X = \mathbb{N}$ gives:

$$
\Big\| \sum_{i\in\mathbb{N}} f_i \Big\|_p \leq \sum_{i\in\mathbb{N}} \|f_i\|_p
$$

Corollary A.1.2. *Consider $g \in L_1(\mathbb{R}^n)$ and $f \in L_p(\mathbb{R}^n)$ such that $1 \leq p \leq \infty$. Then,*

$$
\|f * g\|_p \leq \|g\|_1 \|f\|_p
$$

where:

$$
f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\ d^{(n)}y
$$

is the convolution product.

Proof: The case $p = \infty$ is trivial and the case $p = 1$ follows by Fubini's Theorem. Let $1 < p < \infty$. Because $g \in L_1(\mathbb{R}^n)$ the measure $d\mu(y) = |g(y)| d^{(n)}y$ is finite and Minkowski inequality implies:

$$
\begin{aligned}
\|f * g\|_p &\leq \left\| \int_{\mathbb{R}^n} |f(x-y)|\ |g(y)|\ d^{(n)}y \right\|_{L_p(d^{(n)}x)} \\
&\leq \int_{\mathbb{R}^n} \|f(x-y)\|_{L_p(d^{(n)}x)} |g(y)|\ d^{(n)}y \\
&= \|g\|_{L_1(d^{(n)}x)} \|f\|_{L_p(d^{(n)}x)}
\end{aligned}
$$

and this proves the corollary. $\qquad\square$

Remark A.1.2.   • $(L_1(\mathbb{R}^n), *)$ is a commutative complex Banach algebra.

• $L_p(\mathbb{R}^n)$ is a $L_1(\mathbb{R}^n)$-module with the convolution action.

- The above corollary is a particular case of the *Young inequality*: Consider $r, p \geq 1$, $g \in L_r(\mathbb{R}^n)$, $f \in L_p(\mathbb{R}^n)$ and $1 \leq p \leq q \leq \infty$ such that:

$$\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q}$$

Then,

$$||f * g||_q \leq ||g||_r ||f||_p$$

Respect to the L_p norm, the space of compact support continuous functions $C_0(\mathbb{R}^n)$ is dense in the space of simple functions and because the latter is dense in $L_p(\mathbb{R}^n)$ we have:

$$\overline{C_0(\mathbb{R}^n)}^p = L_p(\mathbb{R}^n)$$

A stronger result, Corollary A.2.3, will be proved later in section A.2.

Lema A.1.3. *Consider $1 \leq p < \infty$ and $f \in L_p(\mathbb{R}^n)$. Define $T_y f \in L_p(\mathbb{R}^n)$ such that $(T_y f)(x) = f(x - y)$ for every $y \in \mathbb{R}^n$. Then, the map $y \mapsto T_y f$ is continuous.*

Proof: First, assume that $f \in C_0(\mathbb{R}^n)$ and let $\epsilon > 0$. If f is identically zero then the claim is trivial. Assume that f is not identically zero. Because f is uniformly continuous there is a real number $\delta > 0$ such that $||h|| < \delta$ implies:

$$|f(x - h) - f(x)| < \epsilon \, Area(support(f) + B(0; \delta))^{-1/p}$$

Then, for every $h \in \mathbb{R}^n$ such that $||h|| < \delta$ we have:

$$||T_h f - f||_p = \left(\int_{\mathbb{R}^n} |f(x - h) - f(x)|^p \, d^{(n)}x \right)^{1/p} \leq \epsilon$$

Now consider $f \in L_p(\mathbb{R}^n)$. There is $g \in C_0(\mathbb{R}^n)$ and a real number $\delta > 0$ such that $||f - g|| < \epsilon/3$ and $||h|| < \delta$ implies $||T_h g - g||_p < \epsilon/3$. Thus,

$$||T_h f - f||_p \leq ||T_h f - T_h g||_p + ||T_h g - g||_p + ||g - f||_p < \epsilon$$

because T_h is an isometry. $\square$

Lema A.1.4. *Consider $1 \leq p \leq \infty$, $f \in L_p(\mathbb{R}^n)$ and $g \in C_0^m(\mathbb{R}^n)$ such that $0 \leq m \leq \infty$. Then $f * g \in C_{unif}^m(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$ and:*

$$D^\alpha(f * g) = f * D^\alpha g$$

for every multi-index α such that $|\alpha| < m$.

Proof: By Corollary A.1.2 we have $f * g \in L_p(\mathbb{R}^n)$. It rest to show that $f * g \in C^m(\mathbb{R}^n)$. First consider $m = 0$. By Hölder's inequality we have:

$$
\begin{aligned}
|(f * g)(x + h) - (f * g)(x)| &= \left| \int_{\mathbb{R}^n} f(x + h - y)g(y) \, d^{(n)}y - \int_{\mathbb{R}^n} f(x - y)g(y) \, d^{(n)}y \right| \\
&= \left| \int_{\mathbb{R}^n} f(x - y)g(y + h) \, d^{(n)}y - \int_{\mathbb{R}^n} f(x - y)g(y) \, d^{(n)}y \right| \\
&= \left| \int_{\mathbb{R}^n} f(x - y)\big(g(y + h) - g(y)\big) \, d^{(n)}y \right| \\
&\leq ||f||_p ||T_{-h}g - g||_q
\end{aligned}
$$

and because of the previous Lemma we have the uniform continuity of $f * g \in C_{unif}(\mathbb{R}^n)$.

Now consider $m = 1$ and $t > 0$. By the mean value Theorem there is $c \in (0, t)$ such that:

$$
\begin{aligned}
|(f * g)(x + te_i) - (f * g)(x)| &= \int_{\mathbb{R}^n} f(y)\big(g(x + te_i - y) - g(x - y)\big)\, d^{(n)}y \\
&= \int_{\mathbb{R}^n} f(y)\big(g(x + te_i - y) - g(x - y)\big)\, d^{(n)}y \\
&= t \int_{\mathbb{R}^n} f(y) \frac{\partial g}{\partial x_i}(x + ce_i - y)\, d^{(n)}y \\
&= t\, f * \frac{\partial g}{\partial x_i}(x + ce_i)
\end{aligned}
$$

Because $\partial g / \partial x_i \in C_0(\mathbb{R}^n)$ then $f * (\partial g / \partial x_i) \in C_{unif}(\mathbb{R}^n)$. We conclude that $\partial(f * g)/\partial x_i$ exists and coincides with $f * (\partial g / \partial x_i)$. Since i was arbitrary we have that $f * g \in C^1_{unif}(\mathbb{R}^n)$ and the general case follows by induction. $\qquad\square$

A.2 Approximations to the Identity and Weyl's Lemma

The convolution action of $L_1(\mathbb{R}^n)$ on $L_p(\mathbb{R}^n)$ doesn't have a unit; i.e. there is no element $\delta \in L_1(\mathbb{R}^n)$ (Dirac delta) such that $\delta * f = f$ for every $f \in L_p(\mathbb{R}^n)$. For this reason we need to approximate it. For any real function φ on $\mathbb{R}^n$ and $\epsilon > 0$ define φ_ϵ such that $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$.

Lema A.2.1. *Consider a positive real function $\varphi \geq 0$ such that $\varphi \in L_1(\mathbb{R}^1)$. Then $||\varphi_\epsilon||_1 = ||\varphi||_1$ for every $\epsilon > 0$ and:*

$$
\lim_{\epsilon \to 0} \int_{\{|x| > \delta\}} \varphi_\epsilon(x)\, d^{(n)}x = 0
$$

for every $\delta > 0$.

Proof: The first claim follows from the definition and a change of variables. For the second, by dominated convergence we have:

$$
\lim_{k \to \infty} \int_{\{|x| > k\}} \varphi(x)\, d^{(n)}x = 0
$$

Then,

$$
\int_{\{|x| > \delta\}} \varphi_\epsilon(x)\, d^{(n)}x = \epsilon^{-n} \int_{\{|x| > \delta\}} \varphi(x/\epsilon)\, d^{(n)}x = \int_{\{|x| > \delta/\epsilon\}} \varphi(x)\, d^{(n)}x \to 0
$$

when ϵ tends to zero. $\qquad\square$

Lema A.2.2. *Consider a positive real function $\varphi \geq 0$ such that $\varphi \in L_1(\mathbb{R}^n)$ and $||\varphi||_1 = 1$.*

- If $f \in L_p(\mathbb{R}^n)$ such that $1 \le p < \infty$ then,

$$\lim_{\epsilon \to 0} ||f * \varphi_\epsilon - f||_p = 0$$

- If $f \in L_\infty(\mathbb{R}^n)$ then for every continuity point x of f we have:

$$\lim_{\epsilon \to 0} f * \varphi_\epsilon(x) = f(x)$$

- If $f \in L_\infty(\mathbb{R}^n)$ and f is uniformly continuous on a domain D then $f * \varphi_\epsilon$ converges uniformly to f on D when ϵ tends to zero.

- If f is continuous and φ has compact support, then $f * \varphi_\epsilon$ converges to f locally uniformly when ϵ tends to zero.

Proof:

- By hypothesis $\varphi_\epsilon dy$ is a probability measure so we can write:

$$|f * \varphi_\epsilon(x) - f(x)| = \left| \int_{\mathbb{R}^n} \big(f(x-y) - f(x)\big)\varphi_\epsilon(y)dy \right|$$

and by Minkowski inequality A.1.1 we have:

$$
\begin{aligned}
||f * \varphi_\epsilon - f||_p &\le \left\| \int_{\mathbb{R}^n} \big|f(x-y) - f(x)\big|\varphi_\epsilon(y)dy \right\|_p \\
&\le \int_{\mathbb{R}^n} ||f(x-y) - f(x)||_{L_p(dx)}\varphi_\epsilon(y)dy \quad\quad (\text{A.1})
\end{aligned}
$$

Let $\eta > 0$. Because of Lemma A.1.3 there is a real number $\delta > 0$ such that $||y|| < \delta$ implies $||T_y f - f||_p < \eta/2$. Proceeding with relation A.1 we have $||f*\varphi_\epsilon - f||_p \le I + II$ where:

$$
\begin{aligned}
I &= \int_{\{||y||<\delta\}} ||f(x-y) - f(x)||_{L_p(dx)}\varphi_\epsilon(y)dy \\
&\le (\eta/2) \int_{\{||y||<\delta\}} \varphi_\epsilon(y)dy \le \eta/2
\end{aligned}
$$

and:

$$
\begin{aligned}
II &= \int_{\{||y||>\delta\}} ||f(x-y) - f(x)||_{L_p(dx)}\varphi_\epsilon(y)dy \\
&\le 2||f||_p \int_{\{||y||>\delta\}} \varphi_\epsilon(y)dy
\end{aligned}
$$

By Lemma A.2.1, there is a real number $\sigma > 0$ such that $\epsilon < \sigma$ implies $II \le \eta/2$. We have proved: For every $\eta > 0$ there is a real number $\sigma > 0$ such that $||f*\varphi_\epsilon - f||_p \le \eta$; i.e. The limit exists and equals zero.

- Suppose that x is a continuity point of f. We proceed exactly as before:

$$|f * \varphi_\epsilon(x) - f(x)| = \left| \int_{\mathbb{R}^n} \big(f(x-y) - f(x) \big) \varphi_\epsilon(y) dy \right|$$

Let $\eta > 0$. There is a real number $\delta > 0$ such that $||y|| < \delta$ implies $|f(x-y) - f(x)| < \eta/2$. Then, $|f * \varphi_\epsilon(x) - f(x)| \leq I + II$ such that:

$$
\begin{aligned}
I &= \int_{\{||y||<\delta\}} |f(x-y) - f(x)| \varphi_\epsilon(y) dy \\
&\leq (\eta/2) \int_{\{||y||<\delta\}} \varphi_\epsilon(y) dy \leq \eta/2
\end{aligned}
$$

and:

$$
\begin{aligned}
II &= \int_{\{||y||>\delta\}} |f(x-y) - f(x)| \varphi_\epsilon(y) dy \\
&\leq 2||f||_\infty \int_{\{||y||>\delta\}} \varphi_\epsilon(y) dy
\end{aligned}
$$

Again, by Lemma A.2.1, there is a real number $\sigma > 0$ such that $\epsilon < \sigma$ implies $II \leq \eta/2$. We have proved: For every $\eta > 0$ there is a real number $\sigma > 0$ such that $|f * \varphi_\epsilon(x) - f(x)| \leq \eta$; i.e. The limit exists and $f * \varphi_\epsilon(x)$ converges to $f(x)$ when ϵ tends to zero.

- Almost verbatim to the above item.

- For the final claim, consider a compact set K of $\mathbb{R}^n$ and a real number $\eta > 0$. Because f is uniformly continuous on K, there is a real number $\delta > 0$ such that $||y|| < \delta$ implies $|f(x-y) - f(x)| < \eta/2$ for every $x \in K$. In the same way as before, $\sup_{x \in K} |f * \varphi_\epsilon(x) - f(x)| \leq I + II$ such that:

$$
\begin{aligned}
I &= \sup_{x \in K} \int_{\{||y||<\delta\}} |f(x-y) - f(x)| \varphi_\epsilon(y) dy \\
&\leq (\eta/2) \int_{\{||y||<\delta\}} \varphi_\epsilon(y) dy \leq \eta/2
\end{aligned}
$$

and:

$$
\begin{aligned}
II &= \sup_{x \in K} \int_{\{||y||>\delta\}} |f(x-y) - f(x)| \varphi_\epsilon(y) dy \\
&\leq 2||f \chi_{K'}||_\infty \int_{\{||y||>\delta\}} \varphi_\epsilon(y) dy
\end{aligned}
$$

where $K' = K + supp(\varphi_\epsilon) = K + \epsilon.supp(\varphi)$ is a compact set hence $||f \chi_{K'}||_\infty < \infty$. By Lemma A.2.1, there is a real number $\sigma > 0$ such that $\epsilon < \sigma$ implies $II \leq \eta/2$. We have proved: For every $\eta > 0$ there is a real number $\sigma > 0$ such that $\sup_{x \in K} |f * \varphi_\epsilon(x) - f(x)| \leq \eta$ and we have the claim.

$\square$

Definition A.2.1. Consider a positive real function $\varphi \in L_1(\mathbb{R}^n)$ such that $||\varphi||_1 = 1$. The function $\epsilon \mapsto \varphi_\epsilon$ is called an *approximation to the identity*.

Corollary A.2.3. *The space $C_0^\infty(\mathbb{R}^n)$ of infinitely differentiable functions with compact support is dense in $L_p(\mathbb{R}^n)$.*

Proof: Let $f \in L_p(\mathbb{R}^n)$ and $\eta > 0$. Consider a large enough radius R such that $||f\chi_{||y||>R}||_p < \eta/2$. Consider also a positive real function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $||\varphi||_1 = 1$. By Lemma A.2.2 there is $\epsilon > 0$ such that $||(f\chi_{||y||\leq R}) * \varphi_\epsilon - f\chi_{||y||\leq R}||_p < \eta/2$. Define $g = (f\chi_{||y||\leq R}) * \varphi_\epsilon$. We have:

$$||g - f||_p = ||g - f\chi_{||y||\leq R} - f\chi_{||y||>R}||_p \leq ||g - f\chi_{||y||\leq R}||_p + ||f\chi_{||y||>R}||_p < \eta$$

Because of Lemma A.1.4 and the fact that:

$$supp(g) \subset \overline{B(0;R)} + supp(\varphi_\epsilon) = \overline{B(0;R)} + \epsilon.supp(\varphi)$$

we conclude that $g \in C_0^\infty(\mathbb{R}^n)$. $\square$

The following is the space of smooth functions whose derivatives decay to zero faster than the inverse of any polynomial:

Definition A.2.2. An infinitely differentiable function f is in the *Schwartz class* $\mathcal{S}(\mathbb{R}^n)$ if:

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty$$

for every pair of multi-indices α, β.

Corollary A.2.4. *The Schwartz class is dense in $L_p(\mathbb{R}^n)$.*

Proof: It follows from the fact that $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$ and Corollary A.2.3. $\square$

The following is the *Weyl's Lemma*:

Lema A.2.5. *Let f be a continuous function on a domain $D \subset \mathbb{C}$ with distributional derivative $f_{\bar{z}}$ locally integrable on D. If $f_{\bar{z}} = 0$ on D then f is holomorphic on D.*

Proof: Consider a positive real function $\varphi \in C_0^1(\mathbb{C})$ with compact support and $||\varphi||_1 = 1$. Consider also a compact set K in D. Because φ has compact support, there is a real number $\sigma > 0$ such that $0 < \epsilon < \sigma$ implies:

$$K + supp(\varphi_\epsilon) = K + \epsilon.supp(\varphi) \subset D$$

Then, $f * \varphi_\epsilon$ is defined on K for every $0 < \epsilon < \sigma$. By Lemma A.1.4, $f * \varphi_\epsilon \in C^1(K)$ is differentiable on K for every $0 < \epsilon < \sigma$. Because φ has compact support and $f_{\bar{z}}$ is the locally integrable derivative in distributional sense:

$$(f * \varphi_\epsilon)_{\bar{z}} = f_{\bar{z}} * \varphi_\epsilon = 0$$

on K for every $0 < \epsilon < \sigma$. Hence, $f * \varphi_\epsilon$ is holomorphic on K for every $0 < \epsilon < \sigma$. By the fourth item of Lemma A.2.2, $(f * \varphi_\epsilon)_{0<\epsilon<\sigma}$ converge uniformly to f on K when ϵ tends to zero. We conclude that f is holomorphic on K and because the compact set was arbitrary, f is holomorphic on D. $\square$

A.3   Quasiconformal mappings

This section is almost verbatim to [IT] and [Ah]. We want to consider orientation-preserving homeomorphisms satisfying the Beltrami equation $f_{\bar{z}} = \mu f_z$. Because of this, f should be at least partially differentiable almost everywhere (a.e.) in the considered domain. This is guaranteed by assuming that f is ACL: A function f on a planar domain D is said to be ACL (absolutely continuous on lines) if the following condition holds for every rectangle $R = [a, b] \times [c, d]$ in D whose sides are parallel either to the real axis or the imaginary axis: As a function of x, $f(\_, y)$ is absolutely continuous on $[a, b]$ for almost everfy $y \in [c, d]$, and as a function of y, $f(x, \_)$ is absolutely continuous on $[c, d]$ for almost every $x \in [a, b]$.

Remark A.3.1. An absolutely continuous function g on an interval I is differentiable at almost every $t \in I$. Hence, when a function f on a domain D is ACL, the partial derivatives f_z and $f_{\bar{z}}$ are well defined and finite at almost every $z \in D$. Moreover they are measurable.

Analytic definition A: Let f be an orientation preserving homeomorphism of a planar domain D into the complex plane. We say that f is quasiconformal (qc) on D if f satisfies the following conditions:

- f is ACL on D.

- There is a constant k with $0 \leq k < 1$ such that:

$$|f_{\bar{z}}| \leq k|f_z|$$

  almost everywhere on D.

Setting $K = (k + 1)/(k - 1)$ we say that f is $K - qc$ on D. We call the infimum of K such that f is $K - qc$ the maximal dilatation of f and denote it by K_f.

Example 1: A conformal mapping of a domain D is quasiconformal on D such that $k = 0$ and $K = 1$. Later we will see that the converse is also true: Every 1-qc map is conformal (Weyl's Lemma).

Example 2: An affine mapping f such that $f(z) = az + b\bar{z} + c$ such that $|b| > |a|$ is quasiconformal such that $k = |b|/|a|$ and $K = (|a| + |b|)/(|a| - |b|)$.

Example 3: For a given K define f such that:

$$f(z) = \begin{cases} z & z \in H \\ x + iKy & z = x + iy \in \mathbb{C} - H \end{cases}$$

Then f is $K - qc$.

Remark A.3.2. Define f such that

$$f(z) = \frac{z}{1 - |z|^2}$$

such that $z \in \Delta$. Then f is an orientation preserving diffeomorphism of the unit disk Δ onto $\mathbb{C}$ but not quasiconformal. Actually, there are no quasiconformal mappings of Δ onto $\mathbb{C}$.

Remark A.3.3. Let f be a $K - qc$ mapping of a domain D onto another domain D' and g be a conformal mapping of D'. Then the composite mapping $g \circ f$ is $K - qc$. In fact, it is easy to see that $g \circ f$ verifies the definition and that $K_{g \circ f} = K_f$.

In general, existence of the partial derivatives f_z and $f_{\bar{z}}$ is not enough to guarantee good properties of f. However, in the case of homeomorphisms we have the following remarkable result [**?**]:

Proposition A.3.1. *If a homeomorphism f of a domain D into $\mathbb{C}$ has the partial derivatives f_x and f_y a.e. on D, then f is totally differentiable a.e. on D.*

Proposition A.3.2. *If f is a quasiconformal mapping of a domain D, then the partial derivatives f_z and $f_{\bar{z}}$ are locally square integrable on D.*

Proof: First, we have a completely additive set function A by attaching the area $A(E)$ of $f(E)$ to every Borel subset E of D. Let J_f be the density function of A with respect to the two-dimensional Lebesgue measure $dxdy$. By Lebesgue's decomposition theorem we have:

$$\int \int_E J_f \; dxdy \leq A(E)$$

for every measurable subset E of D. On the other hand, by Proposition A.3.1 we see that f is totally differentiable at almost $z \in D$, and at such a point z we can show that:

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2$$

By the second item of the quasiconformal definition we have:

$$|f_{\bar{z}}|^2 \leq |f_z|^2 \leq \frac{1}{1 - k^2} J_f$$

almost everywhere on D and we have the result. $\qquad\square$

Proposition A.3.3. *For every quasiconformal mapping f of a domain D, the partial derivatives f_z and $f_{\bar{z}}$ are coincident with those in the sense of distribution. Namely, for every element ϕ of $C_0^\infty(D)$, the set of all smooth functions on D with compact supports, it follows that:*

$$\int \int_D f_z \phi \; dxdy = - \int \int_D f \phi_z \; dxdy$$

$$\int \int_D f_{\bar{z}} \phi \; dxdy = - \int \int_D f \phi_{\bar{z}} \; dxdy$$

Proof: By Proposition A.3.2 and Fubini's theorem, the left hand side of the above identities can be rewritten as repeated integrals. Since f is ACL the assertions follows by integration by parts. $\qquad\square$

Noting Propositions A.3.2 and A.3.3 we consider the following alternative definition:

Analytic definition A': Let f be a homeomorphism of a domain D into $\mathbb{C}$ which preserves orientation. We say that f is quasiconformal on D if the following conditions hold:

- The distributional partial derivatives of f with respect to z and $\bar{z}$ can be represented by locally integrable functions f_z and $f_{\bar{z}}$, respectively, on D.

- There exists a constant k with $0 \leq k < 1$ such that:

$$|f_{\bar{z}}| \leq k|f_z|$$

almost everywhere on D.

Theorem A.3.4. *Definitions A and A$'$ of quasiconformal mappings are equivalent.*

Corollary A.3.5. *Let g be a conformal mapping of a domain D onto another D', and f be a $K - qc$ mapping of D'. Then $f \circ g$ is a $K - qc$ mapping of D.*

Proof: Let $w = u + iv$ be the variable on D'. By assumption, there exists the distributional derivatives f_w and $f_{\bar{w}}$ and they are locally integrable on D'. Since $\phi \circ g^{-1} \in C_0^\infty(D')$ for every $\phi \in C_0^\infty$, there exist the distributional derivatives $(f \circ g)_z$ and $(f \circ g)_{\bar{z}}$ and the coincide with the locally integrable functions $(f_w \circ g).g'$ and $(f_{\bar{w}} \circ g).\overline{g'}$ respectively on D. Moreover:

$$|(f \circ g)_{\bar{z}}| \leq k|(f \circ g)_z|$$

where $k = (K - 1)/(K + 1)$ and we have the result. □.

Corollary A.3.6. *A $1 - qc$ mapping on a domain D is holomorphic on D.*

Proof: A $1 - qc$ mapping mapping f on a domain D verifies $f_{\bar{z}} = 0$ on D as a distribution. By Weyl's Lemma A.2.5 f is holomorphic on D. □

A.4 Mapping Theorem

The following is the Douady-Hubbard approach to the mapping Theorem.

Definition A.4.1. Consider a Beltrami differential $\mu \in L_\infty(\mathbb{C})_1$ with compact support and the unique quasiconformal solution f to the μ-Beltrami equation such that $f(z) - z \in O(1/|z|)$.

Because μ has compact support, it is univalent outside a disk of sufficiently large radius R and we can write:

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \tag{A.2}$$

In particular $f_z - 1 \in L_2(\mathbb{C})$: Because f is quasiconformal we have that $f_z \in L_{2,loc}(\mathbb{C})$ and it rest to verify this property at a neighborhood of ∞. Equation A.2 implies $f_z - 1 \in O(1/|z|^2)$; i.e. There is a constant C such that $|f_z(z) - 1| \leq C/|z|^2$ for every $z \in \mathbb{C}$ such that $|z| > R$. Then:

$$\int_{|z|>R} |g_z(z) - 1|^2 \leq \int_0^{2\pi} \int_R^{+\infty} \frac{C}{r^3} d\theta dr < \infty$$

and we have the claim.

The following is the Douady-Hubbard [Hu] beautiful proof of existence of Beltrami equation solutions and analytic dependence in the case Beltrami differentials with compact support:

Theorem A.4.1. *Consider a compact set $K \subset \mathbb{C}$ and consider the set of Beltrami differentials $B_K(\mathbb{C}) \subset L_\infty(\mathbb{C})_1$ with support contained in K. Then, for every $\mu \in B_K(\mathbb{C})$ there is a unique DH solution f^μ to the μ-Beltrami equation. Moreover, the map $\mu \mapsto f^\mu$ is analytic in $B_K(\mathbb{C})$.*

Proof: Consider the Fourier transform $\mathcal{F} : L_2(\mathbb{C}) \to L_2(\mathbb{C})$. For every map f with distributional partial derivatives on L_2 we have the relations:

$$\mathcal{F}(\partial_z f)(\zeta) = 2\pi\bar{\zeta}i \; \partial_z\mathcal{F}(f)(\zeta)$$

$$\mathcal{F}(\partial_{\bar{z}} f)(\zeta) = 2\pi\zeta i \; \partial_{\bar{z}}\mathcal{F}(f)(\zeta)$$

Hence, there is an operator $\mathcal{L} : L^2(\mathbb{C}) \to L_2(\mathbb{C})$ such that for every map f with distributional partial derivatives on L_2:

$$\partial_z f = \mathcal{L}(\partial_{\bar{z}} f)$$

where $\mathcal{L}$ is explicitly given by:

$$\mathcal{F}(\mathcal{L}(w))(\zeta) = \frac{\bar{\zeta}}{\zeta}\mathcal{F}(w)(\zeta)$$

for every $w \in L_2(\mathbb{C})$. Because of the above expression and the fact that the Fourier transform $\mathcal{F}$ is a unitary operator, it follows that $\mathcal{L}$ is an isometry; i.e. $||\mathcal{L}|| = 1$ in operator norm. Consider $\mu \in B_K(\mathbb{C})$. Then, a DH map $f(z) = z + g(z)$ is a solution of the μ-Beltrami equation if and only if:

$$\partial_{\bar{z}} g = \mu\big(1 + \mathcal{L}(\partial_{\bar{z}} g)\big)$$

and we have the following functional equation:

$$\big(id - \mu\mathcal{L}\big)(\partial_{\bar{z}} g) = \mu$$

Because $||\mu||_\infty < 1$ and $||\mathcal{L}|| = 1$ we have $||\mu\mathcal{L}|| < 1$ hence the operator above has an inverse and we have:

$$\partial_{\bar{z}} g = \big(id - \mu\mathcal{L}\big)^{-1}(\mu) = \mu + \mu\mathcal{L}(\mu) + \mu\mathcal{L}\big(\mu\mathcal{L}(\mu)\big) + \dots$$

Because μ has compact support then $\partial_{\bar{z}} g$ has also compact support and the convolution $(\pi z)^{-1} * \partial_{\bar{z}} g$ is well defined. Because this operator solves the $\partial_{\bar{z}}$ problem on functions with compact support: $\partial_{\bar{z}}\big((\pi z)^{-1} * h\big) = h$, and the fact that this operator acting on compact support functions gives functions in the $O(1/|z|)$ class, we have the remarkable formula:

$$g = \frac{1}{\pi z} * (\partial_{\bar{z}} g) = \frac{1}{\pi z} * \big(id - \mu\mathcal{L}\big)^{-1}(\mu) \tag{A.3}$$

In particular, the formula above makes clear the analytic dependence in $B_K(\mathbb{C})$. $\qquad\square$

Remark A.4.1. The operator $\mathcal{L}$ is actually the operator T in [Ah] defined as a singular integral operator. In the same reference, because the operator P is acting on compact support functions and we are interested in DH solutions, it takes the form of a convolution product. Because we are dealing directly with L_2, in contrast to the Ahlfors approach the Calderón-Zigmund Theorem is not needed in this one.

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