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# On some applications of exchangeable and stationary dependence structures

T E S I S

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# *Abstract*

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Doctor of Mathematics

**On some applications of exchangeable and stationary dependence structures**

by Arrigo COEN

Exchangeable and stationary models arise in a wide variety of problems where statistical analysis is applied. They represent two of the most natural dependence structures that appear in real phenomena. Accordingly, this work considers three different aspects about these dependent structures. The first aspect that we analyze is the construction of multivariate stationary processes with a given stationary distribution. In particular, we study three different models using our construction; these models present t-multivariate, Gaussian and Wishart stationary distribution, respectively. The second aspect that we handle is the computation of ruin probabilities under exchangeable claim amounts scenario. We extend various important risk theory results of the independent framework. Finally, the third aspect that we handle is the implications on the renewal equation if the renewals are exchangeable. We found that the renewal function in this framework can be rewritten as the solution of a new type of equations. Furthermore, we obtain the general solution to them.

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Indeed, once again, I only lacking to say $\sigma$-algebra.

Arrigo Coen
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# Contents

# Abbreviations

| Symbol | Explanation |
|---|---|
| $(\Omega, \mathscr{F}, \mathbb{P})$ | Overall probability space |
| $\mathbb{P}$ | Probability measure |
| $\mathbb{E}$ | Expectation operator |
| $F_X / f_X$ | Distribution/density of the variable $X$ |
| $X \sim F$ | The random variable $X$ is distributed accordingly to $F$ |
| $\overline{F}$ | Tail distribution $\overline{F} = 1 - F$ |
| $F_e$ | Equilibrium distribution |
| $F^{*n}$ | $n$th convolution of the distribution $F$ |
| $\mathbb{R}/\mathbb{R}_+$ | The set of real/non-negative real numbers |
| $\mathbb{N}/\mathbb{N}_+$ | The set of positive/non-negative numbers |
| $\mu$ | de Finetti's measure of an exchangeable sequence |
| $\mathscr{P}$ | The set of all probability measures |
| $\mathscr{Q}$ | The $\sigma$-algebra of the sets $\{Q \in \mathscr{P} : Q(A) \leq y\}$ for some $A \in \mathscr{F}$ and $y \in [0,1]$ |
| $\mathrm{Ber}(q)$ | Bernoulli distribution with mean $p$ |
| $\mathrm{Geo}(p)$ | Geometric distribution with mean $(1-p)/p$ |
| $\mathrm{Exp}(\beta)$ | Exponential distribution with mean $1/\beta$ |
| $\mathrm{Ga}(\alpha, \beta)$ | Gamma distribution with mean $\alpha/\beta$ |
| $\mathrm{PH}(\pi, S)$ | Phase-type distribution with mean $-\pi S^{-1}\mathbf{1}$ |
| $\mathrm{D}(a_1, \ldots, a_p)$ | Dirichlet distribution with mean $(a_1, \ldots, a_p)/\sum a_i$ |
| $\mathrm{W}(\nu, M)$ | Wishart distribution of dimension $p$ with mean $\nu M$ |
| $\mathrm{IW}(\nu, \Psi)$ | Inverse Wishart distribution of dimension $p$ with mean $\Psi/(\nu - p - 1)$ |

| | |
|---|---|
| $\mathrm{N}(\mu, \Sigma)$ | Normal/Gaussian distribution with mean vector $\mu$ and correlation matrix $\Sigma$ |
| $\mathrm{U}(a, b)$ | Uniform distribution on $[a, b]$ |
| $\mathscr{M}_X$ | Moment generating function of the variable $X$ |
| $\varphi_X$ | Characteristic function of the variable $X$ |
| $\mathscr{L}_H$ | Laplace transform of the function $H$ |
| $\psi(u)$ | Ruin probability with initial capital $u$ |
| $R_t^u$ | Reserve process with initial capital $u$ |
| $\mathscr{B}$ | The net profit condition set |
| $\mathscr{C}$ | The adjustment coefficient set |
| CL | Cramér-Lundberg |
| CLEC | Cramér-Lundberg model with exchangeable claims |
| $\mathsf{Var}/\mathsf{Corr}/\mathsf{Cov}$ | Variance/Correlation/Covariance |
| $\Gamma(\cdot)$ | Gamma function |
| $\mathbb{1}_{\{\cdot\}}$ | The indicator function |
| $\propto$ | Proportional to |
| $\triangle$ | Symmetric difference |
| $\mathrm{etr}(\cdot)$ | Exponential trace function |
| $\sigma(\mathscr{A})/\sigma(X)$ | The $\sigma$-algebra of the class $\mathscr{A}$ / of the random variable $X$ |
| a.s. | almost sure |
| i.i.d. | independent and identically distributed |
| $\stackrel{\mathrm{d}}{=}$ | Equality in distribution |

*Dedicated to Babbo and Elí*

# Introducción (Spanish version)

En estadística, la dependencia de las variables juega un importante rol en el comportamiento de un modelo, y por ende influye en su habilidad para representar la realidad. El estudiar estas estructuras nos permite representar de manera más adecuada los fenómenos que estamos analizando. Sin una apropiada estructura de dependencia, la inferencia que un modelo es capaz de capturar puede ser incorrecta.

Las aportaciones de este trabajo se basan en dos tipos de estructuras de dependencia: estacionariedad e intercambiabilidad. Estas simetrías distribucionales representan el comportamiento de procesos que probabilísticamente no son afectados por traslaciones y permutaciones, respectivamente. Estos supuestos son naturales en muchas aplicaciones, por tanto, estas estructuras son útiles para plantear esquemas generales fuera del supuesto de independencia.

Gran parte de la literatura que trata a estas estructuras de dependencia se enfoca en su caracterización, lo cual fue iniciado por el celebrado teorema de representación de Bruno de Finetti [de Finetti, 1937] y la versión que planteó Maitra [Maitra, 1977] para variables aleatorias estacionarias. Sin embargo, fuera de la importancia de la intercambiabilidad para la estadística Bayesiana, pocos trabajos se enfocan en estudiar las implicaciones de estas estructuras en áreas de modelaje de procesos estocásticos. Nuestra motivación es en esta dirección y, en este trabajo, exploraremos las implicaciones de estos supuestos en algunos problemas específicos.

En concreto centraremos nuestra discusión en tres diferentes aspectos de estas estructuras de dependencia

- Aspecto 1: *La construcción de procesos estacionarios multivariados dada una distribución estacionaria, fijada de antemano.*
- Aspecto 2: *El calculo de probabilidades de ruina bajo el supuesto de que los reclamos sean intercambiables.*

- Aspecto 3: *Las implicaciones sobre la ecuación de renovación cuando los tiempos de las renovaciones son intercambiables.*

A continuación damos una breve descripción de cada uno de estos aspectos:

Para enfrentar el primer aspecto, en el Capítulo 1 presentamos una construcción de procesos multivariados estacionarios. La ventaja de nuestro enfoque es que otorga libertad en la selección de la distribución estacionaria, mientras que la mayoría de los planteamientos están restringidos a una sola familia paramétrica de distribuciones multivariadas. En particular, generalizamos las ideas de Pitt and Walker [2005] and Mena and Walker [2009], al caso multivariado. Usando esta construcción, presentaremos tres modelos de procesos multivariados estacionarios; dichos modelos tienen a las distribuciones t-multivariada, Gaussiana y Wishart como distribución estacionaria, respectivamente. Modelos de este tipo son frecuentemente utilizados en aplicaciones; por ejemplo, estos modelos son ampliamente utilizados en finanzas para la estimación de volatilidad.

Para hacer frente al segundo aspecto desarrollamos resultados analíticos sobre la probabilidad de ruina para el modelo de reserva de Cramér-Lundberg con reclamos intercambiables (el modelo CLEC, por sus siglas en inglés), en el Capítulo 3. En este modelo los reclamos tienen una estructura de dependencia dada por un proceso intercambiable. Esta hipótesis es adecuada para el modelaje de ciertas reservas, ya que en algunos casos las observaciones de los reclamos muestran una correlación positiva. Esto ocurre, por ejemplo, en seguros catastróficos que cubren tormentas, ya que las magnitudes de los reclamos están positivamente correlacionadas por la magnitud de la tormenta que los genera.

En el Capítulo 2 presentamos una breve introducción a la teoría clásica de riesgo. Esta introducción nos será útil para analizar extensiones de resultados clásicos para el modelo CLEC, e.g. una generalización de la fórmula de Pollaczek-Khinchine y una generalización de la desigualdad de Lundberg. Es importante mencionar que mostraremos una condición equivalente a la *condición de ganancia neta*, para el caso intercambiable. Esta nueva condición revela importantes puntos a tomar en cuenta por una compñía de seguros cuando la hipótesis de independencia no es satisfecha. Nuestro trabajo puede ayudar a mostrar los riesgos que estaría tomando una compañía al toma cierto portafolio de seguros, bajo el supuesto de reclamos intercambiables. Algunos de nuestros hallazgos pueden ser consultados en Coen and Mena [2015b].

Finalmente, para analizar el tercer aspecto, en el Capítulo 4, investigamos las implicaciones teóricas en un modelo de renovación con renovaciones intercambiables. En dicho capítulo extendemos algunos de los resultados clásicos de teoría de renovación. Para caracterizar estos procesos definimos el concepto de ecuaciones de renovación condicionadas. En particular, exhibiremos la existencia y unicidad de sus soluciones.

Al tener hipótesis menos restrictivas, los modelos de renovación intercambiables pueden ser aplicados a escenarios más generales que los modelos de renovación clásicos. Por ejemplo, pueden ser aplicados en la teoría de confiabilidad para analizar los costos asociados con distintas políticas de mantenimiento, cuyas épocas de renovación tienen cierta dependencia. Nuestros hallazgos sobre ecuaciones de renovación intercambiables forman parte manuscrito que estamos finalizando [Coen and Mena, 2015a].

# Preface (English version)

In statistics, the dependence structure of a model plays an important role in determining its ability to resemble reality. Disentangling such structure allows us to study the intrinsic relations among variables. Consequently, without an adequate dependent structure the inference that the model grants could be weak.

In this work we base our developments on two types of dependence structures: stationarity and exchangeability. In words, these distributional symmetries resemble the behavior of processes that are probabilistically unaffected under shifts and permutations, which appear naturally in many applications. Accordingly, they serve as a natural general framework to settings outside independence.

Great part of the literature regarding these dependence structures focuses on their characterizations, starting with the celebrated representation theorem by Bruno de Finetti [de Finetti, 1937] and Maitra's version [Maitra, 1977] for stationary sequences of random variables. However, besides the impact of exchangeability within the Bayesian statistics literature, only few works are devoted to study and analyze the implications of such dependences structures in other areas of stochastic modeling. Therefore, our motivations lies in this direction and, in this work, we explore the impact of these characteristic features in some specific problems.

Accordingly, in this work we center our discussion to three different aspects about these dependent structures:

- Aspect 1: *The construction of multivariate stationary processes with a given stationary distribution.*
- Aspect 2: *The computation of ruin probabilities under exchangeable claim amounts scenarios.*

- Aspect 3: *The implications of the renewal equation when renewals times are exchangeable.*

Let us briefly describe each of these three aspects:

In order to undertake the first aspect, in Chapter 1 we present a construction technique of multivariate stationary process. The advantage of this approach is that it grants liberty in the selection of the stationary distribution, whereas most existing approaches are restricted to only specific choices of parametric multivariate distributions. In particular, we generalize some ideas of Pitt and Walker [2005] and Mena and Walker [2009], to the multivariate framework. Using this construction we present, in Chapter 1, three multivariate stochastic process models; these models present t-multivariate, Gaussian and Wishart stationary distribution, respectively. Models of this kind are frequently need in applied areas. For instance, they are widely used in financial applications for local volatility estimation and smoothing. Accordingly, part of our ongoing work focuses on the preparation of a manuscript that contains our findings in this subject.

To handle the second aspect we develop analytical results of the ruin probability under the Cramér-Lundberg reserve model with exchangeable claims (CLEC model) in Chapter 3. In this model the dependence structure among claims follows an exchangeable process. This assumption is adequate for a reserve model because in some cases claim data show positive correlated feature; for example, in storm insurance the size of the claim amounts are positive correlated by the magnitude of the storms that generates them. Consequently, in Chapter 2 we give a review of the classical theory that we will extent for the CLEC model. Indeed, we will show explicit equivalences of various results, e.g. a generalization of the Pollaczek-Khinchine formula and a generalized Lundberg inequality. In particular, we will give an equivalent condition to the *net profit condition*, for the exchangeable scenario. This condition unveils important concerns for the insurer when the independence assumption is violated. Our analysis could be applied to measure the consequences of the decisions of an insurance portfolio, since it can grant knowledge about the magnitude of the risks under a more general scenario. Some of our findings in this direction can be found in Coen and Mena [2015b].

Finally, to undertake with the third aspect described above we investigate, in Chapter 4, the theoretic consequence of a renewal model with exchangeable renewals. In this chapter we extend some of the classical renewal theory results. To characterize these processes, we define the concept of conditional renewal function. This functions can be rewritten as the solutions of exchangeable renewal equations, and using these equations we exhibit expressions for them. As a result of its less constraining assumptions, exchangeable renewal models can be applied to more general scenarios, where the classical renewal models can not. For instance, our results could be employed in reliability theory to analyze repair-vs-replace decisions, where the costs associated with various maintenance policies need to be identified, and the renewal epochs show a nonindependent behavior. The findings that we have obtain concerning renewal exchangeable equations are part of a manuscript that we are now finishing for its possible publication in a peer review journal [Coen and Mena, 2015a].

# Chapter 1

# Stationarity, exchangeability and some stochastic processes with multivariate marginal distributions

This chapter presents the basic background on exchangeability and stationarity necessary for subsequent chapters. In particular, the constructions of some stochastic processes with multivariate marginal distributions are analyzed. In Section 1.1 the motivations to study general symmetric structures are explained. Section 1.2 analyzes the exchangeable symmetry, and also discusses some theoretical results on this subject that will be used in Chapters 3 and 4. Section 1.3 focuses on the concept of stationary processes, and gives a review of the relevant literature of different constructions of this kind of process. The final three sections of this chapter present multivariate stationary processes with appealing symmetric properties.

## 1.1 Introduction

One natural feature that appears in many different real process is a symmetric structure. From a probabilistic prospective, this translates to retain certain symmetry in the law driving the random elements that describe such a process. This property should be understood in the broad sense of invariance under a family of measurable transformations. Hence, this motivates the following definition.[1]

[1]We are assuming a fixed probability space $(\Omega, \mathscr{F}, \mathbb{P})$, rich enough to develop all the probabilistic structures presented.

**Definition 1.** *A stochastic process $X = \{X_i : i \in \mathscr{I}\}$ is said to be symmetric to the family of measurable transformations $T = \{T_j : j \in \mathscr{J}\}$ if*

$$T_j(X) \stackrel{d}{=} X,$$

*for all $j \in \mathscr{J}$, where $\stackrel{d}{=}$ denotes equality in distribution.*

An example of a distributional symmetry is given by an independent sequence $X = \{X_n : n \in \mathbb{N}\}$, with respect to the family of transformations defined by $T_k(X) = \{X_{n+k} : n \in \mathbb{N}\}$, for $k \in \mathbb{N}$; in this example, the symmetry is granted because shifts in the index do not affect the probability law of the process. Also, a Markov chain that starts in its stable distribution is symmetric to these transformations.

In general, the property of symmetry could be established from many different angles, as many as families of measurable transformations exists. From a modeler point of view, this structure allow us, in particular, to establish a nonindependent interaction among variables. Using this concept we can reflect the real interplay, against the assumption of independence, which in some cases is unrealistic and is typically imposed to facilitate mathematical computations.

In some instances, the assumption of distributional symmetry is misunderstood as a result of a certain proclivity to see it as a trend around a value. Let us consider the scenario given in Figure 1.1, to exemplify this last point. This figure shows a trajectory of a process with the symmetry *stationarity* (see Definition 8). In this example, the process that generates the trajectory has an stationary distribution which is the mixture of three Gaussians distributions. At first glance, the trajectory exhibited in this figure could be misinterpreted as belonging to a non-stationary process. Consequently, in this case a misjudgment could appear because the symmetric structure is establish around a non visually-symmetric distribution.

By reason of their natural resemblance to real properties, the distributional symmetries of stationarity and exchangeability, have been the focal point in many studies; e.g. Yaglom [1987a,b] and Kallenberg [2005], and references therein. These symmetries are indeed very appealing to model real phenomena.

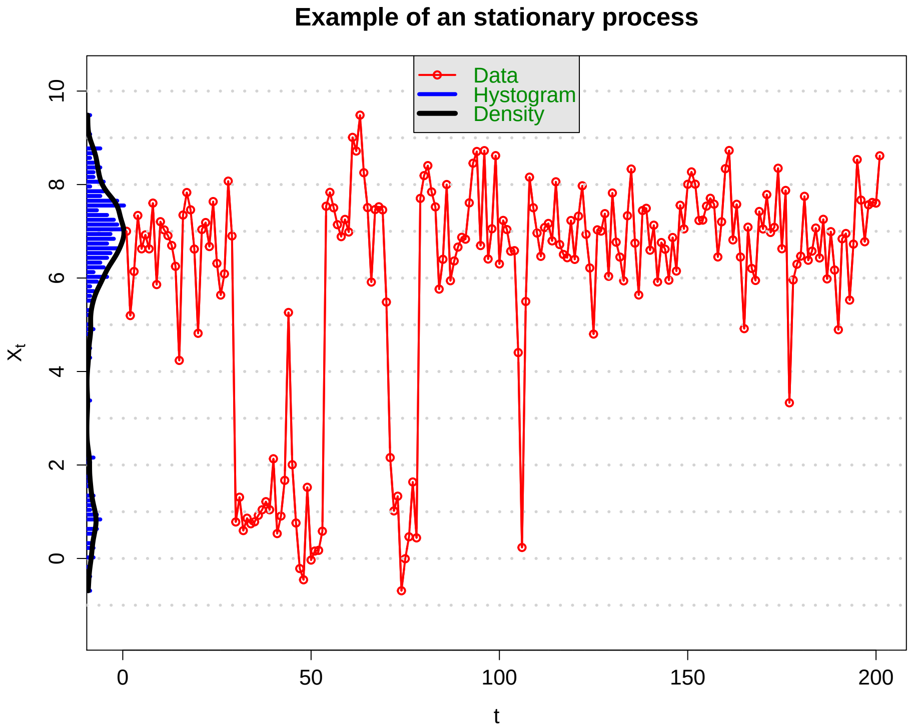



FIGURE 1.1: Example of a stationary process. The figure shows the trajectory of a stationary process, its histogram, and its density estimation. In this example, the stationary distribution was a mixture of three Gaussian distributions, yielding a distribution with multiple modes, and a stationary process with multiple regimes.

## 1.2 Exchangeability

As mention in Kallenberg [1973],

> "*Interchangeability is one of the most natural extensions of the concept of independence, being sufficiently general to provide a unifying framework for the theories of infinite divisibility, empirical distributions and sampling from finite populations, and yet sufficiently restrictive to admit an explicit treatment in terms of canonical representations.*"

To start our analysis of the exchangeable symmetry we define first the concept of *conditional independence*. The equivalence between conditional independence and exchangeability is provided by de Finetti's representation theorem (Theorem 3, below). The definition of conditional independence reflects the existence of a factor such that conditioning on it the variables of a process are independent, which is a common assumption in many models.

**Definition 2.** *Let $\mathscr{G}$ be a $\sigma$-algebra of events and $\{\mathscr{G}_i : i \in I\}$ a set of classes of events indexed by $I$. The* ***classes*** *$\{\mathscr{G}_i : i \in I\}$ are said to be* ***conditionally independent*** *given $\mathscr{G}$ if for every finite $K \subset I$, and all choices of $A_k \in \mathscr{G}_k$ with $k \in K$,*

$$\mathbb{P}\left[\bigcap_{k\in K} A_k \middle| \mathscr{G}\right] = \prod_{k\in K} \mathbb{P}\left[A_k|\mathscr{G}\right], \quad a.s.$$

*Accordingly, a* ***stochastic process*** *$\{X_i : i \in I\}$ is called* ***conditional independent*** *given $\mathscr{G}$ if the sequence of classes $\{\sigma(X_i) : i \in I\}$ is conditionally independent given $\mathscr{G}$.*

If $\mathscr{G} = \{\emptyset, \Omega\}$, then conditional independence given $\mathscr{G}$ coalesces to ordinary (unconditional) independence, while if $\mathscr{G} = \mathscr{F}$, where $\mathscr{F}$ denotes the $\sigma$-algebra of our fixed probability space $(\Omega, \mathscr{F}, \mathbb{P})$, then every sequence of classes of events is conditionally independent given $\mathscr{G}$.

Independent random variables may lose their independence under conditioning. For example, if $X_1$ and $X_2$ are independent with $\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = -1] = p \in (0,1)$ for $i = 1,2$, and we define $S = X_1 + X_2$, then $\mathbb{P}[X_i = 1|S] > 0$, $i = 1,2$, for $S = 0$ or $2$, whereas $\mathbb{P}[X_1 = 1, X_2 = 1|S_2] = 0$ when $S = 0$. On the other hand, dependent random variables may gain independence under conditioning, i.e., become conditionally independent. For instance, in a Markov chain the variables $\{X_1, X_2, \ldots, X_{n-1}\}$ and $\{X_{n+1}, X_{n+2}, \ldots\}$ are conditional independent given $X_n$, but they can be not independent.

To define, in mathematical terms, the concept of exchangeability we first state the following three definitions: *finite permutation*, *$\sigma$-algebra of permutable events* and *tail $\sigma$-algebra.*

**Definition 3.** *A mapping $\pi = \{\pi_i : i \geq 1\}$ from $\mathbb{N}_+$ onto itself is called a* ***finite permutation*** *if $\pi$ is one-to-one and $\pi_n = n$ for all but finite number of integers.*

**Definition 4.** *We will denote by $\mathscr{D}$ the set of all finite permutations $\pi$ and $\mathscr{B}^\infty$ be the class of Borel subsets of $\mathbb{R}^\infty$. For a stochastic process $X = \{X_i : i \geq i\}$ we define the* ***$\sigma$-algebra of permutable events*** *of $X$ by*

$$\mathscr{I} = \left\{X^{-1}(B) : B \in \mathscr{B}^\infty, \mathbb{P}\left[X^{-1}(B) \bigtriangleup (\pi X)^{-1}(B)\right] = 0,\ all\ \pi \in \mathscr{D}\right\},$$

*where $\pi X = \{\pi_i X_i : i \geq 1\}$ for $\pi$ a finite permutation, and $\bigtriangleup$ is the symmetric difference set operator.*

The proof that $\mathscr{I}$ is actually a $\sigma$-algebra follows by the fact that $\triangle$ preserves complementation and union.

**Definition 5.** *The* ***tail*** $\sigma$***-algebra*** *of a stochastic process* $X = \{X_n : n \geq 1\}$ *is given by*

$$\mathscr{T} = \bigcap_{n=1}^{\infty} \sigma\left(X_j : j \geq n\right).$$

*The sets of the tail* $\sigma$*-algebra are called* ***tail events*** *and functions measurable relative to the tail* $\sigma$*-algebra are named* ***tail functions****.*

The assumption of exchangeability is quotidian in many experimental schemes. However, in many cases its implications are not taken in to account. When considering samples of an experiment, it is common to treat the information obtained from every observation point regardless of its position within the sample. This idea implies a symmetric treatment of the observations. So, the next definition formalizes this concept in mathematical terms.

**Definition 6.** *A sequence of random variables* $X = \{X_n : n \geq 1\}$ *is said to be* ***exchangeable*** *if* $X \stackrel{d}{=} \pi X$ *for every* $\pi \in \mathscr{D}$*.*

One motivation for the definition of exchangeability is to express the symmetry of beliefs about the random quantities in the weakest possible way. It says that the labeling of the variables is irrelevant. There are many situations in which this assumption is reasonable; examples of its application can be consulted, for instance, in Koch and Spizzichino [1982] applied to survey sampling, actuarial techniques, operational research problems, among others.

In general, it is not possible to embed a given finite set of exchangeable random variable in an infinite set of exchangeable random variables, or even in a larger finite set. For example, if $\mathbb{P}[X_1 = 1, X_2 = 0] = \mathbb{P}[X_1 = 0, X_2 = 1] = 1/2$, one cannot even adjoin a third random variable and preserve exchangeability. This is important to take into account when establishing an exchangeable model.

The next three theorems have their motivation in de Finetti's representation theorem, and they establish equivalences to exchangeability. Proofs of this results can be consulted in Loève [1978, p. 30], Aldous [1985, p. 20], Schervish [1995, Ch. 1] and Chow and Teicher [1997, p. 232]. First, Theorem 1 relates exchangeability and conditional independence, and also indicates to which $\sigma$-algebras we can condition to obtain this equivalence.

**Theorem 1.** *The process* $\{X_n : n \geq 1\}$ *is exchangeable if and only if it is conditionally independent and identically distributed given some* $\sigma$*-algebra* $\mathscr{G}$ *of events. Moreover,* $\mathscr{G}$ *can be taken to be either the* $\sigma$*-algebra* $\mathscr{I}$ *of permutable events or the tail* $\sigma$*-algebra* $\mathscr{T}$*, and*

$$\mathbb{P}[X_1 < x|\mathscr{I}] = \mathbb{P}[X_1 < x|\mathscr{T}], \quad a.s.$$

The next result shows that the tail $\sigma$-algebra and the $\sigma$-algebra that makes the process conditional independent are almost the same.

**Theorem 2.** *Let the process* $\{X_n : n \in \mathbb{N}\}$ *be conditionally independent given a* $\sigma$*-algebra* $\mathscr{G}$ *of events. Then for any* $T \in \mathscr{T}$ *there exists some* $G \in \mathscr{G}$ *with*

$$\mathbb{P}[G \triangle T] = 0,$$

*where* $G \triangle T = (G \cap T^c) \cup (G^c \cap T)$ *is the symmetric difference between the sets* $G$ *and* $T$*.*

In order to establish the de Finetti's theorem we first define the concepts of *random probability measure* and *empirical distribution.*

**Definition 7.** *Given a measurable space* $(S, \mathscr{A})$*, a stochastic process* $Q = \{Q(A) : A \in \mathscr{A}\} = \{Q(A, \omega) : A \in \mathscr{A}, \omega \in \Omega\}$ *is called a* ***random probability measure*** *over* $(S, \mathscr{A})$ *if:*

*i) The function* $Q(\cdot, \omega)$ *is a probability measure in* $(S, \mathscr{A})$ *for every* $\omega \in \Omega$*.*

*ii) The function* $Q(A, \cdot)$ *is a random variable in* $(\Omega, \mathscr{F}, \mathbb{P})$ *for every* $A \in \mathscr{A}$*.*

**Example 1.** *Given a stochastic process* $\{X_n : n \geq 1\}$*, we define its* $n$***-empirical distribution process*** *by*

$$Q_n(B, \omega) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_B(X_i(\omega)), \tag{1.1}$$

*for any* $B \in \mathscr{F}$ *and* $\omega \in \Omega$*. It follows that the empirical distribution is in fact a random probability measure on* $(\Omega, \mathscr{F})$*, for each* $n \geq 1$*.*

Notice that random measures are random elements that define $\sigma$-algebras over the set $\mathscr{P}$ of probability measures. In this work, an important role is played by the **de Finetti's** $\sigma$**-algebra** $\mathscr{Q}$, which is generated by the sets

$$\mathbb{Q}(x, y) = \{F \in \mathscr{P} | Q(x) \leq y\}, \qquad x, y \in \mathbb{R}.$$

Accordingly, it can be shown that the sets of the form

$$\left\{ Q \in \mathscr{P} : \int_{\mathbb{R}} x dQ(x) \leq y \right\}, \qquad y \in \mathbb{R}$$

are $(\mathscr{P}, \mathscr{Q})$ measurable, see Blum et al. [1958]. In general, if $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then the sets

$$\left\{ Q \in \mathscr{P} : \int_{\mathbb{R}} f(x) dQ(x) \leq y \right\}, \qquad y \in \mathbb{R},$$

are $(\mathscr{P}, \mathscr{Q})$ measurable,see Grandell [1977]. These types of measurable sets will be used in Chapters 3 and 4 to prove properties of our exchangeable models.

Now, we will state a version of the de Finetti's theorem. This result establishes that exchangeability is equivalent to the existent of a random probability measure $Q$, such that conditioning to it the variables are independent with $Q$ as they probability measure. This theorem also shows that $Q$ is the limit of the empirical distributions.

**Theorem 3** (Bruno de Finetti's representation theorem)**.** *The stochastic process $\{X_n : n \geq 1\}$ is exchangeable if and only if there is a random probability measure $Q$ over $(\Omega, \mathscr{F})$, such that conditional on $Q$, $\{X_n : n \geq 1\}$ are independent identically distributed with measure $Q$. Moreover, this implies the existence of a measure $\mu$ over $(\mathscr{P}, \mathscr{Q})$, called the* ***de Finetti's measure****, such that*

$$\mathbb{P}\left[X_1 \leq x_1, \ldots, X_n \leq x_n\right] = \int_{\mathscr{P}} \prod_{i=1}^{n} Q(x_i) \mu(dQ), \qquad (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

*for every $n \geq 1$. Also, the distribution of $Q$ is unique, and the $n$-empirical distributions (1.1) converges to $Q(B)$ almost surely for each $B \in \mathscr{F}$.*

An important point to underline here is that, is that de Finetti's theorem operates over infinite sequences, and cannot be applied to finite exchangeable sequences. To see this last point, let $\{X_1, \ldots, X_n\}$ be exchangeable, with finite variance $\sigma^2$ and correlation $\rho$, then

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \sigma^2 + \sum_{i \neq j} \rho\sigma = n\sigma\left[1 + (n-1)\rho\right].$$

Since this is a positive quantity, we have

$$\rho \leq -\frac{1}{n-1}. \tag{1.2}$$

Thus the members of an infinite exchangeable sequence are always positively correlated. Consequently, if the correlation between variables is negative then they can never be conditional independent and at the same time identically distributed.

Also is important to notice that the marginal distribution of an exchangeable process $\{X_n : n \geq 1\}$ is equal to

$$\begin{aligned} F_{X_i}(x) &= \mathbb{P}\left[X_i \leq x\right] \\ &= \mathbb{E}\left[\mathbb{P}\left[X_i \leq x | Q\right]\right] \\ &= \mathbb{E}\left[Q(x)\right], \end{aligned} \tag{1.3}$$

for any $x \in \mathbb{R}$. In words, this last equation says that the marginal stationary distribution is the mean of the random distribution $Q$.

The next couple of examples give constructions of exchangeable variables with different correlation coefficients.

**Example 2.** *Let us show that any correlation in* $[-1/(n-1), 1]$ *can be attained by a set of n exchangeable random variables. Assuming that* $\{Z_i : 1 \leq i \leq n\}$ *are n independent identically distributed random variables with* $\mathbb{E}[Z_i] = 0$ *and* $\mathbb{E}\left[Z_i^2\right] = 1$*, define*

$$X_i = Z_i + c \sum_{j=1}^{n} Z_j$$

*with* $c \in \mathbb{R}$*. This implies that their joint characteristic function is*

$$\begin{aligned} \varphi_{X_1,\ldots,X_n}(t_1,\ldots,t_n) &= \mathbb{E}\left[\exp\left\{i\left[t_1 X_1 + \ldots + t_n X_n\right]\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{i\left[\left(t_1 + c\sum_{j=1}^{n} t_j\right) Z_1 + \ldots + \left(t_n + c\sum_{j=1}^{n} t_j\right) Z_n\right]\right\}\right] \\ &= \varphi\left(t_1 + c\sum_{j=1}^{n} t_j\right) + \cdots + \varphi\left(t_n + c\sum_{j=1}^{n} t_j\right), \end{aligned}$$

*which is a symmetric function with respect to* $(t_1, \ldots, t_n)$*, where* $\varphi$ *is the common characteristic function of each of the variables* $Z_1, \ldots, Z_n$*. Accordingly,* $\{X_i : 1 \leq i \leq n\}$ *are exchangeable. An easy computation shows that the correlations of* $\{X_i : 1 \leq i \leq n\}$ *are given by*

$$\mathsf{Corr}(X_i, X_j) = 1 - \frac{1}{nc^2 + 2c + 1}.$$

*For different values of* $c$*, this correlation take values in* $[-1/(n-1), 1)$*. Finally, to obtain an exchangeable process with* $\text{Corr}(X_i, X_j) = 1$ *we define the process* $\{X_i : 1 \leq i \leq n\}$*, where all its variables are equal to* $Z_1$*, i.e.,* $X_1 = \ldots = X_n = Z_1$*.*

**Example 3.** *In this example we will construct an infinite exchangeable sequence with correlation value in* $[0, 1]$*. In order to establish our construction we assume that we have a sequence* $\{Z_i : i \geq 0\}$ *of i.i.d. random variables. Thus, we define*

$$X_i = cZ_0 + Z_i, \qquad i \geq 1,$$

*for some* $c \in \mathbb{R}$*. Then,* $\{X_i : i \geq 0\}$ *are exchangeable, because condition to* $Z_0$ *are independent and identically distributed. Notice that* $\text{Corr}(X_i, X_j) = c^2/(c^2+1)$*, which takes values on* $[0, 1)$*, when* $c$ *varies. For a correlation equal to one, we take* $X_i = Z_0$*, for every* $i \geq 1$*.*

One difficulty in the last example is that, in general, we have a lack of analytic expressions for the marginal distributions. To overcome this problem we will use a Bayesian parametric construction in this work. To construct exchangeable sequences we will first fix a parametric family of distributions and then establish a prior distribution over its parameters (see Sections 3.7 to 3.10). Indeed, many of our analysis use tools from the Bayesian statistical theory.

The concept of exchangeability was fist introduced by Bruno de Finetti. In de Finetti [1931] he characterizes mixtures of process with discrete marginal distributions, with only two possible states. He subsequently, de Finetti [1937], generalized this result to any type of mixtures of sequences of independent, identically distributed random variables. Following these ideas, Freedman [1962] shows necessary and sufficient conditions for those stochastic process that can be represented as mixtures of various special families of discrete time processes. Using the theory of extreme points of convex sets, a generalization of this is then presented in Freedman [1963], to the case of continuous time process. We recommend to the reader to consult Kingman [1978], for examples of how to model phenomena with exchangeable processes. Some version of extensions of the central limit theorem within the exchangeable framework can be found in Blum et al. [1958], Chernoff and Teicher [1958] and Klass and Teicher [1987]. There exists a vast literature concerning with analytical properties of exchangeable sequences, for further references see Hewitt and Savage [1955], Kendall [1967], Kallenberg [1973, 1974, 1975], Diaconis and Freedman [1980a,b] and Huang [1990].

## 1.3 Stationarity

There are several real random phenomena sharing certain type of stationary behavior, whose distributions are, to some extend, non affected by shifts in their indexing, typically identified as time or space. An example of this behavior could be consulted in Barndorff-Nielsen et al. [1998], where they study the behavior of a financial time series and show how well can be fitted by an stationary model. Let us establish this concept mathematically.

**Definition 8.** *A stochastic process $X = \{X_t : t \in T\}$, where $T = \mathbb{R}$ or $T = \mathbb{N}$, will be called stationary if all its finite-dimensional distributions remain the same under translations in time, i.e., if*

$$(X_{t_1}, \dots, X_{t_n}) \stackrel{d}{=} (X_{t_1+\tau}, \dots, X_{t_n+\tau}), \quad \forall\, t_i, t_i + \tau \in T.$$

This assumption helps to obtain mathematical tractable models; for instance, a simplification in the estimation procedures could appear because the number of parameters involved are reduced [Ferrari and Wyner, 2003]. In particular, this assumption is fruitful to model multivalued phenomenon where over-parametrization can be specially problematic. Applications of stationary multivariate processes have been studied by Rue and Held [2005] on epidemiology, Kindermann and Snell [1980] on social sciences, Xu [2011] on finance, Winkler [2003] on image analysis, and Vanmarcke [2010] on geology. In general, they are also used in special applications like detection theory, signal processing, spatial statistics, and reliability.

For a recent account of the classification of stationary processes we recommend the reader to look at the book of Kallenberg [2005]. Also important to mention are the studies by Cramér and Leadbetter [2004] and Lindgren [2013], which show many theoretical implications and the wide spectrum o application of these type of processes.

In this work we use different constructions of stationary processes, let us now give a brief account of some approaches relevant for our further developments. In particular, to those constructions of Markovian reversible stationary models.

The problem of constructing stationary models with prescribed marginal distributions has recently received considerable attention. Within the discrete-time literature, two works that serve as cornerstone are Lawrance and Lewis [1977] and Jacobs and Lewis [1977]; both of these contributions focus in constructions with exponential marginal distributions. Many models then follow their ideas, in particular, with marginal distributions such as gamma, Poisson and negative binomial; examples of it could be consulted in Gaver and Lewis [1980], Lawrance [1982], McKenzie [1986, 1988], Lewis et al. [1989], McCormick and Park [1992] and Al-Osh and Aly [1992]. A review of non-Gaussian first order linear autoregressive models can be consulted in Grunwald et al. [1995]. Nowadays, new constructions of stationary processes continues to appear, in particular for the multivariate framework.

In Joe [1996] a construction of ARMA models for convolution-closed families of infinitely divisible distributions is presented. An extension of this work then appear in Jørgensen and Song [1998], for AR(1) time series, using the method of *thinning*. To construct these types of series they establish the relation

$$X_t = A_t(X_{t-1};\alpha) + \epsilon_t, \qquad t = 1,2,\ldots,$$

where $A_t$ is the thinning operator, and the two terms of the right hand side are independent of each other. This generalizes the notion of binomial thinning of a Poisson variable treated by McKenzie [1988], and the beta thinning of the gamma distribution used by Lewis et al. [1989].

Based on this construction, more general frameworks are proposed in Pitt et al. [2002]. They explore the unidimensional discrete time process which fulfils the lineal equation

$$\mathbb{E}[X_n|Y_{n-1}] = \rho Y_{n-1} + (1-\rho)\mu,$$

where $Y$ is an auxiliary process. A previous application of a similar idea is presented in Barndorff-Nielsen [1997] who generates a Markov chain $\{X_n : n \geq 1\}$ via a sequence of unknown latent states $\{Y_n : n \geq 1\}$, related by certain conditional densities $f(y_n|x_n)$ and $f(x_n|y_{n-1})$. They use the Gaussian and the inverse Gaussian distributions to define the transition densities. A lack of a joint density contributes to the absence of analytic expressions for the marginal and stationary distribution of $\{X_n : n \geq 1\}$.

In this work we use frequently the idea of an auxiliary process to construct exchangeable and stationary process. This idea helps to gain control and to understand accurately the interactions among variables because it factorizes the entire distribution of the process as products of conditional distributions. The versatility of this idea could be applied to the construction of discrete and continuous time multivariate processes (see Sections 1.4 to 1.6).

The number of contributions within the multivariate framework has, however, received less attention in the literature. This, is perhaps due to the fact of its multivariate character complexity their joint marginal density and their transition density. Nevertheless, the stationary multivariate processes theory has many important applications. For instance, some appealing references are found in: Greenwood and Williamson [1966] and Nayak [1973] for metallic surfaces; Wong and Tsui [1977] for electrical engineering; Matérn [1960] for forestry problems; and in Panda [1977], Panda and Dubitzki [1979], and Bruce and Schachter [1980] for image analysis.

Due to the above mention complexity one can resolve to a particular family of distributions. Accordingly, for the sake of clarity let us continue our discussion using density functions.

One appealing construction of multivariate process is presented in Fox and West [2011]. They construct a process with stationary Wishart distribution by using the property of being closed under conditionality. Due to the above mention complexity one can resolve to a particular family of distributions. To define the density $f(x_{t-1}, x_t)$ they use an inverse Wishart distribution with a random matrix $\phi_t$. This matrix $\phi_t$ must accomplish the relation:

$$\begin{pmatrix} X_{t-1} & \phi'_t \\ \phi_t & X_t \end{pmatrix} \sim \mathrm{IW}_{2q}\left(n+2, n\begin{pmatrix} S & SF' \\ FS & S \end{pmatrix}\right). \tag{1.4}$$

They show which are the restrictions for $F$ and $S$ to make the process $X$ stationary and reversible.

In Mena and Walker [2009] the construction studied by Pitt et al. [2002], of univariate processes, is extended to the continuous time case. In the remaining sections of this chapter we seek to generalize this idea to some multivariated-value processes. The extension that we will present has the advantage of an easy implementation and interpretation. It is worth to emphasize that this construction works for continuous and discrete time processes.

Let us present our multivariate reversible stationary process construction. The steps of this construction start with the desired (fixed) stationary multivariate density $f(x_t)$. Then, we introduce a latent process $Y = \{Y_t : t \in \mathbb{R}_+\}$, through the conditional multivariate density $f(y_t|x_t)$, in such a way that the domain of this density, as a function of $x_t$, coincides with the support of $f(x_t)$. With the knowledge of $f(x_t)$ and $f(y_t|x_t)$, an application of Bayes theorem leads to a form for $f(x_t|y_t)$, using

$$f(x_t|y_t) \propto f(y_t|x_t)f(x_t).$$

Moreover, it is important to mention that both of the processes $X$ and $Y$ are reversible and also satisfy the independence structure:

$$\mathbb{P}\left[Y_{t+s}|\mathscr{Y}^{(t)},\mathscr{X}^{(t)}\right] = \mathbb{P}\left[Y_{t+s}|X_t\right],$$
$$\mathbb{P}\left[X_{t+s}|\mathscr{Y}^{(t+s)},\mathscr{X}^{(t)}\right] = \mathbb{P}\left[X_{t+s}|Y_{t+s}\right],$$

where $\mathscr{Y}^{(t)} = \{Y_s : 0 \leq s \leq t\}$ and $\mathscr{X}^{(t)} = \{X_s : 0 \leq s \leq t\}$ [see Liu et al., 1994]. The transition density driving a Markovian process with stationary distribution $f(x_t)$ is then constructed by

$$f(x_{t+s}|x_s) = \int f(x_{t+s}|y_s)f(y_s|x_s)dy_s. \tag{1.5}$$

This resemblance precise the type of dependence induced by Gibbs sampler Markov chains, but in continuous time. Clearly, such a process is reversible with stationary distribution $f(x_t)$.

In the next sections we will exhibit three different stationary models that we worked during our research. Other models were developed under the research, but the chosen models exemplify better the ideas of the construction. Each of these sections is devoted to a different multivariate model. The first model can be used to model processes with light tails, the second to model the evolution of a time indexed covariance matrix, and the third model could be used for heavy tailed processes.

## 1.4 Gaussian-Gaussian model

It this section we present a construction of a continuous time process $X = \{X_t : t \in \mathbb{R}_+\}$, such that for every time its invariant distribution is Gaussian given by

$$X_t \sim \mathrm{N}(\mu, \Sigma), \qquad t \in \mathbb{R}_+,$$

where $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$ are fixed. We will induce the dependence in the model using the conditional distribution

$$Y_t|X_0 \sim \mathrm{N}(X_0, \phi_t), \qquad t \in \mathbb{R}_+, \tag{1.6}$$

where $\phi_t$ is a positive defined matrix.

An instance of the potential applications of Gaussian multivariate processes can be consulted in Rue and Held [2005], and references therein.

In the following lines we will find conditions over $\phi_t$ needed to obtain a reversible stationary process. To this end, let us first use Bayes theorem to obtain

$$X_t|Y_t \sim \mathrm{N}\left(C_t[\phi_t^{-1}Y_t + \Sigma^{-1}\mu], C_t\right), \qquad t \in \mathbb{R}_+, \tag{1.7}$$

where $C_t = [\phi_t^{-1} + \Sigma^{-1}]^{-1}$. From equations (1.5) to (1.7) we obtain that the transition density of the process $X$, which is given by

$$X_t|X_0 \sim \mathrm{N}\left(C_t[\phi_t^{-1}X_0 + \Sigma^{-1}\mu], C_t + C_t\phi_t^{-1}C_t^T\right). \tag{1.8}$$

To achieve the restrictions over $\phi_t$ such that Markovianity is preserved, we must find the conditions such that the Chapman-Kolmogorov equations are satisfy. The moment generation function of $X_t$ given $X_0 = x_0$ is

$$\mathscr{L}_{X_t|X_0=x_0}(\lambda) = \exp\left\{\left[x_0^T\phi_t^{-1} + \mu^T\Sigma^{-1}\right]C_t\lambda + \frac{1}{2}\lambda^T\left[C_t + C_t\phi_t^{-1}C_t\right]\lambda\right\},$$

and the Chapman-Kolmogorov equations applied to generator functions are then equivalent to

$$\mathbb{E}\left[\mathscr{L}_{X_{t+s}|X_s}(\lambda)|X_0\right] = \mathscr{L}_{X_{t+s}|X_0}(\lambda), \qquad t,s \in \mathbb{R}_+. \tag{1.9}$$

First notice that

$$\begin{aligned}
&\mathbb{E}\left[\mathscr{L}_{X_{t+s}|X_s}(\lambda)|X_0 = x_0\right] \\
&= \exp\left\{\mu^T\Sigma^{-1}C_t\lambda + \frac{1}{2}\lambda^T\left[C_t + C_t\phi_t^{-1}C_t\right]\lambda\right\} \times \mathscr{L}_{X_t|X_0=x_0}\left(\phi_t^{-1}C_t\lambda\right) \\
&= \exp\left\{\mu^T\Sigma^{-1}C_t\lambda + \frac{1}{2}\lambda^T\left[C_t + C_t\phi_t^{-1}C_t\right]\lambda\right\}
\end{aligned}$$

$$\times \exp\left\{[x_0^T\phi_t^{-1}+\mu^T\Sigma^{-1}]\,C_t\phi_t^{-1}C_t\lambda+\frac{1}{2}\lambda^T C_t\phi_t^{-1}\,[C_t+C_t\phi_t^{-1}C_t]\,\phi_t^{-1}C_t\lambda\right\}.$$

From such expression and equation (1.9) we have that

$$\phi_{s+t}\Sigma^{-1}+1=\left(\phi_t\Sigma^{-1}+1\right)\left(\phi_s\Sigma^{-1}+1\right). \tag{1.10}$$

This implies that $\phi_t$ is given by

$$\phi_t=\Sigma\left(e^{\alpha t}-1\right), \tag{1.11}$$

for some $\alpha$ matrix of dimensions $d\times d$. Substituting the right side of equation (1.11) in to equation (1.8), we obtain

$$X_t|X_0\sim \mathrm{N}\left(e^{-\Sigma\alpha\Sigma^{-1}t}X_0+\left(1-e^{-\Sigma\alpha\Sigma^{-1}t}\right)\mu,\quad \Sigma\left(1-e^{-2\alpha t}\right)\right). \tag{1.12}$$

The constrains over $\alpha$ are given by the restriction of the positive definiteness of the covariance matrix of the densities in equations (1.6) and (1.12). This implies that $\Sigma\left(e^{\alpha t}-1\right)$ and $\Sigma(1-e^{-2\alpha t})$ are positive definite matrices for every $t$.

It is important to emphasize that the Gaussian-Gaussian model is in fact a diffusion process [Stroock and Varadhan, 2006]. Let us find its diffusion coefficients; to obtain them we use the limit formulas of Kloeden and Platen [1992, pp. 68]

$$\begin{aligned} a(x) &= \lim_{t\downarrow 0}\frac{\mathbb{E}\,[X_t|X_0=x]-x}{t} \\ &= -\Sigma\alpha\Sigma^{-1}(x-\mu), \end{aligned}$$

and

$$\begin{aligned} B(s,x)B(s,x)^T &= \lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}\left[(X_t-x)(X_t-x)^T|X_0=x\right]-x\right) \\ &= \lim_{t\downarrow 0}\frac{1}{t}\left(\left(1-e^{-2t\alpha}\right)\Sigma+\left[\left(e^{-t\alpha}-1\right)(x-\mu)\right]\left[\left(e^{-t\alpha}-1\right)(x-\mu)\right]^T\right) \\ &= 2\alpha\Sigma. \end{aligned}$$

Then the stationary process $X$ coincides with that being a solution of the multivariate stochastic differential equation

$$dX_t = -\Sigma\alpha\Sigma^{-1}(x-\mu)dt + \sqrt{2\alpha\Sigma}dW_t, \tag{1.13}$$

which corresponds to the Ornstein-Uhlenbeck multivariate diffusion process. In general, this process is known as the solution of a more general multivariate stochastic differential equation. Nevertheless, we have obtain, the particular case, in equation (1.13) when it is reversible. For a recent account of the theory of diffusions we refer the reader to Rogers and Williams [2000a,b]. Indeed, in the univariate case it is granted the reversibility property, while in the multivariate framework it is not. This example shows the advantage of our construction to obtain the conditions for this property.

## 1.5 Wishart-Wishart model

The Wishart distribution is used, in particular, to model the distribution of the covariance matrix of different phenomena, this clearly happens since its support is the set of positive definite matrices. This distribution originally emerged from the study of the maximum likelihood estimator of the covariance matrix of multivariate Gaussian variables [see Wishart, 1928]. Examples underlying modern uses of covariance matrix-value stochastic processes can be found in Leung et al. [2013], where a currency option pricing model is developed using the Wishart distribution. It is also important to mention the work by Bru [1991] where the Wishart process is defined as the strong solution of a stochastic differential equation, which is a generalization the idea of the Bessel process. Other examples in the same line can be consulted in Fox and West [2011], Gourieroux et al. [2009] and Philipov and Glickman [2006a,b].

The model of this section has the Wishart distribution as its stationary distribution. To be more precise, we fix the stationary distribution to be

$$X_t \sim \mathrm{W}_p(\nu, M), \qquad t, s \in \mathbb{R}_+,$$

where $\nu$ is its freedom degrees, and $M$ its a scale matrix of dimension $p \times p$, which is symmetric and positive definite. Once again, mimicking the construction in the previous section, the relation between the stationary process $X$ and the auxiliary process

$Y$ is assumed to be given by

$$Y_t|X_0 \sim \mathrm{W}_p\left(a, \left(\phi_t X_0 \phi_t\right)^{-1}\right),$$

where $a \geq p$, and $\phi_t$ is a matrix of dimensions $p \times p$ of complete rank. Following the construction in Mena and Walker [2009], these hypothesis imply that

$$X_t|Y_t \sim \mathrm{W}_p\left(\nu + a, \left[\phi_t Y_t \phi_t + M^{-1}\right]^{-1}\right),$$

and the transition density is given by

$$\begin{aligned} f\left(x_t|x_0\right) &= \int f\left(x_t|y_t\right) f\left(y_t|x_0\right) dy_t \\ &= \frac{\left|\phi_t\right|^a |x_t|^{(2a+\nu-p-1)/2} \operatorname{etr}\left(-M^{-1}x_t/2\right)}{2^{(2a+\nu)p/2}\Gamma_p\left(\frac{a}{2}\right)\Gamma_p\left(\frac{a+\nu}{2}\right)} \\ &\quad \times \int \operatorname{etr}\left(-\phi_t(x_t+x_0)\phi_t y_t/2\right)\left|M^{-1}+y_t\phi_t\phi_t\right|^{(a+\nu)/2}\left|y_t\right|^{(a-p-1)/2} dy_t, \end{aligned} \tag{1.14}$$

where $\operatorname{etr}(\cdot)$ stands for the exponential trace function, i.e. $\operatorname{etr}(A) = \exp\{\operatorname{tr}(A)\}$. For this model we have not found yet a simplified expression for the transition density. Part of our future work focuses in develop analytic results for this model using Gibbs' sampling techniques.

## 1.6 Gaussian-Wishart model

The t-multivariate distribution is frequently used to model heavy tailed observations. In many cases, it is applied to data which originally were modeled by Gaussian distributions, but the demand of heavier tails makes it necessary. Examples of its applicability can be consulted in Nason [2001] to define an index over the cluster accuracy, and in Galimberti and Soffritti [2014] where it explores a linear model where the error terms follow a finite mixture of t-multivariate distributions. For a deep analysis of the applications of the t-multivariate distribution we refer the reader to Kotz and Nadarajah [2004].

To construct the model of this section we will use the two previously presented models. For each time $t \in \mathbb{N}$ the process has a conditional distribution $X_t|W_t \sim \mathrm{N}\left(\mu, W_t^{-1}\right)$, where $\mu$ is a fixed vector and the process $W$ has stationary distribution $W_t \sim \mathrm{W}(\nu, M)$ for each $t \in \mathbb{N}$. To obtain the conditional distribution $\mathrm{N}\left(\mu, W_t^{-1}\right)$ we use an auxiliary process $Y$ so that

$$Y_t | X_0, W_0 \sim \mathrm{N}\left(X_0, \phi_t\right), \qquad t \in \mathbb{N}, \tag{1.15}$$

where $\phi_t$ is, for now, a positive definite matrix.

Using that

$$f(x_t | y_t, w_t) \propto f(y_t | x_t, w_t) f(x_t | w_t),$$

we obtain

$$X_t | Y_t, W_t \sim \mathrm{N}\left(C_t \left[\phi_t^{-1} Y_t + W_t \mu\right], C_t\right), \tag{1.16}$$

where $C_t = \left[\phi_t^{-1} + W_t\right]^{-1}$. Applying equations (1.5), (1.15) and (1.16) we obtain the transition conditional distributions of the process $X$ given $W$ are given by

$$X_t | X_0, W_0 \sim \mathrm{N}\left(C_t \left[\phi_t^{-1} X_0 + W_0 \mu\right], C_t + C_t \phi_t^{-1} C_t^T\right). \tag{1.17}$$

A useful characteristic of this model is the control over its general and local variability. We have control over the evolution of the general covariance between the points using the parameters of the process $W$; this happens as a consequence of the role of $W_t$ in equation (1.17). It is important to notice that this gives us control over the tails of the process, making them heavier or lighter. On the other hand, we have control over the local variability of the process; this is granted by changing the values of the parameter $\phi_t$. This parameter controls the variability of the transitions of successive states of the process (see equation (1.17)); for instance, a change of $\phi_t$ could accelerate the velocity of convergence to the stationary distribution.

The following result shows that the process $X$ has an stationary multivariate t distribution.

**Theorem 4.** *The Gaussian-Wishart model has stationary distribution multivariate t, given by*

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right) |M|^{1/2}}{\pi^{-n/2} \Gamma\left(\frac{\nu}{2}\right) \left|1 + M\left(x-\mu\right)\left(x-\mu\right)^T\right|^{-\frac{\nu+1}{2}}}.$$

*Proof.*

$$f(x) = \int f\left(x | w^{-1}\right) f\left(w^{-1}\right) dy$$

$$
\begin{aligned}
&= \int (2\pi)^{-n/2} |w|^{-1/2} \exp\left\{-\left(x-\mu\right)^T w^{-1} \left(x-\mu\right)/2\right\} \\
&\quad \times 2^{-\nu n/2} \Gamma_n^{-1}\left(\frac{\nu}{2}\right) |M|^{-\nu/2} |w|^{-(\nu+p+1)/2} \operatorname{etr}\left\{-M^{-1} w^{-1}/2\right\} dw \\
&= \pi^{-n/2} 2^{-(\nu+1)n/2} \Gamma_n^{-1}\left(\frac{\nu}{2}\right) |M|^{-\nu/2} \\
&\quad \times \int |w|^{-(\nu+1+p+1)/2} \operatorname{etr}\left\{-\left[\left(x-\mu\right)\left(x-\mu\right)^T + M^{-1}\right] w^{-1}/2\right\} dw \\
&= \pi^{-n/2} 2^{-(\nu+1)n/2} \Gamma_n^{-1}\left(\frac{\nu}{2}\right) |M|^{-\nu/2} 2^{(\nu+1)n/2} \Gamma_n^{-1}\left(\frac{\nu+1}{2}\right) \left|\left(x-\mu\right)\left(x-\mu\right)^T + M^{-1}\right|^{-\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right) |M|^{1/2}}{\pi^{-n/2} \Gamma\left(\frac{\nu}{2}\right) \left|1 + M\left(x-\mu\right)\left(x-\mu\right)^T\right|^{-\frac{\nu+1}{2}}}.
\end{aligned}
$$

In the last equality we use that

$$
\Gamma_n(a) = \pi^{(n-1)/2} \Gamma(a) \Gamma_{n-1}\left(a - \frac{1}{2}\right),
$$

and the proof is complete. □

As a consequence of the last result the Gaussian-Wishart model could be use to emulate the behavior of a heavy tailed stationary phenomena. The use of t-multivariate distributions enjoys renewed interest due to applications in mathematical finance, especially through the use of the Student t copula, see Demarta and McNeil [2005] and Chan and Li [2008].

# Chapter 2

# Key results in risk theory

In order to analyze a reserve model with a dependent structure on claim amounts and extend many of the classical results under this framework, we first establish in this chapter the basic background on risk theory that we will use in further chapters.

Among other things, an insurance contract is an instrument against a fortuitous turn of events established between the insured and the insurance company. The insured is the object of some random risks and decides to buy a contract from the insurance company to soften the possible consequences of future events. In order to price adequately this type of contracts, the insurance company must take into account all the important features of the risk and the available knowledge from similar past events. In addition, an arbitration commission must be present to supervise the fair play between the two parts.

Although there are many important objects in an insurance company to be modeled mathematically, we will restrict our attention to its reserve process. The monetary reserve of the company is intrinsically linked with its capability to pay its obligations and avoid bankruptcy. Focusing on the size of the reserve is also important, because insurance laws and regulations could request some threshold to allow the company to operate [see Wahl and Rose, 2011].

In this chapter, we will give a brief and basic analysis of the risk theory framework that will be used on the rest of the work; its layout is as follows. In Section 2.1, we will present the Cramér-Lundberg model and show an interpretation of its assumptions and implications. Section 2.2 defines the problem of ruin, and Section 2.3 gives an analysis of the net profit condition in the classical framework. In Sections 2.4 to 2.6

we will show some classical results of the ruin probabilities; e.g. integro-differential expressions, bounds and equivalent formulations. At the end, in Section 2.8, we review some studies that relax the classical assumptions. All the results shown in this chapter will be applied in Chapter 3 as a benchmark to contrast the behavior of a dependent claim scenario.

## 2.1 Introduction to the classical theory: the Cramér-Lundberg model

One could think about the behavior of the reserve of an insurance company through time as:

$$\text{Reserve at time } t = \text{Initial capital} + \text{Incomes at time } t - \text{Outcomes at time } t.$$

This abstraction, which is not trivial at all, allows us to separately model the different elements that constitute the reserve process. Let us have some words about the general properties of each of these three components:

1. The objective of the initial capital is to provide a buffer which protects the interests of policyholders. This buffer should be sufficiently large to allow to take management actions and regulator actions against the impact of adverse events.

2. Incomes at time $t$ are mainly constituted by the premiums charged to the insured in the time interval $(0, t]$. These premium amounts must be calibrated by taking into account the magnitude and frequency of the possible claims and the market behavior (see Sundt [1991] and Daykin et al. [1994]). We will make the usual assumption that incomes follow a deterministic linear function. This hypothesis is supported on the fact that the insurance companies we model are big enough that premiums arrive almost continuously and they are non-affected by the number of insured quitting their contracts. There is a vast amount of literature centered on how to calculate the constant premium rate [see for instance Goovaerts et al., 1984], but we will not elaborate on this point here.

3. Outcomes at time $t$ are mainly constituted by the claims charged to the insurance company in the time interval $(0,t]$, and modeled by a random process $S = \{S_t : t \geq 0\}$. Many interesting analyses from different viewpoints had been devoted to the study of the intricate and complex behavior of this process. In particular, one natural and usual way to shape this process, that we will use in this work, is given by establishing a counting process $N = \{N_t : t \geq 0\}$ modeling claim times and a stochastic process $Y = \{Y_i : i \in \mathbb{N}\}$ to model claim amounts, independent of each other. In this way, the times at which the claims occur and their severities, are modeled separately. Other approaches do not disaggregate the information in this way, e.g. when the reserve is modeled by some Lévy process. One advantage of this type of analysis is a better understanding of the contribution of each process.

Let us first analyze the claim time process $N$. In Chapter 4 we will generalize these ideas by presenting results of a more general process to model claim times. Therefore, the next definition, that follows the notation of Ross [1996], describes what we mean when referring to counting processes.

**Definition 9.** *A stochastic process $N = \{N_t : t \geq 0\}$ is said to be a* ***counting process*** *if $N_t$ represents the total number of "events" that have occurred up to time $t$. That is, a counting process $N$ must satisfy:*

1. $N_t \geq 0$.
2. *$N$ is integer valued.*
3. *$N_s \leq N_t$, for $s < t$.*
4. *$N_t - N_s$ equals the number of events that have occurred in the interval $(s,t]$, for $s < t$.*

*In particular, a counting process is said to possess independent increments if the numbers of events occurring in disjoint time intervals are independent, and it is said to possess stationary increments if the distribution of the number of events that occur in any time interval depends on it only through its length.*

By far, the most tractable and studied counting process is the *Poisson process*, defined next.

**Definition 10.** *The counting process $N$ is said to be a (homogeneous)* ***Poisson process*** *having rate $\lambda$, $\lambda > 0$, if:*

1. $N_0 = 0$.
2. *$N$ has independent increments.*
3. *The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$. That is, for all $s, t \geq 0$,*

$$\mathbb{P}[N_{t+s} - N_s = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots.$$

The Poisson process is a stochastic counting process that appears in many phenomenon in which there is a large population of individuals who, more or less independently of each other, have a small probability of contributing to the count in the next small time interval. For a thorough treatment of the Poisson process, we refer the reader to Kingman [1993] and Kingman [2006].

*Remark* 2.1. It is important to mention that the study of the homogeneous Poisson process also unveil characteristics of the inhomogeneous process, i.e. when the rate parameter $\lambda$ is a function of time. This interaction between the these two processes happens because because any inhomogeneous process can be transformed into a homogeneous one (and vice versa) by using a time change [for example see Mikosch, 2009, Section 2.1.3]. Similarly, it is worth saying that the Palm–Khintchine theorem states that the superposition of independent equilibrium renewal processes behaves asymptotically like a Poisson process [see Heyman and Sobel, 2004, p. 157]. These two properties contribute to the wildly accepted use of the Poisson process in a variety of models.

Nevertheless, a limitation of the Poisson process is that the jumps are always of unit size. The next definition gives a stochastic process with random size of jumps.

**Definition 11.** *Let $Y = \{Y_i : i \in \mathbb{N}\}$ be a sequence of independent and identically distributed $\mathbb{R}_+$-valued random variables, having distribution function $F$, and suppose that this sequence is independent of $N = \{N_t : t \geq 0\}$, a Poisson random variable with mean $\lambda$. Then, the random process*

$$S_t = \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0,$$

*is said to be a* ***compound Poisson process****, with Poisson parameter* $\lambda$ *and component distribution* $F$.

As is commonly known, the Poisson compound process inherit many properties of the Poisson process [see Bening and Korolev, 2002]. Filip Lundberg uses this process to postulate a continuous time risk model where the aggregate claims in any interval have a compound Poisson distribution, which is presented in the next definition.

**Definition 12.** *The* ***Cramér-Lundberg (CL) reserve model*** $R^u = \{R_t^u : t \geq 0\}$, *is given by*

$$R_t^u = u + pt - \sum_{i=1}^{N_t} Y_i, \qquad t \geq 0, \tag{2.1}$$

*with the assumptions:*

- $u$ *is the initial reserve at time* 0, *and* $p$ *is the constant premium rate.*
- $N$ *is a Poisson process of rate* $\lambda$.
- $Y$ *is a sequence of independent and identically* $F$ *distributed variables with values in* $\mathbb{R}_+$, *also independent of* $N$.

Figure 2.1 shows an example of the path behavior of this type of reserve processes. One reason for using a Poisson process in this definition lies in the fact that many properties of the Poisson process are desirable to model claims times; for example, the process $R^u = \{R_t^u : t \geq 0\}$ posses a memoryless behavior, and also the probability of simultaneous claims is zero. These particular constrains must be take into account when one wants to model real data. Excellent analyzes of this model can be found for example in Bowers et al. [1997], Beard et al. [1969], Dickson et al. [1992], Rolski et al. [1999] and Asmussen and Albrecher [2010].

Many studies are devoted to generalizations of the CL model, and many others demonstrate why it could be used as a suitable approximation to processes that not fulfilling all its hypothesis; for instance, in some cases these processes are used to model reserves despite the fact that their incomes are nondeterministic [see Bühlmann, 1970, Chapter 2]. Furthermore, in Gerber [1984] and Michel [1987] it is shown that the collective risk model could be used to approximate the individual risk model. In some instances there are not enough data to adjust an individual model, although the calibration of the collective model could be efficient.

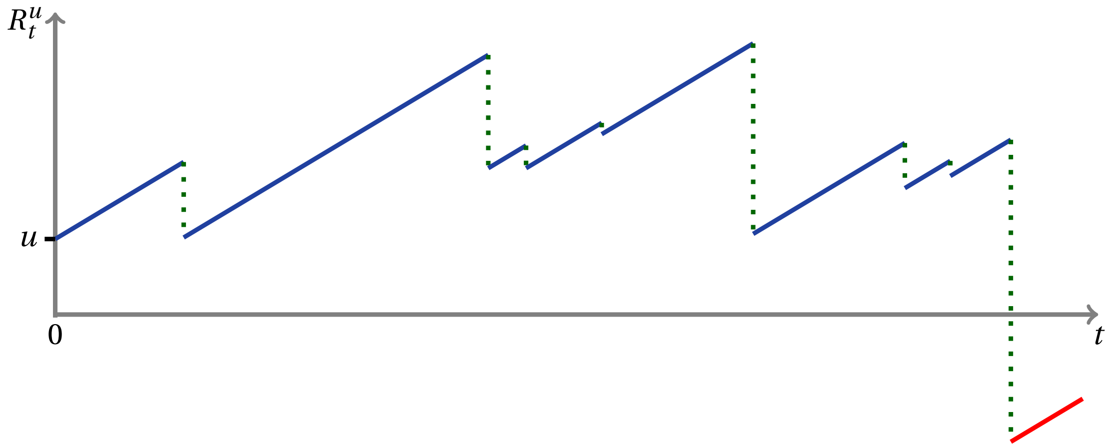



FIGURE 2.1: Example of a possible reserve trajectory for the CL model. Blue lines denotes the periods without the claims, green dots indicate the drops caused by claims, and the red line characterizes a period with a negative reserve.

Several generalizations, mainly featuring more realistic considerations for the counting processes $N$ and more general premium functions, can be easily found in the literature, see, for instance, Thorin [1971b] and the references therein; for further evidences see Grandell [1997] and Bening and Korolev [2002]. However, in most of these studies independence among claims is a persistent assumption.

To continue with our analysis of the reserve trajectories the next proposition provides information about the moments of the reserve process $R^u$; a proof of this result can be consulted in Kaas et al. [2004, p. 47].

**Proposition 5.** *For a general counting process $N$ and $t, s, u > 0$ with $t \leq s$ we have*

$$\mathbb{E}\left[R_t^u\right] = u + pt - \mathbb{E}[N_t]\,\mathbb{E}[Y_1],$$

$$\mathsf{Cov}(R_t^u, R_s^u) = \mathsf{Var}(R_t^u) = \mathbb{E}[N_t]\,\mathsf{Var}(Y_1) + \mathsf{Var}(N_t)(\mathbb{E}[Y_1])^2,$$

$$M_{R_t^u}(r) = \exp\{r\,(u + ct)\}\,M_{N_t}\left(\ln(M_{Y_1}(-r))\right),$$

*where $M_X$ represents the moment generator function of a variable $X$. In particular, when $N$ is a Poisson process,*

$$\mathbb{E}\left[R_t^u\right] = u + pt - \lambda t \mathbb{E}[Y_1],$$

$$\mathsf{Cov}(R_t^u, R_s^u) = \mathsf{Var}(R_t^u) = \lambda t \mathbb{E}\left[Y_1^2\right],$$

$$\mathsf{Corr}(R_t^u, R_s^u) = \sqrt{t/s},$$

$$M_{R_t^u}(r) = \exp\left\{r(u+ct) + \lambda t\left(M_{Y_1}(-r) - 1\right)\right\}.$$

This proposition implies that $\mathbb{E}\left[R_t^u/t\right] \to c - \lambda\mu$ as $t \to \infty$; then a condition towards having solvency is $c - \lambda\mu > 0$, as it implies a positive drift for the reserve trajectories. In Section 2.3 we will elaborate on more closely this condition, and its implications over the ruin probability.

We like also to remark that the covariance between $R_t^u$ and $R_s^u$, for $t < s$, is given only by the variance $R_t^u$. This means that the variability is generated only by their variables in common. Since, we can rewrite $R_s^u$ as $R_t^u + (R_s^u - R_t^u)$, and these two terms are independent of each other.

A reserve model will be useless without an indicator of the solvency of the insurance company. The next section will focus in an indicator of this kind.

## 2.2 Ruin probabilities

It is fundamental for a profitable insurance company to have a positive reserve. This surplus could then be used by the insurance company to shield itself against future time periods when claim payments exceed the premiums collected, or to be invested in order to obtain dividends [see Asmussen and Albrecher, 2010, Ch. VIII]. To analyze the liability of an insurance company we will define what we mean by *ruin* and *probability of ruin*. Here and subsequently, ruin is the event of having a negative reserve; mathematically speaking:

$$\text{ruin} = \{R_t^u < 0 \text{ for some } t > 0\}.$$

While the analogous accounting concept is more complex, for our purposes this definition will suffice.

The probability of ruin is usually studied as a function of the initial capital $u$, even though it is a function of many other factors (e.g. the distribution of claim amounts, premium rates and the distribution of the claims' frequencies). One theoretical reason to define this probability as a function of the initial capital is that the ruin event is settled in terms of other capitals, the forthcoming capitals. Consequently, we can establish an equation for the ruin probability as a function of $u$ (see Section 2.4). One practical reason for this viewpoint is that the initial capital is a variable under the control of the insurance company, whose minimum could be explicitly requested by a regulatory entity to increase the likelihood to fulfill the company's liabilities.

The study of ruin probabilities could be divided in two major subjects: finite-horizon and infinite-horizon. There exists a vast literature concerning both of these topics. However, the ideas that we what to generalize are focus on the infinite-horizon ruin probabilities, and we will not analyze the finite-horizon case. Under these lines of thought the next definition follows.

**Definition 13.** *The (infinite-horizon)* ***ruin probability*** *is defined as*

$$\psi(u) = \mathbb{P}\left[\inf_{t\geq 0} R_t^u < 0\right], \qquad u \geq 0. \tag{2.2}$$

Explicit expressions for equation (2.2) are rare, even under the classical framework; we refer the reader to Section 2.7 to consult some of this special cases. To overcome this deficiency, several research lines had been develop to unveil the behavior of equation (2.2). In the following sections of this chapter, we will analyze the classical results of ruin probabilities that we require for the rest of this work.

## 2.3 The classical net profit condition

In general terms, the next four properties indicate a profitable insurance business:

**P1** More incomes than outcomes are expected.

**P2** If the business lasts endlessly, an enormous amount of money will be obtain.

**P3** The ruin probability is less than one.

**P4** By increasing the initial capital the ruin probability is decreased.

All this properties sound rational and valid not only for an insurance business, but also for any kind of business. Moreover, each of them brings into focus different qualities of a productive insurance company; since an enterprise without those characteristics would not be practical to model because is doomed to fail. It is worth pointing out that any of them implies an avoidance of getting ruined, they only mitigate the risk. In this section, we will analyze them mathematically and study their interactions under the framework of the CL model.

Let us start by establishing property P1 in mathematical terms.

**Definition 14.** *We term the condition*

$$p > \lambda \mathbb{E}[Y_1], \tag{2.3}$$

*the CL net profit condition. For simplicity of notation, we also define the safety loading by*

$$\eta = (p/\lambda \mathbb{E}[Y]) - 1.$$

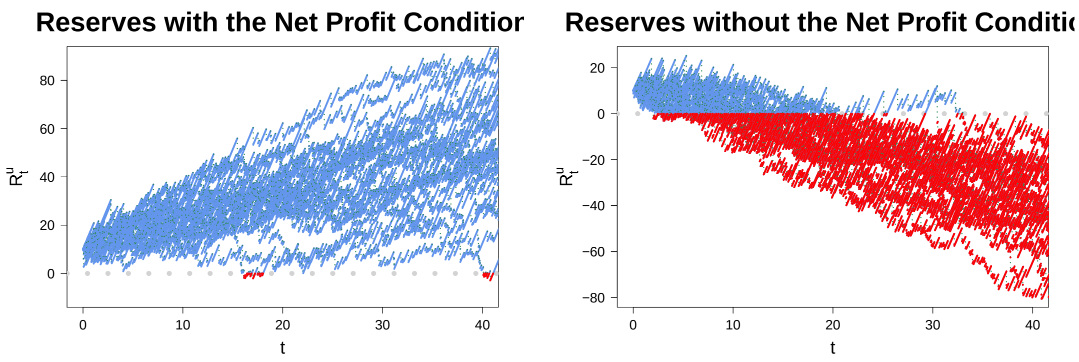
The interpretation of the net profit condition is rather intuitive; for any given lapse of size $t$, the expected number of claims, $\lambda t$, times the expected size of claims, $\mathbb{E}[Y_1]$, must be smaller than the premium income in this lapse, $pt$; this implies equation (2.3). In other words, the average cash flow in the portfolio must be positive; in average, more premium flows into the portfolio than claims costs flow out. As already mentioned, this does not mean that ruin is avoided, since the expectation of a stochastic process says relatively little about the fluctuations of the process.

The idea behind property P2 is that a lucrative enterprise eventually will make a huge amount of money if it continues operating without interruption. Consequently, this property refers to the limit behavior of the reserve. The next proposition characterizes this limit and establish the implication of property P1 on property P2 [see Asmussen and Albrecher, 2010, p. 73].

**Proposition 6.** *The drift and the oscillation of $R^u$ are driven by:*

1. *Regardless the value of $\eta$, $R^u_t/t \xrightarrow{a.s.} p - \lambda \mathbb{E}[Y_1]$ as $t \to \infty$;*
2. *If $\eta > 0$, then $R^u_t \xrightarrow{a.s.} \infty$;*
3. *If $\eta < 0$, then $R^u_t \xrightarrow{a.s.} -\infty$;*
4. *If $\eta = 0$, then $\liminf_{t\to\infty} R^u_t = -\infty$, $\limsup_{t\to\infty} R^u_t = \infty$.*

An important remark on this proposition is that the only scenario for a profitable business is when $\eta > 0$, because in all the other cases the reserve surely reaches big negative numbers in a finite time. Examples of typical trajectories assuming $\eta > 0$




(A) $\eta > 0$. (B) $\eta < 0$.

FIGURE 2.2: Examples of reserve trajectories showing their remarkable different behavior with and without the net profit condition. In the left plot the reserve trajectories satisfies the net profit condition which gives them an upward trend, against the behaviors of their analogues in right plot with a downward trend, given by the lack of the net profit condition.

and $\eta < 0$ are shown in Figure 2.2. A helpful analysis of this limit is given in Mikosch [2004, p. 159].

To continue with property P3, the next corollary shows how the net profit condition implies that the ruin probability is less than one, its proof is straightforward from the last proposition.

**Corollary 6.1.** *The ruin probability $\psi(u)$ is equal to 1 for all $u$ when $\eta \leq 0$, and less than 1 for all $u$ when $\eta > 0$.*

This corollary allows us to think the net profit condition as a bound condition, because if it is satisfied then $\psi$ is always less than one. This corollary will be fundamental in Section 3.3. Therefore, property P1 implies property P2.

Finally, the idea of property P4 comes from the notion that as bigger is our reserve as farther we are of getting ruin. Mathematically, it refers to the limit behavior of $\psi(u)$ when $u \to \infty$. The next theorem establishes the implication of property P1 on property P4 [see Embrechts et al., 1997, p. 31]

**Theorem 7.** *In the CL model, assuming the net profit condition we have the limit behavior*

$$\lim_{u \to \infty} \psi(u) = 0. \tag{2.4}$$

This result has a practical viewpoint. Under the net profit condition, an insurance company could, in principle, take a risk if its initial capital as high as needed to obtain desired ruin probability level. An example of this is presented in Remark 2.2.

Hence, we have shown that property P1 implies all the others. In conclusion, the net profit condition is a necessary assumption to model the reserve process of profitable insurance business, because it warranties the desirable and necessary mention premises. From now on we assume the net profit condition over all the reserve process that we analyze.

## 2.4 Integro-differential and integral equations

In this section we introduce a function which embraces the probability of ruin as a particular case. This function has significant importance on its own (see Dickson [1992] and Dufresne and Gerber [1988]), even though we will only use it to present some formulae for the ruin probability $\psi$.

**Definition 15.** *For $u, y \geq 0$, $G(u, y)$ is defined as the probability that ruin occurs with initial reserve $u$ and deficit immediately after ruin occurs at most $y$.*

The function $G$ is an special case of the Gerber-Shiu penalty function [see Gerber and Shiu, 1998], however this level of generality it is not essential to express our main ideas. The relation between the ruin probability and $G$ is given by

$$\psi(u) = \lim_{y \to \infty} G(u, y), \quad u \geq 0.$$

Using a first step analysis, $G$ can be rewritten as an integro-differential equation, which is the subject of the following result [see Klugman et al., 2012, p. 303].

**Theorem 8.** *The function $G$ satisfies the next integro-differential equation*

$$\frac{\partial}{\partial u} G(u, y) = \frac{\lambda}{c} G(u, y) - \frac{\lambda}{c} \int_0^u G(u - x, y) dF(x) - \frac{\lambda}{c} [F(u + y) - F(u)], \quad u \geq 0, \tag{2.5}$$

*where $F$ is the distribution function of the claims.*

As nearly all equations of this type, closed analytic solutions for equation (2.5) are only known for a few special instances (e.g. claims distributed exponentially). Nevertheless, some properties of $G$ may be determined using this equation; for instance, under suitable conditions, solutions to equation (2.5) could be approximated using analytic methods [Pitts and Politis, 2007] and numerical methods [Butcher, 2008].

On the other hand, as a theoretical result, Theorem 8 has many important implications; in particular the next couple of results, Theorems 9 and 10, could be proven directly from it. Theorem 9 allow us to see the ruin probability as a defective renewal equation [see Rolski et al., 1999, p. 213], and Theorem 10 is used to rewrite $\psi$ as a geometric compound sum.

**Theorem 9.** *The ruin probability $\psi$ satisfies the integro-differential equation*

$$\psi'(u) = \frac{\lambda}{p}\psi(u) - \frac{\lambda}{p}\int_0^u \psi(u-x)\,dF(x) - \frac{\lambda}{p}\overline{F}(u), \quad u \geq 0, \tag{2.6}$$

*and the integral equation*

$$\psi(u) = \frac{\lambda}{p}\left(\int_u^\infty \overline{F}(x)\,dx + \int_0^u \psi(u-x)\overline{F}(x)\,dx\right), \quad u \geq 0, \tag{2.7}$$

*where* $\overline{F}(x) = 1 - F(x)$.

The solution of equation (2.7) can also be expressed as an infinite series of functions, called a Neumann series [see Watson, 1995, Ch. XVI]. A direct proof of Theorem 9 could be also consulted in Asmussen and Albrecher [2010, p. 79] and Grandell [1991, p. 6].

The next theorem give us the distribution of the deficit if the initial capital is zero. For a direct proof of it, without the use of Theorem 8, we refer the reader to Bowers et al. [1997, Section 13.5].

**Theorem 10.**

$$G(0,y) = \frac{\lambda}{c}\int_0^y \overline{F}(x)\,dx, \quad y \geq 0. \tag{2.8}$$

As a particular case of equation (2.8) we obtain that $\psi$ satisfies the following result.

**Corollary 10.1.** *The ruin probability with zero initial reserve is*

$$\psi(0) = \frac{1}{1+\eta} = \frac{\lambda \mathbb{E}[Y_1]}{p} \tag{2.9}$$

The important point to note here is that the ruin probability with initial capital zero only depends on the safety loading, and not on the particular distribution of the claims.

## 2.5 Lundberg's exponential bound

In this section we present an upper exponential bound for the ruin probability. To this end, we need to define the concept of *adjustment coefficient*, which is the first nonnegative root $\gamma$ of the Lundberg's fundamental equation

$$\int_0^\infty e^{\gamma x}\overline{F}(x)\,dx = \frac{p}{\lambda}. \tag{2.10}$$

To exhibit the Lundberg's bound we first establish the next theorem; this result, as Theorem 9, presents an equivalent expression for the ruin probability.

**Theorem 11.** *If there exists a* $\gamma > 0$ *satisfying equation* (2.10)*, then*

$$\psi(u) = \frac{e^{-\gamma u}}{\mathbb{E}\left[\exp\left\{-\gamma R_T^u\right\} | T < \infty\right]}, \qquad u \geq 0, \tag{2.11}$$

*where* $T = \inf\{t \geq 0 : R_t^u < 0\}$ *is the ruin time.*

Some proofs of Theorem 11 use the fact that $\gamma$ is the only constant that makes $\left\{\exp\left\{-\gamma R_T^u\right\} : t \geq 0\right\}$ a martingale [see Bowers et al., 1997, p. 426]. Likewise, the use of martingale techniques is common in many proofs of this kind of bounds; see for instance Dassios and Embrechts [1989], Delbaen and Haezendonck [1985] and [Møller, 1992]. Also this type of techniques have given more concrete proofs to classical results (see for instance Gerber [1979, 1988]).

The next theorem could be proven as a direct consequence of Theorem 11 and the fact that $R_T^u < 0$ given that $T < \infty$.

**Theorem 12** (Lundberg bound)**.** *For the classical risk model,*

$$\psi(u) \leq e^{-\gamma u}, \qquad u \geq 0, \tag{2.12}$$

*where* $\gamma$ *is known as the adjustment coefficient.*

*Remark* 2.2. Theorem 12 gives us a twofold way to manage the magnitudes of risks of an insurance company. Suppose that the company can only handle ruin probabilities smaller than $\alpha$ and the moment generator function of the claims exists. If the initial capital $u$ is prearranged, then a loading factor $\eta$ such that

$$\eta \geq \frac{u\left\{\mathbb{E}\left[\exp\left(-\frac{\ln\alpha}{u}X\right)\right]-1\right\}}{-\mu\ln\alpha}-1,$$

gives us the desired bounded probability. On the other hand, if we have $\eta$ fixed, then initial capitals of the form

$$u \geq \frac{-\ln(\alpha)}{\eta},$$

gives us the now required probabilities. So, Theorem 12 gives us information about needed restrictions for the parameters.

Taylor [1976] has used differential and integral inequalities to derive an improvement to equation (2.12), obtaining

$$\psi(u) \leq \left(\sup_{x\in[0,\omega]} \frac{e^{rx}\int_x^{\infty}\overline{F}(s)ds}{\int_x^{\infty}e^{rs}\overline{F}(s)ds}\right), \qquad u \geq 0,$$

where $\omega = \sup\{x : F(x) < 1\}$ is the maximum claim size. For a deeper treatment of this type of bounds, we refer the reader to Willmot [1994].

One of the main drawback of the mentioned bounds is that they only exist for claims with light-tail distributions. Light-tail distributions are characterized by having tails lighter than an exponential function, against heavy tail distributions that have tails heavier than any exponential function. Many interesting studies focus on the heavy tail analysis, e.g. Resnick [2007] and Embrechts et al. [1997]. In particular, we would like to mention the woks of Asmussen and Kroese [2006], Asmussen and Binswanger [1997] and Asmussen et al. [2000] on this subject; they, in particular, tackle the ruin estimation problem using variance reduction techniques.

## 2.6 Pollaczek-Khinchine formula and Laplace transformation of $\psi$

In this section, we present two useful expressions for the ruin probability. These expressions have many things in common; for instance, both of them give us analytic algorithms to find approximations for the ruin probability. The Pollaczek-Khinchine formula and Laplace transformation had several works devoted to their implications and generalizations, although we will restrict our discussion to those aspects that we will use in Chapter 3. Let us first discuss the Pollaczek-Khinchine formula.

The Pollaczek-Khinchine formula [Khintchine, 1932, Pollaczek, 1930], also known as Beekman's formula [Beekman, 1968, 1985], gives a simple and general algorithm to compute the ruin probabilities using a convolution approach. To establish its algorithm, we need to define the *equilibrium distribution* $F_e$, also named *integrated tail distribution*, given by

$$F_e(x) = \frac{1}{\mathbb{E}[Y_1]} \int_0^x \overline{F}(y)\,dy, \qquad x \geq 0. \tag{2.13}$$

**Theorem 13** (The Pollaczek-Khinchine formula)**.** *For each* $u \geq 0$*,*

$$\psi(u) = \left(1 - \frac{\lambda \mathbb{E}[Y_1]}{p}\right) \sum_{n=1}^{\infty} \left(\frac{\lambda \mathbb{E}[Y_1]}{p}\right)^n \overline{(F_e)^{*n}}(u), \tag{2.14}$$

*where* $(F_e)^{*n}$ *denotes the* $n$*-convolution*[1] *of* $F_e$*.*

Proofs of this result can be consulted in Rolski et al. [1999, p. 166] and Deelstra and Plantin [2014, p. 32]. In general, equation (2.14) is proved by seeing that ruin is equivalent to the event that the maximum aggregate loss is greater than the initial capital, or by analyzing the Laplace transform of a compound geometric sum and comparing it to the transform of the ruin probability.

The idea of a compound geometric sum is key to the Monte Carlo estimation of the infinite time ruin probability. Even if one could simulate the reserve process $R^u$, one would not obtain an estimator of $\psi$; this happens because one will need to let the simulation program to run until the end of time to only find one non-ruin trajectory. Theorem 13 tell us that the mean of variables $\mathbb{1}_{\{X_1+\ldots+X_M>u\}}$ is an estimator of $\psi$, where $X_i \sim F_e$ are independent and also independent of $M \sim \text{Geo}(\eta/(1+\eta))$. See Section 3.6 for an extension of these ideas.

Another valuable use of the Pollaczek-Khinchine formula is the Panjer's recursion [Panjer, 1981, 1986]. This recursion uses a discretization of $F_e$ to approximate equation (2.14); we refer the reader to Dufresne and Gerber [1989] for a comparison of this method against others, and to Dickson [1995] for a review of it.

[1] In Section 4.1 we give a closer analysis to the convolution operator.

Also it is important to mention that equation (2.14) could be used for heavy tail claim frameworks; in particular, it does not depend on the existence of the Lundberg coefficient (see equation (2.10)).

Let us now continue with analyzing the Laplace transform of the ruin probability. In this work we refer to the expression

$$L_{\psi}(s) = \int_0^{\infty} \psi(u) e^{-su} du, \qquad s > 0, \tag{2.15}$$

as the Laplace transform of $\psi$. Note that this integral exists and is finite for all values $s > 0$. A useful survey of the Laplace transform is given in Widder [1945]. For a thorough treatment of this topic we refer the reader to Schiff [1999].

**Theorem 14.** *The Laplace transform of equation* (2.15) *can be expressed as*

$$L_{\psi}(s) = \frac{1}{s} - \frac{p - \lambda \mathbb{E}[Y_1]}{ps - \lambda(1 - l_Y(s))}, \qquad s > 0. \tag{2.16}$$

A direct proof of Theorem 14 is presented in Rolski et al. [1999, p. 165]; others apply the Laplace transform to both sides of equation (2.14).

Methods to invert numerically equation (2.16) are remarkably easy to implement, in some cases the programs to compute it are compose only by a small number of lines [see Abate and Whitt, 1992]. An implementation of the inversion formula to approximate $\psi$ is presented by Lima et al. [2002], they exhibit a numerical analysis of its behavior under diverse claim distributions; in particular, they extend this framework to the case of *Erlang*(2) inter-claims times. Likewise, an analytic application of Theorem 14 is presented in Thorin [1973], where the Laplace transform is used to obtain an expression of the ruin probability for hyperexponential distributed claims. For a deeper discussion of the use of inversion techniques, we refer the reader to Embrechts et al. [1993].

Summarizing, the results of Theorem 13 and Theorem 14 give us analytic and numerical expressions for the ruin probability. Unlike equations (2.6) and (2.7), where $\psi$ is only expressed as the solution of a mathematical statement, equations (2.14) and (2.15) present $\psi$ in terms of $u$, $p$ and $F$. They are analytic results because in some instances we could insert these terms in the formula and obtain a tractable expression, and they are numerical results because both provide us algorithms to approximate $\psi$. These benefits have been used to find or approximate $\psi$ by many authors, and they are also extended in our exchangeable model (see Sections 3.5 and 3.6).

## 2.7 Some analytic expressions for the ruin probability

Some known expressions for the ruin probability $\psi$ are:

- If $Y_i \sim \text{Exp}(\beta)$ then
$$\psi(u) = \frac{1}{1+\eta}\exp\left\{-\frac{\eta\beta u}{1+\eta}\right\}.$$

- If $Y_i \sim \text{Ga}(\alpha,\alpha)$, with $\alpha < 1$ [see Grandell and Segerdahl, 1971, Thorin, 1973], then
$$\psi(u) = \frac{\eta(1-R/\alpha)\exp\{-Ru\}}{1+(1+\eta)R-(1+\eta)(1-R/\alpha)} + \frac{\alpha\eta\sin(\alpha\pi)}{\phi}\cdot I,$$
where
$$I = \int_0^\infty \frac{x^\alpha \exp\{-(x+1)\alpha u\}}{[x^\alpha\{1+\alpha(1+\eta)(x+1)\}-\cos(\alpha\pi)]^2+\sin^2(\alpha\pi)}dx.$$

- If $Y_i \sim f(y) = q\alpha e^{\alpha y} + (1-q)\beta e^{\beta y}$ [see Panjer and Willmot, 1992], then
$$\psi(u) = \frac{1}{(1+\eta)(r_2-r_1)}\left[(\rho-r_1)\exp\{-r_1u\}+(r_2-\rho)\exp\{-r_2u\}\right],$$
where
$$r_1 = \frac{\rho+\eta(\alpha+\beta)-\left[\{\rho+\eta(\alpha+\beta)\}^2-4\alpha\beta\eta(1+\eta)\right]^{1/2}}{2(1+\eta)},$$
$$r_2 = \frac{\rho+\eta(\alpha+\beta)+\left[\{\rho+\eta(\alpha+\beta)\}^2-4\alpha\beta\eta(1+\eta)\right]^{1/2}}{2(1+\eta)},$$
and
$$p = \frac{q\alpha^{-1}}{q\alpha^{-1}+(1-q)\beta^{-1}}, \qquad \rho = \alpha(1-p)+\beta p.$$
For more general mixtures of exponential distributions see Cramér [1955].

- If $Y_i \sim \text{PH}(\pi, S)$ [see Asmussen and Albrecher, 2010, p. 264], then
$$\psi(u) = -\frac{\lambda}{p}\pi' S^{-1}\exp\left\{(S+S\mathbf{1}\lambda\pi' S^{-1}/p)u\right\}\mathbf{1}$$

where $\pi$ is the vector of initial probabilities, $S$ is the subgenerator matrix, and $\mathbf{1}$ is a vector of ones [see also Bladt, 2005].

All this expressions will play an important roll in our analysis of the exchangeable model in Chapter 3.

## 2.8 Relaxation of the classical independent claims assumption

The wide variety of risk products and their complexity, together with current international needs such as the Solvency II directive, require models that relax the strong requirement of independence among claims. The European solvency system is discussed in e.g. Linder and Ronkainen [2004] and Sandström [2005, 2007]. Specific topics of the solvency project are discussed in Djehiche and Hörfelt [2005] and Vesa et al. [2007]. Indeed, It is easy to find insurance contracts where positively dependent claims might induce an increment of the aggregated claims. Let us mention three examples of this behavior:

- Using data from Friedman and Companies [1987] and Müller and Pflug [2001] observe that claims caused by tornadoes exhibit a significative positive correlation.
- Nikoloulopoulos and Karlis [2008] show that the intensity of an earthquake and the time elapsed from the previous one are positively related.
- Kiladis and Diaz [1989] show that climate patterns in eastern Australia are heavily dependent on the Southern Oscillation Index.

These examples have a direct connection to climate damage insurance. Indeed, violations of the independence assumption are noted by various authors, e.g. Ambagaspitiya [2009], Biard et al. [2008] and Abbas et al. [2012].

Current literature offers only a few risk models that incorporate dependence among claims. For example: Albrecher and Boxma [2004] allow the time between claims to depend on the previous claim magnitude. Gerber [1982] and Promislow [1991] assume a linear ARMA model for the claim sizes process. Mikosch and Samorodnitsky [2000a] use a stationary ergodic stable process to model claims, and analyze their asymptotic behavior. Cossette and Marceau [2000] model dependence through a Poisson shock process. However, it is important to mention that in most of these generalizations, the complexity inherent to their dependence structures makes difficult the obtention of explicit formulae.

In the majority of studies of reserves with dependent structures, only obtain bounds and/or asymptotic results for the ruin probability are obtained. For instance, in the case of light-tailed claim sizes, Müller and Pflug [2001] present a Lundberg limiting result by assuming conditions on the probability-generating function, Nyrhinen [1998, 1999] obtain rough exponential estimates for ruin probabilities using large deviations techniques, and Albrecher and Kantor [2002] study the behavior of the Lundberg exponent as a function of a dependence measure using three simulation methods of the ruin probability. In the other hand, for heavy-tailed claims, Asmussen et al. [1999] extend a classical result on the distribution tail of the maximum of a random walk, and Mikosch and Samorodnitsky [2000a,b] evaluate the asymptotic behavior of the ruin probability for some stationary ergodic stable processes. See also Asmussen and Albrecher [2010, Ch. XIII] and the references therein.

In the next chapter we will present our results over the ruin probability of a reserve process with exchangeable claims.

# Chapter 3

# The CLEC model

When we seek to relax the classical independent claims approach a natural idea is to establish an exchangeable claim process. The assumption of exchangeability among claims is adequate for some insurance business; it resembles that the sequence of the claims amounts does not have a particular trend and allows, at the same time, to establish a dependent structure. The purpose of this chapter is to develop the theoretic consequences of equation (2.1) under this framework. This model will be call the Cramér-Lundberg model with exchangeable claims (CLEC).

An estimation of the CLEC process was first explored by Mena and Nieto-Barajas [2010]; however, besides some finite-horizon simulation scenarios, little has been said about the probability of ruin. Hence, here we look for various analytical results equivalent to those available for the ruin probability under the CL model. In particular, we observe that, due to the identically distributed property featuring exchangeable sequences, some of the marginal properties carry over to the exchangeable scenario. Most importantly, explicit equivalences of various results, e.g. a generalization of the Pollaczek-Khinchine formula, a generalized Lundberg inequality, etc., are obtained. One significant ingredient in our proposal is the Bayesian construction of exchangeable sequences, which brings out an appealing modeling component to the classical CL scheme.

The layout in this chapter is as follows: In Section 3.1, we define the CLEC model and present various results for its moments. Section 3.2 analyze the ruin probabilities for this model and gives the connection to ruin probabilities of the CL model. In Section 3.3 an equivalent condition to the net profit condition of the classical framework is obtain. Sections 3.4 to 3.6 extents classical results to the ruin probabilities, and these results are then applied to some specific claim scenarios in Sections 3.7 to 3.10. Some concluding remarks are deferred to Section 3.11.

Part of the results presented in this chapter can be found in Coen and Mena [2015b].

## 3.1 Introduction to the CLEC model

Let us start with the definition of the CLEC model.

**Definition 16.** *The Cramer-Lundberg reserve model with exchangeable claims (**CLEC**) model, $R^u = \{R^u_t : t \geq 0\}$, is given by*

$$R^u_t = u + pt - \sum_{i=1}^{N_t} Y_i \qquad t \geq 0, \tag{3.1}$$

*with the assumptions:*

- *$u$ is the initial reserve at time 0, and $p$ is the constant premium rate.*
- *$N = \{N_t : t \geq 0\}$ is a Poisson process of rate $\lambda$.*
- *$Y = \{Y_i : i \in \mathbb{N}\}$ is an exchangeable sequence, with de Finetti's measure $\mu$.*
- *$Y$ and $N$ are independent of each other.*

One can notice that by taking de Finetti's measure as a point measure the CLEC model includes the CL model as an special case. An important characteristic to mention is that under the CLEC model there are infinite different types of exchangeable process that have the same marginal claim distribution, which gives it more flexibility to model different types of dependency. This approach adds a natural modeling feature to risk models.

A useful property of the CLEC model is its time homogeneous structure. Under this model, the distribution of a sequence of $n$ claims at a given set of times remains the same under time shifts, meaning that

$$\mathbb{P}\left[R^u_{t_1} \leq x_1, \ldots, R^u_{t_n} \leq x_n | R^u_{t_0} = x_0\right] = \mathbb{P}\left[R^u_{t_1+h} \leq x_1, \ldots, R^u_{t_n+h} \leq x_n | R^u_{t_0+h} = x_0\right]$$

where $x_i, t_i, h > 0$ for all $i = 1, 2, \ldots, n$.

Let us now analyze the behavior of the moments of the CLEC model. For the sake of comparison, we will compare them to their analogues in the classical model with the same claim distribution. Here and subsequently, we will add the subindex $CL$ to quantities referring to the Cramer-Lundberg model, e.g. $R^u_{t,CL}$, $\psi_{CL}$, etc.

While the expected values of the reserve coincide for the two models,

$$\begin{aligned}\mathbb{E}\left[R^u_t\right] =& u + pt - \mathbb{E}\left[\sum_{i=1}^{N_t} Y_i\right]\\ =& u + pt - \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} Y_i \middle| N_t\right]\right]\\ =& u + pt - \mathbb{E}\left[N_t\right]\mathbb{E}\left[Y_1\right]\\ =& \mathbb{E}\left[R^u_{t,CL}\right],\end{aligned}$$

we have that its covariances vary, because for $t$ and $s$, with $t \leq s$, they are related by

$$\begin{aligned}\mathsf{Cov}(R^u_t, R^u_s) =& \mathbb{E}\left[N_t\mathbb{E}\left[Y_1|Q\right] + N_t(N_s-1)\mathbb{E}\left[Y_1Y_2|Q\right]\right] - (\mathbb{E}\left[N_t\mathbb{E}\left[Y_1|Q\right]\right])^2\\ =& \mathsf{Cov}(R^u_{t,CL}, R^u_{s,CL}) + \lambda^2 t^2 \mathsf{Cov}(Y_1, Y_2),\end{aligned}$$

since $\mathsf{Cov}(R^u_{t,CL}, R^u_{s,CL}) = \lambda t \mathbb{E}\left[Y_i^2\right]$. In particular, notice that the difference between the two covariances grows as $t^2$ over time; this shows some bigger dispersion of the CLEC reserve. Clearly, as a particular case

$$\mathsf{Var}(R^u_t) = \mathsf{Var}(R^u_{t,CL}) + \lambda^2 t^2 \mathsf{Cov}(Y_1, Y_2).$$

One common mistake is to overload the variances of the claims in order to accommodate the independent claims assumption. Indeed, there are cases where selecting a heavy-tail distribution for the claims is not an adequate choice, and the claim behaviors would be better captured by a dependent scheme.

For $t < s$ the correlation for the CLEC model is given by

$$\mathsf{Corr}(R^u_t, R^u_s) = \mathsf{Corr}(R^u_{t,CL}, R^u_{s,CL})\sqrt{\frac{\mathbb{E}\left[Y_1^2\right] + \lambda s \mathsf{Cov}(Y_1, Y_2)}{\mathbb{E}\left[Y_1^2\right] + \lambda t \mathsf{Cov}(Y_1, Y_2)}} \tag{3.2}$$

where $\text{Corr}(R^u_{t,CL}, R^u_{s,CL}) = \sqrt{t/s}$ (see Proposition 5). Unlike the $\text{Corr}(R^u_{t,CL}, R^u_{s,CL})$, the correlation for the CLEC model depends on the covariance of the exchangeable claims. Also, it is easy to check that the square root in equation (3.2) is bigger than one, which means that the CLEC correlation dominates the CL correlation. In general, we have

$$\sqrt{\frac{\lambda \text{Cov}(Y_1, Y_2)}{\mathbb{E}\left[Y_1^2\right] + \lambda \min(t,s)\text{Cov}(Y_1, Y_2)}} \leq \text{Corr}(R^u_t, R^u_s) \leq 1.$$

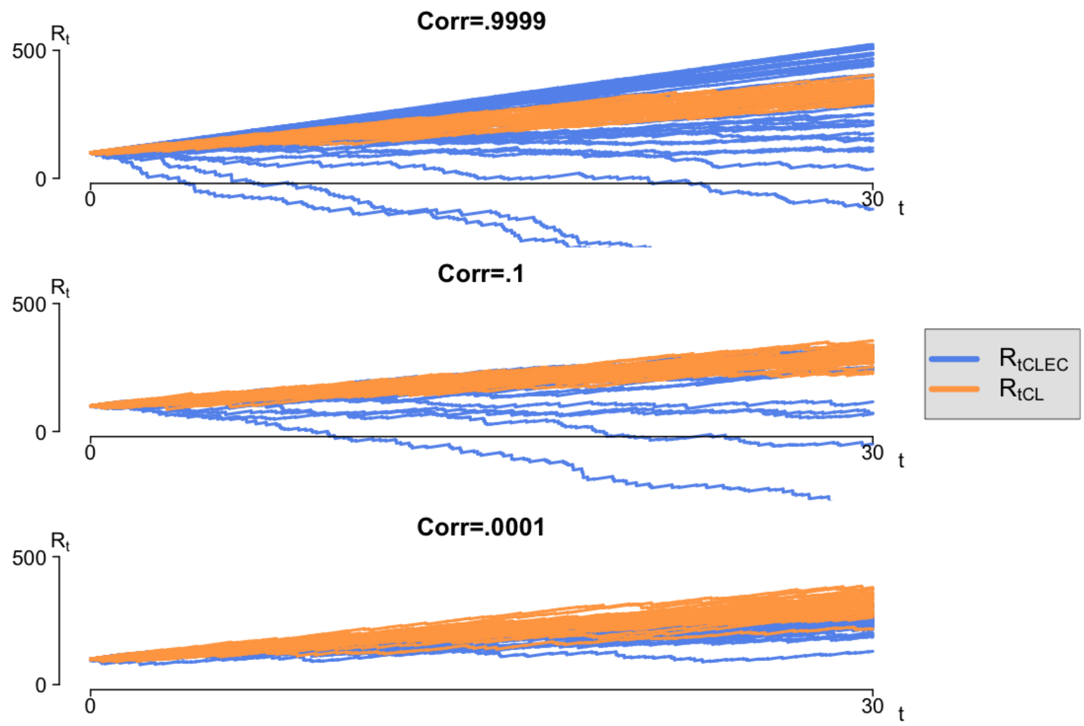



FIGURE 3.1: Different behaviors of the reserve process under distinct correlations. Blue trajectories correspond to the CLEC model and the orange ones corresponds to the CL model. All trajectories have the same claim distribution, but in cases with CLEC model different de Finetti measures.

Figure 3.1 is show the different behaviors of the reserve process under distinct correlations, although with the same marginal distribution for the two models. A remarkable bigger tendency to get ruin is exhibit when the correlation is bigger; in general, a more spread behavior is obtained when the correlation is increased.

Depending on the application in mind, the current literature of Bayesian nonparametrics offers several choices for the de Finetti's measure $\mu$, e.g. the Dirichlet process [Ferguson, 1973]; the class of Gibbs-type priors [De Blasi et al., 2015]; the logistic normal distribution [Lenk, 1988], etc. However, depending on the context, sometimes it is more convenient to restrict the support of $\mu$ to a parametric family. Clearly, within this latter scenario, one is bounded to a parametric model choice, besides the choice for the restricted, finite dimensional distribution, $\mu$.

It is worth noticing that the CLEC reserve process no longer has independent increments, but rather classifies as a conditional Markov process with exchangeable increments [van Handel, 2009]. For this reason, some results available for the CL would not necessarily follow in this framework (e.g. the Cramér's asymptotic ruin formula).

## 3.2 Ruin probabilities for the CLEC model

In this section we will establish our main result, which is a link between the ruin probabilities under the exchangeable claims model to ruin probabilities with independent claims. To this end we need to fix some new concepts; in general, many of them are generalizations of classical ones, but their scope allow us to analyze the new interactions appearing in the CLEC model.

**Definition 17.** *Following the same notation as in Section 1.2, let us define the* ***conditional ruin probability*** *by*

$$\psi(u|Q) = \mathbb{P}\left[\inf_{t\geq 0} R_t^u < 0 \middle| Q\right] \tag{3.3}$$

*for every $u \geq 0$ and $Q \in \mathscr{P}$.*

It is worth pointing out that a conditional ruin probability could be a random entity if $Q$ is a random distribution. In view that the claims are conditional independent, the above quantity is in fact the ruin probability corresponding to a CL model with $Q$ as it claim distribution. The link between $\psi(u|Q)$ and $\psi(u)$ is the subject of the following result.

**Theorem 15.** *Let $\mu$ be the de Finetti's measure for the exchangeable claims in the CLEC model, then*

$$\psi(u) = \int_{\mathscr{P}} \psi(u|Q)\mu(dQ). \tag{3.4}$$

*Proof.* The proof follows from de Finetti's representation theorem and an application of the fundamental property of conditional expectation,

$$\psi(u) = \mathbb{P}\left[\inf_{t\geq 0} R_t^u < 0\right]$$

$$=\mathbb{E}\left[\mathbb{P}\left[\inf_{t\geq 0} R_t^u < 0 \middle| Q\right]\right]$$

$$=\int_{\mathscr{P}} \psi(u|Q)\mu(dQ).$$

□

*Remark* 3.1. An interpretation of the last result is that in the CLEC model the probability of ruin is a weighted average of independent ruin probabilities of the CL models, where the weight structure is given through the de Finetti's measure.

Theorem 15 is a key result because it allows us to extend properties of the CL model to the CLEC model. For example, using the fact that $\psi(u|Q)$ is a nonincreasing function of $u$, it follows that the ruin probability for the CLEC model is also a nonincreasing function. Also, this theorem reveals information about the interaction between $\mu$ and $\psi$, which clearly brings relevance to the selection of an adequate measure $\mu$ and therefore an appealing modeling feature to accommodate a particular dependence. In the next sections we will analyze the implications of Theorem 15, and develop extensions of classical results using it.

## 3.3 The net profit condition set

One premise under which the CL model is used is the *net profit condition*, that is

$$p > \lambda\mathbb{E}[Y_1], \tag{3.5}$$

which is necessary to avoid bankruptcy with probability one (see Section 2.3). Within the classical model, if this condition is accomplished, then the ruin probability decays to zero as the initial capital increases, i.e.

$$\psi_{CL}(u) \to 0 \qquad \text{as} \qquad u \to \infty.$$

This reflects a desired property of any potential insurance business; namely the fact that with enough initial capital one is able to cover any risk level. Hence, in this section we will find which is an equivalent condition to the classical net profit condition under the CLEC model.

Let us first denote by $\mathscr{B}$ the set

$$\mathscr{B} = \{Q \in \mathscr{P} : \lambda \mathbb{E}[Y_1|Q] < p\},$$

and name it the *net profit condition set*. It is worth pointing out that this set is measurable with respect to the de Finetti's measure $\mu$ (see Section 1.2). Then $\mathscr{B}$, is the set of distributions that accomplish the classical net profit condition (3.5). This set allows us to rewrite equation (3.4) as

$$\begin{aligned} \psi(u) &= \int_{\mathscr{B}^c} \psi(u|Q)\mu(dQ) + \int_{\mathscr{B}} \psi(u|Q)\mu(dQ) \\ &= \int_{\mathscr{B}} \psi(u|Q)\mu(dQ) + \mu\left(\mathscr{B}^c\right), \end{aligned} \tag{3.6}$$

using the fact that $\psi(u|Q) = 1$ for $Q \in \mathscr{B}^c$.

Before establishing a net profit condition for the CLEC model, let us highlight two important properties differentiating the CLEC model from the classical independent claims scenario. First, we show that the probability of ruin is bounded from below with a quantity that could be different than zero (compare this against equation (2.4)), then we exhibit the limit behavior of $\psi(u)$ as $u$ tends to infinity.

**Proposition 16.** *The ruin probability in the CLEC model has the lower bound* $\mu(\mathscr{B}^c)$.

*Proof.* From equation (3.6) we obtain

$$\psi(u) \geq \mu\left(\mathscr{B}^c\right),$$

using the fact $\psi(u|Q)$ is nonnegative, and this is the desired conclusion.

□

This bound is also the limit of the function $\psi$, as the next proposition shows.

**Proposition 17.**

$$\lim_{u\to\infty} \psi(u) = \mu\left(\mathscr{B}^c\right).$$

*Proof.*

$$\lim_{u\to\infty} \psi(u) = \lim_{u\to\infty} \int_{\mathscr{B}} \psi(u|Q)\mu(dQ) + \mu\left(\mathscr{B}^c\right)$$

$$
\begin{aligned}
&= \int_{\mathscr{B}} \lim_{u\to\infty} \psi(u|Q)\mu(dQ) + \mu\left(\mathscr{B}^c\right) \\
&= \int_{\mathscr{B}} 0\,\mu(dQ) + \mu\left(\mathscr{B}^c\right) \\
&= \mu\left(\mathscr{B}^c\right),
\end{aligned}
$$

where the third equality uses Theorem 7. □

From Proposition 17, we can compare the limiting behavior of the ruin probabilities of the CL and CLEC models, as the initial surplus increases; see Figure 3.2. In the CL model this probability could be arbitrary small if the net profit condition (3.5) is satisfied, however, in the CLEC model this value could be far from zero, when only working under (3.5), see Figure 3.2(A). This observation is crucial when one is choosing a model, because for the CL model there is always, given the classical net profit condition is fulfilled, an initial capital big enough to reduce the probability of ruin to any fixed level. Instead, for the CLEC model there might be cases where no matter how big we make the initial capital, the ruin probability might never be reduced beyond a certain level. As illustrated in Figure 3.2(A), this happens to levels below the quantity $\mu(\mathscr{B}^c)$, while if $\mu(\mathscr{B}^c) = 0$ the ruin probability of CLEC decays to zero as the initial capital increases, see Figure 3.2(B). This clearly suggest a re-definition of condition (3.5) needs to take place.

**Definition 18.** *We term the condition*

$$
\mu\left(\mathscr{B}^c\right) = 0, \tag{3.7}
$$

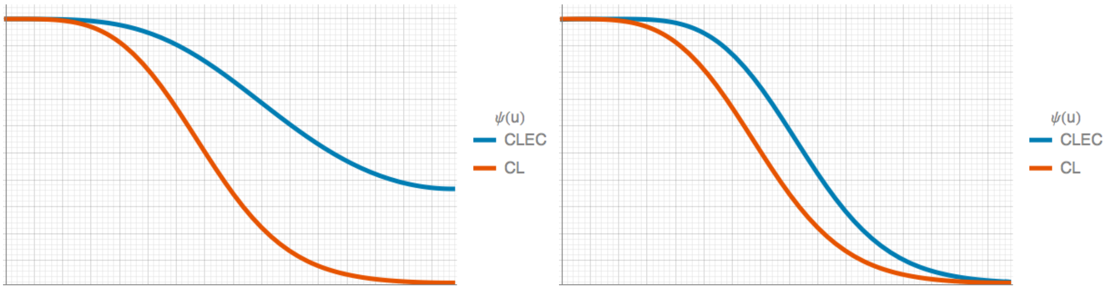
*the CLEC net profit condition.*

Indeed, this new condition ensures a measure zero to all conditional CL models not satisfying (3.5). The next proposition shows a relation between the CLEC net profit condition and the classical net profit condition.

**Proposition 18.** *The CLEC net profit condition implies the classical net profit condition.*

*Proof.* If we assume the condition of equation (3.7), then

$$
\mathbb{E}[Y_i] = \int \mathbb{E}[Y_i|Q]\,\mu(dQ)
$$




(A) $\mu(\mathscr{B}^c) > 0$. (B) $\mu(\mathscr{B}^c) = 0$.

FIGURE 3.2: Example of the general behavior of the ruin probability with and without the condition of (3.7). Is important to mention that even under this condition the ruin probabilities under the CLEC model still been bigger that the ones of the CL model. A particular case of this behavior is show in Figure 3.5.

$$
\begin{aligned}
&= \int_{\mathscr{B}} \mathbb{E}[Y_i|Q]\,\mu(dQ) \\
&< \int_{\mathscr{B}} \frac{p}{\lambda}\mu(dQ) \\
&= \frac{p}{\lambda}.
\end{aligned}
$$

□

The opposite implication is not always true; this is exhibit in the examples of Sections 3.7 to 3.10.

In the next three sections we look for various analytical results equivalent to those available for the ruin probabilities under the CL model.

## 3.4 Lundberg inequality and behavior at $u = 0$

As noted in Asmussen and Albrecher [2010], the asymptotic behavior of the ruin probability is affected by the dependence among claims. In particular, they observe that under certain conditions

$$
\lim_{u\to\infty} \frac{\log \psi(u)}{-\gamma\epsilon} = 1,
$$

when the aggregated claims could be expressed as a sum of dependent random variables, where $\gamma, \epsilon > 0$ satisfy some conditions [see also Glynn and Whitt, 1994]. This shows a reduction on the asymptotic speed of the ruin probability, when compared against the independent scheme. A classical result of the independent, light-tail, case is the Cramér-Lundberg condition [see Cai, 2006]. This condition assumes the existence of a constant $R > 0$, called the *adjustment coefficient*, satisfying Lundberg equation (see equation (2.10)), and gives an exponential tail behavior for the ruin probability. To display the analogous behavior for the exchangeable model we need to define the set of random distributions $\mathscr{C}$ by

$$\mathscr{C} = \left\{Q \in \mathscr{P} : \theta(R, Q) = 0 \text{ for some } R > 0\right\},$$

where the function $\theta$ is given by

$$\theta(r, Q) = \lambda\left(M_{Y|Q}(r) - 1\right) - pr,$$

where $r > 0$, $Q \in \mathscr{P}$, and $M_{Y|Q}$ denotes the moment-generating function of the distribution $Q$. The region $\mathscr{C}$ is called the the *adjustment coefficient set.* We can now generalize the Lundberg inequality for the CLEC model.

**Proposition 19** (Lundberg Inequality)**.** *The probability of ruin is bounded from above by*

$$\psi(u) \leq \int_{\mathscr{B}\mathscr{C}} e^{-R(Q)u}\mu(dQ) + \int_{\mathscr{B}\mathscr{C}^c} \psi(u|Q)\mu(Q) + \mu\left(\mathscr{B}^c\right), \qquad u \geq 0, \tag{3.8}$$

*where $R(Q)$ is the solution to the equation $\theta(R, Q) = 0$ for a fixed $Q$.*

*Proof.* From the classical result we know that

$$\psi(u|Q) \leq e^{-R(Q)u}$$

for $Q \in \mathscr{B}\mathscr{C}$ [see, e.g., Cai, 2006], which implies

$$\begin{aligned}
\psi(u) &= \int_{\mathscr{P}} \psi(u|Q)\mu(dQ) \\
&= \int_{\mathscr{B}\mathscr{C}} \psi(u|Q)\mu(dQ) + \int_{\mathscr{B}\mathscr{C}^c} \psi(u|Q)\mu(dQ) + \mu\left(\mathscr{B}^c\right) \\
&\leq \int_{\mathscr{B}\mathscr{C}} e^{-R(Q)u}\mu(dQ) + \int_{\mathscr{B}\mathscr{C}^c} \psi(u|Q)\mu(dQ) + \mu\left(\mathscr{B}^c\right).
\end{aligned}$$

□

Passing to the study of the ruin probability at the extreme point zero, and in order to compare the behaviors of the CL and CLEC models, we use as a benchmark a fixed claim marginal distribution. At the outset, we will assume that only the classical net profit condition (3.5) is satisfied. This will help us to unveil the difference between both approaches while keeping the same marginal distribution, namely the consequences of using an independent claims model in the presence of dependent claims. However, if a business is to be undertaking under the CLEC model, condition (3.7) must be achieved.

Under the framework mention, the next proposition compares the behavior of both ruin probabilities at zero.

**Proposition 20.** *Under the classical net profit condition we have*

$$\psi(0) - \psi_{CL}(0) = \mu\left(\mathscr{B}^c\right) - \frac{\lambda}{p}\mathbb{E}\left[Y_1 \mathbb{1}_{\mathscr{B}^c}\right], \tag{3.9}$$

*where* $\mathbb{1}_{\{\cdot\}}$ *denotes the indicator function.*

*Proof.* For the CL model we have that $\psi_{CL}(0) = \lambda \mathbb{E}\left[Y_1\right]/p$ (see equation (2.9)), which implies

$$\begin{aligned}
\psi(0) &= \int_{\mathscr{B}} \psi(0|Q)\mu(dQ) + \mu\left(\mathscr{B}^c\right) \\
&= \int_{\mathscr{B}} \frac{\lambda \mathbb{E}\left[Y_1|Q\right]}{p}\mu(dQ) + \mu\left(\mathscr{B}^c\right) \\
&= \frac{\lambda}{p}\int_{\mathscr{B}} \mathbb{E}\left[Y_1|Q\right]\mu(dQ) + \mu\left(\mathscr{B}^c\right) \\
&= \frac{\lambda}{p}\mathbb{E}\left[Y_1 \mathbb{1}_{\mathscr{B}}\right] + \mu\left(\mathscr{B}^c\right) \\
&= \psi_{CL}(0) + \mu\left(\mathscr{B}^c\right) - \frac{\lambda}{p}\mathbb{E}\left[Y_1 \mathbb{1}_{\mathscr{B}^c}\right].
\end{aligned}$$

□

In the case that $\mu\left(\mathscr{B}^c\right) = 0$ we obtain $\psi_{CLEC}(0) = \psi_{CL}(0)$; this happens because all the ruin probabilities taking into account by the de Finetti's measure start in the value $\lambda \mathbb{E}\left[Y_1\right]/p$.

From relation equation (3.9), it is clear that, when working under (3.5) only, there are cases in which $\psi_{CL}(0) > \psi(0)$ but at some point, say $u^*$, this relation flips to $\psi_{CL}(u) < \psi(u)$ for all $u > u^*$. Figure 3.5 also illustrates this point.

## 3.5 Laplace transform

Similar to the CL model, there are cases in which one might not have an analytic expression for the ruin probability of the CLEC model. In such cases we might resort to alternative expressions of the ruin probability. In this direction, this section and the next give expressions for the Laplace transform and the Pollaczek-Khintchine formula for the ruin probability (see Section 2.6).

Let us denote by $L_\psi$ the Laplace transformation of $\psi$, that is

$$L_\psi(s) = \int_0^\infty e^{-su}\psi(u)\,du. \tag{3.10}$$

**Proposition 21** (Laplace transform)**.** *The Laplace transform for the ruin probability of the CLEC model is given by*

$$L_\psi(s) = \frac{1}{s} - \int_{\mathscr{B}} \frac{p - \lambda\mathbb{E}[Y_1|Q]}{ps - \lambda\left(1 - L_Q(s)\right)}\mu(dQ), \tag{3.11}$$

*where $L_Q$ denotes the Laplace transform of the distribution $Q$.*

*Proof.* Following Rolski et al. [1999] we can see that for $Q \in \mathscr{B}$ (see Theorem 14)

$$L_{\psi(\cdot|Q)}(s) = \frac{1}{s} - \frac{p - \lambda\mathbb{E}[Y_1|Q]}{ps - \lambda\left(1 - L_Q(s)\right)},$$

that implies

$$\begin{aligned}
L_\psi(s) &= \int_0^\infty e^{-su}\psi(u)\,du \\
&= \int_0^\infty \int_{\mathscr{B}} e^{-su}\psi(u|Q)\mu(dQ)\,du + \int_0^\infty \int_{\mathscr{B}^c} e^{-su}\mu(dQ)\,du \\
&= \int_{\mathscr{B}} \int_0^\infty e^{-su}\psi(u|Q)\,du\,\mu(dQ) + \frac{\mu(\mathscr{B}^c)}{s} \\
&= \int_{\mathscr{B}} \frac{1}{s} - \frac{p - \lambda\mathbb{E}[Y_1|Q]}{ps - \lambda\left(1 - L_Q(s)\right)}\mu(dQ) + \frac{\mu(\mathscr{B}^c)}{s}
\end{aligned}$$

$$= \frac{1}{s} - \int_{\mathscr{B}} \frac{p - \lambda \mathbb{E}[Y_1|Q]}{ps - \lambda\left(1 - L_Q(s)\right)} \mu(dQ),$$

which establishes the formula. □

## 3.6 Pollaczek-Khintchine formula

In this section we well give the extension of the Pollaczek-Khintchine formula to the exchangeable framework, and an algorithm to approximate ruin probabilities through simulation. As in the classical model, the knowledge of how to simulate reserve trajectories could be not enough to approximate the infinite time ruin probabilities (see section 2.6). As is done in the CL framework, this formula could be use to approximate directly the ruin probabilities by restrict the infinite number of terms in its sum to a finite number, since they are decreasing.

**Proposition 22** (Pollaczek-Khintchine formula)**.** *The ruin probability corresponding to the CLEC model can be represented as the following infinite sum*

$$\psi(u) = 1 - \int_{\mathscr{B}} \frac{\rho_Q}{1+\rho_Q} \sum_{m=0}^{\infty} (1+\rho_Q)^{-m} F_{I,Q}^{*m}(u) \mu(dQ), \qquad u \geq 0, \tag{3.12}$$

*where*

$$\rho_Q = \frac{p}{\lambda \mathbb{E}[Y_1|Q]} - 1 \qquad \text{and} \qquad F_{I,Q}(u) = \frac{1}{\mathbb{E}[Y_1|Q]} \int_0^u \mathbb{P}\left[Y_1 > y|Q\right] dy, \quad u \geq 0.$$

*Proof.* By a simple application of the formula to the independent case [Rolski et al., 1999] (see Theorem 13) it is obvious that, for $Q \in \mathscr{B}$ we have

$$\psi(u|Q) = 1 - \frac{\rho_Q}{1+\rho_Q} \sum_{m=0}^{\infty} (1+\rho_Q)^{-m} F_{I,Q}^{*m}(u), \qquad u \geq 0,$$

which leads to the stated result. □

Indeed, instead of simulating many paths of the reserve process to then calculate a Monte Carlo estimation of the ruin probability, we can resort to Algorithm 1 below to obtain a estimator $\widehat{\psi}$ of $\psi$. An important ingredient of this algorithm are the values for $m$ and $n$, which represent the number of times we simulate $Q$ and the number of trajectories of $R^u$ for each $Q$, respectively.

---

**Algorithm 1** Estimator of $\psi(u)$

---

**for** $j = 1$ to $m$ **do**
  Simulate $Q_j$ using $\mu|_{\mathscr{B}}$.
  **for** $i = 1$ to $n$ **do**
    Simulate $M_{ij}$ using $Geom\left(\frac{\rho_{Q_j}}{1+\rho_{Q_j}}\right)$.
    Compute $S_{ij} = X_1 + \cdots + X_{M_{ij}}$ using $X_k \sim F_{I,Q_j}(u)$.
    **if** $S_{ij} > u$ **then**
      Make $Z_{ij} = 1$.
    **else**
      $Z_{ij} = 0$.
    **end if**
  **end for**
**end for**
**return** $\widehat{\psi(u)} = \frac{1}{mn}\sum_{ij} Z_{ij} + \mu(\mathscr{B}^c)$.

---

## 3.7 Exponential–Bernoulli model

To fix ideas, we devote the next four sections to develop some relevant models. In order to induce the dependence in the exchangeable claims, $Y$, we degenerate the de Finetti's measure to parametric family of distributions. That is, we assume that the claim sequence is constructed as $Y_i|Z \sim f(\cdot|Z)$ and $Z \sim F_Z$.

Also we construct exchangeable processes using the ideas of Mena and Nieto-Barajas [2010]; they suggest a way to build an exchangeable sequence with a required marginal for $Y_i$. Their method goes as follows: given the desired choice of marginal distribution for $Y$, say $F_Y$, introduce a latent variable $Z$ via the conditional distribution $F_{Z|Y}$, having parametrical support coinciding with the support of $F_Y$. With these two distributions one can easily obtain the marginal for $Z$ and, through Bayes theorem, the conditional $F_{Y|Z}$.

In this *Exponential–Bernoulli* model $\mu$ is a point measure that assigns probability $q \in (0,1)$ to the exponential distribution with mean $1/\alpha$ and probability $1-q$ to the exponential distribution with mean $1/\beta$, for some fixed $\alpha, \beta > 0$. Clearly, this model could be rephrase as $Y_i|Z = 0 \sim \text{Exp}(\alpha)$ and $Y_i|Z = 1 \sim \text{Exp}(\beta)$, with $Z \sim \text{Ber}(q)$. This implies that the marginal density is

$$
\begin{aligned}
f(y) &= \int Q(y)\mu(dQ) \\
&= q\alpha e^{-\alpha y} + (1-q)\beta e^{-\beta y}, \qquad (3.13)
\end{aligned}
$$

and the covariance between two claims is

$$\mathrm{Cov}(Y_i, Y_j) = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 q(1-q).$$

For the CL model the net profit condition (3.5) reduces to

$$p > \lambda \left[\frac{q}{\alpha} + \frac{1-q}{\beta}\right],$$

and for the CLEC model the condition $\mu(\mathscr{B}^c) = 0$ is equivalent to

$$p > \lambda \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\},$$

which is a stronger condition.

Let us assume that the net profit condition for the CLEC model is satisfied in order to compare the ruin probability under both modelling scenarios. In the CL model the solution for the Exp($\beta$) claim distribution is well known and given by

$$\psi(u|Q = \mathrm{Exp}(\beta)) = \begin{cases} 1 & \beta \leq \frac{\lambda}{p} \\ \frac{\lambda}{\beta p} \exp\left\{-u\left(\beta - \frac{\lambda}{p}\right)\right\} & \beta > \frac{\lambda}{p}, \end{cases} \tag{3.14}$$

and thus by applying Theorem 15, the ruin probability of the CLEC for the Exponential-Bernoulli model is

$$\psi(u) = q\frac{\lambda}{\alpha p} \exp\left\{-u\left(\alpha - \frac{\lambda}{p}\right)\right\} + (1-q)\frac{\lambda}{\beta p} \exp\left\{-u\left(\beta - \frac{\lambda}{p}\right)\right\}.$$

To obtain the ruin probability of the CL model with the same marginal distribution given in equation (3.13), which is the mixture of two exponential distributions, Panjer and Willmot [1992] used the Laplace transform inversion, and get

$$\psi_{CL}(u) = \frac{1}{(1+\theta)(r_2 - r_1)}\{(\rho - r_1)e^{-r_1 u} + (r_2 - \rho)e^{-r_2 u}\},$$

where

$$r_1 = \frac{\rho + \theta(\alpha+\beta) - \left[\{\rho + \theta(\alpha+\beta)\}^2 - 4\alpha\beta\theta(1+\theta)\right]^{1/2}}{2(1+\theta)},$$

$$r_2 = \frac{\rho + \theta(\alpha+\beta) + \left[\{\rho + \theta(\alpha+\beta)\}^2 - 4\alpha\beta\theta(1+\theta)\right]^{1/2}}{2(1+\theta)}$$

and

$$\rho = \frac{(1-q)\alpha^2 + \beta^2 q}{(1-q)\alpha + \beta q}, \qquad \theta = \frac{\alpha\beta p}{\lambda((1-q)\alpha + \beta q)} - 1.$$

See Figure 3.3 for a typical behavior of the ruin probabilities under this model.

Furthermore, the general expression for the CLEC ruin probability corresponding to the mixture of $m$ exponential distributions, is given by

$$\psi(u) = \sum_{i=1}^{m} q_i \frac{\lambda}{\alpha_i p} \exp\left\{-u\left(\alpha_i - \frac{\lambda}{p}\right)\right\}, \qquad u \geq 0, \tag{3.15}$$

where $\alpha_1, \ldots, \alpha_m$ are the parameters of the exponentials distributions and $q_1, \ldots, q_m$ are the weights of the mixture, such that $\sum_{i=1}^{m} q_i = 1$ and for $i = 1, \ldots, m$, $0 \leq q_i \leq 1$. Is important to note that this last equation also works for $m = \infty$.

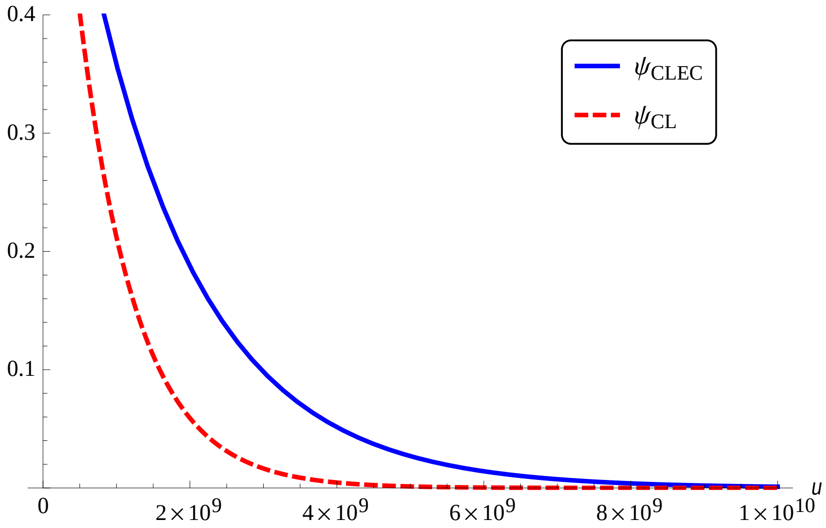



FIGURE 3.3: Example of the ruin probabilities for the Exponential-Binomial model. The solid line corresponds to the ruin probability of the CLEC model whereas the dashed line for the CL model.

## 3.8 Erlang–Geometric model

Let us fix the marginal claim distribution to be $Y \sim \text{Exp}(\beta)$ with mean $1/\beta$ and the conditional distribution $Z|Y \sim \text{Po}(\phi Y)$ for a fixed number $\phi > 0$. This implies that $Y|Z \sim \text{Erl}(z+1, \beta+\phi)$ with mean $(z+1)/(\beta+\phi)$, and $Z$ is geometric distributed with mass probability function given by

$$f(z) = \left(\frac{\beta}{\beta+\phi}\right)\left(\frac{\phi}{\beta+\phi}\right)^{z} \mathbb{1}_{\{0,1,2,\ldots\}}(z). \tag{3.16}$$

For the CLEC model we use the explicit solutions of the Erlang claims distribution, see Section 2.7. In Asmussen and Albrecher [2010], a tractable expression for the ruin probability under the classic independent risk model with Phase-type claims is given (see Section 2.7). Using the fact that the Erlang distribution is a special case of the Phase-type class of distributions, the conditional probability of ruin is given by

$$\psi(u|Z) = \begin{cases} He^{Du}\mathbf{1} & 0 \le Z \le \kappa - 2 \\ 1 & \kappa - 2 < Z, \end{cases} \tag{3.17}$$

where $\kappa$ is the largest integer less or equal to $p(\beta+\phi)/\lambda$,

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}_{Z\times 1}, \qquad D = \begin{pmatrix} -\beta & \beta & 0 & 0 & \cdots & 0 \\ 0 & -\beta & \beta & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & -\beta & \beta & 0 \\ 0 & 0 & 0 & 0 & -\beta & \beta \\ \frac{\lambda}{p} & \frac{\lambda}{p} & \frac{\lambda}{p} & \frac{\lambda}{p} & \frac{\lambda}{p} & \frac{\lambda}{p} - \beta \end{pmatrix}_{Z\times Z},$$

$$\text{and} \qquad H = \left(\tfrac{\lambda}{p\beta}, \tfrac{\lambda}{p\beta}, \ldots, \tfrac{\lambda}{p\beta}\right)_{1\times Z}.$$

Following Theorem 15, we have

$$\begin{aligned}
\psi(u) &= \int_{\mathscr{P}} \psi(u|Z)\, d\mu(Z) \\
&= \sum_{z=0}^{\kappa-2} He^{Du}\mathbf{1}\left(\frac{\beta}{\beta+\phi}\right)\left(\frac{\phi}{\beta+\phi}\right)^{z} + \sum_{z=\kappa-1}^{\infty}\left(\frac{\beta}{\beta+\phi}\right)\left(\frac{\phi}{\beta+\phi}\right)^{z} \\
&= \sum_{z=0}^{\kappa-2} He^{Du}\mathbf{1}\left(\frac{\beta}{\beta+\phi}\right)\left(\frac{\phi}{\beta+\phi}\right)^{z} + \left(\frac{\phi}{\beta+\phi}\right)^{\kappa-1}
\end{aligned}$$

In this model is important to remark that

$$\mu\left(\mathscr{B}^c\right) = \left(\frac{\phi}{\beta+\phi}\right)^{\kappa-1},$$

and

$$\mathrm{Corr}(Y_1, Y_2) = \frac{\phi}{\beta + \phi}.$$

Hence, the parameter $\phi$ controls the correlation of the exchangeable process $Y$, which ranges in the interval $[0,1)$. Figure 3.4 shows a comparison of the ruin probabilities for the CL and CLEC models, for this model under different values of $\phi$.

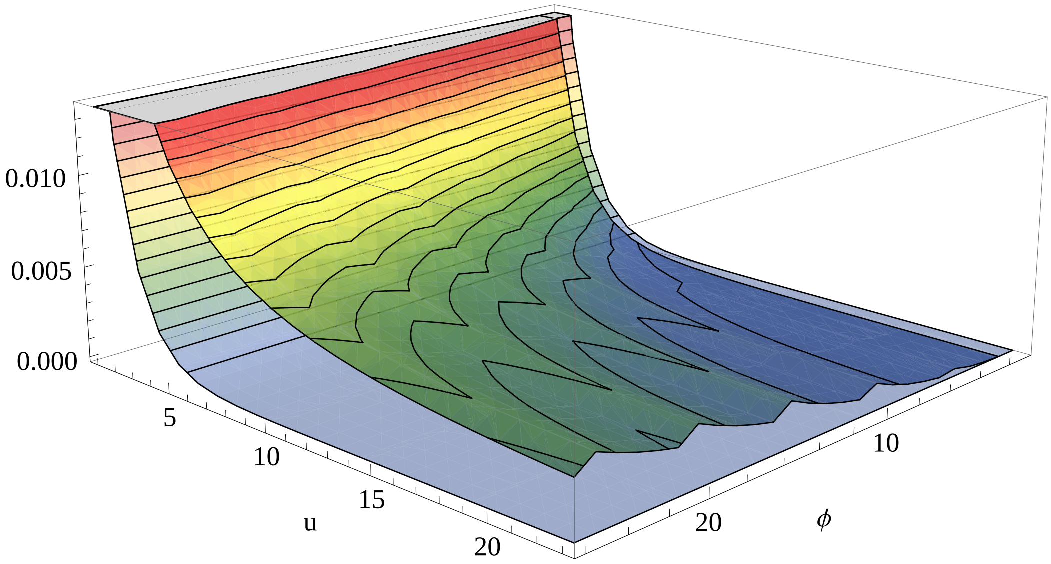


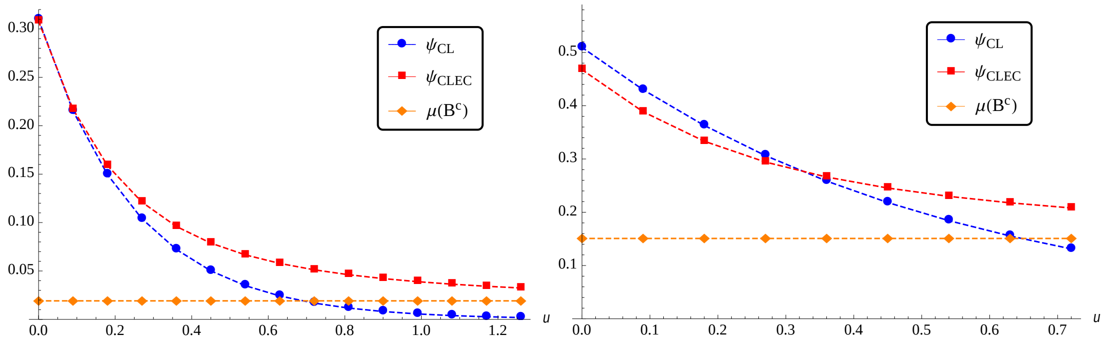

FIGURE 3.4: Example of the probabilities of ruin for the Erlang-Geometric model. The top graphic is the probability of ruin for the CLEC model, and the bottom one is the analogous for the CL model. The figure shows the different behavior for distinct values of $\phi$.

## 3.9 Exponential–Gamma model

For the Exponential-Gamma model we assume that the claims have conditional distribution given by $Y_i|Z \sim \mathrm{Exp}(Z)$, with $Z \sim \mathrm{Ga}(a,b)$, where this latter refers to a Gamma distribution with mean $a/b$. As before, the advantage of this model is that some of the expressions are analytic; for instance, the distribution of claims has density

$$f(y) = \frac{ab^a}{(y+b)^{a+1}} \mathbb{1}_{[0,\infty)}(y).$$

That is, a Pareto distribution with mean $b/(a-1)$, for $a > 1$. Note that the Pareto distribution is heavy tailed, and in the case $a \in (0,1)$ its mean becomes infinite. Applying




(A) $(a,b,p,\lambda) = (5.8, 3.4, 4.2, 7.6)$. (B) $(a,b,p,\lambda) = (3.8, 6.3, 1.3, 2.6)$.

FIGURE 3.5: Example of the behavior of the ruin probability for the *Exponential-Gamma model* for different parameters. The probability of ruin on the CLEC model (squares) is represented by $\psi_{CLEC}$, and $\psi_{CL}$ represent the analogue for the CL model (circles), both with the same parameters for each graphic. The straight line (diamonds) represents the limit behavior of the CLEC model. In both graphics the CLEC net profit condition (3.7) is not accomplished.

equation (3.14) and Theorem 15, we obtain

$$\begin{aligned}\psi(u) &= \int_{\mathscr{P}} \psi(u|Z)\,d\mu(Z)\\ &= \int_0^{\frac{\lambda}{p}} 1 \times \frac{b^a z^{a-1} e^{-bz}}{\Gamma(a)} dz + \int_{\frac{\lambda}{p}}^{\infty} \frac{\lambda}{zp} \exp\left\{-u\left(z - \frac{\lambda}{p}\right)\right\} \times \frac{b^a z^{a-1} e^{-bz}}{\Gamma(a)} dz\\ &= \lambda b^a (b+u)^{1-a} e^{-\frac{\lambda u}{p}} \frac{\Gamma\left(a-1, \frac{(b+u)\lambda}{p}\right)}{p\Gamma(a)} + 1 - \frac{\Gamma\left(a, \frac{b\lambda}{p}\right)}{\Gamma(a)}.\end{aligned}$$

Figure 3.5 shows the evolution of the ruin probabilities for both, the CLEC and CL models, as the initial surplus increases. In this model the function $\psi$ starts at the value

$$\psi(0) = \frac{b\lambda\Gamma\left(a-1, \frac{b\lambda}{p}\right)}{p\Gamma(a)} + 1 - \frac{\Gamma\left(a, \frac{b\lambda}{p}\right)}{\Gamma(a)},$$

and decrease with an exponential rate to the value

$$\lim_{u\to\infty} \psi(u) = 1 - \frac{\Gamma\left(a, \frac{b\lambda}{p}\right)}{\Gamma(a)}.$$

Notice that the last two equations change remarkably under the independence hypothesis. If we assume independence among claims, provided only the classical net profit condition is satisfied, we obtain

$$\psi_{CL}(0) = \frac{\lambda\mathbb{E}[Y_1]}{p} = \frac{\lambda b}{p(a-1)},$$

$$\lim_{u\to\infty} \psi_{CL}(u) = 0,$$

and so

$$\psi(0) - \psi_{CL}(0) = \frac{b\lambda\Gamma\left(a-1, \frac{b\lambda}{p}\right)}{p\Gamma(a)} - \frac{\Gamma\left(a, \frac{b\lambda}{p}\right)}{\Gamma(a)} + 1 - \frac{b\lambda}{(a-1)p}$$

This last quantity could be positive or negative; for instance, for the values $a = 4$, $b = 3$, $\lambda = 2$, and $p = 10$ is negative, and for $a = 43$, $b = 4$, $\lambda = 7$, and $p = 3$ is positive. This tell us that, when seen as function of the initial surplus, in some cases the ruin probabilities for the CL model could start above the corresponding to the CLEC model, but at some point the two probabilities intercept, and then after this point the ruin probabilities of the CLEC model are the biggest. Of course, this last behavior does not occur when at the outset the parameters are chosen such that (3.7) is attained.

## 3.10 Phase-type–Dirichlet model

This final model uses one of the most general claim distributions with available analytical expressions for the ruin probabilities, under the classical CL framework. Indeed, the Phase-type distributions embrace a great variety of distributions on the positive real line, enjoying a good balance between generality and tractability. They can be seen to represent the absorption times of certain Markov jump processes with $\{1,\ldots,m\}$ transient states and an absorbing one $m+1$, see Bladt [2005] and the references therein. An appealing feature of the class of Phase-type distributions is that it is weakly dense in the space of all $\mathbb{R}_+$-valued distributions, see Asmussen [2003]. To construct a parametric example over these distributions, we define the Phase-type-Dirichlet model where the claims have conditional distribution given by $Y_i|Z \sim \mathrm{PH}(Z,S)$, with $Z \sim \mathrm{D}(a_1,\ldots,a_p)$. Where by $X \sim \mathrm{PH}(\pi,S)$, we mean a Phase-type random variable with density

$$f(x) = \pi' e^{Sx}(-S\mathbf{1}), \qquad \text{for all } x \geq 0 \tag{3.18}$$

where the parameter $\pi = (\pi_1, \ldots, \pi_m)$ represents the vector of initial probabilities on the transient states, $S$ is the subgenerator $(m \times m)$-matrix of the underlying Markov chain and $\mathbf{1}$ is a $m$-vector of ones. The matrix $S$ is defined by

$$\Lambda = \begin{pmatrix} S & -S\mathbf{1} \\ \mathbf{0} & 0 \end{pmatrix}$$

where $\Lambda$ is the intensity matrix of the Markov chain and $\mathbf{0}$ is a transpose $m$-vector of zeros. Without loss of generality we are assuming that this Markov chain has a zero probability of start at the absorbing state $m+1$.

Similarly, $D(a_1, \ldots, a_m)$ denotes a Dirichlet distribution with parameters $a_1, \ldots, a_m > 0$, with density given by

$$f(z_1, z_2, \ldots, z_m) = \frac{\Gamma(a_1 + \ldots + a_m)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_m)} z_1^{a_1 - 1} \ldots z_m^{a_m - 1}, \tag{3.19}$$

for nonnegative numbers $z_1, \ldots, z_n$ such that $\sum_{i=1}^m z_i = 1$, and zero otherwise. This distribution has expected value $\boldsymbol{A}/a$ with $\boldsymbol{A} = (a_1, \ldots, a_m)$, and $a = \sum_{i=1}^m a_i$.

In an analogue manner of previous models, the distributions of equations (3.18) and (3.19) will help us to define an exchangeable sequence, though in this case with Phase-type marginal distributions. To this end, we will first proof some auxiliary results.

**Lemma 1.** *If* $Y|Z \sim \mathrm{PH}(Z, S)$ *and* $Z \sim \mathrm{D}(a_1, \ldots, a_m)$*, then the marginal* $Y \sim \mathrm{PH}(\boldsymbol{A}/a, S)$.

*Proof.* Let $\Delta = \{z = (z_1, \ldots, z_m) : 0 \le z_1, \ldots, z_m, \sum_{i=1}^m z_i = 1\}$

$$\begin{aligned} f(y) &= \int f(y|z) f(z) dz \\ &= \int_\Delta z^T e^{Sy}(-S\mathbf{1}) \frac{\Gamma(a_1 + \ldots + a_m)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_m)} z_1^{a_1 - 1} \ldots z_m^{a_m - 1} dz \\ &= \left[ \int_\Delta z^T \frac{\Gamma(a_1 + \ldots + a_m)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_m)} z_1^{a_1 - 1} \ldots z_m^{a_m - 1} dz \right] e^{Sy}(-S\mathbf{1}), \end{aligned} \tag{3.20}$$

where the vector between brackets in equation (3.20) is given by

$$\begin{aligned} &\left[ \int_\Delta z^T \frac{\Gamma(a_1 + \ldots + a_m)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_m)} z_1^{a_1 - 1} \ldots z_m^{a_m - 1} dz \right]_i \\ &\qquad = \int_\Delta z_i^T \frac{\Gamma(a_1 + \ldots + a_m)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_m)} z_1^{a_1 - 1} \ldots z_n^{a_m - 1} dz \end{aligned}$$

$$= \int_{\Delta} z_i \frac{\Gamma(a_1 + \ldots + a_m)}{\Gamma(a_1) \cdot \ldots \cdot \Gamma(a_m)} z_1^{a_1 - 1} \ldots z_n^{a_m - 1} dz$$

$$= a_i / a, \tag{3.21}$$

which leads to the stated result. □

This property, of closure under marginalization, tell us essentially that if we are able to obtain analytic expression for the ruin probability in the CL model with Phase-type claims hence we are potentially able to do it for the CLEC model as well, subject to a marginalization of $Z$.

Before depicting the ruin probability, let us first, display some relevant moment properties for the claim reserving processes.

**Lemma 2.** *If $Y_i|Z \sim \mathrm{PH}(Z,S)$ are conditional independent random variables given $Z$, and $Z \sim \mathrm{D}(a_1,\ldots,a_m)$ then*

1. $\mathbb{E}[Y_i] = -\boldsymbol{A}S^{-1}\boldsymbol{1}/a$, *and* $\mathsf{Var}(Y_i) = 2\boldsymbol{A}S^{-2}\boldsymbol{1}/a - (\boldsymbol{A}S^{-1}\boldsymbol{1}/a)^2$.
2. $\mathsf{Cov}(Y_1, Y_2) = (S^{-1}e)^T \mathsf{Var}(Z)(S^{-1}e)$.

*In the last expression the covariance matrix* $\mathsf{Var}(Z)$ *is given by* $\mathsf{Var}(Z)_{ij} = -a_i a_j / a^2(a+1)$ *and* $\mathsf{Var}(Z)_{ii} = -a_i(a-a_i)/a^2(a+1)$, *for* $1 \le i < j \le m$, *and* $a = \sum_{i=1}^m a_i$.

*Proof.*

1. It follows directly from Lemma 1 and the moment expressions for a Phase-type distribution [see Bladt, 2005].
2. Using the properties of conditional independence we obtain

$$\begin{aligned}
\mathsf{Cov}(Y_1, Y_2) =& \mathbb{E}[\mathsf{Cov}(Y_1, Y_2|Z] + \mathsf{Cov}(\mathbb{E}[Y_1|Z], \mathbb{E}[Y_2|Z]) \\
=& \mathsf{Cov}(\mathbb{E}[Y_1|Z], \mathbb{E}[Y_2|Z]) \\
=& \mathsf{Cov}(-ZS^{-1}e, -ZS^{-1}e) \\
=& \mathsf{Var}(-ZS^{-1}e) \\
=& (S^{-1}e)^T \mathsf{Var}(Z)(S^{-1}e),
\end{aligned}$$

and the proof ends using the the second moments of $Z$.

□

Asmussen and Albrecher [2010] showed that under the classic independent risk model, with phase-type claims, the probability of ruin is given by

$$\psi(u|Z) = \begin{cases} -\frac{\lambda}{p}\pi' S^{-1} \exp\left\{u\left(S + S\mathbf{1}\lambda\pi' S^{-1}/p\right)\right\}\mathbf{1} & p > -\lambda\pi' S^{-1}\mathbf{1} \\ 1 & p \leq -\lambda\pi' S^{-1}\mathbf{1}. \end{cases} \tag{3.22}$$

Analogously to the previous models, in order to take advantage of equation (3.22), we construct the exchangeable sequence with the above conditional representation of the Phase-type distribution via a Dirichlet random variable. Hence, the ruin probability becomes

$$\psi(u) = \int_{\mathscr{P}} \psi(u|Q)\mu(dQ)$$

$$= \int_{\Delta\cap\{p\leq\lambda z^T S^{-1}\mathbf{1}\}} \frac{\Gamma(a_1+\ldots+a_m)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_m)} z_1^{a_1-1}\ldots z_m^{a_m-1} dz$$

$$+ \int_{\Delta\cap\{p>\lambda z^T S^{-1}\mathbf{1}\}} \left(-\frac{\lambda}{p}\pi' S^{-1} \exp\left\{u\left(S + S\mathbf{1}\lambda\pi' S^{-1}/p\right)\right\}\mathbf{1}\right) \frac{\Gamma(a_1+\ldots+a_m)}{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_m)} z_1^{a_1-1}\ldots z_m^{a_m-1} dz$$

The complexity of the region $\Delta \cap \{p > \lambda\pi' S^{-1}\mathbf{1}\}$ prevent us to obtain a compact general analytic expression, but for a fixed array of parameters we could give the formula for $\psi$. For example, if we denote by $\text{Be}(\alpha,\beta)$ a beta density with mean $\alpha/(\alpha+\beta)$ and define an exchangeable sequence with conditional distribution $Y_i|Z \sim \text{PD}(\pi, S)$ for $i = 1,2,\ldots$ with $\pi = (Z, 1-Z)^T$ and $Z \sim \text{Be}(\alpha,\beta)$, we obtain the following expression for the ruin probability

$$\psi(u) = \int \psi(u|Z=z)dF_Z(z)$$

$$= \int_{[0,1]\cap\{r>\lambda\pi' S^{-1}\mathbf{1}\}} \left(ve^{(T+(-S\mathbf{1})v)u}\mathbf{1}\right) \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha,\beta)} dz + \int_{[0,1]\cap\{r\leq\lambda\pi' S^{-1}\mathbf{1}\}} \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha,\beta)} dz,$$

which can be calculated for a fixed set of parameters.

## 3.11 Concluding remarks

The CLEC model provides with an appealing alternative to the CL model that relaxes the important assumption of independence among claims. Indeed, it does not complicate the obtention of ruin probability results when contrasted to the independent claims framework.

An important observation, is that when working under the classical net profit condition, the ruin probability for the CLEC model might have a positive lower bound, when seen as a function of the initial surplus. Such observation might be seen as a warning message, namely that under dependent claim scenarios, there might not exist enough reserve to undertake a certain insurance business, at least not when working with the usual net profit condition. Therefore, it is advisable to work under the CLEC model from start together with its new net profit condition (3.7).

The exchangeability dependence scenario treated here for the CL model might be also valid for more general frameworks, such as that where the reserve is model by a more general exchangeable increments process. Indeed, more general results, such as those encountered in the Gerber-Shui Theory [Kyprianou, 2013]; part of our ongoing research focuses in these subjects. Furthermore, in the next chapter we analyze implications of a more general claim times regime.

# Chapter 4

# Exchangeable renewal equations

A renewal process is used to model occurrences of events happening at random times, where the times between the occurrences can be approximated by independent and identically distributed nonnegative random variables. Some applications of such a models can be typically found in: insurance risk models, queues, counter process, inventory systems, traffic flow, evolutionary genetic mechanisms, and economic structures.

A natural generalization of these models is given by considering a counting process for which the interarrival times are exchangeable. One of the advantages of the exchangeable scheme is that it allows us to establish different dependence schemes over the same fixed marginal renewal distribution. The study of this generalization was motivated by CLEC reserve model, because we want to establish a richer structure in the time occurrences of claims.

These kind of models are sometimes referred as mixed renewal models, but we prefer the term exchangeable to emphasize our viewpoint on their connection to the de Finetti's measure. In general, most of the literature of exchangeable renewal processes is devoted to their characterization and distributional properties. For a deep discussion of their distributional properties we refer the reader to Kallenberg [1973, 1974]. Accordingly, our contribution focuses in extending the idea of a renewal equation to the exchangeable setting. This could resolve the unrealistic assumption of independent renewals.

To present our analysis over exchangeable renewal processes, in Section 4.1 we will show the basic background on renewal theory that we will use. In Section 4.2 we present the new definitions that we apply in the exchangeable framework. We develop in Section 4.3 our main results over exchangeable renewal equations. Finally, in Section 4.4 we show further results over this exchangeable model.

Part of the results presented in this chapter can be found in Coen and Mena [2015a].

## 4.1 Classical renewal equation theory

In this section we will give a review of the classical renewal theory results that we will need for the rest of this chapter. Many proofs of them can be found in the literature; for instance, some classical references of renewal theory are Cox [1962], Doob [1948] and Smith [1958], and more recent treatments appear in Çinlar [1975], Asmussen [2003], Heyman and Sobel [2004], Gut [2009] and Gallager [2013].

A classical renewal process is a stochastic model for events that occur randomly in time. Its basic mathematical assumption is that the times between the successive arrivals are independent and identically distributed. This type of processes have a very rich and interesting mathematical structure and can be used as a foundation for building more realistic models.

Let us start by defining the renewal property for the independent framework, in mathematical terms.

**Definition 19.** *Let $T = \{T_i : i \in \mathbb{N}\}$ be nonnegative sequence of i.i.d. random variables with marginal density $F$, such that $F(0) < 1$. We define their **adding process** $S = \{S_n : n \in \mathbb{N}\}$ by*

$$S_0 = 0, \quad \textit{and} \qquad S_n = T_1 + T_2 + \cdots + T_n \quad n \geq 1.$$

*Accordingly, we define the **renewal process** $N = \{N_t : t \geq 0\}$ by*

$$N_t = \sup\{n \in \mathbb{N} : S_n \leq t\} \quad t \geq 0.$$

One interpretation of these variables is to think of $N_t$ as the number of arrivals to a system in the interval $(0, t]$, $T_n$ as the interarrival time of the $n$-arrival, and $S_n$ as the arrival time of the $n$-arrival. Figure 4.1 reviews the connection between the sample values of these random variables.

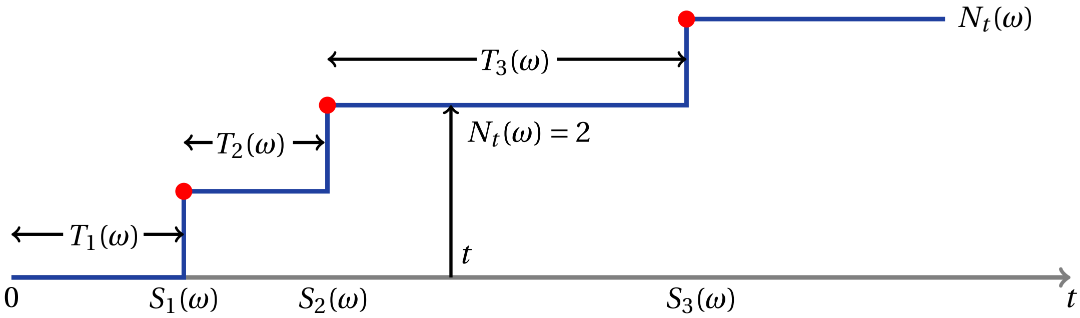



FIGURE 4.1: A sample function of an arrival process for a given sample point $\omega$ with its arrival epochs $\{S_n(\omega)\}_{n\in\mathbb{N}}$, its interarrival times $\{T_n(\omega)\}_{n\in\mathbb{N}}$, and its counting process $\{N_t(\omega)\}_{t\geq 0}$. The sample function of the counting process is the step function illustrated with a unit step at each arrival epoch.

Two important implications of the last definition are:

$$\{S_n \leq t\} = \{N_t \geq n\} \qquad \text{and} \qquad \{S_n > t\} = \{N_t < n\}.$$

Consequently, the renewal processes can be specified in three different ways: first, by the joint distributions of the arrival epochs $\{S_n\}_{n\in\mathbb{N}}$; second, by the joint distributions of the interarrival times $\{T_n\}_{n\in\mathbb{N}}$; and, third, by the joint distributions of the counting random variables, $\{N_t\}_{t\geq 0}$.

An frequently used operator in the theory of renewal processes is the convolution operator, because in many instances we need to express the distribution of a sum. Recall that the convolution of two distributions $F$ and $G$ of nonnegative random variables is defined by

$$F * G(t) = \int_0^t F(t-s)dG(s) = \int_0^t G(t-s)dF(s), \qquad t \geq 0.$$

The convolution $F * G$ gives the distribution function of the sum of two independent random variables with distribution functions $F$ and $G$ respectively. Let $F$ be the distribution for the $T_i$. We will write $F^{*n}$ for the convolution of $F$ with itself $n$-times, i.e., for the distribution of $T_1 + \ldots + T_n$. For convenience we will write $F^{*0}$ for the trivial distribution function associated to the random variable which is identically 0. Henceforth, we will follow this notation.

Much of renewal theory results concern with determining the properties of the *renewal function*,

$$U(t) = \mathbb{E}[N_t], \qquad t \geq 0, \tag{4.1}$$

which is the expected number of renewals up to time $t$. Its importance comes from the fact that in the independent framework the renewal function determines the probabilistic behavior of the renewal process. Accordingly, a big part of the literature about the classical renewal theory is concerned with determining the properties of this function. The following proposition shows an expression of $U$ in terms of the distribution of the renewals, $F$.

**Proposition 23.** *Following the notation of Definition 19, the classical renewal function (4.1) is determined by*

$$U(t) = \sum_{n=1}^{\infty} F^{*n}(t), \qquad t \geq 0. \tag{4.2}$$

*Proof.* We can rewrite the renewal process $N$ as

$$N_t = \sum_{n=1}^{\infty} I_n,$$

where

$$I_n = \begin{cases} 1 & \text{if the } n\text{th renewal ocurred in } [0,t] \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\begin{aligned} \mathbb{E}[N_t] &= \mathbb{E}\left[\sum_{n=1}^{\infty} I_n\right] \\ &= \sum_{n=1}^{\infty} \mathbb{E}[I_n] \\ &= \sum_{n=1}^{\infty} \mathbb{P}[S_n \leq t] \\ &= \sum_{n=1}^{\infty} F^{*n}(t), \end{aligned}$$

□

To obtain the equivalence between $U$ and the distribution of the renewals $F$, let us first obtain an expression for their Laplace transforms; by definition, the Laplace transform of the renewal function is given by

$$\mathscr{L}_U(s) = \int_0^\infty e^{-st}\, dU(t), \qquad s \geq 0.$$

The next result shows the relationship between the Laplace transform of $U$ and the one of $F$.

**Proposition 24.** *Using the notation of Definition 19, we have that the Laplace transform of $U$ and the one of $F$ are related by*

$$\mathscr{L}_U(s) = \frac{\mathscr{L}_F(s)}{1 - \mathscr{L}_F(s)}, \qquad s \geq 0. \tag{4.3}$$

*Proof.* Since $F$ is a distribution function and we are assuming assume $F(0) < 1$, when ever $s$ is nonnegative[1], then

$$0 < \mathscr{L}_F(s) < 1.$$

So, taking Laplace transformations of both sides equation (4.2) yields

$$\begin{aligned}
\mathscr{L}_U(s) &= \sum_{n=1}^{\infty} \mathscr{L}_{F^{*n}}(s) \\
&= \sum_{n=1}^{\infty} [\mathscr{L}_F(s)]^n \\
&= \frac{\mathscr{L}_F(s)}{1 - \mathscr{L}_F(s)}.
\end{aligned}$$

□

An equivalent expression to equation (4.3) is given by

$$\mathscr{L}_F(s) = \frac{\mathscr{L}_U(s)}{1 - \mathscr{L}_U(s)}. \tag{4.4}$$

So, equation (4.3) and equation (4.4) express the equivalence of $U$ and $F$. Therefore, the renewal function $U$ completely characterizes the renewal process $N$; this happens because $U$ determines $\mathscr{L}_U$, which determines $\mathscr{L}_F$ by equation (4.4), which in turn characterizes the renewal process $N$.

[1]This also works even if $s$ is complex but its real part when is positive.

Now, we will continue with the analysis of $N$ as the solution of a integral equation. An equation of the form

$$g(t) = h(t) + \int_0^t g(t-x)dF(x), \qquad t \geq 0,$$

is called a renewal-type equation. In convolution notation the above states that

$$g = h + g * F.$$

Either iterating the above expression, or applying Laplace transforms we can show that a renewal-type equation has the solution

$$g(t) = h(t) + \int_0^t h(t-x)dm(x), \qquad t \geq 0,$$

where $m(x) = \sum_{n=1}^{\infty} F^{*n}(x)$.

To show the integral expression of $U$, defined in equation (4.1), we need to present the concept of *renewal argument*; in general, many results can be efficiently obtained using this idea. This is simply the observation that at any renewal epoch, another renewal process starts which is a probabilistic replica of the original process. To explain this concept, let us consider a renewal process $\{N_t\}_{t\geq 0}$ with generating sequence $\{T_n\}_{n\in\mathbb{N}}$, then fix a positive integer $k$, and suppose that $S_k = s_k$. Now define a generating sequence $\{T'_n\}_{n\in\mathbb{N}}$ where for each sample path $\omega$, $T'_n(\omega) = T_{k+n}(\omega)$, and let $\{N'_{t-s_k}\}_{t\geq s_k}$ be its renewal counting process. For any $t' = t - s_k > 0$,

$$\begin{aligned}
\mathbb{P}\left[N'(t') \geq j\right] &= \mathbb{P}\left[T'_1 + \ldots + T'_j \leq t'\right] \\
&= \mathbb{P}\left[T_{k+1} + \ldots + T_{k+j} \leq t'\right] \\
&= \mathbb{P}\left[T_1 + \ldots + T_j \leq t'\right] \\
&= \mathbb{P}\left[N(t') \geq j\right],
\end{aligned}$$

since the $T$'s are i.i.d. . Consequently, this last expression is in fact the renewal argument in mathematical terms.

This implies that for $k = 1$

$$N_t = \begin{cases} 0 & \text{if } t < T_1 \\ 1 + N'(t - T_1) & \text{if } t \geq T_1, \end{cases}$$

or equivalently

$$N_t \stackrel{\mathrm{d}}{=} \begin{cases} 0 & \text{if } t < T_1 \\ 1 + N(t - T_1) & \text{if } t \geq T_1. \end{cases} \tag{4.5}$$

We will prove the next theorem using a renewal argument; it can be also proved inverting the Laplace transformations of equation (4.4). This theorem characterizes the utility function as a renewal equation.

**Theorem 25.** *A renewal function $U$ satisfies the renewal equation*

$$U(t) = F(t) + \int_0^t U(t-x)dF(x), \qquad t \geq 0, \tag{4.6}$$

*and the only solution of equation* (4.6) *that has an Laplace transform is given by equation* (4.2).

*Proof.* From equation (4.5),

$$\mathbb{E}[N_t | T_1] = \begin{cases} 0 & \text{if } T_1 > t \\ 1 + U(t - T_1) & \text{if } T_1 \leq t. \end{cases} \tag{4.7}$$

Taking expectations of the last expression yields

$$U(t) = \int_0^t [1 + U(t-x)]dF(x), \qquad t \geq 0,$$

which completes the proof. □

The expressions given in equations (4.2) and (4.6) are useful to unveil the values of $U$, and with them the behavior of its renewal process. In some cases we can apply the theory of differential equations to obtain analytical expression from this equations, whereas in other cases we can approximate the values. Indeed, the equation (4.2) allow us to approximate $U$ by calculate enough terms of the convolutions $F^{*n}$, to approximate the infinite sum. On the other hand, the equation (4.6) gives an expression we where $U(t)$ only depends on $U(t-x)$, $x \in [0, t)$, so recursive computation of $U$ is possible. In the next sections we will extend this results to the exchangeable framework.

## 4.2 Exchangeable renewal processes

Now, we will show the mathematical implications of an exchangeable structure among renewals. We will name such a counting process by exchangeable renewal process. This is a widely accepted assumption in the literature; for instance, applications of the exchangeable renewal process can be consulted in Shanthikumar [1987] for modeling components lifetimes, Kuczura [1973] for modeling streams of customers, Segerdahl [1970] for modeling the occurrence of rare events, and Cook et al. [1999] for modeling disease activity in studies of chronic conditions. In these last references the phenomena of study do not fulfill the independence among arrivals assumption. A exchangeable renewal model could be applied to phenomena where the probability laws of the interarrivals are symmetrical, with a positive correlation, and not necessary independent.

In order to analyze the concept of *conditional equation*, let us first define a exchangeable renewal process.

**Definition 20.** *Let $T = \{T_i : i \in \mathbb{N}\}$ be nonnegative sequence of exchangeable random variables with de Finetti's measure $\mu$ (see section 1.2), with marginal density $F$, such that $F(0) < 1$. We define their **exchangeable adding process** $S = \{S_n : n \in \mathbb{N}\}$ by*

$$S_0 = 0 \qquad S_n = T_1 + T_2 + \cdots + T_n \quad n \geq 1.$$

*Accordingly, we define the **exchangeable renewal process** $N = \{N_t : t \geq 0\}$ by*

$$N_t = \sup\{n : S_n \leq t\} \quad t \geq 0.$$

Each of this process definitions resemble the same structure as its homologous in the independent scheme. Nevertheless, in the exchangeable case we have a richer dependence structure to establish among the variables interactions.

It is important to remark the difference between exchangeable renewal processes and Cox[2] processes. A Cox process is a stochastic process which is a generalization of a Poisson process where the time-dependent intensity $\lambda(t)$ is itself a stochastic process, whereas exchangeable renewal process are not restricted to the Poisson distribution. The only case where the two definitions intersect, is in the case where the

[2] The process is named after the statistician David Cox, who first published the model in Cox [1955].

de Finetti's measure focuses all its mass in exponential distributions; known as mixed Poisson process.

We continue to have the relation

$$\{N_t = n\} = \{S_n \leq t < S_{n+1}\}. \tag{4.8}$$

A way of rephrasing equation (4.8) is to think of $t \to N_t$ as the inverse function of $n \to S_n$. This idea is used to convert results on $S$ to results on $N$ in the independent framework; for instance, limit theorems for one of the processes may be derived from the corresponding limit theorems for the other [see Gut, 2009, Ch. 2]. Part of our ongoing research focuses in obtaining limit results for the exchangeable renewal process.

**Definition 21.** *We denote by $U$ the* ***(exchangeable) renewal function****, which is given by*

$$U(t) = \mathbb{E}[N_t], \quad t \geq 0, \tag{4.9}$$

*Remark* 4.1. In this sense we will speak of a renewal function, without refer only to the independent case, but rather to the more general exchangeable case.

As already mentioned, one of the main reasons for focusing on this function is that uniquely determines de distributional properties of the renewal process in the classical framework. However, in the exchangeable case this uniqueness does not happen. A counterexample of this is given in the next example.

**Example 4.** *As previously noticed, the independent framework is a special case of the exchangeable when the de Finetti's allocates all its mass on a single distribution. With this in mind, we will show that we can construct an exchangeable renewal process, different to the Poisson process, but with the same exchangeable renewal function as the Poisson process. Let us define the exchangeable renewal process $N$ by the next conditional scheme*

$$T_i|\Delta \sim \mathrm{Exp}(\Delta)$$

$$\Delta \sim \mathrm{U}(0, 2\lambda),$$

*where $T_i$ are the interarrivals. Under these assumptions we obtain that*

$$\mathbb{E}[N_t] = \mathbb{E}[\mathbb{E}[N_t|Q]]$$

$$= \int_{\mathscr{P}} \mathbb{E}[N_t|Q]\,\mu(dQ)$$

$$= \int_0^{2\lambda} \mathbb{E}[N_t|\Delta]\,\frac{1}{2\lambda}d\Delta$$

$$= \int_0^{2\lambda} t\Delta\frac{1}{2\lambda}d\Delta$$

$$= \lambda t.$$

*Then, this exchangeable renewal process have the same renewal function as a Poisson process with parameter $\lambda$. In this example the intearrivals density is given by*

$$f_{T_i}(t) = \frac{1 - e^{-2t\lambda}(1+2x\lambda)}{2x^2\lambda}, \qquad t \geq 0$$

*which is completely different to the exponential distribution of parameter $\lambda$.*

Indeed, the renewal function does not characterize the exchangeable renewal process; in this framework we need a richer structure. To define such structure, we need to analyze the connection between the renewal function and the de Finetti's measure.

**Theorem 26.** *If $U$ is a renewal function with de Finetti's measure $\mu$, then*

$$U(t) = \sum_{n=1}^{\infty} \int Q^{*n}(t)\mu(dQ), \qquad t \geq 0.$$

*Proof.* Conditioning with $Q$ we obtain,

$$\begin{aligned} U(t) =& \mathbb{E}[\mathbb{E}[N_t|Q]] \\ =& \int \mathbb{E}[N_t|Q]\,\mu(dQ) \\ =& \int \sum_{n=1}^{\infty} Q^{*n}(t)\mu(dQ) \\ =& \sum_{n=1}^{\infty} \int Q^{*n}(t)\mu(dQ), \end{aligned}$$

in the third equality we use the fact that $\mathbb{E}[N_t|Q]$ is the classical renewal function with renewals distributed $Q$, and this is known to be equal to $\sum_{n=1}^{\infty} Q^{*n}(t)$ (see Proposition 23). □

As in the independent framework, it is important to known that $N_t$ has finite expectation; which is the subject of the next result.

**Theorem 27.** *If $U$ is a renewal function, then $U(t) < \infty$ for all $t \geq 0$.*

*Proof.* For a non random distribution $F$ we have $F^{*(n+m)} = F^{*n} * F^{*m}$, and

$$F^{*(n+m)}(t) = \int_0^t F^{*m}(t-x)dF^{*n}(x)$$
$$\leq F^{*n}(t)F^{*m}(t).$$

Because the variables $\{T_i : i \in \mathbb{N}\}$ are nonnegative, it follows that there exists a $r$ such that $\mathbb{P}[T_1 + \ldots + T_r > t] > 0$, for every fixed $t \geq 0$. In the case that the support of $F$ is $(0,\infty)$ then $r = 1$ is adequate. Using the fact any natural $n$ could be represented as $mr + k$, then $\{1,2,\ldots\} = \{mr + k : m \in \{0,1,\ldots,\}, k \in \{1,2,\ldots,r\}\}$,

$$\begin{aligned} F^{*n}(t) =& F^{*(mr+k)}(t) \\ =& F^{*(r+(m-1)r+k)}(t) \\ \leq& F^{*r}(t)F^{*(m-1)r+k}(t) \\ &\vdots \\ \leq& [F^{*r}(t)]^m F^{*k}(t). \end{aligned}$$

This implies that

$$\begin{aligned} U(t) &= \sum_{n=1}^{\infty} \mathbb{P}\left[\sum_{i=1}^{n} T_i \leq t\right] \\ &= \sum_{n=1}^{\infty} \int Q^{*n}(t)\mu(dQ) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{r} \int Q^{*(mr+k)}(t)\mu(dQ) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{r} \mathbb{P}\left[\sum_{i=1}^{mr+k} T_i \leq t\right] \\ &\leq \sum_{m=1}^{\infty} \sum_{k=1}^{r} \mathbb{P}\left[\sum_{i=1}^{mr} T_i \leq t\right] \mathbb{P}\left[\sum_{i=1}^{k} T_i \leq t\right] \end{aligned}$$

$$\leq \sum_{m=1}^{\infty} \sum_{k=1}^{r} \left[ \mathbb{P}\left[ \sum_{i=1}^{r} T_i \leq t \right] \right]^m \mathbb{P}\left[ \sum_{i=1}^{k} T_i \leq t \right]$$

$$= \sum_{m=1}^{\infty} \left[ \mathbb{P}\left[ \sum_{i=1}^{r} T_i \leq t \right] \right]^m \sum_{k=1}^{r} \mathbb{P}\left[ \sum_{i=1}^{k} T_i \leq t \right]$$

$$< \infty.$$

□

The last result implies that the number of arrivals in any finite interval is expected to be finite. This is a desirable property from a modeler point of view, since an infinite number of arrivals could be inadequate. Likewise, the last proof shows that

$$\mathbb{P}\left[ \sum_{i=1}^{n} T_i \leq t \right] \to 0 \qquad \text{as} \qquad n \to \infty,$$

which is equivalent to

$$\mathbb{E}\left[ Q^{*n}(t) \right] \to 0 \qquad \text{as} \qquad n \to \infty.$$

These last expression implies that the probability of a infinite number of arrivals converge to zero. So, an exchangeable renewal model has the characteristic of finite number of arrivals in a finite interval.

## 4.3 Conditional renewal equations

When one is working with exchangeable process it is useful to have expressions for conditional expectations with respect to elements of the $\sigma$-algebra of distributions $\mathscr{Q}$ (see section 1.2). Accordingly, we will present in this section two generalizations of the concept of a exchangeable renewal function; the *conditional renewal function*, and the *conditional renewal set function*. Moreover, these new definitions will be used to show our main result of this chapter, Theorem 29.

**Definition 22.** *We define the* ***conditional renewal function*** *as*

$$U(t|Q) = \mathbb{E}\left[N_t|Q\right], \qquad t \geq 0, Q \in \mathscr{P}, \tag{4.10}$$

*where $\mathscr{P}$ is the set of all nonnegative distribution functions of the support of $\mu$.*

For a non random distribution $F$, the mapping $t \mapsto U(t|F)$ is the classical renewal function with renewals distributed as $F$. Under these notations we obtain the next result.

**Theorem 28.** *The relation between the conditional renewal function and the renewal function is given by*

$$U(t) = \int U(t|Q)\mu(dQ), \qquad t \geq 0.$$

*Proof.* Conditioning over the values of the random distribution $Q$, we obtain:

$$\begin{aligned} U(t) =& \mathbb{E}[N_t] \\ =& \mathbb{E}[\mathbb{E}[N_t | Q]] \\ =& \int U(t|Q)\mu(dQ). \end{aligned}$$

The last equality follows by the de Finetti's representation theorem (see Theorem 3). □

Let us obtain an equation for the conditional renewal function. By conditioning to $T_1$ and $Q$, we obtain

$$\begin{aligned} U(t) =& \mathbb{E}[N_t] \\ =& \mathbb{E}[\mathbb{E}[N_t|Q]] \\ =& \int \int_0^\infty \mathbb{E}[N_t | T_1 = x, Q]\, dQ(x)\mu(dQ) \\ =& \int \int_0^t \mathbb{E}[N_t | T_1 = x, Q]\, dQ(x)\mu(dQ) \\ =& \int \int_0^t [1 + U(t-x|Q)]\, dQ(x)\mu(dQ) \\ =& \int \int_0^t 1 dQ(x)\mu(dQ) + \int \int_0^t U(t-x|Q)\, dQ(x)\mu(dQ) \\ =& \int Q(t)\mu(dQ) + \int \int_0^t U(t-x|Q)\, dQ(x)\mu(dQ) \\ =& \mathbb{E}[Q(t)] + \mathbb{E}[Q * U(t|Q)] \end{aligned}$$

And so we obtain the equation

$$U(t) = \mathbb{E}[Q(t)] + \mathbb{E}[Q * U(t|Q)] \tag{4.11}$$

This last equation motivates the next definition, which we will use to define conditional renewal equations.

**Definition 23.** *Let us introduce the* ***conditional renewal set function*** *as*

$$U(t|\mathcal{G}) = \mathbb{E}[N_t|Q \in \mathcal{G}]. \tag{4.12}$$

*for all* $t \geq 0$ *and* $\mathcal{G} \in \mathcal{Q}$.

It is important to notice the relation among Definition 21, Definition 22 and Definition 23. To this end, let us point out two special cases of the last definition. First, by an abuse of notation, for $\mathcal{G} = \{Q\}$ the conditional renewal function and the conditional renewal set function coincide if we let $U(t|Q) = U(t|\{Q\})$. Second, the conditional renewal set function and the renewal function coincide for $\mathcal{G} = \mathcal{P}$, i.e.,

$$U(t) = U(t|\mathcal{P}), \qquad t \geq 0.$$

This last equation not only implies that our notation is consistent, but also that the conditional renewal function grant us all the probabilistic information about the exchangeable renewal process. Let us exemplify this last point.

**Example 5.** *Let us define the conditional renewal set function by*

$$U(t|Q) = \begin{cases} \alpha t & \textit{with probability } p \\ \beta t & \textit{with probability } 1-p, \end{cases} \tag{4.13}$$

*for some fixed numbers* $\alpha, \beta > 0$ *and* $p \in (0,1)$. *This implies that*

$$Q \sim \begin{cases} \text{Exp}(\alpha) & \textit{with probability } p \\ \text{Exp}(\beta) & \textit{with probability } 1-p \end{cases} \tag{4.14}$$

*Consequently the de Finetti's measure $\mu$ is a measure that assigns probability $p$ to the exponential distribution with mean $1/\alpha$ and probability $1-p$ to the exponential distribution with mean $1/\beta$. Furthermore, under these assumptions we obtain that*

$$U(t) = p\alpha t + (1-p)\beta t, \qquad t \geq 0.$$

*So, the conditional renewal function characterizes completely the renewal process.*

Under these notations we can rewrite the conditional renewal function as

$$U(t|\mathscr{P}) = \mathbb{E}\left[Q(t)\right] + \mathbb{E}\left[Q * U(t|Q)\right], \qquad t \geq 0 \tag{4.15}$$

which is a generalization of equation (4.6). Furthermore, this last equation is a particular case of

$$U(t|\mathscr{G}) = \mathbb{E}\left[Q(t)\right] + \mathbb{E}\left[Q * U(t|Q); Q \in \mathscr{G}\right], \qquad t \geq 0, \mathscr{G} \in \mathscr{Q}. \tag{4.16}$$

The importance of this last expression lies in the fact that it contains all the needed information about the exchangeable renewal process. In the exchangeable framework the dependence structure is characterize by the de Finetti's measure, and since the classical renewal function is only a function of time, it do not have a rich enough structure to characterize the dependence among renewals. Instead, the conditional renewal function grant us this information in its second argument, $\mathscr{G}$. This function characterizes the de Finetti's measure.

The next theorem is our main result of this chapter. In this theorem we generalize the type of equation given in equation (4.16), and find its solution in terms of the conditional renewal function. Hereafter, we call this class of equations *conditional renewal equations.*

**Theorem 29.** *Let $a(t)$ be a bounded positive function over bounded intervals. Let $A(t|\mathscr{G}) : \mathbb{R}^+ \times \mathscr{Q} \to \mathbb{R}^+$. Then there is one and only one solution to the equation*

$$A(t|\mathscr{G}) = a(t) + \mathbb{E}\left[Q * A(t|Q); Q \in \mathscr{G}\right], \qquad t \geq 0, \mathscr{G} \in \mathscr{Q}, \tag{4.17}$$

*with*

$$\sup_{0 \leq s \leq t, \mathscr{G} \in \mathscr{Q}} |A(s|\mathscr{G})| < \infty,$$

*and the solution is given by*

$$A(t|\mathscr{G}) = a(t) + \mathbb{E}\left[a * U(t|Q); Q \in \mathscr{G}\right], \qquad t \geq 0, \mathscr{G} \in \mathscr{Q}. \tag{4.18}$$

*Proof.* First, let us see that the function

$$A'(t|\mathscr{G}) = a(t) + \mathbb{E}\left[a * U(t|Q); Q \in \mathscr{G}\right], \qquad t \geq 0, \mathscr{G} \in \mathscr{Q}, \tag{4.19}$$

is bounded over bounded intervals for fixed $\mathscr{G} \in \mathscr{Q}$. For a fix $T \in \mathbb{R}_+$, we have

$$\begin{aligned}
\sup_{0\leq t\leq T} A'(t|\mathscr{G}) &= \sup_{0\leq t\leq T} \left(a(t) + \mathbb{E}\left[a * U(t|Q); Q \in \mathscr{G}\right]\right) \\
&= \sup_{0\leq t\leq T} \left(a(t) + \int_{\mathscr{G}} a * U(t|Q)\mu(dQ)\right) \\
&\leq \sup_{0\leq t\leq T} a(t) + \int_{\mathscr{G}} \sup_{0\leq t\leq T} a * U(t|Q)\mu(dQ) \\
&= \sup_{0\leq t\leq T} a(t) + \int_{\mathscr{G}} \sup_{0\leq t\leq T} \int_0^t a(t-x)dU(x|Q)\mu(dQ) \\
&\leq \sup_{0\leq t\leq T} a(t) + \int_{\mathscr{G}} \int_0^T \sup_{0\leq t\leq T} a(t)dU(x|Q)\mu(dQ) \\
&= \sup_{0\leq t\leq T} a(t) + \sup_{0\leq t\leq T} a(t) \int_{\mathscr{G}} \int_0^T dU(x|Q)\mu(dQ) \\
&= \sup_{0\leq t\leq T} a(t) \left(1 + \int_{\mathscr{G}} U(T|Q)\mu(dQ)\right) \\
&\leq \sup_{0\leq t\leq T} a(t) \left(1 + \int U(T|Q)\mu(dQ)\right) \\
&= \sup_{0\leq t\leq T} a(t)\left(1 + U(T)\right).
\end{aligned}$$

This implies that

$$\sup_{0\leq t\leq T \ \ \mathscr{G}\in\mathscr{Q}} A'(t|\mathscr{G}) < \infty$$

Let see that $A'$ accomplishes equation (4.17). For the definition of $A'$, we see that

$$A'(t|\mathscr{P}) = a(t) + \mathbb{E}\left[a * U(t|Q)\right],$$

$$\begin{aligned} A'(t|Q) =& a(t) + a * U(t|Q) \\ =& a(t) + a * \sum_{n=1}^{\infty} Q^{*n}(t). \end{aligned}$$

These last expressions implies:

$$\begin{aligned} A'(t|\mathscr{G}) =& a(t) + \mathbb{E}\left[a * U(t|Q); Q \in \mathscr{G}\right] \\ =& a(t) + \int_{\mathscr{G}} a * U(t|Q)\mu(dQ) \\ =& a(t) + \int_{\mathscr{G}} a * \left(Q(t) + \sum_{n=2}^{\infty} Q^{*n}(t)\right)\mu(dQ) \\ =& a(t) + \int_{\mathscr{G}} a * \left(Q(t) + Q * \sum_{n=2}^{\infty} Q^{*(n-1)}(t)\right)\mu(dQ) \\ =& a(t) + \int_{\mathscr{G}} a * \left(Q(t) + Q * \sum_{n=1}^{\infty} Q^{*n}(t)\right)\mu(dQ) \\ =& a(t) + \int_{\mathscr{G}} Q * \left(a(t) + a * \sum_{n=1}^{\infty} Q^{*n}(t)\right)\mu(dQ) \\ =& a(t) + \int_{\mathscr{G}} Q * A'(t|Q)\mu(dQ) \\ =& a(t) + \mathbb{E}\left[Q * A'(t|Q); Q \in \mathscr{G}\right]. \end{aligned}$$

Therefore, $A'$ fulfills equation (4.17). To end the proof let us prove that is this solution is unique. Assume that the function $B$, bounded over intervals, is also solution to equation (4.17), i.e.,

$$B(t|\mathscr{G}) = a(t) + \mathbb{E}\left[Q * B(t|Q); Q \in \mathscr{G}\right].$$

As already mentioned the evaluation on $Q$ and $\mathscr{P}$ plays an important roll. In the case of equation (4.17) we obtain that

$$\begin{aligned} A(t|\mathscr{P}) =& a(t) + \mathbb{E}\left[Q * A(t|Q)\right], \\ A(t|Q) =& a(t) + Q * A(t|Q), \end{aligned}$$

and analogously for $B$. These last expressions imply

$$\begin{aligned}
|A(t|Q) - B(t|Q)| =& |Q * A(t|Q) - Q * B(t|Q)| \\
=& |Q * [A(t|Q) - B(t|Q)]| \\
& \vdots \\
=& |Q^{*n} * [A(t|Q) - B(t|Q)]| \\
=& \left| \int_0^t A(t-x|Q) - B(t-x|Q) dQ^{*n}(x) \right| \\
\le& \int_0^t |A(t-x|Q) - B(t-x|Q)| \, dQ^{*n}(x) \\
\le& Q^{*n}(t) \sup_{0 \le s \le t} |A(s|Q) - B(s|Q)| \\
\le& Q^{*n}(t) \sup_{0 \le s \le t, \mathscr{G} \in \mathscr{Q}} |A(s|\mathscr{G}) - B(s|\mathscr{G})|,
\end{aligned}$$

and so

$$\begin{aligned}
|A(t|\mathscr{G}) - B(t|\mathscr{G})| =& |\mathbb{E}[Q * A(t|Q) - Q * B(t|Q); Q \in \mathscr{G}]| \\
\le& \mathbb{E}[|Q * [A(t|Q) - B(t|Q)]|; Q \in \mathscr{G}] \\
\le& \mathbb{E}\left[ Q^{*n}(t) \sup_{0 \le s \le t, \mathscr{G} \in \mathscr{Q}} |A(s|\mathscr{G}) - B(s|\mathscr{G})|; Q \in \mathscr{G} \right] \\
\le& \mathbb{E}\left[ Q^{*n}(t); Q \in \mathscr{G} \right] \sup_{0 \le s \le t, \mathscr{G} \in \mathscr{Q}} |A(s|\mathscr{G}) - B(s|\mathscr{G})|,
\end{aligned}$$

and we obtain the uniqueness using that $A$ and $B$ are bounded and $\mathbb{E}[Q^{*n}(t)] \to 0$ as $n \to \infty$. This proves the theorem. □

This last theorem give us a new characterization of a exchangeable renewal process, but at the same time allows us to establish the particular scheme that the renewal process will follow.

Let us have some words about the value of this last result. To this end, let us analyze the equation (4.17) in the case of $\mathscr{G} = \mathscr{P}$, i.e.,

$$A(t) = a(t) + \mathbb{E}[Q * A(t|Q)], \qquad t \ge 0.$$

This last equation expresses that a change in the process $A$ is driven by the sum of two terms: $a(t)$, a given rate of change, plus an expected value. This second term is the mean value of the convolution of a random distribution $Q$ and the particular behavior of $A(t)$ given this distribution. Therefore, the solution of an exchangeable renewal equation is not restricted to a particular choice of distribution for the renewal, since its convolution part depends on $Q$. In other words, the behavior of the solution depends on the selection of a random distribution $Q$.

In the classical independent framework, many quantities of interest of the renewal process can be described by a renewal equation, and Theorem 29 gives the general solutions for the exchangeable framework. Indeed, it does not complicate the obtention of exchangeable renewal processes results when contrasted to the independent claims framework.

## 4.4 Other results of conditional renewal processes

The results that we present in this section grant us information about the behavior of the exchangeable renewal processes. They are extensions of useful results of the independent framework. Accordingly, the next proposition shows an expression for the expected value of the next renewal after the time $t$ (see Figure 4.2).

**Proposition 30.** *If $N$ is an exchangeable renewal processes with de Finetti's measure $\mu$, then*

$$\mathbb{E}\left[S_{N_t+1}\right] = \mathbb{E}\left[Y_1\right] + \int \mathbb{E}\left[Y_1|Q\right] U(t|Q)\mu(dQ), \qquad t \geq 0.$$

*Proof.* We known from the renewal theory on the independent framework [see Heyman and Sobel, 2004, p. 121], that for a fixed $Q$,

$$\mathbb{E}\left[S_{N_t+1}|Q\right] = \mathbb{E}\left[Y_1|Q\right]\left[U(t|Q)+1\right].$$

Then, the proof follows directly from de Finetti's representation and simple application of the fundamental property of conditional expectation.

$$\begin{aligned}\mathbb{E}\left[S_{N_t+1}\right] &= \mathbb{E}\left[\mathbb{E}\left[S_{N_t+1}|Q\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[Y_1|Q\right]\left[U(t|Q)+1\right]\right]\end{aligned}$$

$$= \mathbb{E}[Y_1] + \int \mathbb{E}[Y_1|Q]\, U(t|Q)\mu(dQ).$$

□

The next proposition gives us a lower bound for the renewal function.

**Proposition 31.** *If $U$ is an exchangeable renewal function with de Finetti's measure $\mu$, then*

$$U(t) \geq \int_{\mathscr{D}} \frac{t}{\mathbb{E}[Y_1|Q]}\mu(dQ) - 1, \qquad t \geq 0,$$

*where*

$$\mathscr{D} = \left\{Q \in \mathscr{P} : \mathbb{E}[Y_1|Q] < \infty\right\},$$

*is the set of random distributions with finite first moment.*

*Proof.* We known from the renewal theory on the independent framework [see Heyman and Sobel, 2004, p. 121], that for a fixed $Q$,

$$U(t|Q) \geq \frac{t}{\mathbb{E}[Y_1]} - 1 \qquad t \geq 0,$$

when $\mathbb{E}[Y_1|Q] < \infty$, and

$$U(t|Q) \geq -1 \qquad t \geq 0,$$

in the case of $\mathbb{E}[Y_1|Q] = \infty$. This gives

$$\begin{aligned} U(t) &= \mathbb{E}[U(t|Q)] \\ &= \int_{\mathscr{D}} U(t|Q)\mu(dQ) + \int_{\mathscr{D}^c} U(t|Q)\mu(dQ) \\ &\leq \int_{\mathscr{D}} \left(\frac{t}{\mathbb{E}[Y_1]} - 1\right)\mu(dQ) - \mu(\mathscr{D}) \\ &= \int_{\mathscr{D}} \frac{t}{\mathbb{E}[Y_1|Q]}\mu(dQ) - 1. \end{aligned}$$

□

Let us denote by $L_U$ the Laplace transformation of $U$, that is

$$L_U(s) = \int_0^\infty e^{-st} U(t)\,dt, \qquad s \geq 0. \tag{4.20}$$

The next result shows a way to calculate $L_U$ using the Laplace transforms of the random distributions.

**Proposition 32.** *Following the notation of equation* (4.20) *we have that*

$$L_U(s) = \int \frac{L_Q(s)}{1-L_Q(s)}\mu(dQ), \qquad s \geq 0, \tag{4.21}$$

*where $L_Q$ is the Laplace transform of the distribution $Q$.*

*Proof.*

$$\begin{aligned} L_U(s) &= \int_0^\infty e^{-st}U(t)dt \\ &= \int_0^\infty e^{-st}\sum_{n=1}^{\infty}\int Q^{*n}(t)\mu(dQ)dt \\ &= \int \sum_{n=1}^{\infty}\int_0^\infty e^{-st}Q^{*n}(t)dt\mu(dQ) \\ &= \int \sum_{n=1}^{\infty}[L_Q(s)]^n\mu(dQ) \\ &= \int \frac{L_Q(s)}{1-L_Q(s)}\mu(dQ), \end{aligned}$$

in the last equality we use that for any $Q$ distribution, and we are assuming that $Q(0) < 1$, whenever $s$ is nonnegative, $0 < L_Q(s) < 1$ □

Proposition 32 provides a criterion for obtain analytic expressions for $U$, by applying the inverse Laplace transform method on equation (4.21). Moreover, in cases when no analytic formulae is obtained, this last result allow us to use Laplace approximation techniques to obtain an estimator of $U$.

The final result that we will present gives a representation of the distribution of three variables related with the exchangeable renewal process. These variables characterize the local behavior of $N_t$. The random variable $A_t = t - S_{N_t}$, called the deficit; it represent the time since the last renewal before $t$. The random variable $B_t = S_{N_t+1} - t$, called the excess; it represents the time from $t$ until the next renewal occurs. Finally, the random variable $L_t = S_{N_t+1} - S_{N_t}$, called the spread; it represents the length of the interrenewal time in progress at time $t$.[3] Two important relations among this variables are:

$$L_t = A_t + R_t$$
$$= T_{N_t+1}.$$

Figure 4.2 presents a graphic representation of the relationship among these processes. Our definitions agree with classical ones, but the exchangeable assumption allow us to use them to model more general phenomena.

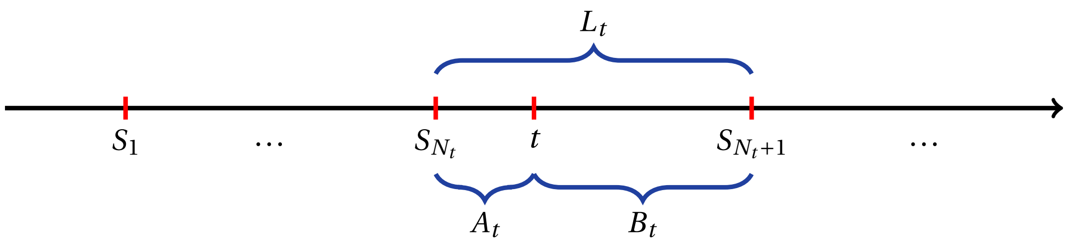



FIGURE 4.2: Graphic representation of the relationship among the processes $A_t$, $B_t$ and $L_t$.

**Proposition 33.** *Using the above notation, the variables $A_t$, $B_t$, and $L_t$ have the following representation for their distribution functions, under the exchangeable framework*

$$F_{A_t}(z) = \begin{cases} \int \int_{t-z}^{t} Q^c(t-x)dU(x|Q)\mu(dQ) & \text{if } z \le t \\ 1 & \text{if } z > t, \end{cases} \tag{4.22}$$

$$F_{B_t}(z) = \int Q(t+z)\mu(dQ) - \int_0^t Q^c(t+c-x)dU(x|Q)\mu(dQ), \tag{4.23}$$

[3] In the literature this variables are commonly also known under different names. $A_t$ is also called the backward recurrence time, or the age process. $B_t$ is also called, also, the residual life, or the forward recurrence time. Likewise, $L_t$ is also known as the recurrence time, or the length process.

*and*

$$F_{L_t}(z) = \int\int_{t-z}^{t} [Q(z) - Q(t-u)]\,dU(t|Q)\mu(dQ) + \begin{cases} 0 & \text{if } z \le t \\ \int Q(z) - Q(t)\mu(dQ) & \text{if } z > t, \end{cases} \tag{4.24}$$

*where $dU(t|Q)$ is the measure obtained with respect to the first argument, $t$.*

*Proof.* Let us first notice that the conditional distributions of the variables $A_t$, $B_t$ and $L_t$ coincide with the non-conditional distributions of the independent framework, when condition to the de Finetti's random distribution $Q$. So, using the results of the independent framework (see Karlin and Taylor [1975] and Haviv [2013]), we obtain that the conditional distribution of $A_t$ given $Q$ is

$$\mathbb{P}[A_t \le z|Q] = \begin{cases} \int_{t-z}^{t} Q^c(t-x)\,dU(x) & \text{if } z \le t \\ 1 & \text{if } z > t. \end{cases}$$

Therefore, applying the de Finetti's representation theorem we obtain

$$\begin{aligned} F_{A_t}(z) &= \int \mathbb{P}[A_t \le z|Q]\,\mu(dQ) \\ &= \begin{cases} \int\int_{t-z}^{t} Q^c(t-x)\,dU(x|Q)\mu(dQ) & \text{if } z \le t \\ 1 & \text{if } z > t. \end{cases} \end{aligned}$$

which establishes equation (4.22). The proofs of equations (4.23) and (4.24) follow similarly using

$$\mathbb{P}[B_t \le z|Q] = Q(t+z) - \int_0^t Q^c(t+c-x)\,dU(x|Q),$$

and

$$\mathbb{P}[L_t \le z|Q] = \int_{t-z}^{t} [Q(z) - Q(t-u)]\,dU(t|Q) + \begin{cases} 0 & z \le t \\ Q(z) - Q(t) & z > t. \end{cases}$$

□

The results presented in this chapter unveil some of the properties of the exchangeable renewal process. In general, these model are applicable to quite a range of practical probability problems, where the independence assumption is not fulfilled. These models have the particular benefit of a easy interpretation, while at the same time they do not complicate the obtention of theoretic results, compared to other dependence structures.

We are trying to generalize the results of Thorin [1970, 1971a], where is analyze the behavior of the ruin probabilities under a classical renewal process directing the claim arrivals, using the analysis presented in this chapter and the previous one.

# Conclusions

In this work, we have shown some new results over exchangeable and stationary processes. These distributional properties relax the sometimes unrealistic assumption of independence, without compromising much its mathematical treatment. Furthermore, we present the particular advantages of our models to resemble dependent structures.

As mention in the preface, this work considers three different aspects about these dependent structures. Let us mention our conclusions and future work, on each of this aspects.

The first aspect that we analyzed is the construction of multivariate stationary processes with a given stationary distribution. We have seen that our generalization of the ideas of Pitt and Walker [2005] and Mena and Walker [2009] is effective to construct multivariate reversible stationary processes. Indeed, the parametric models that we present, using this construction, could be used to resemble a wide variety of different behaviors, by changing their parameter values.

By studding the Gaussian-Gaussian model we obtained an appealing construction of reversible Ornstein-Uhlenbeck multivariate diffusion processes. Our construction scheme has the advantage of an easy implementation and interpretation, compared to the usual constructions.

On the other hand, the Wishart-Wishart model could be use to model the evolution of a covariance matrix. This process has the Wishart distribution as its stationary distribution. Accordingly, this model can be used to analyze the volatility currency option pricing, and also the correlation matrix of interest rates.

We also present a heavy tailed stationary processes, the Gaussian-Wishart model. Its heavy tail property is proved in Theorem 4. This process has t-multivariate stationary distribution. The use of this distribution is enjoying nowadays a vast interest due to its use in mathematical finance, which makes this model appealing to applications.

The second aspect that we study is the computation of ruin probabilities under exchangeable claim amounts scenario. We extend various important results of the independent framework to the exchangeable one.

The motivation to analyze the CLEC model was that among the driving assumptions in classical collective risk models, the independence among claims is frequently violated by real applications. Therefore, there is an evident need of models that relax such a restriction.

The main result that we obtain is that the ruin probability under the exchangeable claims model can be represented as the expected value of the ruin probabilities corresponding to certain independent claims cases. This allows us to extend some classical results to our dependent claims scenario. The main tool that we use was the de Finetti's representation theorem for exchangeable random variables, and as a consequence a natural Bayesian modeling feature for risk processes becomes available.

An important observation, is that when working under the classical net profit condition, the ruin probability for the CLEC model might have a positive lower bound, when seen as a function of the initial surplus. Such observation might be seen as a warning message, namely that under dependent claim scenarios, there might not exist enough reserve to undertake a certain insurance business, at least not when working with the usual net profit condition. Therefore, it is advisable to work under the CLEC model from start, together with its new net profit condition.

Finally, the third aspect that we analyze is the implications of the renewal equation if the renewals are exchangeable. We characterize the exchangeable renewal process in terms of the conditional renewal function. In particular, we show that this function can be rewritten as the solution of a new type of equations; exchangeable renewal equations. Furthermore, we obtain the general solution to this type of equations.

One of the advantages of the exchangeable scheme is that it allows us to establish different dependence schemes over the same fixed marginal renewal distribution. Consequently, this model could be use to resemble the real interplay among renewals.

The results presented over this aspect unveil some of the properties of the exchangeable renewal process. In general, these models are applicable to quite wide range of practical probability problems, where the independence assumption is not fulfilled. These models have the particular benefit of a easy interpretation, while at the same time they do not complicate the obtention of theoretic results, compared to other dependence structures.

Also is important to mention that we are trying to generalize the results of Thorin [1970, 1971a], where is analyze the behavior of the ruin probabilities under a classical renewal process directing the claim arrivals.

As a conclusion, the exchangeable and stationary models arise in a wide variety of problems where statistical analysis is applied; they represent two of the most natural dependence structures. Accordingly, the results of this work can be applied to unveil important characteristics of real phenomena.

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