UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

PROGRAMA DE POSGRADO EN CIENCIAS MATEMÁTICAS

THE MENGER AND ROTHBERGER PROPERTIES IN $C_p(X,2)$

TESIS

QUE PARA OPTAR POR EL GRADO DE:

DOCTOR EN CIENCIAS MATEMÁTICAS

PRESENTA:

DANIEL BERNAL SANTOS

DIRECTOR DE TESIS:

DR. ÁNGEL TAMARIZ MASCARÚA,

FACULTAD DE CIENCIAS, UNAM

MIEMBROS DEL COMITÉ TUTOR:

DR. ALEJANDRO ILLANES MEJÍA, INST. DE MAT., UNAM

DR. ADALBERTO GARCÍA MÁYNEZ Y CERVANTES, INST. DE MAT., UNAM

FACULTAD DE CIENCIAS, UNAM, MÉXICO, D.F., OCTUBRE DE 2015

UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

POSGRADO EN CIENCIAS MATEMÁTICAS

FACULTAD DE CIENCIAS

THE MENGER AND ROTHBERGER PROPERTIES IN $C_p(X,2)$

Autor:
M. EN C. DANIEL BERNAL-SANTOS

Director de Tesis:
DR. ÁNGEL TAMARIZ-MASCARÚA

Octubre 2015

Agradecimientos

A Dios por darme la oportunidad de vivir.
A mis padres y hermanos, que con sus consejos he podido llegar al día de hoy.
A mi querida esposa Vero, que gracias a su apoyo y paciencia logré culminar un paso mas en mi preparación profesional.
A mi director de tesis, Dr. Ángel Tamariz Mascarúa, que sin sus enseñanzas, sugerencias y preguntas hubiese sido imposible este trabajo.
A los Alejandro's (Darío y Dorantes) y Reynaldo, por su atención y sugerencias en el seminario.
Al CONACyT por otorgarme la beca 322416/233751.
Finalmente, a mis files amigos; el Blacky†, el Chockys, el Canelo†, la Güera(†), la Mayca(†), el Pepereto, el Goldduck y los que faltaron.

Introducción

W. Hurewicz en [Hur26] introdujo una propiedad topológica del tipo cubierta y demostró que esta propiedad es equivalente a la propiedad E introducida por M.K. Menger en [Men24]. Dicha propiedad está localizada entre las propiedades de σ-compacidad y Lindelöf. W. Hurewicz conjeturó que la propiedad E de Menger caracterizaba la σ-compacidad de los espacios métricos, sin embargo, A. W. Miller en [MF88] muestra la existencia de un subespacio de los números reales de tamaño ω_1 con la propiedad E, contradiciendo de esta manera la *Conjetura de Hurewicz.*

En los primeros artículos de investigación, los espacios (métricos) con la propiedad E son llamados *espacios Hurewicz* por Lelek en [Lel69]. En la más reciente literatura, M. Scheepers y M. Sakai han llamado a estos espacios *espacios Menger* o espacios con la *propiedad Menger* [Sak09; Sch99a]. El presente trabajo continuará con la terminología de estos dos últimos autores.

En la actualidad la propiedad Menger difícilmente es estudiada de manera singular, la propiedad Menger, en conjunto con otras variaciones de esta misma propiedad, trajo consigo la introducción de muchas otras propiedades del tipo cubierta. Marion Scheepers en [Sch99a], y sus subsecuentes artículos, inició la era moderna de la *combinatoria infinita en topología,* o bien *principios de selección en matemáticas.* En estos trabajos se hace una generalización de la propiedad Menger mediante el uso de algunos *"operadores de selección"* (S_{fin}, S_1 y U_{fin}). Cada uno de estos operadores generalizan algunas propiedades del tipo cubierta ya conocidas como la propiedad *Hurewicz, Rothberger, Gerlitz-Nagy* y *γ-espacio,* entre otras. Finalmente, entre los años 2008 y 2012, de los principios de selección de M. Scheepers fue extraida la clase de los espacios *selectivamente separables.* Dicha clase fue extensamente estudiada dejando algunos problemas que permanecen abiertos [Bel+08].

En C_p-teoría uno de los grandes problemas sin resolver es *caracterizar los espacios X para los cuales $C_p(X)$ es Lindelöf.* Debido a que la propiedad Menger implica Lindelöf es natural plantear el siguiente problema: ¿bajo qué condiciones en X, $C_p(X)$ es Menger? Este problema ya ha sido atendido. Hurewicz prueba que este último hecho sucede sólo en el caso trivial; es decir, cuando X es finito [Arh92, Theorem II.2.10]. Sin embargo no se puede decir lo mismo del subespacio $C_p(X,2)$ de $C_p(X)$, de hecho para cualquier espacio discreto X, $C_p(X,2)$ es un espacio compacto y por tanto Menger. Otro ejemplo no tan trivial es el conjunto de Cantor 2^ω, $C_p(2^\omega,2)$ es Menger ya que éste es numerable. Más aún, $C_p(X,2)$ puede llegar a ser σ-compacto. En [CCTM03] A. Contreras-Carreto y Á. Tamariz-Mascarúa prueban que si X es un subespacio de $C_p(Y)$, donde Y es compacto, y X' (el conjunto de puntos no aislados de X) es compacto, entonces $C_p(X,2)$ es σ-compacto.

El presente trabajo es una compilación de los resultados obtenidos durante mi estancia doctoral los cuales hablan acerca de la propiedad Menger y Rothberger en el espacio $C_p(X,2)$ cuando X pertenece a alguna de las siguientes clases: subespacios de $C_p(Y)$, espacios simples, GO-espacios y Ψ-espacios (véase Chapter 2, Chapter 3 o bien, [BS15b; BSTM15; BS15a]) .

El primer capítulo está basado en [Win95] (uno de mis artículos favoritos), éste presenta una lista de propiedades básicas de los espacios Menger y ejemplos concernientes a estos mismos, las pruebas de estos resultados, en su mayoría, fueron dadas por L. Wingers. En la última sección del capítulo 1 se presentan dos temas importantes acerca de la propiedad Menger. El primero, el producto de espacios Hurewicz, y el segundo, la conjetura de Hurewicz.

R.Z. Buzyakova en [Buz04] prueba que $C_p(X)$ es Lindelöf para cada subespacio primero numerable numerablemente compacto de ordinales X. En el capítulo 2 mostramos que si X es un subespacio primero numerable de ordinales, entonces $C_p(X,2)$ es Menger si y sólo si $C_p(X,2)$ es Lindelöf y X', el conjunto de puntos no aislados de X, es numerablemente compacto (Theorem 2.32). Este resultado en conjunto con el teorema de Buzyakova antes mencionado tienen como consecuencia que $C_p(X,2)$ es Menger para cada subespacio primero numerable numerablemente compacto de ordinales X. De manera más general, haciendo uso de las técnicas usadas por Buzyakova en [Buz07] logramos probar el siguiente resultado: Si L es un GO-espacio primero numerable sin puntos aislados, entonces $C_p(X,2)$ es Menger si y sólo si $C_p(L,2)$ es Lindelöf y L es numerablemente compacto (Theorem 2.28).

Finalmente, el capítulo 3 (como el capítulo 2) contiene una sección dedicada a los Ψ-espacios y el espacio de funciones $C_p(\Psi(\mathscr{A}),2)$. A. Dow y P. Simon en [DS06] prueban dos hechos, el primero es que si $\mathfrak{b} > \omega_1$, entonces $C_p(\Psi(\mathscr{A}),2)$ no es Lindelöf para cada familia casi ajena maximal (mad) $\mathscr{A}$, y el segundo, que, asumiendo $\diamondsuit$, existe una familia mad $\mathscr{A}$ para la cual $C_p(\Psi(\mathscr{A}),2)$ es Lindelöf. Más adelante M. Huršák, P.J. Septycki y Á. Tamariz-Mascarúa prueban, suponiendo la hipótesis del continuo (CH), la existencia de una familia mad $\mathscr{A}$ para la cual $C_p(\Psi(\mathscr{A}), 2)$ es Lindelöf [HSTM05]. En el capítulo 3 logramos mejorar este último resultado demostrando que si se supone CH, existe una familia mad $\mathscr{A}$ para la cual $C_p(\Psi(\mathscr{A}), 2)$ es Rothberger (Theorem 3.35).

Contents

Notation and terminology

In this doctoral thesis, all notation and terminology follow references [Eng89; Arh92]. All spaces under consideration are assumed to be Tychonoff, i.e., $T_{3\frac{1}{2}}$. The set of ordinals strictly less than an ordinal α equipped with its order topology will be denoted simply by α. And $\mathbb{N}$ denotes $\omega\setminus\{0\}$. For a subset A of a topological space X, $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ denote the closure and interior in X of A, respectively. If there is no possibility of confusion, we will simply write $\mathrm{cl}(A)$ and $\mathrm{int}(A)$. A subset A of a topological space is *nowhere dense* if $\mathrm{int}(\mathrm{cl}(A)) = \emptyset$. By βX we denote the *Stone-Čech compactification* of a space X. Given a space X, X' denotes the set of non-isolated points of X. And by a cover of a set X we mean a family of sets whose union contains X.

For spaces X and Y, $C_p(X,Y)$ is the subspace of Y^X consisting of the continuous functions from X to Y (i.e., $C(X,Y)$ endowed with the topology of pointwise convergence). As usual, $C_p(X)$ will mean $C_p(X,\mathbb{R})$. For a space X, $n\in\omega$, points $x_0,\dots,x_n\in X$, $f\in C_p(X)$ and a positive real number δ, we will denote by $[f;x_0,\dots,x_n;\delta]$ the set

$$\{g\in C_p(X) : \forall i(0\le i\le n \to |f(x_i)-g(x_i)|<\delta)\}.$$

Recall that for every space X and every discrete space Y, there exists a zero-dimensional space Z such that $C_p(X,Y)$ is homeomorphic to $C_p(Z,Y)$. So, where reference is made to $C_p(X,Y)$ where Y is discrete, we will assume that X is a zero-dimensional space.

Let $iw(X)$ be the minimal cardinal κ such that X has a weaker Tychonoff topology of weight κ; evidently, the statement $iw(X)=\omega$ is equivalent to saying that X has a weaker separable metrizable topology. A family $\mathcal{N}$ of subsets of a space X is a *network* for X if every open subset of X is a union of a subfamily of $\mathcal{N}$. In other words, a network is like a base, only its elements need not be open. The *network weight* of X, denoted by $nw(X)$, is the least cardinality of a network for X. The spaces with countable network weight are called *cosmic*. A family $\mathscr{P}$ of non-empty subsets of a space X is said to be a π-*network* at $x \in X$ if every neighborhood of x contains some member of $\mathscr{P}$.

A space X has *countable tightness*, which is denoted by $t(X) \le \omega$, if for any $x \in X$ and $A \subset X$ with $x \in \text{cl}(A)$, there is a countable set $B \subset A$ such that $x \in \text{cl}(B)$. For any space X, $t^*(X) = \sup\{t(X^n) : n \in \omega\}$.
Given a metric space X with metric d and a subset A of X, the *diameter* of A is $\delta(A) = \sup\{d\left(x, y\right) : x, y \in A\}$.
Given spaces X and Y and $f : X \to Y$, the n-th power $f^n : X^n \to Y^n$ of the map f is defined by $f^n(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$.
For any space X and every real valued function $f : X \to \mathbb{R}$, $\text{supp}(f)$ denotes the set $\text{cl}(\{x \in X : f(x) \neq 0\})$.
We say that a space S is a Σ-*product* of spaces from a class of topological spaces $\mathscr{C}$ if there is a family $\{X_t : t \in T\}$ of spaces in $\mathscr{C}$ and a point $a \in X = \prod\{X_t : t \in T\}$ such that S is equal to the set $\{x \in X : |\{t \in T : x(t) \neq a(t)\}| \le \omega\}$.

For any set X, $[X]^{<\omega}$ will denote the set of all finite subsets of X.

CHAPTER

1 Menger spaces

A space X is said to have the *Menger property* (or simply X is *Menger*) if for every sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of open covers of X, there exists a sequence of finite sets $\langle \mathscr{F}_n : n \in \omega \rangle$ such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a cover of X and $\mathscr{F}_n \subset \mathscr{U}_n$ for every $n \in \omega$.

1.1 Basic properties on the Menger property

Similar to compactness, the Menger property does not depend on the universe in which it is immersed; that is:

Proposition 1.1. *Let Y be a subspace of X. Then Y is Menger if and only if for every sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of covers of Y consisting of open subsets of X, there exists a sequence $\langle \mathscr{F}_n : n \in \omega \rangle$ of finite sets such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a cover of Y and $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$.*

As we have already mentioned in the Introducción:

Proposition 1.2. *Every σ-compact space is Menger and every Menger space is a Lindelöf space.*

Proof. Let $X = \bigcup_{n\in\omega} K_n$ be a space where each K_n is a compact space and let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. For each $n \in \omega$ we choose a finite subset $\mathscr{F}_n \subset \mathscr{U}_n$ whose union contains K_n. Then the union of the sequence $\langle \mathscr{F}_n : n \in \omega \rangle$ forms a cover for X.

If X is a Menger space and $\mathscr{U}$ is a cover of X, then for the constant sequence $\langle \mathscr{U}_n : n \in \omega \rangle$, where $\mathscr{U}_n = \mathscr{U}$ for each $n \in \omega$, there is a sequence of finite sets $\langle \mathscr{F}_n : n \in \omega \rangle$ such that

$X = \bigcup_{n\in\omega} \bigcup \mathscr{F}_n$ and $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$. Then $\bigcup_{n\in\omega} \mathscr{F}_n$ is a countable cover of X contained in $\mathscr{U}$. □

The following two examples show that the classes of topological spaces mentioned in Proposition 1.2 are pairwise contained properly. The first example shows a Lindelöf space which is not Menger.

Example 1.3. The space of irrational numbers ω^ω is a non Menger Lindelöf space.

Proof. If we define, for each $n \in \omega$ and $s \in \omega^n$, $U_s^n = \{f \in \omega^\omega : s \subset f\}$, then, for each $n \in \mathbb{N}$, the collection $\mathscr{U}_n = \{U_s^n : s \in \omega^n\}$ is an open cover of ω^ω. Now, let $\mathscr{F}_n \subset \mathscr{U}_n$ be a finite subset for each $n \in \mathbb{N}$. Then, since the set $\{\mathrm{Im}(s) : U_s^n \in \mathscr{F}_n\}$ is finite, we can choose $k_n \in \omega \setminus \{\mathrm{Im}(s) : U_s^n \in \mathscr{F}_n\}$. Then, the function $f : \omega \to \omega$ defined by $f(n) = k_n$ does not belong to any element of $\bigcup_{n\in\omega} \mathscr{F}_n$. □

The second example shows a Menger space which is not σ-compact.

Example 1.4. The one-point Lindelöfication of any uncountable discrete space is a non σ-compact Menger space.

Proof. Let κ be an uncountable cardinal number and let $D(\kappa)$ be the set κ with the discrete topology. The Lindelöfication of $D(\kappa)$ is the set $L_\kappa = D(\kappa) \cup \{\infty\}$, where $\infty \notin \kappa$, with the following topology: every point of $D(\kappa)$ is isolated and the open neighborhoods of ∞ are the subsets of L_κ that contain ∞ and have countable complement. Every compact subset of L_κ is finite, and hence, L_κ is not σ-compact. If $\langle \mathscr{U}_n : n \in \omega \rangle$ is a sequence of open covers of L_κ, there exists an element $U_0 \in \mathscr{U}_0$ such that $\infty \in U_0$. Then $L_\kappa \setminus U_0$ is countable. In consequence we can choose, for each $n \in \mathbb{N}$, an element $U_n \in \mathscr{U}_n$ such that $L_\kappa \setminus U_0 \subset \bigcup_{k\in\mathbb{N}} U_k$. In this manner, $\langle \{U_n\} : n \in \omega \rangle$ is the required sequence. □

Some other properties of Menger spaces are as follows.

Proposition 1.5 ([Win95])**.** *Any closed subspace of a Menger space is a Menger space.*

Proof. Let X be a Menger space and let $S \subset X$ be a closed subspace of X. Let $\langle \mathscr{U}_n^S : n \in \omega \rangle$ be a sequence of open covers of S. We may assume that the members of $\mathscr{U}_n^S$ are open in X. If $\mathscr{U}_n = \mathscr{U}_n^S \cup \{X \setminus S\}$, then $\langle \mathscr{U}_n : n \in \omega \rangle$ is a sequence of open covers in X and since X is a Menger space, there is a sequence $\langle \mathscr{V}_n : n \in \omega \rangle$ with $\mathscr{V}_n \in [\mathscr{U}_n]^{<\omega}$ such that $X = \bigcup_{n\in\omega} \bigcup \mathscr{V}_n$. Then, if $\mathscr{V}_n^S = \mathscr{V}_n \setminus \{X \setminus S\}$, $\langle \mathscr{V}_n^S : n \in \omega \rangle$ is the required sequence. □

Proposition 1.6 ([Win95])**.** *If X is Menger and $f : X \to Y$ is a continuous surjection, then Y is Menger.*

Proof. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of Y. If $\mathscr{V}_n = \{f^{-1}[U] : U \in \mathscr{U}_n\}$, then $\langle \mathscr{V}_n : n \in \omega \rangle$ is a sequence of open covers of X. Since X is Menger, there is a sequence $\langle \mathscr{G}_n : n \in \omega \rangle$ of finite sets such that $\bigcup\{\mathscr{G}_n : n \in \omega\}$ is a cover of X and $\mathscr{G}_n \subset \mathscr{V}_n$ for each $n \in \omega$. Let $\mathscr{F}_n = \{f[V] : V \in \mathscr{G}_n\}$. Then, $Y = \bigcup_{n\in\omega} \bigcup \mathscr{F}_n$ and $\mathscr{F}_n \in [\mathscr{U}_n]^{<\omega}$ for each $n \in \omega$. □

A surjective function $f : X \to Y$ is *perfect* if it is closed and $f^{-1}(y)$ is compact for each $y \in Y$.

Proposition 1.7. *If $f : X \to Y$ is a perfect function and Y is Menger, then X is Menger.*

Proof. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. We can suppose that each $\mathscr{U}_n$ is closed under finite unions; that is, if $\mathscr{F}$ is a finite subset of $\mathscr{U}_n$, then $\bigcup \mathscr{F} \in \mathscr{U}_n$. For any $n \in \omega$ we define $\mathscr{V}_n = \{Y \setminus f[X \setminus U] : U \in \mathscr{U}_n\}$. Then, since f is perfect, each $\mathscr{V}_n$ is an open cover of Y. Indeed, since f is closed, each element of $\mathscr{V}_n$ is open. Given $y \in Y$, $f^{-1}(y)$ is a compact subspace of X, then there is an element $U \in \mathscr{U}_n$ such that $f^{-1}(y) \subset U$ and hence, $y \in Y \setminus f[X \setminus U] \in \mathscr{V}_n$.
Now, since Y is Menger, there is a sequence of finite sets $\langle \mathscr{G}_n : n \in \omega \rangle$ with $Y = \bigcup_{n\in\omega} \bigcup \mathscr{G}_n$ and $\mathscr{G}_n \subset \mathscr{V}_n$. We choose a finite subset $\mathscr{F}_n \subset \mathscr{U}_n$ such that $\mathscr{G}_n = \{Y \setminus f[X \setminus U] : U \in \mathscr{F}_n\}$. It is not difficult to show that $X = \bigcup_{n\in\omega} \bigcup \mathscr{F}_n$. □

Corollary 1.8. *If X is Menger and Y is a compact space, then $X \times Y$ is Menger.*

Proof. Simply observe that the projection function over the first coordinate $\pi_X : X \times Y \to X$ is a perfect function. □

For the purposes of the next chapter, the following equivalent formulation of Menger spaces will be useful.

Lemma 1.9. *A space X is Menger if and only if for any sequence of open covers $\langle \mathscr{U}_n : n \in \omega \rangle$ such that, for every $n \in \omega$, $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$, there exists a sequence of finite sets $\langle \mathscr{F}_n : n \in \omega \rangle$ such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a cover of X and $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$.*

Proof. We only have to show the sufficiency. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. By recursion we define a new sequence $\langle \mathscr{U}'_n : n \in \omega \rangle$ such that, for each $n \in \omega$, $\mathscr{U}'_{n+1}$ refines $\mathscr{U}'_n$ and $\mathscr{U}_{n+1}$. Let $\mathscr{U}'_0 = \mathscr{U}_0$. Suppose that $\mathscr{U}'_0, \ldots, \mathscr{U}'_n$ have been defined. We define $\mathscr{U}'_{n+1} = \{U \cap V : U \in \mathscr{U}_n \wedge V \in \mathscr{U}'_n\}$. Then the sequence $\langle \mathscr{U}'_n : n \in \omega \rangle$ satisfies the required properties. Finally, note that every finite choice of this new sequence induce a finite choice of original sequence. □

A G_δ*-set* is a countable intersection of open sets. The following lemma shows that every Menger space is contained in some G_δ-set.

Lemma 1.10 ([Win95])**.** *A space X is Menger if and only if for each sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence of finite subsets $\langle \mathscr{F}_n : n \in \omega \rangle$ such that*

$$X = \bigcap_{n\in\omega} \bigcup_{m\geq n} \bigcup \mathscr{F}_m$$

and $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$.

Proof. Suppose that X is Menger and let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. For each $n \in \omega$, $\langle \mathscr{U}_m : m \geq n \rangle$ is a sequence of open covers of X. Since X is Menger, for each $n \in \omega$, there is a sequence of finite subsets $\langle \mathscr{F}^n_m : m \geq n \rangle$ whose union forms a cover of X and $\mathscr{F}^n_m \subset \mathscr{U}_m$ for each $m \geq n$. For each $m \in \omega$, if $\mathscr{F}_m = \bigcup_{k\leq m} \mathscr{F}^k_m$, then $\mathscr{F}_m$ is a finite subset of $\mathscr{U}_m$. And $X = \bigcup_{m\geq n} \bigcup \mathscr{F}^n_m \subset \bigcup_{m\geq n} \bigcup \mathscr{F}_m$ for all $n \in \omega$; that is, $X = \bigcap_{n\in\omega} \bigcup_{m\geq n} \bigcup \mathscr{F}_m$.
The reciprocal is trivial. □

Definition 1.11. If X is a space, then $\mathscr{M}(X)$ is the set of all Menger subspaces of X.

Theorem 1.12 ([Win95])**.** *A space X is a Menger space if and only if every cover $\{G_H : H \in \mathscr{M}(X)\}$ of X, where $H \subset G_H$ and G_H is a G_δ-set, has a countable subcover.*

Proof. The necessity is obvious. We will proceed to prove the sufficiency. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. For each $M \in \mathscr{M}(X)$, by Lemma 1.10, there is a sequence $\langle \mathscr{F}^M_n : n \in \omega \rangle$ with $\mathscr{F}^M_n \in [\mathscr{U}_n]^{<\omega}$ such that for all $n \in \omega$, $M \subset \bigcup_{m\geq n} V^M_m$, where $V^M_m = \bigcup \mathscr{F}^M_m$. If $O^M_n = \bigcup_{m\geq n} V^M_m$, then $G_M = \bigcap_{n\in\omega} O^M_n$ is a G_δ-set containing M. By hypothesis, the cover $\{G_M : M \in \mathscr{M}(X)\}$ has a countable subcover $\{G_{M_n} : n \in \omega\}$.
For each $n \in \omega$, let $\mathscr{F}'_n = \bigcup_{k\leq n} \mathscr{F}^{M_k}_n$. It is clear that $\mathscr{F}'_n \in [\mathscr{U}_n]^{<\omega}$ and it is not difficult to show that $X = \bigcap_{n\in\omega} \bigcup_{m\geq n} \bigcup \mathscr{F}'_m$. □

Corollary 1.13 ([Win95])**.** *If $X = \bigcup_{n\in\omega} X_n$ and X_n is Menger for each $n \in \omega$, X is Menger.*

We shall need the following results.

Corollary 1.14 ([Tel72])**.** *If X is Menger and Y is a σ-compact space, then $X \times Y$ is Menger.*

Proof. By Corollary 1.8, $X \times Y$ is a countable union of Menger spaces and, by Corollary 1.13, $X \times Y$ is Menger. □

A space X is a *P-space* if all G_δ-sets in X are open.

Corollary 1.15 ([Win95])**.** *Let $f : X \to Y$ be a closed, continuous and surjective function such that $f^{-1}(y)$ is Menger for any $y \in Y$. If Y is a Lindelöf P-space, then X is Menger.*

Proof. Let $\mathscr{M} = \{f^{-1}(y) : y \in Y\} \subset \mathscr{M}(X)$. For each $M \in \mathscr{M}$, let G_M be a G_δ-set in X containing M. Since f is closed and Y is a P-space, there is an open set V_M in Y such that $M \subset f^{-1}[V_M] \subset G_M$. Indeed, suppose that $G_M = \bigcap_{n\in\omega} O_n^M$, where every O_n^M is an open subset of X and $M = f^{-1}(y)$. Since f is closed, $Y \setminus f\left[X \setminus O_n^M\right]$ is an open subset of Y containing y for every $n \in \omega$, but Y is a P-space, then we can choice an open subset V_M of Y such that $y \in V_M \subset Y \setminus f\left[X \setminus O_n^M\right]$ for every $n \in \omega$. In this manner, $M \subset f^{-1}[V_M] \subset G_M$. Then, since f is surjective, the collection $\mathscr{V} = \{V_M : M \in \mathscr{M}\}$ forms an open cover of Y, but Y is Lindelöf and, hence, $\mathscr{V}$ has a countable subcover $\mathscr{V}' = \{V_{M_n} : n \in \omega\}$. In this manner, the family $\{f^{-1}\left[V_{M_n}\right] : n \in \omega\}$ is a cover of X and therefore, $\{G_{M_n} : n \in \omega\}$ covers X. The conclusion follows from Theorem 1.12. □

Corollary 1.16 ([Win95])**.** *A P-space is Menger if and only if it is Lindelöf.*

Proof. If X is a Menger space, by Proposition 1.2, X is Lindelöf. Now, if X is a Lindelöf P-space, the identity function $i : X \to X$ satisfies the conditions of Corollary 1.15 and so X is Menger. □

Define a preorder $\leq^*$ on ω^ω by $f \leq^* g$ if and only if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We write $f <^* g$ if $f \leq^* g$ and $f \neq g$. A subset $D \subset \omega^\omega$ is *dominating* if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. And a subset $B \subset \omega^\omega$ is *unbounded* if there is no $g \in \omega^\omega$ with $f \leq^* g$ for each $f \in B$. Also, we define the $\mathfrak{d}$ominating and un$\mathfrak{b}$ounded cardinals as follows:

$$\mathfrak{d} = \min\{|D| : D \subset \omega^\omega \ \wedge \ D \text{ is dominating}\},$$

$$\mathfrak{b} = \min\{|B| : B \subset \omega^\omega \ \wedge \ B \text{ is unbounded}\}.$$

Since every dominating subset of ω^ω is unbounded, $\mathfrak{b} \leq \mathfrak{d}$. The following result is a generalization of Proposition 1.2.

Proposition 1.17 ([Win95])**.** *If $X = \bigcup_{\alpha\in\kappa} X_\alpha$ is Lindelöf, X_α is Menger for each $\alpha \in \kappa$ and $\kappa < \mathfrak{b}$, then X is Menger.*

Proof. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. Since X is Lindelöf, we can suppose that every $\mathscr{U}_n$ is countable, say $\{U_m^n : m \in \omega\}$. For each $\alpha \in \kappa$, since X_α is Menger, by Lemma 1.10 we can choose a function $f_\alpha : \omega \to \omega$ such that $X_\alpha \subset \bigcap_{n\in\omega} \bigcup_{m\geq n} \bigcup_{k\leq f_\alpha(m)} U_k^m$. Given that $\kappa < \mathfrak{b}$, the family $B = \{f_\alpha : \alpha \in \kappa\}$ is not unbounded. Then, there exists a bound $f \in \omega^\omega$ for B; that is, $g \leq^* f$ for each $g \in B$. Let $\mathscr{F}_n = \{U_m^n : m \leq f(n)\}$. If $x \in X$, $x \in X_\alpha$ for some $\alpha \in \kappa$. Since $f_\alpha \leq^* f$, we can choose $n \in \omega$ such that $f_\alpha(s) \leq f(s)$ for each $s \geq n$. Now, for each $n \in \omega$, there is $m \geq n$ and $k \leq f_\alpha(m)$ such that $x \in U_k^m$. Then choose $m \geq n$ and $k \leq f_\alpha(m)$ for which $x \in U_k^m$. Hence $U_k^m \in \mathscr{F}_m$ and the union of the sequence $\langle \mathscr{F}_n : n \in \omega \rangle$ forms a cover of X. □

Then, if $|X| < \mathfrak{b}$ and X is Lindelöf, X is Menger. But the following proposition proves more than this.

Proposition 1.18 ([Win95])**.** *If X is Lindelöf and $|X| < \mathfrak{d}$, then X is Menger.*

Proof. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X. Since X is Lindelöf, we can suppose that each $\mathscr{U}_n$ can be written as $\{U^n_m : m \in \omega\}$. For each $x \in X$, define $f_x : \omega \to \omega$ as $f_x(n) = \min\{k \in \omega : x \in U^n_k\}$. Then, since $|X| < \mathfrak{d}$, the family $\{f_x : x \in X\}$ is not dominating; that is, there exists $f \in \omega^\omega$ such that $f \not\leq^* f_x$ for each $x \in X$. In consequence, for each $x \in X$, there is $m \in \omega$ such that $f_x(m) < f(m)$ and, by definition, $x \in U^m_{f_x(m)}$. This shows that $X = \bigcup_{n\in\omega} \bigcup_{k<f(n)} U^n_k$. □

The following example show the existence of a Lindelöf space of cardinality $\mathfrak{d}$ not Menger.

Example 1.19 ([Win95])**.** If $D \subset \omega^\omega$ is a dominating family, then D is not Menger.

Proof. For each $x \in D$ and $n \in \omega$, let $U^n_x = \{f \in D : \forall i \in \omega(i \leq n \to f(i) = x(i))\}$. If $\mathscr{U}_n = \{U^n_x : x \in D\}$, then $\langle \mathscr{U}_n : n \in \omega \rangle$ is a sequence of open covers of D. Let $\langle \mathscr{F}_n : n \in \omega \rangle$ be any sequence of finite sets where $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$. Without loss of generality assume $\mathscr{F}_n \neq \emptyset$. If $g \in \omega^\omega$ is defined by $g(n) = \max\{x(n) : U^n_x \in \mathscr{F}_n\} + 1$, then for any $m \in \omega$, $g \notin \bigcup \mathscr{F}_m$, since for any $x \in \bigcup \mathscr{F}_m$, $x(m) < g(m)$. Given that D is dominating, there exists an $f \in D$ and $N \in \omega$ such that $f(i) \geq g(i)$ for every $i \geq N$. Then $f \notin \bigcup \mathscr{F}_m$ for any $m \geq N$ and hence, $D \neq \bigcap_{n\in\omega} \bigcup_{m\geq n} \bigcup \mathscr{F}_m$. By Lemma 1.10, D is not Menger. □

The following proposition resolves the study of the Menger property on ω-powers of a space.

Proposition 1.20. *For any space X, X^ω is Menger if and only if X is compact.*

Proof. Suppose that X^ω is Menger and X is not compact. Since X is Lindelöf, X is not countably compact. Then X contains a closed countable discrete subspace D. In this manner D^ω is a closed subspace of X^ω homeomorphic to ω^ω. But this is a contradiction to Proposition 1.5 and to Example 1.3. □

As it happens with the Lindelöf property, the Menger property is not productive. In Section 1.3 we will show an example (assuming the Continuum Hypothesis) of a Menger space X such that X^2 is not Menger.

1.2 Zero-dimensional Menger spaces

In this section we present some results regarding the zero-dimensional spaces and their relations with dominating subsets of ω^ω are shown.

A space is *zero-dimensional* if it has a base consisting of clopen sets (open and closed sets). The following theorem gives a characterization of zero-dimensional Lindelöf spaces with the Menger property.

Theorem 1.21 ([Win95]). *Let X be a zero-dimensional Lindelöf space. Then X is Menger if and only if X cannot be mapped continuously onto a dominating subset of ω^ω.*

Proof. The necessity is clear because of Example 1.19 and Proposition 1.6. Suppose that X is not Menger. Since X is a zero-dimensional Lindelöf space, we can find a sequence of countable open covers $\langle \mathscr{U}_n : n \in \omega\rangle$ consisting of clopen sets such that for any sequence of finite sets $\langle \mathscr{F}_n : n \in \omega\rangle$, where $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$, $X \neq \bigcap_{n\in\omega}\bigcup_{m\geq n}\bigcup\mathscr{F}_m$. Let $\mathscr{U}_n = \{U^n_m : m \in \omega\}$.
For each $x \in X$, let $f_x \in \omega^\omega$ be defined by $f_x(n) = \min\{m \in \omega : x \in U^n_m\}$. Let $D = \{f_x : x \in X\}$ and $F : X \to D$ be defined by $F(x) = f_x$. Clearly F is a surjection. If D is not dominating, by similar arguments to those given in the proof of Proposition 1.18, we can find a sequence of finite sets $\langle \mathscr{F}_n : n \in \omega\rangle$ such that $X = \bigcup_{n\in\omega}\bigcup\mathscr{F}_n$ and $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$, contradicting the supposition. Then D is a dominating subset of ω^ω. We will finish the proof by showing the continuity of F.
Let $x \in X$ and let V be an open subset of $D \subset \omega^\omega$ containing $F(x)$. We can suppose that $V = \{f \in D : \forall m \in \omega(m \leq n \to f(m) = f_x(m))\}$ for some $n \in \omega$. Let $U_m = U^m_{f_x(m)} \setminus \bigcup_{k<f_x(m)} U^m_k$. Then $U_x = \bigcap_{m\leq n} U_m$ is an open neighborhood of x. If $x' \in U_x$ and $m \leq n$, $x' \in U^m_{f_x(m)} \setminus \bigcup_{k<f_x(m)} U^m_k$, and hence $f_{x'}(m) = f_x(m)$. This shows that $F[U_x] \subset V$. □

The last theorem of this section characterizes the zero-dimensional metric separable spaces with the Menger property.

Theorem 1.22 ([Win95]). *A zero-dimensional metric separable space is Menger if and only if it is not homeomorphic to a dominating subset of ω^ω.*

Proof. The necessity is clear because of Example 1.19. Let X be a zero-dimensional metric separable non Menger space. Since X is a zero-dimensional metric separable space, we can find a sequence of open covers $\langle \mathscr{U}_n : n \in \omega\rangle$ such that for any sequence of finite sets $\langle \mathscr{F}_n : n \in \omega\rangle$ where $\mathscr{F}_n \subset \mathscr{U}_n$ for each $n \in \omega$, $X \neq \bigcap_{n\in\omega}\bigcup_{m\geq n}\bigcup\mathscr{F}_m$. We can suppose without loss of generality that:

- $\mathscr{U}_n = \{U^n_m : m \in \omega\}$,
- U^n_m is clopen and the diameter of U^m_n, $\delta\left(U^n_m\right)$, is less than $\frac{1}{n+1}$ for each $n, m \in \omega$.

For each $x \in X$, let $f_x \in \omega^\omega$ be defined by $f_x(n) = \min\{m \in \omega : x \in U^n_m\}$. Let $D = \{f_x : x \in X\}$ and $F : X \to D$ be defined by $F(x) = f_x$. With the same arguments used in the proof of Theorem 1.21 we can prove that D is dominating and F is a continuous surjection. We next show that F is injective.

Let $x, y \in X$ be two different points. Let $n \in \omega$ be such that $\frac{1}{n+1} < \frac{d(x,y)}{2}$. With this choice it is not difficult to prove that $U^n_{f_x(n)} \cap U^n_{f_y(y)} = \emptyset$. In this manner, $f_x(n) \neq f_y(n)$ and hence, $F(x) \neq F(y)$.

Finally, we prove that F is open. First we give a base at each point of X. Let $\mathscr{B}_x = \{B^x_n : n \in \omega\}$ be defined by recursion as follows:

$$B^x_0 = U^n_{f_x(0)} \setminus \bigcup_{m<f_x(0)} U^0_m$$

and $B^x_{n+1} = (U^{n+1}_{f_x(n+1)} \setminus \bigcup_{m<f_x(n+1)} U^{n+1}_m) \cap B^x_n$. We claim that $\mathscr{B}_x$ is a base at x. Indeed, let $r > 0$ and $n \in \omega$ be such that $\frac{1}{n} < r$. Since $x \in B^x_n \subset U^n_{f_x(n)}$ and $\delta(U^n_{f_x(n)}) < \frac{1}{n+1} < r$, $U^n_{f_x(n)} \subset B(x,r) := \{y \in X : d(x,y) < r\}$. It follows that $x \in B^x_n \subset B(x,r)$. So, $\mathscr{B}_x$ is a base at x. For each $x \in X$ and $n \in \omega$ the set $V^x_n = \{f \in D : \forall m \in \omega (m \leq n \to f_x(m) = f(m))\}$ is open in D. Then, to prove that F is open it is enough to show that $F[B^x_n] = V^x_n$ for each $x \in X$ and each $n \in \omega$. But this is immediate since the following statements are equivalent:

- $y \in B^x_n$;
- $y \in U^m_{f_x(m)} \setminus \bigcup_{k<f_x(m)} U^m_k$ for each $m \leq n$;
- $f_y(m) = f_x(m)$ for each $m \leq n$;
- $F(y) \in V^x_y$. □

1.3 Product of Menger spaces and the Hurewicz Conjecture

This section is an analysis of two topics concerning the Menger property. The first is the product of Menger spaces. It is well-known that the product of two Lindelöf spaces is not necessarily Lindelöf; for example the Sorgenfrey line. In this section we present, assuming the Continuum Hypothesis, an example of a Menger space whose square is not Menger. The other topic is the Hurewicz Conjecture. We present two counter-examples of this conjecture. The first under Continuum Hypothesis and the second within ZFC.

Product of Menger spaces

In order to present a space with the Menger property whose square is not Menger, we need the existence of a Luzin set in $\mathbb{R}$.

Definition 1.23. A subset L of $\mathbb{R}$ is a *Luzin set* if L is an uncountable dense set in $\mathbb{R}$ and for each nowhere dense subset A in $\mathbb{R}$, $A \cap L$ is countable.

Lemma 1.24 (CH)**.** *Every open interval U in $\mathbb{R}$ contains an uncountable subset whose intersection with every closed nowhere dense subset in U is countable.*

Proof. Let $\{F_\alpha : \alpha \in \omega_1\}$ be the collection of all closed nowhere dense subsets of U. Since U is homeomorphic to $\mathbb{R}$, by Baire's Theorem, U cannot be covered by a countable family of closed nowhere dense subsets. Then, by recursion, we can choose, for each $\alpha \in \omega_1$, $x_\alpha \in U \setminus (\bigcup_{\beta \in \alpha} F_\beta \cup \{x_\beta : \beta \in \alpha\})$. Let $X = \{x_\alpha : \alpha \in \omega_1\}$. We claim that X is the required set. Indeed, let M be a closed nowhere dense subset in U. Then M is equal to F_α for some $\alpha \in \omega_1$, and hence $M \cap X \subset \{x_\beta : \beta \in \alpha\}$ is countable. □

Corollary 1.25 (CH)**.** *There is a Luzin set.*

Proof. Let $\mathscr{B}$ be a countable base of $\mathbb{R}$ consisting of open intervals. For every $U \in \mathscr{B}$, let L_U be a subset of U given by Lemma 1.24, and define $L = \bigcup_{U \in \mathscr{B}} L_U$. Since $\mathscr{B}$ is a base, L is dense in $\mathbb{R}$. Let M be a nowhere dense subset of $\mathbb{R}$. Then for each $U \in \mathscr{B}$, $\mathrm{cl}_{\mathbb{R}}(M) \cap U$ is a closed nowhere dense subset in U. In consequence, $\mathrm{cl}_{\mathbb{R}}(M) \cap U$ has countable intersection with L_U. Therefore, $L \cap M \subset \bigcup_{U \in \mathscr{B}} (\mathrm{cl}_{\mathbb{R}}(M) \cap L_U)$ is countable. □

There are other models of ZFC, besides ZFC+CH, where Luzin sets exist. The following theorem shows the existence of a Menger space whose square is not Menger in models of ZFC+(there exists a Luzin set).

Theorem 1.26. *The existence of a Luzin set implies the existence of a Menger space X whose square $X \times X$ is not Menger.*

Proof. Let L be a Luzin set and let us define a space X_L to be the set L with the topology of subspace of the Sorgenfrey's line. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of open covers of X_L. Fix a dense subset $D = \{d_n : n \in \omega\}$ of the subspace L of $\mathbb{R}$. For each $n \in \omega$, let $x_n \in \mathbb{R}$ be such that $X_L \cap [d_n, x_n)$ is contained in some element $U_n \in \mathscr{U}_n$. In this manner, the set $Y = \mathbb{R} \setminus \bigcup_{n \in \omega} [d_n, x_n)$ is nowhere dense in $\mathbb{R}$ (because L is dense in $\mathbb{R}$). Then $Y \cap X_L = Y \cap L$ is countable. If $Y \cap X_L = \{z_n : n \in \omega\}$, we can choose $V_n \in \mathscr{U}_n$ containing z_n for each $n \in \omega$. Then $\langle \{U_n, V_n\} : n \in \omega \rangle$ is the required sequence.
To prove that $X_L \times X_L$ is not Menger it is enough to see that $X_L \times X_L$ is not normal (every Menger regular space is a Lindelöf regular space, and hence, is a normal space [Eng89]). For this, simply note two facts: X_L is separable and the set $\Delta = \{(x, -x) : x \in X_L\}$ is a closed discrete subspace of X^2. Therefore, by Jones' Lemma [Eng89, Problem 1.7.12(c)], $X_L \times X_L$ is not normal. □

Example 1.27 (CH)**.** There is a Menger space whose square is not Menger.

Proof. Use Corollary 1.25 and Theorem 1.26. □

There is another construction, using Martin's Axiom (MA), of a Menger space whose square admits a mapping onto the irrational numbers (see [O'F86]). Therefore, under MA, there exists a Menger space whose square is not Menger (see Proposition 1.6). We are not aware of the existence of such example within ZFC.

The Hurewicz Conjecture

The W. Hurewicz conjecture is: *A subset of the real line is Menger if and only if it is σ-compact.* In this part we present two counter-examples to this conjecture. By *space of real numbers* we refer to some subspace of $\mathbb{R}$. Remember that a subset A of a space X is *perfect* if it is closed and without isolated points. We need to remember the following theorem (see [Eng89, Problem 1.7.11]).

Theorem 1.28 (Cantor-Bendixson)**.** *In a second countable space every closed subspace can be written as a union of a countable set and a perfect set.*

Since every uncountable perfect subset of $\mathbb{R}$ contains a subspace homeomorphic to the Cantor set 2^ω [Eng89, Problem 4.5.5(b)], by Cantor-Bendixson's Theorem, every closed subspace of $\mathbb{R}$ is finite, countable or has size $\mathfrak{c}$. With this fact and, again, by Cantor-Bendixson's Theorem we have the following lemma.

Theorem 1.29. *Every uncountable σ-compact space of real numbers has size $\mathfrak{c}$.*

A space X is *Rothberger* if for each sequence of open covers $\langle \mathcal{U}_n : n \in \omega \rangle$, there exists $U_n \in \mathcal{U}_n$ for each $n \in \omega$ such that $X = \bigcup_{n\in\omega} U_n$. Every closed subspace of a Rothberger space is Rothberger and every Rothberger space is Menger. This and other basic properties are proved in Chapter 3.
The Cantor set 2^ω is not Rothberger. Indeed, if $\pi_n : 2^\omega \to 2$ is the n-th projection function, then the sequence $\langle \{\pi_n^{-1}(0), \pi_n^{-1}(1)\} : n \in \omega \rangle$ witnesses the failure of the Rothberger property (see also [Sch99a, Theorem 2.3]). With these facts we can prove the following theorem.

Theorem 1.30 (CH)**.** *There is a Menger space of real numbers which is not σ-compact.*

Proof. Let L be a Luzin set endowed with the topology of subspace of $\mathbb{R}$. This space exists by Corollary 1.25. We will prove that L is Rothberger. Let $D = \{d_n : n \in \omega\}$ be a dense subset of L and let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of open covers of L. For each $n \in \omega$ we choose $U_{2_n} \in \mathcal{U}_{2n}$ such that $d_n \in U_{2n}$. Then $A = \mathbb{R} \setminus \bigcup_{n\in\omega} U_{2n}$ is nowhere dense. So, we can write $A \cap L = \{x_n : n \in \omega\}$. Finally, choose $U_{2n+1} \in \mathcal{U}_{2n+1}$ containing x_n. In this manner $L = \bigcup_{n\in\omega} U_n$. This proves that L is Rothberger and hence Menger.

Now, L is not σ-compact. Indeed, otherwise, by Cantor-Bendixson's Theorem, L contains an uncountable perfect set, but every uncountable perfect set contains a copy of the Cantor set 2^ω (see [Eng89, Problem 4.5.5(a)]). Since L is Rothberger, 2^ω is Rothberger which is impossible [Sch99a, Theorem 2.3]. $\square$

Now we present the counter-example to the Hurewicz Conjecture within ZFC. This was given by W. Miller and H. Fremlin in [MF88]. We need a characterization of the cardinal $\mathfrak{b}$.

Lemma 1.31 ([KV84]). *Let* $\mathfrak{b}_1 = \min\{|B| : B \subset \omega^\omega \text{ is unbounded } \wedge B \text{ is well-ordered by } <^* \wedge \forall f \in B(n < m \rightarrow f(n) < f(m))\}$. *Then* $\mathfrak{b} = \mathfrak{b}_1$.

Proof. Clearly $\mathfrak{b} \leq \mathfrak{b}_1$. Let $F = \{f_\alpha : \alpha \in \mathfrak{b}\}$ be an unbounded subset of ω^ω. Since the set S of strictly increasing functions from ω into ω clearly is dominating we can with recursion pick $g_\beta \in S$ for $\beta \in \mathfrak{b}$ such that $f <^* g_\beta$ for each $f \in \{g_\gamma : \gamma \in \beta\} \cup \{f_\beta\}$. Then $B = \{g_\beta : \beta \in \mathfrak{b}\}$ shows that $\mathfrak{b} \geq \mathfrak{b}_1$. $\square$

Theorem 1.32. *There is a Menger space of real numbers which is not σ-compact.*

Proof. If $\omega_1 = \mathfrak{c}$, the example is given by Theorem 1.30. Then, we can suppose that $\omega_1 < \mathfrak{c}$. By Theorem 1.29, it is enough to prove the existence of a Menger subspace of $\mathbb{R}$ of size ω_1. But, if $\omega_1 < \mathfrak{b}$, by Proposition 1.18, any subset of size ω_1 of $\mathbb{R}$ is Menger. Then, assume $\omega_1 = \mathfrak{b}$.
First we will see some facts of ω^ω. Observe that ω^ω is homeomorphic to the subspace of irrational numbers $\mathbb{P}$ of $\mathbb{R}$ [Eng89, Problem 4.3.G]. Let $K \subset \omega^\omega$ be a compact subspace. Then the n-th projection $\pi_n[K]$ over the n-th factor of ω^ω is compact and hence is finite. In this manner, we can choose $f \in \omega^\omega$ such that $K \subset \prod_{n\in\omega}[0, f(n)]$. Then, if we define, for each $f \in \omega^\omega$, $C_f = \{g \in \omega^\omega : g <^* f\}$, by the previous argument, for each compact K of ω^ω, there is $f \in \omega^\omega$ such that $K \subset C_f$.
Another fact: if $F \subset \omega^\omega$ is countable, F is not unbounded and hence there is $f \in \omega^\omega$ such that $g <^* f$ for each $g \in F$. As a consequence of the two previous facts, if E is a σ-compact subspace of ω^ω, there is $f \in \omega^\omega$ such that $E \subset C_f$.
Now, we are going to construct the required space. Let B be an unbounded subset of ω^ω well-ordered by $<^*$ with order type equal to ω_1 (see Lemma 1.31). Identifying B with some subset of $\mathbb{P}$, we define X to be the subspace $B \cup \mathbb{Q}$ of $\mathbb{R}$ where $\mathbb{Q}$ is the set of rational numbers. Recall that $\omega_1 < \mathfrak{c}$, and by Theorem 1.29, X is not a σ-compact subspace of $\mathbb{R}$.
First we will prove an important property of X. Let U be an open subset of $\mathbb{R}$ containing $\mathbb{Q}$ and $E = \mathbb{R} \setminus U$. Since every closed subset of $\mathbb{R}$ is σ-compact, E can be written as $\bigcup_{n\in\omega} K_n$ where each K_n is compact. Note that $K_n \subset \mathbb{P}$. By the previous paragraphs there is $f \in \omega^\omega$ such that $\bigcup_{n\in\omega} F_n \subset C_f$. Since B is unbounded, C_f contains only a countable subset of B. Therefore $X \setminus U$ is countable. This shows that for every open subset U of $\mathbb{R}$ containing $\mathbb{Q}$, $X \setminus U$ is countable.

We will finish the proof by showing that X is Menger. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of covers of X consisting of open subsets of $\mathbb{R}$. We choose $U_n \in \mathscr{U}_n$ for each $n \in \omega$ such that $\mathbb{Q} \subset \bigcup_{n\in\omega} U_n$. By the previous property of X, $X \setminus \bigcup_{n\in\omega} U_n$ is countable and hence, we can choose $V_n \in \mathscr{U}_n$ in such a way that $X \setminus \bigcup_{n\in\omega} U_n \subset \bigcup_{n\in\omega} V_n$. Then $\langle \{U_n, V_n\} : n \in \omega \rangle$ is the required sequence. $\square$

CHAPTER

2 The Menger property in $C_p(X,2)$

A space X is said to have the *Menger property* (or simply X is *Menger*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X, there exists a sequence of finite sets $\langle \mathcal{F}_n : n \in \omega \rangle$ such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a cover of X and $\mathcal{F}_n \subset \mathcal{U}_n$ for every $n \in \omega$.

Recall that where reference is made to $C_p(X,Y)$ where Y is discrete, we will assume that X is a zero-dimensional space.

2.1 General results about the Menger property in $C_p(X,2)$

First we give a consequence of tightness type in X when $C_p(X,2)$ is Menger or Lindelöf. The following is shown in [Arh92, Theorem I.4.1]:

($\star$) *If $C_p(X)$ is a Lindelöf space, then each finite power of X has countable tightness.*

A family $\mathscr{P}$ of non-empty subsets of a space X is said to be a *π-network* at $x \in X$ if every neighborhood of x contains some member of $\mathscr{P}$.

Definition 2.1. A space X has *countable supertightness* at $x \in X$ if any π-network at x consisting of finite subsets of X contains a countable π-network at x. If X has this property at each of its points we say that X has *countable supertightness*, and we denote this fact by $st(X) \leq \omega$.

Clearly, countable supertightness implies countable tightness. With this new notion of tightness, we obtain the following result which was motivated by ($\star$):

Proposition 2.2. *If $C_p(X,2)$ is Lindelöf, then $st(X^n) \leq \omega$ for any $n \in \omega$.*

Proof. Fix $k \in \mathbb{N}$, a point $x = (x_1, \dots, x_k) \in X^k$ and a π-network $\mathscr{P}$ at x consisting of finite subsets of X^k. We take open neighborhoods $U_1, \dots, U_k$ such that, for each $i, j \in \{1, \dots, k\}$, $x_i \in U_i$, $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $U = U_1 \times \cdots \times U_k$. We can suppose that each member of $\mathscr{P}$ is contained in U. Since the space $C_p(X,2)$ is Lindelöf, the closed subspace

$$\Phi = \{f \in C_p(X,2) : \forall i (1 \leq i \leq k \to f(x_i) = 1)\}$$

of $C_p(X,2)$ is Lindelöf. For each $F \in \mathscr{P}$, we define $H_F = \bigcup \{\pi_i[F] : i \in \{1, \dots, k\}\}$, where π_i is the projection of X^k over the i-th coordinate, and $V_F = \{f \in C_p(X,2) : \forall x (x \in H_F \to f(x) = 1)\}$.

Given $f \in \Phi$, for each $i \in \{1, \dots, k\}$, there is an open subset $V_i \subset U_i$ such that $x_i \in V_i$ and $f[V_i] \subset \{1\}$. Since $\mathscr{P}$ is a π-network, there is $F \in \mathscr{P}$ such that $F \subset V_1 \times \cdots \times V_k$. So, $f[\pi_i[F]] \subset \{1\}$ for each $i \in \{1, \dots, k\}$ and consequently $f \in V_F$. This shows that $\{V_F : F \in \mathscr{P}\}$ is an open cover of Φ. Therefore, there is a countable subset $\mathscr{P}'$ of $\mathscr{P}$ such that $\{V_F : F \in \mathscr{P}'\}$ forms an open cover of Φ. Let us prove that $\mathscr{P}'$ is a π-network at x.

Let $W = W_1 \times \cdots \times W_k$ be an open subset of X^k which contains x. We can assume that $W_i = W_j$ if $x_i = x_j$ and $W_i \subset U_i$ for each $i, j \in \{1, \dots, k\}$. We choose $f \in C_p(X,2)$ such that

$$f[X \setminus \bigcup_{i=1}^{k} W_i] \subset \{0\}$$

and $f(x_i) = 1$ for each $i \in \{1, \dots, k\}$. Thus $f \in \Phi$, and consequently, there is $F \in \mathscr{P}'$ such that $f \in V_F$. Now, if $(y_1, \dots, y_k) \in F$, since $F \subset U$, $y_i \in U_i$ for each $i \in \{1, \dots, k\}$. Moreover, due to the fact that $f \in V_F$, $y_1, \dots, y_k \in \bigcup_{i=1}^{k} W_i$. However, $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$, then $y_i \in W_i$ for each $i \in \{1, \dots, k\}$. This shows that $F \subset W$. $\square$

A space X has *countable fan tightness* if for any $x \in X$ and any sequence $\langle A_n : n \in \omega \rangle$ of subsets of X such that $x \in \bigcap_{n \in \omega} \mathrm{cl}(A_n)$, we can choose a finite set $B_n \subset A_n$ for each $n \in \omega$ in such a way that $x \in \mathrm{cl}(\bigcup \{B_n : n \in \omega\})$.

M. Sakai introduces the following notion.

Definition 2.3 ([Sak12])**.** A space X has *countable fan tightness for finite sets* if for each point $x \in X$ and each sequence $\langle \mathscr{P}_n : n \in \omega \rangle$ of π-networks at x consisting of finite subsets of X, there is, for each $n \in \omega$, a finite subfamily $\mathscr{G}_n \subset \mathscr{P}_n$ such that $\bigcup \{\mathscr{G}_n : n \in \omega\}$ is a π-network at x.

The following equivalent formulation for countable fan tightness for finite sets will be useful.

Lemma 2.4. *A space X has a countable fan tightness for finite sets if and only if for each point $x \in X$ and any decreasing sequence $\langle \mathscr{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ of π-networks at x, there are, for each $n \in \omega$, finite subfamilies $\mathscr{G}_n \subset \mathscr{P}_n$ such that $\bigcup \{\mathscr{G}_n : n \in \omega\}$ is a π-network at x.*

Proof. The necessity is clear. We show the sufficiency. Let $\langle \mathscr{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ be a sequence of π-networks at $x \in X$. For each $n \in \omega$, we define $\mathscr{P}'_n = \bigcup_{n \leq k} \mathscr{P}_k$. Then, by hypothesis, there is a sequence of finite sets $\langle \mathscr{F}'_n : n \in \omega \rangle$, where $\mathscr{F}'_n \subset \mathscr{P}'_n$, such that $\bigcup_{n\in\omega} \mathscr{F}'_n$ is a π-network at x. Hence, if we define $\mathscr{F}_n = (\bigcup_{k \leq n} \mathscr{F}'_k) \cap \mathscr{P}_n$, $\langle \mathscr{F}_n : n \in \omega \rangle$ is the required sequence. □

Making a modification of the proof of Proposition 2.2 we have the following.

Proposition 2.5. *If the space $C_p(X,2)$ is Menger, then X^n has countable fan tightness for finite sets for any $n \in \omega$.*

Proof. We fix a $k \in \mathbb{N}$, a point $x = (x_1, \ldots, x_k) \in X^k$ and a sequence $\langle \mathscr{P}_n : n \in \omega \rangle$ of π-networks of X^k at x consisting of finite subsets of X. We take open subsets $U_1, \ldots, U_k$ of X such that, for each $i, j \in \{1, \ldots, k\}$, $x_i \in U_i$, $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $U = U_1 \times \cdots \times U_k$. We can suppose that, for every $n \in \omega$, each member of $\mathscr{P}_n$ is contained in U. Since the space $C_p(X,2)$ is Menger, the closed subspace

$$\Phi = \{f \in C_p(X,2) : \forall i (1 \leq i \leq k \to f(x_i) = 1)\}$$

of $C_p(X,2)$ is Menger. For each $F \in [X^k]^{<\omega}$, we define $H_F = \bigcup \{\pi_i[F] : i \in \{1, \ldots, k\}\}$, where π_i is the projection of X^k over the i-th coordinate, and we set $V_F = \{f \in C_p(X,2) : \forall x (x \in H_F \to f(x) = 1)\}$. For each $n \in \omega$, let

$$\mathscr{U}_n = \{V_F : F \in \mathscr{P}_n\}.$$

Given $f \in \Phi$, for each $i \in \{1, \ldots, k\}$, there is an open subset $V_i \subset U_i$ such that $x_i \in V_i$ and $f[V_i] \subset \{1\}$. Since $\mathscr{P}_n$ is a π-network, there is $F \in \mathscr{P}_n$ such that $F \subset V_1 \times \cdots \times V_k$. So $f[\pi_i[F]] \subset \{1\}$ for each $i \in \{1, \ldots, k\}$. Thus $f \in V_F \in \mathscr{U}_n$. This implies that $\mathscr{U}_n$ is an open cover of Φ. Therefore, since Φ is Menger, there is a sequence of finite sets $\langle \mathscr{F}_n : n \in \omega \rangle$ such that $\bigcup_{n\in\omega} \mathscr{F}_n$ forms a cover of Φ and $\mathscr{F}_n \subset \mathscr{U}_n$ for every $n \in \omega$. Choosing a finite subset $\mathscr{P}'_n \subset \mathscr{P}_n$ such that $\mathscr{F}_n$ is equal to $\{V_F : F \in \mathscr{P}'_n\}$ and using similar arguments to the ones used in the proof of Proposition 2.2, we can prove that $\bigcup_{n\in\omega} \mathscr{P}'_n$ is a π-network at x. □

The converses of Proposition 2.2 and Proposition 2.5 are false. The following example can be found in [Arh92, Example II.1.7].

Example 2.6. Let X be the well-known "double arrow" compact space; that is, X is the set $[0,1] \times 2$ endowed with the topology generated by the lexicographic order. For each $a \in (0,1)$, we define $f_a : X \to 2$ as follows:

$$f_a(x) = \begin{cases} 0, & \text{if } x \leq (a,0); \\ \\ 1, & \text{if } x \geq (a,1). \end{cases}$$

Then, the subspace $A = \{f_a : a \in (0,1)\}$ is closed and discrete in $C_p(X,2)$. Hence, $C_p(X,2)$ is not a Menger space. However, X^n has countable fan tightness for finite sets (because X^n satisfies the first axiom of countability) for each $n \in \omega$.

Proposition 1.20 leaves out the possibility that $C_p(X,2)^\omega \cong C_p(X,2^\omega)$ has the Menger property when X is not a discrete space.

Proposition 2.7. *For any zero dimensional space X, $C_p(X,2)^\omega$ is Menger if and only if X is a discrete space.*

Proof. Since $C_p(X,2)$ is dense in 2^X, then $C_p(X,2)$ is compact if and only if X is a discrete space. The conclusion follows from Proposition 1.20. □

We will only analyze finite powers of the spaces $C_p(X,2)$, and for this we have the following:

Proposition 2.8. *For any space X, $C_p(X,2)^n$ is Menger for any $n \in \omega$ if and only if $C_p(X,k)$ is Menger for any $k \in \omega$.*

Proof. This is immediate of the fact that $C_p(X,2)^n$ is homeomorphic to $C_p(X,2^n)$ for any $n \in \omega$ and the fact that a closed subspace of a Menger space is Menger. □

Definition 2.9. A space X is called an *Eberlein-Grothendieck-space*, or an *EG-space*, if it is homeomorphic to a subspace of $C_p(Y)$ for some compact space Y. We say that X is *Eberlein compact* if X is a compact EG-space.

M. Sakai [Sak12, Lemma 2.8] shows that $C_p(X)$ has countable fan tightness for finite sets if and only if X^n is Menger for each $n \in \omega$. Making use of this fact, it is clear that, indeed, all EG-spaces have countable fan tightness for finite sets. With everything we have already said, it is natural to conjecture that the spaces X with $C_p(X,2)$ Menger are subspaces of $C_p(Y)$ where Y^n is Menger for each $n \in \omega$. Corollary 2.12 shows that this is true as long as X' is compact. Before we present this result some notation is needed.

Definition 2.10. We are going to say that a non-empty class of topological spaces $\mathscr{S}$ is *complete* if the following conditions hold:

(a) every closed subspace of a member of $\mathscr{S}$ belongs to $\mathscr{S}$,

(b) every continuous image of a member of $\mathscr{S}$ belongs to $\mathscr{S}$,

(c) if Y is compact and $X \in \mathscr{S}$, then $X \times Y \in \mathscr{S}$, and

(d) if $\{Y_n : n \in \omega\}$ is a sequence of subspaces of a space X and $Y_n \in \mathscr{S}$ for each $n \in \omega$, then the subspace $\bigcup_{n\in\omega} Y_n$ of X belong to $\mathscr{S}$.

Some consequences of the following theorem will be useful in the next chapter.

Theorem 2.11. *Let $\mathscr{S}$ be a complete class of topological spaces. Let X be a subspace of $C_p(Y)$ where $Y^k \in \mathscr{S}$ for each $k \in \omega$. If X' is compact, then $C_p(X,2)^n \in \mathscr{S}$ for each $n \in \omega$.*

Proof. We only show that $C_p(X,2) \in \mathscr{S}$; the $(n \geq 2)$-cases are shown similarly. For each $n \in \mathbb{N}$, we define

$$F_n = \{\varphi \in 2^X : \exists (y_1,\dots,y_n) \in Y^n \forall f \in X'(\varphi[[f;y_1,\dots,y_n;1/n]] = \{\varphi(f)\})\},$$

where $[f;y_1,\dots,y_n;1/n] = \{g \in C_p(Y) : \forall i(1 \leq i \leq n \to |f(y_i)-g(y_i)| < 1/n)\}$.
Each $F_n \in \mathscr{S}$. Indeed, for each $n \in \mathbb{N}$, F_n coincides with $\pi_2[S_n]$, where π_2 is the projection of $Y^n \times 2^X$ over 2^X and

$$S_n = \{(y_1,\dots,y_n,\varphi) \in Y^n \times 2^X : \forall f \in X'(g \in [f;y_1,\dots,y_n;1/n] \to \varphi(f) = \varphi(g))\}.$$

Now, note that $Y^n \times 2^X \in \mathscr{S}$. Then, to prove that $F_n \in \mathscr{S}$, since all continuous images of elements of $\mathscr{S}$ belong to $\mathscr{S}$, it is sufficient to show that $S_n \in \mathscr{S}$. And to do this we proceed as follows: Let $(y_1^0,\dots,y_n^0,\varphi_0) \in Y^n \times 2^X \setminus S_n$. This means that there are $f_0 \in X'$ and $g_0 \in X$ such that $g_0 \in [f_0;y_1^0,\dots,y_n^0;1/n]$ and $\varphi_0(f_0) \neq \varphi_0(g_0)$. Then, the open set

$$(\prod_{i=1}^{n} |f_0 - g_0|^{-1}[[0,1/n)]) \times \{\varphi \in 2^X : \varphi(f_0) = \varphi_0(f_0) \wedge \varphi(g_0) = \varphi_0(g_0)\}$$

of $Y^n \times 2^X$ contains the point $(y_1^0,\dots,y_n^0,\varphi_0)$ and does not intersect S_n. Therefore, S_n is a closed subset of $Y^n \times 2^X$. So, $S_n \in \mathscr{S}$.

Now we will show that $C_p(X,2)$ is equal to $\bigcup_{n\in\mathbb{N}} F_n$ and, since the countable union of elements of $\mathscr{S}$ belongs to $\mathscr{S}$, our theorem will be proved.

Claim 1. $C_p(X,2) \subset \bigcup_{n\in\mathbb{N}} F_n$. In fact, fix a function $\varphi \in C_p(X,2)$. Since φ is continuous, for each $f \in X'$ we can take a neighborhood U_f of f in $C_p(Y)$ such that $\varphi(g) = \varphi(f)$ if $g \in U_f \cap X$. Now, for each $f \in X'$ there are $n_f \in \omega$ and $y_1^f,\dots,y_{n_f}^f \in Y$ for which

$$f \in [f;y_1^f,\dots,y_{n_f}^f;1/n_f] \cap X \subset U_f \cap X.$$

Since X' is a compact space, there are points $f_0,\dots,f_k \in X'$ such that

$$X' \subset [f_0;y_1^{f_0},\dots,y_{n_{f_0}}^{f_0};1/(2n_{f_0})] \cup \dots \cup [f_k;y_1^{f_k},\dots,y_{n_{f_k}}^{f_k};1/(2n_{f_k})].$$

For each $f \in X'$ we take

$$V_f = [f;y_1^{f_0},\dots,y_{n_{f_0}}^{f_0},y_1^{f_1},\dots,y_{n_{f_1}}^{f_1},\dots,y_1^{f_k},\dots,y_{n_{f_k}}^{f_k};1/l] \cap X$$

with $l = 2(n_{f_0}+\dots+n_{f_k})$. It is evident that the collection $\mathcal{V} = \{V_f : f \in X'\}$ covers X'.

Claim 1.1. The collection $\mathcal{V}$ refines $\{U_f \cap X : f \in X'\}$. Indeed, if $f \in X'$, f must belong to

$$[f_j; y_1^{f_j}, \ldots, y_{n_{f_j}}^{f_j}; 1/(2n_{f_j})]$$

for some $j \in \{1, \ldots, k\}$. Then, if $g \in V_f$ we have

$$|g(y_i^{f_j}) - f_j(y_i^{f_j})| \leq |g(y_i^{f_j}) - f(y_i^{f_j})| + |f(y_i^{f_j}) - f_j(y_i^{f_j})| < \frac{1}{l} + \frac{1}{2n_{f_j}} \leq \frac{1}{n_{f_j}}.$$

Therefore, $g \in [f_j; y_1^{f_j}, \ldots, y_{n_{f_j}}^{f_j}; 1/(n_{f_j})] \cap X$, and the later set is contained in $U_{f_j} \cap X$.

Now we prove that φ belongs to F_l. First note the following: if $f \in X'$, $g \in X$ and they satisfy $|f(x) - g(x)| < 1/l$ for all $x \in \{y_1^{f_0}, \ldots, y_{n_{f_0}}^{f_0}, \ldots, y_1^{f_k}, \ldots, y_{n_{f_k}}^{f_k}\}$, then $g \in V_f$ and, consequently, $f, g \in U_h \cap X$ for some $h \in X'$. Because of the choice of U_h, $\varphi(g) = \varphi(h) = \varphi(f)$. Therefore, for each $f \in X'$, if $g \in [f; y_1^{f_0}, \ldots, y_{n_{f_0}}^{f_0}, \ldots, y_1^{f_k}, \ldots, y_{n_{f_k}}^{f_k}; 1/l]$, $\varphi(f) = \varphi(g)$. This shows that $\varphi \in F_l$.

Claim 2. For each $n \in \mathbb{N}$, $F_n \subset C_p(X,2)$. We will prove that each element of F_n is a continuous function. Let $\varphi \in F_n$ and $f \in X$. If f is an isolated point of X, then φ is continuous at f. Suppose $f \in X'$. By definition of F_n, there is $(y_1, \ldots, y_n) \in Y^n$ such that $\varphi[[f; y_1, \ldots, y_n; 1/n] \cap X] = \{\varphi(f)\}$. Since $[f; y_1, \ldots, y_n; 1/n] \cap X$ is an open subset of X containing f, φ is continuous at f. □

A countable open cover $\mathcal{U}$ of a space X is a γ-*cover* if it is infinite and for each $x \in X$ the set $\{U :\in \mathcal{U} : x \notin U\}$ is finite. The collection of all γ-covers of X is denoted by Γ.
The following notion was also introduced in [Sch99a].

$\mathrm{U}_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of members of $\mathcal{A}$, there exists, for each $n \in \omega$, a finite subset $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathcal{B}$.

A space X is *Hurewicz* if it is Lindelöf and satisfies $\mathrm{U}_{fin}(\Gamma, \Gamma)$. It is not difficult to prove that closed subspaces of Hurewicz spaces are Hurewicz, any countable union of Hurewicz subspaces of a space is a Hurewicz subspace [Jus+96] and continuous images of Hurewicz spaces are Hurewicz. And, in [Tal11, Theorem 8], it was shown that preimages of Hurewicz spaces under perfect mappings are Hurewicz. Thus the class of Hurewicz is a complete subclass of the class of topological spaces. The class of Menger spaces and the class of Lindelöf spaces are also complete, so:

Corollary 2.12. *Let X be a subspace of $C_p(Y)$ where Y^k is Menger (resp., Hurewicz, Lindelöf) for each $k \in \omega$. If X' is compact, then $C_p(X,2)^n$ is Menger (resp., Hurewicz, Lindelöf) for each $n \in \omega$.*

Given a space X, $C_p^*(X,\omega)$ denotes the subspace of $C_p(X)$ consisting of all bounded functions with values in ω.

Corollary 2.13. *Let X be a zero-dimensional space and suppose that X' is compact. Then the following statements are equivalent:*

(a) $C_p(X,2)^n$ *is Menger for each* $n \in \omega$*;*

(b) $X \subset C_p(Y)$ *for some space Y such that Y^n is Menger for each* $n \in \omega$*;*

(c) $C_p^*(X,\omega)$ *is Menger.*

Proof. By Proposition 1.5 and Corollary 1.13, the equivalence of (a) and (c) is immediate from the facts that

$$C_p^*(X,\omega) = \bigcup_{n\in\omega} C_p(X,n)$$

and $C_p(X,2)^n$ is homeomorphic to $C_p(X,2^n)$ for each $n \in \omega$. (b) implies (a) follows from Corollary 2.12. And the proof of (a) implies (b) is as follows: For each $x \in X$, we define $\widetilde{x} : C_p(X,2) \to 2$ as $\widetilde{x}(f) = f(x)$. It is not difficult to show that the function $x \mapsto \widetilde{x}$ is an embedding of X into $C_p(C_p(X,2))$. Then $Y = C_p(X,2)$ is the required space. □

A subspace Y of a space X is *bounded* in X if for every continuous function $f : X \to \mathbb{R}$, $f \restriction Y$ is a bounded function, or equivalently, if every locally finite family $\mathscr{O}$ of non-empty open subsets of X, where $Y \cap O \neq \emptyset$ for each $O \in \mathscr{O}$, is finite.

Since $\mathbb{Q}$ and ω^ω are second countable, $C_p(\mathbb{Q},2)$ and $C_p(\omega^\omega,2)$ are Lindelöf [Arh92, Theorem I.1.3]. The following result rules out the possibility that $C_p(\mathbb{Q},2)$ and $C_p(\omega^\omega,2)$ satisfy the Menger property.

Theorem 2.14. *If $C_p(X,2)$ is Menger, then X' is bounded in X.*

Proof. We proceed by contradiction. Suppose that there exists an infinite locally finite family $\{O_n : n \in \omega\}$ of non-empty open subsets of X such that $O_n \cap X' \neq \emptyset$ for each $n \in \omega$. We can suppose without loss of generality that each element of the sequence is open and closed, and that any two different elements of this sequence are disjoint. Let $Y = X \setminus \bigcup_{n\in\omega} O_n$. Since the family $\{O_n : n \in \omega\}$ is locally finite and every O_n is open and closed, Y is open and closed. Moreover, the family $\{O_n : n \in \omega\} \cup \{Y\}$ forms a partition of X in clopen subsets of X. Then $C_p(X,2)$ is homeomorphic to

$$\left(\prod_{n\in\omega} C_p(O_n,2)\right) \times C_p(Y,2).$$

For each $n \in \omega$, since $C_p(X,2)$ is Menger, by Proposition 1.5, $C_p(O_n,2)$ is Menger, and hence, Lindelöf. On the other hand, since each O_n contains a non-isolated point of X,

$C_p(O_n,2)$ is a proper dense subspace of 2^{O_n}. So, since $C_p(O_n,2)$ is Lindelöf, then $C_p(O_n,2)$ is not countably compact; in particular, it contains a countable discrete closed subspace D_n. In this manner, $\prod_{n\in\omega} D_n$ is a closed subspace of $C_p(X,2)$, and given that $C_p(X,2)$ is Menger, $\prod_{n\in\omega} D_n$ is Menger, which is impossible since it is homeomorphic to ω^ω (see Example 1.3). □

Corollary 2.15. *If X is a normal space and $C_p(X,2)$ is Menger, then X' is countably compact.*

Proof. By Theorem 2.14, X' is bounded in X. Since X is a normal space and X' is a closed subset of X, X' is pseudocompact. Again, by the normality of X, X' is countably compact. □

Corollary 2.16. *Let X be a Lindelöf space. Then $C_p(X,2)^n$ is Menger for any $n\in\omega$ if and only if X' is compact and $X\subset C_p(Y)$ for some space Y such that Y^n is Menger for each $n\in\omega$.*

Proof. If $C_p(X,2)^n$ is Menger for any $n\in\omega$, by Corollary 2.13, $X\subset C_p(Y)$ for some space Y such that Y^n is Menger for each $n\in\omega$. Furthermore, applying Corollary 2.15, X' is countably compact and hence compact since X is a Lindelöf space. The proof of the converse is a consequence of Corollary 2.12. □

The following is a consequence of Theorem III.4.23 in [Arh92].

Proposition 2.17. *If X is separable, then every countably compact subspace of $C_p(X)$ is compact.*

Corollary 2.18. *Let X be a space with $iw(X)=\omega$. Then the following statements are equivalent.*

(a) *X' is compact and $X\subset C_p(Y)$ for some space Y such that Y^n is Menger for any $n\in\omega$;*

(b) *$C_p(X,2)^n$ is Menger for any $n\in\omega$ and X is a normal space.*

Proof. By Corollary 2.12, (a) implies that $C_p(X,2)^n$ is Menger for any $n\in\omega$. Moreover, since X is completely regular and X' is compact, then X is normal [GJ60, Problem 3D,5]. Now suppose (b), since X is homeomorphic to some subspace of $C_p(C_p(X,2))$, to prove (a) it is sufficient to show that X' is compact. The normality of X and Corollary 2.15 imply that X' is a countably compact space. Given that $iw(X)=\omega$, $C_p(X,2)$ is separable (see proof of Theorem I.1.5 in [Arh92]). By Proposition 2.17, X' is compact. □

On metric spaces we have the following.

Theorem 2.19. *Let X be a metrizable space. Then the following statements are equivalent.*

(a) *$C_p(X,2)$ is Menger;*

(b) *$C_p(X,2)^n$ is Menger for each $n \in \omega$;*

(c) *$C_p(X,2)$ is σ-compact;*

(d) *X' is compact.*

Proof. Since every metrizable space is an EG-space (see [Arh92, Theorem IV.1.25]), X is an EG-space. Then, if X' is countably compact, it is compact being X metrizable; so, X' is Eberlein compact, and by Corollary 4.12 in [CCTM03], $C_p(X,2)$ is σ-compact. This proves that (d) implies (c). Clearly (c) implies (b) and (b) implies (a). Finally, by Corollary 2.15, (a) implies (d). □

2.2 The Menger property in $C_p(L,2)$ when L is a GO-space

A space L is a *GO-space* (Generalized Ordered space) if it is a subspace of a linearly ordered topological space (LOTS). Observe that if L is a countable GO-space, then L is zero-dimensional, separable and metrizable (and $C_p(L,2)$ is Lindelöf). Then, by Theorem 2.19 we obtain:

Proposition 2.20. *Let L be a countable GO-space. Then the following statements are equivalent.*

(a) *$C_p(L,2)$ is Menger;*

(b) *$C_p(L,2)$ is σ-compact;*

(d) *L' is compact.*

Now we are going to characterize the Menger property in $C_p(L,2)$ when L is an uncountable GO-space (without isolated points). We will follow some notations, terminology and constructions due to R.Z. Buzyakova in [Buz07]. First we will review a construction of the Dedekind completion of a given GO-space L.

Definition 2.21. An ordered pair $\langle A,B\rangle$ of disjoint closed subsets of a GO-space L is called a *Dedekind section* if $A\cup B = L$, $\sup A$ does not exist, $\inf B$ does not exist, and A is to the left of B; that is, for every $a \in A$ and $b \in B$, $a < b$ holds. A pair $\langle L,\emptyset\rangle$ ($\langle\emptyset,L\rangle$) is also a Dedekind section if $\sup L$ ($\inf L$) does not exist.

Definition 2.22. The *Dedekind completion* of a GO-space L, denoted by cL, is constructed as follows. The set cL is the union of L and the set of all Dedekind sections of L. The order on cL is natural: the order on cL among elements of L coincides with the order on L of these elements. If $x \in L$ and $y = \langle A,B\rangle \in cL\setminus L$ then x is less (greater) than y if $x \in A$ ($x \in B$). If $x = \langle A_1,B_1\rangle$ and $y = \langle A_2,B_2\rangle$ are elements of $cL\setminus L$, then x is less than y if A_1 is a proper subset of A_2. Consider now cL with the order topology generated by the order just defined. We will denote by ∞ and $-\infty$ the supremum and infimum, respectively, of cL.

Observe that for every GO-space L, cL is a compact linearly ordered topological space. For a given GO-space L we consider the space $T(L)$:

Definition 2.23. An element $x \in cL$ is in $T(L)$ if and only if $x \in cL \setminus L$, or $x \in L$ and either x is the smallest or the greatest element of L or x has an immediate successor in L. We endowed $T(L)$ with the following topology: points of $T(L)$ that are in L are declared isolated. The other points inherit base neighborhoods from the Dedekind completion cL, that is, for every $x \in T(L) \setminus L$ a neighborhood base at x is the family $\{U \cap T(L) : x \in U \in \tau(cL)\}$, where $\tau(cL)$ is the topology of cL.

Observe that $T(L)$ is a GO-space. Indeed, $T(L)$ can be obtained from cL as follows. For each $x \in L$ that has an immediate succesor x^+ in L, insert a new point p_x between x and x^+. If $x \in L$ is the smallest element of L, we add a point $p_{-\infty}$ to the left of x; and if $x \in L$ is the greatest element of L, we add a point p_∞ to the right of x. Denote by L' the resulting space. Then cL' is a compact linearly ordered topological space containing cL as a closed subspace. The subspace of cL' that consists of all inserted points p_x's and $cL \setminus L$ is a copy of $T(L)$. Thus we can think of $T(L)$ as a GO-space with the order inherited from cL'.

Example 2.24.

- If $L = \mathbb{Q}$ is the subspace of rationals of $\mathbb{R}$, then cL is the two-point compactification of $\mathbb{R}$. Since no point of L has an immediate successor in L, $T(L)$ is the subspace of cL consisting of $\mathbb{P}$, the set of irrationals, with the two points at "infinity" of the compactification cL.

- If $L = \omega_1$, then $cL = \omega_1 + 1$. Observe that every ordinal in ω_1 has an immediate successor in ω_1. Then $T(L)$ is the Lindelöfication of the ω_1-sized discrete space.

- For every ordinal τ, let $\tau_\omega = \{\alpha \leq \tau : \mathrm{cof}(\alpha) \leq \omega\}$. Then $c\tau_\omega$ is $\tau + 1$ and $T(\tau_\omega)$ is the set $\tau + 1$ with the following topology: every point in $\tau + 1$ with cofinality less than or equal to ω is an isolated point and and a local base for any of the remaining point α is given by all neighborhoods of α in the LOTS $\tau + 1$.

- Let L be the "double arrows space of Alexandroff"; that is, L is the subspace $A^+ \cup A^-$, where $A^+ = [0,1) \times \{0\}$ and $A^- = (0,1] \times \{1\}$, of the space $[0,1] \times 2$ with the lexicographic order topology. Let S any subset of $(0,1)$ and $L_S = L \setminus ((S \times \{0\}) \cup (S \times \{1\}))$. Note that cL_S is L and every point of $([0,1) \setminus S) \times \{0\}$ has an immediate successor in L_S. Then $T(L_S)$ can be obtained from $[0,1]$ by retaining the standard topology on points of $S \cup \{0\}$ and declaring all others points isolated.

R.Z. Buzyakova presents in [Buz07] more examples of $T(L)$ for some particular GO-spaces.

If $x_1,\dots,x_n \in cL$ and $-\infty \le x_1 \le \dots \le x_n \le \infty$, then by $f = f^0_{x_1,\dots,x_n}$ we denote the function from L to 2 defined by

$$f[(x_i, x_{i+1}] \cap L] = \{i \mod 2\},$$

for each $i \in \{1,\dots,n\}$. The rightmost formula is simply $\{1\}$ if n is odd and $\{0\}$ otherwise. The functions $f^1_{x_1,\dots,x_n}$ are defined similarly by switching places between $\{0\}$ and $\{1\}$ in the above formulas.

Definition 2.25. A function f from X to 2 belongs to $S_p(L,2)$ if and only if there exists $-\infty \le x_1 \le \dots \le x_n \le \infty$ in $T(L)$ such that $f = f^0_{x_1,\dots,x_n}$ or $f = f^1_{x_1,\dots,x_n}$.

Observe that $[-\infty, x_1] \cap L$, $(x_1, x_2] \cap L, \dots, (x_n, \infty] \cap L$ are clopen subsets of L because $x_1,\dots,x_n \in T(L)$. Therefore $f^0_{x_1,\dots,x_n}$ is continuous and $S_p(L,2) \subset C_p(L,2)$. The topology of $S_p(L,2)$ is the topology inherited from $C_p(L,2)$. For each $n \in \mathbb{N}$, we define

$$S_n = \{f \in S_p(L,2) : \exists x_1,\dots,x_n \in T(L)(f = f^0_{x_1,\dots,x_n} \vee f = f^1_{x_1,\dots,x_n})\}.$$

Observe that $S_p(L,2) = \bigcup_{n\in\mathbb{N}} S_n$, and $S_p(L,2) = C_p(L,2)$ if L is countably compact.
We are going to denote by S^* the subspace $\{f^1_x : x \in T(L)\}$ of S_1.

For any $f, g \in C_p(L,2)$, the addition $f+g$ is taken mod 2.

Lemma 2.26 ([Buz07])**.** *Let L be a GO-space. Then for any $f \in S_p(L,2)$, there exist $f_1,\dots,f_k \in S^*$ such that $f = f_1 + \dots + f_k$.*

More properties of S^* are given in [Buz07]. One of these is the following:

Theorem 2.27 ([Buz07])**.** *The subspace S^* of $S_p(L,2)$ is homeomorphic to $T(L)$.*

For a countably compact GO-space L, R.Z. Buzyakova proves in [Buz07] that $C_p(L,2)$ is Lindelöf if and only if $T(L)$ is Lindelöf. We show that in fact this is a sufficient condition in order to have $C_p(L,2)$ Menger.

Theorem 2.28. *Let L be a first countable GO-space without isolated points. The following statements are equivalent.*

(a) *$C_p(L,2)$ is Lindelöf and L is countably compact,*

(b) *$T(L)$ is Lindelöf and L is countably compact,*

(c) *$T(L)^n$ is Menger for each $n \in \omega$ and L is countably compact,*

(d) *$C_p(L,2)^n$ is Menger for each $n \in \omega$,*

(e) *$C_p^*(L,\omega)$ is Menger,*

(f) *$C_p(L,2)$ is Menger.*

Proof. The equivalence $(a) \leftrightarrow (b)$ is Theorem 4.1 in [Buz07]. We suppose (b) and we are going to prove (c). Let us show that $T(L)$ is a P-space. For each $n \in \omega$, let U_n be an open subset of $T(L)$. We are going to prove that $F = \bigcap_{n\in\omega} U_n$ is open. Take any x in this intersection. If $x \in L$ then x is isolated in $T(L)$. If $x \notin L$ then, due to the countably compactness of L, x is unreachable by nontrivial countable sequences in cL, and therefore, in $T(L)$. In both cases, we conclude that x is in the interior of F. This shows that $T(L)$ is a P-space. Then $T(L)^n$ is a P-space for $n \in \omega$. Applying Noble's theorem [Nob69] (i.e., a countable power of a Lindelöf P-space is Lindelöf), $T(L)^\omega$ is Lindelöf, and hence, $T(L)^n$ is Lindelöf for any $n \in \omega$. But Lindelöf property agrees with Menger property in P-spaces (see Corollary 1.16). Then $T(L)^n$ is Menger for any $n \in \omega$.

Now suppose (c). Given that $T(L)$ is homeomorphic to S^* (see Theorem 2.27) and that the countable union of Menger spaces is Menger, the topological sum $\bigoplus_{k\in\omega}(S^*)^k$ is Menger. Moreover, every finite power of this space is Menger. Besides, if we define the continuous function $\mathscr{F} : \bigoplus_{k\in\omega}(S^*)^k \to S_p(L,2)$ as $\mathscr{F}(F) = f_1 + \cdots + f_k$ where $F = (f_1,\ldots,f_k) \in (S^*)^k$, then, by Lemma 2.26, $\mathscr{F}$ is surjective. Then, for each $n \in \omega$, the function $\mathscr{F}^n : (\bigoplus_{k\in\omega}(S^*)^k)^n \to S_p(L,2)^n$ defined by $\mathscr{F}^n(F_1,\ldots,F_n) = (\mathscr{F}(F_1),\ldots,\mathscr{F}(F_n))$ is a surjective continuous function. Thus, $S_p(L,2)^n = C_p(L,2)^n$ is Menger. This shows that (c) implies (d).

(d) implies (e) is trivial. Since $C_p(L,2)$ is a closed subspace of $C_p^*(L,\omega)$, by Proposition 1.5, (e) implies (f). Finally, if we suppose (f), then, by Corollary 2.15, $L = L'$ is countably compact and clearly $C_p(L,2)$ is Lindelöf. This proves that (f) implies (a). □

Corollary 2.29. *Let L be a first countable countably compact GO-space without isolated points. Then, $C_p(L,2)$ is Lindelöf if and only if $C_p(L,2)$ is Menger.*

Problem 2.30. Determine when $C_p(L,2)$ is Menger assuming L is a first countable GO-space (without any restriction about the isolated points in L).

2.3 The Menger property in $C_p(X,2)$ when X is a subspace of ordinals

By a *subspace of ordinals* we are refering to a subspace of an ordinal α. As we have already said, the set of ordinals lower than an ordinal α endowed with its order topology is denoted by α. As a corollary of Proposition 2.20 we have the following.

Corollary 2.31. *Let $\alpha \in \omega_1$. Then the following statements are equivalent.*

(a) $C_p(\alpha,2)$ *is Menger,*

(b) $C_p(\alpha,2)$ *is σ-compact,*

(c) α *is a succesor ordinal.*

If X is normal and $C_p(X,2)$ is Menger, then X' is countably compact and X has countable fan tightness (see Corollary 2.15 and Proposition 2.2). Then, when X is a subspace of ordinals and $C_p(X,2)$ is Menger, X' must be countably compact, X is first countable and, obviously, $C_p(X,2)$ is Lindeöf. In the following statements we see that these properties are enough to obtain the Menger property in $C_p(X,2)$.

The proof of the following theorem was suggested to the referee of [BSTM15] by Professor Piotr Szewczak. His proof is simpler than the one we gave in a previous version of this work.

Theorem 2.32. *Let X be a subspace of ordinals and $n \in \mathbb{N}$. Then $C_p(X,2)^n$ is Menger if and only if $C_p(X,2)^n$ is Lindelöf and X' is countably compact.*

Proof. Remember that $C_p(X,2)^n$ is homeomorphic to $C_p(X,2^n)$. We are going to show our theorem when $n = 1$ (the proof for 2^n instead of 2 is similar). It is obvious that if $C_p(X,2)$ is Menger, then $C_p(X,2)$ is Lindelöf. Moreover, by Corollary 2.15, X' is countably compact. Reciprocally, we suppose that $C_p(X,2)$ is Lindelöf and X' is countably compact. We will show that $C_p(X,2)$ is Menger. We will prove this fact by induction over $\alpha = \sup X$. Let us assume that the statement is true for every $\beta < \alpha$, that is, if Z is a first countable subspace of ordinals, Z' is countably compact, $\beta = \sup Z$ and $C_p(Z,2)$ is Lindelöf, then $C_p(Z,2)$ is Menger. Let $\delta = \sup X'$.

Case I. If $\delta < \alpha$, then $X' \subset Z = X \cap (\delta+1)$, Z is clopen in X and $X \setminus Z$ is clopen and discrete in X. Therefore $X = Z \oplus (X \setminus Z)$ and $C_p(X,2) \cong C_p(Z,2) \times 2^{X\setminus Z}$. Given that $C_p(Z,2)$ is a closed subspace of $C_p(X,2)$, we deduce that $C_p(Z,2)$ is Lindelöf. Note that $Z' = X'$ so it is countably compact. From the inductive hypothesis we have that $C_p(Z,2)$ is Menger and by compactness of $2^{X\setminus Z}$ the space $C_p(X,2)$ is also Menger.

Case II. If $\delta = \alpha$ and there is in X a strictly increasing countable sequence $\langle \alpha_n : n \in \omega \rangle$ which converges to α. Then by $\delta = \alpha$ and countable compactness of X' we infer that $\alpha \in X'$. Let us observe that the family $\{X \cap (\alpha_n, \alpha] : n \in \omega\}$ forms a base for X at α. Since every $f \in C_p(X,2)$ is continuous at α and $\alpha \in X'$, there is $k \in \omega$ such that $f[(\alpha_k,\alpha]] = \{f(\alpha)\}$. Let $A_n^j = \{f \in C_p(X,2) : \forall x \in X (x \in (\alpha_n,\alpha] \to f(x) = j)\}$ for every $n \in \omega$ and $j \in 2$. We have that

$$C_p(X,2) = \bigcup \{A_n^j : n \in \omega \wedge j \in 2\}$$

Every A_n^j is homeomorphic to $C_p(Z_n,2)$ where $Z_n = X \cap (\alpha_n + 1)$. Since Z_n is clopen in X, we can easily verify that $C_p(Z_n,2)$ is Lindelöf and Z_n' is countably compact. Now it follows from our inductive assumption that $A_n^j \cong C_p(Z_n,2)$ is Menger. Because $C_p(X,2) = \bigcup\{A_n^j : n \in \omega \wedge j \in 2\}$ we conclude that $C_p(X,2)$ is Menger.

Case III. If $\delta = \alpha$ and the cofinality of α in X is not countable. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of countable open covers of $C_p(X,2)$ consisting of basic open sets. Let us observe that each element $U \in \mathscr{U}_n$ is of the form $U = \prod_{x\in X} U(x) \cap C_p(X,2)$, where $U(x) \neq 2$ only if $x \in X_U \subset X$, where X_U is finite. Let us observe that there is some $\beta \in X \cap \alpha$ such that $\bigcup\{X_U : \exists n\,(n \in \omega \wedge U \in \mathscr{U}_n)\} \subset X \cap (\beta + 1) = Z$. Clearly $\beta = \sup Z$. Then

$$(*) \quad \forall U\, \forall x\, (\exists n\,(n \in \omega \wedge U \in \mathscr{U}_n) \wedge x \in X \setminus Z \to U(x) = 2).$$

It is easy to see that Z is a clopen subset of X and $Z' = X' \cap (\beta + 1)$. Hence Z' is countably compact being a closed subset of the countably compact space X'. Since $C_p(X,2)$ is homeomorphic to $C_p(Z,2) \times C_p(X \setminus Z,2)$ we have that $C_p(Z,2)$ is a closed subspace of $C_p(X,2)$ and so it is Lindelöf. By inductive assumption, $C_p(Z,2)$ is Menger. Now let $\mathscr{U}_n' = \{U \cap C_p(Z,2) : U \in \mathscr{U}_n\}$ for every $n \in \omega$. Then $\langle \mathscr{U}_n' : n \in \omega \rangle$ is a sequence of open cover of $C_p(Z,2)$. Therefore, there are $\mathscr{V}_n' \in [\mathscr{U}_n']^{<\omega}$ such that $\bigcup\{\mathscr{V}_n' : n \in \omega\}$ covers $C_p(Z,2)$. For every $n \in \omega$ pick $\mathscr{V}_n \in [\mathscr{U}_n]^{<\omega}$ such that $\mathscr{V}_n' = \{U \cap C_p(Z,2) : U \in \mathscr{V}_n\}$. By $(*)$ we have that $\bigcup\{\mathscr{V}_n : n \in \omega\}$ covers $C_p(X,2)$. □

It is shown in [Buz04] that for every countably compact first countable subspace X of ordinals, $C_p(X,2)^n$ is Lindelöf for each $n \in \omega$. Therefore:

Corollary 2.33. *For any countably compact first countable subspace X of ordinals, $C_p(X,2)^n$ is Menger for each $n \in \omega$.*

It is shown in [Buz06] that for every first countable subspace X of ordinals with countable extent we have that $C_p(X,2)^n$ is Lindelöf for each $n \in \omega$. So, we obtain:

Corollary 2.34. *Let X be a first countable subspace of ordinals with countable extent. Then $C_p(X,2)^n$ is Menger for each $n \in \omega$ if and only if X' is countably compact.*

Corollary 2.15 shows that when X is normal, if $C_p(X,2)$ is Menger, then X' is countably compact. With the same hypotheses we cannot imply the compactness of X'. Indeed, by Corollary 2.33, $C_p(\omega_1,2)$ is Menger.

Observe, on the one hand, that the ordinal number $X = \omega \cdot \omega$ is a countable, metrizable ordinal subspace such that $C_p(X,2)$ is Lindelöf but $C_p(X,2)$ is not Menger (see Theorem 2.32). So, it is not possible to add the statement "$C_p(X,2)$ is Lindelöf" in the list of equivalent statements neither in Theorem 2.19 nor in Proposition 2.20, nor in Corollary 2.31 (compare with Corollary 2.29). On the other hand, the converse of Corollary 2.33

is not true. Indeed, $C_p(\omega,2)^n$ is Menger for any $n \in \omega$ and ω is not countably compact. A non discrete example of the same fact is the countable metrizable ordinal subspace $Y = (\omega \cdot \omega + 1) \setminus \{\omega\}$.

Moreover, it is natural to conjecture that the class of subspaces of ordinals X for which $C_p(X,2)$ is Menger is equal to the class of ordinal subspaces which are the topological sum of two subspaces, one of them a discrete subspace and the other a first countable countably compact ordinal subspace. This is not true. In fact, consider

$$X = \{\omega_1 \cdot n : n \leq \omega\} \cup \bigcup_{n\in\omega} \{(\omega_1 \cdot n) + m : m \in \omega\}.$$

We have that $C_p(X,2)$ is Menger but X cannot be expressed as the sum of a discrete subspace plus a countably compact first countable ordinal subspace. Indeed, if X is the topological sum $Z \oplus Y$, where Z is a discrete space and Y is countably compact space. First note that $\omega_1 \cdot \omega \in Y$. Since Y is open in X and $\omega_1 \cdot \omega \in Y$, there is $n \in \omega$ such that $[\omega_1 \cdot n, \omega_1 \cdot \omega] \cap X \subset Y$. Then $\{\omega_1 \cdot n + m : m \in \omega\}$ is an infinite closed discrete subspace of Y, which is imposible. Using similar arguments to the ones used in the Case II of proof of Theorem 2.32 we obtain that $C_p(X,2)$ is Menger (even σ-compact).
Also, it is natural to conjecture that Corollary 2.33 is valid for any GO-space (or LOTS) not only for subspaces of ordinals, but Example 2.6 shows that this is false.

In [Buz06, Question 3.3] R. Z. Buzyakova asks if $C_p(X,2)$ is Lindelöf when X is a first countable subspace of ordinals and X' have countable extend. Then, is natural asks if $C_p(X,2)$ is Lindelöf when X is a first countable subspace of ordinals and X' is countably compact. We ask this question in the following form (see Theorem 2.32):

Problem 2.35. Is there a first countable ordinal subspace X with X' countably compact such that $C_p(X,2)$ is not Menger?

2.4 The Menger property in $C_p(X,2)$ when X is a countable simple space

A space X is called *simple* if X has exactly one non-isolated point. For any filter $\mathscr{F}$ on ω, we define the space $\omega \cup \{\mathscr{F}\}$ as follows: any $n \in \omega$ is declared isolated and the sets $A \cup \{\mathscr{F}\}$, where $A \in \mathscr{F}$, form a base of neighborhoods of $\mathscr{F}$. Any countable simple space is homeomorphic to $\omega \cup \{\mathscr{F}\}$ for a filter $\mathscr{F}$ on ω.

It is proved in [Arh92, Proposition III.3.3] that if A_τ denotes the one-point compactification of the discrete space of cardinality τ, then A_τ is Eberlein compact. And therefore $C_p(A_\tau,2)$ is Menger. It is also shown in [Arh92, Example III.1.7] that if $\mathscr{F} \in \omega^*$, $\omega \cup \{\mathscr{F}\}$ is not an EG-space; that is, $\omega \cup \{\mathscr{F}\}$ cannot be embedded in a space $C_p(Y)$ where Y is compact.

The following results shows that, under certain conditions, $\omega \cup \{\mathscr{F}\}$ can be embedded in a space $C_p(Y)$ for some space Y for which Y^n is Menger for every $n \in \omega$. These conditions are set in the following definitions.

Definition 2.36 ([Laf89])**.** An ultrafilter $\mathscr{F} \in \omega^*$ is a *strong P-point* if for any sequence $\langle \mathscr{C}_n : n \in \omega \rangle$ of compact subspaces of $\mathscr{F}$ (considering $\mathscr{F}$ as a subset of 2^ω with the product topology) there is an interval partition $\langle I_n : n \in \omega \rangle$ of 2^ω such that for each choice of $X_n \in \mathscr{C}_n$ we have

$$\bigcup_{n \in \omega} (I_n \cap X_n) \in \mathscr{F}.$$

Given a filter $\mathscr{F}$ on ω we define $\mathscr{F}^{<\omega}$ to be the filter on $[\omega]^{<\omega} \setminus \{\emptyset\}$ generated by $\{[F]^{<\omega} \setminus \{\emptyset\} : F \in \mathscr{F}\}$. Note that the filter $\mathscr{F}^{<\omega}$ on $[\omega]^{<\omega} \setminus \{\emptyset\}$ is not an ultrafilter even if $\mathscr{F}$ is.

Definition 2.37 ([BHV13])**.** A filter $\mathscr{F}$ on a countable set S is a P^+-filter if for any $\subset$-descending sequence $\langle X_n : n \in \omega \rangle \subset \mathscr{F}^+$, there is an $X \in \mathscr{F}^+$ such that $X \subset^* X_n$ for all $n \in \omega$, where $\mathscr{F}^+ = \{X \subset S : S \setminus X \notin \mathscr{F}\}$.

The elements of $\mathscr{F}^+$ are called *positive sets* (with respect to $\mathscr{F}$). Then, a filter $\mathscr{F}$ is a P^+-filter if every decreasing sequence of positive sets has a positive pseudointersection. The definition of a strong P-point that we will use is the following.

Theorem 2.38 ([BHV13])**.** *An ultrafilter $\mathscr{F} \in \omega^*$ is a strong P-point if and only if $\mathscr{F}^{<\omega}$ is a P^+-filter.*

The following result was conjectured by M. Huršák and is the key to characterize the Menger property in $C_p(\omega \cup \{\mathscr{F}\}, 2)$.

Proposition 2.39. *Let $\mathscr{F}$ be a filter on ω. The space $X = \omega \cup \{\mathscr{F}\}$ has countable fan tightness for finite sets if and only if $\mathscr{F}^{<\omega}$ is a P^+-filter.*

Proof. First note that $\mathscr{P} \in (\mathscr{F}^{<\omega})^+$ if and only if $\mathscr{P}$ is a π-network at $\mathscr{F}$ in X. Suppose that $\mathscr{F}^{<\omega}$ is a P^+-filter. Let $\langle \mathscr{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ be a decreasing sequence of π-networks at $\mathscr{F}$ (see Lemma 2.4). Given that $\mathscr{F}^{<\omega}$ is a P^+-filter, and $\langle \mathscr{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ is a decreasing sequence of positive sets with respect to $\mathscr{F}^{<\omega}$, $\langle \mathscr{P}_n : n \in \omega \rangle$ has a positive pseudointersection $\mathscr{P} \in (\mathscr{F}^{<\omega})^+$. Since $\mathscr{P} \setminus \mathscr{P}_0$ is finite, $\mathscr{P} \cap \mathscr{P}_0$ is a π-network at $\mathscr{F}$. Then, if we suppose that $\bigcap_{n \in \omega} \mathscr{P}_n = \{p_n : n \in \omega\}$, and define $\mathscr{K}_n = (\mathscr{P} \cap \mathscr{P}_n \setminus \mathscr{P}_{n+1}) \cup \{p_n\}$ for each $n \in \omega$, $\bigcup_{n \in \omega} \mathscr{K}_n = \mathscr{P} \cap \mathscr{P}_0$ is a π-network at $\mathscr{F}$. Observe that $\mathscr{K}_n$ is finite because $(\mathscr{P} \cap \mathscr{P}_n \setminus \mathscr{P}_{n+1}) \cup \{p_n\} \subseteq (\mathscr{P} \setminus \mathscr{P}_{n+1}) \cup \{p_n\}$ and $\mathscr{P}$ is a pseudointersection of the family $\{\mathscr{P}_n : n < \omega\}$.

Reciprocally, suppose that $\omega \cup \{\mathscr{F}\}$ has countable fan tightness for finite sets. Let $\langle \mathscr{P}_n : n \in \omega \rangle$ be a decreasing sequence of positive sets. Since $\langle \mathscr{P}_n : n \in \omega \rangle$ is a sequence of π-networks at $\mathscr{F}$, there is a sequence of finite sets $\langle \mathscr{F}_n : n \in \omega \rangle$ such that $\mathscr{P} = \bigcup_{n \in \omega} \mathscr{F}_n$

is a π-network at $\mathscr{F}$ and $\mathscr{F}_n \subset \mathscr{P}_n$ for each $n \in \omega$. Then $\mathscr{P}$ is a positive set and, since $\mathscr{P} \setminus \mathscr{P}_{n+1} \subset \mathscr{F}_0 \cup \cdots \cup \mathscr{F}_n$, $\mathscr{P}$ is a pseudointersection of $\langle \mathscr{P}_n : n \in \omega \rangle$. □

Corollary 2.40. *Let $\mathscr{F} \in \omega^*$. The subspace $\omega \cup \{\mathscr{F}\}$ of $\beta\omega$ has countable fan tightness for finite sets if and only if $\mathscr{F}$ is a strong P-point.*

Theorem 2.41. *Let $\mathscr{F}$ be a filter on ω and $X = \omega \cup \{\mathscr{F}\}$. Then, the following statements are equivalent:*

(a) *$C_p(X,2)$ is Menger;*

(b) *$C_p(X,2)^n$ is Menger for any $n \in \omega$;*

(c) *$C_p^*(X,\omega)$ is Menger;*

(d) *$\mathscr{F}^{<\omega}$ is a P^+-filter.*

Proof. If $C_p(X,2)$ is Menger, then, by Proposition 2.5 and Proposition 2.39, $\mathscr{F}^{<\omega}$ is a P^+-filter. Now assume that $\mathscr{F}^{<\omega}$ is a P^+-filter in ω. We are going to show that $C_p(X,2)$ is Menger (the proof for n instead of 2 is similar). For each $k \in 2$ and $F \subset \omega$, we define $A_F^k = \{f \in 2^\omega : \forall x(x \in F \to f(x) = k)\}$. Note that $C_p(X,2)$ is homeomorphic to the subspace $\bigcup\{A_F^k : F \in \mathscr{F} \wedge k \in 2\}$ of 2^ω. Then, to see that $C_p(X,2)$ is Menger, by Corollary 1.13, it is enough to show that $\bigcup\{A_F^k : F \in \mathscr{F}\}$ is Menger for each $k \in 2$. However, $\bigcup\{A_F^k : F \in \mathscr{F}\}$ is homeomorphic to $\bigcup\{A_F^m : F \in \mathscr{F}\}$ for $k, m \in 2$. So, it is enough to prove that $A = \bigcup\{A_F^0 : F \in \mathscr{F}\}$ is Menger. To simplify the notations, we write A_F to mean A_F^0 and, if F is a single point x, we write A_x instead of A_F.

Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of countable covers of A such that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$ for each $n \in \omega$ (see Lemma 1.9). We can suppose that each $\mathscr{U}_n$ is closed under finite unions. For each open subset of A, let $Y_U = \{H \in [\omega]^{<\omega} : A_H \subset U\}$. And we define $Z_n = \bigcup_{U \in \mathscr{U}_n} Y_U$ for each $n \in \omega$. In view of the fact that $\mathscr{U}_{n+1}$ refines $\mathscr{U}_n$, $Z_{n+1} \subset Z_n$. Moreover, Z_n is a positive set. Indeed, if $F \in \mathscr{F}$, since A_F is compact, there is an element $U \in \mathscr{U}_n$ containing A_F. Given that $A_F = \bigcap_{x \in F} A_x$ and A_x is compact, there is $H \in [F]^{<\omega}$ such that $A_H = \bigcap_{x \in H} A_x \subset U$. Then $[F]^{<\omega} \cap Z_n \neq \emptyset$. Now, since $\mathscr{F}^{<\omega}$ is a P^+-filter, the sequence of positive sets $\langle Z_n : n \in \omega \rangle$ has a positive pseudointersection $\widetilde{Z} \in (\mathscr{F}^{<\omega})^+$. Suppose that $\bigcap_{n \in \omega} Z_n = \{b_n : n \in \omega\}$, then we define, for each $n \in \omega$, $\mathscr{P}_n = (\widetilde{Z} \cap Z_n \setminus Z_{n+1}) \cup \{b_n\}$. In the same way as in Proposition 2.39 we infer that $\mathscr{P}_n$ is finite for every $n \in \omega$. In this manner $Z = \bigcup_{n \in \omega} \mathscr{P}_n = \widetilde{Z} \cap Z_0 \in (\mathscr{F}^{<\omega})^+$. For each $n \in \omega$ and $H \in \mathscr{P}_n$, we choose $U_H \in \mathscr{U}_n$ such that $A_H \subset U_H$. We define $\mathscr{F}_n = \{U_H : h \in \mathscr{P}_n\}$ for each $n \in \omega$. Given $f \in A$, there is $F \in \mathscr{F}$ such that $f \in A_F$. Since Z is a positive set, $[F]^{<\omega} \setminus \{\emptyset\}$ intersects Z and, hence, intersects some $\mathscr{P}_n$. Consequently, if $H \in ([F]^{<\omega} \setminus \{\emptyset\}) \cap \mathscr{P}_n$, then $A_F \subset A_H \subset U_H$. This proves that $f \in U_H$. That is, $\bigcup_{n \in \omega} \mathscr{F}_n$ is a cover of A.

The implication (b) → (c) is a consequence of Corollary 1.13 and the equality $C_p^*(X,\omega) = \bigcup_{n\in\omega} C_p(X,n)$. The implication (c) → (b) is a consequence of Proposition 1.5 and the fact that each $C_p(X,n)$ is a closed subset of $C_p^*(X,\omega)$. □

As a consequence of Theorem 2.38 and Theorem 2.41 we conclude:

Corollary 2.42. *Let $\mathscr{F} \in \omega^*$ and X the subspace $\omega \cup \{\mathscr{F}\}$ of $\beta\omega$. Then the following statements are equivalent*

(a) *$C_p(X,2)$ is Menger;*

(b) *$C_p(X,2)^n$ is Menger for any $n \in \omega$;*

(c) *$C_p^*(X,\omega)$ is Menger;*

(d) *$\mathscr{F}$ is a strong P-point.*

As previously mentioned, Example III.1.7 in [Arh92] shows that $\omega \cup \{\mathscr{F}\}$ is not an EG-space and, by the Theorem 4.16 in [CCTM03], $C_p(\omega \cup \{\mathscr{F}\},2)$ is not σ-compact. Then, by Corollary 2.42, if $\mathscr{F}$ is a strong P-point, $C_p(\omega \cup \{\mathscr{F}\},2)$ is a Menger space which is not σ-compact.

2.5 The Menger property in $C_p(\Psi(\mathscr{A}),2)$

An *almost disjoint family* of subsets of ω is a collection $\mathscr{A}$ of subsets of ω such that each element in $\mathscr{A}$ is infinite, and if $A, B \in \mathscr{A}$, $|A \cap B| < \aleph_0$. An almost disjoint family $\mathscr{A}$ is maximal if it is not proper subfamily of an another almost disjoint family. For an infinite maximal almost disjoint family (mad) $\mathscr{A}$ on ω, a Ψ-space is a space $\Psi(\mathscr{A})$ whose underlying set is $\omega \cup \mathscr{A}$ and the topology is given by: All points of ω are isolated, and the neighborhood base at $A \in \mathscr{A}$ consists of all sets $\{A\} \cup A \setminus F$ where F is a finite subset of ω.

Definition 2.43. A mad family $\mathscr{A}$ is *Mrówka* if the Stone-Cěch compactification $\beta\Psi(\mathscr{A})$ of $\Psi(\mathscr{A})$ coincides with the one-point compactification of $\Psi(\mathscr{A})$.

A. Dow shows in [DS06] that if $\mathfrak{b} > \omega_1$, for each mad family $\mathscr{A}$, $C_p(\Psi(\mathscr{A}),2)$ is not Lindelöf and, hence, in this case, $C_p(\Psi(\mathscr{A}),2)$ is not Menger. M. Hrušák, P.J. Szeptycki and Á. Tamariz-Mascarúa show in [HSTM05], assuming the Continuum Hypothesis, the existence of a Mrówka mad family $\mathscr{A}$ such that $C_p(\Psi(\mathscr{A}),2)$ is Lindelöf.

For a mad family $\mathscr{A}$ and $j \in 2$, we define the closed subspace $\sigma_n^j(\mathscr{A}) = \{f \in C_p(\Psi(\mathscr{A}),2) : |f^{-1}(j) \cap \mathscr{A}| \le n\}$ of $C_p(\Psi(\mathscr{A}),2)$. If $\mathscr{A}$ is a Mrówka family, then

$$C_p(\Psi(\mathscr{A}),2) = \bigcup_{n\in\omega,\, j\in 2} \sigma_n^j(\mathscr{A}).$$

For every $n\in\omega$, $\sigma_n^0(\mathscr{A})$ is homeomorphic to $\sigma_n^1(\mathscr{A})$. We are going to write $\sigma_n(\mathscr{A})$ instead of $\sigma_n^1(\mathscr{A})$. Thus, by Corollary 1.13:

Lemma 2.44. *If $\mathscr{A}$ is a Mrówka mad family then $C_p(\Psi(\mathscr{A}),2)$ is Menger if and only if $\sigma_n(\mathscr{A})$ is Menger for each $n\in\omega$.*

To characterize when $\sigma_n(\mathscr{A})$ is Menger, we need certain terminology and notation. For $a,b\in\mathscr{P}(\omega)$, $a\triangle b$ will denote their symmetric difference; that is $a\triangle b=(a\cup b)\setminus(a\cap b)$. Given a mad family $\mathscr{A}$ and $\mathscr{Y}\subset\mathscr{P}(\omega)$, we will say that $\mathscr{A}^n$ is *concentrated* on $\mathscr{Y}$ [HSTM05], if for each open U of the Cantor set 2^ω containing $\chi_{\mathscr{Y}}=\{\chi_y:y\in\mathscr{Y}\}$, there is a countable $\mathscr{B}\subset\mathscr{A}$ such that $\chi_{\cup x}\in U$ for all $x\in[\mathscr{A}\setminus\mathscr{B}]^n$. And we will say that $\mathscr{A}^n+[\omega]^{<\omega}$ is *concentrated* on $\mathscr{Y}$ if for each open subset U of 2^ω containing $\chi_{\mathscr{Y}}$, there is a countable subset $\mathscr{B}\subset\mathscr{A}$ such that $\chi_{(\cup x)\triangle b}\in U$ for all $x\in[\mathscr{A}\setminus\mathscr{B}]^n$ and for all $b\in[\omega]^{<\omega}$.

Lemma 2.45. *Let $\mathscr{A}$ be a mad family. If $\mathscr{A}^{n+1}+[\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ and $\sigma_n(\mathscr{A})$ is Menger, then $\sigma_{n+1}(\mathscr{A})$ is Menger.*

Proof. The proof depends on two claims.

Claim 1. If V is an open subset of $\sigma_{n+1}(\mathscr{A})$ containing $\sigma_n(\mathscr{A})$, then there is a countable subset $\mathscr{B}\subset\mathscr{A}$ such that $f^{-1}(1)\cap\mathscr{B}\neq\emptyset$ for any $f\in\sigma_{n+1}(\mathscr{A})\setminus V$.

Indeed, since $\sigma_0(\mathscr{A})$ is a countable subset of $\sigma_n(\mathscr{A})$, we can choose a sequence of finite functions $s_k\subset\Psi(\mathscr{A})\times 2$ such that $\sigma_0(\mathscr{A})\cap[s_k]\neq\emptyset$ and $\sigma_0(\mathscr{A})\subset\bigcup_{k\in\omega}[s_k]\subset V$, where $[s_k]=\{f\in\sigma_{n+1}(\mathscr{A}):s_k\subset f\}$ for each $k\in\omega$. Note that $s_k^{-1}(1)\subset\omega$ and $s_k\restriction\mathscr{A}$ is the constant zero for each $k\in\omega$. We define the open subset U of 2^ω to be $\bigcup_{k\in\omega}\{f\in 2^\omega:s_k\restriction\omega\subset f\}$ and note that $\chi_{[\omega]^{<\omega}}\subset U$. Then, by hypothesis, there is a countable subset $\mathscr{B}'\subset\mathscr{A}$ such that $\chi_{\bigcup x\triangle b}\in U$ for all $x\in[\mathscr{A}\setminus\mathscr{B}']^{n+1}$ and for all $b\in[\omega]^{<\omega}$. Let $\mathscr{B}=\mathscr{B}'\cup\bigcup_{k\in\omega}(s_k^{-1}(0)\cap\mathscr{A})$ and show that $\mathscr{B}$ is the required set by Claim 1. Let $f\in\sigma_{n+1}(\mathscr{A})\setminus V$ and $x=f^{-1}(1)\cap\mathscr{A}$. Since V contains $\sigma_n(\mathscr{A})$, $|x|=n+1$. Proceed by contradiction, suppose that $x\cap\mathscr{B}=\emptyset$. We choose $b\in[\omega]^{<\omega}$ such that $f^{-1}(1)\cap\omega=\bigcup x\triangle b$. By the choice of $\mathscr{B}$, $\chi_{\bigcup x\triangle b}\in U$ and consequently, there is $k\in\omega$ such that $s_k^{-1}(1)\subset\bigcup x\triangle b=\omega\cap f^{-1}(1)$ and $s_k^{-1}(0)\cap\omega\subset\omega\setminus(\omega\cap f^{-1}(1))=f^{-1}(0)\cap\omega$. Given that $x\cap s_k^{-1}(0)=\emptyset$, $s_k^{-1}(0)\subset f^{-1}(0)$. Then $f\in[s_k]$ which is a contradiction, and Claim 1 is proved.

Claim 2. If V is an open subset of $\sigma_{n+1}(\mathscr{A})$ containing $\sigma_n(\mathscr{A})$, then there is a countable subset Y of $\sigma_1(\mathscr{A})$ such that $\sigma_{n+1}(\mathscr{A})\setminus V\subset\bigcup_{h\in Y}(h+\sigma_n(\mathscr{A}))$, where $h+\sigma_n(\mathscr{A})=\{h+g:g\in\sigma_n(\mathscr{A})\}$.

Let $\mathscr{B}$ be a countable subset of $\mathscr{A}$ given by Claim 1, and define $Y=\{f\in\sigma_1(\mathscr{A}):f^{-1}(1)\cap\mathscr{A}\subset\mathscr{B}\}$. Then Y is countable. Let $f\in\sigma_{n+1}(\mathscr{A})\setminus V$. Again, by the choice of $\mathscr{B}$, there is an element $a\in f^{-1}(1)\cap\mathscr{B}$. We define a continuous function $g:\Psi(\mathscr{A})\to 2$ as follows

$$g(x) = \begin{cases} 1, & \text{if } x \in a \cup \{a\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then $g \in Y$ and $f+g \in \sigma_n(\mathscr{A})$ and consequently $f = g + (f+g) \in \bigcup_{h \in Y}(h + \sigma_n(\mathscr{A}))$. This concludes the proof of Claim 2.

Now, we are going to finish the proof of our lemma. Let $\langle \mathscr{U}_k : k \in \omega \rangle$ be a sequence of covers of $\sigma_{n+1}(\mathscr{A})$. Since $\sigma_n(\mathscr{A})$ is Menger, there is a finite subset $\mathscr{F}'_k \subset \mathscr{U}_k$ for each $k \in \omega$ such that $\sigma_k(\mathscr{A}) \subset \bigcup\bigcup_{k\in\omega} \mathscr{F}'_k$. Then, by Claim 2, there is a countable subset $Y \subset \sigma_1(\mathscr{A})$ such that $\sigma_{k+1}(\mathscr{A}) \setminus \bigcup\bigcup_{k\in\omega} \mathscr{F}'_k \subset \bigcup_{h\in Y}(h+\sigma_n(\mathscr{A}))$. Since $\sigma_n(\mathscr{A})$ is homeomorphic to $h+\sigma_n(\mathscr{A})$ for each $h \in Y$ and Y is countable, $\bigcup_{h\in Y}(h+\sigma_n(\mathscr{A}))$ is Menger. Then, there is a finite subset $\mathscr{F}''_k \subset \mathscr{U}_k$ for each $k \in \omega$ such that $\bigcup_{k\in\omega} \mathscr{F}''_k$ is a cover of $\sigma_{n+1}(\mathscr{A}) \setminus \bigcup\bigcup_{k\in\omega} \mathscr{F}'_k$. Therefore, the sequence $\langle \mathscr{F}'_k \cup \mathscr{F}''_k : k \in \omega \rangle$ is the required choice. □

As we have already mentioned in the previous paragraphs, $\sigma_0(\mathscr{A})$ is countable. Then, by Lemma 2.45, if $\mathscr{A}^k + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ for each $k \leq n$, then $\sigma_n(\mathscr{A})$ is Menger. However, in [HSTM05, Corollary 4.3] M. Hrušák, M., P.J. Szeptycki and Á. Tamariz-Mascarúa proves the following two results.

Proposition 2.46 ([HSTM05])**.** *Let $\mathscr{A}$ be a mad family and $n \in \omega$. Then, $\mathscr{A}^n + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ if and only if $\mathscr{A}^n$ is concentrated on $[\omega]^{<\omega}$.*

Corollary 2.47 ([HSTM05])**.** *Suppose that $\mathscr{A}$ is a mad family and $n \in \omega$. Then, $\sigma_n(\mathscr{A})$ is Lindelöf if and only if $\mathscr{A}^k$ is concentrated on $[\omega]^{<\omega}$ for all $k \leq n$.*

These last two results with Lemma 2.45 imply the following result.

Proposition 2.48. *Let $\mathscr{A}$ be a mad family and $n \in \omega$. Then the following statements are equivalent.*

(a) *$\sigma_n(\mathscr{A})$ is Lindelöf;*

(b) *$\sigma_n(\mathscr{A})$ is Menger;*

(c) *$\mathscr{A}^k$ is concentrated on $[\omega]^{<\omega}$ for every $k \leq n$.*

Proof. It is clear that (b) implies (a) and by Corollary 2.47, (a) implies (c). Finally, Lemma 2.45 and Proposition 2.46 prove that (c) implies (b). □

A corollary of the previous result is:

Theorem 2.49. *Let $\mathscr{A}$ be a Mrówka mad family. Then the following are equivalent.*

(a) $C_p(\Psi(\mathscr{A}),2)$ *is Lindelöf;*

(b) $C_p(\Psi(\mathscr{A}),2)$ *is Menger;*

(c) $\mathscr{A}^n$ *is concentrated on* $[\omega]^{<\omega}$ *for every* $n \in \omega$.

In Chapter 3 appear a more general theorem of previous result. Theorem 4.5 in [HSTM05] shows, assuming CH, the existence of a Mrówka mad family $\mathscr{A}$ for which $\mathscr{A}^n$ is concentrated on $[\omega]^{<\omega}$ for all $n \in \omega$. Then we have the following:

Theorem 2.50 (CH)**.** *There is a mad family* $\mathscr{A}$ *such that* $C_p(\Psi(\mathscr{A}),2)$ *is Menger.*

In [BSTM15] we ask the following: Let $\mathscr{A}$ be the Mrówka mad family whose existence is guaranteed by Theorem 4.5 in [HSTM05]. Is $C_p(\Psi(\mathscr{A}),2)^n$ Menger for every $n \geq 2$? But in the next chapter we gives a positive answer to this question (see Theorem 3.35).

In general we have a problem as follow.

Problem 2.51. Are there a topological space X and a natural number $n > 2$ such that $C_p(X,2)$ is Menger and $C_p(X,2)^n$ is not Menger?

2.6 Other results and observations

Definition 2.52 ([Sch99b])**.** A space X is said to be *selective separable* if for every sequence $\langle D_n : n \in \omega \rangle$ of dense subsets of X, there is a sequence of finite sets $\langle F_n : n \in \omega \rangle$ such that $\bigcup_{n\in\omega} F_n$ is dense in X and $F_n \subset D_n$ for every $n \in \omega$.

For a space X, let $PR(X)$ be the space of all finite subsets of X with the *Pixley-Roy topology*, that is, the topology whose base is the family of all sets of the form

$$[A : U] = \{B \in PR(X) : A \subset B \subset U\},$$

where $A \in PR(X)$ and U is an open subset of X containing A. M. Sakai [Sak12] shows that $PR(X)$ is selective separable if and only if X is countable and X^n has a countable fan tightness for any $n \in \omega$. Also he proved that the countable fan tightness for finite sets property is preserved under finite products for simple spaces. As a corollary of Corollary 2.40 we have the following:

Corollary 2.53. *Let* $\mathscr{F} \in \omega^*$. *Then* $PR(\omega \cup \{\mathscr{F}\})$ *is selective separable if and only if* $\mathscr{F}$ *is a strong P-point.*

Another result in this sense is the following.

Proposition 2.54. *If* $\mathscr{F}$ *is a filter on* ω *of character less than* $\mathfrak{d}$, *then* $\omega \cup \{\mathscr{F}\}$ *has countable fan tightness for finite sets.*

Proof. Let $\{U_\alpha : \alpha < \kappa\}$ be a local base at $\mathscr{F}$, where $\kappa < \mathfrak{d}$. Let $\langle \mathscr{P}_n : n \in \omega \rangle$ be a sequence of π-networks consisting of finite sets at $\mathscr{F}$. We can suppose that $\mathscr{P}_n = \{P^n_m : m \in \omega\}$ for each $n \in \omega$. We define, for each $\alpha < \kappa$, the function $f_\alpha : \omega \to \omega$ as $f_\alpha(n) = \min\{k \in \omega : P^n_k \subset U_\alpha\}$. Then, the family $\{f_\alpha : \alpha < \kappa\}$ is not cofinal, that is, there is a function $f \in \omega^\omega$ such that $f \not\leq^* f_\alpha$ for each $\alpha < \kappa$. For each $n \in \omega$, let $\mathscr{F}_n = \{P^n_m : m \leq f(n)\}$. Then the sequence $\langle \mathscr{F}_n : n \in \omega \rangle$ is the required sequence. □

In [Bla10, Theorem 9.25] it is shown that all filter of character less than $\mathfrak{d}$ is a P-point. In view of Proposition 2.39 and Proposition 2.54 we have the following corollary.

Corollary 2.55. *All ultrafilters on ω of character less than $\mathfrak{d}$ are strong P-points.*

CHAPTER

3 The Rothberger property in $C_p(X,2)$

A space X is said to have the *Rothberger property* (or simply X is *Rothberger*) if for every sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of open covers of X, there exists $U_n \in \mathscr{U}_n$ for each $n \in \omega$ such that $X = \bigcup_{n\in\omega} U_n$. Some basic properties about Rothberger spaces are:

Proposition 3.1 ([Jus+96])**.**

(a) *Every closed subspace of a Rothberger space is Rothberger.*

(b) *The continuous image of a Rothberger space is Rothberger.*

(c) *The countable union of Rothberger spaces is Rothberger.*

A subset A of a space X is *meager* (or the first category) if it is the union of many nowhere dense sets. Let $\mathscr{M}$ be the family of meager subsets of $\mathbb{R}$. Then, we define the cardinal $\operatorname{cov}(\mathscr{M}) = \min\{|M| : M \subset \mathscr{M} \wedge \mathbb{R} = \bigcup M\}$. Obviously $\operatorname{cov}(\mathscr{M}) \leq \mathfrak{c}$ and, by the Category Baire Theorem [Eng89], $\omega_1 \leq \operatorname{cov}(\mathscr{M})$. Since the set of irrational numbers $\mathbb{P}$ and $\mathbb{Q}$ are dense subspaces of $\mathbb{R}$, $\mathbb{Q}$ is meager and $\mathbb{P}$ is homeomorphic to ω^ω, it is not difficult to prove that $\operatorname{cov}(\mathscr{M}) = \min\{|M| : \omega^\omega = \bigcup M \wedge \forall m(m \in M \to m \text{ is meager in } \omega^\omega)\}$.
A family of functions $G \subset \omega^\omega$ can be *guessed* by a function $f \in \omega^\omega$ if for each $g \in G$, the set $\{n \in \omega : f(n) = g(n)\}$ is infinite.
Fix $g \in \omega^\omega$ and let E_g be the set $\{f \in \omega^\omega : |\{n \in \omega : g(n) = f(n)\}| < \omega\}$. If we define $D_n = \{f \in \omega^\omega : \forall k \in \omega(f(k) = g(k) \to k < n)\}$ for each $n \in \omega$. Then, each D_n is a closed nowhere dense subset of ω^ω. Hence E_g is a meager subset of ω^ω. Moreover, if $G \subset \omega^\omega$ cannot be guessed, then $\omega^\omega = \bigcup_{g\in G} E_g$. With this we have that $\operatorname{cov}(\mathscr{M}) \leq \min\{|G| : G \subset \omega^\omega \wedge G \text{ cannot be guessed}\}$. The other inequality is also true but the proof is more difficult

(see [Bar87, Theorem 1.7]) and therefore we have the following nice characterization of $\mathrm{cov}(\mathcal{M})$.

Theorem 3.2 ([Bar87])**.** $\mathrm{cov}(\mathcal{M}) = \min\{|G| : G \subset \omega^\omega \wedge \forall g \in \omega^\omega \exists f \in G(|\{n \in \omega : g(n) = f(n)\}| < \omega)\}$.

With this characterization we can prove the following:

Theorem 3.3. *If X is a Lindelöf space and $|X| < \mathrm{cov}(\mathcal{M})$, then X is Rothberger.*

Proof. Let $\langle \mathcal{U}_n : n \in \omega \rangle$ be a sequence of countable open covers of X, say $\mathcal{U}_n = \{U^n_m : m \in \omega\}$. For each $x \in X$, we define $f_x : \omega \to \omega$ defined by $f_x(n) = \min\{m : x \in U^n_m\}$. Then the family $\{f_x : x \in X\}$ can be guessed by function $f \in \omega^\omega$ (see Theorem 3.2). It is not difficult to prove that $X = \bigcup_{n\in\omega} U^n_{f(n)}$. □

There is an analogous analysis for the Rothberger property similar to Section 1.2 (see [MF88, Theorem 5] and [Mil81; Bar87; Mil82]).

Recall that a space of reals X is *null* if for each positive real ϵ, there exists an open cover $\{I_n : n \in \omega\}$ of X such that $\sum_{n\in\omega} \delta(I_n) < \epsilon$. In his 1919 paper [Bor19], Borel introduced the following stronger property: A set of reals X is *strongly null* (or has *strong measure zero*) if, for each sequence $\langle \epsilon_n : n \in \omega \rangle$ of positive reals, there exists an open cover $\{I_n : n \in \omega\}$ of X such that $\delta(I_n) < \epsilon_n$ for each $n \in \omega$. But Borel was unable to construct a nontrivial (that is, an uncountable) example of a strongly null set. He therefore conjectured that there is no such example. Sierpiński (1928) observed that every Luzin set is strogly null, thus the Continuum Hypothesis implies that *the Borel Conjecture* is false. Sierpiński asked whether the property of being strongly null is preserved undertaking homeomorphic or even continuous images. The answer, given by Rothberger (1941) in [Rot41], is negative under the Continuum Hypothesis. Then Rothberger introduced the topological version of strongly null defining the *Rothberger spaces* which are preserved undertaking continuous images. Clearly the Rothberger property implies strongly null. Moreover Fremlin and Miller in [MF88] proved that for a metric space (X,d), the Rothberger property is the same as having strong measure zero with respect to all metrics which generate the same topology as the one defined by d. The question of consistency of the Borel Conjecture was settled in 1976, when Laver in his work [Lav76] showed that the Borel Conjecture is consistent.

3.1 Consequences of the Rothberger property in $C_p(X,2)$

Similar to Chapter 2 we only analyze the finite power of $C_p(X,2)$ (see Proposition 1.20).

Proposition 3.4. *For any space X, $C_p(X,2)^n$ is Rothberger for any $n \in \omega$ if and only if $C_p(X,k)$ is Rothberger for any $k \in \omega$.*

Proof. This is immediate from the fact that $C_p(X,2)^n$ is homeomorphic to $C_p(X,2^n)$ for any $n \in \omega$, and the fact that a closed subspace of a Rothberger space is Rothberger. □

Observe that the class of Rothberger spaces is not complete. As we have already mentioned, the compact metrizable Cantor space is not Rothberger. So, the analogous result of Corollary 2.12 for Rothberger spaces cannot be deduced from Theorem 2.11. See Problem 3.18, below.

We begin with some consequences of the Rothberger property in $C_p(X,2)$ concerning some types of tightness. M. Sakai introduces the following notion.

Definition 3.5 ([Sak12])**.** A space X has *strong countable fan tightness for finite sets* if for each point $x \in X$ and each sequence $\langle \mathscr{P}_n \subset [X]^{<\omega} : n \in \omega \rangle$ of π-networks at x, there is, for each $n \in \omega$, $P_n \subset \mathscr{P}_n$ such that $\{P_n : n \in \omega\}$ is a π-network at x.

Clearly, strong countable fan tightness for finite sets implies countable fan tightness.

Proposition 3.6. *If the space $C_p(X,2)$ is Rothberger, then X^n has strong countable fan tightness for finite sets for any $n \in \omega$.*

Proof. We fix a $k \in \mathbb{N}$, a point $x = (x_1, \dots, x_k) \in X^k$ and a sequence $\langle \mathscr{P}_n : n \in \omega \rangle$ of π-networks at x consisting of finite subsets of X. We take open subsets $U_1, \dots, U_k$ of X such that, for each $i, j \in \{1, \dots, k\}$, $x_i \in U_i$, $U_i = U_j$ if $x_i = x_j$, and $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$. Let $U = U_1 \times \cdots \times U_k$. We can suppose that, for every $n \in \omega$, each member of $\mathscr{P}_n$ is contained in U. Since the space $C_p(X,2)$ is Rothberger, the closed subspace

$$\Phi = \{f \in C_p(X,2) : \forall i (1 \le i \le k \to f(x_i) = 1)\}$$

of $C_p(X,2)$ is Rothberger. For each $F \in [X^k]^{<\omega}$, we define $H_F = \bigcup\{\pi_i[F] : i \in \{1, \dots, k\}\}$, where π_i is the projection of X^k over the i-th coordinate, and we set $V_F = \{f \in C_p(X,2) : \forall x (x \in H_F \to f(x) = 1)\}$. For each $n \in \omega$, let

$$\mathscr{U}_n = \{V_F : F \in \mathscr{P}_n\}.$$

Given $f \in \Phi$, for each $i \in \{1, \dots, k\}$, there is an open subset $V_i \subset U_i$ such that $x_i \in V_i$ and $f[V_i] \subset \{1\}$. Since $\mathscr{P}_n$ is a π-network, there is $F \in \mathscr{P}_n$ such that $F \subset V_1 \times \cdots \times V_k$. So $f[\pi_i[F]] \subset \{1\}$ for each $i \in \{1, \dots, k\}$. Thus $f \in V_F \in \mathscr{U}_n$. This implies that $\mathscr{U}_n$ is an open cover of Φ. Therefore, since Φ is Rothberger, there is, for each $n \in \omega$, $U_n \in \mathscr{U}_n$ such that $\langle U_n : n \in \omega \rangle$ forms a cover of Φ. Choose a $P_n \in \mathscr{P}_n$ such that $U_n = V_{P_n}$. Let us prove that $\{P_n : n \in \omega\}$ is a π-network at x. Let $W = W_1 \times \cdots \times W_k$ be an open subset which contains x. We can assume that $W_i = W_j$ if $x_i = x_j$ and $W_i \subset U_i$ for each $i, j \in \{1, \dots, k\}$. We choose $f \in C_p(X,2)$ such that

$$f[X \setminus \bigcup_{i=1}^{k} W_i] \subset \{0\}$$

and $f(x_i) = 1$ for each $i \in \{1,\dots,k\}$. Thus $f \in \Phi$, and consequently, there is $n \in \omega$ such that $f \in V_{P_n}$. Now, if $(y_1,\dots,y_k) \in P_n$, since $P_n \subset U$, $y_i \in U_i$ for each $i \in \{1,\dots,k\}$. Moreover, due to the fact that $f \in V_{P_n}$, we have that $y_1,\dots,y_k \in \bigcup_{i=1}^k W_i$. However, $U_i \cap U_j = \emptyset$ if $x_i \neq x_j$, then $y_i \in W_i$ for each $i \in \{1,\dots,k\}$. This shows that $P_n \subset W$. □

Theorem 3.7. *If $C_p(X,2)$ is Rothberger, then X is pseudocompact.*

Proof. We prove that X is bounded in X. We proceed by contradiction. Suppose that there exists an infinite locally finite family $\{O_n : n \in \omega\}$ of non-empty open subsets of X. We can suppose without loss of generality that each element of the sequence is open and closed, and that any two of its elements are disjoint. Let $Y = X \setminus \bigcup_{n\in\omega} O_n$. Since the sequence $\{O_n : n \in \omega\}$ is locally finite and avery O_n is closed and open, Y is open and closed. Moreover, the family $\{O_n : n \in \omega\} \cup \{Y\}$ forms a partition of X in open and closed subsets of X. Then $C_p(X,2)$ is homeomorphic to

$$(\prod_{n\in\omega} C_p(O_n,2)) \times C_p(Y,2).$$

For each $n \in \omega$, $|C_p(O_n,2)| \geq 2$. Th.en $\prod_{n\in\omega} C_p(O_n,2)$ contains a closed copy of 2^ω. But 2^ω is not Rothberger (see [Sch99a, Theorem 2.3] or the arguments below Theorem 1.29). □

Corollary 3.8. *If $C_p(X,2)$ is Rothberger and X is a normal space, then X is countably compact.*

With this corollary we can characterize the Rothberger property in $C_p(X,2)$ when X is a metrizable space.

Corollary 3.9. *Let X be a metrizable space. Then the following statements are equivalent.*

(a) *$C_p(X,2)^n$ is Rothberger for every $n \in \omega$;*

(b) *X is compact;*

(c) *$C_p(X,2)$ is countable.*

Proof. Corollary 3.8 shows (a) implies (b). If X is compact, then X is second countable. Then, $C_p(X,2)$ is countable. This shows (b) implies (c). And (c) implies (a) is obvious. □

So, if X is the subspace $\{[(x,k)] : k \in \omega \wedge x \in \{0\} \cup \{1/n : n \in \mathbb{N}\}\}$ of the metrizable hedgehog $J(\aleph_0)$ (see [Eng89], Example 4.1.5), $C_p(X,2)$ is Menger (see Theorem 4.14 in [BS15b]) but it is not Rothberger. By the way, using the same Theorem 4.14 in [BS15b], if α is a countable limit ordinal, then $C_p(\alpha,2)$ is Lindelöf and it is not Menger.

3.2 The Rothberger property in $C_p(X,2)$ and ω-monolithicity

Recall that the *network weight* of X, denoted by $nw(X)$, is the minimal cardinal of a network for X. The spaces with a countable network weight are called *cosmic*.

Definition 3.10. Given a cardinal number κ, a space X is κ*-monolithic* if $nw(\mathrm{cl}(A)) \leq \kappa$ for any $A \subset X$ with $|A| \leq \kappa$.

The next theorem shows some examples of spaces X for which $C_p(X,2)$ is Rothberger.

Theorem 3.11. *Let $n \in \omega$. If X is a countably compact ω-monolithic space and $C_p(X,2)^n$ is Lindelöf, then $C_p(X,2)^n$ is Rothberger.*

Proof. We only show that $C_p(X,2)$ is Rothberger; the $(n \geq 2)$-cases are shown similarly. Let $\langle \mathscr{U}_n : n \in \omega \rangle$ be a sequence of countable open covers of $C_p(X,2)$ consisting of canonical open sets. For each $n \in \omega$, suppose that $\mathscr{U}_n = \{[s^n_m] : m \in \omega\}$ where $s^n_m \subset X \times 2$ is a finite function and $[s^n_m] = \{f \in C_p(X,2) : s^n_m \subset f\}$. Let $A = \bigcup_{n,m\in\omega} \mathrm{dom}(s^m_n)$. Since A is countable and X is ω-monolithic, $\mathrm{cl}(A)$ is cosmic, but X is countably compact, then $\mathrm{cl}(A)$ is a compact cosmic space. Therefore, by Corollary 3.9, $C_p(\mathrm{cl}(A),2)$ is Rothberger.

Since $\mathrm{cl}(A)$ is a compact subspace of X, every continuous function $h : \mathrm{cl}(A) \to 2$ has a continuos extension $\widetilde{h} : X \to 2$. Therefore $\langle \mathscr{V}_n : n \in \omega \rangle$ is a sequence of open covers of $C_p(\mathrm{cl}(A),2)$ where $\mathscr{V}_n = \{V^n_m : m \in \omega\}$ and $V^n_m = \{h \in C_p(\mathrm{cl}(A),2) : s^n_m \subset h\}$. Thus, there is $g \in \omega^\omega$ such that $\bigcup_{n\in\omega} V^n_{g(n)} = C_p(\mathrm{cl}(A),2)$. Let $f \in C_p(X,2)$. Then there is $n \in \omega$ such that $f \restriction \mathrm{cl}(A) \in V^n_{g(n)}$. So, $s^n_{g(n)} \subset f \restriction \mathrm{cl}(A) \subset f$. Hence, $f \in [s^n_{g(n)}]$. This shows that $\bigcup_{n\in\omega}[s^n_{g(n)}] = C_p(X,2)$. □

A space is *Sokolov* (or has the *Sokolov property*) if for any sequence $\langle F_n : n \in \omega \rangle$ where F_n is a closed subspace of X^n for any $n \in \omega$, there is a continuous function $f : X \to X$ such that $nw(f[X]) \leq \omega$ and $f^n[F_n] \subset F_n$.

We compile some known facts about the Sokolov space in the following theorem. They were proved by Sokolov in [Sok86; Sok93] or can easily be deduced from his results.

Lemma 3.12 ([Tka05])**.**

(a) *Any closed subspace of a Σ-product of second countable spaces is Sokolov.*

(b) *If X is a Sokolov space and $t^*(X) \leq \omega$, then $C_p(X)$ is Lindelöf.*

(c) *Every Sokolov space is ω-monolithic.*

(d) *Every Sokolov space is normal.*

By Theorem 3.7 and Proposition 3.6, if $C_p(X,2)$ is Rothberger, then X is pseudocompact and $t^*(X) \leq \omega$. These last two properties plus the Sokolov property imply that $C_p(X,2)$ is Rothberger:

Theorem 3.13. *Let X be a Sokolov space. Then $C_p(X,2)^n$ is Rothberger for any $n \in \omega$ if and only if X is pseudocompact and $t^*(X) \leq \omega$.*

Proof. If $C_p(X,2)$ is Rothberger, by Proposition 3.6 and Theorem 3.7, $t^*(X) \leq \omega$ and X is pseudocompact.
Reciprocally, if X is pseudocompact and $t^*(X) \leq \omega$, by Lemma 3.12(b)-(d), $C_p(X)$ is Lindelöf and X is a countably compact ω-monolithic space. The conclusion follows from Proposition 3.4, Proposition 3.1(a) and Theorem 3.11. □

The compact subspaces of Σ-products of real lines are called *Corson compact*. Moreover, for any set A, $\Sigma 2^A$ denotes the Σ-product of the two element discrete space around the constant function zero; that is,

$$\Sigma 2^A = \{x \in 2^A : |\{\alpha \in A : x(\alpha) \neq 0\}| \leq \omega\}.$$

The space $C_p(\Sigma 2^A)$ is a Lindelöf space [Arh92].

Corollary 3.14. *For any Corson compact X, $C_p(X,2)^n$ is Rothberger for any $n \in \omega$.*

Proof. For any Σ-product of real lines Y, $t^*(Y) \leq \omega$ (see [Arh92]). Then $t^*(X) \leq \omega$. Applying Lemma 3.12(a) and Theorem 3.13, $C_p(X,2)^n$ is Rothberger for any $n \in \omega$. □

Every Eberlein compact space is Corson compact [Arh92], so:

Corollary 3.15. *For every Eberlein compact X, $C_p(X,2)^n$ is Rothberger for every $n \in \omega$.*

Let κ be a cardinal number. A space X is κ-*stable* if for each continuous image Y of X, $iw(Y) \leq \kappa$ if and only if $nw(Y) \leq \kappa$. We have the following relation between κ-monolithicity and κ-stability:

Theorem 3.16 ([Arh92])**.** *The space $C_p(X)$ is κ-monolithic if and only if X is κ-stable.*

With this theorem and Theorem 3.11 we have the following.

Corollary 3.17. *Let X be a subspace of $C_p(Y)$ where Y is ω-stable and Y^k is Lindelöf for each $k \in \omega$. If X is a compact space, then $C_p(X,2)^n$ is Rothberger for each $n \in \omega$.*

Proof. By Theorem 3.16, $C_p(Y)$ is ω-monolithic and, hence, X is ω-monolithic. Since Y^k is Lindelöf for each $k \in \omega$, by Corollary 2.12, $C_p(X,2)^n$ is Lindelöf for each $n \in \omega$. The rest of the proof follows the proof that we gave in Theorem 3.11. □

From the hypotheses of Corollary 3.17 and the structure of Corollary 2.12, the following natural problem arises.

Problem 3.18. Let X be a compact subspace of $C_p(Y)$ where Y^n is Rothberger for every $n \in \omega$. Is $C_p(X,2)$ then Rothberger if X is compact?

In [Arh92, Theorem IV.8.16] the following theorem is proved with additional axioms.

Theorem 3.19 ([Arh92] MA+¬CH)**.** *Every zero-dimensional compact space X for which $C_p(X)$ is Lindelöf, is ω-monolithic.*

Corollary 3.20 (MA+¬CH)**.** *If X is a compact space with $C_p(X)$ Lindelöf, then $C_p(X,2)^n$ is Rothberger for any $n \in \omega$.*

Proof. By Theorem 3.19, X is ω-monolithic. And since $C_p(X)$ is Lindelöf, $C_p(X,2)^n$ is Lindelöf for any $n \in \omega$. The conclusion follows from Theorem 3.11. □

E.A. Reznichenko showed that assuming MA+¬CH, every compact zero-dimensional space X with $C_p(X,\mathbb{R})$ Lindelöf is ω-monolithic (see Theorem 3.19). This leads to the conjecture that, perhaps, strong covering properties of a suitable $C_p(X,Y)$ might imply ω-monolithicity of X. One might, e.g., ask whether Reznichenko's result can be generalized:

Question 3.21. Assume X is a zero-dimensional compact space and that $C_p(X,2)^n$ is Rothberger for every $n \in \omega$. Does this imply that X is ω-monolithic?

Theorem 3.36 gives a consistent counterexample.

3.3 The Rothberger property in $C_p(X,2)$ when X is a simple space

Every simple space is normal. Then, if X is a simple space and $C_p(X,2)$ is Rothberger, by Corollary 3.8, X is countably compact. But the only countably compact simple spaces are the one-point compactifications of discrete spaces.
Given a set X, $D(X)$ denotes the set X with the discrete topology and $A(X)$ denotes the one-point compactification of the discrete space $D(X)$. Of course, for any set X, $A(X)$ is homeomorphic to $A(|X|)$. Recall that the set of ordinals lower than an ordinal α endowed with its order topology is denoted by α.

Theorem 3.22. *Let X be a simple space. Then, the following statements are equivalent.*

(1) *X is the one point compactification of a discrete space;*

(2) *$C_p(X,2)$ is Rothberger;*

(3) *$C_p(X,2)^n$ is Rothberger for any $n \in \omega$.*

Proof. The implication (3) → (2) is obvious. The implication (2) → (1) follows from the remarks made in the paragraph preceding this theorem.
(1) → (3): Let κ be the cardinality of X. By Corollary 3.14, it is enough to see that $A(\kappa)$ is a Corson compact space. Indeed, for every $\alpha \in \kappa$, we define $f_\alpha : \kappa \to 2$ as

$$f_\alpha(x) = \begin{cases} 1, & \text{if } x = \alpha; \\ \\ 0, & \text{otherwise.} \end{cases}$$

And let $g : \kappa \to 2$ be the constant function 0. Then $K = \{f_\alpha : \alpha \in \kappa\} \cup \{g\}$ is a subspace of $\Sigma 2^\kappa$ homeomorphic to $A(\kappa)$. □

3.4 The Rothberger property in $C_p(X,2)$ when X is a GO-space

For each ordinal α, we define $\alpha_\omega = \{\beta \leq \alpha : \mathrm{cof}(\beta) \leq \omega\}$. We consider α_ω with the topology inherited from α.

Proposition 3.23. *Any countably compact first countable subspace of ordinals is homeomorphic to α_ω for some ordinal α.*

Proof. Let X be a countably compact first countable subspace of ordinals. Fix an ordinal α such that $|X| < |\alpha_\omega|$. By recursion we define a sequence $\langle x_\gamma : \gamma \in \alpha_\omega \rangle$ of elements of X. Let $x_0 = \min X$ and for each $\gamma \in \alpha_\omega$ define $x_\gamma = \min H_\gamma$ where $H_\gamma = \{x \in X : \forall \beta(\beta < \gamma \to x > x_\beta)\}$, if $H_\gamma \neq \emptyset$, and $x_\gamma = x_0$ in another case. Let δ be the minimal ordinal γ in α_ω such that $H_\gamma = \emptyset$. By the countable compactness of X, δ is a successor ordinal; say $\nu + 1$. Then $X = \{x_\beta : \beta \leq \nu\}$ and X is homeomorphic to ν_ω. □

In [Tka11], V.V. Tkachuk proved that any first countable countably compact subspace of ordinals has the Sokolov property.

Theorem 3.24. *Let X be a subspace of ordinals. Then the following statements are equivalent.*

(a) *$C_p(X,2)^n$ is Rothberger for each $n \in \omega$;*

(b) *X is first countable and countably compact;*

(c) *X is homeomorphic to α_ω for some ordinal α.*

Proof. If $C_p(X,2)$ is Rothberger, then X has countable tightness and, by Corollary 3.8, X is countably compact. This proves (a) implies (b). By Proposition 3.23, (b) and (c) are equivalent. Finally, Theorem 2.5 in [Tka11] shows that X is Sokolov and, by Theorem 3.13, $C_p(X,2)^n$ is Rothberger for every $n \in \omega$. □

Now, we will characterize the Rothberger property in $C_p(L,2)$ when L is any GO-space. Before this we need some results. The first is about the space S^* defined in Chapter 2.

Lemma 3.25 ([Buz07]). *S^* is closed in $C_p(L,2)$.*

For any notions (or classes) $\mathscr{A}$ and $\mathscr{B}$, in [Sch99a], Marion Scheepers introduced two general notions or selection principles:

$\mathrm{S}_1(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathscr{U}_n : n \in \omega \rangle$ of members of $\mathscr{A}$, there exists, for each $n \in \omega$, $U_n \in \mathscr{U}_n$ such that $\{U_n : n \in \omega\} \in \mathscr{B}$.

Given an open cover $\mathscr{U}$ of space X, we say that $\mathscr{U}$ is a *ω-cover* if for every finite subset F there is an element U of $\mathscr{U}$ such that $F \subset U$. The collection of all ω-covers of X is denoted by Ω. In [Sak88] M. Sakai proved the following lemma.

Lemma 3.26. *Let X be a space. Then X^n is Rothberger for every $n \in \omega$ if and only if X satisfies the selection principle $\mathrm{S}_1(\Omega,\Omega)$.*

An open cover $\mathscr{U}$ of a space X is called *γ-cover* if for each $x \in X$, the set $\{U \in \mathscr{U} : x \notin U\}$ is finite. The set of all γ-covers of X is denoted by Γ. The spaces that satisfy the principle $\mathrm{S}_1(\Omega,\Gamma)$ are called *γ-spaces* by J. Gerlits and Zs. Nagy in [GN82]. Observe that

$$\mathrm{S}_1(\Omega,\Gamma) \rightarrow \mathrm{S}_1(\Omega,\Omega).$$

Corollary 1.16 shows that the Menger and Lindelöf properties agree in P-spaces. The next lemma is proved in [GN82, p. 155] as a remark.

Lemma 3.27 ([GN82]). *If X is a Lindelöf P-space, then X satisfies $\mathrm{S}_1(\Omega,\Gamma)$.*

With this we can prove the following.

Theorem 3.28. *Let L be a GO-space. The following statements are equivalent.*

(a) *$T(L)$ is Lindelöf and L is countably compact;*

(b) *$T(L)^n$ is Rothberger for every $n \in \omega$ and L is countably compact;*

(c) *$C_p(L,2)^n$ is Rothberger for every $n \in \omega$.*

Proof.

(a) implies (b). Let us show that $T(L)$ is a P-space. Let $S = \bigcap_{n\in\omega} U_n$, where each U_n is open in $T(L)$. Fix $x \in S$. If $x \in L$, then $\{x\}$ is open in $T(L)$, hence x is in the interior of S. Suppose that $x \in cL$. Assume $x \in (-\infty, \infty)$. Since base neighborhoods at x in $T(L)$ are from the subspace topology on cL, for each $n \in \omega$ we can fix an interval (a_n, b_n) containing x such that $(a_n, b_n) \subset U_n$. We may assume that $a_n, b_n \in L$ and the $(n+1)$-st interval is inside the n-th interval. Then $x \in \bigcap_{n\in\omega}(a_n, b_n) \subset S$. If x is not in the interior of S then either $a_n \to x$ or $b_n \to x$. Since $x \notin L$, we arrived at a contradiction with the countable compactness.
Since $T(L)$ is a Lindelöf P-space, by Lemma 3.27, $T(L)$ satisfies $\mathrm{S}_1(\Omega, \Gamma)$ and, hence, $\mathrm{S}_1(\Omega, \Omega)$. Then, by Lemma 3.26, every finite power of $T(L)$ is Rothberger.

(b) implies (c). Given that $T(L)$ is homeomorphic to S^* (see Theorem 2.27) and the countable union of Rothberger spaces is Rotheberger (see Proposition 3.1(c)), the topological sum $\bigoplus_{n\in\omega}(S^*)^n$ is Rothberger. On the other hand, if we define the continuous function $\mathscr{F} : \bigoplus_{n\in\omega}(S^*)^n \to S_p(L, 2)$ as $\mathscr{F}(F) = f_1 + \cdots + f_k$ where $F = (f_1, \ldots, f_k)$, by Lemma 2.26, $\mathscr{F}$ is surjective. Moreover, if we define $Z = \bigoplus_{n\in\omega}(S^*)^n$, then Z^n is Rothberger for every $n \in \omega$. In this manner the n-th power function $\mathscr{F}^n : Z^n \to S_p(L, 2)^2$ of $\mathscr{F}$ defined as $\mathscr{F}^n(F_1, \ldots, F_n) = (\mathscr{F}(F_1), \ldots, \mathscr{F}(F_n))$ is an onto continuous function. Then, by Proposition 3.1(b), $S_p(L, 2)^n$ is Rothberger for every $n \in \omega$. But $S_p(L, 2) = C_p(L, 2)$ if L is countably compact. Therefore $C_p(L, 2)^n$ is Rothberger for every $n \in \omega$.

(c) implies (a). If $C_p(L, 2)$ is Rothberger, by Corollary 3.8, L is countably compact. Since $C_p(L, 2)$ is Rothberger, $C_p(L, 2)$ is Lindelöf and applying Lemma 3.25 and Theorem 2.27, we conclude that $T(L)$ is Lindelöf. □

3.5 The Rothberger property in $C_p(\Psi(\mathscr{A}), 2)$

In Chapter 2 the following was proved.

Theorem 3.29 ([BS15b]). *Let $\mathscr{A}$ be a mad family and $n \in \omega$. Then, the following statements are equivalent.*

(a) $C_p(\Psi(\mathscr{A}), 2)$ *is Lindelöf;*

(b) $C_p(\Psi(\mathscr{A}), 2)$ *is Menger;*

(c) $\mathscr{A}^k$ *is concentrated on* $[\omega]^{<\omega}$ *for every* $k \leq n$.

We will enlarge this proposition with one more statement concerning the Rothberger property in $C_p(\Psi(\mathscr{A}), 2)$. Before this we need some notations and terminology.

For a mad family $\mathscr{A}$, $n \in \omega$ and $j \in n$, we define the subspace

$$^n\sigma^j_m(\mathscr{A}) = \{f \in C_p(\Psi(\mathscr{A}), n) : \forall i \in n(i \neq j \to |f^{-1}(i) \cap \mathscr{A}| \leq m)\}$$

of $C_p(\Psi(\mathscr{A}), n)$. It is not hard to see that this subspace is closed.

If $\mathscr{A}$ is a Mrówka mad family, then

$$C_p(\Psi(\mathscr{A}), n) = \bigcup_{m \in \omega, j \in n} {}^n\sigma^j_m(\mathscr{A}).$$

For every $m \in \omega$ and $i, j \in n$, $^n\sigma^i_m(\mathscr{A})$ is homeomorphic to $^n\sigma^j_m(\mathscr{A})$. We are going to write $^n\sigma_m(\mathscr{A})$ instead of $^n\sigma^0_m(\mathscr{A})$. Thus, by Proposition 3.1(a) and Proposition 3.1(c):

Lemma 3.30. *If $\mathscr{A}$ is a Mrówka mad family, then $C_p(\Psi(\mathscr{A}), n)$ is Rothberger if and only if $^n\sigma_m(\mathscr{A})$ is Rothberger for each $m \in \omega$.*

For each $n \in \omega$, we define

$$\mathrm{Q}(n) = \{g \in n^\omega : |\mathrm{supp}(g)| < \omega\}.$$

With this terminology we introduce the following property, which is a generalization of when a mad family concentrates on $[\omega]^{<\omega}$ (see [HSTM05], the original definition is equivalent to the case $\bigstar^2_m(\mathscr{A})$). For a mad family $\mathscr{A}$ and $m, n \in \omega$, we define

$\bigstar^n_m(\mathscr{A})$: For each open subset U of n^ω containing $\mathrm{Q}(n)$, there exists a countable subset $\mathscr{B} \subset \mathscr{A}$ such that

$$\{g \in n^\omega : \exists \widehat{g} \in C_p(\Psi(\mathscr{A}), n)(\widehat{g} \restriction \omega = g \wedge \mathrm{supp}(\widehat{g}) \cap \mathscr{A} \in [\mathscr{A} \setminus \mathscr{B}]^m)\} \subset U.$$

The following generalized version of Theorem 4.2 in [HSTM05] holds:

Lemma 3.31. *Let $\mathscr{A}$ be a mad family and $n, m \in \omega$. If $^n\sigma_m(\mathscr{A})$ is Lindelöf, then the property $\bigstar^n_k(\mathscr{A})$ is satisfied for all $k \leq m$.*

Proof. Suppose that the property $\bigstar^n_k(\mathscr{A})$ is false for some $k \leq m$. So, we may fix an open set U in n^ω, a pairwise disjoint family $\{y_\alpha : \alpha \in \omega_1\} \subset [\mathscr{A}]^k$ and $\{g_\alpha : \alpha \in \omega_1\} \subset C_p(\Psi(\mathscr{A}), n)$ such that

(i) $\mathrm{Q}(n) \subset U$, and

(ii) for each $\alpha \in \omega_1$, $\mathrm{supp}(g_\alpha) \cap \mathscr{A} = y_\alpha$ and $g_\alpha \restriction \omega \notin U$.

Since $\{y_\alpha : \alpha \in \omega_1\}$ are pairwise disjoint, any complete accumulation point of $\{g_\alpha : \alpha \in \omega_1\}$ must be in ${}^n\sigma_0$. Moreover, since U contains $\mathrm{Q}(n)$, there is an open subset V in ${}^n\sigma_m(\mathscr{A})$ containing ${}^n\sigma_0$ such that $f \restriction \omega \in U$ for each $f \in V$. Indeed, we can fix a set $\mathscr{F}$ consisting of finite functions such that $U = \{g \in n^\omega : \exists s \in \mathscr{F}(s \subset g)\}$, then $V = \{f \in {}^n\sigma_m(\mathscr{A}) : \exists s \in \mathscr{F}(s \subset f)\}$ is the required open set.
Thus, the open set V contains any complete accumulation point of $\{g_\alpha : \alpha \in \omega_1\}$ and, by (ii), $g_\alpha \notin V$ for each $\alpha \in \omega_1$. This means that the uncountable set $\{g_\alpha : \alpha \in \omega_1\}$ has no complete accumulation points in ${}^n\sigma_m(\mathscr{A})$ which is a contradiction. $\square$

We need the following terminology for the proof of the next lemma. For each $n \in \omega$ and each $t \in \omega^n$ we define

$$ {}^n\sigma_t(\mathscr{A}) = \{f \in C_p(\Psi(\mathscr{A}), n) : \forall i \in n (i \neq 0 \to |f^{-1}(i) \cap \mathscr{A}| \leq t(i))\}. $$

The order $\preceq$ will denote the lexicographic order on ω^n. Observe that if $m \in \omega$ and $t \in \omega^n$ is the constant function m, then ${}^n\sigma_m(\mathscr{A}) = {}^n\sigma_t(\mathscr{A})$.

Lemma 3.32. *Let $\mathscr{A}$ be a mad family, $n \in \omega$, $t_0 \in \omega^n$ and $p = \sum_{i=1}^{n-1} t_0(i)$. If $\bigstar_p^n(\mathscr{A})$ is satisfied and ${}^n\sigma_t(\mathscr{A})$ is Rothberger for every $t \prec t_0$, then ${}^n\sigma_{t_0}(\mathscr{A})$ is Rothberger.*

Proof. We adapt, for our purposes, the respective part of the proof of Lemma 8.2 from [BS15b]. The proof depends on two claims.

Claim 1. If V is an open subset of ${}^n\sigma_{t_0}(\mathscr{A})$ containing ${}^n\sigma_t(\mathscr{A})$ for each $t \prec t_0$, then there is a countable subset $\mathscr{B} \subset \mathscr{A}$ such that for any $f \in {}^n\sigma_{t_0}(\mathscr{A}) \setminus V$, there is $1 \leq i < n$ with $f^{-1}(i) \cap \mathscr{B} \neq \emptyset$.

Indeed, since ${}^n\sigma_0(\mathscr{A})$ is a countable subset of ${}^n\sigma_{t_0}(\mathscr{A})$, we can choose a sequence of finite functions $s_k \subset \Psi(\mathscr{A}) \times n$ such that ${}^n\sigma_0(\mathscr{A}) \cap [s_k] \neq \emptyset$ and ${}^n\sigma_0(\mathscr{A}) \subset \bigcup_{k\in\omega}[s_k] \subset V$, where $[s_k] = \{f \in {}^n\sigma_{t_0}(\mathscr{A}) : s_k \subset f\}$ for each $k \in \omega$. Note that $s_k^{-1}(i) \subset \omega$ for each $1 \leq i < n$ and, thus, $s_k \restriction \mathscr{A}$ is the constant zero function for each $k \in \omega$. We define the open subset U of n^ω to be $\bigcup_{k\in\omega}\{f \in n^\omega : s_k \restriction \omega \subset f\}$ and note that $\mathrm{Q}(n) \subset U$. Let $\mathscr{B}'$ be a countable subset of $\mathscr{A}$ given by $\bigstar_p^n(\mathscr{A})$. Let $\mathscr{B} = \mathscr{B}' \cup \bigcup_{k\in\omega}(s_k^{-1}(0) \cap \mathscr{A})$ and let us show that $\mathscr{B}$ is the required set in Claim 1. Let $f \in {}^n\sigma_{t_0}(\mathscr{A}) \setminus V$ and $x = \mathrm{supp}(f) \cap \mathscr{A}$. Since V contains ${}^n\sigma_t(\mathscr{A})$ for each $t \prec t_0$, $|x| = p$. Now, we proceed by contradiction supposing that $x \cap \mathscr{B} = \emptyset$. Then $\mathrm{supp}(f) \cap \mathscr{A} \in [\mathscr{A} \setminus \mathscr{B}]^p$. By the choice of $\mathscr{B}$, $f \restriction \omega \in U$ and consequently, there is $k \in \omega$ such that $s_k \restriction \omega \subset f \restriction \omega$ and, since $x \cap s_k^{-1}(0) = \emptyset$ and $s_k \restriction \mathscr{A}$ is the constant zero, $s_k \subset f$. Thus $f \in V$, which is impossible, and Claim 1 is proved.

Claim 2. If V is an open subset of ${}^n\sigma_{t_0}(\mathscr{A})$ containing ${}^n\sigma_t(\mathscr{A})$ for each $t \prec t_0$, then there is a countable set $Y \subset C_p(\Psi(\mathscr{A}), n)$ such that ${}^n\sigma_{t_0}(\mathscr{A}) \setminus V \subset \bigcup_{h\in Y, t\prec t_0}(h + {}^n\sigma_t(\mathscr{A}))$, where

$h + {}^n\sigma_t(\mathscr{A}) = \{h + g : g \in {}^n\sigma_t(\mathscr{A})\}$ and addition is taken mod n.

Let $\mathscr{B}$ be a countable subset of $\mathscr{A}$ given by Claim 1. Fix $1 \le j < n$ and let $r_j(i)$ be 1 if $i = j$ and 0 otherwise. Define $Y = \bigcup_{j=1}^{n-1}\{f \in {}^n\sigma_{r_j}(\mathscr{A}) : f^{-1}(j) \cap \mathscr{A} \subset \mathscr{B}\}$. It is not difficult to show that Y is countable.
Let $f \in {}^n\sigma_{t_0}(\mathscr{A}) \setminus V$. By the choice of $\mathscr{B}$, there is $1 \le i < n$ and an element $a \in f^{-1}(i) \cap \mathscr{B}$. We define a continuous function $g : \Psi(\mathscr{A}) \to n$ as follows

$$g(x) = \begin{cases} n - i, & \text{if } x \in a \cup \{a\}; \\ \\ 0, & \text{otherwise.} \end{cases}$$

If $t_1 \in \omega^n$ is defined as $t_1(l) = t_0(l)$ if $l \neq i$ and $t_1(i) = t_0(i) - 1$, we obtain that $f + g \in {}^n\sigma_{t_1}(\mathscr{A})$ and $t_1 \prec t_0$. Let $h \in C_p(\Psi(\mathscr{A}), n)$ be the additive inverse function of g. Observe that $h \in Y$. Consequently, $f = h + (f + g) \in \bigcup_{h \in Y, t \prec t_0}(h + {}^n\sigma_t(\mathscr{A}))$. This concludes the proof of Claim 2.

Now, we are going to finish the proof of our lemma. Let $\langle \mathscr{U}_k : k \in \omega\rangle$ be a sequence of covers of ${}^n\sigma_{t_0}(\mathscr{A})$ and $\{P_t : t \preceq t_0\}$ a partition of ω into infinite sets. Since for each $t \prec t_0$, ${}^n\sigma_t(\mathscr{A})$ is Rothberger, there is, for each $k \in P_t$, $U_k \in \mathscr{U}_k$ such that ${}^n\sigma_t(\mathscr{A}) \subset \bigcup_{k \in P_t} U_k = V_t$. Then, by Claim 2, there is a countable set Y such that ${}^n\sigma_{t_0}(\mathscr{A}) \setminus \bigcup_{t \prec t_0} V_t \subset \bigcup_{h \in Y, t \prec t_0}(h + {}^n\sigma_t(\mathscr{A}))$. Since ${}^n\sigma_t(\mathscr{A})$ is homeomorphic to $h + {}^n\sigma_t(\mathscr{A})$ for each $h \in Y$ and Y is countable, $\bigcup_{h \in Y, t \prec t_0}(h + {}^n\sigma_t(\mathscr{A}))$ is Rothberger (see Proposition 3.1(c)). Then, there is $U_k \in \mathscr{U}_k$ for each $k \in P_{t_0}$ such that $\bigcup_{k \in P_{t_0}} U_k$ covers ${}^n\sigma_{t_0}(\mathscr{A}) \setminus \bigcup_{t \prec t_0} V_t$. Therefore, the sequence $\{U_k : k \in \omega\}$ is the required choice. □

With lemma we have the main theorem of this section.

Theorem 3.33. *Let $\mathscr{A}$ be a mad family and $n \in \omega$. Then the following statements are equivalent.*

(a) *$C_p(\Psi(\mathscr{A}), 2)^n$ is Lindelöf;*

(b) *$C_p(\Psi(\mathscr{A}), 2)^n$ is Menger;*

(c) *$C_p(\Psi(\mathscr{A}), 2)^n$ is Rothberger;*

(d) *The property $\bigstar_m^{2^n}(\mathscr{A})$ is satisfied for all $m \in \omega$.*

Proof. First observe that $C_p(\Psi(\mathscr{A}), 2)^n$ is homeomorphic to $C_p(\Psi(\mathscr{A}), 2^n)$. The implication (d) → (c) is proved as follows: by Lemma 3.30 it is sufficient to show that ${}^{2^n}\sigma_m(\mathscr{A})$ is Rothberger for each $m \in \omega$. Indeed, fix $m \in \omega$ and $t_m \in \omega^{2^n}$ to be the constant function m. Since ${}^{2^n}\sigma_0$ is countable, this is Rothberger, and if we suppose that ${}^{2^n}\sigma_t$ is Rothberger for

each $t \prec t_0$ for some $t_0 \preceq t_m$, by hypothesis and Lemma 3.32, ${}^{2^n}\sigma_{t_0}(\mathscr{A})$ is Rothberger. By induction, ${}^{2^n}\sigma_{t_m}(\mathscr{A}) = {}^{2^n}\sigma_m(\mathscr{A})$ is Rothberger.
The implications (c) → (b) and (b) → (a) are clear. Finally, if $C_p(\Psi(\mathscr{A}),2^n)$ is Lindelöf, the closed subspace ${}^{2^n}\sigma_m(\mathscr{A})$ of $C_p(\Psi(\mathscr{A}),2^n)$ is Lindelöf for each $m \in \omega$ and, by Lemma 3.31, $\bigstar^{2^n}_m(\mathscr{A})$ is satisfied for each $m \in \omega$. This proves that (a) → (d). □

Evidently, $\mathscr{A}^m + [\omega]^{<\omega}$ is concentrated on $[\omega]^{<\omega}$ if and only if $\bigstar^2_m(\mathscr{A})$ is satified. By Proposition 2.46 and Theorem 3.33 we obtain:

Corollary 3.34. *Let $\mathscr{A}$ be a mad family and $n \in \omega$. Then, the following statements are equivalent.*

(a) $C_p(\Psi(\mathscr{A}),2)$ *is Lindelöf;*

(b) $C_p(\Psi(\mathscr{A}),2)$ *is Menger;*

(c) $C_p(\Psi(\mathscr{A}),2)$ *is Rothberger;*

(d) $\mathscr{A}^m$ *is concentrated on* $[\omega]^{<\omega}$ *for every* $m \in \omega$.

As was shown in [HSTM05], every finite power of $C_p(\Psi(\mathscr{A}),2)$ is Lindelöf, where $\mathscr{A}$ is the family constructed in Theorem 2.50 (and Theorem 4.5 in [HSTM05]). Theorem 3.33 then gives the following:

Theorem 3.35 (CH)**.** *There is a Mrówka mad family $\mathscr{A}$ such that $C_p(\Psi(\mathscr{A}),2)^n$ is Rothberger for each $n \in \omega$.*

The following gives a consistent negative answer to Question 3.21.

Theorem 3.36 (CH)**.** *There is a Mrówka mad family $\mathscr{A}$ such that $C_p(\beta\Psi(\mathscr{A}),2)^n$ is Rothberger for every $n \in \omega$.*

Proof. It is sufficient to observe that the function

$$\phi^{2^n}_m : {}^{2^n}\sigma_m(\mathscr{A}) \to \{g \in C_p(\beta\Psi(\mathscr{A}),2^n) : \forall i \in 2^n (i \neq 0 \to |g^{-1}(i) \cap \mathscr{A}| \leq m)\}$$

defined by $\phi^{2^n}_m(f) = \widetilde{f}$ is an onto continuous function where $\widetilde{f}$ is the continuous extension of $f : \Psi(\mathscr{A}) \to 2^n$ to $\beta\Psi(\mathscr{A})$. □

Bibliography

A.V. ARHANGEL'SKIĬ

[Arh92] *Topological Funtion Spaces*, Mathematics and its Applications (Soviet Series), Kluwer Academic Publishers, 1992.

D. BARMAN AND A. DOW

[BD11] *Selective separability and SS^+*, Topology Proceedings, **37**, 2011, pp. 181–204.

T. BARTOSZYŃSKI

[Bar87] *Combinatorial aspects of measure and category*, Fundamenta Mathematicae, **127**, 1987, pp. 225–239.

A. BELLA, M. BONANZINGA, M.V. MATVEEV AND V.V. TKACHUK

[Bel+08] *Selective separability: general facts and behavior in countable spaces*, Topology Proceedings, **32**(1), 2008, pp. 15–30.

D. BERNAL-SANTOS

[BS15a] *The Rothberger property on* $C_p(\Psi(\mathscr{A}), 2)$, Commentationes Mathematicae Universitatis Carolinae to appear, 2015.

[BS15b] *The Rothberger property on* $C_p(X, 2)$, Topology and its Applications, **196**, 2015, pp. 106–119.

D. BERNAL-SANTOS AND Á. TAMARIZ-MACARÚA

[BSTM15] *The Menger property on* $C_p(X, 2)$, Topology and its Applications, **183**, 2015, pp. 110–126.

A. BLASS

[Bla10] *Combinatorial Cardinal Characteristics of the Continuum*, Handbook of Set Theory (M. Foreman and A. Kanamori, eds.) Springer, Netherlands, 2010, pp. 395–489.

A. BLASS, M. HRUŠÁK AND J. VERNER

[BHV13] *On strong P-points*, Proceedings of the American Mathematical Society, **141**-(8), 2013, pp. 2875–2883.

E. BOREL

[Bor19] *Sur la classification des ensemble de mesure nulle*, Bulletin de la Société Mathématique de France, **47**, 1819, pp. 97–125.

R.Z. BUZYAKOVA

[Buz04] *In search for Lindelöf C_p's*, Commentationes Mathematicae Universitatis Carolinae, **45**(1), 2004, pp. 145–151.

[Buz06] *Spaces of continuous characteristics functions*, Commentationes Mathematicae Universitatis Carolinae, **47**(4), 2006, pp. 599–608.

[Buz07] *Function spaces over GO-spaces: Part I*, Topology and its Applications, **154**, 2007, pp. 917–924.

A. CONTRERAS-CARRETO AND Á. TAMARIZ-MACARÚA

[CCTM03] *On some Generalizations of compactness in spaces $C_p(X,2)$ y $C_p(X,\mathbb{Z})$*, Boletín de la Sociedad Matemática Mexicana, **9**(3), 2003, pp. 291–308.

A. DOW AND P. SIMON

[DS06] *Spaces of continuous functions over Ψ-space*, Topology and its Applications, **153**(13), 2006, pp. 2260–2271.

R. ENGELKING

[Eng89] *General Topology*, Sigma Series in Pure Mathematics, Vol. 6, Heldermann Verlag Berlin, 1989.

J. GERLITS AND ZS.NAGY

[GN82] *Some properties of $C(X)$, I*, Topology and its Applications, **14**, 1982, pp. 151–161.

L. GILLMAN AND M. JERISON

[GJ60] *Rings of Continuous Functions*, Editorial Board, Springer-Verlag, 1960.

M. HRUŠÁK, P.J. SZEPTYCKI, AND Á. TAMARIZ-MASCARÚA

[HSTM05] *Spaces of functions defined on Mrówka spaces*, Topology and its Applications, **148**, 2005, pp. 239–252.

W. HUREWICZ

[Hur26] *Über eine Verallgemeinerung des Borelschen Theorems*, Mathemathische Zeitschrift, **24**, 1926, pp. 401–421.

W. JUST, W. MILLER, M. SCHEEPERS AND J. SZEPTYCKI

[Jus+96] *The combinatorics of open covers II*, Topology and its Applications, **73**, 1996, pp. 241–266.

LJ. D.R. KOCINAC

[Koc04] *Selected results on selection principles*, Procceedings of the 3rd Seminar on Geometry and Topology (Sh. Rezapour, ed.) July 15-17, Tabriz, Iran, 2004, pp. 71–104.

K. KUNEN AND E. VAUGHN

[KV84] *Handbook Set-theoretic Topology*, Elsevier Science Publishers B.V., North-Holland, 1984.

C. LAFLAMME

[Laf89] *Forcing with filters and complete combinatorics*, Annals of Pure and Applied Logic, **42**(2), 1989, pp. 125–163.

R. LAVER

[Lav76] *On the consistensy of Borel's conjeture*, Acta Mathematica, **37**, 1976, pp. 151–169.

A. LELEK

[Lel69] *Some covering properties of spaces*, Fundamenta Mathematicae, **64**(2), 1969, pp. 209–218.

M.K. MENGER

[Men24] *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte der Wiener Akademie, **133**, 1924, pp. 421–444.

A.W. MILLER AND D.H. FREMLIN

[MF88] *On some properties of Hurewicz, Menger and Rothberger*, Fundamenta Mathematicae, **129**, 1988, pp. 17–33.

A.W. MILLER

[Mil81] *Some properties of measure and category*, Transactions of the American Mathematical Society, Vol. **266**(1), 1981, pp. 93–114.

[Mil82] *A characterization of the least cardinal for which the Baire category theorem fails*, Proceedings of the American Mathematical Society, **86**(3), 1982, pp. 498–502.

N. NOBLE

[Nob69] *Products with closed projections*, Transactions of the American Mathematical Society, **140**, 1969, pp. 381–391.

J.M. O'FARREL

[O'F86] *Construction of a Hurewicz metric space square is not a Hurewicz space*, Fundamenta Mathematicae, **127**, 1986, pp. 41–43.

F. ROTHBERGER

[Rot41] *Sur des families indenombrables de suites de nombres naturels, et le problémes concernant la propieté C*, Proceedings of the Cambridge Philosophical Society, **37**, 1941, pp. 109–126.

M. SAKAI

[Sak88] *Property C" and function spaces*, Proceedings of the American Mathematical Society, **104**, 1988, pp. 917–919.

[Sak06] *Special subsets of reals characterizing local properties of function spaces*, Selection Principles and Covering Properties in Topology (L. Kočinac, ed), Quaderni di Matematica 18, Seconda Universita de Napoli, Caserta, 2006, pp. 195–225.

[Sak09] *Menger subsets of Sorgenfrey line*, Proceedings of the American Mathematical Society, **137**(9), 2009, pp. 3129–3138.

[Sak12] *Selective separability of Pixley Roy hyperspaces*, Topology and its Applications, **159**, 2012, pp. 1591–1598.

M. SCHEEPERS

[Sch99a] *Combinatorics of open covers I: Ramsey theory*, Topology and its Applications, **69**, 1999, pp. 1295–1306.

[Sch99b] *The combinatorics of open covers VI. Selectors for sequences of dense sets*, Quaestiones Mathematicae, **22**(1), 1999, pp. 109–130.

[Sch03] *Selection principles and covering properties in topology*, Note in Matematica, **22**, 2003, pp. 3–41.

G.A. SOKOLOV

[Sok86] *On Lindelöf spaces of continuous functions*, in Russian, Matem. Zametki, **39**-(6), 1986, pp. 887–894.

[Sok93] *Lindelöf property and the iterated continuous function spaces*, Fundamenta Mathematicae, **143**, 1993, pp. 87–95.

D. TALL

[Tal11] *Lindelöf spaces which are Menger, Hurewicz, Alster, productive, or D*, Topology and its Applications, **158**, 2011, pp. 2556–2563.

R. TELGAŃSKY

[Tel72] *Concerning product of paracompact spaces*, Fundamenta Mathematicae, **74**, 1972, pp. 153–159.

V.V. TKACHUK

[Tka05] *A nice class extracted from C_p-theory*, Commentationes Mathematicae Universitatis Carolinae, **46**(3), 2005, pp. 503–513.

[Tka11] *Countably compact first countable subspace of ordinals have the Sokolov property*, Quaestiones Mathematicae, **34**, 2011, pp. 225–234.

B. TSABAN

[Tsa06] *Some new directions in infinite-combinatorial topology*, Set theory (J. Bagaria and S. Todorčevic, eds.) Trends in Mathematics, Birkhauser, 2006, pp. 225–255.

[Tsa07] *Selections principles and special sets of reals*, Open Problems in Topology II (E. Pearl, ed.) Elsevier B.V., 2007, pp. 91–108.

B. TSABAN AND L. ZDOMSKYY

[TZ12] *Hereditarily Hurewicz spaces and Arhangel'skiĭ sheaf amalgamations*, Journal of the European Mathematical Society, **12**, 2012, pp. 353–372.

L. WINGERS

[Win95] *Box products and Hurewicz spaces*, Topology and its Applications, **64**, 1995, pp. 9–21.

Index