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On the k-kernel problem and its complexity in some classes of digraphs.

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PRESENTA:

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“ Science has fallen into many errors - errors which have been fortunate and useful rather than otherwise, for they have been the steppingstones to truth. ”

Jules Verne

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When I wrote the acknowledgements for my bachelor's thesis I made an real effort to mention as many people as I could. Primarily, because there were more than a few persons whose influence was essential in that period of my life, but also because I though I was never going to write one of those sections again.

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Last, but not least, my family. Thank you for always being there.

Preface

The notion of kernel of a digraph was introduced by von Neumann and Morgenstern in 1944 and has been extensively studied since then. A well known result about kernels, due to Chvátal, states that the problem to determine whether a given graph has a kernel is NP-complete. Also, there has been a considerable amount of work devoted to find new sufficient conditions for the existence of kernels in digraphs. An example is a result by Duchet, which will be mentioned in the introduction of this work as Theorem 1.2.1.

As it is natural in mathematics, many ideas related to others arise throughout the years. An offspring of the concept of kernel is the notion of k-kernel. Pavol Hell and César Hernández proved that the problem of determining if a given digraph has a 3-kernel is NP-complete even when restricted to cyclically 3-partite digraphs with circumference 6. Also, they found new sufficient conditions for a cyclically 3-partite digraph to have a 3-kernel. The second chapter of this work is devoted to generalize this results for k-kernels.

Also, once results have been obtained, it is usual to ask if some of the hypotheses can be dropped and, if not the case, if weaker versions of them could be used instead. Some digraphs are acyclic (contain no directed cycles), some are symmetric (every arc is a directed 2-cycle), and every digraph is somewhere in between those two families. Since every acyclic digraph and every symmetric digraph have k-kernels but not every digraph does, it is not too crazy to ask the following two questions:

1. How many symmetric arcs can a digraph have and still have a k-kernel?

2. How few symmetric arcs must a digraph have in order to guarantee the existence of a k-kernel?

The third chapter contains some the work we have done around the second of those questions. We propose a conjecture that, if true, would generalize to

k-kernels a result about the existence of kernels in digraphs given by Duchet. While we have only obtained some partial results towards proving the general case, complete solutions for 3 and 4-kernels are given here, as well as examples showing that the hypotheses are sharp.

Outline

This work is divided in two parts. The first one, contained in the second chapter, generalizes some of the results in [6]. Most of it is dedicated to proving the following result:

Theorem 2.1.13. *The k-kernel problem for cyclically k-partite digraphs is NP-complete, even when restricted to cyclically k-partite digraphs with circumference $2k$.*

We reduce the k-coloring problem to the k-kernel problem restricted to cyclically k-partite digraphs with circumference $2k$. The objective of most of the results in this section is to develop a procedure to construct, for any given a graph G, a digraph $D_{G,k}$ with the following property:

Corollary 2.1.13. *There is a bijective correspondence between the k-colorings of G and the k-kernels of $D_{G,k}$.*

The polynomial reduction makes use of a family of cyclically k-partite digraphs with circumference $2k$ without a k-kernel. The elements of this family have a particular property: in every class of the partition there is one sink. However, if we restrict the number of classes in a cyclically k-partite with sinks, it is possible to guarantee the existence of a k-kernel. If $\mathcal{Z}$ is the family of cyclically k-partite digraphs such that there are at least $k-2$ elements of the partition without sinks, we can state the result as follows:

Theorem 2.2.1. *The digraphs in $\mathcal{Z}$ have a k-kernel.*

The chapter ends by showing that the hypotheses of this results are optimal with regard to the number of classes without sinks.

The third chapter, which contains the second part of the work, is devoted to finding new sufficient conditions for the existence of k-kernels. It is shown that if a digraph is such that at least a certain proportion of the arcs of every cycle is symmetric, then the existence of a k-kernel is guaranteed, for $k = 3$ and $k = 4$.

The results that summarize the work done there are the following:

Corollary 3.1.4. *Let D be a digraph. If every directed cycle B in D has at least $\frac{1}{2}\ell(B) + 1$ symmetric arcs, then D has a 3-kernel.*

Corollary 3.2.6. *Let D be a digraph. If every directed cycle B in D has at least $\frac{2}{3}\ell(B) + 1$ symmetric arcs, then D has a 4-kernel.*

Finally, a general conjecture is presented, as well as a family of digraphs that show the sharpness of the hypotheses.

Contents

Chapter 1

Introduction

The present chapter contains the basic concepts and definitions that we will use throughout this work. We also introduce the notation that will be used in the text. For a set X and a natural number n, we will denote with $\binom{X}{n}$ to the collection of all subsets of X with n elements.

Let f be a function from A to B and $S \subset B$. We use $f^{-1}[S]$ to denote the inverse image of the set S under f.

1.1. Graphs and Digraphs

Even though the primary objects we will study in this work are digraphs, it is natural to begin with the concept of graph rather than with the notion of digraph. A *graph* G is an ordered pair $(V(G), E(G))$, where $V(G) \neq \varnothing$ and $E(G) \subseteq \binom{V}{2}$. The set V is called the set of vertices or *vertex set* and E the set of edges or *edge set*. When it does not cause confusion, we will simply write V and E.

Normally, we represent the vertices of a graph with points and the edges with lines joining the corresponding points. If $x, y \in V(G)$, we denote the edge $\{x, y\}$ with xy. Notice that xy and yx represent the same edge. A graph is depicted in Figure 1.1. A *walk* W in G is a sequence of vertices $v_1 v_2 \cdots v_k$ such that $v_i v_{i+1} \in E$ for every $1 \leq i \leq k-1$. With this notation, we say that W is a walk from v_1 to v_k, or simply a $v_1 v_k$-walk.

A digraph is, intuitively, a graph where we add a direction to the edges. More formally, a *digraph* D is an ordered pair $(V(D), A(D))$, where $V(D)$ is the set of vertices or *vertex set* of D and $A(D)$ the set of arcs or

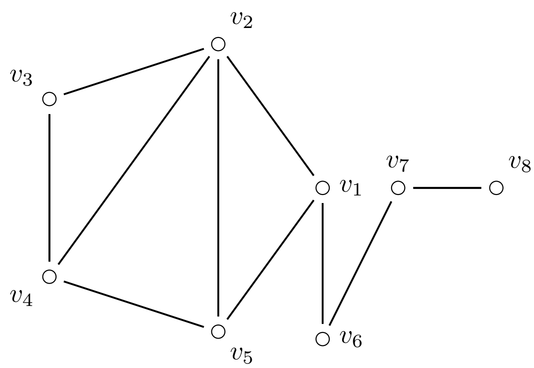

Figure 1.1: A graph G.

arc set of D, where $A(D) \subseteq V(D) \times V(D) \setminus \Delta(V(D))$, with $\Delta(V(D)) = \{(x,x) : x \in V(D)\}$. The elements of $\Delta(V(D))$ are called loops, but we will work with loopless digraphs so we do not include them in the definition. When there is no confusion about which digraph we are referring to, we will simply write V and A. The number $|V(D)|$ is called the *order* of D, while $|A(D)|$ is the *size* of D. We will also use $\|D\|$ to denote the size of D.

If H is a digraph, we say it is a *subdigraph* of D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. We say H is an *induced subdigraph* of D if it is a subdigraph of D and $A(H) = A(D) \cap [V(H) \times V(H)]$. If S is a subset of V, we use $D[S]$ to denote the induced subdigraph of D whose vertex set is S.

If D is a digraph such that $A(D) = V(D) \times V(D) \setminus \Delta(V(D))$, we say that D is the *complete* digraph of order $|V(D)|$. If G is a graph and $E(G) = \binom{V}{2}$ we call it the *complete* graph of order $|V(G)|$.

For an arc $(x,y) \in A$, the vertex x will be called the *tail* of the arc and the vertex y will be called the *head*. We will say that an arc (x,y) is *symmetric* if (y,x) is also and element of A. If $(x,y) \in A$, we say that x is an *in-neighbor* of y and that y is an *out-neighbor* of x. The set of all the out-neighbors of a vertex x in a digraph D is denoted by $N_D^+(x)$, while we use $N_D^-(x)$ for the set of in-neighbors of x. These sets are called the *out-neighborhood* and *in-neighborhood*, respectively. The *out-degree* (*in-degree*) of a vertex x is the cardinality of $N_D^+(x)$ ($N_D^-(x)$). Also, we will use $N_D^+[x]$ and $N_D^-[x]$ to denote the sets $N_D^+(x) \cup \{x\}$ and $N_D^-(x) \cup \{x\}$, respectively. A vertex $x \in V(D)$ such that $N_D^+(x) = \varnothing$ is called a *sink*.

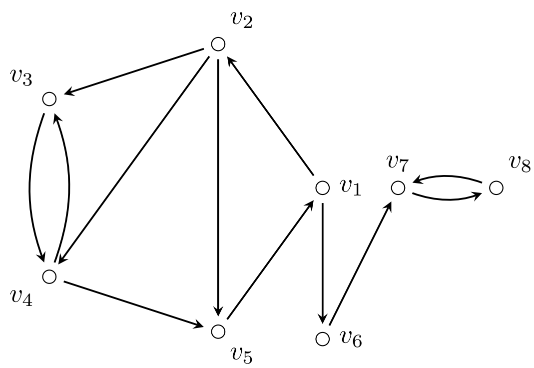

Figure 1.2: A digraph D. The arcs (v_3, v_4) and (v_7, v_8) are symmetric.

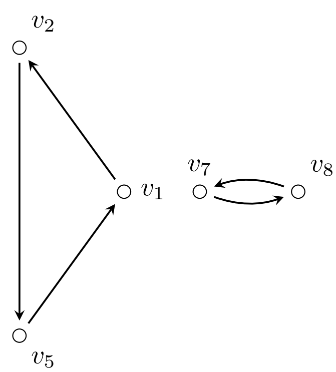

Figure 1.3: The subdigraph of D induced by $\{v_1, v_2, v_5, v_7, v_8\}$.

Let X and Y be subsets of V. We say that Y *absorbs* X if for every $x \in X$, there is a $y \in Y$ such that $(x, y) \in A$. We say that X is *independent* if $X \times X \cap A(D) = \varnothing$. In Figure 1.2, v_5 absorbs v_2 and v_4, and the set $\{v_2, v_6, v_8\}$ is independent.

A *(directed) walk* W is an alternated sequence of vertices and arcs, denoted by $W = (v_1, a_1, v_2, \ldots, v_{k-1}, a_{k-1}, v_k)$, such that $a_{i+1} = (v_i, v_{i+1})$ for $1 \leq i \leq k-1$. Notice that, for our definition of digraph, it is enough to write the sequence of vertices to identify a walk. We say that W is *closed* if $v_1 = v_k$. A *(directed) trail* is a walk such that $a_i \neq a_j$ whenever $i \neq j$. Usually, when we talk about trails, we only write the vertices of the trail. Again, a *closed* *trail* is a trail where the first and last vertices coincide. A *cycle* is a closed directed trail where all the intermediate vertices are different. A *(directed) path* is a trail where $v_i \neq v_j$ whenever $i \neq j$. A digraph is *acyclic* if it contains no cycle. The directed cycle of length n is denoted by C_n.

In Figure 1.2, $(v_2, v_3, v_4, v_5, v_2, v_3)$ is a walk, (v_2, v_3, v_4, v_3) is a trail, the sequence (v_1, v_2, v_3, v_4) is a path and $(v_1, v_2, v_3, v_4, v_5, v_1)$ is a cycle.

Por a path $P = (v_1, v_2, \ldots, v_n)$ and $v_s, v_{s+t} \in V(P)$, we will use denote $(v_s, v_{s+1}, \ldots, v_{s+t})$ by $v_s P v_{s+t}$. The *length* of a path P is the cardinality of $A(P)$ and we denote it by $\ell(P)$. Also, the length of a cycle C is the cardinality of $A(C)$ and is also denoted by $\ell(C)$. If $x, y \in V$, then $\mathcal{P}(x, y)$ will denote the set of all paths from x to y. The *circumference* of a digraph D is the length of the largest cycle in D. The *distance* from x to y in D is $\min\{\ell(P) : P \in \mathcal{P}(x, y)\}$ and is denoted by $d_D(x, y)$. When there is no confusion, we will simply write $d(x, y)$. Notice that it is not necessarily true that $d(x, y) = d(y, x)$. For example, in Figure 1.3 we have $d(v_2, v_5) = 1$ but $d(v_5, v_2) = 2$.

It is natural to think that there are ways to connect the concepts of graph and digraph. Let G be a graph and D a digraph. An *orientation* of G is a digraph $\vec{G}$ such that $V(G) = V(\vec{G})$ and for every edge $xy \in E(G)$, either (x, y) or (y, x) is an element of $A(\vec{G})$. The *underlying* graph of D is the graph such that $V(D) = V(G)$ and for every arc $(x, y) \in A(D)$, we have $xy \in E(G)$. The graph in Figure 1.1 is the underlying graph of the digraph in Figure 1.2, while Figure 1.5 is an orientation of Figure 1.4.

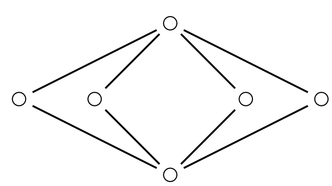

Figure 1.4:

We say that a graph is *connected* if for every $x, y \in V(G)$, there is a xy-walk in G. A digraph D is connected if its underlying graph is. The digraph in Figure 1.2 is connected, while the one depicted in Figure 1.3 is not.

More information about graphs can be found in [9] and in [10], while for digraphs [8] is a good reference.

1.2. k-kernels

Let k and l be positive integers. A natural generalization of the concepts of independence is the k-independence. We say that a set $S \subseteq V$ is *k-independent* if for every pair of different vertices $x, y \in S$, we have $d(x, y) \geq k$. A set $T \subseteq V$ is *l-absorbent* if for every $y \in V$ there exists $x \in T$ such that $d(y, x) \leq l$. A set $K \subseteq V$ that is k-independent and l-absorbent is called a *(k, l)-kernel*. A *k-kernel* is a $(k, k-1)$-kernel. A 2-kernel is simply called a *kernel*. In this work we will focus on k-kernels.

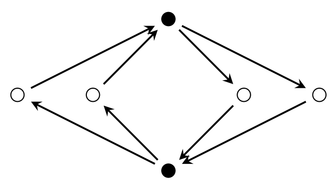

Figure 1.5: The black vertices form a kernel of the digraph.

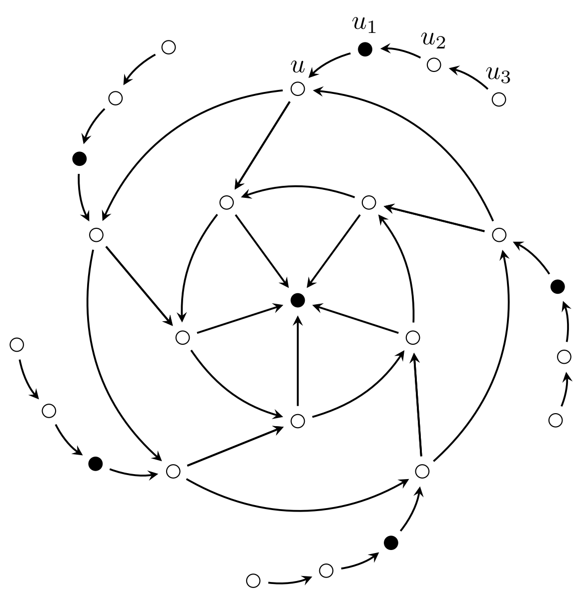

Figure 1.6: The black vertices form a 3-kernel of the digraph.

We also generalize the concepts of in-neighborhood and out-neighborhood. For a vertex $x \in V(D)$, the set $\{y \in V(D) : d_D(x,y) \leq k\}$ will be denoted by $N_D^{+k}(x)$, while we will use $N_D^{-k}(x)$ to denote $\{y \in V(D) : d_D(y,x) \leq k\}$. These sets are called the *k-out-neighborhood* and the *k-in-neighborhood*, respectively. In Figure 1.6, the vertices u_1 and u_2 are in $N_D^{-2}(u)$, while u_3 is not since $d(u_3, u) = 3$.

A digraph D is said to be *kernel-perfect* if every induced subdigraph of D has a kernel. We state a classical result about kernels as Theorem 1.2.1, a result by Duchet that is proven in [4].

Theorem 1.2.1. *If every directed cycle in D has at least one symmetric arc, then D is kernel-perfect.*

In particular, if every directed cycle in D has at least one symmetric arc, then D has a kernel. In Section 3.1, a proof of a generalization of this result for 3-kernels is found, while the corresponding version for 4-kernels is in Section 3.2 . A general conjecture is also stated in that chapter.

We will also use the concept of k-closure of a digraph. For a positive integer k, the *k-closure* of a digraph D, denoted by $\mathcal{C}^k(D)$, is a digraph whose vertex set is $V(D)$ and whose arc set is formed by the arcs (x,y) for every $x, y \in V(D)$ such that $d_D(x,y) \leq k$.

For further reading about kernels, refer to [8].

1.3. Homomorphisms, cyclic partitions and complexity

Let D and H be digraphs. A *homomorphism* from D to H is a function $f : V(D) \to V(H)$ such that $(f(x), f(y)) \in A(H)$ whenever $(x,y) \in A(D)$. This is also called a H-coloring of D since we get a k-coloring of the digraph (in the traditional sense) by taking H as the complete digraph with k vertices.

We can also consider homomorphisms between graphs simply by using edges instead of arcs in the definition above.

We say that a digraph D is *cyclically k-partite* if there is a partition of V in k sets $\{V_0, V_1, \ldots, V_{k-1}\}$ such that if $(x,y) \in A$, then there is $0 \leq i \leq k-1$ such that $x \in V_i$ and $y \in V_{i+1}$, with indices taken modulo k. It is easy to see that a digraph D is cyclically k-partite if and only if there is a homomorphism from D to C_k. The digraph in Figure 1.7 is cyclically 4-partite

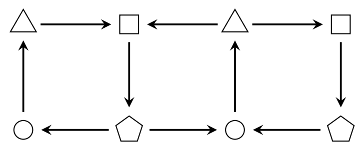

Figure 1.7: The partition classes are represented with pentagons, squares, triangles and circles.

Even though it is not explicitly proven in [1], it is easy to see that, if D is a connected cyclically k-partite digraph, then only one cyclic k-partition of D exists.

The formal definitions and basic results about algorithmic complexity are omitted here, but can be found in [2]. A deeper and more complete exposition about homomorphisms is given in [1]. The main result about complexity that we will use, due to Hell and Nešetřil, is

Theorem 1.3.1. *The H-coloring problem is*

- *P if H is bipartite.*
- *NP-complete otherwise.*

and can also be found in [1].

Let n be a positive integer. A k-coloring of G is an homomorphism from G to the complete graph with k-vertices. A digraph is k-colorable if its underlying graph is k-colorable.

Let D be digraph and $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ a family of disjoint subsets of V such that $\bigcup_{i=1}^{k} = V$. We say that $\mathcal{P}$ is a k-partition of D if every V_i is an independent set. When such a partition exists, we say that D is *k-partite*. A 2-partite digraph is called *bipartite*.

There is also a connection between partitions and kernels. A result in this direction is due to Neumann-Lara and can be found in [5]. The result is stated below as Theorem 1.3.2.

Theorem 1.3.2. *Every bipartite digraph has a kernel.*

Since a kernel is a 2-kernel and a bipartite digraph is a 2-partite digraph, a natural thing to ask is whether every k-partite digraph has a k-kernel. This is not true in general. Also, it is a simple observation that every bipartite digraph is also cyclically 2-partite. Again, we can ask whether cyclically k-partite digraphs have a k-kernel. It is also not true and the digraphs O_k and E_k, which are described in Section 2.1, are counterexamples.

In [6], it was proven by Hell and Hernández-Cruz that determining whether a digraph has a 3-kernel is NP-complete even when restricted to cyclically 3-partite digraphs with circumference 6. The work in Section 2.1 is devoted to prove that determining whether a digraph has a k-kernel is NP-complete even when restricted to cyclically k-partite digraphs with circumference $2k$. Nevertheless, something else can be said about k-kernels in cyclically k-partite digraphs.

Also in [6], it is proved that every digraph that admits a cyclic 3-partition of its vertex set such that there is at least one element of the partiton without sinks has a 3-kernel. In Section 2.2 is the proof of a generalization of that result and counterexamples that show that the number of the partition classes is optimal are provided.

Take a k-colorable digraph D and ϕ the corresponding n-coloring. Notice that from the definition of homomorphism it follows that if $x, y \in V(D)$ are such that $\phi(x) = \phi(y)$, then $(x, y) \notin A(D)$. Hence, sets $\{\phi^{-1}[\{x\}] : x \in K_n\}$ are a partition of $V(D)$ and each $\phi^{-1}[\{x\}]$ is an independent set.

The problem of determining if a given graph G is k-colorable is known as the k-coloring problem. In this work we will use a consequence of Theorem 1.3.1, stated below

Theorem 1.3.3. *The k-coloring problem is NP-complete.*

Chapter 2

Cyclically k-partite digraphs

2.1. The complexity of the k-kernel problem

We find infinite families of cyclically k-partite digraphs with circumference $2k$ whose members have no k-kernel. We have to consider the odd and even values of k in separate cases. For the odd case, the family members are the natural generalizations of the example found in [7]. For every $k \in \mathbb{N}$, where $k \geq 3$, we define the following sets:

- $U_k := \{u_i : 1 \leq i \leq k, i \in \mathbb{N}\}$
- $V_k := \{v_i : 1 \leq i \leq k, i \in \mathbb{N}\}$

Consider the directed cycle C with vertex set $V(C) = U_k \cup V_k$ and arcs $(u_k, v_1), (v_k, u_1)$ and $(u_i, u_{i+1}), (v_i, v_{i+1})$ for $1 \leq i \leq k-1$. For every integer r that satisfies $1 \leq r \leq \left\lfloor \frac{k}{2} \right\rfloor$, create a directed path of length $k-2$ and add the arc from u_{2r+1} to the initial vertex of this path. Repeat this step for the vertices v_{2r} with the integer r satisfying the same condition as before. We will call those digraphs O_k. The digraph O_5 is depicted in Figure 2.1.

For the even case, consider the same cycle C as before and add the arcs (u_i, v_{i+1}) for odd i, and (v_i, u_{i+1}) for even i, with $1 \leq i \leq k-1$, to obtain a $u_1 v_k$-directed path. For each vertex v of this path, create a directed path of length $k-2$ and add the arc from v to the initial vertex of the path. We will call those digraphs E_k. The digraph E_4 is shown in Figure 2.2.

Lemma 2.1.1. *Let $k \geq 3$ be an integer. The graph O_k has no k-kernel.*

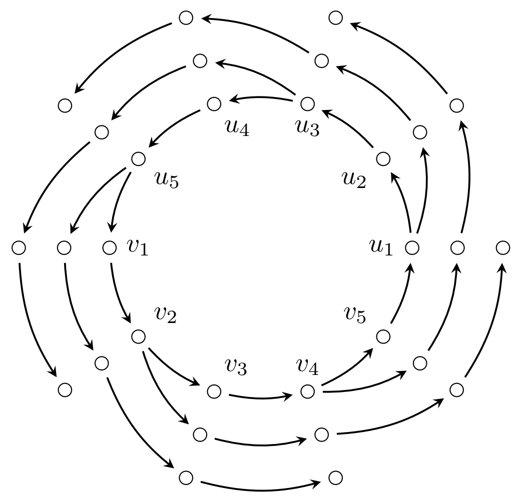

Figure 2.1: The digraph O_5, a cyclically 5-partite digraph without a 5-kernel.

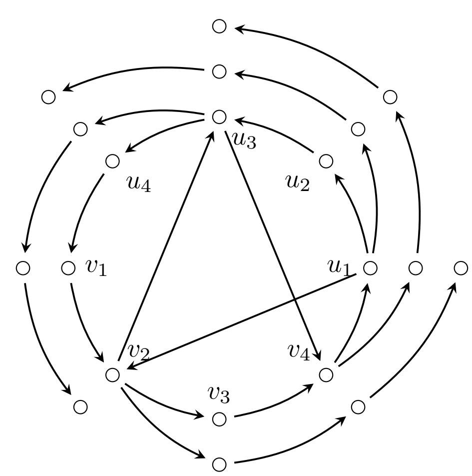

Figure 2.2: The digraph E_4, a cyclically 4-partite digraph without a 4-kernel.

Proof. If the set $K \subseteq V(O_k)$ is a k-kernel of the digraph O_k, then all the sinks of O_k must be elements of K. Therefore, it is impossible for the vertices u_{2r+1} and v_{2r}, with $1 \leq r \leq \lfloor \frac{k}{2} \rfloor$, to be in K, because of the $k-1$ independence of the k-kernel. In order to absorb the remaining vertices of the cycle, at least one vertex of the cycle must be included in K. We will assume without loss of generality that $v_1 \in K$. Observe that $d(v_k, v_1) > k$, $d(v_1, v_k) = k - 1$ and $d(u_i, v_1) \leq k - 1$ for every $2 \leq i \leq k$. Hence, $u_i \notin K$ for every $1 \leq i \leq k$ and therefore v_k is not $(k-1)$-absorbed by K, a contradiction. ■

Lemma 2.1.2. *For every even $k \in \mathbb{N}$, where $k \geq 4$, the graph E_k has no k-kernel.*

Proof. If the set $K \subseteq V(E_k)$ is a k-kernel of the digraph E_k, then all the sinks of E_k must be elements of K. This implies that at least one vertex of the cycle C must be an element of K. Also, if two vertices of C are in K, one of them must be an element of U_k and the other one an element of V_k.

If $u_{2r} \in K$, with $1 \leq r \leq \frac{k}{2}$, then every vertex u_s, with $s \leq 2r - 1$, and every vertex v_l, with $2r + 1 \leq l$, would be $(k-1)$-absorbed by K. Since all the sinks of E_k must be elements of K, the vertex v_{2r} is not in K. It is clear that the vertex v_{2r-1} is not $(k-1)$-absorbed by a sink of E_k. Since E_k is a cyclically k-partite digraph, the distance from v_{2r-1} to w, where w is any vertex in the same class as u_{2r} satisfies $d(v_{2r-1}, w) \equiv 1 \pmod{k}$. But $N^+(v_{2r-1}) = \{v_{2r}\}$, so $d(v_{2r-1}, u_{2r}) \geq k + 1$, which means v_{2r-1} cannot be $(k-1)$-absorbed by u_{2r}. Thus, in order for v_{2r-1} to be absorbed, another vertex $v_t \in V_k$ must be in K. Recall that t must be odd. Also, it must be at most $2r - 1$, otherwise $d(v_t, u_{2r}) \leq k - 1$ contradicting the k-independece of K. But $d(u_{2r}, v_t) \leq k - 1$ whenever $t \leq 2r - 1$. That means that $V_k \cap K = \varnothing$, implying that v_{2r-1} is not $(k-1)$-absorbed by K, a contradiction. Therefore, $U_k \cap K = \varnothing$.

If v_1 is in K, then $V_k \cap K = \{v_1\}$. Since v_3 is not $(k-1)$-absorbed by a sink of E_k and no vertex of U_k can also be in K, the vertex v_3 is not $(k-1)$-absorbed by K. Suppose now that a vertex v_{2r-1} is an element of K, with $2 \leq r \leq \frac{k}{2}$. The vertex u_{2r-2} must be absorbed by a vertex in V_k since no sink of E_k absorbs it. The distance between u_{2r-2} and v_{2r-1} must satisfy $d(u_{2r-2}, v_{2r-1}) \equiv 1 \pmod{k}$. The only element in $N^+(u_{2r-2})$ is v_{2r-1}, so $d(u_{2r-2}, v_{2r-1}) \geq k + 1$, which means K does not $(k-1)$-absorb u_{2r-2}, a contradiction.

■

It is easy to see that any k-kernel of a digraph D includes every sink of D. Consider an integer $k \geq 3$ and a cyclically k-partite digraph D. If $S \subseteq V(D)$ is the set of all the sinks of D and S is not a k-kernel of D, then the set $T \subseteq V(D)$ of all the vertices not $(k-1)$-absorbed by S is not empty. Let H be a k-cycle. We define a new digraph $\hat{D}$, formed by adding H to D, along with an arc from each vertex of T to some vertex of H in such way that the resulting graph is cyclically k-partite.

Lemma 2.1.3. *Let D be a cyclically k-partite digraph and $S \subseteq V(D)$ the subset of all the sinks of D, where $k \geq 3$. If S is not a k-kernel of D, then the digraph $\hat{D}$ has exactly k different k-kernels and each of them includes exactly one vertex of the k-cycle.*

Proof. Let H be the cycle $(w_1, \ldots, w_k, w_1)$, and let $V_1, V_2, \ldots, V_k$ be the elements of the cyclic k-partition of $V(D)$. We claim that the set $K_i = \{w_i\} \cup (V_i \cap T) \cup S$ is a k-kernel of $(\hat{D})$.

In order to prove it, take x a vertex in $V(D) \setminus K_i$. If x is $(k-1)$-absorbed by a sink or $x = w_j$ for some $1 \leq j \leq k$, then it is $(k-1)$-absorbed by K_i. If $x \in T \setminus K_i$, then x is an element of V_s for some $1 \leq s \leq k$, with $s \neq i$. Hence the arc (x, w_j) is in $A(\hat{D})$ for some $j \neq i+1$. But clearly $d(x, w_i) = 1 + d(w_j, w_i) \leq 1 + k - 2 = k - 1$, implying that x is $(k-1)$-absorbed by w_i and proving our claim.

To see that those are all the k-kernels of $\hat{D}$, it suffices to see that any k-kernel of $\hat{D}$ is equal to K_i for some i. Consider K any k-kernel of $\hat{D}$. We know that K is k-independent and $(k-1)$-absorbent, so it must contain a unique vertex w_i of H, and the k-independence of K guarantees that only one vertex of the cycle can be in K. Since all the sinks of $\hat{D}$ must be included in any k-kernel of $\hat{D}$, we know that $\{w_i\} \cup S \subseteq K$. Consider now a vertex $y \in V_i \cap T$ and let z be a vertex of T such that $d(y, z) \leq k - 1$. Clearly, since $z \in T$ and $z \notin V_i$, we have that $d(z, w_i) \leq k - 1$, implying that $z \notin K$. This means that no vertex in T can $(k-1)$-absorb y, so y must be an element of K.

This shows that $K_i = \{w_i\} \cup (V_i \cap T) \cup S \subseteq K$. But k-kernels are maximal k-independent sets, therefore $K = K_i$.

■

Corollary 2.1.4. *Let $k \geq 3$ be an integer. The digraph $\hat{O}_k$ has exactly k different k-kernels, each one containing a different vertex from the k-cycle.*

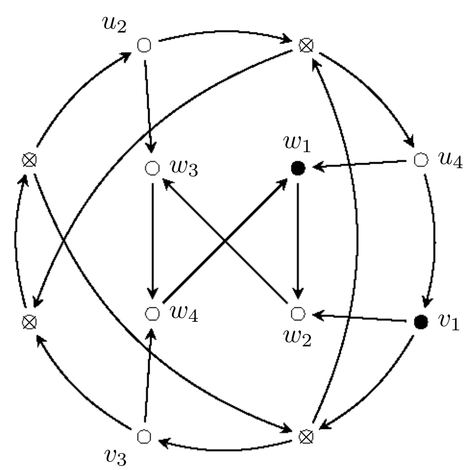

Figure 2.3: The digraph $\hat{E}_4$. A 4-kernel is formed by the black vertices and the sinks of the digraph.

Corollary 2.1.5. *Let $k \geq 4$ be an integer. The digraph $\hat{E}_k$ has exactly k different k-kernels, each one containing a different vertex from the k-cycle.*

Both corolaries follow directly from Lemma 2.1.3. The digraphs shown in Figure 2.3 and Figure 2.4 are the digraphs $\hat{E}_4$ and $\hat{O}_5$, where some of the vertices have been omitted. Each crossed vertex has a tail of the corresponding length.

Consider now two cyclically k-partite digraphs D_1 and D_2. We know that each of the digraphs $\hat{D}_1$ and $\hat{D}_2$ have exactly k different k-kernels. We will describe a special way to join these two digraphs. We will call H_1 and H_2 to the k-cycles we added to the digraphs D_1 and D_2 in order to construct $\hat{D}_1$ and $\hat{D}_2$. The way in which $\hat{D}_1$ and $\hat{D}_2$ will be connected is such that the resulting digraph has a k-kernel consisting of a kernel of each digraph and a few more vertices.

Let $k \geq 3$ be an integer and consider a directed cycle of length $2k$ with vertex set $\{x_1, x_2, \ldots, x_{2k}\}$. For every $i \in \{2, 4, 5, \ldots, k, k+1\}$, create a directed path with $k-1$ vertices and add the arc from x_i to the first vertex of this path. Create a directed path of length $k-3$, add the arc from the terminal vertex of this path to x_1 and, for every new vertex v in this path, create a directed path of length $k-2$ and add the arc from v to the initial vertex of the path. The digraph just described, which we will call S_k, is

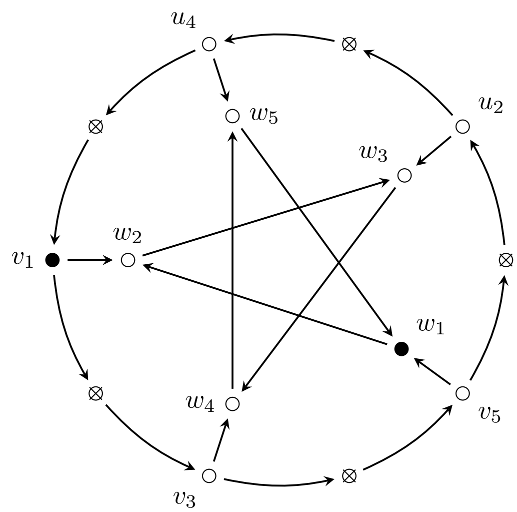

Figure 2.4: The digraph $\hat{O}_5$. A 4-kernel is formed by the black vertices and the sinks of the digraph.

depicted in Figure 2.5 for $k = 5$. As in previous figures, the pending tails of length $k-1$ have been substituted by a $\times$ over the corresponding vertex.

Now, consider the digraphs $\hat{D}_1$, $\hat{D}_2$ and k disjoint copies of the digraph S_k. We can assume that $V(H_1) = \{v_1, v_2, \ldots, v_k\}$ and $V(H_2) = \{u_1, u_2, \ldots, u_k\}$. First, connect each vertex $u_i \in H_2$ to one copy of S_k by adding the arc from u_i to the only vertex in S_k with in-degree zero. Next, add the arc from x_{k+1} to v_i. The resulting digraph is called $J(\hat{D}_1, \hat{D}_2)$. The case $k = 4$ of the construction is shown in Figure 2.6, where the vertices of the digraphs D_1 and D_2 were omitted or, in other words, only the vertices and arcs of the cycles H_1 and H_2 are shown.

Notice that, straightforwardly from its construction, the digraph $J(\hat{D}_1, \hat{D}_2)$ is cyclically k-partite and has circumference $2k$. It has other interesting properties that follow from the fact that the k-kernels of the digraphs $\hat{D}_1$ and $\hat{D}_2$ are determined by the vertices of the cycles H_1 and H_2 that they include. For example, as it is proven in the second of the following lemmas, it has a k-kernel.The first one, Lemma 2.1.6, is concerned with how we can choose the k-kernels of $\hat{D}_1$ and $\hat{D}_2$ in order to make them compatible and extend them to a k-kernel of $J(\hat{D}_1, \hat{D}_2)$. Both lemmas are worded with the notation used in the construction of $J(\hat{D}_1, \hat{D}_2)$.

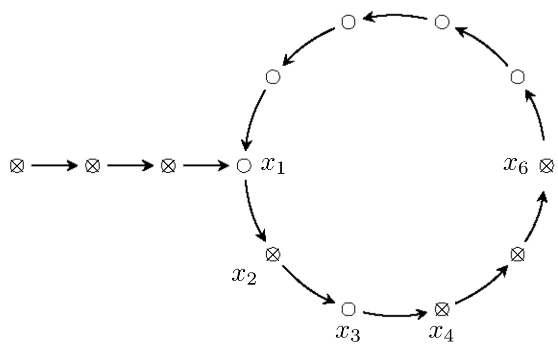

Figure 2.5: The digraph S_5

Lemma 2.1.6. *Let K be a k-kernel of $J(\hat{D}_1, \hat{D}_2)$. If $v_i \in K$, then the vertex u_i is not an element of K. Conversely, if $u_i \in K$, then the vertex v_i is not included in K.*

Proof. Suppose that the vertex v_i is in K and consider the copy of S_k attached to v_i. Then, the vertices $x_{k+1}, x_k, \ldots, x_3$ cannot be included in K due to the k-independence of K. The vertex x_2 is at distance $k-1$ from a sink, so it cannot be an element of K. Also, if x_1 is not included in K, then no other vertex in the digraph can $(k-1)$-absorb it (because all the vertices that could are $x_k, x_{k-1}, \ldots, x_2$). Then, x_1 must be included in K and, since $d(u_i, x_1) = k-1$, the vertex $u_i \notin K$.

On the other hand, if $u_i \in K$, then the vertex x_1 cannot be included in K. Also, the vertices $x_2, x_4, \ldots, x_{k-1}$ cannot be elements of K, since they are all at distance $k-1$ of a sink of the digraph. This shows that the only remaining vertex of the digraph that can $(k-1)$-absorb x_1 is x_3, so it must be included in K. Therefore, in order to preserve the k-independence of K, the vertex v_i cannot be included in K.

■

Lemma 2.1.7. *Let K_1 be a k-kernel for $\hat{D}_1$ containing the vertex v_i and K_2 a k-kernel for $\hat{D}_2$ containing the vertex u_j, where $i \neq j$. The set $K_1 \cup K_2$ can be extended in a unique way to a k-kernel of $J(\hat{D}_1, \hat{D}_2)$.*

Proof. Let us use S_v and S_u to denote the copies of S_k in the digraph $J(\hat{D}_1, \hat{D}_2)$ that are adjacent, respectively, to v_i and u_j. First, we must add to

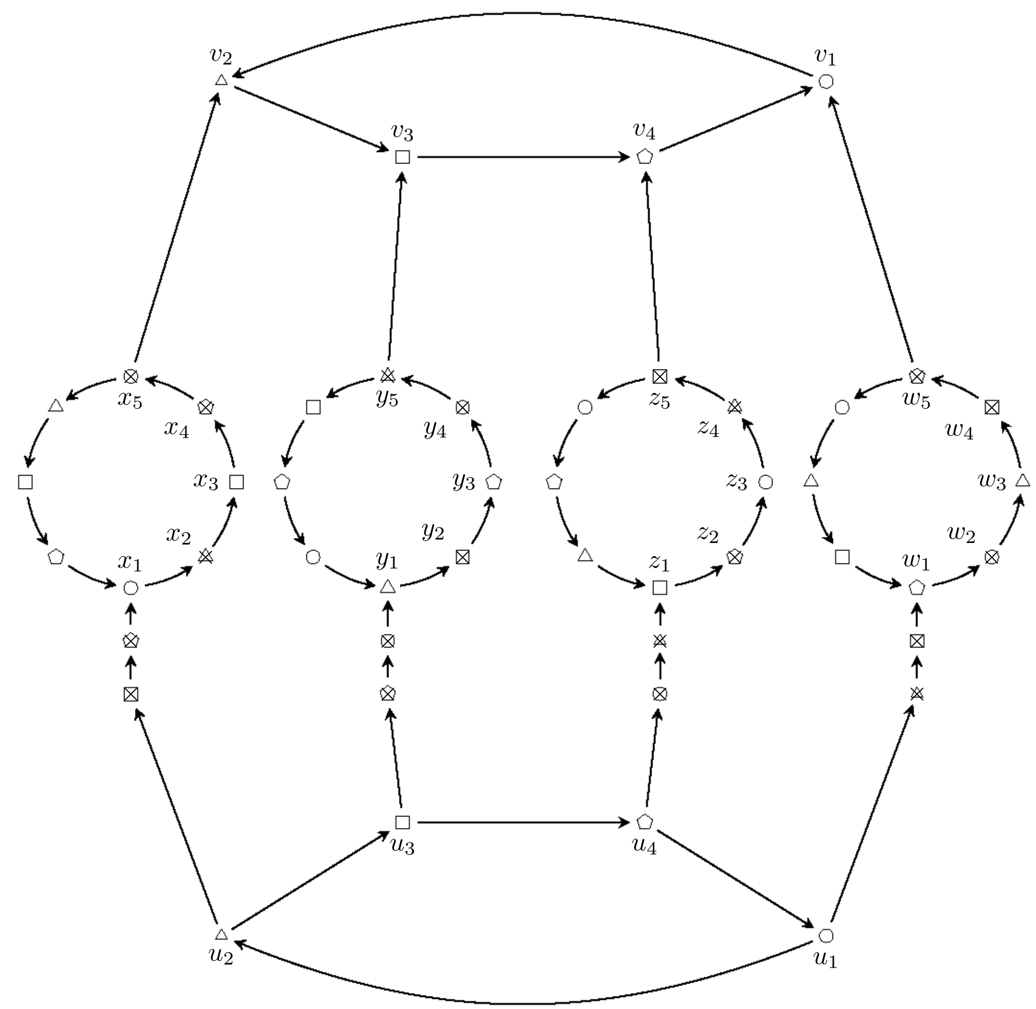

Figure 2.6: A representations of the digraph $J(\hat{D}_1, \hat{D}_2)$ when D_1 and D_2 are cyclycally 4-partite

.

$K_1 \cup K_2$ all the sinks in the copies of S_k. Since $v_i \in K_1 \cup K_2$, following the proof of Lemma 2.1.6, the vertex x_1 of S_v must be included in the extension of $K_1 \cup K_2$ to a k-kernel of the digraph. Notice that we have $(k-1)$-absorbed all the vertices in S_v by adding its sinks and x_1 to $K_1 \cup K_2$.

Now, we have to choose vertices from the directed cycles in the remaining copies of S_k in order to $(k-1)$-absorb the cycles. Since we are dealing with directed cycles of length $2k$ which, at most, have $k-1$ already $(k-1)$-absorbed vertices, we have to choose exactly two diametrically opposed vertices from the directed cycle in order to $(k-1)$-absorb it. The vertex x_{2k} of each directed cycle must be either included in the k-kernel or $(k-1)$-absorbed by another vertex. Since the vertex x_k is at distance $k-1$ from a sink of the digraph, its diametrically opposed vertex, namely x_{2k}, cannot be included in K. The only vertices that may be included in K and can $(k-1)$-absorb x_{2k} are x_1 and x_3, in view of the fact that the vertices $x_2, x_4, \ldots, x_{k-1}$ cannot be included in K. But including x_1 in K would imply that the vertex diametrically opposed to it, namely x_{k+1}, must also be included in K, which is impossible since there is a sink at distance $k-1$ from x_{k+1}. Therefore, we have $x_3 \in K$, implying that the other vertex in each directed cycle that is also in K is x_{k+3}.

Clearly, the set formed by $K_1 \cup K_2$ with all the remaining sinks in the copies of S_k, the vertex x_1 in S_v and the vertices x_3 and x_{k+3} in all the remaining copies of S_k are a $(k-1)$-absorbent set. It only remains to check the k-independence of the set, but due to the way the vertices were chosen, it suffices to notice that the vertex u_j is at distance at least k of any other vertex chosen. This last observation is easy to see, since the only vertices that might be close to u_j are x_3 and x_{k+3} in S_u, but $d(u_j, x_3) = k+1$ and $d(u_j, x_{k+3}) = 2k+1$ and there is no directed path from x_3 or x_{k+3} to the vertex u_j.

We have proven that K is both k-independent and $(k-1)$-absorbent, so it is a K kernel of $J(\hat{D}_1, \hat{D}_2)$. The construction shows that it is the unique k-kernel of the digraph.

■

Now, let G be a graph and k an integer greater or equal to 3. Considering F as $\hat{O}_k$ if k is odd and $\hat{E}_k$ if k is even, we construct a digraph $D_{G,k}$ of order $\mathcal{O}(|V| + |E|)$ as follows:

1. Let $\vec{G}$ be an acyclic orientation of G.
2. For every vertex $v \in V(\vec{G})$, construct a copy of F and label it F_v.

3. For every arc $(u, v) \in A(\vec{G})$, join F_u and F_v with $J(F_u, F_v)$.

The following results are independent of the choice of the acyclic orientation of G. In the wording of the following lemmas, G will be a graph and k an integer, with $k \geq 3$.

Lemma 2.1.8. *The digraph $D_{G,k}$ is cyclically k-partite and has circumference $2k$.*

Proof. Since we construct $D_{G,k}$ by choosing an acyclic orientation of G, the only cycles in $D_{G,k}$ are the ones in the copies of the digraph F (which have length at most $2k$), implying that $D_{G,k}$ has circumference $2k$. Clearly, the copies of F are cyclically k-partite and, for two adjacent vertices u and v, such partitions are compatible when F_u and F_v are joined by $J(F_u, F_v)$. By joining the compatible partite sets we get a k-partition of $D_{G,k}$. ∎

For the following lemma, C_v will denote the k-cycle of F_v, where $v \in V(\vec{G})$.

Lemma 2.1.9. *If $D_{G,k}$ has a k-kernel K, then exactly one vertex from C_v must be included in K for every $v \in V(\vec{G})$.*

Proof. Since $\vec{G}$ is acyclic, every vertex $x \in V(G)$ is either a sink of $\vec{G}$ or has a directed path starting at x and ending in a sink of $\vec{G}$. The fact that $D_{G,k}$ has a k-kernel implies that $K \cap V(F_v)$ is a k-kernel of F_v and $|K \cap C_v| = 1$ for every $v \in V(\vec{G})$ that is a sink.

Let $S \subseteq V(G)$ be the set of sinks of $\vec{G}$ and $u \in N^-_{\vec{G}}(S)$. We know that $(u, v) \in A(\vec{G})$ for some sink v. The proof of Lemma 2.1.6 shows that the vertex in $K \cap C_v$ forces other vertices in $J(F_u, F_v)$ to also be in K and that $V(F_u) \setminus V(C_u)$ can not be k-absorbed by them. Hence, $K \cap V(F_u)$ is a k-kernel of F_u and Corollaries 2.1.4 and 2.1.5 imply that a vertex from C_u must be included in $K \cap F_u$. Lemma 2.1.7 guarantees that $K \cap J(F_u, F_v)$ is uniquely determined by $K \cap (V(C_u) \cup V(C_v))$. By considering now $N^{-m}_{\vec{G}}(S)$ and applying induction over m, we conclude the proof. ∎

Lemma 2.1.10. *Let K be a k-kernel for $D_{G,k}$ and $C_v = \{v_1, v_2, \ldots, v_k\}$ the cycle of length k in F_v. Then the function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(v) = i$, where v_i is the vertex of C_v included in K, is well defined and is a k-coloring of G.*

Proof. The first statement follows from Lemma 2.1.9 and the second from Lemma 2.1.6. ■

Lemma 2.1.11. *If $f : V(G) \to \{1, 2, \ldots, k\}$ is a k-coloring of G, then the set $\{v_{f(v)}\}_{v \in V(G)}$ consisting of exactly one vertex of each cycle of length k in each F_v can be extended in a unique way to a k-kernel of $D_{G,k}$.*

Proof. From Corollary 2.1.4 and Corollary 2.1.5, it follows that there is a unique k-kernel for F_v containing the vertex $v_{f(v)}$ for every $v \in V(G)$, which we will denote with K_v. If there is a vertex $u \in V(G)$ such that $uv \in E(G)$ and $(u, v) \in A(\overrightarrow{G})$, then clearly $f(u) \neq f(v)$. Also, due to Lemma 2.1.7, there is a unique way to extend $K_u \cup K_v$ to a kernel of $J(F_u, F_v)$. By joining all such extensions for every arc in $\overrightarrow{G}$ we obtain a k-kernel of $D_{G,k}$ that is uniquely determined by the set $\{v_{f(v)}\}_{v \in V(G)}$. ■

A direct consequence of Lemma 2.1.10 and Lemma 2.1.11 is the following corolary.

Corollary 2.1.12. *There is a bijective correspondence between the k-colorings of G and the k-kernels of $D_{G,k}$.*

We are now ready to prove the main result of this section.

Theorem 2.1.13. *The k-kernel problem for cyclically k-partite digraphs is NP-complete, even when restricted to cyclically k-partite digraphs with circumference $2k$.*

Proof. A polynomial reduction of the k-coloring problem to the k-kernel problem in cyclically k-partite digraphs with circumference $2k$ is given by Corolary 2.1.12. Given K, a subset of $V(D)$ from a given digraph D, it can be verified if K is a k-kernel of D in polynomial time, proving our claim. ■

The following theorem, which can be found in [3], gives us a relation between the $(k-1)$-closure of a digraph D and the existence of a k-kernel of D:

Theorem 2.1.14. *Let D be a digraph and $k \geq 2$ an integer. Then $K \subseteq V(D)$ is a k-kernel of D if and only if K is a kernel of $\mathcal{C}^{k-1}(D)$.*

Finally, as a direct consequence of Theorem 2.1.13 and Theorem 2.1.14, we obtain the following corolary:

Corollary 2.1.15. *The kernel problem restricted to the class of k-colorable digraphs is NP-complete, where $k \geq 3$.*

Proof. Let G be a graph. Corollary 2.1.4 implies that G has a k-coloring if and only if $D_{G,k}$ has a k-kernel. Since $D_{G,k}$ is cyclically k-partite, the $(k-1)$-closure of D is k-colorable. Also, from Theorem 2.1.14, we have that the $(k-1)$-closure of $D_{G,k}$ has a kernel if and only if $D_{G,k}$ has a k-kernel if and only if G has a k-coloring. Therefore, there is a polynomial reduction of the k-coloring problem to the kernel problem restricted to the class of k-colorable digraphs. ■

2.2. Sufficient conditions

Let us denote by $\mathcal{Z}$ the family of digraphs that admit a cyclically k-partition of its vertex set such that there are at least $k-2$ elements of the partition without sinks, where k is a positive integer greater than or equal to 3.

Theorem 2.2.1. *The digraphs in $\mathcal{Z}$ have a k-kernel.*

Proof. Let $V_1, V_2, \ldots, V_k$ be the sets in which V is cyclically k-partitioned. If D has no sinks, then any of the V_i is a k-kernel. If only one of the V_i has sinks, then it is a k-kernel of the digraph. Therefore, we assume that V_1 and V_s are the elements of the partition that have sinks, where $2 \leq s \leq k$.

We recursively define a family of subdigraphs of D as follows.

- $D_0 = D$.
- $X_0 = \{v \in V(D) : d^+_D(v) = 0\}$.
- $D_{i+1} = D[V(D_i) \setminus N^{-(k-1)}_D[X_i]]$.
- $X_{i+1} = \left\{v \in V(D_{i+1}) : N^{+(k-1)}_D(v) \cap V(D_i) = \varnothing\right\}$.

First, we notice that $X = \bigcup_{i \in \mathbb{N}} X_i$ is k-independent in D. If $x, y \in X$, then there exist positive integers n, m such that $x \in X_n$ and $y \in X_m$. Without loss of generality, we can assume that $n \leq m$. If $n = m$, then $N^{+(k-1)}_D(x) \cap V(D) = \varnothing$ and $N^{+(k-1)}_D(y) \cap V(D) = \varnothing$, implying that $k \leq d(x, y)$ and $k \leq d(y, x)$.

If $n < m$, then we have $k \leq d(x,y)$, since $N_D^{+(k-1)}(x) \cap V(D) = \varnothing$. Now, due to the construction of the digraph D_{n+1}, we know that every vertex $z \in V(D)$ such $d(z,x) \leq k-1$ is not a vertex of D_{n+1}. Therefore, since $y \in V(D_m) \subseteq V(D_{n+1})$, we have that $d(y,x) \geq k$. This proves that X is k-independent.

We will now show that $X \subseteq V_1 \cup V_s$. We proceed by contradiction. Since $X_0 \subseteq V_1 \cup V_s$, there must be a minimum positive integer m such that $\bigcup_{i \leq m-1} X_i \subseteq V_1 \cup V_s$ but $\bigcup_{i \leq m} X_i \cap V_j \neq \varnothing$ for some $2 \leq j \leq k$, with $s \neq j$. If $x \in \bigcup_{i \leq m} X_i \cap V_j$ for some $2 \leq j \leq k$ with $j \neq s$, then $N_D^+(x) \neq \varnothing$ and $N_D^+(x) \cap V(D_{m-1}) = \varnothing$. Let $y \in N_D^+(x)$. In order to erase y, the set $\bigcup_{i \leq m-1} X_i$ must contain a vertex z such that $d(y,z) \leq k-1$. Since x itself is not $(k-1)$-absorbed by z, we know that $d(x,z) > k-1$. It follows from this observations that $d(y,z) = k-1$, implying that z and x are in the same element of the cyclic k-partition of D, which means $z \in V_j \cap \bigcup_{i \leq m-1} X_i$, a contradiction due to the choice of m.

The way we constructed the set X shows that the set X must be included in every k-kernel of the digraph D (if one exists). Let $D' = \bigcap_{i \in \mathbb{N}} D_i$. If $V(D') = \varnothing$, then X is a k-kernel of the digraph D.

Let us suppose that $V(D') \neq \varnothing$ and let $V_i' = V_i \cap V(D')$. Clearly, if $Y \subseteq V(D')$ is a k-independent set in D, then $X \cup Y$ is also k-independent in D. Notice also that $d_{D'}^+(v) \neq 0$ for every $v \in V_i'$, with $2 \leq i \leq k$ and $i \neq s$ (otherwise, there would be a vertex of X not included in $V_1 \cup V_s$).

We must consider two separate cases: when $s = 2$ and $s \neq 2$. First, we assume that $s = 2$. A direct consequence of this is that $N_{D'}^{+(k-1)}(v) \cap V_1' \neq \varnothing$, hence $N_D^{+(k-1)}(v) \cap V_1 \neq \varnothing$. If we take $x \in V_2'$, we know that $N_D^{+(k-1)} \cap V(D') \neq \varnothing$. Let y be a vertex of $V(D')$ that is also in $N_D^{+(k-1)}(x)$. If $y \in \bigcup_{i=3}^k V_i'$, then there is a vertex z such that $d(x,z) \leq k-1$. On the other hand, if $y \in V_1'$, then $d(x,y) = k-1$. This shows that every vertex of $V(D')$ is either in V_1' or is $(k-1)$-absorbed (in D) by V_1'.

The set $K = X \cup V_1'$ is clearly k-independent. Any vertex in $V(D) \setminus K$ is either $(k-1)$-abosorbed by a vertex of X or is in $V(D')$. In the latter case, it is either a vertex of V_1' or is $(k-1)$-absorbed by V_1'. This proves that K is a k-kernel of D.

Suppose now that $s \neq 2$. Now, we will denote the sets V_1', $\bigcup_{i=2}^{s-1} V_i'$, V_s' and $\bigcup_{i=s+1}^k V_i'$ by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$, respectively. Now, for every $S \subseteq V(D')$, consider the function $\Phi_S : \mathcal{A} \cup \mathcal{C} \to \{0, \alpha, \beta, \gamma\}$ such that:

1. If $v \in \mathcal{A}$

- $\Phi_S(v) = \alpha$ if $S \cap N_D^{+(k-1)}(v) \cap \mathcal{B} \neq \varnothing$.
- $\Phi_S(v) = \beta$ if $S \cap N_D^{+(k-1)}(v) \cap \mathcal{B} = \varnothing$ and $S \cap N_D^{+(k-1)}(v) \cap \mathcal{C} \neq \varnothing$.
- $\Phi_S(v) = \gamma$ if $S \cap N_D^{+(k-1)}(v) \cap \mathcal{B} = \varnothing$, $S \cap N_D^{+(k-1)}(v) \cap \mathcal{C} = \varnothing$ and $S \cap N_D^{+(k-1)}(v) \cap \mathcal{D} \neq \varnothing$.

2. If $v \in \mathcal{C}$

- $\Phi_S(v) = \alpha$ if $S \cap N_D^{+(k-1)}(v) \cap \mathcal{D} \neq \varnothing$.
- $\Phi_S(v) = \beta$ if $S \cap N_D^{+(k-1)}(v) \cap \mathcal{D} = \varnothing$ and $S \cap N_D^{+(k-1)}(v) \cap \mathcal{A} \neq \varnothing$.
- $\Phi_S(v) = \gamma$ if $S \cap N_D^{+(k-1)}(v) \cap \mathcal{D} = \varnothing$, $S \cap N_D^{+(k-1)}(v) \cap \mathcal{A} = \varnothing$ and $S \cap N_D^{+(k-1)}(v) \cap \mathcal{B} \neq \varnothing$.

3. $\Phi_S(v) = 0$ otherwise.

Notice that every vertex $x \in V(D')$ satisfies $N_D^{+(k-1)}(x) \cap V(D') \neq \varnothing$, which means that $\Phi_{V(D')}(v) \neq 0$ for every $v \in V(D')$. If $\Phi^{-1}[\{\gamma\}] \cap \mathcal{A} = \varnothing$, then every vertex $v \in V(D')$ is either a vertex of $\mathcal{C}$ or $N_D^{+(k-1)}(v) \cap \mathcal{C} \neq \varnothing$. This implies that every vertex of D' is k-absorbed by C in D, which means $X \cup C$ is a k-kernel of D. The case $\Phi^{-1}[\{\gamma\}] \cap \mathcal{C} = \varnothing$ is analogous.

Let us assume that $\Phi^{-1}[\{\gamma\}] \cap \mathcal{A} \neq \varnothing$ and $\Phi^{-1}[\{\gamma\}] \cap \mathcal{C} \neq \varnothing$. We define a family of sets recursively as follows:

- $S_0 = V(D')$
- $\Gamma_0 = \Phi_{S_0}[\{\gamma\}]$
- $S_{i+1} = S_i \setminus N_D^{-(k-1)}[\Gamma_i]$
- $\Gamma_{i+1} = \Phi_{S_{i+1}}^{-1}[\{0, \gamma\}]$ if

$$\Phi_{S_{i+1}}^{-1}[\{0, \gamma\}] \cap \mathcal{A} \cap S_{i+1} \neq \varnothing \neq \Phi_{S_{i+1}}^{-1}[\{0, \gamma\}] \cap \mathcal{C} \cap S_{i+1}.$$

  Otherwise, $\Gamma_{i+1} = \varnothing$.

Let $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$. In order to prove that $X \cup \Gamma$ is k-independent in D, it suffices to show that the set Γ is k-independent in D. If $x, y \in \Gamma$, then we have $n, m \in \mathbb{N}$ such that $x \in \Gamma_n$ and $y \in \Gamma_m$ and are minimum with that property. Without loss of generality, we can suppose $n \leq m$ and we can also assume that $x \in \mathcal{A}$ and $y \in \mathcal{C}$.

We know that $\Phi_{S_n}(x) \in \{0, \gamma\}$, because $x \in \Gamma_n$. Whether $\Phi_{S_n}(x) = 0$ or $\Phi_{S_n}(x) = \gamma$, we know $S_n \cap N_D^{+(k-1)}(x) \cap \mathcal{C} = \varnothing$, hence $d(x, y) \geq k$. If $n < m$, then $S_m \subseteq S_{n+1}$, which means $d(y, x) \geq k$. If $n = m$, then $\Phi_{S_n}(y) \in \{0, \gamma\}$, so $S_n \cap N_D^{+(k-1)}(y) \cap \mathcal{A} = \varnothing$ and hence $d(y, x) \geq k$. This shows that the set Γ is k-independent.

Since the vertex set of the digraph D is finite, there exists $n \in \mathbb{N}$ such that $\Gamma_m = \varnothing$ for every $m \geq n$. If $S_n = \varnothing$, then the set $K = X \cup \Gamma$ is a k-kernel. To prove this last assertion, take $v \in V(D) \setminus K$. If $v \in V(D) \setminus V(D')$, it is $(k-1)$-absorbed by vertex in X. If $v \in V(D')$, there exists $i \in \mathbb{N}$ such that $v \in S_i$ but $v \notin S_{i+1}$. Since $v \notin K$, then $v \in N_D^{-(k-1)}[\Gamma_i]$, so it is $(k-1)$-absorbed by K, which shows that K is a k-kernel of D.

It only remains to check what happens when $S_n \neq \varnothing$. By considering the smallest such n, we have that $\Gamma_n = \varnothing$ implies

$$\Phi_{S_{i+1}}^{-1}[\{0, \gamma\}] \cap \mathcal{A} \cap S_{i+1} = \varnothing \quad \text{or} \quad \Phi_{S_{i+1}}^{-1}[\{0, \gamma\}] \cap \mathcal{C} \cap S_{i+1} = \varnothing.$$

In the first case, the set $K = X \cup \Gamma \cup (S_n \cap \mathcal{C})$ is a k-kernel of D. It suffices to see that every vertex $v \in S_n$ is $(k-1)$-absorbed by K. If $v \notin \Gamma \cup (S_n \cap \mathcal{C})$, then $v \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$.

- If $v \in S_n \cap \mathcal{B}$. It is easy to see that every vertex v in $\mathcal{B} \cap S_n$ satisfies $d^+_{D'[S_n]}(v) \neq 0$. It follows straightforwardly that there exists a vertex $x \in S_n \cap \mathcal{C}$ such that $d_D(v, x) \leq k - 1$.

- If $v \in S_n \cap \mathcal{A}$. Given the fact that $\Phi_{S_n}^{-1}[\{0, \gamma\}] \cap \mathcal{A} \cap S_n = \varnothing$, we have $\Phi_{S_n}(v) \in \{\alpha, \beta\}$. If $\Phi_{S_n}(v) = \beta$, there exists a vertex $x \in S_n \cap \mathcal{C}$ such that $d_D(v, x) \leq k - 1$. If $\Phi_{S_n}(v) = \alpha$, then there is a vertex $y \in S_n \cap \mathcal{B}$ such that $d_D(v, y) \leq k - 1$. The vertex y is in $S_n \cap \mathcal{B}$, hence there is a vertex $x \in S_n \cap \mathcal{C}$ such that $d_D(y, x) \leq k - 1$. Since the digraph D is cyclically k-partite, we know that $d_D(v, x) \leq k - 1$.

- If $v \in S_n \cap \mathcal{D}$. It is also clear that every vertex v in $\mathcal{B} \cap S_n$ satisfies $d^+_{D'[S_n]}(v) \neq 0$. For this reason, there is a vertex $y \in S_n \cap \mathcal{A}$ with $d_D(v, y) \leq k - 1$. The fact that $y \in S_n \cap \mathcal{A}$ and that D is cyclically k-partite guarantee the existence of a vertex $x \in \mathcal{C}$ such that $d_D(y, x) \leq k - 1$ and $d_D(v, x) \leq k - 1$.

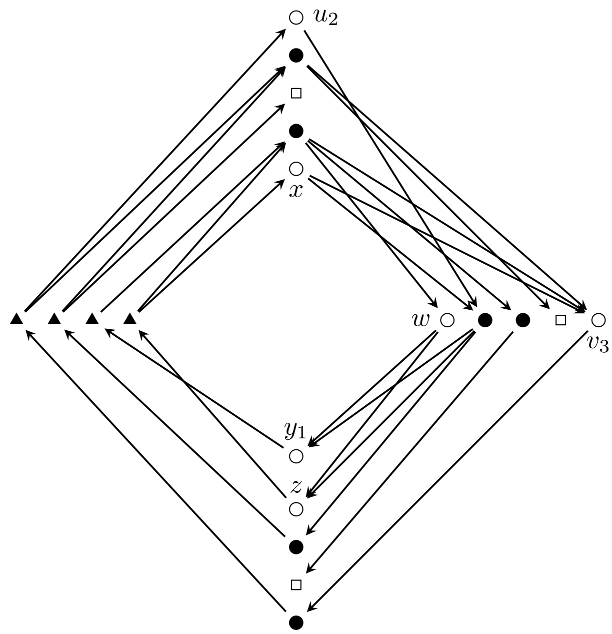

Figure 2.7: The digraph B_4. The vertices represented with squares are the sinks of the digraph, while the vertices 3-absorbed by them are filled with black.

This shows that $K = X \cup \Gamma \cup (S_n \cap \mathcal{C})$ is a k-kernel of D. The case $\Phi^{-1}_{S_{i+1}}[\{0, \gamma\}] \cap \mathcal{C} \cap S_{i+1} = \varnothing$ is analogous. This proves that every digraph in $\mathcal{Z}$ has a k-kernel.

■

In order to see that the number of elements of the cyclic k-partition of D without sinks can not be improved, consider the digraph B_4 depicted in Figure 2.7.

Notice that there is only one element in the cyclic 4-partition of B_4 without sinks.

Lemma 2.2.2. *The digraph B_4 does not have a 4-kernel.*

Proof. First, let us see that adding the unabsorbed vertices in one class of the partition to the sinks of the digraph does not give us a 4-kernel. Let S be the set of sinks of B_4.

- The set $S \cup \{x, u_2\}$ is not a 4-kernel, because the vertex y_1 is not 3-absorbed. Since the only vertex of B_4 not 3-absorbed by $S \cup \{x, u_2\}$ is y_1 and the set $S \cup \{x, u_2, y_1\}$ is not 4-independent, the set $S \cup \{x, u_2\}$ cannot be extended to a 4-kernel of B_4.

- The set $S \cup \{w, v_3\}$ is not a 4-kernel, because the vertex u_2 is not 3-absorbed. Since the only vertex of B_4 not 3-absorbed by $S \cup \{w, v_3\}$ is $u_2$and the set $S \cup \{w, v_3, u_2\}$ is not 4-independent, the set $S \cup \{w, v_3\}$ cannot be extended to a 4-kernel of B_4.

- The set $S \cup \{z, y_1\}$ is not a 4-kernel, because the vertex v_3 is not 3-absorbed. Since the only vertex of B_4 not 3-absorbed by $S \cup \{z, y_1\}$ is v_3 and the set $S \cup \{z, y_1, v_3\}$ is not 4-independent, the set $S \cup \{z, v_1\}$ cannot be extended to a 4-kernel of B_4.

Clearly S is contained in any 4-kernel of B_4, so it suffices to show that S cannot be extended to a 4-kernel of B_4. Suppose that K is a 4-kernel of B_4. The 4-independence ensures that $R = K \setminus S \subseteq \{x, u_2, y_1, z, v_3, w\}$.

The only vertices in R that can 3-absorb the vertex x are y_1, z and v_3. This means that any 4-kernel of B_4 must contain at least one of these vertices. Thanks to the previous observations and that $d(z, x) = d(y_1, x) = 2$, $d(x, v_3) = 1$ and $d(z, v_3) = d(y_1, v_3) = 3$, we have that only one vertex in $\{x, y_1, z, v_3\}$ can be included in a 4-kernel of B_4.

If $x \in K$, then the 4-independence of K implies $\{z, w, y_1, v_3\} \cap K = \varnothing$. Since $S \cup \{x, u_2\}$ cannot be extended to a 4-kernel of B_4, we have that $u_2 \notin K$. This means that $K \subseteq S \cup \{x\}$, which clearly contains no 4-kernel of B_4.

If $y_1 \in K$ or $z \in K$, then the vertex v_3 is neither in K (due to the 4-independence of K) nor is it 3-absorbed by K. If $v_3 \in K$, then $u_2 \notin K$ because $d(v_3, u2) = 3$. Since the only remaining vertex that can absorb u_2 is w, we have that $w \in K$, contradicting the fact that the set $S \cup \{w, v_3\}$ cannot be extended to a 4-kernel of B_4.

In any case, we have a contradiction, which means the digraph B_4 does not have a 4-kernel.

■

Let N be the element of the cyclic 4-partition of B_4 without sinks and k an integer greater or equal than 5. The key observation about the digraph B_4 that will allow us to build counterexamples for every $k \geq 5$ is this: Every vertex in N is 3-absorbed by one of the sinks of B_4. Label the vertices in N with $\{a, b, c, d\}$. For every vertex $s \in N$ consider a directed path T_s with vertex set $V(T_x) = \{s_1, s_2, \ldots, s_{k-4}, s_{k-3}\}$ and arc set $A(T_s) = \{(s_i, s_{i+1} : 1 \leq i \leq k-4\}$. Also, consider the sets

- $A_1 = \{(z, y) \in A(B_4) : z, y \notin N\}$.
- $A_2 = \{(z, s_1) : z \in V(B_4) \setminus N, s \in N \text{ and } (z, s) \in A(B_4)\}$.
- $A_3 = \{(s_{k-3}, y) : y \in V(B_4) \setminus N, s \in N \text{ and } (s, y) \in A(B_4)\}$.

The digraph B_k has vertex set $V(B_k) = [V(B_4) \setminus N] \cup [\bigcup_{s \in N} V(T_s)]$ and arc set $A(B_k) = A_1 \cup A_2 \cup A_3 \cup [\bigcup_{s \in N} A(T_s)]$.

Since the vertices in N are 3-absorbed by the sinks of B_4, the vertices of the paths T_s with $s \in N$ are $(k-1)$-absorbed by the sinks of B_k. If S_k is the set of sinks of B_k, any k-kernel of the digraph B_k would be contained in $S_k \cup \{x, u_2, y_1, z, v_3, w\}$. Hence, it is easy to see that there is a bijection between the 4-kernels of B_4 and the k-kernels of B_k, which means B_k has no k-kernel. This examples show that the number of elements of the cyclic k-partition of D without sinks can not be improved. In Figure 2.8 we can see the digraph B_6, where only the vertices in one of the trajectories T_s have been labeled.

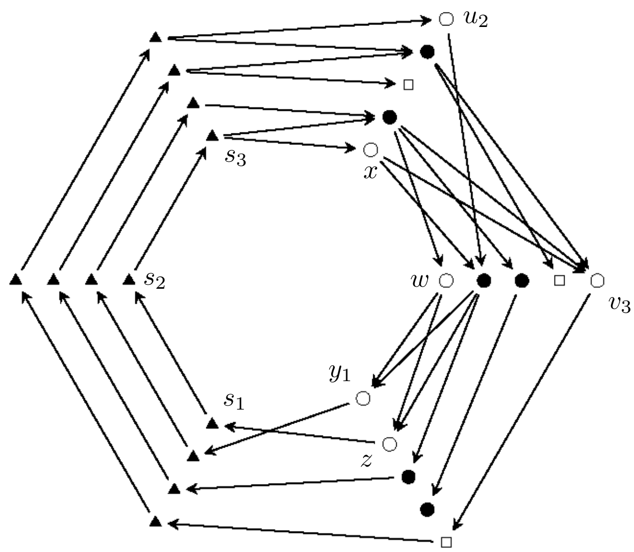

Figure 2.8: The digraph B_6. The vertices filled with black are the vertices 5-absorbed by the sinks of the digraph.

Chapter 3

A Generalization of Duchet.

3.1. 3-kernels

It is convenient now to remember Theorem 1.2.1, Duchet's result, which states that a sufficient condition for the existence of a kernel in a digraph D is that every cycle in D has a symmetric arc.

Here we present a generalization of Duchet for 3-kernels. The idea is to prove that the 2-closure of a digraph whose cycles have a at least certain proportion of symmetric arcs satisfies the hypothesis of Theorem 1.2.1. The important part is to notice that a cycle of the 2-closure may not be a cycle of D, but the fact that there is a cycle in the $\mathcal{C}^2(D)$ gives us some information about the structure of D. First, we prove in general a result for a particular type of configuration of arcs in D.

Lemma 3.1.1. *Let D be a digraph and k a positive integer such that every cycle C of D has at least $\frac{k-2}{k-1}\ell(C)+1$ symmetric arcs and B be a cycle of $H=\mathcal{C}^{k-1}(D)$. If the paths of D that give rise to the arcs of $A(C)\setminus A(D)$ are mutually internally disjoint, then B has a symmetric arc.*

Proof. Let $B=(v_1,v_2,\ldots,v_n,v_1)$ be a cycle of $\mathcal{C}^{k-1}(D)$ that satisfies the hypothesis. For $1\leq i\leq n-1$, let T_i be the v_iv_{i+1}-path in D that originates the arc (v_i,v_{i+1}) in H. Similarly, we will use T_n to denote v_nv_1-path in D that gives rise to the arc (v_n,v_1). Let $\mathcal{T}=\{T_i : 1\leq 1i]le1n\}$. Also, for every $j\in\{2,3,\ldots,k-1\}$, we will denote with m_j to the number elements in $\mathcal{T}$ of length j.

It is clear that by joining the paths in $\mathcal{T}$ with the arcs in $A(C) \cap A(D)$ in the natural way we get a cycle of length

$$n - \left(\sum_{j=2}^{k-1} m_j \right) + \sum_{j=2}^{k-1} j m_j = n + \sum_{j=2}^{k-1} (j-1) m_j.$$

Also, since the length of C is n, we have that

$$m_{k-1} \leq n, m_{k-1} + m_{k-2} \leq n, \ldots, \sum_{j=2}^{k-1} m_j \leq n,$$

so the addition of these inequalities yields

$$\sum_{j=2}^{k-1} (j-1) m_j \leq (k-2) n.$$

By adding $\sum_{j=2}^{k-1} (k-2)(j-1) m_j$ on both sides of the inequality, we have

$$\sum_{j=2}^{k-1} (j-1) m_j + \sum_{j=2}^{k-1} (k-2)(j-1) m_j \quad \leq \quad (k-2) n + \sum_{j=2}^{k-1} (k-2)(j-1) m_j$$

Performing algebraic manipulations on both sides we get the following

$$\begin{aligned} \sum_{j=2}^{k-1} (k-1)(j-1) m_j &\leq (k-2) \left[n + \sum_{j=2}^{k-1} (j-1) m_j \right] \\ (k-1) \sum_{j=2}^{k-1} (j-1) m_j &\leq (k-2) \left[n + \sum_{j=2}^{k-1} (j-1) m_j \right] \\ \sum_{j=2}^{k-1} (j-1) m_j &\leq \frac{(k-2)}{(k-1)} \left[n + \sum_{j=2}^{k-1} (j-1) m_j \right] \end{aligned}$$

The last inequality and the hypothesis about the number of symmetric arcs in the cycles of D imply that at least one arc of B is symmetric.

■

We will now prove a generalization of Theorem 1.2.1 for 3-kernels.

Lemma 3.1.2. *Let D be a digraph. If every directed cycle B in D has at least $\frac{1}{2}\ell(B)+1$ symmetric arcs then, every C_3 in $H = \mathcal{C}^2(D)$ has at least one symmetric arc.*

Proof. Let $C = (v_0, v_1, v_2, v_0)$ be a 3-cycle of H. If every arc of C is also an arc of D, then C is symmetric. If only two of the arc in C are arc of D, we can assume without loss of generality that $(v_0, v_1), (v_1, v_2) \in A(D)$. Since $(v_2, v_0) \in A(H)$, there exists $w \in V(D)$ such that (v_2, w) and (w, v_0) are arcs of D. If $w = v_1$, we have that (v_1, v_2) is a symmetric arc in D and, therefore, in H. On the other hand, if $w \neq v_1$, then (w, v_0, v_1, v_2, w) is a 4-cycle in D, and from the main hypothesis we derive that $(v_0 v_1)$, (v_1, v_2) or both are symmetric in H.

Suppose now that only (v_0, v_1) is an arc of D. Then there are vertices $v, w \in V(D)$ such that $(v_1, v), (v, v_2), (v_2, w)$ and (w, v_0) are arcs of D. If $v = w$, then (v_0, v_1, v, v_0) is a triangle of D and the arc (v_0, v_1) is symmetric in H. If $v \neq w$, then $(v_0, v_1, v, v_2, w, v_0)$ is a 5-cycle in D, which at least has four symmetric arcs. If one of those is (v_0, v_1), we are done. Otherwise, the pairs $(v_1, v), (v, v_2)$ and $(v_2, w), (w, v_0)$ are symmetric. The first case implies that (v_1, v_2) is symmetric and the latter that (v_2, v_0) is symmetric.

Finally, let us consider the case where none of the arcs in C is an arc of D. Let u, v, w be vertices of D such that $(v_0, u), (u, v_1), (v_1, v), (v, v_2), (v_2, w)$ and (w, v_0) are arcs of D. If $u = v_2$, it is easy to see that the both (v_1, v_2) and (v_2, v_0) are symmetric arcs in H. The cases $v = v_0$ and $w = v_1$ are analogous.

Assume that the vertices u, v and w are different from v_0, v_1 and v_2. If $u = v = w$, then every arc in C is symmetric. If $u = v$ but $u \neq w$, then (v_0, v, v_2, w, v_0) is a 4-cycle and as such it has at least three symmetric arcs, implying that (v_0, v), (v, v_2) or both are symmetric arcs in H. The cases $v \neq u = w$ and $v = w \neq u$ are analogous. If u, v and w are all different, then Lemma 3.1.1 guarantees the existence of a symmetric arc in C.

■

Theorem 3.1.3. *Let D be a digraph. If every directed cycle B in D has at least $\frac{1}{2}\ell(B) + 1$ symmetric arcs, then every cycle in $H = \mathcal{C}^2(D)$ has a symmetric arc.*

Proof. Let C be a cycle in H. We proceed by induction on the length of C. The case when C has length three is covered by Lemma 3.1.2.

Suppose then that $C = (v_1, v_2, \ldots, v_n, v_0)$ is an n-cycle in H. For every arc $(v_i, v_{i+1}) \in A(C) \setminus A(D)$ there is a vertex $v_{i(i+1)} \in V(D)$ such that (v_i, v_{ij}, v_j) is a directed path in D. If $v_{i(i+1)} \neq v_{j(j+1)}$ for every $i \neq j$, and $v_{i(i+1)} \neq v_j$ for every $1 \leq i, j \leq n$, then Lemma 3.1.1 gives us the existence of a symmetric arc of C.

Thus, we can assume that $v_{i(i+1)} = v_j$ for some $1 \leq i, j \leq n$ or that $v_{i(i+1)} = v_{j(j+1)}$ for some $i \neq j$. If $v_{i(i+1)} = v_{(i+1)(i+2)}$ for some $1 \leq i \leq n$, then $(v_i, v_{i(i+1)}, v_{i+2})$ is a path of length two in D, implying that $(v_i, v_{i+2}) \in A(H)$. We can assume without loss of generality that $i = 1$, hence $C' = (v_1, v_3) \cup v_3 C v_1$ is a cycle of length $n - 1$ in H, just as depicted in Figure 3.1. The induction hypothesis implies that C' has at least one symmetric arc. If C' has a symmetric arc other than (v_1, v_3), then C has a symmetric arc. Thus, we can assume that (v_1, v_3) is a symmetric arc in H. We have two possibilities. If $(v_3, v_1) \in A(D)$, then (v_1, v_{12}, v_3, v_1) is a triangle in D. The main hypothesis guarantees that (v_1, v_{12}) or (v_{12}, v_3) is symmetric, implying that (v_1, v_2) or (v_2, v_3) is symmetric. If $(v_3, v_1) \notin A(D)$, there is a vertex $w \in V(D)$ such that (v_3, w, v_1) is a path in D. If $w = v_2$ or $w = v_{12}$, then (v_1, v_2) and (v_2, v_3) are symmetric. Otherwise, $(v_1, v_{12}, v_3, w, v_1)$ is a 4-cycle in D and has at least three symmetric arcs. Hence, (v_1, v_{12}) or (v_{12}, v_3) is symmetric and, therefore, (v_1, v_2) or (v_2, v_3) is a symmetric arc of C.

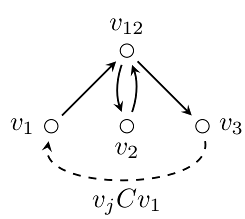

Figure 3.1: The case $v_{12} = v_{23}$.

If $v_{i(i+1)} = v_j$ for some $1 \leq i, j \leq n$, then we can suppose that $j \notin \{i-1, i+2\}$, otherwise (v_{i-1}, v_i) or (v_{i+1}, v_{i+2}) would be a symmetric arc of C. Now, let $1 \leq i \neq j \leq n$ be such that $v_{i(i+1)} = v_{j(j+1)}$ or $v_{i(i+1)} = v_j$ and $|j - i|$ is minimum with this property. We can assume without loss of generality that $i = 1$.

If $v_{12} = v_{j(j+1)}$ (see Figure 3.2), then we have already observed that $j = 2$ implies the existence of a symmetric arc in H, so $j \geq 3$. Let P be the walk obtained from v_2Cv_j by replacing every arc $(v_i, v_{i(i+1)}) \in A(C) \setminus A(D)$ with the path $(v_i, v_{i(i+1)}, v_{i+1})$ of D. Since we chose $|j-i|$ minimum, P is a path in D and $C' = P \cup (v_j, v_{12}, v_2)$ is a cycle in D of length $k+j$, where $k = |A(v_2Cv_j) \setminus A(D)|$. From the main hypothesis we derive that C' has at least $\frac{k+j}{2}+1$ symmetric arcs in D. If there is an arc $(v_i, v_{i+1}) \in A(P) \cap A(C)$ that is symmetric in D, then we have found a symmetric arc of C. Otherwise, we have $\frac{k+j}{2}+1$ symmetric arcs in the remaining $k+1$ pairs of arcs of C'. But, $k \leq j-2$ and hence $k \leq \frac{k+j-2}{2}$. We obtain that $k+1 \leq \frac{k+j}{2}$. Hence, the Pidgeonhole Principle implies that either a pair of arcs $(v_i, v_{i(i+1)})$ and $(v_{i(i+1)}, v_{(i+1)})$ of C' are symmetric in D or the arcs (v_j, v_{12}) and (v_{12}, v_2) are symmetric in D. In the former case, the arc (v_i, v_{i+1}) is a symmetric arc of C.

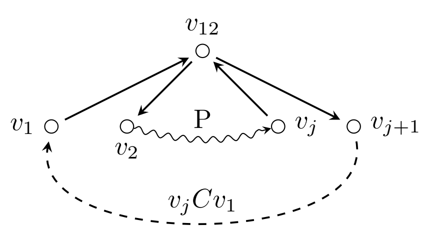

Figure 3.2: The case $v_{12} = v_{j(j+1)}$.

In the latter case, let us observe that (v_1, v_{12}, v_{j+1}) is a path in D, and hence $C'' = (v_1, v_{j+1}) \cup v_{j+1}Cv_1$ is a directed cycle in H of length less than n. Thus, we can derive from the induction hypothesis that C'' has at least one symmetric arc. If such symmetric arc is not (v_1, v_{j+1}), then we have already found a symmetric arc of C. So, $(v_{j+1}, v_1) \in A(H)$, and we have two cases. If $(v_{j+1}, v_1) \in A(D)$, then $(v_1, v_{12}, v_{j+1}, v_1)$ is a cycle of D. Hence, the arcs (v_1, v_{12}) and (v_{12}, v_{j+1}) are symmetric in D. We can conclude that $(v_2, v_1) \in A(H)$ and $(v_{j+1}, v_j) \in A(H)$. Thus, we may assume that there is a vertex $x \in V(D)$ such that (v_{j+1}, x, v_1) is a path in D. If $x = v_{12}$, then (v_2, v_{12}, v_1) is a path in D, and (v_2, v_1) is a symmetric arc of H. If $x \neq v_{12}$, then $(v_1, v_{12}, v_{j+1}, x, v_1)$ is a cycle in D. Again, at least one of the arcs (v_1, v_{12}) or (v_{12}, v_{j+1}) is symmetric in D. This implies that $(v_2, v_1) \in A(H)$ or $(v_{j+1}, v_j) \in A(H)$, as desired.

If $v_{12} = v_j$ (see Figure 3.3), then we have already observed that $j \in \{n, 3\}$ implies the existence of a symmetric arc of C. Thus, since D is loopless, we can consider $j \notin \{1,2,3,n\}$. By an argument similar to the previous case, we obtain the path P replacing every arc $(v_i, v_{i(i+1)}) \in V(C) \setminus V(D)$ for the path $(v_i, v_{i(i+1)}, v_{i+1})$ in v_2Cv_j. And again, we construct the cycle $C' = P \cup (v_j, v_2)$ in D of length $k+j-1$, where $k = |A(v_2Cv_j) \setminus A(D)|$. Let us observe that $k \leq j-2$, thus $k \leq \frac{k+j-2}{2}$ and $k+1 \leq \frac{k+j}{2}$. It follows from the main hypothesis that there are at least $\frac{k+j-1}{2}+1$ symmetric arcs in C'. Let us observe that, if the arc (v_j, v_2) is symmetric in D, then there are at least $\frac{k+j-1}{2}$ symmetric arcs in P. But this implies that either there is an arc of $A(v_2Cv_j) \cap A(D)$ that is symmetric in D, or that there exists i, with $2 \leq i \leq j-1$, such that the arcs $(v_i, v_{i(i+1)})$ and $(v_{i(i+1)}, v_{(i+1)})$ are symmetric in D. In any case, C has at least one symmetric arc.

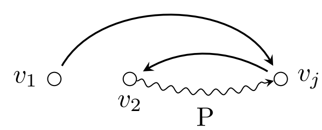

Figure 3.3: The case $v_{12} = v_j$.

Since in any case the cycle C has a symmetric arc, the result follows from the Principle of Mathematical Induction.

■

Since every cycle of $\mathcal{C}^2(D)$ has a symmetric arc, it is kernel perfect due to Theorem 1.2.1. By applying Theorem 2.1.14 to the digraph D we get that D has a 3-kernel. This is stated in the following corollary .

Corollary 3.1.4. *Let D be a digraph. If every directed cycle B in D has at least $\frac{1}{2}\ell(B)+1$ symmetric arcs, then D has a 3-kernel.*

3.2. 4-kernels

Now, we will now prove a similar result for 4-kernels. We need a few previous technical lemmas to do so.

Observation 3.2.1. Let D be a digraph. If every directed cycle C in D has at least $\frac{2}{3}\ell(C)+1$ symmetric arcs, then every cycle with length at most five is symmetric. Also, every cycle of length greater than five than has at least five symmetric arcs.

Lemma 3.2.2. *Let D be a digraph such that every directed cycle C in D has at least $\frac{2}{3}\ell(C)+1$ symmetric arcs and $u, v \in V(D)$. If P is a directed uv-path, Q is a directed vu-path and $\|P\| + \|Q\| \leq 5$, then every arc in $A(P) \cup A(Q)$ is symmetric.*

Proof. Clearly, if $\|P\| = 1$ or $\|Q\| = 1$, the result follows from Observation 3.2.1. Let $S_P = V(P) \setminus \{u, v\}$ and $S_Q = V(Q) \setminus \{u, v\}$. Suppose that $\|P\| = 2 = \|Q\|$. If $S_P \cap S_Q = \varnothing$, we have the desired result by Observation 3.2.1. If $S_P \cap S_Q \neq \varnothing$, then Q is the path obtained by reversing the arrows of P, which means every arc in $A(P) \cup A(Q)$ is symmetric.

Finally, assume without loss of generality that $\|P\| = 3$ and $\|Q\| = 2$. Take $P = (u, x, y, v)$ and $Q = (v, z, u)$. If $z = x$, then (u, x) is symmetric and (x, y, v, x) is a C_3, which by 3.2.1 is symmetric. A similar argument works when If $z = y$. In any case, every arc in $A(P) \cup A(Q)$ is symmetric.

■

Lemma 3.2.3. *Let D be a digraph such that every directed cycle C in D has at least $\frac{2}{3}\ell(C)+1$ symmetric arcs and $u, v \in V(D)$. If P is a directed uv-path, Q is a directed vu-path and $\|P\| + \|Q\| \leq 6$, then every arc in $A(P) \cup A(Q)$ is symmetric but at most one.*

Proof. The cases where $\|P\|+\|Q\| \leq 5$ are covered by Lemma 3.2.3. Suppose that $\|P\| + \|Q\| = 6$. If $\|P\| = 1$ or $\|Q\| = 1$, then PQ is a C_6 and has at least five symmetric arcs. Take $S_P = V(P) \setminus \{u, v\}$ and $S_Q = V(Q) \setminus \{u, v\}$. If $S_P \cap S_Q = \varnothing$, we have that $uPvQu$ is a C_6 the result follows directly. We can thus assume that $S_P \cap S_Q \neq \varnothing$. If $\|P\| = 2$ and $\|Q\| = 4$, take $P = (u, w, v)$ and $Q = (v, x, y, z, u)$. If $w = x$, we have that the arc (v, x) is symmetric and that (u, x, y, z, u) is a C_5, so every arc in $A(P) \cup A(Q)$ is symmetric. If $w = z$, a similar argument yields the same result. If $w = y$, then (u, y, z, u) and (v, x, y, v) are directed triangles, so they are symmetric and the results follows.

Finally, suppose that $\|P\| = 3 = \|Q\|$, $P = (u, z, w, v)$ and $Q = (v, x, y, u)$. If $z = y$ and $w = x$, then Q is the path obtained by reversing the arrows of P, which means every arc in $A(P) \cup A(Q)$ is symmetric. If $z = x$ and $w = y$, then (u, x, y, u) and v, x, y, v are directed triangles, which are symmetric in D. If $z = y$ and $w \neq x$, then (u, y) is symmetric and (v, x, y, w, v) is a C_4, hence every arc in $A(P) \cup A(Q)$ is symmetric. Very similar arguments solve the remaining cases.

■

Lemma 3.2.4. *Let D be a digraph. If every directed cycle C in D has at least $\frac{2}{3}\ell(C)+1$ symmetric arcs then, every C_3 in $H = \mathcal{C}^3(D)$ has at least one symmetric arc.*

Proof. Let $C = (v_1, v_2, v_3, v_1)$ be a 3-cycle of H. If $A(C) \subseteq A(D)$, then every arc of C is symmetric. If $|A(C) \cap A(D)| = 2$, we can suppose that $(v_2, v_3), (v_3, v_1) \in A(D)$. Since $(v_1, v_2) \in A(H) \setminus A(D)$, there is a path T of length at most three from v_1 to v_2. Either $v_3 \in V(T)$ or $v_3 \notin V(T)$. In any case, the arc (v_3, v_1) is in a cycle in D of length at most five and therefore is symmetric.

If $|A(C) \cap A(D)| = 1$, we can assume that $(v_3, v_1) \in A(D)$. Let T_1 be the v_1v_2-path of length at most three in D and T_2 the v_2v_3-path of length at most three in D. If $v_3 \in V(T_1)$, then the arc (v_3, v_1) is in a cycle in D of length at most three, implying it is symmetric. If $v_1 \in V(T_2)$, then the arc (v_2, v_1) is an arc of H, implying it is symmetric. Suppose that neither $v_3 \in V(T_1)$ nor $v_1 \in V(T_2)$. If $V(T_1) \cap V(T_2) = \varnothing$, then $T_1T_2v_1$ is a C_7 in D and has at least 6 symmetric arcs, so either the arc (v_3, v_1) is symmetric or both (v_1, v_2) and (v_2, v_3) are symmetric. If $V(T_1) \cap V(T_2) \neq \varnothing$, then the arc (v_3, v_1) is in a cycle in D of length at most five and it is therefore symmetric.

We can now assume that $A(C) \cap A(D) = \varnothing$. Let T_i be the shortest v_iv_{i+1}-path of length at most three for $1 \leq i \leq n-1$ and take $S_i = V(T_i) \setminus \{v_i, v_{i+1}\}$. Also, let T_n be the shortest v_nv_1-path of length at most three and take $S_n = V(T_n) \setminus \{v_1, v_n\}$. If $S_i \cap V(C) \neq \varnothing$ for some $i \in \{1, 2, 3\}$, then clearly C has a symmetric arc.

Suppose now that $S_i \cap S_j = \varnothing$ for every $i \neq j$. In this case, Lemma 3.1.1 gives us the existence of a symmetric arc of C. Finally, we must check what happens when there exist $i \neq j$ such that $S_i \cap S_j \neq \varnothing$. We can assume without loss of generality that $i = 1$ and $j = 2$. We must check all the different ways in which T_1 and T_2 can intersect. Notice that, since $S_1 \cap S_2 \neq \varnothing$, the distance from v_1 to v_3 is at most four. If $d(v_1, v_3) \leq 3$, then the arc (v_1, v_3) is symmetric. It only remains to check when $d(v_1, v_3) = 4$. In this case, we can assume that $T_1 = (v_1, x_1, y_1, v_2)$ and $T_2 = (v_2, x_2, y_2, v_3)$, where $y_1 = x_2$ and the remaining vertices are all different. If $S_3 \cap (S_1 \cup S_2) = \varnothing$, then $A = v_1T_1y_1T_2v_3T_3v_1$ is a cycle of length six or seven. If its length is six, the main hypothesis implies that A has at least five symmetric arcs. If its length is seven, the main hypothesis implies that A has at least six symmetric arcs. In either case, either (v_1, v_2) or (v_2, v_3) is symmetric by the Pidgeonhole Principle.

Suppose that $S_3 \cap (S_1 \cup S_2) \neq \varnothing$. First, take $T_3 = (v_3, x_3, v_1)$. If $x_3 = x_2$, then $v_1T_1x_2T_3v_1$ is a C_3 in D, it is symmetric in H and this means $(v_2, v_1) \in A(H)$, so it is symmetric. If $x_3 = y_2$, then $v_1T_1x_2T_2y_2T_3v_1$ is a C_4 in D, which is symmetric in H, $(v_2, v_1) \in A(H)$, so it is a symmetric arc. If $x_3 = x_1$, an argument analogous to the one used in the previous case shows that (v_3, v_2) is a symmetric arc of H.

Finally, take $T_3 = (v_3, x_3, y_3, v_1)$. If $y_3 \in S_1 \cup S_2$, arguments analogous to the ones used in the case where T_3 has length two work. Thus, we can assume that $y_2 \notin S_1 \cup S_2$ and $x_3 \in S_1 \cup S_2$. If $x_3 = x_2$, then $v_1T_1x_2T_3v_1$ is a C_4 in D, it is symmetric in H so $(v_2, v_1) \in A(H)$, and (v_1, v_2) is symmetric. If $x_3 = y_2$, then $v_1T_1x_2T_2y_2T_3v_1$ is a C_5 in D, it is symmetric in H, $(v_2, v_1) \in A(H)$, hence (v_1, v_2) is symmetric. Again, if $x_3 = x_1$, an argument analogous to the previous one shows that (v_3, v_2) is symmetric.

■

Now, we can prove an analogue of Theorem 3.1.3 for 4-kernels. It is not surprising, specially if one compares Lemma 3.1.2 and Lemma 3.2.4, that the basic structure of the proof of Theorem 3.2.5 is very similar to the one of Theorem 3.1.3. Nevertheless, working with 4-kernels means we have to work with longer paths in the digraph, which involves a few difficulties that are not present in the case of 3-kernels.

We will work with a cycle in the 4-closure of a digraph D and the paths in D that originate the arcs in that cycle. The proof consists of four main parts. First, we check what happens when all the paths are internally disjoint. This is easy thanks to Lemma 3.1.1.

After that, we start working assuming that two of those paths are not internally disjoint. We check what happens when those paths correspond to consecutive arcs in the cycle in the second part. A special case, which we will call an ω-configuration, arises here.

Finally, we study the ω-configurations along with the case where the paths correspond to arcs that are not consecutive in the cycle.

Theorem 3.2.5. *Let D be a digraph. If every directed cycle B in D has at least $\frac{2}{3}\ell(B) + 1$ symmetric arcs, then every cycle in $H = \mathcal{C}^3(D)$ has a symmetric arc.*

Proof. Let C be a cycle in H. We proceed by induction on the length of C. The case when C has length three is covered by Lemma 3.2.4.

Suppose then that $C = (v_1, v_2, \dots, v_n, v_0)$ is an n-cycle in H. For every arc $(u, v) \in A(C)$ there is a directed uv-path in D of length at most three (possibly the same arc). For every $1 \leq i \leq n-1$, let T_i be such directed path and T_n be the directed path from v_n to v_1, and take $S_i = V(T_i) \setminus \{v_i, v_{i+1}\}$ for $1 \leq i \leq n-1$ and $S_n = V(T_n) \setminus \{v_1, v_n\}$. If $S_i \cap S_j = \varnothing$ and $S_i \cap V(C) = \varnothing$ for every $1 \leq i < j \leq n$, then Lemma 3.1.1 gives us the desired result.

If $S_i \cap V(C) \neq \varnothing$, then, for some $0 \leq j \leq n$, there is a $v_j \in S_i$. Without loss of generality, we can assume that $|j - i|$ is minimum with such property and that $i = 1$. This means $(v_1, v_j) \in A(H)$. If $j = n$, then $(v_1, v_n) \in A(H)$ and it is symmetric. If $j = 3$, then $(v_3, v_2) \in A(H)$ and it is symmetric. Hence, we can assume that $j \notin \{1, 2, 3, n\}$.

It is easy to see that (v_1, v_j) and (v_j, v_2) are arcs of H (Figure 3.4). The cycle $C' = v_1 v_j C v_1$ is a cycle in H of length less than n, so it has a symmetric arc by induction hypothesis. If such symmetric arc is and arc of C, we are done. Otherwise, there is a directed path P of length at most three from v_j to v_1. Since the length of the path $v_1 T_1 v_j$ is at most two, an application of Lemma 3.2.2 with P and $v_1 T_1 v_j$ gives us that the arcs of $v_1 T_1 v_j$ are symmetric. On the other hand, the cycle $C'' = v_2 C v_j v_2$ is a cycle in H of length less than n, hence it has a symmetric arc. We can assume that (v_j, v_2) is the symmetric arc, otherwise we are done. Since (v_j, v_2) is symmetric, there is a directed path Q from v_2 to v_j of length at most three. Again, applying Lemma 3.2.2 to Q and $v_j T_1 v_2$ proves that the arcs of $v_j T_1 v_2$ are symmetric. Since $T_1 = v_1 T_1 v_j T_1 v_2$ and all its arcs are symmetric, we have that (v_1, v_2) is a symmetric arc of C.

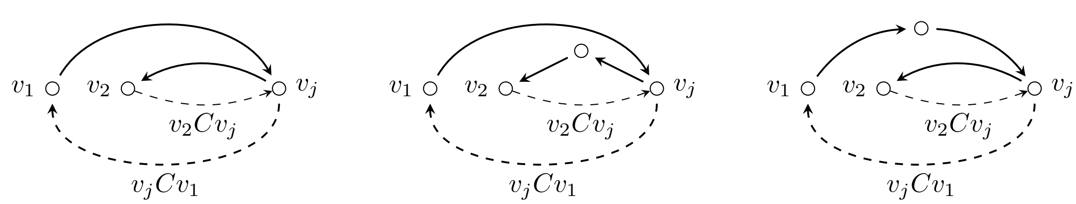

Figure 3.4: The case $v_j \in S_1$.

Suppose now that $S_i \cap V(C) = \varnothing$ for every $i \in \{1, 2, \dots, n\}$ but $S_i \cap S_j \neq \varnothing$

for some $1 \leq i < j \leq n$. First, take the case where v_i and v_j are consecutive vertices in the cycle. Without loss of generality, take $i = 1$ and $j = 2$. We will check all the possible ways in which T_1 and T_2 can intersect.

Notice that every time $d(v_1, v_3) \leq 3$ we will have that $(v_1, v_3) \in A(H)$. Hence, $v_1 v_3 C v_1$ is a cycle of length less than n and the induction hypothesis gives us the existence of a symmetric arc which we can assume to be (v_1, v_3). This is because we would have a symmetric arc of C otherwise. The fact that (v_1, v_3) is symmetric implies there is a path P of length at most 3 from v_3 to v_1. We will use this fact whenever we can in all the following cases.

- If $\|T_1\| = 2 = \|T_2\|$ (Figure 3.5). The only possible way in which they can intersect is when $T_1 = (v_1, x, v_2)$ and $T_2 = (v_2, x, v_3)$. In this case, Lemma 3.2.2 implies that (v_1, x) is symmetric, so the arc v_1, v_2 is a symmetric arc of C.

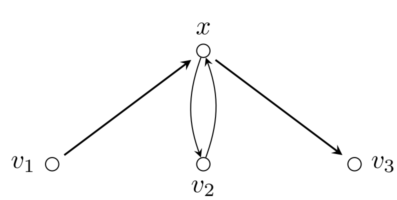

Figure 3.5: $\|T_1\| = 2 = \|T_2\|$.

- If $\|T_1\| = 2$ and $\|T_2\| = 3$ (Figure 3.6). Let $T_1 = (v_1, x, v_2)$ and $T_2 = (v_2, y, z, v_3)$.

  If $y = x$, then take $Q = (v_1, x, z, v_3)$. By applying Lemma 3.2.3 we derive that either (v_1, x) is symmetric, in which case (v_1, v_2) is a symmetric arc of C, or both (x, z) and (z, v_3) are symmetric, implying that (v_2, v_3) is a symmetric arc of C.

  If $z = x$, then take $Q = (v_1, z, v_3)$. Here, Lemma 3.2.2 gives us that (v_1, x) is symmetric, (v_2, y, x, v_1) is a path of length 3 and therefore the arc (v_1, v_2) is symmetric.

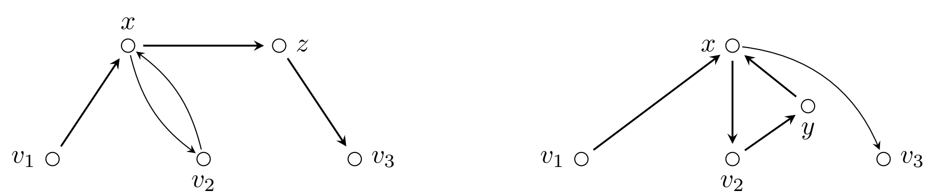

Figure 3.6: $\|T_1\| = 2$ and $\|T_2\| = 3$.

- If $\|T_1\| = 3$ and $\|T_2\| = 2$. This is very similar to the previous case. Let $T_1 = (v_1, x, y, v_2)$ and $T_2 = (v_2, z, v_3)$.

  If $z = y$, take $Q = (v_1, x, y, v_3)$. If we use Lemma 3.2.2 we can see that either both arcs in (v_1, x, y) are symmetric, and hence the arc (v_1, v_2) is a symmetric arc of C, or (z, v_3) is symmetric, implying (v_2, v_3) is a symmetric arc in C.

  If $z = x$, then $Q = (v_1, x, v_3)$. Lemma 3.2.2 guarantees that every arc in Q is symmetric, so (v_2, x, v_1) is a directed path in D and, therefore, the arc (v_1, v_2) is symmetric.

- If $\|T_1\| = 3 = \|T_2\|$. Let $T_1 = (v_1, x, y, v_2)$ and $T_2 = (v_2, z, w, v_3)$.

  If $z = x$ and $w = y$(Figure 3.7 (a)), take $Q = (v_1, x, y, v_3)$. Now, Lemma 3.2.3 guarantees that either (v_1, x) or (y, v_3) is symmetric. In the first case we have that (v_2, x, v_1) is a path in D and (v_1, v_2) is symmetric. In the second case, (v_3, y, v_2) is a path in D and (v_1, v_2) is symmetric.

  If $z = y$ and $w = x$ (Figure 3.7 (b)), take $Q = (v_1, x, v_3)$ and apply Lemma 3.2.2. We get that (v_1, x) is symmetric and this means (v_1, v_2) is a symmetric arc of C.

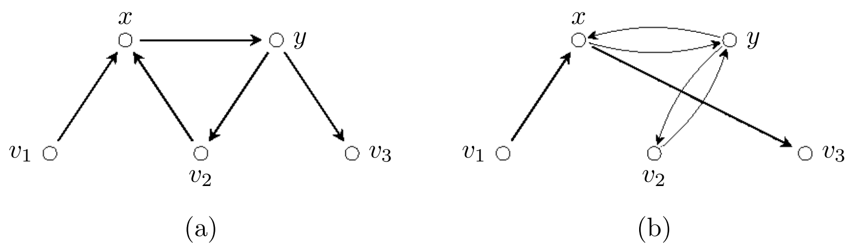

Figure 3.7: $\|T_1\| = 2$ and $\|T_2\| = 3$.

If $z = x$ and $w \neq y$ (Figure 3.8 (a)), take $Q = (v_1, x, w, v_3)$. Notice that (x, y, v_2, x) is a C_3 of D, so it is symmetric. Applying now Lemma 3.2.3 gives us that either (v_1, x) is symmetric, implying that (v_1, v_2) is a symmetric arc of C, or both (z, w) and (w, v_3) are symmetric, hence (v_2, v_3) is a symmetric arc of C.

If $z \neq x$ and $w = y$ (Figure 3.8 (b)), take $Q = (v_1, x, y, v_3)$. In a way similar to the previous case, (v_2, z, y, v_2) is a C_3 of D, so it is symmetric. An application of Lemma 3.2.3 gives us that either (v_1, v_2) or (v_2, v_3) is a symmetric arc of C.

If $z \neq y$ and $w = x$ (Figure 3.8 (c)), we have $Q = (v_1, x, v_3)$. Here, an application of Lemma 3.2.2 gives us that (v_1, x) is symmetric. This means that (v_2, z, x, v_1) is a path in D and therefore (v_1, v_2) is symmetric.

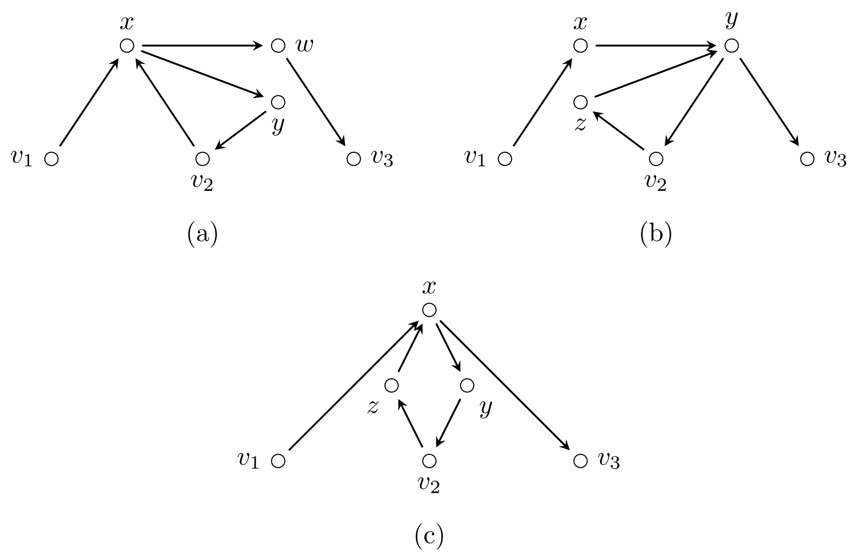

Figure 3.8: $\|T_1\| = 3 = \|T_2\|$

The only remaining case is when $T_1 = (v_1, x, y, v_2)$, $T_2 = (v_2, z, w, v_3)$ and we have $z = y$ and $w \neq x$. Let use call this case an ω configuration and say that T_1 and T_2 intersect in an ω configuration. The arcs (x, y) and (y, w) will be called the *inner arcs* of the ω configuration formed by T_1 and T_2, and we will use $\iota(T_1, T_2)$ to denote the set $\{(x, y), (y, w)\}$. The symmetric arc (v_2, y) will be called the *spike* of the ω configuration (see Figure 3.9)formed by T_1 and T_2 and we will use $\sigma(T_1, T_2)$ to denote the set $\{(v_2, y), (y, v_2)\}$. The arcs (v_1, x) and (w, v_3) will be called the *outer* arcs and the set $\{(v_1, x), (w, v_3)\}$ will be denoted by $\epsilon(T_1, T_2)$.

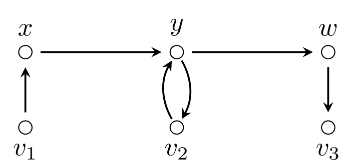

Figure 3.9: The ω configuration.

Since in this case we do not have a directed path from v_1 to v_3 of length less than 3, we must proceed in a different manner. We can assume that whenever there are T_i and T_j such that $S_i \cap S_j \neq \varnothing$ either there are $i \leq l_1 < l_2 \leq j$ such that $S_{l_1} \cap S_{l_2} \neq \varnothing$ with $i \neq l_1$ or $j \neq l_2$, or v_i and v_j are consecutive in C and the intersection between T_i and T_j is a ω configuration.

Let $I = \{v_k, v_{k+1}, \ldots, v_{k+t}\}$ a set of consecutive vertices of the cycle C, where the subscripts are taken in the natural way induced by the cycle. We say that I is a ω-block if the following conditions are satisfied:

1. $(v_r, v_{r+1}) \notin A(D)$.

2. The intersection between T_r and T_{r+1} is a ω configuration for every $k \leq r \leq k+t$.

3. Either $(v_{k-1}, v_k) \in A(D)$ or $S_{k-1} \cap S_k = \varnothing$.

4. Either $(v_{k+t}, v_{k+t+1}) \in A(D)$ or $S_{k+t} \cap S_{k+t+1} = \varnothing$.

An ω-block $I = \{v_k, v_{k+1}, \ldots, v_{k+t}\}$ will be called proper if $S_i \cap S_j = \varnothing$ when i and j are not consecutive, like in Figure 3.10. Otherwise, I will be called improper. An example of an improper ω-block can be seen in Figure 3.11). Clearly, improper ω-blocks have at least four vertices.

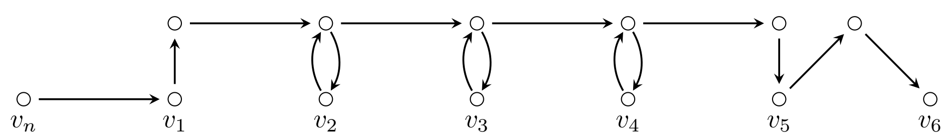

Figure 3.10: The set $\{v_1, v_2, v_3, v_4, v_5\}$ is a proper ω-block.

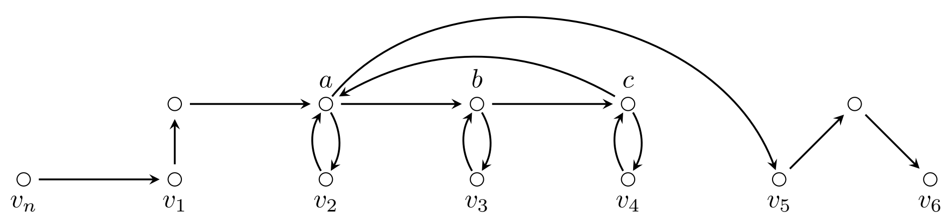

Figure 3.11: The set $\{v_1, v_2, v_3, v_4, v_5\}$ is an improper ω-block.

Let $I = \{v_k, v_{k+1}, \dots, v_{k+t}\}$ be an improper ω-block. This means that there are integers k_1, k_2 such that $k \leq k_1 < k_1 + 1 < k_2 \leq k + t$ and $S_{k_1} \cap S_{k_2} \neq \varnothing$. We can assume that $k_2 - k_1$ is minimum with such property and that $k_1 = 1$ and $k_2 = j$. Let $T_1 = (v_1, x, y, v_2)$ and $T_j = (v_j, z, w, v_{j+1})$. Since $z \in T_{j-1}$, the minimality of $j - 1$ guarantees that $z \notin \{x, y\}$. This means that either $w = x$ or $w = y$.

If $w = x$, take the cycle B in D that is induced by the arc set $\{(z, x)\} \cup E$, where

$$E = \bigcup_{r=1}^{j-1} \iota(T_r, T_{r+1}).$$

Since $2 < j$, we have that $E \neq \varnothing$. In Figure 3.11, the cycle (a, b, c, a) is the cycle B. If $\ell(B) \leq 5$, then B is symmetric and and every arc in E is symmetric, so the arc (v_2, v_3) is a symmetric arc in C. If $\ell(B) \geq 6$, then it has at least five symmetric arcs. Clearly $|A(B) \setminus E| = 1$, so there is a symmetric arc in E and, therefore, a symmetric arc in C. The case where $x = y$ is analogous.

From now on, we can assume that if two consecutive vertices v_i, v_{i+1} satisfy $S_i \cap S_{i+1} \neq \varnothing$, then T_i and T_{i+1} intersect in a ω configuration and there exists an ω-block I such that $v_i, v_{i+1} \in I$. Also, we can suppose that every ω-block is proper.

First, suppose that whenever there are T_i and T_j such that $S_i \cap S_j \neq \varnothing$, we have $i + 1 = j$ (with indices taken in the natural way along the cycle). Clearly, this means that for every arc $(v_i, v_{i+1}) \in A(C)$, exactly one of the following conditions is fulfilled:

- $(v_i, v_{i+1}) \in A(D)$.
- $S_i \cap S_j$ for every $j \neq i$.
- There is 6 an ω-block I such that $v_i, v_{i+1} \in I$.

Let $\mathfrak{L}, \mathfrak{K}, \Omega, \mathfrak{O}$ and α be defined as follows:

- $\mathfrak{L} = |\{T_i : 2 \leq i \leq j, \|T_i\| = 3\}|$.
- $\mathfrak{K} = |\{T_i : 2 \leq i \leq j, \|T_i\| = 2\}|$.
- Ω is the set of all the ω-blocks contained in $V(C)$.

- $\mathfrak{O} = |\Omega|$.
- $\alpha = \{(x, y) \in A(C) \setminus A(D) : \{x, y\} \not\subset I, I \in \Omega\}$.

It is easy to see that B, the cycle induced by

$$[A(C) \cap A(D)] \cup \left[\bigcup_{(v_i, v_{i+1}) \in \alpha} A(T_i) \right] \cup \left[\bigcup_{I \in \Omega} \iota_I \right] \cup \left[\bigcup_{I \in \Omega} \epsilon_I \right],$$

has length $n + 2\mathfrak{O} + 2\mathfrak{L} + \mathfrak{K}$.

Since $\mathfrak{O} + \mathfrak{L} \leq n$ and $\mathfrak{O} + \mathfrak{L} + \mathfrak{K} \leq n$, we have the following inequalities:

$$\begin{aligned} 2\mathfrak{O} + 2\mathfrak{L} + \mathfrak{K} &\leq 2n \\ 6\mathfrak{O} + 6\mathfrak{L} + 3\mathfrak{K} &\leq 2n + 4\mathfrak{O} + 4\mathfrak{L} + 2\mathfrak{K} \\ 2\mathfrak{O} + 2\mathfrak{L} + \mathfrak{K} &\leq \frac{2}{3}(n + 2\mathfrak{O} + 2\mathfrak{L} + \mathfrak{K}) \end{aligned}$$

The main hypothesis implies that B has at least $\frac{2}{3}(n + 2\mathfrak{O} + 2\mathfrak{L} + \mathfrak{K}) + 1$ symmetric arcs, so C has at least one symmetric arc.

Suppose now that there are positive integers i, j such that $S_i \cap S_j \neq \varnothing$ and $1 \leq i < j < n$. Again, we must check every way in which S_i and S_j may intersect. Suppose that $j - i$ is minimum with these properties and, without loss of generality, that $i = 1$. First, let us check the most direct cases.

1. If $\|T_1\| = 2 = \|T_j\|$ (Figure 3.12). The only possible way in which they can intersect is when $T_1 = (v_1, x, v_2)$ and $T_j = (v_j, x, v_{j+1})$. Let $P_1 = (v_1, x, v_{j+1})$ and $P_2 = (v_j, x, v_2)$. Clearly, the arcs (v_j, v_2) and (v_1, v_{j+1}) are arcs of H. Take $B_1 = v_1 v_{j+1} C v_1$ and $B_2 = v_2 C v_{j+1} v_2$. Clearly, B_1 and B_2 have length less than n and, by induction hypothesis, they have a symmetric arc. We can assume that the symmetric arc in B_1 is (v_1, v_{j+1}) and the one in B_2 is (v_j, v_2), since otherwise we would have a symmetric arc in C. This means that there is a directed $v_{j+1}v_1$-path and a directed $v_2 v_j$- path in D. Let us call them Q_1 and Q_2 respectively. Now, applying Lemma 3.2.2 gives us that the arcs in T_1 and T_j are symmetric, so (v_1, v_2) and (v_j, v_{j+1}) are symmetric arcs in C.

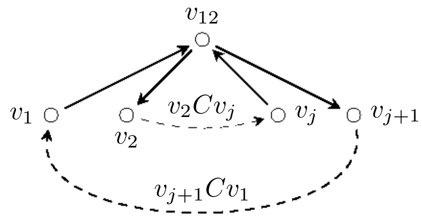

Figure 3.12: The case $\|T_1\| = 2 = \|T_j\|$.

2. If $\|T_1\| = 2$ and $\|T_j\| = 3$. Let $T_1 = (v_1, x, v_2)$ and $T_j = (v_j, y, z, v_{j+1})$.

   If $x = y$ (see Figure 3.13 (a)), then we can see that the arcs (v_j, x) and (x, v_2) are symmetric arcs of D just like in the previous case. Now, since the arc $(v_1, v_{j+1}) \in A(H)$, we have that $B_1 = v_1v_{j+1}Cv_1$ is a cycle of length less than n, so it has a symmetric arc and we can assume it is (v_1, v_{j+1}). Hence, there is a path Q_1 from v_{j+1} to v_1 of length at most three. By applying Lemma 3.2.3, we get that either (v_1, x) is a symmetric arc of D and so (v_1, v_2) is a symmetric arc in C, or both (x, y) and (y, v_{j+1}) are symmetric arcs of D and thus $(v_j, vj + 1)$ is a symmetric arc in C. The case $x = z$ (Figure 3.13 (b))is very similar.

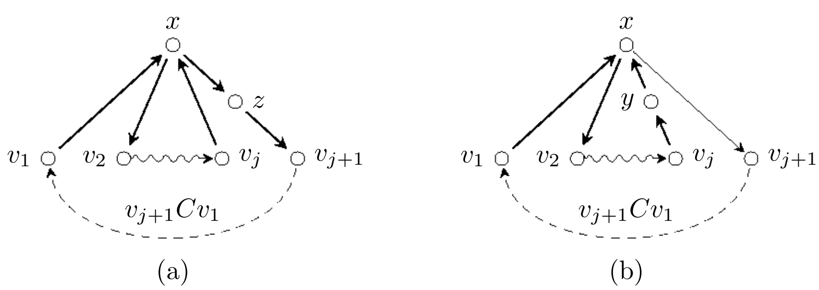

Figure 3.13: $\|T_1\| = 2$ and $\|T_2\| = 3$.

3. If $\|T_1\| = 3$ and $\|T_j\| = 2$. Let $T_1 = (v_1, x, y, v_2)$ and $T_j = (v_j, z, v_{j+1})$. This case is similar to the previous one.

4. If $\|T_1\| = 3 = \|T_j\|$. Let $T_1 = (v_1, x, y, v_2)$ and $T_j = (v_j, z, w, v_{j+1})$.

If $z = y$ and $w = x$ (Figure 3.14). Here, we have that $(v_1, v_{j+1}), (v_j, v_2) \in A(H)$. By taking $B_1 = v_1 v_{j+1} C v_1$ and $B_2 = v_2 C v_j v_2$ we can show that the arcs $(v_1, x), (x, v_{j+1}), (v_j, y)$ and (y, v_2) are symmetric arcs of D just like we did before, so both arcs (v_1, v_2) and (v_j, v_{j+1}) are symmetric arcs in C.

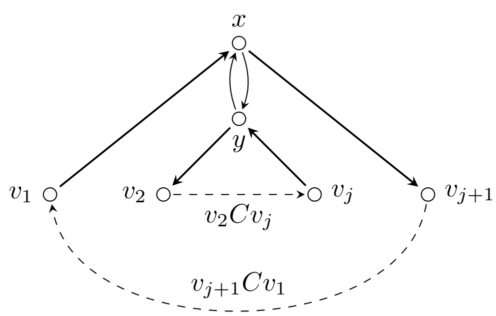

Figure 3.14: The case $z = y$ and $w = x$.

If $z \neq y$ and $w = x$ (Figure 3.15). Here we have $(v_1, v_{j+1}), (v_j, x), (x, v_2) \in A(H)$. By taking $B_1 = v_1 v_{j+1} C v_1$ we can conclude that (v_1, x) is a symmetric arc in D. Now, take $B_2 = v_j x v_2 C v_j$. It is easy to see that B_2 has length less than n, so it has a symmetric arc. Again, we can assume that the symmetric arc is either (v_j, x) or (x, v_2). If (v_j, x) is symmetric, there is a directed path from x to v_j of length at most three in D. Let Q be such directed path. Here, Lemma 3.2.2 guarantees that both (v_j, z) and (z, x) are symmetric arcs in D, and so (v_j, v_{j+1}) is a symmetric arc in C. The case where If (x, v_2) is symmetric is analogous.

If $z = y$ and $w \neq x$ (see Figure 3.16). This case is solved similarly to the case $z \neq y$ and $w = x$.

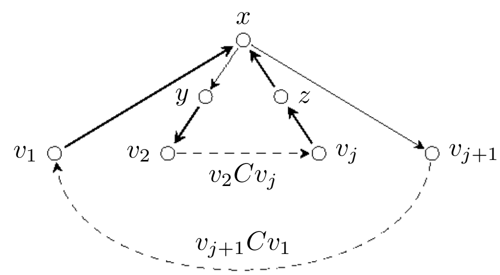

Figure 3.15: The case $z \neq y$ and $w = x$.

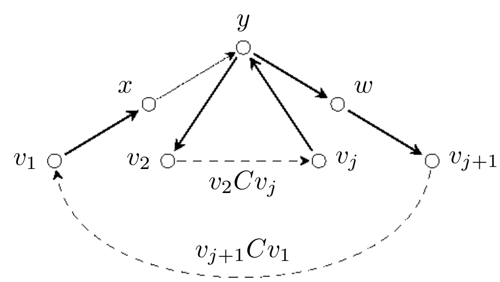

Figure 3.16: The case $z = y$ and $w \neq x$.

The three remaining cases are $z = x$ and $w = y$, $z \neq x$ and $w = y$ and finally $z = x$ and $w \neq y$. This configurations are depicted in Figure 3.17 (a), (b) and (c), respectively. It is straightforward to check that the result is true when $j = 3$ or $j = n - 1$, so we can assume that $3 < j < n - 1$.

Notice that if $S_1 \cap S_2 \neq \varnothing$ and $S_{j-1} \cap S_j \neq \varnothing$, the T_1 and T_2 intersect in an ω-configuration, just like T_{j-1} and T_j. Here, the minimality of $j - 1$ implies that the only possible case is $w = x$ and $z \neq y$, which is already covered. This means that $S_1 \cap S_2 \neq \varnothing$ and $S_{j-1} \cap S_j \neq \varnothing$ cannot occur simultaneously.

Also, if either $S_1 \cap S_2 \neq \varnothing$ or $S_{j-1} \cap S_j \neq \varnothing$, then the case $z = x$ and $w = y$ is excluded due to the minimality of $j - 1$.

First, suppose that $S_1 \cap S_2 \neq \varnothing$ and therefore T_1 and T_2 intersect in an ω-configuration. Let $T_2 = (v_2, y, u, v_3)$.

If $z = x$ and $w \neq y$, we have that $(v_1, v_{j+1}), (v_j, y), (y, v_3) \in A(H)$. Let $B_1 = v_1v_{j+1}Cv_1$ and $B_2 = v_jyv_3Vv_j$. Since B_1 and B_2 are cycles with length less than n, each has a symmetric arc.

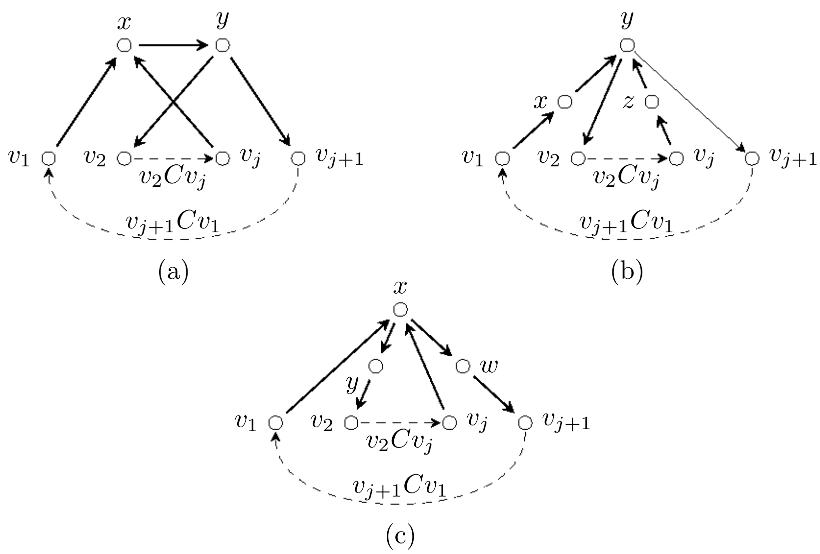

Figure 3.17: The three remaining cases.

If an arc of B_2 other than (v_j, y) or (y, v_3) is symmetric, we are done, thus we can consider that one of them is the symmetric arc. If (y, v_3) is symmetric, then there is a directed path of length at most three from v_3 to y. Call that directed path P. If we take $Q = (y, u, v_3)$, we have that the arcs (y, u) and (u, v_3) are symmetric arcs in D thanks to Lemma 3.2.2, so the arc (v_2, v_3) is a symmetric arc in C. Hence, assume that (v_j, y) is symmetric. In an analogous way, we get that (v_j, x) and (x, y) are symmetric arcs in D.

We can assume that the symmetric arc in B_1 is (v_1, v_{j+1}), since otherwise we would be done. This means there is a directed path from v_{j+1} to v_1 in D, so by Lemma 3.2.3 we have that either (v_1, x) is symmetric or both (x, w) and (w, v_{j+1}) are. In the former case, we have (v_1, v_2) is a symmetric arc in C. In the later, we get that (v_j, v_{j+1}) is the symmetric in C.

If $z \neq y$ and $w = x$, we also have that $(v_1, v_{j+1}), (v_j, y), (y, v_3) \in A(H)$. Again, take $B_1 = v_1 v_{j+1} C v_1$ and $B_2 = v_j y v_3 C v_j$ and, by the same reasons, assume that the symmetric arc in B_1 is (v_1, v_{j+1}), and either (v_j, y) or (y, v_3) is the symmetric arc in B_2.

In this case, Lemma 3.2.2 guarantees that both (v_1, x) and (x, v_{j+1}) are symmetric arcs in D. In the same way as in the previous case, if (y, v_3) is symmetric, then (v_2, v_3) is a symmetric arc in C. On the other hand, if (v_j, y)

is symmetric, Lemma 3.2.3 gives us that either (x, y) is a symmetric arc in D and thus (v_1, v_2) is a symmetric arc in C, or both (v_j, z) and (z, x) are symmetric in D, implying that (v_j, v_{j+1}) is a symmetric arc in C.

Now, suppose that $S_{j-1} \cap S_j \neq \varnothing$ and T_{j-1} and T_j intersect in an ω-configuration. Let $T_{j-1} = (v_2, u, z, v_3)$. It cannot be that $z = x$, because it would contradict the minimality of $j - 1$. So it remains to see what happens when $z \neq x$ and $w = y$. In this case, we have that $(v_{j-1}, z), (z, v_2), (v_1, v_{j+1}) \in A(H)$. Take $B_1 = v_1 v_{j+1} C v_1$ and $B_2 = v_{j-1} z v_2 C v_{j-1}$. Both B_1 and B_2 have a symmetric arc, and we can assume that (v_1, v_{j+1}) is the symmetric arc in B_1 and the one in B_2 is either (v_{j-1}, z) or (z, v_2).

If (v_{j-1}, z) is the symmetric arc in B_2, then an application of Lemma 3.2.2 yields that (v_{j-1}, v_j) is a symmetric arc in C. If (z, v_2) is the symmetric arc in B_2, again Lemma 3.2.2 gives us that (z, y) and (y, v_2) are a symmetric arcs in D.

On the other hand, since (v_1, v_{j+1}) is the symmetric arc in B_1, applying Lemma 3.2.3 shows that either (y, v_{j+1}) is a symmetric arc in D and thus (v_j, v_{j+1}), or both (v_1, x) and (x, y) are symmetric arcs in D, hence (v_1, v_2).

For the last part of the proof, we can assume that $S_1 \cap S_2 = \varnothing$ and $S_{j-1} \cap S_j = \varnothing$. The observation about proper ω-blocks that will be the key for the following arguments is this: if one arc in ι_I is symmetric, then there is a symmetric arc in C.

Notice that in the three remaining cases there is a directed path from v_j to v_2 of length three contained in $A(T_1) \cup A(T_j)$. Call it P. There is as well a path of length three from v_1 to v_{j+1} contained in $A(T_1) \cup A(T_j)$, which we will call Q. Also there are i, k such that $1 < i < k < j$ and $S_i \cap S_k \neq \varnothing$, then $i + 1 = k$ and there is a proper ω-block I such that $i, k \in I$ and $I \subseteq V(C')$, where $C' = C[\{v_2, v_3, \ldots, v_{j-1}\}]$. Just like before, this means that for every $(v_i, v_{i+1}) \in A(C')$ exactly one of the following conditions is fulfilled:

- $(v_i, v_{i+1}) \in A(D)$.
- $S_i \cap S_j = \varnothing$ for every $j \neq i$.
- There is an ω-block I such that $v_i, v_{i+1} \in I$.

Let $\mathfrak{L}, \mathfrak{K}, \Omega, \mathfrak{O}$ and α be defined as follows:

- $\mathfrak{L} = |\{T_i : 2 \leq i \leq j, \|T_i\| = 3\}|$.
- $\mathfrak{K} = |\{T_i : 2 \leq i \leq j, \|T_i\| = 2\}|$.

- Ω be the set of all the ω-blocks contained in $V(C')$.
- $\mathfrak{O} = |\Omega|$.
- $\alpha = \{(x, y) \in A(C') : \{x, y\} \not\subset I, I \in \Omega\}$.

Simple calculations show that the cycle induced by

$$[A(C') \cap A(D)] \cup \left[\bigcup_{(v_i, v_{i+1}) \in \alpha} A(T_i) \right] \cup \left[\bigcup_{I \in \Omega} \iota_I \right] \cup \left[\bigcup_{I \in \Omega} \epsilon_I \right] \cup A(P),$$

which we will call B_2, has length $\ell(B_2) = j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} + 1$.

Since $\mathfrak{O} + \mathfrak{L} \leq j - 2$ and $\mathfrak{O} + \mathfrak{L} + \mathfrak{K} \leq j - 2$, we have the following inequalities:

$$\begin{aligned}
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq 2j - 4 \\
3\mathfrak{K} + 6\mathfrak{L} + 6\mathfrak{O} &\leq 2j + 2\mathfrak{K} + 4\mathfrak{L} + 4\mathfrak{O} - 4 \\
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} - 2) \\
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O}) - \frac{4}{3} \\
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} + 1 - 1) - \frac{4}{3} \\
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} + 1) - \frac{2}{3} - \frac{4}{3} \\
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} + 1) - 2
\end{aligned}$$

By adding 3 to the right side of the inequality, we get

$$\begin{aligned}
\mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} &\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 2\mathfrak{O} + 1) - 2 + 3 \\
&\leq \frac{2}{3}(j + \mathfrak{K} + 2\mathfrak{L} + 1) + 1
\end{aligned}$$

which is the lower bound for the number of symmetric arcs in D of the cycle B. If an arc other than the ones in $A(P)$ is symmetric, then C has a symmetric arc. Hence, assume that the three symmetric arcs of B_2 are the ones in $A(P)$.

On the other hand, consider the cycle $B_1 = v_1 v_{j+1} C v_1$. It has length lesser than n, so it has a symmetric arc and we can assume it is v_1, v_{j+1}. Here, an application of Lemma 3.2.3 yields that at least two arc in Q are symmetric. It is very simple now to verify that the symmetric arc in P plus the two symmetric arcs in Q guarantee the existence of a symmetric arc in C.

■

Again, the fact that every cycle of $\mathcal{C}^3(D)$ has a symmetric arc implies it is kernel perfect due to Duchet's result. By applying Theorem 2.1.14 to the digraph D we get that D has a 4-kernel, so we have the following result:

Corollary 3.2.6. *Let D be a digraph. If every directed cycle B in D has at least $\frac{2}{3}\ell(B) + 1$ symmetric arcs, then D has a 4-kernel.*

3.3. Is it true for k-kernels?

Now, we propose the following conjecture for the general case:

Conjecture 3.3.1. If every directed cycle B in a digraph D has at least $\frac{k-2}{k-1}\ell(B) + 1$ symmetric arcs, then D has a k-kernel.

It is convenient now to give examples to see the importance of the $+1$ in the bound for the number of symmetric arcs since it is necessary for the result to be true. Let k be an integer, with $k \geq 3$ and V_k, U_k and W_k be disjoint sets with $k-1$ elements. We will use v_i, u_i and w_i to denote its elements, respectively, for $1 \leq i \leq k-1$. Let $E = e_v, f_v, e_u, f_u, e_w, f_w$ be a set disjoint of V_k, U_k and W_k.

Let H_k be the digraph such that its vertex set is $V(H_k) = V_k \cup U_k \cup W_k \cup E$ and its arc set is formed by:

- The arcs (v_i, v_{i+1}), (u_i, u_{i+1}) and (w_i, w_{i+1}), for every $1 \leq i \leq k-2$.

  The arcs (v_i, v_{i-1}), (u_i, u_{i-1}) and (w_i, w_{i-1}), for every $2 \leq i \leq k-1$.
- The arcs (v_{k-1}, u_1), (u_{k-1}, w_1) and (w_{k-1}, v_1).

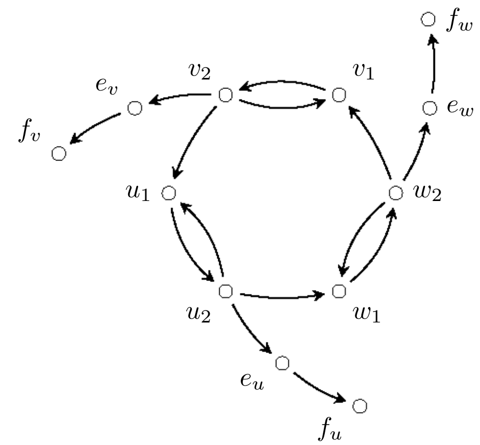

Figure 3.18: The digraph H_3

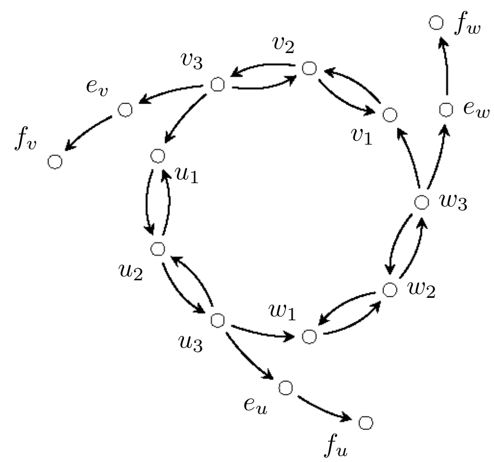

Figure 3.19: The digraph H_4

- The arcs (s_{k-1}, e_s) and (e_s, f_s), for every $s \in \{v, u, w\}$.

Notice that the only cycle in H_k of length greater than two is $C = (v_1, v_2, \ldots, v_{k-1}, u_1, u_2, \ldots, u_{k-1}, w_1, w_2, \ldots, w_{k-1}, v_1)$ and has length $3(k-1)$ and has exactly $3(k-2) = \frac{k-2}{k-1}(3(k-1)) = \frac{k-2}{k-1}\ell(C)$ symmetric arcs. Nevertheless, H_k has no k-kernel.

Proposition 3.3.2. *The digraph H_k has no k-kernel.*

Proof. Suppose that K is a k-kernel of H_k. Clearly, $\{f_v, f_u, f_w\} \subseteq K$, since they are the sinks of H_k. The sinks of H_k clearly $(k-1)$-absorb every vertex in $V(H_k)$ except v_1, u_1 and w_1, so at least one of them must be included in K. Thanks to the symmetry of H_k, we can assume that $v_1 \in K$. Since $d(w_1, v_1) = k-1$, we have that w_1 is $(k-1)$-absorbed by v_1. Since $d(v_1, u_1) = k-1$, the vertex u_1 cannot be included in K, but neither is it $(k-1)$-absorbed by a vertex in K, contradicting the fact that K was a k-kernel of H_k.

■

This shows that a digraph D such that every cycle of D has at least $\frac{k-2}{k-1}\ell(C)$ symmetric are does not necessarily have a k-kernel, which shows that we cannot drop the $+1$ in the hypothesis.

Conclusions

The results obtained in the second chapter show that bounding the length of the cycles in a digraph not only does not guarantee the existence of a k-kernel, but that determining if a given digraph has one is still very complicated, algorithmically speaking. Nonetheless, in [6] it is proven that a digraph that contains only cycles of length 3 has a 3-kernel. This points towards a line of research that seems to be very interesting.

By analyzing the proof of Theorem 3.2.5, it becomes clear that we need to find a new strategy if we want to prove the proposed conjecture. It is not hard to see that several ω-like configurations will emerge if the same technique is used to study other particular cases of the conjecture, as well as the general case.

However, provided the conjecture is true, it may become a very useful tool in the study of k-kernels. A possible line of work, side by side with the conjecture, is to find (whenever possible) generalizations to k-kernels of the results that make use of Duchet's.

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