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SOBRE COLORACIONES COMPLETAS: GRÁFICAS Y DISEÑOS
(ON COMPLETE COLORINGS: GRAPHS AND DESIGNS)

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

PRESENTA:
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Part I

Español / Spanish

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Politécnica de Cataluña.

A Loli

Prefacio

El trabajo de investigación contenido en esta tesis está enmarcado dentro del área de las Matemáticas Discretas. El eje principal que se aborda es un tipo de coloración llamado "coloración completa" (esto es, cada par de colores se encuentran) al colorear diversas estructuras de incidencia: gráficas, diseños de bloques y geometrías finitas.

Todos los resultados expuestos fueron desarrollados a lo largo del trabajo doctoral y debido a que el material fue muy extenso decidimos (en acuerdo con mi asesora de doctorado) elaborar esta tesis recopilando los artículos de investigación que realizamos, mismos que están publicados, aceptados, en proceso de revisión o en preparación.

Cabe resaltar que la tesis está escrita en inglés (idioma universal en el que se encuentran reportados los resultados relevantes en matemáticas dentro del ámbito internacional).

A continuación se dan algunas especificaciones de los artículos recopilados:

Capítulo 2: Los resultados de este capítulo fueron un trabajo conjunto con Gabriela Araujo-Pardo los cuales están contenidos en la nota "On $\omega\psi$-perfection of graphs" publicado en la revista "Electronic Notes in Discrete Matematics" [AR13], en el artículo "On $\omega\psi$-perfect graphs" el cual está en revisión [AR], y en el artículo "A new characterization of trivially perfect graphs" el cuál está publicado en la revista "Electronic Journal of Graph Theory and Applications" [Rub15].

En el artículo "On $\omega\psi$-perfect graphs" [AR] y en el acta "On $\omega\psi$-perfection of graphs" [AR13] trabajé ampliamente en las demostraciones de los teoremas y en colaboración con mi asesora le dimos estructura y redactamos ambos escritos; posteriormente elaboré el artículo "A new characterization of trivially perfect graphs" [Rub15] en el cual yo he sido el único autor.

Capítulo 3: Los resultados de este capítulo fueron un trabajo conjunto con Gabriela Araujo-Pardo, Juan José Montellano-Ballesteros y Ricardo Strausz, cuyos resultados están contenidos en el artículo "On the pseudoachromatic index of the complete graph II" publicado en la revista "Boletín de la Sociedad Matemática Mexicana" [AMRS14] y en el artículo "On the pseudoachromatic index of the complete graph III" el cual está en preparación [AMRS].

En los artículos "On the pseudoachromatic index of the complete graph II" [AMRS14] y "On the pseudoachromatic index of the complete graph III" [AMRS] he aportado contribuciones desde mi tesis de licenciatura y tesina de maestría; en colaboración con mis coautores trabajamos ampliamente en las demostraciones de los teoremas, les dimos estructura y redactamos los artículos.

Capítulo 4: Los resultados de este capítulo fueron un trabajo conjunto con Gabriela Araujo-Pardo, György Kiss y Adrián Vázquez-Ávila los cuales están contenidos en el artículo "Achromatic and pseudoachromatic indices of designs" el cuál está en preparación [AKRV].

En el artículo "Achromatic and pseudoachromatic indices of designs" [AKRV] he contribuído en la elaboración del problema, trabajamos ampliamente en las demostraciones de los teoremas y en la estructura del artículo, ahora estamos redactándolo.

Capítulo 5: Los resultados de este capítulo fueron un trabajo conjunto con Oswin Aichholzer, Gabriela Araujo-Pardo, Natalia García-Colín, Thomas Hackl, Dolores Lara y Jorge Urrutia, los cuales están contenidos en el artículo "Geometric achromatic and pseudoachromatic indices", aceptado para su publicación en la revista indizada "Graphs and Combinatorics" [AAG+].

El artículo "Geometric achromatic and pseudoachromatic indices" [AAG+] lo iniciamos trabajando en un taller de geometría combinatoria, posteriormente afinamos las demostraciones, primero con Dolores Lara y posteriormente con todo el grupo logrando interesantes contribuciones.

Capítulo 6: Los resultados de este capítulo fueron un trabajo conjunto con György Kiss, los cuales están contenidos en el artículo "A note on m-factorizations of complete multigraphs arising from designs" aceptado para su publicación en la revista indizada por matemáticas "Ars Mathematica Contemporanea" [KR15].

El artículo "A note on m-factorizations of complete multigraphs arising from designs" [KR15] se realizó durante una estancia de investigación que realizé bajo la supervisión de György Kiss en la universidad del ELTE. Trabajamos ampliamente en las demostraciones de los teoremas, estructuramos y redactamos el artículo.

Referencias

[AAG$^+$] O. Aichholzer, G. Araujo–Pardo, N. García–Colín, T. Hackl, D. Lara, C. Rubio–Montiel, and J. Urrutia. Geometric achromatic and pseudoachromatic indices. *Graphs Combin.*

[AKRV] G. Araujo–Pardo, Gy. Kiss, C. Rubio–Montiel, and A. Vázquez–Ávila. Achromatic and pseudoachromatic indices of designs. In preparation.

[AMRS] G. Araujo–Pardo, J. J. Montellano–Ballesteros, C. Rubio–Montiel, and R. Strausz. On the pseudoachromatic index of the complete graph III. In preparation.

[AMRS14] G. Araujo–Pardo, J. J. Montellano–Ballesteros, C. Rubio–Montiel, and R. Strausz. On the pseudoachromatic index of the complete graph II. *Bol. Soc. Mat. Mex. (3)*, 20(1):17–28, 2014.

[AR] G. Araujo–Pardo and C. Rubio–Montiel. On $\omega\psi$-perfect graphs. In review.

[AR13] G. Araujo–Pardo and C. Rubio–Montiel. The $\omega\psi$-perfection of graphs. In *The VII Latin-American Algorithms, Graphs, and Optimization Symposium*, volume 44 of *Electron. Notes Discrete Math.*, pages 163–168. Elsevier Sci. B. V., Amsterdam, 2013.

[KR15] Gy. Kiss and C. Rubio–Montiel. A note on m-factorizations of complete multigraphs arising from designs. *Ars Math. Contem.*, 8(1):165–175, 2015.

[Rub15] C. Rubio–Montiel. A new characterization of trivially perfect graphs. *Electron. J. Graph Theory Appl.*, 3(1):22–26, 2015.

Resumen

A continuación se dará una descripción breve de cada uno de los capítulos de este trabajo:

Capítulo 1: *Introducción.*

Contiene las definiciones básicas de teoría de gráficas enfocadas en el componente principal de esta tesis: coloraciones completas. Además se describen los objetos combinatorios llamados diseños y el caso especial de las geometrías finitas.

Capítulo 2: *Caracterizando gráficas ab-perfectas.*

En este capítulo se estudia una generalización de las llamadas "gráficas perfectas". Damos caracterizaciones de las gráficas ab-perfectas en términos de parámetros relacionados con coloraciones completas. Además se extienden algunas definiciones a gráficas infinitas.

Capítulo 3: *Índices acromático y pseudoacromático de la gráfica completa.*

En este capítulo se exhiben coloraciones en aristas de la gráfica completa, las cuales prueban muchos de los valores exactos del "índice acromático" y del "índice pseudoacromático" de la gráfica completa (parámetros estrechamente relacionadas). También se mejora la cota superior del índice pseudoacromático de la gráfica completa.

Capítulo 4: *Índices acromático y pseudoacromático de diseños.*

Un (v,κ)-diseño es una pareja $(\mathcal{P},\mathcal{B})$; donde $\mathcal{P}$ es un v-conjunto de puntos y $\mathcal{B}$ es una colección de κ-subconjuntos de $\mathcal{P}$ llamados bloques. Cada 2-subconjuntos de

$\mathcal{P}$ aparece en precisamente un bloque. En este capítulo se extiende la noción de los índices acromático y pseudoacromático a los diseños y se presentan resultados en los espacios proyectivos y afines finitos.

Capítulo 5: *Índices acromático y pseudoacromático geométricos.*

Una gráfica geométrica es una gráfica dibujada en el plano tal que sus vértices son puntos en posición general y sus aristas son segmentos rectilíneos. En este capítulo se extiende la noción de los índices acromático y pseudoacromático a las gráficas geométricas y se presentan resultados para las gráficas geométricas completas.

Capítulo 6: *Sobre m-factorizaciones de multigráficas completas que provienen de los espacios proyectivos afines.*

Se presentan algunas nuevas familias infinitas de m-factorizaciones simples e indescomponibles de la multigráfica completa. Las construcciónes provienen de las geometrías finitas.

Capítulo 7:

En el capítulo final incluímos una discusión sobre las distintas contribuciones de esta tesis al conocimiento y desarrollo del área de la Matemática Discreta y Combinatoria.

Part II

English / Inglés

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Firstly, I would like to express my gratitide to my advisor –Gabriela Araujo– for her assistance in carrying out this thesis and for all her support during my PhD. I also would like to express my gratitide to my tutor committee –Gabriela Araujo, Juan José Montellano and Camino Balbuena–, my synod of the 2nd Candidacy Exam –Gabriela Araujo, Ricardo Strausz and Mika Olsen– and my synod of this thesis –Gabriela Araujo, Juan José Montellano, Eugenia O'Reilly, Gelasio Salazar and Daniel Pellicer– for their help and comments on the development of this text.

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Part of the research was conducted at the Institute of Mathematics, the Faculty of Sciences, the Mathematical Innovation Center and the Faculty of Sciences Juriquilla Campus, all these centers belonging to the National Autonomous University of Mexico; at the Department of Geometry of the Eötvös Loránd University; and at the Department of Applied Mathematics III of the Polytechnic University of Catalonia.

To Loli

Preface

The research contained in this thesis is in the area of Discrete Mathematics. The main topic is a type of coloring called "complete coloring" (ie, each pair of colors are incident) used for coloring various incidence structures: graphs, block designs and finite geometries.

All results presented have been developed during the PhD and since the material was very extensive we decided (in agreement with my main advisor) to develop this thesis by compiling my research papers, which have been published, accepted or are being reviewed or are in preparation.

Below there are some specifications of the papers included:

Chapter 2: The results of this chapter were a joint work with Gabriela Araujo-Pardo which are contained in the paper "On $\omega\psi$-perfection of graphs" published in the proceedings "Electronic Notes in Discrete Mathematics" [AR13], in the paper "On $\omega\psi$-perfect graphs" which is in review [AR], and in the paper "A new characterization of trivially perfect graphs" published in the "Electronic Journal of Graph Theory and Applications" journal [Rub15].

In the papers "On $\omega\psi$-perfect graphs" [AR] and "On $\omega\psi$-perfection of graphs" [AR13] I worked extensively in the proofs of theorems and in collaboration with my advisor we gave structure and we wrote both of them; subsequently I elaborated the article "A new characterization of trivially perfect graphs" [Rub15] in which I was the sole author.

Chapter 3: The results of this chapter were a joint work with Gabriela Araujo-Pardo, Juan José Montellano-Ballesteros and Ricardo Strausz which are contained in the paper "On the pseudoachromatic index of the complete graph II" published in the "Boletín de la Sociedad Matemática Mexicana" journal [AMRS14] and in the paper "On the pseudoachromatic index of the complete graph III" which is in preparation [AMRS].

In the papers "On the pseudoachromatic index of the complete graph II" [AMRS14] and "On the pseudoachromatic index of the complete graph III" [AMRS] I have made contributions since my undergraduate thesis and master's thesis; in collaboration with my coauthors worked extensively in the proofs of theorems, we gave structure and write both of them.

Chapter 4: The results of this chapter were a joint work with Gabriela Araujo-Pardo, György Kiss and Adrián Vázquez-Ávila which are contained in the paper "Achromatic and pseudoachromatic indices of designs" which is in preparation [AKRV].

In the paper "Achromatic and pseudoachromatic indices of designs" [AKRV] I contributed in part in the elaboration of the problem. My coauthors and me worked extensively in the proofs of theorems and structure of the article. We are now writing it.

Chapter 5: The results of this chapter were a joint work with Oswin Aichholzer, Gabriela Araujo-Pardo, Natalia García-Colín, Thomas Hackl, Dolores Lara, and Jorge Urrutia which are contained in the paper "Geometric achromatic and pseudoachromatic indices". This paper is accepted for publication in the "Graphs and Combinatorics" indexed journal [AAG+].

The paper "Geometric achromatic and pseudoachromatic indices" [AAG+] was started at a workshop in combinatorial geometry, later we refine the proofs, first with Dolores Lara and later with the rest of the group. The contributions of the group were very interesting.

Chapter 6: The results of this chapter were a joint work with György Kiss, which are contained in the paper "A note on m-factorizations of complete multigraphs arising from designs" accepted for publication in the "Ars Mathematica Contemporanea" indexed journal [KR15].

The paper "A note on m-factorizations of complete multigraphs arising from designs" [KR15] was made during a research visit that I made under the supervision of György Kiss at ELTE University. We work extensively in the proofs of theorems, structured and drafted the paper.

References

[AAG$^+$] O. Aichholzer, G. Araujo–Pardo, N. García–Colín, T. Hackl, D. Lara, C. Rubio–Montiel, and J. Urrutia. Geometric achromatic and pseudoachromatic indices. *Graphs Combin.*

[AKRV] G. Araujo–Pardo, Gy. Kiss, C. Rubio–Montiel, and A. Vázquez–Ávila. Achromatic and pseudoachromatic indices of designs. In preparation.

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[KR15] Gy. Kiss and C. Rubio–Montiel. A note on m-factorizations of complete multigraphs arising from designs. *Ars Math. Contem.*, 8(1):165–175, 2015.

[Rub15] C. Rubio–Montiel. A new characterization of trivially perfect graphs. *Electron. J. Graph Theory Appl.*, 3(1):22–26, 2015.

Abstract

Below is a brief description of each of the chapters of this work:

Chapter 1: *Introduction.*

We will give the basic definitions of the Graph Theory focused on the major component of this thesis: Complete Colorings. In addition to describing the combinatorial objects called Designs and the special case of the Finite Projective Geometries.

Chapter 2: *Characterizing ab-perfect graphs.*

In this chapter, we will cover a generalization of graph perfection. We will characterize the ab-perfect graphs for several parameters related to complete colorations and we will extend some definitions to infinite graphs.

Chapter 3: *Achromatic and pseudoachromatic indices of the complete graph.*

In this chapter we will exhibit closely related edge-colorings of the complete graph and prove many exact values of the achromatic and pseudoachromatic indices of the complete graph of order n. We will also improve the upper bound of the pseudoachromatic index of the complete graph.

Chapter 4: *Achromatic and pseudoachromatic indices of designs.*

A (v, κ)-design is a pair $(\mathcal{P}, \mathcal{B})$; $\mathcal{P}$ is a v-set of points and $\mathcal{B}$ is a collection of κ-subsets of $\mathcal{P}$ called blocks. Each 2-subset of $\mathcal{P}$ appears in precisely one block. In this chaptes, we will extend the notion of the achromatic and pseudoachromatic indices for designs and present results for the finite projective and affine spaces.

Chapter 5: *Geometric achromatic and pseudoachromatic indices.*

A geometric graph is a graph drawn in the plane such that its vertices are points in general position, and its edges are straight-line segments. In this chapter, we will extend the notion of the achromatic and pseudoachromatic indices for geometric graphs and we present results for complete geometric graphs.

Chapter 6: *On m-factorizations of complete multigraphs arising from designs.*

Some new infinite families of simple, indecomposable m-factorizations of the complete multigraph are presented. Most of the constructions come from finite geometries.

Chapter 7:

In this concluding chapter we include a discussion on how the thesis provides contributions to knowledge in Discrete Mathematics and Combinatorics.

Chapter 1

Introduction

A *graph* G is a nonempty set V of *vertices* together with a set E of 2-subsets of V called *edges*. Each edge $\{u, v\}$ of E is commonly denoted by uv or vu. The number of vertices of a graph G is its *order*; its number of edges is its *size*. Graphs are finite, infinite, countable and so on according to their order. Except in Subsection 2.1.2, our graphs will be finite. To indicate that a graph G has *vertex set* V and *edge set* E, we sometimes write $G = (V, E)$. To emphasize that V is the vertex set of a graph G, we often write V as $V(G)$. For the same reason, we also write E as $E(G)$.

If uv is an edge of G, then u and v are *adjacent vertices*. Two adjacent vertices are referred to as *neighbors* of each other. The set of neighbors of a vertex v is called the *neighborhood* of v and is denoted by $N_G(v)$ (or only by $N(v)$ if the graph G under discussion is clear). The set $N_G[v] = N_G(v) \cup \{v\}$ (or only $N[v]$) is called the *closed neighborhood* of v.

The *degree* of a vertex v in a graph G is the number of vertices in G that are adjacent to v. The degree of a vertex v is denoted by $deg_G(v)$ (or $deg(v)$). A vertex of degree 0 is referred to as an *isolated vertex*. The largest degree among the vertices of G is called the *maximum degree* of G and it is denoted by $\Delta(G)$. The *minimum degree* of G is denoted by $\delta(G)$. If every vertex of G has degree r, then G is *r-regular*.

A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ (briefly denoted by $H \subseteq G$). If $V(H) = V(G)$, then H is a *spanning subgraph* of G. For a nonempty subset A of $V(G)$, the *subgraph* $\langle A \rangle_G$ *of* G *induced by* A (or anly $\langle A \rangle$) has A as its vertex set and two vertices u and v in A are adjacent in $\langle A \rangle_G$ if and only if u and v are adjacent in G. A subgraph H of a graph G is called an *induced subgraph* (for short denoted by $H \leq G$) if there is a nonempty subset A of $V(G)$ such that H is isomorphic to $\langle A \rangle_G$. Then given $S \subseteq V(G)$, $G \setminus S$ is the subgraph of G induced by $V(G) \setminus S$. If we consider a set $\{v_1, \ldots, v_a\} \subseteq V(G)$ then we write $\langle v_1, \ldots, v_a \rangle := \langle \{v_1, \ldots, v_a\} \rangle$. A graph G without an induced subgraph H is called *H-free*. A graph H_1-free, H_2-free, $\ldots$, H_a-free is denoted by $(H_1, H_2, \ldots, H_a)$-free.

For two (not necessarily distinct) vertices u and v in a graph G, a *$u - v$ walk* W in G is a sequence of vertices in G, beginning at u and ending at v such that consecutive vertices in W are adjacent in G. Such a walk W in G can be expressed as $W = (u = v_0, v_1, \ldots, v_k = v)$, where $v_i v_{i+1} \in E(G)$ for $0 \leq i \leq k-1$. Two vertices u and v in a graph G are *connected* if G contains a $u - v$ walk. The graph G itself is *connected* if every two vertices of G are connected. A graph G that is not connected is a *disconnected* graph. A connected subgraph H of a graph G is a *component* of G if H is not a proper subgraph of a connected subgraph of G. A walk whose initial and terminal vertices are distinct is an *open walk*; otherwise, it is a *closed walk*. An open walk in a graph G in which no vertex is repeated is called a *path*. A nontrivial closed walk of a graph G in which no vertex is repeated is a *cycle* in G. A cycle in G that contains every vertex of G is called a *Hamiltonian cycle* of G. The graph that is itself a cycle of order $n \geq 3$ is denoted by C_n and the graph that is a path of order n is denoted by P_n. A graph is *complete* if every two distinct vertices in the graph are adjacent. The complete graph of order n is denoted by K_n and K_0 is defined as $\emptyset$ –when it is convenient we may denote $\emptyset$ by K_0–. The *complete multigraph* λK_v has v vertices and λ edges joining each pair of vertices. Note that $1K_v$ is K_v.

By a *factor* of a graph G, we mean a spanning subgraph of G. A k-regular factor is called a *k-factor*. A 1-factor in a graph G is also called a *perfect matching* in G. A *factorization* $\mathcal{F}$ of a graph G is a collection of factors of G such that every edge of G belongs to exactly one factor in $\mathcal{F}$. Therefore, if each factor in $\mathcal{F}$ is nonempty, then the edge sets of the factors produce a partition of $E(G)$. A *k-factorization* of a graph G is a factorization of G into k-factors. A graph G is k-factorable if there exists a

k-factorization of G.

The *stability number* $\beta(G)$ is the cardinality of the largest set of pairwise nonadjacent vertices. A maximal complete subgraph of G is also called a *clique* of G. The total number of cliques of G is denoted by $m(G)$. The maximum order of a clique of G is called the *clique number* of G and is denoted by $\omega(G)$. Given two graphs G and H, the set of edges with one vertex in G and one in H is denoted as $V(G)V(H)$-edges. We denote by $G \oplus H$ the *join* between graphs (or *directed sum*) $\oplus$, which is defined as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup (V(G)V(H)$-edges). If G is a graph, $G \oplus \emptyset$ is defined as G. For k mutually vertex-disjoint graphs G_1, G_2,..., G_k, the *union* $G = G_1 \cup G_2 \cup \cdots \cup G_k$ of these k graphs is defined by $V(G) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$. If $G_1 = G_2 = \cdots = G_k = H$, then $G = G_1 \cup G_2 \cup \cdots \cup G_k$ is denoted as $G = {}^kH$. Note that ${}^1H = H$. ${}^0H$ is defined as $\emptyset$.

A *vertex coloring* (for short *coloring*) of a graph G with k colors is a surjective function that assigns to each vertex of G a color from the set $[k] := \{1, 2, \ldots, k\}$. A coloring is *proper* if any two adjacent vertices have different colors, and it is *complete* if every pair of colors appears on at least one pair of adjacent vertices. The *chromatic number* $\chi(G)$ *of* G is the smallest number k for which there exists a proper coloring of G with k colors. It is not hard to see that any proper coloring of G with $\chi(G)$ colors is a complete coloring. The *achromatic number* $\alpha(G)$ of G is the largest number k for which there exists a proper and complete coloring of G with k colors. The *pseudoachromatic number* $\psi(G)$ of G is the largest number k for which there exists a complete coloring of G with k colors. Clearly we have that

$$\omega(G) \leq \chi(G) \leq \alpha(G) \leq \psi(G). \tag{1.1}$$

For convenience, in this thesis $0 \in \mathbb{N}$. See [BCL79, BM76, CZ09, Die05, Har69] for a more detailed introduction.

1.1 On complete colorings

In this section we collect some interesting facts and results from the achromatic and pseudoachromatic numbers. Interesting results on these invariants can be found in [BRT93, CZ09, Edw00, GK74, HM76, Xu91] and the references therein.

The achromatic number was introduced by Harary, Hedetniemi and Prins in 1967 [HHP67] (see also [Hed66]), and the pseudoachromatic number was introduced by Gupta in 1969 [Gup69]. It turns out that the exact determination of the numbers is quite difficult, see [CE97, EM95, Yeg01, YG80, Zak06] to know about the computational complexity. Several authors have studied these parameters, some known results are the following:

Harary, Hedetniemi and Prins proved that for any graph G and for every integer k with $\chi(G) \leq k \leq \alpha(G)$ there is a proper and complete coloring of G with k colors (see [HHP67, CZ09]).

Gupta proved the inequalities type Nordhaus-Gaddum for the chromatic, achromatic and pseudoachromatic numbers which give lower and upper bounds on the sum and the product of the chromatic number of a graph and its complement, in terms of the order of the graph (see [Gup69, NG56]).

Yegnanarayanan, Balakrishnan and Sampathkunar prove that if $2 \leq a \leq b \leq c$ then there exists a graph G with chromatic number a, achromatic number b, and pseudoachromatic number c (see [YBS00, Bha79, COTZ10]).

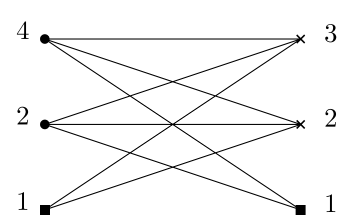

Figure 1.1: Different colorations of $K_{3,3} - e$.

Figure 1.1 shows a graph $G = K_{3,3} - e$, where all of these parameters are different: G has $\chi(G) = 2$, $\alpha(G) = 3$ (a proper and complete coloring with the points $\bullet$, $\blacksquare$ and

$_\times$) and $\psi(G) = 4$ (a complete coloring with the numbers 1, 2, 3 and 4), see [Rub09] for details.

1.2 Block designs

In this section we collect some concepts and results from block design theory. For a detailed introduction to block designs we refer to [And90, CvL91].

Let v, b, κ, r and λ be positive integers with $v > 1$ and $v > \kappa$. Let $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a triple consisting of a set $\mathcal{P}$ of v distinct objects, called points of D, a set $\mathcal{B}$ of b distinct objects, called blocks of D, and an incidence relation I, a subset of $\mathcal{P} \times \mathcal{B}$. We say that x is incident to y (or y is incident to x) if and only if the ordered pair (x, y) is in I. D is called a $2-(v, b, \kappa, r, \lambda)$ *design* if it satisfies the following axioms:

1. Each block of D is incident to exactly κ distinct points of D.
2. Each point of D is incident to exactly r distinct blocks of D.
3. If x and y are distinct points of D, then there are exactly λ blocks of D incident to both x and y.

We say that two blocks are *incident* if there is a point that is incident to both. The parameters of a $2-(v, b, \kappa, r, \lambda)$ design are not all independent. The two basic equations connecting them are the following:

$$vr = b\kappa \quad \text{and} \quad r(\kappa - 1) = \lambda(v - 1). \tag{1.2}$$

These necessary conditions are not sufficient, for example, no $2-(43, 43, 7, 7, 1)$-design exists [Tar00].

A $2-(v, b, \kappa, r, \lambda)$ design is called a balanced incomplete block design and it is also denoted by (v, κ, λ)-design (if $\lambda = 1$, for short, it is denoted by (v, κ)-design). 'Balance' refers to the Property 3, and 'incomplete' refers to the condition $\kappa < v$.

1.2.1 Correspondence

Let $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a (v, κ, λ)-design, where $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is the set of its blocks. Identify the points of D with the vertices of the complete multigraph $G = \lambda K_v$. Then in the natural way, the set of points of each block of D induces in G a subgraph isomorphic to K_κ. For $B_i \in \mathcal{B}$, G_i denotes the complete subgraph of order κ of G induced by B_i. The graphs G_i are constructed in such a way that the sets $E(G_i)$ form a partition of the set $E(G)$.

In this way when we said that a graph G isomorphic to λK_n is a *representation of the design* D we will understand that $V(G)$ is identified with the points of D and that there is a family of subgraphs (blocks) $\{G_1, \ldots, G_b\}$ of G, such that for each block B_i of D, G_i is the subgraph induced by the set of points of B_i.

1.2.2 Resolvability

A *resolution class* (or, a parallel class) of a (v, κ, λ)-design D is a partition of the point-set of the design into blocks (then κ divides v). In general, an *f-resolution class* of a design is a collection of blocks, which together contain every point of the design exactly f times. A *resolution* of a design is a partition of the block-set of the design into r resolutions (since each point of D is incident to exactly r distinct blocks of D). A (v, κ, λ)-design with a resolution is called *resolvable.*

Necessary conditions for the existence of a resolvable (v, κ, λ)-design are $\lambda(v-1) \equiv 0 \pmod{(\kappa-1)}$, $v \equiv 0 \pmod{\kappa}$ and $b \geq v + r - 1$, (see [Bos42]). In [RW71] and [HRW72] it is proved that there exists a resolvable (v, κ)-design for $\kappa = 3$ and $\kappa = 4$ respectively when the previous conditions are met.

1.3 Projective and affine spaces

In this section we collect the basic properties of the projective and affine spaces. For a more detailed introduction we refer to the book [Hir79].

Let V_{n+1} be an $(n+1)$-dimensional vector space over the finite field of q elements, $\mathrm{GF}(q)$. The *n-dimensional projective space* $\mathrm{PG}(n, q)$ is the geometry whose

k-dimensional subspaces for $k = 0, 1, \dots, n$ are the $(k+1)$-dimensional subspaces of V_{n+1} with the zero deleted. A k-dimensional subspace of $\mathrm{PG}(n,q)$ is called *k-space.* In particular subspaces of dimension zero, one and two are respectively a *point,* a *line* and a *plane,* while a subspace of dimension $n-1$ is called a *hyperplane.*

The relation $\sim$

$$\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists\, 0 \neq \alpha \in \mathrm{GF}(q) : \ \mathbf{x} = \alpha \mathbf{y}$$

is an equivalence relation on the elements of $V_{n+1} \setminus \mathbf{0}$ whose equivalence classes are the points of $\mathrm{PG}(n,q)$. Let $\mathbf{v} = (v_0, v_1, \dots, v_n)$ be a vector in $V_{n+1} \setminus \mathbf{0}$. The equivalence class of $\mathbf{v}$ is denoted by $\overline{\mathbf{v}}$. The homogeneous coordinates of the point represented by $\overline{\mathbf{v}}$ are $(v_0 \colon v_1 \colon \dots \colon v_n)$. Hence two $(n+1)$-tuples $(x_0 \colon x_1 \colon \dots \colon x_n)$ and $(y_0 \colon y_1 \colon \dots \colon y_n)$ represent the same point of $\mathrm{PG}(n,q)$ if and only if there exists $0 \neq \alpha \in \mathrm{GF}(q)$ such that $x_i = \alpha y_i$ holds for $i = 0, 1, \dots, n$.

A k-space contains those points whose representing vectors $\mathbf{x}$ satisfy the equation $\mathbf{x}A = \mathbf{0}$, where A is an $(n+1) \times (n-k)$ matrix of rank $n-k$ with entries in $\mathrm{GF}(q)$. In particalur a hyperplane contains those points whose homogeneous coordinates $(x_0 \colon x_1 \colon \dots \colon x_n)$ satisfy a linear equation

$$u_0 x_0 + u_1 x_1 + \cdots + u_n x_n = 0$$

where $u_i \in \mathrm{GF}(q)$ and $(u_0, u_1, \dots, u_n) \neq \mathbf{0}$.

The number of k-spaces in an n-dimensional vector space over $\mathrm{GF}(q)$ (with $k \leq n$) is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})} & \text{if } k \geq 1, \\ 1 & \text{if } k = 0 \end{cases}$$

The proof of the following proposition is straightforward:

Proposition 1.

- *The number of k-spaces in* $\mathrm{PG}(n,q)$ *is* $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.
- *The number of k-spaces of* $\mathrm{PG}(n,q)$ *through a given i-space in $PG(n,q)$ $(i \leq k)$ is* $\begin{bmatrix} n-i \\ k-i \end{bmatrix}_q$.

- *In particular the number of k-spaces of* $\mathrm{PG}(n,q)$ *through* 2 *distinct points in* $\mathrm{PG}(n,q)$ *($1 \leq k$) is* $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

If $\mathcal{H}_\infty$ is any hyperplane of $\mathrm{PG}(n,q)$, then the *n-dimensional affine space over* $\mathrm{GF}(q)$ is $\mathrm{AG}(n,q) = \mathrm{PG}(n,q) \setminus \mathcal{H}_\infty$. The subspaces of $\mathrm{AG}(n,q)$ are the subspaces of $\mathrm{PG}(n,q)$ with the points of $\mathcal{H}_\infty$ deleted in each case. The hyperplane $\mathcal{H}_\infty$ is called the *hyperplane at infinity of* $\mathrm{AG}(n,q)$, and for $k = 0, 1, \ldots, n-2$ the k-dimensional subspaces in $\mathcal{H}_\infty$ are called the *k-spaces at infinity of* $\mathrm{AG}(n,q)$. Let $1 < i < n$ be an integer. Two i-spaces of $\mathrm{AG}(n,q)$ are called *parallel*, if the corresponding i-spaces of $\mathrm{PG}(n,q)$ intersect $\mathcal{H}_\infty$ in the same $(i-1)$-space. The parallelism is an equivalence relation on the set of i-spaces of $\mathrm{AG}(n,q)$. As a straightforward corollary of Proposition 1 we get the following:

Proposition 2. *In* $\mathrm{AG}(n,q)$ *each equivalence class of parallel i-spaces contains q^{n-i} subspaces.*

1.3.1 Spreads and packings

Following terminology from geometry, an *i-spread* $\mathcal{S}^i$ of $\mathrm{PG}(n,q)$ or of $\mathrm{AG}(n,q)$ (a resolution class) is a set of pairwise disjoint i-spaces which gives a partition of the points of the geometry. A *i-packing* $\mathcal{P}^i$ of $\mathrm{PG}(n,q)$ or of $\mathrm{AG}(n,q)$ (a resolution) is a set of pairwise disjoint i-spreads which gives a partition of the set of i-spaces of the geometry.

1.3.2 Examples

Projective and affine spaces provide examples of (v, κ, λ)-designs.

Example 3. *Let $i < n$ be positive integers. The projective space* $\mathrm{PG}(n,q)$ *can be considered as a 2-design $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of* $\mathrm{PG}(n,q)$*, $\mathcal{B}$ is the set of i-spaces of* $\mathrm{PG}(n,q)$ *and* I *is the set theoretical inclusion. The parameters of D are* $v = \frac{q^{n+1}-1}{q-1}$, $b = \begin{bmatrix} n+1 \\ i+1 \end{bmatrix}_q$, $\kappa = \frac{q^{i+1}-1}{q-1}$, $r = \begin{bmatrix} n \\ i \end{bmatrix}_q$ *and* $\lambda = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q$.

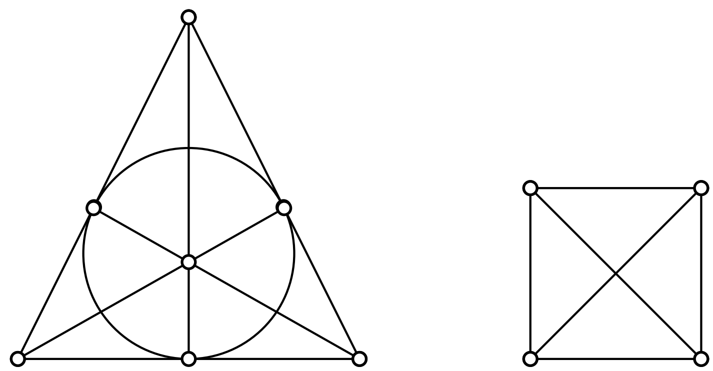

Figure 1.2: PG(2, 2) and AG(2, 2).

Example 4. *Let $i < n$ be positive integers. The affine space* $\mathrm{AG}(n, q)$ *can be considered as a 2-design* $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$*, where* $\mathcal{P}$ *is the set of points of* $\mathrm{AG}(n, q)$*,* $\mathcal{B}$ *is the set of* i*-spaces of* $\mathrm{AG}(n, q)$ *and* I *is the set theoretical inclusion. The parameters of* D *are* $v = q^n$*,* $b = q^{n-i}\begin{bmatrix} n \\ i \end{bmatrix}_q$*,* $\kappa = q^i$*,* $r = \begin{bmatrix} n \\ i \end{bmatrix}_q$ *and* $\lambda = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q$*.*

In Sections 3 and 4, Examples 3 and 4 will be denoted by $\mathrm{PG}(n, q)$ and $\mathrm{AG}(n, q)$, respectively for $i = 1$. In Section 6, Examples 3 and 4 will be denoted by $\mathrm{PG}^{(i)}(n, q)$ and $\mathrm{AG}^{(i)}(n, q)$, respectively.

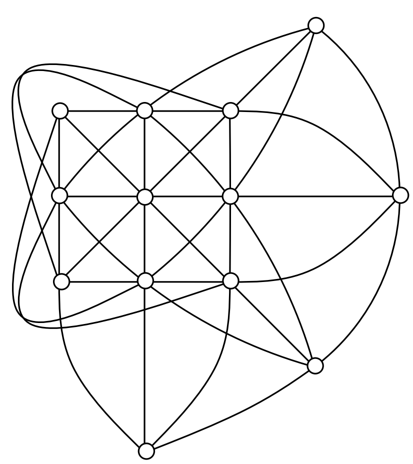

Figure 1.3: PG(2, 3).

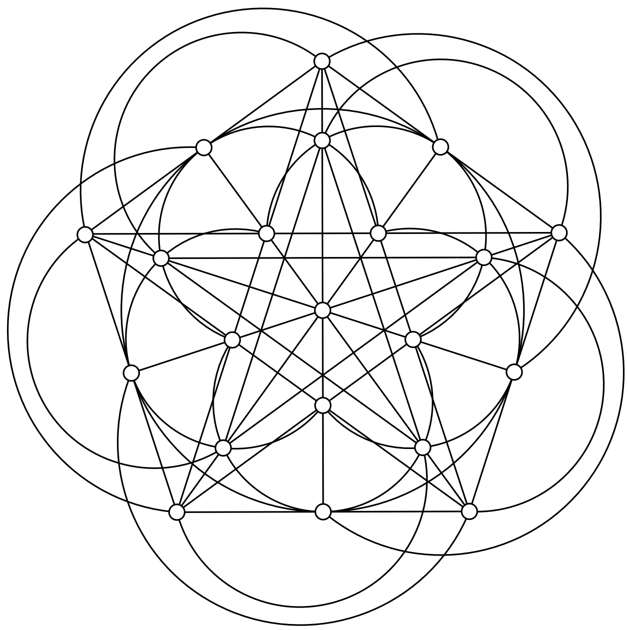

Figure 1.4: PG(2, 4).

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Chapter 2

Characterizing *ab*-perfect graphs

A coloring of G is called *pseudo-Grundy* if it is a coloring having the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i (see [Ber73, CZ09]). Consequently, every pseudo-Grundy coloring is a complete coloring. The *pseudo-Grundy number* $\gamma(G)$ is the maximum k for which a pseudo-Grundy coloring of G with k colors exists. A *Grundy* coloring is a proper pseudo-Grundy coloring. The *Grundy number* $\Gamma(G)$ is the maximum k for which a Grundy coloring of G exists (see [CZ09]).

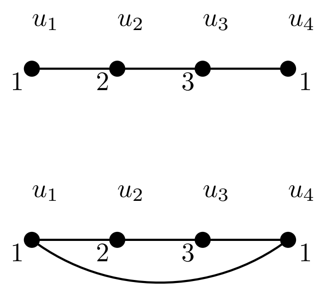
Clearly,

$$\omega(G) \leq \chi(G) \leq \Gamma(G) \leq a(G) \leq \psi(G) \tag{2.1}$$

where a is the pseudo-Grundy number γ or the achromatic number α (see Equation 1.1).

Figure 2.1 (Up) shows a graph with a Grundy coloring with 3 colors, and Figure 2.1 (Down) shows a graph with a pseudo-Grundy coloring with 3 colors.

Recall that a *greedy coloring* ς of a graph G is a proper coloring obtained from an ordering ϕ: $v_1, v_2, \ldots, v_n$ of the vertices of G in some manner, by defining $\varsigma(v_1) = 1$, and once colors have been assigned to $v_1, v_2, \ldots, v_t$ for some integer t with $1 \leq t < n$, $\varsigma(v_{t+1})$ is defined as the smallest color not assigned to any neighbor of v_{t+1} belonging to the set $\{v_1, v_2, \ldots, v_t\}$. The coloring ς so produced is then a Grundy coloring of G. That is, every greedy coloring is a Grundy coloring and then Grundy colorings always



ω	χ	Γ	γ	α	ψ
2	2	3	3	3	3
2	2	2	3	2	3

Figure 2.1: A Grundy coloring of P_4 with 3 colors.

exist for a graph G.

The Grundy number was introduced by Grundy in 1939 (see [Gru39]) and the pseudo-Grundy number by Chartrand and Zhang in 2009 (see [CZ09] page 442, and see also the variant *pseudo-Grundy fuction* in Berge's book of 1973 [Ber73] page 312).

Let a, b be two parameters of a graph G. The graph G is called *ab-perfect* if for every induced subgraph H of G, $a(H) = b(H)$. This definition extends the usual notion of *perfect graph* introduced by Berge [Ber61] in 1961; with this notation a perfect graph is denoted by $\omega\chi$-perfect. The concept of the ab-perfect graphs was introduced by Christen and Selkow in [CS79] and extended in [AR, BIEM12, Rub15, RZ01, Yeg01] (see also [AR13]).

Some important known results related to this are the following: Lóvasz [Lov72b] proved in 1972 that G is $\omega\chi$-perfect if and only if its complement is $\omega\chi$-perfect; Chudnovsky, Robertson, Seymour and Thomas [CRST06] proved in 2006 that G is $\omega\chi$-perfect if and only if G and its complement are C_{2k+1}-free for all $k \geq 2$. Christen and Selkow proved in [CS79] that for any graph G the following are equivalent: G is $\omega\Gamma$-perfect, G is $\chi\Gamma$-perfect, and G is P_4-free. They also proved in [CS79] that for any graph G the following are equivalent: G is $\omega\alpha$-perfect, G is $\chi\alpha$-perfect, and G is $(P_4, P_3 \cup K_2, {}^3K_2)$-free.

Theorem 5 (Seinsche [Sei74])**.** *A P_4-free graph is $\omega\chi$-perfect.*

By the hereditary property of the H-free graphs we have the following remark:

Remark 6. *If G is an H-free graph then all induced subgraph of G are also H-free.*

In Section 2.1, characterizations are given of the families of βm-perfect graphs, $\omega\gamma$-perfect graphs and $\chi\gamma$-perfect graphs and we further extended some definitions to locally finite graphs and countable graphs. The results of this section are contained in [Rub15]. In Section 2.2, characterizations are given of the families of $\omega\psi$-perfect graphs and $\chi\psi$-perfect graphs. The results of this section are contained in [AR, AR13].

2.1 Trivially perfect graphs

A βm-perfect graph G is called *trivially perfect*, i.e. if for every induced subgraph $\beta(G)$ (the stability number) equals $m(G)$ (the number of cliques).

Since there must be $\beta(G)$ distinct cliques containing the members of a maximum stable set, clearly,

$$\beta(G) \leq \theta(G) \leq m(G), \tag{2.2}$$

where θ denotes the *clique cover*: the least number of cliques of G whose union covers $V(G)$ (see Fig 2.2).

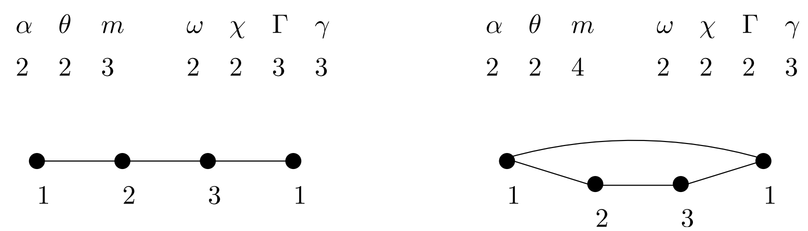


Figure 2.2: Left; a Grundy coloring of P_4 with 3 colors. Right; a pseudo-Grundy coloring of C_4 with 3 colors.

Since a graph G is $\omega\chi$-perfect if and only if its complement is $\omega\chi$-perfect (see [Ber73, Ber75, Lov72a]) then:

Theorem 7. *(Lóvasz [Lov72b]) A graph G is $\omega\chi$-perfect if and only if G is $\beta\theta$-perfect.*

By Equation (2.2), a βm-perfect graph is "trivially" perfect (see [Gol78, Gol80]). The trivially perfect graphs are also called *comparability graphs of trees* by Wolk (see [Wol62, Wol65]); or *quasi-threshold graphs* by Ma (see [MWW89, YCC96]).

2.1.1 Characterizations for finite graphs

There exist several characterizations of trivially perfect graphs, e.g. [AR13, AR, Gol78, Wol62, Wol65, YCC96]. We will use the following equivalence to prove Theorem 13:

Theorem 8 (Golumbic [Gol78])**.** *A graph G is trivially perfect if and only if G is (C_4, P_4)-free.*

Corollary 9. *A graph G is θm-perfect if and only if G is βm-perfect.*

Proof. ($\Rightarrow$) Since $\theta(C_4) = \theta(P_4) = 2$, $m(C_4) = 4$ and $m(P_4) = 3$ then G is (C_4, P_4)-free, then the implication follows.

($\Leftarrow$) This implication is immediate from Equation (2.2). □

It is not difficult to note the following remarks:

Remark 10. *If G is a connected P_4-free graph, then $diam(G) \leq 2$.*

Lemma 11 (Wolk [Wol65])**.** *If G is a connected (C_4, P_4)-free graph of order n, then $\Delta(G) = n - 1$.*

Proof. Let x be a vertex of maximum degree. Suppose that $deg(x) < n - 1$. So x has a non-neighbor z. Since G is connected, there is a path between x and z, and since G is P_4-free, this path is $x - y - z$ for some y. Since $deg(x) \geq deg(y)$ and z is adjacent to y and not to x, there must be a vertex u adjacent to x and not to y. Then $\{u, x, y, z\}$ induces a P_4 or C_4, a contradiction. So $deg(x) = n - 1$. □

Theorem 12. *A connected graph G is (C_4, P_4)-free if and only if G is a complete graph or there exist a set of connected (C_4, P_4)-free graphs $\{G_1, \ldots, G_k\}$ for some $k \geq 2$ and $m \geq 1$ such that $G = K_m \oplus \bigcup_{i=1}^{k} G_i$.*

Proof. Assume that G is a (C_4, P_4)-free graph. By Lemma 11, there exists $w_1 \in V(G)$, such that $deg(w_1) = n - 1$. If $G \setminus \{w_1\}$ is disconnected, then $G = K_1 \oplus \bigcup_{i=1}^{k} G_i$ where $K_1 = w_1$ and $G_1, \ldots, G_k$ are the components of $G \setminus \{w_1\}$. As each component is an induced subgraph of G then, by Remark 6 it is (C_4, P_4)-free graph, hence, the implication is true. Now, if $G \setminus \{w_1\}$ is connected and also, since it is an induced subgraph of G, yet again, by Remark 6, it is (C_4, P_4)-free graph. By Lemma 11 there exists $w_2 \in V(G \setminus \{w_1\})$, such that $deg_{G\setminus\{w_1\}}(w_2) = n - 2$. Newly, if $G \setminus \{w_1, w_2\}$ is disconnected, then $G = K_2 \oplus \bigcup_{i=1}^{k} G_i$ where $K_2 = \langle w_1, w_2 \rangle$ and $G_1, \ldots, G_k$ are the components of $G \setminus \{w_1, w_2\}$. Each component is an induced subgraph of G and applying the same argument than before the implication is true. As G is a finite graph we can repeat this procedure obtaining a complete graph K_m with $V(K_m) = \{w_1, \ldots, w_m\}$, and $\{G_1, \ldots, G_k\}$ a set of connected (C_4, P_4)-free graphs which are the components of $G \setminus \{w_1, \ldots, w_m\}$, and $G = K_m \oplus \bigcup_{i=1}^{k} G_i$ with $G \setminus K_m$ a disconnected graph for $k \geq 2$.

To prove the converse we take a set $X = \{v_1, v_2, v_3, v_4\}$ of vertices in $V(G)$ and we analize three cases:

- If $X \subseteq V(G_i)$ for some $i \in [k]$ then, by hypothesis, $\langle X \rangle$ is a (C_4, P_4)-free graph.
- If X is in two or more components of $\{G_1, \ldots, G_k\}$ then $\langle X \rangle$ is disconnected and then it is a (C_4, P_4)-free graph.
- If there exists v_i in $V(K_m)$ for some $i \in [4]$ then we have a vertex of degree 3 in $\langle X \rangle$ and neither P_4 nor C_4 have a vertex of degree 3.

□

In the following result, one should note that the finiteness of G is not necessary for the proof; the finiteness of $\omega(G)$ is sufficient.

Theorem 13. *For any graph G the following are equivalent: $\langle 1 \rangle$ G is (C_4, P_4)-free, $\langle 2 \rangle$ G is $\omega\gamma$-perfect, and $\langle 3 \rangle$ G is $\chi\gamma$-perfect.*

Proof. To prove $\langle 1 \rangle \Rightarrow \langle 2 \rangle$ assume that G is (C_4, P_4)-free. Let ς be a pseudo-Grundy coloring of G with $\gamma(G)$ colors. We will prove by induction on t that for $t \leq \gamma(G)$, G contains a complete subgraph of t vertices with the t highest colors of ς. This proves (for $t = \gamma(G)$) that G is $\omega\gamma$-perfect since every induced subgraph of G is (C_4, P_4)-free (see Remark 6).

For $t = 1$, there exists a vertex with color $\gamma(G)$, then the assertion is trivial. Let us now suppose that we have $t - 1$ vertices $v_1, \ldots, v_{t-1}$ in the $t - 1$ highest colors such that they are the vertices of a complete subgraph, and define V_i as the set of vertices colored $\gamma(G) - (t-1)$ by ς and adjacent to v_i ($1 \leq i < t$). Since ς is a pseudo-Grundy coloring, no V_i is empty. Any two such sets are comparable with respect to inclusion: otherwise there must be vertices p in $V_i \setminus V_j$ and q in $V_j \setminus V_i$ and the subgraph induced by $\{p, v_i, v_j, q\}$ would be isomorphic to C_4 or P_4. Therefore the $t-1$ sets V_i are linearly ordered with respect to inclusion, and there is a k ($1 \leq k < t$) with

$$V_k = \bigcap_{1 \leq i < t} V_i.$$

Thus there is a vertex v_t in V_k which is colored with $\gamma(G) - t + 1$ by ς and is adjacent to each of the v_i ($1 \leq i < t$).

The proof of $\langle 2 \rangle \Rightarrow \langle 3 \rangle$ is immediate from Equation (2.2).

To prove $\langle 3 \rangle \Rightarrow \langle 1 \rangle$ note that if $H \in \{C_4, P_4\}$ then $\chi(H) = 2$ and $\gamma(H) = 3$ hence the implication is true (see Fig 2.2). $\square$

Corollary 14. *Every $\chi\gamma$-perfect graph is $\omega\chi$-perfect.*

2.1.2 Extensions for infinite graphs

We presuppose here the axiom of choice. The definitions of pseudo-Grundy coloring with k colors and of proper coloring with k colors of a finite graph are generalizable to any cardinal number. The *chromatic number* $\chi(G)$ of a graph G is defined as the smallest cardinal κ such that the graph has a proper coloring with κ colors. The *clique number* $\omega(G)$ of a graph as the supremum of the cardinalities of the complete subgraphs of the graph G (see [CS79]). Similarly, for any ordinal number μ (such that $|\mu| = \kappa$), a *pseudo-Grundy* coloring of a graph with κ colors is a coloring of the vertices of the graph with the elements of μ such that for any $\mu'' < \mu'$ and any vertex v colored μ' there is a vertex colored μ'' adjacent to v. The *pseudo-Grundy number* $\gamma(G)$ of a graph G is the supremum of the cardinalities κ for which there is a pseudo-Grundy coloring of the graph with μ colors such that $|\mu| = \kappa$. A graph is *locally finite (or countable)* if all its vertices have finite (or countable) degrees (see [Die05]).

Next we proof a generalization of Theorem 13 for some classes of infinite graphs. Afterwards we show that there exists a graph, not belonging to these classes, for which the theorem does not hold.

Theorem 15. *The statements* $\langle 1\rangle$, $\langle 2\rangle$ *and* $\langle 3\rangle$ *of Theorem 13 are equivalent for each locally finite graph and for each countable graph.*

Proof. To prove $\langle 1\rangle \Rightarrow \langle 2\rangle$, let H be an induced subgraph of G. If $\omega(H)$ is finite, we can use the proof of Theorem 13 to show that $\gamma(H) = \omega(H)$. Otherwise, if $\omega(H)$ is infinite, then $\gamma(H) = \omega(H)$, because $\gamma(H)$ is at most the supremum of the degrees of the vertices of H, which is at most $\aleph_0$, if G is locally finite or countable.

The implications $\langle 2\rangle \Rightarrow \langle 3\rangle$ and $\langle 3\rangle \Rightarrow \langle 1\rangle$ hold for any graph, finite or not. □

The following example can be found in [CS79] but first recall some definitions of Set Theory, see [JW96] for a more detailed introduction (see also [Die05]):

- To begin with, we recall that ω is the smallest ordinal greater than every natural number and ω_1 is the first uncountable ordinal.

- Additionally, a limit ordinal is an ordinal number that neither zero nor a successor ordinal, $\lim \omega_1 = \{\mu < \omega_1 \colon \mu \text{ is a limit ordinal}\}$, then $|\lim \omega_1| = |\omega_1| = \aleph_1$ and if $\mu \in \lim \omega_1$, $|\mu| = |\omega| = \aleph_0$.

- Finally, for every $\mu \in \lim \omega_1$ there exists a bijection $f_\mu \colon \omega \to \mu$ and an injection $F \colon (\lim \omega_1) \times \omega \to \omega_1$ such that $(\mu, n) \mapsto f_\mu(n) \in \mu$ and $F\,|_{\{\mu\}\times\omega} = f_\mu$.

Let G be the non-countable, locally countable graph formed by the disjoint union of $\aleph_1$ complete countable subgraphs of $\aleph_0$ vertices, where its components are indexed with the countable limit ordinals, and their vertices with the natural numbers, that is

$$G = \bigsqcup_{\mu \in \lim \omega_1} K_{|\mu|}.$$

Clearly $\omega(G) = \chi(G) = \aleph_0$, and G is (C_4, P_4)-free.

Color the n-th vertex of the μ-th component with $F(\mu, n)$. Each $\mu \in \lim \omega_1$ is used as a color in the $(\mu + 1)$-th component. Since for each $\mu \in \lim \omega_1$, $f_\mu(n)$ is injective, the function F defines a coloring with $|\lim \omega_1|$ colors. Since $f_\mu(n)$ is surjective for each $\mu \in \lim \omega_1$, the function F is a pseudo-Grundy coloring with $|\lim \omega_1|$ colors.

2.2 On $\omega\psi$-perfect graphs

In 2001, Yegnanarayanan [Yeg01] states the following:

1. For any finite graph G the following are equivalent: $\langle 1 \rangle$ G is $\omega\psi$-perfect, $\langle 2 \rangle$ G is $\chi\psi$-perfect, $\langle 3 \rangle$ G is $\alpha\psi$-perfect and $\langle 4 \rangle$ G is C_4-free.

2. Every $\alpha\psi$-perfect graph is $\chi\alpha$-perfect.

3. Every $\chi\alpha$-perfect graph is $\omega\chi$-perfect.

Unfortunately, (1.) is false (a counterexample is P_4 because it is C_4-free but not $\omega\psi$-perfect, see Figure 2.2); i.e., $\langle 4 \rangle$ does not necessarily imply $\langle 1 \rangle$). Consequently (2.) and (3.) are not well founded. However, while (2.) is false (newly the counterexample is P_4, see Figure 2.2), (3.) is true (see Theorem 16). Note that if a graph G is $\omega\psi$-perfect then immediatly it is $\omega\chi$-perfect (see Equation 2.1). That is, G is perfect in the usual sense and it is well known that G does not allow odd cycles (except triangles) or their complements (see [CRST06]) and then the condition C_4-free is not sufficient. With the aim of finding the correctness of these statements we obtain the following results:

Theorem 16. *Let G be a graph and let $a \in \{\Gamma, \alpha, \psi\}$. G is χa-perfect if and only if G is ωa-perfect. Moreover, if G is χa-perfect then it is $\omega\chi$-perfect.*

In the following two theorems we prove some equivalences between graphs. Note that in Theorem 17 we analyze some equivalence when G is a connected graph, and in Theorem 18 we analize the same, but in this case G is any graph (not necessarily connected).

Theorem 17. *For any connected graph G the following are equivalent: $\langle 1 \rangle$ G is $\omega\psi$-perfect, $\langle 2 \rangle$ G is $\chi\psi$-perfect, $\langle 3 \rangle$ G is $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free and $\langle 4 \rangle$ $G = K_{n_1} \oplus$*

$(K_{n_2} \cup K_{n_3} \cup {}^{n_4}K_1)$ or $G = K_{n_1} \oplus (G' \cup {}^{n_2}K_1)$ for $n_1 \in \mathbb{Z}^+$ and $n_2, n_3, n_4 \in \mathbb{N}$ where G' is a connected non complete $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free graph.

Theorem 18. *For any graph G the following are equivalent: $\langle 1 \rangle$ G is $\omega\psi$-perfect, $\langle 2 \rangle$ G is $\chi\psi$-perfect, $\langle 3 \rangle$ G is $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free and $\langle 4 \rangle$ $G = K_{n_1} \cup K_{n_2} \cup {}^{n_3}K_1$ or $G = G' \cup {}^{n_2}K_1$ for $n_1 \in \mathbb{Z}^+$ and $n_2, n_3 \in \mathbb{N}$, where G' is a connected non complete $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free graph.*

2.2.1 Relationship

In Figure 2.3 we show a directed transitive graph D where the vertices of D represent the classes of ab-perfect graphs and they are labeled at its label is ab respectively for $a, b \in \{\omega, \chi, \Gamma, \alpha, \psi\}$. If two classes are equivalent they define the same vertex (see Theorem 16). If a class of ab-perfect graphs implies a class of cd-perfect graphs then there is an arrow from vertex ab to vertex cd. To prove that D does not contain another arrow we have the following counterexamples: P_4, C_4, C_5 and $P_3 \cup K_2$ (see Tables 2.1, 2.2, 2.3, 2.4).

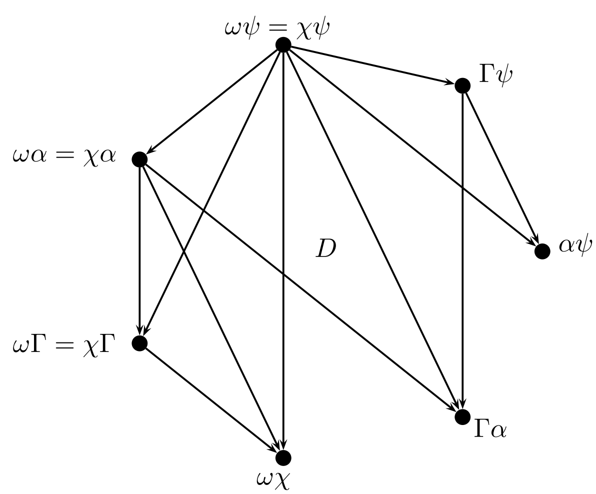


Figure 2.3: Relationship between ab-perfect graphs with $a, b \in \{\omega, \chi, \Gamma, \alpha, \psi\}$.

$\alpha\psi \nrightarrow \omega\psi$	$\Gamma\psi \nrightarrow \omega\psi$	$\Gamma\alpha \nrightarrow \omega\psi$	$\omega\chi \nrightarrow \omega\psi$
$\alpha\psi \nrightarrow \omega\alpha$	$\Gamma\psi \nrightarrow \omega\alpha$	$\Gamma\alpha \nrightarrow \omega\alpha$	$\omega\chi \nrightarrow \omega\alpha$
$\alpha\psi \nrightarrow \omega\Gamma$	$\Gamma\psi \nrightarrow \omega\Gamma$	$\Gamma\alpha \nrightarrow \omega\Gamma$	$\omega\chi \nrightarrow \omega\Gamma$

Table 2.1: P_4 is a counterexample in these cases.

$\omega\alpha \nrightarrow \omega\psi$	$\omega\alpha \nrightarrow \Gamma\psi$	$\omega\alpha \nrightarrow \alpha\psi$	$\omega\chi \nrightarrow \Gamma\psi$	$\Gamma\alpha \nrightarrow \Gamma\psi$
$\omega\Gamma \nrightarrow \omega\psi$	$\omega\Gamma \nrightarrow \Gamma\psi$	$\omega\Gamma \nrightarrow \alpha\psi$	$\omega\chi \nrightarrow \alpha\psi$	$\Gamma\alpha \nrightarrow \alpha\psi$

Table 2.2: C_4 is a counterexample in these cases.

$\alpha\psi \nrightarrow \omega\chi$	$\Gamma\psi \nrightarrow \omega\chi$	$\Gamma\alpha \nrightarrow \omega\chi$

Table 2.3: C_5 is a counterexample in these cases.

$\alpha\psi \nrightarrow \Gamma\psi$	$\alpha\psi \nrightarrow \Gamma\alpha$	$\omega\Gamma \nrightarrow \Gamma\alpha$	$\omega\chi \nrightarrow \Gamma\alpha$	$\omega\Gamma \nrightarrow \omega\alpha$

Table 2.4: $P_3 \cup K_2$ is a counterexample in these cases.

2.2.2 Results

Proof of Theorem 16. Assume that G is χa-perfect. G is P_4-free because $\chi(P_4) = 2$ and $a(P_4) = 3$ (see Figure 2.1). By Theorem 5, G is $\omega\chi$-perfect, then for all induced subgraphs H we have that $\omega(H) = \chi(H) = a(H)$, i.e., G is ωa-perfect. The converse is immediate from (2.1). □

We will prove a short, but useful lemma:

Lemma 19. *If $H = K_{m_1} \cup K_{m_2} \cup {}^{m_3}K_1$ where $m_1 \in \mathbb{Z}^+$ and $m_2, m_3 \in \mathbb{N}$ then $\omega(H) = \psi(H)$.*

Proof. Note that $\omega(H) = \max\{m_1, m_2\}$, suppose that $\max\{m_1, m_2\} < \psi(H)$ and let $\varsigma\colon V(H) \to [\psi(H)]$ be a complete coloring of H with $\psi(H)$ colors. Note that each color class has at least two vertices in $K_{m_1} \cup K_{m_2}$, in other case, as $deg(v) \leq \max\{m_1, m_2\} - 1$ for any $v \in V(H)$, ς would not be complete. Then $\psi(H) \leq \frac{m_1+m_2}{2} \leq \max\{m_1, m_2\}$ which is a contradiction and we conclude that $\psi(H) = \max\{m_1, m_2\}$. □

Proof of Theorem 17. The proof of $\langle 1\rangle \Rightarrow \langle 2\rangle$ is immediate from (2.1).

To prove $\langle 2\rangle \Rightarrow \langle 3\rangle$ note that if $H \in \{C_4, P_4, P_3 \cup K_2, {}^3K_2\}$ then $\chi(H) = 2$ and $\psi(H) = 3$ hence the implication is true.

We prove $\langle 3\rangle \Rightarrow \langle 4\rangle$ using Theorem 12. We know that G is complete, or there exists $k \geq 2$ and $n_1 \in \mathbb{Z}^+$ such that all the $\{G_1, \ldots, G_k\}$ are connected (C_4, P_4)-free graphs, $G = K_{n_1} \oplus \bigcup_{i=1}^{k} G_i$ and $G \setminus K_{n_1}$ is disconnected. Without loss of generality, if $|E(G_i)| \geq 2$ for three distinct values of i then ${}^3K_2$ is an induced subgraph of G which is not possible by hypothesis. Then we have that $G = K_{n_1} \oplus (G_1 \cup G_2 \cup {}^{n_4}K_1)$ for $n_4 \in \mathbb{N}$. Now, if G_1 is a connected non complete graph, as we noted in the proof of Lemma 11 G_1 has P_3 as induced subgraph, in which case necessarily $G_2 \cong K_1$ since in other case we have $P_3 \cup K_2$ as induced subgraph of G. Using Remark 6 we have that G_1 is a $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free graph hence the implication is true.

Finally, we will divide the proof of $\langle 4\rangle \Rightarrow \langle 1\rangle$ into two cases and in both proofs we will use induction over $|V(G)|$:

Case 1) Let $G = K_{n_1} \oplus (K_{n_2} \cup K_{n_3} \cup {}^{n_4}K_1)$, by induction, if $|V(G)| = 1$ then $G \cong K_1$ is $\omega\psi$-perfect. Assume that if $|V(G)| \leq m$ then G is $\omega\psi$-perfect and take a graph $G = K_{n_1} \oplus (K_{n_2} \cup K_{n_3} \cup {}^{n_4}K_1)$ such that $n_1 + n_2 + n_3 + n_4 = m + 1$. If H is an induced connected subgraph of G such that $|V(H)| < |V(G)|$ then clearly $H = K_{m_1} \oplus (K_{m_2} \cup K_{m_3} \cup {}^{m_4}K_1)$ where $m_i \leq n_i$ for all $i \in \{1, \ldots, 4\}$ and $m_1 + m_2 + m_3 + m_4 \leq m$, then by induction hypothesis H is $\omega\psi$-perfect, in particular $\omega(H) = \psi(H)$.
Now, if H is a disconnected induced subgraph of G such that $|V(H)| < |V(G)|$ then $H = K_{m_1} \cup K_{m_2} \cup {}^{m_3}K_1$ where $m_1 \in \mathbb{Z}^+$ and $m_2, m_3 \in \mathbb{N}$ and by Lemma 19, $\omega(H) = \psi(H)$.
Now, we will prove that we also have that $\omega(G) = \psi(G)$. Let $n = \max\{n_2, n_3, a\}$ with $a = 1$ if $n_4 > 0$ and $a = 0$ if $n_4 = 0$. If $n = 0$ then $G = K_{n_1}$ and it is $\omega\psi$-perfect. Suppose that $n \neq 0$, clearly $\omega(G) = n_1 + n$. Suppose that $\psi(G) > n_1 + n$ and let $\varsigma \colon V(G) \to [\psi(G)]$ a complete coloring of G with $\psi(G)$ colors. First, we suppose also that $n_4 > 0$; let u be a vertex of ${}^{n_4}K_1$. As the neighbours of u are n_1 then u meets at most n_1 different chromatic classes, then there must exist another vertex v such that $\varsigma(u) = \varsigma(v)$ and clearly $N(u) \subseteq N[v]$. Then $\psi(G) = \psi(G \setminus u)$, but $\omega(G) = \omega(G \setminus u)$ and $G \setminus u$ is an induced subgraph of G, contradicting the induction hypothesis. Then we

have that $G = K_{n_1} \oplus (K_{n_2} \cup K_{n_3})$ and $n = \max\{n_2, n_3\}$. Let $u \in V(K_{n_2} \cup K_{n_3})$ such that $\varsigma(u) \notin \{\varsigma(v) \colon v \in V(K_{n_1})\}$. As the neighbours of u are at most $n_1 + n - 1$ then u meets at most $n_1 + n - 1$ different chromatic classes, then there must exist another vertex v in $V(K_{n_2} \cup K_{n_3})$ such that $\varsigma(u) = \varsigma(v)$ and $\psi(G) \leq n_1 + \frac{n_2+n_3}{2} \leq n_1 + n$ which is a contradiction. Then $\omega(G) = \psi(G)$, hence G is $\omega\psi$-perfect.

Case 2) Let $G = K_{n_1} \oplus (G' \cup {}^{n_2}K_1)$, if $|V(G)| = 4$ then $G = K_1 \oplus P_3 = K_2 \oplus ({}^2K_1)$ and by the previous case G is $\omega\psi$-perfect. Now, assume that any graph with this structure and order less or equal than m is $\omega\psi$-perfect, and let $G = K_{n_1} \oplus (G' \cup {}^{n_2}K_1)$ be such that $n_1 + n' + n_2 = m + 1$, where $n' = |V(G')|$. If H is an induced connected subgraph of G such that $|V(H)| < |V(G)|$ then by Remark 6, H is a $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free graph, hence by $\langle 3 \rangle \Rightarrow \langle 4 \rangle$ we have that

1. $H = K_{n_5} \oplus (K_{n_6} \cup K_{n_7} \cup {}^{n_8}K_1)$ or $H = K_{n_5} \oplus (H' \cup {}^{n_6}K_1)$ for $n_5 \in \mathbb{Z}^+$ and $n_6, n_7, n_8 \in \mathbb{N}$ where H' is a connected graph $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free.

2. If $H = K_{n_5} \oplus (K_{n_6} \cup K_{n_7} \cup {}^{n_8}K_1)$ then using the previous case we conclude that H is $\omega\psi$-perfect, in particular, $\omega(H) = \psi(H)$.

3. If $H = K_{n_5} \oplus (H' \cup {}^{n_6}K_1)$ then by induction hypothesis H is $\omega\psi$-perfect, in particular, $\omega(H) = \psi(H)$.

Furthermore, if H is an induced disconnected subgraph of $G = K_{n_1} \oplus (G' \cup {}^{n_2}K_1)$ such that $|V(H)| < |V(G)|$ then H is an induced disconected subgraph of $G' \cup {}^{n_2}K_1$. Then if $H \leq {}^{n_2}K_1$, $\omega(H) = \psi(H)$. Now, if $H \not\leq {}^{n_2}K_1$ then $\psi(H) = \psi(H \setminus {}^{n_2}K_1)$ and $\omega(H) = \omega(H \setminus {}^{n_2}K_1)$. Since $H \setminus {}^{n_2}K_1 \leq G'$ and G' is a proper induced connected subgraph of G, by arguments like those in 1., 2. and 3. G' is $\omega\psi$-perfect, in particular $\omega(H \setminus {}^{n_2}K_1) = \psi(H \setminus {}^{n_2}K_1)$, then $\omega(H) = \psi(H)$.
Finally, we will prove that $\omega(G) = \psi(G)$. Note that $\omega(G) = \omega(G') + n_1$ and $\omega(G') \geq 2$. Suppose that $\omega(G') + n_1 < \psi(G)$. Let $\varsigma \colon V(G) \to [\psi(G)]$ a complete coloring of G with $\psi(G)$ colors. First, we suppose that $n_2 > 0$; let u be a vertex of ${}^{n_2}K_1$. As the neighbours of u are n_1 then u meets at most n_1 different chromatic class from u, then there exists another vertex v such that $\varsigma(u) = \varsigma(v)$ and clearly $N(u) \subseteq N[v]$. Then $\psi(G) = \psi(G \setminus u)$ but $\omega(G) = \omega(G \setminus u)$ and $G \setminus u$ is an induced subgraph of G with

$\omega(G \setminus u) < \psi(G \setminus u)$ which is newly a contradiction. Then, if $\omega(G') + n_1 < \psi(G)$ we have that $G = K_{n_1} \oplus G'$. Now, if there exist u and v in K_{n_1} such that $\varsigma(u) = \varsigma(v)$, clearly $N(u) \subseteq N[v]$ then $\psi(G) = \psi(G \setminus u)$ but $\omega(G) = \omega(G \setminus u)$ and $G \setminus u$ is an induced subgraph of G, which newly is a contradiction. Hence $\varsigma(u) \neq \varsigma(v)$ for all $u, v \in V(K_{n_1})$. Now, since G' is not a complete graph, $\omega(G') \geq 2$ and by induction hypothesis $\psi(G) \geq \omega(G') + n_1$ then there exist at least two colors i and j that do not appear in vertices of K_{n_1} and there exists $xy \in E(G')$ such that $\varsigma(x) = i$ and $\varsigma(y) = j$. Then $\psi(G') = \psi(G) - n_1$ and on the other hand $\omega(G') = \omega(G) - n_1$. By argumens like those in 1., 2. and 3. this is a contradiction. Hence G is $\omega\psi$-perfect. □

Proof of Theorem 18. The proof of $\langle 1 \rangle \Rightarrow \langle 2 \rangle$ is immediate from (2.1).

To prove $\langle 2 \rangle \Rightarrow \langle 3 \rangle$ note that if $H \in \{C_4, P_4, P_3 \cup K_2, {}^3K_2\}$ then $\chi(H) = 2$ and $\psi(H) = 3$, hence the implication is true.

To prove $\langle 3 \rangle \Rightarrow \langle 4 \rangle$ we only note that if G is connected we apply Theorem 17. In other case let $\{G_1, \ldots, G_n\}$ be the connected components of G, then at most two of them are different of K_1 because in other case we have ${}^3K_2$ as induced subgraph of G which it is not possible. When one of them is a non complete graph then the rest of them are isolated vertices because in other case we have $P_3 \cup K_2$ as an induced subgraph of G. Then we have two different cases, either $G = K_{n_1} \cup K_{n_2} \cup {}^{n_3}K_1$ or $G = G' \cup {}^{n_2}K_1$ where $n_1 \in \mathbb{N}$ and $n_2, n_3 \in \mathbb{Z}^+$ and G' is connected non complete $(C_4, P_4, P_3 \cup K_2, {}^3K_2)$-free graph and the result follows.

Finally, the arguments to proof $\langle 4 \rangle \Rightarrow \langle 1 \rangle$ are exactly the same that those used to proved the Theorem 17, only it is important to note that if $G = K_{n_1} \cup K_{n_2} \cup {}^{n_3}K_1$ then Lemma 19 states that $\omega(G) = \psi(G)$, and is also necessary to prove the equality for all induced subgraph but this analysis is similar that those made in the Lemma 19. To prove the $\omega\psi$-perfectness of G when $G = G' \cup {}^{n_2}K_1$ we also use similar arguments to those in the proof of Theorem 17. □

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Chapter 3

Achromatic and pseudoachromatic indices of the complete graph

It is central in Graph Theory to study the behavior of any parameter in complete graphs. In this chapter we endeavour to determine the pseudoachromatic number of the line graph of the complete graph $\psi_1(K_n) := \psi(L(K_n))$, also known as the pseudoachromatic *index* of the complete graph, and its close relation with $\alpha_1(K_n) := \alpha(L(K_n))$, the achromatic index of such a graph (see [AMS11, Bou78, Jam89, San05, TRJL88]).

The pseudoachromatic index of two infinite classes of complete graphs have been found so far, namely $\psi_1(K_{q^2+q+1})$ for any odd prime power q [Bou78], and $\psi_1(K_{q^2+2q+2})$ when q is a power of 2 [AMS11]. And the achromatic index of an infinite class of complete graphs have been found so far, namely $\alpha_1(K_{q^2+q+1})$ for any odd prime power q [Bou78]. Table 3.1 can be found in [HPW04] in which we describe the first values of the achromatic index of the complete graph.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$\alpha_1(K_n)$	1	3	3	7	8	11	14	18	22	27	32	39	39

Table 3.1: Exact values for $\alpha_1(K_n)$, $2 \leq n \leq 14$.

In particular, we prove the following:

Theorem 20. *If $x \geq 2$ is an integer, then*

$$\psi_1(K_m) \leq \begin{cases} g_m(x) - 1 & m \in \{4x^2 - x, \ldots, 4x^2 + 3x - 3\} \\ g_m(x) & m \in \{4x^2 + 3x - 2, 4x^2 + 3x - 1\} \\ f_m(x+1) & m \in \{4x^2 + 3x, \ldots, 4(x+1)^2 - (x+1) - 1\}. \end{cases}$$

where $g_m(x) := 2x(m - x - 1) + 1$ *and* $f_m(x+1) := \left\lfloor \frac{\binom{n}{2}}{x+1} \right\rfloor$.

On the other hand, supported in the combinatorial structure of projective planes of even order, we exhibit optimal complete edge-colorings of the complete graphs with some particular numbers of vertices; we show the following

Theorem 21. *Let $q \geq 4$ be a power of 2. If $n = q^2 + q + 1$ and $a \in \{0, 1, 2\}$ then*

$$\psi_1(K_{n+q-a}) = \alpha_1(K_{n+q-a}) = q(n + q - 2a).$$

Theorem 22. *Let $q \geq 4$ be a power of 2. If $n = q^2 + q + 1$ and $a \in \{3, 4, \ldots, \frac{q}{2} + 1\}$ then*

$$\psi_1(K_{n+q-a}) \geq q(n + q - 2a).$$

Putting Theorem 20 and Theorem 22 together, we obtain the following set of exact values for the psudoachromatic index of the complete graph:

Corollary 23. *Let $q \geq 4$ be a power of 2. If $a \in \{3, 4, \ldots, \left\lceil \frac{1+\sqrt{4q+9}}{2} \right\rceil - 1\}$ and $n = q^2 + q + 1$ then*

$$\psi_1(K_{n+q-a}) = q(n + q - 2a).$$

Theorem 21, when $q = 4$ and $a = 0$, states that $\alpha_1(K_{25}) = 100$. This value was given earlier by Jamison [Jam89] using some type of diagrams. For $q = 2$, Theorems 22 works for $\psi_1(K_{q^2+q+1-a})$ and $a \in \{0, 1\}$, and that give the same values that in Table 3.1.

In Section 3.1 we prove Theorem 20 and the upper bounds of Theorem 21 and Corollary 23. In Section 3.2 we exhibit close related edge-colorings of the complete graph to prove Theorem 21 and Theorem 22. The results of this chapter are contained in [AMRS14, AMRS]).

3.1 Upper bound

We use the following result due to Jamison [Jam89]:

Lemma 24. *If $x \geq 2$ is an integer, then*

$$\psi_1(K_m) \leq \begin{cases} g_m(x) & \text{if } m \in \{4x^2 - x, \dots, 4x^2 + 3x - 1\}, \\ f_m(x+1) & \text{if } m \in \{4x^2 + 3x, \dots, 4(x+1)^2 - (x+1) - 1\} \end{cases}$$

where $g_m(x) := 2x(m - x - 1) + 1$ *and* $f_m(x+1) := \left\lfloor \frac{\binom{m}{2}}{x+1} \right\rfloor$. □

Lemma 24 can be deduced by the following facts: Given $\varsigma\colon V(G) \to [\psi_1(K_m)]$ a complete coloring of the line graph G of K_m with $\psi_1(K_m)$ colors, let x be the size of the smallest chromatic class H in the colored graph G. It follows that the number of chromatic classes, $\psi_1(K_m)$, is at most $\lfloor \binom{m}{2}/x \rfloor$. On the other hand, it is not hard to see that the number of vertices, which are not in H, but are adjacent to some vertex in H is at most $2x(m-x-1)$. Thus, since ς is complete the number of chromatic classes $\psi_1(K_m)$ is at most $2x(m-x-1)+1$. Therefore

$$\psi_1(K_m) \leq \max\left\{\min\{g_m(x), f_m(x) \colon x \in \mathbb{N}\}\right\}.$$

From here, and by a detailed and technical analysis, the order between the values of $2x(m-x-1)+1$ and $\lfloor \binom{m}{2}/x \rfloor$ in terms of m and x can be obtained, and it yields Lemma 24.

In Table 3.2 the first values of the pseudoachromatic index of the complete graph are described; they follow from Table 3.1 and Lemma 24 except for $m = 4$ and $m = 12$, which are easy to prove. For $m = 12$, if we suppose that $\psi_1(K_{12}) = 33$ then each chromatic class is a matching.

n	2	3	4	5	6	7	8	9	10	11	12	13
$\psi_1(K_m)$	1	3	4	7	8	11	14	18	22	27	32	39

Table 3.2: Exact values for $\psi_1(K_m)$, $2 \le m \le 13$.

Corollary 25. *Let q be even, $n = q^2 + q + 1$ and $a \in \{0, 1, \ldots, \frac{q}{2} + 1\}$, then*

$$\psi_1(K_{n+q-a}) \le f_{n+q-a}(\tfrac{q}{2} + 1).$$

If $a \in \{0, 1, \ldots, \left\lceil \frac{1+\sqrt{4q+9}}{2} \right\rceil - 1\}$, then

$$\psi_1(K_{n+q-a}) \le q(n + q - 2a).$$

Proof. Since $4\left(\frac{q}{2}\right)^2 + 3\left(\frac{q}{2}\right) = q^2 + \frac{3q}{2} \le n + q - a \le q^2 + \frac{7q}{2} + 2 = 4\left(\frac{q+2}{2}\right)^2 - \left(\frac{q+2}{2}\right) - 1$, then by Lemma 24,

$$\psi_1(K_{n+q-a}) \le f_{n+q-a}(\tfrac{q}{2} + 1) = \left\lfloor \frac{(n+q-a)(n+q-a-1)}{2(\frac{q+2}{2})} \right\rfloor.$$

After some easy calculations we see that

$$\left\lfloor \frac{(n+q-a)(n+q-a-1)}{q+2} \right\rfloor = q(n+q-2a) + \left\lfloor \frac{a^2-a}{q+2} \right\rfloor.$$

On the other hand,

$$\frac{a^2-a}{q+2} < 1 \Leftrightarrow a^2 - a - (q+2) < 0 \Leftrightarrow \left(a - \frac{1-\sqrt{4q+9}}{2}\right)\left(a - \frac{1+\sqrt{4q+9}}{2}\right) < 0$$

so, $\frac{1-\sqrt{4q+9}}{2} < a < \frac{1+\sqrt{4q+9}}{2}$ and then $a \in \{0, \ldots, \left\lceil \frac{1+\sqrt{4q+9}}{2} \right\rceil - 1\}$ and the result follows. □

The following theorem improves Lemma 24:

Proof of Theorem 20. Let $x \geq 2$ and let $n \in \{4x^2-x, \dots, 4x^2+3x-3\}$. By Lemma 24, $\psi_1(K_n) \leq g_n(x)$. We will prove that $\psi_1(K_n) \leq g_n(x) - 1$, to do this, we suppose that $\psi_1(K_n) = g_n(x)$ and finally arrive to a contradiction. Let $\varsigma\colon E \to [g_n(x)]$ be a complete coloring with $2x(n-x-1)+1$ colors. First of all we will prove that no chromatic class can have less than x edges. Suppose there exist a color class C with s edges such that $s < x$, then C will be adjacent to at most $\binom{2s}{2} - s + 2s(n-2s) = 2s(n-s-1)$ edges, but $2s(n-s-1) < 2x(n-x-1)$, in consequence C could not be adjacent to all other color classes. Then, each color class has at least x edges. Suppose now that there exist a color class C with exactly x edges then it is clear that C is adjacent to exactly $2x(n-x-1)$ other edges and also they must all have different colors, otherwise C does not meet all the other color classes. Note that the only way to get this is when C is a matching. Since each color class has at least x edges, the number of color classes with more than x edges is at most $\binom{n}{2} - xg_n(x)$, hence there are at least $g_n(x) - \{\binom{n}{2} - xg_n(x)\}$ color classes with x edges. For the rest of the proof, we need at least two color classes of size x, but as we see in the following this is true. In fact

$2 \leq g_n(x) - \{\binom{n}{2} - x(g_n(x))\}$ if and only if $n^2-(4x^2+4x+1)n+4x^3+8x^2+2x+2 \leq 0$,

$$\text{i.e., } \left(n - \frac{4x^2+4x+1-\sqrt{D_1}}{2}\right)\left(n - \frac{4x^2+4x+1+\sqrt{D_1}}{2}\right) \leq 0,$$

where $4x^2+2x-3/2 < \sqrt{D_1} = \sqrt{16x^4+16x^3-8x^2-7} < 4x^2+2x-1$, which is equivalent to $\sqrt{D_1} = 4x^2+2x-3/2+\epsilon$ for some $0 < \epsilon < 1/2$, and then

$$n \in \left[x + \frac{5}{4} - \frac{\epsilon}{2}, 4x^2+3x-\frac{1}{4}+\frac{\epsilon}{2}\right] \cap \{4x^2-x, \dots, 4x^2+3x-3\},$$

$$\text{ie, } n \in \{4x^2-x, \dots, 4x^2+3x-3\}.$$

Let C be a color class of size x. This class is a matching with $2x$ vertices and there are $\binom{2x}{2} - x$ edges in the induced subgraph $\langle C \rangle$ that are not in C. Therefore, each one of this edges has different color and each one of this colors is in a class with more than

x edges because they are adjacent to two edges of C.

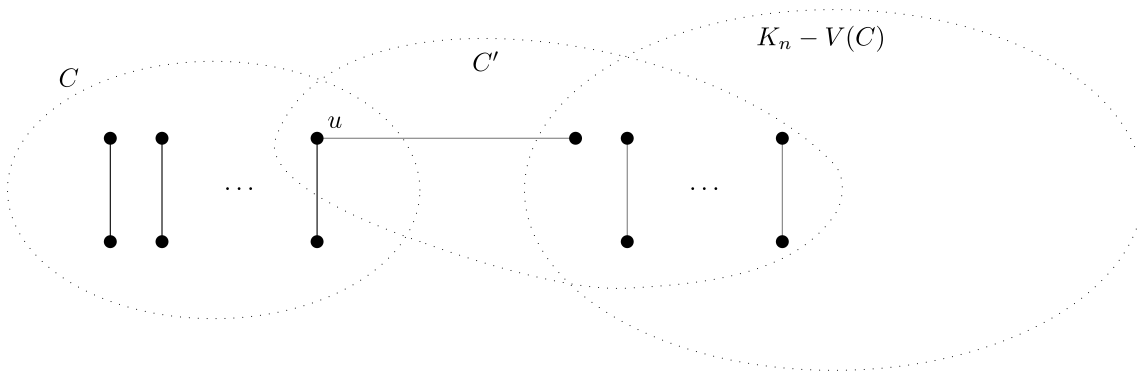

Figure 3.1: K_n

Let C' be another color class of size x. C' meets C in a vertex u. In Fig 3.1 we give a diagram of K_n indicating the vertices of C and C'. The $2x-2$ edges through u in $\langle C'\rangle \setminus C'$ meet C' in two vertices, therefore its color classes are larger than x. As mentioned before, there are at least $g_n(x) - \{\binom{n}{2} - x(g_n(x))\}$ color classes of size x, then there are at most $\binom{2x}{2} - x + (2x-2)(g_n(x) - \{\binom{n}{2} - x(g_n(x))\} - 1)$ chromatic classes of size greater than x and hence we have the following:

$$\binom{n}{2} - x(g_n(x)) \geq \binom{2x}{2} - x + (2x-2)(g_n(x) - \{\binom{n}{2} - x(g_n(x))\} - 1).$$

In other words, it must meet the following:

$$(2x-1)n^2 - (8x^3+4x^2-6x-1)n + 8x^4 + 12x^3 - 12x^2 - 2x \geq 0$$

$$\text{i.e., } \left(n - \frac{8x^3+4x^2-6x-1-\sqrt{D_2}}{4x-2}\right)\left(n - \frac{8x^3+4x^2-6x-1+\sqrt{D_2}}{4x-2}\right) \geq 0$$

where $\sqrt{D_2} = \sqrt{64x^6 - 144x^4 + 80x^3 - 4x^2 + 4x + 1} = 8x^3 - 9x + 5 + r_2$ and then

$$\left(n - (x + \frac{5}{4} + r_3)\right)\left(n - (4x^2 + 3x - \frac{9}{4} + r_4)\right) \geq 0,$$

$$\text{i.e., } n \in \left[4x^2 + 3x - \frac{9}{4} + r_4, \infty\right) \cap \{4x^2 - x, \ldots, 4x^2 + 3x - 3\} = \emptyset.$$

Then we have a contradiction and we conclude that if if $x \geq 2$ is an integer and $n \in \{4x^2 - x, \ldots, 4x^2 + 3x - 3\}$ then $\psi_1(K_n) \leq g_n(x) - 1$. □

The rest of the chapter will prove the lower bound of Theorem 21 and Theorem 22 given by edge-colorations.

3.2 Lower bound: edge-colorations

Since $\psi_1(K_{n+q-a}) \geq \alpha_1(K_{n+q-a})$ in order to prove Theorem 21 it suffices to show that $\alpha_1(K_{n+q-a}) \geq q(n+q-2a)$. Similary, in order to prove Theorem 22 we need to show that $\psi_1(K_{n+q-a}) \geq q(n+q-2a)$. We will do this by exhibiting, for each $a \in \{0,1,2\}$, a proper and complete edge-coloring of K_{n+q-a} with $q(n+q-2a)$ colors, and for $a \in \{3, 4, \ldots, \frac{q}{2}+1\}$, a complete edge-coloring of K_{n+q-a} with $q(n+q-2a)$ colors. For the construction of such an edge-coloring we need some definitions and remarks.

3.2.1 Basic terminology

A *projective plane* consists of a set of n points, a set of lines, and an incidence relation between points and lines having the following properties:

1. Given any two distinct points there is exactly one line incident to both of them.

2. Given any two distinct lines there is exactly one point incident to both of them.

3. There are four points, such that no line is incident to more than two of them.

Such a plane has $n = q^2 + q + 1$ points (for some number q) and n lines; each line contains $q+1$ points and each point belongs to $q+1$ lines. The number q is called the *order* of the projective plane. A projective plane of order q is called Π_q. It is not hard to prove that the projective space $\mathrm{PG}(2, q)$ is a projective plane, then if q is a prime power there exists a projective plane of order q, namely $\mathrm{PG}(2,q)$, which is called the *algebraic projective plane* since it arises from finite fields (see Section 1.3).

In the remainder of this section we exclusively work with $\mathrm{PG}(2,q)$ for q a power of two, but the proofs also work for any projective plane of even order q.

Let $\mathbb{P}$ be the set of points of $\mathrm{PG}(2,q)$ and let $\mathbb{L} = \{\mathbb{l}_1, \ldots, \mathbb{l}_n\}$ be the set of lines of $\mathrm{PG}(2,q)$. Now identify the points of $\mathrm{PG}(2,q)$ with the set of vertices of the complete graph K_n. In a natural way the set of points of each line of $\mathrm{PG}(2,q)$ induces in K_n a subgraph isomorphic to K_{q+1}. For each line $\mathbb{l}_i \in \mathbb{L}$ let $l_i = (V(l_i), E(l_i))$ be the subgraph of K_n induced by the set of $q+1$ points (vertices) of $\mathbb{l}_i$. By the properties of the projective plane for every pair $\{i,j\} \subseteq [n]$, $|V(l_i) \cap V(l_j)| = 1$ and then $\{E(l_1), \ldots, E(l_n)\}$ is a partition of $E(K_n)$. In this way when we said that a graph G isomorphic to K_n is a *representation of the projective plane* $\mathrm{PG}(2,q)$ we will understand that $V(G)$ is identified with the points of $\mathrm{PG}(2,q)$ and that there is a family of subgraphs (lines) $\{l_1, \ldots, l_n\}$ of G, such that for each line $\mathbb{l}_i$ of $\mathrm{PG}(2,q)$ l_i is the subgraph induced by the set of points of $\mathbb{l}_i$.

Let m be a positive integer. Given an edge-coloring $\Gamma\colon E(K_m) \to C$ we say that a vertex $x \in V(K_m)$ is an *owner* of a set of colors $C' \subseteq C$ whenever for every $c \in C'$ there is $y \in V(K_m)$, such that $\Gamma(xy) = c$; and given a subgraph G of K_m we say that G is an *owner* of a set of colors $C' \subseteq C$, if each vertex of G is an owner of C'. With this notation, Γ is a complete edge-coloring, if for every pair of colors in C there is a vertex in K_m, which is an owner of both colors.

Lemma 26. *Let $n = q^2+q+1$, with q a prime power, and let t be a positive integer. Let G be a subgraph of K_{n+t} isomorphic to K_n and assume that G is a representation of* $\mathrm{PG}(2,q)$*. Let $\Gamma\colon E(K_{n+t}) \to C$ be an edge-coloring of K_{n+t}. Suppose that each line l_i of G is an owner of a set of colors $C_i \subseteq C$. Then for every pair of colors $\{c_1, c_2\} \subseteq \bigcup_{i=1}^{q^2+q+1} C_i$ there is $x \in V(G)$, which is an owner of c_1 and c_2.*

Proof. Let $\{c_1, c_2\} \subseteq \bigcup_{i=1}^{q^2+q+1} C_i$. If there is $i \in [q^2+q+1]$, such that $\{c_1, c_2\} \subseteq C_i$, since l_i is the owner of C_i it follows that each $x \in V(l_i)$ is an owner of c_1 and c_2. If $c_1 \in C_i$

and $c_2 \in C_j$ then there is $x \in V(G)$ such that $x = V(l_i) \cap V(l_j)$, and since l_i and l_j are owners of C_i and C_j, respectively, x is an owner of c_1 and c_2. □

Now, we define different edge-colorations for some special graphs that will be used later.

It is well know (see [BM76], [Har69]) that any complete graph of even order r admits a 1-factorization and that any complete graph of odd order r admits a 2-factorization by hamiltonian cycles.

Definition 27. *Let r be an even integer. An edge-coloring $\Gamma\colon E(K_r) \to [r-1]$ will be said to be of* Type 1 *if for every $i \in \{1, 2, \dots, r-1\}$ the set $\{xy \in E(K_r)\colon \Gamma(xy) = i\}$ is a perfect matching of K_r.*

Definition 28. *Let r be an odd integer. An edge-coloring $\Gamma\colon E(K_r) \to [r]$ will be said to be of* Type 2 *if we obtain Γ in the following way: Let G be the graph (isomorphic to K_{r+1}) obtained by adding a new vertex x_0 and all the $x_0V(K_r)$-edges. Let Γ' be an edge-coloring of Type 1 of G and, for every $e \in E(K_r)$, let $\Gamma(e) := \Gamma'(e)$.*

Definition 29. *Let r be an odd integer. An edge-coloring $\Gamma\colon E(K_r - xy) \to [r-2]$ will be said to be of* Type 3 *if we obtain Γ in the following way: Let G be the graph (isomorphic to K_{r-1}) obtained by deleting from K_r a vertex x and all the $xV(K_{r-1})$-edges. Let Γ' be an edge-coloring of Type 1 of G and, for every $e \in E(K_r - xy)$, let $\Gamma(e) := \Gamma'(e)$ if $e \in E(G)$, and $\Gamma(xw) := \Gamma'(yw)\ \forall w \in V(G) \setminus \{y\}$.*

Definition 30. *Let r be an odd integer. An edge-coloring $\Gamma_i\colon E(C_r) \to \{i, i+\frac{r-1}{2}\}$ will be said to be of* Type 4 *if we obtain Γ_i in the following way: Let G be the graph obtained by deleting the edge $xy \in E(C_r)$. Let $\Gamma_i'\colon E(G) \to \{i, i+\frac{r-1}{2}\}$ be a proper edge-coloring of G (remember that* proper *means that each vertex has different colors in its edges) and, for every $e \in E(C_r)$, let $\Gamma_i(e) := \Gamma_i'(e)$ if $e \in E(G)$, and $\Gamma_i(xy) := \Gamma_i'(xw)$ for $w = N(x) \setminus \{y\}$. Observe that x is owner of one color.*

Definition 31. *Let r be an odd integer. An edge-coloring $\Gamma\colon E(K_r) \to [r-1]$ will be said to be of* Type 5 *in x_0 if we obtain Γ in the following way: Let $\{G_1, \dots, G_{\frac{r-1}{2}}\}$ be a 2-factorization of K_r such that $G_i = C_r\ \forall i \in \{1, \dots, \frac{r-1}{2}\}$. Let Γ_i be a edge-coloring of*

G_i of Type 4 for $x = x_0$ and, for every $e \in E(K_r)$, let $\Gamma(e) := \Gamma_i(e)$ if $e \in G_i$. Observe that x_0 is owner of $\frac{r-1}{2}$ colors.

3.2.2 The proper and complete edge-colorings

Proof of Theorem 21. To prove the lower boud of this theorem we will exhibit a complete proper edge-coloring of K_{n+q-a} with $q(n+q-2a)$ colors, for $n = q^2+q+1$. Let C be a set of $q(n+q-2a)$ colors and let $\{C_1, C_2, \dots, C_{q^2+q+2}\}$ be a partition of C in the following way: for $1 \le i \le (a+1)q+1$, C_i is a set of $q-1$ colors, and for $(a+1)q+2 \le i \le q^2+q+2$, C_i is a set of $q+1$ colors.

Let G be a subgraph of K_{n+q-a} isomorphic to K_n and let $H = K_{n+q-a} \setminus V(G)$. Clearly H is isomorphic to K_{q-a} and $K_{n+q-a} = G \oplus H$. Let G be a representation of $\mathrm{PG}(2,q)$, let $\{l_1, \dots, l_{q^2+q+1}\}$ be the set of lines of G, and let $V(H) = \{h_1, \dots, h_{q-a}\}$.

The remainder of the proof is divided into 3 cases given by $a = 0$, $a = 1$ and $a = 2$:

1. Case $a = 0$.

   Let $v_0 \in V(G)$ and let L_{v_0} be the set of lines of G through v_0. Without loss of generality, let $L_{v_0} = \{l_i \colon i \in [q+1]\}$ and $L_{v_0}^c = \{l_i \colon q+2 \le i \le q^2+q+1\}$.

   For each $l_i \in L_{v_0}$ let $p_i \in V(l_i) \cap V(l_{q+2}) \subseteq V(G)$. Let $E_1 = \{v_0 p_i \colon i \in [q]\} \subseteq E(G)$ and $E_2 = \{h_i p_i \colon i \in [q]\} \cup \{v_0 p_{q+1}\} \subseteq E(K_{n+q})$. In Figure 3.2 we give a description of K_{n+q}.

   We colo the edges of K_{n+q} in the following way:

   i) For each $l_i \in L_{v_0}^c$ let $\Gamma_i \colon E(l_i) \to C_i$ be an edge-coloring of Type 2. For each $l_i \in L_{v_0}^c$ and for each $x \in V(l_i)$ let $c(x, l_i)$ be the only color $c \in C_i$, such that for every $y \in V(l_i)$, $\Gamma_i(xy) \neq c$. Observe that $\bigcup_{x \in V(l_i)} c(x, l_i) = C_i$. For each $x \in V(G) \setminus \{v_0\}$ let $c(x) = \{c(x, l_i) \colon x \in V(l_i) \text{ and } l_i \in L_{v_0}^c\}$. Given that for each $x \in V(G) \setminus \{v_0\}$ there are q lines l in $L_{v_0}^c$, such that $x \in V(l)$, $c(x)$ is a set of q colors.

   For each $x \in V(G) \setminus \{v_0, p_1, \dots, p_q\}$, we color the set of q edges $\{xh \colon h \in V(H)\}$ with the set of colors $c(x)$. For each $p_j \in \{p_1, \dots, p_q\}$ we color the set of q edges

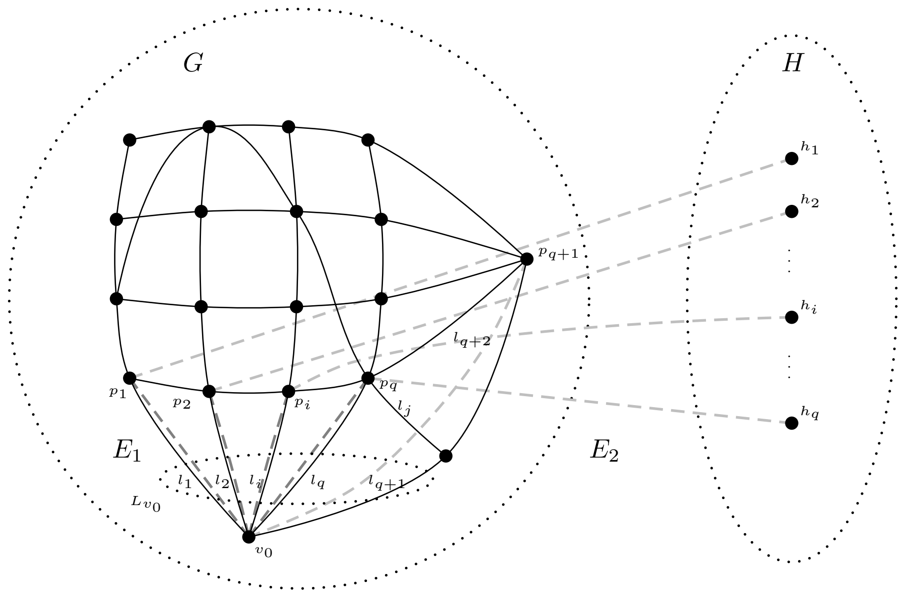

Figure 3.2: K_{n+q}.

$\{p_jh\colon h \in V(H) \setminus \{h_j\}\} \cup \{p_jv_0\}$ with the set of colors $c(p_j)$. Observe that we are coloring edges in E_1 and avoid coloring edges in E_2.

In this way, each line $l_i \in L^c_{v_0}$ is an owner of C_i; and at this point we have assigned a color to each edge of $E_1 \cup \bigcup_{l \in L^c_{v_0}} E(l)$ and, except for those edges in E_2, to all the $(V(G) \setminus \{v_0\})V(H)$-edges. The coloring so far is proper.

ii) For each $l_i \in L_{v_0}$ let l'_i be the subgraph of G obtained by deleting v_0 from l_i. Let $\Gamma_i\colon E(l'_i) \to C_i$ be an edge-coloring of Type 1. For each $i \in [q+1]$ let $J_i = \{v_0z\colon z \in V(l'_i) \setminus \{p_i\}\}$ (observe that $J_i = \{v_0z\colon z \in V(l_i)\} \setminus (E_1 \cup E_2)$). For $i \in [q]$ we color the set of $q-1$ edges in J_i with the set of colors C_{i+1} and color the set of $q-1$ edges J_{q+1} with the set of colors C_1. In this way each line l_i in L_{v_0} is an owner of C_i, and since for every pair $\{i, j\} \subseteq [q+1]$, $V(l'_i) \cap V(l'_j) = \emptyset$ there is no pair of edges of the same color incident with the same vertex.

Now we shall assign colors to the edges in $E_2 \cup E(H)$ and to all the $\{v_0\}V(H)$-edges.

iii) Let $H' = H \oplus v_0$ and let $\Gamma_{q^2+q+2}\colon E(H') \to C_{q^2+q+2}$ be an edge-coloring of Type 2 and for each $x \in V(H')$ let $c(x, H')$ be the only color $c \in C_{q^2+q+2}$, such that for every $y \in V(H')$ $\Gamma_{q^2+q+2}(xy) \neq c$. For each $x \in V(H')$ there is $y_x \in \{p_i\colon 1 \leq i \leq q+1\}$, such that $xy_x \in E_2$. Let $\Gamma(xy_x) = c(x, H')$. In this way H' is an owner of C_{q^2+q+2}.

We have already assigned a color to each edge in K_{n+q}. By the construction it follows that the resultant edge-coloring Γ of K_{n+q} is a proper edge-coloring. Let $\{c_1, c_2\} \subseteq C$. If $\{c_1, c_2\} \subseteq \bigcup_{i=1}^{q^2+q+1} C_i$, since every line l_i in G is an owner of C_i, it follows by Lemma 26 that there is $x \in V(G)$, which is an owner of both colors. Analogously, if $\{c_1, c_2\} \subseteq C_{q^2+q+2}$, since H' is an owner of C_{q^2+q+2}, every $x \in V(H')$ is an owner of both colors.

Let us suppose $c_1 \in \bigcup_{i=1}^{q^2+q+1} C_i$ and $c_2 \in C_{q^2+q+2}$. If $c_1 \in C_j$ with $j \in [q+1]$, v_0 is an owner of c_1, and since $v_0 \in V(H')$, v_0 is also an owner of c_2. If $c_1 \in C_j$ with $q+2 \leq j \leq q^2+q+1$ then there is a vertex $x \in V(l_j)$ such that $c(x, l_j) = c_1$ and, by construction, there is $y \in V(H')$ such that $\Gamma_j(xy) = c_1$. Hence, y is an owner of c_1 and since $y \in V(H')$ y is an owner of c_2. Therefore, Γ is a complete proper edge-coloring of K_{n+q} and the lemma follows for $a = 0$.

2. Case $a = 1$.

Let $\{v_0, v_1\} \subseteq V(G)$ and ℓ^* be the unique line of G, such that $\{v_0, v_1\} \subseteq V(\ell^*)$. For $i \in \{0, 1\}$ let $L_{v_i} = \{\ell_j^{v_i}\colon j \in [q]\}$ be the set of lines $l \neq \ell^*$ of G such that $v_i \in V(l)$. Without loss of generality suppose that $\ell_i^{v_0} = l_i$, $\ell_i^{v_1} = l_{q+i}$ for $i \in [q]$, and $\ell^* = l_{2q+1}$. Let $L^c = \{l_{2q+2}, \ldots, l_{q^2+q+1}\}$.

Observe that for every $x \in V(G) \setminus V(\ell^*)$ there is a unique pair $\ell_i^{v_0} \in L_{v_0}$ and $\ell_j^{v_1} \in L_{v_1}$ such that $x = V(\ell_i^{v_0}) \cap V(\ell_j^{v_1})$. This allows us to assign coordinates to the vertices in $V(G) \setminus V(\ell^*)$ in the following way: for each pair $\ell_i^{v_0} \in L_{v_0}$ and $\ell_j^{v_1} \in L_{v_1}$ let $p_{(i,j)} \in V(\ell_i^{v_0}) \cap V(\ell_j^{v_1})$. Thus $V(G) \setminus V(\ell^*) = \{p_{(i,j)}\colon i, j \in [q]\}$.

Let $E_1 = \{v_0 p_{(i,q)}\colon i \in [q-1]\}$; $E_2 = \{v_1 x\colon x \in V(\ell^*) \setminus \{v_0, v_1\}\}$; $E_3 = \{v_1 p_{(q,i)}\colon i \in [q-1]\}$ and $E_4 = \{h_i p_{(i,q)}\colon i \in [q-1]\} \cup \{v_0 p_{(q,q)}\} \cup \{v_1 p_{(q,q)}\}$.

In Figure 3.3 we illustrate K_{n+q-1}.

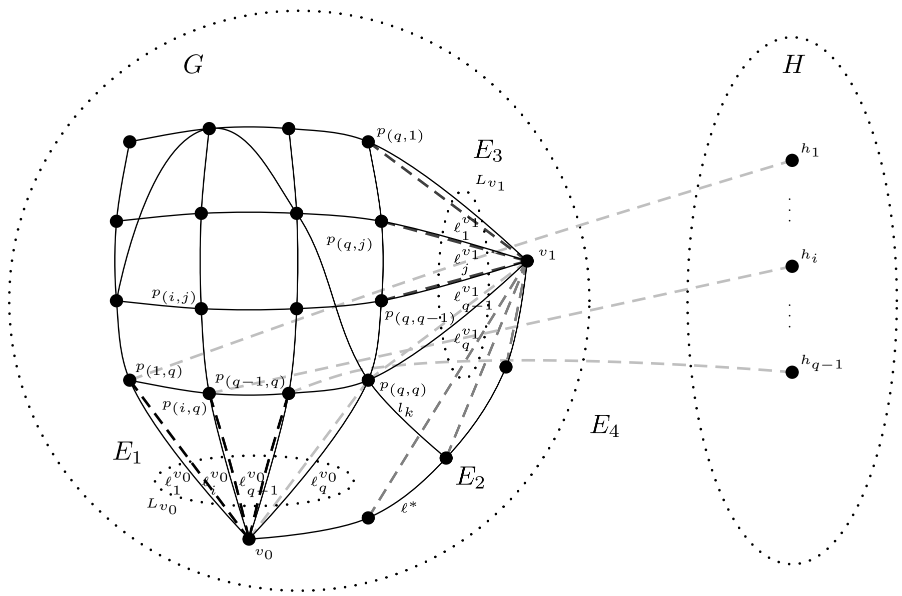

Figure 3.3: K_{n+q-1}.

We color the edges of K_{n+q-1} in the following way:

i) For each $l_i \in L^c$ let $\Gamma_i \colon E(l_i) \to C_i$ be an edge-coloring of Type 2. For each $l_i \in L^c$, and for each $x \in V(l_i)$, let $c(x, l_i)$ be the only color $c \in C_i$, such that for every $y \in V(l_i)$, $\Gamma_i(xy) \neq c$. Observe that $\bigcup_{x \in V(l_i)} c(x, l_i) = C_i$; for each $x \in V(G) \setminus \{v_0, v_1\}$ let $c(x) = \{c(x, l_i) \colon x \in V(l_i) \text{ and } l_i \in L^c\}$.

For each $x \in V(G) \setminus V(\ell^*)$, $c(x)$ is a set of $q-1$ colors (since there are $q-1$ lines l in L^c such that $x \in V(l)$). Since $x = p_{(i,j)}$ for some $i, j \in [q]$, on one hand, if $(i,j) = (q,q)$ or $j \neq q$ we color the set of $q-1$ edges $\{xh \colon h \in V(H)\}$ with the set of colors $c(x)$. On the other hand, if $j = q$ and $i \in [q-1]$, we color the set of $q-1$ edges $\{xh \colon h \in V(H) \setminus \{h_i\}\} \cup \{p_{(i,q)}v_0\}$ with the set of colors $c(x)$ (here we are coloring edges of E_1 and avoid coloring the edges of E_4).

If $x \in V(\ell^*) \setminus \{v_0, v_1\}$, $c(x)$ is a set of q colors (since there are q lines l in L^c such that $x \in V(l)$) we color the set of q edges $\{xh \colon h \in V(H)\} \cup \{v_1 x\}$ with the set of colors $c(x)$ (here we are coloring edges of E_2).

Observe that in this way each line l_i in L^c is an owner of C_i. At this point we have assigned a color to each edge in $E_1 \cup E_2 \cup \bigcup_{l \in L^c} E(l)$ and, except for those edges in E_4, to all the $(V(G) \setminus \{v_0, v_1\})V(H)$-edges.

ii) For each $\ell_i^{v_0} \in L_{v_0}$ let $\overline{\ell}_i^{v_0}$ be the subgraph of G obtained by deleting v_0 from $\ell_i^{v_0}$. Let $\Gamma_i \colon E(\overline{\ell}_i^{v_0}) \to C_i$ be an edge-coloring of Type 1. For each $\ell_i^{v_0} \in L_{v_0}$ let $J_i = \{v_0 z \colon z \in V(\overline{\ell}_i^{v_0}) \setminus \{p_{(i,q)}\}\}$ (observe that $J_i = \{v_0 z \colon z \in V(\ell_i)\} \setminus (E_1 \cup E_4)$). For $i \in [q-1]$, we color the set of $q-1$ edges J_i with the set of colors C_{1+i}, and coloring the set of $q-1$ edges J_q with the set of colors C_1. In this way each line $\ell_i^{v_0} = l_i$ in L_{v_0} is the owner of C_i and since for every pair $\{i, j\} \subseteq [q]$, $V(\overline{\ell}_i^{v_0}) \cap V(\overline{\ell}_j^{v_0}) = \emptyset$, there is no pair of edges of the same color incident with the same vertex.

Analogously, for each $\ell_i^{v_1} \in L_{v_1}$ let $\overline{\ell}_i^{v_1}$ be the subgraph of G obtained by deleting v_1 from $\ell_i^{v_1}$. Let $\Gamma_{q+i} \colon E(\overline{\ell}_i^{v_1}) \to C_{q+i}$ be an edge-coloring of Type 1 (recall that $\ell_i^{v_1} = l_{q+i}$ for $i \in [q]$). For each $\ell_i^{v_1} \in L_{v_1}$ let $J_i = \{v_1 z \colon z \in V(\overline{\ell}_i^{v_1}) \setminus \{p_{(q,i)}\}\}$(observe that $J_i = \{v_1 z \colon z \in V(\ell_i)\} \setminus E_3$). For $i \in [q-1]$, we color the set of $q-1$ edges J_i with the set of colors C_{q+1+i}, and the set of $q-1$ edges J_q with the set of colors C_{q+1}.

Let $\overline{\ell}^*$ be the subgraph of G obtained by deleting v_1 from ℓ^* (recall that $\ell^* = l_{2q+1}$). Let $\Gamma_{2q+1} \colon E(\overline{\ell}^*) \to C_{2q+1}$ be an edge-coloring of Type 1. We color the set of $q-1$ edges $E_3 = \{v_1 p_{(q,i)} \colon i \in [q-1]\}$ with the set of colors C_{2q+1}. Since $\{p_{(q,i)} \colon i \in [q-1]\} \cap V(\overline{\ell}^*) = \emptyset$ there is no pair of edges of the same color incident with the same vertex.

In this way, each line l_i in $L_{v_0} \cup L_{v_1}$ is an owner of C_i, and $\ell^* = l_{2q+1}$ is an owner of C_{2q+1}. Now it just remains to assign colors to the edges in $E_4 \cup E(H)$, to the edge $v_1 v_0$ and to all the $(\{v_0, v_1\})V(H)$-edges.

iii) Let $H' = H \oplus v_0 \oplus v_1$ and let $\Gamma_{q^2+q+2} \colon E(H') \to C_{q^2+q+2}$ be an edge-coloring of Type 2 and for each $x \in V(H')$, let $c(x, H')$ be the only color $c \in C_{q^2+q+2}$ such that for every $y \in V(H')$ $\Gamma_{q^2+q+2}(xy) \neq c$. For each $x \in V(H')$ there is

$y_x \in \{p_{(i,q)} \colon i \in [q]\}$, such that $xy_x \in E_4$. Let $\Gamma(xy_x) = c(x, H')$; in this way, H' is an owner of C_{q^2+q+2}.

We have already assigned a color to each edge in K_{n+q-1}. By the construction it follows that the resulting edge-coloring Γ is a proper edge-coloring of K_{n+q-1}. Let $\{c_1, c_2\} \subseteq C$. If $\{c_1, c_2\} \subseteq \bigcup_{i=1}^{q^2+q+1} C_i$ since every line l_i in G is an owner of C_i by Lemma 26 it follows that there is $x \in V(G)$ which is an owner of both colors. Analogously, if $\{c_1, c_2\} \subseteq C_{q^2+q+2}$, since H' is an owner of C_{q^2+q+2}, every $x \in V(H')$ is an owner of both colors. Let us suppose $c_1 \in \bigcup_{i=1}^{q^2+q+1} C_i$ and $c_2 \in C_{q^2+q+2}$. If $c_1 \in C_j$, with $j \in [q]$, v_0 is an owner of c_1 and since $v_0 \in V(H')$, v_0 is also an owner of c_2. If $c_1 \in C_j$ with $q+1 \leq j \leq 2q+1$, v_1 is an owner of c_1 and since $v_1 \in V(H')$, also is an owner of c_2. If $c_1 \in C_j$, with $2q+2 \leq j \leq q^2+q+1$, there is a vertex $x \in V(l_j)$ such that $c(x, l_j) = c_1$ and, by construction, there is $y \in V(H')$ such that $\Gamma_j(xy) = c_1$. Hence y is an owner of c_1 and since $y \in V(H')$, y is an owner of c_2. Therefore Γ^* is a complete proper edge-coloring of K_{n+q-1} and the lemma follows for $a = 1$.

3. Case $a = 2$.

   Let $\{v_0, v_1, v_2\} \subseteq V(G)$ be three non-colinear points in $\mathrm{PG}(2,q)$. For $\{i,j\} \subset \{0,1,2\}$, with $i \neq j$, let $\ell_{i,j}$ be the line in G such that $\{v_i, v_j\} \subseteq V(\ell_{i,j})$. For $i \in \{0,1,2\}$ let $L_{v_i} = \{\ell^j_{v_i} \colon j \in [q-1]\}$ be the set of lines l of G, such that $v_i \in V(l)$ and $l \notin \{\ell_{i,j} \colon \{i,j\} \subset \{0,1,2\}; i \neq j\}$.

   Now we assign coordinates to the set of vertices $(V(\ell_{0,1}) \cup V(\ell_{0,2}) \cup V(\ell_{1,2})) \setminus \{v_0, v_1, v_2\}$ in the following way:

   For each $j \in [q-1]$ let $p^j_{0,1} \in V(\ell_{0,1}) \cap V(\ell^j_{v_2})$, $p^j_{0,2} in V(\ell_{0,2}) \cap V(\ell^j_{v_1})$ and $p^j_{1,2} \in V(\ell_{1,2}) \cap V(\ell^j_{v_0})$.

   Since $\{l_1, \ldots, l_{q^2+q+1}\}$ is the set of lines of G, without loss of generality, for each $j \in [q-1]$ and each $i \in \{0,1,2\}$, $\ell^j_{v_i} = l_{(q-1)i+j}$; $\ell_{0,1} = l_{3q-2}; \ell_{0,2} = l_{3q-1}$ and $\ell_{1,2} = l_{3q}$. Let $L^c = \{l_{3q+1}, \ldots, l_{q^2+q+1}\}$.

Let $E_0 = \{v_0 p^j_{1,2} \colon j \in [q-1]\}$; $E_1 = \{v_1 p^j_{0,2} \colon j \in [q-1]\}$; $E_2 = \{v_2 p^j_{0,1} \colon j \in [q-1]\}$ and $E_3 = \{v_1 h \colon h \in V(H)\} \cup \{v_0 v_1\}$. In Figure 3.4 we illustrate K_{n+q-2}.

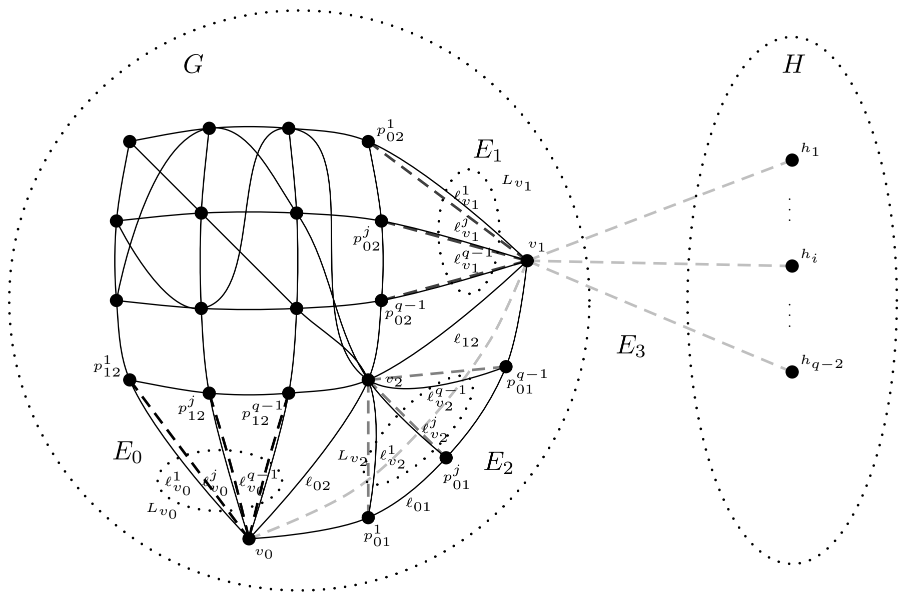

Figure 3.4: K_{n+q-2}.

We color the edges of K_{n+q-2} in the following way:

i) For each $l_i \in L^c$ let $\Gamma_i \colon E(l_i) \to C_{i+1}$ be an edge-coloring of Type 2. For each line $l_i \in L^c$, and for each $x \in V(l_i)$, let $c(x, l_i)$ be the only color $c \in C_{i+1}$, such that for every $y \in V(l_i)$, $\Gamma_i(xy) \neq c$. Observe that $\bigcup_{x \in V(l_i)} c(x, l_i) = C_i$. For each $x \in V(G) \setminus \{v_0, v_1, v_2\}$ let $c(x) = \{c(x, l_i) \colon x \in V(l_i) \text{ and } l_i \in L^c\}$.

Let $x \in V(G) \setminus \{v_0, v_1, v_2\}$. If $x \in V(G) \setminus (V(\ell_{0,1}) \cup V(\ell_{0,2}) \cup V(\ell_{1,2}))$, $c(x)$ is a set of $q-2$ colors (since there are $q-2$ lines l in L^c such that $x \in V(l)$). We color the set of $q-2$ edges $\{xh \colon h \in V(H)\}$ with the set of colors $c(x)$. If $x \in (V(\ell_{0,1}) \cup V(\ell_{0,2})) \cup V(\ell_{1,2}) \setminus \{v_0, v_1, v_2\}$, $c(x)$ is a set of $q-1$ colors (since there are $q-1$ lines l in L^c, such that $x \in V(l)$) and $x = p^j_{i,k}$ for some $j \in [q-1]$ and $\{i,k\} \subset \{0,1,2\}$. Let $r = \{0,1,2\} \setminus \{i,k\}$ and coloring the set of $q-1$ edges $\{xh \colon h \in V(H)\} \cup \{v_r p^j_{i,k}\}$ with the set of colors $c(x)$ (here we are coloring edges of E_r, with $r \in \{0,1,2\}$).

In this way, each line l_i in L^c is an owner of C_{i+1}. At this point we have assigned a color to each edge in $\big(\bigcup_{i\in\{0,1,2\}} E_i\big) \cup \bigcup_{l\in L^c} E(l)$, and to all the $(V(G)\setminus\{v_0,v_1,v_2\})V(H)$-edges.

ii) Let $i \in \{0,1,2\}$. For each $\ell^j_{v_i} \in L_{v_i}$ let $\ell^{j'}_{v_i}$ be the subgraph of G obtained by deleting v_i from $\ell^j_{v_i}$. Let $\Gamma_{(q-1)i+j}\colon E(\ell^{j'}_{v_i}) \to C_{(q-1)i+j}$ be an edge-coloring of Type 1 (recall that for each $j \in [q-1]$ and $i \in \{0,1,2\}$, $\ell^j_{v_i} = l_{(q-1)i+j}$). For each $\ell^j_{v_i} \in L_{v_i}$ let $J^j_i = \{v_iz\colon z \in V(\ell^{j'}_{v_i}) \setminus \{p^j_{s,k}\}; \{s,k\} = \{0,1,2\}\setminus\{i\}$ (observe that $J^j_i = \{v_iz\colon z \in V(\ell^j_{v_i})\} \setminus E_i$). Similarly as before, for $j \in [q-2]$ and $i \in \{0.1.2\}$, we color the set of $q-1$ edges in J^j_i with the set of colors $C_{(q-1)i+j+1}$, and the set of $q-1$ edges J^{q-1}_i with the set of colors $C_{(q-1)i+1}$. In this way, for $i \in \{0,1,2\}$, each line l_j in L_{v_i} is the owner of C_j, and since for every $i \in \{0,1,2\}$ and every pair $\{k,j\} \subseteq [q-1]$, $V(\ell^{j'}_i) \cap V(\ell^{k'}_i) = \emptyset$, there is no pair of edges of the same color incident with the same vertex.

Let $\ell'_{0,1}$ be the subgraph of G obtained by deleting v_0 from $\ell_{0,1}$. Let $\Gamma_{3q-2}\colon E(\ell'_{0,1}) \to C_{3q-2}$ be an edge-coloring of Type 1 (recall that $\ell_{0,1} = l_{3q-2}$). Let $\ell'_{0,2}$ be the subgraph of G obtained by deleting v_0 from $\ell_{0,2}$, and let $\Gamma_{3q-1}\colon E(\ell'_{0,2}) \to C_{3q-1}$ be an edge-coloring of Type 1. Similarily, let $\ell'_{1,2}$ be the subgraph of G obtained by deleting v_1 from $\ell_{1,2}$; and let $\Gamma_{3q}\colon E(\ell'_{1,2}) \to C_{3q}$ be an edge-coloring of Type 1.

We color the set $\{v_0p^j_{0,1}\colon j \in [q-1]\}$ with the set of colors C_{3q-1} (observe that $\Gamma[E(\ell'_{0,1})] = C_{3q-2}$); the set $\{v_0p^j_{0,2}\colon j \in [q-1]\}$ with C_{3q-2} and we color E_3 with C_{3q}. Since $V(\ell'_{0,1}) \cap V(\ell'_{0,2}) = \emptyset$ and $V(\ell'_{1,2}) \cap (V(H) \cup \{v_0,v_1\}) = \emptyset$ there is no pair of edges of the same color incident with the same vertex.

In this way, each line l_i in G is an owner of C_i. Now it just remains to assign colors to the edges in $E(H)$ and $\{v_1p^j_{1,2}\colon j \in [q-1]\}$, to all the $(\{v_0,v_2\})V(H)$-edges and to the edges v_0v_2, v_1v_2.

iii) Let $H' = H \oplus v_0 \oplus v_2$ and let $\Gamma_{3q+1}\colon E(H') \to C_{3q+1}$ be an edge-coloring of Type 1 and we color the set of $q-1$ edges $\{v_1p^j_{1,2}\colon j \in [q-1]\}$ with C_{3q+1}. In this way, the graph $H' \oplus v_1$ is an owner of C_{3q+1} (although the edge v_1v_2 has no color assigned yet). We color v_1v_2 with any color in C which neither v_1 nor v_2 are owners.

We have already assigned a color to each edge in K_{n+q-2}. By the construction it follows that the resulting edge-coloring Γ is a proper edge-coloring of K_{n+q-2}. Let $\{c_1, c_2\} \subseteq C$. If $\{c_1, c_2\} \subseteq \bigcup_{i \neq 3q+1} C_i$, since every line l_i in G is an owner of C_i if $i < 3q+1$, and every line l_i is an owner of C_{i+1} if $i > 3q+1$, by Lemma 26 it follows that there is $x \in V(G)$ which is an owner of both colors. Analogously, if $\{c_1, c_2\} \subseteq C_{3q+1}$, since $H' \oplus v_1$ is an owner of C_{3q+1}, there is $x \in V(H' \oplus v_1)$ which is an owner of both colors. Let us suppose $c_1 \in \bigcup_{i \neq 3q+1} C_i$ and $c_2 \in C_{3q+1}$. If $c_1 \in C_j$, with $j \in [3q]$, we have that for some $i \in \{0, 1, 2\}$, v_i is an owner of c_1, and since $v_i \in V(H' \oplus v_1)$, v_i is also an owner of c_2. If $c_1 \in C_j$, with $3q+2 \leq j \leq q^2+q+2$, there is $x \in V(l_j)$ such that $c(x, l_j) = c_1$ and, by construction, there is $y \in V(H' \oplus v_1)$, such that $\Gamma_j(xy) = c_1$. Hence y is an owner of c_1 and since $y \in V(H' \oplus v_1)$, y is an owner of c_2. Therefore Γ is a complete proper edge-coloring of K_{n+q-2} and the lower bound on theorem follows.

Finally, from Corollary 25 we know that $\psi_1(K_{n+q-a}) \leq q(n+q-2a)$ for q an even number, $n = q^2+q+1$ and $a \in \{0, 1, 2\}$. On the other hand, since there exist a projective plane of ordes q when q is a power of 2, it follows that $q(n+q-2a) \leq \alpha_1(K_{n+q-a})$. By equation (1.1), $\alpha_1(K_{n+q-a}) \leq \psi_1(K_{n+q-a})$, and the result follows. □

3.2.3 The complete edge-colorings

Proof of Theorem 22. To prove this theorem we will exhibit a complete edge-coloring of K_{n+q-a} with $q(n+q-2a)$ colors for $n = q^2+q+1$. Let $\mathcal{C}$ be a set of $q(n+q-2a)$ colors and let $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\}$ be a partition of $\mathcal{C}$ in the following way: $\mathcal{C}_i$ is a set of q colors, for $1 \leq i \leq q-2a+3$; $\mathcal{C}_i$ is a set of $q-1$ colors, for $q-2a+4 \leq i \leq a(q-1)+q+1$; $\mathcal{C}_i$ is a set of $q+1$ colors, for $a(q-1)+q+2 \leq i \leq q^2+q$ and $\mathcal{C}_n$ is a set of $q-1$ colors.

Let G be a subgraph of K_{n+q-a} isomorphic to K_n and let $H = K_{n+q-a} \setminus V(G)$ be. Clearly H is isomorphic to K_{q-a} and $K_{n+q-a} = G \oplus H$. Let G be a representation of $\mathrm{PG}(2, q)$ and let $L = \{l_1, \ldots, l_n\}$ be the set of lines of G.

Let $V(H) = \{h_1, \ldots, h_{q-a}\}$, let $v_0 \in V(G)$ be and let ℓ^* be a line l of G such that $v_0 \notin V(\ell^*)$.

Let W, U and V be a partition of $V(\ell^*)$ such that $W = \{w_1, \ldots, w_{q-2a+3}\}$, $U = \{u_1, \ldots, u_{a-2}\}$ and $V = \{v_1, \ldots, v_a\}$, then $\ell^* = \langle W \rangle \oplus \langle U \rangle \oplus \langle V \rangle$. Let $L_0 = \{\ell_x : x \in \ell^* \text{ and } v_0 \in \ell_x\} \subseteq L$ the set of lines through v_0. Let L_W, L_U and L_V be a partition of L_0 such that $L_W = \{\ell_x : x \in W\}$, $L_U = \{\ell_x : x \in U\}$ and $L_V = \{\ell_x : x \in V\}$.

For $i \in \{1, \ldots, a\}$, let $L_i = \{\ell_j^{v_i} : v_i \in \ell_j^{v_i}, j \in [q-1], \ell_j^{v_i} \neq \ell^*, \ell_j^{v_i} \notin L_0\}$ the set of lines through v_i different to ℓ^* and ℓ_{v_i}. For $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, q-1\}$, let $Z = \{z_j^{v_i} \in \ell_j^{v_i} \cap \ell_{v_{i+1}} : i \neq a\} \cup \{z_j^{v_a} \in \ell_j^{v_a} \cap \ell_{v_1}\} \subseteq V(G)$. Let $Y = V(G) \setminus \{Z \cup V(\ell^*) \cup \{v_0\}\}$.

Without loss of generality, let $l_i = \ell_{w_i}$ be for $i \in \{1, \ldots, q-2a+3\}$, $l_{i+q-2a+3} = \ell_{u_i}$ for $i \in \{1, \ldots, a-2\}$, $l_{i+q-a+1} = \ell_{v_i}$ for $i \in \{1, \ldots, a\}$ and let $l_{i(q-1)+2+j} = \ell_j^{v_i}$ for $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, q-1\}$. Finally, let $L' = \{l_i : a(q-1)+q+2 \leq i \leq q^2+q\}$ and $\ell^* = l_n$. In Fig 3.5 we illustrate K_{n+q-a}.

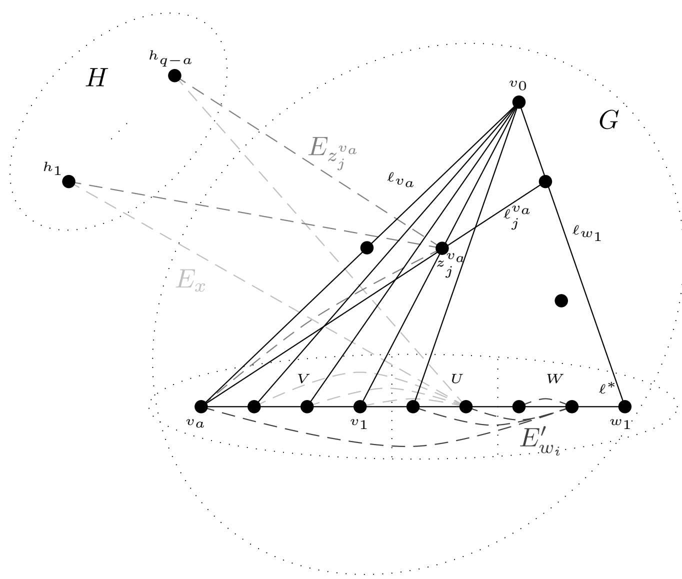

Figure 3.5: K_{n+q-a}.

With the aim to define some subset of edges of K_{n+q-a} we first define a special function h:

For $i \in \{1, \dots, q-2a+3\}$ and $j \in \{1, \dots, \frac{q}{2} - a + 1\}$, let

$$h(i+j) = \begin{cases} i+j & \text{if } i+j \leq q-2a+3, \\ i+j-(q-2a+3) & \text{if } i+j > q-2a+3, \end{cases}$$

and let

$$E'_{w_i} = \{w_i w_{h(i+1)}, \dots, w_i w_{h(i+\frac{q}{2}-a+1)}, w_i u_1, \dots, w_i u_{a-2}, w_i v_a\}$$

be a set of $\frac{q}{2}$ edges.

Now, we will define other subsets of edges of K_{n+q-a}.

For $x \in U \cup W$, let

$$E_x = \{xv_1, \dots, xv_{a-1}, xh_1, \dots, xh_{q-a}\}$$

be a set of $q-1$ edges. For each $z_j^{v_i}$, let

$$E_{z_j^{v_i}} = \{z_j^{v_i} v_i, z_j^{v_i} h_1, \dots, z_j^{v_i} h_{q-a}\}$$

be a set of $q-a+1$ edges. Let

$$E' = \{v_0 v_a, v_0 u_1, \dots, v_0 u_{a-2}, v_a u_1, \dots, v_a u_{a-2}\} \cup E(\langle U \rangle)$$

be a set of $\binom{a}{2}$ edges.

We color the edges of K_{n+q-a} in the following way:

1. For $l_i \in L_W$, let $\Gamma_i \colon E(l_i) \to \mathcal{C}_i$ be an edge-coloring of Type 5 in w_i and let $\mathcal{C}(w_i)$ be the subset of $\frac{q}{2}$ colors of $\mathcal{C}_i$ which w_i is not owner then we assignt to te set E'_{w_i} exactly the colors of $\mathcal{C}(w_i)$.

   In this way, each line $l_i \in L_W$ is an owner of $\mathcal{C}_i$, and we have assigned a color to each edge of $\bigcup_{i=1}^{q-2a+3} (E(l_i) \cup E'_i)$.

2. For each $l_i \in L_U \cup L_V$ let l'_i be the subgraph of G obtained by deleting the

edge v_0u_i from l_i if $i \in \{q-2a+4, \dots, q-a+1\}$ and the edge v_0v_i from l_i if $i \in \{q-a+2, \dots, q+1\}$. Let $\Gamma_i \colon E(l'_i) \to \mathcal{C}_i$ be an edge-coloring of Type 3.

For each $l_j \in L_i$, let $l'_j = l_j \setminus E_{z_j^{v_i}}$ be and let $\Gamma_j \colon E(l'_j) \to \mathcal{C}_j$ be an edge-coloring of Type 3.

Now, each line l_j in L_i is an owner of $\mathcal{C}_j$, and at this point we have assigned a color to each edge of $\bigcup_{i=1}^{q-2a+3} (E(l_i) \cup E'_i) \cup \bigcup_{i=q-2a+4}^{a(q-1)+q+1} (E(l'_i)$.

3. For each $l_i \in L'$ let $\Gamma_i \colon E(l_i) \to \mathcal{C}_i$ be an edge-coloring of Type 2. For each $l_i \in L'$, and for each $x \in V(l_i)$, let $c(x, l_i)$ be the only color $c \in \mathcal{C}_i$ such that for every $y \in V(l_i - x)$, $\Gamma_i(xy) \neq c$. Observe that $\bigcup_{x \in V(l_i)} c(x, l_i) = \mathcal{C}_i$. For each x in L' let $c(x) = \{c(x, l_i) \colon x \in V(l_i) \text{ and } l_i \in L'\}$ be.

   For each y in Y there are $a+1$ lines $l \notin L'$ such that $y \in V(l)$, then $c(y)$ is a set of $q-a$ colors. Color the set of $q-a$ edges $\{yh_1, \dots, yh_{q-a}\}$ with the set of colors $c(y)$.

   For each z in Z there are a lines $l \notin L'$ such that $z \in V(l)$, then $c(z)$ is a set of $q-a+1$ colors. Color the set of $q-a+1$ edges E_z with the set of colors $c(z)$.

   For each $x \in U \cup V$ there are 2 lines $l \notin L'$ such that $x \in V(l)$, then $c(x)$ is a set of $q-1$ colors. Color the set of $q-1$ edges E_x with the set of colors $c(x)$.

   Now it just remain to assign colors to the edges $H \oplus < V > \oplus \{v_0\}$ and E'.

4. Let $H' = H \oplus < V > \oplus v_0$ be and let $\Gamma_n \colon E(H' - v_0v_a) \to \mathcal{C}_n$ be an edge-coloring of Type 3. In this way, H' is an owner of $\mathcal{C}_n$.

5. Let $\Gamma \colon E' \to \{c\}$ be a edge-coloring where $c \in \mathcal{C}$.

   We have already assigned a color to each edge in K_{n+q-a}. If $\{c_1, c_2\} \subseteq \bigcup_{i=1}^{n-1} C_i$, then since every line l_i in G is an owner of C_i, by Lemma **??** it follows that there is $x \in V(G)$ which is an owner of both colors. Analogously, if $\{c_1, c_2\} \subseteq C_n$, since H' is an owner of C_n, there is $x \in V(H')$ which is an owner of both colors. Let us suppose $c_1 \in \bigcup_{i=1}^{n-1} C_i$ and $c_2 \in C_n$. If $c_1 \in C_j$ with $1 \leq j \leq a(q-1)+q+1$, there is a vertex $x \in V \cap V(l_j)$ and x is an owner of c_1, and since $x \in V(H')$, x is also an owner of c_2. If $c_1 \in C_j$ with $a(q-1)+q+2 \leq j \leq q^2+q$, there is a vertex $x \in V(l_j)$ such that $c(x, l_j) = c_1$ and, by construction, there is $y \in V(H')$ such that $\Gamma_j(xy) = c_1$. Hence y is an owner of c_1 and since $y \in V(H')$, y is an owner of c_2. Therefore Γ is a complete edge-coloring of K_{n+q-a} and the theorem follows.

□

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Chapter 4

Achromatic and pseudoachromatic indices of designs

Given a (v, κ)-design $D = (\mathcal{V}, \mathcal{B})$, a *block-coloring* (for short *coloring*) of D with k colors is a surjective function $\varsigma \colon \mathcal{B} \to [k]$. A coloring of D with k colors is *proper*, if any two different blocks $B, B' \in \mathcal{B}$ with $B \cap B' \neq \emptyset$ satisfy $\varsigma(B) \neq \varsigma(B')$. The *chromatic index* $\chi_1(D)$ of D is the smallest k such that there exists a proper coloring with k colors of D.

A coloring of a design D is *complete*, if each pair of colors appears on at least one point of D. The *achromatic index* $\alpha_1(D)$ of D is the largest number k for which there exists a complete and proper coloring of D with k colors. The *pseudoachromatic index* $\psi_1(D)$ of D is the largest number k for which there exists a complete coloring of D with k colors.

Clearly we have that

$$\chi_1(D) \leq \alpha_1(D) \leq \psi_1(D). \tag{4.1}$$

The well-known Erdős-Faber-Lovász Conjecture (for short EFL) [Erd76, Erd81] states (or, more precisely, a particular case of an equivalent formulation of it is) that

$$\chi_1(D) \leq v,$$

for D any (v,κ)-design. It is important to say that the EFL Conjecture is open even for the $(v,3)$-designs (Steiner triple systems $STS(v)$). In fact, in [CC82] it is proved that $\chi_1(D) < \frac{\kappa v}{\kappa-1}$, for any (v,κ)-design D; moreover, the main result of [CC82] states that $\chi_1(D) \leq v$, for cyclic designs $D = (\mathbb{Z}_v, \mathcal{B})$ (that is the mapping $i \mapsto i+1$ is an automorphism). For general results on the EFL Conjecture see [ARV, AV, CL88, CC82, Fab10, Kah92, KM08, MS10, RS07a, RS07b], and references therein.

If D is a $(v,2)$-design, then $\chi_1(D)$, $\alpha_1(D)$ and $\psi_1(D)$ are the usual *chromatic*, *achromatic* and *pseudoachromatic indices* respectively of the complete graph K_v (see [Gup69, HHP67] and Chapter 3). For instance, Vizing's Theorem verifies the EFL Conjecture, since $\chi_1(K_v)$ is at most v.

The main result of [BJV89] verified the EFL conjecture for finite projective spaces (see Section 1.3.2). It is not hard to see that $\chi_1(\Pi_q) = \alpha_1(\Pi_q) = \psi_1(\Pi_q) = v$, where $v = q^2+q+1$ is the number of points in any projective plane Π_q of order q (see Subsection 3.2.1).

The achromatic index of $STS(v)$ has been studied before. The following two theorems are proved in [CC83] (see also [RC92]):

Theorem 32. *For any $STS(v)$, $\alpha_1(STS(v)) \leq \frac{1}{\sqrt{2}}v^{1.5} + \Theta(n)$.*

Theorem 33. *For infinitely many v, there exists an $STS(v)$, such that $\alpha_1(STS(v)) \geq \frac{1}{2}v^{1.5} + \Theta(n)$.*

In the next section we give upper bounds for the achromatic and pseudoachromatic indices for some types of designs. In Section 4.2 we give the exact values for the achromatic and pseudoachromatic indices for the affine plane. The results of this chapter are contained in [AKRV].

4.1 Upper bound of $\psi_1(D)$

We obtain the following upper bound for the pseudoachromatic index for designs:

Theorem 34. *Let D be a (v,κ)-design. Then*

$$\psi_1(D) \leq \frac{\sqrt{v}(v-1)}{\kappa-1} < \frac{v^{1.5}}{\kappa-1}.$$

Proof. If r is the number of blocks incident to a point then the number of incidences of blocks is $v\binom{r}{2} \geq \binom{\psi_1(D)}{2}$ so that $\psi_1(D)^2 - \psi_1(D) \leq vr(r-1)$. Solving this inequality we get

$$\psi_1(D) \leq \frac{1+\sqrt{1+4vr(r-1)}}{2}.$$

Since $\sqrt{1+4vr(r-1)} \leq \sqrt{4vr^2}-1$, we get $\psi_1(D) \leq \sqrt{v}r$ and the result follows. □

For $k=3$, Theorem 34 improves Theorem 32.

4.1.1 Projective plane with resolvable lines

We use the next proposition in Theorem 36.

Proposition 35. *Let $\kappa > 1$ and $\gamma > 0$ be integers, then $(k-1)^{2\gamma} \equiv 1 \mod k$.*

Proof. Since $(k-1)^2 \equiv 1 \mod k$ then $(k-1)^{2\gamma} \equiv 1^\gamma \mod k$ and the result follows. □

Theorem 36. *If $\kappa-1$ is a prime power, $q=(\kappa-1)^{2\gamma+1}$ and there exists a resolvable $(q+1,\kappa)$-design D', then there exista a (v,κ)-design D such that*

$$\alpha_1(D) \geq \frac{\lfloor\sqrt{v}\rfloor v}{\kappa-1}$$

Proof. By Proposition 35 we have that $(k-1)^{2\gamma+1} \equiv k-1 \mod k(k-1)$ and then $q+1 \equiv \kappa \mod \kappa(\kappa-1)$. If there exists D' (note that there are $\frac{q}{\kappa-1}$ pairwise disjoint resolution classes) we can define a new (q^2+q+1,κ)-design D obtained from $\mathrm{PG}(2,q)$ and D' taking each line of $\mathrm{PG}(2,q)$ as the resolvable design D'. If resolvable classes of each line have different colors, then the coloring is proper. The coloring is complete due to the properties of $\mathrm{PG}(2,q)$, therefore

$$\alpha_1(D) \geq \frac{qv}{\kappa-1} = \frac{\lfloor\sqrt{v}\rfloor v}{\kappa-1}, \text{ where } v = q^2+q+1.$$

□

When $\kappa = 3$ (or $\kappa = 4$) and $q = 2^{2\gamma+1}$ ($q = 3^{2\gamma+1}$ respectively), then there exist the corresponding resolvable design and the projective plane of order q; moreover, for $\kappa = 3$ Theorem 36 implies Theorem 33 for $c = 1/2$.

4.2 Affine plane

In this section we shall prove that $\psi_1(\mathrm{A}_q) = \lfloor\frac{(q+1)^2}{2}\rfloor$ (see Theorems 38 and 41).

4.2.1 Upper bound of $\psi_1(\mathrm{A}_q)$

The next lemma is an immediate consequence of the properties of the affine plane A_q.

Lemma 37. *Any set of $q+2$ lines of the affine plane A_q contains two lines in the same parallel class.* □

As a consecuence of the Lemma 37 we have the following upper bound for $\psi_1(\mathrm{A}_q)$.

Theorem 38.

$$\psi_1(\mathrm{A}_q) \leq \lfloor\tfrac{(q+1)^2}{2}\rfloor.$$

Proof. We proceed by contradiction. Assume that there exists a complete coloring using at least $\lfloor\frac{(q+1)^2}{2}\rfloor + 1$ colors for A_q. This coloring must have at most $q^2 + q - \left(\lfloor\frac{(q+1)^2}{2}\rfloor + 1\right)$ chromatic classes of cardinality greater than one (remember that the affine plane has $q^2 + q$ lines). Thus, there are at least

$$h(q) = \lfloor\tfrac{(q+1)^2}{2}\rfloor + 1 - \left(q^2 + q - \left(\lfloor\tfrac{(q+1)^2}{2}\rfloor + 1\right)\right)$$

chromatic classes of size one. Hence, there are at least $q+2$ lines intersecting pairwise since

$$h(q) = 1 - \left((q+1)^2 - 2\left\lfloor\frac{(q+1)^2}{2}\right\rfloor\right) + q + 2 = \begin{cases} q+2 & \text{if } q \text{ is even,} \\ q+3 & \text{if } q \text{ is odd.} \end{cases}$$

This contradicts Lemma 37 and therefore the theorem follows. □

4.2.2 Lower bound of $\psi_1(\mathrm{A}_q)$

In Theorem 41 we prove that Theorem 38 is tight. To derive the lower bound we use a certain complete coloring of A_q with $\lfloor\frac{(q+1)^2}{2}\rfloor$ colors. Before this, we need the following simple lemmas:

Lemma 39. *If l and l' are two lines of A_q in different parallel class, then $l \cap l' \neq \emptyset$.* □

Lemma 40. *If l and l' are two non-intersecting lines of A_q, then l and l' are in the same parallel class.* □

Theorem 41.

$$\psi_1(\mathrm{A}_q) \geq \lfloor\tfrac{(q+1)^2}{2}\rfloor.$$

Proof. As A_q is resolvable, there are $q+1$ spreads $\mathscr{S}_0, \ldots, \mathscr{S}_q$ given by the parallel clases (see Subsection 1.3), each one with q lines; that is $\mathscr{S}_i = \{l_{i,1}, \ldots, l_{i,q}\}$ for $i = 0, \ldots, q$ (each line having q points). Consider the following arrangement of lines:

$$M = \begin{pmatrix} l_{0,1} & l_{0,2} & \cdots & l_{0,q-1} & l_{0,q} \\ l_{1,1} & l_{1,2} & \cdots & l_{1,q-1} & l_{1,q} \\ l_{2,1} & l_{2,2} & \cdots & l_{2,q-1} & l_{2,q} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{q-1,1} & l_{q-1,2} & \cdots & l_{q-1 q-1} & l_{q-1,q} \\ l_{q,1} & l_{q,2} & \cdots & l_{q,q-1} & l_{q,q} \end{pmatrix}$$

We start by coloring the set of lines $S_0 = \{l_{0,1}, l_{1,2} \ldots, l_{q-1,q}, l_{q,q}\}$, that is, the lines in the main upper diagonal and the last element in the lower main diagonal. We assign colors to the elements of S_0 using the following function:

$$f_0 \colon S_0 \to \{1, \ldots, q+1\}, \text{ given by}$$

$$f_0(l_{i,i+1}) = i+1, \text{ for } i \in \{0, \ldots, q-1\}, \text{ and } f_0(l_{q,q}) = q+1$$

Note that these lines are is in different parallel classes. Hence, the partial coloring to this point is complete (by Lemma 39). The number of colors we have used so far is $N_0 = q+1$.

Let $S_{i,j} = \{l_{j,i}, l_{i-1,j}\}$, for $i \in \{1,\ldots,q-1\}$ and $j \in \{i+1,\ldots,q\}$. Note that the lines of $S_{i,j}$ are in different parallel classes (since $j \neq i-1$), also $S_{i,j} \cap S_{r,s} = \emptyset$, if $i \neq r$ or $j \neq s$. We assign $\binom{q}{2}$ colors to the lines of $\bigcup\limits_{i,j} S_{i,j}$ as follows:

$$f_{i,j}\colon S_{i,j} \to \{q+2,\ldots,q+1+\tbinom{q}{2}\}, \text{ given by}$$

$$f_{i,j}(l_{i-1,j}) = f_{i,j}(l_{j,i}) = q+1+j-i+\tbinom{q}{2} - \tbinom{q+1-i}{2}.$$

If we replace the lines in M with their corresponding colors assigned until now, we have the following matrix:

$$\begin{pmatrix} 1 & q+2 & \cdots & 2q-1 & 2q \\ l_{1,1} & 2 & \cdots & 3q-3 & 3q-2 \\ q+2 & l_{2,2} & \cdots & 4q-6 & 4q-5 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2q-1 & 3q-3 & \cdots & l_{q-1q-1} & q \\ 2q & 3q-2 & \cdots & q+1+\binom{q}{2} & q+1 \end{pmatrix}$$

We are assigning colors to the lines in the upper triangle and the lines in the lower triangle. The number of colors we have used so far is $N_1 = N_0 + \binom{q}{2}$.

This leaves the lines $\{l_{1,1},\ldots,l_{q-1,q-1}\}$ uncolored, which are in different parallel classes. Let S_i be the set of two lines $\{l_{i,i}, l_{q-i,q-i}\}$, for $i \in \{1,\ldots,\lceil\frac{q-1}{2}\rceil\}$. We assign $\lfloor\frac{q-1}{2}\rfloor$ colors to the lines of $\bigcup\limits_{i} S_i$ as follows:

$$f_i\colon S_i \to \{\tfrac{q^2+q+2}{2}+1,\ldots,\tfrac{q^2+q+2}{2}+\lfloor\tfrac{q-1}{2}\rfloor\} \text{ such that:}$$

$$f_0(l_{i,i}) = f_0(l_{q-i,q-i}) = \tfrac{q^2+q+2}{2}+i.$$

We are assigning colors to the lines of the lower diagonal of M. Note that the lines of S_i are in different parallel classes. Hence, the partial coloring so far is complete by

Lemma 39. Therefore, the number of colors used are

$$N_2 = N_1 + \left\lfloor \frac{q-1}{2} \right\rfloor = \left\lfloor \frac{(q+1)^2}{2} \right\rfloor .$$

Note that if q is odd, then all lines of M are colored. However, if q is even, then the line $l_{\left\lceil \frac{q-1}{2} \right\rceil, \left\lceil \frac{q-1}{2} \right\rceil}$ is uncolored, but this is not a problem, since we can assign trivially the color 1. □

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Chapter 5

Geometric achromatic and pseudoachromatic indices

In this chapter we extend the notions of pseudoachromatic and achromatic indices to geometric graphs, and present upper and lower bounds for the case of complete geometric graphs. The results of this chapter are contained in [AAG+].

5.1 Preliminaries

Let $G = (V, E)$ be a simple graph. A *geometric embedding* of a graph G is a function that maps V to a set S of points in the plane, and E to a set of (possibly crossing) straight-line segments whose endpoints belong to S. A geometric graph G is the image of a particular geometric embedding of G. Throughout this chapter we assume that all sets of points of S are in general position, that is, no three points are on a common line. For brevity we refer to the points in S as vertices of G, and to the straight-line segments connecting two points in S as edges of G. Note that any set of points in the plane induces a complete geometric graph. We say that two edges of G *intersect* if they have a common endpoint or they cross. Two edges are *disjoint* if they do not intersect. A coloring of the edges of G is *proper* if every pair of edges of the same color is disjoint. A coloring is *complete* if each pair of colors appears on at least one pair of intersecting edges.

The *chromatic index* $\chi_1(\mathsf{G})$ of G is the smallest number k for which there exists a proper coloring of the edges of G using k colors. The *achromatic index* $\alpha_1(\mathsf{G})$ of G is the largest number k for which there exists a complete and proper coloring of the edges of G using k colors. The *pseudoachromatic index* $\psi_1(\mathsf{G})$ of G is the largest number k for which there exists a complete coloring of the edges of G using k colors.

We extend these definitions to graphs in the following way. Let G be a graph. The *geometric chromatic index* $\chi_g(G)$ of G is the largest value k for which a geometric embedding H of G exists such that $\chi_1(\mathsf{H}) = k$. Likewise, the *geometric achromatic index* $\alpha_g(G)$ and the *geometric pseudoachromatic index* $\psi_g(G)$ of G, are defined as the smallest value k for which a geometric embedding H of G exists such that $\alpha_1(\mathsf{H}) = k$ and $\psi_1(\mathsf{H}) = k$, respectively.

From the above definitions we get that for any graph G

$$\chi_1(G) \leq \chi_g(G), \tag{5.1}$$

$$\chi_1(G) \leq \alpha_1(G) \leq \psi_1(G) \leq \psi_g(G), \tag{5.2}$$

$$\alpha_g(G) \leq \psi_g(G); \tag{5.3}$$

and for geometric graphs we obtain

$$\chi_1(\mathsf{G}) \leq \alpha_1(\mathsf{G}) \leq \psi_1(\mathsf{G}). \tag{5.4}$$

Consider the cycle C_n of length $n \geq 3$. In this case $\chi_1(C_n)$ is equal to 2 if n is even, and is equal to 3 if n is odd. On the other hand, it is not hard to see that $\chi_g(C_n) = n-1$ if n is even and $\chi_g(C_n) = n$ if n is odd. However, $\alpha(C_n) = \alpha_1(C_n) = \max\{k\colon k\lfloor \frac{k}{2} \rfloor \leq n\} - s(n)$, where $s(n)$ is the number of positive integer solutions to $n = 2x^2 + x + 1$; and $\psi(C_n) = \psi_1(C_n) = \psi_g(C_n) = \max\{k\colon k\lfloor \frac{k}{2} \rfloor \leq n\}$ (see [CZ09, HM76, Yeg00]).

It is known that if G is a planar graph then there always exists a geometric embedding j, where no two edges of $j(G)$ intersect, except possibly in a common endpoint [Fár48]. Therefore, $\psi_1(G) = \psi_1(j(G)) = \psi_g(G)$ and $\alpha_1(G) = \alpha_1(j(G)) \geq \alpha_g(G)$.

However, $\chi_1(G) = \chi_1(j(G)) \leq \chi_g(G)$ (for instance, and as we mentioned before, $\chi_1(C_4) = 2$ and $\chi_g(C_4) = 3$). In general, $\alpha_1(G)$ is not comparable to $\alpha_g(G)$, for example, $\alpha_1(K_n) = \Theta(n^{3/2})$ (see [Jam89]) but $\alpha_g(K_n) = \Theta(n^2)$ as we shall see in Theorem 44.

The chromatic index of a geometric graph G has been studied before. Let l be a positive integer and $I(S)$ the graph in which one vertex corresponds to one subset of S of size l, and one edge corresponds to two vertices of G whose respective convex hulls intersect. This graph was defined in [ADH⁺05], where the authors study its chromatic number for the case when $l = 2$. If we denote by K_n the complete geometric graph with vertex set S, then for the case $l = 2$, $\chi(I(S)) = \chi_1(\mathsf{K}_n)$. In the same paper the authors define and study the number $i(n) = \max\{\chi(I(S)) \colon S \subset \mathbb{E}^2$ in general position, $|S| = n\}$. Note that for the case $l = 2$ it happens that $i(n) = \chi_g(K_n)$. Recall that by K_n we denote the complete graph on n vertices. The following theorem appears in [ADH⁺05].

Theorem 42. *For each $n \geq 3$: i) If the vertices of K_n are in convex position then $\chi_1(\mathsf{K}_n) = n$, ii) $n \leq \chi_g(K_n) \leq cn^{3/2}$for some constant $c > 0$.*

In this chapter we prove:

Theorem 43. *Let G be a geometric graph of order n. The pseudoachromatic index $\psi_1(\mathsf{G})$ of G is at most $\lfloor \frac{n^2+n}{4} \rfloor$.*

Theorem 44. *For each $n \geq 3$:*

i) If the vertices of K_n are in convex position then

$$\alpha_1(\mathsf{K}_n) = \psi_1(\mathsf{K}_n) = \lfloor \tfrac{n^2+n}{4} \rfloor,$$

ii) $0.0710n^2 - \Theta(n) \leq \psi_g(K_n) \leq 0.1781n^2 + \Theta(n)$.

5.2 Points in convex position

In this section we prove Claim i) of Theorem 44. In Subsection 5.2.1 we present an upper bound for $\psi_1(\mathsf{G})$ for any geometric graph G; and then in Subsection 5.2.2 we exclusively work with point sets in convex position, and derive a tight lower bound for $\alpha_1(\mathsf{K}_n)$.

5.2.1 Upper bound: $\psi_1(\mathsf{G}) \leq \lfloor \frac{n^2+n}{4} \rfloor$

The following theorem is a consequence of the fourth issue of [Erd46] (see also [Čer05]).

Theorem 45. *Any geometric graph with n vertices and $n+1$ edges, contains two disjoint edges.*

Using this theorem we obtain the following result.

Corollary 46. *Let G be a geometric graph of order n. There are at most n chromatic classes of size one in any complete coloring of G.* □

This corollary immediately implies an upper bound on $\psi_1(\mathsf{G})$.

Proof of Theorem 43. We proceed by contradiction. Assume there exists a geometric graph G for which a complete coloring using at least $\lfloor \frac{n^2+n}{4} \rfloor + 1$ colors exist. This coloring must have at most $\binom{n}{2} - \left(\lfloor \frac{n^2+n}{4} \rfloor + 1 \right)$ chromatic classes of cardinality larger than one. Thus, the number of chromatic classes of size one is at least $\lfloor \frac{n^2+n}{4} \rfloor + 1 - \left(\binom{n}{2} - \lfloor \frac{n^2+n}{4} \rfloor - 1 \right)$, that is,

$$1 - \left(\binom{n+1}{2} - 2 \left\lfloor \frac{\binom{n+1}{2}}{2} \right\rfloor \right) + n + 1 = \begin{cases} n+1 & \text{if } \binom{n+1}{2} \text{ is odd,} \\ n+2 & \text{if } \binom{n+1}{2} \text{ is even.} \end{cases} \tag{5.5}$$

This contradicts Corollary 46 and therefore the theorem follows. □

5.2.2 Tight lower bound: $\alpha_1(\mathsf{G}) \geq \lfloor \frac{n^2+n}{4} \rfloor$

In this subsection we prove that the bound presented in Theorem 43 is tight. To derive the lower bound we use a complete geometric graph induced by a set of points in convex position. We call this type of graph a *complete convex geometric graph*. The crossing pattern of the edge set of a complete convex geometric graph depends only on the number of vertices, and not on their particular position. Without loss of generality we therefore assume that the point set of the graph corresponds to the vertices of a regular polygon. In the remainder of this section we exclusively work with this type of graphs. To simplify the proof of the main statement of this section, in the following we will define different sets of edges and prove some important properties of these sets.

Let G be a complete convex geometric graph of order n, and let $\{1, \ldots, n\}$ be the vertices of the graph listed in clockwise order. For the remainder of this section it is important to bear in mind that all sums are taken modulo n; for the sake of simplicity we will avoid writing this explicitly. We denote by $e_{i,j}$ the edge between the vertices i and j. We call an edge $e_{i,j}$ a *halving edge* if in both of the two open semi-planes defined by the line containing $e_{i,j}$, there are at least $\lfloor \frac{n-2}{2} \rfloor$ points of G. Using this concept we obtain the following definition.

Definition 47. *Let $i, j, k \in \{1, \ldots, n\}$, such that $e_{i,j}$ and $e_{j+1,k}$ do not intersect. We call the pair of edges $(e_{i,j}, e_{j+1,k})$ a* halving pair of edges *(*halving pair*, for short) if at least one of $e_{i,j+1}$, $e_{i,k}$, or $e_{j,k}$ is a halving edge. This halving edge is called the* witness *of the halving pair.*

See Figure 5.1 for an example of a halving pair $(e_{i,j}, e_{j+1,k})$, with $e_{i,k}$ as witness. Note that a halving pair may have more than one witness.

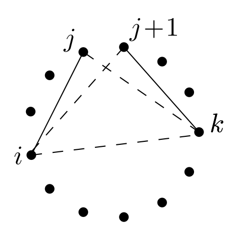

Figure 5.1: Example for $n = 13$. Edges of the halving pair $(e_{i,j}, e_{j+1,k})$ are shown solid, dashed edges represent the possible halving edges, of which $e_{i,k}$ is a halving edge (the witness) in the shown example.

We say that an edge e intersects a pair of edges (f, g) if e intersects at least one of f or g. We say that two pairs of edges intersect if there is an edge in the first pair which intersects the second pair.

Lemma 48. *Let* G *be a complete convex geometric graph of order* n*. i) Each two halving edges intersect. ii) Any halving edge intersects any halving pair of edges. iii) Any two halving pairs intersect.*

Proof. To prove Claim i) assume that there are two halving edges which do not intersect. These edges divide the set of vertices of G into two disjoint sets of size at least $\lfloor \frac{n-2}{2} \rfloor$ and one set of size at least 4 (the vertices of the two halving edges). Then, the total number of vertices is:

$$2\left\lfloor \frac{n-2}{2} \right\rfloor + 4 = \begin{cases} n+1 & \text{if n is odd} \\ n+2 & \text{if n is even} \end{cases} \tag{5.6}$$

This is a contradiction, which proves Claim i).

Observe that the convex hull of each halving pair $(e_{i,j}, e_{j+1,k})$ defines a quadrilateral $(i, j, j+1, k)$ (see Figure 5.1). The halving edge witnessing the halving pair is contained in the corresponding convex hull: it is either the edge $e_{i,k}$, or one of the diagonals of the quadrilateral. It is easy to see that if either $e_{i,k}$ or one of the diagonals is intersected by an edge f, then f also intersects at least one edge of the pair $(e_{i,j}, e_{j+1,k})$. Using this observation we prove the remaining two cases by contradiction.

Assume there exists a halving edge and a halving pair which do not intersect, or two halving pairs which do not intersect. Then their corresponding halving edges (witnesses) do not intersect either, because they are contained in the quadrilaterals. This contradicts Claim i), and thus proves Claim ii) and iii). □

Definition 49. *Let* G *be a complete convex geometric graph of even order* n*. We call an edge* $e_{i,j}$ *an* almost-halving edge *if* $e_{i,j+1}$ *is a halving edge.*

Please observe that this definition and the following lemma are only stated (and valid) for even n.

Lemma 50. *Let* G *be a complete convex geometric graph of even order* n*. Let* f *be an almost-halving edge,* e *a halving edge, and* E *a halving pair. Then i)* f *and* e *intersect, ii)* f *and* E *intersect.*

Proof. We prove Claim i) by contradiction. If e and f do not intersect, then they divide the set of vertices of G into three sets: one of size at least $\frac{n-2}{2}$, one of size at least $\frac{n-2}{2} - 1$, and one of size at least 4. In total the number of vertices is (at least):

$$2\left(\frac{n-2}{2}\right) - 1 + 4 = n + 1 \tag{5.7}$$

This is a contradiction, which proves Claim i).

To prove Claim ii) we use Claim i): the halving edge witnessing E must intersect f. On the other hand such a halving edge is inside the convex hull of E (see Figure 5.1). From these two observations it follows that E and f intersect. □

We need two more concepts from the literature. A *straight-line thrackle* [LPS97] of G is a subset of edges of G with the property that any two distinct edges intersect (they have a common endpoint or they cross). Theorem 45 implies that the size of any straight-line thrackle of G is at most n. In the following we always refer to a straight-line thrackle as thrackle, since we are only working with geometric embeddings of graphs.

Given a set $J \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$, a *circulant graph* $C_n(J)$ of G is defined as the graph with vertex set equal to $V(\mathsf{G})$ and $E(C_n(J)) = \{e_{i,j} \in E(\mathsf{G}) \colon j - i \equiv k \mod n, \text{ or } j - i \equiv -k \mod n,\ k \in J\}$. See Figure 5.2 (left) for an example of a circulant graph $C_n(J)$ with $J = \{\lfloor \frac{n}{2} \rfloor - 1\}$ and $n = 13$.

The following theorem provides the lower bound on the achromatic index.

Theorem 51. *Let* G *be a complete convex geometric graph of order* n*. The achromatic index of* G *satisfies the following bound:*

$$\alpha_1(\mathsf{G}) \geq \lfloor \tfrac{n^2+n}{4} \rfloor.$$

Proof. Consider the following partition of the set of edges of G:

$$E(\mathsf{G}) = E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \bigcup E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup \left(\bigcup_{i \in I} E(C_n(\{i, \lfloor \tfrac{n}{2} \rfloor - 1 - i\})) \right)$$

where $I = \{1, \ldots, \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor\}$.

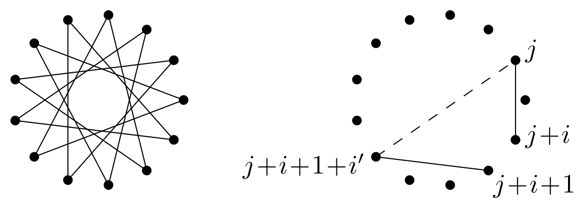

Figure 5.2: Examples for $n = 13$. Left: A circulant graph $C_n\left(\left\{\lfloor \frac{n}{2} \rfloor - 1\right\}\right)$. Right: A pair of edges (solid) with same color from $E(C_n(\{i, i'\}))$, with $i = 2$ and some fixed j. The witness of the halving pair is shown dashed.

Observe that the first term is a circulant graph of halving edges and thus, by Lemma 48, its set of edges defines a thrackle. This thrackle is maximal (containing n edges) if n is odd but it is not maximal (containing only $\frac{n}{2}$ edges) if n is even. Note further, that for fixed i the third term is either the union of two circulant graphs of size n, or one circulant graph of size n (only in the case when $i = \lfloor \frac{n}{2} \rfloor - 1 - i$).

If n is odd, then, in total, the edge set of G is partitioned into $\frac{n-1}{2}$ circulant graphs, each of them of size n. If n is even, then the partition consists of $\frac{n}{2} - 1$ circulant graphs of size n, plus one circulant graph of size $\frac{n}{2}$. Using this partition we give a coloring on the edges of G, and prove that this coloring is proper and complete.

We start by coloring all circulant graphs in the third term of the partition, except for $i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$. In the following we set $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$ and therefore refer to $C_n(\{i, \lfloor \frac{n}{2} \rfloor - 1 - i\})$ as $C_n(\{i, i'\})$. For every $i \in I \setminus \left\{\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor\right\}$ we assign colors to $C_n(\{i, i'\})$ using the following function.

$$f_i \colon E(C_n(\{i, i'\})) \longrightarrow \{(i-1)n + 1, \ldots, (i-1)n + n\} \text{ such that:}$$

$$\begin{aligned} e_{j,j+i} &\mapsto (i-1)n + j, \text{ and} \\ e_{j+i+1,j+i+1+i'} &\mapsto (i-1)n + j, \end{aligned}$$

for $j \in \{1, \ldots, n\}$. See Figure 5.2 (right) for an example with $i = 2$.

The first rule colors the edges of $C_n(\{i\})$, while the second rule colors the edges of $C_n(\{i'\})$. For fixed i and j both rules assign the same color. Therefore, the chromatic classes are pairs of edges, one edge ($e_{j,j+i}$) from $C_n(\{i\})$ and one edge ($e_{j+i+1,j+i+1+i'}$) from $C_n(\{i'\})$. Observe, that all these pairs are halving pairs ($e_{j,j+i}, e_{j+i+1,j+i+1+i'}$) of G, because the edge $e_{j,j+i+1+i'} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ is halving. Hence, the partial coloring so far is complete (by Lemma 48) and proper (because the two edges in each color class do not intersect). The number of colors we have used so far is $N_1 = n\left(\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor - 1\right)$.

So far a subset of edges of the third term of the above partition of the set of edges of G is colored. This leaves the following parts of the partition uncolored:

$$E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \bigcup E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$.

These remaining circulant graphs differ for n even or odd. Further, the two cases $i = i'$ and $i \neq i'$ need to be distinguished (for the remainder of the third term). This basically results in the four cases $n \equiv x \mod 4$, for $x \in \{0, 1, 2, 3\}$. In a nutshell, to color the remaining edges, the thrackle, $E(C_n\{\lfloor \frac{n}{2} \rfloor\}))$ will be colored first (if n is even together with one half of $E(C_n(\{\lfloor \frac{n}{2} \rfloor - 1\})))$. Then (the remaining half of) the circulant graph $C_n(\{\lfloor \frac{n}{2} \rfloor - 1\})$ together with $C_n(\{i, i'\})$ ($i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$) is colored. In each step we will prove, that the (partial) coloring is proper and complete.

1. Case n is odd. To color the maximal thrackle, $E(C_n(\{\lfloor \frac{n}{2} \rfloor\}))$, we assign colors to its edges using the function

   $$f \colon E(C_n(\{\lfloor \tfrac{n}{2} \rfloor\})) \longrightarrow \{N_1 + 1, \ldots, N_1 + n\} \text{ such that:}$$

   $$e_{j,j+\lfloor \frac{n}{2} \rfloor} \mapsto N_1 + j,$$

   for each $j \in \{1, \ldots, n\}$. Observe that $E(C_n(\{\lfloor \frac{n}{2} \rfloor\}))$ is a set of n halving edges. See Figure 5.3 (left) for an example of such a thrackle.

   The coloring so far is proper, because each new chromatic class has size one.

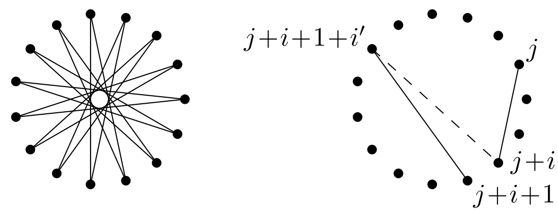

Figure 5.3: Examples for $n = 15$. Left: A circulant graph $C_n\left(\left\{\lfloor \frac{n}{2} \rfloor\right\}\right)$ of halving edges if n is odd. Right: Halving pair (solid) with color $N_2 + j$ from $E(C_n(\{i, i'\}))$, with $n \equiv 3 \mod 4$ and some fixed j. The witness of the halving pair is shown dashed.

Further, each chromatic class so far consists of either a halving edge or a halving pair. Hence, by Lemma 48, the coloring is also complete. It is easy to see that we are using $N_2 = N_1 + n = n\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ colors so far.

The remaining uncolored edges are partitioned into

$$E(C_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor$ and $i' = \lfloor \frac{n}{2} \rfloor - 1 - i$. These two circulant graphs will be colored together. Let $i'' = \lfloor \frac{n}{2} \rfloor - 1$. $C_n(\{i''\})$ consists of n edges. The size of $E(C_n(\{i, i'\}))$ depends on the two cases $i = i'$ and $i \neq i'$.

(a) $i = i'$: As n is odd, $n \equiv 3 \mod 4$. The circulant graph $C_n(\{i, i'\}) = C_n(\{i\})$ is of size n. Thus, $2n$ edges remain uncolored. We assign n colors to the $2n$ edges of $C_n(\{i, i''\})$ as follows:

$$f_i \colon E(C_n(\{i, i''\})) \longrightarrow \{N_2 + 1, \ldots, N_2 + n\}, \text{ such that}$$

$$\begin{aligned} e_{j,j+i} &\mapsto N_2 + j, \\ e_{j+i+1,j+i+1+i''} &\mapsto N_2 + j \end{aligned}$$

for $j \in \{1, \ldots, n\}$.

Each new chromatic class consists of a pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i''})$ of edges. See Figure 5.3 (right) for an example of such a pair. Because the edge $e_{j+i,j+i+1+i''} = e_{j+i,j+i+\lfloor \frac{n}{2} \rfloor}$ is a halving edge, the pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i''})$ is a halving pair. Therefore, all edges are colored and each chromatic class consists of either a halving edge or a halving pair. By Lemma 48 the coloring is complete and proper (as the edges of halving pairs are disjoint).

The total number of colors used is $N_3 = N_2 + n = n(\lfloor \frac{\lfloor \frac{n}{2} \rfloor - 1}{2} \rfloor + 1)$, that is $N_3 = \left\lfloor \frac{n^2+n}{4} \right\rfloor$ colors, as $n \equiv 3 \mod 4$ in this case.

(b) $i \neq i'$: As n is odd, $n \equiv 1 \mod 4$. The circulant graph $C_n(\{i, i'\})$ is of size $2n$. Thus, $3n$ edges remain uncolored. We assign n colors to the $2n$ edges of $C_n(\{i, i'\})$ and $\left\lfloor \frac{n}{2} \right\rfloor$ colors to the n edges of $C_n(\{i''\})$ as follows:

$$f_i \colon E(C_n(\{i, i', i''\})) \longrightarrow \{N_2 + 1, \dots, N_2 + n + \lfloor \tfrac{n}{2} \rfloor\}, \text{ such that}$$

$$\begin{aligned} e_{j,j+i} &\mapsto N_2 + j, \\ e_{j+i+1,j+i+1+i'} &\mapsto N_2 + j, \end{aligned}$$

for $j \in \{1, \dots, n\}$, and

$$\begin{aligned} e_{j,j+i''} &\mapsto N_2 + n + j, \\ e_{j+i''+1,j+i''+1+i''} &\mapsto N_2 + n + j, \end{aligned}$$

for $j \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. See Figure 5.4 (left and middle) for examples.

Each new chromatic class consists of a pair of edges. These pairs are either $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ or $(e_{j,j+i''}, e_{j+i''+1,j+i''+1+i''})$ combined from the edges of $C_n(\{i, i'\})$ or $C_n(\{i''\})$, respectively. The pair $(e_{j,j+i}, e_{j+i+1,j+i+1+i'})$ is a halving pair with the halving edge $e_{j,j+i+1+i'} = e_{j,j+\lfloor \frac{n}{2} \rfloor}$ as witness, and $(e_{j,j+i''}, e_{j+i''+1,j+i''+1+i''})$ is a halving pair with the halving edges $e_{j,j+i''+1} =$

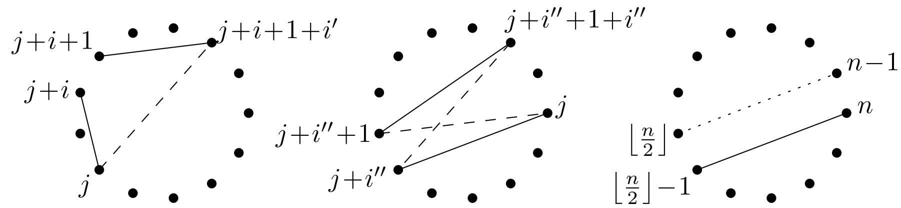

Figure 5.4: Examples with $n = 13$, for n is odd and $i \neq i'$: $n \equiv 3 \mod 4$. Left: Halving pair with color $N_2 + j$ from $E(C_n(\{i, i'\}))$. Middle: Halving pair with color $N_2 + n + j$ from $E(C_n(\{i''\}))$. Both for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed. Right: The single remaining edge $e_{n,\lfloor\frac{n}{2}\rfloor-1}$ (solid) is combined with the halving edge $e_{\lfloor\frac{n}{2}\rfloor,n-1}$ (dotted), colored with color $N_1 + \lfloor\frac{n}{2}\rfloor$.

$e_{j,j+\lfloor\frac{n}{2}\rfloor}$ and $e_{j+i'',j+i''+1+i''} = e_{j+i'',j+i''+\lfloor\frac{n}{2}\rfloor}$ as witnesses. Each chromatic class so far consists of either a halving edge or a halving pair. Hence, the coloring is complete (by Lemma 48) and proper (as the edges of halving pairs are disjoint).

Note, that a single edge, $e_{n,\lfloor\frac{n}{2}\rfloor-1}$ of $C_n(\{i''\})$, remains uncolored. We add this edge to the chromatic class (with color $N_1 + \lfloor\frac{n}{2}\rfloor$) containing the halving edge $e_{\lfloor\frac{n}{2}\rfloor,n-1} = e_{\lfloor\frac{n}{2}\rfloor,\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{2}\rfloor}$. See Figure 5.4 (right). Observe, that $e_{n,\lfloor\frac{n}{2}\rfloor-1}$ and $e_{\lfloor\frac{n}{2}\rfloor,n-1}$ are disjoint, thus the coloring remains proper. Further, adding an edge to an existing chromatic class of a complete coloring, maintains the completeness of the coloring.

As all edges are colored, the total number of colors used is $N_3 = N_2 + n + \lfloor\frac{n}{2}\rfloor = n(\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{2}\rfloor + 1) + \lfloor\frac{n}{2}\rfloor$, that is $N_3 = \left\lfloor\frac{n^2+n}{4}\right\rfloor$, as $n \equiv 1 \mod 4$ in this case.

2. Case n is even. Recall that only N_1 chromatic classes exist so far, each containing a halving pair of edges. The thrackle $E(C_n(\{\lfloor\frac{n}{2}\rfloor\})) = E(C_n(\{\frac{n}{2}\}))$ is not maximal in this case. See Figure 5.5 (left). To get a maximal thrackle we add half the edges of $C_n(\{\frac{n}{2} - 1\})$ to $C_n(\{\frac{n}{2}\})$. Note that $E(C_n(\{\frac{n}{2} - 1\}))$ is a set of almost-halving edges in the case of n even.

   Let the thrackle $E(C'_n(\{\frac{n}{2}-1\})) = \{e_{1,\frac{n}{2}}, \ldots, e_{\frac{n}{2},n-1}\}$ and the thrackle $E(C''_n(\{\frac{n}{2}-$

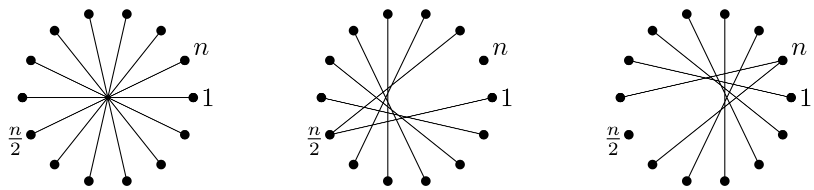

Figure 5.5: Examples with $n = 14$, for the case when n is even. Left: The thrackle, $E(C_n(\frac{n}{2}\}))$, of the $\frac{n}{2}$ halving edges. Middle: The thrackle, $E(C'_n(\{\frac{n}{2}-1\}))$, of the first $\frac{n}{2}$ almost-halving edges of $E(C_n(\{\frac{n}{2}-1\}))$. Right: The thrackle, $E(C''_n(\{\frac{n}{2}-1\}))$, of the second $\frac{n}{2}$ almost-halving edges of $E(C_n(\{\frac{n}{2}-1\}))$.

$1\})) = \{e_{\frac{n}{2}+1,n}, \ldots, e_{n,\frac{n}{2}-1}\}$ define the two halves of $C_n(\{\frac{n}{2}-1\})$ with $\frac{n}{2}$ almost-halving edges each. See Figure 5.5 (middle and right). It is easy to see that $E(C'_n(\{\frac{n}{2}-1\}))$ is a thrackle (all its edges intersect each other). Further, by Lemma 50, each almost-halving edge intersects each halving edge. Thus, $E(C_n(\{\frac{n}{2}\}) \cup C'_n(\{\frac{n}{2}-1\}))$ is a maximal thrackle of size n. The following function assigns one color to each edge of this maximal thrackle.

$$f\colon E\left(C_n(\{\frac{n}{2}\}) \cup C'_n(\{\frac{n}{2}-1\})\right) \longrightarrow \{N_1+1, \ldots, N_1+n\}, \text{ such that}$$

$$\begin{aligned} e_{j,j+\frac{n}{2}} &\mapsto N_1 + j, \\ e_{j,j+\frac{n}{2}-1} &\mapsto N_1 + \frac{n}{2} + j \end{aligned}$$

for each $j \in \{1, \ldots, \frac{n}{2}\}$.

The coloring so far is proper, because each new chromatic class has size one. Further, each chromatic class consists of either a halving edge, a halving pair, or an almost-halving edge. The almost-halving edges used so far form a thrackle and thus, intersect each other. Hence, by Lemmas 48 and 50, the coloring is also complete. It is easy to see that we are using $N_2 = N_1 + n = n\lfloor\frac{n-2}{4}\rfloor$ colors so far.

The remaining uncolored edges are partitioned into

$$E(C'''_n(\{\lfloor \tfrac{n}{2} \rfloor - 1\})) \bigcup E(C_n(\{i, i'\}))$$

where $i = \lfloor \frac{n-2}{4} \rfloor$ and $i' = \frac{n}{2} - 1 - i$ (as n is even). These two circulant graphs will be colored together. For brevity, let $i'' = \frac{n}{2} - 1$ and $C'''_n(\{i''\})$ be the set of the remaining $\frac{n}{2}$ almost-halving edges. The size of $E(C_n(\{i, i'\}))$ depends on the two cases $i = i'$ and $i \neq i'$.

(a) $i = i'$: As n is even, $n \equiv 2 \mod 4$. The circulant graph $C_n(\{i, i'\}) = C_n(\{i\})$ is of size n. Thus, $n + \frac{n}{2}$ edges remain uncolored. We assign $\frac{n}{2} + \lfloor \frac{n}{4} \rfloor$ colors to the $n + \frac{n}{2}$ edges of $C_n(\{i\}) \cup C'''_n(\{i''\})$ as follows:

$$f_i \colon E(C_n(\{i\}) \cup C'''_n(\{i''\})) \longrightarrow \{N_2 + 1, \ldots, N_2 + \tfrac{n}{2} + \lfloor \tfrac{n}{4} \rfloor\}, \text{ such that}$$

$$\begin{aligned} e_{\frac{n}{2}+j, \frac{n}{2}+j+i''} &\mapsto N_2 + j, \\ e_{\frac{n}{2}+j+i''+1, \frac{n}{2}+j+i''+1+i} = e_{j,j+i} &\mapsto N_2 + j \end{aligned}$$

for $j \in \{1, \ldots, \frac{n}{2}\}$, and

$$\begin{aligned} e_{\frac{n}{2}+j, \frac{n}{2}+j+i} &\mapsto N_2 + \tfrac{n}{2} + j, \\ e_{\frac{n}{2}+j+i+1, \frac{n}{2}+j+i+1+i} &\mapsto N_2 + \tfrac{n}{2} + j \end{aligned}$$

for $j \in \{1, \ldots, \lfloor \frac{n}{4} \rfloor\}$. See Figure 5.6 (left and middle) for examples.

Each new chromatic class consists of a halving pair of edges from $E(C_n(\{i\}) \cup C'''_n(\{i''\}))$, either $(e_{\frac{n}{2}+j, \frac{n}{2}+j+i''}, e_{\frac{n}{2}+j+i''+1, \frac{n}{2}+j+i''+1+i})$ with the halving edge $e_{\frac{n}{2}+j, \frac{n}{2}+j+i''+1} = e_{\frac{n}{2}+j, \frac{n}{2}+j+\frac{n}{2}}$ as witness, or $(e_{\frac{n}{2}+j, \frac{n}{2}+j+i}, e_{\frac{n}{2}+j+i+1, \frac{n}{2}+j+i+1+i})$ with, again, the halving edge $e_{\frac{n}{2}+j, \frac{n}{2}+j+i+1+i} = e_{\frac{n}{2}+j, \frac{n}{2}+j+\frac{n}{2}}$ as witness.

Each chromatic class so far consists of either a halving edge, a halving pair, or one of the $\frac{n}{2}$ almost-halving edges that form a thrackle. Hence, the coloring is complete (by Lemmas 48 and 50) and proper (as the edges of

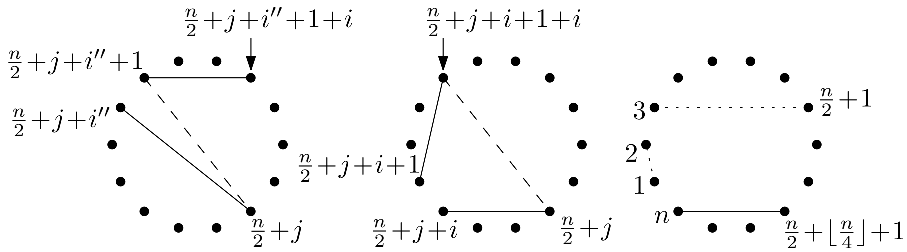

Figure 5.6: Examples with $n = 14$, for the case when n is even and $i = i'$: $n \equiv 2 \mod 4$. Left: Halving pair with color N_2+j. Middle: Halving pair with color $N_2+\frac{n}{2}+j$. Both for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed. Right: The single remaining edge $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ (solid) is combined with the halving pair $(e_{1,2}, e_{3,\frac{n}{2}+1})$ (dotted), colored with color 1.

halving pairs are disjoint).

Note, that a single edge, $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ of $C_n(\{i\})$, remains uncolored. We add this edge to the chromatic class (with color 1) containing the halving pair $(e_{1,2}, e_{3,\frac{n}{2}+1})$ (see Figure 5.6 (right)). Observe, that $e_{\frac{n}{2}+\lfloor\frac{n}{4}\rfloor+1,n}$ and $(e_{1,2}, e_{3,\frac{n}{2}+1})$ are disjoint. Thus, the coloring remains proper. Further, adding an edge to an existing chromatic class of a complete coloring, maintains the completeness of the coloring.

As all edges are colored, the total number of colors used is $N_3 = N_2 + \frac{n}{2} + \lfloor\frac{n}{4}\rfloor = n\lfloor\frac{n-2}{4}\rfloor + \frac{n}{2} + \lfloor\frac{n}{4}\rfloor$, that is $N_3 = \left\lfloor\frac{n^2+n}{4}\right\rfloor$, as $n \equiv 2 \mod 4$ in this case.

(b) $i \neq i'$: As n is even, $n \equiv 0 \mod 4$. The circulant graph $C_n(\{i,i'\})$ is of size $2n$. Thus, $2n + \frac{n}{2}$ edges remain uncolored. We assign $\frac{n}{2} + 3\frac{n}{4}$ colors to the $2n + \frac{n}{2}$ edges of $C_n(\{i,i'\}) \cup C''_n(\{i''\})$ as follows:

$$f_i\colon E(C_n(\{i,i'\}) \cup C''_n(\{i''\})) \longrightarrow \{N_2+1,\ldots,N_2+\tfrac{n}{2}+3\tfrac{n}{4}\}, \text{ such that}$$

$$\begin{aligned} e_{\frac{n}{2}+j,\frac{n}{2}+j+i''} &\mapsto N_2+j, \\ e_{\frac{n}{2}+j+i''+1,\frac{n}{2}+j+i''+1+i} = e_{j,j+i} &\mapsto N_2+j, \end{aligned}$$

$$e_{3\frac{n}{4}+j,3\frac{n}{4}+j+i''} \mapsto N_2+\tfrac{n}{4}+j,$$
$$e_{3\frac{n}{4}+j+i''+1,3\frac{n}{4}+j+i''+1+i'} \mapsto N_2+\tfrac{n}{4}+j,$$

for each $j \in \{1,\ldots,\frac{n}{4}\}$, and

$$e_{\frac{n}{4}+j,\frac{n}{4}+j+i} \mapsto N_2+\tfrac{n}{2}+j,$$
$$e_{\frac{n}{4}+j+i+1,\frac{n}{4}+j+i+1+i'} \mapsto N_2+\tfrac{n}{2}+j$$

for each $j \in \{1,\ldots,3\frac{n}{4}\}$. See Figure 5.7 for an example of these three different types of pairs of edges.

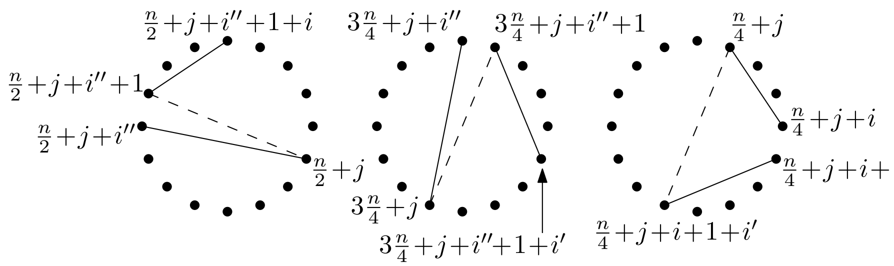

Figure 5.7: Examples with $n=16$, for the case when n is even and $i \neq i'$: $n \equiv 0 \mod 4$. Left: Halving pair with color N_2+j. Middle: Halving pair with color $N_2+\frac{n}{4}+j$. Right: Halving pair with color $N_2+\frac{n}{2}+j$. All for fixed j. Halving pairs are shown solid, witnesses of the halving pairs are shown dashed.

Each new chromatic class consists of a halving pair of edges from $C_n(\{i,i'\}) \cup C''_n(\{i''\})$, either $(e_{\frac{n}{2}+j,\frac{n}{2}+j+i''}, e_{\frac{n}{2}+j+i''+1,\frac{n}{2}+j+i''+1+i})$ with the halving edge $e_{\frac{n}{2}+j,\frac{n}{2}+j+i''+1} = e_{\frac{n}{2}+j,\frac{n}{2}+j+\frac{n}{2}}$ as its witness (see Figure 5.7 (left)), $(e_{3\frac{n}{4}+j,3\frac{n}{4}+j+i''}, e_{3\frac{n}{4}+j+i''+1,3\frac{n}{4}+j+i''+1+i'})$ with halving edge $e_{3\frac{n}{4}+j,3\frac{n}{4}+j+i''+1} = e_{3\frac{n}{4}+j,3\frac{n}{4}+j+\frac{n}{2}}$ as its witness (see Figure 5.7 (middle)), or $(e_{\frac{n}{4}+j,\frac{n}{4}+j+i}, e_{\frac{n}{4}+j+i+1,\frac{n}{4}+j+i+1+i'})$ with, again, the halving edge $e_{\frac{n}{4}+j,\frac{n}{4}+j+i+1+i'} = e_{\frac{n}{4}+j,\frac{n}{4}+j+\frac{n}{2}}$ as witness (Figure 5.7 (right)). Each chromatic class so far consists of either a halving edge, a halving pair, or one of the $\frac{n}{2}$ almost-halving edges that form a thrackle. Hence, the coloring is complete (by Lemmas 48 and 50) and proper (as the edges of halving pairs are disjoint).

As all edges are colored, the total number of colors used is $N_3 = N_2 + \frac{n}{2} + 3\frac{n}{4} = n\lfloor \frac{n-2}{4} \rfloor + \frac{n}{2} + 3\frac{n}{4}$, that is $N_3 = \left\lfloor \frac{n^2+n}{4} \right\rfloor$, as $n \equiv 0 \mod 4$ in this case.

□

Proof of Theorem 44 i). Using Theorem 51 we get that $\left\lfloor \frac{n^2+n}{4} \right\rfloor \leq \alpha_1(\mathsf{G})$, and by Theorem 43 and Equation 5.4 we conclude that $\alpha_1(\mathsf{G}) = \psi_1(\mathsf{G}) = \left\lfloor \frac{n^2+n}{4} \right\rfloor$. □

5.3 On $\psi_g(K_n)$

In this section we consider point sets in general position in the plane, and present lower and upper bounds for the geometric pseudoachromatic index.

5.3.1 Upper bound for $\psi_g(K_n)$

It seems natural that there should be a relationship between the rectilinear crossing number of a graph, and its geometric achromatic and pseudoachromatic indices. Using known bounds for the rectilinear crossing number, we obtain the following results. Let with $deg(v)$ denote the degree of vertex v.

Lemma 52. *Let* G *be a geometric graph of size* m*, with* $m = \sum_{v \in V} \binom{deg(v)}{2}$*. Denote by* $\overline{cr}(\mathsf{G})$ *the number of edge crossings in* G*. Then:*

$$\psi_1(\mathsf{G}) \leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G}))}}{2} \right\rfloor.$$

Proof. The number of incidences between pairs of edges is $I \geq \binom{\psi_1(\mathsf{G})}{2}$, so that $\psi_1(\mathsf{G})(\psi_1(\mathsf{G})-1) \leq 2I = 2(m+\overline{cr}(\mathsf{G}))$. Solving this inequality we get $\psi_1(\mathsf{G}) \leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G}))}}{2} \right\rfloor$. □

Theorem 53. *Let* G *be a graph of size* m*, with* $m = \sum_{v \in V} \binom{deg(v)}{2}$*. Denote by* $\overline{cr}(G)$ *its rectilinear crossing number. Then:* $\psi_g(G) \leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(G)}}{2} \right\rfloor$.

Proof. Let G_0 be a geometric graph of G such that $\overline{cr}(\mathsf{G}_0) = \overline{cr}(G)$. By Lemma 52 we have the following:

$$\begin{aligned}\psi_g(G) &= \min\{\psi_1\{\mathsf{G}\}\colon \mathsf{G} \text{ is a geometric graph of } G\} \leq \psi_1\{\mathsf{G}_0\} \\ &\leq \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(\mathsf{G}_0))}}{2} \right\rfloor = \left\lfloor \frac{1+\sqrt{1+8(m+\overline{cr}(G))}}{2} \right\rfloor .\end{aligned}$$

□

In [ÁCF+10] the following result was proven:

Theorem 54. $\overline{cr}(K_n) \leq c\binom{n}{4} + \Theta(n^3)$ *for* $c = 0.380488$.

Using the above theorem we obtain the following upper bound on $\psi_g(K_n)$.

Theorem 55. *The geometric pseudoachromatic index of* K_n *has the following upper bound:*

$$\psi_g(K_n) \leq 0.1781n^2 + \Theta(n).$$

Proof. Since $|E(L(K_n))| = \sum_{v\in V(K_n)} \binom{deg(v)}{2} = n\binom{n-1}{2}$, by Theorems 53 and 54,

$$\begin{aligned}\psi_g(K_n) &\leq \frac{1}{2}\sqrt{8\overline{cr}(K_n)) + \Theta(n^3)} + \Theta(1) \leq \frac{1}{2}\sqrt{8\frac{c}{4!}n^4 + \Theta(n^3)} + \Theta(1) \\ &= \sqrt{\frac{c}{12}}n^2 + \Theta(n) \leq 0.1781n^2 + \Theta(n),\end{aligned}$$

where $c = 0.380488$. □

5.3.2 Lower bound for $\psi_g(K_n)$

In order to obtain a lower bound for $\psi_g(K_n)$, we divide the plane into seven regions and then use this partition of the plane to construct a partition of the edges of the graph. We utilize a specific configuration $\mathcal{L}$ of lines, defined as follows; see also Figure 5.8 for a drawing of the configuration. Let S be a set of $m = 13n + 6 + r$ points in general position in the plane ($r < 13$). Choose horizontal lines l_1, l_2, and l_3 (listed top-down) so that when writing A', B' for the set of points between l_1 and l_2, and between the lines l_2 and l_3, respectively, we have $|A'| = 12n + 6$ and $|B'| = n + r$. Let l_4, l_5, l_6 be concurrent lines that divide the set A' into 6 parts [Ced64] (at most there are 6 points contained in the concurrent lines), each containing at least $2n$ points in its interior. We call those six the sets as A, B, C, D, E, F and they are listed in clockwise order. Denote by p the point of intersection of the three lines and take $G \subseteq B'$ such that $|G| = n$.

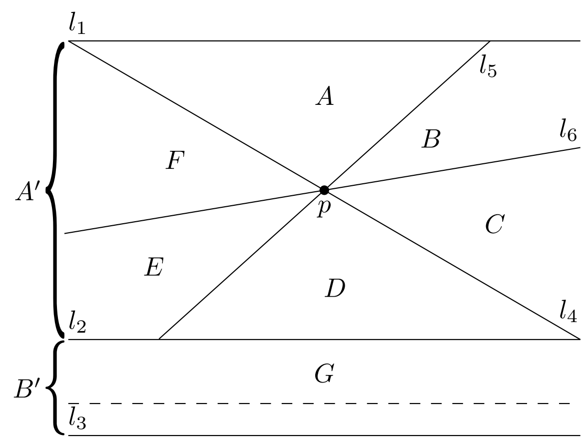

Figure 5.8: The line configuration $\mathcal{L}$.

Let $A = \{a_1, \ldots, a_{2n}\}$, $B = \{b_1, \ldots, b_{2n}\}$, $C = \{c_1, \ldots, c_{2n}\}$, $D = \{d_1, \ldots, d_{2n}\}$, $E = \{e_1, \ldots, e_{2n}\}$, $F = \{f_1, \ldots, f_{2n}\}$, and $G = \{g_1, \ldots, g_n\}$. For $i, j \in \{1, \ldots, 2n\}$, we construct three sets of graphs:

- The subgraphs $\mathsf{X}_{i,j}$ with vertex set $\{a_i, b_j, d_i, e_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{a_i b_j, b_j d_i, d_i e_j, e_j a_i\} \cup \begin{cases} \{a_i g_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{d_i g_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd} . \end{cases}$$

  Let $\mathsf{X}'_{i,j} \leq \mathsf{X}_{i,j}$ be the subgraph C_4 induced by vertices a_i, b_j, d_i, e_j.

- The subgraphs $\mathsf{Y}_{i,j}$ with vertex set $\{b_i, c_j, e_i, f_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{b_ic_j, c_je_i, e_if_j, f_jb_i\} \cup \begin{cases} \{b_ig_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{e_ig_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd .} \end{cases}$$

Let $\mathsf{Y}'_{i,j} \leq \mathsf{Y}_{i,j}$ be the subgraph induced by vertices b_i, c_j, e_i, f_j.

- The subgraphs $\mathsf{Z}_{i,j}$ with vertex set $\{c_i, d_j, f_i, a_j, g_{\lceil \frac{j}{2} \rceil}\}$ and edges

$$\{c_id_j, d_jf_i, f_ia_j, a_jc_i\} \cup \begin{cases} \{c_ig_{\frac{j}{2}}\} & \text{if } j \text{ is even} \\ \{f_ig_{\frac{j+1}{2}}\} & \text{if } j \text{ is odd .} \end{cases}$$

Let $\mathsf{Z}'_{i,j} \leq \mathsf{Z}_{i,j}$ be the subgraph induced by vertices c_i, d_j, f_i, a_j.

Please note that each subgraph $\mathsf{X'}_{i,j}, \mathsf{Y'}_{i,j}$ and $\mathsf{Z'}_{i,j}$, is a quadrilateral, not necessarily convex.

Lemma 56. *The point p is inside each of the polygons induced by the graphs $\mathsf{X'}_{i,j}, \mathsf{Y'}_{i,j}$ and $\mathsf{Z'}_{i,j}$, defined above.*

Proof. Consider the polygon induced by $\mathsf{X'}_{i,j}$ and recall that $V(\mathsf{X'}_{i,j}) = \{a_i, b_j, d_i, e_j\}$. Let $\ell(a_id_i)$ be the line induced by the segment a_id_i, and let Π_e be the semiplane defined by $\ell(a_id_i)$ containing p. Without loss of generality, e_j is in Π_e; let Π_a be the semiplane induced by $\ell(d_ie_j)$ and containing a_i, then $p \in \Pi_e \cap \Pi_a$. Furthermore, the segment a_ie_j is in one of the two open semiplanes defined by l_5, and d_i is in the opposite one. Therefore, p is inside $\mathsf{X'}_{i,j}$. Analogously, p is inside $\mathsf{Y'}_{i,j}$ and also inside $\mathsf{Z'}_{i,j}$. □

Lemma 57. *Any two graphs from the set $\{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$ intersect.*

Proof. We proceed by contradiction. Let $\mathsf{G}_{i,j}, \mathsf{H}_{i',j'} \in \{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$, and assume that $\mathsf{G}_{i,j}$ and $\mathsf{H}_{i',j'}$ do not intersect. In particular, $\mathsf{G}'_{i,j}$ does not intersect $\mathsf{H}'_{i',j'}$. Since p is inside both polygons, without loss of generality $\mathsf{H}'_{i',j'}$ is inside $\mathsf{G}'_{i,j}$. Therefore, the edge $h_{i',j'} \in E(\mathsf{H}_{i,j}) \setminus E(\mathsf{H}'_{i,j})$ intersects $\mathsf{G}'_{i,j}$ because $h_{i',j'}$ has a vertex in $\mathsf{H}'_{i',j'}$ (in the interior of $\mathsf{G}'_{i,j}$) and another vertex in the set G (in the exterior of $\mathsf{G}'_{i,j}$). The theorem follows. □

Proof of Theorem 44 ii). For every geometric embedding of K_n one can construct the configuration $\mathcal{L}$. By construction the partition A, B, C, D, E, F has $\lfloor 2\frac{n-6}{13} \rfloor$ points, and K_n contains $3\frac{4}{169}n^2 - \Theta(n)$ edge disjoint graphs $\mathsf{G}_{i,j}$ ($\mathsf{G}_{i,j} \in \{\mathsf{X}_{i,j}, \mathsf{Y}_{i,j}, \mathsf{Z}_{i,j}\}$). We assign a different color to each of these graphs. By Lemma 57 each two of these subgraphs intersect, therefore $0.0710n^2 - \Theta(n) \leq \psi_g(K_n)$. □

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Chapter 6

On m-factorizations of complete multigraphs arising from affine spaces

The *complete multigraph* λK_v has v vertices and λ edges joining each pair of vertices. An m-*factor* of a multigraph G is a set of pairwise vertex-disjoint m-regular subgraphs, which induce a partition of the vertices. An m-*factorization* of G is a set of pairwise edge-disjoint m-factors such that these m-factors induce a partition of the edges. An m-factorization is called *simple* if the m-factors are pairwise distinct (if no m-factor is repeted). Furthermore, an m-factorization of λK_v is *decomposable* if there exist positive integers μ_1 and μ_2 such that $\mu_1 + \mu_2 = \lambda$ and the factorization is the union of m-factorizations of $\mu_1 K_v$ and of $\mu_2 K_v$, otherwise it is called *indecomposable*. There is no direct correspondence between simplicity and indecomposability.

Many papers deal with m-factorizations of graphs and multigraphs. This is an interesting problem in its own right, but it is motivated by several applications, too. In particular, if $m = 1$ then a one-factorization of K_v corresponds to a schedule of a round robin tournament. For a comprehensive survey on one-factorizations we refer to [Wal97]. A special case of 2-factorizations is the famous *Oberwolfach problem* (see for example [Als96, BMR09]). Several authors investigated 3-factorizations of λK_v with a certain automorphism group (see for example [ABM04, KMT01]). In general, decompositions of λK_v is also a widely studied problem (see for example [BHMS11, BR08, FH11, Smi10]). As m increases, the structure of an arbitrary m-factor of λK_v can be much more complicated and the existence problem becomes much more difficult. In this chapter we restrict ourselves to construct factorizations in which all factors are regular graphs of degree m whose connected components are complete graphs on $(m+1)$ vertices. In the case $m = 1$ an indecomposable one-factorization of λK_{2n} is denoted by IOF$(2n, \lambda)$. Only a few conditions on the parameters are known: if IOF$(2n, \lambda)$ exists, then $\lambda < 1 \cdot 3 \cdot \ldots \cdot (2n-3)$ [BW89]; each IOF$(2n, \lambda)$ can be embedded in a simple IOF$(2s, \lambda)$, provided that $\lambda < 2n < s$ [CCR85]. Six infinite classes of indecomposable one-factorizations have been constructed so far, namely a simple IOF$(2n, n-1)$ when $2n-1$ is a prime [CCR85], IOF$(2(\lambda+p), \lambda)$ where $\lambda > 2$ and p is the smallest prime wich does not divide λ [AD91] (an improvement of this result can be found in [Chu01]), a simple IOF$(2^h+2, 2)$ where h is a positive integer [Son01], IOF$(q^2+1, q-1)$ where q is an odd prime number [KSS01], a simple IOF$(q^2+1, q+1)$ for any odd prime power q [Kis02], and a simple IOF(q^2, q) for any even prime power q [Kis02]. Most of these constructions arise from finite geometry.

The aim of this chapter is to construct new simple and indecomposable m-factorizations of λK_v for different values of m, λ and v. In Section 6.1 we collect some definitions and we describe a general construction method of m-factorizations which is based on spreads of block designs. In Sections 6.2, affine spaces is the key object. We present several new multigraph factorizations using configurations of that structure. The results of this chapter are contained in [KR15].

6.1 Preliminaries

It follows from the properties of a resolvable (v, κ, λ)-design D that a resolution class of D gives a $(\kappa-1)$-factor of λK_v and a resolution of D gives a $(\kappa-1)$-factorization of λK_v (see Section 1.2.2). Hence we get the following well-known fact.

Lemma 58 (Basic Construction). *The existence a resolvable (v, κ, λ)-design is equivalent to the existence of a $(\kappa-1)$-factorization of the complete multigraph λK_v.*

In PG(n, q) or AG(n, q)), an *f-fold i-spread*, $\mathcal{S}_f^i$, is a set of pairwise disjoint i-

dimensional subspaces such that every point of the geometry is contained in exactly f subspaces of $\mathcal{S}_f^i$. An *i-packing*, $\mathcal{P}^i$, of $\mathrm{PG}(n,q)$ (or of $\mathrm{AG}(n,q)$) is a set of spreads such that each i-dimensional subspace of the geometry is contained in exactly one of the spreads in $\mathcal{P}^i$, i.e., the spreads give a partition of the i-dimensional subspaces of the geometry. The i-spreads, f-fold i-spreads and i-packings induce a resolution class, an f-resolution class and a resolution in $\mathrm{PG}^{(i)}(n,q)$ (or in $\mathrm{AG}^{(i)}(n,q)$), respectively.

It is easy to construct spreads and packings in $\mathrm{AG}^{(i)}(n,q)$, because each parallel class of i-spaces is an i-spread. The situation is much more complicated in $\mathrm{PG}^{(i)}(n,q)$. There are only few constructions of spreads known. The following theorem summarizes the known existence conditions.

Theorem 59 ([Hir79], Theorems 4.1 and 4.16)**.**

- *There exists an i-spread in $\mathrm{PG}^{(i)}(n,q)$ if and only if $(i+1)|(n+1)$.*
- *Suppose that i, l and n are positive integers such that $(l+1)|\gcd(i+1,n+1)$. Then there exists an f-fold i-spread in $\mathrm{PG}^{(i)}(n,q)$, where $f=(q^{i+1}-1)/(q^{l+1}-1)$.*

6.2 Factorizations arising from finite affine spaces

In this section, we investigate the spreads and packings of $\mathrm{AG}(n,q)$ and the corresponding factorizations of multigraphs. In each case we apply Lemma 58, so we identify the points of $\mathrm{AG}(n,q)$ with the vertices of the complete multigraph.

Theorem 60. *Let q be a prime power, $i<n$ be a positive integer and $\lambda_i=\left[{n-1 \atop i-1}\right]_q$. Then there exists a simple (q^i-1)-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$. $\mathcal{F}^i$ is decomposable if and only if there exists an f-fold $(i-1)$-spread in $\mathrm{PG}^{(i-1)}(n-1,q)$ for some $1\leq f<\lambda_i$.*

Proof. Consider the n-dimensional affine space as $\mathrm{AG}(n,q)=\mathrm{PG}(n,q)\setminus\mathcal{H}_\infty$ where $\mathcal{H}_\infty$ is isomorphic to $\mathrm{PG}(n-1,q)$. Take the design $D=\mathrm{AG}^{(i)}(n,q)$ and apply Lemma 58. If Π^{i-1} is an $(i-1)$-space of $\mathcal{H}_\infty$, then the set of the q^{n-i} parallel affine i-spaces through Π^{i-1} is an i-spread of D. This spread induces a (q^i-1)-factor. We denote the set of $(i-1)$-spaces of $\mathcal{H}_\infty$ as Π_j^{i-1} for $j\in\{1,\ldots,g\}$ where g is the number of $(i-1)$-spaces of $\mathcal{H}_\infty$. And we denote the i-spreads through Π_j^{i-1} as F_j^i. If $\Pi_1^{i-1},\Pi_2^{i-1},\ldots,\Pi_g^{i-1}$

are distinct $(i-1)$-spaces of $\mathcal{H}_\infty$ and they form an f-fold spread, then $f = (g(q^i - 1))/(q^n - 1)$, and the union of the corresponding (q^i-1)-factors F^i_j, for $j = 1, 2, \ldots, g$, gives a $(q^i - 1)$-factorization of fK_{q^n}. Distinct $(i-1)$-spaces of $\mathcal{H}_\infty$ obviously define distinct $(q^i - 1)$-factors, so this factorization is simple. In particular if we consider all $(i-1)$-spaces of $\mathcal{H}_\infty$, then

$$g = \begin{bmatrix} n \\ i \end{bmatrix}_q, \ f = \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{q^i - 1}{q^n - 1} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q = \lambda_i,$$

hence the union of the corresponding factors gives a simple $(q^i - 1)$-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$.

Suppose that $\mathcal{F}^i$ is decomposable, then there exist two positive integers μ_1 and μ_2 such that $\mu_1 + \mu_2 = \lambda_i$ and $\mathcal{F}^i$ can be written as the union $\mathcal{F}^i = \mathcal{F}_1 \cup \mathcal{F}_2$; $\mathcal{F}_1$ and $\mathcal{F}_2$ are (q^i-1)-factorizations of $\mu_1 K_{q^n}$ and $\mu_2 K_{q^n}$, respectively, having no $(q^i - 1)$-factors in common, since $\mathcal{F}^i$ is simple. For $h = 1, 2$, the relation $\mu_h \binom{q^n}{2} = \binom{q^i}{2} q^{n-i} |\mathcal{F}_h|$ holds, hence $\mu_h(q^n - 1) = (q^i - 1)|\mathcal{F}_h|$. Without loss of generality we can set $\mathcal{F}_1 = \cup_{j=1}^{f_1} F^i_j$ with $f_1 = \mu_1(q^n-1)/(q^i-1)$, and $\mathcal{F}_2 = \mathcal{F}^i \setminus \mathcal{F}_1$, $f_2 = |\mathcal{F}_2|$.

Let u_1 and u_2 be two affine points and let w be the point at infinity of the line u_1u_2. Since $\mathcal{F}_h$ is a factorization of $\mu_h K_{q^n}$, there are exactly μ_h factors of $\mathcal{F}_h$ containing the edge $[u_1, u_2]$, say $F^i_{j_1}, F^i_{j_2}, \ldots, F^i_{j_{\mu_h}}$. The edge $[u_1, u_2]$ belongs to $F^i_{j_s}$ if and only if $w \in \Pi^{i-1}_{j_s}$ for every $1 \leq s \leq \mu_h$. This happens if and only if $\cup_{j=1}^{f_h} \Pi^{i-1}_j$ contains each point of $\mathcal{H}_\infty$ exactly μ_h times, which means that $\cup_{j=1}^{f_h} \Pi^{i-1}_j$ is a μ_h-fold spread in $\mathcal{H}_\infty$, for every $h = 1, 2$. It is thus proved that if $\mathcal{F}^i$ is decomposable, then $\mathrm{PG}^{i-1}(n-1, q)$ posesses an f-fold spread for some $1 \leq f < \lambda_i$.

Conversely, suppose that there exists a μ_1-fold spread in $\mathrm{PG}^{i-1}(n-1,q)$ for some $1 \leq \mu_1 < \lambda_i$. Let $\mathcal{F}_1 = \cup_{j=1}^{f_1} F^i_j$ be a μ_1-fold spread in $\mathcal{H}_\infty$. Then $|\mathcal{F}_1| = f_1 = \mu_1(q^n - 1)/(q^i - 1)$. Let $\mathcal{T}$ be the set of all $(i-1)$-dimensional subspaces in $\mathcal{H}_\infty$ and let $\mathcal{F}_2 = \mathcal{T} \setminus \mathcal{F}_1$. Then $|\mathcal{T}| = \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q$,

$$|\mathcal{F}_2| = \begin{bmatrix} n \\ i \end{bmatrix}_q - \mu_1(q^n-1)/(q^i-1) = \left(\begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q - \mu_1 \right) \frac{q^n - 1}{q^i - 1},$$

so if $\mu_2 = \left[{n-1 \atop i-1}\right]_q - \mu_1$, then $\mathcal{F}_2$ is a μ_2-fold spread in $\mathcal{H}_\infty$ and $1 \leq \mu_2 < \lambda_i$ holds.

As we have already seen, $\mathcal{F}_h$ defines a (q^i-1)-factorization of $\mu_h K_{q^n}$ for $h = 1, 2$. Then $\mathcal{F}^i = \mathcal{F}_1 \cup \mathcal{F}_2$, because $\mu_1 + \mu_2 = \lambda_i$. Hence the (q^i-1)-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$ is decomposable. □

Corollary 61. *If* $\gcd(i,n) > 1$ *then the* (q^i-1)*-factorization* $\mathcal{F}^i$ *described in Theorem 60 of* $\lambda_i K_{q^n}$ *is decomposable.*

Proof. Let $1 < l+1$ be a divisor of $\gcd(i,n)$. Then it follows from Theorem 59 that there exists an $(q^i-1)/(q^{l+1}-1)$-fold spread in $\mathcal{H}_\infty$, so $\mathcal{F}^i$ is decomposable. □

To decide the decomposability of $\mathcal{F}^i$ in the case $\gcd(i,n) = 1$ is a hard problem in general. We prove its indecomposability in the following important case.

Theorem 62. *The* $(q^{n-1}-1)$*-factorization* $\mathcal{F}^{n-1}$ *of* $(q^{n-1}-1)/(q-1)K_{q^n}$ *is indecomposable.*

Proof. It is enough to prove that if $\cup_{j=1}^{g} \Pi_j^{n-2}$ is an f-fold $(n-2)$-spread in $\mathcal{H}_\infty$, then $\cup_{j=1}^{g} \Pi_j^{n-2}$ consists of all $(n-2)$-dimensional subspaces of $\mathcal{H}_\infty$, because this implies $f = \lambda_{n-1}$, so the statement follows from Theorem 60.

Each Π_j^{n-2} contains exactly $(q^{n-1}-1)/(q-1)$ points, thus the standard double counting of the point-space pairs $p \in \Pi_j^{n-2}$ in $\mathcal{H}_\infty$ gives

$$g\frac{q^{n-1}-1}{q-1} = f\frac{q^n-1}{q-1},$$

hence

$$f = \frac{g(q^{n-1}-1)}{q^n-1}.$$

But $\gcd(q^n-1, q^{n-1}-1) = q-1$ and f is an integer, so $g \geq (q^n-1)/(q-1)$ which implies $g = (q^n-1)/(q-1)$, hence $f = \lambda_{n-1}$. □

In particular if $n = 2$, we get the following.

Corollary 63. *If* q *is a prime power then there exists a simple and indecomposable* $(q-1)$*-factorization of* K_{q^2}.

If $q = 2^r$ then each $(q^i - 1)$-factor in $\mathcal{F}^i$ is the vertex-disjoint union of 2^{r-i} complete graphs on 2^i vertices. It is well-known that these graphs can be partitioned into one-factors in many ways (it was proved by Hartman and Rosa [HR85], that there is no cyclic one-factorization of K_{2^i} for $i \geq 3$, namely, An m-factorization of a graph G into k copies of a graph H is a *cyclic factorization* if H is drawn in an appropriate manner so that rotating H through an appropriate angle $k-1$ times produces this m factorization), hence Theorem 60 implies several one-factorizations of $\lambda_i K_{2^r}$.

Each of the one-factorizations arising from $\mathcal{F}^i$ is simple. This follows since distinct $(i-1)$-dimensional subspaces define distinct $(q^i - 1)$-factors of $\mathcal{F}^i$, and the since one-factors of $\lambda_i K_{q^n}$ arising from distinct $(q^i - 1)$-factors of $\mathcal{F}^i$ are distinct because they are the union of q^{n-i} one-factors on q^i vertices of a connected component.

There are both decomposable and indecomposable one-factorizations among these examples. We show it in the smallest case $q = 2$, $n = 3$. Let $\mathcal{F}^2$ be the 3-factorization of $3K_8$ induced by $\mathrm{AG}(3, 2)$.

Let $\mathrm{PG}(3, 2) = \mathrm{AG}(3, 2) \cup \mathcal{H}_\infty$. Then $\mathcal{H}_\infty$ is isomorphic to the Fano plane. Let its points be $0, 1, 2, 3, 4, 5$ and 6 such that for $j = 0, 1, \ldots, 6$, the triples $L_j = (j, j+1, j+3)$ form the lines of the plane, where the addition is taken modulo 7. Now the 3-factors of $\mathcal{F}^2$ can be described in the following way. Let a be a fixed point in $\mathrm{AG}(3, 2)$. Then L_j defines a 3-factor F_j^2 whose connected components are complete graphs $K_{2^i} = K_4$. Let $L_{j,a}$ be the complete graph containing a, and let $L_{j,\bar{a}}$ be the other component of F_j^2.

$\mathcal{H}_\infty$ defines one-factors and a one-factorization of K_8 in the following obvious way. The edge joining two points of $\mathrm{AG}(3, 2)$, say b and c, belong to the one-factor G_s for some $s \in \mathcal{H}_\infty$ if and only if b, c and s are collinear points in $\mathrm{PG}(3, 2)$. Then $\mathcal{G} = \cup_{s=0}^{6} \mathcal{G}_s$ is a one-factorization of K_8.

We can define a decomposable one-factorization of $3K_8$ in the following way. Take $L_{j,a}$ and $L_{j,\bar{a}}$ and let $s \in L_j$ be any point. Then G_s gives a one-factor of $L_{j,a}$ and a one-factor of $L_{j,\bar{a}}$. Hence $\mathcal{G}_j = \cup_{s \in L_j} G_s$ is the union of three one-factors of $3K_8$, and $\mathcal{G}' = \cup_{j=0}^{6} \mathcal{G}_j$ is a one-factorization of $3K_8$.

In $\mathcal{H}_\infty$ there are three lines through the point s, hence $\mathcal{G}'$ contains each one-factor G_s three times. Thus $\mathcal{G}'$ is decomposable, because it is obviously the union of three copies of $\mathcal{G}$.

But we can define an indecomposable one-factorization, too. Let L_j be a line in $\mathcal{H}_\infty$, take $L_{j,a}$ and $L_{j,\overline{a}}$ and let M_j^1 be the one-factor which contains the following pairs of points in AG$(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and b,c,j are collinear in PG$(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and $b,c,j+1$ are collinear in PG$(3,2)$.

Let M_j^2 be the one-factor which contains the following pairs of points in AG$(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and $b,c,j+1$ are collinear in PG$(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and $b,c,j+3$ are collinear in PG$(3,2)$.

Finally let M_j^3 be the one-factor which contains the following pairs of points in AG$(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and $b,c,j+3$ are collinear in PG$(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and b,c,j are collinear in PG$(3,2)$.

Then $\mathcal{M}_j = \cup_{t=1}^3 M_j^t$ is a union of three one-factors of $3K_8$, and $\mathcal{M} = \cup_{j=0}^6 \mathcal{M}_j$ is a one-factorization of $3K_8$.

Suppose that this 1-factorization is decomposable, then it contains a 1-factorization $\mathcal{E}$ of K_8. Therefore, $\mathcal{E}$ is the union of seven one-factors. We may assume without loss of generality, that M_0^1 belongs to $\mathcal{E}$. It contains an edge through a, say (a,b), and a pair (c,d) such that the lines ab and cd are parallel in AG$(3,2)$. There are two more lines in the parallel class of ab, say ef and gh. It follows from the definition of the one-factors that exactly one of them contains the pairs (e,f) and (a,b), another one contains the pairs (e,f) and (c,d), and a third one contains the pairs (e,f) and (g,h). But $\mathcal{E}$ contains each pair exactly once, hence it must contain the one-factor containing the pairs (e,f) and (g,h). But this is a one-factor of type M_0^t, where $t \neq 1$. Hence $\mathcal{E}$ contains M_0^t where $t = 2$ or 3. If we repeat the previous argument, we get that $\mathcal{E}$ must contain M_0^l for all $t \in \{1,2,3\}$. Thus $\mathcal{E}$ is the union of triples of type M_j^t, $t = 1,2,3$, but this is a contradiction, because $\mathcal{E}$ consists of seven one-factors.

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Chapter 7

Conclusion

There is little doubt that the most studied and best known subject within Graph Theory is Coloring:

Graph Coloring is arguably the most popular subject in Graph Theory.

Noga Alon (1993)

Plane graphs and their colorings have been the main topic of intensive research since the beginnings of graph theory because of their connection to the *four color problem*. As stated originally the four color problem asked whether it is always possible to color the regions of a plane map with four colors such that regions which share a common boundary (and not just a point) receive different colors. Then graph colorings has become a subject of great interest, largely because of its diverse theoretical results, its numerous applications, and its unsolved problems.

This thesis is devoted to this important subject, specifically in one branch of graph colorings that has received attention recently: *complete colorings* [CZ09]. The paper's collection provides contributions to knowledge in the following research areas:

Chapter 2: This chapter is about perfection in graphs where perfection means the equality, for a graph G and every induced subgraph of G, of two parameters among the lenght of the maximum clique, the chromatic number, the pseudo-Grundy chromatic number [Rub15], the Grundy chromatic number, the achromatic number and the pseudoachromatic number [AR, AR13]. This concept of perfection shelters the germinal concept, introduced by C. Berge in 1961 [Ber61], and presents other ways to capture the relation between graph coloring parameters. We shall mention that this concept was introduced earlier by Christen and Selkow in 1979 [CS79] and extended by V. Yegnanarayanan in 2001 [Yeg01].

Chapter 3: In this chapter we calculate achromatic and pseudoachromatic indices of those complete graphs whose sizes are of the form $(2^\gamma+1)^2-a$, for $a \in \{0,1,2\}$ and $\gamma > 1$ [AMRS14]; and the pseudoachromatic index for $a \in \{3,4,\dots,\left\lceil\frac{1+\sqrt{4q+9}}{2}\right\rceil-1\}$ [AMRS]. The main contribution, and the main difficulty, lies on proving the lower bounds, since the matching upper bounds follow rather easily from a theorem by Jamison [Jam89], which is improved here. These bounds are obtained by nontrivial constructions, based on the structure of projective planes [Bou78].

Chapter 4: This chapter is about complete colorings in block designs [And90]. We bounded the achromatic and the pseudoachromatic indices of block design, we improve some known theorems of [CC83] and we give some exact values for particular designs (see [AKRV]).

Chapter 5: In this chapter we studied the achromatic and pseudoachromatic indices for geometric graphs. The main result in this chapter is determining the achromatic and the pseudoachromatic indices for the convex geometric complete graph and bounding when points are in general position. We introduce the parameter "geometric achromatic and pseudoachromatic indices" of an abstract graph (see [AAG$^+$]).

Chapter 6: This chapter is about simple and indecomposable κ-factorizations of the complete multigraph λK_v [Wal97]. The existence of a κ-factorization of the complete multigraph λK_v is equivalent to the existence of a resolvable (v,k,λ)-design. Here examples of simple κ-factorizations of λK_v for some values of κ and λ are given. We show that in some cases these κ-factorizations are indecomposable and these examples are obtained from affine and projective geometries [KR15].

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Contents