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AI-MAXIMAL INDEPENDENT FAMILIES AND SOME RESULTS ON WEAKLY PSEUDOCOMPACT SPACES

TESIS

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Invento matemáticas, por lo tanto
pienso y existo.

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A los sinodales Adalberto García Maynez, Fernando Hernández Hernández, Michael Hrušák y Roberto Pichardo Mendoza.

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Introducción

Es un honor para mi presentar la siguiente tesis de Doctorado, fruto de un continuo trabajo académico de ocho semestres bajo la supervisión del Dr. Ángel Tamariz Mascarúa. El objetivo de este trabajo es presentar los resultados que obtuve durante mi estancia doctoral.

El capítulo 1 está dedicado a estudiar resultados preliminares acerca de los espacios resolubles, espacios maximales y espacios submaximales.

En el capítulo 2 presento los resultados originales que obtuve durante la primera parte de mi estancia doctoral. Fruto de este trabajo fue la redacción del artículo de investigación [14], mismo que ya fue aceptado para su publicación en la revista Topology and its Applications. En este capítulo, se estudian las consecuencias de la existencia de familias ai-maximales y se dan dos condiciones suficientes que garantizan su existencia.

En particular demostré que bajo la hipótesis del continuo, no existen familias ai-maximales y que la existencia de dichas familias implica la existencia de un espacio Baire, irresoluble y sin puntos aislados. En [32], se demuestra que la existencia de un espacio Baire, irresoluble y sin puntos aislados es equiconsistente con la existencia de un cardinal medible, lo que implica que la existencia de familias ai-maximales de cardinalidad máxima implica la existencia de cardinales medibles en algún modelo consistente con ZFC.

La motivación para estudiar familias ai-maximales surgió después de estudiar espacios resolubles y descubrir que en [28], los autores inventaron un método que utiliza familias maximales independientes para demostrar que existen espacios ω-resolubles que no son maximalmente resolubles. En consecuencia, pude modificar dicho método para determinar condiciones suficientes y condiciones necesarias para la existencia de familias ai-maximales independientes.

Pavlov anunció en [37, Theorem 3.16] que, bajo la Hipótesis del Continuo, cualquier espacio Baire ccc sin puntos aislados y T_1 es resoluble. En un reciente trabajo demostré que bajo la Hipótesis del Continuo, la existencia de un cardinal medible es equivalente a la existencia de un espacio Baire crowded ccc casi irressoluble y T_1. Esta afirmación demuestra que la proposición de Pavlov [37, Theorem 3.16] es incorrecta. Aunado a esto, pude probar lo siguiente. 1.- Cualquier espacio crowded ccc con cardinalidad menor que el primer cardinal débilmente inaccesible es casi resoluble. 2.- Si 2^ω es menor que el primer cardinal débilmente inaccesible entonces cualquier espacio T_2 crowded ccc es casi resoluble. 3.- $V = L$ implica que todo espacio crowded es casi resoluble. Estos resultados están recopilados en el manuscrito [13].

El capítulo 3 está dedicado a estudiar resultados preliminares acerca de las compactaciones de los espacio Tychonoff, las realcompactaciones y los espacios pseudocompactos.

En el capítulo 4 presento lo resultados originales que obtuve acerca de los espacios débilmente pseudocompactos. En particular, obtuve una caracterización de este tipo de espacios usando subanillos regulares; caractericé los espacios débilmente pseudocompactos en la clase de los espacios localmente pseudocompactos y demostré que los espacios que no son localmente pseudocompactos no son G_δ densos en ninguna compactación simple.

Saber si el espacio ω^{ω_1} es débilmente pseudocompacto aún no se ha podido determinar, pero pude demostrar una condición necesaria para que esto último suceda, a saber, que la retícula de compactaciones de ω^{ω_1} no es b retícula. Todos los resultados que obtuve sobre espacios débilmente pseudocompactos han sido recopilados en el artículo [15].

Chapter 1

Preliminaries on almost resolvable spaces

1.1 Resolvable spaces

In [23], Hewitt introduced the concept of *resolvable spaces*, defining them as those topological spaces which contain two disjoint dense subsets. An easy example is the real line $\mathbb{R}$, indeed, the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{P}$, form a partition of $\mathbb{R}$ in two dense subsets.

More generally, if α is a cardinal and $\alpha \geq 1$, we will say that a space X is *α-resolvable* if there is a family of α disjoint dense subsets. Observe that 2-resolvable is equivalent to resolvable and all spaces are 1-resolvable. It is easy to see that if Y is an α-resolvable subspace of X then $\operatorname{cl}_X Y$ is α-resolvable, because every dense subset of Y is also dense in $\operatorname{cl}_X Y$.

Example 1.1. In the open interval $(0,1)$ the set $D = \{p/2^q : 0 < p < 2^q, p, q \in \omega \text{ and } p \text{ odd}\} \subseteq \mathbb{Q} \cap (0,1)$ is dense. If we take $n > 1$ and define $D_m = \{p/2^{nq+m} : 0 < p < 2^{nq+m}, p, q \in \omega \text{ and } p \text{ odd}\}$ for every $m < n$, then $\{D_m : m < n\}$ is a family of n disjoint dense subsets in $(0,1)$. Therefore, $(0,1)$ is n-resolvable for every $n \geq 2$ and so is $[0,1]$.

We will see in Corollary 1.6, that if a space X is n-resolvable for every $n \geq 2$, then X is ω-resolvable. The next theorem says that the union of α-resolvable spaces is α-resolvable.

Theorem 1.2. *[9] Let $\alpha \geq 1$ be a cardinal and let X be a space of the form $X = \cup_{j \in J} X_j$ with each X_j, α-resolvable. Then X is α-resolvable.*

Proof. Consider the family $\mathcal{S} = \{\mathcal{A} \subseteq \mathcal{P}(X) : \mathcal{A} \text{ is a family of pairwise disjoint } \alpha\text{-resolvable spaces}\}$. Then $\mathcal{S}$ is nonempty and $(\mathcal{S}, \subseteq)$ is a partially ordered set. Take a chain $\mathcal{C}$ in $\mathcal{S}$. Then $\cup\mathcal{C}$ is an element of $\mathcal{S}$. Applying Zorn's Lemma, we can find a maximal family $\mathcal{A}$ of pairwise disjoint α-resolvable subspaces of X.

If the open set $U = X \setminus \mathrm{cl}_X(\cup\mathcal{A})$ is nonempty, there is $j \in J$, such that $U \cap X_j \neq \emptyset$. Then $U \cap X_j$ is α-resolvable, and $\mathcal{A} \cup \{U \cap X_i\}$ is a family of pairwise disjoint α-resolvable spaces contradicting the maximality of $\mathcal{A}$.

Therefore $\cup\mathcal{A}$ is dense in X. Now, for every element $A \in \mathcal{A}$, there is a family $\{D_\xi^A : \xi < \alpha\}$ of pairwise disjoint dense subsets of A. Define

$$D_\xi = \cup_{A\in\mathcal{A}} D_\xi^A, \text{ for every } \xi < \alpha.$$

Take a nonempty open set V in X. Since $\cup\mathcal{A}$ is dense in X, there is $A \in \mathcal{A}$ such that $V \cap A$ is nonempty, hence $V \cap A_\xi$ is nonempty for every $\xi < \alpha$. Then $V \cap D_\xi$ is nonempty for every $\xi < \alpha$, and this shows that $\{D_\xi : \xi < \alpha\}$ is a family of pairwise disjoint dense subsets of X. □

Corollary 1.3. Let X be a space. For every cardinal $\alpha \geq 1$, there is a unique partition $\{Y_\alpha, Z_\alpha\}$ of X such that every α-resolvable subset of X is contained in Y, Y is closed and α-resolvable, and no subset of Z is α-resolvable.

Proof. Define $Y = \cup\{Z \subseteq X : Z \text{ is } \alpha\text{-resolvable}\}$. Because of the last theorem, Y is α-resolvable. Since $\mathrm{cl}_X Y$ is α-resolvable, Y is closed. Then, $\{Y, X \setminus Y\}$ is the desired partition. □

Corollary 1.4. Let $n \geq 2$ be a finite integer. Let X be a n^2-resolvable space. If $X = D \cup E$ and D is not n-resolvable, then some nonempty open subset of E is n-resolvable.

Proof. Since X is n^2-resolvable, we can write $X = \cup_{1\leq i\leq n} E_i$, where $\{E_i : 1 \leq i \leq n\}$ is a family of pairwise disjoint subspaces such that each E_i is a union of n many disjoint dense subsets of X.

Observe that $D = \cup_{1\leq i\leq n}(D \cap E_i)$. Since D is not n-resolvable, there is a nonempty open set U and an i_0 such that $U \cap D \cap E_{i_0}$ is empty. Then $U \cap E_{i_0} \subseteq U \cap E$. Since E_{i_0} is a union of n many disjoint dense subsets of X and U is open, then $U \cap E$ is a nonempty open n-resolvable subset of E. □

Theorem 1.5. *[5] If X is n-resolvable for every $n \geq 2$, then there are disjoint dense subsets D and $X \setminus D$ of X such that $X \setminus D$ is k-resolvable for every finite cardinal $k \geq 2$.*

Proof. Since X is 2-resolvable, we can write $X = E \cup F$ where E and F are disjoint and dense in X. For each $n \geq 2$, apply Corollary 1.3 to obtain $E = Y_n \cup Z_n$ where $Y_n \cap Z_n = \emptyset$, no subset of Z_n is n-resolvable, every n-resolvable subset of E is contained in Y_n and Y_n is n-resolvable and closed. Find an open set U_n such that $U_n \cap E = Z_n$.

For each $n \geq 2$, define $V_n = \cup_{m \leq n} U_m$, and observe that $V_n \cap E = Z_n$ for every $n \in \omega$. Indeed, if $V_n \cap E \cap Y_n \neq \emptyset$, then $U_m \cap Y_n$ is nonempty for some $m \leq n$. Then, $U_m \cap Y_n$ is n-resolvable. But, $Z_m = U_m \cap E \supseteq U_m \cap Y_n$. Then $U_m \cap Y_n$ is not m-resolvable, because it is a subspace of Z_m. Since $m \leq n$, we have a contradiction.

Now, take $n_0 \geq 2$. Since V_{n_0} is n_0^2-resolvable (because X is n_0^2-resolvable), and $Z_{n_0} = V_{n_0} \cap E$ does not contain nonempty open n_0-resolvable subsets, then, applying the last corollary, we have that $V_{n_0} \cap F$ is n_0-resolvable.

Let $V_0 = \cup_{n \geq 2} V_n$. Then, $V_0 \cap F$ is k-resolvable for every $k \geq 2$. To see this, take $k \geq 2$. Then, $V_m \cap F$ is k-resolvable for every $m \geq k$. Since $V_p \subseteq V_k$ for every $p \leq k$, $V_p \cap F$ is k-resolvable for every $p \leq k$. Then, $V_0 \cap F$ is a union of k-resolvable subspaces. It is easy to see that $\mathrm{cl}_X V_0 \cap F$ is also k-resolvable for every $n \geq 2$. Since $(X \setminus \mathrm{cl}_X V_0) \cap E$ is an open subset of Y_n, for every $n \geq 2$, then $(X \setminus \mathrm{cl}_X V_0) \cap E$ is n-resolvable for every $n \geq 2$.

Define $D = (\mathrm{cl}_X V_0 \cap E) \cup ((X \setminus \mathrm{cl}_X V_0) \cap F)$. Then D and $X \setminus D$ are dense in X and $X \setminus D = (\mathrm{cl}_X V_0 \cap F) \cup ((X \setminus \mathrm{cl}_X V_0) \cap E)$ is n-resolvable for every $n \geq 2$. □

Corollary 1.6. [25] If X is n-resolvable for every finite cardinal $n \geq 2$, then X is ω-resolvable.

Proof. There are disjoint dense subsets D and $X \setminus D$ of X such that $X \setminus D$ is k-resolvable for every finite cardinal $k \geq 2$. Define $D_0 = D$.

Let $n \in \omega$ and suppose that $\{D_0, D_1, \ldots, D_n, X \setminus (\cup_{m \leq n} D_m)\}$ is a family of disjoint dense subsets of X such that $X \setminus (\cup_{m \leq n} D_m)$ is k-resolvable for every finite cardinal $k \geq 2$.

There are disjoint dense subsets D_{n+1} and $(X \setminus (\cup_{m \leq n} D_m)) \setminus D$ of $X \setminus (\cup_{m \leq n} D_m)$ such that $(X \setminus (\cup_{m \leq n} D_m)) \setminus D$ is k-resolvable for every finite cardinal $k \geq 2$. Hence $\{D_0, D_1, \ldots, D_{n+1}, X \setminus (\cup_{m \leq n+1} D_m)\}$ is a family of disjoint dense subsets of X such that $X \setminus (\cup_{m \leq n+1} D_m)$ is k-resolvable for every finite cardinal $k \geq 2$.

Therefore, $\{D_n : n \in \omega\}$ is a family of pairwise disjoint dense subsets of X. □

Corollary 1.7. The space $\mathbb{Q}$ is ω-resolvable and so is $\mathbb{R}$.

Observe that if a topological space X has an isolated point x, then every dense subset D of X contains x, therefore X is not resolvable. We will say that a topological space X is *crowded*, if X contains no isolated points. So, the study of resolvable spaces is interesting just in the realm of crowded spaces.

The *dispersion character* of X, denoted by $\Delta(X)$, is the least cardinality of a nonempty open subset of X. If U is a nonempty open subset of a crowded space X, then the cardinality of every family of disjoint dense subsets of X is less than or equal to $|U|$. Therefore we have proved the following proposition.

Proposition 1.8. *If X is a crowded space and X is α-resolvable, then $\alpha \leq \Delta(X)$.*

The *weight* of a space X is defined as the smallest cardinal number of the form $|\mathcal{B}|$ such that $\mathcal{B}$ is a base for X; this cardinal is denoted by $w(X)$. A family $\mathcal{N}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighborhood U of x, there is an $M \in \mathcal{N}$ such that $x \in M \subseteq U$. The *network weight* of a space X is defined as the smallest cardinal number of the form $|\mathcal{N}|$ such that $\mathcal{N}$ is a network for X; this cardinal is denoted by $nw(X)$. It is easy to prove that $nw(X) \leq w(X)$ and $nw(X) \leq |X|$.

Theorem 1.9. *[17] For every compact space X we have $nw(X) = w(X)$.*

Proof. Let $nw(X) = \kappa$ and let $\mathcal{N}$ be a network for X of cardinality κ. If k is finite, then X is discrete of cardinality κ and $w(X) = nw(X)$.

Suppose that $\kappa \geq \omega$. Let $\mathcal{B} = \{M \in \mathcal{N} :$ there exist $N \in \mathcal{N}$ and disjoint open sets $U_1, U_2 \in \tau_X$ such that $M \subseteq U_1, N \subseteq U_2\}$. Let $\mathcal{B}_0$ be the family of finite intersections of $\mathcal{B}$. Therefore, $\mathcal{B}_0$ is a base for a topology σ in X.

Let x, y, be two distinct points of X. There are open subsets $U_1, U_2 \in \tau_X$ such that $x \in U_1, y \in U_2$. Since $\mathcal{N}$ is a network, there exist $N_1, N_2 \in \mathcal{N}$ such that $x \in N_1 \subseteq U_1, y \in N_2 \subseteq U_2$. Then $N_1, N_2 \in \sigma$. Therefore (X, σ) is Hausdorff. Observe that the identity function i from X to (X, σ) is continuous. Since (X, τ_X) is compact, i is closed, and therefore, i is a homeomorphism.

Since $|\mathcal{B}_0| = |\mathcal{N}|$, we have $w((X, \sigma)) \leq \kappa$. Hence $w(X) \leq \kappa$. □

The following theorem is going to be useful.

Theorem 1.10 (Disjoint refinement lemma)**.** *[10] Let α be an infinite cardinal and let $\{A_\xi : \xi < \alpha\}$ be a family of sets such that $|A_\xi| \geq \alpha$ for $\xi < \alpha$. Then there is a family of pairwise disjoint sets $\{B_\xi : \xi < \alpha\}$ such that:*

1. *$B_\xi \subseteq A_\xi$ for $\xi < \alpha$,*

2. *$|B_\xi| = \alpha$ for $\xi < \alpha$.*

Proof. Let $x_0^0 \in A_0$. Let $\beta < \alpha$ and suppose that for $\xi' \leq \xi < \beta$ we have defined $x_\xi^{\xi'} \in A_{\xi'}$ such that $x_\xi^{\xi'} \in A_{\xi'} \setminus \{x_\zeta^{\zeta'} : \zeta' \leq \zeta < \xi\}$ and $x_\beta^{\zeta'} \neq x_\beta^{\zeta''}$ for $\zeta' < \zeta'' \leq \beta$.

Choose $x_\beta^{\beta'} \in A_{\beta'} \setminus \{x_\xi^{\xi'} : \xi' \leq \xi < \beta\}$ for $\beta' \leq \beta$ and $x_\beta^{\beta'} \neq x_\beta^{\beta''}$ for $\beta' < \beta'' \leq \beta$. Define

$$B_\xi = \{x_\beta^\xi : \xi \leq \beta < \alpha\}$$

Then the family $\{B_\xi : \xi < \alpha\}$ is as required. □

Proposition 1.11. *Let X be a crowded space and α be an infinite cardinal. If X has a network $\mathcal{N}$ such that $|\mathcal{N}| \leq \alpha$ and for every $S \in \mathcal{N}$, $|S| \geq \alpha$, then X is α-resolvable.*

Proof. Set $\mathcal{N} = \{S_\xi : \xi < \alpha\}$. By the disjoint refinement lemma, there is a family of pairwise disjoint sets $\{B_\xi : \xi < \alpha\}$ such that:

1. $B_\xi \subseteq S_\xi$ for $\xi < \alpha$,

2. $|B_\xi| = \alpha$ for $\xi < \alpha$.

For each $\xi < \alpha$, let $\{b_\xi^\gamma : \gamma < \alpha\}$ be an enumeration of B_ξ without repetitions. Now, for each $\gamma < \alpha$, define $D_\gamma = \{b_\xi^\gamma : \xi < \alpha\}$. We are going to see that D_γ is dense for every $\gamma < \alpha$. Take $\gamma_0 < \alpha$. Let U be a nonempty open set of X. Since $\mathcal{N}$ is a network, there is ξ_0 such that $S_{\xi_0} \subseteq U$ and $B_{\xi_0} \subseteq U$. Therefore $b_{\xi_0}^{\gamma_0} \in D_{\gamma_0} \cap U$. Since $\{B_\xi : \xi < \alpha\}$ is a pairwise disjoint family, $\{D_\gamma : \gamma < \alpha\}$ is a pairwise disjoint family of dense subsets of X. □

The next theorem shows that there is a Tychonoff crowded space X such that X is $\Delta(X)$-resolvable.

Theorem 1.12. *Let X be a crowded Hausdorff locally compact space, then X is $\Delta(X)$-resolvable.*

Proof. Let $\mathcal{A} = \{U \in \tau_X : \Delta(X) = |U|, \operatorname{cl}_X U$ is compact and $|\operatorname{cl}_X U| = |U|\}$. It is easy to see that $\mathcal{A}$ is nonempty. Indeed, take an open set V such that $V = \Delta(X)$. If W is an open subset of V such that $\operatorname{cl}_X W$ is compact and $\operatorname{cl}_X W \subseteq V$, then $|W| \leq |\operatorname{cl}_X W| \leq |V| = \Delta(X)$, but $|W| \geq \Delta(X)$. Therefore $|W| = |\operatorname{cl}_X W| = \Delta(X)$.

Let $U \in \mathcal{A}$ and let $\mathcal{B}$ be a base for U with $|\mathcal{B}| = w(U)$. By Theorem 1.9, $w(\mathrm{cl}_X U) \leq |\mathrm{cl}_X U|$. Then

$$|\mathcal{B}| = w(U) \leq w(\mathrm{cl}_X U) \leq |\mathrm{cl}_X U| = |U| = \Delta(X).$$

Since $|B| \geq \Delta(X)$ for every $B \in \mathcal{B}$, Proposition 1.11 establishes that U is $\Delta(X)$-resolvable. Therefore $\cup\mathcal{A}$ is $\Delta(X)$-resolvable. Observe that $\cup\mathcal{A}$ is dense. Therefore X is $\Delta(X)$-resolvable. □

Proposition 1.13. *Let (X, ρ) be a metric space, then $\omega(X) = d(X)$.*

Proof. If $\mathcal{B}$ is a base for X. For every $B \in \mathcal{B}$, take $x_B \in B$. Then, $\{x_B : B \in \mathcal{B}\}$ is a dense subset of X. Therefore $\omega(X) \leq d(X)$.

Now, let A be a dense subset of X such that $|A| = d(X)$. Denote by $\mathcal{B}$ the family of all balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$, such that $x \in A$ and $r \in \mathbb{Q}$. Then, $|\mathcal{B}| \leq |A|$.

We are going to prove that $\mathcal{B}$ is a base for (X, ρ). Take a point $y \in X$ and U an open set in X such that $y \in U$. There is $r \in \mathbb{R}$ such that $B(y, r) \subseteq U$. Since A is dense, there is $x \in A$ such that $x \in B(y, r/3)$. Take a rational number r_0 such that $r/3 < r_0 < r/2$. Then $y \in B(x, r_0)$. Take $z \in B(x, r_0)$. Then $d(z, y) \leq d(z, x) + d(x, y) < r_0 + r/3 < r$. Therefore,

$$B(x, r_0) \subseteq B(y, r) \subseteq U.$$

Hence $\mathcal{B}$ is a base for (X, ρ) and $w(X) \leq d(X)$. □

Theorem 1.14. *If X is a crowded metric space, then X is $\Delta(X)$-resolvable.*

Proof. Let X be a crowded metric space and $\mathcal{A} = \{U \in \tau_X : \Delta(X) = |U|\}$. It is easy to see that $\mathcal{A}$ is nonempty.

Let $U \in \mathcal{U}$ and $\mathcal{B}$ be a base for U with $|\mathcal{B}| = w(U)$. By theorem 1.13, $w(U) = d(U)$. Then,

$$|\mathcal{B}| = w(U) = d(U) \leq |U| = \Delta(X).$$

Since $|B| \geq \Delta(X)$ for every $B \in \mathcal{B}$, the proposition 1.11 establishes that U is $\Delta(X)$-resolvable. Therefore $\cup\mathcal{A}$ is $\Delta(X)$-resolvable. Observe that $\cup\mathcal{A}$ is dense. Therefore X is $\Delta(X)$-resolvable. □

1.2 Maximal spaces

A crowded space X is *irresolvable* if X is not resolvable, more generally, given a cardinal α, we say that X is α*-irresolvable* if X is not α-resolvable.

We are going to study irresolvable spaces using maximal spaces.

Definition 1.15. [16] Call a space X *maximal* if its topology, τ_X, is maximal in the collection of all crowded topologies on X. If T is a separation axiom, call X maximal T if τ_X is maximal in the collection of all crowded topologies satisfying T on X.

Proposition 1.16. *[16] If $i \in \{1, 2, 3, 3.5\}$, then maximal T_i spaces are irresolvable.*

Proof. Let (X, τ) be a topological space. Suppose that D, E are disjoint dense subsets of (X, τ). Consider the topology τ' on X generated by $\tau \cup \{D, X \setminus D\}$. Since τ is crowded, so is τ'. Hence D is an element of τ' but D is not an element of τ, therefore τ' properly contains τ.

If $i \in \{1, 2\}$ and τ is T_i, then τ' is T_i. Hence (X, τ) is not maximal T_i if $i \in \{1, 2\}$.

Now suppose that τ is T_3. Take $x \in X$ and $x \in A \cap D$, where $A \in \tau_X$. There is $B \in \tau_X$ such that $x \in B$ and $\mathrm{cl}_X B \subseteq A$. If $y \notin A \cap D$, then $y \in (X \setminus \mathrm{cl}_X B) \cup (X \setminus D)$, therefore $y \notin \mathrm{cl}_{\tau'}(B \cap D)$. Hence, $\mathrm{cl}_{\tau'}(B \cap D) \subseteq A \cap D$. Since $x \in \mathrm{cl}_{\tau'}(B \cap D)$, τ' is T_3. If $x \in A \setminus D$, where $A \in \tau_X$, the proof is analogous. Hence (X, τ) is not maximal T_3.

Now suppose that τ is Tychonoff. Take $x \in X$ and $x \in A \cap D$, where $A \in \tau_X$. There is a continuous function $g : (X, \tau) \to [0, 1]$ such that $g(x) = 0$ and $g[X \setminus A] \subseteq \{1\}$. Since $\tau \subseteq \tau'$, g is also continuous when considered as a function from (X, τ') into $[0, 1]$ and the function $f : (X, \tau') \to [0, 1]$ defined by $f(y) = 0$ if $y \in D$ and $f(y) = 1$ if $y \in X \setminus D$ is continuous. Therefore $(g + f)(x) = 0$ and $(g + f)[X \setminus (A \cap D)] \subseteq [1, 2]$. Hence (X, τ') is Tychonoff and (X, τ) is not maximal Tychonoff. $\square$

The following theorem constructs a countable crowded Tychonoff irresolvable space.

Theorem 1.17. *[23]*

1. *A space is maximal if and only if it is maximal T_1.*

2. *Maximal T_2 spaces are maximal.*

3. *There exists a maximal Tychonoff space.*

Proof. (1). If X is a maximal space, define

$$\tau' = \{U \setminus F : U \in \tau_X \text{ and } F \text{ is a finite subset of } U\}.$$

If $x \in X$, then $X\setminus\{x\}$ is open, so τ' is a T_1 crowded topology on X containing τ_X. Therefore τ_X is T_1.

(2). If σ is a crowded topology on X such that $\tau_X \subseteq \sigma$, then σ is T_2, and so $\tau_X = \sigma$.

(3). Consider the set $\mathcal{P} = \{\tau : \tau$ is a crowded Tychonoff topology on $\omega\}$. Then $\mathcal{P}$ is nonempty and is partially ordered by $\subseteq$. Now, if $\mathcal{C}$ is a chain in $\mathcal{P}$, then $\cup\mathcal{C}$ is closed under finite intersections. Hence $\cup\mathcal{C}$ is base for some topology, $\vee\mathcal{C}$, on ω. To see that $\vee\mathcal{C}$ is Tychonoff, take $x \in U$ for some $U \in \cup\mathcal{C}$. There is $\tau \in \mathcal{C}$ such that $U \in \tau$. There is a continuous function $f : (X, \tau) \to [0, 1]$ such that $f(x) = 0$ and $f[X \setminus U] \subseteq \{1\}$, since $\tau \subseteq \vee\mathcal{C}$, f is continuous in $\vee\mathcal{C}$. Then $\vee\mathcal{C}$ is a T_3 topology on ω. Hence $\mathcal{P}$ has a maximal element by Zorn's Lemma. □

Example 1.18. ω has a maximal Tychonoff topology τ. Hence (ω, τ), is a crowded Tychonoff irresolvable space. It is easy to see that $\Delta((\omega, \tau)) = \omega$.

It is interesting to study the unique partition given by corollary 1.3 for a maximal space. Actually, if a space (X, τ) is maximal and $\alpha = 2$, then $Y_2 = \emptyset$. Indeed, take a crowded subspace $Z \subseteq X$. Suppose that D and E are disjoint dense subsets of Z. Then the topology τ' on X generated by $\tau\cup\{D\}$ is crowded because D is dense in Z. By the maximality of $X, \tau = \tau'$. Therefore D is an open subset of Z whose intersection with E is empty. Hence, X contains no nonempty resolvable subspace.

We will call a crowded space X hereditarily irresolvable if X contains no nonempty resolvable subspace. We just proved the following proposition.

Proposition 1.19. *Maximal spaces are hereditarly irresolvable.*

A space X is *open hereditarly irresolvable* (*OHI*) if it is crowded and every nonempty open subspace of X is irresolvable.

A subspace A of a space X is *nowhere dense* if $\operatorname{int}_X \operatorname{cl}_X A = \emptyset$. A space X is *nodec* if every nowhere dense subset of X is closed. Observe that if X is a nodec space and A is a nowhere dense subset of X, then every subset of A is also nowhere dense, hence every subset of A is closed, therefore A is discrete.

Lemma 1.20. *[16] Let X be a crowded space. Then X is an OHI space if and only if for every subset A of X, if $\operatorname{int}_X A$ is empty, then A is nowhere dense*

Proof. Assume that X is OHI. Suppose that $\operatorname{int}_X A$ is empty. Let U be a nonempty open subset of X. Then $U \setminus A$ is dense in U. Since U is not resolvable, $U \cap A$ is not dense in U. Hence, there is $p \in U$ and there is a open subset V of X such that $p \in V$ and $V \cap U \cap A = \emptyset$, this implies that $p \in X \setminus \operatorname{cl}_X A$. Hence, $\operatorname{int}_X \operatorname{cl}_X(A) = \emptyset$. Then A is nowhere dense.

Now let U be a nonempty open subset of X. Let A be dense in U, then $U \subseteq \operatorname{int}_X \operatorname{cl}_X A$ and A is not nowhere dense. By hypothesis, A has nonempty interior, so $U \setminus A$ is not dense in U and U is not resolvable. □

A topological space X is *extremely disconnected*, if $\operatorname{cl}_X U$ is open for every open subset U in X.

Maximal spaces are extremely disconnected, indeed, let U be a nonempty open subset of (X, τ). Then, the topology τ' generated by $\tau \cup \{\operatorname{cl}_X U\}$ is a crowded topology on X. Therefore, $\tau' = \tau$ and $\operatorname{cl}_X U$ is open in X.

1.3 Submaximal spaces

We will call a space X *submaximal* if every dense subset of X is open in X. Observe that a submaximal space cannot contain two disjoint dense subsets.

Proposition 1.21. *Maximal spaces are submaximal*

Proof. Assume that (X, τ) is a maximal space. Let D be a dense subset of X. We have seen that the topology τ' generated by $\tau \cup \{D\}$ is crowded, therefore $\tau' = \tau$ and D is open. □

Observe that if A is a nowhere dense subset of a submaximal space X, then $\operatorname{int}_X A$ is empty and $X \setminus A$ is dense in X, therefore A is closed. Hence, submaximal spaces are nodec.

Proposition 1.22. *Crowded submaximal spaces are open hereditarly irresolvable.*

Proof. Let U be a nonempty open subset of a crowded submaximal space X. If D is dense in U, then $D \cup (X \setminus \operatorname{cl}_X U)$ is dense in X, so it is open, then D is open, therefore U is submaximal, hence irresolvable. □

Proposition 1.23. *A crowded space X is submaximal if and only if X is open hereditarly irresolvable and nodec.*

Proof. Suppose that X is OHI and nodec. Let D be a dense subset of X. Then $\operatorname{int}(X \setminus D) = \emptyset$. By 1.20, $X \setminus D$ is nowhere dense. Since X is nodec, D is open. □

A space X is *σ-discrete* (*strongly σ-discrete*) if X is can be written as a countable union of discrete subspaces(closed discrete subspaces).

Theorem 1.24. *[3] Let X be a crowded submaximal space. Then X is σ-discrete if and only if is strongly σ-discrete.*

Proof. Let D be a discrete subset of X. Since X is crowded, $\operatorname{int}_X D$ is empty, and so D is nowhere dense, on the other hand, X is nodec and thus, D is closed. Therefore, if X is a countable union of discrete subsets then X is a countable union of closed discrete subsets. $\square$

The following characterization of submaximal spaces is useful.

Proposition 1.25. *[3] A space X is submaximal if and only if every subset A of X is the intersection of an open subset and a closed subset of X.*

Proof. Suppose that X is submaximal. Let A be a subset of X. It is easy to see that $U = A \cup (X \setminus \operatorname{cl}_X A)$ is dense in X, therefore it is open. Hence $A = U \cap \operatorname{cl}_X A$.

Suppose that every subset A of X is the intersection of an open subset and a closed subset of X. Let D be a dense subset of X. Then $D = U \cap C$, where U is open in X and C is closed in X. Hence C is equal to X and U is equal to D. Therefore D is open in X. $\square$

A space X is *ccc* if every pairwise disjoint family of nonempty open subsets of X is countable. Observe that if D is dense in a ccc space X then D is also ccc. Indeed, let $\{U_s : s \in S\}$ be a family of pairwise nonempty open disjoint subsets of D. For every $s \in S$, let V_s be an open subset of X such that $U_s = V_s \cap D$. Since D is dense, $\{V_s : s \in S\}$ is a family of pairwise nonempty open disjoint subsets of X. Therefore, $|S| \leq \omega$.

Theorem 1.26. *[3] Every regular submaximal ccc space X is a Q-set, that is, every subset of X is a G_δ set in X.*

Proof. By proposition 1.25, it is enough to prove that every closed subset of X is a G_δ set in X.

Let F be a closed subset of X. Suppose that $X \setminus F$ is nonempty and define

$$\gamma = \{\operatorname{cl}_X U : U \text{ is a nonempty open subset of } X \text{ and } F \cap \operatorname{cl}_X U = \emptyset\}.$$

Since X is regular, $X \setminus F = \cup\gamma$. By Zorn's Lemma, there is a maximal pairwise disjoint subfamily ξ of γ. By hypothesis, $|\xi| \leq \omega$. Let $V = \cup\xi$.

Observe that V is F_σ and $V \cup F$ is dense in X. Then, $V \cup F$ is open in X. Therefore $F = (V \cup F) \setminus V$ and F is G_δ. □

In section 2.1 we are going to construct crowded ccc Tychonoff submaximal spaces which are not maximal. In order to do so, we will use the following well known results.

Lemma 1.27. *A space X is extremely disconnected if and only if whenever U and V are open disjoint subsets of X, we get $\operatorname{cl}_X U \cap \operatorname{cl}_X V = \emptyset$.*

Proof. Suppose that X is extremely disconnected. Let U, V be open subsets of X and suppose that $\operatorname{cl}_X U \cap \operatorname{cl}_X V \neq \emptyset$. Since $\operatorname{cl}_X U$ is open, $\operatorname{cl}_X U \cap V \neq \emptyset$ and $U \cap V$ is not empty.

Now, let U be an open subset of X. Then $U, X \setminus \operatorname{cl}_X U$ are open disjoint subsets of X so, by hypothesis, $\operatorname{cl}_X U \cap \operatorname{cl}_X(X \setminus \operatorname{cl}_X U) = \emptyset$, but $\operatorname{cl}_X U \cup \operatorname{cl}_X(X \setminus \operatorname{cl}_X U) = X$. Then $\operatorname{cl}_X U$ is open. □

Lemma 1.28. *If X is a regular infinite space and $x \in X$ is a non isolated point with a local countable base, then X is not extremely disconnected.*

Proof. Let $\{B_n : n \in \omega\}$ be a countable local base at x such that $B_{n+1} \subseteq B_n$ for every $n \in \omega$. Let $x_0 \in B_0 \setminus \{x\}$ and $x_n \in B_n \setminus (\{x_m : m < n\} \cup \{x\})$. Since X is regular, there is a family $\{U_n : n \in \omega\}$ of pairwise disjoint open sets in X such that $x_n \in U_n$ for every $n \in \omega$. Therefore $W = \cup\{U_{2n} : n \in \omega\}$ and $V = \cup\{U_{2n+1} : n \in \omega\}$ are open disjoint sets such that $x \in \operatorname{cl}_X W \cap \operatorname{cl}_X V$. Therefore, X is not extremely disconnected. □

Lemma 1.29. *Let Y and Z be Hausdorff spaces and let $f : Y \to Z$ be an open continuous surjective function. If A is a dense extremely disconnected subspace of Y, then $f[A]$ is extremely disconnected.*

Proof. Suppose that A is dense in Y and $f[A]$ is not extremely disconnected. There exist $y \in f[A]$ and nonempty open disjoint subset U, V of $f[A]$ such that $y \in \operatorname{cl}_{f[A]} U \cap \operatorname{cl}_{f[A]} V$. Take open sets U_1, V_1 in Z such that $U_1 \cap f[A] = U$ and $V_1 \cap f[A] = V$. Since f is surjective, $f[A]$ is dense in Z. Then, $U_1 \cap V_1$ is empty. Pick $x \in A$ with $f(x) = y$. Let W be an open set in Y such that $x \in W$. Then, $f[W]$ is an open set in Z such that $y \in f[W]$, therefore, $f[W] \cap U_1 \neq \emptyset$ and $f[W] \cap V_1 \neq \emptyset$, which implies that $W \cap f^{\leftarrow}[U_1] \neq \emptyset$ and $W \cap f^{\leftarrow}[V_1] \neq \emptyset$. As a consequence

$$x \in \operatorname{cl}_Y(f^{\leftarrow}[U_1]) \cap \operatorname{cl}_Y(f^{\leftarrow}[V_1]) \cap A = \operatorname{cl}_A(f^{\leftarrow}[U_1] \cap A) \cap \operatorname{cl}_A(f^{\leftarrow}[U_1] \cap A)$$

because A is dense in Y. But $f^{\leftarrow}[U_1] \cap A$ and $f^{\leftarrow}[V_1] \cap A$ are disjoint and open in A. Therefore A is not extremely disconnected. □

Theorem 1.30. *Assume that X_t is a Hausdorff space with more than one point for every $t \in T$ with T infinite. Then the product space $X = \prod\{X_t : t \in T\}$ does not contain a dense extremely disconnected subspace.*

Proof. Suppose that A is a dense extremely disconnected subspace of X. Let S be a countable subset of T. The projection $\pi_S : X \to X_S = \prod\{X_t : t \in S\}$ is an open continuous function. By Lemma 1.29, X_S has a dense extremely disconnected subspace B.

Now, let $t \in S$, since $B_t = \pi_t[B]$ is extremely disconnected and dense in X_t and X_t is Hausdorff with more than one point, there is a nonempty clopen set U_t in B_t such that $V_t = B_t \setminus U_t$ is a nonempty clopen set in B_t. Let $f_t : B_t \to \{0,1\}$ be defined by $f_t(x) = 0$ if $x \in U_t$ and $f_t(x) = 1$ otherwise. Then, f_t is open, continuous and surjective.

Define $Y = \prod\{B_t : t \in S\}$ and observe that $B \subseteq Y$. Let $f : Y \to \{0,1\}^S$ be defined by

$$\pi_t(f(x)) = f_t(x(t))$$

for every $x \in Y$ and every $t \in S$.

Then, f is an open continuous surjective function. By Lemma 1.29, $f[B]$ is a dense extremely disconnected subspace of the second countable regular space $\{0,1\}^S$. Hence $f[B]$ is a second countable regular space with no isolated points, by 1.28, $f[B]$ is not extremely disconnected. $\square$

Corollary 1.31. No maximal space can be embedded as a dense subspace into an infinite product of Hausdorff disconnected spaces.

Proof. Recall that maximal spaces are extremely disconnected. $\square$

A space X is *Baire* if for every collection $\{U_n : n \in \omega\}$ of open dense subsets of X, the intersection $\cap\{U_n : n \in \omega\}$ is dense in X. Observe that if U is an open subspace of a Baire space, then U is Baire.

Theorem 1.32. *[1] The following conditions are equivalent.*

1. *There is a crowded Hausdorff irresolvable Baire space.*
2. *There is a Hausdorff maximal non-σ-discrete space.*
3. *There is a crowded Hausdorff submaximal non-σ-discrete space.*

Proof. Suppose that X is a crowded Hausdorff Baire irresolvable space. By 1.3, X has a nonempty open hereditarily irresolvable subspace Z. Since Z is open, Z is Baire and we may assume that X itself is open hereditarily irresolvable. Take any crowded maximal topology σ on X such that $\tau_X \subseteq \sigma$.

Therefore (X, σ) is Hausdorff maximal. Suppose that (X, σ) is σ-discrete. Then $X = \cup\{X_n : n \in \omega\}$, where X_n is discrete in (X, σ) and $X_m \cap X_j = \emptyset$, whenever $m \neq j$. If $\operatorname{int}_{\tau_X} X_n$ is nonempty, then $\operatorname{int}_\sigma X_n$ is nonempty, wich is a contradiction, since X_n is discrete in the crowded space (X, σ). Therefore, every X_n has empty interior in X. By 1.20, X_n is nowhere dense in X. Therefore X is not Baire. Thus, (X, σ) is a maximal non-σ-discrete space.

Every maximal space in submaximal.

Suppose that X is a crowded Hausdorff submaximal non-σ-discrete space and suppose the existence of a nonempty open σ-discrete subspace of X. By Zorn's Lemma, there is a maximal pairwise disjoint family $\mathcal{A}$ of nonempty open σ-discrete subspaces. Define $Y = \operatorname{cl}_X(\cup\mathcal{A})$. Since $\operatorname{int}_X(Y \setminus \cup\mathcal{A})$ is empty, by lemma 1.20, $Y \setminus \cup\mathcal{A}$ is nowhere dense. Since X is nodec, $Y \setminus \cup\mathcal{A}$ is closed and discrete. For each $U \in \mathcal{A}$, there is a family $\{D_U^n : n \in \omega\}$ of pairwise disjoint discrete subspaces of U. For each $n \in \omega$, define $D_n = \cup_{U \in \mathcal{A}} D_U^n$. It is easy to see that each D_n is discrete. Therefore $\cup\mathcal{A}$ is σ-discrete, and so is Y. Therefore $Z = X \setminus Y$ is a nonempty open subspace of X. If V is a nonempty open σ-discrete subspace of Z, by the maximality of $\mathcal{A}$, there is $U \in \mathcal{A}$, such that $V \cap U$ is nonempty, but $U \subseteq Y$. Therefore, Z contains no nonempty open σ-discrete subset.

Observe that Z is Baire. Indeed, suppose that $\{U_n : n \in \omega\}$ is a collection of open dense subsets of Z. Since Z is submaximal, $Z \setminus U_n$ is closed and discrete for every $n \in \omega$. So the set $P = \cup\{Z \setminus U_n : n \in \omega\}$ is a σ-discrete subspace of Z. Hence, every nonempty open subset of Z meets $Z \setminus P$, this means that $\cap\{U_n : n \in \omega\}$ is dense in Z. $\square$

Chapter 2

Almost resolvable spaces and ai-maximal independent families

2.1 Almost resolvable spaces

The following definition was introduced by Bolstein[5].

A topological space X is *almost resolvable* if X is the union of a countable collection of subsets each of them with empty interior. Otherwise, we will say that X is *almost irresolvable*.

Observe that every resolvable space is almost resolvable. Indeed if A, B are disjoint dense subsets of a topological space X, the interiors $\operatorname{int}_X A$ and $\operatorname{int}_X B$ are empty, then X is almost resolvable. Also, if A is a nonempty open subspace of an almost resolvable space, then A is almost resolvable.

Proposition 2.1. *[14, Dorantes, Pichardo, Tamariz] If X is almost irresolvable and $\Delta(X) = |X|$, then $|X|$ has uncountable cofinality.*

Proof. Without loss of generality, let us assume that the underlying set of X is the cardinal κ. Our argument will be by contrapositive so assume that $\{\alpha_n : n \in \omega\}$ is an increasing sequence of ordinals whose supremum is κ and such that $\alpha_0 = 0$. Define, for each integer n, $Y_n := [\alpha_n, \alpha_{n+1})$ to obtain a countable cover of X. Since X is almost irresolvable, $\operatorname{int} Y_m \neq \emptyset$, for some m, and therefore $\Delta(X) < |X|$. $\square$

By example 1.18, ω has a crowded Tychonoff irresolvable topology τ. It is clear that $\triangle((\omega, \tau)) = \omega$. By the previous proposition, (ω, τ) is almost resolvable not resolvable.

Definition 2.2. Let X be a topological space. X will be called *almost ω-resolvable* if there exists $\{Y_n : n < \omega\}$, a cover of X, such that $\bigcup_{i<n} Y_i$ has empty interior for each $n < \omega$. All spaces which lack this kind of cover will be called *almost ω-irresolvable*. Thus any space which is almost ω-resolvable is almost resolvable.

Proposition 2.3. *If X contains an almost ω-resolvable (almost resolvable) dense subspace, then X is almost ω-resolvable (almost resolvable).*

Proof. Suppose that D is a dense subspace of X such that $D = \cup\{D_n : n \in \omega\}$ and $\mathrm{int}_D(\cup\{D_m : m \leq n\})$ is empty for every $n \in \omega$. Therefore $X = (X \setminus D) \cup (\cup\{D_n : n \in \omega\})$ and it is easy to see that X is almost ω-resolvable. □

Theorem 2.4. *[40] If X is the union of almost ω-resolvable (almost resolvable) subspaces, then, X is almost ω-resolvable (almost resolvable)*

Proof. Suppose that $X = \cup\{X_j : j \in J\}$ where each X_j is almost ω-resolvable. Let $\mathcal{A} = \{A_s : s \in S\}$ be a maximal family of pairwise disjoint, nonempty almost ω-resolvable subspaces of X. For each $s \in S$, let $A_s = \cup\{A_s^n : n \in \omega\}$, where $\mathrm{int}_{A_s}(\cup\{A_s^m : m \leq n\})$ is empty for every $n \in \omega$. Without lost of generality we can suppose $A_s^n \cap A_s^m = \emptyset$ for every $n \neq m$.

Suppose that $B = X_j \setminus \mathrm{cl}_X(\cup\mathcal{A})$ is nonempty for some $j \in J$. Since B is open in X_j, B is almost ω-resolvable and $B \cap A = \emptyset$ for every $A \in \mathcal{A}$, contradicting the maximality of $\mathcal{A}$. Hence $X_j \setminus \mathrm{cl}_X(\cup\mathcal{A})$ is empty for every $j \in J$, therefore $\cup\mathcal{A}$ is dense in X.

Now, define $Z = X \setminus \cup\mathcal{A}$ and $Y_n = \cup\{A_s^n : s \in S\}$. Since $\cup\mathcal{A}$ is dense, $\mathrm{int}_X Z$ is empty.

Suppose that U is a nonempty open set in X. Then, $U \cap A_t$ is nonempty for some $t \in S$. Let $n \in \omega$, since A_t is almost ω-resolvable, there is $a \in (U \cap A_t) \setminus \cup\{A_t^m : m \leq n\}$. Observe that $(U \cap A_t) \cap Y_m = A_t^m$ for every $m \in \omega$. Therefore $a \in (U \cap A_t) \setminus \cup\{Y_m : m \leq n\}$. Hence $Z \cup (\cup\{Y_m : m \leq n\})$ has empty interior for every $n \in \omega$.

For almost resolvable, the proof is analogous. □

Note that if X is a submaximal space and $A \subseteq X$ has void interior, then all its subsets are closed in X. Hence A is closed discrete.

Proposition 2.5. *[14, Dorantes, Pichardo, Tamariz] If X is crowded submaximal, the following are equivalent.*

1. *X is almost ω-resolvable.*

2. *X is almost resolvable.*

3. *X is strongly σ-discrete.*

4. *X is σ-discrete.*

Proof. By Theorem 1.24, (3) and (4) are equivalent. It is clear that (1) implies (2). To prove that (1) follows from (4) assume that $\{D_n : n < \omega\}$ is a cover of X consisting of discrete subspaces. By letting $Y_n = D_n \setminus \bigcup_{i<n} D_i$, $n \in \omega$, we obtain $\{Y_k : k \in \omega\}$, a partition of X into discrete subspaces. Since X is crowded, each set $\bigcup_{k<n} Y_k$ has empty interior so (1) holds. Since every set with void interior is closed and discrete, (2) implies (3). $\square$

A topological space X will be called *open hereditarly almost irresolvable* (OHAI, for short) if every nonempty open subspace of X is almost irresolvable.

Remark 2.6. Since resolvability implies almost resolvability, if a space X is OHAI, then X is OHI.

Lemma 2.7. *Every almost irresolvable space has a nonempty OHAI open subspace.*

Proof. Let X be an almost irresolvable space and let $Y = \cup\{Z \subseteq X : Z$ is almost resolvable$\}$. By Theorem 2.4, Y is almost resolvable. Since Y is dense in $\mathrm{cl}_X Y$, by Proposition 2.3, $\mathrm{cl}_X Y$ is almost resolvable, then Y is closed. Therefore $X \setminus Y$ is a nonempty OHAI open subspace of X. $\square$

The following theorem can be found in [2].

Proposition 2.8. *[2, Corollary 5.4] In the class of Baire spaces, resolvability and almost resolvability are the same concept.*

The following definition is well known.

Definition 2.9. An *ideal* on a nonempty set S is a collection I of subsets of S such that:

1. $\emptyset \in I$ and $S \notin I$,

2. if $X \in I$ and $Y \in I$, then $X \cup Y \in I$,

3. if $X, Y \subseteq S, X \in I$, and $Y \subseteq X$, then $Y \in I$.

Examples 2.10.

1. The trivial ideal, $I = \{\emptyset\}$.
2. Let X_0 be a nonempty subset of S. If $X_0 \neq S$, the ideal $I = \{X \subseteq S : X \subseteq X_0\}$ is called the principal ideal on S generated by X_0.
3. Let S be an infinite set and let $I = \{X \subseteq S : X \text{ is finite}\}$. Then I is an ideal on S.
4. If X is a topological space, define $I = \{Y \subseteq X : Y \text{ is nowhere dense in } X\}$. Then I is the ideal of nowhere dense subsets of X.

Definition 2.11. An ideal I on X is *σ-complete* if for every countable subfamily $\{X_n : n \in \omega\} \subseteq I$, the union $\cup\{X_n : n \in \omega\}$ belongs to I.

Definition 2.12. Let X be space. A subspace A of X is *meager* in X if $A = \cup\{A_n : n \in \omega\}$, where A_n is nowhere dense in X for every $n \in \omega$. All other subsets of X are called *second category* in X.

Example 2.13. Observe that every Baire space is second category in itself. Let X be a Baire space and $I = \{Y \subseteq X : Y \text{ is first category in } X\}$, then I is a σ-complete ideal in X.

Theorem 2.14. *[14, Dorantes, Pichardo, Tamariz] The following statements are equivalent for any submaximal crowded topological space X.*

1. *X is OHAI.*
2. *X is Baire.*
3. *If I is the collection of all subsets of X with empty interior, then I is a σ-complete ideal on X.*
4. *X has no nonempty open σ-discrete subspaces.*

Proof. Let I be the collection of all subsets of X with empty interior and let $A \in I$. By 1.20, A is nowhere dense. Since X is nodec, A is closed and discrete. Now, if A is closed and discrete, A is nowhere dense, and A has empty interior. Therefore I coincides with the ideal of nowhere dense subsets of X.

Let U be a nonempty open subset of X. Since X is crowded and submaximal, U is crowded and submaximal. By 2.5, U is almost resolvable if and only if U is σ-discrete. Hence (1) $\Leftrightarrow$ (4).

Suppose that $\mathcal{D}$ is a countable family of dense open subsets of X, such that $U = X \setminus \mathrm{cl}_X(\cap\mathcal{D})$ is nonempty. Observe that $\{X \setminus D : D \in \mathcal{D}\}$ is a subfamily of I. But $U \subseteq \cup\{X \setminus D : D \in \mathcal{D}\}$. Therefore I is not σ-complete and this proves (3) $\to$ (2).

If U is a nonempty open σ-discrete subspace then U is a countable union of nowhere dense subsets of X, in particular, U is not Baire and then X is not Baire. Therefore (2) $\to$ (4).

If I is not σ-complete, there is a family $\{A_n : n \in \omega\} \subseteq I$, such that $U = \text{int}_X(\cup\{A_n : n \in \omega\})$ is nonempty. Since U is open in $X, \text{int}_U(U \cap A_n)$ is empty for every $n \in \omega$. Therefore $\{U \cap A_n : n \in \omega\}$ is a countable cover of U formed by subsets of U with empty interior in U. Hence (1) $\to$ (3). $\square$

Definition 2.15. An ideal on a set S is *σ-saturated* if I is σ-complete, $\{s\} \in I$ for every $s \in S$, and $\mathcal{P}(S) \setminus I$ contains no uncountable pairwise disjoint family.

Theorem 2.16 (Dorantes)**.** *Let X be a crowded space and define $I = \{Y \subseteq X : \text{int}_X Y = \emptyset\}$. Then,*

1. *I is an ideal on X if and only if X is OHI.*
2. *I is a σ-complete ideal on X if and only if X is OHAI.*
3. *I is a σ-saturated ideal on X if and only if X is OHAI and ccc.*

Proof. Observe that $\emptyset \in I, X \notin I$ and if $A \subseteq Y$ with $Y \in I$ then $A \in Y$. Then I is an ideal if and only if $Y \in I$ and $Z \in I$ implies $Y \cup Z \in I$. Since X is crowded, $\{x\} \in I$ for every $x \in X$. Also, I contains the ideal of nowhere dense subsets of X. Recall that OHAI implies OHI.

1. Suppose that U is a nonempty open resolvable subspace of X. There is $D \subseteq U$ such that D and $U \setminus D$ are dense in U. Then $\text{int}_X D = \text{int}_X(U \setminus D) = \emptyset$. But, $U \notin I$, hence, I is not an ideal.

   Suppose that X is OHI. If $A \in I$, by 1.20, A is nowhere dense, then I coincides with the ideal of nowhere dense sets in X.

2. Suppose that X is OHAI. Then I is an ideal. Let $\{Y_n : n \in \omega\}$ be a countable subfamily of I. If U is open in X and $U \subseteq \cup\{Y_n : n \in \omega\}$, then $\{U \cap Y_n : n \in \omega\}$ is a countable cover of U such that $\text{int}_U(U \cap Y_n) = \emptyset$ for every $n \in \omega$. Since X is OHAI, U is empty and $\cup\{Y_n : n \in \omega\}$ belongs to I. Therefore, I is a σ-complete ideal on X.

   Suppose that X is not OHAI. There exist a nonempty open set U and countable cover $\{U_n : n \in \omega\}$ of U, such that $\text{int}_U(U_n) = \emptyset$ for every $n \in \omega$. Then $\text{int}_X(U_n) = \emptyset$ for every $n \in \omega$. Hence, $\{U_n : n \in \omega\} \subseteq I$, but $\cup\{U_n : n \in \omega\}$ does not belong to I. Hence, I is not σ-complete.

3. Suppose that X is OHAI. Then, I is a σ-complete ideal on X. Observe that $\mathcal{P}(S)\setminus I$ is the collection of all subsets of X with nonempty interior. Therefore X is ccc if and only if $\mathcal{P}(S)\setminus I$ contains no uncountable pairwise disjoint family. Therefore I is a σ-saturated ideal on X if and only if X is ccc.

□

Definition 2.17. [26] An *Ulam matrix* is a collection $\{A_{\alpha,n} : \alpha \in \omega_1, n \in \omega\}$ of subsets of ω_1 such that:

1. if $\alpha \neq \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for every $n \in \omega$;
2. for each α, the set $\omega_1 \setminus (\cup\{A_{\alpha,n} : n \in \omega\})$ is at most countable.

In other words, an Ulam matrix has ω_1 rows and ω columns. Each column consists of pairwise disjoint sets and the union of each row contains all but countably many elements of ω_1.

Lemma 2.18. *[26] An Ulam matrix exists.*

Proof. For each $\xi \in \omega_1$, let $f_\xi : \xi \to \omega$ be an injective function. We can do this because ξ is countable for every $\xi \in \omega_1$. For $\alpha \in \omega_1$ and $n \in \omega$, define $A_{\alpha,n} \subseteq \omega_1$ by

$$\xi \in A_{\alpha,n} \text{ if and only if } f_\xi(\alpha) = n.$$

Let $\alpha, \beta \in \omega_1$. If $\xi \in A_{\alpha,n} \cap A_{\beta,n}$ for some $n \in \omega$, then $f_\xi(\alpha) = n = f_\xi(\beta)$. Therefore, $\alpha = \beta$ and property (1) of 2.17 is verified.

Let $\alpha \in \omega_1$. Suppose that $\alpha \in \xi$. Then

$$\xi \in A_{\alpha, f_\xi(\alpha)}.$$

Therefore $\{\xi \in \omega_1 : \alpha \in \xi\}$ is contained in $\cup\{A_{\alpha,n} : n \in \omega\}$. Hence,

$$\omega_1 \setminus (\cup\{A_{\alpha,n} : n \in \omega\}) \subseteq \alpha + 1.$$

The last inequality implies that property (2) is also verified. □

Lemma 2.19. *[26] There is no σ-saturated ideal on ω_1.*

Proof. Let $\{A_{\alpha,n} : \alpha \in \omega_1, n \in \omega\}$ be an Ulam matrix. Suppose that I is a σ-complete ideal in ω_1 such that I contains all singletons. Let $\alpha \in \omega_1$. By 2.17(2), the set $\omega_1 \setminus (\cup\{A_{\alpha,n} : n \in \omega\})$ belongs to I. By definition 2.9, the set $\cup\{A_{\alpha,n} : n \in \omega\}$ does not belong to I. Therefore, there exists $n_\alpha \in \omega$ such that $A_{\alpha,n_\alpha} \notin I$.

Now, for every $n \in \omega$, define $W_n = \{\alpha \in \omega_1 : n = n_\alpha\}$. Since $\omega_1 = \cup\{W_n : n \in \omega\}$, there is $m \in \omega$ such that W_m is uncountable.

Define

$$\mathcal{A} = \{A_{\alpha,m} : \alpha \in W_m\}.$$

By 2.17(1), $\mathcal{A}$ is pairwise disjoint family, then, $\mathcal{A}$ is uncountable and by construction $\mathcal{A} \subseteq \mathcal{P}(\kappa) \setminus I$.

By definition 2.15, I is not σ-saturated. Therefore, there is no σ-saturated ideal on ω_1. $\square$

Corollary 2.20. If Z is a crowded ccc almost irresolvable space then $|Z| > \omega_1$.

Proof. By 2.7, there is a nonempty open OHAI subspace X of Z. Every finite crowded space is almost resolvable, then $|X| \geq \omega$. By 2.16(3), there is a σ-saturated ideal on X, then $|X| \neq \omega$. By 2.19, $|X| \neq \omega_1$. Hence, $\omega_1 < |X| \leq |Z|$. $\square$

Definition 2.21. An ideal I on a nonempty set S is a *prime ideal* on S, if for every $X \subseteq S$, either $X \in I$ or $S \setminus X \in I$.

Lemma 2.22. *[26] If I is a σ-saturated ideal on S, then either there exists $Z \subseteq S$, such that $I \restriction Z = \{X \subseteq Z : X \in I\}$ is a prime ideal on Z, or there exists a σ-saturated ideal on some cardinal $\kappa \leq 2^\omega$.*

Proof. Suppose that I is a σ-saturated ideal on S such that for every $Z \subseteq S$, $I \restriction Z = \{X \subseteq Z : X \in I\}$ is not a prime ideal on Z. Observe that $I \restriction Z$ is an ideal if and only if $Z \notin I$. Then, for every $Z \subseteq S$, either $Z \in I$ or there is a subset $X_Z \subseteq Z$ such that $X_Z \notin I$ and $Z \setminus X_Z \notin I$.

Now, define $T_0 = \{S\}$, and suppose that for some $n \in \omega, n > 0$, we have defined T_n such that:

1. if $A \in T_n$, then $A \notin I$,
2. if $A, B \in T_n$ and $A \neq B$, then $A \cap B = \emptyset$,
3. if $A \in T_n$, then there is $B \in T_{n-1}$ such that $A \subseteq B$ and
4. $\cup T_n = S$ for every $n \in \omega$.

Define $T_{n+1} = \{X_Z, Z \setminus X_Z : Z \in T_n\}$. It is easy to see that T_{n+1} satisfies 1, 2, 3 and 4. Therefore, we can define T_n for every $n \in \omega$.

Let $T = \cup\{T_n : n \in \omega\}$ and define $X \leq Y$ if $Y \subseteq X$ for every $X, Y \in T$. Then $(T, \leq)$ is a partially ordered set. A subset $C \subseteq T$ is a chain if $(C, \leq)$ is linearly ordered. By Zorn's Lemma, T contains maximal chains.

If C is a chain in T, then for every $n \in \omega$, $C \cap T_n$ has at most one element, because two different elements of T_n are disjoint, and if C is maximal chain, then $C \cap T_n$ has exactly one element for every $n \in \omega$.

Observe that T is countable. Let $\{C_\alpha : \alpha < \kappa\}$, with $\kappa \leq 2^\omega$, be an enumeration of all the maximal chains of T such that $Z_\alpha = \cap C_\alpha$ is nonempty.

Let $\alpha \neq \beta$. If for every $n \in \omega$, $C_\alpha \cap T_n = C_\beta \cap T_n$, then $C_\alpha = C_\beta$. Therefore, there is $n \in \omega$ such that $C_\alpha \cap T_n \neq C_\beta \cap T_n$. Take $A \in C_\alpha \cap T_n$ and $B \in C_\beta \cap T_n$. Hence, $Z_\alpha \cap Z_\beta \subseteq A \cap B = \emptyset$.

Now, take $x \in X$. Observe that for every $n \in \omega$, there is $A^x_n \in T_n$ such that $x \in A^x_n$. Hence $\mathcal{A}^x = \{A^x_n : n \in \omega\}$ is a chain in T. Suppose that C is a chain in T such that $\mathcal{A}^x \subseteq C$. If $A \in C$, there is $m \in \omega$ such that $A \in T_m$, if $A \neq A^x_m$, then, $A \cap A^x_m = \emptyset$ and C is not a chain. Hence $C \subseteq \mathcal{A}^x$. Therefore, $\mathcal{A}^x$ is a maximal chain. Since $x \in \cap\mathcal{A}^x$, there is $\alpha < \kappa$ such that $\mathcal{A}^x = C_\alpha$.

If for some $\alpha < \kappa, Z_\alpha \notin I$, then X_{Z_α} and $Z_\alpha \setminus X_{Z_\alpha}$ are contained in $\cap C_\alpha$. Hence $C_\alpha \cup \{X_{Z_\alpha}\}$ and $C_\alpha \cup \{Z_\alpha \setminus X_{Z_\alpha}\}$ are chains. By the maximality of $C_\alpha, \{X_{Z_\alpha}, Z_\alpha \setminus X_{Z_\alpha}\} \subseteq C_\alpha$, which is a contradiction. Hence, $Z_\alpha \in I$, for every $\alpha < \kappa$.

By the previous three paragraphs, $\{Z_\alpha : \alpha < \kappa\}$ is a partition of S into κ elements of I.

Define $f : S \to \kappa$ by

$$f(x) = \alpha \text{ if and only if } x \in Z_\alpha.$$

Let $J = \{Z \subseteq \kappa : f^{\leftarrow}[Z] \in I\}$. It is easy to see that J is an ideal on κ. Suppose that $\{Z_p : p \in \omega\}$ is a countable subcollection of J. Then, $\{f^{\leftarrow}[Z_p] : p \in \omega\}$ is a countable subcollection of I. Hence, $f^{\leftarrow}[\cup_{p\in\omega} Z_p] \in I$, therefore $\cup_{p\in\omega} Z_p$ belongs to J. Then, J a σ-complete ideal on κ. Let $\beta \in \kappa$. Then, $Z_\beta = f^{\leftarrow}[\{\beta\}]$ and $\{\beta\} \in J$.

Assume that $\{A_u : u \in U\} \subseteq \mathcal{P}(\kappa) \setminus J$ and $A_u \cap A_v = \emptyset$ for all $u \neq v$. Then, $\{f^{\leftarrow}[A_u] : u \in U\}$ is a pairwise disjoint family contained in $\mathcal{P}(S) \setminus I$. Since I is σ-saturated, U is countable. We conclude that J is a σ-saturated ideal on κ. $\square$

Theorem 2.23 (Dorantes)**.** *If Z is a Hausdorff crowded ccc almost irresolvable space, then there exists a σ-saturated ideal on some cardinal $\kappa \leq 2^\omega$.*

Proof. By 2.7, Z has an open OHAI subspace X. By 2.16, $I = \{Y \subseteq X : \operatorname{int}_X Y = \emptyset\}$ is a σ-saturated ideal on X. Let Y be an arbitrary subset of X. We are going to prove that $I \restriction Y = \{X \subseteq Y : X \in I\}$ is not a prime ideal on Y.

If $Y \in I$, then, $Y \in I \restriction Y$, so $I \restriction Y$ is not an ideal on Y. Suppose that $Y \notin I$. Then, $\operatorname{int}_X Y \neq \emptyset$. Since X is Hausdorff without isolated points, there are nonempty open disjoint sets V, W in X such that $V \cup W \subseteq Y$. Then, $V \notin I$ and since $W \subseteq Y \setminus V$, we obtain, $Y \setminus V \notin I$. Hence $I \restriction Y$ is not a prime ideal on Y. By 2.22, there exists a σ-saturated ideal on $\kappa \leq 2^\omega$. □

Since there is no σ-complete ideal on ω, we have the following theorem:

Theorem 2.24 (Dorantes)**.** *If X is a Hausdorff crowded ccc almost irresolvable space, then the Continuum Hypothesis fails.*

Proof. By 2.23, there exists a σ-saturated ideal on some cardinal $\kappa \leq 2^\omega$. Then, $\kappa \neq \omega$ and by 2.19, $\kappa \neq \omega_1$. Therefore $\omega_1 < \kappa \leq 2^\omega$. □

The following definitions are well known.

Definition 2.25. The *cofinality* $cf(\kappa)$ of an infinite cardinal κ is the smallest cardinal λ such that κ can be represented as the union of some family, of cardinality at most λ, of sets of cardinality less than κ.

A cardinal κ is *regular* if $cf(\kappa) = \kappa$, otherwise κ is *singular*.

It is easy to see that if κ is an infinite cardinal, $cf(\kappa)$ is less than or equal to κ. Then κ is singular if and only if $cf(\kappa) < \kappa$.

Definition 2.26. Let κ be an uncountable cardinal. An ideal I on X is *κ-complete* if for every subfamily $\{X_\alpha : \alpha < \gamma\} \subseteq I$, with $\gamma < \kappa$, the union $\cup\{X_\alpha : \alpha < \gamma\}$, belongs to I. Observe that σ-complete is equivalent to ω_1-complete.

Lemma 2.27. *[26, Lemma 10.5] Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ, and let I be such ideal. Then I is κ-complete.*

Proof. Suppose that I is not κ-complete. There is a collection $\{X_\alpha : \alpha < \gamma\} \subseteq I$ such that $\gamma < \kappa$ and the union $X = \cup\{X_\alpha : \alpha < \gamma\}$ is not an element of I.

For each $\alpha < \gamma$ define $Y_\alpha = X_\alpha \setminus \cup\{X_\beta : \beta < \alpha\}$. Then, Y_α is an element of I for every $\alpha < \gamma$ and $\cup\{Y_\alpha : \alpha < \gamma\} = X$.

Define $f : X \to \gamma$ by

$$f(x) = \alpha \text{ if and only if } x \in Y_\alpha.$$

Let $J = \{Z \subseteq \gamma : f^{\leftarrow}[Z] \in I\}$. It is easy to see that J is a σ-saturated ideal on γ. This contradicts the choice of κ as the least cardinal with the property that there is a σ-saturated ideal on κ. □

Remark 2.28. If I is a σ-saturated and κ-complete ideal on κ, then κ is regular.

Proof. If κ is singular then $\kappa = \cup\{\beta_\alpha : \alpha < \lambda\}$ and $\beta_\alpha < \kappa$ for every $\alpha < \lambda$ and $\lambda < \kappa$. Since all the singletons belong to I, $\beta_\alpha \in I$ for every $\alpha < \lambda$. Since I is κ-complete, $\kappa \in I$, which is a contradiction. □

The following definition can be found in [26]

Definition 2.29. An uncountable cardinal κ is *weakly inaccessible* if κ is a regular limit cardinal.

Lemma 2.30. *[26, Lemma 10.14] Let κ, λ be infinite cardinals. If $\kappa = \lambda^+$, then there is no κ-complete σ-saturated ideal on κ.*

Proof. For each $\xi \in \lambda^+$, let $f_\xi : \xi \to \lambda$ be an injective function. We can do this because $|\xi| \leq \lambda$. Define $A_{\alpha,\eta}$, for $\alpha \in \lambda^+$ and $\eta \in \lambda$ by

$$\xi \in A_{\alpha,\eta} \text{ if and only if } f_\xi(\alpha) = \eta.$$

We are going to prove that $\{A_{\alpha,\eta} : \alpha \in \lambda^+ \eta \in \lambda\}$ is a collection of subsets of λ^+ such that:

1. if $\alpha \neq \beta \in \lambda^+$, then $A_{\alpha,\eta} \cap A_{\beta,\eta} = \emptyset$ for every $\eta \in \lambda$;
2. for each $\alpha \in \lambda^+$, the set $\lambda^+ \setminus (\cup\{A_{\alpha,n} : \eta \in \lambda\})$ has cardinality less than or equal to λ.

Let $\alpha, \beta \in \lambda^+$. If $\xi \in A_{\alpha,\eta} \cap A_{\beta,\eta}$ for some $\eta \in \lambda$, then $f_\xi(\alpha) = \eta = f_\xi(\beta)$. Therefore, $\alpha = \beta$ and property (1) is verified.

Let $\alpha \in \lambda^+$. Suppose that $\alpha \in \xi$. Then

$$\xi \in A_{\alpha, f_\xi(\alpha)}.$$

Therefore $\{\xi \in \lambda^+ : \alpha \in \xi\}$ is contained in $\cup\{A_{\alpha,\eta} : \eta \in \lambda\}$. Hence,

$$\lambda^+ \setminus (\cup\{A_{\alpha,\eta} : \eta \in \lambda\}) \subseteq \alpha + 1.$$

The last inequality implies that property (2) is also verified.

Now, suppose that I is a κ-complete ideal on κ such that I contains all singletons.

Let $\alpha \in \lambda^+$. By (2), the set $\lambda^+ \setminus (\cup\{A_{\alpha,n} : \eta \in \lambda\})$ belongs to I. By definition 2.9, the set $\cup\{A_{\alpha,\eta} : \eta \in \lambda\}$ does not belong to I. Since I is κ-complete, there exists $\eta_\alpha \in \lambda$ such that $A_{\alpha,\eta_\alpha} \notin I$.

Now, for every $\eta \in \lambda$, define $W_\eta = \{\alpha \in \lambda^+ : \eta = \eta_\alpha\}$. Since $\lambda^+ = \cup\{W_\eta : \eta \in \lambda\}$ and λ^+ is regular, there is $\mu \in \lambda$ such that W_μ has cardinality λ^+.

Define

$$\mathcal{A} = \{A_{\alpha,\mu} : \alpha \in W_\mu\}.$$

By (1), $\mathcal{A}$ is pairwise disjoint family, then, $\mathcal{A}$ is uncountable and by construction $\mathcal{A} \subseteq \mathcal{P}(\kappa) \setminus I$.

By definition 2.15, I is not σ-saturated. Therefore, there is no κ-complete σ-saturated ideal on κ. □

Corollary 2.31. [26] Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ, and let I be such ideal. Then I is κ-complete and κ is weakly inaccessible.

Proof. I is a κ-complete σ-saturated ideal on κ [Lemma 2.27]. We are going to prove that κ is weakly inaccessible. Let λ be a cardinal such that $\lambda < \kappa$. Then $\lambda^+ \leq \kappa$, but $\lambda^+ \neq \kappa$ [Lemma 2.30]. Therefore κ is a limit cardinal. Since κ is regular [Remark 2.28], κ is weakly inaccessible. □

Theorem 2.32 (Dorantes)**.** *Let Z be a T_2 crowded ccc space. If 2^ω is less than the first weakly inaccessible cardinal, then Z is almost resolvable.*

Proof. Suppose that Z is a crowded ccc almost irresolvable T_2 space. Then there exists a σ-saturated ideal on $\kappa \leq 2^\omega$ [Theorem 2.23]. Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ. Then κ is weakly inaccessible [Corollary 2.31]. Obviously $\kappa \leq 2^\omega$. Hence Z is almost resolvable. □

Corollary 2.33 (Dorantes)**.** If 2^ω is less than the first weakly inaccessible cardinal, then:

1. every T_2 crowded Baire ccc space is resolvable,

2. every T_2 crowded submaximal ccc space is strongly σ-discrete.

Proof.

1. If X is a T_2 crowded Baire ccc space, then X is almost resolvable, by Proposition 2.8, X is resolvable,
2. If X is a T_2 crowded submaximal ccc space, then X is almost resolvable, by Proposition 2.5, X is strongly σ-discrete.

$\square$

Corollary 2.34 (Dorantes)**.** If the Continuum Hypothesis holds then:

1. every T_2 crowded ccc space is almost resolvable,
2. every T_2 crowded Baire ccc space is resolvable,
3. every T_2 crowded submaximal ccc space is strongly σ-discrete.

Theorem 2.35 (Dorantes)**.** *Let Z be a crowded ccc space. If the cardinality of Z is less than the first weakly inaccessible cardinal, then Z is almost resolvable.*

Proof. Suppose that Z is a crowded ccc almost irresolvable space. By 2.7, there is a crowded ccc OHAI subspace $X \subseteq Z$. By 2.16, $I = \{Y \subseteq X : \operatorname{int}_X Y = \emptyset\}$ is a σ-saturated ideal on X. Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ. Then κ is weakly inaccessible [Corollary 2.31]. Obviously $\kappa \leq |Z|$. Hence, the cardinality of Z is greater or equal to the first inaccessible cardinal. $\square$

Corollary 2.36 (Dorantes)**.** Let Z be a crowded ccc space and suppose that the cardinality of Z is less than the first weakly inaccessible cardinal,

1. if Z is Baire, then Z is resolvable,
2. if Z is submaximal, then Z is strongly σ-discrete.

Corollary 2.37 (Dorantes)**.** Suppose that there are no weakly inaccessible cardinals, then:

1. every crowded ccc space is almost resolvable,
2. every crowded Baire ccc space is resolvable,
3. every crowded submaximal ccc space is strongly σ-discrete.

2.2 Baire ccc irresolvable spaces

Definition 2.38. A space X is *regular card-homogeneous* if $\triangle(X) = |X|$ and $|X|$ is a regular cardinal.

Theorem 2.39 (Dorantes)**.** *Let X be a set. Suppose that $I \subseteq \mathcal{P}(X)$ is a prime σ-saturated ideal on X, then there is a T_1 ccc submaximal connected OHAI homogeneous topology τ on X. If, in addition I is $|X|$-complete, then $\Delta((X,\tau)) = |X|$.*

Proof. Define $\tau = \{\emptyset\} \cup \mathcal{P}(X) \setminus I$. Then τ is a topology on X. Indeed, $\emptyset, X \in \tau$. Suppose that $\{U_s : s \in S\} \subseteq \tau$, if U_{s_0} is nonempty for some $s_0 \in S$, then, $U_{s_0} \notin I$. since $U_{s_0} \subseteq \cup\{U_s : s \in S\}$, $\cup\{U_s : s \in S\} \notin I$. Now, take $A, B \in \tau$ and suppose that $A \cap B$ is nonempty. Since I is prime, $X \setminus A \in I$ and $X \setminus B \in I$. Therefore, $X \setminus A \cup X \setminus B \in I$. Hence $X \setminus (A \cap B) \in I$ and $A \cap B \notin I$.

Consider the topological space (X, τ). Let $x \in X$. Since I is σ-saturated, $\{x\} \in I$, then $X \setminus \{x\} \notin I$, hence $\{x\}$ is closed in X. Since $\{x\}$ is not an element of I, $\{x\}$ is not open, hence X is crowded.

Observe that $I = \{Y \subseteq X : \operatorname{int}_X Y = \emptyset\}$. By 2.16, X is OHAI and ccc.

Let D be a dense subspace of X. Then, $\operatorname{int}_X(X \setminus D)$ is empty, therefore, $X \setminus D \in I$, hence $D \notin I$ and D is open. Then, X is submaximal. Observe that for every subset $A \subseteq X$, then either A is open or $X \setminus A$ is open. Also, if $A \subseteq X$ and A is clopen, then $A = \emptyset$ or $A = X$, therefore X is connected.

We are going to prove that X is homogeneous. Let $x, y \in X$ such that $x \neq y$. Let $f : X \to X$ be a bijection such that $f(x) = y, f(y) = x$ and $f(z) = z$ for every $z \in X \setminus \{x, y\}$. We are going to prove that f is a homeomorphism.

Let $U \notin I$. If $U \subseteq U \setminus \{x, y\}$, or $x, y \in U$, then $f[U] = U$ and $f^{\leftarrow}[U] = U$.

Without lost of generality suppose that $f(y) = x \in U$ and $y \notin U$. Since $\{x\} \in I$, $(U \setminus \{x\}) \notin I$, hence $(U \setminus \{x\}) \cup \{y\}$ does not belong to I. It is easy to see that $f[U] = (U \setminus \{x\}) \cup \{y\}$. Also, $y \in (U \setminus \{x\}) \cup \{y\}$ and $f[(U \setminus \{x\}) \cup \{y\}] \subseteq U$.

Therefore, f is continuous and open. Hence, X is a T_1 ccc submaximal connected OHAI homogeneous space.

Now, if I is $|X|$-complete, then every nonempty open subset of (X, τ) has cardinality $|X|$. $\square$

Corollary 2.40 (Dorantes)**.** Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ, and let I be such ideal. Then, either there exists a T_1 ccc submaximal connected OHAI homogeneous space Z

such that Z is regular card-homogeneous or there exists a σ-saturated ideal on $\kappa \leq 2^\omega$.

Proof. By Lemma 2.27 and Remark 2.28, I is κ-complete and κ is regular.

Suppose that there exists $X \subseteq \kappa$, such that $I \restriction X = \{Z \subseteq X : Z \in I\}$ is a prime ideal on X. Then, $I \restriction X$ is a prime σ-saturated ideal on X. By corollary 2.39, there is a T_1 crowded ccc submaximal OHAI space Z such that $\Delta(Z) = |Z| \leq |X|$. By 2.16, there is a σ-saturated ideal on Z, since κ is the least cardinal having a σ-saturated ideal, $|Z| \geq \kappa$.

Otherwise, by 2.22, there exists a σ-saturated ideal on $\kappa \leq 2^\omega$. □

Theorem 2.41. *[Dorantes] If 2^ω is less than the first weakly inaccessible then the following conditions are equivalent:*

1. *there is a T_1 ccc submaximal connected OHAI homogeneous space Z such that Z is regular card-homogeneous,*
2. *there is a T_1 ccc Baire connected OHAI homogeneous space Z such that Z is regular card-homogeneous,*
3. *there is a T_1 crowded ccc OHAI space Z such that Z is regular card-homogeneous, and*
4. *there exists a σ-saturated ideal on some set S.*

Proof. By 2.14, a submaximal space X is Baire if and only if X is OHAI, then $1 \Rightarrow 2$.

It is clear that $2 \Rightarrow 3$.

If 3 holds, then, by 2.16(3), there exists a σ-saturated ideal on some set S.

Suppose (4). Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ, and let I be such ideal. Then I is κ-complete and κ is weakly inaccessible [Corollary 2.31].

By Corollary 2.40, either there exists a T_1 crowded ccc submaximal connected irresolvable space Z such that $\Delta(Z) = |Z|$ with $|Z|$ is regular, or there exists a σ-saturated ideal on some $\lambda \leq 2^\omega$.

Obviously $\kappa \leq \lambda \leq 2^\omega$. Since κ is weakly inaccessible, there is no a σ-saturated ideal on $\lambda \leq 2^\omega$. Hence, (1) holds. □

Corollary 2.42 (Dorantes)**.** If there exists a σ-saturated ideal on some set S and 2^ω is less than the first weakly inaccessible, then there is a Baire crowded connected ccc OHAI T_1 space Z such that Z is regular card-homogeneous.

It turns out that we can construct T_1 crowded ccc Baire almost irresolvable spaces using measurable cardinals, but first we need some definitions.

Definition 2.43. [26] Let S be an infinite set. A nontrivial *σ-additive measure* on S is a real valued function μ on $\mathcal{P}(S)$ such that:

1. $\mu(\emptyset) = 0$ and $\mu(S) = 1$;
2. if $X \subseteq Y$ then $\mu(X) \leq \mu(Y)$;
3. $\mu(\{x\}) = 0$ for all $x \in X$;
4. if $\{X_n : n \in \omega\}$ is a pairwise disjoint family of subsets of S, then

$$\mu(\cup\{X_n : n \in \omega\}) = \Sigma\{\mu(X_n) : n \in \omega\}.$$

Let μ be a nontrivial σ-additive measure on a set S and consider the ideal of null sets:

$$I_\mu = \{X \subseteq S : \mu(X) = 0\}.$$

Then I_μ is a nontrivial σ-saturated ideal on S. Indeed, the empty set belongs to I_μ and S does not. If $X \in I_\mu$ and $Y \in I_\mu$ then $\mu(X \cup Y) \leq \mu(X) + \mu(Y) = 0$, then $X \cup Y$ is an element of I_μ. If $Y \in I_\mu$ and $X \subseteq Y$ then $\mu(X) = 0$, hence $X \in I_\mu$. Therefore I_μ is an ideal. Let $x \in X$, then $\mu(\{x\}) = 0$ and $x \in I_\mu$. Hence I_μ is a nontrivial ideal.

Let $\{X_n : n \in \omega\}$ be a countable family of elements of I_μ. Then $\mu(\cup\{X_n : n \in \omega\}) \leq \Sigma\{\mu(X_n) : n \in \omega\} = 0$. Hence, the union $\cup\{X_n : n \in \omega\}$ belongs to I_μ, therefore I_μ is σ-complete.

Suppose that $J = \{X_\alpha : \alpha < \omega_1\}$ is an uncountable pairwise disjoint subfamily of $\mathcal{P}(S) \setminus I_\mu$. For each $n \in \omega$ define

$$J_n = \{\alpha < \omega_1 : \mu(X_\alpha) = 1/n\}.$$

It is easy to see that $\omega_1 = \cup\{J_n : n \in \omega\}$. Then, there is $n_0 \in \omega$ such that J_{n_0} is uncountable. Take a countable subfamily J'_{n_0} of J_{n_0}. Then

$$\mu(\cup\{J : J \in J'_{n_0}\}) = \Sigma\{\mu(J) : J \in J'_{n_0}\} = \Sigma\{1/n : J \in J'_{n_0}\} > \infty.$$

Hence, every pairwise disjoint subfamily of $\mathcal{P}(S) \setminus I_\mu$ is countable. By definition 2.15, I is a σ-saturated ideal on S. Therefore we have proved:

Theorem 2.44. *[26] If μ is a nontrivial σ-additive measure on a set S then I_μ is a nontrivial σ-saturated ideal on S.*

Corollary 2.45 (Dorantes)**.** If there exists a nontrivial σ-additive measure on a set S and if 2^ω is less than the first weakly inaccessible then there exists a submaximal crowded ccc OHAI T_1 space Z such that Z is regular card-homogeneous.

Proof. Immediate from Theorem 2.44 and Corollary 2.42. □

Definition 2.46. [26, Definition 10.8] An uncountable cardinal κ is *real-valued measurable* if there exists a nontrivial κ-additive measure μ on κ.

Theorem 2.47. *[26, Corollary 10.7] The least cardinal that carries a nontrivial σ-additive measure is real-valued measurable.*

Definition 2.48. [26, Definition 10.3] An uncountable cardinal κ is *measurable* if there is a κ-complete prime ideal I on κ such that $\{x\} \in I$ for every $x \in \kappa$.

Theorem 2.49 (Dorantes)**.** *The existence of a measurable cardinal κ implies the existence of a a ccc submaximal connected OHAI homogeneous T_1 space Z such that $\Delta(Z) = |Z| = \kappa$.*

Proof. Suppose that κ is a measurable cardinal. By Definition 2.48, there is a κ-complete prime ideal I on κ such that $\{x\} \in I$ for every $x \in \kappa$. We will see that there exists a nontrivial σ-additive measure μ on κ such that $I = I_\mu$.

Define $\mu : \mathcal{P}(\kappa) \to \mathbb{R}$ by $\mu(X) = 0$ if $X \in I$ and $\mu(X) = 1$ if $X \notin I$ for every $X \subseteq \kappa$. Then:

1. $\mu(\emptyset) = 0$ and $\mu(\kappa) = 1$;
2. if $X \subseteq Y$ and $X \in I$, then $\mu(X) = 0 \leq \mu(Y)$; if $X \subseteq Y$ and $X \notin I$, then $Y \notin I$ and $\mu(X) \leq 1 = \mu(Y)$;
3. since $\{x\} \in I$ for every $x \in \kappa$, $\mu(\{x\}) = 0$ for every $x \in \kappa$;
4. suppose that $\{X_n : n \in \omega\}$ is a pairwise disjoint family of subsets of κ. If $\{X_n : n \in \omega\} \subseteq I$, then $\cup\{X_n : n \in \omega\}$ belongs to I, hence $\mu(\cup\{X_n : n \in \omega\}) = 0 = \Sigma\{\mu(X_n) : n \in \omega\}$. Now suppose that there is m such that $X_m \notin I$, since I is a prime ideal $\kappa \setminus X_m$ is an element of I, hence $X_p \subseteq \kappa \setminus X_m$ and $X_p \in I$ for every $p \neq m, p \in \omega$. Therefore $\mu(\cup\{X_n : n \in \omega\}) = 1 = \Sigma\{\mu(X_n) : n \in \omega\}$.

Hence, μ is a nontrivial σ-additive measure μ on κ such that $I = I_\mu$. Then, I is a σ-saturated ideal [Theorem 2.44]. Since I is a prime ideal, applying 2.39, we obtain the desired result. □

The next Theorem appears in [26].

Theorem 2.50. *[26, Corollary 10.10] If κ carries a κ-complete σ-saturated ideal, then either κ is measurable or $\kappa \leq 2^\omega$.*

The following theorem is well known.

Theorem 2.51. *If 2^ω is less than the first weakly inaccessible then the following conditions are equivalent:*

1. *there is a measurable cardinal,*
2. *there is a real-valued measurable cardinal,*
3. *there exists a σ-saturated ideal on some set S.*

Proof. If κ is a measurable cardinal, then κ carries a nontrivial σ-additive measure, by Theorem 2.47, there is real-valued measurable cardinal.

If there is a real-valued measurable cardinal κ, then there exists a σ-saturated ideal on κ [Theorem 2.44].

Finally, suppose (9). Let κ be the least cardinal with the property that there is a σ-saturated ideal on κ, and let I be such ideal. Then I is κ-complete and κ is weakly inaccessible [Corollary 2.31].

Then, either κ is measurable or $\kappa \leq 2^\omega$ [Theorem 2.50]. Since we are assuming that 2^ω is less than the first weakly inaccessible, κ is measurable. □

We have an improvement of this theorem.

Theorem 2.52. *If 2^ω is less than the first weakly inaccessible then the following conditions are equivalent:*

1. *there is a measurable cardinal,*
2. *there is a real-valued measurable cardinal,*
3. *there exists a submaximal crowded ccc OHAI T_1 space Z such that Z is regular card-homogeneous,*
4. *there exists a Baire crowded ccc OHAI T_1 space,*
5. *there exists a Baire crowded ccc almost irresolvable T_1 space,*

6. *there exists a Baire crowded ccc irresolvable T_1 space,*

7. *there exists a Baire crowded ccc almost irresolvable space,*

8. *there exists a crowded ccc almost irresolvable space,and*

9. *there exists a σ-saturated ideal on some set S.*

Proof. If there is a real-valued measurable cardinal κ, by Corollary 2.45, there exists a submaximal crowded ccc OHAI T_1 space Z such that Z is regular card-homogeneous.

If X is a submaximal OHAI space, then X is Baire [Theorem 2.14].

(4) $\Rightarrow$ (5), (5) $\Rightarrow$ (7) and (7) $\Rightarrow$ (8) are immediate.

In the class of Baire spaces, resolvability and almost resolvability are the same concept [2, Corollary 5.4]. Hence (5) and (6) are equivalent.

If there exists a crowded ccc almost irresolvable, then there exists a σ-saturated ideal on some set S [Theorem 2.16]. $\square$

Since the Continuum Hypothesis implies that 2^ω is less than the first weakly inaccessible then we have the following Corollary:

Corollary 2.53. Assuming the Continuum Hypothesis, the following conditions are equivalent:

1. there is a measurable cardinal,

2. there is a real-valued measurable cardinal,

3. there exists a submaximal crowded ccc OHAI T_1 space Z such that Z is regular card-homogeneous,

4. there exists a Baire crowded ccc OHAI T_1 space,

5. there exists a Baire crowded ccc almost irresolvable T_1 space,

6. there exists a Baire crowded ccc irresolvable T_1 space,

7. there exists a Baire crowded ccc almost irresolvable space,

8. there exists a crowded ccc almost irresolvable space,and

9. there exists a σ-saturated ideal on some set S.

A.V. Arhangel'skiĭ and P.J. Collins asked in [3, Problem 7.4] if every regular submaximal space ccc is strongly σ-discrete. Assuming the existence of a measurable cardinal, then by Theorem 2.49, there is a T_1 crowded ccc submaximal OHAI space Z, by 2.5, Z is not strongly σ-discrete. In this case, Z is Baire [Theorem 2.14], then it is proved that Pavlov's claim [37, Theorem 3.16] is incorrect.

However, assuming CH, by 2.24, every T_2 crowded ccc is almost resolvable, by 2.5, every T_2 crowded ccc submaximal is strongly σ-discrete.

In contrast with the previous affirmation, it is worth noting that, under Martin's Axiom ($\mathrm{MA}(\omega_1) + \neg CH$), every T_2 crowded ccc space is almost resolvable [7, Theorem 4.1].

Definition 2.54. Let I be an ideal over a set S. A family $\mathcal{R} \subseteq P(S) \setminus I$ is *I-dense* if for every $X \in \mathcal{P}(S) \setminus I$, there is an $S \in \mathcal{R}$ such that $S \setminus X \in I$; $\mathcal{R}$ is *I-proper* if for any finite subfamily $\mathcal{R}' \subseteq \mathcal{R}$ either $\cap\mathcal{R}' = \emptyset$ or $\cap\mathcal{R}' \in \mathcal{P}(S) \setminus I$; $\mathcal{R}$ is *I-almost disjoint* if for every distinct $S, S' \in R, S \cap S' \in I$. Let $A \in \mathcal{P}(S) \setminus I$. An *$I$-partition* of A is a maximal I-almost disjoint family of subsets of A. An I-partition P_2 of A is a refinement of an I-partition P_1 of A, denoted by $P_1 \leq P_2$, if every $X \in P_2$ is a subset of some $Y \in P_2$.

The ideal I is *weakly precipitous* if I is σ-complete and whenever A does not belong to I, and $\{P_n : n \in \omega\}$ are I-partitions of A such that $P_0 \leq P_1 \leq \ldots \leq P_n \leq \ldots$, then there exists a sequence of sets $W_0 \supseteq W_1 \supseteq \ldots W_n \supseteq \ldots$, such that $W_n \in P_n$ for each $n \in \omega$ and $\cap\{W_n : n \in \omega\} \neq \emptyset$.

Proposition 2.55. *[32, Proposition 2.9] Let I be an ideal on some cardinal κ. If I has an I-dense and I-proper family, then every I-almost disjoint family has cardinality at most κ. If, in addition, I is σ-complete, then I is weakly precipitous.*

Theorem 2.56. *[32, Theorem 2.0] If κ is a regular cardinal that carries a weakly precipitous ideal, then there is a measurable cardinal in some inner model of ZFC.*

We have the next proposition.

Proposition 2.57 (Dorantes)**.** *If X is a crowded OHAI space then $I = \{Y \subseteq X : \mathrm{int}_X Y = \emptyset\}$ has an I-dense and I-proper family and I is weakly precipitous.*

Proof. I is σ-complete [Theorem 2.16(2)]. It is easy to see that the topology of X, τ_X, is both I-dense and I-proper, then I is weakly precipitous [Proposition 2.55]. $\square$

Kunen, Szymański and F. Tall proved the following theorem.

Theorem 2.58. *[32, Theorem 2.1] The following conditions are equivalent:*

1. *There is a Baire irresolvable space,*
2. *there is a σ-complete ideal I, such that I has an I-dense and I-proper family.*

We have an improvement of the last theorem.

Theorem 2.59 (Dorantes)**.** *The following conditions are equivalent:*

1. *There is a Baire irresolvable space,*
2. *there is a crowded almost irresolvable space and*
3. *there is a σ-complete ideal I, such that I has an I-dense and I-proper family.*

Proof. It is easy to see that (1)$\Rightarrow$(2) [2, Corollary 5.4].

Suppose (2), then there is a crowded OHAI space. By 2.57, (3) holds. By [32, Theorem 2.1], (3) $\Rightarrow$ (1).

Another interesting way to prove (2) $\Rightarrow$ (1) is the following. If Z is crowded almost irresolvable, there is a crowded OHAI subspace X of Z. Then $I = \{Y \subseteq X : \operatorname{int}_X Y = \emptyset\}$ is σ-complete [Theorem 2.16, (2)]. Suppose that $\mathcal{D}$ is a countable family of dense subsets of X. Observe that $\{X \setminus D : D \in \mathcal{D}\}$ is a subfamily of I. Then $\cup\{X \setminus D : D \in \mathcal{D}\} \in I$, hence $\operatorname{int}_X(X \setminus \cap\mathcal{D}) = \emptyset$. Therefore $\cap\mathcal{D}$ is dense and X is Baire. ☐

Kunen, Szymański, and Tall proved in [32, Corollary 3.6] the following:

Theorem 2.60. *[32] The following conditions are pairwise equiconsistent*

1. *there is a measurable cardinal,*
2. *there is a real-valued measurable cardinal and*
3. *there is a OHI Baire space X such that X is regular card-homogeneous.*

We have an improvement of the last Theorem.

Theorem 2.61. *The following statements are pairwise equiconsistent:*

1. *There is a measurable cardinal,*
2. *there is a real-valued measurable cardinal,*

3. *there is a Baire crowded ccc almost irresolvable T_1 space Z such that Z is regular card-homogeneous and*

4. *there is a crowded almost irresolvable space Z such that Z is regular card-homogeneous.*

Proof. Suppose that there is a crowded almost irresolvable space Z such that Z is regular card-homogeneous. Let X be a nonempty OHAI subspace of Z. Then X is regular card-homogeneous. Hence, $I = \{Y \subseteq X : \mathrm{int}_X Y = \emptyset\}$ is weakly precipitous [Proposition 2.57]. Since the cardinality of X is regular, there is a measurable cardinal in some inner model of ZFC [Theorem 2.56]. □

Proposition 2.62. *[32] If $V = L$, then any crowded space X, such that X is regular card-homogeneous, is almost resolvable.*

In case that the space X is ccc we can omit the regular card-homogeneous hypothesis.

Theorem 2.63. *If $V = L$, then every crowded ccc space is almost resolvable.*

Proof. $V = L$ implies the Continuum Hypothesis and there are no measurable cardinals, then then every crowded ccc space is almost resolvable [Theorem 2.52]. □

Kunen and Tall proved in [33] the following proposition:

Proposition 2.64. *[33] If there is a weakly precipituous ideal on some cardinal κ, then there is a measurable cardinal in some inner model.*

Then, we have the following Theorem:

Theorem 2.65. *The following statements are pairwise equiconsistent:*

1. *There is a measurable cardinal,*

2. *there is a real-valued measurable cardinal,*

3. *there is a Baire ccc irresolvable space and*

4. *there is a Baire irresolvable space and*

5. *there is a crowded almost irresolvable.*

Proof. Suppose (2), by Theorem 2.61, there is a Baire ccc almost iressolvable space. In the class of Baire spaces, resolvability and almost resolvability are the same concept [2, Corollary 5.4]. Hence (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). It is easy to see that (3) $\Rightarrow$ (4).

Suppose that there is a crowded almost irresolvable space Z. Let X be a nonempty OHAI subspace of Z. Hence, $I = \{Y \subseteq X : \operatorname{int}_X Y = \emptyset\}$ is weakly precipitous [Proposition 2.57]. Therefore, there is a measurable cardinal in some transitive model of ZFC [Proposition 2.64]. $\square$

Corollary 2.66. If V=L, then:

1. every crowded space is almost resolvable,
2. every crowded Baire space is resolvable,
3. every crowded submaximal space is strongly σ-discrete.

O. T. Alas et al. asked the following question:

Question 2.67. [1, Question 4.2] Is it true in ZFC that every homogeneous submaximal (regular) space is strongly σ-discrete?.

We have a partial answer. If κ is a measurable cardinal, there is a prime σ-saturated ideal on κ, then there is a T_1 ccc submaximal connected OHAI homogeneous space Z such that $\Delta(Z) = |Z| = \kappa$. Hence Z is not strongly σ-discrete [Proposition 2.5]. Observe that κ^+ is not measurable [26, Lemma 10.4]. This example, also shows that the next theorem cannot be extended to T_1 spaces if there are measurable cardinals.

Theorem 2.68. *[1, Theorem 3.13] Let κ be a non-measurable cardinal. If X is a crowded locally homogeneous Hausdorff space with $|X| \leq \kappa$, then X is almost resolvable.*

2.3 Maximal independent families

In this section we are going to construct irresolvable dense subspaces of the Cantor Cube 2^κ for every infinite cardinal κ.

Let S be a set. As usual, $\mathcal{P}(S)$ is the power set of S and $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\omega)$. The inverse image $f^{\leftarrow}[A]$ of $A \subseteq Y$ under the function $f : X \to Y$ is $\{x \in X : f(x) \in A\}$.

For any cardinal κ, denote by $D(\kappa)$ the discrete space of cardinality κ. If κ is finite $D(\kappa) = \kappa$.

Definition 2.69. The collection of all partial functions from S into T is denoted by $\mathrm{Fn}(S,T)$, i.e., $p \in \mathrm{Fn}(S,T)$ if and only if $p \subseteq S \times T$ is a finite function. In particular, when $T = 2$, elements of $\mathrm{Fn}(S,2)$ are normally called *conditions*. If $p \in \mathrm{Fn}(S,T)$, $\mathrm{dom}(p) = \{s \in S : (s,t) \in p \text{ for some } t \in T\}$. Define $[p] := \{f \in T^S : p \subseteq f\}$ for each $p \in \mathrm{Fn}(S,T)$.

Lemma 2.70. *Let κ be a cardinal, the family $\{[p] : p \in Fn(S, D(\kappa))\}$ is the canonical base for the topological product $D(\kappa)^S$ and $w(2^S) \leq |S|$.*

Proof. Let $V = \cap_{\alpha\in F}\pi_\alpha^{\leftarrow}[\{p_\alpha\}]$ be a canonical open set in $D(\kappa)^S$, where F is a finite subset of S and $p_\alpha \in D(\kappa)$ for every $\alpha \in F$. Define $p \in \mathrm{Fn}(S, D(\kappa))$ by $p(\alpha) = p_\alpha$ for every $\alpha \in F$. Observe that $[p] = V$. Since $|\mathrm{Fn}(S,2)| = |S|$, $w(2^S) \leq |S|$. □

Theorem 2.71 (The Hewitt-Marczewski-Pondiczery Theorem)**.** *If κ is an infinite cardinal, then $d(D(\kappa)^{2^\kappa}) \leq \kappa$.*

Proof. Let T be the topological product 2^κ. Then $|T| = 2^\kappa$ and $w(T) \leq \kappa$. Take a base $\mathcal{B}$ for T such that $|\mathcal{B}| \leq \kappa$ and denote by $\mathcal{T}$ the collection of all finite families of pairwise disjoint members of $\mathcal{B}$, it is clear that $|\mathcal{T}| \leq \kappa$.

Define

$$D = \{f : T \to D(\kappa) : \text{there is } \mathcal{A}_f \in \mathcal{T} \text{ such that}$$

$$f \text{ is constant in every member of } \mathcal{A}_f \text{ and in } T \setminus \cup\mathcal{A}_f\}$$

For every $n \in \omega$, define $D_n = \{f \in D : |A_f| = n\}$, and observe that $|D_n| = \kappa^{n+1} = \kappa$. Since $D = \cup\{D_n : n \in \omega\}$, the cardinality of D is less or equal to κ.

We are going to prove that D is dense in the topological product $D(\kappa)^T$. Let $p \in \mathrm{Fn}(T, D(\kappa))$. Since T is Hausdorff, for every $t \in \mathrm{dom}(p)$, there is $U_t \in \mathcal{B}$ such that $t \in U_t$ and if $t, s \in \mathrm{dom}(p)$ and $t \neq s$, then $U_t \cap U_s = \emptyset$. Therefore, $\{U_t : t \in \mathrm{dom}(p)\}$ is an element of $\mathcal{T}$.

Now, define $f : T \to D(\kappa)$ by

$$f(s) = p(t) \text{ if } s \in U_t \text{ for some } t \in \ \mathrm{dom}(p) \text{ and}$$

$$f(t) = 0 \text{ if } t \in T \setminus \cup\{U_t : t \in \ \mathrm{dom}(p)\}.$$

It is clear that $f \in D$. Since $f(t) = p(t)$ for every $t \in \mathrm{dom}(p)$, $f \in [p]$. Therefore, D is dense in in the topological product $D(\kappa)^T$. □

Corollary 2.72. Let κ be an infinite cardinal. If $\{X_s : s \in S\}$ is a family of topological spaces such that $d(X_s) \leq \kappa$ for every $s \in S$ and $|S| \leq 2^\kappa$, then $d(\Pi\{X_s : s \in S\}) \leq \kappa$.

Proof. Suppose that the spaces X_s are nonempty and $|S| = 2^\kappa$. For every $s \in S$ let A_s be a dense subspace of X_s such that $|A_s| = d(X_s)$ and $f_s : D(\kappa) \to A_s$ be an onto function. Since $D(\kappa)$ is discrete, f_s is continuous for each $s \in S$.

Define $f : D(\kappa)^S \to \Pi\{A_s : s \in S\}$ by

$$\pi_s(f(g)) = f_s(g(s))$$

for every $g \in D(\kappa)^S$ and $s \in S$. Since f_s is continuous for each $s \in S$, f is continuous. Therefore, $d(\Pi\{A_s : s \in S\}) \leq d(D(\kappa)^S)$. By the Hewitt-Marczewski-Pondiczery Theorem, $d(\Pi\{A_s : s \in S\}) \leq \kappa$, since $\Pi\{A_s : s \in S\}$ is dense in $\Pi\{X_s : s \in S\}, d(\Pi\{X_s : s \in S\}) \leq \kappa$. □

Theorem 2.73. *Let κ be an infinite cardinal. If $\{X_s : s \in S\}$ is a family of topological spaces such that $d(X_s) \leq \kappa$ for every $s \in S$, then any family of pairwise disjoint open subsets of the Cartesian product $X = \Pi_{s\in S} X_s$ has cardinality $\leq \kappa$.*

Proof. Let $\{U_t : t \in T'\}$ be a family of pairwise disjoint nonempty open subsets of X. Suppose that $|T'| > \kappa$.

If $|T'| > 2^\kappa$ take a subset T of cardinality 2^κ in T'. If $|T'| \leq 2^\kappa$ take $T = T'$.

For every $t \in T$ there exist a finite set $S_t \subseteq S$ and a family of open sets $W_t^s \subseteq X_s$ for every $s \in S_t$ such that $V_t = \cap\{\pi^{\leftarrow}[W_s^t] : s \in S_t\}$ is contained in U_t. The set $S_0 = \cup\{S_t : t \in T\}$ also has cardinality $\leq 2^\kappa$. By 2.72, the Cartesian product $\Pi_{s\in S_0} X_s$ contains a dense subset A of cardinality $\leq \kappa$.

Observe that $\{\pi_{S_0}[V_t] : t \in T\}$ is a family of nonempty open disjoint sets in $\Pi_{s\in S_0} X_s$. Indeed, if $f \in \pi_{S_0}[V_{t_1}] \cap \pi_{S_0}[V_{t_2}]$ with $t_1, t_2 \in T$, then, there exist $g_1 \in V_{t_1}$ and $g_2 \in V_{t_2}$ such that $f = \pi_{S_0}(g_1) = \pi_{S_0}(g_2)$, since $S_{t_1}, S_{t_2} \subseteq S_0$, $f \in U_{t_1} \cap U_{t_2}$. Therefore $t_1 = t_2$. Since A is dense in $\Pi_{s\in S_0} X_s$, $|T| \leq |A| \leq \kappa$, which is a contradiction. □

Corollary 2.74. In the Cartesian product of separable spaces any family of pairwise disjoint non empty open sets is countable.

Example 2.75. For any cardinal κ the Cantor cube 2^κ is ccc, and if D is dense in 2^κ, then D is also ccc.

For a cardinal κ, $[S]^\kappa$ denotes the collection of all subsets of S which have cardinality κ. Similarly, $[S]^{<\kappa}$ is the family of all subsets of S whose cardinality is less than κ. Given a cardinal λ, we denote by $\kappa^{<\lambda}$ the cardinality of the set $[\kappa]^{<\lambda}$.

Definition 2.76. Let S be a set and let $\mathcal{C} = \{(C^0_\alpha, C^1_\alpha) : \alpha < \lambda\}$ be a nonempty family of pairs of subsets of S such that each unordered pair $\{C^0_\alpha, C^1_\alpha\}$ is a partition of S.

1. For each nonempty $p \in \mathrm{Fn}(\lambda, 2)$ we define
$$\mathcal{C}(p) := \bigcap\{C^{p(\alpha)}_\alpha : \alpha \in \operatorname{dom} p\},$$
and $\mathcal{C}(\emptyset) = S$.
2. We say that $\mathcal{C}$ is *independent* if $\mathcal{C}(p) \neq \emptyset$ for each $p \in \mathrm{Fn}(\lambda, 2)$.
3. $\mathcal{C}$ will be called *uniform* if for all $p \in \mathrm{Fn}(\lambda, 2)$ we have $|\mathcal{C}(p)| = |S|$.
4. $\mathcal{C}$ is *separating* if for each pair of distinct points $x, y \in S$ there exist $\alpha < \lambda$ and $i < 2$ such that $x \in C^i_\alpha$ and $y \in C^{1-i}_\alpha$.

Remark 2.77. To avoid trivialities we will consider only infinite independent families on infinite sets.

Let $p, q \in \mathrm{Fn}(S, 2)$ be arbitrary. We say that $p \leq q$ iff $q \subseteq p$. p and q will be called *compatible* (in symbols, $p \mid q$) if $p \cup q$ is a function; otherwise they are *incompatible* (in symbols, $p \perp q$). A subset of $\mathrm{Fn}(S, 2)$ in which any two different elements are incompatible will be called an *antichain*. The following lemma is part of the folklore and is used through all the chapter.

Lemma 2.78. *Let $\mathcal{C}$ be an independent family of size λ. For all $p, q \in$ $Fn(\lambda, 2)$ the following holds:*

1. *$\mathcal{C}(p) \subseteq \mathcal{C}(q)$ if and only if $p \leq q$.*
2. *$\mathcal{C}(p) \cap \mathcal{C}(q) \neq \emptyset$ if and only if p and q are compatible.*

Proof. Assume that $p \not\leq q$ and fix $(\alpha, i) \in q \setminus p$. If $\alpha \in \operatorname{dom} p$, then $(\alpha, 1-i) \in p$ and therefore $\mathcal{C}(p) \cap \mathcal{C}(q) = \emptyset$. When $\alpha \notin \operatorname{dom} p$, we let $r = p \cup \{(\alpha, 1-i)\}$ to obtain $\mathcal{C}(r) \subseteq \mathcal{C}(p)$ and $\mathcal{C}(r) \cap C(q) = \emptyset$; in particular, $\mathcal{C}(p) \not\subseteq \mathcal{C}(q)$. If $p \leq q$, then p is an extension of q and $\mathcal{C}(p) \subseteq \mathcal{C}(q)$.

If $p \perp q$, then for some $(\alpha, i) \in p$, we get $(\alpha, 1-i) \in q$ and therefore $\mathcal{C}(p) \cap \mathcal{C}(q) = \emptyset$. On the other hand, if $p \mid q$, then $p \cup q$, is an extension of p and q, and $\mathcal{C}(p) \cap \mathcal{C}(q) = \mathcal{C}(p \cup q)$. $\square$

The constructions described in the following two paragraphs will be used constantly in this thesis and are well known.

Given an independent family $\mathcal{C} = \{(C^0_\alpha, C^1_\alpha) : \alpha < \lambda\}$ on a set S, there is a topology for S which has $\{\mathcal{C}(p) : p \in \mathrm{Fn}(\lambda, 2)\}$ as a base. Indeed, for any $\alpha < \lambda, S = \mathcal{C}(\{(\alpha, 0)\}) \cup \mathcal{C}(\{(\alpha, 1)\})$ and $\mathcal{C}(p) \cap \mathcal{C}(q) = \mathcal{C}(p \cup q)$. The topological space which results of endowing S with this topology will be denoted by $S_{\mathcal{C}}$.

Remark 2.79. Another space that can be naturally associated to $\mathcal{C}$ is the following: for each $x \in S$ let $d_x : \lambda \to 2$ be defined by $d_x(\alpha) = 0$ if and only if $x \in C^0_\alpha$. Then $D_{\mathcal{C}}$ will denote the subspace $\{d_x : x \in S\}$ of the topological product 2^λ.

The next proposition is well known and establishes a connection between the spaces studied in the previous paragraphs.

Proposition 2.80. *Let $\mathcal{C}$ be an independent family of size λ on a set S. Then:*

1. *For all $x \in S$ and for all $p \in Fn(\lambda, 2), d_x \in D_{\mathcal{C}} \cap [p]$ is equivalent to $x \in \mathcal{C}(p)$.*
2. *$\mathcal{C}$ is uniform if and only if $\Delta(S_{\mathcal{C}}) = |S_{\mathcal{C}}|$.*
3. *$D_{\mathcal{C}}$ is dense in the product space 2^λ.*
4. *The map $h : S_{\mathcal{C}} \to D_{\mathcal{C}}$ given by $h(x) = d_x$ is continuous.*
5. *$\mathcal{C}$ is separating if and only if h is a homeomorphism.*

Proof.

1. Let $x \in S$ and $p \in \mathrm{Fn}(\lambda, 2)$. Then $d_x \in [p]$ if and only if $p \subseteq d_x$, when and only when $p(\alpha) = d_x(\alpha)$, for every $\alpha \in \operatorname{dom} p$, and this is equivalent to $x \in C^{p(\alpha)}_\alpha$ for every $\alpha \in \operatorname{dom} p$, if and only if $x \in \mathcal{C}(p)$.
2. By (1), $|\mathcal{C}(p)| = \kappa$ for all $p \in \mathrm{Fn}(\lambda, 2)$, if and only if $\Delta(S_{\mathcal{C}}) = \kappa$
3. Let $p \in \mathrm{Fn}(\lambda, 2)$. Since $\mathcal{C}$ is independent, we can take $y \in \mathcal{C}(p)$. Then $d_y \in [p] \cap D_{\mathcal{C}}$.
4. Let $p \in \mathrm{Fn}(\lambda, 2)$. By 2, $x \in \mathcal{C}(p)$ if and only if $d_x \in [p]$. Therefore $h[\mathcal{C}(p)] = [p] \cap D_{\mathcal{C}}$. Thus, h is continuous and open.
5. By 4, we have that h is a homeomorphism if and only if h is injective. Suppose h is injective, take two different points $x, y \in S$, then $d_x = h(x) \neq h(y) = d_y$, hence there is $\alpha < \lambda$ such that $d_x(\alpha) = 1 - d_y(\alpha)$.

Therefore $x \in C_\alpha^{d_x(\alpha)}$ and $y \in C_\alpha^{1-d_y(\alpha)}$, and $\mathcal{C}$ is separating. Now Suppose $\mathcal{C}$ is separating, take two different points $x, y \in S$. There exist $\alpha < \lambda$ and $i < 2$ such that $x \in C_\alpha^i$ and $y \in C_\alpha^{1-i}$. Therefore $d_x(\alpha) = i \neq 1 - i = d_y(\alpha)$, and h is injective.

□

Corollary 2.81. If $\mathcal{C}$ is a separating independent family of cardinality λ, on S, then $S_\mathcal{C}$ is Tychonoff ccc and crowded.

Proof. By the previous theorem, $S_\mathcal{C}$ is homeomorphic to a dense subspace of 2^λ, then $S_\mathcal{C}$ is Tychonoff, since 2^λ is crowded and ccc, then so is $S_\mathcal{C}$. □

In this chapter we have constructed dense subspaces of Cantor cubes, 2^λ, using independent families and now we are going to construct independent families using dense subspaces of Cantor cubes.

In order to fulfill the promise, take $Y = \{y_\alpha : \alpha < \kappa\}$ be a dense subset of 2^λ (we are assuming that $\alpha \neq \beta$ implies $y_\alpha \neq y_\beta$). For each $\xi < \lambda$ and $i < 2$ define $B_\xi^i := \{\alpha < \kappa : y_\alpha(\xi) = i\}$ and take $\mathcal{B}_Y := \{(B_\xi^0, B_\xi^1) : \xi < \lambda\}$. The following lemma is well known.

Lemma 2.82. *Let Y be a dense subset of cardinality κ on the product space 2^λ. The family $\mathcal{B}_Y$ is a separating independent family on κ of cardinality λ such that $D_{\mathcal{B}_Y} = Y$*

Proof. It is easy to see that the unordered pair $\{B_\xi^0, B_\xi^1\}$ is a partition of κ.

Let $p \in Fn(\lambda, 2)$. Since $[p]$ is an open subset of 2^λ, there is $\alpha < \lambda$ such that $y_\alpha \in [p] \cap Y$. Since $p \subseteq y_\alpha, \alpha \in \mathcal{B}_Y(p)$.

By the remark 2.79, $d_\alpha(\xi) = 0$ if and only if $\alpha \in B_\xi^0$, but this happens if and only if $y_\alpha(\xi) = 0$. Then $d_\alpha = y_\alpha$ for every $\alpha < \kappa$. Then $D_{\mathcal{B}_Y} = Y$. □

We will see how to construct irresolvable spaces using independent families.

Definition 2.83. We will say that an independent family $\mathcal{C}$ on S is *maximal independent*, if it is not properly contained in any other independent family on S.

The next proposition is well known.

Proposition 2.84. *Let $\mathcal{C}$ be a separating independent family $\mathcal{C}$ of cardinality λ on S. Then $\mathcal{C}$ is maximal independent if and only if $D_\mathcal{C}$ is an irresolvable subset of 2^λ.*

Proof. Assume that $D_{\mathcal{C}}$ is resolvable. Let U, V two disjoint dense subsets of $D_{\mathcal{C}}$. Let $A = \{x \in S : d_x \in U\}$. We are going to prove that $\mathcal{C}' = \mathcal{C} \cup \{(A, S \setminus A)\}$ is an independent family on S which properly contains $\mathcal{C}$.

First observe that $(A, S \setminus A)$ is not an element of $\mathcal{C}$. Otherwise, there would be $\alpha < \lambda$ and $i \in \{0,1\}$ such that $A = C^i_\alpha$. Let $p = \{(\alpha, i)\}$, since V is dense, there exists $x \in S$ such that $d_x \in [p] \cap V$. By 2.80(1), $x \in \mathcal{C}(p) = A$, and again by 2.80, $d_x \in U$, which implies that U and V are not disjoint.

Then $\mathcal{C}'$ properly contains $\mathcal{C}$. Now Let $p \in \mathrm{Fn}(\lambda, 2)$. Then $[p] \cap U$ and $[p] \cap V$ are nonempty, then $\mathcal{C}(p) \cap A$ and $\mathcal{C}(p) \cap S \setminus A$ are nonempty. Therefore $\mathcal{C}'$ is an independent family on S which properly contains $\mathcal{C}$.

Now suppose that $\mathcal{C}$ is not maximal and let $\mathcal{C}'$ be an independent family on S such that $\mathcal{C}$ is properly contained in $\mathcal{C}'$. Let $(A, S \setminus A) \in \mathcal{C}' \setminus \mathcal{C}$. Define $U = \{d_x \in D_{\mathcal{C}} : x \in A\}$. Then U and $D_{\mathcal{C}} \setminus U$ are disjoint dense subsets. Indeed, let $p \in \mathrm{Fn}(\lambda, 2)$. Since $\mathcal{C}'$ is independent, there exist $x \in \mathcal{C}(p) \cap A$ and $y \in \mathcal{C}(p) \cap S \setminus A$, then $d_x \in [p] \cap U$ and $d_y \in [p] \cap D_{\mathcal{C}} \setminus U$. □

Let κ be an infinite cardinal. There is a dense subset Y of cardinality κ in 2^{2^κ}. Then $\mathcal{B}_Y$ is an independent family on κ of cardinality 2^κ.

Lemma 2.85. *There exists a uniform independent family of subsets of κ of cardinality 2^κ.*

Proof. Define $P = [\kappa]^{<\omega} \times [[\kappa]^{<\omega}]^{<\omega}$. That is, P is the set of all pairs $(F, \mathcal{F})$ where F is a finite subset of κ and $\mathcal{F}$ is a finite set of finite subsets of κ. Since $[\kappa]^{<\omega}$ has cardinality κ, $\mathcal{P}$ has cardinality κ.

Take a enumeration $\mathcal{P}(\kappa) = \{U_\alpha : \alpha < 2^\kappa\}$ of all subsets of κ. For each $\alpha < 2^\kappa$, let

$$C^0_\alpha = \{(F, \mathcal{F}) \in P : F \cap U_\alpha \in \mathcal{F}\} \text{ and } C^1_\alpha = P \setminus C^0_\alpha$$

and define $\mathcal{C} = \{(C^0_\alpha, C^1_\alpha) : \alpha < 2^\kappa\}$.

Let $p \in \mathrm{Fn}(2^\kappa, 2)$. For each pair of distinct elements $\alpha, \beta \in \mathrm{dom}(p)$, let $\xi_{\alpha,\beta}$ be an element of κ such that either $\xi_{\alpha,\beta} \in U_\alpha \setminus U_\beta$ or $\xi_{\alpha,\beta} \in U_\beta \setminus U_\alpha$.

Define

$$G = \{\xi_{\alpha,\beta} : \alpha, \beta \in \mathrm{dom}(p), \alpha \neq \beta\}.$$

For each $\xi \in \kappa \setminus G$, define $F_\xi = G \cup \{\xi\}$. Observe that $F_\xi \cap U_\alpha \neq F_\xi \cap U_\beta$ when $\alpha, \beta \in \mathrm{dom}(p), \alpha \neq \beta$. Therefore $F_\xi \cap U_\alpha = F_\xi \cap U_\beta$ if and only if $\alpha = \beta$. Define $\mathcal{F}_\xi = \{F_\xi \cap U_\alpha : p(\alpha) = 0\}$.

Claim: $(F_\xi, \mathcal{F}_\xi) \in \mathcal{C}(p)$. Take $\alpha \in \mathrm{dom}(p)$. If $p(\alpha) = 0, F_\xi \cap U_\alpha \in \mathcal{F}_\xi$, then, $(F_\xi, \mathcal{F}_\xi) \in C^0_\alpha$. If $p(\alpha) = 1, F_\xi \cap U_\alpha \notin \mathcal{F}_\xi$, then, $(F_\xi, \mathcal{F}_\xi) \in C^1_\alpha$. Hence $(F_\xi, \mathcal{F}_\xi) \in \mathcal{C}(p)$. Therefore $\mathcal{C}(p)$ has cardinality κ.

If $\alpha \neq \beta, C^0_\alpha \setminus C^0_\beta$ is not empty and also $C^0_\alpha \setminus C^1_\beta$ is not empty, hence $\mathcal{C}$ is a uniform independent family on κ of cardinality 2^κ. □

2.4 Ai-maximal independent families

Definition 2.86. Let $\mathcal{C}$ be an independent family of size λ on a set S.

1. $\mathcal{C}$ is *ai-maximal independent* if for every partition $\{Y_n : n < \omega\}$ of S there exist $p \in \mathrm{Fn}(\lambda, 2)$ and $m < \omega$ such that $\mathcal{C}(p) \subseteq Y_m$.
2. We say that $\mathcal{C}$ is *aωi-maximal independent* if for every partition $\{Y_n : n < \omega\}$ of S there exist $p \in \mathrm{Fn}(\lambda, 2)$ and $m < \omega$ satisfying $\mathcal{C}(p) \subseteq \bigcup_{i \leq m} Y_i$.

We are going to see how to construct almost irresolvable spaces from ai-maximal independent families.

Proposition 2.87. *[22, Proposition 2.6] Let $\mathcal{C}$ be an independent family of size λ on a cardinal κ. Then $\mathcal{C}$ is ai-maximal independent iff $D_\mathcal{C}$ is an almost irresolvable subspace of 2^λ.*

Proof. Suppose that $\mathcal{C}$ is an ai-maximal independent family. Take a countable partition $\{U_n : n \in \omega\}$ of $D_\mathcal{C}$. For each $n \in \omega$, define $B_n = \{\alpha < \kappa : d_\alpha \in U_n\}$. Then $\{B_n : n \in \omega\}$ is a partition of κ. By hypothesis, there exist $p \in \mathrm{Fn}(\lambda, 2)$ and $m \in \omega$ such that $\mathcal{C}(p) \subseteq B_m$. Then $[p] \cap D_\mathcal{C} \subseteq U_m$. Indeed, if $d_\alpha \in [p] \cap D_\mathcal{C}$ then $\alpha \in \mathcal{C}(p) \subseteq B_m$, by definition of B_m, we have that $d_\alpha \in U_m$. Hence $\mathrm{int}_{D_\mathcal{C}}(U_m)$ is not empty and $D_\mathcal{C}$ is almost irresolvable.

Now assume that $D_\mathcal{C}$ is almost irresolvable. Take a countable partition $\{Y_n : n < \omega\}$ of κ. For each $n \in \omega$, define U_n to be the subset $\{d_\alpha : \alpha \in Y_n\}$ of $D_\mathcal{C}$. The collection $\{U_n : n \in \omega\}$ is a countable partition of $D_\mathcal{C}$. Since $D_\mathcal{C}$ is almost irresolvable, there is $m \in \omega$ such that $\mathrm{int}_{D_\mathcal{C}}(U_m)$ is not empty. Take $p \in Fn(\lambda, 2)$ such that $[p] \cap D_\mathcal{C} \subseteq U_m$. Hence $\mathcal{C}(p) \subseteq Y_m$. □

Similarly, [22, Proposition 2.7] states that $\mathcal{C}$ is aωi-maximal independent iff $D_\mathcal{C}$ is an almost ω-irresolvable subspace of 2^λ.

Corollary 2.88 (Dorantes)**.** There are no ai-maximal independent families under the Continuum Hypothesis.

Proof. If $\mathcal{C}$ is an ai-maximal independent family, then $D_\mathcal{C}$ is a dense subspace of the product space, 2^λ for some λ. Then $D_\mathcal{C}$ is a Hausdorff crowded, ccc, almost irresolvable space. By 2.24 CH fails. □

We are going to study some consequences of the existence of ai-maximal independent families.

Recall that if $Y = \{y_\alpha : \alpha < \kappa\}$ is a dense subset of 2^λ and if for each $\xi < \lambda$ and $i < 2$ we define $B^i_\xi := \{\alpha < \kappa : y_\alpha(\xi) = i\}$, then $\mathcal{B} := \{(B^0_\xi, B^1_\xi) : \xi < \lambda\}$ is a separating independent family such that $D_\mathcal{B} = Y$. In particular, we have the following.

Proposition 2.89. *For each dense almost ω-irresolvable subspace Y of 2^λ of size κ there exists an aωi-maximal independent family on κ of size λ.*

Definition 2.90. Let $\mathcal{C}$ be an independent family of size 2^κ on a set S, where $\kappa := |S|$. We say that $\mathcal{C}$ is a *nice independent family on S* if the following conditions hold:

1. $\mathcal{C}$ is separating,
2. each element of $[S_\mathcal{C}]^{<\kappa}$ is closed discrete in $S_\mathcal{C}$,
3. if $A \in [S_\mathcal{C}]^\kappa$, then either A is closed discrete in $S_\mathcal{C}$ or $\mathcal{C}(p) \subseteq A$ for some $p \in \mathrm{Fn}(2^\kappa, 2)$.

A nice independent family on S which is, at the same time, ai-maximal independent will be called a *nice ai-maximal independent family*. Similarly, a *nice aωi-maximal independent family* is a nice independent family which is aωi-maximal independent.

Proposition 2.91. *Suppose that $\mathcal{C}$ is a nice independent family on a set S. Then $D_\mathcal{C}$ is submaximal.*

Proof. Let D be a dense subset of $D_\mathcal{C}$. Then $D_\mathcal{C} \setminus D$ has empty interior, because D is dense, therefore $D_\mathcal{C} \setminus D$ is closed. □

Nice independent families produce spaces with interesting properties:

Proposition 2.92. *If $\mathcal{C}$ is a nice independent family, then every subset of $X_\mathcal{C}$ is a G_δ, i.e., $X_\mathcal{C}$ is a Q-set space*

Proof. As we noted above, $X_\mathcal{C}$ is submaximal and since $\mathcal{C}$ is separating, $X_\mathcal{C}$ is Tychonoff and ccc. By proposition 1.26, $X_\mathcal{C}$ is a Q-set space. This proves (1). □

The following result states that we can always modify a suitable independent family to obtain a uniform nice independent family.

Theorem 2.93. *[14, Dorantes, Pichardo, Tamariz] Let $\mathcal{B}$ be a uniform independent family on κ of size 2^κ. There is a uniform nice independent family $\mathcal{C}$ on κ such that if $p \in Fn(2^\kappa, 2)$ and $Y \in [\kappa]^\kappa$ satisfy $\mathcal{B}(p) \subseteq Y$, then there exists $q \in Fn(2^\kappa, 2)$ with $\mathcal{C}(q) \subseteq Y$.*

Proof. Enumerate $\mathcal{B} = \{(B^0_\xi, B^1_\xi) : \xi < 2^\kappa\}$ and $[\kappa]^\kappa = \{F_\xi : \xi < 2^\kappa\}$. Now partition 2^κ into two pieces, I_0 and I', such that $|I_0| = \kappa^{<\kappa}$ and $|I'| = 2^\kappa$. Also fix a partition $\{J_{A,\alpha} : A \in [\kappa]^{<\kappa}, \alpha \in \kappa \setminus A\} \subseteq [I_0]^\omega$ of I_0 into countable pieces. For all $A \in [\kappa]^{<\kappa}$, $\alpha \in \kappa \setminus A$, and $\xi \in J_{A,\alpha}$ define

$$C^0_\xi := (B^0_\xi \cup A) \setminus \{\alpha\} \text{ and } C^1_\xi := \kappa \setminus C^0_\xi = (B^1_\xi \setminus A) \cup \{\alpha\}.$$

Thus, define

$$\mathcal{B}_0 = \{(C^0_\xi, C^1_\xi) : \xi \in I_0\} \cup \{(B^0_\xi, B^1_\xi) : \xi \in 2^\kappa \setminus I_0\},$$

then $\mathcal{B}_0$ is a separating uniform independent family of size 2^κ.

Let $\mathbb{P} := \mathrm{Fn}(2^\kappa, 2)$.

If $\mathcal{B}_0(q) \subseteq F_0$ for some $q \in \mathbb{P}$, define $p_0 = q, I_1 = I_0, K_0 = \emptyset, f_0 = \emptyset$ and $\mathcal{B}_1 = \mathcal{B}_0$.

If $\mathcal{B}_0 \nsubseteq F_0$ for all $q \in \mathbb{P}$, let K_0 be a subset of cardinality κ of $I', p_0 = \emptyset$ and $f_0 : K_0 \to \kappa$ be a bijection. We want to make F_0 closed discrete in $\mathcal{B}_1$.

For each $\xi \in K_0$, define:

$$C^0_\xi = (B^0_\xi \cup F_0) \setminus \{f_0(\xi)\} \text{ and } C^1_\xi = \kappa \setminus C^0_\xi = (B^1_\xi \setminus F_0) \cup \{f_0(\xi)\}.$$

Observe that if $i \in F_0$, there is $\xi \in K_0$ such that $f_0(\xi) = i$, and

$$C^1_\xi \cap F_0 = \{i\}.$$

Define $I_1 = I_0 \cup K_0$ and

$$\mathcal{B}_1 = \{(C^0_\xi, C^1_\xi) : \xi \in I_1\} \cup \{(B^0_\xi, B^1_\xi) : \xi \in 2^\kappa \setminus I_0\},$$

We are going to prove that $\mathcal{B}_1$ is a uniform independent family of size 2^κ.

Suppose that $\mathcal{B}_1(p)$ has cardinality less than κ for some $p \in \mathbb{P}$. Then $A = \mathcal{B}_1(p) \in [\kappa]^{<\kappa}$, take $\alpha \in \kappa \setminus A$ and $\xi \in J_{A,\alpha} \setminus \mathrm{dom}(p)$. Then,

$$\mathcal{B}_1(p) \subseteq C^0_\xi.$$

Let $p^* = p \cup \{(\xi, 1)\}$, therefore

$$\mathcal{B}_1(p^*) = \emptyset.$$

Observe that $\mathcal{B}_0(p^*) \setminus F_0 \subseteq \mathcal{B}_1(p^*) = \emptyset$.

Then $\mathcal{B}_0(p^*) \subseteq F_0$, which is a contradiction. Then $\mathcal{B}_1$ is uniform independent family on κ.

Assume that for some $\alpha < 2^\kappa$ we have constructed

(i) a sequence $\{p_\beta : \beta < \alpha\} \subseteq \mathbb{P}$,

(ii) a collection $\{K_\beta : \beta < \alpha\}$ of subsets of I' such that $|K_\beta| \in \{0, \kappa\}$ for each $\beta < \alpha$,

(iii) a bijection $f_\beta : K_\beta \to \kappa$ for each $\beta < \alpha$ with $|K_\beta| = \kappa$, and

(iv) a family $\{\{C^0_\xi, C^1_\xi\} : \xi \in K_\beta\}$ of partitions of κ for each $\beta < \alpha$

in such a way that the following holds for each $\beta < \alpha$:

(1β) If we let $I_\beta = I_0 \cup \bigcup_{\xi<\beta} K_\xi$ and

$$\mathcal{B}_\beta = \{(C^0_\xi, C^1_\xi) : \xi \in I_\beta\} \cup \{(B^0_\xi, B^1_\xi) : \xi \in 2^\kappa \setminus I_\beta\},$$

then $\mathcal{B}_\beta$ is a uniform independent family of size 2^κ.

(2β) If $\mathcal{B}_\beta(q) \subseteq F_\beta$ for some $q \in \mathbb{P}$, then $K_\beta = \emptyset$ and $\mathcal{B}_\beta(p_\beta) \subseteq F_\beta$.

(3β) When $\mathcal{B}_\beta(q) \not\subseteq F_\beta$ for all $q \in \mathbb{P}$, then

(a) $p_\beta = \emptyset$,

(b) K_β is a subset of $I' \setminus \bigcup_{\xi<\beta}(K_\xi \cup \operatorname{dom} p_\xi)$ with $|K_\beta| = \kappa$, and

(c) for each $\xi \in K_\beta$:

$$C^0_\xi = (B^0_\xi \cup F_\beta) \setminus \{f_\beta(\xi)\} \text{ and } C^1_\xi = \kappa \setminus C^0_\xi = (B^1_\xi \setminus F_\beta) \cup \{f_\beta(\xi)\}.$$

Define $I_\alpha = I_0 \cup (\cup_{\xi<\alpha} K_\xi)$ and

$$\mathcal{B}_\alpha = \{(C^0_\xi, C^1_\xi) : \xi \in I_\alpha\} \cup \{(B^0_\xi, B^1_\xi) : \xi \in 2^\kappa \setminus I_\alpha\}.$$

Suppose that F_α contains $\mathcal{B}_\alpha(p)$ for some $p \in \mathbb{P}$. Define $p_\alpha = p, K_\alpha = \emptyset$. Therefore $\mathcal{B}_\alpha(p_\alpha) \subseteq F_\alpha$ and (2α) holds.

Now, suppose that $F_\alpha \not\supseteq \mathcal{B}_\alpha(p)$ for every $p \in \mathbb{P}$. Since $K = (\cup\{\operatorname{dom}(p_\zeta) : \zeta < \alpha\}) \cup (\cup\{K_\zeta : \zeta < \alpha\})$ has cardinality less than 2^κ, we can choose a subset K_α of cardinality κ in $I' \setminus K$. Set $p_\alpha = \emptyset$ and $f_\alpha : K_\alpha \to \kappa$ a bijection.

For each $\xi \in K_\alpha$, define:

$$C^0_\xi = (B^0_\xi \cup F_\alpha) \setminus \{f_\alpha(\xi)\} \text{ and } C^1_\xi = \kappa \setminus C^0_\xi = (B^1_\xi \setminus F_\alpha) \cup \{f_\alpha(\xi)\}.$$

Therefore, (3α) holds.

If $\alpha = \beta + 1$ for some $\beta < 2^\kappa$, then, $I_\alpha = I_\beta \cup K_\beta$.

If $K_\beta = \emptyset$, then $\mathcal{B}_\alpha = \mathcal{B}_\beta$. Therefore, $\mathcal{B}_\alpha$ is a uniform independent family of size 2^κ and (1α) holds.

Suppose that K_β is nonempty. By (2β), $\mathcal{B}_\beta(q) \not\subseteq F_\beta$ for all $q \in \mathbb{P}$. Suppose that $\mathcal{B}_\alpha(p)$ has cardinality less than κ for some $p \in \mathbb{P}$. Then $A = \mathcal{B}_\alpha(p) \in [\kappa]^{<\kappa}$, take $\gamma \in \kappa \setminus A$ and $\xi \in J_{A,\gamma} \setminus \mathrm{dom}(p)$. Then,

$$\mathcal{B}_\alpha(p) \subseteq C^0_\xi.$$

Let $p^* = p \cup \{(\xi, 1)\}$, therefore

$$\mathcal{B}_\alpha(p^*) = \emptyset.$$

By (3β)(c), $\mathcal{B}_\beta(p^*) \setminus F_\beta \subseteq \mathcal{B}_\alpha(p^*) = \emptyset$.

Then $\mathcal{B}_\beta(p^*) \subseteq F_\beta$, which is a contradiction. Then $\mathcal{B}_\alpha$ is a uniform independent family of size 2^κ on κ and (1α) holds.

Now, suppose that α is a limit ordinal.

Let $p \in \mathbb{P}$. Observe that

$$B_\alpha(p) = (\cap\{C^{p(\xi)}_\xi : \xi \in \mathrm{dom}\, p \cap I_\alpha\}) \cap (\cap\{B^{p(\xi)}_\xi : \xi \in \mathrm{dom}\, p \setminus I_\alpha\}).$$

Since $\mathrm{dom}\, p \cap I_\alpha$ is finite, there is $\beta < \alpha$ such that $\mathrm{dom}\, p \cap I_\alpha \subseteq I_\beta \subseteq I_\alpha$. Therefore,

$$B_\alpha(p) = (\cap\{C^{p(\xi)}_\xi : \xi \in \mathrm{dom}\, p \cap I_\beta\}) \cap (\cap\{B^{p(\xi)}_\xi : \xi \in \mathrm{dom}\, p \setminus I_\beta\}).$$

Hence

$$B_\alpha(p) = B_\beta(p),$$

implying that B_α is a uniform independent family of size 2^κ on κ and (1α) holds.

Hence, (1α),(2α) and (3α) are satisfied for every $\alpha < 2^\kappa$, and the transfinite construction of $\mathcal{C} = \mathcal{B}_{2^\kappa}$ is completed.

$\mathcal{C}$ is separating. Indeed, if $\alpha, \beta \in \kappa$ and $\alpha \neq \beta$, take $\xi \in J_{\{\alpha\},\beta}$. Then

$$\alpha \in C^0_\xi \text{ and } \beta \in C^1_\xi.$$

By construction, $\mathcal{C}$ is uniform.

If $A \in [\kappa]^{<\kappa}$, take any $\alpha \in \kappa \setminus A$ and $\xi \in J_{A,\alpha}$, then $A \subseteq C^0_\xi$ and $\alpha \in C^1_\xi$. Therefore $\kappa \setminus A$ is open in $\kappa_{\mathcal{C}}$. To see that A is discrete, take $\gamma \in A$, and $\xi \in J_{A\setminus\{\gamma\},\gamma}$ then

$$A \cap C^1_\xi = \{\gamma\}.$$

Take $F \in [\kappa]^\kappa$, then $F = F_\beta$ for some $\beta < 2^\kappa$. If (2β) holds, then, $K_\beta = \emptyset$ and $\mathcal{B}_\beta(p_\beta) \subseteq F_\beta$. Therefore $\mathcal{C}(p_\beta) = \mathcal{B}_\beta(p_\beta) \subseteq F_\beta$. Suppose that (3β) holds, if $i \in F_\beta$, there is $\xi \in K_\beta$ such that $f_\beta(\xi) = i$, and

$$C^1_\xi \cap F_\beta = \{i\}.$$

Therefore, F_β is discrete. If $i \in \kappa \setminus F_\beta$, there is $\xi \in K_\beta$ such that $f_\beta(\xi) = i$, and

$$C^1_\xi \cap F_\beta = \emptyset \text{ and } i \in C^1_\xi.$$

Therefore F_β is closed and discrete in $\kappa_{\mathcal{C}}$.

Hence, $\mathcal{C}$ is a uniform nice independent family on κ

Finally, suppose that $Y \subseteq \kappa$ and $p \in \mathbb{P}$ satisfy $\mathcal{B}(p) \subseteq Y$. Then $Y = F_\beta$ for some $\beta < 2^\kappa$, because $\mathcal{B}$ is uniform. It suffices to show the existence of a condition $r \in \mathbb{P}$ with $\mathcal{B}_\beta(r) \subseteq \mathcal{B}(p)$. Indeed, if this is the case, then at stage β the assumptions in (2β) hold and therefore $\mathcal{B}_\beta(p_\beta) \subseteq Y$. One easily verifies that $\mathcal{B}_\beta(p_\beta) = \mathcal{B}_\gamma(p_\beta)$ whenever $\beta \leq \gamma \leq 2^\kappa$. In particular, $\mathcal{C}(p_\beta) \subseteq Y$.

In order to find the condition r that we mentioned in the previous paragraph, we need the following claim.

Claim. For each $\delta \in I_\beta \cap \operatorname{dom} p$ and any finite set $H \subseteq 2^\kappa$ with $\operatorname{dom} p \subseteq H$ there exist $\delta', \delta'' \in I_\beta \setminus H$ such that $\delta' \neq \delta''$ and

$$C^{p(\delta)}_\delta \cap C^0_{\delta'} \cap C^1_{\delta''} \subseteq B^{p(\delta)}_\delta.$$

To prove the claim we will consider two cases. If $\delta \in I_0$, then $\delta \in J_{b,\alpha}$ for some $b \in [\kappa]^{<\kappa}$ and $\alpha \in \kappa \setminus b$. Thus any pair of different points $\delta', \delta'' \in J_{b,\alpha} \setminus H$ will work. On the other hand, when $\delta \in I_\beta \setminus I_0$, there exists $\xi < \beta$ with $\delta \in K_\xi$. Set $\alpha := f_\xi(\delta)$ and notice that we only need to take $\delta' \in K_\xi \setminus H$ and $\delta'' \in J_{\emptyset,\alpha} \setminus H$.

Using finite recursion we define, for each $\delta \in I_\beta \cap \operatorname{dom} p$, a pair of ordinals $\delta', \delta'' \in I_\beta$ satisfying the conclusion of the Claim and such that

$$r := p \cup \{(\xi', 0) : \xi \in I_\beta \cap \operatorname{dom} p\} \cup \{(\xi'', 1) : \xi \in I_\beta \cap \operatorname{dom} p\}$$

is a function. Therefore $\mathcal{B}_\beta(r) \subseteq \mathcal{B}(p)$. □

Corollary 2.94. For any infinite cardinal κ, there is a crowded, submaximal, ccc space X such that $\Delta(X) = |X|$.

Proof. By 2.85 there is a uniform independent family $\mathcal{B}$ on κ of cardinality 2^κ. By 2.93, there is a nice uniform independent family $\mathcal{C}$ on κ of cardinality 2^κ. By 2.91, $D_{\mathcal{C}}$ is submaximal, and since $D_{\mathcal{C}}$ is dense in product space 2^κ, $D_{\mathcal{C}}$ is crowded and ccc. Since $\mathcal{C}$ is uniform $\Delta(D_{\mathcal{C}}) = |D_{\mathcal{C}}|$ □

Corollary 2.95. For any cardinal κ, the existence of a uniform ai-maximal (respectively, aωi-maximal) independent family on κ of size 2^κ implies the existence of a uniform nice ai-maximal (respectively, aωi-maximal) independent family on κ.

Proof. Suppose that $\mathcal{B}$ is a uniform ai-maximal independent family on κ with $|\mathcal{B}| = 2^\kappa$ and let $\mathcal{C}$ be the uniform nice independent family given by the previous theorem. Assume that $\{Y_n : n < \omega\}$ is a partition of κ and fix $p \in \mathrm{Fn}(2^\kappa, 2)$ and $m < \omega$ in such a way that $\mathcal{B}(p) \subseteq Y_m$. Hence, there is $q \in \mathrm{Fn}(2^\kappa, 2)$ such that $\mathcal{C}(q) \subseteq Y_m$.

Similar arguments apply in the case where $\mathcal{B}$ is aωi-maximal independent. $\square$

Definition 2.96. Let $\mathcal{C} = \{(C^0_\xi, C^1_\xi) : \xi < \lambda\}$ be an independent family on a set S.

1. For each $p \in \mathrm{Fn}(\lambda, 2)$ we define
$$\mathcal{C}\lceil p := \{(C^0_\xi \cap \mathcal{C}(p), C^1_\xi \cap \mathcal{C}(p)) : \xi \in \lambda \setminus \operatorname{dom} p\}.$$
2. We say that $\mathcal{C}$ is *globally ai-maximal independent on S* if $\mathcal{C}\lceil p$ is ai-maximal independent on $\mathcal{C}(p)$ for all $p \in \mathrm{Fn}(\lambda, 2)$.

Remark 2.97. Let $\mathcal{C}$, S, and λ be as in the definition. For each $r \in \mathrm{Fn}(\lambda, 2)$, $\mathcal{C}(r)$ is an open and closed subspace of $X_\mathcal{C}$ which has $\{\mathcal{C}(r) \cap \mathcal{C}(p) : p \in \mathrm{Fn}(\lambda \setminus \operatorname{dom} r, 2)\}$ as a base for its topology. Therefore $X_{\mathcal{C}\lceil r} = \mathcal{C}(r)$.

Observe that if $\mathcal{C}$ is a nice independent family on κ, then $\mathcal{C} \restriction r$ is a nice independent family on $\mathcal{C}(r)$, for all $r \in \mathrm{Fn}(2^\kappa, 2)$.

Proposition 2.98. *[14, Dorantes, Pichardo, Tamariz] Let $\mathcal{C}$ be an independent family on a set S. Then $\mathcal{C}$ is globally ai-maximal independent iff $X_\mathcal{C}$ is OHAI.*

Proof. Let $\lambda := |\mathcal{C}|$. When $X_\mathcal{C}$ is OHAI and $p \in \mathrm{Fn}(\lambda, 2)$, $\mathcal{C}(p)$ is almost irresolvable; hence Remark 2.97 implies that $\mathcal{C}\lceil p$ is ai-maximal independent.

For the other implication assume that $X_\mathcal{C}$ is not OHAI and fix a family, $\{Y_n : n \in \omega\}$, of pairwise disjoint subsets of $X_\mathcal{C}$ whose union, Y, is a non-empty open subset of X, but each Y_n has empty interior. Then there is $p \in \mathrm{Fn}(\lambda, 2)$ with $\mathcal{C}(p) \subseteq Y$ and therefore $\{\mathcal{C}(p) \cap Y_n : n \in \omega\}$ witnesses that $\mathcal{C}(p)$ is almost resolvable. $\square$

As a consequence of the work done, we obtained:

Proposition 2.99. *If $\mathcal{C}$ is an ai-maximal independent family on κ of size λ, then $\mathcal{C}\lceil r$ is globally ai-maximal independent on $\mathcal{C}(r)$, for some $r \in Fn(\lambda, 2)$.*

Proof. Since $X_{\mathcal{C}}$ is almost irresolvable, Lemma 2.7, implies the existence of a finite function $r \in \mathrm{Fn}(\lambda, 2)$ such that $\mathcal{C}(r)$ is OHAI. According to 2.97, $X_{\mathcal{C}\lceil r}$ is OHAI, by the previous proposition, $\mathcal{C}\lceil r$ is globally ai-maximal independent. □

The next theorem can be found in [20].

Theorem 2.100. *[20] Every almost resolvable space X is the union of a resolvable space X_1 and a first category set X_2 with X_1 closed in X and $X_1 \cap X_2 = \emptyset$.*

Corollary 2.101. If X is Baire almost resolvable, then X is resolvable.

Proof. If X is Baire, X does not have nonempty open sets of first category. □

For a cardinal κ, we will say that a topological space X satisfies $(\dagger_\kappa)$ if X is a dense subspace of 2^{2^κ} with $\Delta(X) = |X| = \kappa$.

Theorem 2.102. *[14, Dorantes, Pichardo, Tamariz] The following statements are equivalent for any cardinal κ.*

1. *There is an almost irresolvable space which satisfies $(\dagger_\kappa)$.*
2. *There is a Baire submaximal space which satisfies $(\dagger_\kappa)$.*
3. *There is a Baire OHI space which satisfies $(\dagger_\kappa)$.*
4. *There is a Baire irresolvable space which satisfies $(\dagger_\kappa)$.*
5. *There is a Baire almost irresolvable space which satisfies $(\dagger_\kappa)$.*
6. *There is a Baire almost ω-irresolvable space which satisfies $(\dagger_\kappa)$.*
7. *There is an almost ω-irresolvable space which satisfies $(\dagger_\kappa)$.*

Proof. Let us prove that (2) follows from (1). If (1) holds, Corollary 2.95 guarantees the existence of a uniform nice ai-maximal independent family $\mathcal{C}$ on κ. According to Proposition 2.99, there is a condition r for which $\mathcal{C}\lceil r$ is globally ai-maximal independent on κ. Since $|\mathcal{C}(r)| = \kappa$ and $\mathcal{C}\lceil r$ is a nice independent family on $\mathcal{C}(r)$, we will assume, without loss of generality, that $\mathcal{C}$ is globally ai-maximal independent on κ. Thus $X_{\mathcal{C}}$ is Baire submaximal (Proposition 2.98 and Theorem 2.14) and $\Delta(X_{\mathcal{C}}) = \kappa$. Using the fact that

$\mathcal{C}$ is separating, we have that $X_{\mathcal{C}}$ is homeomorphic to $D_{\mathcal{C}}$ (see the discussion following Lemma 2.78) and therefore $D_{\mathcal{C}}$ is the space needed in (2).

Implications (2)→(3)→(4) and (5)→(6)→(7) are straightforward.

By 2.101, (4) implies (5).

Finally, if (7) is true, Corollary 2.95 gives the existence of a uniform nice aωi-maximal independent family $\mathcal{C}$ on κ. By Proposition 2.5, X_C is a space whose existence is claim in (1). □

As a consequence of Theorem 2.102 we obtained that the consistency strength of the existence of a family as described in part (6) is greater than the existence of a measurable cardinal:

Corollary 2.103. If κ carries a uniform aωi-maximal independent family of size 2^κ, then κ is measurable in an inner model of ZFC.

Proof. For such a κ we obtain the existence of a dense subspace Y of 2^{2^κ} which is Baire and satisfies $\Delta(Y) = |Y| = \kappa$. It is proved in [32], [33], and [26, Theorem 22.33] that the existence of a space with these characteristics implies the conclusion of the corollary. □

Theorem 2.104. *If an infinite cardinal κ carries a uniform aωi-maximal independent family of size 2^κ, then the following holds:*

1. *κ has uncountable cofinality,*
2. *$\kappa \neq \omega_1$, and*
3. CH *fails, i.e., $\mathfrak{c} > \omega_1$.*

Proof. (1) is a corollary of Proposition 2.1.

To prove (2) and (3) assume that κ is as described in the hypothesis. Corollary 2.95 guarantees the existence of a uniform nice ai-maximal independent family $\mathcal{C}$ on κ. Then $X_{\mathcal{C}}$ is a regular, crowded, submaximal, ccc, almost irresolvable space. By 2.20, $\kappa \neq \omega_1$ and by 2.24 CH fails. □

Note that a corollary of the previous result is that if CH holds, then no cardinal κ carries a uniform aωi-maximal independent family of size 2^κ. The same conclusion is consistent with $\neg$CH. Indeed, [7, Theorem 4.1] states that if there are no Souslin trees, then every ccc crowded Hausdorff space is almost ω-resolvable (see the discussion following Definition 2.86).

In [30, Page 79] and [32, Theorem 3.3] it is shown that if κ is measurable and the ground model satisfies CH, then the generic extension yield by $\mathrm{Fn}(\kappa, 2, \omega_1)$ contains a Baire OHI space X with $\Delta(X) = |X|$ (compare with part (3) of Theorem 2.102). But in the generic extension no cardinal κ carries a uniform aωi-maximal independent family of size 2^κ because CH holds in it.

(1) ([40, Questions 5.8]) *Is the topology generated by the union of a chain of almost ω-resolvable topologies for a set X always almost ω-resolvable?*

Assume that κ carries a uniform ai-maximal independent family of size 2^κ. Corollary 2.95 provides us with a nice ai-maximal independent family $\mathcal{C} = \{(C^0_\alpha, C^1_\alpha) : \alpha < 2^\kappa\}$. For each integer n let $\mathcal{C}_n := \mathcal{C} \setminus \{(C^0_\alpha, C^1_\alpha) : n \leq \alpha < \omega\}$. Denote by τ_n the topology which has $\{\mathcal{C}_n(p) : p \in \mathrm{Fn}(2^\kappa \setminus n, 2)\}$ as a base. Then $\{\tau_n : n < \omega\}$ is an increasing sequence of topologies. Moreover, $\{C^0_n, C^1_n\}$ is a partition of (κ, τ_n) into two disjoint dense sets for all $n < \omega$ and therefore τ_n is resolvable. On the other hand, the topology generated by $\bigcup_n \tau_n$ coincides with the topology of $X_\mathcal{C}$ and so it is almost ω-irresolvable.

(2) ([3, Problem 7.4]) *Is every regular ccc submaximal space strongly σ-discrete?*

Let $\mathcal{C}$ be a nice ai-maximal independent family. Thus $X_\mathcal{C}$ is crowded and submaximal. Also, Proposition 2.5 implies that this space is not σ-discrete. Since $X_\mathcal{C}$ is homeomorphic to $D_\mathcal{C}$, a dense subspace of the product 2^{2^κ}, we have that $X_\mathcal{C}$ is Tychonoff and ccc.

There is a ZFC answer for [36, Problem 3.8]: *Does any submaximal space contain a dense maximal space?* According to [, Theorem 4.1], 2^{2^κ} has a dense subspace Y which is submaximal. On the other hand, an immediate consequence of [1, Corollary 2.2] is that no dense subspace of 2^{2^κ} is maximal. Therefore Y is a submaximal space which contains no dense maximal space.

2.5 Some combinatorics

The following result suggests that if one adds enough random reals, the generic extension may contain an ai-maximal independent family.

Theorem 2.105. *[14, Dorantes, Pichardo, Tamariz] Let $\mathcal{B}$ be a uniform independent family of size 2^κ on a cardinal κ and let $\mathcal{C}$ be the family which was constructed in the proof of Theorem 2.93. If $m : \mathcal{P}(\kappa) \to [0,1]$ is a σ-additive measure such that $m(\mathcal{C}(p)) = 2^{-|p|}$, for each $p \in Fn(2^\kappa, 2)$, then $\mathcal{C}$ is globally ai-maximal independent.*

Proof. Denote by I the ideal of null sets, i.e., $x \in I$ iff $m(x) = 0$. Since m is σ-additive, I is ω_1-complete so, according to Theorem 2.14, we only need to show that I coincides with the collection of all subsets of $X_{\mathcal{C}}$ with void interior (one easily verifies that if this is the case, then the argument given in the proof of Theorem 2.14 to show that I is σ-saturated still works). Set $\mathbb{P} := \mathrm{Fn}(2^\kappa, 2)$.

Observe that if $A \subseteq X_{\mathcal{C}}$ and $p \in \mathbb{P}$ satisfy $\mathcal{C}(p) \subseteq A$, then $m(A) > 0$. Hence all null sets have empty interior.

Now let A be a subset of $X_{\mathcal{C}}$ with empty interior. If $A = \emptyset$, $m(A) = 0$ so let us assume that $A \neq \emptyset$. Since all finite subsets of $X_{\mathcal{C}}$ are closed, we have that $\kappa \setminus A$ is infinite. Our plan is to show that $m(A) \leq 2^{-i}$ for all $i < \omega$.

For the rest of the argument we will follow the notation introduced in the proof of Theorem 2.93. Let $n < \omega$ be arbitrary.

Suppose first that $|A| < \kappa$. Fix a set $H \subseteq \kappa \setminus A$ with $|H| = n$ and for each $\alpha \in H$ let $\overline{\alpha} \in J_{A,\alpha}$ be arbitrary. Thus, if we let $p := \{(\overline{\alpha}, 0) : \alpha \in H\}$, then $A \subseteq \mathcal{C}(p)$ and $|p| = n$. Clearly $m(A) \leq 2^{-n}$.

When $|A| = \kappa$, there exists $\beta < 2^\kappa$ with $A = F_\beta$. Let $H \subseteq K_\beta$ be such that $|H| = n$ and $f_\beta[H] \subseteq \kappa \setminus A$. Thus $q := H \times \{0\} \in \mathbb{P}$ and $A \subseteq \mathcal{C}(q)$. □

This section will end with a combinatorial characterization of the existence of uniform ai-maximal independent families.

Given a poset $\mathbb{P}$, we will denote by $B(\mathbb{P})$ its Boolean completion, i.e., $B(\mathbb{P})$ is a complete Boolean algebra which contains $\mathbb{P}$ as a dense subset. As usual, given a set $S \subseteq B(\mathbb{P})$, $\bigvee S$ and $\bigwedge S$ represent the supremum and the infimum of S in $B(\mathbb{P})$, respectively.

Remark 2.106. If $b \in B(\mathbb{P})$, then $b = \bigvee\{p \in \mathbb{P} : p \leq b\}$ and therefore $B(\mathbb{P}) = \{\bigvee S : S \subseteq \mathbb{P}\}$.

The following fact (see, for example, [31, II Exercise 19]) will be used later.

Remark 2.107. If $S, T \subseteq \mathbb{P}$, then $\bigvee S \leq \bigvee T$ iff for all $p \in S$ and for each $q \leq p$ there exists $r \in T$ with $r \mid q$.

Theorem 2.108. *[14, Dorantes, Pichardo, Tamariz] The following are equivalent for any infinite cardinal κ.*

1. *There exists a uniform ai-maximal independent family on κ of size 2^κ.*
2. *κ carries an ω_1-complete ideal I for which the quotient Boolean algebra $\mathcal{P}(\kappa)/I$ is isomorphic to $B(Fn(2^\kappa, 2))$ and $[\kappa]^{<\kappa} \subseteq I$.*

Proof. We will show first that (2) implies (1). Set $\mathbb{P} := \mathrm{Fn}(2^\kappa, 2)$ and suppose that $f : B(\mathbb{P}) \to \mathcal{P}(\kappa)/I$ is an isomorphism.

Let $\{A_\xi : \xi < 2^\kappa\}$ be an enumeration of I where each A_ξ is listed infinitely many times.

Let $\xi < 2^\kappa$ be arbitrary. Fix $B^0_\xi \subseteq \kappa$ such that $f(\{(\xi,0)\}) = [B^0_\xi]$, where $[B^0_\xi]$ denotes the equivalence class of B^0_ξ modulo I, and let $B^1_\xi := \kappa \setminus B^0_\xi$. Since $\{(\xi,1)\}$ is the Boolean complement of $\{(\xi,0)\}$ in $B(\mathbb{P})$, we have that $f(\{(\xi,1)\}) = [B^1_\xi]$. Define $C^0_\xi := B^0_\xi \setminus A_\xi$ and $C^1_\xi := B^1_\xi \cup A_\xi$.

We will argue that $\mathcal{C} := \{(C^0_\xi, C^1_\xi) : \xi < 2^\kappa\}$ is ai-maximal independent.

We are going to prove that $\mathcal{C}$ is uniform independent. Let $p \in \mathbb{P}$ be arbitrary and observe that if $\xi < 2^\kappa$ and $i < 2$, then $[C^i_\xi] = [B^i_\xi]$. Therefore the equality $f(p) = [\mathcal{C}(p)]$ follows from the fact $p = \bigwedge\{\{(\alpha, p(\alpha))\} : \alpha \in \operatorname{dom} p\}$. In particular, $\mathcal{C}(p) \notin I$ and so $|\mathcal{C}(p)| = \kappa$.

Let $\{Y_n : n \in \omega\}$ be a partition of κ. Since I is a proper ω_1-complete ideal, there is $m < \omega$ with $Y_m \notin I$. Let $b \in B(\mathbb{P})$ and $p \in \mathbb{P}$ be so that $f(b) = [Y_m]$ and $p \le b$. Thus $[\mathcal{C}(p)] = f(p) \le [Y_m]$, i.e., $\mathcal{C}(p) \setminus Y_m \in I$. For some $\xi \in 2^\kappa \setminus \operatorname{dom} p$ we get $A_\xi = \mathcal{C}(p) \setminus Y_m$ and so $q := p \cup \{(\xi, 0)\}$ satisfies $\mathcal{C}(q) \subseteq Y_m$.

Assume (1). Proceeding as in the proof of (1)$\to$(2) in Theorem 2.102, there is a nice independent family $\mathcal{C}$ on κ for which $X_{\mathcal{C}}$ is Baire submaximal and $\Delta(X_{\mathcal{C}}) = \kappa$. Thus I, the ideal of nowhere dense subsets of $X_{\mathcal{C}}$, is an ω_1-complete ideal on κ and coincides with the collection of all subsets of $X_{\mathcal{C}}$ with empty interior. Moreover, each element of I is closed and $[\kappa]^{<\kappa} = [X_{\mathcal{C}}]^{<\kappa} \subseteq I$.

For each $x \subseteq \kappa$ let $x^* := \{p \in \mathbb{P} : \mathcal{C}(p) \subseteq x\}$. Define $h : \mathcal{P}(\kappa) \to B(\mathbb{P})$ by $h(x) := \bigvee x^*$. We will show that the following holds:

(a) for all $x, y \in \mathcal{P}(\kappa)$, $x \setminus y \in I$ iff $h(x) \le h(y)$; and

(b) h is onto.

Notice that if (a) and (b) are true, then h induces an isomorphism from $\mathcal{P}(\kappa)/I$ onto $B(\mathbb{P})$.

Observe that if $p \in x^*$ and $q \le p$, then $q \in x^*$. Therefore we apply Remark 2.107 to obtain that $h(x) \le h(y)$ iff for each $p \in x^*$ there is $q \in y^*$ with $p \mid q$.

Let us prove (a). Suppose that $x \setminus y \in I$ and let $p \in x^*$ be arbitrary. Then $x \setminus y$ is closed, $\mathcal{C}(p) \nsubseteq x \setminus y$, and $\mathcal{C}(p) \subseteq x$. Hence $\mathcal{C}(p) \setminus x = \emptyset$ and $\mathcal{C}(p) \setminus (x \setminus y) = (\mathcal{C}(p) \setminus x) \cup (\mathcal{C}(p) \cap y) = \mathcal{C}(p) \cap y$ is a nonempty open set. There is $q \in \mathbb{P}$ so that $\mathcal{C}(q) \subseteq \mathcal{C}(p) \cap y$. Clearly $q \in y^*$ and $q \mid p$ (Lemma 2.78). By the observation made in the previous paragraph: $h(x) \le h(y)$.

Now suppose that $x \setminus y \notin I$. Then $\mathcal{C}(p) \subseteq x \setminus y$ for some $p \in \mathbb{P}$. In particular, $p \in x^*$. Notice that for all $q \in y^*$ we have $\mathcal{C}(q) \subseteq y$ and thus

$\mathcal{C}(p) \cap \mathcal{C}(q) = \emptyset$, i.e., $p \perp q$ (Lemma 2.78). This shows that $h(x) \not\leq h(y)$ and so (a) is proved.

According to Remark 2.106, h is onto if for each $S \subseteq \mathbb{P}$ there is $x \subseteq \kappa$ such that $h(x) = \bigvee S$. So let $S \subseteq \mathbb{P}$ be arbitrary and define $x := \bigcup\{\mathcal{C}(p) : p \in S\}$. Clearly, $S \subseteq x^*$ and hence $\bigvee S \leq h(x)$. We will use Remark 2.107 to show that $h(x) \leq \bigvee S$. If $p \in x^*$, then $\mathcal{C}(p) \subseteq x$ and hence $\mathcal{C}(p) \cap \mathcal{C}(q) \neq \emptyset$ for some $q \in S$. Thus $p \mid q$ according to Lemma 2.78. □

It is worth noticing that the argument given for (1)→(2) in the previous theorem shows that the existence of an ω_1-complete ideal, I, on κ for which the quotient $\mathcal{P}(\kappa)/I$ is isomorphic to $B(\mathrm{Fn}(2^\kappa, 2))$ implies the existence of an ai-maximal independet family on κ of size 2^κ.

2.6 Questions

This section is dedicated to some interesting problems.

Problem 2.109. Are the following statements consistent with ZFC?

1. There are cardinals κ which carry an aωi-independent family of size 2^κ?

2. There are cardinals κ which carry an aωi-independent family of size λ with $\lambda < 2^\kappa$?

3. For some cardinal λ, 2^λ contains a dense almost ω-irresolvable subspace but no dense almost irresolvable subspace?

Problem 2.110. Is it true that if κ does not carry an aωi-maximal independent family of size 2^κ, then all Baire dense subspaces of 2^{2^κ} are ω-resolvables? (Theorem 2.102 guarantees that if the cardinality of the subspace is κ, then the answer is "yes.")

If $\mathcal{B}$ is an arbitrary independent family on a cardinal κ of size 2^κ, the construction described in the proof of Proposition 2.90 shows how to modify $\mathcal{B}$ to obtain a nice independent family $\mathcal{C}$ on κ. One may wonder if this process preserves algebraic structures.

Problem 2.111. Is it true that if $D_{\mathcal{B}}$ is a topological subgroup of the product 2^{2^κ}, then so is $D_{\mathcal{C}}$?

We can give two remarks regarding this question.

First, it is proved in [3, Corollary 8.16] that the cardinality of a ccc nodec topological group is not greater than $\mathfrak{c}$. Therefore $D_{\mathcal{C}}$ is not a topological subgroup when $\kappa > \mathfrak{c}$ independently of the properties that $\mathcal{B}$ has.

Second, [1, Corollary 3.8] states that if Y is a homogeneous submaximal space with $|Y| = \Delta(Y)$ and $|Y|$ is non-measurable cardinal, then Y is strongly σ-discrete. Thus $X_{\mathcal{C}}$ is not homogeneous when $\mathcal{B}$ is ai-maximal independent and κ is non-measurable (see the proof of Corollary 2.95 and Proposition 2.5).

Chapter 3

Preliminaries on weakly pseudocompact spaces

3.1 The Stone-Čech compactification

Throughout this chapter all topological spaces are considered Tychonoff and with more than one point. A *zero set* in X is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$, where f is a real valued continuous function defined on X, and a cozero set is the complement of a zero set. A pair (aX, a), is a *compactification* of a space X, if aX is compact and $a : X \to aX$ is an embedding such that $a[X]$ is dense in aX. To simplify notation, we will write aX instead of (aX, a). We will denote by $C(X, Z)$ the set of all continuous functions from X into Z.

The next theorem will enable us to construct compactifications

Theorem 3.1 (Embedding Theorem)**.** *Let X be a space and R be a subset of $\mathbb{R}$. If F is a subset of $C(X, R)$ such that:*

1. *if x, y are distinct points in X, there is $f \in F$ such that $f(x) \neq f(y)$ and*

2. *if $x \in U \subseteq X$, where U is open in X, then there is $f \in F$ such that $f(x) \notin \mathrm{cl}_R\, f[X \setminus U]$.*

Then, the evaluation map $e_F : X \to R^F$, defined by $\pi_f(e_F(x)) = f(x)$ for every $x \in X$ and every $f \in F$, is an embedding.

Proof. We know that a function g defined on a space Z to a product space $\Pi\{Z_j : j \in J\}$ is continuous if and only if $\pi_j \circ g$ is continuous for every projection π_j. Since $\pi_f \circ e_F = f$, for every $f \in F, e_F$ is continuous.

Claim 1. e_F is injective. Take x_0 and x_1 two different points in X. There is a continuous function $f \in F$ such that $f(x_0) \neq f(x_1)$. Therefore $\pi_f(e_F(x_0)) = f(x_0) \neq f(x_1) = \pi_f(e_F(x_1))$. Hence $e_F(x_0) \neq e_F(x_1)$

Claim 2. e_F is open onto $e_F[X]$. Suppose that V is an open subset of X and x is an element of V. There is a continuous function $f \in F$ such that $f(x) \notin \mathrm{cl}_R\, f[X \setminus V]$. So, there is an open set $W \subseteq R$ such that $f(x) \in W$ and $W \cap f[X \setminus V] = \emptyset$. Therefore:

$$e_F(x) \in e_F[X] \cap \pi_f^{\leftarrow}[W] \subseteq e_F[V].$$

Indeed, take $y \in X$ such that $e_F(y) \in \pi_f^{\leftarrow}[W]$. Then $\pi_f(e_F(y)) \in W$, that is, $f(y) \in W$ and $f(y) \notin f[X \setminus V]$. Therefore, $y \in V$. Since $e_F[X] \cap \pi_f^{\leftarrow}[W]$ is open in $e_F[X]$, we conclude that $e_F[V]$ is open in $e_F[X]$.

Hence e_F is an embedding. □

In the next theorem we are going to construct a special compactification of a space X.

Theorem 3.2. *[43] Let X be a space. There is a compactification βX of X such that for each continuous function $f : X \to [0,1]$, there is a unique continuous function $\beta f : \beta X \to [0,1]$ such that $f = \beta f \circ \beta$.*

Proof. Define $\beta : X \to [0,1]^{C(X,[0,1])}$ by $\pi_f(\beta(x)) = f(x)$ for every $x \in X$ and $f \in C(X,[0,1])$. By the Embedding Theorem, β is an embedding.

Let $\beta X = \mathrm{cl}\, \beta[X]$. Then $\beta : X \to Y$ is an embedding such that $\beta[X]$ is dense in βX.

Now, if $f : X \to [0,1]$ is a continuous function, then, by the definition of β, we have

$$f = (\pi_f \restriction \beta X) \circ \beta.$$

Finally, if $g : \beta X \to [0,1]$ is a continuous function such that $f = g \circ \beta$, then

$$g(\beta(x)) = f(x) = (\pi_f \restriction \beta X)(\beta(x))$$

for every $x \in X$. Since $\beta[X]$ is dense in βX, $g = \pi_f \restriction \beta X$ and the theorem is established. □

If X is a space, the compactification βX, constructed in the last Theorem, is called the *Stone-Čech compactification* of X.

Corollary 3.3. If X is compact, then βX is homeomorphic to X.

Proof. If X is compact then $\beta[X]$ is compact and $\beta : X \to \beta[X]$ is a homeomorphism. □

If aX and bX are two compactifications of a space X, we will say that $aX \leq bX$, if there is a continuous function $f : bX \to aX$ such that $f \circ b = a$.

Theorem 3.4. *[37] Let X be a space. If $f : X \to K$ is a continuous function from X to a compact space K, then there is a continuous function $\tilde{f} : \beta X \to K$ such that $\tilde{f} \circ \beta = f$.*

Proof. Let f, X and K be as in the statement of the theorem. Define $\alpha : K \to [0,1]^{C(K,[0,1])}$ by $\pi_g(\alpha(t)) = g(t)$ for every $t \in K$ and $g \in C(K,[0,1])$. By the Embedding Theorem, we can see that the subspace $\alpha[K]$ is homeomorphic to K.

Observe that, for every $g \in C(K,[0,1])$, the composition $g \circ f$ is an element of $C(X,[0,1])$, then $\beta(g \circ f) : \beta X \to [0,1]$ is the only continuous function such that $\beta(g \circ f) \circ \beta = g \circ f$.

Therefore, we can define

$$h : \beta X \to [0,1]^{C(K,[0,1])}$$

by

$$\pi_g(h(p)) = \beta(g \circ f)(p) \text{ for every } p \in \beta X \text{ and for every } g \in C(K,[0,1]).$$

Then h is continuous.

If $x \in X$ and $g \in C(K,[0,1])$ then

$$\pi_g(h(\beta(x))) = \beta(g \circ f)(\beta(x)) = g(f(x)) = \pi_g(\alpha(f(x)))$$

Therefore $h(\beta(x)) = \alpha(f(x))$ for every $x \in X$. Hence, $h[\beta[X]] \subseteq \alpha[K]$.

Since $\alpha[K]$ is compact, we also have the following:

$$h[\beta X] = h[\mathrm{cl}_{\beta X}\, \beta[X]] \subseteq \mathrm{cl}\, h[\beta[X]] \subseteq \mathrm{cl}\, \alpha[K] = \alpha[K].$$

Therefore $h : \beta X \to \alpha[K]$. Now, define $\tilde{f} = (\alpha^{\leftarrow}) \circ h$

It is easy to see that $\tilde{f} \circ \beta = f$. □

Corollary 3.5. If aX is a compactification of X, then there is a continuous function $\pi : \beta X \to aX$ such that $\pi \circ \beta = a$. Therefore $\beta X \geq aX$.

Proof. The embedding $a : X \to aX$ is a continuous function from X to the compact space aX. Then, $\tilde{a} : \beta X \to aX$ is a continuous function such that $\tilde{a} \circ \beta = a$. □

Proposition 3.6. *[37] If aX is a compactification of X such that for each continuous function $f : X \to [0,1]$ there is a continuous function $f_a : aX \to [0,1]$ such that $f = f_a \circ a$, then $aX \geq \beta X$.*

Proof. Define $h : aX \to [0,1]^{C(X,[0,1])}$ by

$$\pi_f(h(p)) = f_a(p)$$

for every $f \in C(X,[0,1])$ and every $p \in aX$. Then h is continuous.

Let $x \in X$ and $f \in C(X,[0,1])$. Then

$$\pi_f(h(a(x))) = f_a(a(x)) = f(x) = \pi_f(\beta(x)).$$

Therefore $h(a(x))) = \beta(x)$ for every $x \in X$, and so $h[a[X]] \subseteq \beta[X]$.

We also have the following:

$$h[aX] = h[\mathrm{cl}_{aX}\, a[X]] \subseteq \mathrm{cl}\, h[a[X]] \subseteq \mathrm{cl}\, \beta[X] = \beta X.$$

We conclude that $h : aX \to \beta X$ is such that $h \circ a = \beta$. $\square$

We will say that two compactifications, aX and bX, of a space X are *equivalent* if there is a homeomorphism $h : aX \to bX$ such that $h \circ a = b$. In this case we will write $aX \equiv_X bX$. Observe that if $aX \geq bX$ and $bX \geq aX$ then aX is equivalent to bX. Indeed, there are continuous functions $f : aX \to bX$ and $g : bX \to aX$ such that $f \circ a = b$ and $a = g \circ b$. Therefore $(g \circ f) \restriction a[X] = id_{aX} \restriction a[X]$, where id_Y is the identity homeomorphism defined on a space Y. Since $a[X]$ is dense in $aX, g \circ f = id_{aX}$. In the same way, $f \circ g = id_{bX}$. Therefore $f = g^{\leftarrow}$ and f is a homeomorphism.

A subspace $X \subseteq Y$ is *C^*-embedded* in a space Y if for every continuous function $f : X \to [0,1]$, there is a continuous function $g : Y \to [0,1]$ such that $g \restriction X = f$.

Corollary 3.7. βX is the only compactification, up to equivalence, of X such that $\beta[X]$ is C^*-embedded in βX.

When dealing with compactifications, the next proposition is useful. Recall that a continuous function between topological spaces, $f : X \to Y$, is *perfect* if f is onto, closed and for every $y \in Y$, the fiber $f^{\leftarrow}[\{y\}]$ is compact.

Proposition 3.8. *[37] Let X and Y be spaces. If $f \in C(X,Y)$, S is dense in X, and $f \restriction S$ is a perfect function from S onto $f[S]$, then $f[X \setminus S] \subseteq Y \setminus f[S]$.*

Proof. Suppose that there is $x \in X \setminus S$ such that $f(x) \in f[S]$. Let $T = S \cup \{x\}$. Since $f \restriction S$ is a perfect function, $(f \restriction S)^{\leftarrow}(f(x)) = f^{\leftarrow}(f(x)) \cap S$ is a compact set which we will denote by K. Hence K is closed in T. There is an open subset U in T such that $K \subseteq U$ and $x \notin \mathrm{cl}_T\, U$. Since S is dense in X, S is dense in T. Hence $\mathrm{cl}_T\, U \cup \mathrm{cl}_T(S \setminus U) = T$ and $f(x) \in f[\mathrm{cl}_T(S \setminus U)] \subseteq \mathrm{cl}_{f[S]}\, f[S \setminus U] = f[S \setminus U]$, where the last equality holds because $f \restriction S$ is a closed function. Therefore, there exists $s \in S \setminus U$ such that $f(s) = f(x)$. Then $s \in K \setminus U$, which is a contradiction. □

Proposition 3.9. *Let X be a space. There is a set $\mathcal{K}(X)$ of compactifications of X such that any two distinct members of $\mathcal{K}(X)$ are non equivalent and any compactification of X is equivalent to some member of $\mathcal{K}(X)$.*

If X is a space, then the pair $(\mathcal{K}(X), \leq)$ is a partially ordered set.

Proposition 3.10. *[37] Let X be a space. Then every nonempty subset of $\mathcal{K}(X)$ has a least upper bound in $(\mathcal{K}(X), \leq)$.*

Proof. Let S be a nonempty subset of $\mathcal{K}(X)$. Define

$$F = \{g \circ a : g \in C(aX, [0,1]), aX \in S\}.$$

Then $F \subseteq C(X, [0,1])$. We will see that F satisfies the hypothesis 1) and 2) of the Embedding Theorem.

Take any $aX \in S$. Let x, y be distinct points of X, then $a(x) \neq a(y)$, because $a : X \to aX$ is an embedding. There is a continuous function $g \in C(aX, [0,1])$ such that $g(a(x)) \neq g(a(y))$. Then F satisfies 1).

Suppose that $x \in U \subseteq X$, where U is open in X. Again, since a is an embedding, $a(x) \in a[U] \subseteq aX$, and $a[U]$ is open in $a[X]$. There is a continuous function $g \in C(a[X], [0,1])$ such that $g(a(x)) = 0$ and $g[a[X] \setminus a[U]] \subseteq \{1\}$. Also, observe that $(g \circ a)[X \setminus U] \subseteq g[a[X] \setminus a[U]]$. Then F satisfies 2).

Now, define $e : X \to [0,1]^F$ by

$$\pi_{g \circ a}(e(x)) = g(a(x))$$

for every $x \in X$ and every $g \circ a \in F$. By the Embedding Theorem, e is an embedding. Let $eX = (\mathrm{cl}_{[0,1]^F}\, e[X], e)$. Then eX is a compactification of X.

Claim 1: eZ is an upper bound of S in $(\mathcal{K}(X), \leq)$. To see this, take an arbitrary $aX \in S$. Define $\gamma : aX \to [0,1]^{C(aX,[0,1])}$ by

$$\pi_f(\gamma(y)) = f(y).$$

By the Embedding Theorem, $\gamma[aX]$ is homeomorphic to aX.

We want to see that $eX \geq aX$. In order to achieve this, define

$$h : [0,1]^F \to [0,1]^{C(aX,[0,1])}$$

by

$$\pi_f(h(z)) = \pi_{f\circ a}(z)$$

for every $z \in [0,1]^F$ and every $f \in C(aX,[0,1])$. Then h is continuous.

If $x \in X$, then $\pi_f(h(e(x))) = \pi_{f\circ a}(e(x)) = f(a(x)) = \pi_f(\gamma(a(x)))$. Therefore $h(e(x)) = \gamma(a(x))$ for every $x \in X$, hence $h[e[X]] \subseteq \gamma[a[X]] \subseteq \gamma[aX]$ and

$$h[eX] = h[\operatorname{cl} e[X]] \subseteq \operatorname{cl} h[e[X]] \subseteq \operatorname{cl} \gamma[aX] = \gamma[aX].$$

Therefore $h \restriction eX : eX \to \gamma[aX]$ is continuous and $\gamma^{\leftarrow} \circ h \restriction eX : eX \to aX$ is also continuous. In the last paragraph we established that $h(e(x)) = \gamma(a(x))$ for every $x \in X$, hence $\gamma^{\leftarrow}(h(e(x))) = a(x)$ for every $x \in X$, this implies

$$(\gamma^{\leftarrow} \circ h \restriction eX) \circ e = a$$

Hence $eX \geq aX$.

Claim 2: eX is the least upper bound of S in $(\mathcal{K}(X), \leq)$. To see this, suppose that bX is a compactification of X such that $bX \geq aX$ for every $aX \in S$. We are going to prove that $bX \geq eX$. Indeed, for each $aX \in S$, we have $bX \geq aX$, then there is a continuous function $j_{aX} : bX \to aX$ such that $j_{aX} \circ b = a$. Now, define $h : bX \to [0,1]^F$ by

$$\pi_{g\circ a}(h(w)) = g(j_{aX}(w)),$$

for every $w \in bX$ and every $g \circ a \in F$. Then h is continuous.

Now, take $x \in X$. Then

$$\pi_{g\circ a}(h(b(x))) = g(j_{aX}(b(x))) = g(a(x)) = \pi_{g\circ a}(e(x)),$$

hence $h(b(x)) = e(x)$ for every $x \in X$. Henceforth, $h[b[X]] \subseteq e[X]$ and

$$h[bX] = h[\operatorname{cl} b[X]] \subseteq \operatorname{cl} h[b[X]] \subseteq \operatorname{cl} e[X] = eX.$$

Observe that $h \restriction bX : bX \to eX$ is a continuous function such that $h \circ b = e$. Then $bX \geq eX$.

In conclusion, eX is the least upper bound of S in $(\mathcal{K}(X), \leq)$. $\square$

We will say that $\mathcal{K}(X)$ is a *lattice* if every two compactifications have a lower bound in $(\mathcal{K}, \leq)$. If this is the case, then, by the previous theorem, every two compactifications have a greatest lower bound.

Recall that a space X is locally compact if every point has a compact neighborhood. The following proposition is well known.

Proposition 3.11. *Let A be a subspace of a locally compact space X. Then A is locally compact if and only if A is open in $\mathrm{cl}_X A$.*

Proof. Suppose that A is locally compact. Take a point $x \in A$ and let $Z = \mathrm{cl}_X A$. There is an open set $U \subseteq A$ such that $x \in U$ and $\mathrm{cl}_A U$ is compact. Therefore $\mathrm{cl}_A U$ is closed in Z and $\mathrm{cl}_A U = \mathrm{cl}_Z U$. There is an open set $V \subseteq Z$, such that $U = V \cap A$. Since A is dense in Z, we have $V \subseteq \mathrm{cl}_Z V = \mathrm{cl}_Z(V \cap A) = \mathrm{cl}_Z U = \mathrm{cl}_A U \subseteq A$. Hence $x \in V \subseteq A$ and A is open in Z.

Let A be an arbitrary subset of X. We are going to prove that $Z = \mathrm{cl}_X A$ is locally compact. Take $x \in Z$. There is an open set $U \subseteq X$ such that $x \in U$ and $\mathrm{cl}_X U$ is compact. Then $\mathrm{cl}_X U \cap Z$ is compact. Thus, $x \in U \cap Z$ and $\mathrm{cl}_Z(U \cap Z) = \mathrm{cl}_X(U \cap Z) \cap Z \subseteq \mathrm{cl}_X U \cap Z$. Therefore $\mathrm{cl}_Z(U \cap Z)$ is compact.

Now, suppose that A is open in Z. Take a point $x \in A$ and fix an open set $U \subseteq Z$ such that $x \in U$ and $\mathrm{cl}_Z U$ is compact. There is an open set $W \subseteq X$ such that $x \in W \subseteq \mathrm{cl}_Z W \subseteq U \cap A$. Therefore $\mathrm{cl}_Z W$ is compact and $\mathrm{cl}_A(W \cap A) = \mathrm{cl}_Z(W \cap A) \cap A = \mathrm{cl}_Z(W) \cap A = \mathrm{cl}_Z W$. The second equality holds because A is dense in Z. Therefore, $x \in W \cap A$ and $\mathrm{cl}_A(W \cap A)$ is compact. Hence A is locally compact. $\square$

Definition 3.12. For a space X, a compactification aX of X is a *one point compactification* of X if $|aX \setminus a[X]| = 1$.

Proposition 3.13. *[37]*

1. *If a space X has a one point compactification, then X is locally compact and not compact.*

2. *If a space X is locally compact and not compact, then X has a one point compactification aX such that $bX \geq aX$ for every compactification bX of X.*

Proof. (1). Let aX be a one point compactification of X. Then $aX \setminus a[X]$ is closed, so $a[X]$ is open. Therefore $a[X]$ is locally compact and so is X. Since $a[X]$ is a proper dense subset of $aX, a[X]$ is not compact and neither is X.

(2). Let $w \notin X$ and $Y = X \cup \{w\}$. Define

$$\tau_Y = \{W \subseteq Y : W \cap X \text{ is open in } X \text{ and if } w \in W \text{ then } X \setminus W \text{ is compact}\}.$$

It is straightforward to see that τ_Y is a compact Hausdorff topology on Y such that X is dense in Y. So $aX = (Y, id)$ is a one point compactification of X.

Let bX be a compactification of X. Define $f : bX \to aX$ by $f(b(x)) = x$ for every $x \in X$, and $f(z) = w$ for every $z \in bX \setminus b[X]$. To see that f is continuous, let $z \in bX \setminus b[X]$ and $f(z) \in W$ for some W open in aX. Since $f(z) = w, bX \setminus W$ is compact and therefore $T = bX \setminus b[X \setminus W]$ is open in bX. Observe that $z \in bX \setminus b[X] \subseteq T$. Then $f[T] \subseteq W$. So, f is continuous at z.

Since $b[X]$ is open in bX and $f[b[X]] = X$, then f is continuous in $b[X]$. Also, $f \circ b = id_X$, therefore $bX \geq aX$. □

We will say that $\mathcal{K}(X)$ is a *complete lattice* if every nonempty subset $S \subseteq \mathcal{K}(X)$ has a greatest lower bound.

Proposition 3.14. *Let X be a space. If $\mathcal{K}(X)$ has a smallest element, then $\mathcal{K}(X)$ is a complete lattice.*

Proof. Let aX be the smallest element of $\mathcal{K}(X)$. Let S be a nonempty subset of $\mathcal{K}(X)$. Let $T = \{bX : bX \leq cY \text{ for every } cY \in S\}$. Then $aX \in T$. By 3.10, T has a greatest upper bound dX in $\mathcal{K}(X)$. It is easy to see that dX is the greatest lower bound of S in $\mathcal{K}(X)$. □

We will see that if X is not locally compact, then $\mathcal{K}(X)$ is not a complete lattice.

Proposition 3.15. *[37] For a non compact space X, the following conditions are equivalent.*

1. *$\mathcal{K}(X)$ is a complete lattice.*

2. *X is locally compact.*

3. *X has a one point compactification.*

4. *For every compactification (aX, a) of X, $a[X]$ is open in aX.*

5. *There is a compactification (aX, a) of X such that $a[X]$ is open in aX.*

Proof. Suppose that $\mathcal{K}(X)$ is a complete lattice. Then, $\mathcal{K}(X)$ has a smallest element aX. We are going to prove that aZ is a one point compactification of X. Since X is not compact, $aX \setminus a[X]$ is nonempty. We will show that $|aX \setminus a[X]| \leq 1$. Suppose that p, q are points in $aX \setminus a[X]$ and define $Y = (aX \setminus \{p, q\}) \cup \{r\}$ where $r \notin Z$. Let $f : aX \to Y$ be defined by $f(z) = z$ if $z \in aX \setminus \{p, q\}$ and $f(p) = f(q) = r$. Give Y the quotient topology induced by f.

It is easy to see that Y is Hausdorff, indeed, observe that $f^{\leftarrow}[aX \setminus \{p, q\}] = aX \setminus \{p, q\}$. Therefore, $aX \setminus \{p, q\}$ is open in Y. Therefore, every two points in $aX \setminus \{p, q\}$ have disjoint neighborhoods in Y. Now if $b \in aX$, there are disjoint open sets U, V in aX such that $b \in U$ and $p, q \in V$. Therefore $b \in f[U], r \in f[V]$ and $f[U], f[V]$ are disjoint open sets in Y.

Also, observe that $f \restriction (aX \setminus \{p, q\})$ is a homeomorphism. Since $X \cap \{p, q\}$ is empty, X is a subspace of Y. Since $a[X]$ is dense in aX and f is continuous, $f[a[X]] = a[X]$ is dense in Y. Hence $bX = (Y, id_X)$ is a compactification of X and $aX \geq bX$. Since aX is the smallest element in $\mathcal{K}(X), bX \geq aX$. So, there is a homeomorphism $g : bX \to aX$, such that $g \circ id_X = a$. Observe that $g \circ f : aX \to aX$ is continuous and $g \circ f \circ a = id_X$. Then $g \circ f = id_{aX}$. So $p = (g \circ f)(p) = (g \circ f)(q) = q$. Hence $aX \setminus a[X]$ has just one point and aX is a one point compactification of X. So, 1 implies 3.

Suppose that X has a one point compactification aX. Then, $a[X]$ is open in aX. Hence 3 implies 5.

If $a[X]$ is open in some compactification aX, by 3.11, $a[X]$ is locally compact. Then, 5 implies 2.

Let (aX, a) be an arbitrary compactification of X. By Proposition 3.11, $a[X]$ is locally compact if and only if $a[X]$ is open in $\mathrm{cl}_{aX}\, a[X] = aX$. Hence 2 is equivalent to 4.

If X is locally compact, then $\mathcal{K}(X)$ has a smallest element. By Proposition 3.14, 2 implies 1. □

Corollary 3.16. A non compact, locally compact space X has a unique, up to equivalence, one point compactification.

Proof. Let aX be the one point compactification of X constructed in 3.13(2) and let bX be a one point compactification. By construction of aX, we have that there is a continuous function $f : bX \to aX$ such that $f \circ b = a$. By 3.8, $f[bX \setminus b[X]] \subseteq aX \setminus a[X]$. Therefore f is a bijection. Since bX is compact, f is closed and therefore f is a homeomorphism. Hence $bX \equiv_X aX$. □

A basic reference about the Stone-Čech compactification is [43]. Other references are [37] and [17]. A recent M.S. thesis, written in Spanish, studying different types of compactifications is [12].

3.2 The Hewitt realcompactification

Definition 3.17. A space X is called *realcompact* if there is no space Y which satisfies the following two conditions:

1. There exists an embedding $r : X \to Y$ such that $r[X] \neq \mathrm{cl}_Y\, r[X] = Y$.
2. For each continuous function $f : X \to \mathbb{R}$, there is a continuous function $g : Y \to \mathbb{R}$, such that $g \circ r = f$.

Theorem 3.18. *[17] A space X is compact if and only if X is realcompact and pseudocompact.*

Proof. If X is compact, then it is pseudocompact. If $r : X \to Y$ is an embedding, then $r(X) = \mathrm{cl}_Y\, r(X)$ and so X is realcompact.

Suppose that X is pseudocompact and not compact. Observe that $\beta : X \to \beta X$ is an embedding such that $\beta[X] \neq \mathrm{cl}_{\beta X}\, \beta[X] = \beta X$. Now take a continuous function $f : X \to \mathbb{R}$. By hypothesis, f is bounded. Therefore, there exists a compact subset $K \subseteq \mathbb{R}$ such that $f : X \to K$. By 3.2, there is a continuous function $\beta f : \beta X \to K \subseteq \mathbb{R}$, such that $\beta f \circ \beta = f$. By definition, X is not realcompact. $\square$

Theorem 3.19. *[17] A space X is realcompact if and only if X is homeomorphic to a closed subspace of a power $\mathbb{R}^\kappa$ for some cardinal κ.*

Proof. Let X be a realcompact space. Define $r : X \to \mathbb{R}^{C(X,\mathbb{R})}$ by

$$\pi_f(r(x)) = f(x),$$

for every $x \in X$ and every $f \in C(X, \mathbb{R})$. By the Embedding Theorem, r is an embedding. Define $Y = \mathrm{cl}_{\mathbb{R}^{C(X,\mathbb{R})}}\, r[X]$. Now, if $f : X \to \mathbb{R}$ is a continuous function, then, by definition of r, we have

$$f = (\pi_f \restriction Y) \circ r.$$

By definition of realcompact spaces, $r[X] = \mathrm{cl}\, r[X]$. Hence, X is homeomorphic to a closed subspace $r[X]$ of the Cartesian product $\mathbb{R}^{C(X,\mathbb{R})}$.

Suppose that X is a closed subspace of a power $\mathbb{R}^\kappa$ for some cardinal κ. Suppose that $r : X \to Y$ is an embedding such that $\mathrm{cl}_Y\, r[X] = Y$ and for each continuous function $f : X \to \mathbb{R}$, there is a continuous function $g : Y \to \mathbb{R}$, such that $g \circ r = f$. Since $\pi_\alpha \restriction X : X \to \mathbb{R}$ is continuous for every $\alpha < \kappa$, there is a continuous function $g_\alpha : Y \to \mathbb{R}$, such that $g_\alpha \circ r = \pi_\alpha \restriction X$, for every $\alpha < \kappa$.

We are going to prove that $r[X] = Y$. Define $F : Y \to \mathbb{R}^\kappa$ by

$$\pi_\alpha(F(y)) = g_\alpha(y),$$

for every $y \in Y$ and every $\alpha < \kappa$. Then F is continuous. Now, take $x \in X$. Then

$$\pi_\alpha(F(r(x))) = g_\alpha(r(x)) = \pi_\alpha(x).$$

Therefore $F(r(x)) = x$ for every $x \in X$. We also have

$$F[Y] = F[\operatorname{cl} r[X]] \subseteq \operatorname{cl} F[r[X]] \subseteq \operatorname{cl} X = X.$$

Therefore $F : Y \to X$. Observe that $r(F(r(x))) = r(x)$ for every $x \in X$. Then $r \circ F : Y \to Y$, when restricted to $r[X]$ coincides with $id_{r[X]}$. Since $r[X]$ is dense in Y, $r \circ F = id_Y$. Therefore, $Y = r[F[Y]] \subseteq r[X]$. Hence $Y = r[X]$. By definition 3.17, X is realcompact. $\square$

Corollary 3.20. Every closed subspace of a realcompact space is realcompact.

Corollary 3.21. The arbitrary product of realcompact spaces is realcompact.

Corollary 3.22. Let X be a topological space and let $\{A_s : s \in S\}$ be a family of subspaces of X. If A_s is realcompact for every $s \in S$, then $\cap_{s\in S} A_s$ is realcompact.

Corollary 3.23. If $f : X \to Y$ is a continuous function of a realcompact space to a space Y, then for every realcompact subspace B of Y, the inverse image $f^{\leftarrow}[B]$ is realcompact.

The next theorem provides us a useful characterization of realcompact spaces.

Theorem 3.24. *[17] A space X is realcompact if and only if for every point $p \in \beta X \setminus \beta[X]$ there exists a continuous function $h : \beta X \to [0,1]$ such that $x \in h^{\leftarrow}[\{0\}] \subseteq \beta X \setminus \beta[X]$.*

Proof. Let X be a realcompact space and suppose that $x_0 \in \beta X \setminus \beta[X]$. The function $e : X \to Y = \beta[X] \cup \{x\}$ defined by $e(x) = \beta(x)$ is an embedding satisfying condition 1 in definition 3.17. Since X is realcompact, there is a continuous function $f : X \to \mathbb{R}$ not satisfying condition 2 in 3.17.

Define $g_1(x) = 1 + \max\{f(x), 0\}$ and $g_2(x) = 1 - \min\{f(x), 0\}$. Clearly, $g_i : X \to \mathbb{R}$ are continuous for $i \in \{1, 2\}$.

Suppose that there are continuous functions $G_i : Y \to \mathbb{R}$ such that $G_i \circ e = g_i$ for $i \in \{1, 2\}$. Define $F = (G_1 - G_2) : Y \to \mathbb{R}$. Now, it is easy to see that $F \circ e = f$. Then f satisfies condition 2 in 3.17, which is a contradiction.

Without loss of generality, we can suppose that g_1 does not satisfy condition 2 in 3.17. Define $g = 1/g_1 : X \to [0, 1]$. Observe that $\beta g(x) \geq 0$ for every $x \in X$.

If $\beta g(x_0) > 0$, define $h : Y \to \mathbb{R}$ by $h(x) = 1/(\beta g(x))$ for every $x \in Y$. Then $h(e(x)) = 1/(\beta g(e(x))) = 1/g(x) = g_1(x)$ for every $x \in Y$. Therefore $h \circ e = g_1$, which is a contradiction. Therefore, $\beta g(x_0) = 0$.

Now, suppose that for every point $z \in \beta X \setminus \beta[X]$ there exists a continuous function $h_z : \beta X \to [0, 1]$ such that $z \in h_z^{\leftarrow}[\{0\}] \subseteq \beta X \setminus \beta[X]$. Then $X = \bigcap\{h_z^{\leftarrow}[(0, 1]] : z \in \beta X \setminus \beta[X]\}$. By 3.22 and 3.23, X is realcompact. $\square$

Theorem 3.25. *For every space X there is a unique (up to homeomorphism) realcompact space υX which satisfies the following two conditions:*

1. *There exists an embedding $\upsilon : X \to \upsilon X$ such that $\mathrm{cl}_{\upsilon X}\, \upsilon(X) = \upsilon X$.*

2. *For every continuous function $f : X \to \mathbb{R}$ there exists a continuous function $\upsilon f : \upsilon X \to \mathbb{R}$ such that $\upsilon f \circ \upsilon = f$.*

   The space υX also satisfies the following condition:

3. *For every continuous function $f : X \to Y$ of X to a realcompact space Y, there is a continuous function $\tilde{f} : \upsilon X \to Y$ such that $\tilde{f} \circ \upsilon = f$.*

Proof. Let $\alpha\mathbb{R}$ be the one point compactification of $\mathbb{R}$.

Let $f : X \to \mathbb{R}$ be a continuous function and take the only continuous function $\beta f : \beta X \to \alpha\mathbb{R}$ such that $\beta f \circ \beta = f$. Observe that $\beta[X] \subseteq (\beta f)^{\leftarrow}[\mathbb{R}]$. Define $\upsilon X = \bigcap\{(\beta f)^{\leftarrow}[\mathbb{R}] : f \in C(X, \mathbb{R})\}$.

The function $\upsilon : X \to \upsilon X$, defined by $\upsilon(x) = \beta(x)$ for every $x \in X$, is a embedding such that $\mathrm{cl}_{\upsilon X}\, \upsilon[X] = \upsilon X$.

Now, if $f_0 : X \to \mathbb{R}$ is a continuous function, $\upsilon X \subseteq (\beta(f_0))^{\leftarrow}[\mathbb{R}]$. Therefore, $\beta(f_0) \restriction \upsilon X : \upsilon X \to \mathbb{R}$. Define $\upsilon f = \beta(f_0) \restriction \upsilon X$. It is easy to see that $\upsilon f \circ \upsilon = f$. Hence υX satisfies conditions 1 and 2.

To see that υX satisfies condition 3, let Y be a realcompact space and $f : X \to Y$ be a continuous function. There is an embedding $e : Y \to \mathbb{R}^\kappa$ such that $e[Y]$ is closed in $\mathbb{R}^\kappa$ for some cardinal κ.

Observe that for each $\alpha < \kappa, \pi_\alpha \circ e \circ f : X \to \mathbb{R}$ is a continuous function. Then, $\upsilon(\pi_\alpha \circ e \circ f) : \upsilon X \to \mathbb{R}$ is a continuous function such that $\upsilon(\pi_\alpha \circ e \circ f) \circ \upsilon = \pi_\alpha \circ e \circ f$. Then we can define $F : \upsilon X \to \mathbb{R}^\kappa$ by

$$\pi_\alpha(F(p)) = (\upsilon(\pi_\alpha \circ e \circ f))(p)$$

for every $p \in \upsilon X$ and every $\alpha < \kappa$.

If $x \in X$, then

$$\pi_\alpha F((\upsilon(x))) = \upsilon(\pi_\alpha \circ e \circ f)(\upsilon(x)) = \pi_\alpha(e(f(x))).$$

Therefore $F(\upsilon(x)) = e(f(x))$ for every $x \in X$ and $F[\upsilon[X]] \subseteq e[Y]$. Hence

$$F[\upsilon X] = F[\operatorname{cl} \upsilon[X]] \subseteq \operatorname{cl} F[\upsilon[X]] \subseteq \operatorname{cl} e[Y] = e[Y].$$

Then $F : \upsilon X \to e[Y]$. Define $G = e^{\leftarrow} \circ F : \upsilon X \to Y$. Now it is easy to see that $G \circ \upsilon = f$.

Suppose that $\upsilon_1 X$ is a realcompact space that satisfies conditions 1 and 2. As in the proof for υX, we can see that $\upsilon_1 X$ satisfies condition 3. Therefore, there are continuous functions $f : \upsilon X \to \upsilon_1 X$ and $g : \upsilon_1 X \to \upsilon X$ such that $f \circ \upsilon = \upsilon_1$ and $g \circ \upsilon_1 = \upsilon$. Therefore, $f \circ g \restriction \upsilon_1[X] = id_{\upsilon_1 X} \restriction \upsilon_1[X]$. Since $\upsilon_1[X]$ is dense in $\upsilon_1 X$, $f \circ g = id_{\upsilon_1 X}$. Analogously, $g \circ f = id_{\upsilon X}$. Therefore $f = g^{\leftarrow}$ and f is a homeomorphism. □

The space υX is called the *Hewitt realcompactification* of X.

Another way to see υX is the following.

Corollary 3.26. Let X be a space. Then

$$\upsilon X \cong \cap\{T \subseteq \beta X : T \text{ is realcompact and } \beta[X] \subseteq T\}.$$

Proof. Define

$$\upsilon_1 X = \cap\{T \subseteq \beta X : T \text{ is realcompact and } \beta[X] \subseteq T\}.$$

By 3.22, $\upsilon_1 X$ is realcompact. Let $\upsilon_1 : X \to \upsilon_1 X$ be defined by $\upsilon_1(x) = \beta(x)$ for every $x \in X$. Then υ_1 is an embedding such that $\operatorname{cl}_{\upsilon_1 X} \upsilon_1[X] = \upsilon_1 X$.

Let $f : X \to \mathbb{R}$ be a continuous function and take the only continuous function $\beta f : \beta X \to \alpha\mathbb{R}$ such that $\beta f \circ \beta = f$. Observe that $\beta[X] \subseteq (\beta f)^{\leftarrow}[\mathbb{R}]$. By 3.23, $(\beta f)^{\leftarrow}[\mathbb{R}]$ is realcompact. Then $\upsilon_1 X \subseteq (\beta f)^{\leftarrow}[\mathbb{R}]$.

Therefore, $\beta f \restriction \upsilon_1 X : \upsilon_1 X \to \mathbb{R}$.

Let $\upsilon_1 f = \beta f \restriction \upsilon_1 X$. It is easy to see that $\upsilon_1 f \circ \upsilon_1 = f$. Hence $\upsilon_1 X$ satisfies conditions 1 and 2 of Theorem 3.25. Hence, $\upsilon_1 X \cong \upsilon X$. □

Definition 3.27. The statement $X \subseteq Y$ is G_δ*-dense* in Y means that each nonempty G_δ-set in Y contains a point in X.

Since every nonempty G_δ subset contains a nonempty zero set and every zero set is G_δ, then $X \subseteq Y$ is G_δ-dense in Y if and only if every nonempty zero set in Y meets X.

We are going to see a characterization of Lindelöf spaces which will be useful to the subsequent theorem.

Proposition 3.28. *[39] The following are equivalent for a topological space X.*

1. *X is Lindelöf;*
2. *for every compactification bX of X and every compact $K \subseteq bX \setminus X$, there is a G_δ subset G such that $K \subseteq G \subseteq bX \setminus X$;*
3. *there is a compactification bX of X such that for every compact $K \subseteq bX \setminus X$, there is a G_δ subset G such that $K \subseteq G \subseteq bX \setminus X$.*

Proof. (1) $\Rightarrow$ (2) Assume that X be a Lindelöf space, $bX \in \mathcal{K}(X)$ and K is a compact subset of $bX \setminus X$. For every $x \in X$, there are open disjoint subsets U_x, V_x in bX such that $x \in U_x$ and $K \subseteq V_x$. Then $X \subseteq \cup\{U_x : x \in X\}$. Since X is Lindelöf, there is a countable subset $Y \subseteq X$, such that $X \subseteq \cup\{U_x : x \in Y\}$. Hence, $G = \cap\{V_x : x \in Y\}$ is a G_δ subset such that $K \subseteq G \subseteq bX \setminus X$.

Suppose (3). Let $\{V_s : s \in S\}$ be an open cover of X. For each $s \in S$, there is an open subset U_s of bX such that $V_s = U_s \cap X$. If $bX = \cup\{U_s : s \in S\}$ then there is a finite subcover of X. Suppose that $K = bX \setminus (\cup\{U_s : s \in S\})$ is nonempty. By hypothesis, there is a countable collection $\{W_n : n \in \omega\}$ of open subsets of bX such that $K \subseteq \cap_{n\in\omega} W_n \subseteq bX \setminus X$. Observe that for each $n \in \omega$, $bX \setminus W_n$ is compact and is contained in $\cup\{U_s : s \in S\}$. Therefore, there exists a finite collection $S_n \subseteq S$ such that $bX \setminus W_n \subseteq \cup\{U_s : s \in S_n\}$. Define $T = \cup\{S_n : n \in \omega\}$. It is easy to see that $X \subseteq \cup\{bX \setminus W_n : n \in \omega\} \subseteq \cup\{U_s : s \in T\}$. Observe that T is countable and $X \subseteq \cup\{V_s : s \in T\}$. Hence, X is Lindelöf. $\square$

Theorem 3.29. *Every Lindelöf space is realcompact.*

Proof. Take a Lindelöf space X. Suppose that $p \in \beta X \setminus \beta[X]$. By Proposition 3.28, there is a G_δ subset G such that such that $\{p\} \subseteq G \subseteq \beta X \setminus \beta[X]$. By Theorem 3.24, X is realcompact. $\square$

Corollary 3.30. Let X be a space, then $\upsilon[X]$ is G_δ-dense in υX.

Proof. Suppose that $f : \beta X \to [0,1]$ is a continuous function such that $Z(f) \cap \upsilon[X]$ is empty.

Then $\beta X \setminus Z(f) = f^{\leftarrow}[\mathbb{R} \setminus \{0\}]$. Since $\mathbb{R} \setminus \{0\}$ is Lindelöf, by 3.29, it is realcompact. By 3.23, $\beta X \setminus Z(f)$ is realcompact. By 3.25, $\upsilon X \subseteq \beta X \setminus Z(f)$, therefore $Z(f) \cap \upsilon X$ is empty. □

Theorem 3.31. *[37] Let X be a space and $p \in \beta X$. Then the following are equivalent.*

1. *$p \in \beta X \setminus \upsilon X$.*
2. *There is a zero set Z in βX such that $p \in Z$ and $Z \cap \beta[X] = \emptyset$.*

Proof. Suppose that $p \in \beta X \setminus \upsilon X$. By the construction of υX in 3.25, there is a continuous function $f : X \to \mathbb{R}$ such that $\beta f(p) \in \alpha\mathbb{R} \setminus \mathbb{R}$, where $\alpha\mathbb{R} = \mathbb{R} \cup \{y\}$ is the one point compactification of $\mathbb{R}, y \notin \mathbb{R}$ and $\beta f : \beta X \to \alpha\mathbb{R}$ is the unique continuous function such that $\beta f \circ \beta = f$.

It is easy to verify that the function $g : \alpha\mathbb{R} \to \mathbb{R}$, defined by $g(y) = 0$ and $g(x) = 1/x$ for every $x \in \mathbb{R}$, is continuous.

Therefore $p \in (g \circ \beta f)^{\leftarrow}[\{0\}]$. Let $Z = (g \circ \beta f)^{\leftarrow}[\{0\}]$. Since $\upsilon X \subseteq (\beta f)^{\leftarrow}[\mathbb{R}]$, $Z \subseteq \beta X \setminus \upsilon X$. Therefore, $Z \cap \beta[X] = \emptyset$ and Z is a zero set in βX.

Suppose that there is a zero set Z in βX such that $p \in Z$ and $Z \cap \beta[X] = \emptyset$. By corollary 3.30, $Z \cap \upsilon X$ is empty. Therefore $p \in \beta X \setminus \upsilon X$. □

3.3 Pseudocompact spaces

A space X is *pseudocompact* if every real-valued continuous function is bounded.

Proposition 3.32. *[23] A space X is pseudocompact if and only if X is G_δ-dense in βX.*

Proof. If X is pseudocompact, υX is pseudocompact, then υX is compact, then closed in βX. Hence $\upsilon X = \beta X$. By 3.30, X is G_δ-dense in βX.

Now, suppose that $f : X \to \mathbb{R}$ is a continuous unbounded function. Define $g(a) = \max\{1, f(a)\}$ for every $a \in X$. Then $g : X \to \mathbb{R}$ is a continuous unbounded function such that $Z(g) = \emptyset$ and $1/g : X \to [0,1]$ is continuous. There is a continuous extension $H : \beta X \to [0,1]$ such that $H \restriction X = 1/g$. Take $n \in \omega$. There is $x \in X$, such that $g(x) > n$, then $x \in H^{\leftarrow}[[0, 1/n+1]]$ and we proved that $\{H^{\leftarrow}[0, 1/m+1] : m \in \omega\}$ has the finite intersection property. Since $H^{\leftarrow}[0, 1/m+1]$ is compact for each $m \in \omega\}$, $Z(H)$ is nonempty, but, $Z(H) \cap X = \emptyset$. Therefore X is not G_δ-dense in βX □

Observe that if X is G_δ-dense in βX, then X is G_δ-dense in every compactification. Indeed, take a compactification aX of X. Then there is a continuous function $f : \beta X \to aX$ such that $f \circ \beta = a$. If $Z(g)$ is a nonempty zero set in aX, since f is onto, then $Z(g \circ f)$ is a nonempty zero set in βX. Hence, $Z(g \circ f) \cap \beta[X]$ is nonempty and there is $\beta(x) \in Z(g \circ f) \cap \beta[X]$. Therefore, $g(a(x)) = g(f(\beta(x))) = 0$. Then $Z(g) \cap a[X]$ is nonempty. With the same technique we can also prove that if aX and bX are two compactifications of X such that $aX \leq bX$ and X is G_δ-dense in bX, then X is G_δ-dense in aX.

Proposition 3.33. *Let X be a space. If $A \subseteq X$ is a pseudocompact space and $A \subseteq B \subseteq \mathrm{cl}_X A$, then B is pseudocompact.*

Proof. Suppose that $B \setminus A \neq \emptyset$ and that $f : B \to [0, \to)$ is a continuous unbounded function. For every $n \in \omega$, take $x_n \in A \cap f^{\leftarrow}[(n, \to)]$. We can do this because A is dense in B. Then $f \restriction A : A \to [0, \to)$ is a continuous unbounded function. $\square$

Definition 3.34. Let X be a space. A family $\{Y_s : s \in S\}$ of subsets of X is locally finite if for every $x \in X$, there is an open set U in X such that $x \in U$ and $\{s \in S : U \cap Y_s \neq \emptyset\}$ is finite.

Proposition 3.35. *[23] For a space X the following are equivalent:*

1. *X is pseudocompact.*
2. *Every locally finite family of nonempty open subsets of X is finite.*

Proof. Suppose that $\{U_n : n \in \omega\}$ is an infinite, locally finite family of nonempty open subsets of X. For each $n \in \omega$, take a point $x_n \in U_n$ and a continuous function $f_n : X \to \mathbb{R}$ such that $f_n(x_n) = n$ and $f_n[X \setminus U_n] \subseteq \{0\}$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \Sigma_{n\in\omega} |f_n(x)|.$$

We are going to see that f is continuous. Take $x_0 \in X$ and $\epsilon > 0$. There is an open set U in X such that $x_0 \in U$ and $M = \{n \in \omega : U \cap U_n \neq \emptyset\}$ is finite. For each $i \in M$, there is an open set $W_i \subseteq X$ such that $x_0 \in W_i$ and if $y \in W_i$, then $|f_i(y) - f_i(x_0)| < \epsilon/|M|$. Now, take a point $x \in U \cap (\cap\{W_i : i \in M\})$. Then

$$|f(x) - f(x_0)| = |\Sigma_{n\in\omega}|f_n(x)| - \Sigma_{n\in\omega}|f_n(x_0)|| =$$

$$= |\Sigma_{n\in M}|f_n(x)| - \Sigma_{n\in M}|f_n(x_0)|| = |\Sigma_{n\in M}(|f_n(x)| - |f_n(x_0)|)|$$

$$\leq \Sigma_{n\in M}|f_n(x) - f_n(x_0)| < |M| \cdot \epsilon/|M| = \epsilon.$$

Therefore f is continuous and unbounded.

Now suppose that $f : X \to \mathbb{R}$ is a continuous unbounded function. Take $x_0 \in X$ such that $f(x_0) > 0$. For each $n > 0$ take a point x_n such that $f(x_n) > \max\{n, f(x_{n-1})\}$. It is easy to see that

$$\{f^{\leftarrow}[(f(x_{2n-1}), f(x_{2n+1}))] : n \in \omega, n > 0\}$$

is an infinite, locally finite family of nonempty open subsets of X. $\square$

Definition 3.36. A subset A of a space X is *regular closed* if $\mathrm{cl}_X \,\mathrm{int}_X\, A = A$.

Corollary 3.37. If X is pseudocompact and A is regular closed in X, then A is pseudocompact.

Proof. Let $A = \mathrm{cl}_X \,\mathrm{int}_X\, A$. Suppose that $\mathcal{U} = \{U_n : n \in \omega\}$ is a locally finite family of nonempty open subsets of A. For each $n \in \omega$, let V_n be an open subset of X such that $U_n = V_n \cap A$. Since $V_n \cap A$ is nonempty, $W_n = V_n \cap \mathrm{int}_X\, A \neq \emptyset$ for each $n \in \omega$. Then $\mathcal{W} = \{W_n : n \in \omega\}$ is a family of nonempty open subsets of X. We are going to see that $\mathcal{W}$ is locally finite in X. If $x \in A$, there is an open subset U in X such that $x \in U$ and $M = \{n \in \omega : U \cap U_n \neq \emptyset\}$ is finite. If $U \cap W_m$ is nonempty, then $U \cap V_m \cap A$ is nonempty. Thus, $U \cap U_m$ is nonempty and $m \in M$. Hence $\mathcal{W}$ is locally finite in X.

Since X is pseudocompact, $\mathcal{W}$ is finite, and so is $\mathcal{U}$. $\square$

Mrówka spaces will be used to construct examples in this thesis.

Definition 3.38. A family $\mathcal{A}$ of infinite subsets of ω is *almost disjoint* if the intersection of any two distinct members of $\mathcal{A}$ is finite. A family $\mathcal{A}$ of infinite subsets of ω is *maximal almost disjoint* if for every infinite subset A of ω, there is $B \in \mathcal{A}$ such that $A \cap B$ is finite.

Example 3.39. There is a maximal almost disjoint family of cardinality 2^{ω}.

Enumerate the rationals $\mathbb{Q} = \{q_1, q_2, \ldots\}$. For each irrational number $p \in \mathbb{R} \setminus \mathbb{Q}$, let $S_p = \{q_{n_1^p}, q_{n_2^p}, \ldots\}$ be a pairwise distinct sequence of rational numbers converging to p. Define $A_p = \{n_1^p, n_2^p, \ldots\}$. Since $A_p \cap A_t$ is finite whenever $p, t \in \mathbb{R} \setminus \mathbb{Q}$ satisfy $p \neq t$, the family $\mathcal{B} = \{A_p : p \in \mathbb{R} \setminus \mathbb{Q}\}$ is almost disjoint of cardinality $|\mathbb{R}|$.

Now, let $\mathfrak{A} = \{\mathcal{A} \subseteq \mathcal{P}(\omega) : \mathcal{A}$ is an almost disjoint family on ω and $\mathcal{B} \subseteq \mathcal{A}\}$. Then $\mathfrak{A}$ is non empty.

Let $\mathfrak{C}$ be a chain in $(\mathfrak{A}, \subseteq)$. It is easy to see that $\cup\mathfrak{C}$ is an almost disjoint family on ω. By Zorn's Lemma, $\mathfrak{A}$ has a maximal element $\mathcal{C}$. Then $\mathcal{C}$ is a maximal almost disjoint family of cardinality 2^ω.

Example 3.40 (Mrówka spaces)**.** Let $\mathcal{A}$ be an almost disjoint family. Define

$$\Psi(\mathcal{A}) = \mathcal{A} \cup \omega.$$

Generate a topology on $\Psi(\mathcal{A})$ in the following way: For every $n \in \omega$, $\{n\}$ is a basic neighborhood of n in $\Psi(\mathcal{A})$. If $A \in \mathcal{A}$ and F is a finite subset of A then $\{A\} \cup (A \setminus F)$ is a basic neighborhood of A in $\Psi(\mathcal{A})$.

It is easy to see that $\Psi(\mathcal{A})$ is a Hausdorff, locally compact and zero dimensional space such that ω is dense in $\Psi(\mathcal{A})$.

Proposition 3.41. *Let $\mathcal{A}$ be an almost disjoint family. Then $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is a maximal almost disjoint family.*

Proof. Suppose that $\mathcal{A}$ is a maximal almost disjoint family and let $\mathcal{Z} = \{Z_n : n \in \omega\}$ be an infinite locally finite family of nonempty open subsets of $\Psi(\mathcal{A})$. Let $z_0 \in Z_0 \cap \omega$. Since $\mathcal{Z}$ is locally finite, $M_0 = \{n \in \omega : z_0 \in Z_n\}$ is finite.

Now, let $m > 0$ and suppose that for each $0 < q < m$ we have defined $p_q, z_q \in \omega$ and $M_q \subseteq \omega$ such that:

1. $p_q \in \omega \setminus \cup\{M_r : r < q\}$.
2. $z_q \in Z_{p_q} \cap \omega$.
3. $M_q = \{n \in \omega : z_q \in Z_n\}$.

Let $p_m \in \omega \setminus \cup\{M_q : q < m\}$. Choose $z_m \in Z_{p_m} \cap \omega$ and let $M_m = \{n \in \omega : z_m \in Z_n\}$. Then, it is easy to see that p_m, z_m and M_m satisfy conditions 1, 2, and 3.

Therefore $Z = \{z_m : m \in \omega\}$ is an infinite subset of ω.

Since $\mathcal{A}$ is a maximal almost disjoint family, there is $A \in \mathcal{A}$ such that $A \cap Z$ is infinite. Let F be a finite subset of A. Then $(A \setminus F) \cap Z$ is infinite.

Therefore $\{A\} \cup (A \setminus F)$ is a basic neighborhood of A such that $(\{A\} \cup (A \setminus F)) \cap Z_{p_m} \neq \emptyset$ for every $z_m \in (A \setminus F) \cap Z$.

Hence, $\mathcal{Z}$ is not locally finite in A, which is a contradiction. Then, every locally finite family of nonempty open subsets of $\Psi(\mathcal{A})$ is finite. By Theorem 3.35, $\Psi(A)$ is pseudocompact.

Now, suppose that $\mathcal{A}$ is an almost disjoint family not maximal. There is an infinite subset $A \subseteq \omega$, such that $A \cap B$ is finite for every $B \in \mathcal{A}$.

Hence $\mathcal{Z} = \{\{n\} : n \in A\}$ is an infinite locally finite family of open subsets in $\psi(\mathcal{A})$.

Indeed, if $B \in \mathcal{A}$, then $V = \{B\} \cup B \setminus A$ is an open subset of $\psi(\mathcal{A})$ such that $\{n \in \omega : n \in V\}$ is empty. $\square$

Chapter 4

Weakly pseudocompact spaces

Definition 4.1. [21] A space X is weakly pseudocompact if there is a compactification (aX, a) of X such that $a[X]$ is G_δ-dense in aX.

Then every pseudocompact space is weakly pseudocompact. In this chapter we will identify $a[X]$ to X whenever (aX, a) is a compactification of X.

Theorem 4.2. *[21] Every weakly pseudocompact space is Baire.*

Proof. Assume that X is G_δ-dense in the compactification aX. Let $\{U_n : n \in \omega\}$ be a collection of open dense subsets of X and let U be a nonempty open subset of X. For each $n \in \omega$, there is an open set V_n in aX, such that $U_n = V_n \cap X$ and there is an open set V in aX such that $U = V \cap X$. Then $\{V_n : n \in \omega\}$ is a countable collection of open dense subsets of aX. Since aX is Baire, $D = \cap\{V_n : n \in \omega\}$ is dense in aX. Therefore $D \cap V$ is nonempty and observe that $D \cap V$ is a G_δ subset of aX. By hypothesis, $D \cap V \cap X$ is nonempty, hence $U \cap (\cap\{U_n : n \in \omega\})$ is nonempty. $\square$

Theorem 4.3. *[21] The product of weakly pseudocompact spaces is weakly pseudocompact.*

Proof. Observe that if X_s is G_δ-dense in Y_s for each $s \in S$, then $X = \Pi_{s\in S} X_s$ is G_δ dense in $Y = \Pi_{s\in S} Y_s$. Indeed, take a countable collection $\{U_n : n \in \omega\}$ of open subsets of Y such that $G = \cap\{U_n : n \in \omega\}$ is nonempty. Let $y = \{y_s\}_{s\in S} \in G$. For each $n \in \omega$, there exists a finite subset $F_n \subseteq S$

such that for every $s \in F_n$, there is an open subset V_s^n in Y_s such that $y \in \cap_{s\in F_n} \pi_s^{\leftarrow}[V_s^n] \subseteq U_n$.

Let $s \in S$. If $s \in S \setminus (\cup_{n\in\omega} F_n)$, take $x_s \in X_S$, otherwise define $G_s = \cap\{V_s^n : s \in F_n\}$. Since $y_s \in G_s$, G_s is a nonempty G_δ subset of Y_s. By hypothesis, there is $x_s \in G_s \cap X_s$. Take $x = \{x_s\}_{s\in S}$. Now it is easy to see that $x \in G \cap X$. □

Proposition 4.4. *[21] If X is weakly pseudocompact and Lindelöf then X is compact.*

Proof. Assume that X is Lindelöf and let bX be a compactification of X. If $p \in bX \setminus X$, by 3.28, there is a G_δ subset G in bX such that $p \in G \subseteq bX \setminus X$, then X is not G_δ-dense in bX. □

The next theorem give us a characterization of weakly pseudocompactness in locally compact spaces.

Theorem 4.5. *[21] For a locally compact space X, the following statements are equivalent.*

1. *X is weakly pseudocompact;*
2. *X is G_δ-dense in its one point compactification;*
3. *X is either compact or is not Lindelöf.*

Proof. Suppose that there is a compactification bX of X such that X is G_δ-dense in bX. Let αX be the one point compactification of X. Since $\alpha X \leq bX$, X is G_δ-dense in αX. Hence, 1 implies 2.

By definition, 2 implies 1, and by Proposition 4.4, 1 implies 3.

If X is not G_δ-dense in its one point compactification, $\alpha X \setminus X$ is a G_δ subset of αX, hence X is σ-compact and X is Lindelöf. □

The next example shows that the classes of pseudocompact spaces and weakly pseudocompact spaces are different.

Example 4.6. Let $\mathcal{A}$ be an uncountable almost disjoint family on ω which is not maximal. Then the Mrówka space $\psi(A)$ is locally compact not Lindelöf. Therefore, $\psi(A)$ is weakly pseudocompact non-pseudocompact.

The next theorem studies open subsets of weakly pseudocompact spaces.

Theorem 4.7. *[17] If X is weakly pseudocompact and U is an open subspace of X, then U is either weakly pseudocompact or locally compact Lindelöf.*

Proof. If U is locally compact and not Lindelöf, then U is weakly pseudocompact.

Assume that U is not locally compact. Let bX be a compactification of X such that X is G_δ-dense in bX. Let W be an open subset of bX such that $U = W \cap X$. Define $K = bX \setminus W$ and take $p \in W \setminus U$. We are going to prove that U is G_δ dense in the quotient space $bX/(K \cup \{p\})$.

Let G be a nonempty G_δ subset of $bX/(K \cup \{p\})$ and $\pi : bX \to bX/(K \cup \{p\})$ the natural projection.

We have two cases. If $K \cup \{p\} \subseteq \pi^{\leftarrow}[G]$, then $\pi^{\leftarrow}[G] \setminus K$ is a G_δ subset in bX and contains p. There is $x \in X \cap \pi^{\leftarrow}[G] \setminus K \subseteq U$. Hence $\pi(x) \in G \cap \pi[U] = G \cap U$.

If $(K \cup \{p\}) \cap \pi^{\leftarrow}[G] = \emptyset$, then $\pi^{\leftarrow}[G] \subseteq W$. There is $x \in X \cap \pi^{\leftarrow}[G] \subseteq U$. Hence $\pi(x) \in G \cap \pi[U] = G \cap U$. □

Also, Eckertson asked the following questions.

Questions 4.8.

1. If K is compact and $X \times K$ is weakly pseudocompact, must X be weakly pseudocompact?
2. Is weak pseudocompactness an invariant of perfect maps?
3. Is weak pseudocompactness an inverse invariant of perfect open maps?
4. Can a finite product of non-compact Lindelöf spaces ever be weakly pseudocompact?
5. Are the uncountable products $\omega^{\omega_1}, \mathbb{R}^{\omega_1}$ and $\mathbb{S}^{\omega_1}$, where $\mathbb{S}$ is the Sorgenfrey line, weakly pseudocompact spaces?

In this thesis we improve some of the previous theorems. In Section 4.1, we study locally pseudocompact spaces and the concept of simple pseudocompactness is introduced, some properties of simple pseudocompact spaces and relations to weakly pseudocompactness are studied. In Section 4.2, we introduce the concept of *k-embedded subspace*. In Section 4.3, we give a partial answer to the following question: If Cone(X) is weakly pseudocompact and it is not compact, must X be weakly pseudocompact? (see Theorem 4.60 below). In Section 4.4, we give a positive answer to the question: If $X \times Z$ is weakly pseudocompact and Z is compact, must X be weakly pseudocompact? And using this result, we found weakly pseudocompact spaces whose Hewitt realcompactifications are not weakly pseudocompact. And, in Section 4.5, we prove the main result of this chapter by giving a characterization of weakly pseudocompact spaces in terms of regular subrings of $C^*(X)$.

4.1 Locally pseudocompact spaces

A space X is *locally pseudocompact* if every point of X has a pseudocompact neighborhood.

Observe that if $\mathcal{P}$ is a topological property which is inherited by regular closed subsets and such that every pseudocompact space with $\mathcal{P}$ is compact, then every locally pseudocompact space with $\mathcal{P}$ is locally compact.

There are zero-dimensional countably compact spaces which are not locally compact; an example is provided by that given by Frolík see [19, Example 3.10.19]:

Example 4.9. Let $f : [\beta\omega]^\omega \to \beta\omega$ be a function such that $f(A)$ is an accumulation point of A in $\beta\omega$ for each $A \in [\beta\omega]^\omega$. Define $X_0 = \omega$ and

$$X_\alpha = (\cup_{\gamma<\alpha} X_\gamma) \cup f[[\cup_{\gamma<\alpha} X_\gamma]^\omega]$$

if $0 < \alpha < \omega_1$. Observe that $X = \cup_{\alpha<\omega_1} X_\alpha$ is countably compact and its cardinality is less than or equal to 2^ω. Since all infinite closed subsets of $\beta\omega$ have cardinality 2^{2^ω} (Theorem 3.3, [43]), X is countably compact and it is not locally compact.

Definition 4.10. A subset B of a space X is called *bounded* in X if for every continuous function $f : X \to \mathbb{R}$, $f[B]$ is bounded in $\mathbb{R}$. And a space X is *locally bounded* if every point has a bounded neighborhood in X.

The following lemma is a characterization of the bounded subsets of a space X.

Lemma 4.11. *[21] For $A \subseteq X$, the following are equivalent:*

1. *A is bounded in X and*
2. *$\mathrm{cl}_{\beta X} A \subseteq \upsilon X$.*

Proof. Suppose that there exists $p \in \mathrm{cl}_{\beta X} A \setminus \upsilon X$. By 3.31, there is a continuous function $f : \beta X \to [0,1]$ such that $f(p) = 0$ and $f(x) > 0$ for every $x \in X$. Therefore $1/f \restriction X : X \to [0,1]$ is a continuous function. For each $n \in \omega$, take a point $x_n \in A \cap f^{\leftarrow}[[0,1/n)]$. We can see that $1/f(x_n) > n$. Then $1/f$ is unbounded on A.

Suppose that $\mathrm{cl}_{\beta X} A \subseteq \upsilon X$. Let $f : X \to \mathbb{R}$ be a continuous function. There exists a continuous function $\upsilon f : \upsilon X \to \mathbb{R}$ such that $\upsilon f \restriction X = f$. Then $\upsilon f \restriction \mathrm{cl}_{\beta X} A : \mathrm{cl}_{\beta X} A \to \mathbb{R}$ is a continuous function. Since $\mathrm{cl}_{\beta X} A$ is compact, $\upsilon f \restriction \mathrm{cl}_{\beta X} A$ is bounded. Therefore $f \restriction A = \upsilon f \restriction A$ is bounded. $\square$

A pseudocompact space X is a *pseudocompactification* of a space Y if there is an embedding $\lambda : Y \to X$ such that $\lambda[Y]$ is dense in X.

Definition 4.12. [4] For a space X define $\alpha_0 X = X \cup (\beta X \setminus \upsilon X)$.

Theorem 4.13. *[4] The space $\alpha_0 X$ is a pseudocompactification of X.*

Proof. Suppose that there is a zero set $Z \subseteq \beta(\alpha_0 X) = \beta X$ such that $Z \cap \alpha_0 X = \emptyset$. Then $Z \subseteq \upsilon X \setminus X$. Hence Z is a zero set in υX such that $Z \cap X = \emptyset$, by 3.30, $Z = \emptyset$. Therefore, every nonempty zero set in $\beta(\alpha_0 X)$ meets $\alpha_0 X$, by 3.32, $\alpha_0 X$ is pseudocompact. $\square$

Definition 4.14. [29] For a space X define $\zeta X = X \cup (\beta X \setminus \operatorname{int}_{\beta X} \upsilon X)$.

Theorem 4.15. *[29] The space ζX is a pseudocompactification of X.*

Proof. Observe that $\alpha_0 X \subseteq \zeta X \subseteq \beta X$. By 3.33, ζX is pseudocompact. $\square$

There is a nice characterization of locally pseudocompact spaces.

Theorem 4.16. *[15, Dorantes, Tamariz] For a space X, the following statements are equivalent:*

1. *X is locally pseudocompact;*
2. *$X \subseteq int_{\beta X} \upsilon X$;*
3. *X and $\beta X \setminus \operatorname{int}_{\beta X} \upsilon X$ are disjoint;*
4. *$\zeta X \setminus X$ is compact;*
5. *X has a pseudocompactification with compact remainder;*
6. *X is open in some pseudocompactification;*
7. *X is locally bounded;*
8. *there is a locally compact space Y such that $X \subseteq Y \subseteq \upsilon X$;*
9. *X is an open subset of a pseudocompact space.*

Proof. (6) $\Rightarrow$ (1). Let bX be a pseudocompactification of X with a closed remainder and let $x \in X$. There is a continuous function $f : bX \to [0,1]$ such that $f(x) = 0$ and $f[bX \setminus X] \subseteq \{1\}$. The set $C = f^{\leftarrow}[[0,1/2)]$ is open in bX and $\operatorname{cl}_{bX} C$ is pseudocompact because pseudocompactness is inherited by regular closed subsets. Since $\operatorname{cl}_X C = \operatorname{cl}_{bX} C \cap X$ and $\operatorname{cl}_{bX} C \subseteq X$, then C is an open neighborhood of x with pseudocompact closure in X.

It is easy to see that (1) $\Rightarrow$ (7),(5) $\Rightarrow$ (6), (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (8) and (6) $\Leftrightarrow$ (9).

(7) $\Rightarrow$ (2): Let $x \in X$. There exists an open subset $U \subseteq X$ such that $U \subseteq N \subseteq X$ and N is bounded in X. By lemma 4.11, $\mathrm{cl}_{\beta X} U \subseteq \upsilon X$. Observe that there is an open subset W in βX, such that $U = W \cap X$, and $W \subseteq \mathrm{cl}_{\beta X} W = \mathrm{cl}_{\beta X} U \subseteq \upsilon X$. Therefore, $x \in W \subseteq \mathrm{int}_{\beta X} \upsilon X$. Hence, $X \subseteq \mathrm{int}_{\beta X} \upsilon X$.

(3) $\Rightarrow$ (4). Observe that $\beta X \setminus \mathrm{int}_{\beta X} \upsilon X$ is compact and $\zeta X \setminus X = \beta X \setminus \mathrm{int}_{\beta X} \upsilon X$.

(4) $\Rightarrow$ (5). By 4.15, ζX is a pseudocompactification of X, by hypothesis, the remainder $\zeta X \setminus X$ is compact. $\square$

As a consequence of the previous theorem and Theorem 4.7, we obtain Corollary 3.6 in [35]:

Corollary 4.17. If X is locally pseudocompact, then X is weakly pseudocompact or locally compact Lindelöf.

The next corollary improves Proposition 1.1 in [17].

Corollary 4.18. If X is locally pseudocompact, then X is weakly pseudocompact if and only if either X is compact or X is not Lindelöf

Proof. If X is locally pseudocompact non-Lindelöf, then X is weakly pseudocompact (Corollary 4.17). If X is weakly pseudocompact Lindelöf, by Corollary 4.4, X is compact. $\square$

Example 4.19. [15, Dorantes, Tamariz] A weakly pseudocompact space which is zero-dimensional, locally pseudocompact and it is not locally compact: Let $\mathcal{A}$ be an almost disjoint family in ω which is not maximal and of cardinality $> \aleph_0$. Let S be a zero-dimensional countably compact space which is not locally compact (see Example 4.9). Let $X = \mathcal{A} \cup (\omega \times S)$. We will consider the topology τ in X generated by the following system of basic neighborhoods. A basic neighborhood of a point $A \in \mathcal{A}$ is of the form $\{A\} \cup (B \times S)$ where $B \subseteq A$ and $A \setminus B$ is finite. A basic neighborhood of a point $(n, s) \in \omega \times S$ is of the form $\{n\} \times U$ where U is an open subset of S with $s \in U$. Then (X, τ) is a zero-dimensional, locally pseudocompact, non-pseudocompact, non-Lindelöf space which is not locally compact. By Corollary 4.18, (X, τ) is weakly pseudocompact.

A compactification bX of a space X is *simple* if there is a compact subset $K \subseteq \beta X \setminus X$ such that bX is equivalent to $\beta X/K$.

Definition 4.20. [15, Dorantes, Tamariz] A space X is *simple pseudocompact* if X is G_δ-dense in some simple compactification.

We will say that $(\mathcal{K}(X), \leq)$ is a b-lattice if the simple compactifications of X are dense in $(\mathcal{K}(X), \leq)$; that is, if for each $bX \in \mathcal{K}$, there is a compact set $K \subseteq \beta X \setminus X$ such that $\beta X/K \leq bX$. Observe that if X is locally compact then $\mathcal{K}(X)$ is a b-lattice.

Proposition 4.21. [15, Dorantes, Tamariz] If $\mathcal{K}(X)$ is a b-lattice, then X is weakly pseudocompact if and only if X is simple pseudocompact.

Proof. If X is weakly pseudocompact, then X is G_δ-dense in some compactification Y. Then, there is a compact subset $K \subseteq \beta X \setminus X$ such that $\beta X/K \leq Y$. Therefore, X is G_δ-dense in $\beta X/K$. The inverse implication is obvious. □

Theorem 4.22. *[15, Dorantes, Tamariz] Let X be a non-compact space. The space X is simple pseudocompact if and only if there exists a compact $K \subset \beta X \setminus X$ such that K is not a G_δ-subset of βX and if $\emptyset \neq H \subseteq \beta X$ is a G_δ-subset in βX such that $H \cap K = \emptyset$, then $H \cap X \neq \emptyset$.*

Proof. First we are going to prove the necessity. There is a compact set $K \subset \beta X \setminus X$ such that X is G_δ-dense in $Y = \beta X/K$. Let $\pi : \beta X \to Y$ be the natural projection. If K is a G_δ-subset of βX, then $Y \setminus \pi[\beta X \setminus K]$ is a nonempty G_δ-subset in Y whose intersection with X is empty. Then K is not a G_δ-subset of βX. Let $H \subseteq \beta X$ be a nonempty G_δ-subset disjoint from K. Then $\pi[H]$ is a G_δ-subset in Y. Hence, $\pi[H] \cap X$ is nonempty and therefore H intersects X.

Now we are going to prove the sufficiency. Let $K \subseteq \beta X \setminus X$ be a compact subspace that satisfies the hypotheses. Let $\pi : \beta X \to Y = \beta X/K$ be the natural projection and $U \subseteq Y$ a nonempty G_δ-subspace. We have two choices: $K \subset \pi^{\leftarrow}[U]$ or $\pi^{\leftarrow}[U] \cap K = \emptyset$. In either case the set $H_K = \pi^{\leftarrow}[U] \setminus K$ is a nonempty G_δ-subset such that $H_K \cap K = \emptyset$. By hypothesis, $H_K \cap X \neq \emptyset$, hence $U \cap X \neq \emptyset$. □

Corollary 4.23. If X is simple pseudocompact, then X is locally pseudocompact.

Proof. Let $K \subseteq \beta X \setminus X$ be a compact subspace that satisfies the hypotheses of the last theorem and take $y \in \beta X \setminus K$. Take a G_δ-subset $H \subseteq \beta X$ such that $y \in H$. Then $H \setminus K$ is a nonempty G_δ-subset whose intersection with K is empty. Hence, $(H \setminus K) \cap X \neq \emptyset$ and $H \cap X \neq \emptyset$. Therefore, $\beta X \setminus K \subseteq \upsilon X$.

Since $X \subseteq \beta X \setminus K$, then $X \subseteq \operatorname{int}_{\beta X} \upsilon X$. Now, by applying Theorem 4.16 we finish our proof. $\square$

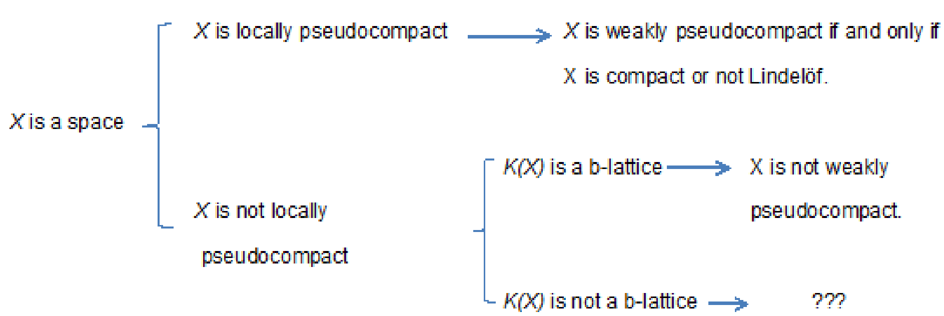
Thus, ω^{ω_1}, $\mathbb{R}^{\omega_1}$ and $\mathbb{S}^{\omega_1}$ are not simple pseudocompact. As a consequence of the next corollary we obtain Theorem 3.2 in [17].

Corollary 4.24. If X is not locally pseudocompact and $\mathcal{K}(X)$ is a b-lattice, then X is not weakly pseudocompact.

We know that $X = D(\omega_1)^\omega$ is weakly pseudocompact. The absolute $E(X)$ is a realcompact space which is not locally compact and so, it is not locally pseudocompact. By [17, Theorem 3.3], $\mathcal{K}(E(X))$ is a b-lattice. Applying the previous corollary, we conclude that $E(X)$ is not weakly pseudocompact (see Example 3.5 in [17]).

Corollary 4.24 does not help us to answer question 4.8.(5) because, as we are going to see below $\mathcal{K}(\omega^{\omega_1})$, $\mathcal{K}(\mathbb{R}^{\omega_1})$ and $\mathcal{K}(\mathbb{S}^{\omega_1})$ are not b-lattices.

What we already have is summarized in the following diagram.



A simple pseudocompact space is locally pseudocompact and weakly pseudocompact, but must a weakly pseudocompact locally pseudocompact space be simple pseudocompact? The answer is in the affirmative.

Theorem 4.25. *[15, Dorantes, Tamariz] If X is pseudocompact and U is an open subspace of X such that $X \subseteq \beta U$. Then, U is either simple pseudocompact or locally compact Lindelöf.*

Proof. If U is locally compact and it is not Lindelöf, then U is G_δ-dense in its one point compactification. Then U is simple pseudocompact.

Now, suppose that U is not locally compact. Let $W \subseteq \beta X = \beta U$ be an open subspace such that $U = W \cap X$ and let $K = \beta U \setminus W$. Take a point $p \in W \setminus U$. Since X is pseudocompact, X is G_δ-dense in βX. So, $\{p\}$ is not a G_δ-subset of $\beta X = \beta U$. Then, U is G_δ-dense in $\beta U/(K \cup \{p\})$. $\square$

Corollary 4.26. Let X be a topological space. Then the following are equivalent.

1. X is simple pseudocompact;
2. X is either compact or locally pseudocompact non-Lindelöf;
3. X is locally pseudocompact weakly pseudocompact.

Proof. By definition and Corollary 4.23, (1) $\Rightarrow$ (3).

Now, suppose that X is locally pseudocompact and non-Lindelöf. Let $\zeta X = X \cup (\beta X \setminus \mathrm{int}_{\beta X}\, \upsilon X)$. By Theorem 4.15, ζX is pseudocompact. Observe that X is open in cX and the last theorem applies. Since X is not Lindelöf, X is simple pseudocompact. This proves (2) $\Rightarrow$ (1).

By Corollary 4.18, it follows that (2) is equivalent to (3). $\square$

So, the free topological sum of ω_1 copies of $\mathbb{R}$ is simple pseudocompact. The Sorgenfrey line is not simple pseudocompact because it is not locally pseudocompact. Moreover, if $\mathcal{A}$ is a non-maximal almost disjoint family on ω of uncountable cardinality, then the Mrówka-Isbell space $\psi(\mathcal{A})$ is a simple pseudocompact space which is not pseudocompact.

When dealing with perfect functions, local pseudocompactness behaves similarly to local compactness.

Theorem 4.27. *Let $f : X \to Y$ be a perfect function between topological spaces. If X is locally pseudocompact then Y is locally pseudocompact.*

Proof. Let X be a locally pseudocompact space. Take $y \in Y$. For every $x \in f^{\leftarrow}[\{y\}]$, there is an open subset $U_x \subseteq X$ such that $x \in U_x$ and $\mathrm{cl}_X\, U_x$ is pseudocompact. Then there is a finite subset $\{x_1, \ldots, x_n\} \subseteq f^{\leftarrow}[\{y\}]$ such that $f^{\leftarrow}[\{y\}] \subseteq \cup_{i=1}^{n} U_{x_i} = W$. Then $\mathrm{cl}_X\, W$ is pseudocompact and $y \in Y \setminus f[X \setminus W] \subseteq f[\mathrm{cl}_X\, W]$. Since pseudocompactness is hereditary to regular closed sets, $\mathrm{cl}_Y(Y \setminus f[X \setminus W])$ is pseudocompact. Therefore, Y is locally pseudocompact. $\square$

Corollary 4.28. Let $f : X \to Y$ be a perfect function between topological spaces. If X is simple pseudocompact then Y is simple pseudocompact.

In the last theorem, we cannot drop the condition that f is closed. Example 3.1 in [17] will help us to see this:

Example 4.29. [15, Dorantes, Tamariz] Consider the spaces $Y = \omega_\omega$ and $X = (\omega_\omega \times \{\omega\}) \cup (\cup_{n<\omega}(\omega_n \times \{n\}))$ as a subspace of $K = (\omega_\omega + 1) \times (\omega + 1)$. The projection $\pi : X \to Y$ defined as $\pi(\alpha, \beta) = \alpha$ is open, continuous and has compact fibers. The space X is locally compact non-Lindelöf; then, it is simple pseudocompact. But Y is Lindelöf non-compact, therefore it is not weakly pseudocompact.

Theorem 4.30. *Let $f : X \to Y$ be an open perfect function from X onto a locally pseudocompact space Y. Then X is locally pseudocompact.*

Proof. Let $x \in X$. There is an open set $U \subseteq Y$ such that $f(x) \in U$ and $\mathrm{cl}_Y U$ is pseudocompact. There is an open set $V \subseteq X$ such that $x \in V$ and $f[V] \subseteq U$. Then $Z = \mathrm{cl}_Y f[V]$ is pseudocompact. We are going to prove that

$$g = f \restriction f^{\leftarrow}[Z]$$

is an open, perfect function onto its image.

If W is an open subset of $f^{\leftarrow}[Z]$, then there is an open subset $W' \subseteq X$ such that $W = W' \cap f^{\leftarrow}[Z]$. Then $f[W] = f[W'] \cap Z$. Indeed, if $z \in f[W'] \cap Z$, there exists $x \in W'$ such that $f(x) = z$. Then $x \in W' \cap f^{\leftarrow}[Z]$, and $z \in f[W]$. Hence g is open. Since $f^{\leftarrow}[Z]$ is closed, g is perfect onto its image. By [19, 3.10.H], $f^{\leftarrow}[Z]$ is pseudocompact. Observe that $x \in V \subseteq \mathrm{cl}_X V \subseteq f^{\leftarrow}[Z]$. Since $\mathrm{cl}_X V$ is a regular closed subset of $f^{\leftarrow}[Z]$, it is pseudocompact. In conclusion, X is locally pseudocompact. □

Corollary 4.31. Let $f : X \to Y$ be an open perfect function from X onto a simple pseudocompact space Y. Then X is simple pseudocompact.

We know that the product of two pseudocompact spaces is not necessarily pseudocompact. Thus, the following proposition is interesting.

Proposition 4.32. [15, Dorantes, Tamariz] Let Z be a locally compact space. If X is simple pseudocompact and is not compact, then $X \times Z$ is simple pseudocompact and is not compact. If, in addition, Z is Lindelöf, then the converse is also true.

Proof. In this proof, Corollary 4.26 will be used.

Suppose that Z is locally compact and X is simple pseudocompact and it is not compact. Then $X \times Z$ is locally pseudocompact and it is not Lindelöf; so, $X \times Z$ is simple pseudocompact and non-compact.

Now assume that $X \times Z$ is simple pseudocompact and that Z is Lindelöf. Thus, X is locally pseudocompact. If X is Lindelöf, then $X \times Z$ is Lindelöf because Z is σ-compact. Since $X \times Z$ is weakly pseudocompact, $X \times Z$ is compact and this leads to a contradiction. Then X is not Lindelöf. Therefore X is simple pseudocompact. □

Proposition 4.33. Let $X = \Pi_{s\in S} X_s$ be a topological product. If X is locally pseudocompact then:

1. all but finitely many X_s are pseudocompact; and
2. each X_s is locally pseudocompact.

   In addition, if X is a non-compact simple pseudocompact space, then we have (1), (2) and
3. some X_s is not Lindelöf.

Proof. Suppose that X is locally pseudocompact. Since the projections are open, each X_s is locally pseudocompact. Take $x \in X$. There is an open subset $U \subseteq X$ such that $x \in U$ and $\mathrm{cl}_X U$ is pseudocompact. Then U contains a basic open set of the form

$$B = \cap_{i=1}^{n} \pi_{s_i}^{\leftarrow}[U_{s_i}]$$

where U_{s_i} is an open subset of X_{s_i} for all $i \in \{1, ..., n\}$.

Therefore, $\mathrm{cl}_X B = \cap_{i=1}^{n} \pi_{s_i}^{\leftarrow}[\mathrm{cl}_{X_s} U_{s_i}] \subseteq \mathrm{cl}_X U$ is pseudocompact. Hence, if $s \neq s_i$, for $i \in \{1, \ldots, n\}$, $X_s = \pi_s[\mathrm{cl}_X B]$ is pseudocompact.

Now, assume that X is simple pseudocompact and non-compact. Then, X is locally pseudocompact and is not Lindelöf (Corollary 4.26). So, (1) and (2) hold. Moreover, if each X_s were Lindelöf, then each X_s would be σ-compact and all but finitely many of X_s would be compact. Then X would be Lindelöf. □

Corollary 4.34. Let γ and κ be two infinite cardinals. Then $\mathbb{R}^\kappa$ and γ^κ are not locally pseudocompact.

It is easy to prove the following proposition.

Proposition 4.35. Let $\mathcal{X}$ be a family of topological spaces. Then the free topological sum of the elements of $\mathcal{X}$, $\oplus\mathcal{X}$, is locally pseudocompact, if and only if each $X \in \mathcal{X}$ is locally pseudocompact.

The following propositions are interesting for themselves.

Proposition 4.36. Let Y be a compactification of X and let $\tilde{t} : \beta X \to Y$ be the continuous extension of the identity function $id_X : X \to X$. If G is a G_δ-subset of βX such that $G \subseteq \beta X \setminus X$ and X is G_δ-dense in Y, then $\tilde{t}[\beta X \setminus G] = Y$.

Proposition 4.37. Let X be a non-compact space. Then the following are equivalent:

1. X is weakly pseudocompact;
2. there is a compactification Y of X such that for every point $y \in Y \setminus X$, if $G \subseteq \beta X$ and G is a G_δ-subspace such that $f^{\leftarrow}[y] \subseteq G$, then $G \cap X \neq \emptyset$, where f is the only continuous extension of the identity.

Proof. Clearly (1) implies (2).

Let us suppose the claim in (2). Let Y be a compactification of X which satisfies (2). If G is a nonempty G_δ-subspace of Y and $y \in G$, then $f^{\leftarrow}[y] \subseteq f^{\leftarrow}[G]$. Hence, $f^{\leftarrow}[G] \cap X \neq \emptyset$. Therefore, $G \cap X \neq \emptyset$. □

4.2 k-embedded subspaces

In this section we are going to prove that the partial ordered set of compactifications $(\mathcal{K}(\omega^{\omega_1}), \leq)$ is not a b-lattice. Also, the concept of k-embeddedness is introduced.

Proposition 4.38. If βX is zero dimensional, then every simple compactification of X is zero dimensional.

Proof. Let $K \subseteq \beta X \setminus X$ be a nonempty compact space. Assume that $\pi : \beta X \to \beta X/K$ is the canonical projection, $p \in \beta X/K$, and $U \subseteq \beta X/K$ is an open subspace such that $p \in U \subseteq \beta X/K$.

Suppose that $K = \pi^{\leftarrow}[p]$. There is a clopen subset $A \subseteq \beta X$ such that $K \subseteq A \subseteq \pi^{\leftarrow}[U]$. Hence $\pi[A]$ is a clopen subspace of $\beta X/K$ such that $p \in \pi[A] \subseteq U$.

Now suppose that $K \cap \pi^{\leftarrow}[p] = \emptyset$. Let $\{x\} = \pi^{\leftarrow}[p]$. There is a clopen subset $A \subseteq \beta X$, such that $x \in A \subseteq \pi^{\leftarrow}[U] \setminus K$. Hence $\pi[A]$ is a clopen subspace of $\beta X/K$ such that $p \in \pi[A] \subseteq U$. □

Corollary 4.39. If X is a strongly zero dimensional space with a connected compactification, then $\mathcal{K}(X)$ is not a b-lattice.

Proposition 4.40. [37, 4.7(g)] X is strongly zero dimensional if and only if for every pair of disjoint zero sets A, B in X, there is a clopen set $C \subseteq X$ such that $A \subseteq C \subseteq X \setminus B$.

Recall the following theorem:

Lemma 4.41 (Arkhangel'skii's Factorization Theorem)**.** *Let D be a dense subspace of the product $X = \Pi_{s \in S} X_s$ such that each X_s has a countable network. Then, for every real-valued continuous function f on D, there* *exists a countable set $K \subseteq S$ and a continuous real-valued function h on $\pi_K[D]$ such that $f = (h \circ \pi_K) \restriction D$.*

Theorem 4.42. *If D is dense in a product of zero-dimensional spaces with countable network, then D is strongly zero-dimensional.*

Proof. Let $\{X_s : s \in S\}$ be a family of zero-dimensional spaces with countable network and let D be a dense subspace of the product $\Pi_{s\in S} X_s$. Let A, B disjoint zero sets in D. There is a continuous function $f : D \to [0,1]$ such that $A = f^{\leftarrow}(0)$ and $B = f^{\leftarrow}(1)$. By Lemma 4.41, there exist a countable subset $T \subseteq S$ and a continuous function $g : Y = \pi_T[D] \to [0,1]$ such that $f = (g \circ \pi_T) \restriction D$. Observe that Y is zero-dimensional and Lindelöf. Therefore, Y is strongly zero-dimensional. So there is a clopen subset $C \subseteq Y$ such that $g^{\leftarrow}(0) \subseteq C \subseteq Y \setminus g^{\leftarrow}(1)$. Hence $\pi_Y^{\leftarrow}[C]$ is a clopen subset of in X such that $A \subseteq \pi_Y^{\leftarrow}[C] \subseteq X \setminus B$. By Proposition 4.40, X is strongly zero-dimensional. □

Corollary 4.43. Let D be a dense subspace of ω^κ. Then, $\mathcal{K}(D)$ is b-lattice if and only if $\kappa < \omega$.

Proof. If $\kappa \geq \omega$, then the interval $[0,1]^\kappa$ is a connected compactification of the strongly zero-dimensional space D. Indeed, the space ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ and ω^κ is homeomorphic to $(\omega^\omega)^\kappa$. Now, if $\kappa < \omega$, then ω^κ is discrete and so D is locally compact, that is, $\mathcal{K}(D)$ is a b-lattice. □

Theorem 4.44. *[15, Dorantes, Tamariz] Let X be a strongly zero-dimensional space. Then $\mathcal{K}(X)$ is a b-lattice if and only if for each compactification Y of X, the closure in Y of the union of non-trivial connected components of $Y \setminus X$ is contained in $Y \setminus X$.*

Proof. Suppose that $\mathcal{K}(X)$ is a b-lattice. Let Y be a compactification of X. Let $\tilde{\beta} : \beta X \to Y$ be the continuous extension of $id_X : X \to X$. There is a compact subset K of $\beta X \setminus X$ such that $\beta X/K \leq Y$. This means that there is a continuous function $\tilde{k} : Y \to \beta X/K$ such that $\tilde{k} \restriction X$ is the identity function. Observe that each element $y \in Y$ is of the form $\tilde{\beta}(x)$, for some $x \in \beta X$, and $\tilde{k}(\tilde{\beta}(x)) = \pi(x)$, where $\pi : \beta X \to \beta X/K$ is the natural quotient map. Also, observe that $(\tilde{k} \circ \tilde{\beta}) \restriction X = \pi \restriction X$. It happens that $(\beta X/K) \setminus X$ is zero-dimensional and $\tilde{k} \restriction \tilde{k}^{\leftarrow}[(\beta X/K) \setminus \{\pi[K]\}]$ is an injective function. Then each non-trivial component in $Y \setminus X$ must be contained in $\tilde{k}^{-1}[\pi[K]]$ which is a compact subset of $Y \setminus X$. Then, $\mathrm{cl}_Y(\bigcup \mathcal{C}) \subseteq Y \setminus X$, where $\mathcal{C}$ is the collection of non-trivial connected components of $Y \setminus X$.

Now, denote by $calC$ the collection of all non-trivial connected components of $Y \setminus X$. Suppose that Y is a compactification of X and that $\mathrm{cl}_Y(\cup\mathcal{C})$ is contained in $Y \setminus X$. Let $\tilde{\beta} : \beta X \to Y$ be the continuous function which extends the identity. Let K be equal to $\tilde{\beta}^{\leftarrow}[\mathrm{cl}_Y(\bigcup\mathcal{C})]$. So K is a compact subset of $\beta X \setminus X$ and the function h from Y to $\beta X/K$ defined as $h(y) = y$ if $y \in Y \setminus \mathrm{cl}_Y(\bigcup\mathcal{C})$ and $h(y) = \pi(K)$ is continuous and $h \restriction X = id_X$. □

Corollary 4.45. [15, Dorantes, Tamariz] If X is a strongly zero dimensional space having a compactification with connected remainder, then $\mathcal{K}(X)$ is b-lattice if and only if X is locally compact.

Proof. Let Y be a compactification of X with connected remainder. If $\mathcal{K}(X)$ is a b-lattice, then there is a compact subset $K \subseteq \beta X \setminus X$ such that $Y \geq \beta X/K$. Since $\beta X/K$ is zero dimensional, $(\beta X/K) \setminus X$ is a point because it is a continuous image of a connected space. Then X is open in $\beta X/K$. □

Definition 4.46. [15, Dorantes, Tamariz] A subspace Y of a space X is said to be *k-embedded* in X if for each compactification bY of Y, there is a compactification cX of X such that $\mathrm{cl}_{cX} Y \leq bY$, that is, there is a continuous function $f : bY \to \mathrm{cl}_{cX} Y$ such that $f \restriction Y = id \restriction Y$.

Observe that if Z, Y and X are topological spaces such that Z is k-embedded in Y and Y is k-embedded in X, then Z is k-embedded in X. We also have:

Lemma 4.47. *Assume that $Z \subseteq Y \subseteq X$ and Z is k-embedded in X. Then, Z is k-embedded in Y.*

Proof. Let aZ be a compactification of Z. By hypothesis, there is a compactification bX of X such that $\mathrm{cl}_{bX} Z \leq aZ$. Observe that $cY = \mathrm{cl}_{bX} Y$ is a compactification of Y. Therefore $\mathrm{cl}_{cY} Z = \mathrm{cl}_{bX} Z \cap cY = \mathrm{cl}_{bX} Z$. Then, Z is k-embedded in Y. □

Theorem 4.48. *[15, Dorantes, Tamariz] If Y is $k-$ and C^*-embedded in X and $\mathcal{K}(X)$ is a b-lattice, then $\mathcal{K}(Y)$ is a b-lattice.*

Proof. Let bY be a compactification of Y. Since Y is k-embedded in X, there is a compactification bX of X such that $\mathrm{cl}_{bX}(Y) \leq bY$. Let $\tilde{t} : bY \to \mathrm{cl}_{bX} Y$ be a continuous function such that $\tilde{t} \restriction Y = id_Y$. Since Y is C^*-embedded in X, then $\beta Y \equiv_Y \mathrm{cl}_{\beta X} Y$. Since $\mathcal{K}(X)$ is a b-lattice, there is a compact $K \subseteq \beta X \setminus X$ such that $\beta X/K \leq bX$. So there is a continuous map $\tilde{k} : bX \to \beta X/K$ such that $\tilde{k}(x) = x$ for every $x \in X$. Moreover, $\tilde{k}[\mathrm{cl}_{bX} Y] = \mathrm{cl}_{\beta X/K} Y$.

Take $M = K \cap \mathrm{cl}_{\beta X} Y$. It happens that M is a compact subset of $\mathrm{cl}_{\beta X} Y$ and $M \cap Y = \emptyset$. So $\mathrm{cl}_{\beta X} Y/M$ is a simple compactification of Y and $\mathrm{cl}_{\beta X} Y/M \equiv_Y \mathrm{cl}_{\beta X/K} Y$. On the other hand,

$$\tilde{f} = (\tilde{k} \restriction \mathrm{cl}_{bX} Y) \circ \tilde{t} : bY \to \mathrm{cl}_{\beta X/K} Y$$

is continuous and $\tilde{f} \restriction Y = id_Y$. Therefore, $\mathcal{K}(Y)$ is a b-lattice. $\square$

Lemma 4.49. *Let X and Y be topological spaces. Let D and H be two subsets of $X \times Y$ such that $\pi_X[D]$ is a C-embedded (resp., C^*-embedded) subset of $\pi_X[H]$. Assume that $y \in Y$ is such that $\pi_X[D] \times \{y\} \subseteq H$. Then, $\pi_X[D] \times \{y\}$ is C-embedded (resp., C^*-embedded) in H. Moreover, if H is dense in $\pi_X[D] \times Y$ and $\pi_X[D]$ is k-embedded in $\pi_X[H]$, then $\pi_X[D] \times \{y\}$ is k-embedded in H.*

Proof. Let $y \in Y$ such that $\pi_X[D] \times \{y\} \subseteq H$ and let $f : \pi_X[D] \times \{y\} \to \mathbb{R}$ be a (bounded) continuous function. Let $h : \pi_X[D] \to \pi_X[D] \times \{y\}$ be the natural homeomorphism, $h(x) = (x, y)$. The function $f' : \pi_X[D] \to \mathbb{R}$ defined by $f'(x) = f((x, y)) = (f \circ h)(x)$ is (bounded) continuous on $\pi_X[D]$. So, there is a (bounded) continuous extension $g' : \pi_X[H] \to \mathbb{R}$ of f'. For each $(a, b) \in H$, we define $g(a, b) = g'(a) = (g' \circ \pi_X)((a, b))$. The function $g : H \to \mathbb{R}$ is (bounded) continuous and extends f. So if $\pi_X[D]$ is a C-embedded (resp., C^*-embedded) subset of $\pi_X[H]$, then, $\pi_X[D] \times \{y\}$ is C-embedded (resp., C^*-embedded) in H.

Now, let $b(\pi_X[D] \times \{y\})$ be a compactification of $\pi_X[D] \times \{y\}$. Since $\pi_X[D]$ and $\pi_X[D] \times \{y\}$ are homeomorphic, there is a compactification $c\pi_X[D]$ of $\pi_X[D]$ such that $c\pi_X[D] \equiv b(\pi_X[D] \times \{y\})$. Let $Z = c\pi_X[D] \times \beta Y$. Since H is dense in $\pi_X[D] \times Y$, Z is a compactification of H. On the other hand, $\mathrm{cl}_Z(\pi_X[D] \times \{y\}) = c\pi_X[D] \times \{y\}$. Then $\pi_X[D] \times \{y\}$ is k-embedded in $X \times Y$. $\square$

As a consequence of Theorem 4.48 and Lemma 4.49 we have:

Corollary 4.50. Let X and Y be topological spaces and let $D, H \subseteq X \times Y$ be such that $\pi_X[D]$ is C^* and k-embedded in $\pi_X[H]$ and H is dense in $\pi_X[D] \times Y$. Then, if $\mathcal{K}(H)$ is a b-lattice, then $\mathcal{K}(\pi_X[D])$ is a b-lattice.

Corollary 4.51. Let $\{X_s : s \in S\}$ be a collection of topological spaces. Let $T \subseteq S$ and $D \subseteq \prod_{s \in S} X_s$. Then, if $\pi_T[D]$ is C^*- and k-embedded in $\prod_{s \in T} X_s$ and $\mathcal{K}(\prod_{s \in S} X_s)$ is a b-lattice, then $\mathcal{K}(\pi_T[D])$ is a b-lattice. In particular, if $\mathcal{K}(\prod_{s \in S} X_s)$ is a b-lattice, then $\mathcal{K}(\prod_{s \in T} X_s)$ is a b-lattice.

Theorem 4.52. *[42] Let X be a space, if there is an infinite sequence $\{x_n : n \in \omega\} \subseteq \beta X \setminus X$ converging to $y \in X$, then $\mathcal{K}(X)$ is not a lattice.*

Proof. It may be assumed that if $m \neq n$ then $x_m \neq x_n$. Define

$$\mathcal{D}_1 = \{\{x\} : x \in \beta X \setminus Y\} \cup \{\{x_{2j}, x_{2j+1}\} : j \in \omega\} \text{ and}$$

$$\mathcal{D}_2 = \{\{x\} : x \in \beta X \setminus Y\} \cup \{\{x_{2j}, x_{2j-1}\} : j \in \omega\}$$

We are going to prove that the quotient space $\beta X/\mathcal{D}_1$ is Hausdorff. Let $\pi_1 : \beta X \to \beta X/\mathcal{D}_1$ be the natural projection.

Take $a, b \in \beta X \setminus (Y \cup \{y\})$. There are open disjoint sets U, V in βX such that $a \in U$ and $b \in V$. Since $Y \cup \{y\}$ is compact, $U' = U \setminus (Y \cup \{y\})$ and $V' = V \setminus (Y \cup \{y\})$ are open in βX and $\pi_1^{\leftarrow}[\pi[U']] = U', \pi_1^{\leftarrow}[\pi[V']] = V'$. Therefore, $\pi[U']$ and $\pi[V']$ are disjoint open sets in $\beta X/\mathcal{D}_1$ such that $\pi(a) \in \pi[U']$ and $\pi(b) \in \pi[V']$.

Take $x_{2j}, x_{2j+1} \in Y$. There are open disjoint sets U, V in βX such that $\{x_{2j}, x_{2j+1}\} \subseteq U$ and $y \in V$. There is $n \in \omega$ such that if $p \geq 2n$, then $x_p \in V$. Then $V' = V \setminus \{x_p : p < 2n\}$ and $U' = U \setminus \{x_p : p < 2n, p \neq 2j, 2j+1\}$ are open disjoint subsets in βX with $\pi_1^{\leftarrow}[\pi[U']] = U', \pi_1^{\leftarrow}[\pi[V']] = V'$.

Therefore, $\pi[U']$ and $\pi[V']$ are disjoint open sets in $\beta X/\mathcal{D}_1$ such that $\{x_{2j}, x_{2j+1}\} \in \pi[U']$ and $\pi(y) \in \pi[V']$. Hence, $\beta X/\mathcal{D}_1$ is Hausdorff. Since π_1 is continuous, $\beta X/\mathcal{D}_1$ is compact. Analogously, $\beta X/\mathcal{D}_2$ is compact. Observe that $\beta X/\mathcal{D}_1$ and $\beta X/\mathcal{D}_2$ are compactifications of X.

Suppose that there is a compactification cX of X such that $cX \leq \beta X/\mathcal{D}_1$ and $cX \leq \beta X/\mathcal{D}_2$. Then, there are continuous functions $g_i : \beta X/\mathcal{D}_i \to cX$ such that $g_i \circ \pi_i \restriction X = c$, for $i \in \{1, 2\}$. Since X is dense in βX, $g_1 \circ \pi_1 = g_2 \circ \pi_2$.

Now,

$$g_1(\pi_1(x_0)) = g_1(\pi_1(x_1)).$$

Suppose that $g_1(\pi_1(x_n)) = g_1(\pi_1(x_{n+1}))$. If $n = 2j$, then

$$g_1(\pi_1(x_{n+1})) = g_2(\pi_2(x_{2(j+1)-1})) = g_2(\pi_2(x_{2(j+1)})) = g_1(\pi_1(x_{n+2})).$$

If $n = 2j - 1$, then

$$g_1(\pi_1(x_{n+1})) = g_1(\pi_1(x_{2j})) = g_1(\pi_1(x_{2j+1})) = g_1(\pi_1(x_{n+2})).$$

Therefore $g_1(\pi_1(x_0)) = g_1(\pi_1(x_n))$ for every $n \in \omega$. Hence $g_1(\pi_1(x_0)) = g_1(\pi_1(y))$, but, by 3.8, $g_1(\pi_1(x_0)) \in cX \setminus c[X]$ and $g_1(\pi_1(y)) = c(y) \in c[X]$, which is a contradiction.

Hence, $\beta X/\mathcal{D}_1$ and $\beta X/\mathcal{D}_2$ don't have a lower bound in $\mathcal{K}(X)$. □

Theorem 4.53. *[42] If X is a first countable space, then X is locally compact if and only if $\mathcal{K}(X)$ is a lattice.*

Proof. If X is locally compact, $\mathcal{K}(X)$ is a complete lattice.

Suppose that X is first countable not locally compact. Then, X is not open in βX. Therefore, there is a point $x \in X \cap \mathrm{cl}_{\beta X}(\beta X \setminus X)$. Let $\{B_n : n \in \omega\}$ be a countable local base of x in X such that $B_n \supseteq B_{n+1}$ for every $n \in \omega$. For each $n \in \omega$, take an open set V_n in βX such that $B_n = V_n \cap X$ and $V_n \supseteq V_{n+1}$ for every $n \in \omega$. For each $n \in \omega$, take $x_n \in V_n \cap (\beta X \setminus X)$.

We are going to see that $\{x_n : n \in \omega\}$ converges to x. Let U, W be open sets in βX such that $x \in U$ and $\mathrm{cl}_{\beta X} U \subseteq W$. There is $n \in \omega$ such that $X \cap V_n = B_n \subseteq U \cap X$. Therefore,

$$\mathrm{cl}_{\beta X} V_n = \mathrm{cl}_{\beta X} B_n \subseteq \mathrm{cl}_{\beta X}(U \cap X) = \mathrm{cl}_{\beta X} U \subseteq W.$$

Hence, $x_m \in W$ for every $m \geq n$. We conclude that $\{x_n : n \in \omega\}$ is a sequence in $\beta X \setminus X$ converging to a point $x \in X$, by 4.52, $\mathcal{K}(X)$ is not a lattice. □

Corollary 4.54. If X is first countable space then the following are equivalent:

1. X is locally compact;
2. $\mathcal{K}(X)$ is a complete lattice;
3. $\mathcal{K}(X)$ a lattice; and
4. $\mathcal{K}(X)$ is a b-lattice.

Proof. Suppose that $\mathcal{K}(X)$ is a b-lattice. Take two compactifications aX and bX of X. There is a simple compactification cX of X such that $cX \leq aX$ and $cX \leq bX\}$ Then $S = \{dX \in \mathcal{K}(X) : dX \leq aX$ and $dX \leq bX\}$ is nonempty. Hence S has a supremum in $\mathcal{K}(X)$. Let eX be the supremum of S. It is easy to see that eX is the greatest lower bound of aX and bX in $\mathcal{K}(X)$. Therefore $\mathcal{K}(X)$ is a lattice.

If $\mathcal{K}(X)$ is a lattice, by the last theorem, X is locally compact.

The equivalence between 1 and 2 is 3.15

If X is locally compact, the one point compactification aX is a simple compactification and $aX \leq bX$ for every compactification bX, then $\mathcal{K}(X)$ is a b-lattice. □

The following result is a consequence of Corollaries 4.54 and 4.51.

Corollary 4.55. For every first countable space X, $\mathcal{K}(X^\kappa)$ is a b-lattice if and only if X is locally compact and either κ is finite or X is compact.

In particular, $\mathcal{K}(\mathbb{S}^\kappa)$ is not b-lattice for every κ, and $\mathcal{K}(\mathbb{R}^\kappa)$ is a b-lattice if and only if $\kappa < \omega$.

4.3 The geometric cone

For a space X, the geometric cone over X is $\mathrm{Cone}(X) = \{0\} \cup (X \times (0,1])$ topologized as follows: every open subset of the Tychonoff product $(X \times (0,1])$ is open in $\mathrm{Cone}(X)$; a basic open neighborhood of 0 is $\{0\} \cup (X \times (0,\epsilon))$, where $\epsilon > 0$.

Observe that X is homeomorphic to the closed subspace $X \times \{1\}$ of $\mathrm{Cone}(X)$. The following theorem is easy to prove.

Proposition 4.56. *Let X be a space. Then,*

1. *X is compact if and only if $Cone(X)$ is compact.*
2. *X is Lindelöf if and only if $Cone(X)$ is Lindelöf.*

Proof. We are going to prove 2. Suppose that $\mathrm{Cone}(X)$ is Lindelöf. Since X is homeomorphic to the closed subspace $X \times \{1\}$ of $\mathrm{Cone}(X)$, X is Lindelöf.

Suppose that X is Lindelöf. Then, $X \times \{1\}$ is Lindelöf. Let $\mathcal{U} = \{U_s : s \in S\}$ be an open cover of $\mathrm{Cone}(X)$. There is $s_0 \in S$ such that $0 \in U_{s_0}$. Hence, there is $\epsilon > 0$ such that $\{0\} \cup (X \times (0,\epsilon)) \subseteq U_{s_0}$. Observe that the subspace $X \times [\epsilon, 1]$ is Lindelöf. Thus, there is a countable subfamily $\{U_m : m \in \omega\}$ of $\mathcal{U}$ such that $X \times [\epsilon,1]$ is contained in $\cup\{U_m : m \in \omega\}$. Therefore, $\mathrm{Cone}(X) \subseteq U_{s_0} \cup (\cup\{U_m : m \in \omega\})$. □

Eckertson posed the next proposition:

Proposition 4.57. *[17, Proposition 2.4] If X is weakly pseudocompact then $Cone(X)$ is weakly pseudocompact.*

Proof. Suppose that X is a G_δ-dense subspace of Y. We are going to prove that $\mathrm{Cone}(X)$ is a G_δ-dense subspace of $\mathrm{Cone}(Y)$. Let G be a nonempty G_δ-subset of $\mathrm{Cone}(Y)$. If $0 \in G$, $G \cap \mathrm{Cone}(X)$ is nonempty. Assume that $0 \notin G$. Then $G \subseteq Y \times (0,1]$. Take a point $(y,r) \in G$. Then, there is a family $\{U_n \times V_n : n \in \omega\}$ of basic open subsets of $Y \times (0,1]$ such that $(y,r) \in \cap\{U_n \times V_n : n \in \omega\} \subseteq G$. Observe that $y \in \cap\{U_n : n \in \omega\}$. Since X is G_δ-dense in Y, there is $x \in X \cap (\cap\{U_n : n \in \omega\})$. It is straightforward to see that $(x,r) \in G$. In conclusion, $G \cap \mathrm{Cone}(X)$ is nonempty.

Now, if Y is compact, so is Cone(Y). Therefore if X is weakly pseudocompact, so is Cone(X). □

Corollary 4.58. Let κ be a cardinal. Then Cone($D(\kappa)$) is weakly pseudocompact if and only if $\kappa > \omega$.

Proof. If $\kappa > \omega$, the discrete space $D(\kappa)$ is locally compact not Lindelöf, then it is weakly pseudocompact. Hence, Cone($D(\kappa)$) is weakly pseudocompact.

If $\kappa \leq \omega$, the space Cone($D(\kappa)$) is a noncompact subspace of R^2. Hence Cone($D(\kappa)$ is Lindelf̈. Therefore, Cone($D(\kappa)$ is not weakly pseudocompact. □

The following question comes naturally.

Question 4.59. If Cone(X) is weakly pseudocompact non-compact, must X be weakly pseudocompact?

We have a partial answer to this question.

Theorem 4.60. *[15, Dorantes, Tamariz] Let X be a non-compact space. If Cone(X) is weakly pseudocompact then $X \times (0, 1]$ is weakly pseudocompact.*

Proof. Suppose that Cone(X) is weakly pseudocompact, since $X \times (0, 1]$ is open in Cone(X), $X \times (0, 1]$ is either weakly pseudocompact or locally compact Lindelöf. If X is Lindelöf, so is Cone(X) [Proposition 4.56]. Hence Cone(X) is compact [Proposition 4.4]. Since X is closed in his cone, X is compact; a contradiction. Henceforth X is not Lindelöf and neither is $X \times (0, 1]$. Therefore, $X \times (0, 1]$ is weakly pseudocompact. □

Theorem 4.61. *For a space X, Cone(X) is pseudocompact if and only if X is pseudocompact.*

Proof. Suppose that X is pseudocompact. Let f : Cone $(X) \to \mathbb{R}$ be a continuous function. There exist $\epsilon > 0$ and $M > 0$, such that

$$f[\{0\} \cup (X \times (0, \epsilon))] \subseteq (f(0) - 1, f(0) + 1),$$

and $f[X \times [\epsilon, 1]] \subseteq (-M, M)$. Then f is bounded.

Now suppose that Cone(X) is pseudocompact. Then $X \times [1/2, 1]$ is pseudocompact, because it is a regular closed subset. Then X is pseudocompact. □

Proposition 4.62. If Cone(X) is locally pseudocompact (locally compact), then X is pseudocompact (respectively, compact).

Proof. There is $\epsilon > 0$ such that $N = \{0\} \cup (X \times (0, \epsilon])$ is a pseudocompact (compact) regular closed neighborhood of 0 in Cone(X). Then $X \times [\epsilon/2, \epsilon]$ is pseudocompact (compact), and X is also pseudocompact (respectively, compact). □

Proposition 4.63. *[17] Let X be a weakly pseudocompact space and let Y be a locally compact space. If $X \times Y$ is not Lindelöf, then $X \times Y$ is weakly pseudocompact.*

Proof. If Y is weakly pseudocompact, so is $X \times Y$. Suppose that Y is not locally pseudocompact. Since Y is locally compact, Y is Lindelöf. If X is compact, then $X \times Y$ is Lindelöf, which is a contradiction. Then X is not compact. Suppose that X is a G_δ-dense subspace in the compactification bX and let αY be the one point compactification of Y with $\alpha Y \setminus Y = \{\infty\}$. Fix points $p \in bX \setminus X$ and $y \in Y$. We are going to prove that $X \times Y$ is G_δ-dense in the compactification $Z = (bX \times \alpha Y)/((bX \times \{\infty\}) \cup \{(p, y)\})$. Let $\pi : X \times Y \to Z$ be the natural projection. Let G be a nonempty G_δ-subset of Z. There is a collection $\{U_n : n \in \omega\}$ of open subsets in Z such that $G = \cap\{U_n : n \in \omega\}$.

Case 1: $\pi[(bX \times \{\infty\}) \cup \{(p, y)\}] \in G$, then $(bX \times \{\infty\}) \cup \{(p, y)\} \subseteq \pi^{\leftarrow}[G]$. Therefore, $(p, y) \in \pi^{\leftarrow}[G]$. For each $n \in \omega$, take open sets $A_n \subseteq bX$ and $B_n \subseteq \alpha Y$ such that $(p, y) \in A_n \times B_n \subseteq \pi^{\leftarrow}[U_n]$. Since $W = \cap\{A_n : n \in \omega\}$ is a nonempty G_δ subset of bX, we can take a point $x \in W \cap X$. Then $(x, y) \in \pi^{\leftarrow}[G]$. Hence $\pi((x, y)) \in G \cap (X \times Y)$.

Case 2: $\pi[(bX \times \{\infty\}) \cup \{(p, y)\}] \notin G$, then $()bX \times \{\infty\}) \cup \{(p, y)\}) \cap \pi^{\leftarrow}[G] = \emptyset$. Hence $\emptyset \neq \pi^{\leftarrow}[G] \subseteq bX \times Y$.

Take a point $(a, b) \in \pi^{\leftarrow}[G]$. Again, for each $n \in \omega$, take open sets $A_n \subseteq bX$ and $B_n \subseteq \alpha Y$ such that $(a, b) \in A_n \times B_n \subseteq \pi^{\leftarrow}[U_n]$. Take a point $x \in (\cap_{n\in\omega} A_n) \cap X$. Then $(x, b) \in \pi^{\leftarrow}[G]$ and therefore $\pi((x, b)) \in G \cap (X \times Y)$. □

Question 4.64. If $X \times (0, 1]$ is weakly pseudocompact, must X be weakly pseudocompact?

We have a partial answer to this question.

Theorem 4.65. *[15, Dorantes, Tamariz] Let X be a locally pseudocompact non-compact space. Then X is weakly pseudocompact if and only if $X \times (0, 1]$ is weakly pseudocompact.*

Proof. If X is weakly pseudocompact, then X is not Lindelöf and so by Proposition 4.63, $X \times (0, 1]$ is weakly pseudocompact.

Now, if $X \times (0,1]$ is weakly pseudocompact, it is not Lindelöf because it is not compact. Since $(0,1]$ is σ-compact, X is not Lindelöf. Therefore X is weakly pseudocompact. $\square$

We have two corollaries of the last theorem.

Corollary 4.66. Let X be a locally pseudocompact non-compact space. Then X is weakly pseudocompact if and only if $\mathrm{Cone}(X)$ is weakly pseudocompact.

Proof. Suppose that $\mathrm{Cone}(X)$ is weakly pseudocompact, by Theorem 4.60 $X \times (0,1]$ is weakly pseudocompact, then X is weakly pseudocompact. $\square$

Corollary 4.67. Let X be a non-compact space. Then X is simple pseudocompact if and only if $X \times (0,1]$ is simple pseudocompact.

Proof. If X is locally pseudocompact and weakly pseudocompact then $X \times (0,1]$ is locally pseudocompact and weakly pseudocompact.

If $X \times (0,1]$ is locally pseudocompact and weakly pseudocompact, then X is locally pseudocompact and it is not Lindelöf. $\square$

Eckertson asked the following question.

Question 4.68. [17, Question 4.1] Can a finite product of non-compact Lindelöf spaces ever be weakly pseudocompact?

We have a partial answer.

Theorem 4.69. *[15, Dorantes, Tamariz] If X and Y are non-compact Lindelöf spaces, then $X \times Y$ is not simple pseudocompact.*

Proof. If $X \times Y$ is simple pseudocompact, since it is not compact, then it is locally pseudocompact non-Lindelöf. Then X and Y are locally pseudocompact Lindelöf spaces, ergo, locally compact Lindelöf spaces and therefore $X \times Y$ is Lindelöf. $\square$

4.4 Products of weakly pseudocompact spaces

Eckertson posed the following question:

Question 4.70. [17, Question 2.6] If Z is compact and $X \times Z$ is weakly pseudocompact, must X be weakly pseudocompact?

We reply in the affirmative:

Theorem 4.71. *[15, A. Dorantes-Aldama, R. Rojas-Hernández] If Z is compact and $X \times Z$ is weakly pseudocompact, then X is weakly pseudocompact.*

Proof. Let Y be a compactification of $X \times Z$ in which it is a G_δ-dense subset. Define the partition $\mathcal{P} = \{\{y\} : y \in Y \setminus (X \times Z)\} \cup \{\{x\} \times Z : x \in X\}$ in Y. Let Y' be the quotient space $Y/\mathcal{P}$ and let $\pi : Y \to Y'$ be the natural proyection.

Claim: The space Y' is Hausdorff.

Indeed, let $y \in Y \setminus (X \times Z)$ and $x \in X$. Since Z is compact, there are disjoint open subsets U, V in Y such that $y \in U$ and $\{x\} \times Z \subseteq V$. Observe that there is an open set $A \subseteq X$ such that $\{x\} \times Z \subseteq A \times Z \subseteq V$. As $A \times Z$ is open in $X \times Z$, there is an open subset $W \subseteq Y$ such that $A \times Z = W \cap (X \times Z)$. It is easy to see that $W \cap V = \pi^{\leftarrow}[\pi[W \cap V]]$ and $x \in \pi[W \cap V]$.

Because $U \cap (X \times Z)$ is open in $X \times Z$, there is an open subset $O \subseteq Y$ such that $\pi_X[U \cap (X \times Z)] \times Z = O \cap (X \times Z)$. Observe that, as $X \times Z$ is dense in Y, $W \cap O = \emptyset$. Then $(O \cup U) \cap (X \times Y) = (\pi_X[U \cap (X \times Z)] \times Z) \cup (U \cap (X \times Z)) = \pi_X[U \cap (X \times Z)] \times Z$. Then $(O \cup U) = \pi^{\leftarrow}[\pi[O \cup U]], y \in \pi[O \cup U]$ and $\pi[W \cap V] \cap \pi[O \cup U] = \emptyset$. Hence Y' is compact Hausdorff.

Observe that X is G_δ-dense in Y'. Henceforth X is weakly pseudocompact. □

It is known that for every space X, we have that X is G_δ-dense in its realcompactification υX. Then if υX is weakly pseudocompact, so is X. Thus, a natural question is: If X is a weakly pseudocompact space, must υX be weakly pseudocompact? We have a negative answer.

Theorem 4.72. *[15, Dorantes, Tamariz] Let Y be a pseudocompact space and let X be a realcompact, locally compact space of non measurable cardinality. Then, X is weakly pseudocompact if and only if $\upsilon(X \times Y)$ is weakly pseudocompact.*

Proof. By [8, Corollary 2.2],

$$\upsilon(X \times Y) = X \times \upsilon Y = X \times \beta Y.$$

By Theorem 4.71, $X \times \beta Y$ is weakly pseudocompact if and only if X is weakly pseudocompact. □

We can see that if $\mathcal{A}$ is a maximal almost disjoint family on ω, then the Mrówka space $\Psi(\mathcal{A})$ is a pseudocompact, non-compact space of non-measurable cardinality. The space $\omega \times \Psi(\mathcal{A})$ is locally compact and it is not Lindelöf, then it is weakly pseudocompact. But $\omega \times \beta\Psi(\mathcal{A})$ is Lindelöf non-compact, and then it is not weakly pseudocompact.

Theorem 4.73. *[15, Dorantes, Tamariz] Let X be a dense k-embedded subspace of Y. If X is weakly pseudocompact, then Y is weakly pseudocompact.*

Proof. Assume that Y is not weakly pseudocompact. Let bX be a compactification of X. Since X is k-embedded in Y, there is a compactification bY of Y such that $cl_{bY}X \leq bX$. Let $\tilde{h} : bX \to \mathrm{cl}_{bY} X$ be a continuous function such that $\tilde{h}(x) = x$ for all $x \in X$. Because X is dense in Y, $cl_{bY}X = bY$. On the other hand, since Y is not weakly pseudocompact, there is a nonempty G_δ-subset G in bY such that $G \cap Y = \emptyset$. Therefore, $\tilde{h}^{-1}[G]$ is a nonempty G_δ-set in bX which has an empty intersection with X. $\square$

Corollary 4.74. If X is G_δ-dense an k-embedded in Y, then X is weakly pseudocompact if and only if Y is weakly pseudocompact.

Example 4.75. Let $\mathcal{A}$ be a maximal disjoint family on ω. Then $\omega \times \Psi(\mathcal{A})$ is not k-embedded in $\omega \times \alpha\Psi(\mathcal{A})$, where $\alpha\Psi(\mathcal{A})$ is the one point compactification of $\Psi(\mathcal{A})$.

Since X is k-embedded in αX, when X is locally compact, the previous example shows that the property of being k-embedded is not productive.

Corollary 4.76. Let α and κ be infinite cardinals such that $\kappa > \omega$. If $\Sigma_\kappa(\mathbb{R}^\alpha)$ is k-embedded in $\mathbb{R}^\alpha$ then $\Sigma_\kappa(\mathbb{R}^\alpha)$ is weakly pseudocompact if and only if $\mathbb{R}^\alpha$ is weakly pseudocompact.

Proof. We know that $\upsilon(\Sigma_\kappa(\mathbb{R}^\alpha)) = \mathbb{R}^\alpha$. Therefore, if $\mathbb{R}^\alpha$ is weakly pseudocompact, then $\Sigma_\kappa(\mathbb{R}^\alpha)$ is weakly pseudocompact.

If $\Sigma_\kappa(\mathbb{R}^\alpha)$ is k-embedded in $\mathbb{R}^\alpha$ then, by Theorem 4.73, if $\Sigma_\kappa(\mathbb{R}^\alpha)$ is weakly pseudocompact, then $\mathbb{R}^\alpha$ is weakly pseudocompact. $\square$

4.5 Gelfand compactifications

In this section we characterize weak pseudocompactness in terms of regular subrings. Recall that $C^*(X)$ is the collection of all bounded real-valued continuous functions defined on X. If $r \in \mathbb{R}$, we will denote by $\bar{r}$ the constant function $\bar{r}(x) = r$ for every $x \in X$.

For a space X and $f \in C^*(X)$, define $||f|| = \sup\{|f(x)| : x \in X\}$. If $f, g \in C^*(X)$, let $\mathrm{d}(f,g) = ||f - g||$. Then d is a complete metric on $C^*(X)$ [37, Proposition 4.5(a)]. We will call d the sup norm metric on $C^*(X)$.

Definition 4.77. [17] A subset $S \subseteq C^*(X)$ is a *subring* of $C^*(X)$ if $\bar{0}$ and $\bar{1}$ belong to S and for $f, g \in S$ both fg and $f - g$ belong to $C^*(X)$.

Definition 4.78. [37] A subring Q of $C^*(X)$ is called regular subring if Q is a complete, with respect to the sup norm metric, subring of $C^*(X)$ such that Q contains all the constant functions and $Z(Q) = \{Z(f) : f \in Q\}$ is a base for the closed sets of X.

Definition 4.79. [17] A nonempty subset $I \subseteq C^*(X)$ is an *ideal* of $C^*(X)$ if $C^*(X) \setminus I$ is nonempty and for every $f \in I$ and every $g \in C^*(X)$, the product fg belongs to I. A ideal $I \subseteq C^*(X)$ is maximal if for $g \in C^*(X) \setminus I$, there are functions $f \in C^*(X)$ and $h \in I$ such that $fg + h = \bar{1}$.

Definition 4.80. [37] Let Q be a regular subring of $C^*(X)$. Denote the set of all the maximal ideals of Q by $m_Q X$. If $f \in Q$, define $S(f) = \{M \in m_Q X : f \in M\}$. Define $\lambda : X \to m_Q X$ by $\lambda(x) = \{f \in Q : f(x) = 0\}$.

Proposition 4.81. *[37, Lemma 4.5j] Let Q be a regular subring of $C^*(X)$ for a space X. Let f, g in Q. Then:*

1. *$S(\bar{1}) = \emptyset$ and $S(\bar{0}) = m_Q X$.*
2. *$S(f) \cup S(g) = S(fg)$.*
3. *$S(f) \cap S(g) \subseteq S(f^2 + g^2)$.*
4. *$\{S(f) : f \in Q\}$ is a closed base for a Compact Hausdorff topology σ in $m_Q X$.*

We will consider $m_Q X$ with the topology σ given by the previous theorem.

Theorem 4.82. *[37, Theorem 4.5m] Let Q be a regular subring of $C^*(X)$ for a space X. Then:*

1. *$\lambda : X \to m_Q X$ is a dense embedding, and if $f \in Q$, then $\lambda[f^{\leftarrow}[\{0\}]] = \lambda[X] \cap S(f)$,*
2. *for every $f \in Q$, there is a unique continuous function $f^e \in C^*(m_Q X)$ such that $f^e \circ \lambda = f$,*

3. $C^*(m_Q X) = \{f^e : f \in Q\}$ *and* $Q = \{f \in C^*(X) : f = F \circ \lambda$ *for some* $F \in C^*(m_Q X)\}$ *and*

4. *for* $f \in Q, S(f) = (f^e)^{\leftarrow}[\{0\}]$.

Lemma 4.83. *Let* X *be a topological space. If* $Q \subseteq C^*(X)$ *is a regular subring and* $F : m_Q X \to \mathbb{R}$ *is a continuous function, then there is a continuous function* $f \in Q$ *such that* $F^{\leftarrow}\{0\} = S(f)$.

Proof. By Theorem 4.82(3), there exists $f \in Q$ such that $F = f^e$. By 4.82(4), $F^{\leftarrow}\{0\} = S(f)$. □

Remark 4.84. Observe that 4.82(4) states that if $f \in Q$ then $S(f)$ is a zero set of $m_Q X$.

Theorem 4.85. *[37, Theorem 4.5(o)] Let* X *be a space and* aX *a compactification of* X*. If* $Q = \{f \circ a : f \in C^*(aX)\}$, *then* $aX \equiv_X m_Q X$.

The next lemma is known and is a motivation for the subsequent theorem.

Lemma 4.86. *[17] A space* X *is pseudocompact if and only if* $f \in C^*(X)$ *and* $Z(f) = \emptyset$ *then* $1/f \in C^*(X)$.

We are ready to present the promised characterization of weak pseudocompactness in terms of regular subrings.

Theorem 4.87. *A space* X *is weakly pseudocompact if and only if there exists a regular subring* Q *of* $C^*(X)$ *such that whenever* $f \in Q$ *and* $Z(f) = \emptyset$, *we obtain* $1/f \in Q$.

Proof. Let X be a weakly pseudocompact space. There is a compactification T of X such that X is G_δ-dense in T. By Theorem 4.85, $\mathbb{Q} = \{f \restriction X : f \in C^*(T)\}$ is a regular subring of $C^*(X)$, and $K \equiv_X m_{\mathbb{Q}} X$. Let $f \in Q$ and suppose that $1/f \notin Q$. Hence $T = \{gf : g \in Q\}$ is a proper ideal of Q such that $f \in T$. Hence, $S(f) \neq \emptyset$. By Remark 4.84, $S(f)$ is a zero set of $m_Q X$, by Theorem 4.82(1), we have

$$\emptyset \neq S(f) \cap \lambda[X] = \lambda[Z(f)].$$

Then $Z(f)$ is nonempty.

Let Q be a regular subring of $C^*(X)$ that satisfies the hypothesis. Let Z be a nonempty zero set of $m_Q(X)$. By Lemma 4.83 there exists $f \in \mathbb{Q}$ such that $Z = S(f)$. Take $M \in S(f)$. Then M is a proper ideal in Q and $f \in M$. Henceforth, $1/f \notin Q$. The hypothesis says that $Z(f) \neq \emptyset$. Then $S(f) \cap \lambda[X] = \lambda[Z(f)] \neq \emptyset$ and X is weakly pseudocompact. □

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