UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
POSGRADO EN CIENCIAS MATEMÁTICAS

ELLIPTIC PROBLEMS WITH CRITICAL AND SUPERCRITICAL NONLINEARITIES

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

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MÉXICO, D. F. NOVIEMBRE 2013

Contents

Abstract

Nosotros consideremos el problema clásico de Lane-Emden-Fowler

$$(\wp_{p,\Theta}) \qquad \begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \Theta, \\ v = 0 & \text{on } \partial\Theta, \end{cases}$$

para ambas, linealidad críticas ($p = 2^*$) y supercríticas ($p > 2^*$), en un dominio acotado suave Θ en $\mathbb{R}^N$, $N \geq 3$. Aquí, como es usual, $2^* := \frac{2N}{N-2}$ es el exponente crítico de Sobolev. En algunos casos particulares uno puede reducir el problema supercrítico a un problema anisotrópico critico de la forma

$$(\wp^*_{a_1,a_2,a_3,\Omega}) \qquad \begin{cases} -\mathrm{div}(a_1(x)\nabla u) + a_2(x)u = a_3(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

en donde Ω es un dominio acotado suave en $\mathbb{R}^n$, $n \geq 3$ y $a_i \in C^0(\overline{\Omega})$ es estrictamente positiva para cada $i = 1, 2, 3$. En esta tesis presentamos resultados acerca de la existencia, no existencia y multiplicidad de soluciones para estos problemas

Chapter 1

Introduction

We consider the classical Lane-Emden-Fowler problem

$$(\wp_{p,\Theta}) \qquad \begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \Theta, \\ v = 0 & \text{on } \partial\Theta, \end{cases}$$

both for critical ($p = 2^*$) and supercritical ($p > 2^*$) nonlinearities, in a bounded smooth domain Θ in $\mathbb{R}^N$, $N \geq 3$. Here, as usual, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent.

In some particular cases one can reduce the supercritical problem to a anisotropic critical problem of the form

$$(\wp^*_{a_1,a_2,a_3,\Omega}) \qquad \begin{cases} -\operatorname{div}(a_1(x)\nabla u) + a_2(x)u = a_3(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, and $a_i \in C^0(\overline{\Omega})$ is strictly positive for $i = 1, 2, 3$.

In this thesis we present results about the existence, non-existence and multiplicity of solutions for these problems. Part of these results were obtained in colaboration with Mónica Clapp and Angela Pistoia and are contained in the papers [13, 14, 15]. Other results are new.

1.1 A brief historical background

Equation $(\wp_{p,\Theta})$ models many physical phenomena. For $p = 2^*$ it arises in fundamental questions in differential geometry like the Yamabe problem or the scalar curvature problem.

Problem $(\wp_{p,\Theta})$ has been widely studied in the last 50 years. The process for understanding it has helped to develop new and interesting techniques which can be applied to a wide variety of problems.

In the subcritical case ($p < 2^*$), the compactness of the Sobolev embedding

$$H_0^1(\Theta) \hookrightarrow L^p(\Theta) \tag{1.1}$$

guarantees that problem $(\wp_{p,\Theta})$ has one positive solution and infinitely many sign changing solutions in every smooth bounded domain Θ.

For $p \geq 2^*$, the embedding (1.1) is no longer compact, so existence of solutions for problem $(\wp_{p,\Theta})$ becomes a delicate issue. In this situation still very little is known concerning the existence and nonexistence of solutions, particularly for the supercritical case.

It is well known that in this case the existence of a solution for problem $(\wp_{p,\Theta})$ depends on the domain. The first result is due to Pohozaev [40], he showed that if the domain Θ satisfies a particular geometric condition then problem $(\wp_{p,\Theta})$ has no nontrivial solution. More precisely

Theorem 1.1 (Pohozhaev, 1965). *If Θ is strictly starshaped, then problem $(\wp_{p,\Theta})$ does not have a nontrivial solution for every $p \geq 2^*$.*

On the other hand, Kazdan and Warner [25] showed that

Theorem 1.2 (Kazdan-Warner, 1975). *If $\Theta = A_{a,b}$ is an annulus, i.e.*

$$\Theta = A_{a,b} := \{x \in \mathbb{R}^N : 0 < a < |x| < b\}$$

then, problem $(\wp_{p,A_{a,b}})$ has infinitely many radial solutions for every $p > 2$.

1.1.1 The critical problem

When $p = 2^*$, the problem $(\wp_{2^*,\Theta})$ is invariant under the group of Möbius transformations, see [17]. This produces a lack of compactness of the associated variational functional, which prevents the straightforward application of the usual variational methods. For this reason, $(\wp_{2^*,\Theta})$ is a quite interesting and challenging problem.

The first nontrivial existence result for the problem $(\wp_{2^*,\Theta})$ is due to Coron [11]. He showed that problem $(\wp_{2^*,\Theta})$ has a positive solution in every domain with a small enough hole. More precisely

Theorem 1.3 (Coron, 1984). *If $0 \notin \Theta$ and $\Theta \supset A_{a,b}$ with a/b large enough then, problem $(\wp_{2^*,\Theta})$ has at least one positive solution.*

A few years later a remarkable result was obtained by Bahri and Coron [3]. They showed that problem $(\wp_{2^*,\Theta})$ has a positive solution in every domain Θ with nontrivial topology. More accurately, they showed that

Theorem 1.4 (Bahri-Coron, 1988). *If $\widetilde{H}_*(\Theta, \mathbb{Z}/2) \neq 0$, then problem $(\wp_{2^*,\Theta})$ has a positive solution.*

Theorem 1.4 shows that the existence of solutions for problem $(\wp_{2^*,\Theta})$ depends also on the topology of the domain and not just on its geometry.

However, the condition $\widetilde{H}_*(\Theta,\mathbb{Z}/2)\neq 0$ is not necessary for the existence of a nontrivial solution. Examples of contractible domains for which problem $(\wp_{2^*,\Theta})$ has at least one nontrivial solution have been given, see for instance [22, 23, 36, 38].

Regarding the multiplicity of solutions for problem $(\wp_{2^*,\Theta})$, many results have been established in specific symmetric domains, for example [36, 29, 18].

In particular, Clapp [12] provided a generalization for the nonautonomous problem $(\wp^*_{1,0,a_3,\Omega})$ of the result given by Kazdan-Warner (Theorem 1.2). For every positive continuous function a_3 she shows that

Theorem 1.5 (Clapp, 2003)**.** *If Ω and a_3 are invariant under a group G of linear isometries of $\mathbb{R}^n$ and the cardinality of all the G-orbits is infinite then, problem*

$$(\wp^*_{1,0,a_3,\Omega})\qquad \begin{cases} -\Delta u = a_3(x)|u|^{2^*-2}u & \textit{in } \Omega,\\ u=0 & \textit{on } \partial\Omega. \end{cases}$$

has infinitely many G-invariant solutions.

For nonsymmetric domains, multiplicity results have been established for problem $(\wp_{2^*,\Theta})$ in domains with a thin enough perforation, see for example [16, 18, 32, 42]. In particular, for this type of domains, there is a remarkable result due to Ge, Musso and Pistoia [24], were they prove that

Theorem 1.6 (Ge-Musso-Pistoia, 2010)**.** *Let $\Theta\subset\mathbb{R}^N$ be a smooth bounded domain, $\xi\in\Theta$ and*

$$\Theta_\epsilon := \Theta\backslash B_\epsilon(\xi)$$

then, the number of solutions for problem $(\wp_{2^*,\Theta_\epsilon})$ goes to infinity as ϵ goes to zero.

But for domains which are neither highly symmetric nor small perturbations of a given domain multiplicity remains largely open. For these domains, Clapp and Pacella [17] gave the first result about the existence of multiple sign changing solutions for problem $(\wp_{2^*,\Theta})$. Using the main result of [17], they showed that there exist many examples of domains Ω having only finite symmetries in which problem $(\wp_{2^*,\Theta})$ has a prescribed number of solutions, one of them being positive and the rest change sign.

This result supports our belief that multiplicity of solutions for problem $(\wp_{2^*,\Theta})$ should hold in noncontractible domains. But the proof of such a general statement is still way out of reach.

1.1.2 The supercritical problem

In the supercritical problem $(\wp_{p,\Theta})$, $p>2^*$, the lack of a Sobolev embedding implies that the standard variational formulation used in the subcritical and critical cases cannot be used.

Unlike in the critical case, in the supercritical case $p > 2^*$ the condition $\widetilde{H}_*(\Theta, \mathbb{Z}/2) \neq 0$ does not guarantee the existence of a solution for problem $(\wp_{p,\Theta})$. This was pointed out by Passaseo in [37, 39], were he showed that for every integer $1 \leq k \leq N-3$ there exists a smooth bounded domain $\Theta_k \subset \mathbb{R}^N$ such that

- Θ_k has the homotopy type of the sphere $\mathbb{S}^k$,
- problem $(\wp_{p,\Theta_k})$ does not have a nontrivial solution for $p \geq 2^*_{N,k} := \frac{2(N-k)}{N-k-2}$,
- problem $(\wp_{p,\Theta_k})$ has infinitely many solutions for $p < 2^*_{N,k}$.

Note that the number $2^*_{N,k} = \frac{2(N-k)}{N-k-2}$ is the Sobolev critical exponent in dimension $(N-k)$.

In addition, Passaseo's domains Θ_k also satisfy that they are invariant under de group $G = O(k+1) \subset O(N)$ and the cardinalities of all the G-orbits in Θ_k are equal to infinity. This shows that there is not an analogous theorem to Theorem 1.5 for the supercritical case.

The first nontrivial existence result for problem $(\wp_{p,\Theta}), p > 2^*$ was obtained by del Pino, Felmer and Musso [20], in the slightly supercritical case, i.e. for $p > 2^*$ but close enough to 2^*.

Many existence results have been established for problem $(\wp_{p,\Theta})$ in the slightly subcritical, i.e. when $p = 2^*_{N,k} - \epsilon, \epsilon$ is positive and small enough, see for instance [1, 4, 5, 32].

There are only a few results about the existence and multiplicity of solutions for problem $(\wp_{p,\Theta})$ when p is the pure supercritical exponent, i.e. when $p = 2^*_{N,k}$ for an integer $1 \leq k \leq N-3$, see for example [26, 46].

In particular, Wei and Yan [46] exhibited domains Θ in which problem $(\wp_{p,\Theta})$ has infinitely many positive solutions. They considered domains Θ of the form

$$\Theta := \{(y, z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Omega\}, \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^{N-k}$ with $\overline{\Omega} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$ which satisfies certain geometric assumptions.

The proof of results [1, 26, 33, 46] uses a procedure which consists on reducing the supercritical problem $(\wp_{p,\Theta})$ to some nonautonomous problem in a domain Ω of lower dimension but with the same exponent p.

We shall also follow this approach to obtain results about the existence and multiplicity of a new type of solutions for the problem $(\wp_{p,\Theta})$ in the pure supercritical exponent.

1.2 Main results of the thesis: The nonautonomous critical problem

In this subsection we state our results for the anisotropic critical problem:

$$(\wp^*_{a_1,a_2,a_3,\Omega}) \qquad \begin{cases} -\mathrm{div}(a_1(x)\nabla u) + a_2(x)u = a_3(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $2^* = \frac{2n}{n-2}$ is the Sobolev critical exponent and $a_i : \overline{\Omega} \to \mathbb{R}$ is a continuous function for $i = 1, 2, 3$ which satisfies that

- $\min\limits_{x\in\overline{\Omega}} a_i(x) > 0$ for $i = 1, 3$ and
- $\min\limits_{x\in\overline{\Omega}} a_2(x) > -\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$.

1.2.1 Multiplicity of solutions in symmetric domains

We shall prove a multiplicity result for problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ which extends the main results in [13, 14, 17]. In order to give the precise statement of this result we need to introduce some notation:

Let $O(n)$ be the group of linear isometries of $\mathbb{R}^n$. If G is a closed subgroup on $O(n)$, we will denote by

$$Gx := \{gx : g \in G\}$$

the G-orbit of x and by $\#Gx$ its cardinality. A domain $\Omega \subset \mathbb{R}^n$ is called G-invariant if $Gx \subset \Omega$ for all $x \in \Omega$ and a function $u : \overline{\Omega} \to \mathbb{R}$ is called G-invariant if u is constant on every orbit Gx with $x \in \overline{\Omega}$. Let S be the best Sobolev constant for the embedding

$$D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n).$$

Now, fix a closed subgroup Γ of $O(n)$ and a nonempty Γ-invariant smooth bounded domain $D \subset \mathbb{R}^n$ such that $\#\Gamma x = \infty$ for all $x \in D$. Also suppose that a_i is an Γ-invariant continuous function for $i = 1, 2, 3$. We shall prove the following result

Theorem 1.7. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ, D, a_1, a_2 and a_3, with the following property: If Ω is a smooth bounded domain that contains D and if it is invariant under the action of a closed subgroup G of Γ for which*

$$\left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}} \#Gx}{a_3(x)^{\frac{n-2}{2}}} \right) > \ell_m$$

holds, then problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ has at least m pairs of G-invariant solutions $\pm u_1, \dots, \pm u_m$ such that u_1 is positive, $u_2, \dots, u_m$ change sign, and

$$\int_\Omega |\nabla u_k|^2 \leq \ell_k S^{\frac{n}{2}} \quad \textit{for every } k = 1, \dots, m.$$

The proof of this result uses variational methods. The main ingredient is a compactness result (Theorem A.1) given in Appendix A, which is an extension of the main result in [12], which in turn is a symmetric version of Struwe's result in [44]. The positive solution is obtained using the classical mountain pass theorem of Ambrosetti and Rabinowitz [2]. Applying a symmetric mountain pass theorem given in [17] we show the existence of multiple solutions which change sign.

Theorem 1.7 is interesting even for the autonomous problem

$$(\wp_{2^*,\Theta}) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Theta, \\ u = 0 & \text{on } \partial\Theta, \end{cases}$$

As we mentioned previously, Bahri and Coron [3] showed that it has at least one positive solution if Θ has nontrivial reduced homology with $\mathbb{Z}/2$-coefficients, see Theorem 1.4. On the other hand, for domains which are neither highly symmetric nor small perturbations of a given domain, multiplicity remains largely open.

Theorem 1.7 extends our results in [13, 14]. The special case for problem $(\wp_{2^*,\Theta})$ where $\Gamma = O(n)$ and $D = A_{a,b}$ was established in [17]. This situation is, however, quite restrictive, particularly in odd dimensions. For example, if $N = 3$, then $\min_{x\in A_{a,b}} \#Gx \leq 2$ for every subgroup $G \neq SO(3), O(3)$. In fact, $\min_{x\in A_{a,b}} \#Gx$ is either 1 or 2 for most subgroups of $O(3)$, cf. [10]. As we shall see, the number ℓ_1 goes to infinity as $b/a \to 1$. Therefore, the main result in [17] will provide solutions in subdomains of $\mathbb{R}^3$ only if b/a is sufficiently large, which is the case already handled by Coron [11] and by Ge, Musso and Pistoia [24].

Theorem 1.7, on the other hand, provides examples in every dimension of domains Ω, having only finite symmetries, in which problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ has a prescribed number of solutions. Specific examples will be given in Chapter 3.

1.2.2 Existence of solutions in punctured domains

We also study a nonautonomous critical problem in punctured domains.

Fix a closed subgroup Γ on $O(n)$ and Ω a smooth bounded Γ-invariant domain in $\mathbb{R}^n$. We denote by

$$\Omega^\Gamma := \{x \in \Omega : gx = x \quad \forall g \in \Gamma\}$$

the set of Γ-fixed points in Ω. Assume $\xi_0 \in \Omega^\Gamma$ and write

$$\Omega_\epsilon := \{x \in \Omega : |x - \xi_0| > \epsilon\},$$

Note that, since $\xi_0 \in \Omega^\Gamma$, Ω_ϵ is also Γ-invariant.

Consider the problem

$$(\wp^*_{Q,\Omega_\epsilon}) \qquad \begin{cases} -\Delta u = Q(x)u^{\frac{n+2}{n-2}} & \text{in } \Omega_\epsilon, \\ u > 0 & \text{in } \Omega_\epsilon, \\ u = 0 & \text{on } \partial\Omega_\epsilon, \end{cases}$$

where $n \geq 3$ and Q is a Γ-invariant continuous function which satisfies $\min_{x\in\overline{\Omega}} Q(x) > 0$.

In Chapter 2 we prove the following result

Theorem 1.8. *Assume that $\nabla Q(\xi_0) \neq 0$. Then there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem $(\wp^*_{Q,\Omega_\epsilon})$ has a Γ-invariant solution u_ϵ which concentrates and blows up at the point ξ_0 as $\epsilon \to 0$.*

Notice that if $\Gamma \equiv 1$ then, Theorem 1.8 shows the existence of one positive solution for the problem $(\wp^*_{Q,\Omega})$ in every smooth bounded, **not necessarily symmetric**, punctured domain. If $Q \equiv 1$, this is Coron's result (Theorem 1.3).

The proof of Theorem 1.8 uses the well-known Lyapunov-Schmidt reduction procedure. With the help of some estimates given in [24] we perform the finite dimensional reduction. Then, we compute the asymptotic expansion of the reduced energy functional and we show that it has a critical point which is stable under C^1-perturbations.

We believe that existence and also multiplicity can be shown for the more general problem $(\wp^*_{a_1,0,a_3,\Omega})$ in a punctured domain, see Section 1.4.

1.3 Main results of the thesis: The supercritical problem

In this section we state our results for the supercritical problem

$$(\wp_{p,\Theta}) \qquad \begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \Theta, \\ v = 0 & \text{on } \partial\Theta, \end{cases}$$

where Θ is a smooth bounded domain $\mathbb{R}^N$, $N \geq 3$ and $p > 2^*$. Here, as before, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

1.3.1 Multiplicity results in domains induced by Hopf maps.

To prove our multiplicity results for problem $(\wp_{p,\Theta})$ we use a reduction procedure, introduced by Ruf and Srikanth in [43]. The results in this subsection will be derived from Theorem 1.7 using the Hopf maps, which we define next.

We are interested in the particular cases when $N = 2, 4, 8$ or 16. In this situation we can write $\mathbb{R}^N = \mathbb{K}\times\mathbb{K}$ where $\mathbb{K}$ is either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$. The set of units $\mathbb{S}_\mathbb{K} := \{\zeta \in \mathbb{K} : |\zeta| = 1\}$ acts on $\mathbb{R}^N$ by multiplication on each coordinate, i.e. $\zeta(z_1, z_2) := (\zeta z_1, \zeta z_2)$.

The Hopf map

$$\hbar_\mathbb{K} : \mathbb{K}\times\mathbb{K} \mapsto \mathbb{R}^{dim\mathbb{K}+1} \equiv \mathbb{R}\times\mathbb{K}$$

is given by

$$\hbar_\mathbb{K}(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2\overline{z_1}z_2).$$

We shall prove the following results.

Theorem 1.9. *Let G be a closed subgroup of $O(dim\mathbb{K}+1)$, Ω be a G-invariant bounded smooth domain in $\mathbb{R}^{dim\mathbb{K}+1}$ all of whose G-invariant orbits are infinite. Set $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$. Then, for $q = 2^*_{N,\dim\mathbb{K}-1}$, the supercritical problem $(\wp_{q,\Theta})$ has infinitely many solutions in Θ which are constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x\in\Omega$.*

Note that, since $0\notin\Omega$, $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$ is homeomorphic to $\Omega\times\mathbb{S}_{\mathbb{K}}$.

Next we consider the case where $\Theta = \hbar_{\mathbb{K}}^{-1}(\Omega)$ and Ω has finite orbits. We fix a closed subgroup Γ of $O(\dim\mathbb{K}+1)$ and a nonempty Γ-invariant bounded smooth domain D in $\mathbb{R}^{\dim\mathbb{K}+1}$ such that $\#\Gamma x = \infty$ for all $x\in D$. We obtain following result.

Theorem 1.10. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ and D, with the following property: If Ω is a smooth bounded domain that contains D and if it is invariant under the action of a closed subgroup G of Γ for which*

$$\min_{x\in\overline{\Omega}}\,(\#Gx)\,|2x|^{\frac{\dim\mathbb{K}-1}{2}} > \ell_m$$

holds, then, for $q = 2^*_{N,\dim\mathbb{K}-1}$, problem $(\wp_{q,\Theta})$ has at least m pairs of solutions $\pm v_1,\ldots,\pm v_m$ in $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$, which are constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x\in\Omega$. In particular, they are $\mathbb{S}_{\mathbb{K}}$-invariant. Moreover, v_1 is positive and $v_2,\ldots,v_m$ change sign.

Theorems 1.9 and 1.10 provides many examples of domains in which some supercritical problems has a prescribed number of solutions. We shall give some of them in Chapter 3.

1.3.2 Multiplicity results in domains of revolution.

Fix $k_1,\ldots,k_m\in\mathbb{N}$ and set $k := k_1+\cdots+k_m$. Let $N\geq k+m+2$ and Ω be a smooth bounded domain in $\mathbb{R}^{N-k}$ such that

$$\overline{\Omega}\subset\{(x_1,\ldots,x_m,x')\in\mathbb{R}^m\times\mathbb{R}^{N-k-m} : x_i>0, i=1,\ldots,m\}. \tag{1.3}$$

Define

$$\Theta := \{(y^1,\ldots,y^m,z)\in\mathbb{R}^{k_1+1}\times\cdots\times\mathbb{R}^{k_m+1}\times\mathbb{R}^{N-k-m} : (|y^1|,\ldots,|y^m|,z)\in\Omega\}. \tag{1.4}$$

In the following we consider $O(N-k-m)$ as the subgroup of $O(N-k)$ which acts on the second factor of $\mathbb{R}^m\times\mathbb{R}^{N-k-m}\equiv\mathbb{R}^{N-k}$. In Chapter 3 we shall prove the following results

Theorem 1.11. *Let G be a closed subgroup of $O(N-k-m)$, Ω be a G-invariant bounded smooth domain as above, all of whose G-orbits are infinite. Then, for $q = 2^*_{N,N-k}$, the supercritical problem $(\wp_{q,\Theta})$ has infinitely many $O(k_1+1)\times\cdots\times O(k_m+1)\times G$-invariant solutions in the domain Θ defined as above.*

For example, if Ω is a solid of revolution around the x_1-axis in $\mathbb{R}^3$ whose closure is contained in $(0,\infty)\times(\mathbb{R}^2\backslash\{0\})$ then, for any k, problem $(\wp_{2^*_{N,3},\Theta})$ has infinitely many $O(k+1)\times O(2)$-invariant solutions in $\Theta := \{(y,z)\in\mathbb{R}^{k+1}\times\mathbb{R}^2 : (|y|,z)\in\Omega\}$.

Next we consider another type of domains Θ. We fix a closed subgroup Γ of $O(N-k-m)$ and a Γ-invariant domain D such that

$$\overline{D}\subset\{(x_1,\dots,x_m,x')\in\mathbb{R}^m\times\mathbb{R}^{N-k-m} : x_i>0, i=1,\dots,m\}.$$

whose Γ-orbits are infinite. Consider the function $\mathfrak{a}:\mathbb{R}^m\times\mathbb{R}^{N-k-m}\to\mathbb{R}$ defined by

$$\mathfrak{a}((x_1,\dots,x_m,x')) = x_1^{k_1}\cdots x_m^{k_m}$$

We prove the following result

Theorem 1.12. *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ, D and $\mathfrak{a}$, with the following property: If Ω is a smooth bounded domain invariant under the action of a closed subgroup G of Γ such that*

$$D\subseteq\Omega\subset\{(x_1,\dots,x_m,x')\in\mathbb{R}^m\times\mathbb{R}^{N-k-m} : x_i>0, i=1,\dots,m\}$$

for which

$$\min_{x\in\overline{\Omega}}(\mathfrak{a}(x)\#Gx)>\ell_m$$

holds, then, for $q=2^*_{N,N-k}$, problem $(\wp_{q,\Theta})$ has at least m pairs of $O(k_1+1)\times\cdots\times O(k_m+1)\times G$-invariant solutions $\pm v_1,\dots,\pm v_m$ in Θ (defined as in 1.4). Moreover, v_1 is positive and $v_2,\dots,v_m$ change sign.

1.3.3 Positive solutions which concentrate along a thin spherical hole

We also establish the existence of a positive solution v_ϵ for problem $(\wp_{p,\Theta_\epsilon})$ in some domains Θ_ϵ obtained by deleting a ϵ-neighborhood of some sphere embedded in Θ and certain supercritical exponents p. These solutions v_ϵ concentrate and blow up along the sphere as $\epsilon\to 0$.

In this section we will state and describe these results. They follow from Theorem 1.8 and the reduction via Hopf maps described in Chapter 3.

Let $N=4,8,16$ and let us consider Θ, an $\mathbb{S}_\mathbb{K}$-invariant bounded smooth domain in $\mathbb{R}^N=\mathbb{K}^2$ such that $0\notin\overline{\Theta}$. Fix a point $z_0\in\Theta$ and for each $\epsilon>0$ small enough let

$$\Theta_\epsilon := \{z\in\Theta : \operatorname{dist}(z,\mathbb{S}_\mathbb{K}z_0)>\epsilon\}$$

where $\mathbb{S}_\mathbb{K}z_0 := \{\vartheta z_0 : \vartheta\in\mathbb{S}_\mathbb{K}\}$. This is again an $\mathbb{S}_\mathbb{K}$-invariant bounded smooth domain in $\mathbb{K}^2$. We consider the supercritical problem

$$(\wp^1_\epsilon)\qquad\begin{cases} -\Delta v = v^{\frac{\dim\mathbb{K}+3}{\dim\mathbb{K}-1}} & \text{in } \Theta_\epsilon,\\ v>0 & \text{in } \Theta_\epsilon,\\ v=0 & \text{on } \partial\Theta_\epsilon.\end{cases}$$

We prove the following result.

Theorem 1.13. *There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, the supercritical problem $(\wp_\epsilon^1)$ has an $\mathbb{S}_\mathbb{K}$-invariant solution v_ϵ which concentrates and blows up along the sphere $\mathbb{S}_\mathbb{K} z_0$ as $\epsilon \to 0$.*

Let Φ be an $[O(m) \times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$ such that $0 \notin \overline{\Phi}$ and $(y_0, 0) \in \Phi$. We write $S_0^{m-1} := \{(y, 0) : |y| = |y_0|\}$ for the $[O(m) \times O(m)]$-orbit of $(y_0, 0)$, and for each $\epsilon > 0$ small enough we set

$$\Phi_\epsilon := \{x \in \Phi : \mathrm{dist}(x, S_0^{m-1}) > \epsilon\}.$$

This is again an $[O(m) \times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$. We consider the supercritical problem

$$(\wp_\epsilon^2) \qquad \begin{cases} -\Delta v = v^{\frac{m+3}{m-1}} & \text{in } \Phi_\epsilon, \\ v > 0 & \text{in } \Phi_\epsilon, \\ v = 0 & \text{on } \partial\Phi_\epsilon. \end{cases}$$

We prove the following

Theorem 1.14. *There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem $(\wp_\epsilon)$ has an $[O(m) \times O(m)]$-invariant solution v_ϵ which concentrates and blows up along the $(m-1)$-dimensional sphere S_0^{m-1} as $\epsilon \to 0$.*

1.3.4 Nonexistence

As we already mentioned the domains Θ_k in Passaseo's examples [37, 39] have the homotopy type of $\mathbb{S}^k$. This shows that the condition $\widetilde{H}_*(\Theta, \mathbb{Z}/2) \neq 0$ is not enough to guarantee a solution for the problem $(\wp_{p,\Theta})$ when $p > 2^*$. But, one may think that perhaps if $\widetilde{H}_*(\Theta, \mathbb{Z}/2)$ is richer, then $(\wp_{p,\Theta})$ will have a nontrivial solution when $p > 2^*$. The next result shows that this is not true in general.

Theorem 1.15. *Given $k = k_1 + \cdots + k_m$ with $k_i \in \mathbb{N}$ and $k \leq N - 3$, and $\varepsilon > 0$ there exists a bounded smooth domain Θ in $\mathbb{R}^N$, such that*

- *Θ has the homotopy type of $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_m}$,*
- *the problem $(\wp_{p,\Theta})$ does not have a nontrivial solution for $p \geq 2^*_{N,k} + \varepsilon$,*
- *the problem $(\wp_{p,\Theta})$ has infinitely many solutions for $p \in (2, 2^*_{N,k})$.*

In particular, if we take all $k_i = 1$, the domain Θ in Theorem 1.15 is homotopy equivalent to the product of k circles. In this case, Θ satisfies not only that the homology is nontrivial but that there are k different cohomology classes in $H^1(\Theta; \mathbb{Z})$ whose cup-product is the generator of $H^k(\Theta; \mathbb{Z})$. Hence, the cup-length of Θ equals $k + 1$.

In Chapter 4 we will give the proof of Theorem 1.15 as well as a nonexistence result which gives more examples of domains Θ in which problem $(\wp_{p,\Theta})$ does not have a nontrivial solution.

1.4 Open problems

In this subsection we indicate some of the open problems which are motivated by this work and which we plan to continue investigating

1. A different kind of reduction that leads to domains in which problem $(\wp_{p,\Theta})$ has solutions for supercritical p has been recently considered in [1, 27]. This reduction is performed by considering domains Θ obtained by rotation; namely, $\Theta := \{(y,z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Omega\}$ where Ω is a domain contained in the half-space $(0,\infty) \times \mathbb{R}^{N-k-1}$, $1 \leq k \leq N-3$. Then, solutions to the problem
$$(\wp_{p,a,\Omega}) \qquad \begin{cases} -\mathrm{div}(a(x)\nabla u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
with $a(x_1, \ldots, x_{N-k}) = x_1^k$, give rise to solutions of problem $(\wp_{p,\Theta})$ for the same exponent p.

   Let $q = \frac{2(N-k)}{N-k-2}$ be the critical exponent in the dimension of Ω. In [27] Kim and Pistoia used the Lyapunov-Schmidt reduction procedure to show that in a punctured domain $\Omega_\epsilon = \{x \in \Omega : \mathrm{dist}(x, \xi_0) \geq \epsilon\}$, $\xi_0 \in \Omega$, $\epsilon > 0$, the critical problem $(\wp_{q,a,\Omega_\epsilon})$ has multiple sign-changing solutions, whose shape resembles a tower of bubbles with alternating sign centered at the point ξ_0, and that the number of such solutions becomes arbitrary large as $\epsilon \to 0$. These solutions give rise to sign changing solutions to the supercritical problem $(\wp_{q,\Theta})$ which are towers of layers concentrating at a k-dimensional sphere in Θ as $\epsilon \to 0$.

   If one wishes to combine this kind of reduction with the one given by the Hopf fibrations in order to produce solutions to supercritical problems in new types of domains, one is lead to consider a more general problem of the form
$$(\wp^*_{a_1,0,a_3,\Omega}) \qquad \begin{cases} -\mathrm{div}(a_1(x)\nabla u) = a_3(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$
where Ω is a smooth bounded domain in $\mathbb{R}^n$, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent in dimension n, and a and b are positive functions on $\overline{\Omega}$.

   I believe that, with some modifications, the argument used by Kim and Pistoia in [27] can be adapted to produce towers of bubbles for problem $(\wp^*_{a_1,0,a_3,\Omega})$ in a punctured domain Ω_ϵ for ϵ small. This would allow us to extend Theorems 1.8, 1.13 and 1.14, as well as the results in [27], and to produce new solutions to supercritical problems. I am currently investigating this question.

2. Apart from the fact that it provides solutions for supercritical problems, problem $(\wp^*_{a,b,\Omega})$ has an interest of its own. Many well-known results for the autonomous equation $a = b \equiv 1$, obtained by variational methods, cannot be extended to the nonautonomous problems in a straightforward manner. The lack of compactness

is stronger in the nonautonomous case because, in general, there are no energy gaps where the Palais-Smale condition holds. Establishing existence for this problem is a challenging question which I wish to continue investigating.

3. I would also like to apply the techniques I have used in my Ph.D. research to other related problems, for example to equations involving the fractional Laplacian. In [8] Capella used the technique for sign-changing solutions developed in [17] to obtain a multiplicity result for the pure critical exponent problem involving the half Laplacian in an annular-shaped domain. I believe that the techniques used in [13, 14] can be adapted to obtain solutions in more general domains, and for supercritical nonlinearities.

4. In [9] Choi, Kim and Lee established existence of multiple bubbling solutions of nonlinear elliptic equations involving the fractional Laplacians and critical exponents. I would like to investigate whether their results can be extended to a nonautonomous problem and to supercritical nonlinearities.

Chapter 2

The nonautonomous critical problem

In this chapter we consider the anisotropic critical problem

$$(\wp^*_{a_1,a_2,a_3,\Omega}) \qquad \begin{cases} -\operatorname{div}(a_1(x)\nabla u) + a_2(x)u = a_3(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- Ω is a smooth bounded domain in $\mathbb{R}^n$,
- $a_i : \overline{\Omega} \to \mathbb{R}$ is a continuous function for $i = 1, 2, 3$ and satisfies that
  - $\min\limits_{x\in\overline{\Omega}} a_i(x) > 0$ for $i = 1, 3$ and
  - $\min\limits_{x\in\overline{\Omega}} a_2(x) > -\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the $-\Delta$ in $H^1_0(\Omega)$.

As we mentioned in the introduction, the solutions to problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ will provide solutions for some supercritical problems, but apart from this property, this problem has an interest of its own.

The aim of this chapter is to prove Theorems 1.7 and 1.8 stated in the introduction.

2.1 Multiplicity in symmetric domains

As before, let $O(n)$ be the group of linear isometries of $\mathbb{R}^n$. For every G closed subgroup of $O(n)$ we will denote by

$$Gx := \{gx : g \in G\}$$

the G-orbit of x and by $\#Gx$ its cardinality. A domain $\Omega \subset \mathbb{R}^n$ is called G invariant if

$$Gx \subset \Omega \text{ for all } x \in \Omega.$$

A function $u : \Omega \to \mathbb{R}$ is called G-invariant if u is constant on every orbit Gx.

Now, fix a closed subgroup Γ on $O(n)$ and a nonempty Γ-invariant smooth domain $D \subset \mathbb{R}^n$ such that $\#\Gamma x = \infty$ for all $x \in D$. Also suppose that a_i is Γ-invariant for $i = 1, 2, 3$. We shall prove that

Theorem 1.7 *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ and D, with the following property: If Ω contains D and if it is invariant under the action of a closed subgroup G of Γ for which*

$$\left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}}\#Gx}{a_3(x)^{\frac{n-2}{2}}}\right) > \ell_m$$

holds, then problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ has at least m pairs of G-invariant solutions $\pm u_1, \ldots, \pm u_m$ such that u_1 is positive, $u_2, \ldots, u_m$ change sign, and

$$\int_\Omega |\nabla u_k|^2 \leq \ell_k S^{\frac{n}{2}} \quad \textit{for every } k = 1, \ldots, m.$$

Theorem 1.7 provides many examples of domains Ω having only finite symmetries in which problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ has a prescribed number of solutions. Specific examples may be obtained as follows:

Example 1. *Let $\Gamma = \mathbb{S}_{\mathbb{C}} := \{c \in \mathbb{C} : |c| = 1\}$. We can think Γ as a subgroup of $O(n)$ in the obvious way. The function*

$$\begin{array}{rcl} \Gamma \times (\mathbb{C} \times \mathbb{R}^{n-2}) & \mapsto & \mathbb{C} \times \mathbb{R}^{n-2} \\ (c, (a, y)) & \mapsto & (ca, y) \end{array}$$

is an action of the group Γ on $\mathbb{C} \times \mathbb{R}^{n-2} = \mathbb{R}^n$. Now, let D_0 be a smooth bounded domain in $\mathbb{R}^{n-1}$ such that

$$D_0 \subset \{(t, y) \in \mathbb{R} \times \mathbb{R}^{n-2} : t > d\}$$

for some $d > 0$, and set

$$D := \{(a, y) \in \mathbb{C} \times \mathbb{R}^{n-2} : (|a|, y) \in \Omega_0\} \subset \mathbb{R}^n.$$

Note that D is Γ-invariant and $\#\Gamma x = \infty \quad \forall x \in D$.

Consider the subgroup

$$G_r := \{e^{2\pi ik/r} : k = 0, \ldots, r-1\} \subset \Gamma,$$

and notice that

$$\#G_r x = r \quad \forall x \in (\mathbb{C}\backslash\{0\}) \times \mathbb{R}^{n-2}.$$

Therefore, if $\Omega \subset (\mathbb{C}\backslash\{0\}) \times \mathbb{R}^{n-2}$ *is an* G_r*-invariant domain and* $\Omega \supset D$*, then*

$$\min_{x\in\Omega} \#Gx = r.$$

So, if we fix $m \in \mathbb{N}$*, then*

$$r \cdot \left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}}}{a_3(x)^{\frac{n-2}{2}}} \right) > \ell_m$$

if r *is large enough. Hence, Theorem 1.7 yields at least* m *pairs of solutions to problem* $(\wp^*_{a_1,a_2,a_3,\Omega})$.

In order to give the proof of Theorem 1.7 we need to introduce some notation and recall some basic notions.

It is well known that, by the principle of symmetric criticality of Palais [35], the G-invariant solutions to problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ are the critical points of the restriction of the functional

$$\begin{aligned} J(u) &:= \frac{1}{2}\int_\Omega (a_1|\nabla u|^2 + a_2 u^2) - \frac{1}{2^*}\int_\Omega a_3|u|^{2^*} \\ &= \frac{1}{2}||u||^2_{a_1,a_2} - \frac{1}{2^*}|u|^{2^*}_{a_3,2^*} \end{aligned}$$

to the space of G-invariant functions

$$H^1_0(\Omega)^G : \{u \in H^1_0(\Omega) : u(gx) = u(x) \text{ for all } g \in G, x \in \Omega\},$$

the non trivial critical points of J are contained in the Nehari manifold

$$\mathcal{N}^G(\Omega) = \{u \in H^1_0(\Omega)^G : ||u||^2_{a_1,a_2} = |u|^{2^*}_{a_3,2^*}\}.$$

We will make use of the following result

Theorem 2.1. *If* Ω *is* G*-invariant, and*

$$\#Gx = \infty \text{ for all } x \in \Omega$$

then, problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ *has infinitely many* G*-invariant solutions. Moreover, there exists a nontrivial* G*-invariant solution* u *to problem* $(\wp^*_{a_1,a_2,a_3,\Omega})$ *which satisfies*

$$J(u) = \inf_{w\in\mathcal{N}^G(\Omega)} J(w).$$

Proof. Since $\#Gx = \infty$ for all $x \in \Omega$, Corollary A.3 shows that the functional J satisfies condition $(PS)^G_c$ in $H^1_0(\Omega)$ for every $c \in \mathbb{R}$.

It is also clear that J satisfies all the other conditions of the mountain pass theorem [2]. Hence, there exists a nontrivial solution u to problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ which satisfies

$$J(u) = \inf_{w\in\mathcal{N}^G(\Omega)} J(w).$$

Moreover, since J is odd, it has an unbounded sequence of critical values.

□

As a corollary of Theorem 2.1 we have the following generalization of Kazdan's and Warner's result (Theorem 1.2)

Corollary 2.2 (Kazdan-Warner)**.** *If $\Omega = A_{a,b}$ is an annulus, then the problem $(\wp^*_{a_1,a_2,a_3,A_{a,b}})$ has infinitely many* ***radial*** *solutions.*

The main ingredient for the proof of Theorem 1.7 is the following mountain pass result for sign changing solutions.

Theorem 2.3. *Let W be a finite dimensional subspace of $H^1_0(\Omega)^G$. If J satisfies condition $(PS)^G_c$ in $H^1_0(\Omega)^G$ for all $c \leq \sup_W J$, then J has at least $\dim(W)-1$ pairs of sign changing critical points $u \in H^1_0(\Omega)^G$ such that $J(u) \leq \sup_W J$.*

Proof. The proof can be found, up to minor modifications, in [17, Theorem 3.7]. □

2.1.1 Proof of Theorem 1.7

Let $P_1(D)$ be the collection of all nonempty Γ-invariant bounded smooth domains contained in D and for every $k \geq 2$ we define

$$P_k(D) := \{(D_1, \dots, D_k) : D_i \in P_1(D), D_i \cap D_j = \emptyset \text{ if } i \neq j\}.$$

Note that $P_k(D) \neq \emptyset$ for all $k \in \mathbb{N}$. Moreover, since $D_i \subset D$ and D_i is Γ-invariant, we have that $\#\Gamma x = \infty$ for all $x \in D_i$. Hence, Theorem 2.1 shows that there exists a nontrivial solution ω_{D_i} to problem $(\wp^*_{a_1,a_2,a_3,D_i})$ which satisfies

$$J(\omega_{D_i}) = \inf_{\omega \in \mathcal{N}^\Gamma(D_i)} J(\omega).$$

In addition, if we extend ω_{D_i} by zero outside D_i, we have that $\omega_{D_i} \in H^1_0(\Omega)^G$ for every bounded G-invariant domain Ω such that $D_i \subset \Omega$ and every $G \subset \Gamma$. Moreover, we have that

$$J(\omega_{D_i}) = \max_{t \geq 0} J(t\omega_{D_i}). \tag{2.1}$$

For every $m \in \mathbb{N}$, let

$$c_m := \inf \left\{ \sum_{i=1}^{m} J(\omega_{D_i}) : (D_1, \dots, D_m) \in \mathcal{P}_m(D) \right\}.$$

Note that $c_1 = J(\omega_D) > 0$ and $J(\omega_{D_i}) \geq c_1$. Next, we will show that

$$c_m > c_{m-1} \qquad \text{for every } m > 1. \tag{2.2}$$

Indeed, if $\varepsilon \in (0, c_1)$ and $(D_1, \dots, D_m) \in \mathcal{P}_m(D)$, we have that $(D_1, \dots, D_{m-1}) \in \mathcal{P}_{m-1}(D)$ and

$$c_{m-1} + \varepsilon \leq \sum_{i=1}^{m-1} J(\omega_{D_i}) + \varepsilon < \sum_{i=1}^{m-1} J(\omega_{D_i}) + J(\omega_{D_m}) = \sum_{i=1}^{m} J(\omega_{D_i}).$$

Therefore,

$$c_{m-1}+\varepsilon \leq \sum_{i=1}^{m} J(\omega_{D_i}) \qquad \text{for every } (D_1,\ldots,D_m)\in \mathcal{P}_m(D), \tag{2.3}$$

which implies 2.2.

We define

$$\ell_k := \left(\frac{1}{n}S^{n/2}\right)^{-1} c_k.$$

We will prove that the sequence (ℓ_k) has the desired property. In order to do this fix $m\in\mathbb{N}$ and let Ω be a bounded smooth domain containing D, invariant under the action of a closed subgroup G of Γ for which

$$\left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}}\#Gx}{a_3(x)^{\frac{n-2}{2}}}\right) > \ell_m. \tag{2.4}$$

We divided the rest of proof into three steps.

Step 1: For every ε small enough, there exist a set $A(\varepsilon):=\{\pm u_1^{\varepsilon},\ldots,\pm u_m^{\varepsilon}\}$ which contains m pairs of G-invariant solutions $\pm u_1^{\varepsilon},\ldots,\pm u_m^{\varepsilon}$ such that u_1^{ε} is positive, $u_2^{\varepsilon},\ldots,u_m^{\varepsilon}$ change sign, and

$$J(u_k^{\varepsilon}) \leq c_m+\varepsilon \qquad \textit{for every } 1\leq k\leq m.$$

Notice that condition (2.4) is equivalent to

$$\left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}}\#Gx}{a_3(x)^{\frac{n-2}{2}}}\right)\frac{1}{n}S^{n/2} > c_m,$$

therefore, we can take $\varepsilon\in(0,c_1)$ such that $c_m+\varepsilon < \left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}}\#Gx}{a_3(x)^{\frac{n-2}{2}}}\right)\frac{1}{n}S^{n/2}$. By the definition of c_m we can choose an element $(D_1,\ldots,D_m)\in\mathcal{P}_m(D)$ such that the associated functions $\{\omega_{D_1},\ldots,\omega_{D_m}\}\subset H_0^1(\Omega)$ satisfy

$$c_m \leq \sum_{i=1}^{m} J(\omega_{D_i}) < c_m+\varepsilon.$$

Now, let W_m the linear subspace of $H_0^1(\Omega)^G$ generated by the set $\{\omega_{D_1},\ldots,\omega_{D_m}\}$. Notice that, since

$$supp\omega_{D_i}\cap supp\omega_{D_j} = D_i\cap D_j=\emptyset,$$

we have that $\dim W_m = m$. Moreover, equation (2.1) shows that

$$d_m := \sup_{W_m} J \leq \sum_{i=1}^{m} J(\omega_{D_i}) < \left(\min_{x\in\overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}}\#Gx}{a_3(x)^{\frac{n-2}{2}}}\right)\frac{1}{n}S^{n/2}.$$

Corollary A.3 implies that J satisfies $(PS)_c^G$ in $H_0^1(\Omega)$ for all $c \leq d_m$, so the mountain pass theorem [2] yields a positive critical point $u_1^\varepsilon \in H_0^1(\Omega)^G$ of J such that $J(u_1^\varepsilon) \leq d_1$. Moreover, Theorem 2.3 yields $m-1$ pairs of sign changing critical points $\pm u_2^\varepsilon, \dots, \pm u_m^\varepsilon \in H_0^1(\Omega)^G$ of J such that

$$J(u_k^\varepsilon) < d_m \leq c_m + \varepsilon \quad \text{for every } 1 \leq k \leq m.$$

Step 2: There exist m pairs of G-invariant solutions $\pm u_1^{'}, \dots, \pm u_m^{'}$ such that $u_1^{'}$ is positive, $u_2^{'}, \dots, u_m^{'}$ change sign, and

$$J(u_k^{'}) \leq c_m \quad \text{\textit{for every} } 1 \leq k \leq m. \tag{2.5}$$

Let $(\varepsilon_r) \subset (0, c_1)$ be a sequence such that $\varepsilon_r \to 0$. Also, for every ε_r, let $A(\varepsilon_r)$ be the set of solutions constructed in step 1.

If there exists $r_0 \in \mathbb{N}$ such that

$$J(w) \leq c_m \quad \text{for every } w \in A(\varepsilon_{r_0}),$$

we have that the proof of step 2 is complete. If this is not the case, we have that for every $r \in \mathbb{N}$ there exists a $w_r \in A(\varepsilon_r)$ such that

$$c_m < J(w_r) \leq c_m + \varepsilon_r$$

This implies that $J(w_r) \to c_m$, and since $\nabla J(w_r) = 0$, we have that (w_r) is a Palais-Smale sequence for J in c_m. Since J satisfies $(PS)_{c_m}^G$, we have that there exists $u_m^{'} \in H_0^1(\Omega)^G$ such that, after passing to a subsequence, we have

$$w_r \to u_m^{'} \text{ in } H_1^0(\Omega)^G$$

Therefore, $u_m^{'}$ is a critical point of J such that

$$J(u_m^{'}) = c_m.$$

On the other hand, note that if Ω satisfies condition (2.4) then it also satisfies that

$$\left(\min_{x \in \overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}} \# Gx}{a_3(x)^{\frac{n-2}{2}}} \right) \frac{1}{n} S^{\frac{n}{2}} > \ell_{m-1} \tag{2.6}$$

Therefore, applying the argument given in step 1, this time for the number $m-1$ and for ε small enough, we obtain $m-1$ pairs of G-invariant solutions $\pm u_1^{'}, \pm u_2^{'}, \dots, \pm u_{m-1}^{'}$ to problem $(\wp^*_{a_1,a_2,a_3,\Omega})$, such that $u_1^{'}$ is positive, $\pm u_2^{'}, \dots, \pm u_{m-1}^{'}$ change sign, and

$$J(u_i^{'}) \leq c_{m-1} + \epsilon < c_m \text{ for every } i = 1, 2, \dots, m-1.$$

Hence, $\pm u_1', \pm u_2', \ldots, \pm u_{m-1}', \pm u_m'$ are m pairs of G-invariant solutions, u_1' is positive and $u_2', \ldots, u_m'$ change sign and

$$J(u_k') \leq c_m \quad \text{for every } 1 \leq k \leq m. \tag{2.7}$$

This proves step 2.

Step 3: There exist m pairs of G-invariant solutions $\pm u_1, \ldots, \pm u_m$ such that u_1 is positive, $u_2, \ldots, u_m$ change sign, and

$$J(u_k) \leq c_k \quad \textit{for every } 1 \leq k \leq m. \tag{2.8}$$

Again, note that if Ω satisfies (2.4) then it also satisfies

$$\left(\min_{x \in \overline{\Omega}} \frac{a_1(x)^{\frac{n}{2}} \# Gx}{a_3(x)^{\frac{n-2}{2}}} \right) \frac{1}{n} S^{\frac{n}{2}} > \ell_k \tag{2.9}$$

for every $k = 1, \ldots, m-1$. So, applying the arguments given in steps 1 and 2 to each k, we obtain k pairs of G-invariant solutions $\pm u_1^k, \pm u_2^k, \ldots, \pm u_k^k$ to problem $(\wp^*_{a_1,a_2,a_3,\Omega})$, such that u_1^k is positive, $\pm u_2^k, \ldots, \pm u_k^k$ change sign, and

$$J(u_i^k) \leq c_k \text{ for every } i = 1, 2, \ldots, k.$$

Setting $u_1 := u_1^1$ and choosing $u_k \in \{u_2^k, \ldots, u_k^k\}$ inductively, such that $u_k \neq u_i$ for every $i = 1, \ldots, k-1$, we obtain m pairs of G-invariant solutions $\pm u_1, \pm u_2, \ldots, \pm u_m$ to problem $(\wp^*_{a_1,a_2,a_3,\Omega})$ such that u_1 is positive, $\pm u_2, \ldots, \pm u_m$ change sign, and

$$J(u_k) \leq c_k \text{ for every } k = 1, 2, \ldots, m. \tag{2.10}$$

Finally, note that, since

$$J(u_k) = \frac{1}{n} \int_\Omega |\nabla u_k|^2 \leq c_k = \frac{1}{n} \ell_m S^{\frac{n}{2}} \tag{2.11}$$

we have that equation (2.10) is equivalent to

$$\int_\Omega |\nabla u_k|^2 \leq \ell_k S^{\frac{n}{2}} \quad \text{for every } k = 1, \ldots, m.$$

this completes the proof.

$\square$

2.2 Existence in punctured domains

In this section we will show the existence of one positive solutions for a nonautonomous critical problem in every smooth bounded domain, not necessary symmetric, but with a small hole. We start with some notation.

Fix a closed subgroup Γ of $O(n)$ and a smooth bounded Γ-invariant domain Ω in $\mathbb{R}^n$. Let $\Gamma x := \{gx : g \in \Gamma\}$ be the Γ-orbit of $x \in \mathbb{R}^n$. We denote by

$$\Omega^\Gamma := \{x \in \Omega : gx = x \quad \forall g \in \Gamma\}$$

the set of Γ-fixed points in Ω. Assume $\xi_0 \in \Omega^\Gamma$ and write

$$\Omega_\epsilon := \{x \in \Omega : |x - \xi_0| > \epsilon\}.$$

Note that, since $\xi_0 \in \Omega^\Gamma$, Ω_ϵ is also Γ-invariant.

We consider the problem

$$(\wp^*_{Q,\epsilon}) \qquad \begin{cases} -\Delta u = Q(x) u^{\frac{n+2}{n-2}} & \text{in } \Omega_\epsilon, \\ u > 0 & \text{in } \Omega_\epsilon, \\ u = 0 & \text{on } \partial\Omega_\epsilon, \end{cases}$$

where $n \geq 3$ and the function $Q \in C^2(\overline{\Omega})$ is Γ-invariant and satisfies $\min_{x \in \overline{\Omega}} Q(x) > 0$.

We will prove the following result.

Theorem 1.8 *Assume that $\nabla Q(\xi_0) \neq 0$. Then there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem $(\wp^*_{Q,\epsilon})$ has a Γ-invariant solution u_ϵ which concentrates and blows up at the point ξ_0 as $\epsilon \to 0$.*

Remark 2.4. *Notice that if $\Gamma \equiv 1$ then, Theorem 1.8 shows the existence of one positive solution for the problem $(\wp^*_Q)$ in every smooth bounded, not necessarily symmetric, punctured domain.*

The proof of Theorem 1.8 uses the well-known Ljapunov-Schmidt reduction, adapted to the symmetric case. In the following section we sketch this reduction, highlighting the places where the symmetries play a role. In subsection 2.2.2 we give an expansion of the reduced energy functional and use it to prove Theorem 1.8.

2.2.1 The finite dimensional reduction

For every bounded domain $\mathcal{U}$ in $\mathbb{R}^n$ we take

$$(u, v) := \int_{\mathcal{U}} \nabla u \cdot \nabla v, \qquad \|u\| := \left(\int_{\mathcal{U}} |\nabla u|^2 \right)^{1/2},$$

as the inner product and its corresponding norm in $H^1_0(\mathcal{U})$. If we replace $\mathcal{U}$ by $\mathbb{R}^n$ these are the inner product and the norm in $D^{1,2}(\mathbb{R}^n)$. We write

$$\|u\|_r := (\int_{\mathcal{U}} |u|^r)^{1/r}$$

for the norm in $L^r(\mathcal{U})$, $r \in [1, \infty)$.

If $\mathcal{U}$ is Γ-invariant for some closed subgroup Γ of $O(n)$ we set

$$H_0^1(\mathcal{U})^\Gamma := \{u \in H_0^1(\mathcal{U}) : u \text{ is } \Gamma\text{-invariant}\}$$

and, similarly, for $D^{1,2}(\mathbb{R}^n)^\Gamma$ and $L^r(\mathcal{U})^\Gamma$.

It is well known that the standard bubbles

$$U_{\delta,\xi}(x) = \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}}, \qquad \alpha_n := [n(n-2)]^{\frac{n-2}{4}}, \quad \delta \in (0,\infty), \quad \xi \in \mathbb{R}^n,$$

are the only positive solutions of the equation

$$-\Delta U = U^p \quad \text{in } \ \mathbb{R}^n,$$

where $p := \frac{n+2}{n-2}$. Thus, the function $W_{\delta,\xi} := \gamma_0 U_{\delta,\xi}$, with $\gamma_0 := [Q(\xi_0)]^{\frac{-1}{p-1}}$, solves the equation

$$-\Delta W = Q(\xi_0) W^p \qquad \text{in } \ \mathbb{R}^n. \tag{2.12}$$

Let

$$\begin{aligned} \psi^0_{\delta,\xi} &:= \frac{\partial U_{\delta,\xi}}{\partial \delta} = \alpha_n \frac{n-2}{2} \delta^{\frac{n-4}{2}} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{n/2}}, \\ \psi^j_{\delta,\xi} &:= \frac{\partial U_{\delta,\xi}}{\partial \xi_j} = \alpha_n (n-2) \delta^{\frac{n-2}{2}} \frac{x_j - \xi_j}{(\delta^2 + |x-\xi|^2)^{n/2}}, \qquad j = 1, \dots, n. \end{aligned} \tag{2.13}$$

The space generated by these $n+1$ functions is the space of solutions to the problem

$$-\Delta \psi = p U^{p-1}_{\delta,\xi} \psi, \qquad \psi \in D^{1,2}(\mathbb{R}^n). \tag{2.14}$$

Note that

$$U_{\delta,\xi} \in D^{1,2}(\mathbb{R}^n)^\Gamma \qquad \text{iff} \qquad \xi \in (\mathbb{R}^n)^\Gamma$$

and, similarly, for every $j = 0, 1, \dots, n$,

$$\psi^j_{\delta,\xi} \in D^{1,2}(\mathbb{R}^n)^\Gamma \qquad \text{iff} \qquad \xi \in (\mathbb{R}^n)^\Gamma.$$

Let Ω be a Γ-invariant bounded smooth domain in $\mathbb{R}^n$, $Q \in C^2(\overline{\Omega})$ be positive and Γ-invariant, and $\xi_0 \in \Omega^\Gamma$. For $\epsilon > 0$ small enough set

$$\Omega_\epsilon := \{x \in \Omega : |x - \xi_0| > \epsilon\}.$$

Consider the orthogonal projection $P_\epsilon : D^{1,2}(\mathbb{R}^n) \to H_0^1(\Omega_\epsilon)$, i.e. if $W \in D^{1,2}(\mathbb{R}^n)$ then $P_\epsilon W$ is the unique solution to the problem

$$-\Delta (P_\epsilon W) = -\Delta W \quad \text{in } \ \Omega_\epsilon, \qquad P_\epsilon W = 0 \quad \text{on } \ \partial\Omega_\epsilon. \tag{2.15}$$

A consequence of the uniqueness is that $P_\epsilon W \in H_0^1(\Omega_\epsilon)^\Gamma$ if $W \in D^{1,2}(\mathbb{R}^n)^\Gamma$.

We denote by $G(x,y)$ the Green function of the Laplace operator in Ω with zero Dirichlet boundary condition and by $H(x,y)$ its regular part, i.e.

$$G(x,y) = \beta_n \left(\frac{1}{|x-y|^{n-2}} - H(x,y) \right),$$

where β_n is a positive constant depending only on n. The following estimates will play a crucial role in the proof of Theorem 1.8.

Lemma 2.5. *Assume that $\delta \to 0$ as $\epsilon \to 0$ and $\epsilon = o(\delta)$ as $\epsilon \to 0$. Fix $\eta \in \mathbb{R}^n$, set $\xi := \xi_0 + \delta\eta$, and define*

$$R(x) := P_\epsilon U_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \alpha_n \delta^{\frac{n-2}{2}} H(x,\xi) + \frac{\alpha_n}{\delta^{\frac{n-2}{2}}(1+|\eta|^2)^{\frac{n-2}{2}}} \frac{\epsilon^{n-2}}{|x-\xi_0|^{n-2}}.$$

Then there exists a positive constant c such that the following estimates hold true for every $x \in \Omega \smallsetminus B(\xi_0,\epsilon)$:

$$\begin{aligned}
|R(x)| &\le c\delta^{\frac{n-2}{2}} \left[\frac{\epsilon^{n-2}(1+\epsilon\delta^{-n+1})}{|x-\xi_0|^{n-2}} + \delta^2 + \left(\frac{\epsilon}{\delta}\right)^{n-2} \right],\\
|\partial_\delta R(x)| &\le c\delta^{\frac{n-4}{2}} \left[\frac{\epsilon^{n-2}(1+\epsilon\delta^{-n+1})}{|x-\xi_0|^{n-2}} + \delta^2 + \left(\frac{\epsilon}{\delta}\right)^{n-2} \right],\\
|\partial_{\xi_i} R(x)| &\le c\delta^{\frac{n}{2}} \left[\frac{\epsilon^{n-2}(1+\epsilon\delta^{-n})}{|x-\xi_0|^{n-2}} + \delta^2 + \frac{\epsilon^{n-2}}{\delta^{n-1}} \right].
\end{aligned}$$

Proof. See Lemma 3.1 in [24]. □

For each $\epsilon > 0$ and $(d,\eta) \in \Lambda^\Gamma := (0,\infty) \times (\mathbb{R}^n)^\Gamma$ set (see (2.12))

$$V_{d,\eta} := P_\epsilon W_{\delta,\xi} = \gamma_0 P_\epsilon U_{\delta,\xi} \qquad \text{with } \delta := d\epsilon^{\frac{n-2}{n-1}}, \quad \xi := \xi_0 + \delta\eta. \tag{2.16}$$

The map $(d,\eta) \mapsto V_{d,\eta}$ is a C^2-embedding of Λ^Γ as a submanifold of $H_0^1(\Omega_\epsilon)^\Gamma$, whose tangent space at $V_{d,\eta}$ is

$$K_{d,\eta}^\epsilon := \operatorname{span}\{P_\epsilon \psi_{\delta,\xi}^j : j = 0,1,\dots,n\}.$$

Note that, since $\xi_0, \eta \in (\mathbb{R}^n)^\Gamma$, also $\xi \in (\mathbb{R}^n)^\Gamma$ and, therefore, $K_{d,\eta}^\epsilon \subset H_0^1(\Omega_\epsilon)^\Gamma$. We write

$$K_{d,\eta}^{\epsilon,\perp} := \{\phi \in H_0^1(\Omega_\epsilon)^\Gamma : (\phi, P_\epsilon \psi_{\delta,\xi}^j) = 0 \ \text{ for } j = 0,1,\dots,n\}$$

for the orthogonal complement of $K_{d,\eta}^\epsilon$ in $H_0^1(\Omega_\epsilon)^\Gamma$, and $\Pi_{d,\eta}^\epsilon : H_0^1(\Omega_\epsilon)^\Gamma \to K_{d,\eta}^\epsilon$ and $\Pi_{d,\eta}^{\epsilon,\perp} : H_0^1(\Omega_\epsilon)^\Gamma \to K_{d,\eta}^{\epsilon,\perp}$ for the orthogonal projections, i.e.

$$\Pi_{d,\eta}^\epsilon(u) := \sum_{j=0}^n (u, P_\epsilon \psi_{\delta,\xi}^j) P_\epsilon \psi_{\delta,\xi}^j, \qquad \Pi_{d,\eta}^{\epsilon,\perp}(u) := u - \Pi_{d,\eta}^\epsilon(u).$$

Let $i_\epsilon^* : L^{\frac{2n}{n+2}}(\Omega_\epsilon) \to H_0^1(\Omega_\epsilon)$ be the adjoint operator to the embedding $i_\epsilon : H_0^1(\Omega_\epsilon) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega_\epsilon)$, i.e. $v = i_\epsilon^*(u)$ if and only if

$$(v, \varphi) = \int_{\Omega_\epsilon} u\varphi \qquad \forall \varphi \in C_c^\infty(\Omega_\epsilon)$$

if and only if

$$-\Delta v = u \quad \text{in } \Omega_\epsilon, \qquad v = 0 \quad \text{on } \partial\Omega_\epsilon. \tag{2.17}$$

Sobolev's inequality yields a constant $c > 0$, independent of ϵ, such that

$$\|i_\epsilon^*(u)\| \leq c\,\|u\|_{\frac{2n}{n+2}} \qquad \forall u \in L^{\frac{2n}{n+2}}(\Omega_\epsilon), \quad \forall \epsilon > 0. \tag{2.18}$$

Note again that

$$i_\epsilon^*(u) \in H_0^1(\Omega_\epsilon)^\Gamma \qquad \text{if} \;\; u \in L^{\frac{2n}{n-2}}(\Omega_\epsilon)^\Gamma.$$

We rewrite problem $(\wp_{Q,\epsilon}^*)$ in the following equivalent way:

$$\begin{cases} u = i_\epsilon^*\left[Q(x) f(u)\right], \\ u \in H_0^1(\Omega_\epsilon), \end{cases} \tag{2.19}$$

where $f(s) := (s^+)^p$ and $p := \frac{n+2}{n-2}$.

We shall look for a solution to problem (2.19) of the form

$$u_\epsilon = V_{d,\eta} + \phi \qquad \text{with } (d,\eta) \in \Lambda^\Gamma \text{ and } \phi \in K_{d,\eta}^{\epsilon,\perp}. \tag{2.20}$$

As usual, our goal will be to find $(d,\eta) \in \Lambda^\Gamma$ and $\phi \in K_{d,\eta}^{\epsilon,\perp}$ such that, for ϵ small enough,

$$\Pi_{d,\eta}^{\epsilon,\perp}[V_{d,\eta} + \phi - i_\epsilon^*(Qf(V_{d,\eta} + \phi))] = 0 \tag{2.21}$$

and

$$\Pi_{d,\eta}^{\epsilon}[V_{d,\eta} + \phi - i_\epsilon^*(Qf(V_{d,\eta} + \phi))] = 0. \tag{2.22}$$

First we will show that, for every $(d,\eta) \in \Lambda^\Gamma$ and ϵ small enough, there exists a unique $\phi \in K_{d,\eta}^{\epsilon,\perp}$ which satisfies (2.21). To this aim we consider the linear operator $L_{d,\eta}^\epsilon : K_{d,\eta}^{\epsilon,\perp} \to K_{d,\eta}^{\epsilon,\perp}$ defined by

$$L_{d,\eta}^\epsilon(\phi) := \phi - \Pi_{d,\eta}^{\epsilon,\perp}\, i_\epsilon^*[Qf'(V_{d,\eta})\phi].$$

It has the following properties.

Proposition 2.6. *For every compact subset D of Λ^Γ there exist $\epsilon_0 > 0$ and $c > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$ and each $(d,\eta) \in D$,*

$$\left\|L_{d,\eta}^\epsilon(\phi)\right\| \geq c\,\|\phi\| \qquad \textit{for all } \phi \in K_{d,\eta}^{\epsilon,\perp}, \tag{2.23}$$

and the operator $L_{d,\eta}^\epsilon$ is invertible.

Proof. The argument given in [24] to prove Lemma 5.1 carries over with minor changes to our situation. □

The following estimates may be found in [28].

Lemma 2.7. *For each $a, b, q \in \mathbb{R}$ with $a \geq 0$ and $q \geq 1$ there exists a positive constant c such that the following inequalities hold*

$$||a+b|^q - a^q| \leq \begin{cases} c \min\{|b|^q, a^{q-1}|b|\} & \text{if } 0 < q < 1, \\ c(|a|^{q-1}|b| + |b|^q) & \text{if } q \geq 1. \end{cases}$$

Again, the argument given to prove similar results in the literature carries over with minor changes to prove the following result. We include it this time to illustrate this fact and also because some of the estimates will be used later on.

Proposition 2.8. *For every compact subset D of Λ^Γ there exist $\epsilon_0 > 0$ and $c > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$ and for each $(d, \eta) \in D$, there exists a unique $\phi^\epsilon_{d,\eta} \in K^{\epsilon,\perp}_{d,\eta} \subset H^1_0(\Omega_\epsilon)^\Gamma$ which solves equation (2.21) and satisfies*

$$\left\|\phi^\epsilon_{d,\eta}\right\| \leq c\epsilon^{\frac{n-2}{n-1}}. \tag{2.24}$$

Moreover, the function $(d, \eta) \mapsto \phi^\epsilon_{d,\eta}$ is a C^1-map.

Proof. Note that $\phi \in K^{\epsilon,\perp}_{d,\eta}$ solves equation (2.21) if and only if ϕ is a fixed point of the operator $T^\epsilon_{d,\eta} : K^{\epsilon,\perp}_{d,\eta} \to K^{\epsilon,\perp}_{d,\eta}$ defined by

$$T^\epsilon_{d,\eta}(\phi) = (L^\epsilon_{d,\eta})^{-1}\Pi^{\epsilon,\perp}_{d,\eta} i^*_\epsilon \left[Qf(V_{d,\eta} + \phi) - Qf'(V_{d,\eta})\phi - Q(\xi_0)(\gamma_0 U_{\delta,\xi})^p\right].$$

We will prove that $T^\epsilon_{d,\eta}$ is a contraction on a suitable ball.

To this aim, we first show that there exist $\epsilon_0 > 0$ and $c > 0$ such that for, each $\epsilon \in (0, \epsilon_0)$,

$$\|\phi\| \leq c\epsilon^{\frac{n-2}{n-1}} \quad \Rightarrow \quad \left\|T^\epsilon_{d,\eta}(\phi)\right\| \leq c\epsilon^{\frac{n-2}{n-1}}. \tag{2.25}$$

From Proposition 2.6 we have that, for some $c > 0$ and ϵ small enough,

$$\left\|(L^\epsilon_{d,\eta})^{-1}\right\| \leq c \qquad \forall (d, \eta) \in D.$$

Using (A.6) we obtain

$$\begin{aligned} \left\|T^\epsilon_{d,\eta}(\phi)\right\| &\leq c \left\|Q\left[f(V_{d,\eta} + \phi) - f'(V_{d,\eta})\phi\right] - Q(\xi_0)(\gamma_0 U_{\delta,\xi})^p\right\|_{\frac{2n}{n+2}} \\ &\leq c \left\|Q\left[f(V_{d,\eta} + \phi) - f(V_{d,\eta}) - f'(V_{d,\eta})\phi\right]\right\|_{\frac{2n}{n+2}} \\ &\quad + c \left\|Qf(V_{d,\eta}) - Q(\gamma_0 U_{\delta,\xi})^p\right\|_{\frac{2n}{n+2}} + c\gamma_0^p \left\|\left[Q - Q(\xi_0)\right] U^p_{\delta,\xi}\right\|_{\frac{2n}{n+2}}. \end{aligned}$$

Using the mean value theorem, Lemma A.10 and the Hölder inequality we have that, for some $t \in (0,1)$,

$$\begin{aligned}\|Q\left[f(V_{d,\eta}+\phi)-f(V_{d,\eta})-f'(V_{d,\eta})\phi\right]\|_{\frac{2n}{n+2}} &\le c\,\|[f'(V_{d,\eta}+t\phi)-f'(V_{d,\eta})]\phi\|_{\frac{2n}{n+2}}\\ &\le c\,\|f'(V_{d,\eta}+t\phi)-f'(V_{d,\eta})\|_{n/2}\,\|\phi\|_{2^*}\\ &\le c(\|\phi\|_{2^*}+\|\phi\|_{2^*}^{\frac{4}{n-2}})\,\|\phi\|_{2^*}\\ &\le c(\|\phi\|_{2^*}^2+\|\phi\|_{2^*}^p).\end{aligned}$$

Moreover, using Lemma 2.5 one can show that

$$\begin{aligned}&\|Qf(V_{d,\eta})-Q(\gamma_0 U_{\delta,\xi})^p\|_{\frac{2n}{n+2}}\le c\left\|(P_\epsilon U_{\delta,\xi})^p-U_{\delta,\xi}^p\right\|_{\frac{2n}{n+2}}\\ &\le\left(c\int_{\Omega_\epsilon}\left|U_{\delta,\xi}^{p-1}(P_\epsilon U_{\delta,\xi}-U_{\delta,\xi})\right|^{\frac{2n}{n+2}}+c\int_{\Omega_\epsilon}|P_\epsilon U_{\delta,\xi}-U_{\delta,\xi}|^{p+1}\right)^{\frac{n+2}{2n}}\\ &\le c\delta,\end{aligned}\tag{2.26}$$

see inequality (6.4) in [24]. Finally, setting $y=\frac{x-\xi}{\delta}=\frac{x-\xi_0}{\delta}-\eta$ and $\widetilde{\Omega}_\epsilon := \{y\in\mathbb{R}^n : \delta y+\xi\in\Omega_\epsilon\}$, and using the mean value theorem, for some $t\in(0,1)$ we obtain

$$\begin{aligned}\left\|[Q-Q(\xi_0)]\,U_{\delta,\xi}^p\right\|_{\frac{2n}{n+2}}&=\left(\int_{\widetilde{\Omega}_\epsilon}|Q(\delta y+\delta\eta+\xi_0)-Q(\xi_0)|^{\frac{2n}{n+2}}\,U^{p+1}(y)dy\right)^{\frac{n+2}{2n}}\\ &=\delta\left(\int_{\widetilde{\Omega}_\epsilon}|\langle\nabla Q(t\delta y+t\delta\eta+\xi_0),y+\eta\rangle|^{\frac{2n}{n+2}}\,U^{p+1}(y)dy\right)^{\frac{n+2}{2n}}\\ &\le c\delta.\end{aligned}\tag{2.27}$$

This proves statement (2.25).

Next we show that we may choose $\epsilon_0>0$ such that, for each $\epsilon\in(0,\epsilon_0)$, the operator

$$T_{d,\eta}^\epsilon : \{\phi\in K_{d,\eta}^{\epsilon,\perp} : \|\phi\|\le c\epsilon^{\frac{n-2}{n-1}}\}\to\{\phi\in K_{d,\eta}^{\epsilon,\perp} : \|\phi\|\le c\epsilon^{\frac{n-2}{n-1}}\}$$

is a contraction and, therefore, has a unique fixed point, as claimed.

If $\phi_1,\phi_2\in\{\phi\in K_{d,\eta}^{\epsilon,\perp} : \|\phi\|\le c\epsilon^{\frac{n-2}{n-1}}\}$, using again the mean value theorem we obtain

$$\begin{aligned}\left\|T_{d,\eta}^\epsilon(\phi_1)-T_{d,\eta}^\epsilon(\phi_2)\right\|&\le c\,\|f(V_{d,\eta}+\phi_1)-f(V_{d,\eta}+\phi_2)-f'(V_{d,\eta})(\phi_1-\phi_2))\|_{\frac{2n}{n+2}}\\ &=c\,\|[f'(V_{d,\eta}+(1-t)\phi_1+\phi_2)-f'(V_{d,\eta})](\phi_1-\phi_2)\|_{\frac{2n}{n+2}}\\ &\le c\,\|f'(V_{d,\eta}+(1-t)\phi_1+\phi_2)-f'(V_{d,\eta})\|_{\frac{n}{2}}\,\|\phi_1-\phi_2\|_{2^*}\end{aligned}$$

for some $t \in [0,1]$, and arguing as before we conclude that

$$\|f'(V_{d,\eta} + (1-t)\phi_1 + \phi_2) - f'(V_{d,\eta})\|_{\frac{n}{2}} \leq c\left(\|(1-t)\phi_1 + \phi_2\|_{2^*} + \|(1-t)\phi_1 + \phi_2\|_{2^*}^{\frac{4}{n-2}}\right)$$
$$\leq c\left(\|\phi_1\|_{2^*} + \|\phi_2\|_{2^*} + \|\phi_1\|_{2^*}^{\frac{4}{n-2}} + \|\phi_2\|_{2^*}^{\frac{4}{n-2}}\right)$$

Hence, if ϵ is sufficiently small, it follows that

$$\left\|T^{\epsilon}_{d,\eta}(\phi_1) - T^{\epsilon}_{d,\eta}(\phi_2)\right\| \leq \kappa \|\phi_1 - \phi_2\|$$

with $\kappa \in (0,1)$.

Finally, a standard argument shows that $(d,\eta) \mapsto \phi^{\epsilon}_{d,\eta}$ is a C^1-map. This concludes the proof. $\square$

Consider the functional $J_\epsilon : H^1_0(\Omega_\epsilon) \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{1}{2}\int_{\Omega_\epsilon} |\nabla u|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} Q|u|^{p+1}.$$

It is well known that the critical points of J_ϵ are the solutions of problem (2.19). We define the reduced energy functional $\widetilde{J}^{\Gamma}_\epsilon : \Lambda^\Gamma \to \mathbb{R}$ by

$$\widetilde{J}^{\Gamma}_\epsilon(d,\eta) := J_\epsilon(V_{d,\eta} + \phi^{\epsilon}_{d,\eta}). \tag{2.28}$$

If $\Gamma = \{1\}$ is the trivial group, we simply write $\widetilde{J}_\epsilon$ instead of $\widetilde{J}^{\Gamma}_\epsilon$ and Λ instead of Λ^Γ.

Next we show that the critical points of $\widetilde{J}^{\Gamma}_\epsilon$ are Γ-invariant solutions of problem (2.19).

Proposition 2.9. *If $(d,\eta) \in \Lambda^\Gamma$ is a critical point of the function $\widetilde{J}^{\Gamma}_\epsilon$, then $V_{d,\eta} + \phi^{\epsilon}_{d,\eta} \in H^1_0(\Omega_\epsilon)^\Gamma$ is a critical point of the functional J_ϵ and, therefore, a Γ-invariant solution of problem* (2.19).

Proof. Assume first that Γ is the trivial group. Then $\Lambda = (0,\infty) \times \mathbb{R}^n$ and the statement is proved using similar arguments to those given to prove Lemma 6.1 in [21] or Proposition 2.2 in [24].

If Γ is an arbitrary closed subgroup of $O(n)$, then Λ^Γ is the set of Γ-fixed points in Λ of the action of Γ on the space $\mathbb{R} \times \mathbb{R}^n$ which is given by $g(t,x) := (t, gx)$ for $g \in \Gamma$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. By the principle of symmetric criticality [35], if $(d,\eta) \in \Lambda^\Gamma$ is a critical point of the function $\widetilde{J}^{\Gamma}_\epsilon$, then (d,η) is a critical point of $\widetilde{J}_\epsilon : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, and the result follows from the previous case. $\square$

2.2.2 The asymptotic expansion of the reduced energy functional

In order to find a critical point of $\widetilde{J}_\epsilon^\Gamma$ we will use the following asymptotic expansion of the functional $\widetilde{J}_\epsilon : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$.

Proposition 2.10. *The asymptotic expansion*

$$\widetilde{J}_\epsilon(d,\eta) = c_0 + Q(\xi_0)^{-\frac{2}{p-1}} F(d,\eta)\epsilon^{\frac{n-2}{n-1}} + o(\epsilon^{\frac{n-2}{n-1}})$$

holds true C^1-uniformly on compact subsets of Λ, where the function $F : (0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ is given by

$$F(d,\eta) := \begin{cases} \alpha d + \beta \frac{1}{(1+|\eta|^2)d} - \gamma \left\langle \frac{\nabla Q(\xi_0)}{Q(\xi_0)}, \eta \right\rangle d & \text{if } n = 3, \\ \beta \left(\frac{1}{(1+|\eta|^2)d}\right)^{n-2} - \gamma \left\langle \frac{\nabla Q(\xi_0)}{Q(\xi_0)}, \eta \right\rangle d & \text{if } n \geq 4. \end{cases} \tag{2.29}$$

for some positive constants c_0, α, β and γ.

Proof. We write

$$\begin{aligned} J_\epsilon(V_{d,\eta} + \phi^\epsilon_{d,\eta}) &= \frac{1}{2}\left\|V_{d,\eta} + \phi^\epsilon_{d,\eta}\right\|^2 - \frac{1}{p+1}\int_{\Omega_\epsilon} Q\left|V_{d,\eta} + \phi^\epsilon_{d,\eta}\right|^{p+1} \\ &= J_\epsilon(V_{d,\eta}) + \gamma_0 \int_{\Omega_\epsilon} (U^p_{\delta,\xi} - (P_\epsilon U_{\delta,\xi})^p)\phi^\epsilon_{d,\eta} \\ &- \gamma_0^p \int_{\Omega_\epsilon} [Q - Q(\xi_0)]\,(P_\epsilon U_{\delta,\xi})^p\,\phi^\epsilon_{d,\eta} + \frac{1}{2}\left\|\phi^\epsilon_{d,\eta}\right\|^2 \\ &- \frac{1}{p+1}\int_{\Omega_\epsilon} Q\left(\left|V_{d,\eta} + \phi^\epsilon_{d,\eta}\right|^{p+1} - |V_{d,\eta}|^{p+1} - (p+1)V^p_{d,\eta}\phi^\epsilon_{d,\eta}\right). \end{aligned}$$

Then, using Hölder's inequality and inequalities (2.24), (2.26) and (2.27) we obtain

$$\begin{aligned} J_\epsilon(V_{d,\eta} + \phi^\epsilon_{d,\eta}) &= J_\epsilon(V_{d,\eta}) + O\left(\epsilon^{\frac{2(n-2)}{n-1}}\right) \\ &= \gamma_0^2\left[\frac{1}{2}\int_{\Omega_\epsilon} U^p_{\delta,\xi}\,(P_\epsilon U_{\delta,\xi}) - \frac{1}{p+1}\int_{\Omega_\epsilon} |P_\epsilon U_{\delta,\xi}|^{p+1}\right] \\ &- \frac{1}{p+1}\gamma_0^{p+1}\int_{\Omega_\epsilon} [Q - Q(\xi_0)]\,|P_\epsilon U_{\delta,\xi}|^{p+1} + O\left(\epsilon^{\frac{2(n-2)}{n-1}}\right). \end{aligned} \tag{2.30}$$

Next, we compute the first summand on the right-hand side of equality (2.30). From

Lemma 2.5 we have that

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_\epsilon} U_{\delta,\xi}^p\left(P_\epsilon U_{\delta,\xi}\right) - \frac{1}{p+1}\int_{\Omega_\epsilon} |P_\epsilon U_{\delta,\xi}|^{p+1} \\
&= \frac{p-1}{2(p+1)}\int_{\Omega_\epsilon} U_{\delta,\xi}^{p+1} - \frac{1}{2}\int_{\Omega_\epsilon} U_{\delta,\xi}^p\left(P_\epsilon U_{\delta,\xi} - U_{\delta,\xi}\right) - \frac{1}{p+1}\int_{\Omega_\epsilon}\left||P_\epsilon U_{\delta,\xi}|^{p+1} - U_{\delta,\xi}^{p+1}\right| \\
&= \frac{p-1}{2(p+1)}\int_{\Omega_\epsilon} U_{1,0}^{p+1} - \frac{1}{2}\int_{\Omega_\epsilon} U_{\delta,\xi}^p\left(P_\epsilon U_{\delta,\xi} - U_{\delta,\xi}\right) + o(\epsilon^{\frac{n-2}{n-1}}) \\
&= \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} U_{1,0}^{p+1} + \frac{1}{2}\int_{\mathbb{R}^n} U_{\delta,\xi}^p \Upsilon_{\delta,\xi}^\epsilon + o(\epsilon^{\frac{n-2}{n-1}}),
\end{aligned}
$$

where

$$
\Upsilon_{\delta,\xi}^\epsilon(x) := \alpha_n \delta^{\frac{n-2}{2}} H(x,\xi) + \alpha_n \frac{1}{\delta^{\frac{n-2}{2}}(1+|\eta|^2)^{\frac{n-2}{2}}}\frac{\epsilon^{n-2}}{|x-\xi_0|^{n-2}}. \tag{2.31}
$$

Setting $x = \xi + \delta y$ we have

$$
\begin{aligned}
&\alpha_n \int_{\mathbb{R}^n} U_{\delta,\xi}^p \Upsilon_{\delta,\xi}^\epsilon \\
&= \alpha_n \int_{\mathbb{R}^n} U_{\delta,\xi}^p(x)(\delta^{\frac{n-2}{2}} H(x,\xi))dx + \alpha_n \int_{\mathbb{R}^n} U_{\delta,\xi}^p(x)\left(\frac{1}{\delta^{\frac{n-2}{2}}(1+|\eta|^2)^{\frac{n-2}{2}}}\frac{\epsilon^{n-2}}{|x-\xi_0|^{n-2}}\right)dx \\
&= \alpha_n \delta^{n-2}\int_{\mathbb{R}^n} U_{1,0}^p(y) H(\delta y + \delta\eta + \xi_0, \delta\eta + \xi_0)dy \\
&+ \alpha_n \frac{1}{(1+|\eta|^2)^{\frac{n-2}{2}}}\int_{\mathbb{R}^n} U_{1,0}^p(y)\left(\frac{\epsilon^{n-2}}{\delta^{n-2}|y-\eta|^{n-2}}\right)dy \\
&= \alpha_n\left(\int_{\mathbb{R}^n} U_{1,0}^p\right) H(\xi_0,\xi_0)\delta^{n-2}(1+o(1)) + \alpha_n g(\eta)\frac{1}{\delta^{n-2}}\epsilon^{n-2}(1+o(1)),
\end{aligned}
$$

where the function $g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$
g(\eta) := \frac{1}{(1+|\eta|^2)^{\frac{n-2}{2}}}\int_{\mathbb{R}^n}\frac{1}{|y-\eta|^{n-2}} U_{1,0}^p(y)dy.
$$

Since $-\Delta U = U^p$ in $\mathbb{R}^n$, an easy computation shows that

$$
g(\eta) = \frac{1}{(1+|\eta|^2)^{\frac{n-2}{2}}} U_{1,0}(\eta) = \alpha_n \frac{1}{(1+|\eta|^2)^{n-2}}.
$$

To compute the second summand on the right-hand side of equality (2.30) we use the Taylor expansion

$$
Q(\delta y + \xi_0 + \delta\eta) = Q(\xi_0) + \delta\langle \nabla Q(\xi_0), y + \eta\rangle + O(\delta^2(1+|y|^2))
$$

, and using conditions (2.16), we obtain

$$\begin{aligned}
&\int_{\Omega_\epsilon} [Q - Q(\xi_0)] \, |P_\epsilon U_{\delta,\xi}|^{p+1} = \int_{\Omega_\epsilon} [Q - Q(\xi_0)] \, U_{\delta,\xi}^{p+1} + o(\epsilon^{\frac{n-2}{n-1}}) \\
&= \int_{\widetilde{\Omega}_\epsilon} (Q(\delta y + \xi_0 + \delta\eta) - Q(\xi_0)) U_{1,0}^{p+1}(y) dy + o(\epsilon^{\frac{n-2}{n-1}}) \\
&= \delta \int_{\mathbb{R}^n} \langle \nabla Q(\xi_0), \eta \rangle U_{1,0}^{p+1}(y) dy + \delta \int_{\mathbb{R}^n} \frac{\langle \nabla Q(\xi_0), y \rangle}{(1+|y|^2)^n} dy + O\left(\epsilon^{\frac{2(n-2)}{n-1}}\right) \\
&= \delta \langle \nabla Q(\xi_0), \eta \rangle \left(\int_{\mathbb{R}^n} U_{1,0}^{p+1} \right) (1 + o(1)),
\end{aligned}$$

because $\int_{\mathbb{R}^n} \frac{\langle \nabla Q(\xi_0), y\rangle}{(1+|y|^2)^n} dy = 0$. Collecting all the previous information we obtain

$$\begin{aligned}
&\widetilde{J}_\epsilon(d,\eta) = J_\epsilon(V_{d,\eta} + \phi^\epsilon_{d,\eta}) \\
&= \begin{cases} c_0 + \gamma_0^2 \left(c_1 H(\xi_0,\xi_0) d + c_2 g(\eta) \frac{1}{d} - c_3 \left\langle \frac{\nabla Q(\xi_0)}{Q(\xi_0)}, \eta \right\rangle d \right) \sqrt{\epsilon} + o(\sqrt{\epsilon}) & \text{if } n = 3, \\ c_0 + \gamma_0^2 \left(c_2 g(\eta) \frac{1}{d^{n-2}} - c_3 \left\langle \frac{\nabla Q(\xi_0)}{Q(\xi_0)}, \eta \right\rangle d \right) \epsilon^{\frac{n-2}{n-1}} + o(\epsilon^{\frac{n-2}{n-1}}) & \text{if } n \geq 4, \end{cases}
\end{aligned}$$

as claimed. □

2.2.3 Proof of Theorem 1.8

We will show that the function F defined in (2.29) has a critical point $(d_0, \eta_0) \in \Lambda^\Gamma = (0,\infty) \times (\mathbb{R}^n)^\Gamma$ which is stable under C^1-perturbations. Then, we deduce from Proposition 2.10 that the functional $\widetilde{J}_\epsilon^\Gamma$ has a critical point in Λ^Γ for ϵ small enough, so the result follows from Proposition 2.9.

Let $n = 3$. Set $\zeta_0 := \frac{\nabla Q(\xi_0)}{Q(\xi_0)}$ and consider the half space $\mathcal{H} := \{\eta \in \mathbb{R}^3 : \alpha - \gamma\langle \zeta_0, \eta \rangle > 0\}$. For each $\eta \in \mathcal{H}$ there exists a unique $d = d(\eta)$, given by

$$d(\eta) = \sqrt{\frac{\beta}{(1+|\eta|^2)(\alpha - \gamma\langle\zeta_0,\eta\rangle)}} \in (0,\infty),$$

such that $F_d(d,\eta) = 0$. Moreover, $F_{dd}(d(\eta),\eta) > 0$ for any $\eta \in \mathcal{H}$. Consider the function $\widetilde{F} : \mathcal{H} \to \mathbb{R}$ defined by

$$\widetilde{F}(\eta) := F(d(\eta),\eta) = 2\beta^2 \sqrt{\frac{\alpha - \gamma\langle\zeta_0,\eta\rangle}{1+|\eta|^2}}.$$

The point

$$\eta_0 := \left(\frac{\alpha - \sqrt{\alpha^2 + \gamma^2 |\zeta_0|^2}}{\gamma |\zeta_0|^2} \right) \zeta_0$$

is a strict maximum point of $\widetilde{F}$. Setting $d_0 := d(\eta_0)$ we deduce from Lemma 5.7 in [30] that (d_0, η_0) is a C^1-stable critical point of the function F. Note that, since $\xi_0 \in \Omega^\Gamma$ and Q is Γ-invariant, $\nabla Q(\xi_0) \in (\mathbb{R}^n)^\Gamma$. Hence, $(d_0, \eta_0) \in \Lambda^\Gamma$.

If $n \geq 4$ arguing as in the previous case we easily conclude that, if

$$\eta_0 := -\frac{\nabla Q(\xi_0)}{|\nabla Q(\xi_0)|}, \qquad d_0 := \left(\frac{(n-2)\beta}{2^{n-2}\gamma} \frac{Q(\xi_0)}{|\nabla Q(\xi_0)|} \right)^{\frac{1}{n-1}},$$

then (d_0, η_0) is a C^1-stable critical point of the function F and $(d_0, \eta_0) \in \Lambda^\Gamma$. This concludes the proof. $\square$

Chapter 3

The supercritical problem.

We consider the supercritical problem

$$(\wp_{p,\Theta}) \qquad \begin{cases} -\Delta v = |v|^{p-2}\, v & \text{in } \Theta, \\ v = 0 & \text{on } \partial\Theta, \end{cases}$$

where Θ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and $p > 2^*$.

The main objective of this chapter is to discuss and prove Theorems 1.9–1.14 stated in the introduction.

3.1 Reduction to a nonautonomous critical problem

3.1.1 Reductions via Hopf maps

Let $N = 4, 8$ or 16 and write $\mathbb{R}^N = \mathbb{K} \times \mathbb{K}$ where $\mathbb{K}$ is either the complex numbers, the quaternions or the Cayley numbers. The set of units $\mathbb{S}_{\mathbb{K}} := \{\zeta \in \mathbb{K} : |\zeta| = 1\}$ acts on $\mathbb{R}^N$ by multiplication on each coordinate, i.e. $\zeta(z_1, z_2) := (\zeta z_1, \zeta z_2)$. Recall that the Hopf map

$$\hbar_{\mathbb{K}} : \mathbb{K} \times \mathbb{K} \mapsto \mathbb{R}^{dim\mathbb{K}+1}$$

is given by

$$\hbar_{\mathbb{K}}(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2\overline{z_1} z_2).$$

We shall make use of the following results, a detailed account is given in Aprendix **??**.

Proposition 3.1. *Let $N = 2, 4, 8, 16$, $p > 2$ and Ω be a smooth bounded domain in $\mathbb{R}^{dim\mathbb{K}+1}$ such that $0 \notin \overline{\Omega}$. If u is a solution of*

$$\begin{cases} -\Delta u = \frac{1}{2|x|}|u|^{p-2}u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.1}$$

then $v = u \circ \hbar_{\mathbb{K}}$ *is an* $\mathbb{S}_{\mathbb{K}}$*-invariant solution of problem*

$$\begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \hbar_{\mathbb{K}}^{-1}(\Omega) := \Theta \\ v = 0 & \text{on } \partial\Theta, \end{cases} \tag{3.2}$$

We will also make use of a variation of this reduction, which was proved by Pacella and Srikanth in [34].

Let $O(m) \times O(m)$ acts on $\mathbb{R}^{2m} \equiv \mathbb{R}^m \times \mathbb{R}^m$ in the obvious way and $O(m)$ act on the last m coordinates of $\mathbb{R}^{m+1} \equiv \mathbb{R} \times \mathbb{R}^m$. Write the elements of $\mathbb{R}^{2m}$ as (y_1, y_2) with $y_i \in \mathbb{R}^m$ and the elements of $\mathbb{R}^{m+1}$ as $x = (t, \zeta)$ with $t \in \mathbb{R}$, $\zeta \in \mathbb{R}^m$.

Proposition 3.2. *Let* $N = 2m$, $m \geq 2$, *and* Θ *be an* $[O(m) \times O(m)]$*-invariant bounded smooth domain in* $\mathbb{R}^{2m}$ *such that* $0 \notin \overline{\Theta}$. *Set*

$$\Omega := \{(t, \zeta) \in \mathbb{R} \times \mathbb{R}^m : \mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|) = (t, |\zeta|) \text{ for some } (y_1, y_2) \in \Theta\}.$$

If $u(t, \zeta) = \mathfrak{u}(t, |\zeta|)$ *is an* $O(m)$*-invariant solution of problem*

$$\begin{cases} -\Delta u = \frac{1}{2|x|} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

then $v(y_1, y_2) := \mathfrak{u}(\mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|))$ *is an* $[O(m) \times O(m)]$*-invariant solution of problem*

$$\begin{cases} -\Delta v = |v|^{p-2}v & \text{in } \Theta, \\ v = 0 & \text{on } \partial\Theta, \end{cases}$$

3.1.2 Reduction via rotations

Next we will present a reduction via rotations which allow us to reduce some supercritical problems to some critical or subcritical *anisotropic problem.*

Fix $k := k_1 + \cdots + k_m$ with $k \leq N - 2$ and $k_i \in \mathbb{N} \cup \{0\}$ for $1 \leq i \leq m$. Let $\Omega \subset \mathbb{R}^{N-k}$ be a smooth bounded domain with

$$\overline{\Omega} \subset \{(x_1, \ldots, x_m, x') \in \mathbb{R}^m \times \mathbb{R}^{N-k-m} : x_i > 0, i = 1, \ldots, m\}. \tag{3.4}$$

Let $H := O(k_1 + 1) \times \cdots \times O(k_m + 1)$ and think H as a subgroup of $O(N)$. Let

$$\begin{aligned} (g_1, \ldots, g_m) \in H \quad &\text{with} \quad g_i \in O(k_i + 1) \text{ and} \\ (y^1, \ldots, y^m, z) \in \mathbb{R}^N \quad &\text{with} \quad y^i \in \mathbb{R}^{k_i+1} \text{ and } z \in \mathbb{R}^{N-k-m}. \end{aligned}$$

The function

$$\begin{gathered} \Gamma \times \mathbb{R}^N \mapsto \mathbb{R}^N \\ (g_1, \cdots, g_m)(y^1, \cdots, y^m, z) \mapsto (g_1 y^1, \cdots, g_m y^m, z) \end{gathered}$$

is an action of the group H on $\mathbb{R}^N$. Therefore, if we set

$$\Theta := \{(y^1, \ldots, y^m, z) \in \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times \mathbb{R}^{N-k-m} : (|y^1|, \ldots, |y^m|, z) \in \Omega\}. \tag{3.5}$$

one has immediately that Θ is H-invariant. Under this assumption the following holds true

Lemma 3.3. *Let Θ, Ω and H as before. Then we have that an H-invariant function $u(y^1, \cdots, y^m, z) = \mathfrak{u}(|y^1|, \cdots, |y^m|, z)$ is a solution to problem*

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if and only if $\mathfrak{u}$ satisfies

$$\begin{cases} -div(\mathfrak{a}(x)\nabla\mathfrak{u}) = \mathfrak{a}(x)|\mathfrak{u}|^{p-2}\mathfrak{u} & \text{in } \Theta, \\ \mathfrak{u} = 0 & \text{on } \partial\Theta, \end{cases}$$

where $\mathfrak{a} : \mathbb{R}^{N-k} \to \mathbb{R}$ is defined by $\mathfrak{a}(x_1, \ldots x_{N-k}) := x_1^{k_1} \ldots x_m^{k_m}$.

3.2 Multiplicity for supercritical problems in symmetric domains

3.2.1 Multiplicity via Hopf fibrations

This subsection is devoted to proof and discuss Theorems 1.9 and 1.10, we recall these results below

Theorem 1.9 *Let G be a closed subgroup of $O(dim\mathbb{K} + 1)$, Ω be a G-invariant bounded smooth domain in $\mathbb{R}^{dim\mathbb{K}+1}$ all of whose G-invariant orbits are infinite. Set $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$. Then, for $q = 2^*_{N,\dim\mathbb{K}-1}$, the supercritical problem $(\wp_{q,\Theta})$ has infinitely many solutions in Θ which are constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x \in \Omega$.*

We fix a closed subgroup Γ of $O(\dim\mathbb{K} + 1)$ and a nonempty Γ-invariant bounded smooth domain D in $\mathbb{R}^{\dim\mathbb{K}+1}$ such that $\#\Gamma x = \infty$ for all $x \in D$. We obtain following result.

Theorem 1.10 *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ and D, with the following property: If Ω contains D and if it is invariant under the action of a closed subgroup G of Γ for which*

$$\min_{x\in\overline{\Omega}} (\#Gx)\,|2x|^{\frac{\dim\mathbb{K}-1}{2}} > \ell_m$$

holds, then, for $q = 2^*_{N,\dim\mathbb{K}-1}$, problem $(\wp_{q,\Theta})$ has at least m pairs of solutions $\pm v_1, \ldots, \pm v_m$ in $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$, which are constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x \in \Omega$. In particular, they are $\mathbb{S}_{\mathbb{K}}$-invariant. Moreover, v_1 is positive and $v_2, \ldots, v_m$ change sign.

Note that the domains Θ in Theorems 1.9 and 1.10 are homeomorphic to $\mathbb{S}_{\mathbb{K}} \times U$, where $\mathbb{S}_{\mathbb{K}} := \{\xi \in \mathbb{K} : |\xi| = 1\}$ is the set of units in $\mathbb{K}$.

Theorems 1.9 and 1.10 provide a new type of domains in which some supercritical problems has multiple solutions. For example

Example 2. *Fix a bounded smooth domain D_0 in $\mathbb{R}^2$ with $\overline{D_0} \subset (0,\infty)\times\mathbb{R}$ and set*

$$D := \{(z,t) \in \mathbb{K}\times\mathbb{R} : (|z|, t) \in D_0\}.$$

Then D is invariant under the action of the group $\Gamma := \mathbb{S}_{\mathbb{C}}$ of unit complex numbers on $\mathbb{K}\times\mathbb{R}$ given by

$$e^{i\theta}(z,t) := (e^{i\theta}z, t)$$

Note also that

$$\#\Gamma x = \infty \quad \forall x \in D.$$

then Theorem 1.9 shows that problem $(\wp_{2^*_{N,\dim\mathbb{K}-1}, \Theta})$ has infinitely many solutions in $\Theta := \hbar_{\mathbb{K}}^{-1}(D)$ which are constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x \in D$.

On the other hand, Theorem 1.10 provides examples of *less symmetrical domains* Θ in where some supercritical problems has a prescribed number of solutions, for instance

Example 3. *Let D as in Example 2 and let G_r the cyclic subgroup of Γ of order r, i.e.*

$$G_r := \{e^{2\pi ik/n} : k = 0, ..., r-1\}.$$

Then $\#G_r x = r$ for every $x \in (\mathbb{K}\setminus\{0\})\times\mathbb{R}$.

Set $\Omega \subset \mathbb{K}\times\mathbb{R}$ an G_r-invariant bounded smooth domain with

$$D \subset \Omega \subset (\mathbb{K}\setminus\{0\})\times\mathbb{R},$$

Notice that

$$\min_{x\in\overline{\Omega}} (\#G_r x) = r,$$

and therefore

$$\min_{x\in\overline{\Omega}} (\#G_r x) \, |2x|^{\frac{\dim\mathbb{K}-1}{2}} = \min_{x\in\overline{\Omega}} r \, |2x|^{\frac{\dim\mathbb{K}-1}{2}} > \ell_m$$

if r is large enough. Hence, Theorem 1.10 yields at least m pairs of solutions to problem $(\wp_{2^*_{N,\dim\mathbb{K}-1}, \Theta})$ in $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$ which are constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x \in \Omega$.

Proof of Theorem 1.9

Let $N = 4, 8$ or 16, $\mathbb{R}^N = \mathbb{K}\times\mathbb{K}$ and fix a closed subgroup G of $O(\dim\mathbb{K}+1)$. Proposition 3.1 shows that if a function u is a solution to problem

$$\begin{cases} -\Delta u = \frac{1}{2|x|}|u|^{2^*_{N,\dim(\mathbb{K}-1)}-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

then, $v = u \circ \hbar_{\mathbb{K}}$ is a solution to problem $(\wp_{2^*_{N,\dim(\mathbb{K}-1)}, \Theta})$ for $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$. It is clear that the function

$$x \mapsto \frac{1}{2|x|}$$

is radial and therefore Γ-invariant. Hence, if Ω is an G-invariant smooth bounded domain in $\mathbb{R}^{\dim\mathbb{K}+1}$ such that

$$\#Gx = \infty \quad \text{for all } x \in \Omega. \tag{3.7}$$

then, Theorem 2.1, with $a_1 \equiv 1$, $a_2 \equiv 0$ and $a_3 \equiv \frac{1}{2|x|}$, shows that problem 3.6 has infinitely many G-invariant solutions.

Finally, since every G-invariant solution u of problem 3.6 is constant on every orbit Gx, then $v = u \circ \hbar_{\mathbb{K}}$ is constant in $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x \in \Omega$.

Proof of Theorem 1.10

Let $N = 4, 8$ or 16, $\mathbb{R}^N = \mathbb{K} \times \mathbb{K}$ and fix a closed subgroup Γ of $O(\dim\mathbb{K}+1)$. Let D a Γ-invariant smooth bounded domain in $\mathbb{R}^{\dim\mathbb{K}+1}$ such that

$$\#\Gamma x = \infty \quad \text{for all } x \in D. \tag{3.8}$$

Again, Proposition **??** shows that if a function u is a solution to problem

$$\begin{cases} -\Delta u = \frac{1}{2|x|}|u|^{2^*_{N,\dim(\mathbb{K}-1)}-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

then, $v = u \circ \hbar_{\mathbb{K}}$ is a solution to problem $(\wp_{2^*_{N,\dim(\mathbb{K}-1)}}, \Theta)$ for $\Theta := \hbar_{\mathbb{K}}^{-1}(\Omega)$.

Hence, if we apply Theorem 1.7 for the domain D and the group Γ, with $a_1 \equiv 1$, $a_2 \equiv 0$ and $a_3 \equiv \frac{1}{2|x|}$, we have that:

There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ and D, with the following property: If Ω contains D and if it is invariant under the action of a closed subgroup G of Γ for which

$$\min_{x\in\overline{\Omega}} (\#Gx)\, |2x|^{\frac{\dim\mathbb{K}-1}{2}} > \ell_m$$

holds, then problem 3.9 has at least m pairs of G-invariant solutions $\pm u_1, \ldots, \pm u_m$. Moreover, u_1 is positive and $u_2, \ldots, u_m$ change sign.

Hence, the function $v_i = u_i \circ \hbar_{\mathbb{K}}$ is a solution to problem $(\wp_{p_{\mathbb{K}}})$ in $\Theta = \hbar_{\mathbb{K}}^{-1}(\Omega)$ which is constant on $\hbar_{\mathbb{K}}^{-1}(Gx)$ for each $x \in \Omega$. for every $i = 1 \ldots m$

3.2.2 Multiplicity via rotations

Proof of Theorems 1.11 and 1.12

The aim of this section is to prove Theorems 1.11 and 1.12 stated in the Introduction. As before, fix $k_1, \ldots, k_m \in \mathbb{N}$ and set $k := k_1 + \cdots + k_m$ and let Ω as in (3.4) and Θ as in (3.5). Consider $O(N-k-m)$ as the subgroup of $O(N-k)$ which acts on the second factor of $\mathbb{R}^m \times \mathbb{R}^{N-k-m} \equiv \mathbb{R}^{N-k}$ and the function $\mathfrak{a} : \mathbb{R}^m \times \mathbb{R}^{N-k-m} \to \mathbb{R}$ defined by

$$\mathfrak{a}((x_1, \ldots, x_m, x')) = x_1^{k_1} \cdots x_m^{k_m}$$

Theorem 1.11 *Let G be a closed subgroup of $O(N-k-m)$, Ω be a G-invariant bounded smooth domain as in 3.4 all of whose G-orbits are infinite. Then, the supercritical problem $(\wp_{2^*_{N,N-k}}, \Theta)$ has infinitely many $O(k_1+1)\times\cdots\times O(k_2+1)\times G$-invariant solutions in the domain Θ defined as (3.5).*

Proof. By Lemma 3.3 we have that if $\mathfrak{u}$ is a G-invariant solution to problem

$$\begin{cases} -div(\mathfrak{a}(x)\nabla\mathfrak{u}) = \mathfrak{a}(x)|\mathfrak{u}|^{2^*_{N,N-k}-2}\mathfrak{u} & \text{in } \Omega, \\ \mathfrak{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

then, $v(y^1, \cdots, y^m, z) = \mathfrak{u}(|y^1|, \cdots, |y^m|, z)$ is a $O(k_1+1)\times\cdots\times O(k_2+1)\times G$-invariant solution to problem $(\wp_{2^*_{N,N-k}}, \Theta)$.

Notice that the function $\mathfrak{a}$ is G-invariant (recall that we are thinking of $G \subset O(N-k-m)$ as the subgroup of $O(N-k)$ which acts on the second factor of $\mathbb{R}^m \times \mathbb{R}^{N-k-m}$).

Since, $\Omega \subset \mathbb{R}^{N-k}$ is G-invariant and all the G-orbits are infinite, then Theorem 2.1, with $a_1 \equiv a_3 \equiv \mathfrak{a}$ and $a_2 \equiv 0$, shows that problem 3.10 has infinitely many G-invariant solutions. This concludes the proof.

$\square$

Next, we will prove

Theorem 1.12 *There exists an increasing sequence (ℓ_m) of positive real numbers, depending only on Γ, D and a, with the following property: If*

$$D \subsetneq \Omega \subset \{(x_1, \ldots, x_m, x') \in \mathbb{R}^m \times \mathbb{R}^{N-k-m} : x_i > 0, i = 1, \ldots, m\}$$

and if Ω is invariant under the action of a closed subgroup G of Γ for which

$$\min_{x\in\overline{\Omega}}(\mathfrak{a}(x)\#Gx) > \ell_m \tag{3.11}$$

holds, problem $(\wp_{2^*_{N,N-k}}, \Theta)$ has at least m pairs of $O(k_1+1)\times\cdots\times O(k_2+1)\times G$-invariant solutions $\pm v_1, \ldots, \pm v_m$ in Θ (defined as in (1.4)). Moreover, v_1 is positive and $v_2, \ldots, v_m$ change sign.

Proof. Applying Lemma 3.3, we have that if $\mathfrak{u}$ is a G-invariant solution to problem

$$\begin{cases} -div(\mathfrak{a}(x)\nabla\mathfrak{u}) = \mathfrak{a}(x)|\mathfrak{u}|^{2^*_{N,N-k}-2}\mathfrak{u} & \text{in } \Omega, \\ \mathfrak{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.12}$$

then, $v(y^1, \cdots, y^m, z) = \mathfrak{u}(|y^1|, \cdots, |y^m|, z)$ is an $O(k_1+1)\times\cdots\times O(k_2+1)\times G$-invariant solution to problem $(\wp_{2^*_{N,N-k}}, \Theta)$. $\square$

Therefore, if $\Omega \subset \mathbb{R}^{N-k}$ satisfies that

$$\min_{x\in\overline{\Omega}} \frac{\mathfrak{a}(x)^{\frac{n}{2}}\#Gx}{\mathfrak{a}(x)^{\frac{n-2}{2}}} = \min_{x\in\overline{\Omega}}(\mathfrak{a}(x)\#Gx) > \ell_m \tag{3.13}$$

then, Theorem 1.7, with $a_1 \equiv a_3 \equiv \mathfrak{a}$ and $a_2 \equiv 0$, shows that problem 3.12 has at least m pairs of G-invariant solutions $\pm u_1, \dots, \pm u_m$ such that u_1 is positive, $u_2, \dots, u_m$ change sign.

Hence, $v_i(y^1, \cdots, y^m, z) = \mathfrak{u}_i(|y^1|, \cdots, |y^m|, z)$ is an $O(k_1+1) \times \cdots \times O(k_2+1) \times G$-invariant solution to problem $(\wp_{2^*_{N,N-k}, \Theta})$ for every $i = 1, 2 \dots, m$.

3.3 Existence of solutions in domain with spherical perforations

3.3.1 Proof of Theorem 1.13

As before, let $N = 4, 8, 16$ and let Θ be an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{R}^N = \mathbb{K}^2$ such that $0 \notin \overline{\Theta}$. Fix a point $z_0 \in \Theta$ and for each $\epsilon > 0$ small enough let

$$\Theta_\epsilon := \{z \in \Theta : \operatorname{dist}(z, \mathbb{S}_{\mathbb{K}} z_0) > \epsilon\}$$

where $\mathbb{S}_{\mathbb{K}} z_0 := \{\vartheta z : \vartheta \in \mathbb{S}_{\mathbb{K}}\}$. This is again an $\mathbb{S}_{\mathbb{K}}$-invariant bounded smooth domain in $\mathbb{K}^2$. We consider the supercritical problem

$$(\wp^1_\epsilon) \qquad \begin{cases} -\Delta v = v^{\frac{\dim \mathbb{K}+3}{\dim \mathbb{K}-1}} & \text{in } \Theta_\epsilon, \\ v > 0 & \text{in } \Theta_\epsilon, \\ u = 0 & \text{on } \partial\Theta_\epsilon. \end{cases}$$

we shall proof that:

Theorem 1.13 *There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, the supercritical problem $(\wp^1_\epsilon)$ has an $\mathbb{S}_{\mathbb{K}}$-invariant solution v_ϵ which concentrates and blows up along the sphere $\mathbb{S}_{\mathbb{K}} z_0$ as $\epsilon \to 0$.*

Proof. Set

$$n := \dim \mathbb{K} + 1, \quad \mathfrak{h}_{\mathbb{K}}(z_0) = \xi_0 \quad \text{and} \quad \mathfrak{h}_{\mathbb{K}}(\Theta) := \Omega \subset \mathbb{R}^{dim\mathbb{K}+1}.$$

Notice that

$$\mathfrak{h}_{\mathbb{K}}(\Theta_\epsilon) = \Omega \backslash B_\epsilon(\xi_0) := \Omega_\epsilon,$$

i.e. $\mathfrak{h}_{\mathbb{K}}(\Theta_\epsilon) = \Omega_\epsilon$ is a punctured domain.

Now, since

$$\frac{n+2}{n-2} = \frac{dim\mathbb{K}+3}{dim\mathbb{K}-1}$$

Proposition **??** shows that if u is a solution to problem

$$\begin{cases} -\Delta u = \frac{1}{2|x|} u^{\frac{n+2}{n-2}} & \text{in } \Omega_\epsilon, \\ u > 0 & \text{in } \Omega_\epsilon, \\ u = 0 & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{3.14}$$

then $v = u \circ \hbar_{\mathbb{K}}$ is a solution to $(\wp_\epsilon^1)$.

Since

$$\nabla \left(\frac{1}{2|x|} \right) (\xi_0) \neq 0$$

then, Theorem 1.8 (with $\Gamma \equiv 1$) shows that there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem Problem (3.14) has a solution u_ϵ which concentrates and blows up at the point ξ_0 as $\epsilon \to 0$.

Therefore, $v_\epsilon = u_\epsilon \circ \hbar_{\mathbb{K}}$ solution to $(\wp_\epsilon^1)$ which is constant on $\hbar_{\mathbb{K}}^{-1}(x)$ for each $x \in \Omega_\epsilon$, i.e. v_ϵ is $\mathbb{S}_{\mathbb{K}}$-invariant.

Finally, since u_ϵ concentrates and blows up at the point ξ_0 as $\epsilon \to 0$, we have that v_ϵ concentrates and blows up along the sphere $\mathbb{S}_{\mathbb{K}} z_0$ as $\epsilon \to 0$.

□

3.3.2 Proof of Theorem 1.14

Let Φ be an $[O(m) \times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$ such that $0 \notin \overline{\Phi}$ and $(y_0, 0) \in \Phi$. We write $S_0^{m-1} := \{(y, 0) : |y| = |y_0|\}$ for the $[O(m) \times O(m)]$-orbit of $(y_0, 0)$, and for each $\epsilon > 0$ small enough we set

$$\Phi_\epsilon := \{x \in \Phi : \operatorname{dist}(x, S_0^{m-1}) > \epsilon\}.$$

This is again an $[O(m) \times O(m)]$-invariant bounded smooth domain in $\mathbb{R}^{2m}$. We consider the supercritical problem

$$(\wp_\epsilon^2) \qquad \begin{cases} -\Delta v = v^{\frac{m+3}{m-1}} & \text{in } \Phi_\epsilon, \\ v > 0 & \text{in } \Phi_\epsilon, \\ u = 0 & \text{on } \partial\Phi_\epsilon. \end{cases}$$

Theorem 1.14 *There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem $(\wp_\epsilon)$ has an $[O(m) \times O(m)]$-invariant solution v_ϵ which concentrates and blows up along the $(m-1)$-dimensional sphere S_0^{m-1} as $\epsilon \to 0$.*

Proof. Let

$$\Omega := \{(t, \zeta) \in \mathbb{R} \times \mathbb{R}^m : \mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|) = (t, |\zeta|) \text{ for some } (y_1, y_2) \in \Phi\},$$

Notice that Ω is Γ-invariant. In effect, if $g \in \Gamma$ and $(t, |\zeta|) \in \Omega$, then

$$(t, |g\zeta|) = (t, |\zeta|) = \mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|)$$

which implies that $(t, g\zeta) \in \Omega$. In addition, if $(y_0, 0) \in S_0^{m-1}$, then we have that

$$\mathfrak{h}_{\mathbb{R}}(|y_0|, |0|) = (|y_0|^2, 0)$$

Set $\xi_0 = (|y_0|^2, 0, \ldots, 0)$. Note that $g\xi_0 = \xi_0$ for every $g \in O(m)$, i.e. $\xi_0 \in \Omega^{O(m)}$

The diagram

$$\begin{array}{ccc} \mathbb{R}^m \times \mathbb{R}^m & \overset{(y_1,y_2)\overset{f_1}{\mapsto}(|y_1|,|y_2|)}{\Longrightarrow} & [0,\infty) \times [0,\infty) \\ & & \Downarrow \hbar_{\mathbb{R}} \\ \mathbb{R} \times \mathbb{R}^m & \overset{(t,\zeta)\overset{f_2}{\mapsto}(t,|\zeta|)}{\Longrightarrow} & \mathbb{R} \times [0,\infty) \end{array}$$

shows that

$$\Omega_\epsilon := \Omega \backslash B_\epsilon(\xi_0) = \{(t,\zeta) \in \mathbb{R} \times \mathbb{R}^m : \mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|) = (t, |\zeta|) \text{ for some } (y_1, y_2) \in \Phi_\epsilon\}.$$

Now, Proposition 3.2 implies that if $u(t,\zeta) = \mathfrak{u}(t,|\zeta|)$ is an $O(m)$-invariant solution of problem

$$\begin{cases} -\Delta u = \frac{1}{2|x|} u^{\frac{m+3}{m-1}} & in\ \Omega_\epsilon, \\ u > 0 & in\ \Omega_\epsilon \\ u = 0 & on\ \partial\Omega_\epsilon, \end{cases} \tag{3.15}$$

then $v(y_1, y_2) := \mathfrak{u}(\mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|))$ is an $[O(m) \times O(m)]$-invariant solution of Problem $(\wp_\epsilon^2)$.

Therefore, using Theorem 1.8 with

$$n = m+1, \quad \xi_0 = (|y_0|^2, 0, \ldots, 0), \quad \Gamma = O(m) \subset O(n), \quad Q(x) = \frac{1}{2|x|},$$

we have that there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, Problem (3.15) has an $O(m)$-invariant solution $u_\epsilon(t,\zeta) = \mathfrak{u}_\epsilon(t,|\zeta|)$ which concentrates and blows up at the point ξ_0 as $\epsilon \to 0$.

Hence, $v_\epsilon(y_1, y_2) = \mathfrak{u}_\epsilon(\mathfrak{h}_{\mathbb{R}}(|y_1|, |y_2|))$ is an $[O(m) \times O(m)]$-invariant solution of Problem $(\wp_\epsilon^2)$ which concentrates and blows up along the $(m-1)$-dimensional sphere S_0^{m-1} as $\epsilon \to 0$

$\square$

Chapter 4

Nonexistence of solutions for the supercritical problem

4.1 Introduction

In this section we present some results about the nonexistence of solutions for the problem

$$(\wp_{p,\Omega,Q}) \qquad \begin{cases} -\Delta u = Q(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$,
- $p \geq 2^*$, where $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.
- $Q \in C^0(\Omega)$ and $\min_{x\in\overline{\Omega}} Q(x) > 0$, (if $Q \equiv 1$ we will write $(\wp_{p,\Omega})$ instead of $(\wp_{p,\Omega,1})$).

It is well known that the existence of a solution for problem $(\wp_{p,\Omega,Q})$ depends on the domain. Pohozhaev's identity [40] implies that if Ω is strictly starshaped then problem $(\wp_{p,\Omega})$ does not have a nontrivial solution for every $p \geq 2^*$. In the critical case, (i.e. when $p = 2^*$), there is a very remarkable result obtained by Bahri and Coron [3]. They showed that problem $(\wp_{2^*,\Omega})$ has a positive solution in every domain Ω with nontrivial topology, see 1.3 .

Remark 4.1. *As we mentioned in the introduction, the condition $\widetilde{H}_*(\Omega, \mathbb{Z}/2) \neq 0$ is not necessary for the existence of a solution, examples of contractible domains for which problem $(\wp_{2^*,\Omega})$ has at least one nontrivial solution have been given, see for instance [22, 23, 36, 38]*

The result obtained by Bahri and Coron suggests the following question : *It is true that the condition $\widetilde{H}_*(\Omega, \mathbb{Z}/2) \neq 0$ guaranties the existence of a positive solution for problem $(\wp_{p,\Omega})$ when $p > 2^*$?*

This question was pointed out by Rabinowitz, as reported by Brezis in [7]. The answer was given by Passaseo [37, 39], he proved that

Theorem 4.2 (Passaseo, 1995). *For every integer $1 \leq k \leq N-3$ there exists a domain Θ_k such that*

- *Θ_k has the homotopy type of $\mathbb{S}^k$,*
- *problem $(\wp_{p,\Theta_k})$ does not have a nontrivial solution for $p \geq 2^*_{N,k} := \frac{2(N-k)}{N-k-2}$,*
- *problem $(\wp_{p,\Theta_k})$ has infinitely many solutions for $p < 2^*_{N,k}$.*

Note that the number $2^*_{N,k} = \frac{2(N-k)}{N-k-2}$ is the Sobolev critical exponent in dimension $(N-k)$.

The domains of Passaseo are defined as follows: Fix an integer $1 \leq k \leq N-3$ and an element $(t,0) \in (0,\infty) \times \mathbb{R}^{N-k-1}$. Take a ball B with center in $(t,0)$ such that $B \subset (0,\infty) \times \mathbb{R}^{N-k-1}$. Then

$$\Theta_k := \{(y,z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in B\}.$$

4.2 Nonexistence results

In this chapter we will describe our nonexistence results for problem $(\wp_{p,\Omega,Q})$, one of them will be a generalization of the Theorem 4.2. We start with some notation

Let Ω be a domain of the form

$$\Omega := \{(y,z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\}, \tag{4.1}$$

where Θ is a bounded smooth domain in $\mathbb{R}^{N-k}$ with $\overline{\Theta} \subset (0,\infty) \times \mathbb{R}^{N-k-1}$

We introduce the following geometric condition, which we will use to guarantee nonexistence.

Definition 1. *We shall say that a domain Θ is doubly starshaped with respect to $\mathbb{R} \times \{0\}$ if there exist two numbers $0 < t_0 < t_1$ such that $t \in (t_0, t_1)$ for every $(t,z) \in \Theta$ and Θ is strictly starshaped with respect to $\xi_0 := (t_0, 0)$ and to $\xi_1 := (t_1, 0)$, i.e.*

$$\langle x - \xi_i, \nu_\Theta(x) \rangle > 0 \qquad \forall x \in \partial\Theta \smallsetminus \{\xi_i\},$$

for each $i = 0, 1$, where $\nu_\Theta(x)$ is the outward pointing unit normal to $\partial\Theta$ at x.

For Ω as in (4.1) and $Q \in \mathcal{C}^1(\overline{\Omega})$ we consider the problem

$$(\wp_{p,\Omega,Q}) \begin{cases} -\Delta u = Q(y,z)\,|u|^{p-2}\,u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

We assume Q to be strictly positive on $\overline{\Omega}$ and radially symmetric in y, i.e. $Q(y,z) = Q(|y|\,,z)$. As before $2_{N,k} = \frac{2(N-k)}{(N-k)-2}$. We prove the following result.

Theorem 4.3. *If Θ is doubly starshaped with respect to $\mathbb{R} \times \{0\}$, $0 \leq k \leq N-3$ and $\langle y, \partial_y Q(y,z)\rangle \leq 0$ and $\langle z, \partial_z Q(y,z)\rangle \leq 0$ for all $(y,z) \in \Omega$. Then Ω (as in (4.1)) satisfies that*

- *Ω has the homotopy type of $\mathbb{S}^k$,*
- *the problem $(\wp_{p,\Omega,Q})$ does not have a nontrivial solution for $p \geq 2^*_{N,k}$,*
- *the problem $(\wp_{p,\Omega,Q})$ has infinitely many solutions for $p \in (2, 2^*_{N,k})$.*

Note that if we take Θ to be a ball B centered at some point $(\tau, 0)$, which is obviously doubly starshaped with respect to $\mathbb{R} \times \{0\}$, then Theorem 4.3 provides the result given by Passaseo in Theorem 4.2.

Also note that the domains in Passaseo's examples [37, 39], as well as those in Theorem 4.3, have the homotopy type of $\mathbb{S}^k$. As we said before, this shows that the condition $\widetilde{H}_*(\Omega, \mathbb{Z}/2) \neq 0$ it is not enough to guarantee a solution for the problem $(\wp_{p,\Omega})$ when $p > 2^*$. But, one may think that perhaps if $\widetilde{H}_*(\Omega, \mathbb{Z}/2)$ is richer then $(\wp_{p,\Omega})$ will have a nontrivial solution when $p > 2^*$. The next result shows that this is not true in general.

Theorem 4.4. *Given $k = k_1 + \cdots + k_m$ with $k_i \in \mathbb{N}$ and $k \leq N-3$, and $\varepsilon > 0$ there exists a bounded smooth domain Ω in $\mathbb{R}^N$, such that*

- *Ω has the homotopy type of $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_m}$,*
- *the problem $(\wp_{p,\Omega})$ does not have a nontrivial solution for $p \geq 2^*_{N,k} + \varepsilon$,*
- *the problem $(\wp_{p,\Omega})$ has infinitely many solutions for $p \in (2, 2^*_{N,k})$.*

In particular, if we take all $k_i = 1$ in Theorem 4.4, the domain Ω is homotopy equivalent to the product of k circles and therefore Ω satisfies that it not only the homology is not nontrivial but there are k different cohomology classes in $H^1(\Omega;\mathbb{Z})$ whose cup-product is the generator of $H^k(\Omega;\mathbb{Z})$. Hence, the cup-length of Ω equals $k+1$.

4.3 Main tools for proving nonexistence

In this subsection we provide a detailed account of the main tools for proving our nonexistence results. We will use the following notation:

Fix $k_1, \ldots, k_m \in \mathbb{N} \cup \{0\}$ with $k := k_1 + \cdots + k_m \leq N - 3$ and let Θ be a bounded smooth domain in $\mathbb{R}^{N-k}$ such that $\overline{\Theta} \subset (0, \infty)^m \times \mathbb{R}^{N-k-m}$. Set

$$\Omega := \{(y^1, \ldots, y^m, z) \in \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times \mathbb{R}^{N-k-m} : \left(\left|y^1\right|, \ldots, \left|y^m\right|, z\right) \in \Theta\}, \tag{4.3}$$

and

$$G := O(k_1 + 1) \times \cdots \times O(k_m + 1) \tag{4.4}$$

Think of G as a subgroup of $O(N)$ acting on $\mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times \mathbb{R}^{N-k-m}$ in the obvious way, i.e.

$$(g_1, \ldots, g_m)(y^1, \ldots, y^m, z) := (g_1 y^1, \ldots, g_m y^m, z) \tag{4.5}$$

for $g_i \in O(k_i + 1)$, $y^i \in \mathbb{R}^{k_i+1}$, $z \in \mathbb{R}^{N-k-m}$. It immediately follows that Ω is G-invariant. The following result holds true.

Proposition 4.5. *Let G, Θ and Ω as above. If $Q(y^1, \cdots, y^m, z) = \mathfrak{Q}(|y^1|, \cdots, |y^m|, z)$ is an G-invariant continuous function in $\overline{\Omega}$ and $0 \leq k \leq N-3$, then problem $(\wp_{p,\Omega,Q})$ has infinitely many G-invariant solutions for $p \in (2, 2^*_{N,k})$.*

Proof. By Lemma ??, we have that a G-invariant function $u(y^1, \cdots, y^m, z) = \mathfrak{u}(|y^1|, \cdots, |y^m|, z)$ is a solution to problem $(\wp_{p,\Omega,Q})$ if and only if $\mathfrak{u}$ is a solution to problem

$$\begin{cases} -div(\mathfrak{b}(z)\nabla\mathfrak{u}) = \mathfrak{c}(z)f(\mathfrak{u}) & \text{in } \Theta, \\ \mathfrak{u} = 0 & \text{on } \partial\Theta, \end{cases} \tag{4.6}$$

where $\mathfrak{b}(z_1, \ldots z_{N-k}) := z_1^{k_1} \ldots z_m^{k_m}$ and $\mathfrak{c}(z) := \mathfrak{Q}(z)\mathfrak{b}(z)$.

It is well known that the solutions of problem (4.6) are the critical points of the functional $J : H_0^1(\Theta) \mapsto \mathbb{R}$, defined by

$$J(v) := \frac{1}{2}\|v\|_{\mathfrak{b}}^2 - \frac{1}{p}|v|_{\mathfrak{c},p}^p, \qquad v \in H_0^1(\Theta)$$

where

$$\|v\|_{\mathfrak{b}}^2 = \int_\Theta \mathfrak{b}(x)\,|\nabla v|^2 \text{ and } |v|_{\mathfrak{c},p}^p = \int_\Theta \mathfrak{c}(x)\,|v|^p.$$

Notice that, since $\overline{\Theta} \subset (0, \infty)^m \times \mathbb{R}^{N-k-m}$, the functions $\mathfrak{b}$ and $\mathfrak{c}$ are continuous and strictly positive in $\overline{\Theta}$. This implies that the functions $\|v\|_{\mathfrak{b}}^2$ and $|v|_{\mathfrak{c},p}^p$ are norms equivalent to those of $H_0^1(\Theta)$ and $L^p(\Theta)$ respectively.

Now, since $\Theta \subset \mathbb{R}^{N-k}$, the Rellich-Kondrachov theorem asserts that $H_0^1(\Theta)$ is compactly embedded in $L^p(\Theta)$ for $p < 2^*_{N-k} = \frac{2(N-k)}{(N-k)-2}$.

Hence, the functional J satisfies the Palais-Smale condition in $H_0^1(\Theta)$. Moreover, it clearly satisfies all other hypotheses of the symmetric mountain pass theorem [2]. So, J has an unbounded sequence of critical values. This implies that problem (4.6) has infinitely many solutions and therefore the problem $(\wp_{p,\Omega,Q})$ has infinitely many G-invariant solutions.

□

The following lemma provides a necessary condition for a function $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\overline{\Omega})$ to be a solution of problem $(\wp_{p,\Omega,Q})$.

Lemma 4.6 (Pucci-Serrin, 1986)**.** *Let $Q \in C^1(\overline{\Omega})$. If $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\overline{\Omega})$ is a solution of problem $(\wp_{p,\Omega,Q})$ and $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$, then*

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \langle \chi, \nu_\Omega\rangle \, d\sigma &= \int_\Omega (div\chi)\left[\frac{1}{p}Q\,|u|^p - \frac{1}{2}|\nabla u|^2\right] dx \\ &\quad + \frac{1}{p}\int_\Omega |u|^p \langle \chi, \nabla Q\rangle \, dx + \int_\Omega \langle \mathrm{d}\chi\,[\nabla u]\,, \nabla u\rangle \, dx \qquad (4.7)\end{aligned}$$

where ν_Ω is the outward pointing unit normal to $\partial\Omega$.

Proof. It follows from the variational identity (4) in Pucci and Serrin's paper [41] □

Remark 4.7. *Since Ω is a **smooth bounded domain**, every weak solution u to problem $(\wp_{p,\Omega,Q})$ is a classical solution, i.e. $u \in \mathcal{C}^2(\Omega)\cap\mathcal{C}^1(\overline{\Omega})$, for more details, see for instance, [45, Appendix B].*

To conclude this subsection we will construct a vector field $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$ which, in combination with Lemma 4.6, will be very useful for the proof of our nonexistence results. To do this, let $\overline{\Theta} \subset (0,\infty)^m \times \mathbb{R}^{N-k-m}$ and Ω as in (4.3). Fix $\tau_1, \ldots, \tau_m \in (0,\infty)$, and for every $1 \le i \le m$, let $\varphi_i : (0,\infty) \mapsto \mathbb{R}$ defined by

$$\varphi_i(t) = \frac{1}{k_i+1}\left[1 - (\frac{\tau_i}{t})^{k_i+1}\right].$$

A direct computation shows that φ_i is strictly increasing in $(0,\infty)$ and solves the following problem

$$\begin{cases} \varphi_i'(t)t + (k_i+1)\varphi_i(t) = 1, & t \in (0,\infty), \\ \varphi_i(\tau_i) = 0. \end{cases}$$

For $y^i \in \mathbb{R}^{k_i+1}\diagdown\{0\}$, we define

$$\chi(y^1, \ldots, y^m, z) := (\varphi_1(|y^1|)y^1, \ldots, \varphi_m(|y^m|)y^m, z). \qquad (4.8)$$

It is clear that $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$. Moreover,

Lemma 4.8. *χ has the following properties*

(P1) $div\chi = N - k$

(P2) $\langle \mathrm{d}\chi(y^1, \dots, y^m, z)\,[\xi]\,, \xi\rangle_{\mathbb{R}^N} \leq \max\{1 - k_1\varphi_1(|y^1|), \dots, 1 - k_m\varphi_m(|y^m|), 1\}\,|\xi|^2$ *for every* $y^i \in \mathbb{R}^{k_i+1} \setminus \{0\}$, $z \in \mathbb{R}^{N-k-m}$, $\xi \in \mathbb{R}^N$.

Proof. We will use the following notation: for every $1 \leq j \leq m$, let $l_j = k_1 + \cdots + k_j + j$ and $(x_1, \dots, x_N) = (y^1, \cdots, y^m, z)$, i.e.

$$y^j = (x_{l_{(j-1)}+1}, \dots, x_{l_j}) \text{ and } z = (x_{(l_m+1)}, \dots, x_N).$$

Hence,

$$|y^j| = \left(\sum_{i=l_{(j-1)}+1}^{l_j} x_i^2\right)^{1/2}.$$

Therefore,

$$\chi(x_1, \cdots, x_N) = (\chi_1(x_1, \cdots, x_N), \dots, \chi_{l_m}(x_1, \cdots, x_N), \dots, \chi_N(x_1, \cdots, x_N))$$

where

$$\chi_i(x_1, \cdots, x_N) = \varphi_j(|y^j|)x_i \quad \text{for } l_{j-1} < i \leq l_j \leq l_m$$

and

$$\chi_i(x_1, \cdots, x_N) = x_i \quad \text{for } l_m < i \leq N.$$

A direct calculation shows that

$$\frac{\partial \chi_i}{\partial x_i}(x_1, \cdots, x_N) = \varphi_j'(|y^j|)\frac{(x_i)^2}{|y^j|} + \varphi_j(|y^j|) \quad \text{for } l_{j-1} < i \leq l_j \leq l_m,$$

and

$$\frac{\partial \chi_i}{\partial x_i}(x_1, \cdots, x_N) = 1 \quad \text{for } l_m < i \leq N.$$

Moreover, since

$$\sum_{i=l_{(j-1)}+1}^{l_j} \frac{\partial \chi_i}{\partial x_i}(x_1, \cdots, x_N) = \varphi_j'(|y^j|)\frac{|y^j|^2}{|y^j|} + \varphi_j(|y^j|) = 1,$$

we conclude that

$$\mathrm{div}\chi(x_1, \cdots, x_N) = \sum_{j=1}^{m}\left(\sum_{i=l_{(j-1)}+1}^{l_j} \frac{\partial \chi_i}{\partial x_i}(x_1, \cdots, x_N)\right) + (N - k - m) = N - k,$$

this proves $(P1)$.

To prove $(P2)$, note that χ is G-equivariant for the G-action defined in (4.5). Indeed, if

$$g := (g_1, \dots, g_m) \in G \text{ and } (y^1, \dots, y^m, z) \in \Omega$$

with

$$y^i \in \mathbb{R}^{k_i+1} \setminus \{0\}, \quad z \in \mathbb{R}^{N-k-m},$$

then

$$\begin{aligned} \chi(g(y^1, \dots, y^m, z)) &= \chi(g_1 y^1, \dots, g_m y^m, z) \\ &= (\varphi_1(\left|g_1 y^1\right|) g_1 y^1, \dots, \varphi_m(|g_m y^m|) g_m y^m, z) \\ &= (\varphi_1(\left|y^1\right|) g_1 y^1, \dots, \varphi_m(|y^m|) g_m y^m, z) \\ &= (g(\varphi_1(\left|y^1\right|) y^1, \dots, \varphi_m(|y^m|) y^m, z)) \\ &= g(\chi(y^1, \dots, y^m, z)). \end{aligned}$$

This implies that

$$g \circ \mathrm{d}\chi(y^1, \dots, y^m, z) = \mathrm{d}\chi(g(y^1, \dots, y^m, z)) \circ g$$

for every $g \in G$. Hence,

$$\begin{aligned} \left\langle \mathrm{d}\chi\left(y^1, \dots, y^m, z\right)[\xi], \xi \right\rangle_{\mathbb{R}^N} &= \left\langle g\left(\mathrm{d}\chi\left(y^1, \dots, y^m, z\right)[\xi]\right), g\xi \right\rangle_{\mathbb{R}^N} \\ &= \left\langle \mathrm{d}\chi\left(g(y^1, \dots, y^m, z)\right)[g\xi], g\xi \right\rangle_{\mathbb{R}^N} \end{aligned}$$

for all $g \in G$ and all $\xi \in \mathbb{R}^N$. Thus, it suffices to show that the inequality (P2) holds for $y^i = (y^i_1, 0, \dots, 0)$ with $y^i_1 > 0$. Set $\chi_i(y^i) := \varphi_i(|y^i|)y^i$. A straightforward computation shows that, for such y^i, $\mathrm{d}\chi_i(y^i)$ is a diagonal matrix whose diagonal entries are $a_{11} = 1 - k_i\varphi_i(y^i_1)$ and $a_{jj} = \varphi_i(y^i_1)$ for $j = 2, \dots, k_i + 1$. Since $\varphi_i(t) < \frac{1}{k_i+1}$ for all $t \in (0, \infty)$, $(P2)$ follows.

□

4.4 Proof of Theorem 4.3 and Theorem 4.4.

In this subsection we will prove our results about the nonexistence of solutions for the problem $(\wp_{p,\Omega,Q})$.

Proof of Theorem 4.3. By Lemma 4.6 we have that if $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ is a solution of problem $(\wp_{p,\Omega,Q})$ and $\chi \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^N)$, then

$$\begin{aligned} \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \langle \chi, \nu_\Omega \rangle \, d\sigma = \int_\Omega (\mathrm{div}\chi)\left[\frac{1}{p} Q\,|u|^p - \frac{1}{2}|\nabla u|^2\right] dx \\ + \frac{1}{p}\int_\Omega |u|^p \langle \chi, \nabla Q \rangle \, dx + \int_\Omega \langle \mathrm{d}\chi[\nabla u], \nabla u \rangle \, dx \end{aligned} \tag{4.9}$$

where ν_Ω is the outward pointing unit normal to $\partial\Omega$. Take χ to be the vector field defined in (4.8) for $m = 1$, $0 \le k \le N-3$ and $\tau_1 = t_0$ as in Definition 1, that is

$$\chi(y,z) := (\varphi(|y|)y, z), \qquad (y,z) \in \left(\mathbb{R}^{k+1} \setminus \{0\}\right) \times \mathbb{R}^{N-k-1}$$

with $\varphi(t) = \frac{1}{k+1}\left[1 - (\frac{t_0}{t})^{k+1}\right]$. Then

$$\mathrm{div}\chi = N - k \tag{4.10}$$

Notice that

$$\varphi(t) \ge 0 \text{ for every } t \in (t_0, \infty). \tag{4.11}$$

and, since $|y| > t_0$ if $(y,z) \in \Omega$, we have that

$$\varphi(|y|) > 0 \text{ if } (y,z) \in \Omega. \tag{4.12}$$

This implies that

$$\langle \chi(y,z), \nabla Q(y,z)\rangle_{\mathbb{R}^N} = \varphi(|y|)\,\langle y, \partial_y Q(y,z)\rangle_{\mathbb{R}^N} + \langle z, \partial_z Q(y,z)\rangle_{\mathbb{R}^N} \le 0 \quad \forall (y,z) \in \Omega. \tag{4.13}$$

By Proposition 4.8 we have that if $y \in \mathbb{R}^{k+1} \setminus \{0\}$ and $z \in \mathbb{R}^{N-k-1}$, then

$$\langle \mathrm{d}\chi(y,z)\,[\xi]\,,\xi\rangle_{\mathbb{R}^N} \le \max\{1 - k\varphi(|y|), 1\}\,|\xi|^2 \quad \text{for every} \quad \xi \in \mathbb{R}^N.$$

Notice that equation (4.11) implies that $1 - k\varphi(t) < 1$ for $t \in (t_0, \infty)$. Therefore, using equation (4.12), we conclude that

$$\langle \mathrm{d}\chi\,(x)\,[\xi]\,,\xi\rangle \le |\xi|^2 \qquad \forall x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{4.14}$$

We will show that

$$\langle \chi(x), \nu_\Omega(x)\rangle > 0 \qquad \forall x \in \partial\Omega \setminus \{g\xi_0, g\xi_1 : g \in O(k+1)\}. \tag{4.15}$$

Since Ω is $O(k+1)$-invariant we have that ν_Ω is $O(k+1)$-equivariant. Thus, it suffices to show that

$$\langle(\varphi(t)t, z), \nu_\Theta(t,z)\rangle > 0 \qquad \text{for all } (t,z) \in \partial\Theta \setminus \{\xi_0, \xi_1\}, \tag{4.16}$$

where $\nu_\Theta(t,z)$ is the outward pointing unit normal to $\partial\Theta$ at (t,z) which we write as $\nu_\Theta(t,z) = (\nu_1(t,z), \nu_2(t,z)) \in \mathbb{R} \times \mathbb{R}^{N-k-1}$. Let $(t,z) \in \partial\Theta$. Since Θ is doubly starshaped we have that

$$(t - t_i)\nu_1(t,z) + \langle z, \nu_2(t,z)\rangle > 0 \qquad \text{if } (t,z) \ne (t_i, 0),\ \text{for } i = 0,1,$$

with t_0, t_1 as in Definition 1. Therefore,

$$\langle(\varphi(t)t, z), \nu_\Theta(t,z)\rangle = \varphi(t)t\nu_1(t,z) + \langle z, \nu_2(t,z)\rangle > (\varphi(t)t - t + t_i)\nu_1(t,z).$$

Set $\psi(t) := \varphi(t)t - t$. Note that $\psi'(t) = -k\varphi(t) < 0$ if $t > t_0$. So, since $t \in (t_0, t_1)$ for every $(t, z) \in \Theta$, we have that

$$\varphi(t_1)t_1 - t_1 = \psi(t_1) \leq \psi(t) \leq \psi(t_0) = -t_0 \qquad \forall (t, z) \in \partial\Theta.$$

If $\nu_1(t, z) \leq 0$, then

$$\langle (\varphi(t)t, z), \nu_\Theta(t, z)\rangle > (\psi(t) + t_0)\nu_1(t, z) \geq 0$$

and if $\nu_1(t, z) \geq 0$, then

$$\langle (\varphi(t)t, z), \nu_\Theta(t, z)\rangle > (\psi(t) + t_1)\nu_1(t, z) \geq \varphi(t_1)t_1\nu_1(t, z) \geq 0.$$

This proves (4.16). Combining properties (4.10), (4.13), (4.14) and (4.15) with identity (4.9) gives

$$\begin{aligned} 0 &< \int_\Omega (\mathrm{div}\chi)\left[\frac{1}{p}Q\,|u|^p - \frac{1}{2}|\nabla u|^2\right]dx + \int_\Omega |\nabla u|^2\,dx \\ &= (N-k)\left(\frac{1}{p} - \frac{1}{2} + \frac{1}{N-k}\right)\int_\Omega |\nabla u|^2\,dx \end{aligned}$$

which implies that $p < 2^*_{N,k}$ if $u \neq 0$.

Proposition 4.5 yields infinitely many solutions for problem $(\wp_{p,\Omega,Q})$ when $p < 2^*_{N,k}$.

$\square$

Proof of Theorem 4.4. Choose $\alpha \in (1, \frac{N-k}{2})$ with $2^*_{N,k} + \varepsilon \geq \frac{2(N-k)}{N-k-2\alpha}$. Fix $\tau_1, \ldots, \tau_m \in (0, \infty)$ and, for the given $k_1, \ldots, k_m$, define χ as in (4.8). Let $0 < \varrho < \tau_i$ be defined by

$$\max\{1 - k_1\varphi_1(\tau_1 - \varrho), \ldots, 1 - k_m\varphi_m(\tau_m - \varrho)\} = \alpha,$$

$\Theta := B_\varrho^{N-k}(\tau)$ be the ball of radius ϱ centered at $\tau = (\tau_1, \ldots, \tau_m, 0)$ in $\mathbb{R}^m \times \mathbb{R}^{N-k-m}$ and Ω be defined as in (4.3). Then Ω has the homotopy type of $\mathbb{S}^{k_1} \times \cdots \times \mathbb{S}^{k_m}$. Moreover, Lemma 4.8 asserts that

$$\mathrm{div}\chi = N-k \qquad \text{and} \qquad \langle \mathrm{d}\chi(x)[\xi], \xi\rangle \leq \alpha\,|\xi|^2 \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{4.17}$$

Since $\varphi_i(t) < 0$ if $t < \tau_i$ and $\varphi_i(t) > 0$ if $t > \tau_i$ we have that, for all but a finite number of points $(x, z) \in \partial\Theta$,

$$\langle (\varphi_1(x_1)x_1, \ldots, \varphi_m(x_m)x_m, z),\ \nu_\Theta(t, z)\rangle = \sum_{i=1}^m \varphi_i(x_i)x_i(x_i - \tau_i) + |z|^2 > 0.$$

Hence,

$$\langle \chi, \nu_\Omega\rangle > 0 \qquad \text{a.e. on } \partial\Omega. \tag{4.18}$$

Combining properties (4.17) and (4.18) with identity (4.9) for $K = 1$ we obtain

$$\begin{aligned} 0 &< \int_\Omega (\mathrm{div}\chi) \left[\frac{1}{p} |u|^p - \frac{1}{2} |\nabla u|^2 \right] dx + \alpha \int_\Omega |\nabla u|^2 \, dx \\ &= (N-k) \left(\frac{1}{p} - \frac{1}{2} + \frac{\alpha}{N-k} \right) \int_\Omega |\nabla u|^2 \, dx \end{aligned}$$

which implies that $p < \frac{2(N-k)}{N-k-2\alpha} \leq 2^*_{N,k} + \varepsilon$ if $u \neq 0$. Consequently, problem $(\wp_{p,\Omega})$ does not have a nontrivial solution in Ω for $p \geq 2^*_{N,k} + \varepsilon$, whereas Proposition 4.5 yields infinitely many solutions for $p < 2^*_{N,k}$. $\square$

Appendix A

Representation of Palais-Smale sequences

A.1 Introduction

Our aim in this appendix is to describe the lack of compactness of the anisotropic critical problem

$$(\wp) \qquad \begin{cases} -\mathrm{div}(a_1(x)\nabla u) + a_2(x)u = a_3(x)|u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(gx) = u(x) & \forall x \in \Omega, g \in G, \end{cases}$$

where

- G is a closed subgroup of $O(N)$.
- Ω is a smooth bounded G-invariant domain in $\mathbb{R}^N$,
- a_i is a G-invariant continuous function for $i = 1, 2, 3$ and satisfies that
  - $\min\limits_{x\in\overline{\Omega}} a_i(x) > 0$ for $i = 1, 3$ and
  - $\min\limits_{x\in\overline{\Omega}} a_2(x) > -\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the $-\Delta$ in $H_0^1(\Omega)$.

Recall that, by the principle of symmetric criticality of Palais [35], the solutions to problem $(\wp)$ are the critical points of the restriction of the functional

$$\begin{aligned} J(u) &:= \frac{1}{2}\int_\Omega (a_1|\nabla u|^2 + a_2 u^2) - \frac{1}{2^*}\int_\Omega a_3|u|^{2^*} \\ &= \frac{1}{2}||u||^2_{a_1,a_2} - \frac{1}{2^*}|u|^{2^*}_{a_3,2^*} \end{aligned}$$

to the space of G-invariant functions

$$H_0^1(\Omega)^G : \{u \in H_0^1(\Omega) : u(gx) = u(x) \text{ for all } g \in G, x \in \Omega\}.$$

Definition 2. *We shall say that a sequence (u_n) in $H_0^1(\Omega)$ is a $G-$invariant Palais-Smale sequence for J at the level c if*

$$u_n \in H_0^1(\Omega)^G, \qquad J(u_n) \to c, \qquad J'(u_n) \to 0 \text{ in } H^{-1}(\Omega).$$

We say that J satisfies the G-invariant Palais-Smale condition $(PS)_c^G$ at the level c in $H_0^1(\Omega)$ if every Palais-Smale sequence for J at the level c has a convergent subsequence in $H_0^1(\Omega)$.

In general, the functional J does not satisfy the Palais-Smale condition for every $c \in \mathbb{R}$, i.e. there exist G-invariant Palais-Smale sequences which do not converge.

In [44], Struwe gave a global compactness result for problem $(\wp)$ in the particular case when $G \equiv 1$ and $a_i \equiv 1$, $i = 1, 2, 3$. In this case, he gave a complete description of the Palais-Smale sequences of the associated variation problem in terms of the solutions of the **limit problem**

$$(\wp_\infty) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{when } |x| \to \infty \end{cases}$$

Roughly speaking, he showed that the Palais-Smale sequences which do not converge, approach to a sum of a solution (possibly trivial) to problem $(\wp)$ plus *nontrivial* solutions of the $(\wp_\infty)$ rescaled by sequences of points in the closure of the domain.

In [12], Clapp generalizes the result given by Struwe to the symmetric non-autonomous problem, (i.e. the particular case of problem $(\wp)$ when G is any closed subgroup of $O(N)$, a_3 is a G-invariant positive continuous function and $a_1 \equiv a_2 \equiv 1$). In this case, she showed that the lack of compactness is produced by solutions of limiting problems of the form

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \mathbb{R}^N, \\ u \to 0 & \text{when } |x| \to \infty \\ u(gx) = u(x) & \text{for all } g \in K \end{cases}$$

concentrating at G-orbits of Ω with orbit type G/K for some closed subgroup K of finite index in G.

In this section we will follow the proof of Clapp and Struwe to give a global compactness result for problem $(\wp)$. In order to give a precise statement let us recall some basic notions. Let $O(N)$ be the group of linear isometries of $\mathbb{R}^N$. If G is a closed subgroup on $O(N)$, we will denote by

$$Gx := \{gx : g \in G\}$$

the G-*orbit* of $x \in \mathbb{R}^N$ and by $\#Gx$ its cardinality. A domain $\Omega \subset \mathbb{R}^N$ is called G invariant if

$$Gx \subset \Omega \text{ for all } x \in \Omega.$$

A function $a : \Omega \to \mathbb{R}$ is called G-invariant if u is constant on every orbit Gx. The subgroup G_x of G defined by

$$G_x := \{g \in G : gx = x\},$$

is the *G-isotropy group of x.*

The main theorem of this chapter is the following

Theorem A.1. *Let (u_n) be a G-invariant Palais-Smale sequence for J at the level c. Then, passing to a subsequence of (u_n) if necessary, there exists a G-invariant solution (possibly trivial) to problem $(\wp)$, an integer $m \geq 0$, closed subgroups $G_1, \dots, G_m$ of finite index in G, sequences $(y_{1,n}), \dots, (y_{m,n})$ in Ω, sequences $(\varepsilon_{1,n}), \dots, (\varepsilon_{m,n})$ in $(0, \infty)$ and non trivial solutions $\widehat{u}_1, \dots, \widehat{u}_m$ to problem $(\wp_\infty)$, with the following properties*

(i) $G_{y_{i,n}} = G_i$ *for all* $n \geq 1$, *and* $y_{i,n} \to y_i$ *for every* $i = 1, \dots, m$.

(ii) $\varepsilon_{i,n}^{-1} dist(y_{i,n}, \partial\Omega) \to \infty$ *and* $\varepsilon_{i,n}|gy_{i,n} - g'y_{i,n}| \to \infty$ *if* $[g'] \neq [g]$ *in* G/G_i *for every* $i = 1, \dots, m$,

(iii) $\widehat{u}_i$ *is* G_i*-invariant for every* $i = 1 \dots m$.

(iv) $$\left\| u_n - u - \sum_{i=1}^{m} \sum_{[g] \in G/G_i} \left(\frac{a_3(y_i)}{a_1(y_i)} \right)^{\frac{2-N}{4}} \varepsilon_{i,n}^{\frac{2-N}{2}} \widehat{u}_i(g^{-1}\varepsilon_{i,n}^{-1}(\cdot - gy_{i,n})) \right\| \to 0 \text{ in } D^{1,2}(\mathbb{R}^N).$$

(v) $$J(u) + \sum_{i=1}^{m} |G/G_i| \left(\frac{a_3(y_i)^{\frac{2-N}{2}}}{a_1(y_i)^{-\frac{N}{2}}} \right) J_\infty(\widehat{u}_i) = c$$

As in [44] and [12], the proof of Theorem A.1 follows from the next lemma.

Lemma A.2. *Let (u_n) be a G-invariant Palais-Smale sequence for J_1 at the level $c > 0$ such that $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)^G$. Then, there exist a closed subgroup K of finite index in G, a sequence (y_n) in Ω, a sequence (ε_n) in $(0, \infty)$, a non trivial solution $\widehat{u}$ to the limit problem $(\wp_\infty)$ and a G-invariant Palais-Smale sequence (v_n) for J_1 with the following properties*

(i) $G_{y_n} = K$ *for all* $n \in \mathbb{N}$, *and* $y_n \to y_0$ *in* $\overline{\Omega}$.

(ii) $\varepsilon_n^{-1} dist(y_n, \partial\Omega) \to \infty$ *and* $\varepsilon_n^{-1}|gy_n - g'y_n| \to \infty$ *if* $[g'] \neq [g]$ *in* G/K

(iii) $\widehat{u}$ *is* K*-invariant,*

(iv) $$v_n = u_n - \sum_{[g] \in G/K} \left(\frac{a_3(y_0)}{a_1(y_0)} \right)^{\frac{2-N}{4}} \varepsilon_n^{\frac{2-N}{2}} \widehat{u}(g^{-1}\varepsilon_n^{-1}(\cdot - gy_n)) + o(1) \text{ in } D^{1,2}(\mathbb{R}^N).$$

(v) $J_1(v_n) = J_1(u_n) - |G/K| \left(\frac{a_3(y_0)^{\frac{2-N}{2}}}{a_1(y_0)^{-\frac{N}{2}}} \right) J_\infty(\widehat{u}) + o(1).$

As in [44], from Theorem A.1 we can conclude the following corollary which is very important for this work.

Corollary A.3. *The functional J satisfies condition $(PS)_c^G$ in $H_0^1(\Omega)$ for every*

$$c < \left(\min_{x \in \overline{\Omega}} \frac{a_1(x)^{\frac{N}{2}} \# Gx}{a_3(x)^{\frac{N-2}{2}}} \right) \frac{1}{N} S^{\frac{N}{2}}. \tag{A.1}$$

In particular, if $\#Gx = \infty$ for all $x \in \overline{\Omega}$, then J satisfies condition $(PS)_c^G$ in $H_0^1(\Omega)$ for every $c \in \mathbb{R}$.

Proof. Let c be given as in the statement of this corollary. Since $\frac{1}{N} S^{\frac{N}{2}}$ is the minimum energy of a non trivial solution $\widehat{u}$ for problem $(\wp_\infty)$, we have that

$$c < \left(\min_{x \in \overline{\Omega}} \frac{a_1(x)^{\frac{N}{2}} \# Gx}{a_3(x)^{\frac{N-2}{2}}} \right) \frac{1}{N} S^{\frac{N}{2}} \leq |G/K| \left(\frac{a_1(y)^{\frac{N}{2}}}{a_3(y)^{\frac{N-2}{2}}} \right) J_\infty(\widehat{u})$$

for every isotropy closed subgroup $K \subset G$ and every $y \in \overline{\Omega}$. Therefore, if (u_n) is a G-invariant Palais-Smale sequence for J at the level c, it follows from statement (v) of Theorem A.1 that $m = 0$. Finally, (iv) shows that (u_n) converge strongly in $H_0^1(\Omega)$.

□

A.2 Main tools for proving Theorem A.1

In order to provide a proof of Theorem A.1 we need some tools which we introduce below

For every closed subgroup H of G, the set

$$(\mathbb{R}^N)^H := \{x \in \mathbb{R}^N : gx = x \text{ for all } g \in G\}$$

is the *fixed point space* of H. Recall that the subgroups H and K of G are called *conjugate* if there exists an element $g \in G$ such that $H = gKg^{-1}$. The *conjugacy class of H* in G is the set

$$(H) := \{gHg^{-1} : g \in G\}.$$

Let

$$\mathbf{C} := \{(H) : H \text{ closed subgroup of } G\}.$$

It is well known that $\mathbf{C}$ is a partially ordered set with the partial order

$$(H) \leq (K) \text{ if and only if } gHg^{-1} \subset K \text{ for some } g \in G.$$

The conjugacy class (G_x) of an isotropy group G_x is called *isotropy class*. Note that

Remark A.4. *If $(K) = (G_x)$ and $K \subset G_x$, then $K = G_x$.*

Proof. Let $g \in G$ such that
$$G_{gx} = gG_xg^{-1} = K. \tag{A.2}$$
The map $x \mapsto gx$ is an isomorphism between $(\mathbb{R}^N)^{G_x}$ and $(\mathbb{R}^N)^K$. On the other hand, since $K \subset G_x$, we have that $(\mathbb{R}^N)^{G_x} \subset (\mathbb{R}^N)^K$. Hence $(\mathbb{R}^N)^{G_x} = (\mathbb{R}^N)^K$. Using equation (A.2) we have that $gx \in (\mathbb{R}^N)^K = (\mathbb{R}^N)^{G_x}$, therefore
$$G_x \subset G_{gx} = K,$$
this ends the proof. □

We shall make use the following result

Lemma A.5. *Every finite-dimensional vector space has only finitely many isotropy classes.*

Proof. See [6, IV.10]

□

Now consider the set
$$V := \{x \in \mathbb{R}^N : \#Gx < \infty\}.$$
Note that V is a linear subspace of $\mathbb{R}^N$. Also, since $Ggx = Gx$ for every $g \in G$, we have that V is G-invariant.

Remark A.6. *For every sequence (w_n) in V, there exist a closed subgroup L of G and a subsequence of (w_n), which we still denote in the same way, with the following properties*

- $G_{w_n} = L \quad \forall n \in \mathbb{N}$
- $(\mathbb{R}^N)^L \subseteq V$

Proof. Let $(w_n) \subset V$. Using Lemma A.5 and passing to a subsequence we have that there exists a closed subgroup L of G such that
$$(G_{w_n}) = (L) \quad \forall n \in \mathbb{N}.$$
On the other hand, since
$$G_{gx} = gG_xg^{-1} \quad \forall g \in G,$$
we have that
$$|(L)| = |(G_{w_n})| \leq \#Gw_n < \infty.$$
Since $|(L)|$ is finite, a subsequence of (w_n) satisfies that
$$G_{w_n} = L \quad \forall n \in \mathbb{N}.$$
Finally, note that if $x \in (\mathbb{R}^N)^L$ then $L \subseteq G_x$, and hence
$$\#Gx = |G/G_x| \leq |G/L| < \infty.$$
This shows that $(\mathbb{R}^N)^L \subseteq V$. □

The following lemma can be find, up to minor modifications, in [19, Lemma 3.2].

Lemma A.7. *Given a sequence (ε_n) in $(0,\infty)$ and a sequence (ξ_n) in $\mathbb{R}^N$ there exists a sequence (y_n) in $\mathbb{R}^N$ and a closed subgroup K of G such that for some subsequence of (ξ_n), which we still denote in the same way, it holds that*

(s1) $\varepsilon_n^{-1} dist(G\xi_n, y_n)$ *is bounded.*

(s2) $G_n = K$ *for all* $n \in \mathbb{N}$.

(s3) If $|G/K| < \infty$, *then* $\varepsilon_n^{-1}|gy_n - g'y_n| \to \infty$ *for any* g, g' *such that* $[g] \neq [g']$ *in* G/K.

(s4) If $|G/K| = \infty$, *then there exists a closed subgroup* K' *of* G *such that* $K \subset K'$, $|G/K'| = \infty$ *and* $\varepsilon_n^{-1}|gy_n - g'y_n| \to \infty$ *for any* g, g' *such that* $[g] \neq [g']$ *in* G/K'.

Proof. Given the sequences (ε_n) and (ξ_n), there are two cases:

Case 1 The sequence $(\varepsilon_n^{-1} dist(\xi_n, V))$ is unbounded.

Case 2 The sequence $(\varepsilon_n^{-1} dist(\xi_n, V))$ is bounded.

Case 1. Passing to a subsequence, we may assume that $\varepsilon_n^{-1} dist(\xi_n, V) \to \infty$. Using Lemma A.5, we may also assume that there exists an isotropy group K such that $(G_{\xi_n}) = (K)$ for every $n \in \mathbb{N}$. Therefore, for every ξ_n there exists $g_n \in G$ such that

$$K = g_n G_{\xi_n} g_n^{-1} = G_{g_n \xi_n}.$$

We define $y_n := g_n \xi_n$. From the definitions of the sequence (y_n) and the closed subgroup K it becomes evident that $(s1)$ and $(s2)$ are satisfied.

Also, using that V is G-invariant, we have that

$$\varepsilon_n^{-1} dist(y_n, V) = \varepsilon_n^{-1} dist(\xi_n, V) \to \infty,$$

therefore, $y_n \notin V$. This implies that

$$|G/K| = |G/G_{y_n}| = \#Gy_n = \infty.$$

Hencce, we have to prove that $(s4)$ holds. Let $V^\perp$ the orthogonal complement of V in $\mathbb{R}^N$ and ϱ_n be the orthogonal projection of y_n onto $V^\perp$. Since $y_n \notin V$ we have that $\varrho_n \neq 0$. Hence, passing to a subsequence, we have that

$$\frac{\varrho_n}{|\varrho_n|} \to \varrho \in V^\perp.$$

We define $K' := G_\varrho$. Also note that, since $y_n \in K$ $\forall n \in \mathbb{N}$, we have that $K \subset K'$. Moreover, using that $\varrho \in V^\perp$, we have

$$|G/K'| = \#G\varrho = \infty.$$

Let $g, g' \in G$ such that $[g] \neq [g']$ in G/K'. Since $|g\varrho - g'\varrho| > 0$, we can take $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\varrho_n}{|\varrho_n|} - \varrho\right| < \frac{1}{4}|g\varrho - g'\varrho| \quad \forall n \geq n_0.$$

Then

$$\begin{aligned}
|g\varrho - g'\varrho| &\leq \left|g\varrho - \frac{g\varrho_n}{|\varrho_n|}\right| + \left|\frac{g\varrho_n}{|\varrho_n|} - \frac{g'\varrho_n}{|\varrho_n|}\right| + \left|\frac{g'\varrho_n}{|\varrho_n|} - g'\varrho\right| \\
&= \left|\frac{g\varrho_n}{|\varrho_n|} - \frac{g'\varrho_n}{|\varrho_n|}\right| + 2\left|\frac{\varrho_n}{|\varrho_n|} - \varrho\right| \\
&\leq \left|\frac{g\varrho_n}{|\varrho_n|} - \frac{g'\varrho_n}{|\varrho_n|}\right| + \frac{1}{2}|g\varrho - g'\varrho| \quad \forall n \geq n_0,
\end{aligned}$$

therefore

$$\frac{1}{2}|g\varrho - g'\varrho||\varrho_n| \leq |g\varrho_n - g'\varrho_n| \quad \forall n \geq n_0.$$

This finally yields

$$\begin{aligned}
\frac{1}{2}|g\varrho - g'\varrho|\varepsilon_n^{-1} dist(y_n, V) &= \frac{1}{2}\varepsilon_n^{-1}|g\varrho - g'\varrho||\varrho_n| \\
&\leq \varepsilon_n^{-1}\, |g\varrho_n - g'\varrho_n| \\
&\leq \varepsilon_n^{-1}\, |gy_n - g'y_n| \quad \forall n \geq n_0.
\end{aligned}$$

Since $\varepsilon_n^{-1} dist(y_n, V) \to \infty$, statement $(s4)$ holds.

Case 2. The sequence $(\varepsilon_n^{-1} dist(\xi_n, V))$ is bounded. Let $\mathfrak{F}$ be the *set of isotropy classes* (G_x) *such that* $x \in V$ *and* $\varepsilon_n^{-1}(dist(\xi_n, (\mathbb{R}^N)^{gG_xg^{-1}}))$ *contains a bounded subsequence for some* $g \in G$. By Lemma A.5 the cardinality of $\mathfrak{F}$ is finite.

We will prove that $\mathfrak{F} \neq \emptyset$. For this purpose, let x_n to be the orthogonal projection of ξ_n onto V. Since $(x_n) \subset V$ using Remark A.6, we can assume that there exist a closed subgroup L of G and a subsequence of (x_n), which we still denote in the same way, with the properties

$$G_{x_n} = L \quad \forall n \in \mathbb{N} \quad \text{and} \quad (\mathbb{R}^N)^L \subseteq V.$$

Therefore

$$\varepsilon_n^{-1} dist(\xi_n, (\mathbb{R}^N)^L) \leq \varepsilon_n^{-1}|\xi_n - x_n| = \varepsilon_n^{-1} dist(\xi_n, V) < c.$$

This shows that $(L) \in \mathfrak{F}$.

Now, since $\mathfrak{F}$ *is a partially ordered non-empty finite set*, there exists an element $(K) \in \mathfrak{F}$ which is a maximal element to the partial order. This means that if $(F) \in \mathfrak{F}$ and $(K) \leq (F)$ then $(H) = (K)$. Moreover, since $(K) \in \mathfrak{F}$, passing to a subsequence of (ξ_n), we may assume that

$$dist(\xi_n, (\mathbb{R}^N)^K) < c < \infty \quad \forall n \in \mathbb{N}.$$

We define y_n to be the orthogonal projection of ξ_n onto $(\mathbb{R}^N)^K$. We immediately find that

$$\varepsilon_n^{-1} dist(G\xi_n, y_n) \leq \varepsilon_n^{-1}|\xi_n - y_n| = \varepsilon_n^{-1} dist(\xi_n, (\mathbb{R}^N)^K) < c \quad \forall n \in \mathbb{N}, \tag{A.3}$$

hence, $(s1)$ holds. Since $(K) \in \mathfrak{F}$, we have $K = G_{x_0}$ for some $x_0 \in V$. This implies that if $x \in (\mathbb{R}^N)^K = (\mathbb{R}^N)^{G_{x_0}}$, then $G_{x_0} \subset G_x$. Therefore

$$\#Gx = |G/G_x| \leq |G/G_{x_0}| = \#Gx_0 < \infty.$$

We conclude that $(\mathbb{R}^N)^K \subset V$. And since

$$(y_n) \subset (\mathbb{R}^N)^K \tag{A.4}$$

using Remark A.6, there exist a closed subgroup $L \subset G$ and a subsequence of (y_n) such that

$$G_{y_n} = L \quad \forall n \in \mathbb{N} \quad \text{and} \quad (\mathbb{R}^N)^L \subseteq V.$$

Note also that equation (A.4) implies that $K \subset L$, and therefore $(\mathbb{R}^N)^L \subset (\mathbb{R}^N)^K$. Then, one has that

$$\varepsilon_n^{-1} dist(\xi_n, (\mathbb{R}^N)^L) = \varepsilon_n^{-1}|\xi_n - y_n| < c \quad \forall n \in \mathbb{N}.$$

So, $(L) \in \mathfrak{F}$. Since (K) is maximal in $\mathfrak{F}$ and $K \subset L$, using Remark A.4, we concluded that

$$K = F = G_{y_n} \quad \forall n \in \mathbb{R}^N.$$

This proves $(s2)$.

Being that $|G/K| < \infty$, we only need to prove $(s3)$. Arguing by contradiction, let us suppose that there exist $g, g' \in K$ such that $[g] \neq [g']$ in G/K and $(\varepsilon_n^{-1}|gy_n - g'y_n|)$ is bounded. Let $g_0 := g^{-1}g' \notin K$ and L to be the subgroup of G generated by $K \cup \{g_0\}$. Let $W_1 = (\mathbb{R}^N)^L$ and W_2 to be the orthogonal complement in $(\mathbb{R}^N)^K$. Write

$$y_n = y_n^1 + y_n^2, \text{ with } y_n^i \in W_i \quad i = 1, 2.$$

Since $g_0 \notin K = G_{y_n}$, we have that $g_0 y_n \neq y_n$ for all $n \in \mathbb{N}$. And therefore, using that $g_0 \in L$ and that $y_n^1 \in W_1$, we obtain

$$0 \neq g_0 y_n - y_n = (g_0 y_n^1 - y_n^1) + (g_0 y_n^2 - y_n^2) = g_0 y_n^2 - y_n^2. \tag{A.5}$$

Hence $y_n^2 \neq 0$ for all $n \in \mathbb{N}$. Passing to a subsequence we obtain that

$$\frac{y_n^2}{|y_n^2|} \to y \in W_2.$$

We will use the sequence $(\varepsilon_n^{-1} y_n^2)$ to arrive a contradiction by showing that it cannot be bounded or unbounded.

First we will suppose that $(\varepsilon_n^{-1}y_n^2)$ is unbounded. Then passing to a subsequence and using equation (*A.5*), we have that

$$\left|\frac{g_0 y_n^2}{|y_n^2|}-\frac{y_n^2}{|y_n^2|}\right| = \frac{\varepsilon_n^{-1}|g_0 y_n - y_n|}{\varepsilon_n^{-1}|y_n^2|} = \frac{\varepsilon_n^{-1}|g y_n - g' y_n|}{\varepsilon_n^{-1}|y_n^2|} \leq \frac{c}{\varepsilon_n^{-1}|y_n^2|} \to 0$$

Therefore $g_0 y = y$, and being that $y \in W_2 \subset (\mathbb{R}^N)^K$, we have that $gy = y$ for all $g \in L$. This means that $y \in (\mathbb{R}^N)^L = W_1$, which is a contradiction.

Let us now suppose that $(\varepsilon_n^{-1}y_n^2)$ is bounded. Since $(y_n^1) \subset (\mathbb{R}^N)^K \subset V$, Remark A.6 implies that there exist a closed subgroup L_1 of G such that $G_{y_n^1} = L_1$ for all $n \in \mathbb{N}$. In addition, using Equation (A.3), results in

$$\varepsilon_n^{-1} dist(\xi_n, \mathbb{R}^N)^{L_1} = \varepsilon_n^{-1}|\xi_n - y_n^1| \leq \varepsilon_n^{-1}|\xi_n - y_n| + \varepsilon_n^{-1}|y_n^2| < c_1 < \infty,$$

for some constant $c_1 > 0$. This shows that $(L_1) \in \mathfrak{F}$. Note also that since $y_n^1 \in W_1$, we have that $K \subset L \subset L_1$. Using again that (K) is a maximal element in $\mathfrak{F}$ and Remark A.4, we concluded that

$$K = L = L_1,$$

which contradicts the fact that $g_0 \notin K$.

Therefore, if $g, g' \in K$ and $[g] \neq [g']$ in G/K then $(|gy_n - g'y_n|)$ is not bounded. This proves $(s3)$ and concludes the proof.

□

Recall that the energy functional $J_\infty : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem $(\wp_\infty)$ is given by

$$J_\infty(u) \;:=\; \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*},$$

Also, we define the functional $J_1 : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_1(u) \;:=\; \frac{1}{2}\int_{\mathbb{R}^N} a_1|\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} a_3|u|^{2^*}$$

The following lemmas are taken from [47].

Lemma A.8. *(Brézis-Lieb Lemma, 1983). Let Ω be an open subset of $\mathbb{R}^N$ and let $(w_n) \subset L^p(\Omega)$, $1 \leq p < \infty$. If*

- *(w_n) is bounded in $L^p(\Omega)$,*
- *$w_n \to w$ almost everywhere on Ω, then*

$$\lim_{n\to\infty}(|w_n|_p^p - |w_n - w|_p^p) = |w|_p^p.$$

Lemma A.9. *If $w_n \rightharpoonup w$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $w \in L^\infty_{loc}(\mathbb{R}^N)$ then*

$$|w_n|^{2^*-2}w_n - |w_n - w|^{2^*-2}(w_n - w) \to |w|^{2^*-2}w \text{ in } (D^{1,2}(\mathbb{R}^N))'.$$

Lemma A.10. *If*

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } H_0^1(\Omega), \\ u_n &\to u \text{ a.e. on } \Omega, \\ u_n &\to u \text{ in } L^2_{loc}(\Omega), \\ J(u_n) &\to c \text{ in } \mathbb{R}, \\ J'(u_n) &\to 0 \text{ in } (H_0^1(\Omega))', \end{aligned}$$

then $J'(u) = 0$ and $v_n := u_n - u$ is such that

$$\begin{aligned} \|v_n\|^2 &= \|u_n\|^2 - \|u\|^2 + o(1), \\ J_1(v_n) &\to c - J(u) \text{ in } \mathbb{R}, \\ J_1'(v_n) &\to 0 \text{ in } (H_0^1(\Omega))', \end{aligned}$$

Proof. Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, it is clear that

$$\begin{aligned} \|v_n\|^2 &= \langle u_n, u_n\rangle - 2\langle u_n, u\rangle + \langle u, u\rangle \\ &= \|u_n\|^2 - \|u\|^2 + o(1). \end{aligned}$$

On the other hand, being that $v_n \to 0$ in $L^2_{loc}(\Omega)$, we obtain

$$J(v_n) - J_1(v_n) = \frac{1}{2}\int_\Omega a_2(x)v_n^2 = o(1), \tag{A.6}$$

moreover, Lemma A.8 implies that

$$\begin{aligned} J(v_n) &= \frac{1}{2}\langle u_n - u, u_n - u\rangle_{a_1,a_2} - \frac{1}{2^*}\int_\Omega a_3|u_n - u|^{2^*} \\ &= \frac{1}{2}(||u_n||_{a_1,a_2} - ||u||_{a_1,a_2}) - \frac{1}{2^*}\int_\Omega a_3|u_n - u|^{2^*} + o(1) \\ &= J(u_n) - J(u) + \frac{1}{2^*}(\int_\Omega a_3|u_n|^{2^*} - \int_\Omega a_3|u|^{2^*} - \int_\Omega a_3|u_n - u|^{2^*}) + o(1) \\ &= c - J(u) + o(1), \end{aligned}$$

which, together with equation (A.6), shows that

$$J_1(v_n) = c - J(u) + o(1).$$

On the other hand, since Ω is bounded, we can combine Rellich-Kondrakov Theorem and Lebesgue's Dominated Convergence Theorem to obtain

$$\int_\Omega a_3|u_n|^{2^*-2}u_n\varphi \to \int_\Omega a_3|u|^{2^*-2}u\varphi, \text{ for every } \varphi \in H_0^1(\Omega). \tag{A.7}$$

Since $u_n \rightharpoonup u$ weakly in $H^1_0(\Omega)$, Equation (A.7) implies that

$$J'(u_n)[\varphi] = \langle u_n, \varphi\rangle_{a_1,a_2} - \int_\Omega a_3|u_n|^{2^*-2}u_n\varphi \longrightarrow \langle u, \varphi\rangle_{a_1,a_2} - \int_\Omega a_3|u|^{2^*-2}u\varphi = J(u)[\varphi].$$

Since also $J'(u_n)[\varphi] \to 0$, we have that $J'(u)[\varphi] = 0$ for every $\varphi \in H^1_0(\Omega)$, e.i $J'(u) = 0$
.

Being that $v_n \to 0$ in $L^2_{loc}(\Omega)$ we have, for n large enough and for every $\varphi \in C^\infty_0(\Omega)$, that

$$\begin{aligned} |J'(v_n)[\varphi] - J'_1(v_n)[\varphi]| &= \left|\int_\Omega a_2 v_n\varphi\right| \\ &\leq \sup_{x\in\Omega}|a_2(x)| \left(\int_\Omega |v_n|^2\right)^2 \left(\int_\Omega |\varphi|^2\right)^2 \\ &\leq C\varepsilon||\varphi|| \end{aligned}$$

Since $C^\infty_0(\Omega)$ is dense in $H^1_0(\Omega)$, we have that $J'(v_n) \to J'_1(v_n)$ in $(H^1_0(\Omega))'$. Finally

$$\begin{aligned} J'(v_n)[\varphi] &= \langle u_n - u, \varphi\rangle_{a_1,a_2} - \int_\Omega a_3|u_n - u|^{2^*-2}(u_n - u)\varphi \\ &= J'(u_n)[\varphi] - J'(u)[\varphi] + \int_\Omega a_3|u_n|^{2^*-2}(u_n)\varphi - \int_\Omega a_3|u|^{2^*-2}u\varphi \\ &\quad + \int_\Omega a_3|u_n - u|^{2^*-2}(u_n - u)\varphi, \end{aligned}$$

hence, using again equation (A.7), we obtain

$$J'(v_n)[\varphi] = J'(u_n)[\varphi] - J'(u)[\varphi] + o(1),$$

and therefore

$$\begin{aligned} J'_1(v_n) &= J'(v_n) + o(1), \\ &= J'(u_n) - J'(u) - o(1), \\ &= o(1), \end{aligned}$$

as we wanted to demonstrate. $\square$

A.3 Proof of Theorem A.1

We will begin with the proof of Lemma A.2 to carry on with the proof of Theorem A.1.

Proof of Lemma A.2. We divided the proof into several steps.
Step 1: Definition of the sequences $(\varepsilon_n),(y_n)$ and the group K.
Let (u_n) be a $G-$invariant Palais-Smale sequence for J_1 at the level $c>0$, equation

$$\frac{1}{N}||u_n||^2_{a_1} = J(u_n) - \frac{1}{2^*}J'(u_n)(u_n) \le |c| + 1 + ||u_n||_{a_1}. \tag{A.8}$$

guaranties that (u_n) is a bounded sequence in $H^1_0(\Omega)^G$ and hence, $J_1'(u_n)(u_n)\to 0$. Therefore

$$\int_\Omega a_3|u_n|^{2^*} = N(J_1(u_n) - \frac{1}{2}J_1'(u_n)(u_n)) \to Nc > 0.$$

Let $\delta := \min\{\frac{Nc}{2}, \frac{(\max\limits_{x\in\Omega} a_3)^{\frac{2-N}{2}}}{(\frac{S}{2}\cdot \min\limits_{x\in\Omega} a_1)^{-\frac{N}{2}}}\} > 0$ and extend $u_n \equiv 0$ outside Ω.

For every $n\in\mathbb{N}$, consider the Levy concentration function $Q_n : [0,\infty)\to[0,\infty)$ defined by

$$Q_n(r) := \sup_{y\in\mathbb{R}^N}\int_{B(y,r)} a_3|u_n|^{2^*}.$$

Notice that Q_n is an increasing continuous function. Moreover, since $Q_n(0)=0$ and $\lim_{r\to\infty} Q_n(r) > \delta$, there exist $\varepsilon_n>0$ and $\xi_n\in\mathbb{R}^N$ such that

$$Q_n(\varepsilon_n) = \sup_{y\in\mathbb{R}^N}\int_{B(y,\varepsilon_n)} a_3|u_n|^{2^*} = \int_{B(\xi_n,\varepsilon_n)} a_3|u_n|^{2^*} = \delta.$$

Now, for the sequences (ε_n) and (ξ_n) we choose K and (y_n) as in Lemma A.7.

By property $(s1)$ of Lemma A.7 there exists a positive constant C_1 such that $\varepsilon_n^{-1}\mathrm{dis}(G\xi_n, y_n) < C_1$ for every $n\in\mathbb{N}$. This implies that there exists $g_n\in G$ such that $B_{\varepsilon_n}(g_n\xi_n)\subset B_{C\varepsilon_n}(y_n)$ for $C:=C_1+1$. Hence, using that a_3 and u_n are G-invariants, one has that

$$\delta = \int_{B_{\varepsilon_n}(\xi_n)} a_3|u_n|^{2^*} = \int_{B_{\varepsilon_n}(g_n\xi_n)} a_3|u_n|^{2^*} \le \int_{B_{C\varepsilon_n}(y_n)} a_3|u_n|^{2^*}. \tag{A.9}$$

Next we will show that $|G/K|<\infty$. If we suppose the contrary, property $(s4)$ of Lemma A.7 guarantees that there exists a closed subgroup K' such that $K\subset K'$, $|G/K'|=\infty$ and $\varepsilon_n^{-1}|gy_n - g'y_n|\to\infty$ for any $[g],[g']\in G/K'$ with $[g]\ne[g']$.

Let $\{g_1,\dots,g_m\}$ in G such that $[g_i]\ne[g_j]$ in G/K' if $i\ne j$. We have that

$$B_{C\varepsilon_n}(g_iy_n)\cap B_{C\varepsilon_n}(g_jy_n) = \emptyset \quad \forall\, i\ne j.$$

for n sufficiently large. Using again that the functions a_3 and u_n are $G-$invariants and inequality (A.9), we obtain

$$m\delta \le \sum_{i=1}^m \int_{B_{C\varepsilon_n}(g_iy_n)} a_3|u_n|^{2^*} \le \int_\Omega a_3|u_n|^{2^*} = Nc + o(1),$$

for every $m \in \mathbb{N}$. Since this is a contradiction, we conclude that $|G/K| < \infty$.

Step 2: Definition of $\widehat{u} \neq 0$ such that $\widehat{u}(gx) = \widehat{u}(x)$ for all $x \in \mathbb{R}^n$ and $g \in K$.

Let us consider the following functions

$$\begin{aligned} \widetilde{u}_n(z) &:= \varepsilon_n^{\frac{N-2}{2}} u_n(\varepsilon_n z + y_n) \\ a_i^n(z) &:= a_i(\varepsilon_n z + y_n) \quad i = 1, 3. \end{aligned}$$

Since u_n is G-invariant and $G_n = K$, we have that $\widetilde{u}_n$ is K-invariant. Indeed, if $g \in K$ we have that

$$\begin{aligned} \widetilde{u}_n(gz) &= \varepsilon_n^{\frac{N-2}{2}} u_n(\varepsilon_n gz + y_n) \\ &= \varepsilon_n^{\frac{N-2}{2}} u_n(g(\varepsilon_n z + y_n)) \\ &= \varepsilon_n^{\frac{N-2}{2}} u_n(\varepsilon_n z + y_n) \\ &= \widetilde{u}_n(z). \end{aligned}$$

for every $z \in \mathbb{R}^N$. Analogously, the equality

$$a_i^n(gz) = a_i(\varepsilon_n gz + y_n) = a_i(g(\varepsilon_n z + y_n)) = a_i^n(z)$$

shows that a_i^n is K-invariant for $i = 1, 3$. We also have that

$$\begin{aligned} \int_\Omega a_3(x)|u_n(x)|^{2^*} dx &= \int_{\Omega_n} a_3^n(z)|\widetilde{u}_n(z)|^{2^*} dz \\ \int_\Omega a_1(x)|\nabla u_n(x)|^2 dx &= \int_{\Omega_n} a_1^n(z)|\nabla \widetilde{u}_n(z)|^2 dz \end{aligned}$$

where

$$\Omega_n := \{z \in \mathbb{R}^n : \varepsilon_n z + y_n \in \Omega\}.$$

Being that $\widetilde{u}_n$ a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ then a subsequence -which will be denoted in the same way- satisfies that

$$\begin{aligned} &\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{weakly in in } D^{1,2}(\mathbb{R}^N) \\ &\widetilde{u}_n \to \widetilde{u} \quad \text{strongly in } L^2_{loc}(\mathbb{R}^N) \\ &\widetilde{u}_n \to \widetilde{u} \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

It follows that $\widetilde{u}$ is K-invariant. Next we will proof that $\widetilde{u} \neq 0$. Arguing by contradiction, let's assume that $\widetilde{u} = 0$. Let $z \in \mathbb{R}^N$ and $\widetilde{h} \in C_0^\infty(\mathbb{R}^N)$ such that

$supp(\widetilde{h}) \subset B_1(z)$. If $h_n(x) = \widetilde{h}(\frac{1}{\varepsilon_n}(x - y_n))$, one has that

$$\begin{aligned} J_1(u_n)[h_n^2 u_n] &= \int_\Omega a_1(x)(\nabla u_n(x) \cdot \nabla(h_n^2(x)u_n(x)))dx - \int_\Omega a_3(x)|u_n(x)|^{2^*} h_n^2(x)dx \\ &= \int_{B_1(z)} a_1^n(z)(\nabla\widetilde{u}_n(z) \cdot \nabla(\widetilde{h}^2(z)\widetilde{u}_n(z)))dz - \int_{B_1(z)} a_3^n(z)|\widetilde{u}_n(z)|^{2^*}\widetilde{h}^2(z)dz. \end{aligned}$$

Since $J_1(u_n)[h_n^2 u_n] \to 0$, we have that

$$\int_{B_1(z)} a_1^n(\nabla\widetilde{u}_n \cdot \nabla(\widetilde{h}^2\widetilde{u}_n)) = \int_{B_1(z)} a_3^n|\widetilde{u}_n|^{2^*}\widetilde{h}^2 + o(1) \tag{A.10}$$

On the other hand, using Hölder and Sobolev inequalities, equation A.10 and the fact $\widetilde{u}_n \to \widetilde{u} = 0$ in $L^2_{loc}(\mathbb{R}^N)$ we have that

$$S\left(\min_{x\in\overline{\Omega}} a_1\right)\left(\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*}\right)^{\frac{2}{2^*}}$$

$$\begin{aligned} &\leq \left(\min_{x\in\overline{\Omega}} a_1\right)\int_{B_1(z)} |\nabla(\widetilde{h}\widetilde{u}_n)|^2 \\ &= \left(\min_{x\in\overline{\Omega}} a_1\right)\int_{B_1(z)} |\widetilde{h}\nabla\widetilde{u}_n + \widetilde{u}_n\nabla\widetilde{h}|^2 \\ &= \left(\min_{x\in\overline{\Omega}} a_1\right)\int_{B_1(z)} (\widetilde{h}^2|\nabla\widetilde{u}_n|^2 + 2\widetilde{h}\widetilde{u}_n\nabla\widetilde{u}_n \cdot \nabla\widetilde{h} + \widetilde{u}_n^2|\nabla\widetilde{h}|^2) \\ &= \left(\min_{x\in\overline{\Omega}} a_1\right)\int_{B_1(z)} \nabla\widetilde{u}_n \cdot \nabla(\widetilde{h}^2\widetilde{u}_n) + o(1) \\ &\leq \int_{B_1(z)} a_1^n(\nabla\widetilde{u}_n \cdot \nabla(\widetilde{h}^2\widetilde{u}_n)) + o(1) \\ &= \int_{B_1(z)} a_3^n|\widetilde{u}_n|^{2^*}\widetilde{h}^2 + o(1) \\ &\leq \left(\max_{x\in\overline{\Omega}} a_3\right)^{\frac{N-2}{N}}\left(\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*}\right)^{\frac{2}{2^*}}\left(\int_{B_1(z)} a_3^n|\widetilde{u}_n|^{2^*}\right)^{\frac{2}{N}} + o(1) \\ &\leq \left(\max_{x\in\overline{\Omega}} a_3\right)^{\frac{N-2}{N}}\left(\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*}\right)^{\frac{2}{2^*}}\delta^{\frac{2}{N}} + o(1), \end{aligned}$$

the last inequality is due

$$\int_{B_1(z)} a_3^n|\widetilde{u}_n|^{2^*} = \int_{B_{\varepsilon_n}(\varepsilon_n z + y_n)} a_3|u_n|^{2^*} \leq \int_{B_{\varepsilon_n}(\xi_n)} a_3|u_n|^{2^*} = \delta.$$

Now, using the definition of δ, one has

$$\begin{aligned} S\left(\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*}\right)^{\frac{2}{2^*}} &\leq \frac{(\max\limits_{x\in\Omega} a_3)^{\frac{N-2}{N}}}{(\min\limits_{x\in\Omega} a_1)} \left(\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*}\right)^{\frac{2}{2^*}} \delta^{\frac{2}{N}} + o(1) \\ &\leq \frac{S}{2}\left(\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*}\right)^{\frac{2}{2^*}} + o(1). \end{aligned}$$

This implies that $\int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*} \to 0$ for every $y \in \mathbb{R}^n$ and for every $\widetilde{h} \in C_0^\infty(\mathbb{R}^N)$ such that $supph \subset B_1(z)$. Hence, if we choose $\widetilde{h} \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp}\widetilde{h} \subset B_1(z)$ and $\widetilde{h} \equiv 1$ in $B_{\frac{1}{2}}(z)$ we have that

$$\int_{B_{\frac{1}{2}}(z)} |\widetilde{u}_n|^{2^*} \leq \int_{B_1(z)} |\widetilde{h}\widetilde{u}_n|^{2^*} \to 0$$

Therefore $\widetilde{u}_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$. This is a contradiction because, using inequality (A.9), we have that

$$0 < \delta \leq \int_{B_{C\varepsilon_n}(y_n)} a_3|u_n|^{2^*} = \int_{B_C(0)} a_3^n|\widetilde{u}_n|^{2^*} \leq \max_{x\in\overline{\Omega}} a_3 \int_{B_C(0)} |\widetilde{u}_n|^{2^*}$$

We conclude that $\widetilde{u} \neq 0$. Since (y_n) is a bounded sequence, a subsequence converge to a point $y_0 \in \mathbb{R}^N$. We define

$$\widehat{u} := \left(\frac{a_3(y_0)}{a_1(y_0)}\right)^{\frac{N-2}{4}} \widetilde{u}.$$

Step 3: It holds true that $\varepsilon_n^{-1} dis(y_n, \partial\Omega) \to \infty$, $y_n \in \Omega$ and $\widehat{u}$ is a solution to problem $(\wp_\infty)$.

Since (ε_n) is bounded, a subsequence satisfies that $\varepsilon_n \to \varepsilon \in [0,\infty)$. We claim that $\varepsilon = 0$. Indeed, if $\varepsilon \neq 0$, for every $\widetilde{h} \in C_0^\infty(\mathbb{R}^N)$, we can consider

$$\begin{aligned} h_n(z) &:= \varepsilon_n^{\frac{2-N}{2}} \widetilde{h}\left(\frac{z-y_n}{\varepsilon_n}\right) \\ h(z) &:= \varepsilon^{\frac{2-N}{2}} \widetilde{h}\left(\frac{z-y_0}{\varepsilon}\right) \end{aligned}$$

Hence, by Lebesgue's dominated convergence, we have that

$$\int_{\mathbb{R}^N} |\nabla(h_n - h)|^2 \to 0.$$

Moreover, using that $u_n \rightharpoonup 0$ in $H_0^1(\Omega)$ and therefore in $D^{1,2}(\mathbb{R}^N)$, we have that

$$\begin{aligned} |\langle \widetilde{u}_n, \widetilde{h}\rangle| = |\,\langle u_n, h_n\rangle\,| &= |\,\langle u_n, h_n\rangle - \langle u_n, h\rangle + \langle u_n, h\rangle\,| \\ &\le |\,\langle u_n, h_n - h\rangle\,| + |\,\langle u_n, h\rangle\,| \\ &\le ||u_n||||h_n - h|| + o(1) \\ &\le C||h_n - h|| + o(1) \\ &= o(1). \end{aligned}$$

Since, this is true for every $\widetilde{h} \in C_0^\infty(\mathbb{R}^N)$ we have that $\widetilde{u} = 0$, which is a contradiction. We conclude that $\varepsilon = 0$.

Let us suppose that the sequence $(\varepsilon_n^{-1} dis(y_n, \Omega))$ is bounded to obtain a contradiction. Passing to a subsequence, we may assume that $\varepsilon_n^{-1} dis(y_n, \Omega) \to d < \infty$. Note that this implies that

$$dist(y_n, \partial\Omega) \to 0. \tag{A.11}$$

After passing to a subsequence, we have the following two cases:

case 1. $y_n \in \overline{\Omega} \quad \forall n \in \mathbb{N}$

case 2. $y_n \in \mathbb{R}^N \backslash \Omega \quad \forall n \in \mathbb{N}$

First we will consider *case 1.*

For every $z \in \partial\Omega$, $\nu(z)$ will denote the inward pointing unit to $\partial\Omega$, and for every y_n let $z_n \in \partial\Omega$ such that $|y_n - z_n| = dist(y_n, \partial\Omega)$. We also may assume that $\nu(z_n) \to \nu$. Since $\partial\Omega$ is a compact smooth domain there exists a positive r_0 such that

$$B_{r_0}(z + r_0\nu(z)) \subset \overline{\Omega} \text{ and } B_{r_0}(z - r_0\nu(z)) \subset \mathbb{R}^N \diagdown \overline{\Omega} \quad \forall z \in \partial\Omega. \tag{A.12}$$

Consider the transformations $\varrho_n^i : \mathbb{R}^N \to \mathbb{R}^N$ and defined by

$$\varrho_n^1(z) := \varepsilon_n z + y_n \text{ and } \varrho_n^2(z) := \frac{1}{\varepsilon_n}(z - y_n)$$

Lets consider the half-space $\mathbb{H} := \{z \in \mathbb{R}^N : \nu \cdot z > -d\}$. We will show that $\widetilde{u} \in D_0^{1,2}(\mathbb{H})$ and that $\widetilde{u}$ is a solution to problem

$$-\Delta u = \frac{a_3(y_0)}{a_1(y_0)}|u|^{2^*-2}u \quad \text{in } \mathbb{H}. \tag{A.13}$$

Fix $x \in \mathbb{R}^n \diagdown \overline{\mathbb{H}}$. Note that $x \cdot \nu(z_n) > -\varepsilon_n^{-1}|y_n - z_n|$ for all $n > n_1$. Hence, there exist $r > 0$ such that

$$x \in B_r(((|\varrho_n^2(z_n)| + r)(-\nu(z_n))) \quad \forall n \ge n_1.$$

Applying the transformation ϱ_n^1, we have that there exists n_2 such that

$$\varrho_n^1(x) \in \varrho_n^1(B_r(((|\varrho_n^2(z_n)| + r)(-\nu(z_n)))) = B_{\varepsilon_n r}(z_n - \varepsilon_n r\nu(z_n)) \subset B_{r_0}(z_n - r_0\nu(z_n))$$

for all $n \geq n_2$. Therefore, using (A.12), for we obtain

$$x \in B_r(((|\varrho_n^2(z_n)| + r)(-\nu(z_n))) \subset \mathbb{R}^N \backslash \Omega_n \quad \forall n \geq n_2.$$

Since supp$\widetilde{u}_n \subset \Omega_n$, we have that if $x \in \mathbb{R}^n \diagdown \overline{\mathbb{H}}$ then

$$\widetilde{u}_n(x) = 0$$

for n sufficiently large. And so, using that $\widetilde{u}_n \to \widetilde{u}$ almost everywhere in $\mathbb{R}^n$, we conclude that $\widetilde{u} \in D_0^{1,2}(\mathbb{H})$. Preceding in a way analogous, for every compact set $\mathbb{K} \subset \overline{\mathbb{H}}$, there exists $r > 0$ and $n_1 \in \mathbb{N}$ such that $\mathbb{K} \subset intB_r((|r - |\varrho_n^2(z_n)|\nu(z_n)))$ for all $n \geq n_1$. Therefore, there exists $n_2 \in \mathbb{N}$, such that

$$\varrho_n^1(B_r((r - |\varrho_n^2(z_n)|)(\nu(z_n))) = B_{\varepsilon_n r}(z_n + \varepsilon_n r\nu(z_n)) \subset B_{r_0}(z_n + r_0\nu(z_n))$$

For all $n \geq n_2$. Hence, using again (A.12) we obtain

$$\mathbb{K} \subset intB_r((r - |\varrho_n^2(z_n)|)(\nu(z_n))) \subset \Omega_n \quad \forall n \geq n_2.$$

If $\widetilde{\phi} \in C_0^\infty(\mathbb{H})$, then $supp\widetilde{\phi} \subset \Omega_n$ for n sufficiently large. Let

$$\phi_n(z) := \varepsilon^{\frac{2-N}{2}}\widetilde{\phi}(\varepsilon_n^{-1}(z - y_n)).$$

It follows that

$$\begin{aligned}
\left| \int_{\Omega_n} a_1^n \nabla\widetilde{u}_n \cdot \nabla\widetilde{\phi} - \int_{\Omega_n} a_3^n |\widetilde{u}_n|^{2^*-2}\widetilde{u}_n\widetilde{\phi} \right| &= |J_1(u_n)[\phi_n]| \\
&\leq ||\nabla J_1(u_n)||||\phi_n|| \\
&\leq ||\nabla J_1(u_n)||||\phi|| \to 0
\end{aligned}$$

and since

$$\int_{\Omega_n} a_1^n \nabla\widetilde{u}_n \cdot \nabla\widehat{\phi} - \int_{\Omega_n} a_3^n |\widetilde{u}_n|^{2^*-2}\widetilde{u}_n\widehat{\phi} \to \int_{\mathbb{H}} a_1(y_0)\nabla\widetilde{u} \cdot \nabla\widehat{\phi} - \int_{\mathbb{H}} a_3(y_0)|\widetilde{u}|^{2^*-2}\widetilde{u}\widehat{\phi},$$

we conclude that

$$\int_{\mathbb{H}} a_1(y_0)\nabla\widetilde{u} \cdot \nabla\widehat{\phi} - \int_{\mathbb{H}} a_3(y_0)|\widetilde{u}|^{2^*-2}\widetilde{u}\widehat{\phi} = 0 \quad \forall\widehat{\phi} \in C_0^\infty(\mathbb{H}).$$

This shows that $\widetilde{u}$ is a solution to problem (A.13). Pohozaev identity [40] shows that $\widetilde{u} = 0$, which is a contradiction.

Therefore, if $y_n \in \overline{\Omega}$ for all $n \in \mathbb{N}$, we have that $(\varepsilon_n^{-1}dis(y_n, \Omega))$ is not bounded. Passing to a subsequence we may suppose that $\varepsilon_n^{-1}dis(y_n, \Omega) \to \infty$.

In *case 2* (when $y_n \in \mathbb{R}^N \backslash \Omega$ for all $n \in \mathbb{N}$), it can be show, in full analogy with *case 1*, that $\varepsilon_n^{-1}dis(y_n, \Omega) \to \infty$.

However, if $y_n \in \mathbb{R}^N \backslash \Omega$, $\varepsilon_n^{-1} dis(y_n, \Omega) \to \infty$ and C is a positive constant, we have that $B_{C\varepsilon}(y_n) \subset \mathbb{R}^N \backslash \Omega$ for n sufficiently large. This implies that $\int_{B_{C\varepsilon_n}(y_n)} |u_n|^{2^*} = 0$, which contradicts equation (A.9).

Also it cannot be that $y_n \in \partial\Omega$, for otherwise $\varepsilon_n^{-1} dis(y_n, \Omega) \to 0$, which is not the case. We concluded that $y_n \in \Omega$ for all $n \in \mathbb{N}$.

Finally, suppose that $\widetilde{\phi} \in C_0^\infty(\mathbb{R}^N)$, and $\mathrm{supp}\widetilde{\phi} \subset B_\rho(0)$. Since $\varepsilon_n^{-1} dis(y_n, \Omega) \to \infty$, we have that $B_{\varepsilon_n \rho}(y_n) \subset \Omega$ and therefore $\widetilde{\phi} \in C_0^\infty(\Omega_n)$ for all $n \geq n_3$. Arguing as above we have that

$$\int_{\mathbb{R}^N} a_1(y_0) \nabla \widetilde{u} \cdot \nabla \widehat{\phi} - \int_{\mathbb{R}^N} a_3(y_0) |\widetilde{u}|^{2^*-2} \widetilde{u} \widehat{\phi} = 0 \quad \forall \widehat{\phi} \in C_0^\infty(\mathbb{R}^N).$$

This implies that $\widetilde{u}$ is a solution to problem

$$-\Delta u = \frac{a_3(y_0)}{a_1(y_0)} |u|^{2^*-2} u \quad \text{in } \mathbb{R}^{\mathbb{N}} \tag{A.14}$$

and therefore, $\widehat{u} = \left(\frac{a_3(y_0)}{a_1(y_0)}\right)^{\frac{N-2}{4}} \widetilde{u}$ is a solution to problem $(\wp_\infty)$.

Step 4: Definition of the sequence (v_n) which satisfies (iv).

Let $r_n := \frac{1}{4} \min\{dist(y_n, \Omega), |g_n - g'(y_n)| \text{ with } [g] \neq [g'] \in G/H\}$. Then we have $r_n \varepsilon_n^{-1} \to \infty$. We choose a radially symmetric function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \phi \leq 1$, $\phi(x) = 1$ if $|x| \leq 1$ and $\phi(x) = 0$ if $|x| \geq 2$. Let us introduce the sequence (v_n) defined by

$$v_n := u_n - \sum_{[g] \in G/H} \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g^{-1}(z - gy_n)) \phi(r_n^{-1}(z - gy_n)).$$

where $a(y_0) := (\frac{a_3(y_0)}{a_1(y_0)})$. Since $\widehat{u}$ is K-invariant and $G_{y_n} = K$ for all $n \in \mathbb{N}$, we have that v_n is well defined, G-invariant and $v \in H_0^1(\Omega)$. Moreover

$$\begin{aligned}
&\left\| v_n - u_n + \sum_{[g]\in G/H} \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g^{-1}(\cdot - gy_n)) \right\|^2 \\
= &\left\| \sum_{[g]\in G/H} \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g^{-1}(\cdot - gy_n))[1-\phi(r_n^{-1}(\cdot - gy_n))] \right\|^2 \\
\leq & \sum_{[g]\in G/H} a(y_0)^{\frac{2-N}{2}} \left\| \varepsilon_n^{\frac{2-N}{2}} \widehat{u}(\varepsilon_n^{-1} g^{-1}(\cdot - gy_n))[1-\phi(r_n^{-1}(\cdot - gy_n))] \right\|^2 \\
= & \; a(y_0)^{\frac{2-N}{2}} |G/K| \left\| \widehat{u}(1-\phi(r_n^1 \varepsilon_n(\cdot))) \right\|^2 \\
\leq & \; C_1 a(y_0)^{\frac{2-N}{2}} |G/K| \left(\int_{|z|>2\varepsilon_n^{-1} r_n} |\nabla \widehat{u}|^2 + (\varepsilon_n^{-1} r_n)^{-2} \int_{\varepsilon_n^{-1} r_n < |z| < 2\varepsilon_n^{-1} r_n} |\widehat{u}|^2 \right) \\
\leq & \; C_2 a(y_0)^{\frac{2-N}{2}} |G/K| \left(\int_{|z|>2\varepsilon_n^{-1} r_n} |\nabla \widehat{u}|^2 + \left(\int_{|z|>2\varepsilon_n^{-1} r_n} |\widehat{u}|^2 \right)^{\frac{2}{2^*}} \right),
\end{aligned}$$

the last inequality follows from Hölder inequality. Since, $\widehat{u} \in D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ and $\varepsilon_n^{-1} r_n \to \infty$, we have that the last expression goes to zero. Therefore (iv) holds.

Step 5: Proof of (v) and that $J_1(v_n) \to 0$.

Let $G/K := \{g_1, \dots g_m\}$. Since $\widetilde{u}_n \rightharpoonup \widetilde{u}$ weakly in $D^{1,2}(\mathbb{R}^N)$ we have that

$$\widetilde{u}_n \circ g_1^{-1} - \widetilde{u} \circ g_1^{-1} \rightharpoonup 0 \text{ weakly in } D^{1,2}(\mathbb{R}^N), \tag{A.15}$$

On the other hand, if $\varphi \in D^{1,2}(\mathbb{R}^N)$, we have that

$$\langle \widetilde{u}(g_i^{-1}(\cdot + \varepsilon_n^{-1}(g_1 y_n - g_i y_n))), \varphi(\cdot) \rangle = \langle \widetilde{u}(\cdot), \varphi(g_1^{-1}(\cdot - \varepsilon_n^{-1}(g_1 y_n - g_i(y_n))))\rangle.$$

Hence, using that $\varepsilon_n^{-1}|g_1 y_n - g_j y_n| \to \infty$ for every $i \neq 1$, we obtain

$$\widetilde{u}(g_i^{-1}(\cdot + \varepsilon_n^{-1}(g_1 y_n - g_i y_n)) \rightharpoonup 0 \text{ weakly in } D^{1,2}(\mathbb{R}^N), \tag{A.16}$$

if $i \neq 1$. Let

$$D_n(z) := u_n(z) - \sum_{i=1}^{m} \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(z - g_i y_n)).$$

Since u_n is G-invariant, it holds that

$$\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) = (\widetilde{u}_n \circ g_1^{-1}) - \sum_{i=1}^{m} \widetilde{u}(g_i^{-1}(z + \varepsilon_n^{-1}(g_1 y_n - g_i y_n))),$$

therefore, using equations (A.15) and (A.16), we obtain

$$\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n) + \widetilde{u} \circ g_1^{-1} \rightharpoonup \widetilde{u} \circ g_1^{-1} \text{ weakly in } D^{1,2}(\mathbb{R}^N) \tag{A.17}$$

$$\begin{aligned}
\|D_n\|_{a_1}^2 &= \int_\Omega a_1 \nabla D_n \cdot \nabla D_n \\
= & \int_{\Omega_n} a_1^n \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1} - \widetilde{u} \circ g_1^{-1}) \cdot \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1} - \widetilde{u} \circ g_1^{-1}) \\
= & \int_{\Omega_n} a_1^n \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1}) \cdot \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1}) \\
- & 2 \int_{\Omega_n} a_1^n \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1}) \cdot \nabla(\widetilde{u} \circ g_1^{-1}) + \int_{\Omega_n} a_1^n \nabla(\widetilde{u} \circ g_1^{-1}) \cdot \nabla(\widetilde{u} \circ g_1^{-1}) \\
= & \int_{\Omega_n} a_1^n \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1}) \cdot \nabla(\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1}) \\
- & \int_{\mathbb{R}^N} a_1(y_0) \nabla(\widetilde{u} \circ g_1^{-1}) \cdot \nabla(\widetilde{u} \circ g_1^{-1}) + o(1) \\
= & \int_\Omega a_1 \nabla(D_n + \varepsilon^{\frac{2-N}{2}} a(y_0) \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n))) \cdot \nabla(D_n + \varepsilon^{\frac{2-N}{2}} a(y_0) \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n))) \\
- & a_1(y_0) \|\widetilde{u}\|^2 + o(1)
\end{aligned}$$

Hence

$$\begin{aligned}
&\|u_n - \sum_{i=1}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))\|_{a_1}^2 = \\
&\|u_n - \sum_{i=2}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))\|_{a_1}^2 - \left(\frac{a_3(y_0)^{\frac{2-N}{2}}}{a_1(y_0)^{-\frac{N}{2}}} \right) \|\widehat{u}\|^2 + o(1)
\end{aligned}$$

Let us denote $a_0(y_0) := \left(\frac{a_3(y_0)^{\frac{2-N}{2}}}{a_1(y_0)^{-\frac{N}{2}}} \right)$. Arguing by induction we have that

$$\|u_n - \sum_{i=1}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))\|_{a_1}^2 = \|u_n\|_{a_1}^2 - m a_0(y_0) \|\widehat{u}\|^2 + o(1)$$

Since, in step (iv) we proof that

$$\|v_n\|_{a_1}^2 = \|u_n - \sum_{i=1}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))\|_{a_1}^2 + o(1), \tag{A.18}$$

we conclude

$$\|v_n\|_{a_1}^2 = \|u_n\|_{a_1}^2 - m a_0(y_0) \|\widehat{u}\|^2 + o(1). \tag{A.19}$$

On the other hand, using equation (A.17), one has that

$$\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n z + g_1 y_n) + \widetilde{u} \circ g_1^{-1} \to \widetilde{u} \circ g_1^{-1} \text{ a.e. on } \mathbb{R}^N.$$

Therefore, Lemma A.8 implies that

$$\begin{aligned}
&\int_\Omega a_3 |D_n|^{2^*} \\
= &\int_{\Omega_n} a_3^n |\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n) + \widetilde{u} \circ g_1^{-1} - \widetilde{u} \circ g_1^{-1}|^{2^*} \\
= &\int_{\Omega_n} a_3^n |\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n) + \widetilde{u} \circ g_1^{-1}|^{2^*} - \int_{\mathbb{R}^N} a_3(y_0) |(\widetilde{u} \circ g_1^{-1})|^{2^*} + o(1) \\
= &\int_\Omega a_3 |D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n))|^{2^*} - \int_{\mathbb{R}^N} a_3(y_0) |\widetilde{u}|^{2^*} + o(1),
\end{aligned}$$

hence

$$\begin{aligned}
&|u_n - \sum_{i=1}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))|_{a_3,2^*}^{2^*} = \\
&|u_n - \sum_{i=2}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))|_{a_3,2^*}^{2^*} - a_0(y_0) |\widehat{u}|_{2^*}^{2^*} + o(1)
\end{aligned}$$

It follows by induction that

$$|u_n - \sum_{i=1}^m \varepsilon_n^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_i^{-1}(\cdot - g_i y_n))|_{a_3,2^*}^{2^*} = |u_n|_{a_3,2^*}^{2^*} - m a_0(y_0) |\widehat{u}|_{2^*}^{2^*} + o(1).$$

Using again equation (A.18), we have that

$$|v_n|_{a_3,2^*}^{2^*} = |u_n|_{a_3,2^*}^{2^*} - m a_0(y_0) |\widehat{u}|_{2^*}^{2^*} + o(1). \tag{A.20}$$

Combining equations (A.19) and (A.20) we obtain

$$J_1(v_n) = J_1(u_n) - |G/K| a_0(y_0) J_\infty(\widehat{u}) + o(1),$$

which concludes the proof of (v).

Now, let $\varphi \in C_0^\infty(\Omega)$ and

$$\varphi_n(z) := \varepsilon^{\frac{N-2}{2}} \varphi(\varepsilon_n z + g_1 y_n).$$

Note that

$$\langle \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)), \varphi \rangle_{a_1} = \int a_1^n \nabla(\widetilde{u} \circ g_1^{-1}) \cdot \int a_1^n \nabla(\varphi_n)$$

also note that, since $\widetilde{u}$ is a solution to problem A.14, we have that

$$\int a_1^n \nabla(\widetilde{u} \circ g_1^{-1}) \cdot \int a_1^n \nabla(\varphi_n) - \int a_3^n |\widetilde{u} \circ g_1^{-1}|^{2^*-2} (\widetilde{u} \circ g_1^{-1}) \varphi_n = o(1)$$

therefore

$$-\langle \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)), \varphi \rangle_{a_1} = -\int a_3^n |\widetilde{u} \circ g_1^{-1}|^{2^*-2} (\widetilde{u} \circ g_1^{-1}) \varphi_n + o(1)$$

Using again Lemma A.9, we obtain

$$\begin{aligned}
& J_1'(D_n)[\varphi] = \langle D_n, \varphi \rangle_{a_1} - \int a_3 |D_n|^{2^*-2} D_n \varphi \\
= \; & \langle D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)), \varphi \rangle_{a_1} - \langle \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)), \varphi \rangle_{a_1} \\
- \; & \int a_3 |D_n|^{2^*-2} D_n \varphi \\
= \; & J'(D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)))[\varphi] \\
+ \; & \int a_3 |D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n))|^{2^*-2} (D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n))) \varphi \\
- \; & \int a_3^n |\widetilde{u} \circ g_1^{-1}|^{2^*-2} (\widetilde{u} \circ g_1^{-1}) \varphi_n - \int a_3 |D_n|^{2^*-2} D_n \varphi + o(1) \\
= \; & J_1'(D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)))[\varphi] \\
+ \; & \int a_3^n |\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n) + \widetilde{u} \circ g_1^{-1}|^{2^*-2} (\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n) + \widetilde{u} \circ g_1^{-1}) \varphi_n \\
- \; & \int a_3^n |\widetilde{u} \circ g_1^{-1}|^{2^*-2} (\widetilde{u} \circ g_1^{-1}) \varphi_n - \int a_3^n |\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n)|^{2^*-2} (\varepsilon_n^{\frac{N-2}{2}} D_n(\varepsilon_n \cdot + g_1 y_n)) \varphi_n + o(1) \\
= \; & J_1'(D_n + \varepsilon^{\frac{2-N}{2}} a(y_0)^{\frac{2-N}{4}} \widehat{u}(\varepsilon_n^{-1} g_1^{-1}(\cdot - g_1 y_n)))[\varphi] + o(1)
\end{aligned}$$

It follows by induction that

$$J_1'(D_n)[\varphi] = J_1'(u_n)[\varphi] + o(1).$$

Finally, using equation (A.18), we conclude that

$$\begin{aligned} J'(v_n)[\varphi] &= J'(u_n)[\varphi] + o(1) \\ &= o(1), \end{aligned}$$

Since this is true for every $\varphi \in C_0^\infty(\Omega)$, we have that

$$\begin{aligned} J_1'(v_n) &= J_1'(u_n) + o(1) \\ &= o(1). \end{aligned}$$

which proves that (v_n) is a Palais-Smale sequence for J_1 and concludes the proof. □

Proof of Theorem A.1. Let (u_n) be a $G-$invariant Palais-Smale sequence for J at the level c. Since

$$\frac{1}{N}||u_n||^2_{a_1,a_2} = J(u_n) - \frac{1}{2^*}J'(u_n)(u_n) \leq |c| + 1 + ||u_n||_{a_1,a_2}.$$

we have that (u_n) is bounded. Therefore, passing to a subsequence if necessary, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H^1_0(\Omega) \\ u_n &\to u \quad \text{in } L^2_{Loc}(\Omega) \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned}$$

Setting $u^1_n := u_n - u$, we have that $u^1_n \rightharpoonup 0$ weakly in $H^1_0(\Omega)^G$. Moreover, Lemma A.10 implies that u is a solution to problem $(\wp)$ and

$$\begin{aligned} \|u^1_n\|^2 &= \|u_n\|^2 - \|u\|^2 + o(1) \\ J_1(u^1_n) &\to c - J(u), \\ J'_1(u^1_n) &\to 0 \text{ in } (H^1_0(\Omega))' \end{aligned}$$

Hence, u^1_n is a G-invariant Palais-Smale sequence for J_1 at the level $c_1 := c - J_1(u)$. If $c_1 \leq 0$, from the equality

$$\frac{1}{N}||u^1_n||^2_{a_1,a_2} = J_1(u^1_n) - \frac{1}{2^*}J'(u^1_n)(u^1_n) + o(1),$$

we have that $c_1 = 0$ and $||u^1_n||_{a_1,a_2} \to 0$, which implies that $u^1_n \to 0$ strongly in $H^1_0(\Omega)$. So, in this case the theorem is satisfied with $m = 0$.

If $c_1 > 0$, from Lemma A.2 we have that there exists a closed subgroup G_1 of finite index in G, a non trivial solution $\widehat{u}_1$ to problem $(\wp_\infty)$, a sequence $(y_{1,n})$ in Ω, a sequence $(\varepsilon_{1,n})$ in $(0,\infty)$ and a G-invariant Palais-Smale sequence u^2_n for J_1 with the following properties

(i) $G_{y_{1,n}} = G_1$ for all $n \in \mathbb{N}$, and $y_{1,n} \to y_1$ in $\overline{\Omega}$.

(ii) $\varepsilon^{-1}_{1,n} dist(y_{1,n}, \partial\Omega) \to \infty$ and $\varepsilon^{-1}_{1,n}|gy_{1,n} - g'y_{1,n}| \to \infty$ if $[g] \neq [g']$ in G/G_1,

(iii) $\widehat{u}_1$ is G_1-invariant.

(iv) $u^2_n = u^1_n - \displaystyle\sum_{[g]\in G/G_1} \left(\frac{a_3(y_1)}{a_1(y_1)}\right) \varepsilon^{\frac{2-N}{2}}_{1,n} \widehat{u}_1(g^{-1}\varepsilon^{-1}_{1,n}(\cdot - gy_{1,n})) + o(1)$ in $D^{1,2}(\mathbb{R}^N)$.

(v) $J_1(u^2_n) = J_1(u_n) - |G/G_1| \left(\frac{a_3(y_1)^{\frac{2-N}{2}}}{a_1(y_1)^{-\frac{N}{2}}}\right) J_\infty(\widehat{u}_1) + o(1)$.

Equality (iv) show that $u_n^2 \rightharpoonup 0$ weakly in $H_0^1(\Omega)^G$. Moreover, since $\widehat{u}_1$ is a solution to problem $(\wp_\infty)$, we have that $J_\infty(\widehat{u}_1) \geq \frac{1}{N} S^{N/2}$. Hence, using (v), we obtain

$$J(u_n) \geq J_1(u_n^1) \geq J_1(u_n^2) + \left(\min_{x\in\overline{\Omega}} \frac{a_3(x)^{\frac{2-N}{2}}}{a_1(x)^{-\frac{N}{2}}} \#Gx \right) \frac{1}{N} S^{N/2} + o(1),$$

therefore

$$\lim_{n\to\infty} J_1(u_n^1) = c_1 \geq \lim_{n\to\infty} J_1(u_n^2) + \left(\min_{x\in\overline{\Omega}} \frac{a_3(x)^{\frac{2-N}{2}}}{a_1(x)^{-\frac{N}{2}}} \#Gx \right) \frac{1}{N} S^{N/2}.$$

Let $\lim_{n\to\infty} J_1(u_n^2) := c_2$. If $c_2 \leq 0$, equality

$$\frac{1}{N} ||u_n^2||_{a_1,a_2}^2 = J_1(u_n^2) - \frac{1}{2^*} J_1'(u_n^2)(u_n^2),$$

shows that $c_2 = 0$ and $||u_n^2||_{a_1,a_2} \to 0$. Therefore $u_n^2 \to 0$ strongly in $H_0^1(\Omega)$. In this case, the theorem is satisfied with m=1.

If $c_2 > 0$, applying again Lemma A.2, this time to the sequence (u_n^2), we have that there exists a closed subgroup G_2 of finite index in G, a no trivial solution $\widehat{u}_2$ to problem $(\wp_\infty)$, a sequence $(y_{2,n})$ in Ω, a sequence $(\varepsilon_{2,n})$ in $(0,\infty)$ and a G-invariant Palais-Smale sequence (u_n^3) for J_1 with the properties (i)-(v).

We can apply the same reasoning to the sequence (u_n^3). The inequality

$$\lim_{n\to\infty} J_1(u_n^i) \geq \lim_{n\to\infty} J_1(u_n^{i+1}) + \left(\min_{x\in\overline{\Omega}} \frac{a_3(x)^{\frac{2-N}{2}}}{a_1(x)^{-\frac{N}{2}}} \#Gx \right) \frac{1}{N} S^{N/2}$$

guaranties that $\lim_{n\to\infty} J_1(u_n^{m+1}) = 0$ for some $m \in \mathbb{N}$, and therefore $||u_n^{m+1}||_{a_1,a_2} \to 0$. This concludes the proof. □

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