UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN CIENCIAS MATEMÁTICAS Y DE LA ESPECIALIZACIÓN EN ESTADÍSTICA APLICADA

TOPICS ON MULTIVARIATE COPULAS AND APPLICATIONS

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN CIENCIAS

PRESENTA:
MARÍA MAGDALENA HERNÁNDEZ CEDILLO

DIRECTOR
DR. JOSÉ MARÍA GONZÁLEZ-BARRIOS MURGUÍA, IIMAS, UNAM

MIEMBROS DEL COMITÉ TUTOR
DRA. MAGALI LOUISE MARIE FOLCH GABAYET, IMUNAM
DRA. ANA MEDA GUARDIOLA, FACULTAD DE CIENCIAS, UNAM

MÉXICO, D. F. 22 DE AGOSTO DE 2013.

To my parents,
Luisa and Carlos.

Contents

Acknowledgements

I would like to express my sincere gratitude to my advisor, Professor José María González-Barrios Murguía, for the continuous support of my Ph. D. study and research, for his patience, motivation, enthusiasm, and his guidance.

I would also like to thank my examiners, Professor Begoña Fernández, Professor Ana Meda, Professor Roger B. Nelsen and Professor Carlo Sempi, I am grateful for their thoughtful and detailed comments and suggestions, and for their very careful revision. I also thank the rest of my Ph. D. study supervisors, Professor Magali Folch, for her inspiration and guidance.

I would like to thank the many people who have taught me mathematics throughout my life.

I would like to acknowledge the financial support of the National Council for Science and Technology, CONACyT.

I thank all of my friends, specially Mirella Ramírez for her kindness, friendship and support.

I wish to thank my siblings for their support, advice and encouragement.

Finally, and most importantly, I thank my parents, María Luisa Cedillo and J. Carlos Hernández for their unconditional love and support.

Introduction

During recent years the research in copulas has given a lot of attention to the construction of bivariate copulas with given properties. Even more recent are the multivariate constructions of copulas with given properties. The idea is to provide new families that allow us to model multivariate data, since the known models are not numerous enough to do so. In this work we will study some of these constructions and we will attempt to provide extensions to higher dimensions to some of the results for bivariate copulas. We also introduce a new construction of copulas, based on a sample, which allows us to carry on statistical procedures.

In Chapter 1 we make a review of the basic concepts of copulas. We give special emphasis to the basic definitions in a multivariate setting. We state the main results for the construction of copulas, the necessary conditions to obtain an Archimedean Copula and some concepts of dependence that will be used later. Most of the notation that will be used throughout the rest of this thesis is given in this chapter.

In Chapter 2 we will try to find an extension of the multivariate patchwork construction given by Durante *et al* [24]. In order to do that, we extend the results in the papers by De Baets and De Meyer [12] and Durante, Saminger-Platz and Sarkoci [23, 24]. Such extensions allow us to provide an alternate proof of the main result in Mesiar and Sempi [55].

In Chapter 3 we define the d-sample copula of order m for integers $d, m \geq 2$, where d is the dimension, based on a sample of a d-distribution function on $\mathbb{R}^d$ or a d-copula on $[0,1]^d$. The sample d-copula definition is grounded on the self similar copula construction and on transformation matrices which are presented in Cuculescu and Theodorecu [8], Fredricks *et al* [34] and Trutschnig and Fernández-Sánchez [69]. We give and prove some of its main properties, and we propose a new statistical methodology with several examples. We also observe that the empirical copula can be obtained from a sample d-copula of order m.

Chapter 4 presents the basic definitions and main results for Archimedean copulas and their diagonals. Then we extend to the multivariate case the result announced by Frank [31] which states that bivariate Archimedean copulas are uniquely determined by their diagonal whenever Frank's condition $\delta'(1-) = 2$ is satisfied. This result appears in Alsina *et al* [2] and in Erdely [26].

Chapter 5 is a review of the construction of a multivariate copula given a d-diagonal presented in the paper of Cuculescu and Theodorescu [8], which is very rich in ideas but does not provide detailed proofs of their results. We also observe that their proof is based on approximations of what we call p-shuffles of δ_{W^d}.

We end this work with conclusions and some insights for future research.

Chapter 1

Preliminaries

We start this section giving the main definitions and several well known results for copulas. Most of the material in this chapter can be found in Nelsen [58].

1.1 Basic concepts

Definition 1.1. *A* **two-dimensional subcopula** *(or* **two-subcopula***, or briefly, a* **subcopula***) is a function C' from $S_1 \times S_2$ to $\mathbf{I}$, where $\mathbf{I} = [0,1]$ and $S_i \subset \mathbf{I}$, $\{0,1\} \subset S_i$ for $i = 1,2$, with the following properties:*

1. *For every u in S_1 and for every v in S_2,*

$$C'(u,0) = 0 = C'(0,v) \tag{1.1}$$

and

$$C'(u,1) = u, \;\; C'(1,v) = v; \tag{1.2}$$

2. *For every u_1, u_2 in S_1 and v_1, v_2 in S_2 such that $u_1 \le u_2$ and $v_1 \le v_2$,*

$$C'(u_2,v_2) - C'(u_1,v_2) - C'(u_2,v_1) + C'(u_1,v_1) \ge 0 \tag{1.3}$$

If $S_1 = S_2 = \mathbf{I}$ we say that C' is a **two-dimensional copula** *or* **two-copula***, or briefly, a copula. We will call the left-hand side of the inequality* (1.3)*, the* C'**-volume** *of $[u_1,u_2] \times [v_1,v_2]$.*

The set of all copulas has lower and upper bounds.

Theorem 1.2. *Let C be a subcopula. Then for every (u,v) in $\operatorname{Dom} C$,*

$$W(u,v) \le C(u,v) \le M(u,v). \tag{1.4}$$

Where $M(u,v) = \min\{u,v\}$ and $W(u,v) = \max\{u+v-1,0\}$.

The functions M and W in (1.4) are themselves copulas and we refer to M as the **Fréchet-Hoeffding upper bound** and W as the **Fréchet-Hoeffding lower bound**. A third important copula that we will frequently find is the **product copula** $\Pi(u,v) = uv$, since it characterizes independence of two random variables.

Remark 1.3. Let C_0 and C_1 be copulas, and let q be any number in $\mathbf{I}$. It is easy to see from definition 1.1 that the weighted arithmetic mean $(1-q)C_0 + qC_1$ is also a copula. Hence we conclude that any convex linear combination of copulas is a copula. We can also easily verify that the geometric mean of two copulas may fail to be a copula, namely, if C is the geometric mean of Π and W, that is $C(u,v) = \Pi(u,v)^{1/2}W(u,v)^{1/2}$, the C-volume of the rectangle $[1/2, 3/4] \times [1/2, 3/4]$ is negative.

Among other important properties of copulas and subcopulas are the following.

Theorem 1.4. *Let C' be a subcopula. Then for every $(u_1, v_1), (u_2, v_2)$ in $\operatorname{Dom} C'$,*

$$|C'(u_2, v_2) - C'(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1| \tag{1.5}$$

Hence C' is uniformly jointly continuous on its domain.

The main result in copula theory is Sklar's theorem, which relates joint distribution functions with its marginals using a copula.

Theorem 1.5 (Sklar's Theorem)**.** *Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all x, y in $\overline{\mathbb{R}}$,*

$$H(x,y) = C(F(x), G(y)). \tag{1.6}$$

If F and G are continuous, then C is unique; otherwise C is uniquely determined on $\operatorname{Ran} F \times \operatorname{Ran} G$. Conversely if C is a copula and F and G are distribution functions, then the function H defined by (1.6) is a joint distribution function with margins F and G.

Definition 1.6. *Let F be a distribution function. Then a* **quasi-inverse** *of F is any function $F^{(-1)}$ such that*

1. If t is in $\operatorname{Ran} F$, then $F^{(-1)}(t)$ is any number x in $\overline{\mathbb{R}}$ such that $F(x) = t$, i.e., for all t in $\operatorname{Ran} F$,

$$F(F^{(-1)}(t)) = t;$$

2. If t is not in $\operatorname{Ran} F$,

$$F^{(-1)}(t) = \inf\{x | F(x) \geq t\} = \sup\{x | F(x) \leq t\}.$$

Note: If F is strictly increasing, it has a single quasi-inverse, which is the ordinary inverse, and in this case we use the customary notation F^{-1}.

Lemma 1.7. *Let H be a distribution function with margins F and G. Then there exists a unique subcopula C' such that*
1. $\operatorname{Dom} C' = \operatorname{Ran} F \times \operatorname{Ran} G$,
2. For all x, y in $\overline{\mathbb{R}}$, $H(x,y) = C'(F(x), G(y))$.
Furthermore, if $F^{(-1)}$ and $G^{(-1)}$ are quasi-inverses of F and G, respectively. Then for any (u,v) in $\operatorname{Dom} C'$,

$$C'(u,v) = H(F^{(-1)}(u), G^{(-1)}(v)). \tag{1.7}$$

When F and G are continuous, the above result holds for copulas as well and provides a method of constructing copulas from joint distribution functions.

1.1.1 Multivariate copulas

Although many of the definitions and theorems have analogous multivariate versions, not all of them do. We will use vector notation for points in $\overline{\mathbb{R}}^n$, e.g., $\underline{\mathbf{a}} = \langle a_1, a_2, \cdots, a_n \rangle$, and we will write $\underline{\mathbf{a}} \leq \underline{\mathbf{b}}$ ($\underline{\mathbf{a}} < \underline{\mathbf{b}}$) when $a_k \leq b_k$ ($a_k < b_k$) for all k. For $\underline{\mathbf{a}} \leq \underline{\mathbf{b}}$ we will let $[\underline{\mathbf{a}}, \underline{\mathbf{b}}]$ denote the n-**box** $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ the cartesian product of n closed intervals. The **vertices** of an n-box are the points $\underline{\mathbf{c}} = \langle c_1, c_2, \cdots, c_n \rangle$, where each c_k is equal to either a_k or b_k. The unit n-**cube** $\mathbf{I}^n$ is the product $\mathbf{I} \times \mathbf{I} \times \cdots \times \mathbf{I}$. An n-**place** real function H is a function whose domain, $\operatorname{Dom} H$, is a subset of $\mathbb{R}^n$ and whose range, $\operatorname{Ran} H$, is a subset of $\mathbb{R}$.

Definition 1.8. *Let $S_1, S_2, \cdots, S_n$ be nonempty subsets of $\overline{\mathbb{R}}$, and let H be an n-place real function such that* $\operatorname{Dom} H = S_1 \times S_2 \times \cdots \times S_n$. *Let $B = [\underline{\mathbf{a}}, \underline{\mathbf{b}}]$ be an n-box all of whose vertices are in* $\operatorname{Dom} H$. *Then the H-**volume of** B is given by*

$$V_H(B) = \sum \operatorname{sgn}(\underline{\mathbf{c}}) H(\underline{\mathbf{c}}) \tag{1.8}$$

where the sum is taken over all vertices $\underline{\mathbf{c}}$ of B, and $\operatorname{sgn}(\underline{\mathbf{c}})$ *is given by*

$$\operatorname{sgn}(\underline{\mathbf{c}}) = \begin{cases} 1, & \text{if } c_k = a_k \text{ for an even number of } k\text{'s}, \\ -1, & \text{if } c_k = a_k \text{ for an odd number of } k\text{'s}. \end{cases}$$

*We will call R a **non trivial** n-**box** if for every $i \in \{1, \dots, n\}$, $-\infty < u_i < v_i < \infty$. For any $1 \leq k \leq n$ and for every $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ define*

$$R_{i_1,\dots,i_k} = \{\langle x_1, \dots, x_n \rangle \in R \mid \text{for every } j \in \{1, \dots, k\}, \text{ either } x_{i_j} = u_{i_j} \text{ or } x_{i_j} = v_{i_j}\}. \tag{1.9}$$

Then we call $R_{i_1,\dots,i_k}$ *an* $(n-k)$-**dimensional face of** R. *If* $k = 1$ *an* $(n-1)$-*dimensional face is usually called simply a* **face of** R, *in this last case we make a distinction. We will denote by*

$$R_i^l = \{\langle x_1, \dots, x_n\rangle \in R \mid x_i = u_i\}$$

and call it the i^{th}*-lower face, and*

$$R_i^u = \{\langle x_1, \dots, x_n\rangle \in R \mid x_i = v_i\}$$

and call it the i^{th}*-upper face.*

Remark 1.9. Of course the domain of the function sgn depends also on the n-box we are using.

Definition 1.10. *An* n*-place real function* H *is* n-**increasing** *if and only if* $V_H(B) \geq 0$ *for all* n*-boxes* B *whose vertices lie in* $\operatorname{Dom} H$.

Definition 1.11. *An* n-**dimensional subcopula** *(or* n-**subcopula***) is a function* C' *from* $S_1 \times S_2 \times \cdots \times S_n$ *to* $\mathbf{I}$, *where* $S_k \subset \mathbf{I}$ *and* $\{0, 1\} \in S_k$ *for* $k = 1, \dots, n$, *with the following properties:*

1. *For every* $\underline{\mathbf{u}}$ *in* $S_1 \times \cdots \times S_n$,

$$C'(\underline{\mathbf{u}}) = 0 \text{ if at least one coordinate of } \underline{\mathbf{u}} \text{ is } 0$$

   and

$$\text{if all coordinates of } \underline{\mathbf{u}} \text{ are } 1 \text{ except possibly } u_k, \text{ then } C'(\underline{\mathbf{u}}) = u_k$$

2. *For every* $\underline{\mathbf{a}}$ *and* $\underline{\mathbf{b}}$ *in* $S_1 \times \cdots \times S_n$ *such that* $\underline{\mathbf{a}} \leq \underline{\mathbf{b}}$,

$$V_{C'}([\underline{\mathbf{a}}, \underline{\mathbf{b}}]) \geq 0 \tag{1.10}$$

When $S_1 = S_2 = \cdots = S_n = \mathbf{I}$ *we say that* C' *is an* n-**dimensional copula** *or* n-**copula**.

Theorem 1.12 (Sklar's theorem in n dimensions)**.** *Let* H *be an* n *dimensional distribution function with margins* $F_1, F_2, \dots, F_n$. *Then there exists an* n*-copula* C *such that for all* $\underline{\mathbf{x}}$ *in* $\overline{\mathbb{R}}^n$,

$$H(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)). \tag{1.11}$$

If $F_1, F_2, \ldots, F_n$ are all continuous, then C is unique; otherwise, C is uniquely determined on $\operatorname{Ran} F_1 \times \operatorname{Ran} F_2 \times \cdots \times \operatorname{Ran} F_n$. *Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are distribution functions, the function H defined by* (1.11) *is an n-dimensional distribution function with margins $F_1, F_2, \ldots, F_n$.*

The extensions of the 2-copulas M, Π and W to n dimensions are denoted M^n, Π^n and W^n, and are given by:

$$M^n(\underline{\mathbf{u}}) = min(u_1, u_2, \ldots, u_n);$$

$$\Pi^n(\underline{\mathbf{u}}) = u_1 u_2 \ldots u_n;$$

$$W^n(\underline{\mathbf{u}}) = \max(u_1 + u_2 + \cdots + u_n - n + 1, 0).$$

Remark 1.13. The functions M^n and Π^n are n-copulas for all $n \geq 2$, but the function W^n fails to be an n-copula for any $n > 2$, see [58]. Nevertheless, we have the following n-dimensional version of the Fréchet-Hoeffding bounds inequality (1.4).

Theorem 1.14. *If C is any n-copula, then for every $\underline{\mathbf{u}}$ in $\mathbf{I}^n$,*

$$W^n(\underline{\mathbf{u}}) \leq C(\underline{\mathbf{u}}) \leq M^n(\underline{\mathbf{u}}). \tag{1.12}$$

Although the Fréchet-Hoeffding lower bound W^n is never a copula for $n > 2$, the left hand inequality in (1.12) is *best possible*, in the sense that for any $n \geq 3$ and any $\underline{\mathbf{u}}$ in I^n, there exists an n-copula C such that $C(\underline{\mathbf{u}}) = W^n(\underline{\mathbf{u}})$:

Theorem 1.15. *For any $n \geq 3$ and any $\underline{\mathbf{u}}$ in $\mathbf{I}^n$, there exists an n-copula C (which depends on $\underline{\mathbf{u}}$) such that*

$$C(\underline{\mathbf{u}}) = W^n(\underline{\mathbf{u}})$$

1.1.2 Archimedean Copulas

Archimedean copulas can easily be constructed and they are specified by a univariate function, so the copulas which are bivariate functions can be written in terms of a univariate function and its pseudo-inverse.

Definition 1.16. *Let φ be a continuous, strictly decreasing function from $\mathbf{I}$ to $[0, \infty]$ such that $\varphi(1) = 0$. The* **pseudo inverse** *of φ is the function $\varphi^{[-1]}$ with* $\operatorname{Dom} \varphi^{[-1]} = [0, \infty]$ *and* $\operatorname{Ran} \varphi^{[-1]} = \mathbf{I}$ *given by*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0), \\ 0, & \varphi(0) \leq t \leq \infty. \end{cases} \tag{1.13}$$

Theorem 1.17. *Let φ be a continuous, strictly decreasing function from* $\mathbf{I}$ *to* $[0, \infty]$ *such that* $\varphi(1) = 0$*, and let* $\varphi^{[-1]}$ *be the pseudo-inverse of* φ *defined by* (1.13)*. Let* C *be the function from* $\mathbf{I}^2$ *to* $\mathbf{I}$ *given by*

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)), \tag{1.14}$$

then C *is a copula if and only if* φ *is convex, that is,* $\varphi(au + (1-a)v) \leq a\varphi(u) + (1-a)\varphi(v)$ $\forall\ 0 \leq a \leq 1$ *and* $\forall\ u, v \in \mathbf{I}$.

Definition 1.18. *Let* φ *and* $\varphi^{[-1]}$ *satisfy the hypotheses of Theorem 1.17. Copulas of the form* (1.14) *are called* **Archimedean copulas***. The function* φ *is called an* **inner generator** *of the copula.*

Note: If $\varphi(0) = \infty$, we say that φ is a **strict generator**. In this case, $\varphi^{[-1]} = \varphi^{-1}$ and $C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ is said to be a **strict** Archimedean copula.

Fundamental properties of Archimedean copulas.

Theorem 1.19. *Let* C *be an Archimedean copula with generator* φ*. Then:*

1. *C is symmetric, that is, $C(u, v) = C(v, u)$ for all u, v in* $\mathbf{I}$*;*

2. *C is associative, that is, $C(C(u, v), w) = C(u, C(v, w))$ for all u, v, w in* $\mathbf{I}$*;*

3. *If $c > 0$ is any constant, then $c\varphi$ is a generator of C.*

4. *The level curves of an Archimedean copula are convex.*

It is easy to show that the diagonal section δ_C of an Archimedean copula C satisfies $\delta_C(u) < u$ for all $u \in (0, 1)$, furthermore, the next theorem, based on Ling [46], gives a useful characterization for Archimedean copulas.

Theorem 1.20. *Let* C *be a copula and let* δ_C *be the diagonal section of* C*. Then* C *is Archimedean if and only if* C *is associative and* $\delta_C(u) < u$ *for all* u *in* $(0, 1)$*.*

Remark 1.21. The Fréchet-Hoeffding upper bound M is not Archimedean since $\delta_M(t) = t$ for all $t \in [0, 1]$. However, W is an Archimedean copula with generator $\varphi(t) = 1 - t$.

1.1.3 Dependence

There are a variety of ways to discuss and to measure dependence. Because the most widely known scale-invariant measures of association are the population versions of Kendall's tau and Spearman's rho, both of which "measure" a form of dependence known as concordance we will begin there.

Definition 1.22. *Let $\langle x_i, y_i\rangle$ and $\langle x_j, y_j\rangle$ denote two observations from a vector $\langle X, Y\rangle$ of continuous random variables. We say that $\langle x_i, y_i\rangle$ and $\langle x_j, y_j\rangle$ are* **concordant** *if $(x_i - x_j)(y_i - y_j) > 0$. Similarly, we say that $\langle x_i, y_i\rangle$ and $\langle x_j, y_j\rangle$ are* **discordant** *if $(x_i - x_j)(y_i - y_j) < 0$.*

That is, a pair of random variables are concordant if large values of one tend to be associated with large values of the other, and small values of one with small values of the other.

The sample version of the measure of association known as Kendall's tau is defined in terms of concordance as follows: Let $\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle\}$ denote a random sample of n observations from a vector $\langle X, Y\rangle$ of continuous random variables. There are $\binom{n}{2}$ distinct pairs $\langle x_i, y_i\rangle$ and $\langle x_j, y_j\rangle$ of observations in the sample, and each pair is either concordant or discordant. Let c denote the number of concordant pairs and d the number of discordant pairs. The **sample version of Kendall's tau** is defined as

$$t = \frac{c-d}{c+d} = \frac{(c-d)}{\binom{n}{2}}. \tag{1.15}$$

The population version of Kendall's tau for a vector $\langle X, Y\rangle$ of continuous random variables with joint distribution function H is defined similarly. Let $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ be independent and identically distributed random vectors, each with joint distribution function H. Then the **population version of Kendall's tau** is defined as the probability of concordance minus the probability of discordance:

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Theorem 1.23. *Let $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ be independent vectors of continuous random variables with joint distributions distribution functions H_1 and H_2, respectively, with common margins F (of X_1 and X_2) and G (of Y_1 and Y_2). Let C_1 and C_2 denote the copulas of $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$, respectively. Let Q be defined as*

$$Q = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Then

$$Q = Q(C_1, C_2) = 4 \iint_{\mathbf{I}^2} C_2(u,v)dC_1(u,v) - 1.$$

We will say that Q is a **concordance function**.

As a corollary we list some of the properties of Q, but first we need a definition.

Definition 1.24. *If C_1 and C_2 are copulas, we say that C_1 is* **smaller** *than C_2 (or C_2 is larger than C_1) and write $C_1 \prec C_2$ if $C_1(u,v) \leq C_2(u,v)$ for all $u, v \in \mathbf{I}$.*

Corollary 1.25. *Let C_1, C_2 and Q be as in Theorem 1.23. Then*

1. *Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$.*
2. *Q is increasing in each argument: if $C_1 \prec C_1'$ and $C_2 \prec C_2'$ for all $\langle u, v\rangle$ in $\mathbf{I}^2$, then $Q(C_1, C_2) \leq Q(C_1', C_2')$.*
3. *For every copula C, $Q(C, C) \in [-1, 1]$.*

Theorem 1.26. *Let X and Y be continuous random variables whose copula is C. Then the population version of Kendall's tau for X and Y is given by*

$$\tau_{X,Y} = \tau_C = Q(C, C) = 4 \iint_{\mathbf{I}^2} C(u,v)dC(u,v) - 1. \tag{1.16}$$

Note: The integral that appears in (1.16) can be interpreted as the expected value of the function $C(U, V)$ of uniform $(0, 1)$ random variables U and V whose joint distribution function is C, that is to say,

$$\tau_C = 4E(C(U, V)) - 1.$$

In general, evaluating the population version of Kendall's tau requires the evaluation of the double integral in (1.16), but for an Archimedean copula the situation is simpler.

Corollary 1.27. *Let X and Y be random variables with an Archimedean copula C with generator φ. The population version τ_C of Kendall's tau for X and Y is given by*

$$\tau_C = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)} dt.$$

Another measure of association based on concordance and discordance is Spearman's rho. To obtain this measure we let $\langle X_1, Y_1\rangle, \langle X_2, Y_2\rangle$ and $\langle X_3, Y_3\rangle$ be three independent random vectors with a common joint distribution function H, with margins F and G and copula C. The population version (we will see later the sample version) $\rho_{X,Y}$ of **Spearman's rho** is defined to be proportional to the probability of concordance minus the probability of discordance for the two vectors $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_3\rangle$, that is, a pair of random vectors with the same margins but one vector has a distribution H, while the components of the other are independent:

$$\rho_{X,Y} = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0]) - P[(X_1 - X_2)(Y_1 - Y_3) < 0].$$

Note: The joint distribution of $\langle X_2, Y_3\rangle$ is $F(x)G(y)$, because X_2 and Y_3 are independent. Thus the copula of $\langle X_2, Y_3\rangle$ is Π, note also that the pair $\langle X_3, Y_2\rangle$ could be used equally as well. So that, using Theorem 1.23 and part 1 of Corollary 1.25 we have

Theorem 1.28. *Let X and Y be continuous random variables whose copula is C. Then the Spearman's rho for Y and Y is given by*

$$\rho_{X,Y} = \rho_C = 3Q(C, \Pi), \tag{1.17}$$

$$= 12 \iint_{\mathbf{I}^2} uv dC(u, v) - 3, \tag{1.18}$$

$$= 12 \iint_{\mathbf{I}^2} C(u, v) du dv - 3. \tag{1.19}$$

Note: The coefficient "3" is a normalization constant, because, as it can be easily verified, $Q(C, \Pi) \in [-1/3, 1/3]$.

1.2 Known results about copula constructions

1.2.1 Some Methods of constructing copulas

The Inversion Method

Given a bivariate distribution function H with continuous margins F and G, we can "invert" H via (1.7) to obtain a copula:

$$C(u, v) = H(F^{(-1)}(u), G^{(-1)}(v)). \tag{1.20}$$

With this copula, new bivariate distributions with arbitrary margins, say F' and G', can be constructed using Sklar's theorem: $H'(x, y) = C(F'(x), G'(y))$.

Geometric Methods

This methodology includes the construction of singular copulas with prescribed support. Some examples are provided in Nelsen [58]. The "ordinal sum" construction consists in rescaling copies of the support of some copulas to fit some subsquares of $\mathbf{I}^2$ as we will see in Chapter 2. For the "shuffles of M" the mass distribution can be obtained by placing the mass for M on $\mathbf{I}^2$, cutting $\mathbf{I}^2$ vertically into a finite number of strips, shuffling the strips with perhaps some of them flipped around their vertical axes of symmetry, and then reassembling them to form the square again, see Nelsen [58]. Another example of construction are "convex sums" which are a continuous analog of the finite convex linear combination (see remark 1.3) of copulas.

Copulas with Prescribed Diagonal Section

We now turn to the construction of copulas having a prescribed diagonal section. The **diagonal section** of a copula C is the function δ_C from I to I defined by $\delta_C(t) = C(t,t)$. Diagonal sections are of interest because if X and Y are random variables with a common distribution function F and copula C, then the distribution functions of the order statistics $\max(X,Y)$ and $\min(X,Y)$ are $\delta_C(F(t))$ and $2t - \delta_C(t)$, respectively.

As a consequence of Theorem 1.4 and Fréchet-Hoeffding bounds (1.4), it follows that if δ is the diagonal section of a copula, then

$$\delta(1) = 1; \tag{1.21a}$$

$$0 \le \delta(t_2) - \delta(t_1) \le 2(t_2 - t_1) \text{ for all } t_1, t_2 \in \mathbf{I} \text{ with } t_1 \le t_2; \tag{1.21b}$$

$$\text{and} \qquad \delta(t) \le t \text{ for all } t \in \mathbf{I} \tag{1.21c}$$

Definition 1.29. *Any function $\delta : \mathbf{I} \to \mathbf{I}$ that satisfies* (1.21a) *to* (1.21c) *will be called simply a* **diagonal**, *while we refer to the function $\delta_C(t) = C(t,t)$ as the* **diagonal section** *of C.*

Now suppose that δ is any diagonal. Is there a copula C whose diagonal section is δ? The answer is yes, as is stated in the following

Theorem 1.30. *Let δ be any diagonal, and set*

$$C(u,v) = \min(u, v, [\delta(u) + \delta(v)]/2). \tag{1.22}$$

Then C is a copula whose diagonal section is δ.

The proof of the above theorem can be found in Fredricks and Nelsen [32].
Copulas of the form given by (1.22) are called **diagonal copulas**.

1.2.2 Patchwork based methods of construction

Nelsen *et al.* [59] studied a method, called copula (or quasi-copula) *diagonal splice*, for creating new copulas or quasi-copulas with a given diagonal section by joining portions of two copulas (or quasi-copulas) with a common diagonal section. This method was introduced by Durante, Mesiar and Sempi [20] for binary aggregation operators and applied to quasi-copulas. The diagonal splice of two functions is given in the next definition.

Definition 1.31. *Let f_1 and f_2 be two functions defined on the square $[0,1]^2$. Then the* **diagonal splice** *of f_1 and f_2 is the function $f_1 \boxslash f_2$ defined by*

$$(f_1 \boxslash f_2) = \begin{cases} f_1(u,v), & \text{if } u \le v, \\ f_2(u,v), & \text{if } u > v. \end{cases}$$

The diagonal splice of two quasi-copulas is always a quasi-copula. The next theorem is presented in Durante, Mesiar and Sempi [20] and proved in Nelsen *et al.* [59].

Theorem 1.32. *Let δ be a diagonal, and Q_1 and Q_2 two quasi-copulas with a common diagonal δ. Then the diagonal splice of Q_1 and Q_2 is also a quasi-copula with diagonal δ.*

Note: The diagonal splice of two copulas Q_1 and Q_2 (respectively proper quasi-copulas) with a common diagonal section δ can be a proper quasi-copula (respectively a copula) as shown in examples 5 and 6 in [59].
The following theorem provides a necessary and sufficient condition for the diagonal splice of two copulas to be a copula.

Theorem 1.33. *Let δ be a diagonal, and C_1 and C_2 two copulas with a common diagonal section δ. Then the diagonal splice $C_1 \boxslash C_2$ is a copula with diagonal δ if, and only if, $C_1(u,v) + C_2(v,u) \le \delta(u) + \delta(v)$ for every $u, v \in \mathbf{I}$ with $u \le v$.*

Notice that if C_1 and C_2 are both symmetric copulas with common diagonal section δ then the condition $C_1(u,v) + C_2(v,u) \le \delta(u) + \delta(v)$ is satisfied.

De Baets and De Meyer [12] established a general framework for constructing copulas that can be regarded as a patchwork-like assembly of arbitrary copulas with non overlapping rectangles as patches based on Proposition 7 of their paper.

Proposition 1.34. *Consider a copula C, a rectangle $R := [u, u'] \times [v, v'] \subset [0, 1]^2$, and a mapping $D : [u, u'] \times [v, v'] \to [0, 1]$. Define the binary operation Q by*

$$Q(x, y) = \begin{cases} D(x, y), & \text{if } \langle u, v\rangle \in [u, u'] \times [v, v'], \\ C(x, y), & \text{elsewhere.} \end{cases}$$

Q is a copula if and only if C and D coincide on the boundaries of R and D is 2-increasing on R.

In 2009 Durante *et al.* [24] presented another method for construction of two-copulas, they call it **Rectangular Patchwork for Bivariate Copulas**. Here we give an alternative enunciation.

Theorem 1.35. *Let C be a copula, let $\{C_j\}_{j\in\mathcal{J}}$ be a family of copulas and let $\{R_j = [u_1^j, v_1^j] \times [u_2^j, v_2^j]\}_{j\in\mathcal{J}}$ be a family of 2-boxes, in this case rectangles in $[0, 1]^2$, such that $R_j \cap R_k \subset \delta(R_j) \cap \delta(R_k)$ for every $j, k \in \mathcal{J}$ with $j \neq k$. Define for every $j \in \mathcal{J}$, $\lambda_j = V_C(R_j)$, and for every $x \in [u_1^j, v_1^j]$ and for every $y \in [u_2^j, v_2^j]$, $R_{j,x} = [u_1^j, x] \times [u_2^j, v_2^j]$ and $R_{j,y} = [u_1^j, v_1^j] \times [u_2^j, y]$. Let $\tilde{C} : [0, 1]^2 \to [0, 1]$ be defined by*

$$\tilde{C}(x, y) = \begin{cases} \lambda_j C_j\left(\frac{V_C(R_{j,x})}{\lambda_i}, \frac{V_C(R_{j,y})}{\lambda_j}\right) + \varphi_j^C(x, y) & \text{if } \langle x, y\rangle \in R_j \text{ and } \lambda_j > 0, \\ C(x, y), & \text{otherwise,} \end{cases} \tag{1.23}$$

where $\varphi_j^C(x, y) = h_{u_2^j}^C(x) + v_{u_1^j}^C(y) - h_{u_2^j}^C(u_1^j)$ with $h_{u_2^j}^C(x) = C(x, u_2^j)$ and $v_{u_1^j}^C(y) = C(u_1^j, y)$. Then $\tilde{C}$ is a copula.

Mesiar and Sempi [55] defined a generalization of the ordinal sums for dimensions greater than or equal to three. For a set of indexes $J \subset \mathbb{N}$, the **ordinal sum** of the n-copulas $\{C_k\}_{k\in\mathcal{J}}$ over a partition of $\mathbf{I}$, $\{[a_k, b_k]\}_{k\in J}$, is given by

$$C(\underline{\mathbf{u}}) = \begin{cases} a_k + (b_k - a_k)C_k\left(\frac{\min(u_1,b_k)-a_k}{b_k-a_k}, \ldots, \frac{\min(u_n,b_k)-a_k}{b_k-a_k}\right) \\ \qquad\qquad \text{if } \min(u_1, \ldots, u_n) \in (a_k, b_k) \text{ for some } k \in J \\ \min(u_1, \ldots, u_n) \qquad \text{elsewhere,} \end{cases} \tag{1.24}$$

for all $\underline{\mathbf{u}} = \langle u_1, \ldots, u_n\rangle \in [0, 1]^n$.

Mesiar and Sempi prove that C in the equation above is a copula for all $n \geq 2$ and, in the case $n = 2$, it coincides with the usual ordinal sum.

Finally a non patchwork-base method is provided by Siburg and Stoimenov [66]. The method consists in gluing rescaled copulas on adjacent n-boxes of $[0,1]^n$ whose union is $[0,1]^n$. The main result is,

Proposition 1.36. *Let $n \geq 2$ and let C_1, C_2 be two n-copulas. Let $0 \leq \theta \leq 1$ and define $R^l_{i,\theta} = [0,1] \times \cdots \times [0,1] \times [0,\theta] \times [0,1] \times \cdots \times [0,1]$, where the interval $[0,\theta]$ is located on the i^{th} coordinate, for some $i \in \{1,2,\ldots,n\}$, similarly define $R^u_{i,\theta} = [0,1] \times \cdots \times [0,1] \times [\theta,1] \times [0,1] \times \cdots \times [0,1]$. Define for every $\underline{\mathbf{x}} = \langle x_1, \ldots, x_i, \ldots, x_n \rangle \in [0,1]^n$*

$$\left(C_1 \circledast_{x_i=\theta} C_2\right)(\underline{\mathbf{x}}) = \begin{cases} \theta C_1\left(x_1, \ldots, \frac{x_i}{\theta}, \ldots, x_n\right) & \text{if} \quad \underline{\mathbf{x}} \in R^l_{i,\theta} \\ (1-\theta) C_2\left(x_1, \ldots, \frac{x_i-\theta}{1-\theta}, \ldots, x_n\right) & \\ +\theta C_1(x_1, \ldots, 1, \ldots, x_n) & \text{if} \quad \underline{\mathbf{x}} \in R^u_{i,\theta}. \end{cases} \tag{1.25}$$

Then $C_1 \circledast_{x_i=\theta} C_2$ is an n-copula.

Chapter 2

Multivariate Patchwork

In this chapter we give an alternative proof of the construction of n-dimensional ordinal sums given in Mesiar and Sempi [55]; we also provide a new methodology to construct n-copulas extending the patchwork methodology of Durante, Saminger-Platz and Sarkoci in [23] and [24]. Finally, we use the gluing method of Siburg and Stoimenov [66] and its generalization in Mesiar *et al* [53] to give an alternative method of patchwork construction of n-copulas, which can be also used in composition with our patchwork method. The main results of this chapter have already been published in [36].

2.1 Introduction

The idea of patching a 2-copula C, or simply copula, in a rectangular region R, by redefining C using another function D on R, is of great interest when modeling some bivariate data. It is well known that in many applications such as Mathematical Finance, Risk Theory, Ecology, etc., the researchers know from previous data what is the behavior of their observations in the tails, but if they try to fit a known model, many times, this model does not agree with these tail behaviors. In this case, it is important to take a base copula C and try to modify it in the regions of interest using some other copulas which have the behavior that we are looking for. This is now possible using the general approach of rectangular patchwork construction.

In this chapter we will generalize these results for n copulas with dimensions $n \geq 3$.

We start with a definition and a generalization of the main results in De Baets and De Meyer [12].

Definition 2.1. *Let* $C : \mathcal{D} \to \mathbf{I}$ *where* $\mathcal{D} \subset \mathbf{I}^n = \times_{i=1}^{n} \mathbf{I}$. *If* $\mathcal{D} = \mathbf{I}^n$ *we will call* C *an* n-**operation**.

i) We will say that C is **increasing** *on $\mathcal{D}$ if and only if for every $\underline{\mathbf{x}}, \underline{\mathbf{z}} \in \mathcal{D}$, where $\underline{\mathbf{x}} \leq \underline{\mathbf{z}}$, we have that $C(\underline{\mathbf{x}}) \leq C(\underline{\mathbf{z}})$, see comments before Definition 1.8.*
ii) We will say that C is **n-increasing** *if C satisfies Definition 1.10.*

Proposition 2.2. *Let $C : \mathcal{D} \to \mathbf{I}$ where $\mathcal{D} = \times_{i=1}^{n} [u_i, v_i] \subset \mathbf{I}^n$. Then C is increasing if and only if for every $i \in \{1, \ldots, n\}$ and for every $u_i \leq x_i \leq v_i$, $C(\cdot, \ldots, \cdot, x_i, \cdot, \ldots, \cdot) : \times_{j=1, j\neq i}^{n} [u_j, v_j] \to \mathbf{I}$ is increasing.*

Definition 2.3. *Let $C : \mathbf{I}^n \to \mathbf{I}$ be an n-operation. Then*
i) C is an **n-conjunctor** *if and only if C is increasing and for every $i \in \{1, \ldots, n\}$ and for every $x_i \in \mathbf{I}$, $C(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$.*
ii) C is an **n-copula** *if and only if C is an n-conjunctor which is also n-increasing.*

Remark 2.4. It is easy to see that if $C : \mathbf{I}^n \to \mathbf{I}$ is an n-conjunctor which is also n-increasing. Then from part i) in Definition 2.3, we have that C satisfies the boundary conditions of a copula and therefore C is an n-copula.
If we define $T_D : \mathbf{I}^n \to \mathbf{I}$ by

$$T_D(x_1, \ldots, x_n) = \begin{cases} \min\{x_1, \ldots, x_n\} & \text{if } \max\{x_1, \ldots, x_n\} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $i \in \{1, \ldots, n\}$ and for every $x_i \in \mathbf{I}$ we have that $T_D(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$. Let $\underline{\mathbf{x}} = \langle x_1, \ldots, x_n \rangle \in \mathbf{I}^n$ and assume that there exists $i \in \{1, \ldots, n\}$ such that $x_i = 1$. Let $\underline{\mathbf{w}} = \langle w_1, \ldots, w_n \rangle \in \mathbf{I}^n$ such that $\underline{\mathbf{x}} \leq \underline{\mathbf{w}}$, then $x_j \leq w_j$ for every $j \in \{1, \ldots, n\} \backslash \{i\}$ and $1 = x_i = w_i$. Then

$$\begin{aligned} 0 &\leq T_D(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) \\ &= \min\{x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n\} \\ &\leq \min\{w_1, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_n\} \\ &= T_D(w_1, \ldots, w_{i-1}, 1, w_{i+1}, \ldots, w_n). \end{aligned}$$

Let $\underline{\mathbf{x}} = \langle x_1, \ldots, x_n \rangle \in \mathbf{I}^n$ such that for every $i \in \{1, \ldots, n\}$, $x_i < 1$ and let $\underline{\mathbf{w}} = \langle w_1, \ldots, w_n \rangle$ such that $\underline{\mathbf{x}} \leq \underline{\mathbf{w}}$. Then $0 = T_D(\underline{\mathbf{x}}) \leq T_D(\underline{\mathbf{w}})$. Therefore, T_D is increasing and an n-conjunctor, according to Definition 2.3. In fact, T_D is the smallest n-conjunctor, observe that T_D is the multivariate version of the "drastic t-norm", see [2].

Proposition 2.5. *Let C be an n-conjunctor, let $R = \times_{i=1}^{n}[u_i, v_i] \subset \mathbf{I}^n$ be a non trivial n-box and let $D : R \to \mathbf{I}$. Define $Q : \mathbf{I}^n \to \mathbf{I}$ by*

$$Q(\underline{\mathbf{x}}) = \begin{cases} D(\underline{\mathbf{x}}) & \textit{if} \quad \underline{\mathbf{x}} \in R, \\ C(\underline{\mathbf{x}}) & \textit{if} \quad \underline{\mathbf{x}} \in [0,1]^n \setminus R. \end{cases} \tag{2.1}$$

If $C = D$ on the boundary of R, that is, ∂R, and D is increasing on R, then Q is an n-conjunctor.

Proof. Let C, D, R and Q as in the hypotheses. If there exist $i, j \in \{1, \ldots, n\}$ with $i \neq j$ such that $v_i < 1$ and $v_j < 1$ then for every $k \in \{1, \ldots, n\}$ and for every $x_k \in \mathbf{I}$, $\langle 1, \ldots, 1, x_k, 1, \ldots, 1\rangle \in \mathbf{I}^n \setminus R$. Then from equation (2.1)

$$Q(1, \ldots, 1, x_k, 1, \ldots, 1) = C(1, \ldots, 1, x_k, 1, \ldots, 1) = x_k.$$

If we assume that there exists $i \in \{1, \ldots, n\}$ such that $v_i < 1$ and for every $j \in \{1, \ldots, n\} \setminus \{i\}$, $v_j = 1$, then for every $y_i \in [u_i, v_i]$, $\langle 1, \ldots, 1, y_i, 1, \ldots, 1\rangle \in \partial R$, and by equation (2.1)

$$Q(1, \ldots, 1, y_i, 1, \ldots, 1) = C(1, \ldots, 1, y_i, 1, \ldots, 1) = y_i.$$

Finally, if for every $i \in \{1, \ldots, n\}$, $v_i = 1$ and $y_k \in [u_k, 1]$ for any $k \in \{1, \ldots, n\}$, then we have that $\langle 1, \ldots, 1, y_k, 1, \ldots, 1\rangle \in \partial R$, and by equation (2.1)

$$Q(1, \ldots, 1, y_k, 1, \ldots, 1) = C(1, \ldots, 1, y_k, 1, \ldots, 1) = y_k.$$

So, Q satisfies the boundary conditions of an n-conjunctor. Now we have to prove that Q is increasing, by Proposition 2.2 it is enough to see that for every $i \in \{1, \ldots, n\}$ and for every $z_i \in \mathbf{I}$ the function $Q(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot)$ is increasing.
So, let $i \in \{1, \ldots, n\}$ be fixed and $z_i \in \mathbf{I}$. We proceed by cases:

First, if $z_i \in [0, u_i] \cup [v_i, 1]$. Denote by $\overset{\circ}{R} = \times_{i=1}^{n}(u_i, v_i)$ the interior of R. Observe that since $C = D$ on ∂R, using equation (2.1), we can redefine Q as follows:

$$Q(\underline{\mathbf{x}}) = \begin{cases} D(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in \overset{\circ}{R}, \\ C(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in [0,1]^n \setminus \overset{\circ}{R}. \end{cases} \tag{2.2}$$

Define the region

$$H_{z_i} = \{\langle x_1, \ldots, x_n\rangle \in \mathbf{I}^n \mid x_i = z_i\} \subset \mathbf{I}^n \setminus \overset{\circ}{R} = \mathbf{I}^n \setminus \bigtimes_{i=1}^{n}(u_i, v_i). \tag{2.3}$$

From equation (2.2), it follows that for every $\langle x_1, \ldots, x_n\rangle \in H_{z_i}$

$$Q(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot) = C(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot),$$

which is increasing by Proposition 2.2.

Second, assume that $z_i \in (u_i, v_i)$, consider two points $\underline{\mathbf{x}}, \underline{\mathbf{w}} \in H_{z_i}$, such that $\underline{\mathbf{x}} \leq \underline{\mathbf{w}}$, that is, $\underline{\mathbf{x}} = \langle x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n\rangle$ and $\underline{\mathbf{w}} = \langle w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n\rangle$ with $x_j \leq w_j$ for every $j \in \{1, \ldots, n\}\backslash\{i\}$. Let

$$H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} := \{\langle z_1, \ldots, z_n\rangle \in H_{z_i} \mid \text{for every } j \in \{1, \ldots, n\}\backslash\{i\},\ x_j \leq z_j \leq w_j\}. \tag{2.4}$$

Here we have to check two cases:

Case 1. If $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} \cap \overset{\circ}{R} = \emptyset$, then $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} \subset \mathbf{I}^n \setminus \overset{\circ}{R}$. So, $Q(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot) =$ $C(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot)$ which is increasing on $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}}$.

Case 2. If $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} \cap \overset{\circ}{R} \neq \emptyset$, in this case we have to check subcases.

Subcase i) Assume that $\underline{\mathbf{x}} \in \overset{\circ}{R}$ and $\underline{\mathbf{w}} \in \overset{\circ}{R}$, then $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} \subset \overset{\circ}{R}$ using equation (2.2), we have that

$$Q(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot) = D(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot)$$

which is increasing on $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}}$ by hypothesis and using Proposition 2.2.

Subcase ii) Assume that $\underline{\mathbf{x}} \notin \overset{\circ}{R}$ and $\underline{\mathbf{w}} \in \overset{\circ}{R}$. Then $\underline{\mathbf{w}} = \langle w_1, \ldots, w_{i-1}, w_i = z_i, w_{i+1}, \ldots, w_n\rangle$ where $u_j < w_j < v_j$ for every $j \in \{1, \ldots, n\}$ and if $\underline{\mathbf{x}} = \langle x_1, \ldots, x_{i-1}, x_i = z_i, x_{i+1}, \ldots, x_n\rangle$ then there exists $j \in \{1, \ldots, n\}\backslash\{i\}$ such that $x_j \notin (u_j, v_j)$. Let

$$J_{\underline{\mathbf{x}}} = \{j \in \{1, \ldots, n\}\backslash\{i\} \mid x_j \notin (u_j, v_j)\}, \tag{2.5}$$

then $J_{\underline{\mathbf{x}}} \neq \emptyset$. Also define

$$J^l_{\underline{\mathbf{x}}} = \{j \in \{1, \ldots, n\}\backslash\{i\} \mid x_j \in [0, u_j]\}, \tag{2.6}$$

and

$$J^u_{\underline{\mathbf{x}}} = \{j \in \{1, \ldots, n\}\backslash\{i\} \mid x_j \in [v_j, 1]\}. \tag{2.7}$$

Then $J_{\underline{\mathbf{x}}} = J^l_{\underline{\mathbf{x}}} \cup J^u_{\underline{\mathbf{x}}}$ which is a disjoint union. If $J^u_{\underline{\mathbf{x}}} \neq \emptyset$, then there exists $j \in \{1, \ldots, n\}$ such that $v_j \leq x_j$, but, $x_j \leq w_j < v_j$, which is a contradiction. So, $J^l_{\underline{\mathbf{x}}} \neq \emptyset$, and in fact, $J_{\underline{\mathbf{x}}} = J^l_{\underline{\mathbf{x}}}$. Therefore,

$$\text{for every } j \in J_{\underline{\mathbf{x}}},\ \ x_j \leq u_j < w_j < v_j. \tag{2.8}$$

Define $\underline{\mathbf{z}} = \langle z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n\rangle$ such that

$$z_j = \begin{cases} u_j & \text{if} \quad j \in J_{\underline{\mathbf{x}}}, \\ x_j & \text{if} \quad j \in J^c_{\underline{\mathbf{x}}} = \{1, \ldots, n\}\backslash J_{\underline{\mathbf{x}}}. \end{cases} \tag{2.9}$$

Then using equation (2.8) $\underline{\mathbf{z}} \in H_{\underline{\mathbf{x}},\underline{\mathbf{w}}}$ and $\underline{\mathbf{z}} \in \partial R$, and also $\underline{\mathbf{x}} \leq \underline{\mathbf{z}} \leq \underline{\mathbf{w}}$. Therefore, using that $C = D$ on ∂R and equation (2.1)

$$\begin{aligned} & Q(w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n) - Q(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \\ = \quad & Q(w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n) - Q(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) \\ & +Q(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) - Q(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \\ = \quad & D(w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n) - D(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) \\ & +C(z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_n) - C(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \quad \geq 0, \end{aligned} \tag{2.10}$$

and $Q(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot)$ is increasing.

Subcase iii) If $\underline{\mathbf{x}} \in \overset{\circ}{R}$ and $\underline{\mathbf{w}} \notin \overset{\circ}{R}$ is completely analogous to subcase ii).

Subcase iv) Assume that $\underline{\mathbf{x}} \notin \overset{\circ}{R}$ and $\underline{\mathbf{w}} \notin \overset{\circ}{R}$. Define $J_{\underline{\mathbf{x}}}$, $J^l_{\underline{\mathbf{x}}}$ and $J^u_{\underline{\mathbf{x}}}$, as in equations (2.5), (2.6) and (2.7), analogously define for $\underline{\mathbf{w}}$, $J_{\underline{\mathbf{w}}}$, $J^l_{\underline{\mathbf{w}}}$ and $J^u_{\underline{\mathbf{w}}}$. Since $\underline{\mathbf{x}} \notin \overset{\circ}{R}$ and $\underline{\mathbf{w}} \notin \overset{\circ}{R}$ we know that

$$J_{\underline{\mathbf{x}}} = J^l_{\underline{\mathbf{x}}} \cup J^u_{\underline{\mathbf{x}}} \neq \emptyset \quad \text{and} \quad J_{\underline{\mathbf{w}}} = J^l_{\underline{\mathbf{w}}} \cup J^u_{\underline{\mathbf{w}}} \neq \emptyset. \tag{2.11}$$

Since $H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} \cap \overset{\circ}{R} \neq \emptyset$, let $\underline{\mathbf{z}} = \langle z_1, \ldots, z_n\rangle \in H_{\underline{\mathbf{x}},\underline{\mathbf{w}}} \cap \overset{\circ}{R}$. Then from (2.4)

$$x_j \leq z_j \leq w_j \quad \text{and} \quad u_j < z_j < v_j \qquad \text{for every} \quad j \in \{1, \ldots, n\}. \tag{2.12}$$

If $J^u_{\underline{\mathbf{x}}} \neq \emptyset$ there exists $j \in \{1, \ldots, n\}\backslash\{i\}$ such that $v_j \leq x_j$, but $x_j \leq z_j < v_j$ which is a contradiction. Hence, $J^l_{\underline{\mathbf{x}}} \neq \emptyset$, and in fact, $J_{\underline{\mathbf{x}}} = J^l_{\underline{\mathbf{x}}}$. So, for every $j \in J_{\underline{\mathbf{x}}}$, $x_j \leq u_j < z_j < v_j$. If $J^l_{\underline{\mathbf{w}}} \neq \emptyset$ there exists $k \in \{1, \ldots, n\}\backslash\{i\}$ such that $w_k \leq u_k$, but $u_k < z_k \leq w_k$ which is a contradiction. Hence, $J^u_{\underline{\mathbf{w}}} \neq \emptyset$, and in fact, $J_{\underline{\mathbf{w}}} = J^u_{\underline{\mathbf{w}}}$. So, for every $k \in J_{\underline{\mathbf{w}}}$, $u_k < z_k < v_k \leq w_k$.

Define $\underline{\mathbf{d}} = \langle d_1, \ldots, d_{i-1}, z_i, d_{i+1}, \ldots, d_n\rangle$ such that

$$d_j = \begin{cases} u_j & \text{if} \quad j \in J_{\underline{\mathbf{x}}}, \\ x_j & \text{if} \quad j \in J^c_{\underline{\mathbf{x}}} = \{1, \ldots, n\}\backslash J_{\underline{\mathbf{x}}}. \end{cases} \tag{2.13}$$

Then using (2.11), (2.12) and (2.13), $\underline{\mathbf{d}} \in \partial R \cap H_{\underline{\mathbf{x}},\underline{\mathbf{w}}}$ and $\underline{\mathbf{x}} \leq \underline{\mathbf{d}} \leq \underline{\mathbf{z}}$.

Define $\underline{\mathbf{e}} = \langle e_1, \ldots, e_{i-1}, z_i, e_{i+1}, \ldots, e_n \rangle$ such that

$$e_k = \begin{cases} v_k & \text{if} \quad k \in J_{\underline{\mathbf{w}}}, \\ w_k & \text{if} \quad k \in J^c_{\underline{\mathbf{w}}} = \{1, \ldots, n\} \backslash J_{\underline{\mathbf{w}}}. \end{cases} \tag{2.14}$$

Then using (2.11), (2.12) and (2.14), $\underline{\mathbf{e}} \in \partial R \cap H_{\underline{\mathbf{x}},\underline{\mathbf{w}}}$ and $\underline{\mathbf{z}} \leq \underline{\mathbf{e}} \leq \underline{\mathbf{w}}$. Therefore,

$$\underline{\mathbf{x}} \leq \underline{\mathbf{d}} \leq \underline{\mathbf{e}} \leq \underline{\mathbf{w}} \qquad \text{with} \qquad \underline{\mathbf{d}} \quad \text{and} \quad \underline{\mathbf{e}} \text{ in } \partial R. \tag{2.15}$$

Then using (2.1) and (2.15)

$$\begin{aligned} & Q(w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n) - Q(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \\ = \; & Q(w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n) - Q(e_1, \ldots, e_{i-1}, z_i, e_{i+1}, \ldots, e_n) \\ & + Q(e_1, \ldots, e_{i-1}, z_i, e_{i+1}, \ldots, e_n) - Q(d_1, \ldots, d_{i-1}, z_i, d_{i+1}, \ldots, d_n) \\ & + Q(d_1, \ldots, d_{i-1}, z_i, d_{i+1}, \ldots, d_n) - Q(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \\ = \; & C(w_1, \ldots, w_{i-1}, z_i, w_{i+1}, \ldots, w_n) - C(e_1, \ldots, e_{i-1}, z_i, e_{i+1}, \ldots, e_n) \\ & + D(e_1, \ldots, e_{i-1}, z_i, e_{i+1}, \ldots, e_n) - D(d_1, \ldots, d_{i-1}, z_i, d_{i+1}, \ldots, d_n) \\ & + C(d_1, \ldots, d_{i-1}, z_i, d_{i+1}, \ldots, d_n) - C(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) \quad \geq 0, \end{aligned} \tag{2.16}$$

So, $Q(\cdot, \ldots, \cdot, z_i, \cdot, \ldots, \cdot)$ is increasing, and by Proposition 2.2 Q is increasing. Therefore, Q is an n-conjunctor. □

Remark 2.6. Observe that if $R = \times_{i=1}^n [u_i, v_i]$ and $S = \times_{i=1}^n [w_i, z_i]$ are two non trivial n-boxes then $R \cap S = \emptyset$ or $R \cap S$ is another n-box, because for every $i \in \{1, \ldots, n\}$

$$[u_i, v_i] \cap [w_i, z_i] = \begin{cases} \emptyset & \text{if} \quad u_i < v_i < w_i < z_i \text{ or } w_i < z_i < u_i < v_i, \\ [w_i, v_i] & \text{if} \quad u_i \leq w_i \leq v_i \leq z_i, \\ [w_i, z_i] & \text{if} \quad u_i \leq w_i < z_i \leq v_i, \\ [u_i, z_i] & \text{if} \quad w_i \leq u_i \leq z_i \leq v_i, \\ [u_i, v_i] & \text{if} \quad w_i \leq u_i < v_i \leq z_i. \end{cases}$$

So, $R \cap S = \Pi_{i=1}^n [u_i, v_i] \cap [w_i, z_i]$ is either the empty set or an n-box.

Definition 2.7. *Let $R = \times_{i=1}^n [u_i, v_i]$ and $S = \times_{i=1}^n [w_i, z_i]$ be two non trivial n-boxes. We will say that R and S are* **adjacent** *if and only if they have a common adjacent face, that is, if there exists $i \in \{1, \ldots, n\}$ such that $v_i = w_i$ or $u_i = z_i$ and for every $j \in \{1, \ldots, n\} \backslash \{i\}$, $u_j = w_j$ and $v_j = z_j$ and $\overset{\circ}{R} \cap \overset{\circ}{S} = \emptyset$.*

Proposition 2.8. *Let* $R = \times_{i=1}^{n}[u_i, v_i]$ *and* $S = \times_{i=1}^{n}[w_i, z_i]$ *be two non trivial* n*-boxes. Let* $\overset{\circ}{R} = \times_{i=1}^{n}(u_i, v_i)$ *be the interior of* R*, and let Vert*(S) *the set of vertices of* S*. Let* $V_{S,R} = Vert(S) \cap \overset{\circ}{R}$*, if we denote by* $|\cdot|$ *the cardinality of a set, then* $|V_{S,R}| \in \{0, 2^0 = 1, 2^1, \ldots, 2^{n-1}, 2^n\}$*, and if* $|V_{S,R}| > 0$ *then* S *can be written as*

$$S = \cup_{k=1}^{2^n/|V_{S,R}|} S_k, \tag{2.17}$$

where every S_k *is an* n*-box. Also, if* $0 < |V_{S,R}| < 2^n$

$$\text{for every } k_1, k_2 \in \{1, \ldots, 2^n/|V_{S,R}|\} \text{ with } k_1 \neq k_2, \quad S_{k_1} \cap S_{k_2} \cap \partial R \neq \emptyset, \tag{2.18}$$

and $S_{k_1} \cap S_{k_2}$ *includes a subset of a* $(n-j)$*-dimensional face of* R *for some* $j \in \{1, \ldots n\}$*. Besides,*

$$\text{for every} \quad k_1, k_2 \in \{1, \ldots, 2^n/|V_{S,R}|\} \text{ with } k_1 \neq k_2, \quad \overset{\circ}{S}_{k_1} \cap \overset{\circ}{S}_{k_2} = \emptyset, \tag{2.19}$$

and for every $k_1 \in \{1, \ldots, 2^n/|V_{S,R}|\}$

$$\text{there exists } k_2 \in \{1, \ldots, 2^n/|V_{S,R}|\}, k_2 \neq k_1, \text{ such that } S_{k_1} \text{ and } S_{k_2} \text{ are adjacent.} \tag{2.20}$$

In the representation of S *in equation* (2.17) *there exists a unique* $k_0 \in \{1, 2, \ldots, 2^n/|V_{S,R}|\}$ *such that* $S_{k_0} \subset R$ *and for the remaining* k'*s,* $S_k \cap \overset{\circ}{R} = \emptyset$*.*

Proof. Let $R = \times_{i=1}^{n}[u_i, v_i]$ and $S = \times_{i=1}^{n}[w_i, z_i]$ be two non trivial n-boxes. Define $V_{S,R} = \text{Vert}(S) \cap \overset{\circ}{R}$. Observe that if $|V_{S,R}| = 0$, then $S \subset (\overset{\circ}{R})^c$, where $(\overset{\circ}{R})^c$ is the complement of $\overset{\circ}{R}$. Observe also that if $|V_{S,R}| = 2^n$ then $S \subset \overset{\circ}{R}$ and $2^n/|V_{S,R}| = 1$. So, equation (2.17) holds defining $S_1 = S$.

We will work with the upper coordinates that define S, in order to facilitate notation and without losing generality.

First, assume that for every $i \in \{1, \ldots, n\}$, $w_i < u_i < z_i < v_i$. Then $V_{S,R} = \text{Vert}(S) \cap \overset{\circ}{R} =$

$\{\langle z_1, \dots z_n\rangle\}$ and $|V_{S,R}| = 1 = 2^0$. We also have that

$$\begin{aligned}
S &= \bigtimes_{i=1}^{n} [w_i, z_i] \\
&= \bigtimes_{i=1}^{n} [w_i, u_i] \cup [u_i, z_i] \\
&= [w_1, u_1] \times \cdots \times [w_{n-1}, u_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [w_{n-1}, u_{n-1}] \times [u_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [u_n, z_n] \\
&\quad \quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\
&\quad \cup [u_1, z_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [w_n, u_n] \\
&\quad \cup [u_1, z_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [u_n, z_n] \\
&= S_1 \cup S_2 \cup S_3 \cup S_4 \cup \cdots \cup S_{2^n - 1} \cup S_{2^n}.
\end{aligned} \tag{2.21}$$

Observe that in the last line of (2.21) we have used an order in the listing, in order to see how it is done, define the intervals $I_{1,i} = [w_i, u_i]$ and $I_{2,i} = [u_i, z_i]$ for every $i \in \{1, \dots, n\}$, observe that from equation (2.21), we can write

$$S = \cup_{\langle j_1, \dots, j_n\rangle \in \{1,2\}^n} \bigtimes_{i=1}^{n} I_{j_i, i}. \tag{2.22}$$

Hence, it is clear that the number of n-boxes in (2.21) is $2^n = 2^n/2^0 = 2^n/|V_{S,R}|$ and (2.17) holds. Besides, in order for the last equality to make complete sense we need to establish an order in $\{1,2\}^n$. Let $\underline{\mathbf{l}} = \langle l_1, \dots, l_n\rangle \in \{1,2\}^n$, define $Q_{\underline{\mathbf{l}}} = \{k \in \{1, \dots, n\} \mid l_k = 2\}$. Now we define $\varphi : \{1,2\}^n \to \{1, 2, 3, \dots, 2^n\}$ by

$$\varphi(\underline{\mathbf{l}}) = 1 + \sum_{k \in Q_{\underline{\mathbf{l}}}} 2^{n-k} \quad \text{for every} \quad \underline{\mathbf{l}} \in \{1,2\}^n.$$

Of course, $Q_{\underline{\mathbf{l}}} = \emptyset$ if and only if $\underline{\mathbf{l}} = \langle 1, 1, \dots, 1\rangle$ and in this case $\varphi(1, 1, \dots, 1) = 1$. Also observe that if $\underline{\mathbf{l}} = \langle 2, 2, \dots, 2\rangle$, then $\varphi(2, 2, \dots, 2) = 1 + \sum_{k=1}^{n} 2^{n-k} = 1 + \sum_{j=0}^{n-1} 2^j = 2^n$. It is not difficult to see that φ is a bijection between $\{1,2\}^n$ and $\{1, 2, 3, \dots, 2^n\}$. The order we propose is the following: we say that if $\underline{\mathbf{r}}, \underline{\mathbf{s}} \in \{1,2\}^n$ then $\underline{\mathbf{r}} < \underline{\mathbf{s}}$ if and only if $\varphi(\underline{\mathbf{r}}) < \varphi(\underline{\mathbf{s}})$. So, using (2.22) we can order the S_i using φ, we will call this order the **binary order**.

Observe that $\langle u_1, \dots u_n\rangle \in S_k$ for every $k \in \{1, 2, 3, \dots, 2^n\}$, hence, (2.18) holds.
Let $k_1, k_2 \in \{1, 2, 3, \dots, 2^n\}$, with $k_1 \neq k_2$, then using the binary order and representation (2.22), there exist $\underline{\mathbf{r}}, \underline{\mathbf{s}} \in \{1, 2\}^n$ with $\underline{\mathbf{r}} \neq \underline{\mathbf{s}}$, such that $k_1 = \varphi(\underline{\mathbf{r}})$ and $k_2 = \varphi(\underline{\mathbf{s}})$; since $\underline{\mathbf{r}} \neq \underline{\mathbf{s}}$ there exists $j \in \{1, 2, \dots, n\}$ such that $r_j = 1$ and $s_j = 2$, or $r_j = 2$ and $s_j = 1$, in either case $\overset{\circ}{I}_{r_j,j} \cap \overset{\circ}{I}_{s_j,j} = (w_j, u_j) \cap (u_j, z_j) = \emptyset$. So, $\overset{\circ}{S}_{k_1} \cap \overset{\circ}{S}_{k_2} = \bigtimes_{i=1}^n \overset{\circ}{I}_{r_i,i} \cap \bigtimes_{i=1}^n \overset{\circ}{I}_{s_i,i} = \emptyset$, and (2.19) holds.
Using the order we establish above it is clear that S_{2m-1} and S_{2m} are adjacent n-boxes for every $m \in \{1, 2, \dots, 2^{n-1}\}$, so, (2.20) holds.
Observe that the only S_k included in R is S_{2^n}, and using representation (2.22), since $[w_i, u_i] \cap (u_i, z_i) = \emptyset$ for every $i \in \{1, \dots, n\}$ then $S_k \cap \overset{\circ}{R} = \emptyset$ for every $1 \leq k < 2^n$.
Second, assume that there exists $j \in \{1, \dots, n\}$ such that $u_j < w_j < z_j < v_j$ and for every $i \in \{1, \dots, n\} \backslash \{j\}$, $w_i < u_i < z_i < v_i$. Without losing generality we can assume that $j = 1$. Then $V_{S,R} = \text{Vert}(S) \cap \overset{\circ}{R} = \{\langle z_1, z_2 \dots z_n\rangle, \langle w_1, z_2, \dots z_n\rangle\}$ and $|V_{S,R}| = 2 = 2^1$. We also have that

$$
\begin{aligned}
S &= [w_1, z_1] \times (\bigtimes_{i=2}^{n} [w_i, z_i]) \\
&= [w_1, z_1] \times (\bigtimes_{i=2}^{n} [w_i, u_i] \cup [u_i, z_i]) \\
&= [w_1, z_1] \times \cdots \times [w_{n-1}, u_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [w_{n-1}, u_{n-1}] \times [u_n, z_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [u_n, z_n] \\
&\quad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\
&\quad \cup [w_1, z_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [u_{n-1}, z_{n-1}] \times [u_n, z_n] \\
&= S_1 \cup S_2 \cup S_3 \cup S_4 \cup \cdots \cup S_{2^{n-1}-1} \cup S_{2^{n-1}}.
\end{aligned} \tag{2.23}
$$

In this case if we call $I_1 = [w_1, z_1]$ and using the same notation as above we have that

$$
S = \cup_{\langle j_2, \dots, j_n\rangle \in \{1,2\}^{n-1}} I_1 \times I_{j_2,2} \times \cdots \times I_{j_n,n}. \tag{2.24}
$$

Hence the number of n-boxes in this case is $2^{n-1} = 2^n/2^1 = 2^n/|V_{S,R}|$ and (2.17) holds, and we use $\varphi : \{1,2\} \to \{1,2,\ldots,2^{n-1}\}$ to establish the order of the n-boxes S_k.

Since $[w_1, z_1] \subset [u_1, v_1]$ and for every $j \in \{2,3,\ldots,n\}$, u_j belongs to the j^{th} coordinate of any S_k for every $k \in \{1,2,\ldots 2^{n-1}\}$, then using equation (1.9),

$$R_{2,3,\ldots,n} \cap S_{k_1} \cap S_{k_2} = \{\langle x_1,\ldots,x_n\rangle \in R \mid \text{for every } j \in \{2,3,\ldots,n\}, x_j = u_j\} \cap S_{k_1} \cap S_{k_2} \neq \emptyset,$$

for every $k_1, k_2 \in \{1,2,\ldots,2^{n-1}\}$ with $k_1 \neq k_2$. So, (2.18) holds.

To see that equation (2.19) holds we use exactly the same arguments as in the first case.

Using the order we established above it is clear that S_{2m-1} and S_{2m} are adjacent n-boxes for every $m \in \{1,2,\ldots,2^{n-2}\}$, so, (2.20) holds.

Observe that the only S_k included in R is $S_{2^{n-1}}$, and since $[w_i, u_i] \cap (u_i, z_i) = \emptyset$ for every $i \in \{1,\ldots,n\}$ then $S_k \cap \overset{\circ}{R} = \emptyset$ for every $1 \le k < 2^{n-1}$.

In the k^{th} step, assume that exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $u_{i_j} < w_{i_j} < z_{i_j} < v_{i_j}$ for every $j \in \{1,2,\ldots,k\}$ and for every $i \in \{1,\ldots,n\} \backslash \{i_1, i_2, \ldots, i_k\}$, $w_i < u_i < z_i < v_i$. We will assume without losing generality that $i_j = j$ for every $j \in \{1,2,\ldots,k\}$.Then $V_{S,R} = \text{Vert}(S) \cap \overset{\circ}{R} = \{\langle x_1, x_2, \ldots, x_n\rangle \in \text{Vert}(S) \,|\, x_i = w_i \text{ or } x_i = z_i \text{ for every } i \in \{1,\ldots,k\},$ and $x_j = z_j$ for every $j \in \{k+1, k+2, \ldots, n\}\}$

and $|V_{S,R}| = 2^k$. We also have that

$$\begin{aligned}
S &= [w_1, z_1] \times \cdots \times [w_k, z_k] \times (\mathop{\times}_{i=k+1}^{n} [w_i, z_i]) \\
&= [w_1, z_1] \times \cdots \times [w_k, z_k] \times (\mathop{\times}_{i=k+1}^{n} [w_i, u_i] \cup [u_i, z_i]) \\
&= [w_1, z_1] \times \cdots \times [w_k, z_k] \times \cdots \times [w_{n-1}, u_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [w_k, z_k] \times \cdots \times [w_{n-1}, u_{n-1}] \times [u_n, z_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [w_k, z_k] \times \cdots \times [u_{n-1}, z_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [w_k, z_k] \times \cdots \times [u_{n-1}, z_{n-1}] \times [u_n, z_n] \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad \cup [w_1, z_1] \times \cdots \times [w_k, z_k] \times \cdots \times [u_{n-1}, z_{n-1}] \times [w_n, u_n] \\
&\quad \cup [w_1, z_1] \times \cdots \times [w_k, z_k] \times \cdots \times [u_{n-1}, z_{n-1}] \times [u_n, z_n] \\
&= S_1 \cup S_2 \cup S_3 \cup S_4 \cup \cdots \cup S_{2^{n-k}-1} \cup S_{2^{n-k}}. \qquad (2.25)
\end{aligned}$$

In this case if we call $I_i = [w_i, z_i]$ for $i \in \{1, 2, \dots .k\}$ and using the same notation as in the first case we have that

$$S = \cup_{\langle j_{k+1},\dots,j_n\rangle \in \{1,2\}^{n-k}} I_1 \times \cdots \times I_k \times I_{j_{k+1},k+1} \times \cdots \times I_{j_n,n}. \tag{2.26}$$

Hence the number of n-boxes in this case is $2^{n-k} = 2^n/2^k = 2^n/|V_{S,R}|$ and (2.17) holds, and we use $\varphi : \{1, 2\} \to \{1, 2, \dots , 2^{n-k}\}$ to establish the order of the n-boxes S_j.
Since $[w_i, z_i] \subset [u_i, v_i]$ for every $i \in \{1, \dots , k\}$ and for every $j \in \{k + 1, k + 2, \dots , n\}$, u_j belongs to the j^{th} coordinate of any S_k for every $k \in \{1, 2, \dots 2^{n-k}\}$, then using equation (1.9),

$$R_{k+1,k+2,\dots,n} \cap S_{k_1} \cap S_{k_2} = \{\langle x_1, \dots , x_n\rangle \in R \mid \text{for every } j \in \{k + 1, k + 2, \dots , n\}, x_j = u_j\} \cap S_{k_1} \cap S_{k_2} \neq \emptyset, \tag{2.27}$$

for every $k_1, k_2 \in \{1, 2, \dots , 2^{n-k}\}$ with $k_1 \neq k_2$. So, (2.18) holds.
To see that equation (2.19) holds we use exactly the same arguments as in the first case.
Using the order we established above it is clear that S_{2m-1} and S_{2m} are adjacent n-boxes for every $m \in \{1, 2, \dots , 2^{n-(k+1)}\}$, so, (2.19) holds.
Observe that the only S_k included in R is $S_{2^{n-k}}$, and since $(w_i, u_i) \cap (u_i, z_i) = \emptyset$ for every $i \in \{1, \dots , n\}$ then $\overset{\circ}{S_k} \cap \overset{\circ}{R} = \emptyset$ for every $1 \leq k < 2^{n-k}$..
Finally, if we let $k = n - 1$ and we proceed as above

$$S = ([w_1, z_1] \times \cdots \times [w_{n-1}, z_{n-1}] \times [w_n, u_n]) \cup ([w_1, z_1] \times \cdots \times [w_{n-1}, z_{n-1}] \times [u_n, z_n]) = S_1 \cup S_2.$$

So, (2.17), (2.18), (2.19) and (2.20) follow immediately, also, only S_2 is included in R and $S_1 \cap \overset{\circ}{R} = \emptyset$. □

Lemma 2.9. *Let $R = \times_{i=1}^n [u_i, v_i]$ and $S = \times_{i=1}^n [w_i, z_i]$ be two adjacent non trivial n-boxes. Let $C : R \to \mathbf{I}$ and $D : S \to \mathbf{I}$ such that $V_C(R) \geq 0$ and $V_D(S) \geq 0$. Define $Q : S \cup R \to \mathbf{I}$ by*

$$Q(\underline{\mathbf{x}}) = \begin{cases} C(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in R, \\ D(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in S \end{cases} \tag{2.28}$$

If $C = D$ on the common face of R and S then Q is well defined and $V_Q(R \cup S) = V_Q(R) + V_Q(S) = V_C(R) + V_D(S) \geq 0$.

Proof. Let $R = \times_{i=1}^{n}[u_i, v_i]$ and $S = \times_{i=1}^{n}[w_i, z_i]$ be two adjacent non trivial n-boxes. By Definition 2.7 there exists $i \in \{1, 2, \ldots, n\}$ such that either $v_i = w_i$ or $u_i = z_i$ and for every $j \in \{1, 2, \ldots, n\} \backslash \{i\}$, $u_j = w_j$ and $v_j = z_j$. Without losing generality we will assume that $i = 1$ and $v_1 = w_1$. Then

$$R = [u_1, v_1] \times \bigtimes_{i=2}^{n}[u_i, v_i] \quad \text{and} \quad S = [v_1, z_1] \times \bigtimes_{i=2}^{n}[u_i, v_i], \tag{2.29}$$

and

$$R \cup S = [u_1, z_1] \times \bigtimes_{i=2}^{n}[u_i, v_i]. \tag{2.30}$$

Also the common face of R and S is the set

$$R \cap S = \{\langle x_1, \ldots, x_n \rangle \mid x_1 = v_1 \text{ and for every } \ j \in \{2, \ldots, n\}, x_j \in [u_j, v_j]\}. \tag{2.31}$$

So, Q is well defined. Let

$$\mathrm{Vert}(R_{u_1}) = \{\underline{\mathbf{c}} \in \mathrm{Vert}(R) \mid c_1 = u_1\} \quad \text{and} \quad \mathrm{Vert}(R_{v_1}) = \{\underline{\mathbf{c}} \in \mathrm{Vert}(R) \mid c_1 = v_1\}. \tag{2.32}$$

Then $\mathrm{Vert}(R) = \mathrm{Vert}(R_{u_1}) \cup \mathrm{Vert}(R_{v_1})$ which is a disjoint union. Define analogously, $\mathrm{Vert}(S_{v_1})$ and $\mathrm{Vert}(S_{z_1})$, then $\mathrm{Vert}(S) = \mathrm{Vert}(S_{v_1}) \cup \mathrm{Vert}(S_{z_1})$ which is a disjoint union. Besides,

$$|\mathrm{Vert}(R_{u_1})| = |\mathrm{Vert}(R_{v_1})| = |\mathrm{Vert}(S_{v_1})| = |\mathrm{Vert}(S_{z_1})| = 2^{n-1}.$$

Define also $\mathrm{Vert}((R \cup S)_{u_1})$ and $\mathrm{Vert}((R \cup S)_{z_1})$ and observe that

$$\mathrm{Vert}((R \cup S)_{u_1}) = \mathrm{Vert}(R_{u_1}), \quad \mathrm{Vert}((R \cup S)_{z_1}) = \mathrm{Vert}(S_{z_1}) \quad \text{and} \quad \mathrm{Vert}(R_{v_1}) = \mathrm{Vert}(S_{v_1}). \tag{2.33}$$

Recall that the function sgn in Definition 1.8, equation (1.8) depends only on the vertices of the n-box T in which the volume is calculated. So, we will denote sgn_T instead of sgn, when the T may change.
We also observe that if $\underline{\mathbf{c}} = \langle c_1, \ldots, c_n \rangle \in \mathrm{Vert}(R_{v_1}) = \mathrm{Vert}(S_{v_1})$. We have that

$$\text{if } \mathrm{sgn}_R(\underline{\mathbf{c}}) = 1, \text{ then } \mathrm{sgn}_S(\underline{\mathbf{c}}) = -1 \quad \text{and} \quad \text{if } \mathrm{sgn}_R(\underline{\mathbf{c}}) = -1, \text{ then } \mathrm{sgn}_S(\underline{\mathbf{c}}) = 1, \tag{2.34}$$

because in R, v_1 is the final point of $[u_1, v_1]$ and in S, v_1 is the initial point of $[v_1, z_1]$, using Definition 1.8. Also, from equations (2.29), (2.30) and (2.33), we have that if $\underline{\mathbf{c}} \in \mathrm{Vert}((R \cup S)_{u_1} = \mathrm{Vert}(R_{u_1})$, then $\mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) = \mathrm{sgn}_R(\underline{\mathbf{c}})$, and if $\underline{\mathbf{c}} \in \mathrm{Vert}((R \cup S)_{z_1} = \mathrm{Vert}(S_{z_1})$, then $\mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) = \mathrm{sgn}_S(\underline{\mathbf{c}})$.

Therefore, if $C = D$ on the vertices of the common face of R and S given in equation (2.31), using equations (2.28), (2.29), (2.33), and (2.34) we have that

$$\begin{aligned}
V_Q(R \cup S) &= V_Q([u_1, z_1] \times \bigtimes_{j=2}^{n} [u_j, v_j]) \\
&= \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(R \cup S)\}} \mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) Q(\underline{\mathbf{c}}) \\
&= \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}((R \cup S)_{z_1})\}} \mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) Q(\underline{\mathbf{c}}) \\
&\quad + \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}((R \cup S)_{u_1})\}} \mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) Q(\underline{\mathbf{c}}) \\
&= \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}((R \cup S)_{z_1})\}} \mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) Q(\underline{\mathbf{c}}) \\
&\quad + \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(S_{v_1})\}} Q(\underline{\mathbf{c}}) \\
&\quad - \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(R_{v_1})\}} Q(\underline{\mathbf{c}}) \\
&\quad + \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}((R \cup S)_{u_1})\}} \mathrm{sgn}_{R \cup S}(\underline{\mathbf{c}}) Q(\underline{\mathbf{c}})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(S_{z_1})\}} \mathrm{sgn}_S(\underline{\mathbf{c}}) D(\underline{\mathbf{c}}) \\
&\quad + \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(S_{v_1})\}} \mathrm{sgn}_S(\underline{\mathbf{c}}) D(\underline{\mathbf{c}}) \\
&\quad + \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(R_{v_1})\}} \mathrm{sgn}_R(\underline{\mathbf{c}}) C(\underline{\mathbf{c}}) \\
&\quad + \sum_{\{\underline{\mathbf{c}} \in \mathrm{Vert}(R_{u_1})\}} \mathrm{sgn}_R(\underline{\mathbf{c}}) C(\underline{\mathbf{c}}) \\
&= V_Q(S) + V_Q(R) \\
&= V_D(S) + V_C(R) \geq 0. \qquad (2.35)
\end{aligned}$$

Observe that if we have a function $C : R \cup S \to \mathbf{I}$ and in definition (2.28) we change D by C, we also have that if $V_C(R) \geq 0$ and $V_C(S) \geq 0$ the result still holds. □

Now we will prove the main result.

Theorem 2.10. *Let $C : \mathbf{I}^n \to \mathbf{I}$ be an n-copula, let $R = \times_{i=1}^n [u_i, v_i] \subset \mathbf{I}^n$ be a non trivial n-box. Let $D : R \to \mathbf{I}$ be a function. Define $Q : \mathbf{I}^n \to \mathbf{I}$ by*

$$Q(x_1, \ldots, x_n) = \begin{cases} D(x_1, \ldots, x_n) & \text{if} \quad \langle x_1, \ldots, x_n \rangle \in R, \\ C(x_1, \ldots, x_n) & \text{if} \quad \langle x_1, \ldots, x_n \rangle \in \mathbf{I}^n \setminus R. \end{cases} \tag{2.36}$$

Then, Q is an n-copula if and only if $D = C$ on ∂R and D is n-increasing.

Note: In Theorem 2.10 we can replace the n-box R by a region of the form $R = [\underline{\mathbf{a}}, 1] \setminus [\underline{\mathbf{b}}, 1]$ with $\underline{\mathbf{a}} \leq \underline{\mathbf{b}}$ and define Q as above. In this case Q will be a copula whenever D is n-increasing, $D = C$ on $\partial R \cap \overset{\circ}{\mathbf{I}}$ and D satisfies the boundary conditions of a copula in $\partial D \cap \partial \mathbf{I}$.

Proof. Let $C : \mathbf{I}^n \to \mathbf{I}$ be an n-copula, let $R = \times_{i=1}^n [u_i, v_i] \subset \mathbf{I}^n$ be a non trivial n-box. Let $D : R \to \mathbf{I}$ be a function. Define Q as in equation (2.36). First, assume that Q is an n-copula, then using Definition 2.3 and Remark 2.4, we have that Q is n-increasing, therefore, $Q|_R$ the restriction of Q to R is also n-increasing, but from equation (2.36), $Q|_R = D$ and D is n-increasing. Besides, any n copula is jointly uniformly continuous, see Nelsen [58], then $C = D$ on ∂R.

Conversely, if we assume that D is n-increasing on R and $C = D$ on ∂R and we define Q as in equation (2.36) then we can redefine Q as follows:

$$Q(x_1, \ldots, x_n) = \begin{cases} D(x_1, \ldots, x_n) & \text{if} \quad \langle x_1, \ldots, x_n \rangle \in \overset{\circ}{R}, \\ C(x_1, \ldots, x_n) & \text{if} \quad \langle x_1, \ldots, x_n \rangle \in \mathbf{I}^n \setminus \overset{\circ}{R}, \end{cases} \tag{2.37}$$

where $\overset{\circ}{R} = \times_{i=1}^n (u_i, v_i)$. Using equation (2.37) we observe that for any $R = \times_{i=1}^n [u_i, v_i] \subset \mathbf{I}^n$ all the vectors of the form $\langle 1, \ldots, 1, x_i, 1, \ldots, 1 \rangle$ with $i \in \{1, \ldots, n\}$ and $x_i \in \mathbf{I}$ belong to $\mathbf{I}^n \setminus \overset{\circ}{R}$. So, using (2.37), $Q(1, \ldots, 1, x_i, 1, \ldots, 1) = C(1, \ldots, 1, x_i, 1, \ldots, 1) = x_i$, for every $i \in \{1, \ldots, n\}$ and for every $x_i \in \mathbf{I}$.

Similarly, if we let $\underline{\mathbf{x}} = \langle x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n \rangle$, then $\underline{\mathbf{x}} \in \mathbf{I}^n \setminus \overset{\circ}{R}$ for every $x_j \in \mathbf{I}$ and for every $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$. So, using (2.37), $Q(\underline{\mathbf{x}}) = C(\underline{\mathbf{x}}) = 0$, for every $\underline{\mathbf{x}} = \langle x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n \rangle$ with $i \in \{1, \ldots, n\}$ and for every $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbf{I}$. Therefore, Q satisfies all the boundary conditions of an n-copula.

Hence, all we have to prove is that Q is n-increasing. So, let $S = \times_{i=1}^{n}[w_i, z_i] \subset \mathbf{I}^n$, we have to see that $V_Q(S) \geq 0$. We will proceed by cases and we will use Proposition 2.8 and Lemma 2.9.

Recall that $V_{S,R} = \text{Vert}(S) \cap \overset{\circ}{R}$, we know from Proposition 2.8 that $|V_{S,R}| \in \{0, 2^0 = 1, 2^1, \ldots, 2^n\}$. If $\text{Vert}(S) = \emptyset$ then $|V_{S,R}| = 0$ and $\text{Vert}(S) \subset (\overset{\circ}{R})^c = \mathbf{I}^n \setminus \overset{\circ}{R}$, and by equation (2.37), $V_Q(S) = V_C(S) \geq 0$, since C is n-increasing.

On the other hand, if $|V_{S,R}| = 2^n$ then $\text{Vert}(S) \subset \overset{\circ}{R} \subset R$, and in this case $S \subset \overset{\circ}{R} \subset R$, and by equation (2.37), $V_Q(S) = V_D(S) \geq 0$, because D is n-increasing on R.

So, we only have to prove that $V_Q(S) \geq 0$ when $2^0 = 1 \leq |V_{S,R}| \leq 2^{n-1}$.

We will see in detail the most difficult case, assume that $|V_{S,R}| = 2^0 = 1$, without losing generality we can assume that S has the representation given in equation (2.21) of Proposition 2.8, that is, $S = \times_{i=1}^{n}[w_i, u_i] \cup [u_i, z_i] = S_1 \cup S_2 \cup \cdots \cup S_{2^n}$. Here, we will take advantage of the order induced by φ in Proposition 2.8. We saw in the proof of Proposition 2.8 that in this case, for every $m \in \{1, 2, \ldots, 2^{n-1}\}$, S_{2m-1} and S_{2m} are adjacent n-boxes. Define

$$T_m^1 = S_{2m-1} \cup S_{2m} \quad \text{for every} \quad m \in \{1, 2, \ldots, 2^{n-1}\}, \ \text{then } S = \cup_{j=1}^{2^{n-1}} T_j^1, \tag{2.38}$$

using equation (2.21). We also have that

$$\begin{aligned}
S &= \times_{i=1}^{n}[w_i, z_i] \\
&= T_1^1 \cup T_2^1 \cup T_3^1 \cup T_4^1 \cup \cdots T_{2^{n-1}-1}^1 \cup T_{2^{n-1}}^1 \\
&= [w_1, u_1] \times \cdots \times [w_{n-2}, u_{n-2}] \times [w_{n-1}, u_{n-1}] \times [w_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [w_{n-2}, u_{n-2}] \times [u_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [u_{n-2}, z_{n-2}] \times [w_{n-1}, u_{n-1}] \times [w_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [u_{n-2}, z_{n-2}] \times [u_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad \cup [u_1, z_1] \times \cdots \times [u_{n-2}, z_{n-2}] \times [w_{n-1}, u_{n-1}] \times [w_n, z_n] \\
&\quad \cup [u_1, z_1] \times \cdots \times [u_{n-2}, z_{n-2}] \times [u_{n-1}, z_{n-1}] \times [w_n, z_n].
\end{aligned} \tag{2.39}$$

It is clear that T_{2m-1}^1 and T_{2m}^1 are adjacent n-boxes for every $m \in \{1, 2, \ldots, 2^{n-2}\}$

Besides, from the proof of this case in Proposition 2.8, we know that

$$T^1_m \cap \overset{\circ}{R} = (S_{2m-1} \cap \overset{\circ}{R}) \cup (S_{2m} \cap \overset{\circ}{R}) = \emptyset \text{ for every } m \in \{1, 2, \ldots, 2^{n-1}-1\}, \tag{2.40}$$

and

$$T^1_{2^{n-1}} \cap \overset{\circ}{R} = (S_{2^n-1} \cap \overset{\circ}{R}) \cup (S_{2^n} \cap \overset{\circ}{R}) = \emptyset \cup \bigtimes_{i=1}^{n} (u_i, z_i) = \bigtimes_{i=1}^{n} (u_i, z_i). \tag{2.41}$$

So, using equation (2.37) and Lemma 2.9, for every $m \in \{1, 2, \ldots, 2^{n-1}-1\}$, we have that

$$V_Q(T^1_m) = V_C(T^1_m) = V_C(S_{2m}) + V_C(S_{2m-1}) \geq 0, \tag{2.42}$$

and for $m = 2^{n-1}$ we have that

$$V_Q(T^1_{2^{n-1}}) = V_C(S_{2^n-1}) + V_D(S_{2^n}) \geq 0. \tag{2.43}$$

We can finish the proof of this case using equations (2.38), (2.40) and (2.41). But, we will keep using Lemma 2.9 to finish the proof, because the remaining cases are necessarily included below. Observe that equation (2.39) represents also the case in which, for every $i \in \{1, 2, \ldots, n-1\}$, $w_i < u_i < z_i < v_i$ and $u_n < z_n < w_n < v_n$. But in this case,

$$|V_{S,R}| = |\mathrm{Vert}(S) \cap \overset{\circ}{R}| = |\{\langle z_1, \ldots, z_{n-1}, z_n\rangle, \langle z_1, \ldots, z_{n-1}, w_n\rangle\}| = 2. \tag{2.44}$$

Now, define

$$T^2_m = T^1_{2m-1} \cup T^1_{2m} \quad \text{for every} \quad m \in \{1, 2, \ldots, 2^{n-2}\}, \text{ then } S = \cup_{j=1}^{2^{n-2}} T^2_j, \tag{2.45}$$

using equation (2.39). We also have that

$$\begin{aligned}
S &= \bigtimes_{i=1}^{n} [w_i, z_i] \\
&= T^2_1 \cup T^2_2 \cup T^2_3 \cup T^2_4 \cup \cdots T^2_{2^{n-2}-1} \cup T^2_{2^{n-2}} \\
&= [w_1, u_1] \times \cdots \times [w_{n-3}, u_{n-3}] \times [w_{n-2}, u_{n-2}] \times [w_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [w_{n-3}, u_{n-3}] \times [u_{n-2}, z_{n-2}] \times [w_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [u_{n-3}, z_{n-3}] \times [w_{n-2}, u_{n-2}] \times [w_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\quad \cup [w_1, u_1] \times \cdots \times [u_{n-3}, z_{n-3}] \times [u_{n-2}, z_{n-2}] \times [w_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad \cup [u_1, z_1] \times \cdots \times [u_{n-3}, z_{n-3}] \times [w_{n-2}, u_{n-2}] \times [w_{n-1}, z_{n-1}] \times [w_n, z_n] \\
&\quad \cup [u_1, z_1] \times \cdots \times [u_{n-3}, z_{n-3}] \times [u_{n-2}, z_{n-2}] \times [w_{n-1}, z_{n-1}] \times [w_n, z_n].
\end{aligned} \tag{2.46}$$

Again, it is clear that T^2_{2m-1} and T^2_{2m} are adjacent n-boxes for every $m \in \{1, 2, \ldots, 2^{n-3}\}$. Besides, as above, we know using equation (2.40) that

$$T^2_m \cap \overset{\circ}{R} = (T^1_{2m-1} \cap \overset{\circ}{R}) \cup (T^1_{2m} \cap \overset{\circ}{R}) = \emptyset \text{ for every } m \in \{1, 2, \ldots, 2^{n-2} - 1\}, \tag{2.47}$$

and using (2.41)

$$T^2_{2^{n-2}} \cap \overset{\circ}{R} = (T^1_{2^{n-1}-1} \cap \overset{\circ}{R}) \cup (T^1_{2^{n-1}} \cap \overset{\circ}{R}) = \emptyset \cup \mathop{\times}_{i=1}^{n} (u_i, z_i) = \mathop{\times}_{i=1}^{n} (u_i, z_i). \tag{2.48}$$

So, using equation (2.37) and Lemma 2.9, we have that

$$V_Q(T^2_m) = V_Q(T^1_{2m}) + V_Q(T^1_{2m-1}) \geq 0 \text{ for every } m \in \{1, 2, \ldots, 2^{n-2}\}. \tag{2.49}$$

And in the case that (2.44) holds, for every $m \in \{1, 2, \ldots, 2^{n-2} - 1\}$, we have that

$$V_Q(T^2_m) = V_C(T^2_m) = V_C(T^1_{2m}) + V_C(T^1_{2m-1}) \geq 0, \tag{2.50}$$

and for $m = 2^{n-2}$ we have that

$$V_Q(T^1_{2^{n-2}}) = V_C(T^1_{2^{n-1}-1}) + V_Q(T^1_{2^{n-1}}) \geq 0. \tag{2.51}$$

Now, of course, we proceed inductively, by defining for $3 \leq k \leq n$

$$T^k_m = T^{k-1}_{2m-1} \cup T^{k-1}_{2m} \quad \text{for every} \quad m \in \{1, 2, \ldots, 2^{n-k}\}, \text{ then } S = \cup_{j=1}^{2^{n-k}} T^k_j, \tag{2.52}$$

where every T^k_m is of the form

$$T^k_m = I_1 \times \cdots \times I_{n-k} \times [w_{n-(k+1)}, z_{n-(k+1)}] \times \cdots \times [w_n, z_n], \tag{2.53}$$

and for every $j \in \{1, 2, \ldots, n-k\}$, either $I_j = [w_j, , u_j]$ or $I_j = [u_j, z_j]$. Again, T^k_{2m-1} and T^k_{2m} are adjacent n-boxes for every $m \in \{1, 2, \ldots, 2^{n-(k+1)}\}$. Besides

$$T^k_m \cap \overset{\circ}{R} = (T^{k-1}_{2m-1} \cap \overset{\circ}{R}) \cup (T^{k-1}_{2m} \cap \overset{\circ}{R}) = \emptyset \text{ for every } m \in \{1, 2, \ldots, 2^{n-k} - 1\}, \tag{2.54}$$

and

$$T^k_{2^{n-k}} \cap \overset{\circ}{R} = (T^{k-1}_{2^{n-k}-1} \cap \overset{\circ}{R}) \cup (T^1_{2^{n-k}} \cap \overset{\circ}{R}) = \emptyset \cup \mathop{\times}_{i=1}^{n} (u_i, z_i) = \mathop{\times}_{i=1}^{n} (u_i, z_i). \tag{2.55}$$

So, using equation (2.37) and Lemma 2.9 we have similar equations as in (2.49), or as in (2.50) and (2.51) depending on the number of vertices in $V_{S,R}$. □

Theorem 1.35 is the main result in the **patchwork construction** of 2-copulas given in Durante *et al* [24]. An alternative shorter proof using the next results is given at the end of this section.

Definition 2.11. *Let $F : \mathcal{D} \to R$ be a function, where $\mathcal{D} \subset R^n$ for some $n \geq 2$. We will say that F is* **modular** *if and only if for any n-box R whose vertices lie in $\mathcal{D}$ we have that $V_F(R) = 0$.*

Using Aczél and Dhombres [1], we give a characterization of modular functions, and a useful Lemma.

Lemma 2.12. *Let $F : \mathcal{D} \to R$ be a function where $\mathcal{D} \subset R^n$ for some $n \geq 2$. Then F is modular if and only if there exist n functions $G_i : R^{n-1} \to R$ such that for every $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle \in \mathcal{D}$*

$$F(\mathbf{x}) = G_1(x_2, x_3, \ldots, x_n) + \cdots + G_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) + \cdots + G_n(x_1, \ldots, x_{n-1}). \tag{2.56}$$

Proof. Let $R = \times_{i=1}^{n}[u_i, v_i]$ be an n-box and let $\underline{c} = \langle c_1, c_2 \ldots, c_n \rangle$ be a vertex of R, define $c_1^* = v_1$ if $c_1 = u_1$ or $c_1^* = u_1$ if $c_1 = v_1$. Then $\underline{c^*} = \langle c_1^*, c_2, \ldots, c_n \rangle$ is another vertex of R, and using equation (1.8), $\text{sgn}(\underline{c})G_1(\underline{c}) + \text{sgn}(\underline{c^*})G_1(\underline{c^*}) = 0$. Repeating the argument for the i^{th} coordinate of $\underline{c}$ respectively, in the remaining $n-1$ functions we have the result. □

The importance of Lemma 2.12 is that the functions $G_1, \ldots, G_n$ in equation (2.56) are completely arbitrary. For example if $n = 3$ and we define $F(x, y, z) = G_1(x, y) + G_2(x, z) + G_3(y, z) + H_1(x) + H_2(y) + H_3(z) + K$ where G_1, G_2, G_3, H_1, H_2 and H_3 are arbitrary functions and K is a constant, then F is modular.

Lemma 2.13. *Let $n \geq 2$, let $R = \times_{i=1}^{n}[u_i, v_i] \subset \mathbf{I}^n$ be an n-box, let $D : R \to R$ be an n-increasing function and let $E : R \to R$ be a modular function. If we define $F : R \to R$ by*

$$F(\underline{\mathbf{x}}) = D(\underline{\mathbf{x}}) + E(\underline{\mathbf{x}}). \tag{2.57}$$

Then F is an n-increasing function.

An useful result about n-increasing functions is the following

Lemma 2.14. *Let $n \geq 2$ and let $R = \times_{i=1}^{n}[u_i, v_i] \subset \mathbf{I}^n$ be a non trivial n-box, let $C : \mathbf{I}^n \to \mathbf{I}$ and $D : \mathbf{I}^n \to \mathbf{I}$ be two n-copulas. Let $\lambda = V_C(R)$ and assume that $\lambda > 0$. Define $E : R \to [0, \lambda]$ by*

$$E(\underline{\mathbf{x}}) = \lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \ldots, \frac{V_C(R_{x_n})}{\lambda}\right) \quad \text{for every} \quad \underline{\mathbf{x}} \in R, \tag{2.58}$$

where for every $i \in \{1, \ldots, n\}$, $u_i \leq x_i \leq v_i$ and

$$R_{x_i} = [u_1, v_1] \times \cdots \times [u_{i-1}, v_{i-1}] \times [u_i, x_i] \times [u_{i+1}, v_{i+1}] \times \cdots \times [u_n, v_n]. \tag{2.59}$$

Then E is an n-increasing function on R.
Besides, if $\underline{\mathbf{x}} = \langle v_1, \ldots, v_{j-1}, x_j, v_{j+1}, \ldots, v_n\rangle$, for some $j \in \{1, \ldots, n\}$ and $u_j \leq x_j < v_j$, then $E(\underline{\mathbf{x}}) = V_C(R_{x_j})$, in particular $E(v_1, v_2, \ldots, v_n) = \lambda$.

Proof. Let $n \geq 2$ and let $R = \times_{i=1}^{n}[u_i, v_i] \subset \mathbf{I}^n$ be a non trivial n-box, let $C : \mathbf{I}^n \to \mathbf{I}$ and $D : \mathbf{I}^n \to \mathbf{I}$ be two n-copulas, such that $\lambda = V_C(R) > 0$. Define the function E as in equation (2.58), first we will observe that E takes values in $[0, \lambda] \subset \mathbf{I}$. Observe that from equation (2.59), for every $\underline{\mathbf{x}} \in R$ and for every $i \in \{1, \ldots, n\}$, R_{x_i} is an n-box and $R_{x_i} \subset R$, then $0 \leq V_C(R_{x_i}) \leq V_C(R) = \lambda$. So, for every $\underline{\mathbf{x}} \in R$,

$$0 \leq \frac{V_C(R_{x_i})}{\lambda} \leq 1 \quad \text{for every} \quad i \in \{1, \ldots, n\}. \tag{2.60}$$

Since D is an n-copula and $0 < \lambda \leq 1$, then E in equation (2.58) takes values on $[0, \lambda] \subset \mathbf{I}$. Assume that $\underline{\mathbf{x}} \in R$ is such that there exists $i \in \{1, \ldots, n\}$ with $x_i = u_i$, observe that in this case R_{u_i} is a trivial n-box, in fact,

$$R_{u_i} = [u_1, v_1] \times \cdots \times [u_{i-1}, v_{i-1}] \times [u_i, u_i] \times [u_{i+1}, v_{i+1}] \times \cdots \times [u_n, v_n],$$

and according to Definition 1.8, R_{u_i} is a face of R. If $\underline{\mathbf{z}} = \langle z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots z_n\rangle$ is a vertex of R_{u_i}, then $z_i = u_i$ and $\underline{\mathbf{z}}$ appears twice when we evaluate $V_C(R_{u_i})$, first appears with positive sign and then appears with negative sign, or vice versa. Therefore, $V_C(R_{u_i}) = 0$. Since, D is an n-copula, then

$$D\left(\frac{V_C(R_{x_1})}{\lambda}, \ldots, \frac{V_C(R_{x_{i-1}})}{\lambda}, \frac{V_C(R_{u_i})}{\lambda}, \frac{V_C(R_{x_{i+1}})}{\lambda}, \ldots, \frac{V_C(R_{x_n})}{\lambda}\right) = 0.$$

Therefore,

$$E(\underline{\mathbf{x}}) = 0 \text{ if there exists } i \in \{1, \ldots, n\}, \text{ such that } x_i = u_i. \tag{2.61}$$

Now, assume that $\underline{\mathbf{x}} \in R$ is such that there exists $i \in \{1, \ldots, n\}$ with $x_i = v_i$, then using equation (2.59), $R_{v_i} = R$. Therefore,

$$\text{if } \ x_i = v_i \text{ for some } \ i \in \{1, \ldots, n\} \text{ then } \frac{V_C(R_{v_i})}{\lambda} = 1. \tag{2.62}$$

Let $i \in \{1, \ldots, n\}$ and let $u_i \leq x_{i_1} \leq x_{i_2} \leq v_i$, then $[u_1, x_{i_1}] \subset [u_i, x_{i_2}]$, and using equation (2.59), $R_{x_{i_1}} \subset R_{x_{i_2}} \subset R$. So, if for some $i \in \{1, \ldots, n\}$

$$u_i \leq x_{i_1} \leq x_{i_2} \leq v_i \quad \text{then} \quad 0 \leq \frac{V_C(R_{x_{i_1}})}{\lambda} \leq \frac{V_C(R_{x_{i_2}})}{\lambda} \leq 1. \tag{2.63}$$

Now assume that $S = \times_{i=1}^{n} [w_i, z_i] \subset R$, then for every $i \in \{1, \ldots, n\}$ we have that $u_i \leq w_i \leq z_i \leq v_i$. So, using equation (2.63) we get that

$$0 \leq \frac{V_C(R_{w_i})}{\lambda} \leq \frac{V_C(R_{z_i})}{\lambda} \leq 1 \quad \text{for every} \quad i \in \{1, \ldots, n\}. \tag{2.64}$$

Define T by

$$T = \mathop{\times}_{i=1}^{n} \left[\frac{V_C(R_{w_i})}{\lambda}, \frac{V_C(R_{z_i})}{\lambda} \right]. \tag{2.65}$$

Then by inequality (2.64) we know that T in equation (2.65) is n-box included in $\mathbf{I}^n$. So, using the definition of E in equation (2.58) and definition (2.65), we have that

$$V_E(S) = \lambda \cdot V_D(T) \geq 0, \tag{2.66}$$

because D is n-increasing. Therefore, from (2.66), E is n-increasing.

Finally, observe that if $\underline{\mathbf{x}} \in R$ is such that there exists $j \in \{1, \ldots, n\}$ such that $u_j \leq x_j < v_j$ and for every $i \in \{1, \ldots, n\} \backslash \{j\}$, $x_i = v_i$, then using equation (2.62), and the fact that D is an n-copula. we have that

$$\begin{aligned} E(\underline{\mathbf{x}}) &= \lambda D \left(\frac{V_C(R_{x_1})}{\lambda}, \ldots, \frac{V_C(R_{x_{j-1}})}{\lambda}, \frac{V_C(R_{x_j})}{\lambda}, \frac{V_C(R_{x_{j+1}})}{\lambda}, \ldots, \frac{V_C(R_{x_n})}{\lambda} \right) \\ &= \lambda D \left(1, \ldots, 1, \frac{V_C(R_{x_j})}{\lambda}, 1, \ldots, 1 \right) \\ &= V_C(R_{x_j}). \end{aligned} \tag{2.67}$$

Observe that if also $x_j = v_j$ then $R_{x_j} = R$ and $E(v_1, v_2, \ldots, v_n) = \lambda$. □

The second result we need to prove Theorem 1.35 is:

Lemma 2.15. *Let $n = 2$ and let $R = \times_{i=1}^{2}[u_i, v_i] \subset [0,1]^2$ be a non trivial n-box, let $C : \mathbf{I}^2 \to \mathbf{I}$ and $D : \mathbf{I}^2 \to \mathbf{I}$ be two copulas. Let $\lambda = V_C(R)$ and assume that $\lambda > 0$. Define $E : R \to [0, \lambda]$ by*

$$E(x_1, x_2) = \lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}\right) \quad \text{for every} \quad \langle x_1, x_2\rangle \in R, \tag{2.68}$$

where

$$R_{x_1} = [u_1, x_1] \times [u_2, v_2] \quad \text{and} \quad R_{x_2} = [u_1, v_1] \times [u_2, x_2]. \tag{2.69}$$

Then there exist two functions $f : [u_1, v_1] \to R$ and $g : [u_2, v_2] \to R$ and a constant K_0, such that if we define $Q : \mathbf{I}^2 \to R$ by

$$Q(x_1, x_2) = \begin{cases} E(x_1, x_2) + f(x_1) + g(x_2) + K_0 & \text{if} \quad \langle x_1, x_2\rangle \in R \\ C(x_1, x_2) & \text{otherwise.} \end{cases}$$

Then Q is a copula.

Proof. Let $n = 2$ and let $R = \times_{i=1}^{2}[u_i, v_i] \subset \mathbf{I}^2$ be a non trivial n-box, let $C : \mathbf{I}^2 \to \mathbf{I}$ and $D : \mathbf{I}^2 \to \mathbf{I}$ be two copulas. Let $\lambda = V_C(R)$ and assume that $\lambda > 0$. Define E as in equation (2.68), by Lemma 2.14 we know that $E : R \to [0, \lambda]$ and that E is 2-increasing. Now let $f : [u_1, v_1] \to \mathbf{R}$ and $g : [u_2, v_2] \to \mathbf{R}$ be two arbitrary functions and let K_0 be an arbitrary constant. If we define $F : R \to \mathbf{R}$ by

$$F(x_1, x_2) = E(x_1, x_2) + f(x_1) + g(x_2) + K_0 \quad \text{for every} \quad \langle x_1, x_2\rangle \in R. \tag{2.70}$$

We know from Lemma 2.12 and Lemma 2.13 that F is a 2-increasing function on R.
Observe that ∂R is given by

$$\begin{aligned} \partial R \;=\; & \{\langle x_1, x_2\rangle \in R \mid x_1 = u_1\} \cup \{\langle x_1, x_2\rangle \in R \mid x_2 = u_2\} \\ & \cup\{\langle x_1, x_2\rangle \in R \mid x_1 = v_1\} \cup \{\langle x_1, x_2\rangle \in R \mid x_2 = v_2\}. \end{aligned} \tag{2.71}$$

In order to apply Theorem 2.10, we need that C and F in equation (2.70) coincide on ∂R given in equation (2.71).
First, observe that $E(u_1, x_2) = 0$ for every $x_2 \in [u_2, v_2]$, as observed in equation (2.61), and for the same reason, $E(x_1, u_2) = 0$ for every $x_1 \in [u_1, v_1]$. Therefore, using equation (2.70), we want to choose, if possible, f, g and K_0 such that the following two equations hold.

$$F(u_1, x_2) = f(u_1)+g(x_2)+K_0 = C(u_1, x_2) \quad \text{and} \quad F(x_1, u_2) = f(x_1)+g(u_2)+K_0 = C(x_1, u_2), \tag{2.72}$$

for every $x_1 \in [u_1, v_1]$ and for every $x_2 \in [u_2, v_2]$. The natural proposal is:

$$f(x_1) = C(x_1, u_2) \text{ and } g(x_2) = C(u_1, x_2) \text{ for every } x_1 \in [u_1, v_1] \text{ and for every } x_2 \in [u_2, v_2]. \tag{2.73}$$

If we substitute the f and g proposed in (2.73) in equation (2.70), and we define $K_0 = -C(u_1, u_2)$ we obtain

$$F(u_1, x_2) = C(u_1, u_2) + C(u_1, x_2) + K_0 = C(u_1, x_2) \tag{2.74}$$

and

$$F(x_1, u_2) = C(x_1, u_2) + C(u_1, u_2) + K_0 = C(x_1, u_2), \tag{2.75}$$

and (2.72) holds, that is, the proposal in equation (2.73) allows that $C = F$ on $\{\langle x_1, x_2\rangle \in R \mid x_1 = u_1\} \cup \{\langle x_1, x_2\rangle \in R \mid x_2 = u_2\}$ of equation (2.71).

Finally, we have to see that proposal in equation (2.73) allows that $C = F$ on $\{\langle x_1, x_2\rangle \in R \mid x_1 = v_1\} \cup \{\langle x_1, x_2\rangle \in R \mid x_2 = v_2\}$.

Let $\underline{\mathbf{x}} = \langle v_1, x_2\rangle$ where $u_2 \leq x_2 \leq v_2$, then we have that

$$\begin{aligned} F(v_1, x_2) &= E(v_1, x_2) + f(v_1) + g(x_2) + K_0 \\ &= V_C(R_{x_2}) + C(v_1, u_2) + C(u_1, x_2) - C(u_1, u_2) \\ &= C(v_1, x_2) - C(v_1, u_2) - C(u_1, x_2) + C(u_1, u_2) \\ &\quad + C(v_1, u_2) + C(u_1, x_2) - C(u_1, u_2) \\ &= C(v_1, x_2). \end{aligned} \tag{2.76}$$

Finally, if $\underline{\mathbf{x}} = \langle x_1, v_2\rangle$ where $u_1 \leq x_1 \leq v_1$, and using the same equations as above,

$$\begin{aligned} F(x_1, v_2) &= E(x_1, v_2) + f(x_1) + g(v_2) + K_0 \\ &= V_C(R_{x_1}) + C(x_1, u_2) + C(u_1, v_2) - C(u_1, u_2) \\ &= C(x_1, v_2) - C(x_1, u_2) - C(u_1, v_2) + C(u_1, u_2) \\ &\quad + C(x_1, u_2) + C(u_1, v_2) - C(u_1, u_2) \\ &= C(x_1, v_2). \end{aligned} \tag{2.77}$$

Therefore from equations (2.76) and (2.77), the proposal in equation (2.73) allows that $F = C$ on ∂R. So, Theorem 2.10 can be applied, and Q given in equation (2.70) is a copula. □

Remark 2.16. Observe that if we use the notation in Durante *et al* (2009), and using equation (2.73) and the definition of K_0 in the previous Lemma,

$$f(x_1) = C(x_1, u_2) = h^C_{u_2}(x_1), g(x_2) = C(u_1, x_2) = v^C_{u_1}(x_2) \text{ and } K_0 = -C(u_1, u_2) = -h^C_{u_2}(u_1).$$

Now we can prove Theorem 1.35.

Proof. (Theorem 1.35) Let C be a copula, let $\{C_j\}_{j\in\mathcal{J}}$ be a family of copulas and let $\{R_j = [u^j_1, v^j_1] \times [u^j_2, v^j_2]\}_{j\in\mathcal{J}}$ be a family of 2-boxes, in this case rectangles in $[0,1]^2$, such that $R_j \cap R_k \subset \partial R_j \cap \partial R_k$ for every $j, k \in \mathcal{J}$ with $j \neq k$. Define for every $j \in \mathcal{J}$, $\lambda_j = V_C(R_j)$, and for every $x \in [u^j_1, v^j_1]$ and for every $y \in [u^j_2, v^j_2]$, $R_{j,x} = [u^j_1, x] \times [u^j_2, v^j_2]$ and $R_{j,y} = [u^j_1, v^j_1] \times [u^j_2, y]$. Let $\tilde{C} : [0,1]^2 \to [0,1]$ defined by

$$\tilde{C}(x,y) = \begin{cases} \lambda_j C_j\left(\frac{V_C(R_{j,x})}{\lambda_i}, \frac{V_C(R_{j,y})}{\lambda_j}\right) + \varphi^C_j(x,y) & \text{if} \quad (x,y) \in R_j \text{ and } \lambda_j > 0, \\ C(x,y), & \quad\quad \text{otherwise,} \end{cases} \tag{2.78}$$

where $\varphi^C_j(x,y) = h^C_{u^j_2}(x) + v^C_{u^j_1}(y) - h^C_{u^j_2}(u^j_1)$.

First we observe that we can assume that every rectangle R_j is non trivial, because otherwise $V_C(R_j) = 0$, and from equation (2.78) then $\tilde{C} = C$ on R_j. We can also assume that for every $j \in \mathcal{J}$, $V_C(R_j) = \lambda_j > 0$, because if $V_C(R_j) = 0$ then from equation (2.78) we have again that $\tilde{C} = C$ on R_j. Therefore, we redefine $\mathcal{J}$ to be the set of indices such that R_j is non trivial and $V_C(R_j) > 0$. In that case, it is clear that $\mathcal{J}$ is at most a countable set, because we are assuming that $\mathring{R}_j \cap \mathring{R}_k = \emptyset$, for every $j, k \in \mathcal{J}$ with $j \neq k$, then $\sum_{j\in\mathcal{J}} V_C(R_j) \leq 1$, which forces $\mathcal{J}$ to be at most a countable set. So, $\mathcal{J}$ can be written as $\mathcal{J} = \{j_1, j_2, \ldots\}$.

Now, we will proceed by induction. Let $k = 1$ and define $R := R_{j_1}$ and $D := C_{j_1}$ in Lemma 2.14, then we know that $\lambda_{j_1} = V_C(R_{j_1}) > 0$, and that $Q_{j_1} := Q$ defined in equation (2.70) is a copula. Here we observe, that Q_{j_1} may be different from C only in $\mathring{R}_{j_1}$.

Now, let $k = 2$ and define $C := Q_{j_1}$, $R := R_{j_2}$ and $D := C_{j_2}$ in Lemma 2.14, then we know that $\lambda_{j_2} = V_C(R_{j_2}) > 0$, and that $Q_{j_2} := Q$ defined in equation (2.70) is a copula. Here we observe, that Q_{j_2} may be different from C only in $\mathring{R}_{j_1} \cup \mathring{R}_{j_2}$, where $\mathring{R}_{j_1} \cap \mathring{R}_{j_2} = \emptyset$.

Inductively, let $k > 2$ and define $C := Q_{j_{k-1}}$, $R := R_{j_k}$ and $D := C_{j_k}$ in Lemma 2.14, then we know that $\lambda_{j_k} = V_C(R_{j_k}) > 0$, and that $Q_{j_k} := Q$ defined in equation (2.70) is a copula. Here we observe, that Q_{j_k} may be different from C only in $\cup^k_{l=1} \mathring{R}_{j_l}$, where $\cap^k_{l=1} \mathring{R}_{j_l} = \emptyset$.

Of course, this procedure leads to equation (2.78), and $\tilde{C}$ is a copula. □

2.2 Multivariate Ordinal Sums of Copulas

In this section we will use our results to prove that the construction of ordinal sums can be extended to $n \geq 3$. This result was first proved in Mesiar and Sempi [55]. We start with a general Proposition.

Proposition 2.17. *Let C_1 be an n-copula for some $n \geq 2$, and let $0 \leq a_1 < b_1 \leq 1$, define $R = \times_{i=1}^{n}[a_1, b_1] = [a_1, b_1]^n$ an n-box in $\mathbf{I}^n$. Define $C : \mathbf{I}^n \to \mathbf{I}$ by*

$$C(\underline{\mathbf{x}}) = \begin{cases} a_1 + (b_1 - a_1)C_1\left(\frac{\min\{x_1,b_1\}-a_1}{b_1-a_1}, \ldots, \frac{\min\{x_n,b_1\}-a_1}{b_1-a_1}\right) & \text{if } \min\{x_1, \ldots, x_n\} \in [a_1, b_1], \\ \min\{x_1, \ldots, x_n\} & \text{elsewhere.} \end{cases} \tag{2.79}$$

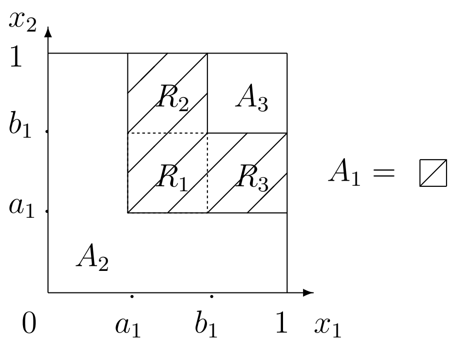
Then C is an n-copula.

Proof. We will proceed by induction. Let $n = 2$, and let $0 \leq a_1 < b_1 \leq 1$. Define C as in equation (2.79). Let $A_1 = \{\langle x_1, x_2\rangle \in \mathbf{I}^2 \mid \min\{x_1, x_2\} \in [a_1, b_1]\}$, then it is clear that

$$A_1^c = \{\langle x_1, x_2\rangle \in \mathbf{I}^2 \mid x_1 < a_1 \text{ or } x_2 < a_1\} \cup \{\langle x_1, x_2\rangle \in \mathbf{I}^2 \mid x_1 > b_1 \text{ and } x_2 > b_1\}$$
$$= A_2 \cup A_3.$$

Observe that A_1 is the union of the $3 = 2^n - 1$ rectangles $R_1 = [a_1, b_1]^2$, $R_2 = [a_1, b_1] \times [b_1, 1]$ and $R_3 = [b_1, 1] \times [a_1, b_1]$, see Figure 1.

Figure 1.- Regions A_1, A_2 and A_3



Define $D : A_1 \to \mathbf{R}$ by

$$D(x_1, x_2) = a_1 + (b_1 - a_1)C_1\left(\frac{\min\{x_1, b_1\} - a_1}{b_1 - a_1}, \frac{\min\{x_2, b_1\} - a_1}{b_1 - a_1}\right). \tag{2.80}$$

If $\langle x_1, x_2\rangle \in R_2$ then $D(x_1,x_2) = a_1+(b_1-a_1)C_1((x_1-a_1)/(b_1-a_1),1) = a_1+(x_1-a_1) = x_1 = \min\{x_1,x_2\}$. Similarly, if $\langle x_1, x_2\rangle \in R_3$ then $D(x_1,x_2) = x_2 = \min\{x_1,x_2\}$. Using Theorem 2.10 we have to see that D is 2-increasing on R_1 and that D and $M(x_1,x_2) = \min\{x_1,x_2\}$ coincide on ∂R_1. We know that C_1 is a 2-copula, so, C_1 is 2-increasing. Define a function $h : R_1 \to \mathbf{I}^2$ by:

$$h(x_1,x_2) = \left(\frac{x_1-a_1}{b_1-a_1}, \frac{x_2-a_1}{b_1-a_1}\right). \tag{2.81}$$

Then it is clear that h is a bijection which takes R_1 onto $\mathbf{I}^2$, and also h is increasing in each coordinate. Let $S = [x_{1,1}, x_{1,2}] \times [x_{2,1}, x_{2,2}]$ where $a_1 \le x_{i,1} \le x_{i,2} \le b_1$ for $i = 1,2$. Then S is a rectangle included in R_1, and the function h in equation (2.81), takes S onto

$$h[S] = \left[\frac{x_{1,1}-a_1}{b_1-a_1}, \frac{x_{1,2}-a_1}{b_1-a_1}\right] \times \left[\frac{x_{2,1}-a_1}{b_1-a_1}, \frac{x_{2,2}-a_1}{b_1-a_1}\right],$$

and using Lemma 2.14 with $E(x_1,x_2) = a_1$, we have from equation (2.80) that

$$V_D(S) = (b_1-a_1)V_{C_1}(h[S]) \ge 0. \tag{2.82}$$

Therefore, from equation (2.82), D is 2-increasing on R_1. Now, we prove that D coincides with M on ∂R_1. Let $\underline{\mathbf{x}} = (x_1,a_1)$ or $\underline{\mathbf{x}} = (a_1,x_2)$, where $a_1 \le x_1, x_2 \le b_1$. Then, since C_1 is a 2-copula

$$D(x_1,a_1) = a_1 + (b_1-a_1)C_1\left(\frac{x_1-a_1}{b_1-a_1}, \frac{a_1-a_1}{b_1-a_1}\right) = a_1 = M(x_1,a_1) \tag{2.83}$$

and

$$D(a_1,x_2) = a_1 + (b_1-a_1)C_1\left(\frac{a_1-a_1}{b_1-a_1}, \frac{x_2-a_1}{b_1-a_1}\right) = a_1 = M(a_1,x_2). \tag{2.84}$$

Finally, if $\underline{\mathbf{x}} = (x_1,b_1)$ or $\underline{\mathbf{x}} = (b_1,x_2)$, where $a_1 \le x_1, x_2 \le b_1$. Then, since C_1 is a 2-copula

$$D(x_1,b_1) = a_1 + (b_1-a_1)C_1\left(\frac{x_1-a_1}{b_1-a_1}, \frac{b_1-a_1}{b_1-a_1}\right) = a_1 + (b_1-a_1)\frac{x_1-a_1}{b_1-a_1} = x_1 = M(x_1,b_1) \tag{2.85}$$

and

$$D(b_1,x_2) = a_1 + (b_1-a_1)C_1\left(\frac{b_1-a_1}{b_1-a_1}, \frac{x_2-a_1}{b_1-a_1}\right) = a_1 + (b_1-a_1)\frac{x_2-a_1}{b_1-a_1} = x_2 = M(b_1,x_2). \tag{2.86}$$

From equations (2.83), (2.84), (2.85) and (2.86) D and M coincide on ∂R_1 and C in equation (2.79) is a 2-copula according to Theorem 2.10.

Let $n > 2$ and let C_1 be an n-copula, let $R_1 = [a_1, b_1]^n$ be an n-box in $\mathbf{I}^n$ and let C be defined as in equation (2.79). Define $D : A_1 \to \mathbf{R}$, where $A_1 = \{\langle x_1, \ldots, x_n\rangle \in \mathbf{I}^n \mid \min\{x_1, \ldots, x_n\} \in [a_1, b_1]\}$, then it is clear that

$$\begin{aligned} A_1^c &= \{\langle x_1, \ldots, x_n\rangle \in \mathbf{I}^n \mid \text{there exists } i \in \{1, \ldots, n\} \text{ such that } x_i < a_1\} \\ &\quad \cup \{\langle x_1, \ldots, x_n\rangle \in \mathbf{I}^n \mid x_i > b_1 \text{ for every } i \in \{1, \ldots, n\}\} \\ &= A_2 \cup A_3. \end{aligned} \tag{2.87}$$

We will observe that in this case A_1 is the union of $2^n - 1$ n-boxes with disjoint interiors, which include R_1. Let $I_{1,i} = [a_1, b_1]$ and $I_{2,i} = [b_1, 1]$ for $i \in \{1, \ldots, n\}$ then

$$[a_1, 1]^n = ([a_1, b_1] \cup [b_1, 1])^n = \cup_{\langle j_1,\ldots,j_n\rangle \in \{1,2\}^n} \bigtimes_{i=1}^{n} I_{j_i,i}.$$

Therefore,

$$\begin{aligned} A_1 &= [a_1, 1]^n \backslash [b_1, 1]^n \\ &= \cup_{\langle j_1,\ldots,j_n\rangle \in \{1,2\}^n} \bigtimes_{i=1}^{n} I_{j_i,i} \backslash \bigtimes_{i=1}^{n} I_{2,i}, \end{aligned}$$

which is a union of $2^n - 1$ n- boxes with disjoint interiors. Define for every $\langle x_1, \ldots, x_n\rangle \in A_1$

$$D(x_1, \ldots, x_n) = a_1 + (b_1 - a_1)C_1\left(\frac{\min\{x_1, b_1\} - a_1}{b_1 - a_1}, \ldots, \frac{\min\{x_n, b_1\} - a_1}{b_1 - a_1}\right). \tag{2.88}$$

Using the same ideas as above and the fact that C_1 is n-increasing, it is not difficult to see that D is n-increasing on A_1. So, using Theorem 2.10 and the note below it, we only have to see that $M(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$ and D coincide on $\partial A_1 \cap \partial(A_2 \cup A_3)$.

Using equation (2.87), if $\langle x_1, \ldots, x_n\rangle \in \partial A_1 \cap \partial(A_2 \cup A_3)$, then there exists $i \in \{1, \ldots, n\}$ such that $x_i = a_1$ and for every $j \in \{1, \ldots, n\}\backslash\{i\}$, $x_j \in [a_1, 1]$, or there exists $i \in \{1, \ldots, n\}$ such that $x_i = b_1$ and for every $j \in \{1, \ldots, n\}\backslash\{i\}$, $x_j \in [b_1, 1]$. In the first case using equation (2.88) and the boundary conditions of C_1, we have that

$$D(x_1, \ldots, x_n) = a_1 + (b_1 - a_1) \cdot 0 = a_1 = \min\{x_1, \ldots, x_n\},$$

and in the second case

$$D(x_1, \ldots, x_n) = a_1 + (b_1 - a_1)C(1, \ldots, 1, \frac{b_1 - a_1}{b_1 - a_1}, 1 \ldots, 1) = b_1 = \min\{x_1, \ldots, x_n\}.$$

Therefore, M and D coincide on $\partial A_1 \cap \partial(A_2 \cup A_3)$ and C defined in equation (2.79) is an n-copula. □

Remark 2.18. In the previous Proposition we define $R = [a_1, b_1]^n$. Let us denote by $\lambda = V_M(R)$, where $M(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$, we will see that $\lambda = b_1 - a_1$. We know that using formula (1.8), we have that

$$V_M(R) = \sum_{\underline{\mathbf{c}} \in \mathrm{Vert}(R)} \mathrm{sgn}(\underline{\mathbf{c}}) M(\underline{\mathbf{c}}). \tag{2.89}$$

We observe that if $\underline{\mathbf{c}} = \langle c_1, \ldots, c_n \rangle \in \mathrm{Vert}(R)$, then for every $i \in \{1, \ldots, n\}$ $c_i = a_1$ or $c_i = b_1$. So, if there exists $i \in \{1, \ldots, n\}$ such that $c_i = a_1$ then $M(\underline{\mathbf{c}}) = a_1$, and the only vertex of R such that $M(\underline{\mathbf{c}}) = b_1$ is $\underline{\mathbf{c}} = \langle b_1, b_1, \ldots, b_1 \rangle =: \underline{\mathbf{b_1}}$. Then using (2.88), we have that

$$\lambda = V_M(R) = b_1 + a_1 \sum_{\underline{\mathbf{c}} \in \mathrm{Vert}(R) \setminus \{\underline{\mathbf{b_1}}\}} \mathrm{sgn}(\underline{\mathbf{c}}). \tag{2.90}$$

Observe that $\sum_{\underline{\mathbf{c}} \in \mathrm{Vert}(R)} \mathrm{sgn}(\underline{\mathbf{c}}) = 0$, this follows using the binomial expansion of $0 = ((-1) + 1)^n$. But, in this case

$$\begin{aligned} 0 &= ((-1) + 1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} \\ &= 1 + \sum_{k=1}^{n} \binom{n}{k} (-1)^k (1)^{n-k}. \end{aligned}$$

Therefore, $\sum_{k=1}^{n} \binom{n}{k} (-1)^k (1)^{n-k} = -1$, and using (2.90) we get that $\lambda = V_M(R) = b_1 - a_1$.

Remark 2.19. It is very important to observe that in the case $n = 2$, the construction of ordinal sums is made by modifying the copula M only on squares that have opposite vertices on the main diagonal, namely, if $R = [a,b]^2 \subset \mathbf{I}^2$ and D is a 2-copula, then the function $C : \mathbf{I}^2 \to \mathbf{I}$ given by

$$C(x_1,x_2) = \begin{cases} a + (b-a)D\left(\frac{x_1-a}{b-a}, \frac{x_2-b}{b-a}\right) & \text{if} \quad \langle x_1, x_2\rangle \in [a,b]^2 \\ M^2(x_1,x_2) = \min\{x_1,x_2\} & \text{elsewhere} \end{cases}$$

is a copula, see for example Nelsen [58]. If we try to extend directly this idea to higher dimensions the result is false, that is, if we take any $n \geq 3$, D an n-copula and $R = [a,b]^n \subset \mathbf{I}^n$ an n-box with opposite vertices on the main diagonal and we define a function $C : \mathbf{I}^n \to \mathbf{I}$ by

$$C(x_1,\ldots,x_n) = \begin{cases} a + (b-a)D\left(\frac{x_1-a}{b-a}, \cdots, \frac{x_n-b}{b-a}\right) & \text{if} \quad \langle x_1,\ldots,x_n\rangle \in [a,b]^n \\ M^n(x_1,\ldots,x_n) = \min\{x_1,\ldots,x_n\} & \text{elsewhere.} \end{cases}$$

Then C is not necessarily an n-copula. To see this, we give an easy example with $n = 3$, $D = \Pi^3$ and $R = [0,1/3]^3$. If we define C as in the above equation, we have that

$$C(x_1,x_2,x_3) = \begin{cases} \frac{1}{3}\Pi^3\left(3x_1,3x_2,3x_3\right) & \text{if} \quad \langle x_1,x_2,x_3\rangle \in [0,1/3]^3 \\ M^3(x_1,x_2,x_3) = \min\{x_1,x_2,x_3\} & \text{elsewhere.} \end{cases} \tag{2.91}$$

In this case, if we take $\langle x_1,x_2,x_3\rangle = \langle 1/3, 1/4, 1/4\rangle$, then $\langle x_1,x_2,x_3\rangle$ is a point on an upper face of the 3-box R, but

$$\frac{1}{3}\Pi^3\left(3\cdot\frac{1}{3}, 3\cdot\frac{1}{4}, 3\cdot\frac{1}{4}\right) = \frac{3}{16} \neq \frac{1}{4} = \min\left\{\frac{1}{3},\frac{1}{4},\frac{1}{4}\right\}.$$

Therefore, by Theorem 2.10, C in equation (2.91) is not a 3-copula.
By Proposition 2.17, we know that if we define

$$C_1(x_1,x_2,x_3) = \begin{cases} \frac{1}{3}\Pi^3\left(\frac{\min\{x_1,1/3\}}{1/3}, \frac{\min\{x_2,1/3\}}{1/3}, \frac{\min\{x_3,1/3\}}{1/3}\right) & \text{if } \min\{x_1,x_2,x_3\} \in [0,1/3], \\ \min\{x_1,x_2,x_3\} & \text{elsewhere.} \end{cases} \tag{2.92}$$

Then C_1 is a 3-copula. Observe that the big difference between equations (2.91) and (2.92) is that in the first line of (2.91) the region is $R = [0,1/3]^3$, and in the first line of equation (2.92) the region is $\mathbf{I}^3 \setminus (1/3,1]^3$, that is, the complement of the 3-box $(1/3,1]^3$.

Of course we can extend Proposition 2.17 to obtain the multivariate version of ordinal sums, as in Mesiar and Sempi [55]. See also the application given in the paper of Durante and Fernández-Sánchez [16].

Theorem A *Let $\{C_j\}_{j\in\mathcal{J}}$ be a family of n-copulas, let $\{[a_j,b_j]\}_{j\in\mathcal{J}}$ where $\mathcal{J}=\{1,\dots,n\}$ or $\mathcal{J}=\{1,2,\dots\}$. Assume that for every $j\in\mathcal{J}$, $0\le a_j<b_j\le 1$, and even more for every $j,j+1\in\mathcal{J}$, $b_j\le a_{j+1}$. Define $C:\mathbf{I}^n\to\mathbf{I}$ by*

$$C(\underline{\mathbf{x}})=\begin{cases} a_j+(b_j-a_j)C_j\left(\frac{\min\{x_1,b_j\}-a_j}{b_j-a_j},\dots,\frac{\min\{x_n,b_j\}-a_j}{b_j-a_j}\right) \\ \qquad\qquad\qquad\qquad \textit{if} \quad \min\{x_1,\dots,x_n\}\in[a_j,b_j]\,\textit{for } j\in\mathcal{J} \\ M^n(\underline{\mathbf{x}})=\min\{x_1,\dots,x_n\} \qquad\qquad \textit{elsewhere.}\end{cases} \tag{2.93}$$

Then C is an n-copula.

Proof. The proof of Theorem A is an easy induction that uses the same arguments that we used on the proof of Theorem 1.35 given in Chapter 1. □

2.3 A Multivariate Patchwork Construction

In this section we provide a multivariate patchwork construction of n-copulas in n-boxes by using the regions determined in multivariate ordinal sums. We will start by taking a 3-copula and a 3-box R with $\langle 1,1,1\rangle$ as one of its vertices.

Theorem 2.20. *Let C and C_1 be two 3-copulas and let $R=[u_1,1]\times[u_2,1]\times[u_3,1]$ where $0<u_i<1$ for $i\in\{1,2,3\}$ and define $\underline{\mathbf{0}}=\langle 0,0,0\rangle$. Assume that $\lambda=V_C(R)>0$, and for every $x_1\in[u_1,1]$, for every $x_2\in[u_2,1]$ and for every $x_3\in[u_3,1]$, define $R_{x_1}=[u_1,x_1]\times[u_2,1]\times[u_3,1]$, $R_{x_2}=[u_1,1]\times[u_2,x_2]\times[u_3,1]$ and $R_{x_3}=[u_1,1]\times[u_2,1]\times[u_3,x_3]$. Let $\tilde{C}:\mathbf{I}^3\to\mathbf{I}$ be defined in $\underline{\mathbf{x}}=\langle x_1,x_2,x_3\rangle$ by*

$$\tilde{C}(\underline{\mathbf{x}})=\begin{cases}\lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda},\frac{V_C(R_{x_2})}{\lambda},\frac{V_C(R_{x_3})}{\lambda}\right)+V_C([\underline{\mathbf{0}},\underline{\mathbf{x}}]\setminus[\underline{\mathbf{u}},\underline{\mathbf{x}}]) & \textit{if } \underline{\mathbf{x}}\in R\\ C(\underline{\mathbf{x}}), & \textit{otherwise,}\end{cases} \tag{2.94}$$

where $[\underline{\mathbf{a}},\underline{\mathbf{b}}]=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]$ for $\underline{\mathbf{a}}=\langle a_1,a_2,a_3\rangle,\underline{\mathbf{b}}=\langle b_1,b_2,b_3\rangle$, and $\underline{\mathbf{u}}=\langle u_1,u_2,u_3\rangle$. Then $\tilde{C}$ is a 3-copula.

Remark 2.21. The 3-box R can be written as $R=[\underline{\mathbf{u}},\underline{\mathbf{1}}]$ where $\underline{\mathbf{1}}=\langle 1,1,1\rangle$.

Proof. Let $D(\underline{\mathbf{x}})=E(\underline{\mathbf{x}})+F(\underline{\mathbf{x}})$ with $E(\underline{\mathbf{x}})=\lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda},\frac{V_C(R_{x_2})}{\lambda},\frac{V_C(R_{x_3})}{\lambda}\right)$ and $F(\underline{\mathbf{x}})=V_C([\underline{\mathbf{0}},\underline{\mathbf{x}}]\setminus[\underline{\mathbf{u}},\underline{\mathbf{x}}])$. By Lemma 2.14 E is a 3-increasing function.

To see that D is also 3-increasing, using Lemma 2.13, we just need to prove that F is a modular function. Let $\underline{\mathbf{x}} \in R$ then,

$$\begin{aligned}
F(\underline{\mathbf{x}}) &= V_C([\underline{\mathbf{0}}, \underline{\mathbf{x}}]) - V_C([\underline{\mathbf{u}}, \underline{\mathbf{x}}]) \\
&= C(\underline{\mathbf{x}}) - \sum_{\underline{\mathbf{c}} \in \mathrm{Vert}([\underline{\mathbf{u}}, \underline{\mathbf{x}}])} \mathrm{sgn}(\underline{\mathbf{c}}) C(\underline{\mathbf{c}}) \\
&= C(x_1, x_2, x_3) - \{C(x_1, x_2, x_3) \\
&\quad -C(u_1, x_2, x_3) - C(x_1, u_2, x_3) - C(x_1, x_2, u_3) \\
&\quad +C(x_1, u_2, u_3) + C(u_1, x_2, u_3) - C(u_1, u_2, x_3) \\
&\quad -C(u_1, u_2, u_3)\} \\
&= C(u_1, x_2, x_3) + C(x_1, u_2, x_3) + C(x_1, x_2, u_3) \\
&\quad -C(x_1, u_2, u_3) - C(u_1, x_2, u_3) - C(u_1, u_2, x_3) \\
&\quad +C(u_1, u_2, u_3), \qquad (2.95)
\end{aligned}$$

but equation (2.95) is a modular function by the observation just below Lemma 2.12.

Now we will prove that $\tilde{C}$ is 3-increasing. Let $\underline{\mathbf{x}}$ be a point in one of the lower faces of R. Without loss of generality let $\underline{\mathbf{x}} = \langle x_1, x_2, u_3 \rangle \in \partial R$ with $u_1 \le x_1 \le 1, u_2 \le x_2 \le 1$. Then $V_C(R_{u_3}) = 0$ and $D(\underline{\mathbf{x}}) = \lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{0}{\lambda}\right) + V_C([0, x_1] \times [0, x_2] \times [0, u_3]) - 0 = C(\underline{\mathbf{x}})$. So, $D = C$ in the lower faces of R.

Using the proof of Theorem 2.10, we can see that $V_{\tilde{C}}(S) = V_C(S \cap \{[0,1]^3 \backslash R\}) + V_D(S \cap R) \ge 0$ for any 3-box $S \subset \mathbf{I}^3$ and so $\tilde{C}$ is 3-increasing.

Finally, we prove that $\tilde{C}$ satisfies the boundary conditions of a copula.

First $\tilde{C}(x_1, x_2, x_3) = C(x_1, x_2, x_3) = 0$ if any of the $x_i = 0$ for $i \in \{1, 2, 3\}$, and $\tilde{C}(1, 1, 1) = \lambda C(1, 1, 1) + V_C([0, 1]^3) - V_C([\underline{\mathbf{u}}, \underline{\mathbf{1}}]) = \lambda + 1 - V_C(R) = 1$ by the definition of λ. Then, since $\tilde{C} = C$ in $\mathbf{I}^3 \backslash R$ we just need to see that $\tilde{C}(\underline{\mathbf{x}}) = D(\underline{\mathbf{x}}) = x_i$, for $\underline{\mathbf{x}} = \langle x_1, 1, 1 \rangle$, $\langle 1, x_2, 1 \rangle, \langle 1, 1, x_3 \rangle$ with $x_i \in [u_i, 1], i = 1, 2, 3$. By the second part of Lemma 2.14 we have $E(x_1, 1, 1) = V_C(R_{x_1})$, $E(1, x_2, 1) = V_C(R_{x_2}), E(1, 1, x_3) = V_C(R_{x_3})$ for $x_i \in [u_i, 1], i = 1, 2, 3$.

Without losing generality let us assume $\underline{\mathbf{x}} = \langle x_1, 1, 1\rangle$, where $u_1 \le x_1 \le 1$, then

$$\begin{aligned} D(\underline{\mathbf{x}}) &= E(x_1,1,1) + F(x_1,1,1) \\ &= V_C(R_{x_1}) + V_C([0,x_1]\times[0,1]\times[0,1]) - V_C([u_1,x_1]\times[u_2,1]\times[u_3,1]) \\ &= V_C(R_{x_1}) + C(x_1,1,1) - V_C(R_{x_1}) \\ &= x_1. \end{aligned} \tag{2.96}$$

Similar results as (2.96) hold if $\underline{\mathbf{x}} = \langle 1, x_2, 1\rangle$ or if $\underline{\mathbf{x}} = \langle 1,1,x_3\rangle$. On the other hand if $\underline{\mathbf{x}} = \langle x_1, 1, 1\rangle$ where $0 \le x_1 \le u_1$, then $\tilde{C}(\underline{\mathbf{x}}) = C(\underline{\mathbf{x}}) = x_1$, since C is a 3-copula. Similar results are obtained for $\underline{\mathbf{x}} = \langle 1, x_2, 1\rangle$ and $\underline{\mathbf{x}} = \langle 1,1,x_3\rangle$ when $0 \le x_2 \le u_2$ and $0 \le x_3 \le u_3$. Therefore, $\tilde{C}$ in equation (2.94) is a 3-copula. □

Remark 2.22. If we let $u_i = 0$ for some $i \in \{1,2,3\}$ in the previous Theorem the result still holds. For example if $u_1 = 0$, then $R = [0,1]\times[u_2,1]\times[u_3,1]$, and if we take $\underline{\mathbf{x}} = \langle 0, x_2, x_3\rangle$ where $u_2 \le x_2 \le 1$ and $u_3 \le x_3 \le 1$, then by definition (2.94) we have that

$$\begin{aligned} \tilde{C}(\underline{\mathbf{x}}) &= \lambda C_1\left(\frac{V_C(R_{x_1=0})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) + V_C([\underline{\mathbf{0}},\underline{\mathbf{x}}]) - V_C([\underline{\mathbf{u}},\underline{\mathbf{x}}]) \\ &= 0 + V_C([0,0]\times[0,x_2]\times[0,x_3]) - V_C([0,0]\times[u_2,x_2]\times[u_3,x_3]) \\ &= 0. \end{aligned}$$

Clearly, Theorem 2.20 can be generalized easily to higher dimensions.

Theorem 2.23. *For every $n \ge 3$ let C and C_1 be two n-copulas and let $R = [u_1,1]\times[u_2,1]\times\cdots\times[u_n,1]$ where $0 \le u_i < 1$ for $i \in \{1,\ldots,n\}$. Assume that $\lambda = V_C(R) > 0$, and for every $i \in \{1,\ldots,n\}$ and for every $x_i \in [u_i,1]$ define $R_{x_i} = [u_1,1]\times\cdots\times[u_{i-1},1]\times[u_i,x_i]\times[u_{i+1},1]\times\cdots\times[u_n,1]$. Let $(C \biguplus_{\underline{\mathbf{u}}} C_1) : \mathbf{I}^n \to \mathbf{I}$ be defined in $\underline{\mathbf{x}} = \langle x_1,\ldots,x_n\rangle$ by*

$$(C \biguplus_{\underline{\mathbf{u}}} C_1)(\underline{\mathbf{x}}) = \begin{cases} \lambda C_1\left(\frac{V_C(R_{x_1})}{\lambda},\ldots,\frac{V_C(R_{x_n})}{\lambda}\right) + V_C([\underline{\mathbf{0}},\underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}},\underline{\mathbf{x}}]) & \text{if } \underline{\mathbf{x}} \in R \\ C(\underline{\mathbf{x}}), & \text{otherwise,} \end{cases} \tag{2.97}$$

where $[\underline{\mathbf{a}},\underline{\mathbf{b}}] = [a_1,b_1]\times\cdots\times[a_n,b_n]$ for $\underline{\mathbf{a}} = \langle a_1,\ldots,a_n\rangle, \underline{\mathbf{b}} = \langle b_1,\ldots,b_n\rangle$, and $\underline{\mathbf{u}} = \langle u_1,\ldots,u_n\rangle$. Then $(C \biguplus_{\underline{\mathbf{u}}} C_1)$ is an n-copula.

Proof. It follows the same steps as the proof of Theorem 2.20 □

Remark 2.24. For every $n \geq 3$ and for every n-copula C we can obtain from equation (2.97) every n-copula C_1. Let C and C_1 arbitrary n-copulas and let $R = \mathbf{I}^n$, then $V_C(R) = 1 = \lambda$, and for every $i \in \{1, \dots, n\}$ and for every $x_i \in [0,1]$, $R_{x_i} = \mathbf{I} \times \cdots \times \mathbf{I} \times [0, x_i] \times \mathbf{I} \cdots \times \mathbf{I}$. So, $V_C(R_{x_i}) = x_i$ for every $i \in \{1, \dots, n\}$ and $\underline{\mathbf{u}} = \underline{\mathbf{0}}$. Therefore, from equation (2.97) we have that $(C \biguplus_{\underline{\mathbf{u}}} C_1)(\underline{\mathbf{x}}) = C_1(\underline{\mathbf{x}})$ for every $\underline{\mathbf{x}} \in \mathbf{I}^n$.

Using Theorem 2.23 we can construct many different new n-copulas. Observe that in the construction of the copula $(C \biguplus_{\underline{\mathbf{u}}} C_1)$ on $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}]$, given in equation (2.97), the copula C remains fixed on $\mathbf{I}^n \setminus R$, and on R we have a rescaled version of the copula C_1. Using Theorem 2.23 we have the following

Definition 2.25. *For $n \geq 3$, let $\mathcal{C}^n$ be the family of n-copulas. For every fixed $\underline{\mathbf{u}} \in [0,1)^n$, we define the function $\biguplus_{\underline{\mathbf{u}}} : \mathcal{C}^n \times \mathcal{C}^n \to \mathcal{C}^n$ via*

$$\biguplus_{\underline{\mathbf{u}}}(C, C_1) = (C \biguplus_{\underline{\mathbf{u}}} C_1). \tag{2.98}$$

Lemma 2.26. *Let $n = 3$, $\underline{\mathbf{u}} \in (0,1)^3$ and Π^3 the product 3-copula and let C_1 and C_2 be 3-copulas. Then*

$$(C_1 \biguplus_{\underline{\mathbf{u}}} C_2) = \Pi^3 \tag{2.99}$$

if and only if $C_1 = C_2 = \Pi^3$.

Proof. Let $n = 3$, $\underline{\mathbf{u}} \in (0,1)^3$ and $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}]$. First, assume that $C_1 = C_2 = \Pi^3$, then $\lambda = V_{\Pi^3}(R) = (1-u_1)(1-u_2)(1-u_3)$. Now, we will see that $V_{\Pi^3}(R_{x_i})/\lambda = (x_i - u_i)/(1-u_i)$ for every $i \in \{1,2,3\}$. Without losing generality we will assume that $i = 1$. Since $R_{x_1} = [u_1, x_1] \times [u_2, 1] \times [u_3, 1]$ then

$$\frac{V_{\Pi^3}(R_{x_1})}{\lambda} = \frac{(x_1 - u_1)(1-u_2)(1-u_3)}{(1-u_1)(1-u_2)(1-u_3)} = \frac{(x_1 - u_1)}{(1-u_1)}.$$

So, using equations (2.94), (2.95) and Definition 2.25, we have that for $\underline{\mathbf{x}} \in R$

$$\begin{aligned}
(C_1 \biguplus_{\underline{\mathbf{u}}} C_2)(\underline{\mathbf{x}}) &= \lambda \Pi^3 \left(\frac{V_{\Pi^3}(R_{x_1})}{\lambda}, \frac{V_{\Pi^3}(R_{x_2})}{\lambda}, \frac{V_{\Pi^3}(R_{x_3})}{\lambda} \right) + V_{\Pi^3}([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) \\
&= \lambda \Pi^3 \left(\frac{(x_1 - u_1)}{(1-u_1)}, \frac{(x_2 - u_2)}{(1-u_2)}, \frac{(x_3 - u_3)}{(1-u_3)} \right) + V_{\Pi^3}([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) \\
&= (x_1 - u_1)(x_2 - u_2)(x_3 - u_3) + x_1 x_2 x_3 - (x_1 - u_1)(x_2 - u_2)(x_3 - u_3) \\
&= x_1 x_2 x_3 \\
&= \Pi^3(\underline{\mathbf{x}}).
\end{aligned}$$

Conversely, assume that equation (2.99) holds for some C_1 and C_2 3-copulas, then from equation (2.94) it is clear that C_1 must be Π^3 in $\mathbf{I}^3 \setminus R$. Then $\lambda = (1-u_1)(1-u_2)(1-u_3)$ and $V_{C_1}([\underline{\mathbf{0}}, \underline{\mathbf{x}}] \setminus [\underline{\mathbf{u}}, \underline{\mathbf{x}}]) = x_1 x_2 x_3 - (x_1 - u_1)(x_2 - u_2)(x_3 - u_3)$. But, from equations (2.94) and (2.99) this implies that $C_2 = \Pi^3$. □

Of course, Lemma 2.26 also holds for $n > 3$.

Example 2.27. Let $n = 3$, let $C_1(\underline{\mathbf{x}}) = \exp\left(- \left[(-\ln(x_1))^\theta + (-\ln(x_2))^\theta + (-\ln(x_3))^\theta \right]^{1/\theta} \right)$ for some $\theta \geq 1$, which is a member of the Gumbel-Hougaard Archimedean family, see Example 4.23 in Nelsen [58], and let $C = \Pi^3$. Let $R = [1/2, 1] \times [1/2, 1] \times [3/4, 1]$, then $\lambda = V_{\Pi^3}(R) = 1/16$, and $V_{\Pi^3}(R_{x_i}) = (1/8)(x_i - 1/2)$ for $i = 1, 2$ with $x_1, x_2 \in [1/2, 1]$, and $V_{\Pi^3}(R_{x_3}) = (1/4)(x_3 - 3/4)$ with $x_3 \in [3/4, 1]$. Besides, $V_{\Pi^3}([\underline{\mathbf{0}}, \underline{\mathbf{x}}]) = x_1 x_2 x_3$ and since $\underline{\mathbf{u}} = \langle 1/2, 1/2, 3/4 \rangle$ then $V_{\Pi^3}([\underline{\mathbf{u}}, \underline{\mathbf{x}}]) = (x_1 - 1/2)(x_2 - 1/2)(x_3 - 3/4)$. Therefore, using Theorem 2.20 if we define $\tilde{C} = (C \biguplus_{\underline{\mathbf{u}}} C_1)$ then $\tilde{C}(\underline{\mathbf{x}}) =$

$$\begin{cases} \frac{1}{16} C_1(2x_1 - 1, 2x_2 - 1, 4x_3 - 3) + x_1 x_2 x_3 - (x_1 - 1/2)(x_2 - 1/2)(x_3 - 3/4) & \text{if } \underline{\mathbf{x}} \in R \\ x_1 x_2 x_3 & \text{otherwise.} \end{cases}$$

Then $\tilde{C}$ is a 3-copula, which behaves like a rescaled version of the Gumbel-Hougaard family on the upper 3-box R and on the rest is the product copula. It is also clear that $\tilde{C}$ is not an ordinal sum.

Example 2.28. We will see that even for $n = 3$, when we take a 3-box not in the upper corner, for instance, a 3-box of the form $R = [0, v_1] \times [0, v_2] \times [0, v_3] \subset I^3$, it might be difficult or impossible to patch a copula C with a rescaled version of a copula D on R. Let us take two 3-copulas C and D such that $\lambda = V_C(R) > 0$, define R_{x_i} for $x_i \in [0, v_i]$ as in equation (2.94) of Theorem 2.20, for every $i \in \{1, 2, 3\}$, and take

$$E(\underline{\mathbf{x}}) = \lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right),$$

which is 3-increasing by Lemma 2.14, then we can find functions $F_{1,2}(x_1, x_2)$, $F_{1,3}(x_1, x_3)$, $F_{2,3}(x_2, x_3)$ from $\mathbf{I}^2$ into R, and functions $H_1(x_1)$, $H_2(x_2)$ and $H_2(x_2)$ from $\mathbf{I}$ into R, such that if we define

$$\varphi(\underline{\mathbf{x}}) = F_{1,2}(x_1, x_2) + F_{1,3}(x_1, x_3) + F_{2,3}(x_2, x_3) + H_1(x_1) + H_2(x_2) + H_3(x_3)$$

for every $\underline{\mathbf{x}} \in R$, then the function $Q : \mathbf{I}^3 \to \mathbf{I}$ defined by

$$Q(\underline{\mathbf{x}}) = \begin{cases} E(\underline{\mathbf{x}}) + \varphi(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in R \\ C(\underline{\mathbf{x}}) & \text{if} \quad \underline{\mathbf{x}} \in \mathbf{I}^3 \backslash R \end{cases}$$

is continuous. However Q is not in general a 3-copula.

We know that the function φ defined above is a modular function by Lemma 2.12, and by Lemma 2.13 the first row in the definition of Q is also 3-increasing. Besides, since C and D are 3-copulas we also know that $E(\underline{\mathbf{x}})$ is continuous. The idea now is try to find an appropriate continuous function φ, such that it makes Q continuous.

First we observe that ∂R is given by the union of $\{\underline{\mathbf{x}} \in R \,|\, \text{there exists } i \in \{1, 2, 3\} \text{ such that } x_i = 0\}$ with $\{\underline{\mathbf{x}} \in R \,|\, \text{there exists } i \in \{1, 2, 3\} \text{ such that } x_i = v_i\}$. Since C is continuous we only have to find φ such it makes coincide the first row with the second row of Q on the upper faces of R, that is, on

$$\{\underline{\mathbf{x}} \in R \,|\, \text{there exists } i \in \{1, 2, 3\} \text{ such that } x_i = v_i\}.$$

In this case we want to find adequate functions in the definition of φ such that the following three conditions hold

$$Q(v_1, x_2, x_3) = C(v_1, x_2, x_3) \quad \text{for every} \quad \langle x_2, x_3\rangle \in [0, v_2] \times [0, v_3],$$

$$Q(x_1, v_2, x_3) = C(x_1, v_2, x_3) \quad \text{for every} \quad \langle x_1, x_3\rangle \in [0, v_1] \times [0, v_3],$$

and

$$Q(x_1, x_2, v_3) = C(x_1, x_2, v_3) \quad \text{for every} \quad \langle x_1, x_2\rangle \in [0, v_1] \times [0, v_2].$$

We define for $\langle x_1, x_2\rangle \in [0, v_1] \times [0, v_2]$

$$F_{1,2}(x_1, x_2) = -\lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, \frac{V_C(R_{x_2})}{\lambda}, 1\right) + C(x_1, x_2, v_3),$$

for $\langle x_1, x_3\rangle \in [0, v_1] \times [0, v_3]$

$$F_{1,3}(x_1, x_3) = -\lambda D\left(\frac{V_C(R_{x_1})}{\lambda}, 1, \frac{V_C(R_{x_3})}{\lambda}\right) + C(x_1, v_2, x_3),$$

and for $\langle x_2, x_3\rangle \in [0, v_2] \times [0, v_3]$

$$F_{2,3}(x_2, x_3) = -\lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) + C(v_1, x_2, x_3).$$

We also define for $x_1 \in [0, v_1], x_2 \in [0, v_2]$ and $x_3 \in [0, v_3]$

$$H_1(x_1) = V_C(R_{x_1}) - C(x_1, v_2, v_3),\ \ H_2(x_2) = V_C(R_{x_2}) - C(v_1, x_2, v_3)$$

and

$$H_3(x_3) = V_C(R_{x_3}) - C(v_1, v_2, x_3).$$

Then if we take $\langle v_1, x_2, x_3\rangle \in R$, we observe that $R_{v_1} = R$, $\lambda = V_C(R) = C(v_1, v_2, v_3)$, and using the boundary properties of D, we have that

$$\begin{aligned}
Q(v_1, x_2, x_3) \quad &= \quad \lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) \\
&\quad -\lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, 1\right) + C(v_1, x_2, v_3) \\
&\quad -\lambda D\left(1, 1, \frac{V_C(R_{x_3})}{\lambda}\right) + C(v_1, v_2, x_3)
\end{aligned}$$

$$\begin{aligned}
&-\lambda D\left(1, \frac{V_C(R_{x_2})}{\lambda}, \frac{V_C(R_{x_3})}{\lambda}\right) + C(v_1, x_2, x_3) \\
&+V_C(R_{v_1}) - C(v_1, v_2, v_3) + V_C(R_{x_2}) - C(v_1, x_2, v_3) + V_C(R_{x_3}) - C(v_1, v_2, x_3) \\
= \quad &-V_C(R_{x_2}) - V_C(R_{x_3}) + C(v_1, x_2, x_3) + \lambda - \lambda + V_C(R_{x_2}) + V_C(R_{x_3}) \\
= \quad &C(v_1, x_2, x_3).
\end{aligned}$$

Therefore, the first condition above holds. Analogously it is easy to see that the other two conditions also hold with the same definition of φ which is clearly continuous. Hence, Q is continuous.
Now, if $\underline{\mathbf{x}} \in R$ is such that $x_i = 0$ for some $i \in \{1,2,3\}$, then it is clear that $V_C(R_{x_i}) = 0$ and since D and C are 3-copulas then $E(\underline{\mathbf{x}}) = 0 = C(\underline{\mathbf{x}})$. However, if we take $\underline{\mathbf{x}} = \langle 0, x_2, x_3\rangle$ with $x_2 \in (0, v_2]$ and $x_3 \in (0, v_3]$, then $E(\underline{\mathbf{x}}) = 0$, $F_{1,2}(0, x_2) = 0$, $F_{1,3}(0, x_3) = 0$, $F_{2,3}(x_2, x_3) = -\lambda D(1, V_C(R_{x_2})/\lambda, V_C(R_{x_3})/\lambda) + C(v_1, x_2, x_3)$, $H_1(0) = 0$, $H_2(x_2) = V_C(R_{x_2}) - C(v_1, x_2, v_3) = C(v_1, x_2, v_3) - C(v_1, x_2, v_3) = 0$ and $H_3(x_3) = V_C(R_{x_3}) - C(v_1, v_2, x_3) = C(v_1, v_2, x_3) - C(v_1, v_2, x_3) = 0$. So, $\varphi(\underline{\mathbf{x}}) = -\lambda D(1, C(v_1, x_2, v_3)/\lambda, C(v_1, v_2, x_3)/\lambda) + C(v_1, x_2, x_3)$, which in general is not zero. Therefore, Q is not a 3-copula although it is continuous.

This last example shows that it is not easy to find a modular function φ, such that the function Q satisfies being a 3-copula. The problem is to find a modular function φ such that the first and second rows in the definition of Q coincide on the lower and upper faces of R. It seems that we can make them coincide if we take only the lower faces or only the upper faces, but not both at the same time, in order to get a 3-copula. Maybe if we define a different function φ we could obtain a 3-copula, but this still remains an open problem.

2.3.1 Patchwork Construction method for dimensions higher than or equal to three

We first observe that the construction of new copulas in Theorem (2.23) is restricted to n-boxes of the form $R = [\underline{\mathbf{u}}, \underline{\mathbf{1}}] \subset \mathbf{I}^n$ where $\underline{\mathbf{u}} = \langle u_1, \ldots, u_n\rangle$ and $\underline{\mathbf{1}} = \langle 1, \ldots, 1\rangle$. The question here is, how to do a construction of a new copula when we want to modify a *base* n-copula C on an arbitrary n-box $R = [\underline{\mathbf{u}}, \underline{\mathbf{v}}] \subset \mathbf{I}^n$ by a *modifying* n-copula D?
In order to answer this question we will propose a new methodology which is based on Theorem 2.23.
Let us assume that $R = [\underline{\mathbf{u}}, \underline{\mathbf{v}}] \subset \mathbf{I}^n$ is a non trivial n-box such that for every $i \in \{1, \ldots, n\}$, $0 < u_i < v_i < 1$. Let $\mathrm{Vert}(R)$ be the set of vertices of R. We will first establish an order in this set of vertices. Let $\underline{\mathbf{c}} \in \mathrm{Vert}(R)$, define a bijection $f : \mathrm{Vert}(R) \to \{1,2\}^n$ given by $f(\underline{\mathbf{c}}) = \langle l_1, l_2, \ldots, l_n\rangle \in \{1,2\}^n$, where $l_i = 1$ if $c_i = u_i$ and $l_i = 2$ if $c_i = v_i$. Define $Q_{f(\underline{\mathbf{c}})} = \{k \in \{1, \ldots, n\} \mid f(\underline{\mathbf{c}})_k = 2\}$, where $f(\underline{\mathbf{c}})_k$ is the k^{th} coordinate of $f(\underline{\mathbf{c}})$. Now we define the composition $\varphi : \mathrm{Vert}(R) \to \{1, 2, \ldots, 2^n\}$ given by

$$\varphi(f(\underline{\mathbf{c}})) = 1 + \sum_{k \in Q_{f(\underline{\mathbf{c}})}} 2^{n-k} \quad \text{for every} \quad \underline{\mathbf{c}} \in \text{Vert}(R).$$

Of course, $Q_{f(\underline{\mathbf{c}})} = \emptyset$ if and only if $\underline{\mathbf{c}} = \langle u_1, u_2, \ldots, u_n \rangle$ and in this case $\varphi(f(u_1, u_2, \ldots, u_n)) = 1$. Also observe that if $\underline{\mathbf{c}} = \langle v_1, v_2, \ldots, v_n \rangle$, then $\varphi(f(v_1, v_2, \ldots, v_n)) = 1 + \sum_{k=1}^{n} 2^{n-k} = 1 + \sum_{j=0}^{n-1} 2^j = 2^n$. It is easy to see that φ is a bijection which establishes an order among the vertices of R, in fact, this order gives the number one to the "lowest" vertex and the number 2^n to the "highest" vertex.
Let C be an n-copula which we will call base copula, let D be an n-copula which we will call modifying copula, and let $R = [\underline{\mathbf{u}}, \underline{\mathbf{v}}] \subset \mathbf{I}^n$ be a non trivial n-box. Then inductively define

- Let $C_1 = (C \uplus_{\underline{\mathbf{u}}} D)$ where $1 = \varphi(\underline{\mathbf{u}})$.
- Given C_k for $1 \leq k < 2^n$, let $\underline{\mathbf{c}} \in \text{Vert}(R)$ such that $\varphi(\underline{\mathbf{c}}) = k + 1$ and define $C_{k+1} = (C_k \uplus_{\underline{\mathbf{c}}} C)$.
- The n-copula C_{2^n} is our target copula.

Observe that if we follow this construction, then in the first step in C_1 we introduce the rescaled version of the n-copula D on $[\underline{\mathbf{u}}, \underline{\mathbf{1}}]$, and in the remaining steps we keep unaltered C_1 at least in the semi open n-box $[\underline{\mathbf{u}}, \underline{\mathbf{v}}) = \times_{i=1}^{n} [u_i, v_i)$.

Example 2.29. Let us assume that $n = 3$ and that we want to construct a 3-copula C which has a desired behavior in each of the eight vertices of I^3. We will use the 3-box given by $R = [0, 1/2]^3$ as an auxiliary tool. We first establish a bijection $h : \text{Vert}(I^3) \to \text{Vert}(R)$, among the vertices of I^3 and the vertices of R. If $\underline{\mathbf{c}} \in \text{Vert}(I^3)$ then the coordinates of $h(\underline{\mathbf{c}})$ are given by

$$h(\underline{\mathbf{c}})_i = \begin{cases} 0 & \text{if} \quad \underline{\mathbf{c}}_i = 0 \\ 1/2 & \text{if} \quad \underline{\mathbf{c}}_i = 1, \end{cases}$$

for $i = 1, 2, 3$. Using the order established at the beginning of this subsection we have Table 1

Table 1.- Order of the vertices of I^3

$\underline{\mathbf{c}}$	$f(\underline{\mathbf{c}})$	$Q_{f(\underline{\mathbf{c}})}$	$\varphi(f(\underline{\mathbf{c}}))$
$\langle 0,0,0\rangle$	$\langle 1,1,1\rangle$	$\emptyset$	1
$\langle 0,0,1\rangle$	$\langle 1,1,2\rangle$	$\{3\}$	$1+2^{3-3}=2$
$\langle 0,1,0\rangle$	$\langle 1,2,1\rangle$	$\{2\}$	$1+2^{3-2}=3$
$\langle 0,1,1\rangle$	$\langle 1,2,2\rangle$	$\{2,3\}$	$1+2^{3-2}+2^{3-3}=4$
$\langle 1,0,0\rangle$	$\langle 2,1,1\rangle$	$\{1\}$	$1+2^{3-1}=5$
$\langle 1,0,1\rangle$	$\langle 2,1,2\rangle$	$\{1,3\}$	$1+2^{3-1}+2^{3-3}=6$
$\langle 1,1,0\rangle$	$\langle 2,2,1\rangle$	$\{1,2\}$	$1+2^{3-1}+2^{3-2}=7$
$\langle 1,1,1\rangle$	$\langle 2,2,2\rangle$	$\{1,2,3\}$	$1+2^{3-1}+2^{3-2}+2^{3-3}=8$

Of course, the order of the vertices of R is the same, that is,

$$\varphi(f(h(\underline{\mathbf{c}}))) = \varphi(f(\underline{\mathbf{c}})) \quad \text{for every} \quad \underline{\mathbf{c}} \in \mathrm{Vert}(I^3).$$

Assume that we have selected eight 3-copulas $\{C_j\}_{j=1}^{8}$, such that if $\underline{\mathbf{c}} \in \mathrm{Vert}(I^3)$ and it satifies that $\varphi(f(\underline{\mathbf{c}})) = j$, then C_j has the desired behavior near the vertex $\underline{\mathbf{c}}$, for every $j \in \{1,\ldots,8\}$. Now we proceed with the construction of a 3-copula which satisfies the desired properties using the 3-box R:

- Let $j=1$, $R^1 = \mathbf{I}^3$ and define $D_1 = C_1$ on R^1, then D_1 is exactly C_1 near $\underline{\mathbf{c}} = \underline{\mathbf{0}} \in \mathrm{Vert}(R)$, $\underline{\mathbf{0}} \in \mathrm{Vert}(I^3)$ and $\varphi(f(\underline{\mathbf{0}})) = 1$.

- Let $j=2$, $R^2 = \mathbf{I}\times\mathbf{I}\times[1/2,1]$, $\underline{\mathbf{u}}^2 = \langle 0,0,1/2\rangle \in \mathrm{Vert}(R)$, then $\varphi(f(h(\underline{\mathbf{u}}^2))) = 2 = j$. Define $D_2 = (D_1 \uplus_{\underline{\mathbf{u}}^2} C_2)$, then from equation (2.97), D_2 behaves like C_1 near $\underline{\mathbf{0}}$ and like C_2 near $\langle 0,0,1\rangle = \underline{\mathbf{c}}$, where $\varphi(f(\underline{\mathbf{c}})) = 2 = j$.

- Inductively, given D_{j-1} for $3 \le j \le 8$, let $R^j = [c_1^j,1]\times[c_2^j,1]\times[c_3^j,1]$, where $\underline{\mathbf{u}}^j = \langle c_1^j, c_2^j, c_3^j\rangle \in \mathrm{Vert}(R)$ is such that $\varphi(f(h(\underline{\mathbf{u}}^j))) = j$. Define $D_j = (D_{j-1} \uplus_{\underline{\mathbf{u}}^j} C_j)$, then from equation (2.97), D_j behaves like C_k near $\underline{\mathbf{c}}^k \in \mathrm{Vert}(I^3)$ for every $k = \{1,2,\ldots,j\}$.

- Then if we define $C := D_8$, C has the desired properties.

The last example has of course generalizations to higher dimensions, and it is of great importance because it allows to model tail dependence, see for example Joe [41] and Nelsen [58]. In fact, in Finance and Risk Theory one of the biggest problems is to model tail dependence for economic variables using copulas, see for example Embrechts *et al* [25], Cherubini *et al* [6], McNeil *et al* [51], Malevergne *et al* [49], Zhang [72], just to mention some references. The use of copulas for modeling dependence has been also used in other areas such as ecology, hidrology, medicine, etc., see for example Dorey and Joubert [14], Erdely and Díaz-Vieira [27] or Salvadori *et al* [63]. The last example provides a method for modeling the tail dependence in dimensions higher than or equal to three.

Of course this methodology also allows us to study the multidimensional versions of the horizontal and vertical sections of 2-copulas and the construction of 2-copulas with given horizontal or vertical sections, see for example see Nelsen [58], Durante *et al* [19] or Klement *et al* [43]. For $n \geq 3$, we can analyze the structure of the $n-1$ dimensional faces of an n-copula, of the form

$$C_i(\underline{\mathbf{x}}) = \{C(x_1, \ldots, x_{i-i}, a_i, x_{i+1}, \ldots, x_n) \mid x_j \in \mathbf{I} \text{ for every } j \in \{1, \ldots, n\} \backslash \{i\}\},$$

where $a_i \in \mathbf{I}$ is fixed and $i \in \{1, \ldots, n\}$; which are clearly equivalent to the horizontal and vertical sections of a 2-copula.

Example 2.30. Let us assume that $n = 3$ and $R = [0, 1/2]^3$ in Example 2.29, and let us select the first six copulas as $C_1 = \cdots = C_6 = \Pi^3$, $C_7 = M^3$ and C_8 is a Gumbel-Hougaard copula with parameter $\theta = 2$, that is,

$$C_8(\underline{\mathbf{x}}) = \exp\left\{-\left[(\ln x_1)^2 + (\ln x_2)^2 + (\ln x_3)^2\right]^{1/2}\right\}.$$

If we proceed with the order given in Table 1 and the steps of Example 2.29, we get that $D_1 = D_2 = \cdots = D_6 = \Pi^3$ by Lemma 2.26, so, we can start at the seventh step.

Let $\underline{\mathbf{u}}^7 = \langle 1/2, 1/2, 0\rangle \in \text{Vert}(R)$, $R^7 = [1/2, 1] \times [1/2, 1] \times \mathbf{I}$ and define $D_7 = (D_6 \uplus_{\underline{\mathbf{u}}^7} C_7)$, after some calculations we have that $\lambda_7 = V_{\Pi^3}(R^7) = 1/4, V_{\Pi^3}(R^7_{x_1}) = x_1/2 - 1/4, x_2/2 - 1/4, V_{\Pi^3}(R^7_{x_3}) = x_3/4, V_{\Pi^3}([\underline{\mathbf{u}}^7, \underline{\mathbf{x}}]) = x_1x_2x_3 - (1/2)(x_2x_3 + x_1x_3) + x_3/4$ and

$$D_7(\underline{\mathbf{x}}) = \begin{cases} \frac{1}{4}\min\{2x_1 - 1, 2x_2 - 1, x_3\} & \\ +\frac{1}{2}(x_2x_3 + x_1x_3) - \frac{1}{4}x_3 & \text{if } \underline{\mathbf{x}} \in R^7 \\ x_1x_2x_3 & \text{otherwise.} \end{cases}$$

For the last step we have $\underline{\mathbf{u}}^8 = \langle 1/2, 1/2, 1/2\rangle \in \text{Vert}(R)$, $R^8 = [1/2, 1] \times [1/2, 1] \times [1/2, 1]$. Define $D_8 = (D_7 \uplus_{\underline{\mathbf{u}}^8} C_8)$, then we get $\lambda_8 = V_{D_7}(R^8) = 1/8, V_{D_7}(R^8_{x_1}) = x_1/2 - 1/4 - C_7(2x_1 - 1, 1, 1/2)/4, V_{D_7}(R^8_{x_2}) = x_2/2 - 1/4 - C_7(1, 2x_2 - 1, 1/2)/4, V_{C_7}(R^8_{x_3}) = x_3/4 -$

$1/8, V_{D_7}([\underline{\mathbf{u}}^8, \underline{\mathbf{x}}]) = D_7(x_1, x_2, x_3) - C_7(2x_1 - 1, 2x_2 - 1, 1/2)/4 + x_3/4 - (x_2x_3 + x_1x_3)/2$ and

$$D_8(\underline{\mathbf{x}}) = \begin{cases} \frac{1}{8}C_8\left(4x_1 - 2 - 2\min\{2x_1 - 1, \frac{1}{2}\}, 4x_2 - 2 - 2\min\{2x_2 - 1, \frac{1}{2}\}, 2x_3 - 1\right) & \\ +\frac{1}{4}\min\{2x_1 - 1, 2x_2 - 1, \frac{1}{2}\} - \frac{1}{4}x_3 + \frac{1}{2}(x_2x_3 + x_1x_3) & \text{if } \underline{\mathbf{x}} \in R^8 \\ \frac{1}{4}\min\{2x_1 - 1, 2x_2 - 1, x_3\} & \\ +\frac{1}{2}(x_2x_3 + x_1x_3) - \frac{1}{4}x_3 & \text{if } \underline{\mathbf{x}} \in [1/2, 1] \times [1/2, 1] \times [0, 1/2] \\ x_1x_2x_3 & \text{otherwise.} \end{cases}$$

We can see that this last copula behaves similar to the copula M in the vertex $\langle 1, 1, 0\rangle$ and similar to a Gumbel-Hougaard copula near the vertex $\langle 1, 1, 1\rangle$.

2.4 An Alternative Patchwork Using Gluing Copulas

In Siburg and Stoimenov [66] a new methodology of constructing n-copulas is proposed. The main idea is to glue two rescaled copulas on adjacent n-boxes of $\mathbf{I}^n$ whose union is $\mathbf{I}^n$, two n-boxes R and S are adjacent if their intersection is a common face. Their main result is the following:

Theorem B *Let $n \geq 2$ and let C_1, C_2 be two n-copulas. Let $0 \leq \theta \leq 1$ and define $R^l_{i,\theta} = \mathbf{I} \times \cdots \times \mathbf{I} \times [0, \theta] \times \mathbf{I} \times \cdots \times \mathbf{I}$, where the interval $[0, \theta]$ is located on the i^{th} coordinate, for some $i \in \{1, 2, \ldots, n\}$, similarly define $R^u_{i,\theta} = \mathbf{I} \times \cdots \times \mathbf{I} \times [\theta, 1] \times \mathbf{I} \times \cdots \times \mathbf{I}$. Define for every $\underline{\mathbf{x}} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbf{I}^n$*

$$\left(C_1 \underset{x_i=\theta}{\circledast} C_2\right)(\underline{\mathbf{x}}) = \begin{cases} \theta C_1\left(x_1, \ldots, \frac{x_i}{\theta}, \ldots, x_n\right) & \text{if } \underline{\mathbf{x}} \in R^l_{i,\theta} \\ (1-\theta)C_2\left(x_1, \ldots, \frac{x_i - \theta}{1-\theta}, \ldots, x_n\right) + \theta C_1(x_1, \ldots, 1, \ldots, x_n) & \text{if } \underline{\mathbf{x}} \in R^u_{i,\theta}. \end{cases} \tag{2.100}$$

Then $C_1 \circledast_{x_i=\theta} C_2$ is an n-copula.

Proof. Observe that the first row in equation (2.100) it is simply C_1 rescaled on $R^l_{i,\theta}$ and the second row is C_2 rescaled on $R^u_{i,\theta}$ plus the value of the rescaled C_1 in $\langle x_1, \ldots, 1, \ldots x_n\rangle$. It is clear that $C_1 \circledast_{x_i=\theta} C_2$ satisfies the boundary conditions of an n-copula. Using that $R^l_{i,\theta}$ and $R^u_{i,\theta}$ are adjacent n-boxes with common face $R_{i,\theta} = \{\underline{\mathbf{x}} \in \mathbf{I}^n \mid x_i = \theta\}$, and observing that on $R_{i,\theta}$ both rows of equation (2.100) coincide, it is clear that $C_1 \circledast_{x_i=\theta} C_2$ is an n-increasing function. Therefore, $C_1 \circledast_{x_i=\theta} C_2$ is an n-copula. □

Of course, the binary operation $\circledast_{x_i=\theta}$, where $i \in \{1, \ldots, n\}$ and $\theta \in \mathbf{I}$, is defined on the family $\mathcal{C}^n$ of all n-copulas. Of course, $\circledast_{x_i=\theta}$ is not a commutative operation. In Proposition 2.20

of Siburg and Stoimenov [66], it is shown that if Π^n is the n-product copula, $i \in \{1, \ldots, n\}$ and $\theta \in \mathbf{I}$. Then

$$C_1 \underset{x_i=\theta}{\circledast} C_2 = \Pi^n \quad \text{if and only if} \quad C_1 = C_2 = \Pi^n. \tag{2.101}$$

This result follows directly from equation (2.100).

Remark 2.31. For dimension $n = 2$ the constructions proposed in Theorem 1.35 in Section 2.1 and the one given in Theorem B are quite different. Let $C = M$, $\mathcal{J} = \{1\}$, $R_1 = [0, 1/2, 1] \times \mathbf{I}$ and $C_1 = \Pi$ in Theorem 1.35. Then $\lambda_1 = V_M(R_1) = 1/2$, $R_{1,x} = [0, x] \times \mathbf{I}$, $R_{1,y} = [0, 1/2], [0, y]$ for every $x \in [0, 1/2]$ and for every $y \in [0, 1]$. So, $V_M(R_{1,x}) = x$ and $V_M(R_{1,y}) = \min\{1/2, y\}$ and $\varphi_1^M(x, y) = h_0^M(x) + v_0^M(y) - h_0^M(0) = 0$. Therefore, using equation (1.23)

$$\begin{aligned}
\tilde{C}(x, y) &= \begin{cases} \frac{1}{2}\Pi(2x, 2\min\{1/2, y\}) & \text{if} \quad \langle x, y\rangle \in [0, 1/2] \times \mathbf{I} \\ \min\{x, y\} & \text{if} \quad \langle x, y\rangle \in [1/2, 1] \times \mathbf{I}\,. \end{cases} \\
&= \begin{cases} 2x \cdot \min\{1/2, y\} & \text{if} \quad \langle x, y\rangle \in [0, 1/2] \times \mathbf{I} \\ \min\{x, y\} & \text{if} \quad \langle x, y\rangle \in [1/2, 1] \times \mathbf{I}\,. \end{cases}
\end{aligned}$$

On the other hand, if we let $C_1 = \Pi$, $C_2 = M$ and $\theta = 1/2$ in Theorem B, then using equation (2.100), we get that

$$\begin{aligned}
\left(\Pi \underset{x=1/2}{\circledast} M\right)(x, y) &= \begin{cases} \frac{1}{2}\Pi(2x, y) & \text{if} \quad \langle x, y\rangle \in [0, 1/2] \times \mathbf{I} \\ \frac{1}{2}\min\{2x - 1, y\} + \frac{1}{2}\min\{1, y\} & \text{if} \quad \langle x, y\rangle \in [1/2, 1] \times \mathbf{I}\,. \end{cases} \\
&= \begin{cases} xy & \text{if} \quad \langle x, y\rangle \in [0, 1/2] \times \mathbf{I} \\ \frac{1}{2}\min\{2x + y - 1, 2y\} & \text{if} \quad \langle x, y\rangle \in [1/2, 1] \times \mathbf{I}\,. \end{cases}
\end{aligned}$$

Of course, $\tilde{C} \neq \left(\Pi \circledast_{x=1/2} M\right)$.

Example 2.32. We will see that even for $n = 3$ the patchwork construction given in Theorem 2.20 is also different from the gluing proposal in Theorem B. In the case where Π^3 is the base copula they may coincide, but in general they are different. Let us glue Π^3 with some 3-copula C_1 in $x_1 = 1/2$, using equation (2.100) we get

$$\left(\Pi^3 \underset{x_1=1/2}{\circledast} C_1\right)(\underline{\mathbf{x}}) = \begin{cases} 1/2 \cdot \Pi^3\left(\frac{x_1}{1/2}, x_2, x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in [0, 1/2] \times \mathbf{I} \times \mathbf{I} \\ (1 - 1/2)C_1\left(\frac{x_1 - 1/2}{1 - 1/2}, x_2, x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in [1/2, 1] \times \mathbf{I} \times \mathbf{I} \\ +1/2\Pi^3(1, x_2, x_3) & \end{cases}$$

$$=\begin{cases} x_1x_2x_3 & \text{if} \quad \underline{\mathbf{x}} \in [0,1/2]\times\mathbf{I}\times\mathbf{I} \\ \frac{1}{2}C_1\left(2x_1-1,x_2,x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in [1/2,1]\times\mathbf{I}\times\mathbf{I} \\ +\frac{1}{2}x_2x_3 & \end{cases} \tag{2.102}$$

Now, using Remark 2.22, let us patch the 3-copula C_1 in $R = [1/2,1]\times\mathbf{I}\times\mathbf{I}$ with the 3-copula Π^3. In this case, $\underline{\mathbf{u}} = \langle 1/2,0,0\rangle$, $\lambda = V_{\Pi^3}(R) = 1/2$, $V_{\Pi^3}(R_{x_1}) = x_1 - \frac{1}{2}, V_{\Pi^3}(R_{x_2}) = \frac{x_2}{2}, V_{\Pi^3}(R_{x_3}) = \frac{x_3}{3}$, $V_{\Pi^3}([\underline{\mathbf{0}},\underline{\mathbf{x}}]) = \Pi^3(x_1,x_2,x_3)$ and $V_{\Pi^3}([\underline{\mathbf{u}},\underline{\mathbf{x}}]) = V_{\Pi^3}([1/2,x_1]\times[0,x_2]\times[0,x_3]) = (x_1-1/2)x_2x_3$, for every $x_1\in[1/2,1]$ and for every $x_2,x_3\in[0,1]$. Then using equation (2.94) in Theorem 2.20 we have that

$$\left(\Pi^3 \biguplus_{\underline{\mathbf{u}}} C_1\right)(\underline{\mathbf{x}}) = \begin{cases} \Pi^3\left(x_1,x_2,x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in [0,1/2]\times\mathbf{I}\times\mathbf{I} \\ 1/2C_1\left(\frac{x_1-1/2}{1/2},\frac{x_2/2}{1/2},\frac{x_3/2}{1/2}\right) + \Pi^3(x_1,x_2,x_3) & \text{if } \underline{\mathbf{x}} \in R = [1/2,1]\times\mathbf{I}\times\mathbf{I} \\ -\mathrm{V}_{\Pi^3}([1/2,x_1]\times[0,x_2]\times[0,x_3]) & \end{cases}$$

$$=\begin{cases} x_1x_2x_3 & \text{if} \quad \underline{\mathbf{x}} \in [0,1/2]\times\mathbf{I}\times\mathbf{I} \\ \frac{1}{2}C_1\left(2x_1-1,x_2,x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in R = [1/2,1]\times\mathbf{I}\times\mathbf{I} \\ +\frac{1}{2}x_2x_3 & \end{cases} \tag{2.103}$$

Then from equations (2.102) and (2.103), clearly, $(\Pi^3 \circledast_{x_1=1/2} C_1) = (\Pi^3 \biguplus_{\underline{\mathbf{u}}} C_1)$. But if we reverse the order and we glue some copula C_1, say $C_1 = \frac{M+\Pi^3}{2}$, with Π^3 in $x = 1/2$ we get from equation (2.100) that

$$\left(C_1 \underset{x_1=1/2}{\circledast} \Pi^3\right)(\underline{\mathbf{x}}) = \begin{cases} 1/2C_1\left(\frac{x_1}{1/2},x_2,x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in [0,1/2]\times\mathbf{I}\times\mathbf{I} \\ (1-1/2)\Pi^3\left(\frac{x_1-1/2}{1-1/2},x_2,x_3\right) & \text{if} \quad \underline{\mathbf{x}} \in [1/2,1]\times\mathbf{I}\times\mathbf{I} \\ +1/2C_1(1,x_2,x_3) & \end{cases}$$

$$=\begin{cases} \frac{1}{4}\min\{2x_1,x_2,x_3\} + \frac{1}{2}x_1x_2x_3 & \text{if} \quad \underline{\mathbf{x}} \in [0,1/2]\times\mathbf{I}\times\mathbf{I} \\ x_1x_2x_3 - \frac{x_2x_3}{4} + \frac{\min\{x_2,x_3\}}{4} & \text{if} \quad \underline{\mathbf{x}} \in [1/2,1]\times\mathbf{I}\times\mathbf{I}, \end{cases} \tag{2.104}$$

and using Theorem 2.20 and Remark 2.22 again, if we patch the same copula C_1 in $R = [1/2,1]\times\mathbf{I}\times\mathbf{I}$ with Π^3 we have that $\underline{\mathbf{u}} = \langle 1/2,0,0\rangle$, $\lambda = V_{C_1}(R) = C_1(1,1,1) - C_1(1/2,1,1) = 1/2$, $V_{C_1}(R_{x_1}) = C_1(x_1,1,1) - C_1(1/2,1,1) = x_1 - 1/2$, $V_{C_1}(R_{x_2}) = C_1(1,x_2,1) - C_1(1/2,x_2,1) = (3x_2 - \min\{1,2x_2\})/4$, $V_{C_1}(R_{x_3}) = C_1(1,1,x_3) - C_1(1/2,1,x_3) = (3x_3 - \min\{1,2x_3\})/4$, $V_{C_1}([\underline{\mathbf{0}},\underline{\mathbf{x}}]\backslash[\underline{\mathbf{u}},\underline{\mathbf{x}}]) = C_1(x_1,x_2,x_3) - (C_1(x_1,x_2,x_3) - C_1(1/2,x_2,x_3)) = C_1(1/2,x_2,x_3)$, for every $x_1 \in [1/2, 1]$ and for every $x_2, x_3 \in [0, 1]$. So, by equation (2.94) in Theorem 2.20 we have that

$$\left(C_1 \biguplus_{\underline{\mathbf{u}}} \Pi^3 \right) (\underline{\mathbf{x}}) = \begin{cases} C_1 (x_1, x_2, x_3) & \text{if } \underline{\mathbf{x}} \in [0, 1/2] \times \mathbf{I} \times \mathbf{I} \\ \frac{1}{2}\Pi^3 \left(\frac{x_1 - 1/2}{1/2}, \frac{(3x_2 - \min\{1, 2x_2\})/4}{1/2}, \frac{(3x_3 - \min\{1, 2x_3\})/4}{1/2} \right) & \text{elsewhere} \\ + C_1(1/2, x_2 x_3) & \end{cases}$$

$$= \begin{cases} \frac{1}{2} \min\{x_1, x_2, x_3\} + \frac{1}{2} x_1 x_2 x_3 & \text{if } \underline{\mathbf{x}} \in [0, 1/2] \times \mathbf{I} \times \mathbf{I} \\ \frac{1}{4} \left(x_1 - \frac{1}{2} \right) (3x_2 - \min\{1, 2x_2\}) (3x_3 - \min\{1, 2x_3\}) & \text{if } \underline{\mathbf{x}} \in [1/2, 1] \times \mathbf{I} \times \mathbf{I} \\ + \frac{1}{2} \min\{1/2, x_2, x_3\} + \frac{1}{4} x_2 x_3 & \end{cases} \tag{2.105}$$

If $\underline{\mathbf{x}} = \langle 1/4, 1/4, 1/4 \rangle$ then from equations (2.104) and (2.105), $(C_1 \circledast_{x_1 = 1/2} \Pi^3)(\underline{\mathbf{x}}) = 9/128$, but $(C_1 \biguplus_{\underline{\mathbf{u}}} \Pi^3)(\underline{\mathbf{x}}) = 9/64$. So, gluing and patching copulas do not always yield the same result. This example also shows that the operations gluing and patching are not commutative.

2.4.1 Proposed Methodology Using Gluing Copulas

Let $R = \times_{i=1}^{n} [u_i, v_i] \subset \mathbf{I}^n$ be a non trivial n-box, we want to define an n-copula C, such that on R it behaves as a rescaled version of another n-copula C_1. We propose to take as a background copula $\Pi^n(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$ the n-product copula and to give a rescaled version of C_1 on R. In order to do so, we can use the gluing method as follows:

- Since R is non trivial then for every $i \in \{1, \ldots, n\}$, $0 \leq u_i < v_i \leq 1$, then there exists $a_i \in [0, 1)$ such that $a_i v_i = u_i$.
- Define $T_1 = \Pi^n \circledast_{x_1 = a_1} C_1$, then T_1 is an n-copula such that for $x_1 \in [0, a_1]$ behaves like Π^n, and for $x_1 \in [a_1, 1]$ behaves like a rescaled version of C_1.
- Inductively, let $T_k = \Pi^n \circledast_{x_k = a_k} T_{k-1}$ for every $k \in \{2, \ldots, n\}$, then T_k is an n-copula such that for $x_k \in [0, a_k]$ behaves like Π^n.
- Define inductively $T_{n+j} = T_{n+j-1} \circledast_{x_j = v_j} \Pi^n$ for every $j \in \{1, 2, \ldots, n\}$.
- Let $C = T_{2n}$, then C has the desired properties.

Example 2.33. We will see that this proposed methodology does not yield to the ordinal sum. In order to do so we will consider the copula Π. Let us define

$$T_1(x_1,x_2) = \left(M \underset{x_1=1/2}{\circledast} \Pi\right)(x_1,x_2) = \begin{cases} \min\left\{x_1, \frac{1}{2}x_2\right\} & \text{if} \quad \langle x_1,x_2\rangle \in [0,1/2]\times \mathbf{I} \\ x_1x_2 & \text{if} \quad \langle x_1,x_2\rangle \in [1/2,1]\times \mathbf{I}\,. \end{cases}$$

In the next step we glue M with the copula T_1 at $x_2 = 1/2$, that is, we define $T_2(x_1,x_2)$ by

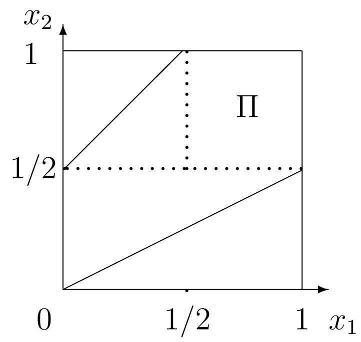
$$\left(M \underset{x_2=1/2}{\circledast} T_1\right)(x_1,x_2) = \begin{cases} \min\left\{\frac{1}{2}x_1, x_2\right\} & \text{if} \quad \langle x_1,x_2\rangle \in \mathbf{I}\times[0,1/2] \\ \min\left\{x_1, \frac{2x_1+2x_2-1}{4}\right\} & \text{if} \quad \langle x_1,x_2\rangle \in [0,1/2]\times[1/2,1] \\ x_1x_2 & \text{if} \quad \langle x_1,x_2\rangle \in [1/2,1]\times[1/2,1]. \end{cases}$$

The resulting copula T_2 behaves like Π on $[1/2,1]^2$, on $\mathbf{I}\times[0,1/2]$ has support the line joining $\langle 0,0\rangle$ and $\langle 1,1/2\rangle$ with mass 1/2 and on $[0,1/2]\times[1/2,1]$ has support the line joining $\langle 0,1/2\rangle$ and $\langle 1/2,1\rangle$ with mass 1/4, see Figure 2 below.
On the other hand the ordinal sum of Π on $[1/2,1]^2$ is given by

$$C(x_1,x_2) = \begin{cases} 2x_1x_2 - x_1 - x_2 + 1 & \text{if} \quad \langle x_1,x_2\rangle \in [1/2,1]^2 \\ \min\{x_1,x_2\} & \text{otherwise.} \end{cases}$$

Of course, the gluing copula T_2 is different from C the ordinal sum.

Figure 2.- Support of the gluing copula T_2



Chapter 3

Sample d-copulas of order m

In this chapter we analyze the construction of d-copulas including the ideas of Cuculescu and Theodorescu [8], Fredricks *et al* [34], Mikusiński and Taylor [56] and Trutschnig and Fernández-Sánchez [69]. Some of these methods use iterative procedures to construct copulas with fractal supports. In this chapter the dimension of the copulas will be denoted by d and in Section 3.2 and Section 3.3, n will always denote the sample size.

In Cuculescu and Theodorescu [8], they introduce a new family of copulas which they call self-similar copulas, in dimension two, using an iterated procedure, they also prove the existence of a d-copula with any given diagonal, we will see in detail this proof in Chapter 5. These ideas were substantially improved in Fredricks *et al* [34] in dimension $d = 2$, and quite recently in Trutschnig and Fernández-Sánchez [69] these results are generalized to $d \geq 3$.

In this chapter we will follow the original ideas given in Cuculescu and Theodorescu [8], giving generalizations to higher dimensions $d \geq 3$.

We start this chapter by stating the main results in Fredericks *et al* [34] in section 3.1 and we relate their concept of transformation matrices to doubly stochastic matrices. Then we generalize the results given in Cuculescu and Theodorescu [8], to any dimension $d > 2$. We also mention that the family of fractal d-copulas is dense in the family of all d-copulas for any $d \geq 2$. Finally we analyze the multivariate extension of Fredricks *et al* [34] given in Trutschnig and Fernández-Sánchez [69].

The main part of this chapter is given in section 3.2, where we introduce the sample d-copula of order m with $m \geq 2$, the main idea is to use the above methodologies to construct a new copula based on a sample. The greatest advantage of the sample d-copula is the fact that it is already an approximating d-copula and that it is easily obtained. A particular case of these d-copulas corresponds to the checkerboard d-copulas defined in Mikusiński and Taylor [56]. We will see that these new copulas provide a nice way to study multivariate data with an approximating copula which is simpler than the empirical multivariate copula, and that the empirical copula is the restriction to a grid of a sample d-copula of order $m = n$. These sample d-copulas can be used to make statistical inference about the distribution of the data, as shown in this section.

We find some of its basic properties and also propose some possible statistical applications of the sample d-copula. We will see that the sample d-copula of order m is a strongly consistent estimator of the d-copula which generated the data.

Section 3.2 also studies some of the basic probability properties of the sample d-copula of order m including its close relation to the multivariate distribution with parameters n the sample size, and v positive parameters whose sum is one and $m \leq v < m^d$. We will also include new applications of the sample d-copula in Statistics.

In section 3.3 we find the minimum number of parameters needed to generate the multivariate distribution associated to the d-sample copula of order m.

Section 3.4 studies the relation between d-copulas and d-dimensionally stochastic matrices. In section 3.5 we include some important remarks.

3.1 d-Copulas with Fractal Supports

In Fredricks *et al* [34] using techniques of iterated function systems (IFS) the authors construct for dimension $d = 2$ a large class of copulas. They first consider a **transformation matrix**, that is a real nonnegative matrix $T_{m_1 \times m_2} = (t_{ij})_{\langle i,j \rangle \in I_{m_1} \times I_{m_2}}$,where $I_m = \{1, \ldots, m\}$, such that $\max\{m_1, m_2\} \geq 2$, $\sum_{i,j} t_{ij} = 1$, $\sum_{i \in I_{m_1}} t_{ij} > 0$ for every $j \in I_{m_2}$ and $\sum_{j \in I_{m_2}} t_{ij} > 0$ for every $i \in I_{m_1}$. Define two partitions of $[0,1]$, $\{p_0, p_1, \ldots, p_{m_1}\}$ and $\{q_0, q_1, \ldots, q_{m_2}\}$, by letting $p_0 = 0 = q_0$, and for $i \in I_{m_1}$ let $p_i = \sum_{i'=1}^{i} \sum_{j \in I_{m_2}} t_{i'j}$, and for $j \in I_{m_2}$ let $q_j = \sum_{j'=1}^{j} \sum_{i \in I_{m_1}} t_{ij'}$. Define

$$R_{ij} = (p_{i-1}, p_i] \times (q_{j-1}, q_j] \quad \text{for every} \quad \langle i, j \rangle \in I_{m_1} \times I_{m_2},$$

where for $i = 1$ or $j = 1$ we take closed intervals instead of right open intervals. Of course, $\{R_{ij}\}_{\langle i,j \rangle \in I_{m_1} \times I_{m_2}}$ is a partition of I^2. Let C be a copula and define a transformation $T(C)$ using the partition of I^2 and the transformation matrix T, where for each $\langle i, j \rangle \in I_{m_1} \times I_{m_2}$, $T(C)$ spreads mass t_{ij} on R_{ij} rescaling the whole mass of C, that is, if $\langle u, v \rangle \in R_{ij}$ let

$$T(C)(u,v) = \sum_{i'<i, j'<j} t_{i'j'} + \frac{u-p_{i-1}}{p_i - p_{i-1}} \sum_{j'<j} t_{ij'} + \frac{v - q_{j-1}}{q_j - q_{j-1}} \sum_{i'<i} t_{i'j} + t_{ij} C\left(\frac{u-p_{i-1}}{p_i-p_{i-1}}, \frac{v-q_{j-1}}{q_j-q_{j-1}}\right), \tag{3.1}$$

where empty sums are defined to be zero. Then $T(C)$ is always a copula. If we define iteratively

$$T^2(C) = T(T(C)) \quad \text{and} \quad T^{k+1}(C) = T(T^k(C)) \quad \text{for every} \quad k > 2.$$

In fact, $T^k(C) = (\otimes^k T)(C)$, where $\otimes^k$ is the tensor product of T with itself k times. It is easy to see by induction that if T is a transformation matrix of dimension $m_1 \times m_2$ then $\otimes^k T$ is also a transformation matrix of dimension ${m_1}^k \times {m_2}^k$ for every $k \geq 2$. Then we have that for any transformation matrix T there exists a unique copula copula C_T, such that $T(C_T) = C_T$. Moreover, $C_T = \lim_{k\to\infty} T^k(C)$ for any copula C. Since C_T does not depend on the copula C, we may restrict to the limit of the sequence $\{T^k(\Pi)\}_{k\geq 1}$. In fact, they call C *invariant* if $C = C_T$ for some transformation matrix T.

They also observe that if $\pi_1 = \{p_0, p_1, \ldots, p_{m_1}\}$ and $\pi_2 = \{q_0, q_1, \ldots, q_{m_2}\}$ are any partitions of $[0,1]$, and we define $t_{ij} = (p_i - p_{i-1})(q_j - q_{j-1})$ for every $\langle i,j\rangle \in I_{m_1} \times I_{m_2}$, then $T = (t_{ij})_{\langle i,j\rangle \in I_{m_1}\times I_{m_2}}$ is a transformation matrix which generates the partitions π_1 and π_2 and has $C_T = \Pi^2$ the product copula.

Recall that for every $k \geq 2$ a square real matrix $\mathbf{P} = (p_{ij})_{i,j=1}^k$ is a **doubly stochastic matrix** if and only if $p_{ij} \geq 0$ and $\sum_{j=1}^k p_{ij} = \sum_{i=1}^k p_{ij} = 1$ for every $i, j \in \{1, 2, \ldots, k\}$.

Define

$$\mathcal{T} = \{T_{m_1\times m_2} \mid T_{m_1\times m_2} \text{ is a transformation matrix, with } m_1, m_2 \geq 2 \text{ and } t_{ij} \in \mathbb{Q} \text{ for every } \langle i,j\rangle \in I_{m_1}\times I_{m_2}\}.$$

Then we have the following result

Lemma 3.1. *Let $T_{m_1\times m_2} \in \mathcal{T}$ then there exist $k \geq 2$ and $P_{k\times k} = (p_{ij})_{i,j=1}^k$ a double stochastic matrix, such that if we define $S_{k\times k} = (p_{ij}/k)_{i,j=1}^k$ then $T(\Pi^2) = S(\Pi^2)$.*

The proof follows directly by considering k the least common multiple of the denominators of $(t_{ij})_{\langle i,j\rangle \in I_{m_1}\times I_{m_2}}$.

The last lemma can be used in every single step of the construction of C_T. However, it is clear that the support of the limit copula C_T may be different from C_S. In fact, in section 3.2

we will use only square matrices T with rational entries, and the first step in the construction of C_T. So, we can think of T as a doubly stochastic matrix times a positive integer. For some results on doubly stochastic matrices see for example Sherman [65] or Marcus [50].

Example 3.2. As an easy example of Lemma 2.1 consider the transformation matrix

$$T = \begin{pmatrix} 0 & 1/3 \\ 2/3 & 0 \end{pmatrix}.$$

Then if we define

$$S = \begin{pmatrix} 0 & 0 & 1/3 \\ 1/6 & 1/6 & 0 \\ 1/6 & 1/6 & 0 \end{pmatrix}.$$

We have that $P = 3 \cdot S$ is a doubly stochastic matrix and $T(\Pi^2) = S(\Pi^2)$.

Now we generalize the results given in Cuculescu and Theodorescu [8].
Recall that a d-**dimensional square matrix P**, for $d \geq 2$, is an array of real numbers of the form $\mathbf{P} = (p_{i_1 i_2 \cdots i_d})_{i_1,\dots i_d=1}^k$ for some $k \geq 2$. We will say that **P** is d-**dimensionally stochastic** if and only if $0 \leq p_{i_1 i_2 \cdots i_d} \leq 1$ for every $i_1, i_2, \dots, i_d \in \{1, \dots, k\}$, and for every $1 \leq j_1 < j_2 < \cdots < j_{d-1} \leq d$ we have that

$$\sum_{i_{j_1}=1}^k \sum_{i_{j_2}=1}^k \cdots \sum_{i_{j_{d-1}}=1}^k p_{i_1 i_2 \cdots i_d} = 1, \tag{3.2}$$

where the remaining index is fixed and taken in $\{1, \dots, k\}$. In this case it is clear that

$$\sum_{i_1=1}^k \sum_{i_2=1}^k \cdots \sum_{i_d=1}^k p_{i_1 i_2 \cdots i_d} = k.$$

Of course a 2-dimensionally stochastic matrix is a doubly stochastic matrix. Let $C^1_{i_1,i_2,\dots,i_d} = p_{i_1 i_2 \cdots i_d}/k$ for every $i_1, \dots, i_d \in \{1, \dots k\}$, and for every $A \in \mathcal{B}(\mathbf{I}^d)$ and for every $n \geq 1$ define

$$\mu_n(A) = k^{dn} \sum_{i_1,\dots,i_d=1}^{k^n} C^n_{i_1,\dots,i_d} \lambda^d \left(\left(\left[\frac{i_1 - 1}{k^n}, \frac{i_1}{k^n} \right] \times \cdots \times \left[\frac{i_d - 1}{k^n}, \frac{i_d}{k^n} \right] \right) \cap A \right), \tag{3.3}$$

where for every $i_1, \dots i_d \in \{1, \dots, k^n\}$, for every $i'_1, \dots i'_d \in \{1, \dots, k\}$ and for every $n \geq 1$,

$$C^{n+1}_{k(i_1-1)+i'_1,\dots,k(i_d-1)+i'_d} = \frac{p_{i'_1 \cdots i'_d}}{k^{d-1}} \cdot C^n_{i_1,\dots,i_d}. \tag{3.4}$$

Here $\mathcal{B}(\mathbf{I}^d)$ is the borel σ-algebra and λ^d is the Lebesgue measure. Then we have a multivariate extension of Cuculescu and Theodorescu [8], its proof follows as in Trutschnig and Fernández-Sánchez [69].

Proposition 3.3. *Let $d \geq 2$, let $\mathbf{P}$ be a d-dimensional square matrix of order $k \geq 2$, which is d-dimensionally stochastic. Let $n \geq 1$ and define μ_n as in equation (3.3), then $(\mathbf{I}^d, \mathcal{B}(\mathbf{I}^d), \mu_n)$ is a probability space. Besides, if we define*

$$C_n(u_1, \ldots, u_d) = \mu_n([0, u_1] \times \cdots \times [0, u_d]) \quad \textit{for every} \quad u_1, \ldots u_d \in \mathbf{I}\,. \tag{3.5}$$

Then C_n is a d-copula for every $n \geq 1$. If we define $\mu_{\mathbf{P}} = \lim_{n\to\infty} \mu_n$, then $\mu_{\mathbf{P}}$ exists with respect to weak convergence and it is a probability measure on $(\mathbf{I}^d, \mathcal{B}(\mathbf{I}^d))$. Evenmore, $\mu_{\mathbf{P}}$ induces a d-copula $C_{\mathbf{P}}$ by defining

$$C_{\mathbf{P}}(u_1, \ldots, u_d) = \mu_{\mathbf{P}}([0, u_1] \times \cdots \times [0, u_d]) \quad \textit{for every} \quad u_1, \ldots u_d \in \mathbf{I}\,. \tag{3.6}$$

If $\mathbf{P}$ includes zeros then $\mu_{\mathbf{P}}$ is a singular measure.

In the case $d = 2$ and $k = 2$, we have that if $0 < a < 1$ and

$$\mathbf{P} = \begin{pmatrix} a/2 & (1-a)/2 \\ (1-a)/2 & a/2 \end{pmatrix}.$$

then $2P$ is doubly stochastic and $C_{\mathbf{P}}$ is a singular copula if $a \neq 1/2$ as observed in Cuculescu and Theodorescu [8], see also [18]. Observe also that a 2-dimensionally stochastic matrix or doubly stochastic matrix depends only on one parameter. We will study this problem in detail in Section 3.3.

Now we will see that for $d \geq 2$ the set of d-copulas given in (3.6) is dense in the family of all copulas with respect to the supremum distance, when we consider the set of d-dimensionally stochastic matrices.

Theorem 3.4. *Let C be a d-copula for some $d \geq 2$, then for every $\epsilon > 0$ there exists $\mathbf{P}$ a d-dimensionally stochastic matrix such that if we construct the copula $C_{\mathbf{P}}$ defined in equation (3.6)*

$$d_{sup}(C, C_{\mathbf{P}}) = \sup_{u_1,\ldots,u_d \in [0,1]} |C(u_1, \ldots, u_d) - C_{\mathbf{P}}(u_1, \ldots, u_d)| < \epsilon. \tag{3.7}$$

The proof of this theorem follows from Mikusiński and Taylor [56], since in each step of the construction of $C_{\mathbf{P}}$ we obtain a checkerboard approximation. Evenmore, they prove that the convergence of the checkerboard approximations to the d-copula C holds in a stronger mode denoted by ∂-convergence which implies uniform convergence.
In Cuculescu and Theodorescu [8], for dimensions greater than or equal to three they only say " For $q \geq 2$ copulas analogous to $\mu_{\mathbf{P}}$ may also be defined (particularly one concentrated on Menger's sponge)...". Here q is the dimension. This statement is not correct as we will see in an example but in [69] the authors provide an example of a 3-copula which has a *Menger's sponge like set* support.

Example 3.5. Recall that Menger's sponge is a fractal construction on $\mathbf{I}^3$ which starts by dividing $\mathbf{I}$ in three equal intervals $[0,1/3],[1/3,2/3]$ and $[2/3,1]$, and then divide the whole cube in 27 cubes of volume 1/27. The first step consists in removing seven cubes

$$C_1 = [1/3,2/3]\times[0,1/3]\times[1/3,2/3] \quad C_2 = [0,1/3]\times[1/3,2/3]\times[1/3,2/3]$$

$$C_3 = [1/3,2/3]\times[1/3,2/3]\times[0,1/3] \quad C_4 = [1/3,2/3]\times[1/3,2/3]\times[1/3,2/3]$$

$$C_5 = [1/3,2/3]\times[1/3,2/3]\times[2/3,1] \quad C_6 = [2/3,1]\times[1/3,2/3]\times[1/3,2/3],$$

and

$$C_7 = [1/3,2/3]\times[2/3,1]\times[1/3,2/3].$$

The result of the first step is a cube where all the middle thirds have been removed, so it has square middle holes that go all the way through on each face. The construction follows inductively by repeating the above procedure to every sub-cube that has not been removed. Let $\mathbf{P} = (p_{i_1 i_2 i_3})_{i_1,i_2,i_3=1}^3$ be a square 3-matrix of order $k=3$. If we want to construct a 3-copula which has as support Menger's sponge then we would have to require that

$$p_{212} = p_{122} = p_{221} = p_{222} = p_{223} = p_{322} = p_{223} = 0.$$

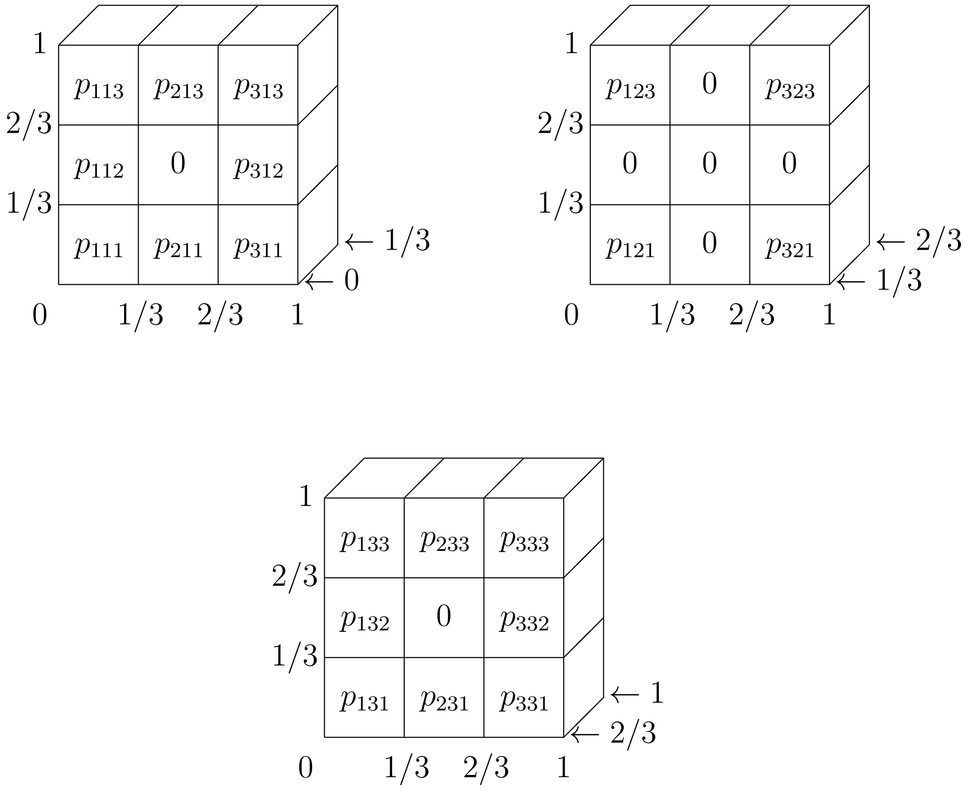
See Figure 3.

$$B_1 = [1/3,2/3]\times\mathbf{I}\times\mathbf{I} \quad B_2 = \mathbf{I}\times[1/3,2/3]\times\mathbf{I} \quad \text{and} \quad B_3 = \mathbf{I}\times\mathbf{I}\times[1/3,2/3].$$

If C is any 3-copula then

$$V_C(B_i) = \frac{1}{3} \quad \text{for every} \quad i \in \{1,2,3\}.$$

Figure 3: Support of μ_1 for **P** in the case of Menger's sponge



But, this implies that

$$V_C(B_1) = p_{211} + p_{213} + p_{231} + p_{233} = \frac{1}{3} \quad V_C(B_2) = p_{121} + p_{123} + p_{321} + p_{323} = \frac{1}{3}$$

and

$$V_C(B_3) = p_{112} + p_{312} + p_{132} + p_{332} = \frac{1}{3}.$$

So, $V_C(B_1) + V_C(B_2) + V_C(B_3) = 1$, that is the whole mass. Hence the remaining entries of the matrix **P** satisfy

$$p_{111} = p_{112} = p_{311} = p_{313} = p_{131} = p_{133} = p_{331} = p_{333} = 0.$$

Which implies that the Menger's sponge cannot be the support of a 3-copula.

Finally, we give the generalization of transformation matrices in dimension d found in Trutschnig and Fernández-Sánchez [69]. Let $I_n = \{1, 2, \ldots, n\}$ for $n \geq 1$. For $d \geq 2$, let $m_1, \ldots, m_d \in \mathbb{N}$ and define $\mathcal{I}^d = \times_{i=1}^{d} I_{m_i}$. Let τ be a probability measure on $(\mathcal{I}^d, 2^{\mathcal{I}^d})$, then we call τ a **generalized transformation matrix** if for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, m_j\}$

$$\sum_{\underline{\mathbf{i}} \in \mathcal{I}^d, i_j = k} \tau(\underline{\mathbf{i}}) > 0. \tag{3.8}$$

Observe that equation (3.8) is a natural extension of the conditions of transformation matrices in Fredricks *et al* [34], and that τ can be written as a d-dimensional matrix T, by writing

$$\tau(\underline{\mathbf{i}}) = t_{i_1, i_2, \ldots, i_d} \quad \text{if} \quad \underline{\mathbf{i}} = \langle i_1, i_2, \ldots, i_d \rangle \in \mathcal{I}^d. \tag{3.9}$$

In the remaining of this section we will only use the case $m_1 = m_2 = \cdots = m_d = m \geq 2$. In this case, if all $\tau(\underline{\mathbf{i}})$ are rational, we observe that equation (3.8) is equivalent to saying that τ induces the existence of an $m_0 \geq m$ and a d-dimensional square matrix T_0 such that $m_0 \cdot T_0$ is a d-dimensionally stochastic matrix, as defined after Example 2.2. This is a consequence of an obvious multivariate extension of Lemma 2.1 for any τ probability measure with rational values on $(\mathcal{I}^d, 2^{\mathcal{I}^d})$ in equation (3.9).

All the results at the beginning of this section about the construction of copulas in Fredricks *et al* [34], can be easily generalized to dimensions $d \geq 3$.

Let $m \geq 2$ and for every $\underline{\mathbf{i}} = \langle i_1, i_2, \ldots, i_d \rangle \in \mathcal{I}_m := \times_{j=1}^d I_m$ define

$$R_{\underline{\mathbf{i}}} = \left(\frac{i_1 - 1}{m}, \frac{i_1}{m} \right] \times \left(\frac{i_2 - 1}{m}, \frac{i_2}{m} \right] \times \cdots \times \left(\frac{i_d - 1}{m}, \frac{i_d}{m} \right], \tag{3.10}$$

where if for some $j \in \{1, \ldots, d\}$, $i_j = 1$, then we take closed intervals instead of left open intervals. Then $\{R_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}} \in \mathcal{I}_m}$ is a partition of $\mathbf{I}^d$ that we will call the **uniform partition of order** m of $\mathbf{I}^d$.

Let C be a d-copula and define for every $\underline{\mathbf{i}} = \langle i_1, i_2, \ldots, i_d \rangle \in \mathcal{I}_m$

$$t_{i_1, i_2, \ldots, i_d} = V_C(R_{\underline{\mathbf{i}}}) \quad \text{and} \quad T^C = (t_{i_1, i_2, \ldots, i_d})_{i_1, \ldots, i_d = 1}^m . \tag{3.11}$$

Then T^C is a square d-dimensional matrix with nonnegative entries, which generates the checkerboard approximation given in [56] and it is a generalized transformation matrix, because by equation (3.9), if we take any $j \in \{1, \ldots, d\}$ and any $k \in \{1, \ldots, m\}$ then by the definition of d-copula

$$\begin{aligned}
\sum_{\underline{\mathbf{i}} \in \mathcal{I}_m, i_j = k} \tau(\underline{\mathbf{i}}) &= \sum_{i_1=1}^{m} \cdots \sum_{i_{j-1}=1}^{m} \sum_{i_{j+1}=1}^{m} \cdots \sum_{i_d=1}^{m} V_C(R_{\langle i_1, \ldots, i_{j-1}, k, i_{j+1}, \ldots, i_d \rangle}) \\
&= V_C\left(\mathbf{I} \times \cdots \times \mathbf{I} \times \left[\frac{k-1}{m}, \frac{k}{m}\right] \times \mathbf{I} \cdots \times \mathbf{I}\right) \\
&= C(1, \ldots, 1, k/m, 1, \ldots, 1) - C(1, \ldots, 1, (k-1)/m, 1, \ldots, 1) \\
&= \frac{1}{m} > 0. \qquad (3.12)
\end{aligned}$$

Observe that in equation (3.12) for every d-copula C, for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, m\}$, $\sum_{\underline{\mathbf{i}} \in \mathcal{I}_m, i_j = k} \tau(\underline{\mathbf{i}}) = 1/m$ only depends on m.

Also observe that $m \cdot T^C$ is a d-dimensionally stochastic square matrix.

Now, if we have $T = (t_{i_1, \ldots, i_d})_{i_1, \ldots, i_d = 1}^{m}$ a generalized transformation square d-dimensional matrix, define $p_{1,0} = p_{2,0} = \cdots = p_{d,0} = 0$, and for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, m\}$ define

$$p_{j,k} = \sum_{i_j=1}^{k} \sum_{i_1=1}^{m} \cdots \sum_{i_{j-1}=1}^{m} \sum_{i_{j+1}=1}^{m} \cdots \sum_{i_d=1}^{m} t_{i_1, \ldots, i_d}. \qquad (3.13)$$

Then $0 = p_{j,0} < p_{j,1} < \cdots < p_{j,m-1} < p_{j,m} = 1$ is the partition with $m+1$ points of $[0,1]$ induced by T, which corresponds to the j^{th} coordinate.

If we take a d-copula C and we define $T(C)(u_1, \ldots, u_d)$ for $\langle u_1, \ldots, u_d \rangle \in \mathbf{I}^d$ using a similar formula as the one used in dimension 2 in equation (3.1), then $T(C)$ is always a d-copula, see also Trutschnig and Fernández-Sánchez [69]. In particular if $C = \Pi^d$ the product d-copula $T(\Pi^d)(u_1, \ldots, u_d)$ has a simpler expression. For example if $d = 3$ and $\langle u_1, u_2, u_3 \rangle \in$

$R_{\langle i_1,i_2,i_3\rangle} = R_{\underline{\mathbf{i}}} = (p_{1,i_1-1}, p_{1,i_1}] \times (p_{2,i_2-1}, p_{2,i_2}] \times (p_{3,i_3-1}, p_{3,i_3}]$ for some $\underline{\mathbf{i}} \in \mathcal{I}_m$ then

$$\begin{aligned} T(\Pi^3)(u_1,u_2,u_3) &= \sum_{i<i_1,j<i_2,k<i_3} t_{i,j,k} + \left(\frac{u_1-p_{1,i_1-1}}{p_{1,i_1}-p_{1,i_1-1}}\right)\sum_{j<i_2,k<i_3} t_{i_1,j,k} \\ &+ \left(\frac{u_2-p_{2,i_2-1}}{p_{2,i_2}-p_{2,i_2-1}}\right)\sum_{i<i_1,k<i_3} t_{i,i_2,k} + \left(\frac{u_3-p_{3,i_3-1}}{p_{3,i_3}-p_{3,i_3-1}}\right)\sum_{i<i_1,j<i_2} t_{i,j,i_3} \\ &+ \left(\frac{u_1-p_{1,i_1-1}}{p_{1,i_1}-p_{1,i_1-1}}\right)\left(\frac{u_2-p_{2,i_2-1}}{p_{2,i_2}-p_{2,i_2-1}}\right)\sum_{k<i_3} t_{i_1,i_2,k} \\ &+ \left(\frac{u_1-p_{1,i_1-1}}{p_{1,i_1}-p_{1,i_1-1}}\right)\left(\frac{u_3-p_{3,i_3-1}}{p_{3,i_3}-p_{3,i_3-1}}\right)\sum_{j<i_2} t_{i_1,j,i_3} \\ &+ \left(\frac{u_2-p_{2,i_2-1}}{p_{2,i_2}-p_{2,i_2-1}}\right)\left(\frac{u_3-p_{3,i_3-1}}{p_{3,i_3}-p_{3,i_3-1}}\right)\sum_{i<i_1} t_{i,i_2,i_3} \\ &+ t_{i_1,i_2,i_3}\left(\frac{u_1-p_{1,i_1-1}}{p_{1,i_1}-p_{1,i_1-1}}\right)\left(\frac{u_2-p_{2,i_2-1}}{p_{2,i_2}-p_{2,i_2-1}}\right)\left(\frac{u_3-p_{3,i_3-1}}{p_{3,i_3}-p_{3,i_3-1}}\right). \end{aligned} \tag{3.14}$$

Observe that from equation (3.14) it is clear that $T(\Pi^3)$ is a 3-copula which assigns uniform mass t_{i_1,i_2,i_3} to each box $R_{\langle i_1,i_2,i_3\rangle}$ for every $\langle i_1,i_2,i_3\rangle \in \{1,\ldots,m\}$. Therefore, the generalized transformation matrix T can be thought as the weighted density of the 3-copula $T(\Pi^3)$, given by $t_{\underline{\mathbf{i}}}/\lambda^d(R_{\underline{\mathbf{i}}})$ for every $\underline{\mathbf{i}} \in \mathcal{I}_m$, induced by the partitions and the 3-boxes that they generate. Of course the d-dimensional case includes 2^d terms that can be easily generalized.

Even if equation (3.14) seems quite complicated, it is easy to program in a computer when we have the 3-dimensional generalized transformation square matrix T of order m. We have written a short program in language **R** which computes $T(\Pi^3)(u_1,u_2,u_3)$ for any given vector $\langle u_1,u_2,u_3\rangle \in \mathbf{I}^3$, see the site https://sites.google.com/site/probstatsr.

3.2 Sample d-Copula of Order m

Now we use the ideas of section 3.1 to define the **sample d-copula of order m** in two settings.

3.2.1 Sample d-Copula of Order m for a d-Copula C

Let $m \geq 2$ and assume that we take an independent sample of size n, where $n \geq m$, from a d-copula C, let us denote the sample by

$$U_n = \{\underline{\mathbf{x_1}}, \cdots, \underline{\mathbf{x_n}}\}, \tag{3.15}$$

where $\underline{\mathbf{x_k}} = \langle x_{k,1} \ldots x_{k,d} \rangle \in \mathbf{I}^d$ for every $k \in \{1, \ldots, n\}$.
Define for every $\underline{\mathbf{i}} = \langle i_1, \ldots, i_d \rangle \in \mathcal{I}_m$ using equation (3.10)

$$s^n_{i_1,\ldots,i_d} = \frac{|R_{\underline{\mathbf{i}}} \cap U_n|}{n}, \tag{3.16}$$

where $|\cdot|$ denotes the cardinality of a set. Define

$$S^n_m = \left(s^n_{i_1,\ldots,i_d}\right)^m_{i_1,\ldots,i_d=1}. \tag{3.17}$$

Then it is clear that S^n_m is a square d-dimensional matrix such that

$$\sum_{i_1,\ldots,i_d=1}^{m} s^n_{i_1,\ldots,i_d} = 1. \tag{3.18}$$

Define

$$\mathcal{S}^+ = \{S^n_m \mid S^n_m \text{ is a generalized transformation matrix and} \, n, m \in \mathbb{N} \text{ with } n \geq m \geq 2\}. \tag{3.19}$$

If we assume that $S^n_m \in \mathcal{S}^+$ then define for every $j \in \{1, \ldots, d\}$ the partitions of $\mathbf{I}$, $\pi^n_j := \{p^n_{j,0}, \cdots, p^n_{j,m}\}$ given in equation (3.13), and define the **sample d-copula of order** m, denoted by C^n_m by

$$C^n_m(u_1, \ldots, u_d) = \begin{cases} S^n_m(\Pi^d)(u_1, \ldots, u_d) & \text{if} \quad S^n_m \in \mathcal{S}^+ \\ \Pi^d(u_1, \ldots, u_d) & \text{if} \quad S^n_m \notin \mathcal{S}^+, \end{cases} \tag{3.20}$$

for every $\langle u_1, \ldots, u_d \rangle \in \mathbf{I}^d$, where $S^n_m(\Pi^d)(u_1, \ldots, u_d)$ is defined as in equation (3.14).
If we are given a sample from a d-copula C of size $n \geq m$, but we do not have any information about C except the sample, then the terms $s^n_{i_1,\ldots,i_d}$ from the d-dimensional matrix S^n_m, give us the relative frequencies of the sample vectors that belong to $R_{\underline{\mathbf{i}}}$ for every $\underline{\mathbf{i}} \in \mathcal{I}_m$, see equation (3.10), which gives us a partition of $\mathbf{I}^d$. So, it seems natural to spread these

frequencies uniformly on the transformed version of $R_{\underline{\mathbf{i}}}$ under the partitions π_j, that is why we select Π^d the product d-copula to define the sample d-copula in equation (3.20). This idea is very common in Statistics, for example, the empirical distribution function assigns uniform mass $1/n$ to each observed point or vector. On the other hand if S_m^n is not a generalized transformation matrix, as defined above, we define the sample d-copula as Π^d, the reason for this selection is the fact that for dimension $d = 2$ and $m = 2$, if S_2^n is not a transformation matrix, then there exists at least one column or one row such that the sum of its entries is zero, and as observed in Fredricks *et al* [34], if T is a column or row vector then $T(\Pi^2) = \Pi^2$, in the remaining case, that is when only one entry in T is non zero, then we could define $T = (1)$, and in this case $T(\Pi^2) = \Pi^2$, even when T is not a transformation matrix. For higher dimensions $d \geq 3$, we refer the reader to the gluing method in Siburg and Stoimenov [66].

Now we analyze some of the main properties of S_m^n.

Proposition 3.6. *Let $m \geq 2$, let C be a d-copula and let $U_n = \{\underline{\mathbf{x_1}}, \cdots, \underline{\mathbf{x_n}}\}$ be an independent sample of size $n \geq m$ from C, define $q_{\underline{\mathbf{i}}} = V_C(R_{\underline{\mathbf{i}}})$ for every $\underline{\mathbf{i}} \in \mathcal{I}_m$. Then the square d-dimensional matrix S_m^n has associated a multinomial distribution with parameters n and $\{q_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}} \in \mathcal{I}_m^C}$, where $\mathcal{I}_m^C = \{\underline{\mathbf{i}} \in \mathcal{I} \,|\, q_{\underline{\mathbf{i}}} > 0\}$. Besides, we have that $0 \leq q_{\underline{\mathbf{i}}} \leq 1/m$ and $\sum_{\underline{\mathbf{i}} \in \mathcal{I}_m, i_j = k} q_{\underline{\mathbf{i}}} = 1/m$ for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, m\}$.*

Proof. Since $\{R_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}} \in \mathcal{I}_m}$ in equation (3.10) is a partition of $\mathbf{I}^d$, then for every $k \in \{1, 2, \ldots, n\}$ there exists a unique $\underline{\mathbf{i}} \in \mathcal{I}_m$ such that $\underline{\mathbf{x_k}} \in R_{\underline{\mathbf{i}}}$. Observe that from equation (3.18) $\sum_{\underline{\mathbf{i}} \in \mathcal{I}_m} n \cdot s_{\underline{\mathbf{i}}}^n = n$. Now, define $q_{\underline{\mathbf{i}}} = t_{i_1, \ldots, i_d}$, as in equation (3.11), for every $\underline{\mathbf{i}} = \langle i_1, \ldots, i_d \rangle \in \mathcal{I}_m$. If $q_{\underline{\mathbf{i}}} = 0$ then $P(\underline{\mathbf{x_k}} \in R_{\underline{\mathbf{i}}}) = 0$ for every $k \in \{1, \ldots, n\}$. So, if we let $\mathcal{I}_m^C = \{\underline{\mathbf{i}} \in \mathcal{I}_m \,|\, q_{\underline{\mathbf{i}}} > 0\}$, then using the independence of the sample

$$P\left(S_m^n = \left(s_{i_1,\ldots,i_d}^n\right)_{i_1,\ldots,i_d=1}^m\right) = \left(\frac{n!}{\Pi_{\underline{\mathbf{i}} \in \mathcal{I}_m^C} \; (n \cdot s_{\underline{\mathbf{i}}}^n)!}\right) \Pi_{\underline{\mathbf{i}} \in \mathcal{I}_m^C} \; q_{\underline{\mathbf{i}}}^{n \cdot s_{\underline{\mathbf{i}}}^n}. \tag{3.21}$$

Therefore, S_m^n has the desired distribution.

The restrictions on the values of the $q_{\underline{\mathbf{i}}}$ follow from equation (3.12). □

Depending on C we can have some simplifications in Proposition 3.6, for example:

Corollary 3.7. *If $C = \Pi^d$ in Proposition 3.6, then*

$$P\left(S_m^n = \left(s_{i_1,\ldots,i_d}^n\right)_{i_1,\ldots,i_d=1}^m\right) = \left(\frac{n!}{\Pi_{\underline{\mathbf{i}}\in\mathcal{I}_m}\ (n\cdot s_{\underline{\mathbf{i}}}^n)!}\right)\left(\frac{1}{m^d}\right)^n, \tag{3.22}$$

and if $C = M^d$, where $M^d(u_1,\ldots,u_d) = \min\{u_1,\ldots,u_d\}$, then

$$P\left(S_m^n = \left(s_{i_1,\ldots,i_d}^n\right)_{i_1,\ldots,i_d=1}^m\right) = \left(\frac{n!}{\Pi_{\underline{\mathbf{i}}\in\mathcal{I}_m^{M^d}}\ (n\cdot s_{\underline{\mathbf{i}}}^n)!}\right)\left(\frac{1}{m}\right)^n. \tag{3.23}$$

Proof. If $C = \Pi^d$ just observe that $q_{\underline{\mathbf{i}}} = 1/m^d > 0$ for every $\underline{\mathbf{i}} \in \mathcal{I}_m$, and if $C = M^d$ then $q_{\underline{\mathbf{i}}} = 1/m$ if and only if $R_{\underline{\mathbf{i}}} = ((k-1)/m, k/m]^d$ for some $k \in \{1,\ldots,m\}$. So, in this case, $I_m^M = \{\underline{\mathbf{i}} \in \mathcal{I}_m \,|\, \underline{\mathbf{i}} = \langle k,\ldots k\rangle$ for some $k \in \{1,\ldots,m\}\}$. □

Now, we state a result about the values of n and m.

Lemma 3.8. *Let $m \geq 2$ and let $C = \Pi^d$ the product d-copula, and assume that the sample size of the sample U_n satisfies that $n = m$. Then*

$$P(S_m^m \in \mathcal{S}^+) = \frac{(m!)^d}{(m^d)^m}. \tag{3.24}$$

Proof. First, let $d = 2$ and $m = n$, in this case, $I_m = \{1,\ldots,m\}^2$ and if we define $S_m^m = (s_{i_1,i_2}^m)_{i_1,i_2=1}^m$ as in equation (3.17), then $s_{i_1,i_2}^m = |R_{\langle i_1,i_2\rangle} \cap U_m|/m$. So, there are at most m vectors, say $\underline{\mathbf{i}}^1,\ldots,\underline{\mathbf{i}}^m \in \mathcal{I}_m$ such that $|R_{\underline{\mathbf{i}}^l} \cap U_m|/m = 1/m > 0$ for every $l \in \{1,\ldots,m\}$. From Fredricks *et al* [34], we know that $S_m^m \in \mathcal{S}^+$ if each column and each row of S_m^m have a positive element. But, since there are at most m entries in S_m^m which are different from zero, then there must be exactly one entry different from zero in each row and in each column. Since S_m^m is a square matrix of order m, we can do this in $m!$ forms. So using Corollary 3.2 equation (3.22)

$$P(S_m^m \in \mathcal{S}^+) = m!\cdot\left(\frac{m!}{1!\cdots 1!}\right)\left(\frac{1}{m^2}\right)^m = \frac{(m!)^2}{m^{2m}}.$$

For $d > 2$ we proceed in a similar way. We know that S_m^m is a d-dimensional square matrix of order m, so, $S_m^m \in \mathcal{S}^+$ if and only if there is exactly one entry of S_m^m different from zero in each coordinate. In this case, proceeding as in the case $d = 2$, in the first coordinate we can select the non-zero entry in m^{d-1} forms, for the second coordinate we have $(m-1)^{d-1}$ forms, etc. Then using Corollary 3.2, equation (3.22) again

$$P(S_m^m \in \mathcal{S}^+) = \Pi_{l=1}^m (l)^{d-1} \cdot \left(\frac{m!}{1! \cdots 1!}\right) \left(\frac{1}{m^d}\right)^m = \frac{(m!)^d}{(m^d)^m},$$

which finishes the proof. □

Remark 3.9. From the proof of Lemma 3.3, it is clear that if the sample size n is less than m, that is, $n < m$, then $\mathcal{S}^+ = \emptyset$, that is why we asked for the condition $n \geq m$ in the definition of a sample d-copula of order m.

Now we give some asymptotic results about C_m^n.

Theorem 3.10. *Let $m \geq 2$, $n \geq m$ and let U_n be an independent sample of size n from a d-copula C for some fixed $d \geq 2$. Define C_m^n as in equation (3.20). Let S_m^n the d-dimensional square matrix induced by the sample U_n given in equations (3.16) and (3.17). Then for every $\underline{\mathbf{i}} = \langle i_1, \ldots, i_d \rangle \in \mathcal{I}_m$ with m fixed,*

$$\lim_{n\to\infty} s_{i_1,\ldots,i_d}^n = V_C(R_{\underline{\mathbf{i}}}) \quad \textit{almost surely.} \tag{3.25}$$

The elements in the partitions $\{p_{j,0}^n, p_{j,1}^n, \ldots, p_{j,m}^n\}$ given in equation (3.13) satisfy that for every $j \in \{1, \ldots, d\}$ and for every $k \in \{0, 1, \ldots, m\}$,

$$\lim_{n\to\infty} p_{j,k}^n = \frac{k}{m} \quad \textit{almost surely.} \tag{3.26}$$

Therefore, if we define the grid $K_m = \{0, 1/m, 2/m, \ldots, (m-1)/m, 1\}^d$, the sample d-copula C_m^n is such that

$$\lim_{n\to\infty} C_m^n(u_1, \ldots, u_d) = C(u_1, \ldots, u_d) \quad \textit{for every} \quad \langle u_1, \ldots, u_d \rangle \in K_m \quad \textit{almost surely.} \tag{3.27}$$

Finally, if we also let $m \to \infty$ with values of $m \approx n^{1/2d}$ we have that

$$C_m^n \quad \textit{converges uniformly and almost surely to} \quad C. \tag{3.28}$$

Proof. Let $m \geq 2$ and $d \geq 2$ be fixed integers, let C be a d-copula and let U_n be a random sample from C of size $n \geq m$. Let $s_{i_1,\ldots,i_d}^n$ be defined as in equation (3.16), and observe that $s_{i_1,\ldots,i_d}^n$ can be written as

$$s_{i_1,\ldots,i_d}^n = \sum_{j=1}^n \frac{1_{R_{\langle i_1,\ldots,i_d \rangle}}(\underline{\mathbf{x_j}})}{n} \quad \text{for every} \quad \underline{\mathbf{i}} = \langle i_1, \ldots, i_d \rangle \in \mathcal{I}_m, \tag{3.29}$$

where 1_A is the indicator function of A. Using the strong law of large numbers (SLLN), we have that for every $\underline{\mathbf{i}} = \langle i_1, \ldots, i_d \rangle \in \mathcal{I}_m$

$$\lim_{n\to\infty} s^n_{i_1,\ldots,i_d} = P(\underline{\mathbf{x}} \in R_{\langle i_1,\ldots,i_d\rangle}) = V_C(R_{\underline{\mathbf{i}}}) \quad \text{almost surely,} \tag{3.30}$$

which proves (3.25). Now using equations (3.12), (3.13) and (3.25), we have that for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, m\}$,

$$\begin{aligned} \lim_{n\to\infty} p^n_{j,k} &= \sum_{i_j=1}^{k} \sum_{i_1=1}^{m} \cdots \sum_{1_{j-1}=1}^{m} \sum_{i_{j+1}=1}^{m} \cdots \sum_{i_d=1}^{m} \lim_{n\to\infty} s^n_{i_1,\ldots,i_d} \\ &= \sum_{i_j=1}^{k} \sum_{i_1=1}^{m} \cdots \sum_{1_{j-1}=1}^{m} \sum_{i_{j+1}=1}^{m} \cdots \sum_{i_d=1}^{m} V_C(R_{\langle i_1,\ldots,i_d\rangle}) \\ &= \sum_{i_j=1}^{k} \frac{1}{m} = \frac{k}{m} \quad \text{almost surely.} \end{aligned} \tag{3.31}$$

So, (3.26) holds. Now, using equations (3.20), (3.1) , (3.14) and their generalizations, it is clear that (3.27) holds.

Finally, equation (3.28) follows from the upper bounds given by the Pólya urn scheme, see Table 5 below, and the multivariate normal approximation of the multinomial distribution.

□

Observe that from equation (3.27), if we let $C_m = \lim_{n\to\infty} C^n_m$, then C_m coincides with μ_1 in equation (3.3), for $k = m$ and $\mathbf{P} = (p_{i_1,\ldots,i_d})^m_{i_1,\ldots,i_d=1}$, where $p_{i_1,\ldots,i_d} = m \cdot V_C(R_{\langle i_1,\ldots,i_d\rangle})$ for every $\langle i_1, \ldots, i_d \rangle \in \mathcal{I}_m$.

In the definition of C^n_m the sample d-copula of order m given in equation (3.20), it is very important to check when the d-dimensional square matrix S^n_m belongs to $\mathcal{S}^+$, in terms of the sample size n and the generating copula C. In order to evaluate $P(S^n_m \in \mathcal{S}^+)$, we will use $C = \Pi^d$ and a simulation procedure to approximate its value. We already have an exact value of $P(S^m_m \in \mathcal{S}^+)$, given in Lemma 3.3, when $n = m$, which is the limit case. We give a preliminary study of $P(S^n_m \in \mathcal{S}^+)$ for $d = 2$ and with $m = 2, 3, 4$ and for $d = 3, 4$ with $m = 2, 3$ for $n \geq m$, in the case $C = \Pi$, using 100,000 simulations. Observe that the case $C = \Pi^d$ is the uniform case, hence the most "spread" case among the d-copulas. See Tables 1 and 2, where the values of $P(S^n_m \in \mathcal{S}^+)$ for several values of $n \geq m$ are approximated via simulations. Observe that even for small values of n the probability of obtaining a generalized transformation matrix is close to one. Also observe that probabilities, in the limit case $n = m$, given in Lemma 3.3 are approximated very accurately.

In order to compare behaviors, we obtained 100,000 simulations in dimensions $d = 2$ and $d = 3$ and different sample sizes n from several families, such as M^2, M^3, W^2, Frank, Clayton, Normal with different parameters to compare the behavior of $P(S_m^n \in \mathcal{S}^+)$ to the samples coming from the product copula of dimension $d = 2$ and $d = 3$. Some of these results can be seen in Table 3 and Table 4. From Tables 1 and 3 we observe that for very small values of n the product copula gives smaller probabilities of $P(S_m^n \in \mathcal{S}^+)$ than the other distributions, we also have that M^2 produces the largest probabilities compare to the other distributions. However, for values of n between 30 and 50 the probabilities are quite similar for all the distributions. Similar observations can be obtained from Tables 2 and 4.

We simulated several extra examples with copulas with Spearman's rho varying from -1 to 0, and we obtained very similar results. For example for $d = 2$, W^2 gives very similar results as M^2, as expected.

For a further exploration of these results we recommend to see the algorithms to generate samples of d-copulas in Mai and Scherer [48].

Another way of finding an upper bound for $P(S_m^n \in \mathcal{S}^+)$ is to use the Pólya approach. Consider k boxes and $n \geq k$ balls, for each ball we select uniformly one of the boxes and the ball is placed inside that box, we repeat independently the procedure for the n balls. We want to find the probability that at the end of this procedure there are no empty boxes, let us call this event E_k^n. This probability is known as the Maxwell-Boltzmann occupancy problem formula given by

$$P\left(E_k^n\right) = \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \left(1 - \frac{j}{k}\right)^n, \tag{3.32}$$

see for example Mahmoud [47] page 37. Observe that if we have a sample of size n from the copula product Π^d and we take $2 \leq m \leq n$, then the m^d boxes used in the construction of the empirical copula of size m have the same probability $1/m^d$. If we assume that $n \geq m^d$ and $k = m^d$, in the occupancy problem above it is clear that the matrix $S_m^n \in \mathcal{S}^+$ if every box has at least one ball (observation). Then $P(E_k^n) \leq P(S_m^n \in \mathcal{S}^+)$. So, if we find a value of n, depending on k, such that $P(E_k^n) \approx 1$, then we have that S_m^n is generalized transformation matrix with very high probability. We proposed to use $n(k)$ the minimum value of n such that $P(E_k^{n(k)}) \geq 0.99999995$, if we use the language **R** and we obtain a probability satisfying this condition it is reported as 1.

Table 1.- Approximations of $P(S_m^n \in \mathcal{S}^+)$ for $d = 2$ and $m = 2, 3, 4$

value of n	$d = 2$ and $m = 2$	$d = 2$ and $m = 3$	$d = 2$ and $m = 4$
2	0.24980	-	-
3	0.56013	0.04890	-
4	0.76562	0.19765	0.00867
5	0.88051	0.37881	0.05392
10	0.99588	0.89792	0.61116
15	0.99984	0.98655	0.89724
20	1	0.99849	0.97408
25	1	0.99979	0.99378
30	1	0.99998	0.99870
35	1	1	0.99950
40	1	1	0.99990
45	1	1	0.99995
50	1	1	1

Table 2.- Approximations of $P(S_m^n \in \mathcal{S}^+)$ for $d = 3, 4$ and $m = 2, 3$

value of n	$d = 3$ and $m = 2$	$d = 3$ and $m = 3$	$d = 4$ and $m = 2$	$d = 4$ and $m = 3$
2	0.12505	-	0.06192	-
3	0.42239	0.01112	0.31648	0.00249
5	0.82229	0.23426	0.76920	0.14455
10	0.99381	0.85042	0.99238	0.80913
15	0.99980	0.97938	0.99980	0.97177
20	0.99998	0.99759	0.99998	0.99625
25	1	0.99971	1	0.99959
30	1	0.99995	1	0.99997
35	1	1	1	0.99999
40	1	1	1	1

Table 3.- Approximations of $P(S_m^n \in \mathcal{S}^+)$ for $d = 2$ and $m = 4$ under different distributions

value of n	Clayton $\theta = 2$	Frank $\theta = 5$	Normal $\rho = 0.5$	M^2
4	0.01393	0.01272	0.01037	0.09364
5	0.06941	0.06674	0.05859	0.23416
10	0.62622	0.62193	0.61671	0.78161
15	0.90017	0.89780	0.89658	0.94685
20	0.97561	0.97532	0.97600	0.98712
25	0.99433	0.99395	0.99428	0.99694
30	0.99866	0.99860	0.99863	0.99934
35	0.99948	0.99967	0.99967	0.99988
40	0.99991	0.99992	0.99993	0.99997
45	0.99999	0.99999	0.99995	0.99998
50	0.99999	0.99999	0.99999	0.99998

Table 4.- Approximations of $P(S_m^n \in \mathcal{S}^+)$ for $d = 3$ and $m = 3$ under different distributions

value of n	Clayton $\theta = 2$	Frank $\theta = 5$	Normal $\rho = 0.5$	M^3
3	0.02657	0.02285	0.01597	0.21962
5	0.29490	0.28744	0.25843	0.61540
10	0.86428	0.86021	0.85690	0.94941
15	0.98036	0.98036	0.97966	0.99249
20	0.99725	0.99750	0.99737	0.99903
25	0.99971	0.99966	0.99962	0.99989
30	0.99999	0.99995	0.99999	0.99999
35	1	0.99999	1	1
40	0.99999	1	1	1

We obtained the values of $n(k)$ for $1 \leq k \leq 150$, see some values on Table 5, and we fit linear and non linear models to check its behavior. We found that a linear model is a good approximation and that for large values of k the estimated line remains above the real values of $n(k)$. Observe that from Tables 1 and 2 the value of n such that $P(S_m^n \in \mathcal{S}^+)$ is close to one is actually smaller than the values of $n(k)$ where $k = m^d$ in the Pólya urn scheme even for small values of k.

Table 5.- Values of $n(k)$ such that of $P(E_k^{n(k)}) \approx 1$ for k of the form m^d

value of k	4	8	9	16	25	27	32	49	64	81	100
value of $n(k)$	64	142	162	304	491	533	639	1005	1332	1708	2131

In the remaining part of this section we will study some important statistical applications.

Assume that $d = 2$ and recall that the main concordance measures are Kendall's tau and Spearman's rho. If C is a copula we know that

$$\tau_C = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 \quad \text{and} \quad \rho_C = 12\int_0^1\int_0^1 uvdC(u,v) - 3, \tag{3.33}$$

see for example [58], equations 5.1.7 and 5.1.15b. Let $2 \le m \le n$ and let $U_n = \{\underline{\mathbf{x_1}}, \ldots, \underline{\mathbf{x_n}}\}$ be a sample of size n of a copula C, or a modified sample of a joint continuous distribution $H(x,y)$. Define $s^n_{i_1,\ldots,i_d}, S^n_m, \mathcal{S}^+$ and C^n_m as in equations (3.16), (3.17), (3.19) and (3.20), and assume that $S^n_m = (s^n_{ij})^m_{i,j=1}$ is a transformation square matrix of order m. Using the same notation as in Fredricks *et al* [34] it is easy to see that for every $i, j \in \{0, 1, \ldots, m\}$

$$\begin{aligned}\int\int_{R_{ij}} C^n_m(u,v)dC^n_m(u,v) &= \int_{q_{j-1}}^{q_j}\int_{p_{i-1}}^{p_i} C^n_m(u,v)\frac{s_{ij}}{(p_1 - p_{i-1})(q_j - q_{j-1})}dudv \\ &= \sum_{i'<i}\sum_{j'<j} s_{i'j'}s_{ij} + \sum_{j'<j}\frac{s_{ij'}s_{ij}}{2} + \sum_{i'<i}\frac{s_{ij'}s_{ij}}{2} + \frac{s^2_{ij}}{4}, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\int\int_{R_{ij}} uvdC^n_m(u,v) &= \int_{q_{j-1}}^{q_j}\int_{p_{i-1}}^{p_i} uv\frac{s_{ij}}{(p_1 - p_{i-1})(q_j - q_{j-1})}dudv \\ &= \frac{s_{ij}}{4}(p_{i-1} + p_i)(q_{j-1} + q_j). \end{aligned} \tag{3.35}$$

Using (3.34) and (3.35) we can prove the following:

Lemma 3.11. *Let $d = 2$ and let $U_n = \{\underline{\mathbf{x_1}}, \ldots, \underline{\mathbf{x_n}}\}$ be a sample of size n of a copula C, or a modified sample of a joint continuous distribution $H(x,y)$. Define $s^n_{i_1,\ldots,i_d}, S^n_m, \mathcal{S}^+$ and C^n_m as in equations (3.16), (3.17), (3.19) and (3.20), and assume that $S^n_m = (s^n_{ij})^m_{i,j=1}$ is a transformation square matrix of order m. Then*

$$\tau_{C^n_m} = \sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\sum_{i'=i+1}^{m}\sum_{j'=j+1}^{m} s^n_{ij}s^n_{i'j'} - \sum_{i=1}^{m}\sum_{j=2}^{m}\sum_{i'=i+1}^{m}\sum_{j'=1}^{j-1} s^n_{ij}s^n_{i'j'}, \tag{3.36}$$

and

$$\rho_{C^n_m} = 3\left(\sum_{i=1}^{m}\sum_{j=1}^{m} s^n_{ij}(p_{i-1} + p_i)(q_{j-1} + q_j) - 1\right). \tag{3.37}$$

Besides,

$$\tau_{C^n_m} \in \left[-\left(1-\frac{1}{m}\right),\left(1-\frac{1}{m}\right)\right] \quad \textit{and} \quad \rho_{C^n_m} \in \left[-\left(1-\frac{1}{m^2}\right),\left(1-\frac{1}{m^2}\right)\right]. \tag{3.38}$$

Observe that if n is a multiple of m and $s^n_{ii} = 1/m$ for every $i \in \{1, \ldots, m\}$, with $s^n_{ij} = 0$ if $i \neq j$, then the upper bounds in (3.38) are attained. For example if $m = 2$, and we consider a copula C such that $V_C(R_{\langle 1,1\rangle}) = 1/2 = V_C(R_{\langle 2,2\rangle})$. Here we consider two extreme cases let $C_1(u,v) = M^2(u,v) = \min\{u,v\}$ for every $\langle u,v\rangle \in I^2$ and $C_2(u,v) = \max\{0, u+v-1/2\}$ if $\langle u,v\rangle \in R_{\langle 1,1\rangle}$, $C_2(u,v) = 1/2 + \max\{0, u+v-3/2\}$ if $\langle u,v\rangle \in R_{\langle 2,2\rangle}$ and $C_2(u,v) = 0$ otherwise, that is, C_2 is a shuffle of M^2. In this case using (3.33) it is easy to see that $\tau_{C_2} = 0$ and $\rho_{C_2} = 1/2$, and obviously $\tau_{M^2} = \rho_{M^2} = 1$. In general, for any $m > 2$ if we let $C_1 = M^2$ and if we define C_2 to be a shuffle of M that behaves like W^2 on each $R_{\langle i,i\rangle}$ for every $i \in \{1, \ldots, m\}$ then we have that $\tau_{C_2} = 1 - 2/m$ and $\rho_{C_2} = 1 - 2/m^2$, but $\tau_{M^2} = \rho_{M^2} = 1$. Therefore, the upper bounds in (3.38) are the average of the minimum and maximum values of τ_C and ρ_C when we only know that $V_C(R_{\langle i,i\rangle}) = 1/m$ for every $i \in \{1, \ldots, m\}$. Of course for the lower bounds we have a similar result. In order to see how the above methodology of estimation of measures of concordance works, we simulated 10000 samples from the normal copula in dimension $d = 2$ of sizes $n = 100$ and $n = 200$ for different values of ρ between -1 and 1. In Table 6 we report the results of the estimations of ρ, when $n = 200$ with $\rho = 0$ and $\rho = 0.5$, and for $m = 5, 7, 12, 15$. Of course when $\rho = 0$ we are sampling from the product copula Π^2.

Table 6.- Estimations of ρ for $n = 200$ and real values $\rho = 0$ and $\rho = 0.5$

	$\rho = 0$		$\rho = 0.5$	
m	$E(\hat{\rho})$	$\mathrm{Var}(\hat{\rho})$	$E(\hat{\rho})$	$\mathrm{Var}(\hat{\rho})$
5	0.000203	0.004563	0.43255	0.003174
7	-0.000567	0.004817	0.45464	0.003079
12	-0.001174	0.004945	0.47136	0.003124
15	0.000470	0.004974	0.47417	0.003079

In Table 6 we can observe that when $\rho = 0$ the expected value of ρ is close to 0 even for small m, and for the case $\rho = 0.5$ the expected values approach 0.5 from the left when m increases and the variances are stable in both cases. We also observed that the variances decrease when the simple size increases form $n = 100$ to $n = 200$. For positive values of ρ the behavior of the expected values and variances is similar to the case $\rho = 0.5$.

Table 7.- Estimations of ρ for $n = 200$ when $\rho = 1$

m	$E(\hat{\rho})$	$\mathrm{Var}(\hat{\rho})$	$\min(\hat{\rho})$	$\max(\hat{\rho})$	upper bound of $\hat{\rho}$
2	0.74626	0.00002713	0.69532	0.75	0.75
5	0.95760	0.00000292	0.94329	0.95997	0.96
9	0.98614	0.00000058	0.98008	0.98760	0.98765
12	0.99189	0.00000026	0.98574	0.99192	0.99305
15	0.99460	0.00000013	0.99212	0.99542	0.99555

In Table 7 we first observe that if $\rho = 1$ we are sampling from the copula M^2 then the expected values approach 1 quickly when m increases. Evenmore, if we observe the values of the minima and maxima they approach the upper bound of $\rho_{C_m^n}$ in equation (3.38), which reflects in smaller variances. Besides, the worst case in the 10000 simulations when $n = 200$ and $m = 15$ is 0.99212 which is very close to one. For negative values of ρ we obtained similar results.

As a second statistical application we proposed a method for the estimation of a parameter when we are sampling from a parametric family $\{C_\theta | \theta \in \Theta\}$ with $\Theta \subset \mathbf{R}$.

For some parametric families $\{C_\theta | \theta \in \Theta\}$ in dimension d, it is possible to make good estimation of the parameter θ using the sample d copula of order m, even in the case when $m = 2$. For example, in the case of some multivariate parametric Archimedean copulas, we observe that $V_{C_\theta}([0, 1/2]^d)$ is a continuous strictly monotone function f of the parameter $\theta \in \Theta \subset \mathbf{R}$ for any $d \geq 2$. This is the case for example in the families Clayton with $\theta \in (0, \infty)$, Frank with $(0, \infty)$, Ali-Mikhail-Haq with $\theta \in [0, 1)$, Gumbel-Hougaard with $\theta \in [1, \infty)$, etc, see [58] Table 4.1. Then by estimating $V_{C_\theta}([0, 1/2]^d)$ using $s^n_{1,1,\ldots,1}$ in the generalized transformation matrix S^n_m as defined in equation (3.17), we can find a unique value of $\hat{\theta}$ such that $f(\hat{\theta}) = s^n_{1,1,\ldots,1}$. In general, we need to find f^{-1} in order to give $\hat{\theta}$, but in many cases f^{-1} may not have an analytic expression. However, it can be approximated very accurately with a numerical procedure.

Observe that when $m = 2$ from Proposition 3.1, we are trying to estimate the value of $\theta \in \Theta$ such that $f(\theta) = V_{C_\theta}(R_{\underline{i_0}}) = p_{\underline{i_0}} = C_\theta(1/2, 1/2, \ldots, 1/2)$, where $\underline{i_0} = \langle 1, 1, \ldots, 1 \rangle$, based on a sample from C_θ of size n. We know from the basic properties of the multinomial distribution that the number of observations that fall in the d-box $R_{\underline{i_0}}$, let us say $X_{\underline{i_0}}$, is distributed as a binomial with parameters n and $p_{\underline{i_0}}$. Therefore, $s^n_{1,1,\ldots,1} = X_{\underline{i_0}}/n$ is distributed as a rescaled binomial with values in $\{0, 1/n, 2/n, \ldots, 1\}$, and for n large enough $s^n_{1,1,\ldots,1}$ is a good estimator of $f(\theta)$, hence $\hat{\theta} = f^{-1}(s^n_{1,1,\ldots,1})$ is a good estimator of θ. The procedure of estimation follows the next steps:

1.- Find the direct image $f[\Theta] = \{f(\theta) = C_\theta(1/2, \ldots, 1/2) | \theta \in \Theta\} \subset I$ for the family $\{C_\theta | \theta \in \Theta\}$.

2.- Given a sample $U_n = \{\underline{x_1}, \ldots, \underline{x_n}\}$ find the value of $s^n_{1,1,\ldots,1}$ in the construction of the sample

d-copula of order $m = 2$. If $s^n_{1,1,\ldots,1} \in \text{int} f[\Theta]$ proceed with the next steps.

3.- If f^{-1} has an analytic expression define $\hat{\theta} = f^{-1}(s^n_{1,1,\ldots,1})$, and we are done. In other case, if Θ is bounded, give a fine grid of Θ, to approximate $f[\Theta]$, otherwise give a fine grid of a bounded subset Θ_0 of Θ such that $s^n_{1,1,\ldots,1} \in f[\Theta_0]$ and it is close to $f[\Theta]$, and use a linear interpolation to estimate $f^{-1}(s^n_{1,1,\ldots,1}) = \hat{\theta}$.

As an application of this methodology we use the Frank family of copulas for $d = 2$ and $d = 3$. In the case $d = 2$ it is easy to see that $f[\Theta] = (0, 1/4) \cup (1/4, 1/2)$ and $f(\theta) = C_\theta(1/2, 1/2)$ is a strictly increasing function which is symmetric with respect to the point $\langle 0, 1/4 \rangle$. If $d \geq 3$ then $f[\Theta] = [1/2^d, 1/2)$ since $\theta \geq 0$. In these cases f^{-1} has no analytic expression, so, we use the grid construction defined above to estimate θ.

We generated 5000 samples of different sizes $n = 500, 1000, 10000, 50000, 100000$ from the Frank copula with parameters $\theta = 2$ and $\theta = 5$. In Tables 8, 9, 10 and 11 we can see the basic statistics of the estimations for the 5000 samples.

Table 8.- Estimations of θ for the Frank copula with $d = 2$ and $\theta = 2$

n	$E(\hat{\theta})$	$\text{Var}(\hat{\theta})$	$\min(\hat{\theta})$	$\max(\hat{\theta})$
500	2.03938	0.582948	-0.51333	5.54950
1000	2.01002	0.286775	0.35233	3.97850
10000	2.00103	0.027756	1.42433	2.65066
50000	2.00187	0.005590	1.74500	2.27800
100000	1.99985	0.002830	1.81266	2.19766

Table 9.- Estimations of θ for the Frank copula with $d = 3$ and $\theta = 2$

n	$E(\hat{\theta})$	$\text{Var}(\hat{\theta})$	$\min(\hat{\theta})$	$\max(\hat{\theta})$
500	2.00465	0.215213	0.37775	3.63266
1000	2.00282	0.103440	0.94150	3.33600
10000	2.00082	0.010230	1.61925	2.42125
50000	2.00155	0.002118	1.85150	2.20225
100000	2.00017	0.001008	1.88675	2.10825

Table 10.- Estimations of θ for the Frank copula with $d = 2$ and $\theta = 5$

n	$E(\hat{\theta})$	$\text{Var}(\hat{\theta})$	$\min(\hat{\theta})$	$\max(\hat{\theta})$
500	5.17081	1.950315	1.57509	13.84323
1000	5.07970	0.891967	2.29133	10.10000
10000	5.01109	0.083791	3.90100	6.12500
50000	5.00001	0.016473	4.58750	5.53900
100000	5.00227	0.008161	4.69700	5.32500

Table 11.- Estimations of θ for the Frank copula with $d = 3$ and $\theta = 5$

n	$E(\hat{\theta})$	$\text{Var}(\hat{\theta})$	$\min(\hat{\theta})$	$\max(\hat{\theta})$
500	5.07597	0.678271	2.66900	10.07200
1000	5.04366	0.319974	3.16400	7.27900
10000	5.00105	0.031167	4.40900	5.67650
50000	5.00303	0.006224	4.73633	5.28200
100000	5.00040	0.002999	4.82266	5.22500

As can be observed in Tables 8, 9, 10 and 11, the average estimation of θ is good in all cases, and as expected the variances decrease as n increases. The minima and maxima of the estimations are relatively far from each other when the sample size is $n = 500$. So, we do not recommend to use this methodology for small n. It is also very important to observe that, as expected from the binomial distribution and the central limit theorem, the estimation of θ is quite good for $n = 100000$, but if we try to use the empirical distribution function when $d = 3$, we would need an array of 10^{15} terms, which is needed to perform calculations in order to estimate θ, which no computer can handle. However, in our tables the elapsed time for each simulation was 15.98 seconds for $\theta = 2$ and $d = 3$, and 16.25 seconds for $\theta = 5$ and $d = 3$.

As a third application we propose a simple goodness-of-fit test. Let us assume that we take a sample of size n coming from a d-copula C and we take $2 \le m \le n$ a fixed integer. Let $R_{\underline{\mathbf{i}}}$ for $\underline{\mathbf{i}} \in \mathcal{I}_m$ be the partition of I^d in the construction of the sample d-copula, and assume that $S_m^n = \left(s_{i_1,\ldots,i_d}^n\right)_{i_1,\ldots,i_d=1}^m$ is a generalized transformation matrix. From Proposition 3.1 we know that the square d-dimensional matrix S_m^n has a multinomial distribution with parameters n and $\{q_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}} \in \mathcal{I}_m^C}$, where $\mathcal{I}_m^C = \{\underline{\mathbf{i}} \in \mathcal{I}_m \mid q_{\underline{\mathbf{i}}} > 0\}$. Therefore, we want to test the simple hypothesis

$$H_0 : S_m^n \rightsquigarrow \text{Mult}\left(n, \{q_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}} \in \mathcal{I}_m^C}\right), \tag{3.39}$$

against the alternative composite hypothesis $H_1 : S_m^n \not\rightsquigarrow \text{Mult}\left(n, \{q_{\mathbf{i}}\}_{\mathbf{i}\in\mathcal{I}_m^C}\right)$. In the literature there are several proposals for a goodness-of-fit test for the multinomial distribution, see for example [7] or [60]. In order to prove $H_0 \quad \text{vs} \quad H_1$ we used the most common statistics, that is, Pearson's χ^2, which has asymptotically a chi-squared distribution with $k-1$ degrees of freedom, where k denotes de the cardinality of $\mathcal{I}_m^C$.
Here we present a couple of examples:
In the first one we choose the case $d = 2$, $m = 2$, and the Frank copula with $\theta = 10$, and different values of n, in this case $q_{\langle 1,1\rangle} = q_{\langle 2,2\rangle} = 0.43136$ and $q_{\langle 1,2\rangle} = q_{\langle 2,1\rangle} = 0.06844$. In Table 12 we give the basic statistical results of 10000 simulations with values of $n = 5, 25, 100, 250, 500$ and $n = 1000$. As we can observe, even for n as small as 25, the expected values of $s_{i,j}^n$ for $i, j \in \{1, 2\}$ are close to the real ones, with smaller variances as n increases as expected. In Table 13 we present the results of the number of rejections of H_0 at level $\alpha = 0.05$ for $n = 100, 250, 500, 1000$ and the mean of the p-values of the 10000 tests. From these results we can see that the test performs as expected when $\alpha = 0.05$. In order to check the power of the test, depending on θ the parameter of the Frank copula, we performed 10000 tests of H_0 as above, for $n = 1000$ with different values of θ varying from $\theta = 6$ up to $\theta = 15$ taking integer values, the number of rejections of H_0 in order were 9997 for $\theta = 6$, 9815 for $\theta = 7$, 6692 for $\theta = 8$, 1860 for $\theta = 9$, 1147 for $\theta = 11$, 3577 for $\theta = 12$, 6646 for $\theta = 13$, 8816 for $\theta = 14$ and 9709 for $\theta = 15$.

Table 12.- Estimations of $s_{ij}^n, i, j \in 1, 2$ for the Frank copula with $d = 2$, $m = 2$ and $\theta = 10$

n	$E(s_{1,1}^n)$	$\text{Var}(s_{1,1}^n)$	$E(s_{1,2}^n)$	$\text{Var}(s_{1,2}^n)$	$E(s_{2,1}^n)$	$\text{Var}(s_{2,1}^n)$	$E(s_{2,2}^n)$	$\text{Var}(s_{2,2}^n)$
5	0.4359	0.03509	0.0668	0.01200	0.0658	0.01196	0.4315	0.03426
25	0.4312	0.00973	0.0681	0.00263	0.0690	0.00254	0.4316	0.00986
100	0.4431	0.00244	0.0688	0.00063	0.0685	0.00064	0.4319	0.00244
250	0.4312	0.00099	0.0683	0.00025	0.0685	0.00025	0.4320	0.00101
500	0.4312	0.00049	0.0688	0.00013	0.0686	0.00013	0.4314	0.00048
1000	0.4314	0.00025	0.0686	0.00006	0.0687	0.00006	0.4313	0.00024

Table 13.- Rejections of H_0 for the Frank copula with $d = 2$, $m = 2$ and $\theta = 10$

n	Number of rejections	Mean of p-value
100	485	0.50105
250	487	0.49428
500	508	0.49923
1000	492	0.50140

As a second example we generated 10000 simulations of the copula M^2 for $m = 3$ and different values of n between 5 and 1000. In this case $q_{\langle 1,1\rangle} = q_{\langle 2,2\rangle} = q_{\langle 3,3\rangle} = 1/3$ and zero in any other case. On Table 14 we report the basic statistics of the simulations. On Table 15 we report the number of rejections of H_0 for $n \geq 25$, and observe that even for $n = 25$ we obtain nice results.

Table 14.- Estimations of $s^n_{ij}, i, j \in 1, 2$ for the copula M^2 with $d = 2$ and $m = 3$

n	$E(s^n_{1,1})$	$\mathrm{Var}(s^n_{1,1})$	$E(s^n_{2,2})$	$\mathrm{Var}(s^n_{2,2})$	$E(s^n_{3,3})$	$\mathrm{Var}(s^n_{3,3})$
5	0.33288	0.019545	0.33364	0.019482	0.33347	0.019482
25	0.33371	0.008919	0.33326	0.008833	0.33302	0.008880
100	0.33331	0.002206	0.33327	0.002223	0.33340	0.002229
250	0.33340	0.000889	0.33319	0.000889	0.33340	0.000889
500	0.33329	0.000444	0.33331	0.000445	0.33339	0.000445
1000	0.33324	0.000222	0.33336	0.000222	0.33339	0.000223

Table 15.- Rejections of H_0 for the M^2 copula with $d = 2$ and $m = 3$

n	Number of rejections	Mean of p-value
25	477	0.49930
100	543	0.49839
250	496	0.49869
500	498	0.50014
1000	507	0.49868

Observe that the null hypothesis (3.39) does not characterize a unique copula, but only gives the volumes of the d-boxes needed in the construction of a d-sample copula of order m. However, when m is large enough (3.39) approximates closely the underlying copula C, by Theorem 3.5. We also performed simulations for $d = 3$ for different families of 3-copulas obtaining similar results. Of course, we can also use different statistics to test (3.39), for example the ones proposed in [7] or [60].

It is very important to observe that it may be possible to make bayesian inference. By Proposition 1, we know that the square d-dimensional matrix S^n_m needed in the construction of the d sample copula of order m follows a multinomial distribution, with restrictions on the values of the $p_{\underline{\mathbf{i}}}$ for $q_{\underline{\mathbf{i}}} \in \mathcal{I}^C_m$. So, we could try to extend the classical approach of considering a Dirichlet prior for the parameters in order to obtain the posterior distribution based on a sample, as in [3]. But, this is material for future research.

Now we study the general setting of the sample d-copulas.

3.2.2 Sample d-Copula of Order m for a Continuous d-Distribution Function

Let $m, d \geq 2$ be fixed integers and let H be a continuous d-distribution function in R^d. Let $V_n = \{\underline{\mathbf{z_1}}, \ldots, \underline{\mathbf{z_n}}\}$ be a random sample from H of size $n \geq m$. Let $U_n = \{\underline{\mathbf{x_1}}, \ldots, \underline{\mathbf{x_n}}\}$ be the usual **modified sample** or pseudo sample, that is, if $j \in \{1, \ldots, n\}$ and $\underline{\mathbf{z_j}} = \langle z_{j,1}, \ldots, z_{j,d} \rangle \in \mathrm{R}^d$, define for $k \in \{1, \ldots, d\}$

$$R_{j,k} = \sum_{l=1}^{n} 1_{\{z_{l,k} \leq z_{j,k}\}}, \tag{3.40}$$

where $R_{j,k}$ is the rank of the observation $z_{j,k}$ for l varying between 1 and n. Now define for every $j \in \{1, \ldots, n\}$, $\underline{\mathbf{x_j}} = \langle x_{j,1}, \ldots, x_{j,d} \rangle$ where

$$x_{j,k} = \frac{R_{j,k}}{n} \quad \text{and for every} \quad k \in \{1, \ldots, d\}. \tag{3.41}$$

Then

$$U_n = \{\underline{\mathbf{x_1}}, \ldots, \underline{\mathbf{x_n}}\} \subset \{1/n, \ldots, (n-1)/n, 1\}^d \subset \mathbf{I}^d. \tag{3.42}$$

Of course, from the continuity assumption on H, the ranks in the definition of $\underline{\mathbf{x_j}}$ are all different for every $j \in \{1, \ldots, n\}$ almost surely.

Recall that the **empirical d-copula** for the modified sample U_n is defined by

$$C^n(u_1, \ldots, u_d) = \frac{1}{n} \sum_{j=1}^{n} 1_{\{x_{j,1} \leq u_1, \ldots, x_{j,d} \leq u_d\}}, \tag{3.43}$$

for every $\langle u_1, \ldots, u_d \rangle \in \mathbf{I}^d$, see for example Nelsen [58]. Observe that the empirical d-copula is not a d-copula. For example, if $\underline{\mathbf{u}} = \langle u_1, \ldots, u_d \rangle$ and $0 < u_1 < 1/n$ then $C^n(u_1, \ldots, u_d) = 0$. In fact, since $C^n(u_1, \ldots, u_d) = 0$ if for some $j \in \{1, \ldots, d\}$, $u_j = 0$. Then it is well known that the restriction of C^n to the grid $\{0, 1/n, \ldots, (n-1)/n, 1\}$ is a d-subcopula.

For $m \geq 2$, $n \geq m$ and U_n a modified random sample from a continuous d-distribution function H. Define $s^n_{i_1, \ldots, i_d}, S^n_m, \mathcal{S}^+$ and C^n_m as in equations (3.16), (3.17), (3.19) and (3.20). In this case the structure of the modified sample U_n simplifies significantly the structure of the sample d-copula of order m, as can be seen in the following:

Theorem 3.12. *Let U_n be a modified random sample obtained from an original random sample V_n of H a continuous d-distribution function in R^d. Define $s^n_{i_1, \ldots, i_d}, S^n_m, \mathcal{S}^+$ and C^n_m*

as in equations (3.16), (3.17), (3.19) and (3.20). Then

$$S_m^n \in \mathcal{S}^+ \quad \text{for every} \quad 2 \leq m \leq n, \tag{3.44}$$

that is, S_m^n is always a generalized transformation square matrix. Besides, if $2 \leq m \leq n$ and we define for every $j \in \{1, \ldots, d\}$ the partitions $\pi_j^n = \{0 = p_{j,0}, p_{j,1}, \ldots, p_{j,m-1}, p_{j,m} = 1\}$ given in equation (3.13), then

$$p_{j,k} = \frac{\lfloor \frac{k \cdot n}{m} \rfloor}{n} \quad \text{for every } j \in \{1, \ldots, d\} \text{ and for every } k \in \{1, \ldots, m\}, \tag{3.45}$$

where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to a. In particular, when $n = m$, $\pi_j^n = \{0, 1/n, \ldots, (n-1)/n, 1\}$ for every $j \in \{1, \ldots, d\}$.
Even more, when $n = m \geq 2$ the sample d-copula C_n^n is such that

$$C_n^n(u_1, \ldots, u_d) = C^n(u_1, \ldots, u_d) \ \text{for every } \langle u_1, \ldots, u_d \rangle \in \{0, 1/n, \ldots, (n-1)/n, 1\}^d, \tag{3.46}$$

that is, we recover the empirical d-copula given in equation (3.43) on the grid $\{0, 1/n, \ldots, 1\}^d$.

Proof. Let U_n be a modified random sample obtained from an original random sample V_n of H a continuous d-distribution function in $\mathbf{R}^d$ and define $s_{i_1,\ldots,i_d}^n, S_m^n, \mathcal{S}^+$ and C_m^n as in equations (3.16), (3.17), (3.19) and (3.20).
Of course, it is enough to see that equation (3.44) holds for the limit case, that is, when $n = m \geq 2$. So Assume that $n = m \geq 2$, in this case, from equations (3.40) and (3.41), we know that $x_{j,k} = R_{j,k}/n$ for every $j, k \in \{1, \ldots, n\}$. But, since all the ranks are different with probability one, we have that the matrix $S_n^n = (s_{i_1,\ldots,i_d}^n)_{i_1,\ldots,i_d=1}^n$, given in equation (3.17), satisfies that for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, n\}$

$$\sum_{i_1=1}^{n} \cdots \sum_{i_{j-1}=1}^{n} \sum_{i_{j+1}=1}^{n} \cdots \sum_{i_d=1}^{n} s_{i_1,\ldots,i_{j-1},i_j=k,i_{j+1},\ldots,i_d}^n = \frac{1}{n}. \tag{3.47}$$

Therefore, S_n^n is a d-dimensional square matrix which is a generalized transformation matrix, that is, $S_n^n \in \mathcal{S}^+$. So, (3.44) holds.
Now, assume that m is such that $2 \leq m \leq n$ and define for every $j \in \{1, \ldots, d\}$ the partitions $\pi_j^n = \{0 = p_{j,0}, p_{j,1}, \ldots, p_{j,m-1}, p_{j,m} = 1\}$ given in equation (3.13). Then we know that

$$p_{j,k} = \sum_{i_j=1}^{k} \sum_{i_1=1}^{m} \cdots \sum_{i_{j-1}=1}^{m} \sum_{i_{j+1}=1}^{m} \cdots \sum_{i_d=1}^{m} s_{i_1,\ldots,i_{j-1},i_j,i_{j+1},\ldots,i_d}^n.$$

Now using the sample size n, the partition of $\mathbf{I}^d$ given by $\{R_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}}\in\mathcal{I}_m}$, see equation (3.10), and by equation (3.29), we have that there are $\lfloor (k \cdot n)/m \rfloor$ points in the regions defined by $p_{j,k}$, where $\lfloor a \rfloor$ is the greatest integer less than or equal to a. Therefore, (3.45) holds.

Finally, if we assume that $n = m \geq 2$, using the definition of the d-sample copula of order n, C_n^n given in equation (3.20), the definition of the empirical copula in equation (3.43), the partition given in equation (3.10), together with (3.14) and its generalizations. It is easy to see that equation (3.46) also holds. □

Observe that in the last Theorem, if n is a multiple of m, then by equation (3.45), $p_{j,k} = k/m$ for every $j \in \{1, \ldots, d\}$ and for every $k \in \{1, \ldots, m\}$, that is, we recover the original partition of $\mathbf{I}^d$. In the case that n is not a multiple of m the partition given in equation (3.45) is still a good approximation of the original partition given by $\{R_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}}\in\mathcal{I}_m}$ in equation (3.10).

The statistical procedures presented in Section 3.2.1 can be used for modified samples. For example, in the case of the concordance measures Kendall's tau and Spearman's rho, we can observe that if we have two continuous random variables X and Y, such that $Y = f(X)$, where f is a strictly increasing function almost surely, then it is well known that the copula $C_{X,Y}$ is the M^2 copula. But, in this case, it is obvious to see that if we have an independent random sample of size n of $\langle X, Y\rangle$, and we take $m = n$, then $\tau_{C_n^n} = 1-1/n$ and $\rho_{C_n^n} = 1-1/n^2$ with probability one, which correspond to the upper bounds in (3.38). So, even for small values of n both measures are close to one.

In order to see how the estimation procedure in Section 3.2.1 works for modified samples, we generated 10000 samples of different sizes n of a joint distribution with exponential margins and corresponding copula Frank with parameter $\theta = 5$ and $d = 2$. We use $n = 200, 500, 1000, 5000$ and $n = 10000$. In general the results had the same behavior as the one in Table 10, providing good estimators of θ.

As an application of the hypothesis testing of (3.39) with modified samples we generated 10000 samples of $Z = \langle X_1, X_2, X_3\rangle$ of three independent normal variables with corresponding variances 1, 4 and 9, with different sample sizes $n = 500, 1000, 10000$ and $n = 100000$. Then we obtained the modified samples for each simulation, and we calculate the corresponding 3-dimensional transformation matrices S_3^n for $m = 3$. Finally we tested the hypothesis $H_0 : q_{\langle i,j,k\rangle} = 1/27 = 0.0370370$ for $i, j, k \in \{1, 2, 3\}$, corresponding to independence.

Instead of giving large tables we report only the extreme cases for each sample size for the twenty seven 3-boxes included in each test.

For $n = 500$ the minimal expected value observed was 0.03683, the maximal expected value was 0.03727, the minimal value observed was 0.006 and the maximal was 0.078, and the maximal variance was 0.0000554. For $n = 1000$ the minimal expected value observed was 0.03690, the maximal expected value was 0.03726, the minimal value observed was 0.013 and the maximal was 0.065, and the maximal variance was 0.0000277. For $n = 10000$ the minimal expected value observed was 0.03700, the maximal expected value was 0.03706, the minimal value observed was 0.0297 and the maximal was 0.044, and the maximal variance was 0.0000028. For $n = 100000$ the minimal expected value observed was 0.037020, the maximal expected value was 0.037056, the minimal value observed was 0.03499 and the maximal was 0.03935, and the maximal variance was 0.00000028. From these results it is clear that the estimations of the $q_{\underline{\mathbf{i}}}$ largely improve as n increases.

When we implement the hypothesis testing we observed that when $\alpha = 0.05$, the number of rejections of (3.39) was quite small for $n \geq 500$.

We also applied the test with $m = 2$ and $d = 3$ with similar results. It is quite important to provide elapsed times for each individual run, from different sample sizes. In Table 16 we give the average elapsed times of each test using modified samples, and the last column gives us the average times to find the empirical copulas. We observed that for $n \geq 600$ the language **R** can not allocate arrays of the required size (NA). We ran our simulations using the language **R** in a Dell Precision 490 Workstation. It is clear that the elapsed times increase linearly with the sample size n, instead of polynomially as it is the case for the empirical copula.

Table 16.- Elapsed times for different sample sizes n in dimension $d = 3$

n	seconds for $m = 2$	seconds for $m = 3$	seconds for $m = n$
50	0.007	0.03	113.28
100	0.017	0.06	1799.36
500	0.08	0.25	722061
1000	0.15	0.50	NA
10000	1.57	4.87	NA
100000	15.94	48.28	NA

The empirical d-copula has big restrictions in terms of evaluations in computers, for example if we consider a sample size $n = 1000$ in dimension $d = 4$, then we need an array of 10^{12} entries, and in many situations we have to perform calculations with this array, which generally can not be supported in a computer. The sample d copula of order m only needs an array of m^d entries which is more manageable specially for small m. Since we can use in its definition $2 \leq m \leq n$, we recommend to use $m = 2$ as the first approximation, in many instances the sample d-copula of order 2 gives us some preliminary information about the data, as observed in Section 3.2.

The sample d-copula of order m can be used in several statistical procedures, such as goodness of fit tests, tests of symmetry, estimation of one or more parameters in parametric models, etc.

Of course by equation (3.46) in Theorem 3.7, we can also use all the asymptotic results known for the empirical d-copula in the case that $n = m$. In the case that $2 \leq m < n$ we think that the convergence of the sample d-copula of order m has also nice asymptotic properties, but this is a topic for future research.

3.3 Minimal number of parametersof a d-dimensionally stochastic matrix

In this section we find the number of parameters necessary for a square d-dimensional matrix to be a d-dimensional stochastic matrix.

We start this section with the simplest case. Assume that $d = 2$ and $m = 2$ and let $\mathbf{P} = (p_{i_1,i_2})_{i_1,i_2=1}^2$ be a real square matrix which is doubly stochastic, then using equation (3.2) we have that $\mathbf{P}$ satisfies:

$$\sum_{i_1=1}^{2} p_{i_1,j} = 1 \quad \text{for} \quad j = 1, 2, \tag{3.48}$$

and

$$\sum_{i_2=1}^{2} p_{i,i_2} = 1 \quad \text{for} \quad i = 1, 2. \tag{3.49}$$

Solving the equations (3.48) and (3.49) we observe that the solution is not unique, and it is given simply by: If $0 \leq p_{1,1} \leq 1$, then $p_{1,2} = p_{2,1} = 1 - p_{1,1}$ and $p_{2,2} = 1 - p_{1,2} = p_{1,1}$, that is, in this case there is only one parameter let us say $0 \leq p_{1,1} \leq 1$ and its value determines uniquely the values of $p_{1,2}, p_{2,1}$ and $p_{2,2}$.

Now assume that $d = 2$ and $m \geq 3$ and let $\mathbf{P} = (p_{i_1,i_2})_{i_1,i_2=1}^m$ be a real square matrix which is doubly stochastic, then using equation (3.2) we have that $\mathbf{P}$ satisfies:

$$\sum_{i_1=1}^{2} p_{i_1,j} = 1 \quad \text{for} \quad j = 1, 2, \ldots, m, \tag{3.50}$$

and

$$\sum_{i_2=1}^{2} p_{i,i_2} = 1 \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{3.51}$$

Observe that (3.50) and (3.51) is a system of $2m$ equations with m^2 variables. Of course if there is a solution of this system of equations the solution is not unique. First let us observe that all entries of $\mathbf{P}$ are nonnegative and let us define $\mathbf{Q} = (p_{i_1,i_2})_{i_1,i_2=1}^{m-1}$. Then from equations (3.50) and (3.51) we have that

$$p_{m,j} = 1 - \sum_{i_1=1}^{m-1} p_{i_1,j} \quad \text{for every} \quad j \in \{1, \ldots, m-1\} \tag{3.52}$$

and

$$p_{i,m} = 1 - \sum_{i_2=1}^{m-1} p_{i,i_2} \quad \text{for every} \quad i \in \{1, \ldots, m-1\}. \tag{3.53}$$

So, using (3.52) and (3.53) we have that

$$\begin{aligned} p_{m,m} &= 1 - \sum_{j=1}^{m-1} p_{m,j} \\ &= 1 - \sum_{j=1}^{m-1} \left(1 - \sum_{i_1=1}^{m-1} p_{i_1,j} \right) \\ &= -m + 2 + \sum_{j=1}^{m-1} \sum_{i_1=1}^{m-1} p_{i_1,j} \end{aligned}$$

$$= -m+2+\sum_{i=1}^{m-1}\sum_{i_2=1}^{m-1} p_{i,i_2}$$

$$= 1-\sum_{i=1}^{m-1}\left(1-\sum_{i_2=1}^{m-1} p_{i,i_2}\right)$$

$$= 1-\sum_{i=1}^{m-1} p_{i,m}. \tag{3.54}$$

Observe that $p_{i,m}$ and $p_{m,j}$ for $i,j \in \{1,2,\ldots,m\}$ can be expressed in terms of the entries of the matrix $\mathbf{Q}$. Hence, the number of parameters required to obtain the matrix $\mathbf{P}$ is the number of entries of the matrix $\mathbf{Q}$, that is, there are $(m-1)^2 = m^2-2(m-1)-1 = m^d-d(m-1)-1$ parameters and $m^2-(m-1)^2 = 2m-1 = d(m-1)+1$ entries of the matrix $\mathbf{P}$ which are determined uniquely in terms of the parameters.
Now, let $\mathbf{P}' = (p_{i_1,i_2})_{i_1,i_2=1}^{m-1}$ be a real square matrix of order $m-1$ with nonnegative entries which satisfies the following $2(m-1)+1 = d(m-1)+1$ equations:

$$0 \le \sum_{i_1=1}^{m-1} p_{i_1,j} \le 1 \quad \text{for every} \quad j \in \{1,\ldots,m-1\}, \tag{3.55}$$

$$0 \le \sum_{i_2=1}^{m-1} p_{i,i_2} \le 1 \quad \text{for every} \quad i \in \{1,\ldots,m-1\}, \tag{3.56}$$

and

$$m-2 \le \sum_{i_1=1}^{m-1}\sum_{i_2=1}^{m-1} p_{i_1,i_2} \le m-1. \tag{3.57}$$

Then if we extend $\mathbf{P}'$ to a square matrix $\mathbf{P}$ of order m by defining $p_{i,m}, p_{m,j}$ as in equations (3.52), (3.53) and $p_{m,m}$ as in equation (3.54), we clearly have that $\mathbf{P}$ is a doubly stochastic matrix. Finally, we observe that all equations (3.55), (3.56) and (3.57) are required to obtain a doubly stochastic matrix.

Example 3.13. Let us take $d = 2$ and $m = 3$ and consider the following three square matrices

$$\mathbf{P}'_{\mathbf{1}} = \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 0.2 \end{pmatrix} \qquad \mathbf{P}'_{\mathbf{2}} = \begin{pmatrix} 0.6 & 0.1 \\ 0.6 & 0.1 \end{pmatrix} \quad \text{and} \quad \mathbf{P}'_{\mathbf{3}} = \begin{pmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{pmatrix} \tag{3.58}$$

We first observe that the matrix $\mathbf{P_1'}$ satisfies the four equation in (3.55) and (3.56), but it does not satisfy equation (3.57) since $\sum_{i_1=1}^{m-1}\sum_{i_2=1}^{m-1} p_{i_1,i_2} = .8 < m-2 = 1$. So, if we try to define $p_{3,3} = 1 - p_{1,3} - p_{2,3} = 1 - 0.6 - 0.6 = -0.2 < 0$ then $\mathbf{P_1'}$ can not be extended to a doubly stochastic matrix.

In the case of $\mathbf{P_2'}$ we observe that it satisfies equations (3.55), (3.57) and $0 \leq p_{1,2} + p_{2,2} = 0.2 < 1$, but, $p_{1,1} + p_{2,1} = 1.2 > 1$. So, if we try to define $p_{3,1} = 1 - p_{1,1} - p_{2,1} = -0.2 < 0$, then $\mathbf{P_2'}$ can not be extended to a doubly stochastic matrix.

in the case of $\mathbf{P_3'}$ it satisfies (3.55), (3.56) and (3.57) in this case the extended doubly stochastic matrix would be:

$$\mathbf{P_3} = \begin{pmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.6 \end{pmatrix}.$$

Now we will consider higher dimensions, that is, $d > 2$. First, let us consider the case $d = 3$, $m = 2$ and let $\mathbf{P} = (p_{i_1,i_2,i_3})_{i_1,i_2,i_3=1}^{2}$ be a 3-dimensionally stochastic matrix of order $m = 2$. In this case, using equation (3.2), $\mathbf{P}$ defines a system of 6 equations with 8 variables. Define

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} p_{1,1,1} \\ p_{1,1,2} \\ p_{2,1,1} \\ p_{2,1,2} \\ p_{1,2,1} \\ p_{1,2,2} \\ p_{2,2,1} \\ p_{2,2,2} \end{pmatrix}, \tag{3.59}$$

and let $b = (1,1,1,1,1,1,1,1)$. We want to find the general solution of the linear equation $A \cdot x = b$, using Mathematica [71] and the function LinearSolve($A.x, b$) we obtained a general solution given by:

$$\begin{aligned} p_{1,2,2} &= 1 - p_{1,1,1} - p_{1,1,2} - p_{1,2,1}, \\ p_{2,1,2} &= 1 - p_{1,1,1} - p_{1,1,2} - p_{2,1,1}, \\ p_{2,2,1} &= 1 - p_{1,1,1} - p_{1,2,1} - p_{2,1,1}, \\ p_{2,2,2} &= -1 + p_{1,1,2} + p_{1,2,1} + p_{2,1,1} + 2p_{1,1,1}. \end{aligned} \tag{3.60}$$

From (3.60) we can see that the solutions of $A \cdot x = b$ are given in terms of four parameters, that is, the set $Q = \{p_{1,1,1}, p_{1,1,2}, p_{1,2,1}, p_{2,1,1}\}$, notice that the set of parameters is given by all the p_{i_1,i_2,i_3}, with exactly one $i_j = 2 = m$ or without any $i_j = 2 = m$. Also observe that in general the possible solutions do not have any further restrictions, so, for example $x' = (-1, 1, 1, 0, 1, 0, 0, 0)$ is one of the solutions of $A \cdot x = b$, where x' denotes x transpose. Since $\mathbf{P} = (p_{i_1,i_2,i_3})_{i_1,i_2,i_3=1}^2$ is a 3-dimensionally stochastic matrix of order $m = 2$, we have additional restrictions, that is, $0 \leq p_{i_1,i_2,i_3} \leq 1$ for every $i_1, i_2, i_3 \in \{1, 2\}$ then we also have restrictions on the parameters associated to (3.60) given by:

$$\begin{aligned} 0 \leq p_{1,1,1} + p_{1,1,2} + p_{1,2,1} &\leq 1 \\ 0 \leq p_{1,1,1} + p_{1,1,2} + p_{2,1,1} &\leq 1 \\ 0 \leq p_{1,1,1} + p_{1,2,1} + p_{2,1,1} &\leq 1 \\ 1 \leq p_{1,1,2} + p_{1,2,1} + p_{2,1,1} + 2p_{1,1,1} &\leq 2. \end{aligned} \tag{3.61}$$

Conversely, if we define the set Q as above such that $0 \leq p_{1,1,1}, p_{1,1,2}, p_{1,2,1}, p_{2,1,1} \leq 1$, and we assume that the restrictions in (3.61) are satisfied, then by defining $p_{1,2,2}$, $p_{2,1,2}$, $p_{2,2,1}$ and $p_{2,2,2}$ using equation (3.60), $\mathbf{P} = (p_{i_1,i_2,i_3})_{i_1,i_2,i_3=1}^2$ is clearly a 3-dimensionally stochastic matrix.

Notice that again the four restrictions in (3.61) are necessary in order to obtain a 3-dimensionally stochastic matrix. For example if we let $p_{1,1,1} = p_{1,1,2} = p_{1,2,1} = p_{2,1,1} = 0.1$, then using (3.60) we have that $p_{1,2,2} = p_{2,1,2} = p_{2,2,1} = 0.7$, but, $p_{2,2,2} = -0.5$. So, $\mathbf{P} = (p_{i_1,i_2,i_3})_{i_1,i_2,i_3=1}^2$ is not a 3-dimensionally stochastic matrix of order $m = 2$. Observe that in this case $Q = \{p_{1,1,1}, p_{1,1,2}, p_{1,2,1}, p_{2,1,1}\}$ satisfies the first three conditions in (3.61), but the fourth condition is not satisfied.

As a second example in higher dimensions we consider $d = 4$ and $m = 2$, so, let $\mathbf{P} = (p_{i_1,i_2,i_3,i_4})_{i_1,i_2,i_3,i_4=1}^2$ be a 4-dimensionally stochastic matrix of order $m = 2$. In this case, using equation (3.2), $\mathbf{P}$ defines a system of 8 equations with 16 variables. Define

$$A = \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \end{pmatrix}, \tag{3.62}$$

$x = (p_{1,1,1,1}, p_{1,1,1,2}, p_{1,1,2,1}, p_{1,1,2,2}, p_{1,2,1,1}, p_{1,2,1,2}, p_{1,2,2,1}, p_{1,2,2,2}, p_{2,1,1,1}, p_{2,1,1,2},$

$p_{2,1,2,1}, p_{2,1,2,2}, p_{2,2,1,1}, p_{2,2,1,2}, p_{2,2,2,1}, p_{2,2,2,2})$ and $b = (1,1,1,1,1,1,1,1)$. We want to find the general solution of the linear equation $A \cdot x = b$, using Mathematica again we obtained a solution given by:

$$\begin{aligned} p_{2,2,2,2} &= -2 + p_{2,2,1,1} + p_{2,1,2,1} + p_{2,1,1,2} + p_{1,2,2,1} + p_{1,2,1,2} + p_{1,1,2,2} \\ &\quad +2p_{1,2,1,1} + 2p_{1,1,2,1} + 2p_{1,1,1,2} + 3p_{1,1,1,1}, \end{aligned}$$

$$\begin{aligned} p_{2,2,2,1} &= 1 - p_{2,2,1,1} - p_{2,1,2,1} - p_{1,2,2,1} - p_{1,2,1,1} - p_{1,1,2,1} - p_{1,1,1,1}, \\ p_{2,2,1,2} &= 1 - p_{2,2,1,1} - p_{2,1,1,2} - p_{1,2,1,2} - p_{1,2,1,1} - p_{1,1,1,2} - p_{1,1,1,1}, \\ p_{2,1,2,2} &= 1 - p_{2,1,2,1} - p_{2,1,1,2} - p_{1,1,2,2} - p_{1,1,2,1} - p_{1,1,1,2} - p_{1,1,1,1}, \\ p_{2,2,2,1} &= 1 - p_{1,2,2,1} - p_{1,2,1,2} - p_{1,2,1,1} - p_{1,1,2,2} - p_{1,1,2,1} - p_{1,1,1,2} - p_{1,1,1,1} \end{aligned}$$

and $p_{2,1,1,1} = 0$. Mathematica gives the warning *Equations may not give solutions for all "solve" variables*. In this case it is easy to see that if $p_{i_1,i_2,i_3,i_4} = 1/8$ for every $i_1, i_2, i_3, i_4 \in \{1,2\}$, then $\mathbf{P} = (p_{i_1,i_2,i_3,i_4})_{i_1,i_2,i_3,i_4=1}^2$ satisfies the linear equation $A \cdot x = b$, but $p_{2,1,1,1} \neq 0$. So, the solution given by Mathematica of the equations above is not the most general solution of the linear equation $A \cdot x = b$. However, the equations above show a pattern which allows to find the most general solution of $A \cdot x = b$, which is given by:

$$\begin{aligned} p_{2,2,2,2} &= -2 + p_{2,2,1,1} + p_{2,1,2,1} + p_{2,1,1,2} + p_{1,2,2,1} + p_{1,2,1,2} + p_{1,1,2,2} \\ &\quad +2p_{2,1,1,1} + 2p_{1,2,1,1} + 2p_{1,1,2,1} + 2p_{1,1,1,2} + 3p_{1,1,1,1}, \end{aligned}$$

$$\begin{aligned} p_{2,2,2,1} &= 1 - p_{2,2,1,1} - p_{2,1,2,1} - p_{1,2,2,1} - p_{1,2,1,1} - p_{1,1,2,1} - p_{2,1,1,1} - p_{1,1,1,1}, \\ p_{2,2,1,2} &= 1 - p_{2,2,1,1} - p_{2,1,1,2} - p_{1,2,1,2} - p_{1,2,1,1} - p_{1,1,1,2} - p_{2,1,1,1} - p_{1,1,1,1}, \\ p_{2,1,2,2} &= 1 - p_{2,1,2,1} - p_{2,1,1,2} - p_{1,1,2,2} - p_{1,1,2,1} - p_{1,1,1,2} - p_{2,1,1,1} - p_{1,1,1,1}, \\ p_{1,2,2,2} &= 1 - p_{1,2,2,1} - p_{1,2,1,2} - p_{1,2,1,1} - p_{1,1,2,2} - p_{1,1,2,1} - p_{1,1,1,2} - p_{1,1,1,1}. \end{aligned} \tag{3.63}$$

From (3.63) we can see that the solutions of $A \cdot x = b$ are given in terms of eleven parameters, that is, the set $Q = \{p_{1,1,1,1}, p_{1,1,1,2}, p_{1,1,2,1}, p_{1,2,1,1}, p_{2,1,1,1}, p_{1,1,2,2}, p_{1,2,1,2}, p_{1,2,2,1}, p_{2,1,1,2}, p_{2,1,2,1}, p_{2,2,1,1}\}$, notice that the set of parameters is given by all the p_{i_1,i_2,i_3}, with exactly two i_j's $= 2 = m$, or with exactly one $i_j = 2 = m$ or without any $i_j = 2 = m$. Observe also that in the first equation of (3.63), the parameters with exactly two i_j's $= 2 = m$ appear with coefficient 1, the parameters with exactly one $i_j = 2 = m$ appear with coefficient

2 and the parameter without any $i_j = 2 = m$ appears with coefficient 3. This observation will be quite useful later. Since $\mathbf{P} = (p_{i_1,i_2,i_3,i_4})_{i_1,i_2,i_3,i_4=1}^2$ is a 4-dimensionally stochastic matrix of order $m = 2$, we have additional restrictions, that is, $0 \leq p_{i_1,i_2,i_3,i_4} \leq 1$ for every $i_1, i_2, i_3, i_4 \in \{1, 2\}$, and we also have restrictions on the parameters associated to (3.63) given by:

$$\begin{gathered}
0 \leq p_{2,2,1,1} + p_{2,1,2,1} + p_{1,2,2,1} + p_{1,2,1,1} + p_{1,1,2,1} + p_{2,1,1,1} + p_{1,1,1,1} \leq 1 \\
0 \leq p_{2,2,1,1} + p_{2,1,1,2} + p_{1,2,1,2} + p_{1,2,1,1} + p_{1,1,1,2} + p_{2,1,1,1} + p_{1,1,1,1} \leq 1 \\
0 \leq p_{2,1,2,1} + p_{2,1,1,2} + p_{1,1,2,2} + p_{1,1,2,1} + p_{1,1,1,2} + p_{2,1,1,1} + p_{1,1,1,1} \leq 1 \\
0 \leq p_{1,2,2,1} + p_{1,2,1,2} + p_{1,2,1,1} + p_{1,1,2,2} + p_{1,1,2,1} + p_{1,1,1,2} + p_{1,1,1,1} \leq 1 \\
\text{and} \\
2 \leq p_{1,1,2,2} + p_{1,2,1,2} + p_{1,2,2,1} + p_{2,1,1,2} + p_{2,1,2,1} + p_{2,2,1,1} \\
+2p_{1,2,2,2} + 2p_{2,1,2,2} + 2p_{2,2,1,2} + 2p_{2,2,2,1} + 3p_{2,2,2,2} \leq 3. \qquad (3.64)
\end{gathered}$$

Conversely, if we define the set Q as above with $0 \leq p_{i_1,i_2,i_3,i_4} \leq 1$, for every $\langle i_1, i_2, i_3, i_4 \rangle$ and $p_{i_1,i_2,i_3,i_4} \in Q$, and we assume that the restrictions in (3.64) are satisfied, then by defining $p_{1,2,2,2}$, $p_{2,1,2,2}$, $p_{2,2,1,2}$, $p_{2,2,2,1}$ and $p_{2,2,2,2}$ using equation (3.63), $\mathbf{P} = (p_{i_1,i_2,i_3,i_4})_{i_1,i_2,i_3,i_4=1}^2$ is clearly a 4-dimensionally stochastic matrix.

Again, it is not difficult to see that all restriction in (3.64) are necessary in order to obtain a 4-dimensionally stochastic matrix.

As a third example we consider the case $d = 3$ and $m = 3$. Let $\mathbf{P} = (p_{i_1,i_2,i_3})_{i_1,i_2,i_3=1}^3$ be a 3-dimensionally stochastic matrix of order $m = 3$. In this case we have a system of 9 equations with 27 variables. Using Mathematica we found the general solution which is given by

$$\begin{aligned}
p_{1,3,3} &= 1 - p_{1,3,2} - p_{1,3,1} - p_{1,2,3} - p_{1,2,2} - p_{1,2,1} - p_{1,1,3} - p_{1,1,2} - p_{1,1,1} \\
p_{2,3,3} &= 1 - p_{2,3,2} - p_{2,3,1} - p_{2,2,3} - p_{2,2,2} - p_{2,2,1} - p_{2,1,3} - p_{2,1,2} - p_{2,1,1} \\
p_{3,1,3} &= 1 - p_{3,1,2} - p_{3,1,1} - p_{2,1,3} - p_{2,1,2} - p_{2,1,1} - p_{1,1,3} - p_{1,1,2} - p_{1,1,1} \\
p_{3,2,3} &= 1 - p_{3,2,2} - p_{3,2,1} - p_{2,2,3} - p_{2,2,2} - p_{2,2,1} - p_{1,2,3} - p_{1,2,2} - p_{1,2,1} \\
p_{3,3,1} &= 1 - p_{3,2,1} - p_{3,1,1} - p_{2,3,1} - p_{2,2,1} - p_{2,1,1} - p_{1,3,1} - p_{1,2,1} - p_{1,1,1} \\
p_{3,3,2} &= 1 - p_{3,2,2} - p_{3,1,2} - p_{2,3,2} - p_{2,2,2} - p_{2,1,2} - p_{1,3,2} - p_{1,2,2} - p_{1,1,2}
\end{aligned}$$

and

$$
\begin{aligned}
p_{3,3,3} = -3 + p_{3,2,2} + p_{3,2,1} + p_{3,1,2} + p_{3,1,1} + p_{2,3,2} + p_{2,3,1} + p_{2,2,3} + p_{2,1,3} + \\
p_{1,3,2} + p_{1,3,1} + p_{1,2,3} + p_{1,1,3} + \\
2p_{2,2,2} + 2p_{2,2,1} + 2p_{2,1,2} + 2p_{2,1,1} + 2p_{1,2,2} + 2p_{1,2,1} + 2p_{1,1,2} + 2p_{1,1,1}
\end{aligned} \tag{3.65}
$$

Notice that in this case we have twenty parameters and the set of parameters is given by all p_{i_1,i_2,i_3} with exactly one $i_j = 3 = m$ or without any $i_j = 3 = m$. We also observe that in the last equation of (3.65) the parameters with exactly one $i_j = 3 = m$ appear with coefficient 1 and the parameters without any $i_j = 3 = m$ appear with coefficient 2.

Finally, observe that in all cases, that is, (3.60), (3.63) and (3.65) the sum of all entries of $\mathbf{P}$ is always m as expected.

Now we will solve the general case, that is, $d \geq 2$ and $m \geq 2$. But first we will introduce some notation.

Let $I_m = \{1, 2, \ldots, m\}$ for any integer $m \geq 2$, and for any $k \in \{0, 1, 2, \ldots, d\}$, where $d \geq 2$, define

$$
\mathcal{I}^d_{m,k} = \{\, \langle i_1, i_2, \ldots, i_d \rangle \in I^d_m \mid \text{with exactly } (d-k) \text{ coordinates equal to } \; m\}. \tag{3.66}
$$

Now we state and prove our main result:

Theorem 3.14. *Let $d \geq 2$ and $m \geq 2$ be two integers, let $\mathbf{P} = (p_{i_1,i_2,\ldots,i_d})_{\langle i_1,\ldots,i_d\rangle \in I^d_m}$ be a d-dimensionally stochastic square matrix. Then $\mathbf{P}$ satisfies the following $d(m-1)+1$ equations*

$$
0 \leq p_{m,m,\ldots,m} = m + d - md + \sum_{k=2}^{d} \left((k-1) \sum_{\langle i_1,\ldots,i_d\rangle \in \mathcal{I}^d_{m,k}} p_{i_1,\ldots,i_d} \right) \leq 1, \tag{3.67}
$$

$$
\begin{aligned}
0 \leq p_{m,m,\ldots,m,1} &= 1 - \textstyle\sum_{\langle i_1,\ldots,i_{d-1}\rangle \in I^{d-1}_m \setminus \{\langle m,m,\ldots,m\rangle\}} p_{i_1,\ldots i_{d-1},1} \leq 1 \\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots \\
0 \leq p_{m,m,\ldots,m,m-1} &= 1 - \textstyle\sum_{\langle i_1,\ldots,i_{d-1}\rangle \in I^{d-1}_m \setminus \{\langle m,m,\ldots,m\rangle\}} p_{i_1,\ldots i_{d-1},m-1} \leq 1 \\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots \\
0 \leq p_{1,m,m,\ldots,m} &= 1 - \textstyle\sum_{\langle i_2,\ldots,i_d\rangle \in I^{d-1}_m \setminus \{\langle m,m,\ldots,m\rangle\}} p_{1,i_2\ldots i_d} \leq 1 \\
&\;\;\vdots \qquad\qquad\qquad\qquad \vdots \\
0 \leq p_{m-1,m,m,\ldots,m} &= 1 - \textstyle\sum_{\langle i_2,\ldots,i_d\rangle \in I^{d-1}_m \setminus \{\langle m,m,\ldots,m\rangle\}} p_{m-1,i_2\ldots i_d} \leq 1.
\end{aligned} \tag{3.68}
$$

Conversely if we have $Q = \{p_{i_1,i_2,\ldots,i_d} | \langle i_1,\ldots,i_d\rangle \in \cup_{k=2}^{d}\mathcal{I}_{m,k}^{d}\}$ *a set of* $m^d - (d(m-1)+1)$ *parameters on* $[0,1]$ *which satisfy the following* $d(m-1)+1$ *conditions*

$$md - m - d \le \sum_{k=2}^{d}\Big((k-1)\sum_{\langle i_1,\ldots,i_d\rangle\in\mathcal{I}_{m,k}^{d}} p_{i_1,\ldots,i_d}\Big) \le md - m - d + 1$$
$$0 \le \sum_{\langle i_1,\ldots,i_{d-1}\rangle\in I_m^{d-1}\setminus\{\langle m,m,\ldots,m\rangle\}} p_{i_1,\ldots i_{d-1},m-1} \le 1$$
$$\vdots$$
$$0 \le \sum_{\langle i_1,\ldots,i_{d-1}\rangle\in I_m^{d-1}\setminus\{\langle m,m,\ldots,m\rangle\}} p_{i_1,\ldots i_{d-1},m-1} \le 1$$
$$\vdots$$
$$0 \le \sum_{\langle i_2,\ldots,i_d\rangle\in I_m^{d-1}\setminus\{\langle m,m,\ldots,m\rangle\}} p_{1,i_2\ldots i_d} \le 1$$
$$\vdots$$
$$0 \le \sum_{\langle i_2,\ldots,i_d\rangle\in I_m^{d-1}\setminus\{\langle m,m,\ldots,m\rangle\}} p_{m-1,i_2\ldots i_d} \le 1. \tag{3.69}$$

Then if we define $p_{i_1,\ldots,i_d}$ *for* $\langle i_1,\ldots,i_d\rangle \in \cup_{k=0}^{1}\mathcal{I}_{m,k}^{d}$ *using equations (3.67) and (3.68), the resulting square matrix* $\mathbf{P} = (p_{i_1,\ldots,i_d})_{\langle i_1,\ldots,i_d\rangle\in I_m^d}$ *is a d-dimensionally square matrix.*

Proof. We first observe that using definition (3.66), $I_m^d = \cup_{k=0}^{d}\mathcal{I}_{m,k}^{d}$ which is a disjoint union. We also observe that $\mathcal{I}_{m,0}^{d} = \{\langle m,m,\ldots,m\rangle\}$ and $\mathcal{I}_{m,1}^{d} = \{\langle m,m,\ldots,m,1\rangle,\ldots,\langle m,m,\ldots,m,m-1\rangle,\ldots,\langle 1,m,m,\ldots,m\rangle,\ldots,\langle m-1,m,m,\ldots,m\rangle\}$. So, the cardinality of $\mathcal{I}_{m,0}^{d}\cup\mathcal{I}_{m,1}^{d}$ equals to $d(m-1)+1$.

First, define $Q = \{p_{i_1,i_2,\ldots,i_d} | \langle i_1,\ldots,i_d\rangle \in \cup_{k=2}^{d}\mathcal{I}_{m,k}^{d}\}$ a set of $m^d - (d(m-1)+1)$ parameters on $[0,1]$ which satisfy the $d(m-1)+1$ conditions in equation (3.69), and define $p_{i_1,\ldots,i_d}$ for $\langle i_1,\ldots,i_d\rangle \in \cup_{k=0}^{1}\mathcal{I}_{m,k}^{d}$. Observe that in the first equation of (3.69) the parameters with exactly $(d-k)$ $i_j^{'}$s $= m$ appear with coefficient $(k-1)$ following the pattern of all the previous examples. Now, using equations (3.67) and (3.68), it is not difficult to see that if we define the square matrix $\mathbf{P} = (p_{i_1,\ldots,i_d})_{\langle i_1,\ldots,i_d\rangle\in I_m^d}$, then $\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_d=1}^{m} p_{i_1,i_2,\cdots,i_d} = m$, and from (3.68) condition (3.2) is also satisfied, so, $\mathbf{P}$ is a d-dimensionally square matrix.

If we assume that $\mathbf{P} = (p_{i_1,\ldots,i_d})_{\langle i_1,\ldots,i_d\rangle\in I_m^d}$ is a d-dimensionally square matrix. Then using (3.2) we have a linear system of md equations with m^d variables, we can write this linear system as $A\cdot x = b$, where A is a matrix of order $md\times m^d$ of zeros and ones, x is a column vector with m^d entries, which includes all the values of $p_{i_1,\ldots,i_d}$ and b is a column vector of ones of order md, the general solution of this linear equation is given in the equalities of (3.67) and (3.68), and since $\mathbf{P}$ is d-dimensionally stochastic then it is clear that all the inequalities in (3.68) are satisfied, inequality (3.67) follows using (3.68) and similar arguments as in equation (3.54). □

It can be easily checked that equation (3.67) holds in the three examples presented above.

As an example let $d \geq 2$ and $m \geq 2$ be two integers, and let us assume that $p_{i_1,\ldots,i_d} = 1/m^{d-1}$ for every $\langle i_1, \ldots, i_d \rangle \in \cup_{k=2}^{d} \mathcal{I}_{m,k}^{d}$. We will see that in this case $p_{i_1,\ldots,i_d} = 1/m^{d-1}$ for every $\langle i_1, \ldots, i_d \rangle \in I_m^d$.

First, from equation (3.68) we have that

$$p_{m,\ldots,m,1} = 1 - \sum_{\langle i_1,\ldots,i_{d-1} \rangle \in I_m^{d-1} \setminus \{\langle m,\ldots,m \rangle\}} p_{i_1,\ldots,i_{d-1},1} = 1 - (m^{d-1} - 1)\frac{1}{m^{d-1}} = \frac{1}{m^{d-1}}.$$

Similarly,

$$p_{m,\ldots,m,1} = \cdots = p_{m,\ldots,m,m-1} = \cdots = p_{1,m,\ldots,m} = \cdots = p_{m-1,m,\ldots,m} = \frac{1}{m^{d-1}}.$$

Now, from equation (3.67) we have that

$$p_{m,\ldots,m} = m + d - md + \sum_{k=2}^{d} (k-1) \sum_{\langle i_1,\ldots,i_d \rangle \in \mathcal{I}_{m,k}^{d}} \frac{1}{m^{d-1}}.$$

Using definition (3.66) we have that

$$\left| \mathcal{I}_{m,k}^{d} \right| = \binom{d}{k} (m-1)^k,$$

where $|\cdot|$ denotes the cardinality of a set. So,

$$p_{m,\ldots,m} = m + d - md + \frac{1}{m^{d-1}} \sum_{k=2}^{d} (k-1) \binom{d}{k} (m-1)^k 1^{d-k}.$$

But,

$$(k-1)\binom{d}{k} = d\binom{d-1}{k-1} - \binom{d}{k}.$$

So,

$$\begin{aligned}\sum_{k=2}^{d}(k-1)\binom{d}{k}(m-1)^k &= d(m-1)\left[\sum_{k=0}^{d-1}\binom{d-1}{k}(m-1)^k 1^{(d-1)-k}-1\right] \\ &\quad -\left[\sum_{k=0}^{d}\binom{d}{k}(m-1)^k 1^{d-k}-1-d(m-1)\right] \\ &= d(m-1)[m^{d-1}-1]-[m^d-1-d(m-1)].\end{aligned}$$

Therefore, $p_{m,\dots,m} = \frac{1}{m^{d-1}}$.

3.4 d-copulas and d-dimensionally stochastic matrices

Let C be a d-copula and define for every $\underline{\mathbf{i}} = \langle i_1, i_2, \dots, i_d\rangle \in I_m^d$, define $R_{\underline{\mathbf{i}}}$ as in (3.10)

$$p_{i_1,i_2,\dots,i_d} = V_C(R_{\underline{\mathbf{i}}}) \quad \text{and} \quad \mathbf{P^C} = (p_{i_1,i_2,\dots,i_d})_{i_1,\dots,i_d=1}^{m}. \tag{3.70}$$

Then $\mathbf{P^C}$ is a square d-dimensional matrix with nonnegative entries, which generates the checkerboard approximation given in [56]. Besides, if we take any $j \in \{1,\dots,d\}$ and any $k \in \{1,\dots,m\}$ then by the definition of d-copula, and using equation (3.12), we have that

$$\begin{aligned}\sum_{\underline{\mathbf{i}}\in I_m^d, i_j=k} p_{\underline{\mathbf{i}}} &= \sum_{i_1=1}^{m}\cdots\sum_{i_{j-1}=1}^{m}\sum_{i_{j+1}=1}^{m}\cdots\sum_{i_d=1}^{m} V_C(R_{\langle i_1,\dots,i_{j-1},k,i_{j+1},\dots,i_d\rangle}) \\ &= \frac{1}{m}. \end{aligned} \tag{3.71}$$

Observe that in equation (3.71) for every d-copula C, for every $j \in \{1,\dots,d\}$ and for every $k \in \{1,\dots,m\}$, $\sum_{\underline{\mathbf{i}}\in I_m, i_j=k} p_{\underline{\mathbf{i}}} = 1/m$ does not depend on j, k or C, only depends on m. Notice also that if we multiply by m all the terms in (3.71) then we have, using equation (3.2), that $m \cdot \mathbf{P^C}$ is a d-dimensionally stochastic matrix. Therefore, using Theorem 3.14, we have established a relation between d-dimensionally stochastic matrices of order $m \geq 2$ and the volumes under a d-copula C of the d-boxes given in partition (3.10) with $m \geq 2$. In fact, we can rewrite Theorem 3.14 in terms of d-copulas.

Theorem 3.15. *Let $d \geq 2$ and $m \geq 2$ be two integers, let $\mathbf{P} = (p_{i_1,i_2,\dots,i_d})_{\langle i_1,\dots,i_d\rangle\in I_m^d}$ be a d-dimensionally stochastic square matrix. Then there exists a d-copula C such that for every*

$\langle i_1, \ldots, i_d \rangle \in I_m^d$

$$\frac{p_{i_1,\ldots,i_d}}{m} = V_C(R_{\langle i_1,\ldots,i_d \rangle}). \tag{3.72}$$

Conversely, if C is a d-copula and $m \geq 2$ is an integer, and we define $p_{i_1,\ldots,i_d}$ and $\mathbf{P^C}$ as in equation (3.70), then $m \cdot \mathbf{P^C}$ is a d-dimensionally stochastic matrix.

The existence of the d-copula C in equation (3.72) is not unique. For example let $d = m = 2$ and define $\mathbf{P} = (p_{i_1,i_2})_{\langle i_1,i_2 \rangle \in I_2^2}$, where $p_{1,1} = p_{2,2} = 1$ and $p_{1,2} = p_{2,1} = 0$. Then clearly $\mathbf{P}$ is a doubly stochastic matrix. Define $C_1(u,v) = M^2(u,v) = \min\{u,v\}$ for every $\langle u,v \rangle \in \mathbf{I}^2$, which is the Fréchet-Hoeffding upper bound copula, and define $C_2(u,v) = \max\{0, u+v-1/2\}$ if $\langle u,v \rangle \in R_{\langle 1,1 \rangle}$, $C_2(u,v) = 1/2 + \max\{0, u+v-3/2\}$ if $\langle u,v \rangle \in R_{\langle 2,2 \rangle}$ and $C_2(u,v) = 0$ otherwise, that is, C_2 is a shuffle of M^2, see [58]. Then it is immediate to see that

$$V_{C_1}(R_{\langle i_1,i_2 \rangle}) = V_{C_1}(R_{\langle i_1,i_2 \rangle}) = p_{i_1,i_2}/2 \quad \text{for every} \quad \langle i_1, i_2 \rangle \in I_2^2.$$

Another key factor in connection with Theorem 3.14, is the fact that if we have $d \geq 2$, $m \geq 2$ and C a d-copula and we define $p_{i_1,\ldots,i_d} = V_C(R_{\langle i_1,\ldots,i_d \rangle})$ for every $\langle i_1, \ldots, i_d \rangle \in I_m^d$ then $Q = \{m \cdot p_{i_1,i_2,\ldots,i_d} | \langle i_1, \ldots, i_d \rangle \in \cup_{k=2}^d \mathcal{I}_{m,k}^d\}$ is a set of $m^d - (d(m-1)+1)$ parameters on $[0,1]$ which satisfy the $d(m-1)+1$ conditions in equation (3.69) and which determines the remaining entries in the d-dimensional matrix $\mathbf{P^C} = (p_{i_1,\ldots,i_d})_{\langle i_1,\ldots,i_d \rangle \in I_m^d}$. This observation allows to reduce the number of parameters to estimate in the multinomial distribution given in Proposition 3.2.1.

3.5 Final Remarks

In the last years many researchers have been proposing methods of constructing multivariate copulas, see for example [22], [61] and [30]. The idea is to provide new families that allow to model multivariate data, since the known models are not numerous enough to do so.

To find multivariate extensions of known results for 2-copulas is of great importance, and lately several papers have been written to achieve this goal. In the case of ordinal sums, see [2] or [58], we have a multivariate extension given in Mesiar and Sempi [55]. For the shuffles, see [58], we have the extension of Durante and Fernández-Sánchez [17]. For extensions in construction of multivariate copulas with a given diagonal, we may cite [39], [62] and [8].

Another interesting references are [10], [11] and [68], see also a recent note on singular copulas in [18].

The importance of the construction of what the authors called self-similar 2-copulas in Cuculescu and Theodorescu [8], was extended in [34] to construct interesting examples of 2-copulas with given fractal supports. For the multivariate extension of the construction of fractal copulas quite recently in Trutschnig and Fernández-Sánchez [69], using the results in [34], give a method using transformation matrices to construct new interesting d-copulas.

In this chapter we provide in Proposition 3.3 the multivariate generalization of the construction in Cuculescu and Theodorescu [8].

We also proved that the resulting family of d-box invariant fractal d-copulas is dense in every dimension $d \geq 2$ for the sup norm.

In section 3.2 we introduced the sample d-copula of order m , based on the ideas of the transformation matrices given in [34], and its generalization in [69], in two settings: First when the sample is obtained from a d-copula C, and second when the sample comes from a continuous d-distribution function on R^d.

In the first case, we observed that the sample d-copula has very nice properties and we provided some important asymptotic results. We also observe that even for small values of n the d-dimensional square matrix S_m^n, used in the definition of the sample d-copula of order m, C_m^n in equation (3.20), is with high probability a generalized transformation matrix. We also provide interesting statistical applications such as a new methodology for estimation of parameters, a goodness-of-fit test and results about the usual concordance measures.

In the second case, for $2 \leq m \leq n$, we proved that d-dimensional square matrix S_m^n, used in the definition of the sample d-copula of order m, C_m^n in equation (3.20), is always a generalized transformation matrix, which allows us to have a non trivial sample d-copula. We also saw that we can recover the empirical d-copula from C_n^n. We also observed that the statistical applications in the first case can be carried out easily to this case by using the modified samples.

In both cases, the empirical d-copula of order m can be used to study the statistical properties of the sample, and to try to model the d-copula that "better fits" the observations.

The empirical copula first proposed by Deheuvels in [13], which he called "*fonction de dépendance empirique*" has very nice theoretical properties. However, for large samples even in small dimensions it has big problems in applications, because of the limitations of a standard computer. If the sample size n is small as well as the dimension d we can still use all the strong statistical techniques developed for empirical copulas, see for example [5], [29] or [35]. However, if the sample size n is large, let us say $n \geq 100000$, even in small dimensions $d = 2$ or $d = 3$, the statisticians require new tools and methods which can be easily implemented in standard computers, without the need of taking a much smaller subsample. We think the sample d copula of order m may be this new tool.

We believe the sample d-copula of order m may be quite useful in applications, because it is easy to obtain and any computer can handle the arrays needed for its construction, considering medium values of m and values of d not so small, even if the sample size n is quite large. This last fact is a great advantage for any statistician.

See the site **https://sites.google.com/site/probstatsr** where all the programs used in this chapter are available in language **R**.

This chapter has been accepted for publication in [38] and [37].

Chapter 4

Frank's Condition for Multivariate Archimedean Copulas.

In this chapter we study some of the possible constructions of a multivariate copula given a diagonal section. We start with the definition of diagonal in a multivariate setting. We will see that the direct algebraic generalizations of the Bertino copula and the diagonal copula (1.22) do not work except when the diagonal section coincides with the diagonal section of the copula M. The main result of this chapter is the generalization of the Frank's condition for a diagonal to determine uniquely an Archimedean copula. Most of the results in this chapter have been accepted for publishing [28].

A very important result that characterizes diagonal sections of copulas, see Definition 1.29, is the following result of Fredricks and Nelsen [32] or Fredricks and Nelsen [33]:

Theorem 4.1. *Let $\delta : \mathbf{I} \to \mathbf{I}$ be a diagonal. Then there exists a copula C such that $\delta_C(t) = \delta(t)$ for every $t \in \mathbf{I}$.*

In particular, in [32] the authors proved Theorem 1.30, that is, if δ is a diagonal and

$$C(u,v) = \min\left\{u, v, \frac{\delta(u)+\delta(v)}{2}\right\} \quad \text{for every} \quad \langle u,v\rangle \in [0,1]^2 \tag{4.1}$$

then C is a singular copula with diagonal section δ and C is **symmetric**, that is, $C(u,v) = C(v,u)$ for every $\langle u,v\rangle \in \mathbf{I}^2$. Another construction of copulas with given diagonal δ can be found in [4], and it is given by

$$B(u,v) = \min\{u,v\} - \inf_{\min\{u,v\}\le t\le \max\{u,v\}} [t - \delta(t)] \quad \text{for every} \quad \langle u,v\rangle \in \mathbf{I}^2\,. \tag{4.2}$$

B is a singular copula called **Bertino copula** with diagonal δ which is also symmetric.

Next we present the definition of diagonal section of a multivariate copula and its properties.

4.1 Possible extensions of n-copulas with a given diagonal section.

We start this section by defining the diagonal section of an n-operation defined on $\mathbf{I}$ for $n \geq 2$.

Definition 4.2. *Let $n \geq 2$ and let $C : \mathbf{I}^n \to \mathbf{I}$ be a function. We define the* **diagonal section of** C*, denoted by δ_C, as a function $\delta_C : \mathbf{I} \to \mathbf{I}$, such that for every $\underline{\mathbf{t}} = \langle t, t, \ldots, t \rangle \in \mathbf{I}^n$*

$$\delta_C(t) = C(\underline{\mathbf{t}}). \tag{4.3}$$

Using Theorems 2.10.2 and 2.10.7 in Nelsen [58], it is easy to see that the diagonal section of an n-copula for $n \geq 2$ satisfies:

Lemma 4.3. *Let $n \geq 2$ and let C be an n-copula with diagonal section δ_C. Then δ_C satisfies the following conditions:*

$$\delta_C(0) = 0 \qquad \textit{and} \qquad \delta_C(1) = 1. \tag{4.4}$$

$$0 \leq \delta_C(t) - \delta_C(s) \leq n(t - s) \quad \textit{for every} \quad 0 \leq s \leq t \leq 1. \tag{4.5}$$

$$\max\{nt - n + 1, 0\} \leq \delta_C(t) \leq t \quad \textit{for every} \quad 0 \leq t \leq 1. \tag{4.6}$$

The last inequality is sharp.

Remark 4.4. It is very important here to notice that W^n is an n-copula if and only if $n = 2$. However, using the proof of Theorem 2.10.13 in Nelsen [58] it is easy to see that for every $n \geq 3$ there exists an n-copula C such that

$$\delta_C(t) = \delta_{W^n}(t) = \max\{nt - n + 1, 0\} \quad \text{for every} \quad t \in \mathbf{I}. \tag{4.7}$$

For example, for $n = 3$ if we define C the 3-copula with density

$$c(x, y, z) = \begin{cases} 9/4 & \text{for} \quad (x, y, z) \in R_1 \cup R_2 \cup R_3, \\ 0 & \quad \text{elsewhere,} \end{cases}$$

where $R_1 = [0, 2/3] \times [0, 2/3] \times [2/3, 1]$, $R_2 = [0, 2/3] \times [2/3, 1] \times [0, 2/3]$ and $R_3 = [2/3, 1] \times [0, 2/3] \times [0, 2/3]$. Then $\delta_C = \delta_{W^3}$.

Now we will see that (4.1) and (4.2) in the introduction cannot be generalized directly to dimensions higher than 2. We start with equation (4.1) of Fredericks and Nelsen [32] and $n = 3$. Let $\delta : [0,1] \to [0,1]$ satisfying equations (4.4), (4.5) and (4.6), the obvious generalization for this case is:

$$C(u,v,w) = \min\left\{u, v, w, \frac{\delta(u)+\delta(v)+\delta(w)}{3}\right\} \quad \text{for every} \quad \langle u,v,w\rangle \in \mathbf{I}^3\,. \tag{4.8}$$

It is clear from equations (4.4), (4.5) and (4.8) that $C(u,v,w) = 0$ if $u = 0$, $v = 0$ or $w = 0$, so C satisfies condition i) of 3-copulas. Besides, using equations (4.4), (4.5) and (4.8), $C(u,1,1) = \min\{u,1,1,(\delta(u)+2\delta(1))/3\} = \min\{u,(\delta(u)+2)/3\} = u$, for every $u \in [0,1]$. Similarly, $C(1,v,1) = v$ and $C(1,1,w) = w$ for every $v, w \in \mathbf{I}$, so, C satisfies condition ii) of a 3-copula. Also, for every $t \in [0,1]$, $C(t,t,t) = \min\{t,\delta(t)\} = \delta(t)$ using equation (4.6). Now we have to check if C in equation (4.8) is 3-increasing. Let $0 < t < 1$ and consider the 3-box $R = [0,t]\times[t,1]\times[t,1]$, then using equations (1.8), (4.8) and the boundary conditions we have that

$$\begin{aligned} V_C(R) &= C(t,1,1) - C(t,1,t) - C(t,t,1) - C(0,1,1) \\ &\quad + C(t,t,t) + C(0,1,t) + C(0,t,1) - C(0,t,t) \\ &= t - 2C(t,t,1) + \delta(t) \\ &= t - 2\min\left\{t, \frac{2\delta(t)+1}{3}\right\} + \delta(t). \end{aligned} \tag{4.9}$$

Now, $t \le (2\delta(t)+1)/3$ if and only if $\delta(t) \ge (3t-1)/2$. Hence, from equation (4.9)

$$V_C(R) = \begin{cases} \delta(t) - t & \text{if} \quad \delta(t) \ge \frac{3t-1}{2} \\ t - \frac{\delta(t)}{3} - \frac{2}{3} & \text{if} \quad \delta(t) < \frac{3t-1}{2}. \end{cases} \tag{4.10}$$

Observe that using equation (4.6), the first row in equation (4.10) is nonnegative if and only if $\delta(t) = t$.

For the second row of equation (4.10), let $h(t) = t - \delta(t)/3 - 2/3$, then $h(t) \ge 0$ if and only if $\delta(t) \le 3t-2$, but, from equation (4.5) we know that $0 \le \delta(v) - \delta(u) \le 3(v-u)$ for every $0 \le u \le v \le 1$. Let $u = t$ and $v = 1$ in the above equation, then $\delta(1) - 3(1-t) \le \delta(t)$, and using equation (4.4) we have that $\delta(t) \ge 3t-2$. Therefore, the second row of equation (4.10) is nonnegative if and only if $\delta(t) = 3t-2$ and $\delta(t) = 3t-2 < (3t-1)/2$ if and only if $t < 1$. Besides, since $\delta(t) \ge 0$, then $\delta(t) = \max\{0, 3t-2\}$, that is, $\delta(t) = \delta_{W^3}(t)$ for every

$t \in \mathbf{I}$. Observe that in this case, if $t < 1/3$ then $\delta(t) = 0 \geq (3t-1)/2$, and using the first row of equation (4.10) we have that $V_C(R) = \delta(t) - t = -t < 0$, so C in equation (4.8) is not a 3-copula.

Therefore, if $\delta(t)$ is a diagonal different from δ_{M^3} then the function defined in (4.8) is not a 3-copula.

If we define $\delta_1(t) = t = \delta_{M^3}(t)$ for every $t \in \mathbf{I}$ then using equation (4.8)

$$C(u,v,w) = \min\{u,v,w,(u+v+w)/3\} = \min\{u,v,w\} = M^3(u,v,w),$$

which is a 3-copula with diagonal δ_1.

Now, let us see if the natural extension of the Bertino copula in equation (4.2) is an n-copula for $n = 3$. Define for $u, v, w \in \mathbf{I}$

$$B(u,v,w) = \min\{u,v,w\} - \inf_{s \in [\min\{u,v,w\}, \max\{u,v,w\}]} [s - \delta(s)], \tag{4.11}$$

where $\delta : \mathbf{I} \to \mathbf{I}$ satisfying conditions (4.4), (4.5) and (4.6). It is clear from conditions (4.4), (4.5) and (4.6), that B satisfies the boundary conditions of a 3-copula and that B is a symmetric function with diagonal section δ. Let $0 < t < 1$ and consider the 3-box $R = [0,t] \times [t,1] \times [t,1]$. Then

$$\begin{aligned}
V_B(R) &= B(t,1,1) - B(t,1,t) - B(t,t,1) - B(0,1,1) \\
&\quad + B(t,t,t) + B(0,1,t) + B(0,t,1) - B(0,t,t) \\
&= \left(t - \inf_{s\in[t,1]}[s-\delta(s)]\right) - 2\left(t - \inf_{s\in[t,1]}[s-\delta(s)]\right) + \delta(t) \\
&= t - 2t + \delta(t) \\
&= \delta(t) - t.
\end{aligned} \tag{4.12}$$

From condition (4.6), $V_B(R) \geq 0$ if and only if $\delta(t) = t$ for every $t \in \mathbf{I}$, that is, if $\delta(t) = \delta_{M^3}(t)$. Therefore, for any δ different from δ_{M^3} the function B defined in (4.11) is not a 3-copula.

In a more recent paper Durante *et al* [21], the authors proposed a copula of the form $D(x,y) = \min\{x,y\} - \min\{\delta(x) - x, \delta(y) - y\}$, where δ is a diagonal. If we try to generalize this to dimension 3, we would have

$$D(x,y,z) = \min\{x,y,z\} - \min\{\delta(x) - x, \delta(y) - y, \delta(z) - z\}. \tag{4.13}$$

Using (4.4) it is clear that $D(x,y,z) = 0$ if $x = 0$, $y = 0$ or $z = 0$, and $D(x,1,1) = x$, $D(1,y,1) = y$ and $D(1,1,z) = z$. So, D satisfies the boundary conditions of a 3-copula and it is symmetric. Let $0 < t < 1$ and consider the 3-box $R = [0,t] \times [t,1] \times [t,1]$, then using eqution (4.13) we have

$$\begin{aligned} V_D(R) &= D(t,1,1) - D(t,1,t) - D(t,t,1) - D(0,1,1) \\ &\quad + D(t,t,t) + D(0,1,t) + D(0,t,1) - D(0,t,t) \\ &= t - 2t + \delta(t) \\ &= \delta(t) - t. \end{aligned} \tag{4.14}$$

Again, as above, D is a 3-copula if and only if $\delta(t) = \delta_{M^3}(t) = t$ for every $t \in \mathbf{I}$, and in this case $D(x,y,z) = \min\{x,y,z\} = M^3(x,y,z)$ for every $x,y,z \in \mathbf{I}$.

The following result is well known.

Theorem C *Let $\delta : \mathbf{I}^n \to \mathbf{I}$ be a* **diagonal** *for some $n \geq 2$, that is, δ satisfies conditions (4.4), (4.5) and (4.6). Then there exists $C : \mathbf{I}^n \to \mathbf{I}$ an n-copula, such that the diagonal section of C, $\delta_C = \delta$.*

A sketch of the proof of this result is given in Cuculescu and Theoderescu [8] and we will see details of this result in Chapter 5. See also Rychlik [62], Jaworski [39], Jaworski and Rychlik [40] and Mesiar and Navara [54].

4.2 Frank's condition for Archimedean 2-copulas

Now, we recall the construction of Archimedean copulas, see for example Chapter 4 of Nelsen [58]. In the next Theorem we restate Definition 1.16, Theorem 1.17 and Definition 1.18.

Theorem 4.5. *Let $\varphi : \mathbf{I} \to [0,\infty]$ be a* **strictly decreasing function***, that is, if $0 \leq s < t \leq 1$ then $\varphi(s) > \varphi(t)$, such that $\varphi(1) = 0$ and $0 < \varphi(0) \leq \infty$. Define the* **pseudo-inverse** *of φ as the function denoted by $\varphi^{[-1]} : [0,\infty] \to \mathbf{I}$, defined by*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{if } \; 0 \leq t \leq \varphi(0) \\ 0 & \text{if } \; t \geq \varphi(0), \end{cases}$$

where φ^{-1} is the usual inverse of φ. Furthermore, $\varphi^{[-1]}(\varphi(t)) = t$ for every $t \in \mathbf{I}$, and $\varphi\left(\varphi^{[-1]}(t)\right) = \min\{t, \varphi(0)\}$ for every $t \in [0,\infty]$. Then if we define

$$C(u,v) = \varphi^{[-1]}\left(\varphi(u) + \varphi(v)\right) \quad \text{for every} \quad \langle u,v \rangle \in \mathbf{I}^2. \tag{4.15}$$

C in (4.15) is a copula, called **Archimedean copula**, *if and only if φ is a* **convex function**, *that is, for every $0 \leq s \leq t \leq 1$ and for every $a \in \mathbf{I}$ we have that $\varphi(as + (1-a)t) \leq a\varphi(s) + (1-a)\varphi(t)$.*

The function φ in equation (4.15) is called an **Archimedean strict generator** if $\varphi(0) = \infty$ and an **Archimedean non-strict generator** if $0 < \varphi(0) < \infty$. In fact, it is easy to see that the pseudo-inverse of φ coincides with the usual inverse if φ is a strict generator. We also know that if φ is a generator of an Archimedean copula C, and we define $\psi = c \cdot \varphi$, where $c > 0$ is a positive constant, then ψ is also a generator of C. It is also clear from equation (4.15), that if C is an Archimedean copula with generator φ and δ_C is its diagonal section then

$$\delta_C(t) = \varphi^{[-1]}(2\varphi(t)) \qquad \text{for every} \qquad t \in \mathbf{I}\,. \tag{4.16}$$

In the case that φ is a strict generator we can rewrite equation (4.16) as

$$\varphi(\delta(t)) = K\varphi(t) \quad \text{where} \quad K = 2. \tag{4.17}$$

The equation (4.17) for $K \neq 0, 1$ is known in the literature as the Schröder functional equation, and it was studied first by Schröder [64] to solve iterative functional equations. The problem that was proposed by Darsow and Frank [9] is the following: If we know the diagonal section of an Archimedean copula C what can be said about its generator φ?
Sungur and Yang [67] state that for Archimedean copulas its diagonal section determines uniquely the corresponding copula. However, this result does not hold. In fact, in the same year Frank [31] announced in a report of a Symposium on functional equations the following result:

Theorem 4.6 (Frank's Condition)**.** *If C is an Archimedean copula with diagonal section δ_C, and $\delta_C'(1-) = 2$ then C is uniquely determined by its diagonal section.*

In the last Theorem $\delta_C'(1-)$ denotes the left derivative of δ_C at $x = 1$. The proof of this result was first included in Alsina *et al* [2], and it includes an example to see that if Frank's condition is not satisfied then we can construct different Archimedean copulas with the same diagonal section.

4.3 Frank's condition for Archimedean n-copulas.

We start this section by recalling some results about multivariate Archimedean copulas, we start with a basic definition.

Definition 4.7. *Let $g : J \to R$ be a an infinitely differentiable function where J is an interval in R. We will say that g is* **completely monotonic** *if its derivatives alternate in signs, that is, g satisfies*

$$(-1)^k \frac{d^k}{dt^k} g(t) \geq 0 \tag{4.18}$$

for every t in the interior of J and for every $k \in \{0, 1, 2, \ldots\}$.

Observe that from equation (4.18), a completely monotonic function satisfies that $\frac{d^2}{dt^2} g(t) \geq 0$, then g is a convex function, see for example Dudley [15]. Using a result in Widder [70], if g is completely monotonic on $J = [0, \infty)$ and $g(c) = 0$ for some $c \in J$, then g is identically zero on J. Therefore, if the pseudo-inverse $\varphi^{[-1]}$, as defined in Theorem 4.6, of an Archimedean generator φ is completely monotonic, it has to be positive on $[0, \infty)$, and so, φ is a strict generator, that is, $\varphi^{[-1]} = \varphi^{-1}$ the usual inverse.

Example 4.8. Let $f(x) = -\ln(x)$ for $0 < x \leq 1$, since f is a nonnegative function such that, $f^{(k)}(x) = (-1)^k / x^k$ for every $k \geq 1$, then f is completely monotonic according to Definition 4.7. Using the fact that the product of completely monotonic functions is also completely monotonic, see for example Miller and Samko [57], we have that $g(x) = (-\ln(x))^n$ for $0 < x \leq 1$ is completely monotonic for every $n \geq 2$.

The following Theorem was first proved in Kimberling [42], and can be found also in Nelsen [58] and Alsina *et al* [2]

Theorem 4.9. *Let φ be a strict Archimedean generator. The function $C : \mathbf{I}^n \to \mathbf{I}$ defined by*

$$C(u_1, \ldots, u_n) = \varphi^{[-1]}(\varphi(u_1) + \cdots + \varphi(u_n)). \tag{4.19}$$

is an n-copula for all $n \geq 2$ if and only if φ^{-1} is completely monotonic.

It is clear that if we define $\varphi(t) = -\ln(t)$ then $\varphi^{-1}(t) = \exp(-t)$ which is clearly completely monotonic, besides we know that φ is the generator of the product copula $\Pi^n(u_1, \ldots, u_n) = u_1 \cdot u_2 \cdots u_n$, see Nelsen [58].

More recently, McNeil and Nešlehova [52] study necessary and sufficient conditions for a generator to construct d-dimensional copulas, for a fixed $d \geq 2$, one of their results states:

Theorem D *Let φ be a strict Archimedean generator with inverse $\phi = \varphi^{-1}$ which has derivatives up to order d on $(0,1)$. Then φ generates an Archimedean d-copula if and only if $(-1)^k\phi^{(k)}(x) \geq 0$ for $k = 0, 1, \ldots, d$.*

This result is Corollary 2.1 in McNeil and Nešlehová [52].

The following result is a particular case of Theorem 6.6 in Kuczma [44], or Theorem 2.3.12 in Kuczma *et al* [45].

Theorem 4.10. *Let $\gamma : \mathbf{I} \to \mathbf{I}$ be a function such that $0 < \gamma(u) < u$ for every $u \in (0,1)$, and assume that $\gamma'(0+) = \frac{1}{n}$ for some $n \geq 2$ fixed. If $s(u)$ is a solution of the functional equation*

$$s\left(\gamma(u)\right) = \frac{1}{n}s(u) \tag{4.20}$$

such that $s(u)/u$ is monotonic in $(0,1)$, then

$$s(u) = k \lim_{m\to\infty} n^m\gamma^m(u) \tag{4.21}$$

where γ^m is the m-th iteration of γ, that is, the composition of γ with itself m times, and k is any positive constant. Solution (4.21) is continuous, convex, unique (up to multiplicative constants) and for $k > 0$ strictly monotonic in $\mathbf{I}$.

Now we will see that Frank's condition in Alsina *et al* [2] can be extended to n-Archimedean copulas using the last Theorem.

Theorem 4.11. *Let $n \geq 3$ and let C be an n-Archimedean copula whose diagonal δ_C satisfies $\delta_C'(1-) = n$. Then C is uniquely determined by its diagonal.*

Proof. Let φ the strict generator of an n-Archimedean copula C for some $n \geq 3$, then using equation (4.19), we have that its diagonal satisfies that $\delta(u) = \varphi^{-1}(n\varphi(u))$ for every $u \in \mathbf{I}$, which is a continuous and strictly increasing function. Therefore, its inverse δ^{-1} is well defined. Now, take $\gamma : \mathbf{I} \to \mathrm{R}$ defined by $\gamma(u) = 1 - \delta^{-1}(1-u)$ for every $u \in \mathbf{I}$, then γ is continuous, strictly increasing, with $\gamma(0) = 0$ and $0 < \gamma(u) < u$ for every $u \in (0,1)$. If we define $s(u) = \varphi(1-u)$ and substitute the definition of γ and s in Schröder's functional equation we have that $\varphi(\delta(u)) = n\varphi(u)$, we get that this functional equation is equivalent to equation (4.20). So, by requiring that $s(u)/u$ be monotonic and $\lim_{u\to 0}[\gamma(u)/u] = 1/n$ we can apply Theorem 4.10, since the existence of a solution $s(u)$ in (4.20) is guaranteed by the existence of $\varphi(u)$ as a consequence of the Archimedeanity hypothesis. The last condition is fulfilled if γ is right differentiable in zero and $\gamma'(0+) = \frac{1}{n}$ which is equivalent to $\delta'(1-) = n$. Besides, since the generator φ is convex and so it is $s(u) = \varphi(1-u)$, then by Proposition 6.3.2 in Dudley [15], we have that $s(u)/u$ is monotonic. Applying Theorem 4.10, we obtain the following formula for φ in terms of the diagonal δ.

$$\varphi(u) = k \lim_{m\to\infty} n^m[1 - \delta^{-m}(u)], \tag{4.22}$$

where δ^{-m} is the composition of δ^{-1} with itself m times and k is a positive constant, because we require that $\varphi \geq 0$. Hence, from Theorem 4.10 solution (4.22) is unique (up to multiplicative constants which generate the same copula C). □

In Alsina *et al* [2], Section 3.8, a counterexample is given, in order to show that if $n = 2$ and φ is generator for an Archimedean copula C such that $\varphi'(1-) = 0$, or equivalently $\delta_C(1-) < 2$, where δ_C is the diagonal of the copula C, then the diagonal does not characterize uniquely the generator φ. Alsina *et al* [2] provide a parametric family of generators $\{\varphi_{\beta,2} \,|\, 0 \leq \beta \leq 1/(1+8\pi)\}$ such that the diagonal section $\delta_{\beta_1,2} = \delta_{\beta_2,2} = \delta_C$, but $C_{\beta_1,2} \neq C_{\beta_2,2}$ for $\beta_1 \neq \beta_2$. We will see that their upper bound for the values of β can be improved. They define for $0 \leq \beta \leq 1$ and for $0 < x < 1$

$$\varphi_{\beta,2}(x) = (\ln(x))^2 + 2^n\beta \sin\left(\frac{(\ln(x))^2}{2^n}\right) \quad \text{if} \quad 2^{n+1}\pi \leq (\ln(x))^2 \leq 2^{n+2}\pi, \tag{4.23}$$

for $n \in \mathbf{Z}$. Observe that the first term in function $\varphi_{\beta,2}$ in equation (4.23) corresponds to the generator of the Gumbel-Hougaard family with $\theta = 2$, see Nelsen [58], Section 4.2. The second term in equation (4.23) is just a small perturbation of the generator by a periodic function. We also observe that the condition $2^{n+1}\pi \leq (\ln(x))^2 \leq 2^{n+2}\pi$ is equivalent to

$$\exp\left(-\sqrt{2^{n+2}\pi}\right) \leq x \leq \exp\left(-\sqrt{2^{n+1}\pi}\right),$$

then we have that

$$\lim_{n\to\infty} \exp\left(-\sqrt{2^{n+2}\pi}\right) = 0 \quad \text{and} \quad \lim_{n\to-\infty} \exp\left(-\sqrt{2^{n+2}\pi}\right) = 1.$$

Then $\lim_{x\to 0} \varphi_{\beta,2}(x) = \infty$ and $\lim_{x\to 1} \varphi_{\beta,2}(x) = 0$. It is easy to see that $\varphi_{\beta,2}$ in equation (4.23) is twice differentiable, where

$$\varphi'_{\beta,2}(x) = \frac{2\ln(x)}{x}\left(1 + \beta\cos\left(\frac{(\ln(x))^2}{2^n}\right)\right),$$

and

$$\varphi''_{\beta,2}(x) = \frac{2(1-\ln(x))}{x^2}\left(1+\beta\cos\left(\frac{(\ln(x))^2}{2^n}\right)\right) - \frac{4(\ln(x))^2}{2^n x^2}\beta\sin\left(\frac{(\ln(x))^2}{2^n}\right),$$

for $\exp\left(-\sqrt{2^{n+2}\pi}\right) \le x \le \exp\left(-\sqrt{2^{n+1}\pi}\right)$ and for every $n \in \mathbf{Z}$. Evaluating numerically the first and second derivatives of $\varphi_{\beta,2}$ we observed that $\varphi'_{\beta,2}(x) \le 0$ and $\varphi''_{\beta,2}(x) \ge 0$ if and only if $0 < \beta \le 0.062548$, and this value is sharp. However, the upper bound of β in Alsina *et al* [2] is $1/(1+8\pi) = 0.038266$ is not sharp, see [26].

Now we give a counterexample, in order to show that if $n = 3$ and φ is generator for an Archimedean 3-copula C such that $\varphi'(1-) = 0$, or equivalently $\delta_C(1-) < 3$, where δ_C is the diagonal of the 3-copula C, then the diagonal does not characterize uniquely the generator φ. We provide a parametric family of generators $\{\varphi_{\beta,3} \,|\, 0 \le \beta \le K\}$ for some $K > 0$, such that the diagonal section $\delta_{\beta_1,3} = \delta_{\beta-2,3} = \delta_C$, but $C_{\beta_1,3} \ne C_{\beta_2,3}$ for $\beta_1 \ne \beta_2$. We define for $0 \le \beta \le 1$ and for $0 < x < 1$

$$\varphi_{\beta,3}(x) = (\ln(x))^4 + 3^m\beta\sin\left(\frac{(\ln(x))^4}{3^m}\right) \quad \text{if} \quad 3^{m+1}\pi \le (\ln(x))^4 \le 3^{m+2}\pi, \tag{4.24}$$

for $m \in \mathbf{Z}$. Again, the first term in function $\varphi_{\beta,3}$ in equation (4.24) corresponds to the the generator of the Gumbel-Hougaard family with $\theta = 4$, and the second term in equation (4.24) is just a small perturbation of the generator by a periodic function. In this case the condition $3^{m+1}\pi \le (\ln(x))^4 \le 3^{m+2}\pi$ for $m \in \mathbf{Z}$, is equivalent to

$$\exp\left(-(3^{m+2}\pi)^{1/4}\right) \le x \le \exp\left(-(3^{m+1}\pi)^{1/4}\right),$$

then we have that

$$\lim_{m\to\infty} \exp\left(-(3^{m+2}\pi)^{1/4}\right) = 0 \quad \text{and} \quad \lim_{m\to-\infty} \exp\left(-(3^{m+2}\pi)^{1/4}\right) = 1.$$

Then $\lim_{x\to 0}\varphi_{\beta,3}(x) = \infty$ and $\lim_{x\to 1}\varphi_{\beta,3}(x) = 0$. It is easy to see that $\varphi_{\beta,3}$ in equation (4.24) is continuous, and in fact, it is three times differentiable, where

$$\varphi'_{\beta,3}(x) = \frac{4(\ln(x))^3}{x}\left(1+\beta\cos\left(\frac{(\ln(x))^4}{3^m}\right)\right),$$

$$\varphi''_{\beta,3}(x) = \frac{4(\ln(x))^2[3-\ln(x)]}{x^2}\left(1+\beta\cos\left(\frac{(\ln(x))^4}{3^m}\right)\right) - \frac{16(\ln(x))^6}{3^m x^2}\beta\sin\left(\frac{(\ln(x))^4}{3^m}\right),$$

and

$$\begin{aligned} \varphi'''_{\beta,3}(x) &= \frac{24\ln(x) - 36(\ln(x))^2 + 8(\ln(x))^3}{x^3}\left(1 + \beta\cos\left(\frac{(\ln(x))^4}{3^m}\right)\right) \\ &- \frac{48(\ln(x))^5[3 - \ln(x)]}{3^m x^3}\beta\sin\left(\frac{(\ln(x))^4}{3^m}\right) \\ &- \frac{64(\ln(x))^9}{3^{2m}x^3}\,\beta\cos\left(\frac{(\ln(x))^4}{3^m}\right). \end{aligned} \tag{4.25}$$

Evaluating numerically it is easy to see that $\varphi_{\beta,3}(x) > 0$ and $\varphi'_{\beta,3}(x) < 0$ for every $0 < x < 1$ and for every $0 \leq \beta \leq 1$. Besides, $\varphi''_{\beta,3}(x) > 0$ for every $0 < x < 1$ if $0 \leq \beta \leq 0.0115$. Let $\phi_{\beta,3} = \varphi^{-1}_{\beta,3}$, then using the formulas for the differentiation of inverse functions we know that

$$\phi'_{\beta,3}(x) = \frac{1}{\varphi'(\phi(x))} \quad \text{and} \quad \phi''_{\beta,3}(x) = \frac{-\varphi''(\phi(x))}{[\varphi'(\phi(x))]^2}.$$

Therefore, $\phi_{\beta,3}(x) > 0$ and $\phi'_{\beta,3}(x) < 0$ for every $0 < x < \infty$ and for every $0 \leq \beta \leq 1$. We also have that $\phi''_{\beta,3}(x) > 0$ for every $0 < x < \infty$ if $0 \leq \beta \leq 0.0115$. Now for the third derivative of $\phi_{\beta,3}$ we have that

$$\phi'''_{\beta,3}(x) = -\frac{\varphi'''_{\beta,3}(\phi_{\beta,3}(x))}{[\varphi'_{\beta,3}(\phi_{\beta,3}(x))]^4} + \frac{3(\varphi''_{\beta,3}(\phi_{\beta,3}(x)))^2}{[\varphi'_{\beta,3}(\phi_{\beta,3}(x))]^5}. \tag{4.26}$$

Observe that in this case the sign of the third derivative of $\phi_{\beta,3}$ is not necessarily the sign of $\varphi'''_{\beta,3}$. Evaluating numerically $\phi'''_{\beta,3}(x)$ in equation (4.26), we obtain that $\phi'''_{\beta,3}(x) < 0$ for every $0 < x < \infty$ if $0 \leq \beta \leq 0.0001267$. Therefore, using Theorem D, if we define $K = 0.0001267$ then the family of generators $\{\varphi_{\beta,3} \mid 0 \leq \beta \leq K\}$ satisfies the desired conditions.

Chapter 5

Higher dimensional copulas with given diagonal sections

The present chapter is a revision of Section 5, *Higher dimension copulas with given diagonal*, in Cuculescu and Theodorescu's paper [8]. For dimension 2 it is well known that given a diagonal, see Definition 1.29, there is always a 2-copula such that its diagonal coincides with the given one. Such a copula is easily constructed for example the diagonal copula in equation (4.1) (see [32], [33]) and the Bertino copula in equation (4.2), as we have seen in chapter 4. For higher dimensions it is also well known that for every diagonal there exists a d-copula such that its diagonal coincides with the given one, however the construction of such copula is not as clear as in dimension 2. We will complete the proof of Theorem 5.1 below using the ideas of Theodorescu [8]. We first give in Theorem 5.2 the necessary and sufficient conditions on a partition of order $2p+1$ of $\mathbf{I}$ so that the function δ_n given in Theorem 5.4 defines a d-diagonal in $\mathbf{I}$. This conditions are not explicitly stated in Cuculescu and Theodorescu [8]. We also filled the gaps in the proof given in [8]. Another proof can be found in Rychlik [62] and Jaworski [39]. We also notice that for a given d-diagonal, the d-diagonal δ_n defined in Theorem 5.4 is the smallest one satisfying $\delta_n\left(\frac{k}{2^n}\right) = \delta\left(\frac{k}{2^n}\right)$, for every $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots, n\}$.

Theorem 5.1. *Let $\delta : \mathbf{I} \to \mathbf{I}$ be a* **diagonal** *for some $n \geq 2$, that is, δ satisfies conditions (4.4), (4.5) and (4.6). Then there exists $C : \mathbf{I}^n \to \mathbf{I}$ an n-copula, such that the diagonal section of C, $\delta_C = \delta$.*

Let us analyze in detail the main ideas in Cuculescu and Theodorescu (2001) [8], we start with a general Theorem which guarantees the existence of a d-copula C with a specific diagonal.

Theorem 5.2. *Let $p \geq 1$ be a positive integer and let $\tau_p = \{t_0, t_1, \ldots, t_{2p-1}, t_{2p}\}$ be a partition of $[0,1]$ such that*

$$0 = t_0 < t_1 < t_2 < \cdots < t_{2p-2} < t_{2p-1} < t_{2p} = 1. \tag{5.1}$$

Define $\delta_{\tau_p} : \mathbf{I} \to R$ to be a continuous piecewise differentiable function with $\delta_{\tau_p}(0) = 0$, such that for every $i \in \{0, 1, \ldots, p-1\}$, δ_{τ_p} is constant on $[t_{2i}, t_{2i+1}]$ and $\delta'_{\tau_p}(t) = d$ for every $t \in (t_{2i+1}, t_{2i})$, where $d \geq 2$ is an integer.
Then δ_{τ_p} is a d-diagonal if and only if the following conditions are satisfied:

$$\sum_{i=0}^{p-1} [t_{2i+2} - t_{2i+1}] = \frac{1}{d} \tag{5.2}$$

and

$$\sum_{i=0}^{k} d(t_{2i+2} - t_{2i+1}) \leq t_{2k+2} \quad \text{for every} \quad k \in \{0, 1, \ldots, p-1\}. \tag{5.3}$$

Besides, if δ_{τ_p} satisfies the above conditions, there exists a d-copula C_{τ_p} such that its diagonal $\delta_{C_{\tau_p}}(t) = \delta_{\tau_p}(t)$ for every $t \in \mathbf{I}$.

Proof. Let $p = 1$, $d \geq 2$ and let $\tau_1 = \{t_0, t_1, t_2\}$ such that $0 = t_0 < t_1 < t_2 = 1$, then from (5.2) we have that $t_2 - t_1 = 1/d$, that is, $t_1 = 1 - 1/d$, then using the definition of δ_{τ_1} we have that

$$\delta_{\tau_1}(t) = \begin{cases} 0 & \text{if} \quad 0 = t_0 \leq t \leq t_1 = 1 - 1/d \\ d(t - 1 + 1/d) & \text{if} \quad 1 - 1/d = t_1 \leq t \leq t_2 = 1. \end{cases}$$

Therefore, $\delta_{\tau_1} = \delta_{W^d}$, where $\delta_{W^d}(t) = \max\{dt - d + 1, 0\}$ is a d-diagonal, using Remark 4.4, we know that there exists a d-copula C_{τ_1} which assigns uniform mass $1/d$ to the sets $A_i = \times_{j=1}^{d} A_{i,j}$, where $A_{i,i} = [1 - 1/d, 1]$ and for $j \in \{1, 2, \ldots, d\} \backslash \{i\}$, $A_{i,j} = [0, 1 - 1/d]$ for every $i \in \{1, \ldots, d\}$, which are sets with disjoint interiors in $\mathbf{I}^d$, and that the diagonal $\delta_{C_{\tau_1}}(t) = \delta_{\tau_1}(t) = \max\{dt - d + 1, 0\}$ for every $t \in \mathbf{I}$.
Let $p = 2$ and let $\tau_2 = \{t_0, t_1, t_2, t_3, t_4\}$, where $0 = t_0 < t_1 < t_2 < t_3 < t_4 = 1$ is a partition of $[0,1]$ satisfying (5.2), that is,

$$\sum_{i=0}^{1} t_{2i+2} - t_{2i+1} = (1 - t_3) + (t_2 - t_1) = \frac{1}{d}. \tag{5.4}$$

Let δ_{τ_2} the function defined by τ_2 then

$$\delta_{\tau_2}(t) = \begin{cases} 0 & \text{if} \quad 0 = t_0 \le t \le t_1 \\ d(t-t_1) & \text{if} \quad t_1 \le t \le t_2 \\ d(t_2-t_1) & \text{if} \quad t_2 \le t \le t_3 \\ d(t_2-t_1) + d(t-t_3) & \text{if} \quad t_3 \le t \le t_4 = 1 \end{cases}$$

We observe that δ_{τ_2} is a continuous piecewise differentiable function, and from (5.4) we have $\delta_{\tau_2}(1) = 1$. So, $\delta_{\tau_2} : \mathbf{I} \to \mathbf{I}$ is an onto increasing function.
We have to prove that δ_{τ_2} is a d-diagonal, that is, we have to see that conditions (4.4), (4.5) and (4.6) are satisfied.
First, since $\delta_{\tau_2}(0) = 0$ and $\delta_{\tau_2}(1) = 1$, then δ_{τ_2} satisfies (4.4).
Second, let $0 \le s \le t \le 1$, since δ_{τ_2} is increasing we know that $0 \le \delta_{\tau_2}(t) - \delta_{\tau_2}(s)$. So, all we have to do is to see that $\delta_{\tau_2}(t) - \delta_{\tau_2}(s) \le d(t-s)$. Assume for example that $0 \le s \le t_1 < t_2 < t_3 \le t \le 1$ then adding and subtracting the intermediate $t_i's$

$$\begin{aligned} \delta_{\tau_2}(t) - \delta_{\tau_2}(s) &= \delta_{\tau_2}(t) - \delta_{\tau_2}(t_3) + \delta_{\tau_2}(t_3) - \delta_{\tau_2}(t_2) \\ &\quad + \delta_{\tau_2}(t_2) - \delta_{\tau_2}(t_1) + \delta_{\tau_2}(t_1) - \delta_{\tau_2}(s) \\ &= d(t_2-t_1) + d(t-t_3) - d(t_2-t_1) + d(t_2-t_1) - d(t_2-t_1) + d(t_2-t_1) \\ &= d(t-t_3) + d(t_2-t_1) \\ &\le d(t-s). \end{aligned}$$

The remaining cases follow from the same idea by adding and subtracting the intermediate $t_i's$, when they exist. Therefore, δ_{τ_2} satisfies condition (4.5).
Third, it is clear that $\delta_{\tau_2}(t) \ge \max\{0, dt - d + 1\}$ for every $t \in \mathbf{I}$ for every τ_2 partition of $\mathbf{I}$ satisfying conditions (5.2) and (5.3). By observing that δ_{τ_2} is increasing and convex on $[t_0, t_2] = [0, t_2]$ and on $[t_2, t_4] = [t_2, 1]$, and from condition (5.3) we have that $\delta_{\tau_2}(t_2) = d(t_2 - t_1) \le t_2$. So, $\delta_{\tau_2}(t) \le t$ for every $t \in \mathbf{I}$ and δ_{τ_2} satisfies condition (4.6). Hence, δ_{τ_2} is d-diagonal.
Now, we will prove the existence of a d-copula C_{τ_2} such that its diagonal coincides with δ_{τ_2}. Using condition (5.3) let us define

$$\alpha = t_2 - d(t_2 - t_1). \tag{5.5}$$

Then $\alpha \ge 0$, besides since $d \ge 2$ then $d(t_2 - t_1) > t_2 - t_1$, so $t_1 > t_2 - d(t_2 - t_1)$. Therefore

$$t_1 > \alpha. \tag{5.6}$$

On the other hand, using (5.2) and (5.5)

$$
\begin{aligned}
t_2 < t_3 &\iff t_2 < 1 - 1 + t_3 \\
&\iff t_2 < d(t_2 - t_1) + d(1 - t_3) - 1 + t_3 \\
&\iff t_2 - d(t_2 - t_1) < (d - 1)[1 - t_3] \\
&\iff (d - 1)[t_3 - 1] + t_2 - d(t_2 - t_1) < 0 \\
&\iff dt_3 - (d - 1) + \alpha < t_3 \\
&\iff t_3(d - 1) < d - 1 - \alpha \\
&\iff t_3 < 1 - \frac{\alpha}{d - 1}. \qquad (5.7)
\end{aligned}
$$

We also observe, using (5.5), that

$$\alpha + (1 - 1/d)(d(t_2 - t_1)) = \alpha + d(t_2 - t_1) - (t_2 - t_1) = t_2 - (t_2 - t_1) = t_1. \qquad (5.8)$$

Now, let us define $\delta^1 : \mathbf{I} \to \mathrm{R}$ by

$$\delta^1(t) = \frac{\delta_{\tau_2}(\alpha + t(t_2 - \alpha))}{t_2 - \alpha}. \qquad (5.9)$$

Then, by equations (5.5) and (5.6), we have

$$\delta^1(0) = \frac{\delta_{\tau_2}(\alpha)}{t_2 - \alpha} = 0, \qquad (5.10)$$

by equation (5.5),

$$\delta^1(1) = \frac{\delta_{\tau_2}(t_2)}{t_2 - \alpha} = \frac{d(t_2 - t_1)}{d(t_2 - t_1)} = 1, \qquad (5.11)$$

and by equations (5.8) and(5.5),

$$\delta^1(1 - 1/d) = \frac{\delta_{\tau_2}(\alpha + (1 - 1/d)(d(t - 2 - t_1)))}{t_2 - \alpha} = \frac{\delta_{\tau_2}(t_1)}{t_2 - \alpha} = 0. \qquad (5.12)$$

It is clear that the function $\varphi_1(t) = \alpha + t(t_2 - \alpha)$ is continuous strictly increasing on $\mathbf{I}$ and its image is the interval $[\alpha, t_2]$, and from equations (5.10), (5.11) and (5.12), the function defined in (5.9) satisfies that for every $t \in \mathbf{I}$, $\delta^1(t) = \max\{0, dt - d + 1\} = \delta_{\tau_1}(t)$, given in (5.4).

Now, from equations (5.2) and (5.5)

$$\begin{aligned} 1 = d(1-t_3) + d(t_2 - t_1) &\iff d - 1 + d(t_2 - t_1) = dt_3 \\ &\iff d - 1 - t_2 + d(t_2 - t_1) = dt_3 - t_2 \\ &\iff d - 1 - \alpha = dt_3 - t_2 \\ &\iff (d-1)\left(1 - \frac{\alpha}{d-1}\right) = dt_3 - t_2 \\ &\iff d\left(1 - \frac{\alpha}{d-1} - t_3\right) = 1 - \frac{\alpha}{d-1} - t_2. \end{aligned} \tag{5.13}$$

Besides, since $(d-1)/d = 1 - 1/d = 1 - (1 - t_3 + t_2 - t_1) = t_3 - t_2 + t_1$ by equation (5.2). Then

$$\begin{aligned} t_3 &= t_2 + t_3 - t_2 + t_1 - t_1 \\ &= t_2 + \frac{d-1}{d} - t_1 \\ &= t_2 + \left(\frac{d-1}{d}\right)\left(1 - \frac{dt_1}{d-1}\right) \\ &= t_2 + \left(\frac{d-1}{d}\right)\left(1 - \frac{t_2}{d-1}\left(1 - d + (d-1)\right) - \frac{dt_1}{d-1}\right) \\ &= t_2 + \left(\frac{d-1}{d}\right)\left(1 - \frac{t_2}{d-1} + \frac{d(t_2 - t_1)}{d-1} - t_2\right) \\ &= t_2 + \left(1 - \frac{1}{d}\right)\left(1 - \frac{\alpha}{d-1} - t_2\right). \end{aligned} \tag{5.14}$$

Let us define $\delta^2 : \mathbf{I} \to \mathbf{R}$ by

$$\delta^2(t) = \frac{\left[\delta_{\tau_2}\left(t_2 + t\left(1 - \frac{\alpha}{d-1} - t_2\right)\right) - \delta_{\tau_2}(t_2)\right]}{1 - \frac{\alpha}{d-1} - t_2} \tag{5.15}$$

Using equation (5.7) we know that $t_3 < 1 - \alpha/(d-1)$ and by hypothesis $t_2 < t_3$ then $1 - \alpha/(d-1) - t_2 > 0$ in equation (5.15). We have from equation (5.15), that

$$\delta^2(0) = 0, \tag{5.16}$$

using the definition of δ_{τ_2} and equations (5.15) and (5.13), we have

$$\begin{aligned} \delta^2(1) &= \frac{\left[\delta_{\tau_2}\left(t_2 + 1 - \frac{\alpha}{d-1} - t_2\right) - \delta_{\tau_2}(t_2)\right]}{1 - \frac{\alpha}{d-1} - t_2} \\ &= \frac{d(t_2 - t_1) + d\left(1 - \frac{\alpha}{d-1} - t_3\right) - d(t_2 - t_1)}{1 - \frac{\alpha}{d-1} - t_2} \\ &= 1, \end{aligned} \tag{5.17}$$

and using equation (5.14) we have that

$$\begin{aligned} \delta^2(1 - 1/d) &= \frac{\left[\delta_{\tau_2}\left(t_2 + \left(1 - \frac{1}{d}\right)\left(1 - \frac{\alpha}{d-1} - t_2\right)\right) - \delta_{\tau_2}(t_2)\right]}{1 - \frac{\alpha}{d-1} - t_2} \\ &= \frac{\left[\delta_{\tau_2}(t_3) - \delta_{\tau_2}(t_2)\right]}{1 - \frac{\alpha}{d-1} - t_2} \\ &= 0. \end{aligned} \tag{5.18}$$

It is clear that the function $\varphi_2(t) = t_2 + t\left(1 - \frac{\alpha}{d-1} - t_2\right)$ is continuous strictly increasing on $\mathbf{I}$ and its image is the interval $[t_2, 1-\alpha/(d-1)]$, and from equations (5.16), (5.17) and (5.18) the function defined in (5.15) satisfies that for every $t \in \mathbf{I}$, $\delta^2(t) = \max\{0, dt-d+1\} = \delta_{\tau_1}(t)$, given in (5.4).

From the case $p = 1$ we know that there exists d-copulas C_1^* and C_2^* such that $\delta_{C_1^*}(t) = \delta^1(t) = \max\{0, dt - d + 1\} = \delta^2(t) = \delta_{C_2^*}(t)$ for every $t \in \mathbf{I}$.

Observe that from equations (5.1), (5.5), (5.6) and (5.7), we have that

$$0 \leq \alpha < t_1 < t_2 < t_3 < 1 - \alpha/(d-1) \leq 1. \tag{5.19}$$

Define $h_1 : \mathbf{I}^d \to \mathrm{R}^d$ by

$$h_1(x_1, \ldots, x_d) = \langle \alpha + (t_2 - \alpha)x_1, \ldots, \alpha + (t_2 - \alpha)x_d \rangle \text{ for every } \langle x_1, \ldots, x_d \rangle \in \mathbf{I}^d. \tag{5.20}$$

Then h_1 is continuous increasing in each coordinate with $h_1(0, \ldots, 0) = \langle \alpha, \ldots, \alpha \rangle$ and $h_1(1, \ldots, 1) = \langle t_2, \ldots, t_2 \rangle$. So, h_1 sends $\mathbf{I}^d$ onto $[\alpha, t_2]^d$. Using equation (5.8), we also have that $h_1(1 - 1/d, \ldots, 1 - 1/d) = \langle t_1, \ldots, t_1 \rangle$. Let C_1 be the image measure induced by the distribution function C_1^* under h_1. If we define $B_i = \times_{j=1}^d B_{i,j}$, where

$$B_{i,i} = [t_1, t_2] \quad \text{and for every } j \neq i, \;\; B_{ij} = [\alpha, t_1]. \tag{5.21}$$

Then C_1 assigns uniform mass $1/d$ to B_i for every $i \in \{1, \ldots, d\}$.

For every $\langle x_1, \ldots, x_d \rangle \in \mathbf{I}^d$ define $h_2 : \mathbf{I}^d \to \mathrm{R}^{\,d}$ by

$$h_2(x_1, \ldots, x_d) = \left\langle t_2 + \left(1 - \frac{\alpha}{d-1} - t_2\right) x_1, \ldots, t_2 + \left(1 - \frac{\alpha}{d-1} - t_2\right) x_q \right\rangle \ . \tag{5.22}$$

Then h_2 is continuous increasing in each coordinate with $h_2(0, \ldots, 0) = \langle t_2, \ldots, t_2 \rangle$ and $h_2(1, \ldots, 1) = \langle 1 - \alpha/(d-1), \ldots, 1 - \alpha/(d-1) \rangle$. So, h_1 sends $\mathbf{I}^d$ onto $[t_2, 1 - \alpha/(d-1)]^d$. Using equation (5.14), we also have that $h_2(1-1/d, \ldots, 1-1/d) = \langle t_3, \ldots, t_3 \rangle$. Let C_2 be the image measure induced by the distribution function C_2^* under h_1. If we define $C_i = \times_{j=1}^{d} C_{i,j}$, where

$$C_{i,i} = [t_3, 1 - \alpha/(d-1)] \quad \text{and for every } j \neq i, \ \ C_{ij} = [t_2, t_3]. \tag{5.23}$$

Then C_2 assigns uniform mass $1/d$ to C_i for every $i \in \{1, \ldots, d\}$.

For every $i \in \{1, 2, \ldots, d\}$ define $D_i = \times_{j=1}^{d} D_{i,j}$, where

$$D_{i,i} = \left[1 - \frac{\alpha}{d-1}, 1\right] \quad \text{and for every } j \neq i, \ \ D_{i,j} = [0, \alpha]. \tag{5.24}$$

Let ν_α be a measure on $\mathbf{I}^d$ such that for every

$$\nu_\alpha(D_i) = \frac{\alpha}{d-1} \quad \text{uniformly, for every} \quad i \in \{1, 2, \ldots, d\}. \tag{5.25}$$

Observe that from equation (5.19), all the sets in (5.24) are disjoint.

Finally, define $\mu_{\tau_2} : \mathcal{B}(\mathbf{I}^d) \to [0, \infty)$, where $\mathcal{B}(\mathbf{I}^d)$ is the set of Borel sets in $\mathbf{I}^d$, by

$$\mu_{\tau_2} = \nu_\alpha + (t_2 - \alpha) C_1 + \left(1 - \frac{\alpha}{d-1} - t_2\right) C_2. \tag{5.26}$$

We observe that all the d-boxes with positive mass in the definitions of ν_α, C_1 and C_2 have disjoint interiors. So, the total mass of μ_{τ_2} is given by

$$\begin{aligned} \mu_{\tau_2}\left(\mathbf{I}^d\right) &= d\frac{\alpha}{d-1} + (t_2 - \alpha) + \left(1 - \frac{\alpha}{d-1} - t_2\right) \\ &= \alpha\left(\frac{d}{d-1} - 1 - \frac{1}{d-1}\right) + 1 \\ &= 1. \end{aligned} \tag{5.27}$$

Then $(\mathbf{I}^d, \mathcal{B}(\mathbf{I}^d), \mu_{\tau_2})$ is a probability space. Define $C_{\tau_2} : \mathbf{I}^d \to \mathbf{I}$ by

$$C_{\tau_2}(t_1, \ldots, t_d) = \mu_{\tau_2}([\underline{0}, \underline{t}]) \quad \text{for every} \quad \underline{t} = \langle t_1, \ldots, t_d \rangle \in \mathbf{I}^d, \tag{5.28}$$

where $\underline{0} = \langle 0, 0, \ldots, 0 \rangle \in \mathbf{I}^d$.

Now, we have to see that the diagonal section $\delta_{C_{\tau_2}}$ coincides with δ_{τ_2}. To see this, observe that the d-boxes which have positive uniform mass in C_{τ_2} are given in equations (5.21), (5.23) and (5.24). There are exactly $3d$, d-boxes, let $G = \cup_{i=1}^d B_i \cup C_i \cup D_i$, in order to obtain the diagonal of C_{τ_2} we have to analyze the intersections $[0, t]^d \cap G$ for every $t \in \mathbf{I}$. We first observe that using (5.19), (5.21), (5.23) and (5.24),

$$G \cap [0, t]^d = \begin{cases} \emptyset & \text{if} \quad 0 \leq t \leq \alpha \\ \emptyset & \text{if} \quad \alpha \leq t < t_1 \\ \cup_{i=1}^d B_i & \text{if} \quad t_2 \leq t < t_3 \\ \cup_{i=1}^d B_i \cup C_i & \text{if} \quad t = 1 - \frac{\alpha}{d-1} \\ G & \text{if} \quad t = 1. \end{cases}$$

Therefore, using (5.29) and the uniformity of the masses of the d-boxes B_i, C_i and D_i and the definition of C_{τ_2} in equation (5.28), and observing that by equation (5.5), $t_2 - \alpha = d(t_2 - t_1)$, by equation (5.13), $d((1 - \alpha/(d-1)) - t_3) = 1 - \alpha/(d-1) - t_2)$, by equation (5.27), $(t_2 - \alpha) + (1 - \alpha/(d-1) - t_2) = 1 - d\alpha/(d-1)$ and by equation (5.2), we have that

$$\delta_{C_{\tau_2}}(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq t_1 \\ d(t - t_1) & \text{if} \quad t_1 \leq t \leq t_2 \\ d(t_2 - t_1) & \text{if} \quad t_2 \leq t \leq t_3 \\ d(t_2 - t_1) + d((1 - \alpha/(d-1)) - t_3) & \text{if} \quad t = 1 - \alpha/(d-1) \\ 1 = d(t_2 - t_1) + d(1 - t_3) & \text{if} \quad t = 1. \end{cases}$$

So, extending (5.29) by uniformity it is clear that

$$\delta_{C_{\tau_2}}(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq t_1 \\ d(t - t_1) & \text{if} \quad t_1 \leq t \leq t_2 \\ d(t_2 - t_1) & \text{if} \quad t_2 \leq t \leq t_3 \\ d(t_2 - t_1) + d(t - t_3) & \text{if} \quad t_3 \leq t \leq 1, \end{cases}$$

which coincides with $\delta_{\tau_2}(t)$ for every $t \in \mathbf{I}$.

Finally, to see that C_{τ_2} is a d-copula, we observe that by the uniformity of the masses in the $3d$, d-boxes, C_{τ_2} has a density c_{τ_2} with support on G. So, C_{τ_2} is d-increasing. Besides,

$C_{\tau_2}(s_1, 1, \dots, 1) = \mu_{\tau_2}([\underline{0}, \underline{s}])$ where $\underline{s} = \langle s_1, 1, \dots, 1\rangle$, for every $s_1 \in \mathbf{I}$. We will prove that $C_{\tau_2}(s_1, 1, \dots, 1) = s_1$ for every $s_1 \in [0, 1]$ by cases, we will use the definition of B_i, C_i, D_i given in equations (5.21), (5.23) and (5.24):

1) If $0 \leq s_1 \leq \alpha$, then $B_j \cap [\underline{0}, \underline{s}] = \emptyset = C_j \cap [\underline{0}, \underline{s}]$ for every $j \in \{1, \dots, d\}$ and

$$D_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ [0, s_1] \times [0, \alpha] \times \cdots \times [1 - \alpha/(d-1), 1] \times \cdots \times [0, \alpha] & \text{if} \quad j \in \{2, \dots, d\}, \end{cases}$$

where $[1 - \alpha/(d-1), 1]$ is located on the j-th coordinate. Then using the uniformity and equation (5.26)

$$\begin{aligned} C_{\tau_2}(\underline{s}) &= \mu_{\tau_2}\left(\cup_{j=2}^{d} D_j \cap [\underline{0}, \underline{s}]\right) \\ &= \sum_{j=2}^{d} \mu_{\tau_2}(D_j \cap [\underline{0}, \underline{s}]) \\ &= \sum_{j=2}^{d} \frac{s_1}{\alpha}\left(\frac{\alpha}{d-1}\right) \\ &= s_1. \end{aligned}$$

2) If $\alpha \leq s_1 \leq t_1$, then $C_j \cap [\underline{0}, \underline{s}] = \emptyset$ for every $j \in \{1, \dots, d\}$

$$D_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ D_j & \text{if} \quad j \in \{2, \dots, d\}, \end{cases}$$

and

$$B_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ [\alpha, s_1] \times [\alpha, t_1] \times \cdots \times [t_1, t_2] \times \cdots \times [\alpha, t_1] & \text{if} \quad j \in \{2, \dots, d\}, \end{cases}$$

where $[t_1, t_2]$ is located on the j-th coordinate. In this case

$$\begin{aligned} C_{\tau_2}(\underline{s}) &= \mu_{\tau_2}\left(\left(\cup_{j=2}^{d} D_j\right) \cup \left(\cup_{j=2}^{d} B_j \cap [\underline{0}, \underline{s}]\right)\right) \\ &= \sum_{j=2}^{d}\left(\frac{\alpha}{d-1}\right) + \sum_{j=2}^{d}(t_2-\alpha)\left(\frac{s_1-\alpha}{t_1-\alpha}\right)\frac{1}{d} \\ &= \alpha + \frac{(t_2-\alpha)}{t_1-\alpha}\frac{(d-1)}{d}(s_1-\alpha) \\ &= \alpha + \frac{(t_2-t_1)}{t_1-\alpha}(d-1)(s_1-\alpha) \\ &= s_1, \end{aligned}$$

because, using (5.5), $-\alpha = d(t_2-t_1)-t_2$, so, $d(t_2-t_1)-(t_2-t_1) = t_1-\alpha$, then $(t_2-t_1)(d-1) = t_1-\alpha$.

3) If $t_1 \le s_1 \le t_2$. Then $C_j \cap [\underline{0}, \underline{s}] = \emptyset$ for every $j \in \{1, \ldots, d\}$

$$D_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ D_j & \text{if} \quad j \in \{2, \ldots, d\}, \end{cases}$$

and

$$B_j \cap [\underline{0}, \underline{s}] = \begin{cases} [t_1, s_1] \times [\alpha, t_1] \times \cdots \times [\alpha, t_i] & \text{if} \quad j = 1 \\ B_j & \text{if} \quad j \in \{2, \ldots, d\}. \end{cases}$$

In this case, using (5.5),

$$\begin{aligned} C_{\tau_2}(\underline{s}) &= \mu_{\tau_2}\left(\left(\cup_{j=2}^{d} D_j\right) \cup \left(\cup_{j=2}^{d} B_j\right) \cup (B_1 \cap [\underline{0}, \underline{s}])\right) \\ &= \sum_{j=2}^{d}\left(\frac{\alpha}{d-1}\right) + \sum_{j=2}^{d}\left(\frac{t_2-\alpha}{d}\right) + (t_2-\alpha)\left(\frac{s_1-t_1}{t_2-t_1}\right)\frac{1}{d} \\ &= \alpha + \frac{(t_2-\alpha)}{d}\left[d-1+\frac{s_1-t_1}{t_2-t_1}\right] \\ &= \alpha + (t_2-t_1)\left[d-1+\frac{s_1-t_1}{t_2-t_1}\right] \\ &= \alpha + d(t_2-t_1) - (t_2-t_1) + s_1 - t_1 \\ &= t_2 - t_2 + t_1 + s_1 - t_1 \\ &= s_1. \end{aligned}$$

4) If $t_2 \leq s_1 \leq t_3$, then

$$D_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ D_j & \text{if} \quad j \in \{2, \ldots, d\}, \end{cases}$$

$$B_j \cap [\underline{0}, \underline{s}] = B_j \quad \text{for every} \quad j \in \{1, \ldots, d\},$$

and

$$C_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ [t_2, s_1] \times [t_2, t_3] \times \cdots \times [t_3, 1 - \alpha/(d-1)] \times \cdots \times [t_2, t_3] & \text{if} \quad j \in \{2, \ldots, d\}, \end{cases}$$

where $[t_3, 1 - \alpha/(d-1)]$ is located on the j-th coordinate. In this case,

$$\begin{aligned} C_{\tau_2}(\underline{s}) &= \mu_{\tau_2}\left(\left(\cup_{j=2}^d D_j\right) \cup \left(\cup_{j=1}^d B_j\right) \cup \left(\cup_{j=2}^d C_j \cap [\underline{0}, \underline{s}]\right)\right) \\ &= \sum_{j=2}^d \frac{\alpha}{d-1} + (t_2 - \alpha) \sum_{j=1}^d \frac{1}{d} + \sum_{j=2}^d \left(1 - \frac{\alpha}{d-1} - t_2\right)\left(\frac{s_1 - t_2}{t_3 - t_2}\right)\frac{1}{d} \\ &= t_2 + \left(\frac{d-1}{d}\right)\left(1 - \frac{\alpha}{d-1} - t_2\right)\left(\frac{s_1 - t_2}{t_3 - t_2}\right) \\ &= t_2 + s_1 - t_2 \\ &= s_1, \end{aligned}$$

because by equation (5.14), $t_3 - t_2 = ((d-1)/d)(1 - \alpha/(d-1) - t_2)$.

5) If $t_3 \leq s_1 \leq 1 - \alpha/(d-1)$, then

$$D_j \cap [\underline{0}, \underline{s}] = \begin{cases} \emptyset & \text{if} \quad j = 1 \\ D_j & \text{if} \quad j \in \{2, \ldots, d\}, \end{cases}$$

$$B_j \cap [\underline{0}, \underline{s}] = B_j \quad \text{for every} \quad j \in \{1, \ldots, d\},$$

and

$$C_j \cap [\underline{0}, \underline{s}] = \begin{cases} [t_3, s_1] \times [t_2, t_3] \times \cdots \times [t_2, t_3] & \text{if} \quad j = 1 \\ C_j & \text{if} \quad j \in \{2, \ldots, d\}. \end{cases}$$

Using the case 4),

$$
\begin{aligned}
C_{\tau_2}(\underline{s}) &= \mu_{\tau_2}\left(\left(\cup_{j=2}^d D_j\right) \cup \left(\cup_{j=1}^d B_j\right) \cup \left(\cup_{j=2}^d C_j\right) \cup (C_1 \cap [\underline{0}, \underline{s}])\right) \\
&= t_2 + \sum_{j=2}^d \left(1 - \frac{\alpha}{d-1} - t_2\right)\frac{1}{d} + \left(1 - \frac{\alpha}{d-1} - t_2\right)\left(\frac{s_1 - t_3}{1-\alpha/(d-1) - t_3}\right)\frac{1}{d} \\
&= t_2 + \left(\frac{d-1}{d}\right)\left(1 - \frac{\alpha}{d-1} - t_2\right) + \frac{1}{d}\frac{\left(1 - \frac{\alpha}{d-1} - t_2\right)}{\left(1 - \frac{\alpha}{d-1} - t_3\right)}(s_1 - t_3) \\
&= t_2 + (t_3 - t_2) + \frac{1}{d}\frac{\left(1 - \frac{\alpha}{d-1} - t_2\right)}{\left(1 - \frac{\alpha}{d-1} - t_3\right)}(s_1 - t_3) \\
&= t_3 + (s_1 - t_3) \\
&= s_1,
\end{aligned}
$$

where we first used equation (5.14), and then by (5.13), we have that $(1/d)(1-\alpha/(d-1) - t_2) = (1 - \alpha/(d-1) - t_3)$.

6) If $1 - \alpha/(d-1) \le s_1 \le 1$, then

$$
D_j \cap [\underline{0}, \underline{s}] = \begin{cases} [1-\alpha/(d-1), s_1] \times [0,\alpha] \times \cdots \times [0,\alpha] & \text{if} \quad j = 1 \\ D_j & \text{if} \quad j \in \{2, \ldots, d\}, \end{cases}
$$

$$
B_j \cap [\underline{0}, \underline{s}] = B_j \quad \text{for every} \quad j \in \{1, \ldots, d\},
$$

and

$$
C_j \cap [\underline{0}, \underline{s}] = C_j \quad \text{for every} \quad j \in \{1, \ldots, d\}.
$$

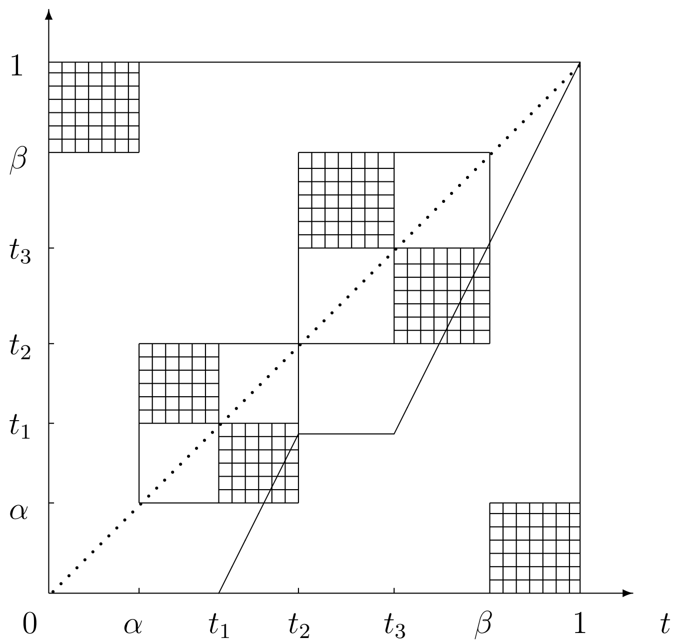
Then using case 4,

$$
\begin{aligned}
C_{\tau_2}(\underline{s}) &= \mu_{\tau_2}\left(\left(\cup_{j=2}^d D_j\right) \cup \left(\cup_{j=1}^d B_j\right) \cup \left(\cup_{j=1}^d C_j\right) \cup (D_1 \cap [\underline{0}, \underline{s}])\right) \\
&= t_2 + \sum_{j=1}^d \frac{\left(1 - \frac{\alpha}{d-1} - t_2\right)}{d} + \frac{\left(s_1 - \left(1 - \frac{\alpha}{d-1}\right)\right)}{\frac{\alpha}{d-1}}\left(\frac{\alpha}{d-1}\right) \\
&= t_2 + \left(1 - \frac{\alpha}{d-1} - t_2\right) + s_1 - 1 + \frac{\alpha}{d-1} \\
&= s_1.
\end{aligned}
$$

Therefore, $C_{\tau_2}(s_1, 1, \ldots, 1) = s_1$ for every $s_1 \in \mathbf{I}$. Of course following the same steps we have that $C_{\tau_2}(1, \ldots, 1, s_j, 1, \ldots, 1) = s_j$ for every $j \in \{2, \ldots, d\}$ and for every $s_j \in \mathbf{I}$. Hence, C_{τ_2} is a d-copula. So, the case $p = 2$ is proved.

As an easy example, assume that $d = 2$ and $p = 2$ define the partition $\tau_2 = \{0 = t_0, t_1 = 0.32, t_2 = 0.47, t_3 = 0.65, t_4 = 1\}$, then $(t_2 - t_1) + (t_4 - t_3) = 1/2$ and $\alpha = 0.17$. In Figure 4 we provide the graph of the support of C_{τ_2} and the diagonal it generates.

Figure 4: Diagonal and Support of C_{τ_2} for the case $d = 2$, $\beta = 1 - \alpha$



We now proceed by induction.

Let $p \geq 3$ and assume that the result holds for every $1 \leq r \leq p - 1$. Let $\tau_p = \{t_0, t_1, \ldots, t_{2p-1}, t_{2p}\}$ be a partition of $\mathbf{I}$ such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_{2p-1} < t_{2p} = 1, \tag{5.29}$$

$$\sum_{i=0}^{p-1} t_{2i+2} - t_{2i+1} = \frac{1}{d}, \tag{5.30}$$

and

$$\sum_{i=0}^{s} d(t_{2i+2} - t_{2i+1}) \leq t_{2s+2} \quad \text{for every} \quad s \in \{0, 1, \ldots, p-1\}. \tag{5.31}$$

Define

$$\alpha = \min_{1\le j\le p-1} t_{2j} - \sum_{l=0}^{j-1} d(t_{2l+2} - t_{2l+1}), \tag{5.32}$$

and the define k to be the minimum index in $\{1, \dots, p-1\}$, such that

$$\alpha = t_{2k} - \sum_{l=0}^{k-1} d(t_{2l+2} - t_{2l+1}). \tag{5.33}$$

Observe that $\alpha \geq 0$ by equation (5.31), we also observe that

$$\alpha < t_1, \tag{5.34}$$

this follows, from equation (5.32), because $\alpha \leq t_2 - d(t_2 - t_1) < t_1$, since $d \geq 2$. On the other hand, since $t_{2p-2} < t_{2p-1}$ by equation (5.29), then using (5.30), (5.32)

$$\begin{aligned}
t_{2p-2} < t_{2p-1} &\iff t_{2p-2} < 1 - 1 + t_{2p-1} \\
&\iff t_{2p-2} < \sum_{i=0}^{p-1} d(t_{2i+2} - t_{2i+1}) - 1 + t_{2p-1} \\
&\iff \alpha \leq t_{2p-2} - \sum_{i=0}^{p-2} d(t_{2i+2} - t_{2i+1}) < d(1 - t_{2p-1}) - 1 + t_{2p-1} \\
&\iff \alpha < (d-1)(1 - t_{2p-1}) \\
&\iff t_{2p-1}(d-1) < d - 1 - \alpha \\
&\iff t_{2p-1} < 1 - \frac{\alpha}{d-1}. \qquad (5.35)
\end{aligned}$$

In this case we can write δ_{τ_p} as

$$\delta_{\tau_p}(t) = \begin{cases}
0 & \text{if} \quad 0 = t_0 \leq t \leq t_1 \\
d(t - t_1) & \text{if} \quad t_1 \leq t \leq t_2 \\
d(t_2 - t_1) & \text{if} \quad t_2 \leq t \leq t_3 \\
d(t_2 - t_1) + d(t - t_3) & \text{if} \quad t_3 \leq t \leq t_4 \\
\quad\vdots & \vdots \qquad \vdots \\
\sum_{i=1}^{j} d(t_{2i} - t_{2i-1}) & \text{if} \quad t_{2j} \leq t \leq t_{2j+1} \\
\sum_{i=1}^{j} d(t_{2i} - t_{2i-1}) + d(t - t_{2j+1}) & \text{if} \quad t_{2j+1} \leq t \leq t_{2j+2} \\
\quad\vdots & \vdots \qquad \vdots \\
\sum_{i=1}^{p-1} d(t_{2i} - t_{2i-1}) & \text{if} \quad t_{2p-2} \leq t \leq t_{2p-1} \\
\sum_{i=1}^{p-1} d(t_{2i} - t_{2i-1}) + d(t - t_{2p-1}) & \text{if} \quad t_{2p-1} \leq t \leq t_{2p} = 1.
\end{cases}$$

Define $\delta^1 : [0,1] \to \mathbf{R}$ by

$$\delta^1(t) = \frac{\delta_{\tau_p}(\alpha + t(t_{2k} - \alpha))}{t_{2k} - \alpha}, \tag{5.36}$$

where α is given in equation (5.32) and $1 \le k \le p-1$ is defined in (5.33). We will see that there exists a partition $\sigma_k = \{s_0, s_1, \ldots, s_{2k-1}, s_{2k}\}$ such that $s_0 = 0 < s_1 < \cdots < s_{2k-1} < s_{2k} = 1$ with $\delta^1 = \delta_{\sigma_k}$. We first observe that from equations (5.34), (5.36) and (5.36) we have that

$$\delta^1(0) = \frac{\delta_{\tau_p}(\alpha)}{t_{2k} - \alpha} = 0. \tag{5.37}$$

We also have that from equations (5.33) and (5.36)

$$\delta^1(1) = \frac{\delta_{\tau_p}(t_{2k})}{t_{2k} - \alpha} = \frac{\sum_{i=0}^{k-1} d(t_{2i+2} - t_{2i+1})}{\sum_{i=0}^{k-1} d(t_{2i+2} - t_{2i+1})} = 1. \tag{5.38}$$

Define $s_0 = 0$

$$s_i = \frac{t_i - \alpha}{t_{2k} - \alpha} \quad \text{for every} \quad i \in \{1, \ldots, 2k\}. \tag{5.39}$$

Then from equation (5.29) we have that

$$0 = s_0 < s_1 < \cdots < s_{2k-1} < s_{2k} = 1, \tag{5.40}$$

that is, $\sigma_k = \{s_0, s_1, \ldots, s_{2k}\}$ is a partition of $\mathbf{I}$. Satisfying

$$\sum_{i=0}^{k-1} s_{2i+2} - s_{2i+1} = \frac{\sum_{i=0}^{k-1} t_{2i+2} - t_{2i+1}}{t_{2k} - \alpha} = \frac{\sum_{i=0}^{k-1} t_{2i+2} - t_{2i+1}}{\sum_{i=0}^{k-1} d(t_{2i+2} - t_{2i+1})} = \frac{1}{d}, \tag{5.41}$$

using equation (5.33) and (5.39). So, σ_k satisfies condition (5.2). Let $0 \le r \le k-1$ then we know from equations (5.32), (5.33) and (5.31), that

$$\begin{aligned} 0 \le \alpha \le t_{2r+2} - \sum_{i=0}^{r} d(t_{2i+2} - t_{2i+1}) \iff & \sum_{i=0}^{r} d(t_{2i+2} - t_{2i+1}) \le t_{2r+2} - \alpha \\ \iff & \sum_{i=0}^{r} d\left(\frac{t_{2i+2} - t_{2i+1}}{t_{2k} - \alpha}\right) \le \left(\frac{t_{2r+2} - \alpha}{t_{2k} - \alpha}\right) \\ \iff & \sum_{i=0}^{r} d(s_{2i+2} - s_{2i+1}) \le s_{2r+2}. \end{aligned} \tag{5.42}$$

Therefore, from (5.42), σ_k satisfies condition (5.3).

From (5.40), (5.41), (5.42) and the induction hypothesis, there exists C_1^* a d-copula such that its diagonal satisfies

$$\delta_{C_1^*}(t) = \delta^1(t) \quad \text{for every} \quad t \in [0,1]. \tag{5.43}$$

Now define $\delta^2 : \mathbf{I} \to \mathbf{R}$ by

$$\delta^2(t) = \frac{\left[\delta_{\tau_p}\left(t_{2k} + t\left(1 - \frac{\alpha}{d-1} - t_{2k}\right)\right) - \delta_{\tau_p}(t_{2k})\right]}{1 - \frac{\alpha}{d-1} - t_{2k}}. \tag{5.44}$$

Then

$$\delta^2(0) = \frac{\left[\delta_{\tau_p}(t_{2k}) - \delta_{\tau_p}(t_{2k})\right]}{1 - \frac{\alpha}{d-1} - t_{2k}} = 0. \tag{5.45}$$

Now, since by equations (5.30) and (5.29)

$$1 = \sum_{i=1}^{k} d(t_{2i} - t_{2i-1}) + \sum_{i=k+1}^{p-1} d(t_{2i} - t_{2i-1}) + d(1 - t_{2p-1}),$$

then

$$d - 1 + \sum_{i=1}^{k} d(t_{2i} - t_{2i-1}) = dt_{2p-1} - \sum_{i=k+1}^{p-1} d(t_{2i} - t_{2i-1}),$$

subtracting t_{2k} on both sides, we get

$$d - 1 - t_{2k} + \sum_{i=1}^{k} d(t_{2i} - t_{2i-1}) = dt_{2p-1} - \sum_{i=k+1}^{p-1} d(t_{2i} - t_{2i-1}) - t_{2k},$$

by equation (5.33)

$$d - 1 - \alpha = dt_{2p-1} - t_{2k} - \sum_{i=k+1}^{p-1} d(t_{2i} - t_{2i-1}),$$

so,

$$(d-1)\left(1 - \frac{\alpha}{d-1}\right) = dt_{2p-1} - t_{2k} - \sum_{i=k+1}^{p-1} d(t_{2i} - t_{2i-1}).$$

Then

$$d\left(1-\frac{\alpha}{d-1}-t_{2p-1}\right)=1-\frac{\alpha}{d-1}-t_{2k}-\sum_{i=k+1}^{p-1}d(t_{2i}-t_{2i-1}). \tag{5.46}$$

Therefore, using equations (5.44), (5.35), (5.36) and (5.46) we have

$$\begin{aligned}
\delta^2(1) &= \frac{\left[\delta_{\tau_p}\left(1-\frac{\alpha}{d-1}\right)-\delta_{\tau_p}(t_{2k})\right]}{1-\frac{\alpha}{d-1}-t_{2k}}\\
&= \frac{\sum_{i=1}^{p-1}d(t_{2i}-t_{2i-1})+d\left(1-\frac{\alpha}{d-1}-t_{2p-1}\right)-\sum_{i=1}^{k}d(t_{2i}-t_{2i-1})}{1-\frac{\alpha}{d-1}-t_{2k}}\\
&= \frac{\sum_{i=k+1}^{p-1}d(t_{2i}-t_{2i-1})+d\left(1-\frac{\alpha}{d-1}-t_{2p-1}\right)}{1-\frac{\alpha}{d-1}-t_{2k}}\\
&= 1. \qquad (5.47)
\end{aligned}$$

Observe that using equations (5.32), (5.33) and (5.29) we have that $t_{2k} \leq t_{2p-2} < t_{2p-1}$. So, by (5.35), in definition (5.44),

$$1-\frac{\alpha}{d-1}-t_{2k} > 1-\frac{\alpha}{d-1}-t_{2p-1} > 0.$$

Let us define

$$u_i=\frac{t_{2k+i}-t_{2k}}{1-\frac{\alpha}{d-1}-t_{2k}} \quad \text{for every} \quad i \in \{2k, 2k+1, \dots, 2p-1\} \tag{5.48}$$

and

$$u_{2p-2k}=\frac{\left(1-\frac{\alpha}{d-1}\right)-t_{2k}}{1-\frac{\alpha}{d-1}-t_{2k}}=1. \tag{5.49}$$

Then using (5.29), (5.48) and (5.49) we have that if we let $\tau_{p-k}=\{u_0, u_1, \dots, u_{2(p-k)}\}$, is a partition of $\mathbf{I}$, such that,

$$0=u_0<u_1<\cdots<u_{2(p-k)-1}<u_{2(p-k)}=1, \tag{5.50}$$

Now, from equation (5.46) we have that

$$\sum_{i=k+1}^{p-1}(t_{2i}-t_{2i-1})+\left(1-\frac{\alpha}{d-1}-t_{2p-1}\right)=\left(\frac{1}{d}\right)\left(1-\frac{\alpha}{d-1}-t_{2k}\right).$$

Therefore, using (5.46), (5.48) and (5.49),

$$\begin{aligned}
\sum_{i=0}^{p-k-1}(u_{2i+2}-u_{2i+1}) &= \sum_{i=0}^{p-k-2}(u_{2i+2}-u_{2i+1})+(u_{2p-2k}-u_{2p-2k-1}) \\
&= \frac{\sum_{i=0}^{p-k-2}(t_{2k+2i+2}-t_{2k+2i+1})+\left(1-\frac{\alpha}{d-1}-t_{2p-1}\right)}{1-\frac{\alpha}{d-1}-t_{2k}} \\
&= \frac{\sum_{i=k+1}^{p-1}(t_{2i}-t_{2i-1})+\left(1-\frac{\alpha}{d-1}-t_{2p-1}\right)}{1-\frac{\alpha}{d-1}-t_{2k}} \\
&= \left(\frac{1}{d}\right)\left(\frac{1-\frac{\alpha}{d-1}-t_{2k}}{1-\frac{\alpha}{d-1}-t_{2k}}\right) \\
&= \frac{1}{d}. \qquad (5.51)
\end{aligned}$$

From equation (5.51), τ_{p-k} satisfies condition (5.2).

Let $r \in \{0,1,\ldots,p-k-1\}$, then $r+k \in \{k,k+1,\ldots,p-1\}$, and by equation (5.32),

$$\sum_{j=0}^{(r+k+1)-1} d(t_{2j+2}-t_{2j+1}) \leq t_{2r+2k+2}-\alpha. \qquad (5.52)$$

Then using (5.48), (5.33) and (5.52)

$$\begin{aligned}
\sum_{j=0}^{r} d(u_{2j+2}-u_{2j+1}) &= \frac{\sum_{j=0}^{r} d(t_{2k+2j+2}-t_{2k+2j+1})}{1-\frac{\alpha}{d-1}-t_{2k}} \\
&= \frac{\sum_{j=0}^{r+k} d(t_{2j+2}-t_{2j+1})-\sum_{j=0}^{k-1} d(t_{2j+2}-t_{2j+1})}{1-\frac{\alpha}{d-1}-t_{2k}} \\
&= \frac{\sum_{j=0}^{r+k} d(t_{2j+2}-t_{2j+1})+\alpha-t_{2k}}{1-\frac{\alpha}{d-1}-t_{2k}} \\
&\leq \frac{t_{2k+2r+2}-\alpha+\alpha-t_2k}{1-\frac{\alpha}{d-1}-t_{2k}} \\
&= u_{2r+2}. \qquad (5.53)
\end{aligned}$$

But, equation (5.53), implies that the partition τ_{p-k} in equation (5.50) satisfies condition (5.3).

From (5.50), (5.51), (5.53) and the induction hypothesis, there exists C_2^* a d-copula such that its diagonal satisfies

$$\delta_{C_2^*}(t) = \delta^2(t) \quad \text{for every} \quad t \in \mathbf{I}\,. \tag{5.54}$$

Now, define $h_1, h_2 : \mathbf{I}^d \to \mathbf{I}^d$ as in equations (5.20) and (5.22), by interchanging t_2 by t_{2k}. Then it is clear that h_1 is continuous strictly increasing and carries $\mathbf{I}^d$ onto $[\alpha, t_{2k}]^d$, and h_2 is continuous strictly increasing and carries $\mathbf{I}^d$ onto $[t_{2k}, \alpha/(1-d)]^d$. Let C_1 and C_2 the measures defined on $\mathcal{B}([0, t_{2k}]^d)$ and on $\mathcal{B}([t_{2k}, 1-\alpha(d-1)]^d)$ respectively, induced by the distribution functions C_1^* and C_2^*, and the functions h_1 and h_2 respectively.
Define D_i for $i \in \{1, 2, \ldots, d\}$ as in equation (5.24), and ν_α as in equation (5.25). Finally, define μ_{τ_p} as in equation (5.26), interchanging t_2 by t_{2k}, that is,

$$\mu_{\tau_p} = \nu_\alpha + (t_{2k} - \alpha) C_1 + \left(1 + \frac{\alpha}{d-1} - t_{2k}\right) C_2. \tag{5.55}$$

Then, as in equation (5.27) we have that $\mu_{\tau_p}(\mathbf{I}^d) = 1$, that is, μ_{τ_p} is a probability measure on the measurable space $(\mathbf{I}^d, \mathcal{B}(\mathbf{I}^d))$. Define $C_{\tau_p} : \mathbf{I}^d \to \mathbf{I}$ by

$$C_{\tau_p}(t_1, \ldots, t_d) = \mu_{\tau_p}([\underline{0}, \underline{t}]) \quad \text{for every} \quad \underline{t} = \langle t_1, \ldots, t_d \rangle \in \mathbf{I}^d\,. \tag{5.56}$$

Then, C_{τ_p} is clearly a distribution function on $\mathbf{I}^d$, and using the same ideas as in the case $p = 2$ we can see that the diagonal section of C_{τ_p} satisfies

$$\delta_{C_{\tau_p}}(t) = C_{\tau_p}(t, t, \ldots, t) = \delta_{\tau_p}(t) \quad \text{for every} \quad t \in [0, 1]. \tag{5.57}$$

Besides, using that

$$0 = t_0 \le \alpha < t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} < t_{2k+1} < \cdots < t_{2p-1} < 1 - \frac{\alpha}{d-1} \le 1 = t_{2p},$$

it is easy to see that C_{τ_p} has uniform $\mathbf{I}$ univariate marginals. Therefore, from (5.57), C_{τ_p} is a d-copula satisfying the conditions of the Theorem. □

Remark 5.3. Let $-\infty < a < b < \infty$ and let $h : [a, b] \to \mathbf{R}$ be a function, such that,

$$h(t) - h(s) \le d(t - s) \quad \text{for every} \quad a \le s \le t \le b. \tag{5.58}$$

If $h(b) = h(a) + d(b - a)$, then $h(t) = h(a) + d(t - a)$ for every $t \in [a, b]$.

To see this, define $\varphi(t) = h(a)+d(t-a)$ for every $t \in [a, b]$. Then $h(a) = \varphi(a)$ and $h(b) = \varphi(b)$. If we assume that there exists $t_0 \in (a, b)$, such that, $h(t_0) < \varphi(t_0) = h(a) + d(t_0 - a)$, then

$$\begin{aligned} h(b) - h(t_0) &= (h(a) + d(b - a)) - h(t_0) \\ &> h(a) + d(b - a) - h(a) - d(t_0 - a) \\ &= d(b - t_0), \end{aligned}$$

which contradicts (5.58). If we assume that there exists $t_0 \in (a, b)$, such that, $h(t_0) > \varphi(t_0) = h(a) + d(t_0 - a)$, then $h(t_0) - h(a) > d(t_0 - a)$ contradicting (5.58), again. So, for every $t \in [a, b]$, $h(t) = \varphi(t) = h(a) + d(t - a)$.

In the next Theorem we prove the statement given in [8] about the minimality of the d-diagonal δ_n.

Theorem 5.4. *Let δ be a d-diagonal for some integer $d \geq 2$. Let $n \geq 0$ and let $k \in \{0, 1, \ldots, 2^n - 1\}$. Define*

$$a_{n,k} = \frac{k+1}{2^n} - \frac{\left[\delta\left(\frac{k+1}{2^n}\right) - \delta\left(\frac{k}{2^n}\right)\right]}{d}, \tag{5.59}$$

and

$$\delta_n(t) = \begin{cases} \delta\left(\frac{k}{2^n}\right) & \text{if} \quad \frac{k}{2^n} \leq t \leq a_{n,k} \\ \delta\left(\frac{k}{2^n}\right) + d(t - a_{n,k}) & \text{if} \quad a_{n,k} \leq t \leq \frac{k+1}{2^n}, \end{cases}$$

for $k \in \{0, 1, \ldots, 2^n - 1\}$. Then δ_n is a d-diagonal.
Evenmore, δ_n is the minimal d-diagonal such that $\delta_n\left(\frac{k}{2^n}\right) = \delta\left(\frac{k}{2^n}\right)$ for every $k \in \{0, 1, \ldots, 2^n\}$. Besides, $\delta_n(t) \to \delta(t)$ uniformly on $\mathbf{I}$.

Proof. Since δ is a d-diagonal, using (4.4) and (4.5), we have that for every $n \geq 0$ and for every $k \in \{0, 1, \ldots, 2^n - 1\}$,

$$0 \leq \frac{\delta((k+1)/2^n) - \delta(k/2^n)}{d} \leq \frac{d((k+1)/2^n - k/2^n)}{d} = \frac{1}{2^n},$$

then by equation (5.59),

$$\frac{k}{2^n} = \frac{k+1}{2^n} - \frac{1}{2^n} \leq a_{n,k} \leq \frac{k+1}{2^n}. \tag{5.60}$$

Also, $\delta_n(a_{n,k}) = \delta(k/2^n) = d(a_{n,k} - a_{n,k}) + \delta(k/2^n)$. So, δ_n in equation (5.60) is well defined. We also observe that from equations (5.59) and (5.60) we have that

$$\delta_n\left(\frac{k}{2^n}\right) = \delta\left(\frac{k}{2^n}\right) \quad \text{and} \quad \delta_n\left(\frac{k+1}{2^n}\right) = \delta\left(\frac{k+1}{2^n}\right), \tag{5.61}$$

for every $n \geq 0$ and for every $k \in \{0, 1, \ldots, 2^n - 1\}$.

Now we will see that $\delta_n(t) \leq \delta(t)$ for every $n \geq 0$ and for every $t \in \mathbf{I}$. Let $n \geq 0$ and let $k \in \{0, 1, \ldots, 2^n - 1\}$. Since δ is increasing by (4.5), then $\delta(k/2^n) \leq \delta((k+1)/2^n)$.

If $\delta(k/2^n) = \delta((k+1)/2^n)$, then $\delta(t) = \delta(k/2^n)$ for every $t \in [k/2^n, (k+1)/2^n]$. Now, by definition (5.59), we have that $a_{n,k} = (k+1)/2^n$, and by (5.60), $\delta_n(t) = \delta(k/2^n)$ for every $t \in [k/2^n, (k+1)/2^n]$. So, $\delta(t) = \delta_n(t)$ for every $t \in [k/2^n, (k+1)/2^n]$.

If $\delta(k/2^n) < \delta((k+1)/2^n) < \delta(k/2^n) + d/2^n$, then by definition (5.59), we have that

$$\frac{k}{2^n} < a_{n,k} < \frac{k+1}{2^n}. \tag{5.62}$$

Then by definition (5.60) and by condition (4.5),

$$\delta_n(t) = \delta\left(\frac{k}{2^n}\right) \leq \delta(t) \quad \text{for every} \quad t \in \left[\frac{k}{2^n}, a_{n,k}\right].$$

Now, if we assume that for some $t \in (a_{n,k}, (k+1)/2^n)$ and $\delta(t) < \delta_n(t) = \delta(k/2^n) + d(t - a_{n,k})$, then by (5.61) and (5.60)

$$\begin{aligned}
\delta\left(\frac{k+1}{2^n}\right) - \delta(t) &> \delta_n\left(\frac{k+1}{2^n}\right) - \delta_n(t) \\
&= \delta\left(\frac{k}{2^n}\right) + d\left(\frac{k+1}{2^n} - a_{n,k}\right) - \delta\left(\frac{k}{2^n}\right) - d(t - a_{n,k}) \\
&= d\left(\frac{k+1}{2^n} - t\right),
\end{aligned}$$

which contradicts (4.5). Hence, $\delta_n(t) \leq \delta(t)$ for every $t \in [k/2^n, (k+1)/2^n]$.

Finally, if $\delta((k+1)/2^n) = \delta(k/2^n) + d/2^n$, then $a_{n,k} = k/2^n$ and $\delta((k+1)/2^n) = \delta(k/2^n) + d((k+1)/2^n - k/2^n)$. So, by Remark 5.3, with $a = k/2^n$ and $b = (k+1)/2^n$, $\delta(t) = \delta(k/2^n) + d(t - a_{n,k}) = \delta_n(t)$ for every $t \in [k/2^n, (k+1)/2^n]$. Therefore, by (4.6),

$$\delta_n(t) \leq \delta(t) \leq t = \delta_{M^d}(t) \quad \text{for every} \quad t \in \mathbf{I}, \text{ and for every } n \geq 0. \tag{5.63}$$

Now, we analyze the extreme cases for δ. First, if we assume that $\delta(t) = \delta_{W^d}(t) = \max\{0, dt - d+1\}$ for every $t \in \mathbf{I}$. Then, in this case $\delta(t) = 0$ if $0 \leq t \leq 1-1/d$, and $\delta(t) = d(t-(1-1/d))$ for $1 - 1/d \leq t \leq 1$. Let $n \geq 0$, then there exists a unique $k_0 \in \{0, 1, \ldots, 2^n - 1\}$ such that $k_0/2^n \leq 1 - 1/d < (k_0+1)/2^n$. Then using equation (5.59) we have that

$$\begin{aligned} a_{n,k_0} &= \frac{k_0+1}{2^n} - \frac{\left[\delta\left(\frac{k_0+1}{2^n}\right) - \delta\left(\frac{k_0}{2^n}\right)\right]}{d} \\ &= \frac{k_0+1}{2^n} - \frac{\left[d\left(\frac{k_0+1}{2^n} - \left(1-\frac{1}{d}\right)\right) - 0\right]}{d} \\ &= 1 - \frac{1}{d}. \end{aligned} \tag{5.64}$$

From the definition of $\delta(t)$ in this case, it is clear that for every $k > k_0$ with $k \in \{0, 1, \ldots, 2^n - 1\}$, we have that $a_{n,k} = k/2^n$, because δ increases with slope d beginning at $t = 1 - 1/d$. So, using definition (5.60) we have that

$$\delta_n(t) = \delta_{W^d}(t) = \max\left\{0, d\left(t - \left(1 - \frac{1}{d}\right)\right)\right\} \quad \text{for every} \quad n \geq 0. \tag{5.65}$$

In fact, using the same arguments as above and (5.63), it is easy to see that for every d-diagonal δ, $\delta_0(t) = \delta_{W^d}(t)$ for every $t \in \mathbf{I}$.

Second, if we assume that $\delta(t) = \delta_{M^d}(t) = t$ for every $t \in \mathbf{I}$. Then from equation (5.59) we have that for every $k \in \{0, 1, \ldots, 2^n - 1\}$,

$$\frac{k}{2^n} < a_{n,k} = \frac{k+1}{2^n} - \frac{1}{2^n d} = \frac{1}{2^n}\left(k + 1 - \frac{1}{d}\right) < \frac{k+1}{2^n}. \tag{5.66}$$

The inequality (5.63) imposes restrictions on the values of $a_{n,k}$ in equation (5.59). For example, if $a_{n,0} < (1/2^n)(1 - 1/d)$, then by equation (5.60) we would have that $\delta_n(t) = d(t - a_{n,0}) > d(t-(1/2^n)(1-1/d))$ for every $t \in [a_{n,0}, 1/2^n]$. But in this case, $\delta_n(1/2^n) > 1/2^n$, contradicting (5.63). Therefore,

$$(1/2^n)(1 - 1/d) \leq a_{n,0} \leq 1/2^n. \tag{5.67}$$

On the other hand, if $a_{n,2^n-1} = 1 = ((2^n-1)+1)/2^n$, then by equation (5.60), we would have that $\delta_n(t) = \delta((2^n - 1)/2^n)$ for every $t \in [(2^{n-1})/2^n, 1]$. But $\delta(1) = 1$, so, $\delta((2^n - 1)/2^n) =$

$1 > (2^n - 1)/2^n$ contradicting (5.63). Therefore,

$$\frac{2^n - 1}{2^n} \le a_{n,2^n-1} < \frac{(2^n-1)+1}{2^n} = 1. \tag{5.68}$$

Observe that from equation (5.59), for every $n \ge 0$ we have that

$$\begin{aligned} \sum_{k=0}^{2^n-1}\left(\frac{k+1}{2^n} - a_{n,k}\right) &= \sum_{k=0}^{2^n-1} \frac{\left[\delta\left(\frac{k+1}{2^n}\right) - \delta\left(\frac{k}{2^n}\right)\right]}{d} \\ &= \frac{\delta(1) - \delta(0)}{d} \\ &= \frac{1}{d}. \end{aligned} \tag{5.69}$$

Besides, using (5.60), if we define $J_\delta = \{k \in \{0, 1, \ldots, 2^n - 1\} \mid k/2^n \le a_{n,k} < (k+1)/2^n\}$ it is clear from (5.69) that

$$\sum_{k \in J_\delta} \left(\frac{k+1}{2^n} - a_{n,k}\right) = \frac{1}{d}. \tag{5.70}$$

From equation (5.70) it is clear that $J_\delta \neq \emptyset$, in fact by equation (5.68), $k = 2^n - 1 \in J_\delta$.
Now, we will find for δ_n an appropriate partition $\tau_p = \{t_0, t_1, \ldots, t_{2p}\}$ of $\mathbf{I}$, such that, $0 = t_0 < t_1 < \cdots < t_{2p-1} < t_{2p} = 1$, for some $1 \le p \le 2^{n-1}$, and that satisfies conditions (5.2) and (5.3) of Theorem 5.2.
Define $H_\delta = \{j \in \{0, 1, \ldots, 2^n - 1\} \mid j/2^n < a_{n,j} < (j+1)/2^n\}$ and $I_\delta = \{k \in \{0, 1, \ldots, 2^n - 1\} \mid a_{n,k} = k/2^n\}$. Then

$$J_\delta = H_\delta \cup I_\delta, \tag{5.71}$$

which is a disjoint union.
Assume that $I_\delta = \{k_1, k_2, \ldots k_s\}$, where $0 < k_1 < k_2 < \cdots < k_s \le 2^n - 1$ with $s \ge 0$, and $H_\delta = \{j_1, j_2, \ldots, j_r\}$, where $0 \le j_1 < j_2 < \cdots < j_r \le 2^n - 1$ with $r \ge 0$. If $s = 0$ or $r = 0$ the corresponding sets are empty. Since $I_\delta \cap H_\delta = \emptyset$, and using (5.71), J_δ can be written as

$$J_\delta = \{l_1, l_2, \ldots, l_{r+s}\}, \quad \text{where} \quad 0 \le l_1 < l_2 < \cdots < l_{r+s} \le 2^n - 1. \tag{5.72}$$

Define for every $i \in \{1, 2, \ldots, r+s\}$

$$w_{2i-1} = \begin{cases} \frac{l_i}{2^n} & \text{if} \quad l_i \in I_\delta \\ a_{n,l_i} & \text{if} \quad l_i \in H_\delta, \end{cases}$$

and

$$w_{2i} = \frac{l_i + 1}{2^n} \quad \text{if} \quad l_i \in J_\delta. \tag{5.73}$$

Let

$$U = \{w_1, w_2, w_3, w_4, \ldots w_{2(r+s)-1}, w_{2(r+s)}\}. \tag{5.74}$$

Then from (5.72) we have that the cardinality of U satisfies that $2(r+s) \leq 2^{n+1}$. First we observe that

$$w_1 = \begin{cases} \frac{l_1}{2^n} & \text{if} \quad l_i = k_1 > 0, \ k_1 \in I_\delta \\ a_{n,l_i} > 0 & \text{if} \quad l_i = j_1 = 0 \in H_\delta. \end{cases}$$

Therefore, $w_1 > 0$.

Observe that if $H_\delta = \{0, 1, 2, \ldots, 2^n - 1\}$, then $I_\delta = \emptyset$ and $J_\delta = H_\delta$. In this case

$$U = \left\{a_{n,0}, \frac{1}{2^n}, a_{n,1}, \frac{2}{2^n}, \ldots, a_{n,2^n-1}, \frac{2^n}{2^n} = 1\right\}.$$

If we define

$$t_0 = 0 < t_1 = a_{n,0} < t_2 = \frac{1}{2^n} < t_3 = a_{n,1} < t_4 = \frac{2}{2^n} < \cdots < t_{2^{n+1}-1} = a_{n,2^n-1} < t_{2^{n+1}} = \frac{2^n}{2^n}.$$

Then $\tau_p = \{t_0, t_1, \ldots, t_{2^{n+1}-1}, t_{2^{n+1}}\}$ is a partition of $\mathbf{I}$, as defined in Theorem 5.2, with $p = 2^n$.

If for some $i \in \{1, 2, \ldots, 2(r+s)\}$ we have that

$$w_{2i} = w_{2i+1} \quad \text{we cancel out the terms } w_{2i} \text{ and } w_{2i+1} \text{ from } U. \tag{5.75}$$

Now define the remaining elements by $\{t_1, t_2, \ldots, t_{2p-1}, t_{2p}\}$, where $t_1 < t_2 < \cdots, t_{2p-1} < t_{2p}$, $t_1 > 0$ and $t_{2p} = 1$ using (5.75) and (5.68). Besides, $p \leq (r+s) \leq 2^n$, and we saw above the upper bound can be found. Therefore, if we take $t_0 = 0$ and we define

$$\tau_p = \{t_0, t_1, t_2, \ldots, t_{2p-1}, t_{2p}\}. \tag{5.76}$$

Then τ_p is a partition that satisfies conditions (5.1), (5.2) and (5.3) of Theorem 5.2, this follows from equation (5.70) and inequality (5.63). So, for every δ d-diagonal and for every $n \geq 0$, and δ_n given in equation (5.60), there exists $p \geq 1$, such that $\delta_n = \delta_{\tau_p}$ as defined in (5.76), where τ_p satisfies the conditions of Theorem 5.2. Therefore, there exists a d-copula C_{τ_p} such that its diagonal satisfies that

$$\delta_{C_{\tau_p}}(t) = \delta_{\tau_p}(t) = \delta_n(t) \quad \text{for every} \quad t \in \mathbf{I}. \tag{5.77}$$

As an explanatory example, assume that δ is a d-diagonal for $d = 2$ and $n = 3$, that is, $2^n = 8$, and assume that δ satisfies:

$$\delta\left(\frac{i}{8}\right) = \begin{cases} 0.01 & \text{if} \quad i = 1 \\ 0.02 & \text{if} \quad i = 2 \\ 0.03 & \text{if} \quad i = 3 \\ 0.28 & \text{if} \quad i = 4 \\ 0.53 & \text{if} \quad i = 5 \\ 0.75 & \text{if} \quad i = 6 \\ 0.75 & \text{if} \quad i = 7 \\ 1 & \text{if} \quad i = 8 \end{cases}$$

Using (5.78) we have that

$$H_\delta = \{0, 1, 2, 5\} = \{j_1, j_2, j_3, j_4\} \quad \text{and} \quad I_\delta = \{3, 4, 7\} = \{k_1, k_2, k_3\},$$

and

$$J_\delta = \{0, 1, 2, 3, 4, 5, 7\} = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7\}.$$

Then using (5.73) and (5.73) we have that

$$w_1 = a_{3,0}, w_3 = a_{3,1}, w_5 = a_{3,2}, w_7 = \frac{3}{8}, w_9 = \frac{4}{8}, w_{11} = a_{3,5} \text{ and } w_{13} = \frac{7}{8},$$

and

$$w_2 = \frac{1}{8}, w_4 = \frac{2}{8}, w_6 = \frac{3}{8}, w_8 = \frac{4}{8}, w_{10} = \frac{5}{8}, w_{12} = \frac{6}{8} \text{ and } w_{14} = \frac{8}{8} = 1,$$

and U in equation (5.74) includes every w_i for $i \in \{1, 2, \ldots, 14\}$. Since

$$w_6 = \frac{3}{8} = w_7 \quad \text{and} \quad w_8 = \frac{4}{8} = w_9.$$

Then condition (5.75) is satisfied by $i = 3$ and $i = 4$. So, we delete four elements of U. If we define using the remaining terms

$$t_0 = 0 < t_1 = a_{3,0} < t_2 = \frac{1}{8} < t_3 = a_{3,1} < t_4 = \frac{2}{8} < t_5 = a_{3,2}$$

$$< t_6 = \frac{5}{8} < t_7 = a_{3,5} < t_8 = \frac{6}{8} < t_9 = \frac{7}{8} < t_{10} = \frac{8}{8} = 1. \tag{5.78}$$

If we define, using (5.78), $\tau_5 = \{t_0, t_1, \ldots, t_8, t_9, t_{10}\}$ is a partition of $[0, 1]$ satisfying the conditions of Theorem 5.2.

Another relevant example is to consider $q = 2$ with $\delta = \delta_{M^2}$ and $n = 2$, in this case

$$\delta\left(\frac{i}{4}\right) = \begin{cases} 0 & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i = 2 \\ 0.5 & \text{if} \quad i = 3 \\ 1 & \text{if} \quad i = 4 \end{cases}$$

Using (5.79) we have that

$$H_\delta = \emptyset \quad \text{and} \quad I_\delta = \{3, 4\} = \{k_1, k_2\}$$

and

$$J_\delta = \{3, 4\} = \{l_1, l_2\}.$$

Then using (5.73) and (5.73) we have that

$$w_1 = \frac{2}{4}, w_3 = \frac{3}{4}, w_2 = \frac{3}{4} \text{ and } w_4 = \frac{4}{4} = 1.$$

and U in equation (5.74) includes every w_i for $i \in \{1, 2, 3, 4\}$. Since

$$w_2 = \frac{3}{4} = w_3.$$

Then condition (5.75) is satisfied by $i = 1$. So we delete two elements of U, that is w_2 and w_3. If we define using the remaining terms

$$t_0 = 0 < t_1 = \frac{2}{4} = \frac{1}{2} < t_2 = \frac{4}{4} = 1,$$

and we define

$$\tau_1 = \{t_0, t_1, t_2\}. \tag{5.79}$$

Then clearly τ_1 is a partition of $\mathbf{I}$, satisfying the conditions of Theorem 5.2, for $p = 1$. So, in general the value of p may vary between 1 and 2^n.

We finally prove that $\lim_{n\to\infty} \delta_n(t) = \delta(t)$ uniformly on $\mathbf{I}$. We know that $\delta_n(0) = 0 = \delta(0)$ and $\delta_n(1) = 1 = \delta(1)$ for every $n \geq 0$. Let $t \in (0, 1)$ and let $n \geq 0$, then there exists a unique $k_0 \in \{0, 1, \ldots, 2^n - 1\}$ such that $k_0/2^n \leq t < (k_0 + 1)/2^n$. So, using definition (5.60), and the fact that δ_n is a d-diagonal, we have that

$$\begin{aligned} |\delta(t)-\delta_n(t)| &\le \left|\delta(t)-\delta\left(\frac{k_0}{2n}\right)\right| + \left|\delta\left(\frac{k_0}{2^n}\right)-\delta_n\left(\frac{k_0}{2^n}\right)\right| + \left|\delta_n\left(\frac{k_0}{2^n}\right)-\delta_n(t)\right| \\ &\le d\left(t-\frac{k_0}{2^n}\right)+0+d\left(t-\frac{k_0}{2^n}\right) \\ &\le \frac{2d}{2^n} \\ &= \frac{d}{2^{n-1}}. \end{aligned} \tag{5.80}$$

So, if we let $\epsilon > 0$ and we take $N \ge 1$, such that $d/2^{N-1} < \epsilon$, then using (5.80), we have $|\delta(t)-\delta_n(t)| < \epsilon$, for every $n \ge N$ and for every $t \in \mathbf{I}$. Therefore, $\delta_n(t)$ converges uniformly to $\delta(t)$ on $\mathbf{I}$.

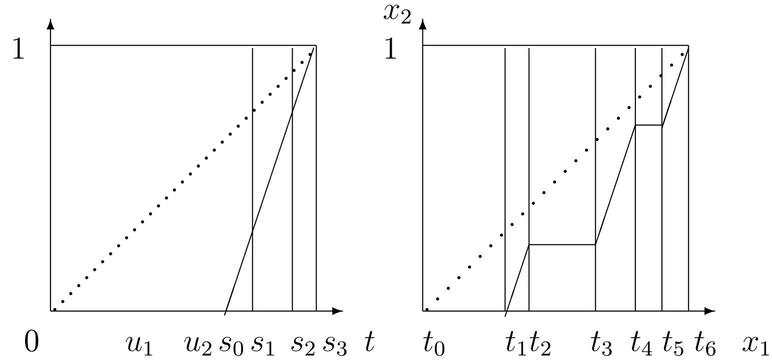
Evenmore, it is not difficult to prove that the sequence $\{\delta_n\}_{n=0}^{\infty}$ increases uniformly to δ on $\mathbf{I}$. □

Now the proof of Theorem 5.1 is straightforward, because, if δ is a d-diagonal then by Theorem by Theorem 5.4, we have a sequence of d-diagonals $\{\delta_n\}_{n=0}^{\infty}$ such that $\delta_n(t)$ converges uniformly to $\delta(t)$ on $\mathbf{I}$. But, for every $n \ge 0$, δ_n satisfies the conditions of Theorem 5.2. So, for every $n \ge 0$ there exists a d-copula C_n such that its diagonal satisfies that $\delta_{C_n} = \delta_n$ for every $n \ge 0$. If we define the supremum distance such that if C, D are d-copulas then $d_{\text{sup}}(C,D) = \sup_{\underline{x}\in\mathbf{I}^d} |C(\underline{x}) - D(\underline{x})|$. Then by compactness of the set of all copulas with supremum distance, there exists C a d-copula and a subsequence $\{C_{n_k}\}_{k=1}^{\infty}$ such that $C_{n_k} \to C$ uniformly. Then C is such that $\delta_C = \delta$.

We also observe that the d-diagonal δ_n given in Theorem 5.4 corresponds to a shuffle of the diagonal $\delta_{W^d}(t)$ defined in equation (4.7). We will say that a shuffle of the diagonal $\delta_{W^d}(t)$ is given by a partition of the interval $[1-1/d, 1]$, namely $1-1/d = s_0 < s_1 < \cdots < s_{p-1} < s_p = 1$ when we select a partition of $[0, 1-1/d]$ $u_0 = 0 < u_1 < u_2 < \cdots < u_{p-1} < u_p = 1-1/d$ such that if we define a partition of $\mathbf{I}$ by taking $t_0 = 0 < t_1 = u_1 < t_2 = u_1 + (s_1 - s_0) < t_3 = u_2 + (s_1 - s_0) < t_4 = u_2 + (s_2 - s_0) < \cdots < t_{2p-1} = u_p + (s_{p-1} - s_0) < t_{2p} = u_p + (s_p - s_0) = 1 - 1/d + (1 - (1 - 1/d)) = 1$, that satisfies (5.2) and (5.3). Then if we define δ_{τ_p}, as in Theorem 5.2, where $\tau_p = \{t_0, t_1, \ldots, t_{2p-1}, t_{2p}\}$, we obtain a d-diagonal.

Example 5.5. Let $d = 3$ and $p = 3$ let $\{2/3 = s_0 < s_1 = 3/4 < s_2 = 9/10 < s_3 = 1\}$ be a partition of $[2/3, 1]$ and define $\{u_0 = 0 < u_1 = 19/60 < u_2 = 17/30 < u_3 = 2/3\}$ to be the partition of $[0, 2/3]$. Then $\tau_p = \{0 = t_0, t_1 = 19/60, t_2 = 4/10, t_3 = 13/20, t_4 = 8/10, t_5 = 9/10, t_6 = 1\}$ which satisfies (5.2) and (5.3). The diagonal δ_{τ_p} is given in Figure 5, and from Theorem 5.4, δ_{τ_p} is the minimal 3-diagonal which passes through the points $\langle 4/10, 1/4\rangle$ and $\langle 8/10, 7/10\rangle$.

Figure 5.- A $s$huffle of δ_{W^d}



Chapter 6

Conclusions

In Chapter 2 we extended the results in De Baets and De Meyer [12] to larger dimensions. These results allowed us to provide alternative proof for the generalization Ordinal Sums in Mesiar and Sempi [55] and to give a general framework for the proofs of the multivariate patchwork-like constructions. We also found a multivariate extension of the bivariate rectangular patchwork construction of [23, 24]. Such constructions can modify a copula only in a box attached to the upper corner of $\mathbf{I}^n$ but we proposed a methodology in which after successive *patches* we can get modifications of a copula in inner n-boxes. The methodology gives us the possibility to model tail dependence, but that is left for future research as well as finding an algorithm to generate a random sample from the *patched* copula.

In Chapter 3 we provided the multivariate generalization of the construction in Cuculescu and Theodorescu [8]. We succeeded in our attempt to construct a new family of copulas: the sample d-copula of order m, based on the ideas of the transformation matrices given in [34], and its generalization in [69]. Such construction is given in two settings: First when the sample is obtained from a d-copula C, and second when the sample comes from a continuous d-distribution function on R^d. In both cases we were able to provide interesting statistical applications such as a new methodology for estimation of parameters, a goodness-of-fit test and results about the usual concordance measures. In the second case, for $2 \leq m \leq n$, we proved that d-dimensional square matrix S_m^n, used in the definition of the sample d-copula of order m, C_m^n in equation (3.20), is always a generalized transformation matrix, which allows us to have a non trivial sample d-copula.

We also proved that the resulting family of d-copulas is dense in every dimension $d \geq 2$ for the sup norm. The main advantage of this new construction can yield possibly useful results when sample sizes are very large. We still need to extend some of the results from the first

setting into the second setting.
Also in this chapter we present the relation of d-dimensionally stochastic matrices and transformation matrices and we find the necessary number of parameters for a square d-dimensional matrix to be a d-dimensional stochastic matrix.

In Chapter 4 we prove that the direct algebraic extensions of the Bertino copula and the diagonal copula do not work. We generalize Frank's condition for a diagonal to determine uniquely an Archimedean copula and its proof, given in Erdely [26]. We were also able to extend the family of generators given in Alsina *et al* [2] which show that when the Frank condition is not satisfied the diagonal does not characterize uniquely the generator and so does not characterize uniquely the Archimedean copula.

Chapter 5 provides a step by step proof of the construction of d-copulas with a given diagonal section. We also make a connection to the concept of shuffles of the diagonal δ_{W^d}.

Bibliography

[1] Aczél, J. and Dhombres, J. (1989). *Functional Equations in Several Variables.* Cambridge University Press, Cambridge, UK.

[2] Alsina, C., Frank, M. J., and Schweizer, B. (2006). *Associative Functions: Triangular norms and copulas.* World Scientific Publishing Co., Singapore.

[3] Berger, J. and Bernardo, J. (1992). Ordered group reference priors with application to the multinomial problem. *Biometrika*, 79-1:25–37.

[4] Bertino, S. (1977). Sulla dissomiglianza tra mutabili cicliche. *Metron*, 35:53–88.

[5] Bouyé, E., Durrleman, V., Nikeghbali, A., Riboulet, G., and Roncalli, T. (2000). *Copulas for finance. A reading guide and some applications.* Paris: groupe de recherche opérationnelle, Crédit Lyonnais, Paris.

[6] Cherubini, U., Luciano, E., and Vecchiato, W. (2004). *Copula Methods in Finance.* John Wiley & Sons, Chichester.

[7] Cressie, N. and Read, T. (1984). Multinomial goodness-of-fit tests. *J. R. Statist. Soc.*, 46:440–464.

[8] Cuculescu, I. and Theodorescu, R. (2001). Copulas: Diagonals, tracks. *Rev. Roumaine Math. Pures Appl.*, 6:731–742.

[9] Darsow, W. and Frank, M. J. (1983). Associative functions and Abel-Schröder systems. *Publ. Math. Debrecen*, 30:253–272.

[10] de Amo, E., Díaz Carrillo, M., and Fernández-Sánchez, J. (2011). Measure-preserving functions and the independence copula. *Mediterr. J. Math.*, 8:431–450.

[11] de Amo, E., Díaz Carrillo, M., and Fernández-Sánchez, J. (2012). Copulas and associated fractal sets. *J. Math. Anal. Appl.*, 386:528–541.

[12] De Baets, B. and De Meyer, H. (2007). Orthogonal grid constructions of copulas. *IEEE Trans. Fuzzy Systems*, 15-6:1053–1062.

[13] Deheuvels, P. (1979). La fonction de dépendance empirique et ses propriétés. Un test non paramétrique d'indépendance. *Roy. Belg. Bull. Cl. Sci.*, 65-5:274–292.

[14] Dorey, M. and Joubert, P. (2005). *Modelling copulas: An overview.* The Staple Inn Actuarial Society, London.

[15] Dudley, R. M. (2002). *Real Analysis and Probability.* Cambridge University Press.

[16] Durante, F. and Fernández-Sánchez, J. (2010a). Multivariate shuffles and approximation of copulas. *Statist. Probab. Lett.*, 80:1827–1834.

[17] Durante, F. and Fernández-Sánchez, J. (2010b). Multivariate shuffles and approximation of copulas. *Statist. Probab. Lett.*, 80:1827–1834.

[18] Durante, F., Fernández-Sánchez, J., and Sempi, C. (2013). A note on the notion of singular copula. *Fuzzy Sets and Systems*, 211:120–122.

[19] Durante, F., Kolesárová, A., Mesiar, R., and Sempi, C. (2007a). Copulas with given diagonal sections: novel constructions and applications. *Int. J. Uncertain. Fuzziness Knowledge-Based Syst.*, 15:397–410.

[20] Durante, F., Mesiar, R., and Sempi, C. (2005). Copulas with given diagonal section: Some new results. In *Procceed. of EUSFLAT-LFA Conference*, pages 931–936.

[21] Durante, F., Mesiar, R., and Sempi, C. (2006). On a family of copulas constructed from the diagonal section. *Soft Comput.*, 10:490–494.

[22] Durante, F., Quesada-Molina, J., and Úbeda-Flores, M. (2007b). On a family of multivariate copulas for aggregation processes. *Inform. Sci.*, 177:5715.5724.

[23] Durante, F., Saminger-Platz, S., and Sarkoci, P. (2008). On representations of 2-increasing binary aggregation functions. *Information Sciences*, 178:4534–4541.

[24] Durante, F., Saminger-Platz, S., and Sarkoci, P. (2009). Rectangular patchwork for bivariate copulas and tail dependence. *Comm. Statist. Theory and Methods*, 28:2515–2527.

[25] Embrechts, P., Höing, A., and Juri, A. (2003). Using copulae to bound the value-at-risk for functions of dependent risks. *Finance Stochat.*, 7:145–167.

[26] Erdely, A. (2007). *Diagonal properties of the empirical copula and applications. Construction of families of copulas with given restrictions.* PhD thesis, UNAM, México.

[27] Erdely, A. and Díaz-Vieira, M. (2010). Nonparametric and semiparametric bivariate modeling of petrophysical porosity-permeability dependence from well log data. In Jaworski, P., Durante, F., Härdle, W. K., and Rychlik, T., editors, *Copula Theory and Its Applications*, pages 267–278.

[28] Erdely, A., González-Barrios, J. M., and Hernández-Cedillo, M. M. (2013). Frank's condition for multivariate archimedean copulas. *Fuzzy Sets and Systems*. In press.

[29] Fermanian, J., Radulović, D., and Wegcamp, M. (2004). Weak convergence of empirical copula. *Bernoulli*, 10-5:847–860.

[30] Fernández-Sánchez, J., Nelsen, R. B., and Úbeda-Flores, M. (2011). Multivariate copulas, quasi-copulas and lattices. *Statist. Probab. Lett.*, 81:1365–1369.

[31] Frank, M. J. (1996). Diagonals of copulas and Schröeder's equation. *Aequationes Math.*, 51:150.

[32] Fredricks, G. A. and Nelsen, R. B. (1997). Copulas constructed from diagonal sections. In Beneš, V. and Štěpán, editors, *Distributions with given marginals and moments problems*, pages 129–136.

[33] Fredricks, G. A. and Nelsen, R. B. (2002). The Bertino family of copulas. In Cuadras, C. M., Fortiana, J., and Rodríguez-Lallena, J. A., editors, *Distributions with given marginals and statistical modelling*, pages 81–91.

[34] Fredricks, G. A., Nelsen, R. B., and Rodríguez-Lallena, J. A. (2005). Copulas with fractal supports. *Insurance Math. Econom.*, 37:42–48.

[35] Genest, C., Rémillard, B., and Beaudoin, D. (2009). Goodness-of-fit tests for copulas: A review and a power study. *Insurance Math. Econom.*, 44:199–213.

[36] González-Barrios, J. M. and Hernández-Cedillo, M. M. (2013a). Construction of multivariate copulas in n-boxes. *Kybernetika*, 49:73–95.

[37] González-Barrios, J. M. and Hernández-Cedillo, M. M. (2013b). Minimal number of parameters of a d-dimensionally stochastic matrix. *Aportaciones Matemáticas de la SMM*. In press.

[38] González-Barrios, J. M. and Hernández-Cedillo, M. M. (2013c). Sample d-copula of order m. *Kybernetika*. In press.

[39] Jaworski, P. (2009). On copulas and their diagonals. *Inform. Sci.*, 179:2863–2871.

[40] Jaworski, P. and Rychlik, T. (2008). On distributions of order statistics for absolutely continuous copulas with applications to reliability. *Kybernetika*, 6:757–776.

[41] Joe, H. (1997). *Multivariate Models and Dependence Concepts*. Chapman& Hall, London.

[42] Kimberling, C. H. (1974). A probabilistic interpretation of complete monotonicity. *Aequationes Math.*, 10:152–164.

[43] Klement, E. P., Kolesárová, A., Mesiar, R., and Sempi, C. (2007). Copulas constructed from the horizontal section. *Comm. Statist. Theor. Meth.*, 36:2901–2911.

[44] Kuczma, M. (1968). *Functional equations in a single variable*. Polish Scientific publishers, Warsaw.

[45] Kuczma, M., Choczewski, B., and Ger, B. (1990). *Iterative functional equations*. Cambridge Universite Press, New York.

[46] Ling, C. H. (1965). Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212.

[47] Mahmoud, H. M. (2008). *Polya urn models*. Chapman & Hall/CRC, New York.

[48] Mai, J. and Scherer, M. (2012). *Simulating copulas: stochastic models, sampling algorithms and applications*. Imperial College Press, London.

[49] Malevergne, Y. and Sornette, D. (2006). *Extreme Financial Risk. From Dependence to Risk Management*. Springer-Verlag, Berlin.

[50] Marcus, M. (1960). Some properties and applications of double stochastic matrices. *Amer. Math. Monthly*, 67:215–221.

[51] McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management*. Princeton University Press, Princeton.

[52] McNeil, A. J. and Nešlehová, J. (2009). Multivariate archimedean copulas, d-monotone functions and l_1-norm symmetric distributions. *Ann. Statist.*, 37:3059–3097.

[53] Mesiar, R., Jágr, V., Juráňová, M., and Komorníková, M. (2008). Univariate conditioning of copulas. *Kybernetika*, 44-6:807–816.

[54] Mesiar, R. and Navara, M. (1999). Diagonals of continuous triangular norms. *Fuzzy Sets and Systems*, 104-1:35–41.

[55] Mesiar, R. and Sempi, C. (2010). Ordinal sums and idempotents of copulas. *Aequat. Math.*, 79:39–52.

[56] Mikusiński, P. and Taylor, M. D. (2010). Some approximations of n-copulas. *Metrika*, 72:385–414.

[57] Miller, K. S. and Samko, S. G. (2001). Completely monotonic functions. *Integr. Transf. and Spec. Funct.*, 4:389–402.

[58] Nelsen, R. B. (2006). *An introduction to copulas (In Springer Series in Statistics).* Springer, New York, second edition.

[59] Nelsen, R. B., Quesada-Molina, J. J., Rodríguez-Lallena, J. A., and Úbeda-Flores, M. (2008). On the construction of copulas and quasi-copulas with given diagonal sections. *Insurance: Math. and Economics*, 42:473–483.

[60] Read, T. and Cressie, N. (1988). *Goodness-of-fit statistics for discrete multivariate data.* Springer Series in Statist., Springer, New York.

[61] Rodríguez-Lallena, J. and Úbeda-Flores, M. (2003). Distribution functions of multivariate copulas. *Statist. Probab. Lett.*, 64:41–50.

[62] Rychlik, T. (1994). Distributions and expectations of order statistics for possible dependent random variables. *J. Multivariate Anal.*, 48:31–42.

[63] Salvadori, G., Michele, C. D., Kottegoda, N. T., and Rosso, R. (2007). *Extremes in Nature. An Approach Using Copulas.* Water Science and Technology Library. Springer, Amsterdam.

[64] Schröder, E. (1870). Über iterierte Funktionen. *Math. Ann.*, 2:296–322.

[65] Sherman, S. (1955). Doubly stochastic matrices and complex vector spaces. *Amer. J. Math.*, 77:245–246.

[66] Siburg, K. F. and Stoimenov, P. A. (2008). Gluing copulas. *Comm. Statist. Theory and Methods*, 37:3124–3134.

[67] Sungur, E. A. and Yang, Y. (1996). Diagonal copulas of archimedean class. *Comm. in Stastist. Theory and Methods*, 7:1659–1676.

[68] Trutschnig, W. (2012). Idempotent copulas with fractal support. *Adv. Comp. Intel.*, 298:161–170.

[69] Trutschnig, W. and Fernández-Sánchez, J. (2012). Idempotent and multivariate copulas with fractal support. *J. Statist. Plan. Infer.*, 142:3086–3096.

[70] Widder, D. V. (1946). *The Laplace transform.* Princeton University Press, Princeton.

[71] Wolfram Research, Inc. (2013). *Mathematica, Version 9.0.* Wolfram, Champaign, Illinois.

[72] Zhang, M. H. (2008). Modelling total tail dependence along diagonals. *Insur. Math. Econ.*, 42:73–80.