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Contents

Introducción a la Tesis

Como el lector ha visto en el título de esta tesis, ésta trata de tres temas distintos que poco (o nada) tienen que ver uno con el otro. Los tres temas son temas de Topología General y se espera que el lector tenga un buen nivel para poder leerla; dos buenos cursos de Topología General deben ser suficientes. Cada una de las tres partes de esta tesis tiene su propia introducción técnica así que aquí nos dedicaremos a hablar de los asuntos más mundanos sobre ella.

Todo empezó con los cursos impartidos por el Dr. Ángel Tamaríz durante mi maestría, con el libro [135]. Tal curso provocó en mi una gran emoción al descubrir una nueva forma de ver la Topología de la que yo conocía por mis estudios de licenciatura. Le pedí a Ángel una tesis de maestría sobre el tema y ya encarrerado, le pedí hacer el doctorado en el mismo tema con él.

Nuestros primeros resultados, de la Parte I, reflejan un intento de resolver problemas combinando la nueva Topología que aprendí en el curso de Ángel y las técnicas de hiperespacios que aprendí durante mi trabajo con el Dr. Alejandro Illanes. En particular el Teorema 4.6 y el Ejemplo 4.12 son ejemplos de la aplicación de esos dos puntos: la inducción transfinita y la intuición geométrica de como se "ve" el espacio de Erdős. Estos resultados se han publicado en el artículo [79].

En la Parte II, Ángel y yo intentamos adentrarnos en los temas conjuntistas que se encuentran en el libro [135]. Sin embargo, necesitábamos la experiencia de alguien familiarizado con los métodos. Por esta razón le pedimos ayuda al Dr. Michael Hrušák. Fruto de esta colaboración redactamos un artículo [80], que a pesar de que no contiene todas las respuestas a las preguntas que nos hicimos, salió bastante bien y nos dió algunas sorpresas (en particular el Ejemplo 9.24).

Completada la investigación de la Parte II, me quedé en Morelia con Michael para aprender más sobre la relación de la Teoría de Conjuntos con la Topología. Al principio intentamos abordar un tema cercano al de los puntos remotos de la Parte II. Sin embargo, el tema no dió los frutos deseados y nos movimos a un tema distinto. El tema de espacios densos en un conjunto numerable es un tema en el que Michael decía que al parecer únicamente él y el Prof. Jan van

Mill estaban interesados. Sin embargo, en esas fechas Michael recibió el borrador del artículo [110] gracias al cual obtuvimos la inspiración necesaria para obtener el Teorema 11.12 y escribir un artículo [81]. Más tarde, el Prof. Jan van Mill estuvo en Morelia trabajando con Michael en el mismo tema. Con la motivación de saber que se estaba trabajando arduamente en el tema en el cubículo de a lado, pude obtener un resultado más (Teorema 12.25) que respondía algunas preguntas. Así concluí exitosamente la escritura de otro artículo [78]. Ambos artículos [81] y [78] que forman la Parte III de la tesis se encuentran en estos momentos en revisión.

Escribir la tesis como está fue algo cansado. A mi siempre me ha molestado ver una prueba incompleta, haciendo referencias a articulos oscuros, que inclusive se te pide un pago si los quieres obtener por internet, eso si están disponibles. He intentado ser lo más explícito posible en las pruebas, incluyendo el mayor número posible de ellas. Sin embargo, no he logrado que la tesis sea completamente autocontenida. Claramente al escribir un trabajo de este nivel, uno tiene que suponer bastante material de Topología General "básica", tal material se encuentra en el Capítulo de Preliminares en la página ix. En la Parte II, dejé dos resultados escenciales sin prueba: el Teorema 6.41 que hubiera necesitado desarrollar mucha teoría y la Proposición 8.14, cuya prueba requeriría desarrollar el artículo original por completo. Debido a cuestiones de tiempo, no fue posible aplicar esta filosofía de la escritura en la Parte III.

Por el mismo camino de ayudar al lector lo más posible, he incluido muchos ejemplos y a veces hecho discusiones de más. Creo que, además de la formalidad que deben estar presentes en los textos matemáticos, los ejemplos y la imaginación de los matemáticos es parte escencial de la escencia de las matemáticas. Sin la motivación del problema, ejemplos bonitos, una intuición de como son los objetos matemáticos y su discusión de como se relacionan unas cosa con otras, las matemáticas carecerían de sentido. La tesis ha sido escrita en inglés ya que considero que así llegaré a un público más amplio.

Los resultados que considero mios están marcados con mi nombre (y los de mis coautores correspondientes). Estos resultados pueden ser resultados que fueron publicados en los artículos de investigación o simplemente resultados que no encontré en ningún lugar y considero de mi invención, aunque sean muy sencillos.

General Introduction

As the reader will notice from the title of this dissertation, we are in fact covering three different topics that have little relation with each other (perhaps none). These three topics are from General Topology and the reader is expected to have a good level in order to understand; two good courses on General Topology will do. Since each of these three parts has its own technical introduction, here we will focus on more trivial matters.

Everything started with the postgraduate courses based on the book [135] and given by Professor Ángel Tamariz while I was a master's student. Those courses woke up in me a new way of looking at topology that I had not seen in my BA studies. I asked Ángel for a master's thesis in the topic and just after completing it I also asked him to become my Ph.D. supervisor.

Our first results from Part I reflect an attempt to solve problems by combining the new Topology I learned at Ángel's course and the hyperspace techniques I learned during my work with Professor Alejandro Illanes. In particular, Theorem 4.6 and Example 4.12 give two instances of this duality: the use of transfinite induction and the geometric intuition of how Erdős space "looks like". These results have been published on the paper [79].

In Part II, Ángel and I were trying to go deeper into some topics related to set theory from the book [135]. However, we needed the experience of an expert in these methods. For this reason we asked Professor Michael Hrušák for help. Thanks to his collaboration we were able to write a paper [80]. Even if this paper does not contain all the answers to our questions, it came out pretty well and gave us some surprises (particularly, Example 9.24).

Having completed research in Part II, I stayed in Morelia with Michael so I could learn more about the relation between Set Theory and Topology. At first we tried to solve a problem related to the remote points from Part II. However, this topic was not fruitful enough and we moved towards a different topic. The topic of countable dense homogeneous spaces is a topic which Michael thought that only Professor Jan van Mill and him were interested in. However, during this time, Michael recieved a preprint [110] that gave us some inspiration to prove

Theorem 11.12 and write a paper [81]. Later, Professor Jan van Mill came to Morelia to work with Michael in that same topic. With the motivation to know that there was work in progress in the office next to mine about the same topic, I was able to obtain a new result (Theorem 12.25) that answered some questions. With this result I was able to write another paper [78]. Both papers [81] and [78] that form the core of Part III are being refereed as this is written.

Writing the dissertation as it is was tiring. I hate when I see an incomplete proof with references to obscure papers, some of which are unavailable or perhaps require online payment in order to gain access. I have tried to be explicit in the proofs, including most of them. However, I could not write a self-contained dissertation. Clearly in a work at this level we have to assume a lot of General Topology's "basic" background, such topics are included in the Preliminaries Chapter in page ix. In Part II, I left two essential results without proof: Theorem 6.41 that would have required to develop a great amount of theory and Proposition 8.14, whose prove would have required to develop the complete original paper. Due to time constrains, it was not possible to apply this philosophy to Part III.

Following the same idea of helping the reader as much as possible, I have included many examples and sometimes made long discussions about some topics. I think that besides all the formality that must be present in a mathematics text, examples and the mathematician's imagination are an essential part of mathematics. Without the mathematical motivation, nice examples, some intuition of how mathematical objects behave and interact with each other, mathematics would have no meaning. This dissertation was written in English because I think that it will arrive to a broader audience in this way.

The results I consider as mine are marked with my name (and my respective coauthors). Some results are from the submitted papers but some of them are results I could not find anywhere so I consider them mine, even if they are very simple.

Preliminaries

Set Theory

Set Theoretic knowledge has become crucial for the General Topologist. We will assume that the reader has at least knowledge of the axioms of ZFC and the axiomatic construction of ordinal numbers. Chapter 1 of [99] will do enough. Of course the other excellent reference for Set Theory is [90]. We will briefly mention some important concepts.

A *partially ordered set* is a set A with a binary relation $\leq$ such that: (a) for all $a \in A$, $a \leq a$; (b) if $a, b \in A$ are such that $a \leq b$ and $b \leq a$, then $a = b$; and (c) if $a, b, c \in A$ are such that $a \leq b$ and $b \leq c$, then $a \leq c$. As usual, we abreviate $(a \leq b) \wedge (a \neq b)$ by $a < b$. A set $D \subset A$ is *dense*[1] *with respect to the order* if every time $a \in A$, then there is $d \in D$ such that $d \leq a$. Two partially ordered sets $\langle A, \leq \rangle$ and $\langle B, \leq \rangle$ are *order isomorphic* if there is a bijection $h : A \to B$ such that $x \leq y$ if and only if $h(x) < h(y)$ for all $x, y \in A$; such an h is called *order isomorphism*. A partially ordered set $\langle A, \leq \rangle$ is *linearly ordered* if for every $x, y \in A$ either $x \leq y$ or $y \leq x$. A partially ordered set is *well-ordered* if every non-empty subset has a minimum.

Ordinals are those defined by von Neumann. Ordinals form a proper class **ON** that is well-ordered by the $\in$ relation. If α and β are ordinals, we will write $\alpha < \beta$ for $\alpha \in \beta$. In particular, every ordinal α is the set of its predecesors $\{\beta : \beta < \alpha\}$. We will identify every natural number n with the set of its predecessors $\{0, \ldots, n-1\}$ (thus, $0 = \emptyset$) so we will consider that each natural number is an ordinal. Recall that there are two types of non-zero ordinals: succesors and limits. The *order type* of a well ordered set $(S, \leq)$ is the unique ordinal α such that $(S, \leq)$ is order-isomorphic to $(\alpha, \in)$. The basic theory of ordinals gives us the Theorems of Induction and Recursion.

Induction Let **O** a class of ordinals such that:

[1] Dense with respect to an order is a different notion from dense subset of a topological space.

(1) $\emptyset \in \mathbf{O}$ and

(2) if β is an ordinal and for all $\alpha < \beta$ we have that $\alpha \in \mathbf{0}$, then $\beta \in \mathbf{O}$.

Then it follows that $\mathbf{O} = \mathbf{ON}$.

Recursion Let $G : \mathbf{V} \to \mathbf{V}$ be a functional where $\mathbf{V}$ is the class of all sets. Then there is a functional $F : \mathbf{ON} \to \mathbf{V}$ such that $F(\alpha) = F(G[\alpha])$.

Cardinals are initial ordinals, this means that an ordinal κ is a cardinal if and only if for every $\alpha < \kappa$ we have that there is no bijection between α and κ. Natural numbers are precisely the finite cardinals (finite by definition) and ω is the first infinite cardinal.

Each cardinal κ has its succesor κ^+. Using this, we can use recursion to list all infinite cardinals in a transfinite list $\{\omega_\alpha : \alpha \in \mathbf{ON}\}$ in the following way: $\omega_0 = \omega$ is the set of natural numbers, $\omega_{\alpha+1} = (\omega_\alpha)^+$ for each $\alpha \in \mathbf{ON}$ and $\omega_\beta = \sup\{\omega_\alpha : \alpha < \beta\}$ when β is a limit ordinal.

An ordinal α is regular if every time $\beta < \alpha$ and $\{\theta_\gamma : \gamma < \beta\} \subset \alpha$, then $\sup\{\theta_\gamma : \gamma < \beta\} < \alpha$. If an ordinal is regular, then it is a cardinal. If κ is a cardinal, then κ^+ is regular. An example of a singular (that is, non-regular) cardinal is ω_ω.

A very good review of the main ideas of ordinals can also be found in Chapter 1 of [29].

Lemma 0.1 If κ is a regular cardinal, then there is a function $\phi : \kappa \to \kappa$ such that $\{\beta < \kappa : \phi(\beta) = \alpha\}$ is cofinal in κ for every $\alpha < \kappa$.

Proof. Let $T = \{\langle\alpha, \beta\rangle \in \kappa \times \kappa : \alpha \leq \beta\}$ be ordered lexicographically, that is, $\langle\alpha_0, \beta_0\rangle < \langle\alpha_1, \beta_1\rangle$ if either $\alpha_0 < \alpha_1$ or both $\alpha_0 = \alpha_1$ and $\beta_0 < \beta_1$. Then T is well-ordered so it has the order type of an ordinal. Since κ is regular, it is not hard to see that such type is indeed κ so there is an order isomorphism $i : \kappa \to T$. Let $\phi : \kappa \to \kappa$ be defined by $\phi = \pi \circ i$, where $\pi : T \to \kappa$ is such that $\pi(\langle\alpha, \beta\rangle) = \beta$. Then it is not hard to see that ϕ is as required. $\square$

The Axiom of Choice plays an important role in mainstream mathematics, particularly in General Topology, since many of its results turn out to be dependent of this axiom (see [77] for more on this). Thus, we will assume the Axiom of Choice (as it would be expected) and use in any of the different following forms.

AC In ZF, the following are equivalent.

(1) The Axiom of Choice: for each set $x \neq \emptyset$ there is a function f with $dom(f) = x$ such that for each $y \in x$, $f(y) \in y$;

(2) The Well-ordering Principle: every set can be well-ordered;

(3) The Kuratowski-Zorn Lemma[2]: if $(\mathcal{X}, \leq)$ is a partially ordered non-empty set such that every totally ordered $\mathcal{C} \subset \mathcal{X}$ has an upper bound in $\mathcal{X}$, then there is a maximal element of $(\mathcal{X}, \leq)$.

We recall some essential facts about filters. If X is a set, then $\mathcal{F} \subset \mathcal{P}(X)$ is a *filter on* X if: (0) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$; (1) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; (2) if $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$. A filter that is not contained in any other filter (so it is maximal with respect to inclusion order) is called an *ultrafilter*.

The *Fréchet filter* is the filter $\mathcal{F}_\omega = \{A \subset \omega : \omega \setminus A \text{ is finite}\}$. A family $\mathcal{A} \subset \mathcal{P}(X)$ has the *finite intersection property* if every time $n < \omega$ and $A_0, \ldots, A_n \in \mathcal{A}$ we have that $A_0 \cap \ldots \cap A_n \neq \emptyset$. Any family $\mathcal{A} \subset \mathcal{P}(X)$ with the finite intersection property is contained in the filter

$$\{F \subset X : \exists n < \omega\ \exists \{A_0, \ldots, A_n\} \subset \mathcal{A}\ (A_0 \cap \ldots \cap A_n \subset F)\}.$$

An important result relating filters to the axiom of choice is that of existence of ultrafilters.

UFT (Ultrafilter Theorem) Every filter is contained in an ultrafilter.

It is known that the Axiom of Choice implies the Ultrafilter Theorem (shown first by Tarski in [155]) but they are not equivalent (see [77, Diagram 2.21] and [89, Hint to excercise 5, p. 132]). Moreover, it is known that the Ultrafilter Theorem is independent of ZFC (see [89, pp. 183–184]).

If X is an arbitrary set, a family $\mathcal{A} \subset \mathcal{P}(X)$ is called *independent* if every time $m, n < \omega$ and $A_0, \ldots, A_m, B_0, \ldots, B_n \in \mathcal{A}$ are all different, then $A_0 \cap \ldots \cap A_m \cap (\omega \setminus B_0) \cap \ldots \cap (\omega \setminus B_n) \neq \emptyset$.

A *tree* is a partially ordered set $(T, \leq)$ where $\{t \in T : t \leq s\}$ is well-ordered for each $s \in T$. A *branch* in T is a totally ordered subset of T maximal with respect to inclusion. The *height* of T is the supremuum of the order types of branches in T.

An issue we must discuss in this section is independence results. The most famous problem that turned out to be independent was the *Continnum Hypothesis*, that is, the following statement.

Continum Hypothesis **CH** is the statement $\mathfrak{c} = \omega_1$.

[2]This is sometimes called just Zorn Lemma. According to [16], Kuratowski actually discovered/invented this principle and later Zorn rediscovered it.

Gödel proved in 1940 that **CH** is consistent with ZFC set theory and Cohen proved in 1963 that $\neg$**CH** is consistent with ZFC as well. So there are indeed some mathematical questions that cannot be solved with the axioms of ZFC alone. Many problems have resulted to be undecidable from ZFC. In this dissertation we will ocassionally mention some results independent of ZFC. For a more complete study of independence, see for example [99] or the more modern [101].

Martin's axiom is the first example of a general set-theoretic quotable principle that has been widely used. See [99, Chapter 2], [101, III.3] or [164] for introductions to Martin's axiom. We will just give the definition of Martin's axiom and briefly mention its relation to some other concepts.

Martin's Axiom Let $\mathbf{P}$ some class of partially ordered sets. Then $\mathbf{MA}(\mathbf{P})$ is the following statement.

If $(P, \leq)$ is a partially ordered set from $\mathbf{P}$, $\kappa < \mathfrak{c}$ and $\{D_\alpha : \alpha < \kappa\}$ is a collection of subsets of P dense with respect to the order, then there exists $G \subset P$ such that

1. if $p, q \in G$, then there is $r \in G$ with $r \leq p$ and $r \leq q$,
2. if $p \in G$ and $q \in P$ is such that $p \leq q$, then $q \in G$, and
3. $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \kappa$.

Also, $\mathbf{MA}$ means $\mathbf{MA}(\mathbf{Q})$ where $\mathbf{Q}$ is the class of all posets and $\mathbf{MA}(\text{countable})$ means $\mathbf{MA}(\mathbf{Q}')$ where $\mathbf{Q}'$ is the class of all countable posets.

It turns out that **MA** is a strong form of the Baire Category Theorem 0.25 below. **MA** follows directly from **CH** but it is also consistent with its negation. We also mention that the inspiration for the invention of **MA** was the method of forcing invented by Cohen to prove the consistency of $\neg$**CH** and in fact the consistency of **MA** is proved by an iteration of forcing (all of this is contained in [99]). See [63] for some consequences of **MA**.

A topic related to independence and Martin's axiom is that of *small cardinals*. These are cardinals that are defined as combinatorial characteristics of the topological space ${}^\omega\omega$. See [39] for a topological introduction to these small cardinals and [19] for a modern and set-theoretically oriented point of view.

The cardinals we will mention in this dissertation are the ones denoted by $\mathfrak{t}$, $\mathfrak{p}$, $\mathfrak{b}$ and $\mathfrak{d}$. We will not give the precise definition of these cardinals since we will only mention them in quoted results and our results do not use the definitions of these cardinals. Perhaps the only essential information about them is the following.

Theorem 0.2 [39]

- In ZFC it is known that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.
- **MA** implies that $\mathfrak{p} = \mathfrak{c}$.
- If one takes a pair of cardinals from $\omega_1, \mathfrak{t}, \mathfrak{b}, \mathfrak{d}, \mathfrak{c}$ then it is consistent that the two are different.

Recently the following has been shown.

Theorem 0.3 [106] $\mathfrak{p} = \mathfrak{t}$ in ZFC.

Finally, let us comment a little about measurable cardinals. Let X be a set, λ an infinite cardinal and $\mathcal{U}$ an ultrafilter on X. We will say that $\mathcal{U}$ is *κ-complete* if for every $\lambda < \kappa$ and every $\{U_\alpha : \alpha < \lambda\} \subset \mathcal{U}$ we have that $\bigcap\{U_\alpha : \alpha < \lambda\} \in \mathcal{U}$. Notice that every ultrafilter in an infinite set is ω-complete. A cardinal κ is called *measurable* if there exists a κ-complete ultrafilter $\mathcal{U}$ on κ.

Measurable cardinals are related to the problem of extending Lebesgue measure. See [90, Chapter 10, pp. 125–138] to read more about this topic. There is also an excellent recent B.S. thesis about the topic [24]. An important point we must stress here is that (1) it is consistent that there are no measurable cardinals, and (2) by Gödel's second incompleteness theorem it is impossible to prove that the consistency of the existence of measurable cardinals (see [103, Theorem IV.5.32]).

General Topology

We will assume that the reader has some maturity in General Topology. In particular, it is assumed that the reader knows facts from two basic courses in General Topology. However, as it is natural that the Topological background of different readers is different, in this Chapter we will give some definitions that may not be of the main stream. We will also give some results that we will use but are not central to our results. All other concepts of General Topology not defined here can be found in [50]. Other good references for General Topology are [165] and [49]. Following van Douwen, a space is crowded if it does not contain isolated points.

Lemma 0.4 [50, Theorem 1.5.4] If X is a space, Y is a Hausdorff space, D is dense in X and $f, g : X \to Y$ are continuous functions such that $f{\restriction_D} = g{\restriction_D}$, then $f = g$.

One of the most important and representative results in General Topology is the following one.

Theorem 0.5 ("Tychonoff theorem", [50, 3.2.4]) The product of compact spaces is compact.

The proof of the Tychonoff theorem uses the Axiom of Choice in the following way.

Proposition 0.6 [77, Theorems 4.68 and 4.70]

(1) The Tychonoff Theorem is equivalent to the Axiom of Choice.

(2) Tychonoff Theorem for Hausdorff spaces $\Leftrightarrow$ UFT $\Leftrightarrow$ for every set S, ${}^S[0,1]$ is compact.

A very interesting discussion of the role of the Axiom of Choice in compactness is contained in Chapters 3.3 and 4.8 of [77].

Compactness of a space can be of course evaluated by using only covers by basic sets. It is surprising that this is true of covers whose elements come from a subbasis (not surprisingly, the standard proof of this fact uses the axiom of choice).

Theorem 0.7 ("Alexander's subbase theorem", see hints in [50, Exercise 3.12.2] or [165, Exercise 17S]) Let X be a Hausdorff space and $\mathcal{B}$ a subbase of X. Then X is compact if and only if every cover of X by members of $\mathcal{B}$ has a finite subcover.

Let X be a space. We will say that $A, B \subset X$ are *completely separated in* X if there exists a continuous function $f : X \to [0,1]$ such that $A \subset f^{\leftarrow}(0)$ and $B \subset f^{\leftarrow}(1)$.

Lemma 0.8 [69, 1.14, Theorem] Let X be a Tychonoff space and $A, B \subset X$. Then A and B are completely separated if and only if they are contained in disjoint zero sets of X.

Recall that a subset A of a topological space X is C*-embedded* in X if every continuous function $f : A \to \mathbb{R}$ can be continuously extended to X. Also, A is C^**-embedded* in X if every continuous *bounded* function $f : A \to \mathbb{R}$ can be continuously extended to X.

Theorem 0.9 ("Urysohn's Extension Theorem", [69, 1.17]) Let X be a Tychonoff space and $Y \subset X$. Then Y is C^*-embedded in X if and only if every two completely separated subsets of Y are also completely separated in X.

Theorem 0.10 ("Taimanov's Theorem", [135, 4.1.m] or [62, §4, 4.6]) Let X be a dense subset of a Tychonoff space T, K a compact space and $f : X \to K$ a continuous function. Then f can be continuously extended to T if and only if for every two disjoint zero sets A, B of X we have $\mathrm{cl}_T(A) \cap \mathrm{cl}_T(B) = \emptyset$.

A function $f : X \to Y$ between topological spaces X and Y is *perfect* if it is closed and for every $y \in Y$ the fiber $f^{\leftarrow}(y)$ is compact.

Proposition 0.11 [135, Theorem 1.8(i)] If $f : X \to Y$ is a continuous function between Hausdorff spaces, $D \subset X$ is dense and $f\restriction_D \colon D \to f[D]$ is perfect, then $f[X \setminus D] \subset Y \setminus f[D]$.

An important concept in General Topology is that of normality. A standard and classical criterion for non-normality is given by the next result.

Theorem 0.12 ("Jones' Lemma", [165, Lemma 15.2, p. 100]) If X is a normal space and D is a closed and discrete subset of X, then $2^{|D|} \leq 2^{d(X)}$.

Given a Tychonoff space X, we will denote its Čech-Stone compactification by βX. In Section 6.1 we will give the definition of this space and some of its most important properties.

If $(X, <)$ is a strict linear order, there is a natural topology that embodies the order. For $x \in X$, define

$$(\leftarrow, x) = \{y \in X : y < x\} \text{ and} \tag{1}$$
$$(x, \rightarrow) = \{y \in X : x < y\}. \tag{2}$$

Consider the topology in $(X, <)$ generated by the set

$$\{(\leftarrow, x) : x \in X\} \cup \{(x, \rightarrow) : x \in X\}.$$

We will say that $(X, \leq)$ is a *linearly ordered space* if we consider it with this topology. For example, in this way ordinals can be though of as topological spaces.

Theorem 0.13 [135, Theorem 2.5.(m)] Any linearly ordered space is hereditarily normal.

Theorem 0.14 [135, Theorem 2.6.(q)(5)] Let κ be a cardinal of uncountable cofinality. Then for every continuous function $f : (\kappa, \in) \to \mathbb{R}$ there exists an $\alpha < \kappa$ and $r \in \mathbb{R}$ such that $f(\beta) = r$ for every $\beta \in [\alpha, \kappa)$.

Given a topological space X, its *weight* $w(X)$ is defined as the smallest infinite cardinal κ such that X has a base of cardinality κ.

Lemma 0.15 Let X be a space of weight κ. If $\mathcal{B}$ is any base of open sets, then there exists a base $\mathcal{B}_0 \subset \mathcal{B}$ with $|\mathcal{B}_0| = \kappa$.

Proof. Let $\mathcal{B}_1$ be a base of X of cardinality κ. For every $U, V \in \mathcal{B}_1$, let $[U, V] = \{W \in \mathcal{B} : U \subset W \subset V\}$. Define $\mathcal{R} = \{(U, V) \in \mathcal{B}_1 \times \mathcal{B}_1 : [U, V] \neq \emptyset\}$ and by the Axiom of Choice, let $f : \mathcal{R} \to \mathcal{B}$ be a function with $f((U, V)) \in [U, V]$ for each $(U, V) \in \mathcal{R}$. Finally, let $\mathcal{B}_0 = f[\mathcal{R}]$. Then it is not hard to see that $\mathcal{B}_0 \subset \mathcal{B}$, $|\mathcal{B}_0| \leq \kappa$ and $\mathcal{B}_0$ is a base. $\square$

Let X be a topological space. The *density* of X, denoted by $d(X)$, is the smallest cardinal κ such that X has a dense subset of cardinality κ and the *cellularity* of X, denoted by $c(X)$, is the supremum of all cardinals κ such that there is a parwise disjoint family of exactly κ open subsets of X.

We will need some facts about metrizable spaces.

Theorem 0.16 [50, 4.1.18] Each compact metrizable space has countable weight.

Theorem 0.17 [50, 4.1.15] In metrizable spaces, the cardinal functions weight, cellularity and density coincide.

Theorem 0.18 [83, Theorem 8.1] If X is a metrizable space and $c(X) = \kappa$, then there is a pairwise disjoint family of open subsets of X of cardinality κ.

Recall that a collection of sets $\mathcal{B}$ of a space X is *discrete* if for every $x \in X$ there is an open set U such that $x \in U$ and $|\{B \in \mathcal{B} : U \cap B \neq \emptyset\}| \leq 1$.

Lemma 0.19 Every metrizable and non-compact space has a countable infinite discrete collection of open subsets.

Proof. Let X be a metrizable non-compact space and d a compatible metric for X. It is well-known that X is not countably compact, so there exists a countable, closed and discrete subset $\{x_n : n < \omega\} \subset X$. Inductively, it is easy to construct a sequence of open subsets $\mathcal{U} = \{U_n : n < \omega\}$ whose closures are pairwise disjoint and such that if $n < \omega$, then $x_n \in U_n$ and U_n has diameter less than $\frac{1}{n+1}$.

We claim that $\mathcal{U}$ is the desired discrete family. Assume that there is a point $x \in X$ such that every open neighborhood of x interesects more than one open set from $\mathcal{U}$. Notice that since the closures of elements of $\mathcal{U}$ are pairwise disjoint, there is at most one $n < \omega$ such that $x \in \mathrm{cl}_X(U_n)$. Thus, we may inductively define a strictly increasing function $\phi : \omega \to \omega$ and for each $n < \omega$ choose a point $y_n \in U_{\phi(n)}$ such that $d(x, y_n) < \frac{1}{n+1}$. Clearly, the sequence $\{y_n : n < \omega\}$ converges to x. Moreover, since $d(x_{\phi(n)}, y_n) < \frac{1}{\phi(n)+1}$ for each $n < \omega$, the sequence $\{x_n : n < \omega\}$ also converges to x. This is a contradiction so the lemma follows. $\square$

A crucial property of metrizable spaces that plays a vital role is *paracompactness*. We first give some definitions about covers and families of sets, let X be a topological space. A family $\mathcal{U} \subset \mathcal{P}(X)$ such that $X = \bigcup \mathcal{U}$ is called a *cover*. If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, we will say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$. If $\mathcal{U} \subset \mathcal{P}(X)$, we will say that $\mathcal{U}$ is *locally finite* if for every $x \in X$ there is an open set V of X such that $x \in V$ and $\{U \in \mathcal{U} : U \cap V \neq \emptyset\}$ is finite. A collection $\mathcal{U} \subset \mathcal{P}(X)$ is σ-locally finite if $\mathcal{U} = \bigcup\{\mathcal{U}_n : n < \omega\}$, where $\mathcal{U}_n$ is locally finite for each $n < \omega$. If $\mathcal{U} \subset \mathcal{P}(X)$ and $x \in \bigcup \mathcal{U}$, then the *star* of x with respect to $\mathcal{U}$ is defined to be the set

$$St(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : x \in U\}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, we will say that $\mathcal{U}$ *star-refines* $\mathcal{V}$ if $\{St(x, \mathcal{U}) : x \in X\}$ is a refinement of $\mathcal{V}$. If d is a compatible metric for X and $\mathcal{U} \subset \mathcal{P}(X)$, the *mesh* of $\mathcal{U}$ is the supremum of the diameters of the elements of $\mathcal{U}$ with respect to d (considering the extended positive reals $(0, \infty]$ so this is well-defined).

Lemma 0.20 [50, 1.1.1] If $\mathcal{G}$ is a locally finite family of subsets of a topological space X, then $\bigcup\{\mathrm{cl}_X(A) : A \in \mathcal{G}\}$ is closed in X.

A space X is *paracompact* if X is Hausdorff and every open cover of X has a locally finite refinement.

Theorem 0.21 ("Stone's theorem", [50, 4.4.1]) Every metrizable space is paracompact.

Theorem 0.22 ("Nagata-Smirnov metrization theorem", [50, 4.4.7]) A regular space is metrizable if and only if it has a σ-locally finite base.

Theorem 0.23 ("Urysohn metrization theorem", [50, 4.2.9]) A second countable regular space is metrizable.

If $f : X \to Y$ is a closed map between topological spaces and $A \subset X$, we let $f^\sharp[A] = X \setminus f[X \setminus A]$, this is called the *small image* of A. Notice that if f is closed and A is closed, then $f^\sharp[A]$ is closed as well.

Recall that a Tychonoff space X is Čech-complete if it is a set of type G_δ in its Čech-Stone compactification.

Proposition 0.24 [50, 4.3.23 and 4.3.24] A subspace Y of a completely metrizable space X is completely metrizable if and only if Y is a set of type G_δ in X.

Theorem 0.25 ("Baire Category Theorem", [50, 3.9.3]) Let X be a Čech-complete space (in particular, if X is a completely metrizable space or a locally compact Hausdorff space). Then every countable sequence of dense open sets of X has dense intersection.

The completeness of a metric can be characterized in the following way.

Theorem 0.26 [50, 4.3.9] A metric space $\langle X, d\rangle$ is complete if and only if for every sequence $\{F_n : n < \omega\}$ of closed sets with $F_{n+1} \subset F_n$ for $n < \omega$ and with diameters with respect to d converging to 0 the intersection $\bigcap\{F_n : n < \omega\}$ is non-empty.

Symbols used

In the following table we will include the notation we will use throughout the text.

$\langle x_0, \ldots, x_{n-1}\rangle$	ordered n-tuple
$\mathcal{P}(X)$	power set of X
$\mathbb{N}$	set of positive integers
ω	set of natural numbers, $\omega = \mathbb{N} \cup \{0\}$
${}^{Y}X$	set of functions from Y to X; when $Y = n \in \mathbb{N}$, it's the set of n-tuples of X
$f^{\leftarrow}[A]$	inverse image, $\{x : f(x) \in A\}$
$\mathbb{R}$	set of real numbers

${}^{\omega}2$	Cantor set as a topological space
$\mathfrak{c}$	cardinality of the continuum, $\mathfrak{c} = 2^{\omega} = \lvert\mathbb{R}\rvert = \lvert{}^{\omega}2\rvert$
$\mathrm{int}_X(A)$	interior of A in the topological space X
$\mathrm{cl}_X(A)$	closure of A in the topological space X
$X \approx Y$	topological spaces X and Y are homeomorphic
$\chi(X, x)$	character of point x in topological space X
$[X]^{<\kappa}$	collection of subsets of X of cardinality strictly less than κ
$[X]^{\leq\kappa}$	collection of subsets of X of cardinality less or equal than κ
$f^{\sharp}[A]$	small image of $A \subset X$ when $f : X \to Y$ is a continuous function, $Y \setminus f[X \setminus A]$

Part I

Disconnectedness Properties of Hyperspaces

Introduction

Given a property **P** defined for topological spaces, a natural question is whether **P** is preserved under topological operations such as subspaces, products or sums. Specifically, we are interested in the case when **P** is a property that talks about some degree of disconnectednes and the operation of taking a *hyperspace*.

In general, by taking a hyperspace we mean the following: given a topological space X, consider some collection $\mathcal{H}(X)$ of subsets of X and give $\mathcal{H}(X)$ some topology. We are interested in the case where

$$\mathcal{H}(X) \subset CL(X) = \{A \subset X : A \text{ is closed and non-empty}\}$$

and the topology in $CL(X)$ called the Vietoris topology. The reason of our interest in this topology is that, in the author's opinion, this is a topology in which the notion of closedness coincides with our intuition. The fact that for metrizable spaces the Vietoris topology in the hyperpace of compact subsets coincides with the one generated by the Hausdorff distance ([88, Theorem 3.1]) can be used to argue in favor of this intuitive feeling.

We shall focus on hyperspaces that have been widely studied such as the hyperspace of compact sets, the hyperspace of finite sets and the so called symmetric products. To read about hyperspaces in other contexts, we refer the reader to [15], [84] [88].

The study of hyperspaces with the Vietoris topology in its generality started with Ernest Michael's paper [111]. In this paper, E. Michael studied the preservation of diverse topological properties under the operation of taking some hyperspaces. Particularly, the author talks about some classical disconnectedness type properties. Thus, we may think that some parts of this section are the continuation of Michael's paper.

The first classes of "*disconnectedness properties*" we will consider are those found in the study of Extensions and Absolutes of Spaces and discussed in the book [135]. These properties are of "extremal disconnectedness" because each of them implies 0-dimensionality (for regular spaces). The preservation of these properties under hyperspaces can be characterized right away, as the reader will notice.

The disconnectedness property that will be the most interesting in the context of taking hyperspaces is the one called "hereditary disconectedness". Hereditary disconnecteness is a property implied by but not equivalent to 0-dimensionality.

The chapter about hereditary disconnectedness is the most complex in this Part. We could not characterize when a hyperspace is hereditarily disconnected. In fact, our analysis will make clear that this property behaves in different ways in seemingly similar cases. We may think that the main Theorem in this Part is Theorem 4.6 and that Example 4.12 illustrates how the converse of this main Theorem does not hold in all cases.

Besides this disconnectedness results, in this Part we will give some partial results of pseudo-compactness and the space $CL(\omega)$, topics in which we could not find strong results but are worth mentioning. Finally, we will give a list of problems about hyperspaces we could not solve and a small discussion about why we found them interesting.

The results of disconnectedness in this chapter have been published on reference [79].

This Part of the dissertation is almost self-contained. Some results and proofs about the Čech-Stone compactification that are used for examples in this Part are only referenced. The spirit of this Part I of the thesis is to focus on the structure of the hyperspaces. However, since Part II of this thesis is related to the Čech-Stone compactification, we have chosen to give the proofs of those results in Part II.

Chapter 1

General Properties of Hyperspaces

In this chapter we will give definitions and basic results on hyperspaces. Let X be a Hausdorff space. The following sets will be our hyperspaces.

$$\begin{aligned} CL(X) &= \{A \subset X : A \text{ is non-empty and closed in } X\}, \\ \mathcal{K}(X) &= \{A \in CL(X) : A \text{ is compact}\}, \\ \mathcal{F}_n(X) &= \{A \subset X : 0 < |A| \leq n\}, \text{ for each } n \in \mathbb{N}, \\ \mathcal{F}(X) &= \bigcup\{\mathcal{F}_n(X) : n \in \mathbb{N}\}. \end{aligned}$$

We will define the topology of our hyperspaces by defining a subbase. For each subset $Y \subset X$ let

$$Y^+ = \{A \in CL(X) : A \subset Y\} \text{ and} \tag{1.1}$$

$$Y^- = \{A \in CL(X) : A \cap Y \neq \emptyset\}. \tag{1.2}$$

We define the Vietoris topology in $CL(X)$ as the one generated by all sets of the form U^+ y U^-, where U is open in X. A base for the Vietoris topology is the one given by the *Vietoris sets*[1], that is, sets of the form[2]:

$$\langle\!\langle U_0, \ldots, U_n \rangle\!\rangle = \left\{A \in CL(X) : A \subset \bigcup_{i=0}^{n} U_i \text{ and } A \cap U_k \neq \emptyset \text{ for all } i \leq n\right\}, \tag{1.3}$$

where $U_0, \ldots, U_n$ are non-empty open subsets of X.

[1] In Mexico we call these sets "Vietóricos".

[2] Notice the similarity between the notation for Vietoris sets $\langle\!\langle U_0, \ldots, U_n \rangle\!\rangle$ and ordered tuples $\langle x_0, \ldots, x_n \rangle$.

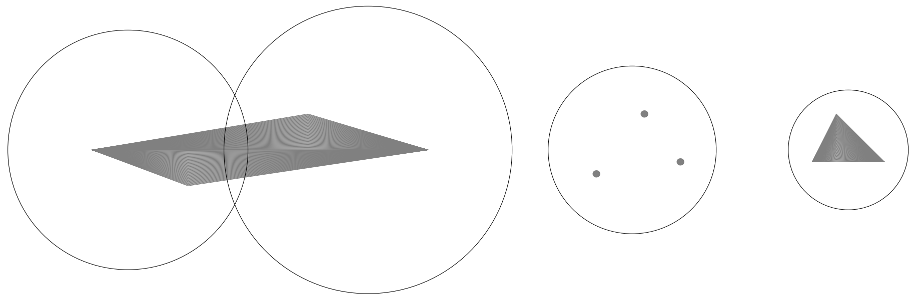

Figure 1.1: A closed subset (in grey) and a Vietoris set neighborhood of it.

The following equation will be important for some proofs.

$$\langle U_0, \ldots, U_n \rangle = (U_0 \cup \ldots \cup U_n)^+ \cap \left(\bigcap \left\{ U_k^- : k \leq n \right\} \right). \tag{1.4}$$

The other sets we have defined are subsets of $CL(X)$ so we will give them the topology as subspaces of $CL(X)$, this topology will also be called Vietoris topology. All these sets will be called *hyperspaces*, and more specifically: $CL(X)$ is the hyperspace of closed subsets, $\mathcal{K}(X)$ is the hyperspace of compact subsets and $\mathcal{F}(X)$ is the hyperspace of finite subsets.

If we are considering the hyperspace $\mathcal{H} \subset CL(X)$ and $U_0, \ldots, U_n$ are non-empty open subsets of X, $\langle U_0, \ldots, U_n \rangle$ will denote the interesection of the Vietoris set defined by Equation 1.3 and $\mathcal{H}$ when no confussion arises.

Given $n \in \mathbb{N}$, the hyperspace $\mathcal{F}_n(X)$ is also known as the *n-th symmetric product* (called *n-th symmetric power* by other authors). The reason for this name is the fact that $\mathcal{F}_n(X)$ is the quotient of ${}^nX$ by an action of the group of permutations; this will be formalized in Lemma 1.2. To prove this result we must first give a nice base for the symmetric product.

Lemma 1.1 Let X be a Hausdorff space, $n < \omega$, $A \in \mathcal{F}_{n+1}(X) \setminus \mathcal{F}_n(X)$ (where $\mathcal{F}_0(X) = \emptyset$) and $\mathcal{U}$ an open subset of $CL(X)$ such that $A \in \mathcal{U}$. Then there are pairwise disjoint open sets $U_0, \ldots, U_n$ such that

$$A \in \langle U_0, \ldots, U_n \rangle \subset \mathcal{U}.$$

Proof. Let $A = \{x_0, \ldots, x_n\}$. Take pairwise disjoint open subsets $V_0, \ldots, V_n$ of X such that $x_k \in V_k$ for each $k \leq n$. Let us consider an Vietoris open subset such that

$$A \in \langle W_0, \ldots, W_s \rangle \subset \mathcal{U} \cap \langle V_0 \ldots, V_n \rangle.$$

For each $k \leq n$, let $U_k = V_k \cap (\bigcap\{W_r : x_k \in W_r\})$. Notice that $U_0, \ldots, U_n$ are pairwise disjoint open subsets such that

$$A \in \langle U_0, \ldots, U_n \rangle \subset \langle W_0, \ldots, W_s \rangle \subset \mathcal{U},$$

which completes the proof. $\square$

Lemma 1.2 Let X be a Hausdorff space and $n \in \mathbb{N}$. The function $f : {}^nX \to \mathcal{F}_n(X)$ defined by $f(\langle x_0, \ldots, x_{n-1}\rangle) = \{x_0, \ldots, x_{n-1}\}$ is a quotient. In particular, the function $x \mapsto \{x\}$ is a homeomorphism between X and $\mathcal{F}_1(X)$.

Proof. First, we will prove that f is continuous. It is enough to prove the continuity in a base of $\mathcal{F}_n(X)$ so by Lemma 1.1 consider $m < n$ and a collection $U_0, \ldots, U_m$ of pairwise disjoint open subsets of X. Then,

$$f^{\leftarrow}[\langle U_0, \ldots, U_m \rangle] = \bigcup\{W_0 \times \cdots \times W_{n-1} : \{W_0, \ldots, W_{n-1}\} = \{U_0, \ldots, U_m\}\},$$

so the continuity of f follows.

To see that f is indeed a quotient, let $\mathcal{U} \subset \mathcal{F}_n(X)$ be such that $f^{\leftarrow}[\mathcal{U}]$ is open, we must prove that $\mathcal{U}$ is open. Let $\{x_0, \ldots, x_m\} \in \mathcal{U}$ with $m < n$. Consider some point in the preimage, $\langle y_0, \ldots, y_{n-1}\rangle$ where $y_i = x_i$ if $i < m - 1$ and $y_i = x_{m-1}$ if $m \leq i < n$. Let $U_0, \ldots, U_{m-1}$ be pairwise disjoint open subsets of X such that $x_i \in U_i$ for $i < m$. Define $V_i = U_i$ if $i < m - 1$ and $V_i = U_{m-1}$ if $m \leq i < n$. As $\langle y_0, \ldots, y_{n-1}\rangle$ is in the open set $f^{\leftarrow}[\mathcal{U}]$, we may choose U_i so that

$$\langle y_0, \ldots, y_{n-1}\rangle \in V_0 \times \cdots \times V_{n-1} \subset f^{\leftarrow}[\mathcal{U}].$$

Now notice that

$$f[V_0 \times \cdots \times V_{n-1}] = \langle U_0, \ldots, U_{m-1} \rangle$$

so

$$\{x_0, \ldots, x_{m-1}\} \in \langle U_0, \ldots, U_{m-1} \rangle \subset \mathcal{U},$$

which proves that $\{x_0, \ldots, x_{m-1}\}$ is an interior point of $\mathcal{U}$. That is, $\mathcal{U}$ is open. $\square$

When meeting a new concept, one must look at some examples that will make the concept more comprehensible. We will now give examples of *geometric models* of hyperspaces. That is, for some spaces X it is possible to know that $\mathcal{H}(X)$ is homeomorphic to some known space.

Example 1.3 Geometric models of hyperspaces.

$\mathcal{F}_{\mathbf{2}}([\mathbf{0},\mathbf{1}])$: By Lemma 1.2, we can obtain $\mathcal{F}_2(X)$ as the quotient of ${}^2[0,1]$ by identifying points of the form $\langle a,b\rangle$ and $\langle b,a\rangle$. In this way, $\mathcal{F}_2([0,1])$ may be represented as the set $\{\langle a,b\rangle \in {}^2\mathbb{R} : 0 \le a \le b \le 1\}$. Thus, $\mathcal{F}_2([0,1])$ is homeomorphic to ${}^2[0,1]$.

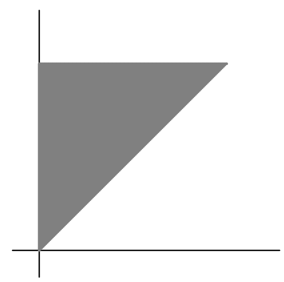

Figure 1.2: $\mathcal{F}_2([0,1])$

$\mathcal{F}_{\mathbf{n}}([\mathbf{0},\mathbf{1}])$: A result of Borsuk and Ulam says that if $n \in \mathbb{N}$, then $\mathcal{F}_n([0,1])$ is homeomorphic to ${}^n[0,1]$ if and only if $n \le 3$. Moreover, $\mathcal{F}_n([0,1])$ cannot be embedded in ${}^n[0,1]$ if $n \ge 4$. Both results mentioned are in [20]. These facts have been exploited by the authors and others in research that is not relevant to the development of this dissertation, see [82].

$\mathcal{K}([\mathbf{0},\mathbf{1}])$: In general, if X is a Peano continuum (a continuous image of the interval $[0,1]$), there is a deep result of infinite-dimensional topology that implies that $\mathcal{K}(X)$ is homeomorphic to the Hilbert cube ${}^\omega[0,1]$. To see the proof of this result, the reader is refered to [115] where the theory needed to prove this result is developed and then the result is proved in [115, Theorem 8.4.5]. Another option is to see Chapter III in [88], where the focus is on the hyperspace techniques used; the infinite-dimensional theory needed is just cited.

$\mathcal{K}({}^\omega\mathbf{2})$: It can be proved that $\mathcal{K}({}^\omega 2)$ is compact, metrizable, 0-dimensional and crowded. By a known result, it follows that $\mathcal{K}({}^\omega 2)$ is homeomorphic to ${}^\omega 2$. See [156, Section 27] for a proof.

$\mathcal{K}(\omega+\mathbf{1})$: If X is a compact, metrizable, 0-dimensional and infinite space such that the set of its isolated points is dense, it can be proved ([88, 8]) that $\mathcal{K}(X)$ is homeomorphic to the *Pelczyński compactum*. The Pelczyński compactum is the only compact, metrizable, 0-dimensional space that has a dense subset of isolated points accumulating at a Cantor set ([156, Section 27, Corollary 2]). In [156, Chapter IV], there is a list of all possible spaces of the type $\mathcal{K}(X)$ when X is a compact, metrizable and 0-dimensional space.

□

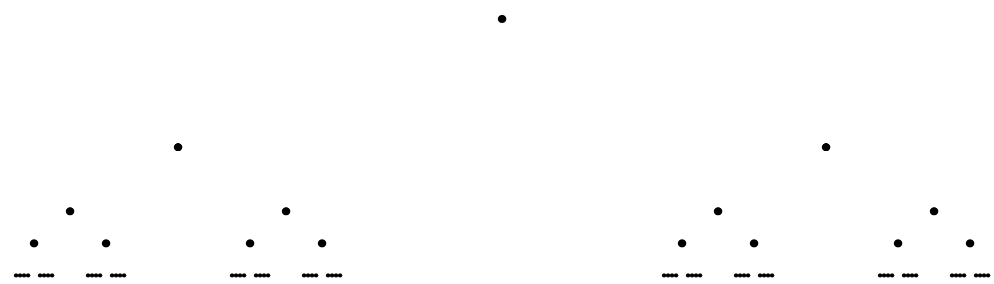

Figure 1.3: The Pelczyński compactum.

The following result gives some conditions of inclusion between hyperspaces. Their proof is easy and is left to the reader.

Lemma 1.4 Let X be Hausdorff, $Y \subset X$ and $n \in \mathbb{N}$.

(a) If Y is dense in X, then $\mathcal{F}(Y)$ is dense in $CL(X)$.

(b) If Y is dense in X, then $\mathcal{F}_n(Y)$ is dense in $\mathcal{F}_n(X)$.

(c) If Y is open in X, then $\mathcal{F}_n(Y)$ is open in $\mathcal{F}_n(X)$.

(d) If Y is closed in X, then $CL(Y)$ is closed in $CL(X)$.

All examples given above are metrizable compacta. We will now study when some separation axioms hold for hyperspaces.

We will leave the discussion of separation axioms on $CL(X)$ for Section 5.3. The reason for doing this is that $CL(X)$ will not be of interest in the context of high disconnectedness properties by Proposition 3.3. We will now focus on $\mathcal{K}(X)$ and its subspaces.

Lemma 1.5 If X is a Hausdorff space, then $\mathcal{K}(X)$ (and any smaller hyperspace) is Hausdorff.

Proof. Let $A, B \in \mathcal{K}(X)$ with $A \neq B$. Choose, without loss of generality, $p \in A \setminus B$ and find two disjoint open subsets U and V such that $p \in U$ and $B \subset V$. Then $A \in U^-$, $B \in V^+$ and $U^- \cap V^+ = \emptyset$. $\square$

Lemma 1.6 Let X and Y be Hausdorff spaces and $f : X \to Y$ a continuous function. Then the function $f^* : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $f^*(A) = f[A]$ is continuous.

Proof. Clearly, f^* is well defined because the continuous image of compact sets is compact. Let U be an open subset of Y, notice that $(f^*)^{\leftarrow}[U^+] = f^{\leftarrow}[U]^+$ and $(f^*)^{\leftarrow}[U^-] = f^{\leftarrow}[U]^-$, so the result follows. $\square$

Recall that a zero set (cozero set) in a space X is a set of the form $f^{\leftarrow}\{0\}$ ($f^{\leftarrow}[(0,1]]$, respectively) where $f : X \to [0,1]$ is continuous. The complement of a zero set is a cozero set. It is also known that a T_1 space is Tychonoff if it has a base of cozero sets ([69, 3.2]).

Lemma 1.7 Let X be Hausdorff, Z be a zero set (cozero set) of X. Then Z^+ and Z^- are zero sets (cozero sets, respectively) of $\mathcal{K}(X)$.

Proof. It is enough to prove the statement of the Lemma for a cozero set U of X because $A^+ = (X \setminus A)^-$ for all $A \subset X$. Let $f : X \to [0,1]$ be such that $U = f^{\leftarrow}[(0,1]]$. Let us consider the associated function $f^* : \mathcal{K}(X) \to \mathcal{K}([0,1])$ defined by $f^*(A) = f[A]$, which is continuous (Lemma 1.6); and the two continuous functions $g_0 : \mathcal{K}([0,1]) \to [0,1]$, $g_1 : \mathcal{K}([0,1]) \to [0,1]$ defined as $g_0(A) = \min A$, $g_1(A) = \max(A)$. Finally, notice that

$$\begin{aligned} U^+ \cap \mathcal{K}(X) &= (g_0 \circ f^*)^{\leftarrow}[(0,1]], \\ U^- \cap \mathcal{K}(X) &= (g_1 \circ f^*)^{\leftarrow}[(0,1]], \end{aligned}$$

which completes the proof. $\square$

Since the Vietoris sets are formed by unions and intersections of subbasic sets (Equation 1.4, page 5) we only need the following result to show that X Tychonoff implies $\mathcal{K}(X)$ is Tychonoff.

Lemma 1.8 Let X be a Hausdorff space, $\mathcal{B}$ a base of X closed under finite unions. Then

$$\langle\!\langle \mathcal{B} \rangle\!\rangle = \{\langle\!\langle U_0, \ldots, U_n \rangle\!\rangle : n < \omega, \forall k \leq n (U_k \in \mathcal{B})\}$$

is a base of $\mathcal{K}(X)$.

Proof. Let $A \in \mathcal{K}(X)$ and $\langle\!\langle U_0, \ldots, U_n \rangle\!\rangle$ a Vietoris set containing it. Since A is compact, there is a finite subset $\mathcal{V} \subset \mathcal{B}$ such that (a) for each $V \in \mathcal{V}$ there is $k \leq n$ such that $V \subset U_k$; (b) $A \subset \bigcup \mathcal{V}$; (c) for each $V \in \mathcal{V}$, $A \cap V \neq \emptyset$ and (d) for each $k \leq n$ there is $V \in \mathcal{V}$ such that $V \subset U_k$. For each $k \leq n$, let $W_k = \bigcup\{V \in \mathcal{V} : V \subset U_k\}$, this is a non-empty open set that is an element of $\mathcal{B}$. It is easy to see that $A \in \langle\!\langle W_0, \ldots, W_n \rangle\!\rangle \subset \langle\!\langle U_0, \ldots U_n \rangle\!\rangle$ and $\langle\!\langle W_0, \ldots, W_n \rangle\!\rangle \in \langle\!\langle \mathcal{B} \rangle\!\rangle$. $\square$

Since the property of being Tychonoff is hereditary and by Lemmas 1.7 and 1.8, we obtain the following characterization.

Corollary 1.9 The following are equivalent for a Hausdorff space X:

(a) X is Tychonoff,

(b) $\mathcal{K}(X)$ is Tychonoff,

(c) $\mathcal{F}(X)$ is Tychonoff,

(d) for all $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is Tychonoff,

(e) there is an $n \in \mathbb{N}$ such that $\mathcal{F}_n(X)$ is Tychonoff.

Also notice that Lemma 1.8 implies the following.

Corollary 1.10 If X is Hausdorff, then $w(X) = w(\mathcal{K}(X))$.

An interesting situation for hyperspaces is that, in general, properties that depend on some base can be easily transfered from a space to its hyperspace, such as in Corollary 1.9. In Chapter 2 we will see this situation does hold for some disconnectedness properties defined with properties of a base. However, the Main Theorem of Section 4 shows that in other cases it is impossible to transfer disconnectedness properties to the hyperspace.

We will need the following convergence result later.

Lemma 1.11 Let X be a Hausdorff space and $\{F_n : n < \omega\} \subset CL(X)$ such that $F_n \subset F_{n+1}$ for all $n < \omega$. Then the sequence $\{F_n : n < \omega\}$ converges to $\mathrm{cl}_X(\bigcup\{F_n : n < \omega\})$ in $CL(X)$.

Proof. Let $C = \mathrm{cl}_X(\bigcup\{F_n : n < \omega\}) \in CL(X)$. We must prove that C is the limit of the sequence $\{F_n : n < \omega\}$ so take a Vietoris set neighborhood $\langle U_0, \ldots, U_m \rangle$ of C. Clearly, for all $n < \omega$ we have that $F_n \subset U_0 \cup \ldots, U_m$. Fix $i \leq n$, then there exists $k(i) < \omega$ such that $F_{k(i)} \cap U_i \neq \emptyset$; otherwise $F_n \subset X \setminus U_i$ for all $n < \omega$ which implies $F \subset X \setminus U_i$, a contradiction. Let $k = \sup\{k(i) : i < \omega\} < \omega$. Since the sequence is strictly increasing, we have that for all $m \geq k$, $F_m \cap U_i \neq \emptyset$. Thus, for all $m \geq k$ we have that $F_m \in \langle U_0, \ldots, U_m \rangle$. This proves the convergence □

As a last example to the beauty of hyperspaces, we present the proof of the following theorem.

Theorem 1.12 If X is a compact Hausdorff space, then $\mathcal{K}(X)$ is compact.

Proof. We will use the Alexander Subbase Theorem 0.7. So let $\mathcal{U} = \{U_t^+ : t \in \mathcal{S}_0\} \cup \{V_t^- : t \in \mathcal{S}_1\}$ be an open cover of $\mathcal{K}(X)$. Let $T = X \setminus \bigcup\{V_t : t \in \mathcal{S}_1\}$, which is a compact subset of X. If $T = \emptyset$, let $t_0 \in \mathcal{S}_0$ be arbitrary. If $T \neq \emptyset$, T must be contained in some element of $\mathcal{U}$, so by the definition of T, there exists $t_0 \in \mathcal{S}_0$ such that $T \subset U_{t_0}$. Now let $K = X \setminus U_{t_0}$, which is a compact subset of X. By the definition of T, K is covered by $\{V_t : t \in \mathcal{S}_1\}$. So by compactness, there is a finite subset $F \in [\mathcal{S}_1]^{<\omega}$ such that $K \subset \bigcup\{V_t : t \in F\}$. From this, it is easy to see that $\mathcal{K}(X)$ is covered by $\{U_{t_0}^+\} \cup \{V_t^- : t \in F\}$. $\square$

Chapter 2

Disconnectedness Properties

In this Chapter we will give definitions of the properties we will study in hyperspaces and how they relate. From now on, we will use the following notation. If X is any space, then

$$\mathcal{CO}(X) = \{U \subset X : U \text{ is open and closed in } X\}$$

is the set of "clopen" subsets of X.

2.1 Properties of weak disconnectedness

Let's start with the properties that are the most natural and have nice geometrical examples.

Definition 2.1 Let X a T_1 space.

(a) We say that X is 0-*dimensional* if for each point $x \in X$ and each closed subset $F \subset X$ with $x \notin F$ there is $U \in \mathcal{CO}(X)$ such that $x \in U$ and $U \cap F = \emptyset$.

(b) We say that X is *totally disconnected* if for any two points x, y with $x \neq y$ there is $U \in \mathcal{CO}(X)$ such that $x \in U$ and $y \notin U$.

(c) We say that X is *hereditarily disconnected* if any $Y \subset X$ with $|Y| > 1$ is disconnected.

Clearly each 0-dimensional space is totally disconnected and each totally disconnected space is hereditarily disconnected. We will now give some examples to show that these definitions are not equivalent. Some of our claims inside the examples will be left unproved because in Chapter 4 we will prove stronger results that imply what we leave with no proofs here. However, in the the examples we will refer the reader to references when proofs or outlines of proofs are given.

Example 2.2 Universal 0-dimensional spaces.

If κ is an infinite cardinal, then the space ${}^{\kappa}2$ is a compact, Hausdorff and 0-dimensional spae of weight κ. In fact, if X is any Hausdorff 0-dimensional space of weight $\leq \kappa$, X can be embedded in ${}^{\kappa}2$. To see this, let $\mathcal{B} = \{B_\alpha : \alpha < \kappa\} \subset \mathcal{CO}(X)$ be a basis of X. For each $\alpha < \kappa$ let $f_\alpha : X \to 2$ be the characteristic function of B_α. It is then not hard to see that the function $\nabla_{\alpha<\kappa} f_\alpha : X \to {}^{\kappa}2$ defined by $(\nabla_{\alpha<\kappa} f_\alpha)(x)(\beta) = f_\beta(x)$ for all $x \in X$ is an embedding. In the countable case, ${}^{\omega}2$ contains a topological copy of every separable, 0-dimensional and metrizable space. Later, in Part II, Proposition 8.6, there is another example of universal space. □

Example 2.3 A space that is totally disconnected but not 0-dimensional

Choose $s \in {}^{\omega}\mathbb{N}$ be such that $s(n) < s(n+1)$ for all $n < \omega$. For example, we may take $s(n) = n + 1$ for all $n < \omega$. Define the function $\phi_s : {}^{\omega}2 \to [0, \infty]$ for $x \in {}^{\omega}2$ as

$$\phi_s(x) = \sum_{n<\omega} \frac{x(n)}{s(n)} = \frac{x(0)}{s(0)} + \frac{x(1)}{s(1)} + \cdots$$

Define $X_s = \{\langle x, t\rangle \in {}^{\omega}2 \times \mathbb{R} : t = \phi_s(x)\}$. Since the projection $\pi : X_s \to {}^{\omega}2$ is a condensation (that is, a continuous bijection), it is easy to see that X_s is a totally disconnected space. To prove that X_s is not 0-dimensional it is necessary to use a technique invented by Erdős in [52]; we refer the reader to [30, Corollary 2] for a complete proof of this fact. The essential property from [30, Corollary 2] is that every $U \in \mathcal{CO}(X_s)$ has an unbounded image under the second projection. It is worth mentioning that the proof that X_s is not 0-dimensional is completely analogous to the technique we will use later in Example 4.12.

We now make some interesting remarks about this space. Notice that X_s is the graph of a 0-dimensional space under a upper semicontinuous function. An intuitive way of thinking about this is that we are taking a 0-dimensional space and the upper semicontinuous function ϕ_s is being declared continuous to define a new topology. These type of constructions are used to study the space called Erdős space, see [32, Theorem 4.15]. Moreover, X_s is related to Continua Theory because, just as Dijkstra mentions in en [30], X_s is homeomorphic to the set of endpoints of the Lelek fan (see [27] for an intuitive description of the Lelek fan). An alternative construction of X_s is given in [51, 1.4.6]. □

Figure 2.1: An embedding of space X_s from Example 2.3 in the plane. Its closure is marked in grey. In this embedding, projection to the first coordinate is one-to-one restricted to X_s.

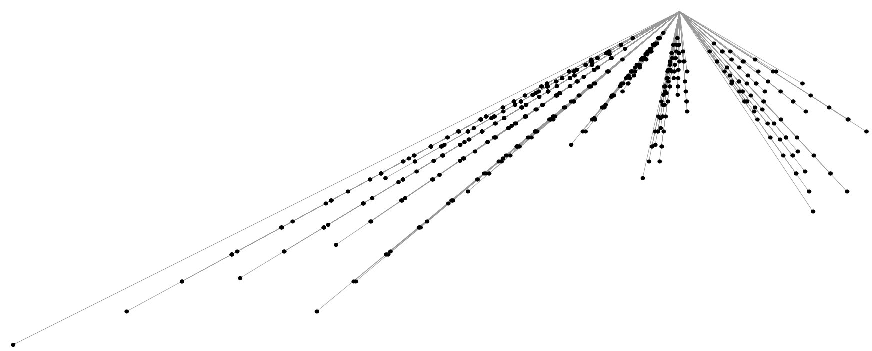

Figure 2.2: Space X_s from Example 2.3 embedded in the Lelek fan (in grey) as its set of endpoints.

Example 2.4 A space that is hereditarily disconnected but not totally disconnected.

Let ϕ_s be as in Example 2.4. Consider the space $Y_s = X_s \cup \{\langle \overline{0}, \infty \rangle\}$, where $\overline{0}$ is the constant 0 function. Using ϕ_s it is possible to show that any pair of points of Y_s can be separated by clopen subsets, except maybe for $p = \langle \overline{0}, 0 \rangle$ and $q = \langle \overline{0}, \infty \rangle$. Since $\{p, q\}$ is not connected, we obtain that Y_s is hereditarily disconnected.

We follow the idea in [51, 1.4.7] in order to see that Y_s is not totally disconnected. If U is a clopen subset of Y_s with $q \in U$, by the definition of the product topology there is $r \in \mathbb{R}$ and V a clopen subset of ${}^{\omega}2$ such that $V \times \{r\} \subset U$.

Thus, we obtain that $W = (V \times [0, r]) \setminus U \in \mathcal{CO}(Y_s)$ is bounded in its second coordinate. By [30, Corollary 2] we obtain that $W = \emptyset$ so $p \in U$. The reader will notice that this argument is used again in Example 4.12. $\square$

Example 2.5 The Knaster-Kuratowski fan.[1]

In Example 2.4 we showed an space that is totally disconnected "modulo one point". We will now describe a hereditarily disconnected space that has no totally disconnected open subsets. Our example is homeomorphic to the famous *Knaster-Kuratowski fan* with its appex removed, see [51, 1.4.C] for the usual description.

Let $C \subset [0, 1]$ be the Cantor set constructed removing middle-thirds as usual, let $Q \subset C$ be the set of endpoints of removed intervals and $P = C \setminus Q$. For each $c \in C$, let

$$L_c = \begin{cases} \{c\} \times ([0,1] \cap \mathbb{Q}), & \text{if } c \in Q, \\ \{c\} \times ([0,1] \setminus \mathbb{Q}), & \text{if } c \in P. \end{cases}$$

Let $\mathbf{F} = \bigcup\{L_c : c \in C\}$, notice that X is dense in the compact set $C \times [0, 1] \subset \mathbb{R}^2$. For each $c_0 \neq c_1$, L_{c_0} and L_{c_1} are separated hereditarily disconnected sets in X, so X is hereditarily disconnected. The proof that $\mathbf{F}$ has no totally disconnected open sets is more difficult. If the reader wants to see the proof of this fact right away, we suggest to see [145, Examples 128 and 129] or the hint in [51, 1.4.C]. However, in Example 4.2 we will prove a stonger property and that proof can be easily adapted to show that $\mathbf{F}$ has no totally disconnected open sets. $\square$

Now we will give the two results that Ernest Michael proved about disconnectedness properties of hyperspaces.

Theorem 2.6 [111, Theorem 4.10] Let X be a T_1 space and $\mathcal{F}(X) \subset \mathcal{H} \subset CL(X)$. If any of X, $\mathcal{F}_n(X)$ ($n \in \mathbb{N}$) or $\mathcal{H}$ is connected, then the following hyperspaces are also connected: X, $\mathcal{F}_m(X)$ for each $m \in \mathbb{N}$ and any $\mathcal{H}'$ satisfying $\mathcal{F}(X) \subset \mathcal{H}' \subset CL(X)$.

Proof. First assume that X is disconnected and infinite (otherwise the conclusion is obvious). Then $X = U \cup V$ where $U, V \in \mathcal{CO}(X) \setminus \{\emptyset, X\}$ are disjoint. If $\mathcal{F}_2(X) \subset \mathcal{H} \subset CL(X)$ then $\mathcal{H} = (U^+ \cup V^+) \cup \langle U, V \rangle$, this is a decomposition

[1] In [145, Examples 128 and 129], the Knaster-Kuratowski fan is refered to as *Cantor's leaky tent* and $\mathbf{F}$ is called *Cantor's teepee*; although it is not clear which space is which in the description, one can infer the correct names from the General Reference Chart in [145, pp. 194–203].

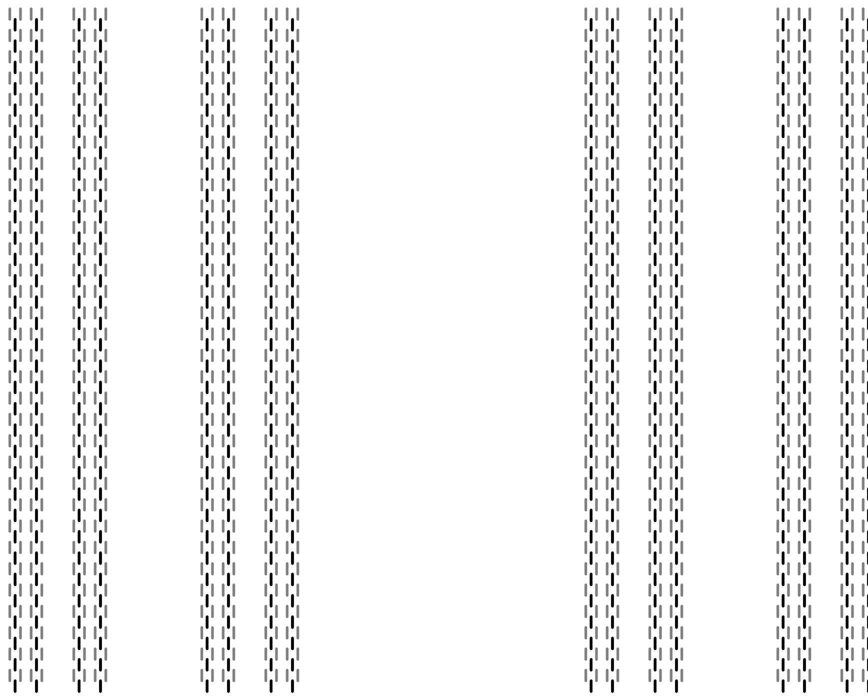

Figure 2.3: Space **F** from Example 2.5.

in two non-empty closed disjoint subsets, so $\mathcal{H}$ is disconnected. Thus, if any one of the hyperspaces in the statement of the Theorem is connected, then X is connected as well.

Now assume that X is connected. By Lemma 1.2, we obtain that for each $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is connected. Since the symmetric products form an increasing chain of connected spaces, we obtain that $\mathcal{F}(X)$ is connected. If $\mathcal{F}(X) \subset \mathcal{H} \subset CL(X)$, we obtain that $\mathcal{H}$ is connected by (a) in Lemma 1.4. $\square$

Theorem 2.7 [111, Proposition 4.13] Let X be a T_1 space. Then the following holds.

(1) X is 0-dimensional if and only if $\mathcal{K}(X)$ is 0-dimensional.

(2) X is totally disconnected if and only if $\mathcal{K}(X)$ is totally disconnected.

(3) X is discrete if and only if $\mathcal{K}(X)$ is discrete.

(4) X has no isolated points if and only if $CL(X)$ has no isolated points.

Proof. Recall that $\mathcal{F}_1(X)$ is a topological copy of X contained in any of our hyperspaces (Lema 1.2) so the implications from right to left are clear.

Assume that X is 0-dimensional. By Lemma 1.8, the set $\langle \mathcal{CO}(X) \rangle$ is a base of $\mathcal{K}(X)$. By Equation 1.4 in page 5, we only need to prove that F^+ y F^- are closed when F is closed. This follows from the facts that $X \setminus F^+ = (X \setminus F)^-$ and $X \setminus F^- = (X \setminus F)^+$. Thus, $\mathcal{K}(X)$ is 0-dimensional.

Now assume that X is totally disconnected. Let $A, B \in \mathcal{K}(X)$ be such that $A \neq B$. Without loss of generality, there exists $p \in A \setminus B$. Since X is totally disconnected and B is compact, there is $U \in \mathcal{CO}(X)$ with $p \in U$ and $U \cap B = \emptyset$. From this it follows that $U^- \in \mathcal{CO}(\mathcal{K}(X))$ is such that $A \in U^-$ and $B \notin U^-$.

If X is discrete, any compact subset is finite. Further, for any finite subset $\{x_0, \ldots, x_n\}$ the Vietoris set $\langle \{x_0\}, \ldots, \{x_n\} \rangle = \{\{x_0, \ldots, x_n\}\}$ is a singleton. If $CL(X)$ has an isolated point, there is a Vietoris set $\langle U_0, \ldots, U_n \rangle$ that is a singleton in $CL(X)$ and it can be easily seen that for all $k \leq n$, U_k is a singleton of an isolated point in X. $\square$

So at least we have the question of when the hyperspace of compact subsets is hereditarily disconnected. This will be the most important question we found in this Part and will be extensively discussed in Chapter 4.

Finally, let us recall scattered spaces. We will not consider them in our test properties for hyperspaces but will be relevant in Theorem 4.10. Recall that a space X is *scattered* if for any nonempty $Y \subset X$, the set of isolated points of Y is nonempty.

It is known that if X is compact and second countable, then X is countable if and only if it is scattered (this follows from the Cantor-Bendixson Theorem, see [94, 6.11]). Compact scattered spaces in general can in fact be characterized in a simple way, we refer the reader to [98, Section 17, pp. 271–284] for more details. We will need the following result.

Lemma 2.8 If X and Y are compact Hausdorff spaces, X is scattered and Y is a continuous image of X, then Y is also scattered.

Proof. Let $f : X \to Y$ be continuous and onto. Assume $K \subset Y$ is nonempty and does not have isolated points. By taking closure, we may assume that K is closed. Using the Kuratowski-Zorn lemma, we can find a closed subset $T \subset X$ that is minimal with the property that $f[T] = K$. Since X is scattered, there exists an isolated point $t \in T$ of T. Notice that $K \setminus \{f(t)\} \subset f[T \setminus \{t\}]$. Also, since K has no isolated points, $K \setminus \{f(t)\}$ is dense in K. But $T \setminus \{t\}$ is compact so it follows that $K \subset f[T \setminus \{t\}]$. This contradicts the minimality of T so such K cannot exist. $\square$

2.2 Quasicomponents

Recall that in a topological space, a *component* is defined to be a maximal connected set. Thus, the components of a space form a decomposition. We are going to need a coarser decomposition that that provided by the components. Recall that if X is any topological space, then the *quasicomponent* of X in $p \in X$ is the following set.

$$\mathcal{Q}(X,p) = \bigcap \{U \in \mathcal{CO}(X) : p \in U\}.$$

Notice that $\mathcal{Q}(X,p)$ is a closed subset of X and contains the component of X at p. The following classic example shows that components and quasicomponents do not coincide in general.

Example 2.9 Components and Quasicomponents do not coincide.

For each $n < \omega$, let $L_n = [0,1] \times \{\frac{1}{n+1}\}$. Define $X = \{p,q\} \cup (\bigcup \{L_n : n < \omega\})$ as a subspace of the plane, where $p = (0,0)$ and $q = (1,0)$. Clearly, the component of X at p is $\{p\}$. However, we now show that $\mathcal{Q}(X,p) = \{p,q\}$. Let $U \in \mathcal{CO}(X)$ be such that $p \in U$. There exists $N < \omega$ such that for all $n \geq N$ we have $U \cap L_n \neq \emptyset$. Since L_n is a topological copy of the unit interval, it is connected and thus $L_n \subset U$ for all $n \geq N$. But since U is closed, $q \in U$. Thus, $\{p,q\} \subset \mathcal{Q}(X,p)$. The other inclusion is easy. □

Once we have the language of components and quasicomponents, we can give characterizations of some disconnected spaces in these terms. Its proof should be clear from the definitions.

Proposition 2.10 Let X be a T_1 space. Then,

(a) X is totally disconnected if and only if all the quasicomponents of X are singletons,

(b) X is hereditarily disconnected if and only if all the components of X are singletons.

Even if components and quasicomponents do not coincide, they do in some special cases. Proposition 2.12 is a key ingredient of our results of Chapter 4 because the compact subsets of hereditarily disconnected spaces will have a stronger form of disconnectedness.

Lemma 2.11 If X is a compact space, then every quasicomponent of X is connected. Particularly, quasicomponents and components coincide in X.

Proof. Let X be a locally compact space and $x \in X$. Assume that $\mathcal{Q}(X,x)$ is disconnected. Then there exist two disjoint open subsets U and V of X such that $\mathcal{Q}(X,x) \subset U \cup V$ and $\mathcal{Q}(X,x) \cap U \neq \emptyset \neq \mathcal{Q}(X,x) \cap V$. Consider the open cover $\mathcal{U} = \{U \cup V\} \cup \{X \setminus W : W \in \mathcal{CO}(X), x \in W\}$. By compactness, there must be a finite subcover $\{U \cup V\} \cup \{X \setminus W_0, \ldots, X \setminus W_n\}$ of $\mathcal{U}$. Thus, $x \in W_0 \cap \ldots \cap W_n \subset U \cup V$. We may assume that $x \in U$, then it is easy to see that $U \cap W_0 \cap \ldots \cap W_n$ is a clopen subset of X that contains x and misses V. This is a contradiction to the definition of V so we have that in fact $\mathcal{Q}(X,x)$ is connected. $\square$

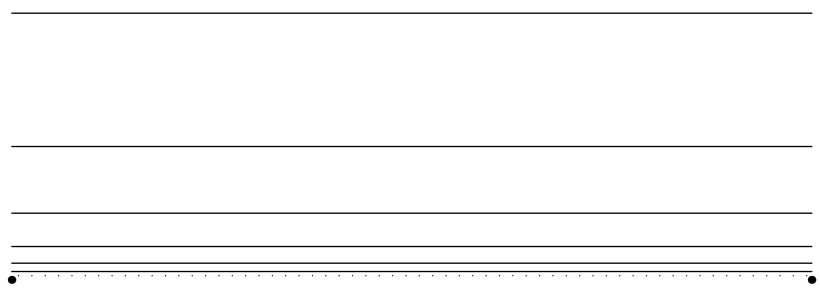

Figure 2.4: Space from Example 2.9.

Proposition 2.12 Every locally compact, Hausdorff and hereditarily disconnected space is 0-dimensional.

Proof. Let X be a locally compact, Hausdorff and hereditarily disconnected space. Since X is locally compact and Hausdorff, there exists a basis $\mathcal{B}$ of open sets such that $\mathrm{cl}_X(U)$ compact for all $U \in \mathcal{B}$. To prove that X is 0-dimensional, it is enough that for each $x \in X$ and $U \in \mathcal{B}$ with $x \in U$ we find a clopen set $V \in \mathcal{CO}(X)$ such that $x \in V \subset U$. Fix such x and U. Since $\mathrm{cl}_X(U)$ is compact and hereditarily disconnected, by Lemma 2.11 we have that $\mathcal{Q}(\mathrm{cl}_X(U), x) = \{x\}$. Thus, for each $y \in \mathrm{bd}_X(U)$, there exists $V_y \in \mathcal{CO}(\mathrm{cl}_X(U))$ such that $x \in V_y$ and $y \notin V_y$. Since $\mathrm{bd}_X(U)$ is compact, there exist an open subcover $\{W(0), \ldots, W(n)\}$ of the cover $\{\mathrm{cl}_X(U) \setminus V_y : y \in \mathrm{bd}_X(U)\}$ of $\mathrm{bd}_X(U)$. Let $V = (\mathrm{cl}_X(U) \setminus W(0)) \cap \ldots \cap (\mathrm{cl}_X(U) \setminus W(n))$, then $V \in \mathcal{CO}(\mathrm{cl}_X(U))$, $x \in V$ and $V \cap \mathrm{bd}_X(U) = \emptyset$. From this, it can easily be seen that $V \in \mathcal{CO}(X)$ and $x \in V \subset U$. This concludes the proof. $\square$

We remark that in [51, 1.4.5], there is a stronger version of Proposition 2.12. As a corollary, the following fact about scattered spaces is easy to prove.

Lemma 2.13 Any compact, Hausdorff and scattered space is 0-dimensional.

Proof. Let X be compact, Hausdorff and scattered. If $C \subset X$, then C has an isolated point so C is connected if and only if C is a singleton. This shows that X is hereditarily disconnected. By Proposition 2.12, X is 0-dimensional. □

Now we will iterate the operation of taking a quasicomponent. This can be viewed intuitively in the following way. Totally disconnected spaces can be thought of as a special class of hereditarily disconnected spaces where points can be separated via clopen subsets immediately. Notice that in order to prove that Example 2.4 is hereditarily disconnected, first we saw that point p could be separated by clopen sets from all other points in X_s and we were left to prove that $\{p, q\}$ is disconnected, which is true in Hausdorff spaces. Also in Example 2.5 we proved that every two sets of the form L_c could be separated by clopen sets and then used that L_c was totally disconnected. Thus, in both cases we needed *two* steps to test the hereditary disconnectedness of the space in question. By iterating the operation of taking a quasicomponent, we can count the number of steps needed to show that a given space is hereditarily disconnected.

For any space X and $p \in X$ we define by transfinite recursion the α- *quasi-component of* X *at* p, $\mathcal{Q}^\alpha(X, p)$, in the following way.

$$\begin{array}{lcll}
\mathcal{Q}^0(X,p) & = & X, & \\
\mathcal{Q}^{\alpha+1}(X,p) & = & \mathcal{Q}(\mathcal{Q}^\alpha(X,p),p), & \text{for each ordinal } \alpha, \\
\mathcal{Q}^\beta(X,p) & = & \bigcap_{\alpha<\beta} \mathcal{Q}^\alpha(X,p), & \text{for each limit ordinal } \beta.
\end{array}$$

We call $\mathfrak{nc}(X,p) = \min\{\alpha : \mathcal{Q}^{\alpha+1}(X,p) = \mathcal{Q}^\alpha(X,p)\}$ the *non-connectivity index of* X *at* p. Intuitively, $\mathfrak{nc}(X,p)$ tells us how many times we must iterate the quasicomponent operation in order to obtain a connected set. So, if X is hereditarily disconnected and $p \in X$, then $\mathfrak{nc}(X,p) = \min\{\alpha : \mathcal{Q}^\alpha(X,p) = \{p\}\}$. Notice that if X is hereditarily disconnected and $|X| > 1$, then X is totally disconnected if and only if $\mathfrak{nc}(X,p) = 1$ for every $p \in X$. Thus, the iterated quasicomponent and the non-connectivity index give us a formal idea of disconnectedness degree of spaces.

A natural question when one sees this definition is whether for every ordinal α there exists a (hereditarily disconnected) space X such that $\mathfrak{nc}(X,x) = \alpha$ for some $x \in X$. The answer is affirmative, we will mention some interesting results in this direction. The proofs of these results are not relevant to the development of the dissertation so we only cite the references where proofs can be found.

The first example is van Douwen's. Since the example is a dendroid explicitly constructed in the plane, we recommend the reader to take a look at it.

Example 2.14 [42] There exists a rational metrizable continuum X and a point $p \in X$ such that $\sup\{\mathfrak{nc}(Y,x) : x \in Y \subset X\} = \omega_1$.

The next three results are from Mihail Ursul and talk about topological groups. We refer the reader to [11] for a general reference to topological groups. It can be proved that if G is a topological group and $g, h \in G$ then $\mathcal{Q}(G,g)$ is homoemorphic to $\mathcal{Q}(G,h)$ via the translation given by $g^{-1}h$ and that $\mathcal{Q}(G,q)$ is a closed normal subgroup ([158, Theorem 12.3]). Thus, the high symmetry of topological groups reflects to its quasicomponents.

The first result gives a subgroup of the plane that needs two applications of the quasicomponent operation in order to become trivial. The second result is more abstract but proves that every Abelian group can be realized as an α-quasicomponent for arbitrary α. Finally, the third result shows that we cannot hope to get a general result for arbitrary (non-Abelian) groups.

Theorem 2.15 [158, Theorem 12.9] There exists a hereditarily disconnected subgroup G of ${}^2\mathbb{R}$ such that $\mathcal{Q}(G,0)$ is topologically isomorphic to $\mathbb{Z}$.

Theorem 2.16 [158, Theorem 12.24] Let α be any ordinal and H some topological Abelian group. Then there exists a topological Abelian group G such that $\mathcal{Q}^\alpha(G,0) = H$.

Theorem 2.17 [158, Example 12.1] A discrete group G is a quasi-component of another group if and only if G is Abelian.

There is one more quasicomponent construction we must consider. If X is any space (no separation axioms required), we can define a quotient space $\mathcal{Q}(X)$ by shrinking each quasicomponent of X to a point (that is, $\mathcal{Q}(X) = \{\mathcal{Q}(X,p) : p \in X\}$ with the quotient topology). We will call $\mathcal{Q}(X)$ the *space of quasicomponents of* X. The following can be easily proved from the definition of quotient topology.

Lemma 2.18 If X is any space, $\mathcal{Q}(X)$ is a Hausdorff totally disconnected space.

2.3 Highly disconnected spaces

In this section we will study some disconnectedness properties that are in some sense extremal.

Definition 2.19 Let X be a Hausdorff space.

(a) X is extremally disconnected (ED) if for every open subset U of X we have that $\mathrm{cl}_X(U)$ is open.

(b) X es basically disconnected (BD) if for every cozero set U of X we have that $\mathrm{cl}_X(U)$ is open.

(c) X is a P-space if every G_δ set of X is open.

There exist Hausdorff ED spaces that are not Tychonoff (for example the Katetov extension of ω, see the definition in [135, p. 307] and [135, Theorem 6.2(b)]). However, for regular spaces, all these properties imply being 0-dimensional (see Propostion 2.24 below). The reader will notice that every metrizable BD space is discrete. This is perhaps why these classes of spaces seem to be counterintuitive in some sense.

In fact, topologists started looking at these properties while studying the Čech-Stone compactification (see the definition in Section 6.1). In particular, $\beta\omega$ is ED, this will be seen in Corollary 6.35. Now we generalize (or perhaps particularize) some disconnectedness properties in the following way.

Definition 2.20 Let X be a Tychonoff space

(a) X is an F-space if every cozero set of X is C^*-embedded in βX.

(b) X is an F'-space if any two disjoint cozero sets of X have disjoint closures.

(c) X is a weak P-space if every countable subset of X is closed and discrete.

Let's briefly discuss the history of these properties. Some of these spaces have their origin in papers of Gillman and Henriksen. First, P-spaces were defined in [67] in an algebraic way, considering rings of real-valued continuous functions defined in topological spaces. After this, in [68], the authors studied other types of rings and defined various classes of spaces according to the corresponding properties of their rings of real-valued continuous functions. Two of these classes of spaces are F-spaces and F'-spaces. For example, F-spaces were defined as those spaces X such that the ring $C(X)$ of real-valued continuous functions defined in X is an F-ring[2], this means that every finitely generated ideal of $C(X)$ is generated by one element (see [69, 14.25]). Gillman and Henriksen studied these

[2]The mathscinet review of [68] says that F-rings were studied first by Irving Kaplanski but it seems that the term was given by Gillman and Henriksen. See also the Introduction and Section 4 of [45].

classes of spaces from the algebraic point of view but the ideas involved turned out to be important in the development of the Čech-Stone compactification. ED and BD spaces appear naturally when one considers Stone spaces of some Boolean Algebras; we shall talk about this in more detail when we construct the Absolute of a Tychonoff space in Section 6.3. The story of weak P-spaces comes from the non-homogeneity problem and we need to give an alternative definition of these spaces in order to make this clear.

The definitions of P-space and weak P-space can be localized (made local) in the following way.

Definition 2.21 Let X be an space and $x \in X$.

(i) We say that x is a *P-point* of X if for every G_δ set G with $x \in G$ we have that $x \in \mathrm{int}_X(G)$.

(ii) We say that x is a *weak P-point* of X if for every countable subset $N \subset X \setminus \{x\}$ we have that $x \notin \mathrm{cl}_X(N)$.

Notice that intuitively, P-points and weak P-points are points that are far away from the outside even in a countable number of steps (although the precise notion of this is different in each case). Every compact space has points that are neither P-points nor weak P-points. Using **CH**, Walter Rudin proved in [142] that $\omega^* = \beta\omega \setminus \omega$ has P-points and thus is not homogeneous under **CH**. Later, Kenneth Kunen introduces the weak P-points in [100] and proves that ω^* has weak P-points in ZFC so it is not homogeneous. We will talk about this in more detail in Section 6.5. The following trivial observation relates our definitions.

Lemma 2.22 A space X is a (weak) P-space if and only if every point of X is a (weak) P-point (respectively).

In this way, we may consider weak P-spaces as the natural generalization of P-spaces. We will need the following characterization for our results in Chapter 3.

Lemma 2.23 A Tychonoff space is a P-space if and only if every zero set is clopen.

Proof. Let X be a Tychonoff space. If X is a P-space, then every zero set is a G_δ set. Thus every zero set is open and is clearly closed by definition. Now assume that all zero sets of X are clopen and let $\{U_n : n < \omega\}$ a sequence of open subsets of X. We must prove that $G = \bigcap\{U_n : n < \omega\}$ is open, so let $x \in G$. Since X is Tychonoff, for each $n < \omega$, there is a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 0$ and $f_n[X \setminus U_n] \subset \{1\}$. Let $f : X \to [0,1]$ be the function defined by $f(x) = \sum_{n<\omega} \frac{f_n(x)}{2^n}$. As mentioned in [50, Corollary 1.5.12], f is a continuous function. Notice that $x \in f^{\leftarrow}(0) \subset G$. Thus, G is open. $\square$

The following diagram (that is part of the one found in [68, p. 390]) shows all the possible relations that one can find in the classes of spaces we have defined in this Section.

$$\begin{tikzcd}
& \mathbf{ED} \arrow[dr, "(b)"] & & \mathbf{F} \arrow[r, "(h)"] & \mathbf{F'} \\
\textbf{discrete} \arrow[ur, "(a)"] \arrow[dr, "(c)"'] & & \mathbf{BD} \arrow[ur, "(g)"] \arrow[r, "(f)"'] & \mathbf{0{-}dim} & \\
& \mathbf{P} \arrow[ur, "(d)"] \arrow[r, "(e)"'] & \textbf{weak P} & &
\end{tikzcd} \tag{2.1}$$

We will also see that F-spaces and P'-spaces are not necessarily disconnected (see Example 2.27). Nevertheless, F'-spaces will be relevant to our study of disconnectedness properties in hyperspaces, see Theorem 3.9.

Proposition 2.24 All implications in Diagram 2.1 hold for Tychonoff spaces.

Proof. For (a), (b) and (c), the implications are clear by definition. For (d), notice that every cozero set in a P-space is clopen because it is an F_σ.

To prove (e), let X be a P-space and $N = \{x_n : n < \omega\} \subset X$ be countable. Fix $y \in X$. For each $n < \omega$, let U_n be an open subset such that $y \in U_n$ and $x_n \notin U_n \setminus \{y\}$. Then $G = \bigcap\{U_n : n < \omega\}$ is a G_δ set, thus open. If $y \in N$, G witnesses that y is an isolated point of N. If $y \notin N$, G witnesses that $y \notin \mathrm{cl}_X(N)$. Then N is a closed and discrete subset of X.

Tychonoff BD spaces are 0-dimensional by regularity and the fact that cozero sets form a base, so we have (f). To prove (g), let U be a cozero set of a BD Tychonoff space X. Let's prove that U is C^*-embedded in X. Since $\mathrm{cl}_X(U)$ is clopen, any continous function defined in $\mathrm{cl}_X(U)$ can be arbitrarily extended to a continous function by defining it to be constant on $X \setminus \mathrm{cl}_X(U)$. Thus, it is enough to assume that U is dense in X. We will now use Taimanov's Theorem 0.10 so let A and B be disjoint zero subsets of U. Since U is a cozero set, there is a continuous function $f : X \to [0,1]$ such that $U = f^{\leftarrow}[(0,1]]$. For each $n < \omega$, let $U_n = \mathrm{cl}_X(f^{\leftarrow}[(\frac{1}{n+2}, 1]])$, clearly $U_n \in \mathcal{CO}(X)$ and moreover, $U = \bigcup\{U_n : n < \omega\}$. In fact it is easy to verify that $A_n = A \cap U_n$ and $B_n = B \cap U_n$ are disjoint zero sets of X for each $n < \omega$. By Lemma 0.8, for each $n < \omega$ there exist disjoint cozero sets V_n and W_n of X such that $A_n \subset V_n$ and $B_n \subset W_n$. Let $V = \bigcup\{V_n : n < \omega\}$ and $W = \bigcup\{W_n : n < \omega\}$, these are disjoint cozero sets of X. Since X is BD it is easy to see that $\mathrm{cl}_X(V) \cap \mathrm{cl}_X(W) = \emptyset$. This implies that $\mathrm{cl}_X(A) \cap \mathrm{cl}_X(B) = \emptyset$. By Taimanov's Theorem 0.10, we obtain that U is C^*-embedded in X.

Finally, for (h), let X be an F-space and U, V two disjoint cozero sets of X. Then $U \cup V$ is a cozero set and the function $f : U \cup V \to [0,1]$ defined by $f[U] \subset \{0\}$ and $f[V] \subset \{1\}$ is continuous. Let $F : X \to [0,1]$ be a continuous extension, then $\mathrm{cl}_X(U) \subset F^{\leftarrow}[0, \frac{1}{3}]$ and $\mathrm{cl}_X(V) \subset F^{\leftarrow}[\frac{2}{3}, 1]$ so $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$. □

Now we will give examples that show that no implication in Diagram 2.1 is an equivalence.

Example 2.25 ED non-discrete spaces

For a ED non-discrete space, it is enough to consider $\beta\omega$, it is ED by Corollary 6.35 and it is clearly not discrete as it is compact and infinite. In fact, $\beta\omega$ is too much: if $p \in \omega^* = \beta\omega\setminus\omega$, it follows from Lemma 6.30 that the space $\omega \cup \{p\} \subset \beta\omega$ is ED. This proves that (a) in Diagram 2.1 is not an equivalence. □

For (b), as we will see in Theorem 6.34, it is enough to consider the Stone space of a σ-complete but non-complete Boolean algebra, for example, the collecton of Borel sets or Lebesgue-measurable sets of the reals. However, studying these examples would be quite technical and perhaps outside our objectives. This is why we will give two examples that will also serve as examples for (c). Both are subsets of linearly ordered spaces and will be very visual.

Example 2.26 Two P-spaces that are suborderable.

The first example is $L = \{\alpha + 1 : \alpha < \omega_1\} \cup \{\omega_1\}$ as a subspace of the linearly ordered $\omega_1 + 1$. Notice that all points of $L \setminus \{\omega_1\}$ are isolated in L. Also, ω_1 is a P-point of L: if $\{(\alpha_n, \omega_1]_L : n < \omega\}$ are basic open neighborhoods of ω_1 and $\alpha = \sup\{\alpha_n : n < \omega\}$, then $(\alpha + 1, \omega_1]_L$ is an open subset of L that contains ω_1 and is contained in $\bigcap\{(\alpha_n, \omega_1]_L : n < \omega\}$. So L is a P-space. The second example appears in [69, 13.18], we describe it next. For $x, y \in {}^{\omega_1}2$ different points, we define $x <_{lex} y$ if $x(\xi) < y(\xi)$, where $\xi = \min\{\alpha < \omega_1 : x(\alpha) \neq y(\alpha)\}$. This defines a strict total ordering on ${}^{\omega_1}2$ that is called *lexicographic order*. Define

$$Q = \{x \in {}^{\omega_1}2 : \{\alpha < \omega_1 : x(\alpha) = 1\} \text{ has a maximum}\}.$$

Now we prove that Q is a P-space and has no isolated points with the topology of the linear order $(Q, <_{lex})$. Since there is a basis of open intervals, it is enough to prove that if $q \in Q$ and $\{x_n : n < \omega\} \subset X$ are such that $x_n <_{lex} x_{n+1} <_{lex} q$

for all $n<\omega$ or $q<_{lex} x_{n+1}<_{lex} x_n$ for all $n<\omega$, then there is $x\in Q$ such that $x_n<_{lex} x<_{lex} q$ for all $n<\omega$ or $q<_{lex} x<_{lex} x_n$ for all $n<\omega$, respectively. Assume that $x_n<_{lex} x_{n+1}<_{lex} q$ for all $n<\omega$, the other case is analogous.

Let $\xi=\max\{\alpha<\omega_1 : q(\alpha)=1\}$, $\xi_n=\max\{\alpha<\omega_1 : x_n(\alpha)=1\}$ for $n<\omega$ and $\xi'=\sup\{\xi_n : n<\omega\}+\xi+1<\omega_1$. Define $x\in {}^{\omega_1}2$ as

$$x(\alpha)=\begin{cases} q(\alpha) & \text{if } \alpha<\xi;\\ 0 & \text{if } \alpha=\xi;\\ 1 & \text{if } \xi<\alpha\leq\xi';\\ 0 & \text{if } \xi'<\alpha<\omega_1.\end{cases}$$

Clearly, $x\in Q$. Since $\xi=\min\{\alpha<\omega_1 : x(\alpha)\neq q(\alpha)\}$ and $x(\xi)=0<1=q(\xi)$, $x<_{lex} q$. Now we have to prove that $x_n<_{lex} x$ for every $n<\omega$. Notice that $x(\xi')=1$ and $x_n(\xi')=0$ for each $n<\omega$ so we know that $x\notin\{x_n : n<\omega\}$.

Fix $n<\omega$ and define $\beta=\min\{\alpha<\omega_1 : x(\alpha)\neq x_n(\alpha)\}$, this is possible by the discussion above. If $\beta<\xi$, then $x(\beta)=q(\beta)$ and $\beta=\min\{\alpha<\omega_1 : q(\alpha)\neq x_n(\alpha)\}$ so $x_n(\beta)<q(\beta)=x(\beta)$, this shows that $x_n<_{lex} x$. If $\beta=\xi$, then we obtain the following: $x_n(\beta)=1=q(\beta)$, $q(\alpha)=x(\alpha)=x_n(\alpha)$ if $\alpha<\beta$ and $x_n(\alpha)\geq q(\alpha)=0$ if $\beta<\alpha$. Thus, $q=x_n$ or $q<_{lex} x_n$, which is a contradiction so it is impossible that $\beta=\xi$. If $\beta>\xi$, notice that $x_n(\alpha)=0=x(\alpha)$ when $\xi'<\alpha<\omega_1$ so in fact $\beta\leq\xi'$; this implies that $x(\beta)=1$ so $x_n(\beta)=0$ and thus $x_n<_{lex} x$. Thus, we have proved that $x_n<_{lex} x$ for all $n<\omega$.

As both L and Q are P-spaces, they are also BD by Proposition 2.24. It is easy to see that L is not ED: the sets $\{\alpha\in L : \alpha \text{ is odd}\}$ and $\{\alpha\in L : \alpha \text{ is even}\}$ are disjoint open subsets of L that have ω_1 in its closure. Now we prove that Q is not ED. Let $p\in Q$ be such that $p(0)=1$ and $p(\alpha)=0$ for all $1\leq\alpha<\omega_1$ and for each $\alpha<\omega_1$, let $p_\alpha\in Q$ be defined as

$$p_\alpha(\beta)=\begin{cases} 0 & \text{if } \beta=0 \text{ or } \alpha+1<\beta,\\ 1 & \text{if } 1\leq\beta\leq\alpha+1.\end{cases}$$

Notice that for all $\alpha<\beta<\omega_1$, $p_\alpha<_{lex} p_\beta<_{lex} p$. It is also easy to see that $p=\sup\{p_\alpha : \alpha<\omega_1\}$. For each $\alpha<\omega_1$, let $I_\alpha=(p_{\alpha+1},p_{\alpha+2})_{<_{lex}}$, which is a non-empty open subset of Q. From this it follows that

$$p\in \mathrm{cl}_Q(\bigcup\{I_\alpha : \alpha \text{ is even}\})\cap\mathrm{cl}_Q(\bigcup\{I_\alpha : \alpha \text{ is odd}\}).$$

So Q is not ED. Another way to prove that L and Q are not ED is quoting the following more general result: if X is a ED space with cellularity strictly smaller than the first measurable cardinal, then any P-point of X is isolated ([135, 6.O]). $\square$

The examples for (d), (e), (f) and (g) are either too difficult to include here in detail or almost trivial. The following is a fast sketch of those examples.

Example 2.27 More examples for Diagram 2.1.

For (d), consider $\beta\omega$, this is a compact ED space (Corollary 6.35), so it is BD by Proposition 2.24. However, in any compact infinite space there must exist points that are limits of countable infinite discrete subsets. Thus, $\beta\omega$ is BD but not a P-space.

For (e), there exists a famous example by Dmitriĭ B. Shakhmatov [152]. This example is a Tychonoff, connected, pseudocompact space that is a weak P-space and such that every countable subset is C^*-embedded. This space cannot be a P-space because every P-space must be 0-dimensional by Propostion 2.24.

For (f) it is enough to consider any metrizable, 0-dimensional, non-discrete space X. In X it is easy to give two disjoint open sets (that must be cozero sets) whose closures intersect. From this it follows that X is not BD.

To see that (g) is not reversible, consider $\beta[0,1)\setminus[0,1)$. By a result of Gillman and Henriksen ([68, Theorem 2.7]), we know that if X is a locally compact, σ-compact space, then $X^* = \beta X \setminus X$ is an F-space. The original proof is algebraic, a topological proof atributed to van Douwen can be found on [113, 1.2.5]. Moreover, it is easy to see that $\beta[0,1)\setminus[0,1)$ is connected and has more than one point, so it cannot be BD by Poposition 2.24. □

Finally, for (h) in Diagram 2.1 we shall give the example from [68, Example 8.14].

Example 2.28 An F'-space that is not a F-space.

Let L be the space described in Example 2.26 and let $L' = \{\alpha+1 : \alpha < \omega_2\} \cup \{\omega_2\}$ as a subspace of the linearly ordered space ω_2+1. Let $L_0 = (L' \times L)\setminus\{\langle\omega_2,\omega_1\rangle\}$, it is easy to see that L_0 is a P-space and thus, BD by Proposition 2.24.

Let $N = \{x_n : n < \omega\}$ be a countable set disjoint from L_0. For each $\alpha < \omega_1$, let $D_\alpha = (N \cup \{\omega_2\}) \times \{\alpha+1\}$. Let $p \in \beta\omega \setminus w$ be chosen arbitrarily. For each $\alpha < \omega$, there is a bijection from D_α to $\omega\cup\{p\}$ that takes $\langle\omega_2,\alpha+1\rangle$ to p; give D_α the topology so that this bijection is a homeomorphism. By the remarks given in Example 2.25, D_α is ED and its only non-isolated point is $\langle\omega_2,\alpha+1\rangle$.

The space we are looking for is $L_1 = L_0 \cup (\bigcup\{D_\alpha : \alpha < \omega_1\})$, so we have to define its topology. Give L_1 the topology such that $U \subset L_1$ is open if and only if $U \cap L_0$ is open in L_0 and for every $\alpha < \omega_1$, $U \cap D_\alpha$ is open in D_α.

If $f : L_1 \to [0,1]$ is defined as $f(\langle x_n, \alpha+1\rangle) = \frac{1}{n+1}$ for all $n < \omega$, $\alpha < \omega_1$ and $f(x) = 0$ if $x \notin \bigcup\{D_\alpha : \alpha < \omega_1\}$, then f is a continuous function that witnesses that L_0 is a zero set of L_1. If $g : L_0 \to \mathbb{R}$ is a continuous function, we can extend it to a continuous function $G : L_1 \to \mathbb{R}$ by defining $G(\langle x_n, \alpha+1\rangle) = g(\langle\omega_2,\alpha+1\rangle) + \frac{1}{n+1}$. Thus, L_0 is C-embedded and C^*-embedded in L_1.

First we argue that L_1 is an F'-space. Let U and V be two disjoint cozero sets of L_1, it is enough to prove that $\mathrm{cl}_{L_1}(U) \cap \mathrm{cl}_{L_1}(V) = \emptyset$. Since L_0 is a closed P-space, $\mathrm{cl}_{L_1}(U \cap L_0) \cap \mathrm{cl}_{L_1}(V \cap L_0) = \emptyset$ by Proposition 2.24. Thus, it is enough to assume that $U \cup V \subset L_1 \setminus L_0$. Moreover, $L_1 \setminus L_0$ is an open discrete subset of L_1, so $\mathrm{cl}_{L_1}(U) \cap \mathrm{cl}_{L_1}(V) \subset \{\omega_2\} \times \omega_1$. However, if $\alpha < \omega_1$, the point $\langle \omega_2, \alpha+1 \rangle$ is contained in the clopen ED subspace D_α. Thus, $\langle \omega_2, \alpha + 1 \rangle \notin \mathrm{cl}_{L_1}(U) \cap \mathrm{cl}_{L_1}(V)$. This completes the proof that $\mathrm{cl}_{L_1}(U) \cap \mathrm{cl}_{L_1}(V) = \emptyset$ so L_1 is an F'-space.

Since $L_1 \setminus L_0 = N \times \{\alpha + 1 : \alpha < \omega_1\}$ is a cozero set of L_1, let us define a continuous function $h : L_1 \setminus L_0 \to [-1, 1]$ and prove that it cannot be extended to L_1. Define

$$h(\langle x_n, \alpha + 1 \rangle) = \begin{cases} 1 & \text{if } \alpha \text{ is odd,} \\ -1 & \text{if } \alpha \text{ is even.} \end{cases}$$

Since $N \times \{\alpha + 1 : \alpha < \omega_1\}$ is discrete, h is continuous. Assume that it can indeed be extended to $H : L_1 \to \mathbb{R}$, we will arrive to a contradiction. Notice that $H(\langle \omega_2, \alpha + 1 \rangle) = 1$ if α is odd and $H(\langle \omega_2, \alpha + 1 \rangle) = -1$ if α is even. Thus, for all $\alpha < \omega_1$, there is $\beta(\alpha) < \omega_2$ such that if $\beta(\alpha) < \gamma < \omega_2$, then $|H(\langle \gamma, \alpha + 1 \rangle) - H(\langle \omega_2, \alpha + 1 \rangle)| < \frac{1}{2}$. Let $\Gamma = \sup\{\beta(\alpha) : \alpha < \omega_1\} < \omega_2$. If $\Gamma < \gamma < \omega_2$ then any neighborhood of $\langle \gamma, \omega_1 \rangle$ has points a, b such that $H(a) > \frac{1}{2}$ and $H(b) < -\frac{1}{2}$. This contradiction[3] proves that h cannot be extended so L_1 is not an F-space. □

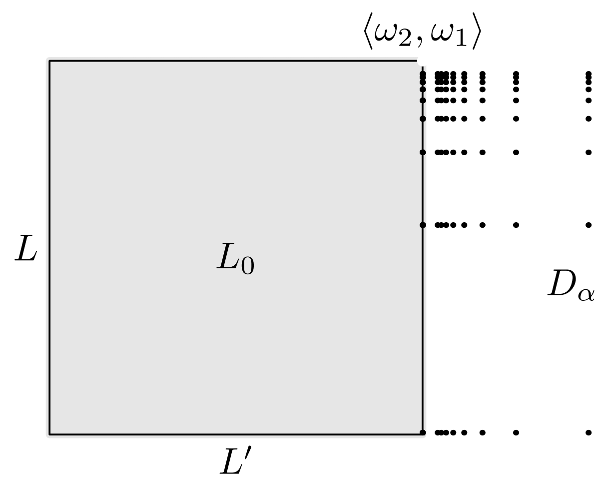

Figure 2.5: Space L_1 from Example 2.28.

[3]This argument is very similar to the one used to prove that the deleted Tychonoff plank is not normal, see [145, 87].

We will need the following result later.

Lemma 2.29 Every F'-space of countable cellularity is ED.

Proof. Let X be an F'-space of countable cellularity and $U \subset X$ an open subset. Define $V = X \setminus \mathrm{cl}_X(U)$ and let $\mathcal{U}$ be a maximal cellular family of cozero sets of X contained in $U \cup V$. Since X is regular, it is easy to prove that $W = \bigcup \mathcal{U}$ is dense. Observe that $\mathcal{U}$ is countable so by [69, 1.14], we have that $W \cup U$ and $W \cup V$ are cozero sets. Since X is an F'-space, we know that $\mathrm{cl}_X(W \cap U) \cap \mathrm{cl}_X(W \cup V) = \emptyset$. By the density of W we obtain that $\mathrm{cl}_X(U) = \mathrm{cl}_X(U \cap W)$ and $V = \mathrm{cl}_X(V \cap W)$, so $\mathrm{cl}_X(U)$ is clopen. $\square$

Chapter 3

Extreme Disconnectedness in Hyperspaces

In this Chapter, we shall study how properties given in Diagram 2.1, p. 24 behave in the context of hyperspaces. The results in this Chapter were published in Sections 2, 3 and 4 of [79].

We begin with two easy facts that show that in this context $CL(X)$ is trivial.

Fact 3.1 If X is an infinite Hausdorff space, then $CL(X)$ contains a convergent sequence.

Proof. Let $N = \{x_n : n < \omega\}$ be an countable infinite subspace of X. If $A_m = \{x_n : n \leq m\}$ for $m < \omega$, then $\{A_m : m < \omega\}$ is a sequence in $CL(X)$ that converges to $\mathrm{cl}_X(N)$ by Lemma 1.11. $\square$

Fact 3.2 If X is an F'-space, then X does not contain convergent sequences

Proof. If $N = \{x_n : n < \omega\}$ is a faithfully indexed sequence that converges to x_0, let U, V be disjoint cozero sets of X such that $\{x_{2n} : n \in \mathbb{N}\} \subset U$ and $\{x_{2n-1} : n \in \mathbb{N}\} \subset V$. Then $x_0 \in \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V)$. $\square$

Thus, we obtain the following.

Proposition 3.3 If X is Hausdorff and $CL(X)$ is an F'-space, then X is finite.

Although $CL(X)$ has no extreme disconnectedness properties (except in trivial cases), it is known that it can be 0-dimensional, see Proposition 5.18.

3.1 A Hyperspace is an F-space if and only if it is a P-space

First, let us show how to detect P-points in symmetric products.

Proposition 3.4 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space and $A \in \mathcal{F}(X)$. The following conditions are equivalent

(a) A is a P-point of $\mathcal{F}(X)$,

(b) A is a P-point of $\mathcal{F}_n(X)$ for each $n \geq |A|$,

(c) every $x \in A$ is a P-point of X.

Proof. Let $A = \{x_0, \ldots, x_m\}$. The implication $(a) \Rightarrow (b)$ is clear because the property of being a P-point is hereditary to subspaces. Assume A is a P-point of $\mathcal{F}_{m+1}(X)$. Let $\{U_i : i < \omega\}$ be a collection of open subsets of X such that $x_0 \in \bigcap_{i<\omega} U_i$. Take $W_0, \ldots, W_m$ pairwise disjoint open subsets of X such that $x_i \in W_i$ for $j \leq m$. For each $i < \omega$, define

$$\mathcal{U}_i = \langle U_i \cap W_0, W_1, \ldots, W_m \rangle.$$

Since A is a P-point in $\mathcal{F}_{m+1}(X)$, by Lemma 1.1, there is a collection $V_0, \ldots, V_m$ consisting of pairwise disjoint open subsets of X such that

$$A \in \langle V_0, \ldots, V_m \rangle \subset \bigcap_{i<\omega} \mathcal{U}_i.$$

We may assume $x_j \in V_j$ for each $j \leq m$. We now prove $V_0 \subset \bigcap_{i<\omega} U_i$. Take $y \in V_0$ and consider the element $B = \{y, x_1, \ldots, x_m\} \in \langle V_0, \ldots, V_m \rangle$. Since $B \in \mathcal{U}_i$ for each $i < \omega$, we get $y \in \bigcap_{i<\omega} U_i$. This proves that x_0 is a P-point of X and by similar arguments, each point of A is a P-point of X. This proves $(b) \Rightarrow (c)$.

Now, let $\{\mathcal{U}_i : i < \omega\}$ be a collection of open subsets of $\mathcal{F}(X)$ that cointain A and assume each point of A is a P-point of X. Using Lemma 1.1, for each $i < \omega$ one may define a collection $U(0, i), \ldots, U(m, i)$ consisting of pairwise disjoint open subsets of X such that for each $j \leq m$, $x_j \in U(j, i)$ and $\langle U(0, i), \ldots, U(m, i) \rangle \subset \mathcal{U}_i$. Each point of A is a P-point so we may take, for each $j \leq m$, an open subset U_j of X such that $x_j \in U_j \subset \bigcap_{i<\omega} U(j, i)$. Thus,

$$A \in \langle U_0, \ldots, U_m \rangle \subset \bigcap_{i<\omega} \mathcal{U}_i,$$

which proves $(c) \Rightarrow (a)$. □

We give the following example of how the construction of hyperspaces can produce complicated spaces from simple ones (compare with the fact that $\mathcal{K}(\omega+1)$ is the Cantor set by Example 1.3).

Example 3.5 A crowded homogeneous P-space using hyperspaces.

Let L be the linearly ordered space from Example 2.26. Let

$$X = \{A \in \mathcal{F}(L) : \omega_1 \in A\},$$

which is a P-space (Proposition 3.4). Let $A \in X$ and let $\langle U_0, \ldots, U_k \rangle$ be a Vietoris set neighborhood of A. We may assume that $U_0, \ldots, U_k$ are pairwise disjoint (Lemma 1.1) and $\omega_1 \in U_0$. Let $\alpha \in U_0 \setminus \{\omega_1\}$, then $A \neq A \cup \{\alpha\} \in \langle U_0, \ldots, U_n \rangle$. Thus, X has no isolated points.

To prove the homogeneity of X it is sufficient to prove the following:

(1) if $A, B \in X$ are such that $|A| = |B|$, then there exists a homeomorphism $H : X \to X$ such that $H(A) = B$,

(2) for every $n \in \mathbb{N}$, there are $A, B \in X$ such that $|A| + 1 = |B| = n + 1$ and a homeomorphism $H : X \to X$ such that $H(A) = B$.

For (1), let $h : L \to L$ be a bijection such that $h[A] = B$ and $h(\omega_1) = \omega_1$. Define $H(P) = h[P]$ for every $P \in X$.

For (2), let $H : Y \to Y$ be defined by

$$H(P) = \begin{cases} P \setminus \{1\}, & \text{if } 1 \in P, \\ P \cup \{1\}, & \text{if } 1 \notin P. \end{cases}$$

Then H is a homeomorphism such that for each $A \in X$ with $0 \notin A$, $|H(A)| = |A| + 1$. It follows that X is homogeneous. □

Next, let us show that the hyperspaces we are considering are extremally disconnected if and only if they are discrete, even in the realm of Hausdorff spaces.

Proposition 3.6 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space and $\mathcal{F}_2(X) \subset \mathcal{H} \subset \mathcal{K}(X)$. Then $\mathcal{H}$ is extremally disconnected if and only if X is discrete.

Proof. Clearly, X discrete implies $\mathcal{H}$ discrete. So assume that X is not discrete, take a non-isolated point $p \in X$ and consider $\mathcal{Z}$ the set of all collections $\mathcal{G}$ such that the elements of $\mathcal{G}$ are pairwise disjoint nonempty open subsets of X and if

$U \in \mathcal{G}$, then $p \notin \mathrm{cl}_X(U)$. By the Kuratowski-Zorn Lemma, we can consider a $\subset$-maximal element $\mathcal{M} \in \mathcal{Z}$. Since X is Hausdorff, $\bigcup \mathcal{M}$ is dense in X.

Let $\mathcal{U} = \bigcup\{U^+ \cap \mathcal{H} : U \in \mathcal{M}\}$. Let $\mathcal{N}$ be the filter of open neighborhoods of p. For each $W \in \mathcal{N}$, there must be $U_W, V_W \in \mathcal{M}$ such that $U_W \neq V_W$, $W \cap U_W \neq \emptyset$ and $W \cap V_W \neq \emptyset$. Let $\mathcal{V} = \bigcup\{\langle U_W, V_W\rangle \cap \mathcal{H} : W \in \mathcal{N}\}$. Then, $\mathcal{U}$ and $\mathcal{V}$ are pairwise disjoint nonempty open subsets of $\mathcal{H}$ but $\{p\} \in \mathrm{cl}_{\mathcal{H}}(\mathcal{U}) \cap \mathrm{cl}_{\mathcal{H}}(\mathcal{V})$. $\square$

Now we will see what happens when a hyperspace is an F'-space. The definition of an F'-space requires the Tychonoff separation axiom (Definition 2.20). By Corollary 1.9 we may assume that the base space of the hyperspaces in question is Tychonoff.

Remark 3.7 Let X be an infinite Tychonoff space. If $\mathcal{K}(X)$ is an F'-space, then $\mathcal{K}(X) = \mathcal{F}(X)$.

Proof. If $Y \in \mathcal{K}(X) \setminus \mathcal{F}(X)$, then $CL(Y) \subset \mathcal{K}(X)$ contains a convergent sequence by Fact 3.1. This contradicts Fact 3.2. $\square$

The following is the most important result in this Chapter.

Proposition 3.8 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Tychonoff space and let $F_2(X) \subset \mathcal{H} \subset \mathcal{K}(X)$. If $\mathcal{H}$ is an F'-space, then X is a P-space.

Proof. Let us assume X is not a P-space, by Lemma 2.23, we may assume there is a continuous function $f : X \to I$ such that $Z = f^{\leftarrow}(0)$ is not clopen. Let $p \in Z \setminus \mathrm{int}_X(Z)$ and consider the following two statements:

(E) There is a neighborhood U of p with $f[U] \subset \{0\} \cup \{\frac{1}{2m} : m \in \mathbb{N}\}$.

(O) There is a neighborhood V of p with $f[V] \subset \{0\} \cup \{\frac{1}{2m-1} : m \in \mathbb{N}\}$.

Notice that since $p \notin \mathrm{int}_X(Z)$, we cannot have (E) and (O) simultaneaously. Assume without loss of generality that (E) does not hold. For each $m \in \mathbb{N}$, let $U_m = f^{\leftarrow}[(\frac{1}{2m+2}, \frac{1}{2m})]$. Then $\{U_m : m \in \mathbb{N}\}$ is a collection of pairwise disjoint cozero sets. Observe that every neighborhood of p intersects some U_m. Also, $f^{\leftarrow}[[0, \frac{1}{2m+2})]$ is a neighborhood of p that misses U_m. Thus,

$$(*)\ p \in \mathrm{cl}_X(\bigcup \{U_m : m \in \mathbb{N}\}) \setminus \bigcup\{\mathrm{cl}_X(U_m) : m \in \mathbb{N}\}.$$

Consider the sets:

$$\begin{aligned} \mathcal{U} &= \bigcup\{U_m^+ \cap \mathcal{H} : m \in \mathbb{N}\}, \\ \mathcal{V} &= \bigcup\{\langle U_m, U_k\rangle \cap \mathcal{H} : m, k \in \mathbb{N}, m \neq k\}, \end{aligned}$$

these are nonempty pairwise disjoint cozero sets by Lemma 1.7 and Equation 1.4, page 5. By $(*)$, it follows that $\{p\} \in \mathrm{cl}_{\mathcal{H}}(\mathcal{U}) \cap \mathrm{cl}_{\mathcal{H}}(\mathcal{V})$, so $\mathcal{H}$ is not an F'-space. $\square$

This allows us to show the following structure result for hyperspaces.

Theorem 3.9 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Tychonoff space and $\mathcal{F}_2(X) \subset \mathcal{H} \subset \mathcal{K}(X)$. Then the following are equivalent:

(a) X is a P-space,

(b) $\mathcal{H}$ is a P-space,

(c) $\mathcal{H}$ is an F'-space.

Proof. First, assume (a). By Lemma 3.4, $\mathcal{F}(X)$ is a P-space and by Remark 3.7, $\mathcal{K}(X) = \mathcal{F}(X)$ so $\mathcal{H} \subset \mathcal{K}(X)$ is a P-space. So (b) holds. The fact that (b) implies (c) follows from Proposition 2.24 and (c) implies (a) by Proposition 3.8. $\square$

3.2 Some spaces such that $\mathcal{K}(X) = \mathcal{F}(X)$

Notice that we have discussed all classes of spaces in Diagram 2.1, p. 24, in the context of hyperspaces, except for weak P-spaces. In this section we will consider weak P-spaces and give some remarks of the class of spaces X such that $\mathcal{K}(X) = \mathcal{F}(X)$. This class includes F'-spaces by the results of the last section and also weak P-spaces.

Fact 3.10 If X is a weak P-space, then $\mathcal{K}(X) = \mathcal{F}(X)$.

Proof. If X is a weak P-space, then every countable subset of X is closed and discrete. If $K \subset X$ is compact and infinite, it contains a countable infinite discrete subset $N \subset K$ and if $x \in \mathrm{cl}_X(N) \setminus N$, then $N \cup \{x\}$ is countable but not discrete. $\square$

We have the following results for weak P-spaces, analogous to those of P-spaces.

Proposition 3.11 Let X be a Hausdorff space and $A \in \mathcal{F}(X)$. The following conditions are equivalent

(a) A is a weak P-point in $\mathcal{F}(X)$,

(b) A is a weak P-point in $\mathcal{F}_n(X)$ for each $n \geq |A|$,

(c) every $x \in A$ is a weak P-point of X.

Proof. Let $A = \{x_0, \ldots, x_m\}$. Notice $(a) \Rightarrow (b)$ because being a weak P-point is hereditary to subspaces.

To prove $(b) \Rightarrow (c)$, assume x_0 is not a weak P-point of X. Let $D = \{y_k : k < \omega\} \subset X \setminus \{x_0\}$ be such that $x_0 \in \mathrm{cl}_X(D)$. Define $B_k = \{y_k, x_1, \ldots, x_m\} \in \mathcal{F}_{m+1}(X)$ for each $k < \omega$. Then, $\{B_k : k < \omega\} \subset \mathcal{F}_{m+1}(X) \setminus \{A\}$ and $A \in \mathrm{cl}_{\mathcal{F}_{m+1}(X)}(\{B_k : k < \omega\})$.

Now we prove $(c) \Rightarrow (a)$. Assume (c) and take $\{B_k : k < \omega\} \subset \mathcal{F}(X) \setminus \{A\}$. For each $k < \omega$, choose $t(k) \in \{0, \ldots, m\}$ such that $x_{t(k)} \notin B_k$. Define $E_r = \{k < \omega : t(k) = r\}$ for each $r \leq m$. So given $r \leq n$, $x_r \notin \bigcup\{B_k : k \in E_r\}$. Since $\bigcup\{B_k : k \in E_r\}$ is countable, there exists an open subset U_r with $x_r \in U_r$ and $U_r \cap (\bigcup\{B_k : k \in E_r\}) = \emptyset$. Finally, let $\mathcal{U} = \langle U_0, \ldots, U_n \rangle$. Then $A \in \mathcal{U}$ and $\mathcal{U} \cap \{B_k : k < \omega\} = \emptyset$. □

Theorem 3.12 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space. Then the following are equivalent

(a) X is a weak P-space,

(b) $\mathcal{K}(X)$ is a weak P-space,

(c) $\mathcal{F}(X)$ is a weak P-space,

(d) $\mathcal{F}_n(X)$ is a weak P-space for all $n \in \mathbb{N}$.

Proof. If we assume (a), by Fact 3.10 we have $\mathcal{K}(X) = \mathcal{F}(X)$, which is a weak P-space by Proposition 3.11. Clearly, (b) implies (c) and (c) implies (d). Finally, (d) and (a) are equivalent by Proposition 3.11. □

We next show that condition $\mathcal{K}(X) = \mathcal{F}(X)$ behaves well under the operation of taking hyperspace in the following way.

Proposition 3.13 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) If X is a Hausdorff space, then $\mathcal{K}(X) = \mathcal{F}(X)$ if and only if every compact subset of $\mathcal{K}(X)$ is finite (that is, $\mathcal{K}(\mathcal{K}(X)) = \mathcal{F}(\mathcal{K}(X))$).

Proof. First, assume $\mathcal{K}(X) = \mathcal{F}(X)$, and let $\mathcal{C} \subset \mathcal{K}(X)$ be compact. Write $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ where $\mathcal{C}_n = \mathcal{C} \cap \mathcal{F}_n(X)$. Notice each $\mathcal{C}_n$ is compact because $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.

Claim. Each $\mathcal{C}_n$ is finite.

Fix $n \in \mathbb{N}$. To prove the claim, consider the natural identification $\pi : {}^nX \to \mathcal{F}_n(X)$ that sends each n-tuple to the set of its coordinates (Lemma 1.2). Also, consider $\pi_k : {}^nX \to X$ the projection onto the kth-coordinate. Since π is perfect, the set $K_n = \pi_k[\pi^{\leftarrow}[\mathcal{C}_n]]$ is a compact subset of X and thus, finite. Now, $\pi^{\leftarrow}[\mathcal{C}_n] \subset K_1 \times \cdots \times K_n$ so $\mathcal{C}_n$ must also be finite. This proves the Claim.

By the Claim, $\mathcal{C}$ is a compact Hausdorff countable space. Since the weight of an infinite compact Hausdorff space is less or equal to its cardinality ([50, 3.1.21]), $\mathcal{C}$ is a compact metric space. Assume $\mathcal{C}$ is infinite, then we can find a faithfully indexed sequence $\{A_n : n < \omega\} \subset \mathcal{C}$ such that $A_0 = \lim A_n$.

Let $A_0 = \{x_0, \ldots, x_s\}$ and take $U_0, \ldots, U_s$ pairwise disjoint open sets such that $x_i \in U_i$ for $i \leq s$. We may thus assume that for every $n < \omega$, $A_n \in \langle U_0, \ldots, U_s \rangle$. For each $n \in \mathbb{N}$, let $k_n \leq s$ be such that $A_n \cap U_{k_n} \neq \{x_{k_n}\}$, we may assume without loss of generality that $k_n = 0$ for every $n \in \mathbb{N}$. Let

$$Y = \bigcup \{A_n \cap U_0 : n < \omega\}.$$

First, if Y is finite, there is an open set V such that $V \cap Y = \{x_0\}$, so the neighborhood $\langle V \cap U_0, U_1, \ldots, U_s \rangle$ intersects the sequence only in A_0, which contradicts the convergence of the A_n. Thus, Y is infinite. We now prove that Y converges to x_0. Let V be an open set such that $x_0 \in V$. Let $k < \omega$ be such that $A_n \in \langle V \cap U_0, U_1, \ldots, U_s \rangle$ for each $n \geq k$. From this it follows that the set

$$Y \setminus \bigcup \{A_n \cap U_0 : n < k\}$$

is a cofinite subset of Y contained in V. Thus, Y is a nontrivial convergent sequence in X. This contradiction implies $\mathcal{C}$ is finite.

The other implication follows from the fact that X is homeomorphic to $\mathcal{F}_1(X) \subset \mathcal{K}(X)$ (Lemma 1.2). □

We end the discussion by showing that weak P-spaces are not the only ones in which the equality compact=finite holds.

Example 3.14 A non weak P-space where all compact subsets are finite.

Let $X = \omega \cup P$, where P is the set of weak P-points of ω^*. It is a famous result of Kunen ([100]) that P is a dense subset of ω^* of cardinality $2^{\mathfrak{c}}$. We claim that $\mathcal{K}(X) = \mathcal{F}(X)$. Every infinite compact space contains a separable compact subspace, so it is sufficient to show that the closure of every infinite countable subset $N \subset X$ is not compact. Since P is a weak P-space closed in

X, $\mathrm{cl}_X(N \cap P) = \mathrm{cl}_P(N \cap P) = N \cap P$ that is compact if and only if it is finite. Thus, we may assume $N \subset \omega$. Since $\omega^* \setminus P$ is also dense in ω^*, $\mathrm{cl}_{\beta\omega}(N) \setminus X \neq \emptyset$. It easily follows that $\mathrm{cl}_X(N)$ is not compact. Notice that X is not a weak P-space because N is dense in X. □

Observe that the space X from Example 3.14 is extremally disconnected because it is a dense subspace of $\beta\omega$ (Proposition 6.30 and Corollary 6.35). We now present an example of a space whose compact subspaces are finite but it is not an F'-space.

Example 3.15 A non F'-space where all compact subsets are finite.

Let $\omega = \bigcup\{A_n : n < \omega\}$ be a partition in infinite subsets. Let $\mathcal{F}_\omega$ be the Fréchet filter (or any filter that contains it) and

$$\mathcal{F} = \{B \subset \omega : \{n < \omega : A_n \setminus B \text{ is finite}\} \in \mathcal{F}_\omega\}.$$

Define the space $X = \omega \cup \{\mathcal{F}\}$ where every point of ω is isolated and the neighborhoods of $\mathcal{F}$ are of the form $\{\mathcal{F}\} \cup A$ with $A \in \mathcal{F}$.

Any infinite compact subspace of X must be a convergent sequence. Let $S \subset \omega$ be infinite. If there exists $m < \omega$ such that $S \cap A_m$ is infinite, let $R = \omega \setminus A_m$. If for each $n < \omega$, $|S \cap A_n| < \omega$ holds, let $R = \omega \setminus S$. In both cases $R \in \mathcal{F}$ and $S \setminus R$ is infinite, so S cannot converge to $\mathcal{F}$.

Also, notice that X is an F'-space if and only if it is extremally disconnected because X has countable cellularity (use Lemma 2.29) and it is easy to see this happens if and only if $\mathcal{F}$ is an ultrafilter. To see $\mathcal{F}$ is not an ultrafilter, for each $n < \omega$, let $A_n = P_n \cup Q_n$ be a partition in infinite subsets. Then $P = \bigcup\{P_n : n < \omega\} \notin \mathcal{F}$, $Q = \bigcup\{Q_n : n < \omega\} \notin \mathcal{F}$ and $\omega = P \cup Q$. However, by maximality an ultrafilter must contain either P or Q. Thus, $\mathcal{F}$ is not an ultrafilter.

Thus, X is a space in which all compact subsets are finite but it is not an F'-space. □

Chapter 4

Hereditarily Disconnected Spaces

In this chapter we will give the main results of Part I. Recall that Theorem 2.7 gave characterizations of 0-dimensionality and total disconnectedness in hyperspaces. In [88, 80.5], Alejandro Illanes and Sam B. Nadler asked whether $\mathcal{K}(X)$ is hereditary disconnected when X is hereditarily disconnected and metrizable. In [133], Elżbieta Pol and Roman Pol answered this question in the negative and gave some other remarks. Our first results generalize some of the results of [133].

We remark that some of the examples in Chapter 2 will be used. Moreover, the arguments missing in those examples can be given in analogous ways to arguments we will provide here.

Our first result gives a method to locate connected sets in a hyperspace.

Lemma 4.1 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space. Assume there is a $K \in \mathcal{K}(X)$ such that for every $U \in \mathcal{CO}(X)$ with $K \subset U$ we have $X = U$. Then

$$\mathcal{C} = \{K \cup \{x\} : x \in X\}$$

is a connected subset of $\mathcal{K}(X)$.

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be open subsets of $\mathcal{K}(X)$ such that $K \in \mathcal{U}$, $\mathcal{C} \subset \mathcal{U} \cup \mathcal{V}$ and $\mathcal{C} \cap \mathcal{U} \cap \mathcal{V} = \emptyset$. Let $U = \{x \in X : K \cup \{x\} \in \mathcal{U}\}$ and $V = X \setminus U$. Clearly, $K \subset U$, we now prove that U is clopen.

First, we prove every point $x \in U$ is in the interior of U, we have two cases. If $x \in K$, let $n < \omega$ and $U_0, \ldots, U_n$ be open subsets of X such that

$$K \in \langle U_0 \ldots, U_n \rangle \subset \mathcal{U}.$$

Notice that $x \in K \subset U_0 \cup \ldots \cup U_n \subset U$. If $x \notin K$, let $V_0, \ldots, V_m, W$ be open subsets of X such that $K \subset V_0 \cup \ldots \cup V_m$, $x \in W$, $W \cap (V_0 \cup \ldots \cup V_m) = \emptyset$ and $K \cup \{x\} \in \langle V_0, \ldots, V_m, W \rangle \subset \mathcal{U}$. Then, $x \in W \subset U$.

Now let $x \in V$, then $K \cup \{x\} \in \mathcal{C} \setminus \mathcal{U} \subset \mathcal{V}$. Let $V_0, \ldots, V_m, W$ be open subsets of X such that $K \subset V_0 \cup \ldots \cup V_m$, $x \in W$, $W \cap (V_0 \cup \ldots \cup V_m) = \emptyset$ and $K \cup \{x\} \in \langle V_0, \ldots, V_m, W \rangle \subset \mathcal{V}$. Then $x \in W \subset V$. This proves V is open and thus, U is closed.

Therefore, U is clopen and contains K so by hypothesis $U = X$. But this implies that $\mathcal{C} \subset \mathcal{U}$. Then $\mathcal{C}$ is connected. □

Using Lemma 4.1, we give a modification of Example 1.1 of [133] showing there was no need to add a Cantor set to the original space.

Example 4.2 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) The hyperspace of the Knaster-Kuratowski fan.

Let $\mathbf{F}$ be the space from Example 2.5. Recall that $\mathbf{F}$ is a hereditarily disconnected space that is not totally disconnected. Let $\pi : \mathbf{F} \to C$ be the projection to the first coordinate (in the plane). We now prove:

Claim 1. There is a compact $G \subset \mathbf{F}$ such that if $c \in C$, $|\pi^{\leftarrow}(c) \cap G| = 1$.

To prove Claim 1, let $D = Q \cup [\mathbb{Q} \cap (I \setminus C)]$ which is a countable dense subset of $[0,1]$. It is a well-known fact that there is a homeomorphism $h : [0,1] \to [0, \frac{1}{2}]$ such that $f[D] = \mathbb{Q} \cap [0, \frac{1}{2}]$ (see Theorem 10.2 in Part III). Let $G = f\restriction_C \subset C \times [0, \frac{1}{2}]$, the graph of the function f restricted to the Cantor set. Claim 1 follows.

Claim 2. Let A, B closed sets of the plane such that $A \cap B \cap \mathbf{F} = \emptyset$, $G \subset A$ and $\mathbf{F} \subset A \cup B$. Then $\mathbf{F} \cap B = \emptyset$.

To prove Claim 2, let $\mathbb{Q} \cap [0,1) = \{q_n : n < \omega\}$ be an enumeration. For each $n < \omega$, let $P_n = C \times \{q_n\}$ and $K_n = \pi[A \cap B \cap P_n]$. Notice that K_n is a compact subset of P because $A \cap B \cap \mathbf{F} = \emptyset$ and $\mathbf{F} \cap P_n = Q \times \{q_n\}$.

Moreover, K_n is nowhere dense in P. To see this, assume W is a nonempty regular open subset of C with $W \cap P \subset K_n$. We have $\mathrm{cl}_C(W \cap P) = \mathrm{cl}_C(W)$ because P is dense in C. Let $x \in W \cap Q$, then $x \in W \subset \mathrm{cl}_C(W \cap P) \subset K_n$. So (x, q_n) is a point of F whose first coordinate is in K_n, this implies $(x, q_n) \in A \cap B \cap \mathbf{F}$, a contradiction.

Since P is completely metrizable, it is a Baire space and the set $Z = P \setminus (\bigcup_{n<\omega} K_n)$ is a dense open subset of P. Fix $c \in Z$. Then for each $n < \omega$, $\langle c, q_n \rangle \notin A \cap B$. Since L_c is dense in $\{c\} \times [0,1]$, $\{c\} \times [0,1] \subset A \cup B$. Now,

$\{c\} \times [0,1]$ is connected so either $\{c\} \times [0,1] \subset A$ or $\{c\} \times [0,1] \subset B$. Since $\langle c, f(c)\rangle \in G \subset A$, we necessarily have $\{c\} \times [0,1] \subset A$. But this implies that $\bigcup\{L_c : c \in Z\}$ is a dense subset of $\mathbf{F}$ contained in A. Then $\mathbf{F} \subset A$ so $\mathbf{F} \cap B = \emptyset$. This proves Claim 2.

By Claim 2 and Lemma 4.1, $\mathcal{C} = \{G \cup \{x\} : x \in \mathbf{F}\}$ is a connected subset of $\mathcal{K}(\mathbf{F})$ with more than one point. We have proved that $\mathcal{K}(\mathbf{F})$ is not hereditarily disconnected. $\square$

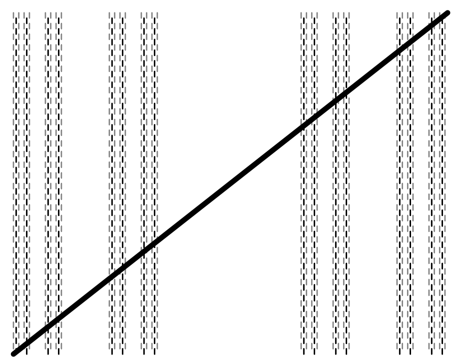

Figure 4.1: Set G in $\mathbf{F}$ from Example 4.2.

Now assume that X is a hereditarily disconnected Hausdorff space that is "almost" totally disconnected. By this, we mean that X is the union of two totally disconnected subsets $X = F \cup Y$. We must ask some other condition on F and Y, because we have examples of connected non-trivial spaces that are the union of two 0-dimensional subspaces (for example, the real line $\mathbb{R}$). We will ask that F is closed. Our Main Theorem 4.6 says that in this case $\mathcal{K}(X)$ is hereditarly disconnected when the quotient space X/F is hereditarily disconnected. Before proving it, we isolate two technical Lemmas we will use often.

Lemma 4.3 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a T_1 space, $T \subset X$ such that

(a) for every $x \in X \setminus T$ there is a $W \in \mathcal{CO}(X)$ such that $x \in W$ and $W \cap T = \emptyset$,

(b) $X \setminus T$ is totally disconnected.

Let $\mathcal{C} \subset \mathcal{K}(X)$ be connected. Then the following holds

($*$) if $Y_1, Y_2 \in \mathcal{C}$, then $Y_1 \setminus T = Y_2 \setminus T$.

Proof. For the sake of producing a contradiction, let us assume $(*)$ does not hold for some $Y_1, Y_2 \in \mathcal{C}$. Let, without loss of generality, $y \in Y_2 \setminus T$ be such that $y \notin Y_1$. For each $x \in Y_1 \setminus T$, let $U_x \in \mathcal{CO}(X)$ such that $x, y \in U_x$ and $U_x \cap T = \emptyset$, this can be done by (a). Since $U_x \subset X \setminus T$ is totally disconnected by (b), let $V_x \in \mathcal{CO}(U_x)$ be such that $x \notin V_x$ and $y \in V_x$. Let $W_x = X \setminus V_x$, observe both $V_x, W_x \in \mathcal{CO}(X)$.

Notice that $T \cup \{x\} \subset W_x$ and $y \notin W_x$. By compactness, there is a finite set $\{x_0, \ldots, x_n\} \subset Y_1 \setminus T$ such that $Y_1 \cup T \subset W_{x_0} \cup \cdots \cup W_{x_n}$. So $W = W_{x_0} \cup \ldots \cup W_{x_n}$ is a clopen subset of X such that $Y_1 \in W^+$ and $Y_2 \notin W^+$. But this contradicts the connectedness of $\mathcal{C}$ so $(*)$ holds. $\square$

Lemma 4.4 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a T_1 space, $T \subset X$ a closed subset and $\emptyset \neq \mathcal{C} \subset \mathcal{K}(X)$ such that

(a) if $Y_1, Y_2 \in \mathcal{C}$, then $Y_1 \setminus T = Y_2 \setminus T$,

(b) if $Y \in \mathcal{C}$, then $Y \cap T \neq \emptyset$.

Define $\Phi : \mathcal{C} \to \mathcal{K}(T)$ by $\Phi(Y) = Y \cap T$. Then Φ is a well-defined, injective and continuous function.

Proof. The function Φ is well-defined by (b) and is injective by (a), we only have to prove the continuity. Let $Y_0 \in \mathcal{C}$. Define $Z = Y_0 \setminus T$. Notice that by (a), $Z = Y \setminus T$ for every $Y \in \mathcal{C}$. If $Z = \emptyset$, Φ is an inclusion that is clearly continuous so assume $Z \neq \emptyset$.

Let $\mathcal{U}$ be an open subset of $\mathcal{K}(T)$ with $\Phi(Y_0) \in \mathcal{U}$. We now prove there is an open subset $\mathcal{V}$ of $\mathcal{K}(X)$ such that $Y_0 \in \mathcal{V}$ and $\Phi[\mathcal{V} \cap \mathcal{C}] \subset \mathcal{U}$. We may assume that $\mathcal{U} = \langle U_1, \ldots, U_n \rangle$ where $U_1, \ldots, U_n$ are nonempty open subsets of T.

Let $V_0 = X \setminus T$. For $1 \leq m \leq n$, let V_m be an open subset of X such that $V_m \cap T = U_m$ and if $U_m \cap \mathrm{cl}_X(Z) = \emptyset$, then also $V_m \cap \mathrm{cl}_X(Z) = \emptyset$. Let $\mathcal{V} = \langle V_0, V_1, \ldots, V_n \rangle$, clearly $Y_0 \in \mathcal{V}$.

Let $Y \in \mathcal{V} \cap \mathcal{C}$. First, if $y \in \Phi(Y)$, then $y \in V_m \cap T$ for some $1 \leq m \leq n$. Thus, $\Phi(Y) \subset U_1 \cup \ldots U_n$. Now, let $1 \leq m \leq n$. If there is a point $y \in U_m \cap \mathrm{cl}_X(Z) \neq \emptyset$, then since $\mathrm{cl}_X(Z) \subset Y$, $y \in U_m \cap \Phi(Y)$. If $U_m \cap \mathrm{cl}_X(Z) = \emptyset$, let $y \in Y \cap V_m$ so that $y \in U_m \cap \Phi(Y)$. In both cases, $U_m \cap \Phi(Y) \neq \emptyset$. This shows $\Phi(Y) \in \mathcal{U}$ and completes the proof. $\square$

The Main Theorem will be proved in two steps. The first step is to add just one point to a totally disconnected space.

Proposition 4.5 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff hereditarily disconnected space and $p \in X$ be such that $X \setminus \{p\}$ is totally disconnected. Then $\mathcal{K}(X)$ is hereditarily disconnected.

Proof. Start with a connected subset $\mathcal{C} \subset \mathcal{K}(X)$. By considering iterated quasi-components, we shall prove that $|\mathcal{C}| = 1$.

For each ordinal α, let $T_\alpha = \mathcal{Q}^\alpha(X, p)$ and $\Gamma = \mathfrak{nc}(X, p)$. Notice that $\{T_\alpha : \alpha < \Gamma\}$ is a strictly decreasing family of closed subsets of X that contain p and $T_\Gamma = \{p\}$. We prove the following two properties by transfinite induction on α:

$(*)_\alpha$ If $Y_1, Y_2 \in \mathcal{C}$, then $Y_1 \setminus T_\alpha = Y_2 \setminus T_\alpha$.

$(\star)_\alpha$ If there exists $Y_0 \in \mathcal{C}$ such that $Y_0 \cap T_\alpha = \emptyset$, then $\mathcal{C} = \{Y_0\}$.

To prove $(*)_0$, just apply Lemma 4.3 to the pair of spaces $T_0 \subset X$. Now, let Y_0 as in $(\star)_0$, so one can find $W \in \mathcal{CO}(X)$ such that $Y_0 \subset W$ and $T_0 \cap W = \emptyset$. But then W^+ is a clopen set so $Y \in W^+$ for all $Y \in \mathcal{C}$. By $(*)_0$, we get $(\star)_0$.

Now assume $(*)_\gamma$ and $(\star)_\gamma$ for every $\gamma \leq \beta$. We now prove $(*)_{\beta+1}$ and $(\star)_{\beta+1}$.

We first consider $(*)_{\beta+1}$. If there exists $Y_0 \in \mathcal{C}$ such that $Y_0 \cap T_\beta = \emptyset$, by $(\star)_\beta$, we have $\mathcal{C} = \{Y_0\}$ and $(*)_{\beta+1}$ is clearly true. So assume that every $Y \in \mathcal{C}$ intersects T_β. By Lemma 4.4, the function $\Phi_\beta : \mathcal{C} \to \mathcal{K}(T_\beta)$ defined by $\Phi_\beta(Y) = Y \cap T_\beta$ is continuous and injective. Let $\mathcal{C}_\beta = \Phi_\beta[\mathcal{C}]$. Using Lemma 4.3 for the pair of spaces $T_{\beta+1} \subset T_\beta$ and the connected subset $\mathcal{C}_\beta$ we get for every $Y_1, Y_2 \in \mathcal{C}$, $(Y_1 \cap T_\beta) \setminus T_{\beta+1} = (Y_2 \cap T_\beta) \setminus T_{\beta+1}$. By $(*)_\beta$, this implies $(*)_{\beta+1}$.

Notice that if there is a $Y_0 \in \mathcal{C}$ such that $Y_0 \cap T_\beta = \emptyset$, then $(\star)_\beta$ implies $(\star)_{\beta+1}$ so assume for every $Y \in \mathcal{C}$, $Y \cap T_\beta \neq \emptyset$. Again we may consider Φ_β and $\mathcal{C}_\beta$ as in the former paragraph. Let $Y_0 \in \mathcal{C}$ such that $Y_0 \cap T_{\beta+1} = \emptyset$. Then one can find $W \in \mathcal{CO}(T_\beta)$ such that $\Phi_\beta[Y_0] \subset W$ and $W \cap T_{\beta+1} = \emptyset$. So W^+ is a clopen set that intersects the connected set $\mathcal{C}_\beta$, therefore, $\Phi_\beta[Y] \in W^+$ for every $Y \in \mathcal{C}$. By $(*)_{\beta+1}$ we conclude $(\star)_{\beta+1}$.

We have left to prove $(*)_\beta$ and $(\star)_\beta$ for a limit ordinal β but these proofs follow from $(*)_\gamma$ and $(\star)_\gamma$ for each $\gamma < \beta$ using that $T_\beta = \bigcap_{\gamma<\beta} T_\gamma$.

Observe that $(*)_\Gamma$ means that if $Y_1, Y_2 \in \mathcal{C}$, then $Y_1 \setminus \{p\} = Y_2 \setminus \{p\}$. By $(\star)_\Gamma$ it easily follows that $|\mathcal{C}| = 1$. So $\mathcal{K}(X)$ is hereditarily disconnected. $\square$

We now procede to prove the main result.

Theorem 4.6 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space. Assume that there is a closed subset $F \subset X$ such that

(a) both F and $X \setminus F$ are totally disconnected,

(b) the quotient X/F is hereditarily disconnected.

Then $\mathcal{K}(X)$ is hereditarily disconnected.

Proof. Let $\mathcal{C} \subset \mathcal{K}(X)$ be a connected subset. Denote by $\pi : X \to X/F$ the quotient map and denote by $\widetilde{F}$ the unique point in $\pi[F]$. Let $\mathcal{D} = \{\pi[C] : C \in \mathcal{C}\}$, this set is connected because $\mathcal{D} = \pi^*[\mathcal{C}]$ where $\pi^* : \mathcal{K}(X) \to \mathcal{K}(X/F)$ is the continuous function defined in Lemma 1.6. Using Proposition 4.5 for $\widetilde{F} \in X/F$ it follows that $\mathcal{D} = \{T\}$ for some $T \in \mathcal{K}(X/F)$. If $\widetilde{F} \notin T$, since π is inyective in $X \setminus F$, $|\mathcal{C}| = 1$. If $\widetilde{F} \in T$, then $Y \cap F \neq \emptyset$ for every $Y \in \mathcal{C}$. Thus, by Lemma 4.4, the function $\Phi : \mathcal{C} \to \mathcal{K}(F)$ given by $\Phi(Y) = Y \cap F$ is continuous and injective. But F is totally disconnected, so by Theorem 2.7, $\mathcal{K}(F)$ is totally disconnected. Thus, $|\mathcal{C}| = |\mathcal{D}| = 1$. □

A natural question here is if the converse to the Main Theorem 4.6 is true. That is, assume $X = Y \cup F$ where both Y, F are totally disconnected, F is closed and $\mathcal{K}(X)$ is hereditarily disconnected, is it true that the quotient X/F must also be hereditarily disconnected? When F is compact, the answer is in the affirmative (Corollary 4.7) but it may not be in general (Case 2 of Example 4.12).

Corollary 4.7 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space. Assume $X = Y \cup T$ where both Y and T are totally disconnected and T is compact. Then $\mathcal{K}(X)$ is hereditarily disconnected if and only if the quotient space X/T is hereditarily disconnected.

Proof. Let $\pi : X \to X/T$ be the quotient and $\widetilde{T}$ the unique point in $\pi[T]$. If X/T is hereditarily disconnected, then $\mathcal{K}(X)$ is hereditarily disconnected by the Main Theorem. If X/T is not hereditarily disconnected, let $R \subset X/T$ be a connected subset with more than one point. Clearly $\widetilde{T} \in R$. Let $F = \pi^{\leftarrow}[R]$, notice $T \subset F$. Define $\mathcal{C} = \{T \cup \{x\} : x \in F\}$ which is connected by Lemma 4.1. Moreover, $|\mathcal{C}| > 1$ because $R \neq \{\widetilde{T}\}$. □

So Corollary 4.7 contains a converse of the statement of the Main Theorem for the case that T is a compact space. In Example 4.12 we present two examples related to the Main Theorem. However, before presenting this, we will present some other results.

First, let us prove that if $\mathcal{K}(X)$ has a connected subset with more than one point, then it must also contain a cannonical one in some sense.

Proposition 4.8 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff hereditarily disconnected space. If $\mathcal{C} \subset \mathcal{K}(X)$ is a connected set with more than one point and $K \in \mathcal{C}$, then there is a closed subset $F \subset X$ with $K \subsetneq F$ such that the set $\mathcal{D} = \{K \cup \{x\} : x \in F\}$ is connected and $|\mathcal{D}| > 1$.

Proof. Consider the set

$$\mathcal{Z} = \{Z \subset X : Z \text{ is closed and for every } Y \in \mathcal{C}, Y \subset Z\}.$$

By the Kuratowski-Zorn lema, there exists a $\subset$-minimal element $F \in \mathcal{Z}$. Notice $K \subset F$. Let $\mathcal{D} = \{K \cup \{x\} : x \in F\}$.

Assume $K = F$. Since K is compact, it is zero-dimensional (Proposition 2.12) and $\mathcal{K}(K)$ is also zero-dimensional (Theorem 2.7). Then $\mathcal{C}$ is a connected subset of $\mathcal{K}(K)$, this implies $|\mathcal{C}| = 1$. This is a contradiction so we have $K \subsetneq F$, which implies $|\mathcal{D}| > 1$.

Let $q : F \to \mathcal{Q}(F)$ be the quotient map onto the space of quasicomponents of F. Consider the continuous function $q^* : \mathcal{K}(F) \to \mathcal{K}(\mathcal{Q}(F))$ from Lemma 1.6. Since $\mathcal{K}(\mathcal{Q}(F))$ is totally disconnected (Lemma 2.18 and Theorem 2.7), $q^*[\mathcal{C}] = \{T\}$ for some compact $T \subset \mathcal{Q}(F)$. Then $G = q^{\leftarrow}[T]$ is such that $G \subset F$ and $\mathcal{C} \subset \mathcal{K}(G)$. By minimality of F, $F = G$. Thus, $q[K] = q^*(K) = T = q[F] = \mathcal{Q}(F)$ so K intersects every quasicomponent of F. From this and Lema 4.1 it easily follows that $\mathcal{D}$ is a connected subset of $\mathcal{K}(X)$. $\square$

We generalize the "countable" in Theorem 1.3 of [133] to "scattered". We start with a useful remark that will help with the proof.

Remark 4.9 If F is hereditarily disconnected and $K \subset F$ is a compact subset such that $\{K \cup \{x\} : x \in F\}$ is connected, then K intersects every quasicomponent of F.

Theorem 4.10 Let X be a Hausdorff hereditarily disconnected space. If $\mathcal{C} \subset \mathcal{K}(X)$ is connected and there exists $T \in \mathcal{C}$ that is scattered, then $|\mathcal{C}| = 1$.

Proof. Assume that $\mathcal{C} \subset \mathcal{K}(X)$ is connected and $|\mathcal{C}| > 1$. By Proposition 4.8, we may assume $\mathcal{C} = \{T \cup \{x\} : x \in F\}$ for some closed subset $F \subset X$ such that $T \subset F$.

We now define a descending transfinite sequence of closed sets F_α (α an ordinal) in the following way. We first take $F_0 = F$. Assume we have already defined F_α. Let $q_\alpha : F_\alpha \to \mathcal{Q}(F_\alpha)$ be the quotient map and let $U_\alpha \subset \mathcal{Q}(F_\alpha)$ be the set of isolated points of $\mathcal{Q}(F_\alpha)$. Define $F_{\alpha+1} = F_\alpha \setminus q_\alpha^{\leftarrow}[U_\alpha]$. Finally, if β is a limit ordinal, let $F_\beta = \bigcap_{\alpha<\beta} F_\alpha$.

We also define for each ordinal α, $T_\alpha = F_\alpha \cap T$ (so that $T_0 = T$) and

$$\mathcal{C}_\alpha = \{T_\alpha \cup \{x\} : x \in F_\alpha\}.$$

By transfinite induction on α we shall prove the following properties

$(0)_\alpha$ If for each $\beta < \alpha$ we have $F_\beta \neq \emptyset$, then for each $\beta < \alpha$, $F_\alpha \subsetneq F_\beta$.

$(1)_\alpha$ (a) For every $Y_1, Y_2 \in \mathcal{C}$, $Y_1 \setminus F_\alpha = Y_2 \setminus F_\alpha$,
(b) For each $Y \in \mathcal{C}$, $T_\alpha \subset Y$,
(c) If $F_\alpha \neq \emptyset$, the function $\Phi_\alpha : \mathcal{C} \to \mathcal{K}(F_\alpha)$ given by $\Phi_\alpha(Y) = Y \cap F_\alpha$ is well-defined, continuous and injective. Moreover, $\mathcal{C}_\alpha = \Phi_\alpha[\mathcal{C}]$.

$(2)_\alpha$ $q_\alpha[T_\alpha] = \mathcal{Q}(F_\alpha)$.

First, notice that $(1)_\alpha$ implies $(2)_\alpha$. To see this, observe that $(1c)_\alpha$ implies $\mathcal{C}_\alpha$ is connected. By Remark 4.9, we get $(2)_\alpha$.

Clearly, $(0)_0$ and $(1)_0$ are true. Assume $(0)_\alpha$, $(1)_\alpha$ and $(2)_\alpha$ hold.

Since T_α is a compact Hausdorff scattered space, it must be 0-dimensional (Lemma 2.13) so by $(2)_\alpha$, Lemma 2.8, Lemma 2.13 and Lemma 2.18, $\mathcal{Q}(F_\alpha)$ is a compact 0-dimensional scattered space. Thus, if $F_\alpha \neq \emptyset$, then also $U_\alpha \neq \emptyset$ and since q_α is onto, $F_{\alpha+1} \subsetneq F_\alpha$. From this $(0)_{\alpha+1}$ follows.

Observe that for each $x \in U_\alpha$, $q_\alpha^{\leftarrow}(x)$ is a clopen quasicomponent of F_α, so it must be an isolated point $\{y\}$. By $(2)_\alpha$, $y \in T_\alpha$. We have obtained

$$(\star)_\alpha \; q_\alpha^{\leftarrow}[U_\alpha] \subset T_\alpha.$$

So we can write

$$(*)_\alpha \; \mathcal{C}_\alpha = \{T_\alpha \cup \{x\} : x \in F_{\alpha+1}\} \cup \{T_\alpha\}.$$

We now prove $(1)_{\alpha+1}$.

First, let $Y_1, Y_2 \in \mathcal{C}$ and $x \in Y_1 \setminus F_{\alpha+1}$. If $x \notin F_\alpha$, by $(1a)_\alpha$, $x \in Y_2 \setminus F_\alpha \subset Y_2 \setminus F_{\alpha+1}$. If $x \in F_\alpha$, by $(*)_\alpha$, we get $T_\alpha \cup \{x\} = T_\alpha \cup \{y\}$ for some $y \in F_{\alpha+1}$ or $T_\alpha \cup \{x\} = T_\alpha$. Notice $x \neq y$ so it must be that $x \in T_\alpha$. Thus, $x \in T \subset Y_2$. We have obtained that $Y_1 \setminus F_{\alpha+1} \subset Y_2 \setminus F_{\alpha+1}$ and by a similar argument, $Y_2 \setminus F_{\alpha+1} \subset Y_1 \setminus F_{\alpha+1}$. This proves $(1a)_{\alpha+1}$.

Condition $(1b)_{\alpha+1}$ is true because of $(1b)_\alpha$ and the fact that $T_{\alpha+1} \subset T_\alpha$.

Assume $F_{\alpha+1} \neq \emptyset$. Notice that by $(2)_\alpha$, $T_{\alpha+1} = T_\alpha \cap F_{\alpha+1} \neq \emptyset$. Then, $(1b)_{\alpha+1}$ implies that for each $Y \in \mathcal{C}$, $Y \cap F_{\alpha+1} \neq \emptyset$. Using this, $(1a)_{\alpha+1}$ and Lemma 4.4 it can be shown that $\Phi_{\alpha+1}$ is a well-defined, continuous and injective function. By similar arguments and $(1c)_\alpha$, we may define a function $\Psi : \mathcal{C}_\alpha \to \mathcal{K}(F_{\alpha+1})$ by $\Psi(Y) = Y \cap F_{\alpha+1}$ this function is continuous and injective. Moreover, the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\Phi_\alpha"'] \arrow[dr, "\Phi_{\alpha+1}"] & \\ \mathcal{C}_\alpha \arrow[r, "\Psi"'] & \mathcal{K}(F_{\alpha+1}) \end{tikzcd}$$

From equation $(*)_\alpha$, we deduce $\Phi_{\alpha+1}[\mathcal{C}] = \Psi[\mathcal{C}_\alpha] = \mathcal{C}_{\alpha+1}$. This proves $(1c)_{\alpha+1}$.

Now, let us assume $(0)_\alpha$, $(1)_\alpha$ and $(2)_\alpha$ for all $\alpha < \gamma$ for some limit ordinal γ. Assume $F_\alpha \neq \emptyset$ for each $\alpha < \gamma$. Fix $\alpha < \gamma$. From $F_\gamma \subset F_{\alpha+1} \subset F_\alpha$ we see that $F_\gamma \neq F_\alpha$. Otherwise, $F_{\alpha+1} = F_\alpha$, which contradicts $(0)_{\alpha+1}$. Thus, we get $(0)_\gamma$.

From $F_\gamma = \bigcap_{\alpha<\gamma} F_\alpha$, $T_\gamma = \bigcap_{\alpha<\gamma} T_\alpha$ and $(1a)_\alpha, (1b)_\alpha$, one can easily deduce $(1a)_\gamma$ and $(1b)_\gamma$. Assume $F_\gamma \neq \emptyset$. By $(2)_\alpha$, $T_\alpha \neq \emptyset$ for each $\alpha < \gamma$. Then by $(0)_\alpha$, the T_α, with $\alpha < \gamma$, form a strictly descending chain of compact nonempty sets, this implies $T_\gamma = \bigcap_{\alpha<\gamma} T_\alpha \neq \emptyset$. By $(1a)_\gamma$ and $(1b)_\gamma$, we can apply Lemma 4.4 to conclude that Φ_γ is well-defined, continuous and injective. Then, it is easy to see that $\Phi_\gamma[\mathcal{C}] = \mathcal{C}_\gamma$. This proves $(1c)_\gamma$.

This completes the induction. Notice that by $(0)_\alpha$, one can define

$$\Gamma = \min\{\alpha : F_\alpha = \emptyset\}.$$

One can show, using $(2)_\alpha$ and the compactness of the T_α, that $\Gamma = \Lambda + 1$ for some ordinal Λ. Observe that $F_\Gamma = F_\Lambda \setminus q_\Lambda^{\leftarrow}[U_\Lambda]$, so every point of F_Λ is isolated. Then, T_Γ is a discrete compact set and thus finite. By $(2)_\Lambda$, $\mathcal{Q}(F_\Lambda)$ must be finite and since it is a space of quasicomponents, $F_\Lambda = \mathcal{Q}(F_\Lambda)$. Thus, $\mathcal{C}_\Lambda = \{T_\Lambda\}$. But $\mathcal{C}_\Lambda$ is the injective image of $\mathcal{C}$ under Φ_Λ. This contradicts $|C| > 1$. Therefore, $|\mathcal{C}| = 1$. $\square$

It is inmediate that the following holds

Corollary 4.11 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Let X be a Hausdorff space. Then the following are equivalent

(a) X is hereditarily disconnected,

(b) for some (equivalently, for each) $n \in \mathbb{N}$, $\mathcal{F}_n(X)$ is hereditarily disconnected,

(c) $\mathcal{F}(X)$ is hereditarily disconnected.

Finally, we will present the example for the Main Theorem 4.6. The first example (Case 1) is an example of this inverse implication. The second example (Case 2) shows that one cannot obtain an inverse of the statement of the Main

Theorem relaxing the requirement of compactness of T to that of being a closed subset of X.

Example 4.12 (Hernández-Gutiérrez and Tamariz-Mascarúa, [79]) Two examples related to the Main Theorem.

We are going to use a special case of Example 2.3 (specifically, when $s(n) = n+1$ for all $n < \omega$). Let $\pi : {}^{\omega}2 \times [0, \infty] \to {}^{\omega}2$ and $h : {}^{\omega}2 \times [0, \infty] \to [0, \infty]$ be the first and second projections, respectively. For each $i < \omega$, let $\rho_i : {}^{\omega}2 \to 2$ be the projection onto the i-th coordinate. We will say a subset $A \subset {}^{\omega}2 \times [0, \infty]$ is bounded if $\sup h[A] < \infty$ and unbounded if it is not bounded (thus, h denotes the "height"). Let $\phi : {}^{\omega}2 \to [0, \infty]$ be the function

$$\phi(t) = \sum_{m<\omega} \frac{t_m}{m+1}.$$

We will consider the spaces $X = \{x \in {}^{\omega}2 : \phi(x) < \infty\}$ and $X_0 = \{(x, \phi(x)) : x \in X\}$. In [30, p. 600], Dijkstra shows that X_0 is homeomorphic to complete Erdős space. Moreover, this space has the following property

(∇) every nonempty clopen subset of X is unbounded.

We will use the basis of ${}^{\omega}2$ formed by the clopen subsets of the form

$$[a_0, \dots, a_n] = \{x \in {}^{\omega}2 : \rho_m(x) = a_m \text{ for all } m \leq n\}$$

where $\{a_0, \dots, a_n\} \subset \{0, 1\}$.

Observe that both X and ${}^{\omega}2 \setminus X$ are dense: for every open set of the form $[a_0, \dots, a_n]$ we may choose $x, y \in [a_0, \dots, a_n]$ such that $x_m = 0 = 1 - y_m$ for each $m > n$; then $x \in X$ and $y \in {}^{\omega}2 \setminus X$.

For each $K \subset {}^{\omega}2$ we define $K_0 = K \times \{\infty\}$ and $Y = X_0 \cup K_0$. Notice that since $\pi{\restriction}_Y$ is $\leq$2-to-1 and $\pi[Y] \subset {}^{\omega}2$ is 0-dimensional, then Y is hereditarily disconnected. By a similar argument, X_0 and K_0 are totally disconnected. We now analyze whether $\mathcal{K}(Y)$ is hereditarily disconnected for two specific examples of K.

<u>Case 1</u>. $K = {}^{\omega}2$.

First, Y/K_0 is connected: if $U \in \mathcal{CO}(Y)$ is such that $K_0 \subset U$, using the compactness of K we get that $X \setminus U$ is bounded, so $Y = U$ by (∇). By Corollary 4.7, $\mathcal{K}(Y)$ is not hereditarily disconnected. In this case, Y/K_0 is homeomorphic to the space of Example 1.4.8 of [51]. By [30, p. 600], Y/K_0 is also homeomorphic to the set of non-ordinary points of the Lelek fan.

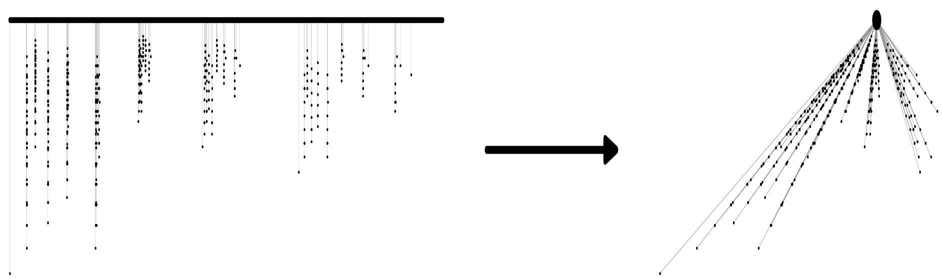

Figure 4.2: In example 4.12, when the set K is compact, if we take the quotient we obtain the endpoints and the appex of the Lelek fan. The image of K is the appex of the Lelek fan.

Case 2. $K = X$

First, we see that Y/K_0 is connected. Observe that since K_0 is not compact, Y/K_0 is not the same quotient as in Case 1 (it is not even first countable at the image of K_0). If $U \in \mathcal{CO}(Y)$ is such that $K_0 \subset U$ and $(x, \infty) \in K_0$, there exists $W \in \mathcal{CO}({}^{\omega}2)$ and $t \in [0, \infty)$ such that $(x, \infty) \in W \times (t, \infty] \subset U$. Thus, $V = (W \times [0, \infty]) \setminus U$ is a bounded clopen subset. By (∇), $V = \emptyset$. Since (x, ∞) was arbitrary, we get $U = Y$. Thus, Y/K_0 is connected. However, we cannot use Corollary 4.7 because K_0 is not compact. In fact, we will show that $\mathcal{K}(Y)$ is hereditarily disconnected. Observe that the proof that Y/K_0 is connected can be modified to show that Y is not totally disconnected, so it is not obvious that $\mathcal{K}(Y)$ is hereditarily disconnected.

A first attempt to prove that $\mathcal{K}(Y)$ is hereditarily disconnected could be showing that any compact subset of Y is scattered and use Theorem 4.10. However, this is false. Recall that $\sum_{m=0}^{\infty} \frac{1}{m+1} = \frac{\pi^2}{6} < \infty$, then the subset

$$P = \{(x, t) \in Y : \text{for each square-free } n \in \mathbb{N}, \rho_{n-1}(x) = 0\}$$

is bounded. As it is pointed out in [30, p. 600], P is homeomorphic to the Cantor set. Fortunately, every compact subset of Y will be "almost everywhere bounded" in the sense of $(0)'_\alpha$ below. We will follow the technique of Theorem 4.10 to prove that $\mathcal{K}(Y)$ is hereditarily disconnected.

Assume that $\mathcal{K}(Y)$ contains a connected subset $\mathcal{C}$ with more than one point. We may assume by Proposition 4.8 that $\mathcal{C} = \{T \cup \{x\} : x \in F\}$ for some closed $F \subset Y$ and some compact $T \subsetneq F$. We now construct a decreasing sequence of closed subsets $F_\alpha \subset Y$ for each ordinal α. Start with $F_0 = F$. If F_α has already been defined, let

$$U_\alpha = \{x \in F_\alpha : \text{there is an open subset } U \subset {}^\omega 2 \text{ and } r \in (0,\infty) \text{ with } \pi(x) \in U \text{ such that if } y \in F_\alpha \cap \pi^{\leftarrow}[U], \text{ then } h(y) < r \text{ or } h(y) = \infty\}.$$

Notice that U_α is open in F_α. Moreover, if $x \in F_\alpha$ and U is like in the definition above for x, then $F_\alpha \cap \pi^{\leftarrow}[U] \subset U_\alpha$. Thus

> $(\star)_\alpha$ Let $x, y \in F_\alpha$ be such that $\pi(x) = \pi(y)$. Then $x \in U_\alpha$ if and only if $y \in U_\alpha$.

So let $F_{\alpha+1} = F_\alpha \setminus U_\alpha$, which is closed. Finally, if β is a limit ordinal, let $F_\beta = \bigcap_{\alpha<\beta} F_\alpha$. We also define for each ordinal α, $T_\alpha = T \cap F_\alpha$ and $\mathcal{C}_\alpha = \{T_\alpha \cup \{x\} : x \in F_\alpha\}$.

We now prove the following properties by transfinite induction.

$(1)_\alpha$ (*a*) For every $Y \in \mathcal{C}$, $T_\alpha \subset Y$,
(*b*) If $Y_1, Y_2 \in \mathcal{C}$, $Y_1 \setminus F_\alpha = Y_2 \setminus F_\alpha$.
(*c*) If $F_\alpha \neq \emptyset$, the function $\Phi_\alpha : \mathcal{C} \to \mathcal{K}(F_\alpha)$ given by $\Phi(Y) = Y \cap F_\alpha$ is well-defined, continuous and injective. Moreover, $\mathcal{C}_\alpha = \Phi_\alpha[\mathcal{C}]$.

$(2)_\alpha$ For each $x \in X$, $F_\alpha \cap \pi^{\leftarrow}(x) \neq \emptyset$ implies $T_\alpha \cap \pi^{\leftarrow}(x) \neq \emptyset$.

$(3)_\alpha$ If $x \in \pi[U_\alpha]$, $F_\alpha \cap \pi^{\leftarrow}(x) = T_\alpha \cap \pi^{\leftarrow}(x)$.

We will procede in the following fashion:

- Step 1: $(1)_0$ is true.
- Step 2: $(1c)_\alpha$ implies $(2)_\alpha$ and $(3)_\alpha$ for each ordinal α.
- Step 3: $(1)_\alpha$ implies $(1)_{\alpha+1}$ for each ordinal α.
- Step 4: If β is a limit ordinal, $(1)_\alpha$ for each $\alpha < \beta$ implies $(1)_\beta$.

This proof is very similar to that of Theorem 4.10, so we will omit some arguments when they follow in a similar way. Step 1 is clear, observe that Φ_0 is the identity function.

Proof of Step 2: Notice that if $F_\alpha = \emptyset$, $(2)_\alpha$ and $(3)_\alpha$ are true, so we may assume $F_\alpha \neq \emptyset$. Thus, $(1c)_\alpha$ implies $\mathcal{C}_\alpha$ is connected.

First, we prove $(2)_\alpha$. Let $x \in X$ and

$$(\bullet)\ Y \cap \pi^{\leftarrow}(x) = \{\langle x, t_0\rangle, \langle x, t_1\rangle\}.$$

Aiming towards a contradiction, assume $\langle x, t_0 \rangle \in F_\alpha$ and $T_\alpha \cap \pi^{\leftarrow}(x) = \emptyset$. Since x is not in the compact set $\pi[T_\alpha]$, there is a $W \in \mathcal{CO}({}^\omega 2)$ such that $x \in W$ and $W \cap \pi[T_\alpha] = \emptyset$. Since $T_\alpha \cup \{\langle x, t_0\rangle\} \in (\pi^{\leftarrow}[W])^-$ and $T_\alpha \notin (\pi^{\leftarrow}[W])^-$, the clopen set $(\pi^{\leftarrow}[W])^-$ separates $\mathcal{C}_\alpha$. This contradiction shows that $(2)_\alpha$ holds.

Next, we prove $(3)_\alpha$. Let $x \in \pi[U_\alpha]$ be such that $\pi^{\leftarrow}(x) \cap F_\alpha \neq \emptyset$. Let us use equation $(\bullet)$ above. By $(2)_\alpha$ and the fact that $T_\alpha \subset F_\alpha$, we only have to show that the case when $\pi^{\leftarrow}(x) \cap F_\alpha = \pi^{\leftarrow}(x)$ and $\pi^{\leftarrow}(x) \cap T_\alpha = \{\langle x, t_0\rangle\}$ is impossible. We will analyze when $t_0 < \infty$, the other possibility being similar.

Since $x \in \pi[U_\alpha]$, there is an open subset $U \subset {}^\omega 2$ and $r \in (0, \infty)$ such that if $y \in F_\alpha$ and $\pi(y) \in U$, then $h(y) \notin [r, \infty)$. Since $T_\alpha \cap ({}^\omega 2 \times [0, r]) = R$ is a nonempty compact set and $x \notin \pi[R]$, there exists $W_0 \in \mathcal{CO}({}^\omega 2)$ such that $x \in W_0$ and $W_0 \cap \pi[R] = \emptyset$. We may assume that $W_0 \subset U$. Let

$$W_1 = (W_0 \times [0, r]) \cap F_\alpha = (W_0 \times [0, r)) \cap F_\alpha$$

which is a clopen subset of F_α. Further, $T_\alpha \cup \{\langle x, t_0\rangle\} \in W_1^-$ and $T_\alpha \notin W_1^-$, this gives a separation of $\mathcal{C}_\alpha$. This is a contradiction so $(3)_\alpha$ follows.

Proof of Step 3: Assume $(1)_\alpha$. By Step 2, $(2)_\alpha$ and $(3)_\alpha$ hold. We may also assume that $F_{\alpha+1} \neq \emptyset$, otherwise $(1)_{\alpha+1}$ is clearly true. First we prove that

$$(*)_\alpha \; \mathcal{C}_\alpha = \{T_\alpha \cup \{x\} : x \in F_{\alpha+1}\} \cup \{T_\alpha\}.$$

The right side of $(*)_\alpha$ is clearly contained in the left side. Let $T_\alpha \cup \{x\} \in \mathcal{C}_\alpha$ with $x \in F_\alpha$. If $x \notin F_{\alpha+1}$, by $(3)_\alpha$, $x \in T_\alpha$. Thus, $T_\alpha \cup \{x\} = T_\alpha$ that is in the right side of $(*)_\alpha$. Thus, $(*)_\alpha$ follows.

We also need that $T_{\alpha+1} \neq \emptyset$. Let $x \in \pi[F_{\alpha+1}]$, by $(2)_\alpha$ there are $x_1, x_2 \in F_\alpha \cap \pi^{\leftarrow}(x)$ such that $x_1 \in F_{\alpha+1}$ and $x_2 \in T_\alpha$. By $(\star)_\alpha$, $x_2 \in T_{\alpha+1}$.

The remaining part of the argument is similar to that of Theorem 4.10, in the part where it is shown that $(1)_{\alpha+1}$ is a consequence of $(0)_\alpha$, $(1)_\alpha$ and $(2)_\alpha$.

The proof of Step 4 is also similar to the part of Theorem 4.10 where it is shown $(1)_\beta$ is the consequence of $(0)_\alpha$,$(1)_\alpha$ and $(2)_\alpha$ for all $\alpha < \beta$ when β is a limit ordinal so we ommit it. This completes the induction.

The key to this example is the following statement:

$(0)_\alpha$ If $F_\alpha \neq \emptyset$, then $U_\alpha \neq \emptyset$.

We shall use the technique Erdős used for for the proof of (∇) (for the original Erdős space, see [52]) to prove $(0)_\alpha$. Assume $F_\alpha \neq \emptyset$ but $U_\alpha = \emptyset$ for some α. We now use induction to find elements $\{x_n : n < \omega\} \subset F_\alpha$, a strictly increasing sequence $\{s_n : n < \omega\} \subset \omega$, $y \in {}^\omega 2 \setminus X$ and a decreasing sequence of open subsets $\{U_n : n < \omega\}$. For each $n < \omega$, call $t_n = \pi(x_n)$ and $y_n = \rho_n(y)$. We find all these with the following properties

(i) $t_n \in U_n$,

(ii) for each $m \leq n$ and $r \leq s_n$, $\rho_r(t_m) = y_r$,

(iii) if $m < n$, then $m + h(x_m) < h(x_n) < \infty$,

(iv) if $m < n$, then $m + h(x_m) < \sum_{m=0}^{n+1} \frac{y_m}{m+1} < \infty$,

(v) $U_n = [y_0, \ldots, y_{s_n}]$.

For $n = 0$ define $s_0 = 0$ and choose $x_0 \in F_\alpha$ arbitrarily. Assume that we have the construction up to n. Since $x_n \notin U_\alpha$, there exists $x_{n+1} \in F_\alpha \cap \pi^{\leftarrow}[U_n]$ such that $n + h(x_n) < h(x_{n+1}) < \infty$. Since

$$\sum_{m<\omega} \frac{\rho_m(t_{n+1})}{m+1} = h(x_{n+1}) < \infty,$$

by the convergence of this series, there exists $s_{n+1} > s_n$ such that

$$n + h(x_n) < \sum_{m=0}^{s_{n+1}} \frac{\rho_m(t_{n+1})}{m+1}.$$

Define $y_m = \rho_m(t_{n+1})$ for $m \in \{s_n+1, \ldots, s_{n+1}\}$. Clearly, conditions $(i)-(v)$ hold. Notice that by (iv), $\phi(y) = \infty$ so in fact $y \in {}^\omega 2 \setminus X$.

By (ii), $\{t_n : n < \omega\}$ converges to y. Moreover, by (iii), $\{x_n : n < \omega\}$ converges to $\langle y, \infty \rangle \notin Y$. Since T_α is compact, there exists $N < \omega$ such that for each $N \leq n < \omega$, $x_n \in F_\alpha \setminus T_\alpha$.

Let $z_n = \langle t_n, \infty \rangle$ for each $n < \omega$. If $N \leq n < \omega$ then by $(2)_\alpha$, $z_n \in T_\alpha$. But $\{z_n : N \leq n < \omega\}$ converges to $\langle y, \infty \rangle \notin T_\alpha$, which is a contradiction. Thus, $(0)_\alpha$ follows.

Obseve that one may also use a similar argument to prove:

$(0)'_\alpha$ U_α is dense in F_α.

We are now ready to produce a contradiction to the assumption that $\mathcal{K}(Y)$ is not hereditarily disconnected. By $(0)_\alpha$, we know that if $F_\alpha \neq \emptyset$, then $F_{\alpha+1} \subsetneq F_\alpha$. Thus, there exists

$$\Gamma = \min\{\alpha : F_\alpha = \emptyset\}.$$

By $(2)_\alpha$ and a compactness argument, it can be proved that $\Gamma = \Lambda + 1$ for some Λ. Then $U_\Lambda = F_\Lambda$, by $(3)_\Lambda$ this implies $F_\Lambda = T_\Lambda$. Thus $\mathcal{C}_\Lambda = \{T_\Lambda\}$. But Φ_Λ is an injective function by $(1c)_\Lambda$ so we have a contradiction. This contradiction proves that $\mathcal{K}(Y)$ is hereditarily disconnected. $\square$

Chapter 5

Miscellanea on Hyperspaces

In this chapter we will present observations on four aspects of hyperspaces. The results presentes in this chapter are not strong enough to be published in a research journal. However, the author thinks that they do reserve to be analysed and mentioned in the dissertation due to their interest.

In the last Section, we will give a summary of our problems on hyperspaces that remain open.

5.1 Symmetric products of ω^*

Recall that if X is any Tychonoff space, then $X^* = \beta X \setminus X$ (see Section 6.1). This section's result is motivated by the following result by van Douwen.

Theorem 5.1 [37, Theorem 2.4] Let κ, τ be nonzero cardinals and Z a first countable noncompact realcompact space without isolated points. Then ${}^{\kappa}(Z^*)$ is homeomorphic to ${}^{\tau}(Z^*)$ if and only if $\kappa = \tau$.

The symmetric product is in some sense a reduced topological product (and in fact becomes more complicated). So it is natural to try to extend Theorem 5.1 to symmetric products. As van Douwen himself points out ([37, Remark 6.6]), the difficult and interesting part of Theorem 5.1 is for finite cardinals. To prove it in its most general situation, van Douwen defined *$\beta\omega$-spaces* to be those spaces X such that if N is a countable discrete subset of X such that $\mathrm{cl}_X(N)$ is compact, then $\mathrm{cl}_X(N)$ is homeomorphic to $\beta\omega$. A $\beta\omega$-space in *non-trivial* if it does contain some copy of $\beta\omega$. By Lemma 6.48, ω^* is a non-trivial $\beta\omega$-space (hence the terminology).

Lemma 5.2 [37, 6.3] ("Number of Factors Lemma") Let κ, τ be nonzero cardinals. For each $\alpha < \tau$, let Z_α be a non-trivial $\beta\omega$-space. Then $\kappa \leq \tau$ if and only if ${}^{\kappa}\beta\omega$ can be embedded in $\prod_{\alpha<\tau} Z_\alpha$.

From the Number of Factors Lemma we can easily infer our result.

Theorem 5.3 (Hernández-Gutiérrez) Let $n, m < \omega$. If $\mathcal{F}_n(\omega^*)$ is homeomorphic to $\mathcal{F}_m(\omega^*)$, then $n = m$.

Proof. Let $h : \mathcal{F}_n(\omega^*) \to \mathcal{F}_m(\omega^*)$ be a homeomorphism and consider $\mathcal{U} = \mathcal{F}_n(\omega^*) \setminus \mathcal{F}_{n-1}(\omega^*)$ and $\mathcal{V} = \mathcal{F}_m(\omega^*) \setminus \mathcal{F}_{m-1}(\omega^*)$. Let $V_0, \ldots, V_{m-1}$ be a collection of nonempty pairwise disjoint clopen subsets of ω^*. Notice that $\mathcal{U}$ and $\mathcal{V}$ are dense open subsets in the corresponding spaces. Thus, there are non-empty pairwise disjoint clopen subsets $U_0, \ldots, U_{n-1}$ of ω^* such that

$$h[\langle U_0, \ldots, U_{n-1}\rangle] \subset \langle V_0 \ldots, V_{m-1}\rangle.$$

Notice that $\langle U_0, \ldots, U_{n-1}\rangle$ is homeomorphic to the product $U_0 \times \ldots \times U_{n-1}$ and $\langle V_0, \ldots, V_{n-1}\rangle$ is homeomorphic to $V_0 \times \ldots \times V_{n-1}$. Since each non-empty clopen subset of ω^* is homeomorpich to ω^*, we may apply Lemma 5.2. From this, we obtain that $n \leq m$. But the argument is symmetric so $m = n$. $\square$

5.2 Pseudocompactness in Hyperspaces

Recall that if X is Hausdorff, then $\mathcal{K}(X)$ is compact if and only if X is compact (Theorem 1.12). It is natural to try to weaken the hypothesis on compactness in the hyperspace and try to find similar equivalences. In this section we will explore weakenings of compactness on hyperspaces.

A Hausdorff space X is *countably compact* if every countable infinite subset of X has a limit point and X is *pseudocompact* if every continuous real-valued function $f : X \to \mathbb{R}$ is bounded. These concepts are natural[1] specializations of the notion of compactness. Every countably compact space is pseudocompact ([50, 3.10.20]) and the converse holds for compact spaces ([50, 3.10.21]). However, there are examples of countably compact spaces that are not compact (for example, ω_1 with the order topology, see [50, 3.10.16]) and pseudocompact spaces that are not countably compact (for example, see 5I or 8.20 in [69]).

It is known that countable compactness and pseudocompactness are not productive properties in general (see [50, 3.20.19]). So it is natural to think that

[1]Actually, in the early 20th century, compact spaces were called bicompact and countably compact spaces were called compact, as mentioned in [50, p. 205] and witnessed in numerous papers of that time.

symmetric products and hyperspaces do not behave well under these properties. We will now give a series of results in this direction. For an exposition of most of them, we recommend the reader to see the recent thesis [59] (in Spanish).

Theorem 5.4 [70, Corollary 2.3] Let X be a Hausdorff space. The following are equivalent.

(i) for every cardinal κ, ${}^{\kappa}X$ is countably compact,

(ii) for every $n < \omega$, ${}^{n}X$ is countably compact,

(iii) $CL(X)$ is countably compact.

Theorem 5.5 [97, Theorem 1.3] Let X be a Tychonoff space and $n < \omega$. Then $\mathcal{F}_n(X)$ is pseudocompact if and only if ${}^{n}X$ is pseudocompact.

Theorem 5.6 [70, Corollary 2.7] Let X be a Tychonoff space. If $CL(X)$ is pseudocompact, then ${}^{n}X$ is pseudocompact for all $n < \omega$.

Theorem 5.7 [70, Example 3.1] There exists a Tychonoff space X such that ${}^{n}X$ is countably compact for all $n < \omega$ but $CL(X)$ is not pseudocompact.

Theorem 5.8 [85, Theorem 5.1] There is a subspace $X \subset \beta\omega$ such that ${}^{\omega}X$ is pseudocompact but $CL(X)$ is not pseudocompact.

As the reader has noticed with these results, much work has been done about hyperspace $CL(X)$ and the symmetric products have an easy characterization for their pseudocompactness. However, we lack results about pseudocompactness of hyperspaces $\mathcal{K}(X)$ and $\mathcal{F}(X)$.

A regular space X is *feebly compact* if every locally finite collection of open subsets of X is finite. Feeble compactness is related to pseudocompactness in the following way.

Lemma 5.9 [135, 1.11(d)] Every feebly compact space is pseudocompact. A Tychonoff space is feebly compact if and only if it is pseudocompact.

Proposition 5.10 (Hernández-Gutiérrez) If X is a regular and infinite space, then $\mathcal{F}(X)$ is not feebly compact so it is not pseudocompact.

Proof. Let $\{U_n : n < \omega\}$ be a family of non-empty open subsets of X with pairwise disjoint closures, this is possible since X is regular. Define

$$\mathcal{U}_n = \langle U_0, \ldots, U_n \rangle \cap \mathcal{F}(X),$$

for each $n < \omega$. Notice that $\mathcal{V} = \{\mathcal{U}_n : n < \omega\}$ is a pairwise disjoint family of open subsets of $F(X)$, let us prove that it is locally finite. Let $A \in F(X)$, define $m = \min\{k < \omega : A \cap \mathrm{cl}_X(U_k) = \emptyset\}$, this is possible as A is finite. Let $V = X \setminus \mathrm{cl}_X(U_M)$, this is non-empty open set. Then $A \in V^+$ and for all $k \geq m$, $V^+ \cap \mathcal{U}_k = \emptyset$. This proves that $\mathcal{V}$ is locally finite. Then $\mathcal{F}(X)$ is not feebly compact. □

There exists a characterization of countable compactness in $\mathcal{K}(X)$ by Milovančević. A Tychonoff space X is called *ω-bounded* if every time $N \subset X$ is countable it follows that $\mathrm{cl}_X(N)$ is compact.

Theorem 5.11 [120, Theorem 2.2] Let X be a Tychonoff space. Then the following are equivalent.

(1) Every σ-compact subset of X has compact closure,

(2) $\mathcal{K}(X)$ is countably compact,

(3) $\mathcal{K}(X)$ is ω-bounded,

(4) $\mathcal{K}(\mathcal{K}(X))$ is countably compact.

It is interesting to notice that Theorem 5.11 has been recently generalized to higher cardinals in [7, Theorem 1.6]. See also [129].

Thus, we only have left the question of pseudocompactness for $\mathcal{K}(X)$. It turns out that this question is far from easy and it is still unsolved. The first observation is the following.

Proposition 5.12 (Hernández-Gutiérrez) Let X be a Tychonoff space. If $\mathcal{K}(X)$ is pseudocompact, then X is pseudocompact.

Proof. If X is not pseudocompact, there is a collection of pairwise disjoint non-empty open subsets $\{U_n : n < \omega\}$ such that $\{\mathrm{cl}_X(U_n) : n < \omega\}$ is a discrete family. For each $n < \omega$ we define

$$\mathcal{U}_n = \langle U_0, \ldots, U_n \rangle \cap \mathcal{K}(X).$$

Notice that $\{\mathcal{U}_n : n < \omega\}$ is a pairwise disjoint family of non-empty open subsets of $K(X)$, we will see that it is discrete. Let $C \in \mathcal{K}(X)$ and notice that $\{n < \omega : C \cap \mathrm{cl}_X(U_n) \neq \emptyset\}$ is a finite set by the compactness of C. Let $m < \omega$ be such that $C \cap \mathrm{cl}_X(U_m) = \emptyset$ and consider $V = X \setminus \mathrm{cl}_X(U_m)$. Notice that $C \in V^+$ and if $k \geq m$, $V^+ \cap \mathcal{U}_k = \emptyset$. This proves that $\mathcal{K}(X)$ is not pseudocompact □

The other direction of Proposition 5.12 is false. For an example, consider any pseudocompact space X where $\mathcal{K}(X) = \mathcal{F}(X)$, then use Proposition 5.10. Of course we have to prove that such spaces exist. We will consider an example that has already been mentioned in Example 2.27 and analyze its hyperspace of compact sets and symmetric products.

First we need a definition. Let κ be an infinite cardinal. If $B \in [\kappa]^{\leq\omega}$, then $\pi_B : {}^{\kappa}[0,1] \to {}^{B}[0,1]$ denotes the projection (restriction) to ${}^{B}[0,1]$. For $X \subset {}^{\kappa}[0,1]$, we will say that X is *ω-dense in* ${}^{\kappa}[0,1]$ if for all $B \in [\kappa]^{\leq\omega}$, the projection $\pi_B[X] = {}^{B}[0,1]$.

Example 5.13 [152] Dmitriĭ B. Shakhmatov's space: There exists a pseudocompact, connected, weak P-space Ш ([2]) that is ω-dense in ${}^{\mathfrak{c}}[0,1]$.

Notice that since Ш is a weak P-space, $\mathcal{K}(\text{Ш}) = \mathcal{F}(\text{Ш})$ is not pseudocompact by Proposition 5.10. In our final result of this section, we will show that the symmetric products of Ш have the same properties, see Proposition 5.16. We need some preliminary results.

Theorem 5.14 ("Arhangel'skiĭ's Factorization Theorem") [8, 0.2.3] Let $\mathcal{S}$ be a non-empty set and $\{X_s : s \in \mathcal{S}\}$ a family of second countable Hausdorff spaces. Assume that $Y \subset \prod\{X_s : s \in \mathcal{S}\}$ is a dense subset and $f : Y \to \mathbb{R}$ is a continuous function. Then there is $B \in [A]^{\leq\omega}$ and a continuous function $g : \pi_B[Y] \to \mathbb{R}$ such that $f = g \circ \pi_B$.

Proposition 5.15 Let κ be an infinite cardinal and $X \subset {}^{\kappa}[0,1]$ be a dense subset. Then X is pseudocompact if and only if X is ω-dense in ${}^{\kappa}[0,1]$.

Proof. First assume that X is pseudocompact and let $A \in [\kappa]^{\leq\omega}$. Since pseudocompactness is preserved under continuous images ([50, 3.10.24]), $\pi_A[X]$ is pseudocompact. Since ${}^{A}[0,1]$ is metrizable, $\pi_A[X]$ is compact ([50, 3.10.21, 4.1.17]). Further, since X is dense in ${}^{A}[0,1]$, $\pi_A[X]$ is dense in ${}^{A}[0,1]$. Thus, $\pi_A[X] = {}^{A}[0,1]$.

Now assume the second part and let $f : X \to \mathbb{R}$ be a continuous function. By the Factorization Theorem 5.14, there is $A \in [\kappa]^{\leq\omega}$ and a continuous function $g : \pi_A[X] \to \mathbb{R}$ such that $f = g \circ \pi_A$. Since $\pi_A[X] = {}^{A}[0,1]$, g is bounded. Thus, f is bounded. □

Proposition 5.16 If $n \in \mathbb{N}$, then $\mathcal{F}_n(\text{Ш})$ is a pseudocompact, connected, weak P-space.

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[2]Ш is the initial of Shakhmatov in the Cyrillic alphabet

Proof. Let $n \in \mathbb{N}$. Notice that since Ш is connected, by Theorem 2.6, $\mathcal{F}_n(Ш)$ is connected. Also, $\mathcal{F}_n(X)$ is a weak P-space by Theorem 3.12. By Theorem 5.5, it is enough to prove that ${}^nШ$ is pseudocompact. It is not difficult to prove from the definition of Ш that ${}^nШ$ is a ω-dense subspace of ${}^{n\times\mathfrak{c}}[0,1]$. By Proposition 5.15 we obtain that ${}^nШ$ is pseudocompact. $\square$

5.3 $CL(X)$ for discrete X

In this section, we will give some remarks about the hyperspace $CL(X)$ when X is a discrete space. As promised in Chapter 1, we will first discuss separation axioms for $CL(X)$.

In the following previously known results we will notice a peculiarity of hyperpace $CL(X)$: if we want $CL(X)$ to have some property **P**, we need that X has a property stronger than **P**.

Proposition 5.17 Let X be a Hausdorff space. Then

(a) $CL(X)$ is Hausdorff if and only if X is regular,

(b) $CL(X)$ is regular if and only if $CL(X)$ is Tychonoff if and only if X is normal.

Proof. Let us start with the proof of (a). First, assume that X is regular, let $A, B \in CL(X)$ with $A \neq B$. Choose, without loss of generality, $p \in A \setminus B$ and find two disjoint open subsets U and V such that $p \in U$ and $B \subset V$. Then $A \in U^-$, $B \in V^+$ and $U^- \cap V^+ = \emptyset$. Then $CL(X)$ is Hausdorff.

Now assume that $CL(X)$ is Hausdorff, we shall prove that X is regular so consider a closed set $F \subset X$ and $p \in X \setminus F$. Let $\mathcal{U}$ and $\mathcal{V}$ be disjoint open subsets of $CL(X)$ with $F \in \mathcal{U}$ and $F \cup \{p\} \in \mathcal{V}$. We may assume that $\mathcal{U}$ and $\mathcal{V}$ are Vietoris sets so $\mathcal{U} = \langle U_0, \ldots, U_n \rangle$ and $\mathcal{V} = \langle V_0, \ldots, V_m \rangle$. Let $U = \bigcup\{U_k : k \leq n\}$, then U is an open set that contains F. Let $V = \bigcap\{V_k : k \leq m, p \in V_k\}$ and let $\{W_0, \ldots, W_r\}$ be an enumeration of $\{V_k - \{p\} : k \leq n, F \cap V_k \neq \emptyset\}$. Notice that $K \cup \{p\} \in \langle V, W_0, \ldots, W_r \rangle \subset \mathcal{V}$. We claim that $U \cap V = \emptyset$. Otherwise, let $q \in U \cap V$. Then it can be easily seen that $F \cup \{q\} \in \mathcal{U} \cap \mathcal{V}$, which is a contradiction. Thus, U and V are disjoint open subsets of X that separate F from p so X is regular.

Now we prove (b). Clearly, $CL(X)$ Tychonoff implies $CL(X)$ regular. Now assume that $CL(X)$ is regular and let $A, B \subset X$ be disjoint closed subsets of X. Notice that B^- is a closed subset of $CL(X)$ since $B^- = X \setminus (X \setminus B)^+$. Further, $A \notin B^-$ so by regularity there is an open set $\mathcal{U}$ with $A \in \mathcal{U}$ and $\mathrm{cl}_{CL(X)}(\mathcal{U}) \cap B^- = \emptyset$. We may assume that $\mathcal{U}$ is a Vietoris set $\langle U_0, \ldots, U_n \rangle$. Let $U = U_0 \cup \cdots \cup U_n$, clearly $A \subset U$ so let us prove $B \cap \mathrm{cl}_X(U) = \emptyset$. Assume this is not the case and let $p \in B \cap \mathrm{cl}_X(U)$. Then it is easy to see that $A \cup \{p\} \in B^- \cap \mathrm{cl}_{CL(X)}(\mathcal{U})$, which is a contradiction. This proves that X is normal.

Finally, assume that X is normal and let us prove that $CL(X)$ is Tychonoff. Let $A \in CL(X)$ and $\mathcal{U}$ an open set in $CL(X)$ with $A \in \mathcal{U}$. We may assume that $\mathcal{U}$ is a Vietoris set $\langle U_0, \ldots, U_n \rangle$. We will define $n+2$ functions $F, G_0, \ldots, G_n$ from $CL(X)$ to $[0,1]$ such that if $G = F \cdot G_0 \cdot \ldots \cdot G_n$ is their product, then $G(A) = 0$ and $G(B) = 1$ whenever $B \in CL(X) \setminus \mathcal{U}$. These arguments are similar to those of Lemma 1.7.

First, by normality, let $f : X \to [0,1]$ be a continuous function such that $f[A] \subset \{0\}$ and $X \setminus (U_0 \cup \ldots \cup U_n) \subset f^{\leftarrow}(1)$. Define $F : CL(X) \to [0,1]$ as $F(Y) = \sup\{f(y) : y \in Y\}$ for all $Y \in CL(X)$. It is not hard to see that F is a continuous function, $F(A) = 0$ and $F(B) = 1$ if $B \not\subset U_0 \cup \ldots \cup U_n$.

For each $m \leq n$, let $p_m \in A \cap U_m$. Then there is a continuous function $g_n : X \to [0,1]$ such that $g_m(p_m) = 0$ and $X \setminus U_m \subset g_m^{\leftarrow}(1)$. Define $G_m : CL(X) \to [0,1]$ by $G_m(Y) = \inf\{g(y) : y \in Y\}$ for all $Y \in CL(X)$. Again, it can be proved that G_m is a continuous function, $G_m(A) = 0$ and $G_m(B) = 1$ if $B \cap U_n = \emptyset$.

Thus, by defining the product $G = F \cdot G_0 \cdot \ldots \cdot G_n$ we obtain that $G : CL(X) \to [0,1]$ is a continuous function with $G(A) = 0$ and $G(B) = 1$ whenever $B \in CL(X) \setminus \mathcal{U}$. Also, by (a), $CL(X)$ is T_1. Thus, $CL(X)$ is Tychonoff. $\square$

As shown in Proposition 3.3, $CL(X)$ does not have high disconnectedness properties. However, it is known when it is 0-dimensional. A Tychonoff space X is *strongly* 0-*dimensional* if for every two disjoint zero sets of X can be separated by a clopen set. Of course a strongly 0-dimensional space is 0-dimensional but the converse is not true in general (not even for metrizable spaces, see [131, Chapter 7, Section 4]).

Proposition 5.18 Let X be a Hausdorff space. Then $CL(X)$ is 0-dimensional if and only if X is normal and strongly 0-dimensional.

Proof. First, assume that X is normal and strongly 0-dimensional. Let $A \in CL(X)$ and let $\mathcal{U}$ be an open set of $CL(X)$ such that $A \in \mathcal{U}$. We may assume that $\mathcal{U}$ is a Vietoris set $\langle U_0, \ldots, U_n \rangle$. The following argument is analogous to the second part of the proof of (b) in Proposition 5.17. Since X is normal, A and $X \setminus (U_0 \cup \ldots \cup U_n)$ can be separated by zero sets, so by hypothesis there is $U \in \mathcal{CO}(X)$ with $A \subset U$ and $U \subset U_0 \cup \ldots \cup U_n$. For each $m \leq n$, let $V_m \in \mathcal{CO}(X)$ with $V_m \cap A \neq \emptyset$ and $V_m \subset U_m$. Then $\mathcal{V} = U^+ \cap (V_0^- \cap \ldots \cap V_n^-) \in \mathcal{CO}(CL(X))$, $A \in \mathcal{V}$ and $\mathcal{V} \subset \mathcal{U}$. Thus, $CL(X)$ is 0-dimensional.

Now assume that $CL(X)$ is 0-dimensional. By Proposition 5.17, X is normal. Let A and B be two disjoint zero sets of X. Then $A \in CL(X)$ and B^- is a closed subset of $CL(X)$ with $A \notin B^-$. Thus, there is $\mathcal{U} \in \mathcal{CO}(CL(X))$ such that $A \in \mathcal{U}$ and $\mathcal{U} \cap B^- = \emptyset$. However, we cannot assume that $\mathcal{U}$ is a Vietoris set (as we did in the first part of the proof of (b) in Proposition 5.17) so we need a more elaborated argument to separate A and B. Let

$$U = \{x \in X : A \cup \{x\} \in \mathcal{U}\},$$

clearly $A \subset U$ and $B \cap U = \emptyset$. We will see that $U \in \mathcal{CO}(X)$, this will prove that X is strongly 0-dimensional.

First, let $p \in U$. Since $A \cup \{p\} \in \mathcal{U}$, there is a Vietoris set $\langle U_0, \ldots, U_n \rangle$ such that $A \cup \{p\} \in \langle U_0, \ldots, U_n \rangle \subset \mathcal{U}$. We may assume that $p \in U_0$ so if $q \in U_0$ then $A \cup \{q\} \in \mathcal{U}$. Then $p \in U_0 \subset U$, which proves that U is open.

Finally, let $p \in X \setminus U$. This means that $A \cup \{p\} \in CL(X) \setminus \mathcal{U}$ so by an argument analogous to that in the previous paragraph, there is an open set V such that $p \in V \subset X \setminus U$. So U is closed as well. This proves that $U \in \mathcal{CO}(X)$ separates A from B. $\square$

Concerning separation axioms, we only have normality left. The problem of normality in $CL(X)$ was a difficult one and was finally solved in the following way.

Theorem 5.19 [159] If X is a Hausdorff space and $CL(X)$ is normal, then X is compact.

The original proof of Theorem 5.19 is in Russian. For a proof in Spanish, we recommend the reader to see [59]. As we will be considering only discrete spaces for the rest of this Section, we will give a proof of the fact that the hyperspace of closed sets of a discrete space is not normal.

Proposition 5.20 Let X be an infinite discrete space. Then $CL(X)$ is not normal.

Proof. Let $X = X_0 \cup X_1$ be a partition of X into two sets of cardinality $|X|$. For each $i \in 2$, let $f_i : X \to X_i$ be a bijection. Consider the space $\mathcal{D} = \{f_0[Y] \cup f_1[X \setminus Y] : Y \in CL(X)\} \subset CL(X)$. We shall prove that $\mathcal{D}$ is a closed and discrete subset of $CL(X)$.

First we prove that $\mathcal{D}$ is discrete. Let $Y \in CL(X)$, we claim that $\langle f_0[Y], f_1[X \setminus Y] \rangle \cap \mathcal{D} = f_0[Y] \cup f_1[X \setminus Y]$. So let $Z \in CL(X)$ be such that $f_0[Z] \cup f_1[X \setminus Z] \in \langle f_0[Y], f_1[X \setminus Y] \rangle$. Then $f_0[Z] \subset f_0[Y]$ and $f_1[X \setminus Z] \subset f_1[X \setminus Y]$ which imply $Z \subset Y$ and $X \setminus Z \subset X \setminus Y$, respectively. Thus $Y = Z$.

Now let $Z \in CL(X) \setminus \mathcal{D}$ and define $Z_i = Z \cap X_i$ for $i \in 2$. Since $Z \notin \mathcal{D}$, then either $f_0^{\leftarrow}[Z_0] \cap f_1^{\leftarrow}[Z_1] \neq \emptyset$ or $f_0^{\leftarrow}[Z_0] \cup f_1^{\leftarrow}[Z_1] \neq X$. In the first case, let $p \in f_0^{\leftarrow}[Z_0] \cap f_1^{\leftarrow}[Z_1]$. Then $Z \in \{f_0(p)\}^- \cap \{f_1(p)\}^-$ and $(\{f_0(p)\}^- \cap \{f_1(p)\}^-) \cap \mathcal{D} = \emptyset$. For the second case, let $q \in X \setminus (f_0^{\leftarrow}[Z_0] \cup f_1^{\leftarrow}[Z_1])$. Then $Z \in CL(X) \setminus \{f_0(q), f_1(q)\}^-$ and $\mathcal{D} \subset \{f_0(q), f_1(q)\}^-$.

Thus, $\mathcal{D}$ is a closed and discrete subset of $CL(X)$ of cardinality $2^{|X|}$. Since $\mathcal{F}(X)$ is a dense subset of $CL(X)$ (Lemma 1.4) of cardinality $|X|$, by Jones' Lemma (Theorem 0.12) we obtain that $CL(X)$ is not normal. □

Since we found a discrete subset of size $2^{|X|}$, we have the following.

Corollary 5.21 Let X be an infinite discrete space. Then $CL(X)$ has weight $2^{|X|}$.

Regarding the normality of $\mathcal{K}(X)$, the problem is still open as far as the author of this dissertation knows. We mention this in Question 5.39.

For the rest of the section, we will restrict to the case when X is discrete. So in this case $CL(X)$ is a 0-dimensional and non-normal space. Notice that in this case $CL(X)$ has a dense set of isolated points, as the following result shows.

Lemma 5.22 Let X be discrete. Then $A \in CL(X)$ is an isolated point if and only if $A \in \mathcal{F}(X)$.

Proof. If $A = \{a_0, \ldots, a_n\}$, then $\langle \{a_0\}, \ldots, \{a_n\} \rangle = \{A\}$ so A is isolated. For the other implication, if $A \in CL(X) \setminus \mathcal{F}(X)$, we next show that A is not an isolated point. Consider any Vietoris set neighborhood $A \in \langle U_0, \ldots, U_n \rangle$. Without loss of generality, we may assume that $|A \cap U_0| = \omega$ and take $x \in A \setminus U_0$. Then $A \setminus \{x\} \in \langle U_0, \ldots, U_n \rangle$ and $\emptyset \neq A \setminus \{x\} \neq A$. □

Thus, we can add finitely many points to $CL(X)$ without changing the space (when X is discrete). We will use this fact later so we emphasize it.

Observation 5.23 If X is an infinite discrete space and F is any finite set (with the discrete topology), $CL(X) \oplus F$ is homeomorphic to $CL(X)$.

So $CL(\omega)$ has a dense set of isolated points. So a natural question is how different the other points of $CL(\omega)$ are from each other. More formally, we will show that $CL(\omega) \setminus \mathcal{F}(\omega)$ is homogeneous. In general, we will calculate the *homogeneity degree* of $CL(X)$ for infinite discrete X. Since the homeomorphism type of a discrete space depends only in its cardinality, we will be considering infinite cardinals with the discrete topology.

Lemma 5.24 If κ and τ are infinite cardinals, $CL(\kappa)\times CL(\tau)$ is homeomorphic to $CL(\kappa+\tau)$.

Proof. Let X, Y be disjoint sets with $|X|=\kappa$, $|Y|=\tau$ and p a point such that $p\notin CL(X\cup Y)$. Give X and Y the discrete topology. Define the function

$$h:(CL(X)\oplus\{p\})\times(CL(Y)\oplus\{p\})\to CL(X\cup Y)\oplus\{p\}$$

in the following way:

$$h(\langle A,B\rangle)=\begin{cases} A\cup B, & \text{if } A\neq p, B\neq p,\\ A, & \text{if } A\neq p, B=p,\\ B, & \text{if } A=p, B\neq p,\\ p, & \text{if } A=p, B=p.\end{cases}$$

Notice that h is bijective and the four domains of definition of h are clopen sets of its domain. From this, it is easy to see that h is a homeomorphism. By observation 5.23 we obtain the result. □

Lemma 5.25 If κ and τ are infinite cardinals, then $CL(\kappa)\oplus CL(\tau)$ is homeomorphic to $CL(\kappa+\tau)$.

Proof. Let us first do the case when $\kappa=\tau$. Let us notice that $CL(\kappa)=(\kappa\setminus\{0\})^+\cup\{0\}^-$. Clearly $(\kappa-\{0\})^+$ is homeomorphic to $CL(\kappa)$. Notice that $\{A\cup\{0\}:A\in CL(\kappa\setminus\{0\})\}\approx CL(\kappa\setminus\{0\})$ by means of the homeomorphism $A\cup\{0\}\mapsto A$. So

$$\{0\}^-=\{\{0\}\}\cup\{A\cup\{0\}:A\in CL(\kappa\setminus\{0\})\}\approx\{0\}\oplus CL(\kappa\setminus\{0\})\approx CL(\kappa),$$

by Observation 5.23. Thus, $CL(\kappa)\approx CL(\kappa)\oplus CL(\kappa)$.

Now we prove the general case, let X and Y be disjoint sets such that $|X|=\kappa$ and $|Y|=\tau$. Give X and Y the discrete topology and take $p\notin CL(X\cup Y)$. Also assume without loss of generality that $\kappa\geq\tau$ so that $\kappa+\tau=\kappa$. Consider the function $h:CL(X)\oplus CL(Y)\to(CL(X)\oplus\{p\})\times(CL(Y)\oplus\{p\})$ given by

$$h(A)=\begin{cases}\langle A,p\rangle, & \text{if } A\subset X,\\ \langle p,A\rangle, & \text{if } A\subset Y.\end{cases}$$

Clearly h is an embedding, let $\mathcal{A}=h[CL(X)\oplus CL(Y)]$. Notice that

$$(CL(X)\oplus\{p\})\times(CL(Y)\oplus\{p\})=(CL(X)\times CL(Y))\cup(\mathcal{A}\cup\{\langle p,p\rangle\}).\qquad(*)$$

By Observation 5.23 and Lemma 5.24, the left side of equation $(*)$ is homeomorphic to $CL(\kappa+\tau)\approx CL(\kappa)$.

The right side of equation $(*)$ is homeomorphic to

$$(CL(\kappa) \times CL(\tau)) \oplus (\mathcal{A} \oplus \{p\}) \approx CL(\kappa) \oplus \mathcal{A}$$

by Observation 5.23 and Lemma 5.24.

Considering the observations made about equation $(*)$, we obtained

$$CL(\kappa) \approx CL(\kappa) \oplus \mathcal{A} \approx CL(\kappa) \oplus CL(\kappa) \oplus CL(\tau) \approx CL(\kappa) \oplus CL(\tau),$$

where we are using that $CL(\kappa) \approx CL(\kappa) \oplus CL(\kappa)$. This completes the proof. □

Proposition 5.26 Let $\kappa \geq \omega$ and $A \in CL(\kappa)$ with $A \neq \kappa$. Then A^- is homeomorphic to $CL(\kappa)$.

Proof. Notice that

$$A^- = \langle A, \kappa \setminus A \rangle \cup A^+ \approx (CL(A) \times CL(\kappa \setminus A)) \oplus CL(A). \tag{$*$}$$

We have different cases depending on whether A or $\kappa \setminus A$ are finite or infinite. If $F \subset X$ is finite, then $CL(F)$ is just a discrete set of $2^{|F|} - 1$ points. Moreover, since the product of a space Y and a discrete space of cardinality $k < \omega$ is just a sum of k copies of Y, we obtain by Lemma 5.25 and the fact that $|\kappa \setminus F| = \kappa$ that

$$CL(F) \times CL(\kappa \setminus F) \approx (2^{|F|} - 1) \times CL(\kappa \setminus F) \approx CL(\kappa \setminus F) \approx CL(\kappa).$$

Thus, if either A is finite or $\kappa \setminus A$ is finite, $CL(A) \times CL(\kappa \setminus A) \approx CL(\kappa)$. By Equation $(*)$, in both of these cases we obtain that $A^- \approx CL(\kappa)$: if A is finite, use Observation 5.23; if $\kappa - A$ is finite, use Lemma 5.25.

If both A and $\kappa \setminus A$ are infinite, Lemmas 5.24 and 5.25 applied to equation $(*)$ directly show that $A^- \approx CL(\kappa)$. □

Proposition 5.27 Let κ be an infinite cardinal and $A \in CL(\kappa) \setminus \mathcal{F}(\kappa)$. Then

$$\chi(CL(\kappa), A) = |A|.$$

Proof. For each $F \in [A]^{<\omega}$, take an enumeration $F = \{x_1, \ldots, x_n\}$ and consider

$$U_F = \langle \{x_1\}, \ldots, \{x_n\}, A \setminus F \rangle.$$

Clearly

$$\mathcal{U} = \{U_F : F \in [A]^{<\omega}\}$$

is a local base of $CL(X)$ at A, and it is of cardinality $|[A]^{<\omega}| = |A|$. Now let $\mathcal{V}$ be a local base of $CL(X)$ at A and assume that $|\mathcal{V}| < |A|$. By Lemma 0.15, we may assume that $\mathcal{V} \subset \mathcal{U}$. Then

$$\mathcal{V} = \{U_F : F \in \mathcal{F}\},$$

For some $\mathcal{F} \subset [A]^{<\omega}$. As we are assuming that $|\mathcal{F}| < |A|$ and each element of $\mathcal{F}$ is finite, there is $y \in A \setminus \bigcup \mathcal{F}$. Then, since A is infinite, $A \setminus \{y\} \in U_F$ for all $F \in \mathcal{F}$. This contradicts the fact that $\mathcal{V}$ is a basis. Thus, $\chi(CL(X), A) = |A|$. □

Proposition 5.28 (Hernández-Gutiérrez) Let κ be an infinite cardinal and $A, B \in CL(\kappa) \setminus \mathcal{K}(\kappa)$. Then the following are equivalent:

(1) $|A| = |B|$,

(2) there is a homeomorphism $H : CL(\kappa) \to CL(\kappa)$ such that $H(A) = B$.

Proof. The implication (2) $\Rightarrow$ (1) is clear from Proposition 5.27. For the other implication, we may assume that $B \neq \kappa$. Notice that $CL(\kappa) = B^+ \cup (\kappa \setminus B)^-$. By Proposition 5.26, $(\kappa \setminus B)^- \approx CL(\kappa)$.

If $A \neq \kappa$, then $CL(\kappa) = A^+ \cup (\kappa \setminus A)^-$, where $(\kappa \setminus A)^- \approx CL(\kappa)$ (again by Proposition 5.26). So there is a homeomorphism $h_0 : (\kappa \setminus A)^- \to (\kappa \setminus B)^-$. Let $h : A \to B$ be a bijection and define $h_1 : A^+ \to B^+$ by $h_1(X) = h[X]$ for every $X \in A^+$. It is easy to see that h_1 is a homeomorphism. Then $H = h_0 \cup h_1 : CL(\kappa) \to CL(\kappa)$ is a homeomorphism with $H(A) = B$.

So now assume that $A = \kappa$. Consider the following sets:

$$\begin{aligned}
\mathcal{U} &= \{X \cup \{0,1\} : X \in CL(\kappa \setminus \{0,1\})\}, \\
\mathcal{V}_0 &= \{X \cup \{0\} : X \in CL(\kappa \setminus \{0,1\})\}, \\
\mathcal{V}_1 &= \{X \cup \{1\} : X \in CL(\kappa \setminus \{0,1\})\}.
\end{aligned}$$

Notice that each one of these sets is clopen in $CL(\kappa)$ and

$$CL(\kappa) = \mathcal{U} \oplus \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{\{0\}, \{1\}, \{0,1\}\}$$

By Lemma 5.25 and Observation 5.23, we have that $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \{\{0\}, \{1\}, \{0,1\}\} \approx CL(\kappa)$. Then, consider any homeomorphism $h_0 : \mathcal{V} \to (\kappa \setminus B)^-$. Given a bijection $h : A \setminus \{0,1\} \to B$, define $h_1 : \mathcal{U} \to B^+$ by $h_1(X) = h[X \setminus \{0,1\}]$, which is a homeomorphism. Then $H = h_0 \cup h_1 : CL(\kappa) \to CL(\kappa)$ is a homeomorphism with $H(A) = B$. □

So in particular, we have the following in the countable case.

Corollary 5.29 $CL(\omega) \setminus \mathcal{F}(\omega)$ is homogeneous.

Regarding homogeneity properties, the next natural question is whether the homogeneous space $CL(\omega) \setminus \mathcal{F}(\omega)$ is in fact a topological group. A famous result of Birkhoff and Kakutani says that a Hausdorff topological group is metrizable if and only if it is first countable (see [11, Theorem 3.3.12]). Then $CL(\omega) \setminus \mathcal{F}(\omega)$ is not a topological group as it is first countable (Proposition 5.27) but not normal (Proposition 5.20) and thus, not metrizable.

As a summary of the results above, we mention that $CL(\omega)$ is a first countable, 0-dimensional, non-normal, crowded, homogeneous space of weight $\mathfrak{c}$. We remark that it has been recently shown that $CL(\omega)$ is strongly 0-dimensional (see [96]).

We next focus on trying to find whether some classes of spaces can be embedded in $CL(\omega)$. This is not a new idea: it was shown in [38] that the existence of subspaces of $CL(\omega)$ that are L-spaces or S-spaces is independent of ZFC (see [139] for an introduction to L-spaces and S-spaces).

Theorem 5.30 (Hernández-Gutiérrez) For any discrete space X there is a closed embedding $e : CL(X) \to CL(X) \setminus \mathcal{F}(X)$.

Proof. Let $A \in CL(X)$ be such that $|A| = |X \setminus A| = |X|$. Consider the following subset of $CL(X) \setminus \mathcal{K}(X)$:

$$\mathcal{A} = \{B \in CL(X) : A \subsetneq B\}.$$

The set $\mathcal{A}$ is closed by the following argument. If $C \notin \mathcal{A}$, we have two cases. The first case is that there is $a \in A \setminus C$, then $(X \setminus \{a\})^+$ is a neighborhood of C that does not intersect $\mathcal{A}$. If $C = A$, then A^+ is a neighborhood of A that does not intersect $\mathcal{A}$. Define a function $h : \mathcal{A} \to CL(X \setminus A)$ as $h(A \cup Y) = Y$. Then it is easy to see that h is a homeomorphism. □

So in some sense the isolated points do not play a role on embeddability. Finally, we will see which ordinals are embeddable in $CL(\omega)$.

Proposition 5.31 Let κ be an infinite cardinal and $\alpha \in \kappa^+$. Then the linearly ordered space $(\alpha, \in)$ can be embedded in $CL(\kappa)$.

Proof. It is enough to show that if α is a limit ordinal, then the linearly ordered space $(\alpha, \in)$ can be embedded in $CL(\kappa)$, where $\kappa = |\alpha|$. Consider an enumeration

$\kappa = \{x_{\beta+1} : \beta < \alpha\} \cup \{x_0\}$. Recursively define:

$$\begin{array}{rcll} X_0 & = & \{x_0\}, & \\ X_{\beta+1} & = & X_\beta \cup \{x_{\beta+1}\}, & \text{for every ordinal } \beta < \alpha, \\ X_\gamma & = & \bigcup_{\beta<\gamma} X_\beta, & \text{for every limit ordinal } \gamma < \alpha. \end{array}$$

Notice that then $\{X_\beta : \beta < \alpha\}$ is a strictly $\subset$-increasing sequence.

Define $h : \alpha \to CL(\kappa)$ as $h(\beta) = X_\beta$. Clearly h is an injective function, we have to prove that it is an embedding. We shall see that the two topologies involved in fact coincide.

First we will see that isolated points in $(\alpha, \in)$ are mapped to isolated points in $CL(\kappa)$. For 0, notice that $X_0 = \langle\{x_0\}\rangle$, which is an open set. If $\beta < \alpha$, then

$$\langle X_\beta, \{x_{\beta+1}\}\rangle \cap h[\alpha] = \{X_{\beta+1}\},$$

so $X_{\beta+1}$ is an isolated point in the image.

Fix $\beta < \alpha$ a limit ordinal.

First consider a basic open neighborhood of X_β, a Vietoris set of the form $\langle U_0, \ldots, U_n\rangle$. Let $x_{\gamma(i)} \in U_i \cap X_\beta$ for each $i \leq n$ and let $\gamma = \sup\{\gamma(i) : i \leq n\} + 1 < \beta$. Then for all $\xi \in (\gamma, \beta]$, we have that $X_\xi \in \langle U_0, \ldots, U_n\rangle$ because $X_\xi \subset X_\beta \subset U_0 \cup \cdots \cup U_n$ and $x_{\gamma(i)} \in X_\xi \cap U_i$ for $i \leq n$. Thus, every open neighborhood of X_β in $CL(\kappa)$ contains an ordered neighborhood of X_β.

Now let $\gamma < \beta$, notice that

$$\langle X_\beta, \{x_{\gamma+1}\}\rangle \cap h[\alpha] = \{X_\xi : \gamma < \xi \leq \beta\},$$

so each ordered neighborhood is also a neighborhood in $CL(\kappa)$. This completes the proof. $\square$

So every countable ordinal can be embedded in $CL(\omega)$. Obviously $\omega_1 + 1$ cannot be embedded in $CL(\omega)$ because $\omega_1 + 1$ is not first countable. However, it is not straightforward to prove that ω_1 cannot be embedded in $CL(\omega)$. Our method to prove this is generalizing Theorem 0.14 to the following result.

Proposition 5.32 Let κ be a cardinal. If $f : (\kappa^+, \in) \to CL(\kappa)$ is a continuous function, there is $\beta < \kappa^+$ such that $f\restriction_{(\beta,\kappa^+)}$ is constant.

Proof. For each $\alpha < \kappa$, define a function $g_\alpha : \kappa^+ \to \{0, 1\}$ by

$$g_\alpha(\beta) = \begin{cases} 0, & \text{if } \alpha \notin f(\beta), \\ 1, & \text{if } \alpha \in f(\beta). \end{cases}$$

Notice that g_α is continuous because $g_\alpha^{\leftarrow}(0) = f^{\leftarrow}[(\kappa \setminus \{\alpha\})^+]$ is clopen. By Theorem 0.14, for each $\alpha < \kappa$, there is $\beta(\alpha) < \kappa^+$ and $t_\alpha \in \{0,1\}$ such that if $\gamma \in [\beta(\alpha), \kappa^+)$, then $g_\alpha(\gamma) = t_\alpha$. Define $B = \{\alpha < \kappa : t_\alpha = 1\}$ and $\beta = \sup\{\beta(\alpha) : \alpha < \kappa\} < \kappa^+$. From this it is easy to see that if $\gamma \in (\beta, \kappa)$, then $f(\gamma) = B$. $\square$

We immediately obtain the following.

Theorem 5.33 (Hernández-Gutiérrez) If α is an ordinal and X is an infinite discrete space, then $(\alpha, \in)$ can be embedded in $CL(X)$ if and only if $\alpha < |X|^+$.

Corollary 5.34 An ordinal with the order topology can be embedded in $CL(\omega)$ if and only if it is countable.

5.4 When is $\mathcal{K}(X)$ C^*-embedded in $CL(X)$?

In this Section, we will mention some results about C-embeddings in hyperspaces.

First let us state the motivation for this problem. Let X be a normal space (so that $CL(X)$ is Tychonoff by Proposition 5.17). Then $CL(X)$ can be embedded in $CL(\beta X)$ by the mapping $e : CL(X) \to CL(\beta X)$ defined by $e(A) = \mathrm{cl}_{\beta X}(A)$. Since $e[CL(X)]$ is dense in $CL(\beta X)$, then $CL(\beta X)$ is a compactification of $CL(X)$. Thus, an interesting question is when $CL(\beta X)$ and $\beta CL(X)$ are equivalent compactifications of $CL(X)$, in the sense of [135, 4.1.(d)]. This problem has been solved in the following way[3].

Theorem 5.35 ([71] and [128]) Let X be a normal space. Then the following are equivalent

(1) $\beta CL(X)$ is equivalent to $CL(\beta X)$,

(2) $CL(X) \times CL(X)$ is pseudocompact,

(3) $CL(X)$ is pseudocompact.

Notice that this problem can be re-stated as follows: "when is $e[CL(X)]$ is C^*-embedded in $CL(\beta X)$? Considering this, the author thinks that determining when $\mathcal{K}(X)$ is C^*-embedded in $CL(X)$ is an interesting problem. The following observation was made.

[3] Natsheh's paper [128] contains an error in the proof of the result but the argument still applies correctly to give the solution.

Proposition 5.36 **(Hernández-Gutiérrez)** Let X be a Hausdorff space. If $\mathcal{K}(X)$ is a normal space that is C^*-embedded in $CL(X)$, then $\mathcal{K}(X)$ is ω-bounded.

Proof. Let $\{T_n : n < \omega\} \subset \mathcal{K}(X)$. By Theorem 5.11, we have to show that $C = \mathrm{cl}_X(\bigcup\{T_n : n < \omega\})$ is compact. Assume this is not the case. For each $n < \omega$, let $K_n = T_0 \cup \cdots \cup T_n$, which is compact. Then by Lemma 1.11, $\{K_n : n < \omega\}$ converges to C in $CL(X)$. This means that $\{K_n : n < \omega\}$ is a countably infinite, closed and discrete space in the normal space $\mathcal{K}(X)$. However, $\mathcal{K}(X)$ is C^*-embedded in $CL(X)$. Thus, for example, the function $\phi : \{K_n : n < \omega\} \to \{-1, 1\}$ such that $\phi(K_n) = (-1)^n$ can be extended to C. This is impossible so we obtain that C is indeed compact. $\square$

The author of this dissertation was not able to solve this problem. However, some other students of Professor Tamariz-Mascarúa got interested in the problem and were able to solve it jointly with Nobuyuki Kemoto. They obtained the following result.

Theorem 5.37 [95] Let κ an ordinal with the order topology. Then $\mathcal{K}(\kappa)$ is C^*-embedded in $CL(\kappa)$ if and only if $\mathbf{cof}(\kappa) > \omega$.

Notice that the question for general spaces remains open (see Question 5.40 below).

5.5 Open Questions

In this section, we mention some interesting questions about the topics of Part I that remain open. We will comment a little about the problems before stating them.

The first two problems concern some properties on $\mathcal{K}(X)$. The spirit is to find equivalents of these properties in $\mathcal{K}(X)$ with some nice property in the base space X. Recall that the following problems have been solved: countable compactness of $\mathcal{K}(X)$ by Theorem 5.11, normality of $CL(X)$ by Theorem 5.19.

Question 5.38 Give a characterization of Tychonoff spaces X such that $\mathcal{K}(X)$ is pseudocompact.

Question 5.39 When is $\mathcal{K}(X)$ a normal space?

The following problem is a generalization of Theorem 5.37. See the discussion before Theorem 5.37 for some background.

Question 5.40 Give a characterization of (normal) spaces X such that $\mathcal{K}(X)$ is C^*-embedded in $CL(X)$.

Another type of space that one would like to embedd in $CL(\omega)$ besides countable ordinals (Corollary 5.34) are Mrówka-Isbell ψ-spaces. Recall that a family $\mathcal{A}$ of subsets of ω is *almost disjoint* ("AD family", for short) if $|A \cap B| < \omega$ every time $A, B \in \mathcal{A}$ with $A \neq B$ (see [99, Chapter II, § 1]).

If $\mathcal{A}$ is an AD family, we can define a space which will be called $\psi(\mathcal{A})$. The underlying set of $\psi(\mathcal{A})$ is $\omega \cup \mathcal{A}$. Every $n \in \omega$ is declared an isolated point and if $A \in \mathcal{A}$, then a basic open neighborhood of A is of the form $\{A\} \cup (A \setminus n)$ where $n \in \omega$. A space of the form $\psi(\mathcal{A})$ is called *Mrówka-Isbell ψ-space*[4] in the literature.

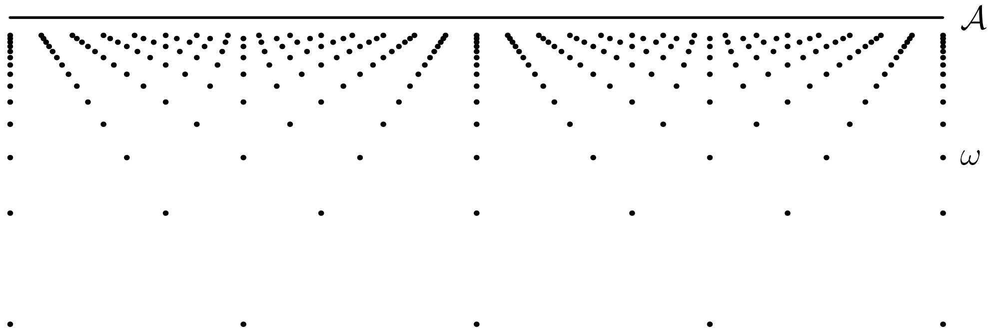

Figure 5.1: A picture of a ψ-space constructed from an AD family $\mathcal{A}$.

See [69, 5I], [135, 1N] or [126, chapter 9, 4.15] for some properties of $\psi(\mathcal{A})$. As far as this author knows, there are no books that treat $\psi(\mathcal{A})$ in a central way. However, there is an excellent B.S. thesis in Spanish with a lot of information on $\psi(\mathcal{A})$, see [4].

It is known that if α is an ordinal less or equal to $\omega \cdot \omega$ (see [90, Definition 2.18, p. 23] or [103, Definition I.8.21, p. 40] to see what this means), then there is an AD family $\mathcal{A}$ such that $\psi(\mathcal{A})$ is homeomorphic to the linearly ordered space $(\alpha, \in)$ (see [4, Proposición 3.2.1] for a nice proof). Thus, the following question seems natural when considering Corollary 5.34.

[4] In [48, Section § 2], Alan Dow and Jerry Vaughan give a small summary of the history of $\psi(\mathcal{A})$. Apparently, some special cases of $\psi(\mathcal{A})$ appear in papers from Alexandroff-Urysohn [2, Chapter V, 1.3] and Katetov [92]. The idea of asking $\mathcal{A}$ to be maximal is due to Mrówka [122]. Gillman and Jerison attribute $\psi(\mathcal{A})$ when $\mathcal{A}$ maximal to Isbell in [69, Notes to Chapter 5, p. 269].

Question 5.41 Let $\mathcal{A}$ be an AD family. When can $\psi(\mathcal{A})$ be embedded in $CL(\omega)$?

In general we can ask the following question.

Question 5.42 For which spaces X is it possible to embedd X in $CL(\omega)$?

The last questions on hyperspaces are the author's hopes to extend the results of Chapter 4. The first one is wishful thinking: one would always like to have nice and easy characterizations, although it might not be possible.

Question 5.43 Give tangible conditions on X so that $\mathcal{K}(X)$ is hereditarily disconnected.

The second one is another hope to see homogeneity playing some important role in the problem.

Question 5.44 Let X be a homogeneous (topological group, perhaps) hereditarily disconnected space. Can one give some characterization of hereditarily disconnectedness of $\mathcal{K}(X)$ in terms of (iterated) quasicomponents and/or the space of quasicomponents $\mathcal{Q}(X)$?

Finally, a related problem, although it does not talk about hyperspaces. Compare with Theorems 2.15, 2.16 and 2.17.

Question 5.45 [158, Question 8, p. 307] Does there exist a hereditarily disconnected subgroup of ${}^3\mathbb{R}$ for which the quasicomponent of the quasicomponent is not zero?

Part II

Spaces of Remote Points

Introduction

It is known that the class of Tychonoff spaces coincides with the class of spaces that can be embedded in compact Hausdorff spaces (Proposition 6.1). In particular, for each Tychonoff space X there exists a space βX called the Čech-Stone compactification of X that has a dense copy of X and that is maximal with respect to this property in some sense ((2) in Theorem 6.4).

A point $p \in \beta X \setminus X$ is called a remote point of X if p is not in the βX closure of any nowhere dense subset of X. So in some sense, remote points of X are far away from X. In this Part, we will study the set $\varrho(X)$ of remote points of X as topological spaces. In particular, we address the following problem: if X and Y are two metrizable spaces, when are $\varrho(X)$ and $\varrho(Y)$ homeomorphic? The results of our research have been published in [80].

What follows is an historical introduction to the theory of remote points. The historical context of remote points goes back to the effort for understanding the space βX. Even for simple spaces X, most of the times βX is a very complicated space. For example, if X is a metrizable and non-compact space, then βX has cardinality $2^{\mathfrak{c}}$ (Proposition 6.49) and none of the points of $\beta X \setminus X$ is of type G_δ (Corollary 6.50). Once such a complicated object is constructed, it is desirable to be able to "see inside" it and be able to understand its structure. For example, since ω (with the discrete topology) is clearly homogeneous, one wonders if it is also the case that the set of non-isolated points of $\beta\omega$, $\beta\omega \setminus \omega$ is homogeneous.

Non-homogeneity of $\beta\omega \setminus \omega$

It turns out that $\beta\omega \setminus \omega$ is highly non-homogeneous. As shown by Zdeněk Frolik in 1967 (Theorem 6.55), $\beta\omega \setminus \omega$ has exactly $2^{\mathfrak{c}}$ different kinds of points in ZFC. Frolik's result shows the non-homogeneity of $\beta\omega \setminus \omega$ but it does not tells us "why" this space is non-homogeneous, since the argument is combinatorial in nature and does not exhibit two different types of points in $\beta\omega \setminus \omega$.

Previously, in 1956 ([142]), Walter Rudin did construct, under **CH**, a special type of points of $\beta\omega \setminus \omega$ defined in a topological way: the famous P-points. Since not all points in a infinite compact Hausdorff space can be P-points, Rudin's result shows that $\beta\omega \setminus \omega$ is not homogenous under CH (Corollary 6.54). However, the existence of P-points without further hypothesis was left open. Even after Frolik's solution to the homogeneity problem in $\beta\omega \setminus \omega$, Rudin's method was perhaps closer to what a General Topologist might wish for.

In 1977-1978, Saharon Shelah proved that there is a model of ZFC in which there are no P-points in $\beta\omega \setminus \omega$ (this was published by Wimmers in [166]). This may sound dissapointing but at the same time, Kenneth Kunen gave ([100]) a proof of the existence (in ZFC) of the so called weak P-points in $\beta\omega \setminus \omega$. As in the case of P-points, there are points in $\beta\omega \setminus \omega$ that are not weak P-points and being a weak P-point is a topological property so $\beta\omega \setminus \omega$ is not homogeneous. See [117] for a longer and more detailed retelling of this story.

Non-homogeneity and remote points

Kunen's result finally settles "why" $\beta\omega \setminus \omega$ is not homogeneous. So a natural question is for what spaces X is X^* homogeneous. Frolik's result on $\beta\omega \setminus \omega$ also shows that if X is not psedocompact, then $\beta X \setminus X$ is not homogeneous (see Theorem 6.56). Also, it is known that if G is a topological group, then βG is homogeneous if and only if G is a pseudocompact ([34]). See, for instance, Example 6.6.

However, once again, Frolik's results do not answer the question of "why" $\beta X \setminus X$ is not homogeneous. Eric van Douwen was one of the advocates of this philosophy and used far points (see definition in Chapter 7, p. 108) in [35] to prove that if X is a nowhere locally compact and metrizable space, then $\beta X \setminus X$ is not homogeneous. Later, in [36] van Douwen proved that if X is a nowhere locally compact and non-pseudocompact space with countable π-weight, then $\beta X \setminus X$ is not homogeneous because it is extremally disconnected at its (many) remote points but has points of non-extremal disconnectedness (Theorem 7.9).

The origin of remote points

In [58], Fine and Gillman proved, assuming **CH**, that there is a point $p \in \beta\mathbb{R}$ such that if D is a discrete subset of $\mathbb{R}$ then $p \notin \mathrm{cl}_{\beta\mathbb{R}}(D)$. Fine and Gillman called such a point a remote point, this does not conflict with the current terminology as every nowhere dense subset of $\mathbb{R}$ is contained in the closure of a discrete subset of $\mathbb{R}$. According to Fine and Gillman's paper (see footnote 4 in [58]), W.F. Eberlein was the first to construct a point $p \in \beta\mathbb{R} - \mathbb{R}$ such that for all *closed* discrete subsets $D \subset \mathbb{R}$ we have $p \notin \mathrm{cl}_{\beta\mathbb{R}}(D)$. We remark that Eberlein's proof used Lebesgue measure and does not use additional assumptions. However, notice that the point found by Eberlein is not necessarily a remote point.

Finally, van Douwen ([36]) and independently Soo Bong Chae and Jeffrey H. Smith ([26]) were able to find remote points in ZFC. Van Douwen proved that if X is a non-pseudocompact space of countable π-weight, then X has $2^{\mathfrak{c}}$ remote points ([36, Theorem 1.5]). Chae and Smith proved that if X is a metrizable space without isolated points, then X has $2^{\mathfrak{c}}$ remote points as well ([26, Theorem 1]). Notice that none of the two results is more general than the other. Moreover, van Douwen gave many applications of remote points in [36], in particular, the non-homogeneity results mentioned above.

Finding remote points

After the existence of remote points for some spaces was proved, one can naturally hope to extend these results for other spaces. We will mention results that in opinion of the author have been significant. In [43], van Douwen and van Mill proved that there non-compact σ-compact spaces without remote points. In [102] Kenneth Kunen, Jan van Mill and Charles F. Mills proved that **CH** is equivalent to the assertion that each non-pseudocompact space with at most $\mathfrak{c}$ continuous and bounded real-valued functions has a remote point. Moreover, it is consistent that there is a separable non-pseudocompact space with no remote points as shown by Alan Dow in [44]. Also, Dow shows in [44] that a pseudocompact space does not have remote points. Dow has been one of the most active researchers in this topic. See [46] where there are references to several of his results. The latest paper about the existence of remote points is [22].

Spaces of remote points

As the reader has noticed, most of research on remote points has been about their existence. However, R. Grant Woods published an interesting result in [167]: **CH** implies every non-compact, locally compact and crowded metrizable space of weight κ has its set of remote points homeomorphic to the set of remote points of $\kappa \times {}^{\omega}2$. Later, Catherine L. Gates extended the results of Woods in [66]. By the time of the publication of [66], van Douwen had anounced his results on [36] so it followed that Wood's results in [167] could be proved in ZFC.

Our results

The objective of this Part of the dissertation is to extend the results of Gates and Woods to metrizable spaces that are not necessarily locally compact. In Chapter 9 we will give the results we were able to obtain. Most of them are about nowhere locally compact and completely metrizable spaces. In Proposition 9.31, the reader will notice a duality between the space of remote points of the rationals and the one of the irrationals. We were not able to completely characterize these spaces, see the Questions scattered in this Chapter.

Chapter 6

The Čech-Stone Compactification and the Absolute

In this chapter, we will give the basic definitions of two of the constructions we will use: the Čech-Stone compactification and the Illiadis Absolute.

6.1 Basics of βX

If X is a Tychonoff space, a *compactification* is a pair $\langle T, e\rangle$ where T is a compact Hausdorff space and $e : X \to T$ is an embedding such that $e[X]$ is dense in X. The *remainder* of X in $\langle T, e\rangle$ is the set $T \setminus e[X]$. In general, a remainder of X is a space Y such that there is a compactification $\langle T, e\rangle$ of X and Y is homeomorphic to $T \setminus X$.

Let us recall Čech's[1] classical construction of the Čech-Stone compactification[2]. Let X be a Tychonoff space and

$$C^*(X) = \{f \in {}^X\mathbb{R} : f \text{ is continuous and bounded}\}.$$

For each $f \in C^*(X)$, let $I_f = [\inf f[X], \sup f[X]]$. By the Tychonoff theorem

[1] There has been some controversy as to whether Tychonoff constructed the Čech-Stone compactification. Čech does give credit to Tychonoff in his paper [28]. However, according to [61], Tychonoff only constructed some other compactification that is not always the Čech-Stone compactification.

[2] In books and papers by US authors this space is called "Stone-Čech compactification", perhaps to honor their co-national Marshal Harvey Stone. We however, take the more natural and less controversial alphabetical order.

0.5, the space $I_X = \prod\{I_f : f \in C^*(X)\}$ is compact. Define $e_X : X \to I_X$ by $e_X(x)(f) = f(x)$ for all $x \in X$ and $f \in C^*(X)$.

It is not hard to prove that e_X is an embedding (for example, using [161, 1.5, p. 4]). Thus, $e_X[X]$ is a homeomorphic copy of X. We define $\beta X = \mathrm{cl}_{I_X}(e_X[X])$. Notice that since βX is a closed subset of the compact space I_X, it is a compact set that contains a dense topolgical copy of X. Notice that if X is already compact, then βX is homeomorphic to X.

Since we are using the Tychonoff Theorem to prove the compactness of βX, at least it is apparent from Proposition 0.6 that we need some version of the Axiom of Choice to construct βX. An interesting discussion of the relation between the Axiom of Choice and the Čech-Stone compactification is contained in Chapter 4.8 of [77].

From all this and the fact that the Tychonoff property is hereditary, the following result follows easily.

Proposition 6.1 Let X be any space. Then the following are equivalent.

(1) X is a Tychonoff space,

(2) X can be embedded in a compact Hausdorff space,

(3) there is an infinite cardinal number κ and an embedding $e : X \to {}^{\kappa}[0,1]$.

We gave a precise definition of βX. However, rather than the explicit definition, it is sometimes desirable to have some topological characterization of βX and forget about the original construction.

Lemma 6.2 Let X be a Tychonoff space. Then $e_X[X]$ is C^*-embedded in βX.

Proof. Let $f : X \to \mathbb{R}$ be continuous and bounded. Define $\beta f : \beta X \to I_f$ as

$$\beta f((x_g)_{g \in C^*(X)}) = x_f,$$

that is, the projection into the coordinate indexed by f. It is easy to prove that $\beta f \circ e_X = f$, so this is the desired continuous extension. □

Lemma 6.3 Let X and Y be Tychonoff spaces such that X is dense and C^*-embedded in Y. If K is a compact Hausdorff space and $f : X \to K$ is a continuous function, then there is a continuous function $F : Y \to K$ such that $F{\restriction}_X = f$.

Proof. Since K is compact, we may assume that $K = e_K[K] \subset I_K$. For each $f \in C^*(X)$, let $f^* \in C^*(Y)$ be such that $f^*{\restriction}_X = f$. Define a function $F : Y \to I_K$

by $F(x)(g) = (g \circ f)^*(x)$ for all $g \in C^*(K)$.

$$\begin{tikzcd} Y \arrow[drr, "(g\circ f)^*"] & & \\ \cup & & \\ X \arrow[r, "f"'] & K \arrow[r, "g"'] & I_g \end{tikzcd}$$

It is not hard to prove that F is continuous and $F\restriction_X = f$. Since X is dense in Y and K is compact, $F[Y] \subset K$. Thus, F is the continuous extension we want. $\square$

The following characterization will allow us to see βX not just as a construction but as an abstract topological object. In fact, in most of our applications, we will assume that X is a dense subset of βX. For example, Theorem 6.23 gives an alternative definition of βX when X is discrete.

Theorem 6.4 Let X be a Tychonoff space and assume that there is a compact Hausdorff space T such that X is a dense subset of T. Then the following are equivalent.

(0) There is as homeomorphism $h : \beta X \to T$ such that $(h \circ e_X)[X]$ is the identity function.

(1) X is C^*-embedded in T.

(2) For every compact Hausdorff space K and each continuous function $f : X \to K$ there exists a continuous function $F : T \to K$ such that $F\restriction_X = f$.

(3) Any two disjoint zero sets in X have disjoint closures in T.

Proof. By Lemma 6.2 (0) implies (1) and by Lemma 6.3, (1) implies (2). If $f : X \to \mathbb{R}$ is bounded, then I_f is a compact Hausdorff space such that $f : X \to I_f$ so (2) clearly implies (1).

To see that (2) implies (0), consider $e_0 : X \to T$ the inclusion and $e_1 = e_X : X \to \beta X$. By Lemmas 6.2 and 6.3, there is a continuous function $f_0 : \beta X \to T$ such that $f_0 \circ e_1 = e_0$. By (2) there is a continuous function $f_1 : T \to \beta X$ such that $f_1 \circ e_0 = e_1$. Notice that then $f_1 \circ f_0 : \beta X \to \beta X$ is a continuous function such that $(f_1 \circ f_0)\restriction_{e_1[X]}$ is the identity in $e_1[X]$.

$$\begin{tikzcd} \beta X \arrow[r, "f_0"] & T \arrow[r, "f_1"] & \beta X \\ & X \arrow[ul, "e_1"] \arrow[u, "e_0"'] \arrow[ur, "e_1"'] & \end{tikzcd}$$

By Lemma 0.4, we obtain that $f_1 \circ f_0$ is the identity in βX. By a similar argument, $f_0 \circ f_1$ is the identity in T. Thus, f_1 is the inverse function of f_0, this shows that both functions are homeomorphisms. Then $h = f_0$ is the homeomorphism we wanted.

Finally, notice that the equivalence of (2) and (3) is Taimanov's Theorem 0.10. $\square$

We will say that a pair $\langle T, e \rangle$ is the Čech-Stone compactification of a Tychonoff space X whenever $e : X \to T$ is a dense embedding, T is a compact Hausdorff space and any of the properties of Theorem 6.4 hold. The spirit is that this pair defines the Čech-Stone compactification modulo homeomorphism. Notice that in most of the cases we will assume that $X \subset \beta X$ and e_X is the inclusion without saying it explicitly.

Let $X^* = \beta X \setminus X$, this is called the *Čech-Stone remainder* of X is βX.

There are many other ways to construct the Čech-Stone compactification. Chapter 1 of [161] contains three different descriptions of βX in terms of ultrafilters of zero sets and spaces of maximal ideals. It also contains references to other different constructions of βX. We also refer the reader to [161] for historical information of these constructions, for example, the diagram in [161, p. 27] is quite interesting.

The classical reference for the Čech-Stone compactification was for many years [69], it takes an algebraic approach. Other interesting sources include [161], [135], [62], [29] and [113].

For an approach on the Čech-Stone compactification that use proximities see [127]. A recent M.S. thesis in Spanish about some aspects of the Čech-Stone compactification and its relation to other compactifications is [33].

Notice that the definition of βX we gave is of some subspace of I_X, which, in general, may be very complicated and very difficult to visualize. Thus, in the spirit of grasping the "nature" of the Čech-Stone compactification, we would like to have examples of spaces X such that βX is known or can be "seen". It turns out that βX is complicated for simple spaces such as ω, $\mathbb{Q}$ or $\mathbb{R}$. We will speak more about these spaces in Section 6.4. The examples we will give here are some "unusual" spaces that will have "simple" compactifications.

Recall that a Tychonoff space X is *pseudocompact* if every continuous function $f : X \to \mathbb{R}$ is bounded. Also, X is *ω-bounded* if every countable subset of X has compact closure. Clearly, an ω-bounded space is pseudocompact.

Example 6.5 Compactifications of ordinals.

First recall that linearly ordered spaces are Tychonoff (Theorem 0.13). Let κ

be an ordinal of uncountable cofinality with the order topology. Notice that κ is a dense subset of the compact space $\kappa + 1$. If $f : \kappa \to \mathbb{R}$ is a continuous function, then by Theorem 0.14 there is $r \in \mathbb{R}$ and $\alpha < \kappa$ such that $f(\beta) = r$ when $\alpha \leq \beta < \kappa$. Thus, we may define a continuous extension $f^* : \kappa + 1 \to \mathbb{R}$ by $f^*(\alpha) = f(\alpha)$ for $\alpha < \kappa$ and $f^*(\kappa) = r$. By (1) in Theorem 6.4, $\kappa + 1 = \beta\kappa$.

We remark that it is known that if X is a linearly ordered space, then βX can be linearly ordered if and only if X is pseudocompact ([138], see [137] for more details of the proof). $\square$

Example 6.6 Σ-products.

Let κ be an uncountable cardinal and consider ${}^{\kappa}2$, this space is clearly homogeneous since it is a topological group. Let

$$\mathbf{\Sigma} = \{x \in {}^{\kappa}2 : \{\alpha < \kappa : x(\alpha) \neq 0\} \text{ is countable}\}.$$

The space $\mathbf{\Sigma}$ is called a Σ-product (see [11] for a general definition). Notice that $\mathbf{\Sigma}$ is in fact a subgroup of ${}^{\kappa}2$ (with the coordinate-wise sum modulo 2) so in particular it is homogeneous. It is easy to prove that $\mathbf{\Sigma}$ is ω-bounded so in particular it is pseudocompact. By [135, Problem 1X], $\mathbf{\Sigma}$ is C^*-embedded in ${}^{\kappa}2$. Thus, $\beta\mathbf{\Sigma} = {}^{\kappa}2$ by Theorem 6.4.

We remark that it is known that if G is a topological group, then βG is homogeneous if and only if G is a pseudocompact (see [34]). For other properties of Σ-products and its generalizations, see [129]. $\square$

Example 6.7 Mrówka-Isbell ψ-spaces[3].

Recall that a family $\mathcal{A}$ of subsets of ω is *almost disjoint* ("AD family", for short) if $|A \cap B| < \omega$ every time $A, B \in \mathcal{A}$ with $A \neq B$ (see [99, Chapter II, § 1]). If $\mathcal{A}$ is an AD family, we can define a space which will be called $\psi(\mathcal{A})$. The underlying set of $\psi(\mathcal{A})$ is $\omega \cup \mathcal{A}$. Every $n \in \omega$ is declared an isolated point and if $A \in \mathcal{A}$, then a basic open neighborhood of A is of the form $\{A\} \cup (A \setminus n)$ where $n < \omega$. A space of the form $\psi(\mathcal{A})$ is called *Mrówka-Isbell ψ-space.*

It is known that $\psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is maximal with respect to the inclusion order (among AD families), see [69, 5I]. Of course the nature of $\psi(\mathcal{A})$ and $\beta\psi(\mathcal{A})$ depends on the combinatorial properties of $\mathcal{A}$. In

[3] In [48, Section § 2], Alan Dow and Jerry Vaughan give a small summary of the history of $\psi(\mathcal{A})$. Apparently, some special cases of $\psi(\mathcal{A})$ appear in papers from Alexandroff-Urysohn [2, Chapter V, 1.3] and Katetov [92]. The idea of asking $\mathcal{A}$ to be maximal is due to Mrówka [122]. Gillman and Jerison attribute $\psi(\mathcal{A})$ when $\mathcal{A}$ maximal to Isbell in [69, Notes to Chapter 5, p. 269].

particular, there is a very interesting result of Mrówka, [124], that states that there exisists a maximal AD family $\mathcal{A}$ such that $|\psi(\mathcal{A})^*| = 1$. See [4, Capítulo 5] for a proof of this result.

Recent work by Alan Dow and Jerry Vaughan gives generalizations of ψ-spaces to uncountable families and give analogous results to Mrówka's classical one, see [48]. $\square$

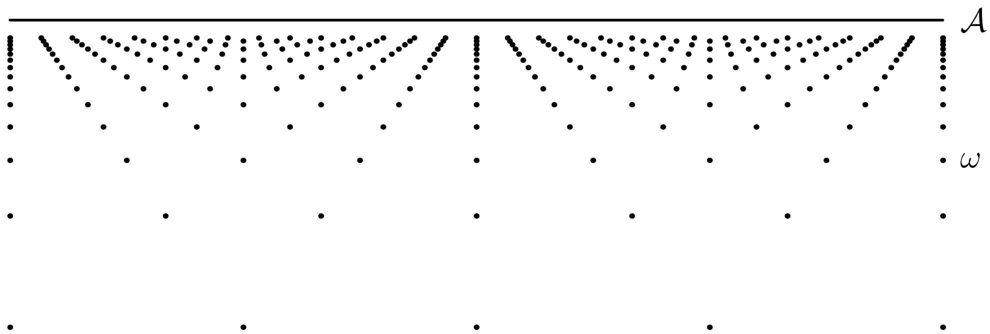

Figure 6.1: A picture of a ψ-space constructed from an AD family $\mathcal{A}$.

The examples we have shown are pseudocompact. Moreover, by the comments given, it is apparent that pseudocompactness is quite important in the study of the Čech-Stone compactification. The following result gives a characterization of pseudocompactness in terms of the Čech-Stone compactification.

Theorem 6.8 [161, Theorem 1.57] Let X be a Tychonoff space. Then X is pseudocompact if and only if every non-empty subset of βX of type G_δ intersects X.

Proof. First, assume that G is a subset of βX of type G_δ contained in X^*. It is not hard to find a zero set of βX contained in G. In other words, there exists a continuous function $f : \beta X \to [0,1]$ such that $f^{\leftarrow}(0) \subset X^*$. Notice that $f > 0$ in X and for all $n < \omega$ there is $x_n \in X$ such that $f(x_n) < \frac{1}{n}$. Define $g : X \to [0,1]$ by $g(x) = \frac{1}{f(x)}$ for all $x \in X$, this is a well-defined continuous function. However, $\lim_{n\to\infty} g(x_n) = \infty$ so X is not pseudocompact.

Now, assume that every non-empty subset of βX of type G_δ intersects X. Let $f : X \to \mathbb{R}$ be a continuous function, we must prove that f is bounded. Let $g : X \to \mathbb{R}$ be defined by $g(x) = \frac{1}{|f(x)|+1}$, then g is a well-defined continuous function that is moreover bounded by 1. Let $\beta g : \beta X \to [0,1]$ be a continuous extension of g by (1) in Theorem 6.4. If $\beta g^{\leftarrow}(0) \neq \emptyset$, by hypothesis, there is $t \in \beta g^{\leftarrow}(0) \cap X$. But then $0 = \beta g(t) = g(t) = \frac{1}{|f(t)|+1} > 0$, which is a contradiction. Thus, there is $0 < \delta < 1$ such that $\beta g(x) > \delta$ for all $x \in \beta X$. In particular, $|f(x)| < \frac{1}{\delta} - 1$ for all $x \in X$ so f is bounded. $\square$

The opposite situation from pseudocompactness is that of realcompactness. A space X is *realcompact* if for every $p \in X^*$ there exist a non-empty set $G \subset \beta X$ of type G_δ of βX such that $p \in G \subset X^*$. It turns out that most of the "simple" spaces are in fact realcompact.

Proposition 6.9 Every Lindelöf space is realcompact. In particular, $\mathbb{R}$, $\mathbb{Q}$, ω and the Sorgenfrey line are realcompact.

Proof. Let $p \in X^*$. For each $x \in X$, since X is Tychonoff, there exists a cozero set $U(x)$ of βX such that $x \in U(x)$ and $p \notin U(x)$. The open cover $\{U(x) : x \in X\}$ of X has a countable subcover $\{U_n : n < \omega\}$. Let $G = \beta X \setminus \bigcup\{U_n : n < \omega\}$. Then G is a set of type G_δ of βX such that $p \in G \subset X^*$. $\square$

In general, we may classify points of X^* as whether or not there are G_δ sets that contain them and miss X. For a Tychonoff space X we define the *Hewitt-Nachbin realcompactification* of X as the space

$$\nu X = \beta X \setminus \bigcup\{G : G \text{ is a set of type } G_\delta \text{ of } \beta X \text{ and } G \subset X^*\}.$$

Notice that $\beta X = \nu X$ if and only if X is pseudocompact and $X = \nu X$ if and only if X is realcompact. However, it is always true that $X \subset \nu X \subset \beta X$. The reader is refered to the book [163] for more on realcompactness and its generalizations. Chapter 8 of [69] also contains a lot of information.

There is a very nice and useful basis of βX. The definition was attributed to Šanin [153] in [50, p. 388]. If X is a Tychonoff space and $U \subset X$ is an open set, we define

$$\mathrm{Ex}_X(U) = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U).$$

We may think that $\mathrm{Ex}_X(U)$ is the maximal extension of U to βX, as the following result shows.

Proposition 6.10 [118] Let X be a Tychonoff space and U, V open subsets of X. Then.

(a) $X \cap \mathrm{Ex}_X(U) = U$,

(b) $\mathrm{cl}_{\beta X}(\mathrm{Ex}_X(U)) = \mathrm{cl}_{\beta X}(U)$,

(c) if W is an open subset of βX such that $W \cap X = U$, then $W \subset \mathrm{Ex}_X(U)$,

(d) $\mathrm{Ex}_X(U) \cap \mathrm{Ex}_X(V) = \mathrm{Ex}_X(U \cap V)$ and $\mathrm{Ex}_X(U) \cup \mathrm{Ex}_X(V) \subset \mathrm{Ex}_X(U \cup V)$,

(e) if X is normal, then $\mathrm{Ex}_X(U) \cup \mathrm{Ex}_X(V) = \mathrm{Ex}_X(U \cup V)$.

Proof. The proof of (a) is straightforward. Since X is dense in βX, by (a) U is dense in $\mathrm{Ex}_X(U)$ and (b) follows from this.

Let W be as in the hypothesis of (c). If there were a point $p \in W \setminus \mathrm{Ex}_X(U)$, then $p \in \mathrm{cl}_{\beta X}(X \setminus U)$ and W is an open set that contains p so $W \cap (X \setminus U) \neq \emptyset$, a contradiction. Thus, (c) holds.

Property (d) holds by properties of the closure operator. Now assume that X is normal and (e) does not hold. Then one can find $p \in (\mathrm{cl}_{\beta X}(X \setminus U) \cap \mathrm{cl}_{\beta X}(X \setminus V)) \setminus \mathrm{cl}_{\beta X}(X \setminus (U \cup V))$. Let W be an open set of βX such that $p \in W$ and $\mathrm{cl}_{\beta X}(W) \cap \mathrm{cl}_{\beta X}(X \setminus (U \cup V)) = \emptyset$. Let $A = (X \setminus U) \cap \mathrm{cl}_{\beta X}(W)$ and $B = (X \setminus V) \cap \mathrm{cl}_{\beta X}(W)$. Then A and B are disjoint closed subsets of X and it is not hard to see that $p \in \mathrm{cl}_{\beta X}(A) \cap \mathrm{cl}_{\beta X}(B)$. Since X is normal, A and B can be separated by disjoint zero sets, so by (3) in Theorem 6.4, $\mathrm{cl}_{\beta X}(A) \cap \mathrm{cl}_{\beta X}(B) = \emptyset$. This is a contradiction so (e) does hold. □

Proposition 6.11 [118] Let X be a Tychonoff space. Then

$$\{\mathrm{Ex}_X(U) : U \text{ is open in } X\}$$

is a base of open subsets of βX.

Proof. Let V be any non-empty open set of βX and let $p \in V$. Let W be an open subset of βX such that $p \in W$ and $\mathrm{cl}_{\beta X}(W) \subset V$. Define $U = W \cap X$, we will show that $p \in \mathrm{Ex}_X(U) \subset V$. By (c) in Proposition 6.10, $p \in W \subset \mathrm{Ex}_X(U)$. Since U is a dense subset of W, by (b) in Proposition 6.10 we obtain that $\mathrm{cl}_{\beta X}(\mathrm{Ex}_X(U)) = \mathrm{cl}_{\beta X}(W) \subset V$. □

We need the following result. It was first known to hold for normal spaces (see [50, 7.1.14]) but aparently it was proved for all Tychonoff spaces in [144, p. 218]. See also [36, Lemma 3.2].

Proposition 6.12 Let X be a Tychonoff space and U an open subset of X. Then $\mathrm{bd}_{\beta X}(\mathrm{Ex}_X(U)) = \mathrm{cl}_{\beta X}(\mathrm{bd}_X(U))$.

Proof. First, let $p \in \mathrm{cl}_{\beta X}(\mathrm{bd}_X(U))$ and let V an open subset of βX such that $p \in V$. Since $V \cap \mathrm{bd}_X(U) \neq \emptyset$, $(V \cap X) \cap \mathrm{bd}_X(U) \neq \emptyset$. Moreover, $V \cap X$ is a non-empty open subset of X so $(V \cap X) \cap U \neq \emptyset$ and $(V \cap X) \cap (X \setminus U) \neq \emptyset$. In particular,

$V \cap U \neq \emptyset$ and $V \cap (X \setminus U) \neq \emptyset$; since V was arbitrary, $p \in \mathrm{cl}_{\beta X}(U) \cap \mathrm{cl}_{\beta X}(X \setminus U)$. By (b) in Proposition 6.10, $p \in \mathrm{cl}_{\beta X}(\mathrm{Ex}_X(U))$. Also, $p \in \mathrm{cl}_{\beta X}(X \setminus U) = \beta X \setminus \mathrm{Ex}_X(U)$. Thus, $p \in \mathrm{cl}_{\beta X}(\mathrm{Ex}_X(U)) \cap \mathrm{cl}_{\beta X}(\beta X \setminus \mathrm{Ex}_X(U)) = \mathrm{bd}_{\beta X}(\mathrm{Ex}_X(U))$.

Now assume that there is $p \in \mathrm{bd}_{\beta X}(\mathrm{Ex}_X(U)) \setminus \mathrm{cl}_{\beta X}(\mathrm{bd}_X(U))$, we shall arrive to a contradiction. Let $V = X \setminus \mathrm{cl}_X(U)$. Let $f : \beta X \to [0,1]$ be a continuous function such that $\mathrm{bd}_X(U) \subset f^{\leftarrow}(0)$ and $f(p) = 1$. Let us prove the following.

$$(*)\ p \in \mathrm{cl}_{\beta X}(U \cap f^{\leftarrow}[(1/2,1]]) \cap \mathrm{cl}_{\beta X}(V \cap f^{\leftarrow}[(1/2,1]]).$$

Let W be an open subset of X with $p \in W$, we may further assume that $W \subset f^{\leftarrow}[(1/2,1]]$. Notice that by the definition of p, $p \in \mathrm{cl}_{\beta X}(U) \cap \mathrm{cl}_{\beta X}(V)$. Thus, there are $x \in W \cap U$ and $y \in W \cap V$. So $x, y \in W$ are such that $x \in U \cap f^{\leftarrow}[(1/2,1]]$ and $y \in V \cap f^{\leftarrow}[(1/2,1]]$. This proves $(*)$.

Since $\mathrm{cl}_X(U) \cap (X \setminus U) = \mathrm{bd}_X(U)$, the function $g : X \to [-1,1]$ defined by $g\restriction_{\mathrm{cl}_X(U)} = f$ and $g\restriction_{X \setminus U} = -f$ is continuous. Let $\beta g : \beta X \to [-1,1]$ be the continuous extension guaranteed by (1) in Theorem 6.4. Notice that by $(*)$, $p \in \mathrm{cl}_{\beta X}(U \cap g^{\leftarrow}[(1/2,1]])$ and $p \in \mathrm{cl}_{\beta X}(V \cap g^{\leftarrow}[[-1,-1/2)])$. These imply that $\beta g(p) \geq 1/2$ and $\beta g(p) \leq -1/2$, respectively. Thus, we obtain a contradiction. This concludes the proof of the result. $\square$

Now that we have our nice base for βX we will return to realcompactness for a while. According to Proposition 6.9 and Theorem 0.17, every separable and metrizable space is realcompact. So one wonders if all metrizable spaces are realcompact. Unfortunately, it may be that there are non-realcompact metrizable spaces, although they have to be really big. More precisely the following is known.

Theorem 6.13 [163, Theorem 13.13] Let X be a non-compact, metrizable space. Then X is realcompact if and only if $w(X)$ is strictly less than the first measurable cardinal.

However, even if there are metrizable spaces that are not realcompact, they are nearly realcompact. A space X is *nearly realcompact* if $\beta X \setminus \nu X$ is dense in X^*. The following has been observed by Woods in [168].

Proposition 6.14 Every metrizable space is nearly realcompact.

Proof. Let X be a metizable space and let V be any open subset of βX such that $V \cap X^*$. By Proposition 6.11, there is an open subset U of X such that $\mathrm{Ex}_X(U) \subset V$ and $\mathrm{Ex}_X(U) \cap X^* \neq \emptyset$. By Lemma 0.19, there exists a discrete collection of open non-empty subsets $\{U_n : n < \omega\} \subset X$. For each $n < \omega$, let $V_n = \bigcup \{U_k : n \leq k < \omega\}$.

Define $G = \bigcap\{\mathrm{Ex}_X(V_n) : n < \omega\}$, then G is a subset of βX of type G_δ. First, notice that by (a) in Proposition 6.10, $\mathrm{Ex}_X(V_n) \cap X = V_n$ so $G \subset X^*$. To see that G is non-empty, choose $x_n \in U_n$ for each $n < \omega$. Then $D = \{x_n : n < \omega\}$ is a closed and discrete subset of X, so clearly there is $p \in \mathrm{cl}_{\beta X}(D) \setminus D$, we will prove now that $p \in G$. For each $k < \omega$, the set $D_k = \{x_n : k \leq n < \omega\}$ is such that $D_k \subset \mathrm{Ex}_X(V_n)$ and $p \in \mathrm{cl}_{\beta X}(D_k)$. Fix $k < \omega$. Notice that D_k and $X \setminus V_k$ are pairwise disjoint closed subsets of X so they are completely separated by the normality of X. Thus, by Lemma 0.8 and (3) in Theorem 6.4, $p \notin \mathrm{cl}_{\beta X}(V_k)$. This means that $p \in \mathrm{Ex}_X(V_k)$. So in fact $p \in G$ and G is non-empty.

Finally, for each $n < \omega$ we have that $\mathrm{Ex}_X(V_n) \subset \mathrm{Ex}_X(U)$ so G is a non-empty subset of βX of type G_δ that is contained in $\mathrm{Ex}_X(U)$. Thus, $V \cap (\beta X - \nu X) \neq \emptyset$. □

Notice that the sequence $D = \{x_n : n < \omega\}$ played a special role in the proof of Proposition 6.14. In fact D is just a copy of ω with the discrete topology and $\mathrm{cl}_{\beta X}(D)$ is just $\beta\omega$. In general, the following holds, its proof of this result should be easy from Theorem 6.4.

Corollary 6.15 If X is Tychonoff and $Y \subset X$ is and C^*-embedded in X, then there is a embedding $h : \beta Y \to \beta X$ such that $h\restriction_Y$ is the identity in Y and $h[\beta Y] = \mathrm{cl}_{\beta X}(Y)$.

We will finally mention that $\beta\omega$ is of great use for studying some properties of βX for non-pseudocompact X. This will be the main topic in Section 6.4. However, to study $\beta\omega$ we need to find a more concrete construction of $\beta\omega$ than the one given in this Section. This will be the topic of Section 6.2 next. We will also use some tools from Section 6.3 to give some properties of $\beta\omega$.

6.2 Stone Spaces of Boolean Algebras

The objects of study in this dissertation are topological spaces but in this moment it will be useful to use some algebraic objects to describe spaces. Both the Čech-Stone compactification and the Absolute are constructed as spaces of ideal points. In the case of the Čech-Stone compactification, we add points at infinity. For the absolute, we can think that we want to blow up each point of the space to some collection of ideal points. Both of these constructions use the concept of an *ultrafilter*: in some informal sense a filter is "a method of convergence to infinity"[4]. The important point is that ultrafilters can be decribed in a general setting by using *Boolean algebras*.

[4]See the Introduction in [29].

A *Boolean algebra* is an ordered set $(\mathcal{B}, \leq)$ with two distinguished elements $0, 1 \in \mathcal{B}$ with[5] $0 \neq 1$ and such that

(·) for every $b \in \mathcal{B}$, $0 \leq b \leq 1$;

(·) $(\mathcal{B}, \leq)$ is a lattice: this means that every two elements $a, b \in \mathcal{B}$ have a least upper bound (also called *supremum*) in $\mathcal{B}$, which we denote by $a \vee b$ and a greatest lower bound (also called *infimum*) in $\mathcal{B}$, which we denote by $a \wedge b$;

(·) $(\mathcal{B}, \leq)$ is a distributive lattice: this means that $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for every $a, b, c \in \mathcal{B}$.

(·) for every $b \in \mathcal{B}$ there exists a unique $b' \in \mathcal{B}$, which we call the complement of b, such that $b \vee b' = 1$ and $b \wedge b' = 0$.

The operations induced by $\wedge$ and $\vee$ are easily seen to be associative and commutative. If $F = \{b_0, \ldots, b_n\} \subset \mathcal{B}$, the symbol $\wedge F$ denotes $b_0 \wedge \ldots \wedge b_n$ and $\vee F$ denotes $b_0 \vee \ldots \vee b_n$ (with their obvious meaning).

An excellent reference for Boolean algebras is the "Handbook of Boolean Algebras", published in three volumes ([98], [73] and [74]). The first volume [98] starts with definitions, basic properties and constructions.

We will be interested in relating Boolean algebras to topology. First we will give some examples. If $(\mathcal{B}, \leq)$ is a Boolean algebra and $b \in \mathcal{B} \setminus \{0\}$ is an element such that $0 < c \leq b$ implies $c = b$ for all $c \in \mathcal{B}$, we say that b is an *atom*.

Example 6.16 $\mathcal{P}(X)$ is a Boolean algebra.

The archetype of a Boolean algebra is without doubts the structure $(\mathcal{P}(X), \subset)$, where X is any non-empty set. Clearly, if $Y_0, Y_1 \in \mathcal{P}(X)$, then $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$. Also, $0 = \emptyset$, $1 = X$ and $Y' = X \setminus Y$ for all $Y \in \mathcal{P}(X)$. Notice that for every $x \in X$, the element $\{x\}$ is an atom. □

Example 6.17 Algebra of clopen sets.

If X is a topological space, then $(\mathcal{CO}(X), \subset)$ is a Boolean algebra. The operations, 0 and 1 are the same as in Example 6.16. Notice that if X is connected, then $\mathcal{CO}(X) = \{\emptyset, X\}$ and if X is discrete, then $\mathcal{CO}(X) = \mathcal{P}(X)$. In some sense (Theorem 6.19), $\mathcal{CO}(X)$ tells us something about X only when X is 0-dimensional, since in this case $\mathcal{CO}(X)$ separates points. If X is 0-dimensional

[5]If a partially ordered set $(\mathcal{B}, \leq)$ satisfies the properties of a Boolean algebra with the exception that $0 = 1$, then $\mathcal{B} = \{0\}$, this one is called trivial Boolean algebra. However, for our purposes, our Boolean algebras will be non-trivial by definition.

and has no isolated points, then it is easy to see that $(\mathcal{CO}(X), \leq)$ has no atoms and is infinite. $\square$

Like in any algebraic (perhaps mathematical) structure, one must say which are the *morphisms* of that structure. A function $f : \mathcal{B}_0 \to \mathcal{B}_1$ between Boolean algebras $(\mathcal{B}_0, \leq)$ and $(\mathcal{B}_1, \leq)$ will be called a *morphism of Boolean algebras* if the following conditions hold:

- $(\cdot)$ $f(0) = 0$, $f(1) = 1$,
- $(\cdot)$ if $a, b \in \mathcal{B}_0$, then $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$, and
- $(\cdot)$ if $a \in \mathcal{B}_0$, then $f(a') = f(a)'$.

Notice that we have made an abuse of notation by not making distintion between the 0 of $\mathcal{B}_0$ and the 0 of $\mathcal{B}_1$ and similarly for 1 and the operational symbols. Also, a morphism of Boolean algebras $f : \mathcal{B}_0 \to \mathcal{B}_1$ is a

- $(\cdot)$ *monomorphism* if $f(b) = 0$ implies $b = 0$ for all $b \in B$,
- $(\cdot)$ *epimorphism* if for all $b \in \mathcal{B}_1$ there is $a \in \mathcal{B}_0$ such that $f(a) = b$,
- $(\cdot)$ *isomorphism* if f is a monomorphism and an epimorphism.

These concepts have of course some motivation from Category Theory (see [1]).

Example 6.18 Maps between spaces.

Let X and Y be topological spaces and $f : X \to Y$ a continuous function. Define $f^* : \mathcal{CO}(Y) \to \mathcal{CO}(X)$ by $f^*(A) = f^{\leftarrow}[A]$ for all $A \in \mathcal{CO}(Y)$. Then it is easy to see that f^* is a morphism of Boolean algebras. So every continuous function induces a morphism of Boolean algebras in the opposite direction. $\square$

Notice that the examples we have given up until now are in essence, clopen subsets of some space. The definition of Boolean algebras is quite abstract, so we may wonder whether there is some way to "see" what Boolean algebras are in a more concrete way. All this in answered in the following way.

Theorem 6.19 (Stone's duality[6]) The category of Boolean algebras and morphisms of Boolean algebras is the category opposite to the category of compact, Hausdorff and 0-dimensional spaces with continuous maps.

[6] As a personal comment, this is the author's favourite mathematical theorem.

Informally, what we mean is that there is a one-to one correpondence between Boolean algebras and compact, Hausdorff and 0-dimensional spaces. This correspondence is not just a bijection; it also preserves structure. However, the correct mathematical statement of Theorem 6.19 would take a lot of space and is outside the interest of this dissertation. We refer the reader to Chapter 2 of [29] or Chapter 3 of [135] for a more detailed statement and proof of Theorem 6.19.

We do give a part of Stone duality that will be interesting to us. In particular, for each Boolean algebra $\mathcal{B}$ we will construct a topological space $\mathbf{st}(\mathcal{B})$ whose algebra of clopen subsets is isomorphic to $\mathcal{B}$ (Theorem 6.22).

Let $(\mathcal{B}, \leq)$ be a Boolean algebra and $\mathcal{U} \subset \mathcal{B}$. Then $\mathcal{U}$ is called an *filter in* $\mathcal{B}$ if (a) $0 \notin \mathcal{U}$, $1 \in \mathcal{U}$; (b) if $a, b \in \mathcal{U}$ then $a \wedge b \in \mathcal{U}$; and (c) if $a \in \mathcal{U}$ and $b \in \mathcal{B}$ is such that $a \leq b$, then $b \in \mathcal{U}$. Moreover, $\mathcal{U}$ is called an *ultrafilter in* $\mathcal{B}$ if $\mathcal{U}$ is a filter and whenever $a \in B$, then either $a \in \mathcal{U}$ or $a' \in \mathcal{U}$.

Example 6.20 Principal ultrafilters.

If X is any non-empty set and $x \in X$, then the collection $\{A \in \mathcal{P} : x \in A\}$ is an ultrafilter in the Boolean algebra $\mathcal{P}(X)$. More generally, if $\mathcal{B}$ is a Boolean algebra and $b \in \mathcal{B} \setminus \{0\}$, then $\{a \in \mathcal{B} : b \leq a\}$ is an ultrafilter if and only if b is an atom. In both cases, these ultrafilters are called *principal ultrafilters* and they are completely determined by one only element of $\mathcal{B}$. $\square$

Notice that this notion of filter extends the regular, set-theoretical one: an ultrafilter in the Boolean algebra $(\mathcal{P}(X), \subset)$ is just an ultrafilter on the set X (see page xi in the Introduction).

The key idea here is: "ultrafilters are points". If $\mathcal{B}$ is a Boolean algebra, we define the *Stone space* of $\mathcal{B}$ as the set

$$\mathbf{st}(\mathcal{B}) = \{\mathcal{U} : \mathcal{U} \text{ is an ultrafilter in } \mathcal{B}\}.$$

We will give the Stone space a topology. For every $b \in \mathcal{B}$, let $\widehat{b} = \{\mathcal{U} \in \mathbf{st}(\mathcal{B}) : b \in \mathcal{U}\}$. The following is not hard to prove.

Proposition 6.21 Let $(\mathcal{B}, \leq)$ be a Boolean algebra. Then

(a) if $b, c \in \mathcal{B}$ are such that $b \neq c$, then $\widehat{b} \neq \widehat{c}$,

(b) $\mathbf{st}(\mathcal{B}) = \widehat{1}$, $\emptyset = \widehat{0}$,

(c) if $a, b \in \mathcal{B}$, then $\widehat{a \vee b} = \widehat{a} \vee \widehat{b}$ and $\widehat{a \wedge b} = \widehat{a} \wedge \widehat{b}$,

(d) if $a \in \mathcal{B}$, then $\widehat{a} = \mathbf{st}(\mathcal{B}) \setminus \widehat{b}$.

Thus, $\{\widehat{b} : b \in \mathcal{B}\}$ is a base of a 0-dimensional topology in $\mathbf{st}(\mathcal{B})$.

We will always assume that $\mathbf{st}(\mathcal{B})$ has the topology defined by the base $\{\widehat{b} : b \in \mathcal{B}\}$.

Theorem 6.22 (**Stone's Representation Theorem**) If $(\mathcal{B}, \leq)$ is a Boolean algebra, then $\mathbf{st}(\mathcal{B})$ is a compact, Hausdorff and 0-dimensional space. Moreover, the map $\lambda : \mathcal{B} \to \mathcal{CO}(\mathbf{st}(\mathcal{B}))$ defined by $\lambda(b) = \widehat{b}$ is an isomorphism of Boolean algebras.

Proof. If $\mathcal{U}, \mathcal{V} \in \mathbf{st}(\mathcal{B})$ are such that $\mathcal{U} \neq \mathcal{V}$, then there exists $b \in \mathcal{U} \setminus \mathcal{V}$. By the definition of an ultrafilter, $b' \in \mathcal{V}$. Also it is easy to see that $b' \notin \mathcal{U}$. Then $\widehat{b}$ and $\widehat{b'}$ are disjoint clopen subsets of $\mathbf{st}(\mathcal{B})$ that separate $\mathcal{U}$ and $\mathcal{V}$. Thus, $\mathbf{st}(\mathcal{B})$ is Hausdorff. By Proposition 6.21, $\mathbf{st}(\mathcal{B})$ is 0-dimensional.

Now consider an open cover $\mathcal{W}$ of $\mathbf{st}(\mathcal{B})$, we must find a finite subcover. We may assume that all elements of $\mathcal{W}$ are basic clopen sets. So $\mathcal{W} = \{\widehat{b} : b \in \mathcal{S}\}$ for some $\mathcal{S} \subset \mathcal{B}$. Assume that $\mathcal{W}$ has no finite subcover, so for each $F \in [\mathcal{S}]^{<\omega}$, $\{\widehat{b} : b \in F\}$ does not cover $\mathbf{st}(X)$. By (c) in Proposition 6.21, this means that for each $F \in [\mathcal{S}]^{<\omega}$, $\vee F \neq 1$. Thus, it is not hard to see that the collection

$$\mathcal{F} = \{c \in \mathcal{B} : \text{there is } F \in [\mathcal{S}]^{<\omega} \text{ with } \wedge F \leq c\}$$

is a filter. By the Kuratowski-Zorn lemma it is easy to find $\mathcal{U} \supset \mathcal{F}$ that is maximal with respect to the property of being a filter. Then it is easy to see that $\mathcal{U}$ is an ultrafilter and $\mathcal{U} \in \mathbf{st}(\mathcal{B}) \setminus (\bigcup \mathcal{W})$. This is a contradiction so $\mathcal{W}$ does have a finite subcover and $\mathbf{st}(\mathcal{B})$ is thus compact.

Finally, let $U \in \mathcal{CO}(\mathbf{st}(\mathcal{B}))$. By compactness, there is $F \in [\mathcal{B}]^{<\omega}$ such that $U = \bigcup \{\widehat{b} : b \in F\} = \widehat{\vee F}$. Thus, $\mathcal{CO}(\mathbf{st}(\mathcal{B})) = \{\widehat{b} : b \in \mathcal{B}\}$. The fact that λ is an isomorphism follows from Proposition 6.21. $\square$

Thus, Theorem 6.22 says that every Boolean algebra is the set of clopen subsets of some topological space. We will now give an example of the use of Theorem 6.22. It turns out that the Čech-Stone compactification of a discrete space can be expressed as a Stone space. This will give us a concrete construction of $\beta\omega$ that gives much more information than the definition we gave above. This point of view will help in Section 6.4.

Theorem 6.23 Let X be a discrete space. Define the function $e : X \to \mathbf{st}(\mathcal{P}(X))$ where $e(p) = \{A \subset X : p \in A\}$ for all $p \in X$. Then $(\mathbf{st}(\mathcal{P}(X)), e)$ is the Čech-Stone compactification of X. Moreover, $e[X]$ is precisely the set of principal ultrafilters.

Proof. The fact that $e[X]$ is the set of principal ultrafilters is easy to prove (see Example 6.20). Thus, for each $x \in X$, $\lambda(\{x\}) = \widehat{\{x\}}$ is the isolated point $\{e(x)\}$. Thus, e is an embedding. To see that $e[X]$ is dense, notice that if $A \in \mathcal{P}(X)$ and $x \in A$, then $e(x) \in \lambda(A)$. Then $e[X]$ is a topological copy of X that is dense in the compact space $\mathbf{st}(\mathcal{P}(X))$. According to (3) in Theorem 6.4, it is enough to prove that if A, B are two completely separated subsets of X, then $\mathrm{cl}_{\mathbf{st}(\mathcal{B})}(e[A]) \cap \mathrm{cl}_{\mathbf{st}(\mathcal{B})}(e[B]) = \emptyset$. But for $Y \subset X$, $e[Y]$ is easily seen to be dense in the clopen subset $\lambda(Y)$ so $\mathrm{cl}_{\mathbf{st}(\mathcal{B})}(e[Y]) = \lambda(Y)$. Notice that $A, B \subset X$ are completely separated if and only if $A \cap B = \emptyset$ (since X is discrete). Finally, if $A \cap B = \emptyset$, then $\lambda(A) \cap \lambda(B) = \lambda(A \cap B) = \emptyset$. This finishes the proof. $\square$

Stone's duality was developed by Marshal Harvey Stone in a series of papers ([149], [150], [151]). The importance of Stone's duality is in the fact that one can look at topological spaces from an algebraic point of view (and the other way around). This point of view of Stone's duality is also related to Category theory (see [1]). Thanks to this duality, many topological problems that may seem untrackable from the pure topological point of view can be attacked by the use of other techniques. A good example of this phenomenon is $\beta\omega$, see [113].

We finally define a special kind of Boolean algebras we will need later. We have already mentioned that a Boolean algebra always has suprema and infima of finite subsets. However, there may be infinite subsets that do not have suprema or infima.

Example 6.24 $\mathcal{CO}(\omega + 1)$.

For each $n < \omega$, let $b_n = \{n\} \in \mathcal{CO}(\omega + 1)$. We will now see that $B = \{b_{2n} : n < \omega\}$ has no supremum. Let $b \in \mathcal{CO}(\omega + 1)$ be any upper bound of B. Since $\omega \in \mathrm{cl}_{\omega+1}(B)$ and b is closed, then $\omega \in b$. Since b is open, there exists $m < \omega$ such that $[m, \omega] \subset b$. Let $k < \omega$ be such that $2k + 1 > m$. Then $c = b \setminus \{2k + 1\}$ is also an upper bound of B and $c < b$. Thus, b is not the supremum of B. $\square$

A Boolean algebra $\mathcal{B}$ is called *complete* if every subset of $\mathcal{B}$ has a supremum. If every countable subset of $\mathcal{B}$ has a supremum, then $\mathcal{B}$ is called *σ-complete*. It is not hard to see that in a σ-complete Boolean algebra every countable subset has an infimum and in a complete Boolean algebra every subset has an infimum.

Example 6.25 Algebras of definable sets.

Let X be a separable and metrizable space. By recursion on an ordinal α, we define the following sets.

$$\begin{aligned}
\mathbf{\Sigma}^0_0(X) &= \Big\{U \subset X : U \text{ is open}\Big\},\\
\mathbf{\Sigma}^0_\alpha(X) &= \Big\{\bigcup \mathcal{A} : \mathcal{A} \subset \bigcup\{\mathbf{\Pi}^0_\beta(X) : \beta < \alpha\}, |\mathcal{A}| \leq \omega\Big\}, \text{ for each } \alpha > 0.\\
\mathbf{\Pi}^0_\alpha(X) &= \Big\{X \setminus A : A \in \Sigma^\alpha_0\Big\}, \text{ for each } \alpha.
\end{aligned}$$

So $\mathbf{\Sigma}^0_0(X)$ are open sets, $\mathbf{\Pi}^0_0(X)$ are closed, $\mathbf{\Sigma}^0_1(X)$ are F_σ and $\mathbf{\Pi}^0_1(X)$ are G_δ . By induction it is possible to prove that for $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) \cup \mathbf{\Pi}^0_\alpha(X) \subset \mathbf{\Sigma}^0_\beta(X)$. Let $\mathcal{BOR}(X) = \bigcup\{\mathbf{\Sigma}^0_\alpha(X) : \alpha < \omega_1\}$. Sets from $\mathcal{BOR}(X)$ are called *Borel* sets of X. It is not hard to see that $\mathcal{BOR}(X)$ is a σ-complete Boolean algebra.

A set $A \subset X$ is said to have the *Baire property* if there is an open set U such that $A \triangle U$ is meager. Equivalently, $A \subset X$ has the Baire property if and only if there is $F \subset X$ of type F_σ such that $A \setminus F$ is meager (see [94, Proposition 8.23]). This is dual to the fact that a set $A \subset \mathbb{R}$ is *Lebesgue measurable* if and only if there is $F \subset \mathbb{R}$ of type F_σ such that $A \setminus F$ is of measure 0 (see [130, Theorem 3.15]).

Then $\{A \subset X : A \text{ has the Baire property}\}$ can be seen to be a σ-complete Boolean algebra that extends $\mathcal{BOR}(X)$. Borel and Baire property are concepts used to measure the complexity of a subset of X, see [94] for more on this.

Also, from the fact that there are sets without the Baire property (for example, a Bernstein set, see [94, Example 8.24]), it can be easily seen that both of these Boolean algebras are not complete. □

Just as the algebra of clopen subsets is the archetype of a Boolean algebra, we have another class of Boolean algebras that cover all complete Boolean algebras.

Let X be a topological space. A set $A \subset X$ is called *regular closed* if $A = \mathrm{cl}_X(\mathrm{int}_X(A))$. Notice that a set $A \subset X$ is regular closed if and only if there is an open set U of X such that $A = \mathrm{cl}_X(U)$.

Let

$$\mathcal{R}(X) = \{A \subset X : A \text{ is regular closed}\}.$$

The following is not hard to prove.

Proposition 6.26 If X is a regular space, then $\mathcal{R}(X)$ is a base of closed neighborhoods of X.

Proposition 6.27 If X is any topological space, then $(\mathcal{R}(X), \subset)$ is a complete Boolean algebra such that

(*a*) $\emptyset$ is the smallest element, X is the greatest one, (*b*) if $\mathcal{G} \subset \mathcal{R}(X)$, then the supremum of $\mathcal{U}$ is given by $\mathrm{cl}_X(\bigcup \mathcal{G})$,

(*c*) if $A, B \in \mathcal{R}(X)$, then the infimum of A and B is given by $\mathrm{cl}_X(\mathrm{int}_X(A \cap B))$ and

(*d*) if $A \in \mathcal{R}(X)$, then the complement of A is given by $\mathrm{cl}_X(X \setminus A) = X \setminus \mathrm{int}_X(A)$.

Notice that in the case of regular closed sets, the infimuum is not just the intersection. So $(\mathcal{R}(X), \subset)$ is not a subalgebra of $(\mathcal{P}(X), \subset)$ in general.

Let $\mathcal{B}_0$ be a Boolean algebra. A *completion* of $\mathcal{B}_0$ is a Boolean algebra monomorphism $e : \mathcal{B}_0 \to \mathcal{B}_1$ such that $\mathcal{B}_1$ is complete and for each $b \in \mathcal{B}_1$ there is $a \in \mathcal{B}_0$ such that $e(a) \leq b$. Recall that in this situation it is said that $e[\mathcal{B}_0]$ is dense in $\mathcal{B}_1$ as an order (see p. ix the Introduction). It turns out that completion is unique in the following sense.

Proposition 6.28 Let $\mathcal{B}_0$ be a Boolean algebra and let $e_1 : \mathcal{B}_0 \to \mathcal{B}_1$ and $e_2 : \mathcal{B}_0 \to \mathcal{B}_2$ be completions of $\mathcal{B}_0$. Then there is a Boolean algebra isomorphism $h : \mathcal{B}_1 \to \mathcal{B}_2$ such that $h \circ e_1 = e_2$.

Proof. For each $b \in \mathcal{B}_1$, let

$$h(b) = \vee\{e_2(c) : c \in \mathcal{B}_1, e_1(c) \leq b\}.$$

By the completeness of $\mathcal{B}_2$, h is well-defined. It is not hard to prove that h has the desired properties. □

Using completions, we can see that complete Boolean algebras are just $\mathcal{R}(X)$ for some X, in the following way.

Theorem 6.29 (a) Every Boolean algebra has a completion.

(b) In particular, if X is a 0-dimensional Hausdorff space, then $\mathcal{R}(X)$ is the completion of $\mathcal{CO}(X)$.

(c) If $\mathcal{B}$ is a complete Boolean algebra, then $\mathcal{R}(\mathbf{st}(\mathcal{B})) = \mathcal{CO}(\mathbf{st}(\mathcal{B}))$.

Proof. By Theorem 6.22, (b) implies (a). Notice that $\mathcal{CO}(X)$ is a dense suborder of $\mathcal{R}(X)$ when X is 0-dimensional. Thus, the inclusion of $\mathcal{CO}(X)$ in $\mathcal{R}(X)$ is the completion of $\mathcal{CO}(X)$ in this case. To see (c), by Theorem 6.22, $\mathcal{B}$ is isomorphic to $\mathcal{CO}(\mathbf{st}(\mathcal{B}))$, so both the identity $\mathcal{CO}(\mathbf{st}(\mathcal{B})) \to \mathcal{CO}(\mathbf{st}(\mathcal{B}))$ and the inclusion $\mathcal{CO}(\mathbf{st}(\mathcal{B})) \to \mathcal{R}(\mathbf{st}(\mathcal{B}))$ are completions. By uniqueness of completion (Proposition 6.28) we obtain that $\mathcal{R}(\mathbf{st}(B)) = \mathcal{CO}(\mathbf{st}(B))$. □

We will study spaces X such that $\mathcal{CO}(X) = \mathcal{R}(X)$ next in Section 6.3.

6.3 Extremally Disconnected Spaces and the Absolute

A space X will be called *extremally disconnected* (*ED* for short) if X is regular and for every open subset U of X we have that $\mathrm{cl}_X(U)$ is open. Notice that regularity implies 0-dimensionality of ED spaces.

As the reader will see later, ED spaces appear naturally in our work. Clearly all discrete spaces are ED but we will soon see examples of non-discrete (or even crowded) ED spaces. First let us give some important properties of ED spaces.

Proposition 6.30 If X is an ED space and $Y \subset X$ is either open or dense in X, then Y is ED.

Proof. Let V be an open subset of Y. Then there is an open subset U of X such that $V = U \cap Y$. Notice that in both cases V is a dense subset of U (if Y is open, then even $V = U$). So $\mathrm{cl}_X(V) = \mathrm{cl}_X(U)$ which is open since X is ED. Finally, notice that $\mathrm{cl}_Y(V) = \mathrm{cl}_X(V) \cap Y$ in both cases. Thus, $\mathrm{cl}_Y(V)$ is open. $\square$

Lemma 6.31 Let X be a regular space. Then X is ED if and only if for every two open subsets U and V of X we have that $U \cap V = \emptyset$ implies $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$.

Proof. First, assume that X is ED and U and V are disjoint open subsets. Then $\mathrm{cl}_X(U)$ and $\mathrm{cl}_X(V)$ are open so $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V)$ is a clopen subset of $\mathrm{bd}_X(U) \cap \mathrm{bd}_X(V)$, which is nowhere dense. Thus, $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$.

Now assume that any two disjoint open subsets of X have disjoint closures. Let U be any non-empty open subset of X and let $V = X \setminus \mathrm{cl}_X(U)$. Then $U \cap V = \emptyset$ so $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$. But $X = \mathrm{cl}_X(U) \cup V$ and $V \cap \mathrm{cl}_X(U) = \emptyset$ so $V = \mathrm{cl}_X(V)$. Then $\mathrm{cl}_X(U)$ is also open. $\square$

Following Lemma 6.31, we can localize the notion of being ED. If X is a Tychonoff space and $p \in X$, we will say that X is *extremally disconnected at* p (*ED at* p) if whenever U and V are disjoint open subsets of X we have that $p \notin \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V)$. Extremal disconnectedness can be tested locally in dense subsets by the following result.

Lemma 6.32 Let D be a dense subset of a Tychonoff space X. If X is ED at each point of D, then D is ED.

Proof. Consider two disjoint open subsets of D, $U \cap D$ and $V \cap D$ where U and V are non-empty open subsets of X. Notice that U and V must be disjoint.

Since X is ED at each point of D, we have that $\mathrm{cl}_D(U \cap D) \cap \mathrm{cl}_D(V \cap D) \subset \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) \subset X \setminus D$. Then $\mathrm{cl}_D(U \cap D) \cap \mathrm{cl}_D(V \cap D) = \emptyset$ so by Lemma 6.31, D is ED. $\square$

We also have a property of extension of continuous functions for subspaces of ED spaces as follows.

Proposition 6.33 Let X be a Tychonoff ED space and let $D \subset X$ be dense. Then D is C^*-embedded in X.

Proof. By Taimanov's Theorem 0.10 it is enough to prove that if A and B are disjoint zero sets of D, then $\mathrm{cl}_X(A) \cap \mathrm{cl}_X(B) = \emptyset$. By Lemma 0.8, A and B are completely separated in D so in particular, they can be separated by open subsets of D. But D is dense so there are disjoint open subsets U and V of X such that $A \subset U$ and $B \subset V$. Since X is ED, $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$ so $\mathrm{cl}_X(A) \cap \mathrm{cl}_X(B) = \emptyset$. This concludes the proof of the Proposition. $\square$

It turns out that ED spaces are just those spaces X such that $\mathcal{CO}(X) = \mathcal{R}(X)$. Thus, this fact shows that ED spaces are dual to complete Boolean algebras by Theorem 6.29. We will next show this, along with an analogue result for basically disconnected spaces for the sake of completeness of Part I of this thesis. Recall that in Part I we defined a space X to be *basically disconnected* (BD) if for every cozero set U of X we have that $\mathrm{cl}_X(U)$ is open.

Theorem 6.34 Let $\mathcal{B}$ be a Boolean algebra. Then $\mathbf{st}(\mathcal{B})$ is ED (BD) if and only if $\mathcal{B}$ is complete (σ-complete, respectively).

Proof. First assume that $\mathcal{B}$ is a complete Boolean algebra. Let $U \subset \mathbf{st}(\mathcal{B})$ be any open set. By Theorem 6.29, $\mathrm{cl}_{\mathbf{st}(\mathcal{B})}(U)$ is clopen. Thus, $\mathbf{st}(\mathcal{B})$ is ED.

Now assume that $\mathbf{st}(\mathcal{B})$ is ED. Let $\mathcal{B}_0 \subset \mathcal{B}$, we have to show that $\mathcal{B}_0$ has a supremuum in $\mathcal{B}$. Let

$$U = \mathrm{cl}_{\mathbf{st}(\mathcal{B})}(\bigcup\{\widehat{B} : B \in \mathcal{B}_0\}).$$

By hypothesis, U is clopen. Notice that U is then an upper bound for $\{\widehat{B} : B \in \mathcal{B}_0\}$ in the Boolean algebra $\mathcal{CO}(\mathcal{B})$. If $V \in \mathcal{CO}(\mathbf{st}(B))$ were another upper bound, then $\bigcup\{\widehat{B} : B \in \mathcal{B}_0\} \subset V$ and taking closure we obtain that $U \subset V$. Thus, if $B_0 \in \mathcal{B}$ is such that $U = \widehat{B_0}$, then $B_0 = \vee\mathcal{B}_0$.

Now we prove the analogue result for BD spaces. The proof that $\mathbf{st}(B)$ is BD implies $\mathcal{B}$ is σ-complete is completely analogous to proving that $\mathbf{st}(B)$ is ED implies $\mathcal{B}$ is complete.

So assume that $\mathcal{B}$ is σ-complete and let U be a cozero set of $\mathbf{st}(\mathcal{B})$. Since cozero sets are F_σ and closed subsets of $\mathbf{st}(\mathcal{B})$ are compact, it is not hard to find a countable family $\{B_n : n < \omega\} \subset \mathcal{B}$ such that $U = \bigcup\{\widehat{B_n} : n < \omega\}$. Let $B \in \mathcal{B}$ be such that $B = \vee\{B_n : n < \omega\}$. Notice that this implies that $U \subset \widehat{B}$. Also, U is dense in $\widehat{B}$, otherwise there is some non-empty clopen set $V \subset \widehat{B} \setminus U$, which contradicts the definition of B. Thus, $\mathrm{cl}_{\mathbf{st}(\mathcal{B})}(U) = \widehat{B}$ is open. Notice that this same argument can be modified to give another proof of the fact that $\mathcal{B}$ is complete implies $\mathbf{st}(\mathcal{B})$ is ED. □

So for example, the argument in Example 6.24 essentially showed that a convergent sequence is not ED. By the characterization of βX when X is discrete given in Theorem 6.23, we obtain the following.

Corollary 6.35 If X is a discrete space, then βX is ED.

However, there is a more general result than Corollary 6.35.

Theorem 6.36 Let X be a Tychonoff space. Then βX is ED if and only if X is ED.

Proof. If βX is ED, then X is ED by Proposition 6.30. So assume that X is ED. Let U be an open subset of βX. Then $U \cap X$ is open and thus, $\mathrm{cl}_X(U \cap X) \in \mathcal{CO}(X)$. Clopen subsets are clearly zero sets so $\mathrm{cl}_X(U \cap X)$ and $Z = X \setminus \mathrm{cl}_X(U \cap X)$ are disjoint zero sets of X. By (3) in Theorem 6.4, $\mathrm{cl}_{\beta X}(\mathrm{cl}_X(U \cap X))$ and $\mathrm{cl}_{\beta X}(Z)$ are disjoint with union βX. Thus, $\mathrm{cl}_{\beta X}(\mathrm{cl}_X(U \cap X)) = \mathrm{cl}_{\beta X}(U)$ is open. This proves that βX is ED. □

A natural question one may ask is which well-known spaces are ED. It is not hard to convince oneself that a non-discrete metrizable space is not ED by constructing two disjoint open sets with closures that intersect. Non-discrete ED spaces are in fact hard to find. Basically, Theorem 6.34, Corollary 6.35 and Proposition 6.30 give the only methods known to the author to construct compact ED spaces.

Now we will give another important construction we will use in our results. For a Tychonoff space X, define the *Gleason space* of X by $\mathbf{G}(X) = \mathbf{st}(\mathcal{R}(X))$. Define the following subspace of $\mathbf{G}(X)$:

$$EX = \{\mathcal{U} \in \mathbf{G}(X) : \bigcap \mathcal{U} \neq \emptyset\}.$$

Let $\mathcal{U} \in \mathbf{G}(X)$ and consider $p, q \in X$ with $p \neq q$. There are $A, B \in \mathcal{R}(X)$ with $p \in A$, $q \in B$ and $A \cap B = \emptyset$. Since $A \wedge B = \emptyset$ in the Boolean algebra $\mathcal{R}(X)$,

$\mathcal{U}$ contains at most one of A or B. Since $\mathcal{U}$ is an ultrafilter, this shows that $|\bigcap \mathcal{U}| \leq 1$.

Thus, we may define the function $k_X : EX \to X$ so that $k_X(\mathcal{U})$ is such that $\bigcap \mathcal{U} = \{k_X(\mathcal{U})\}$. The pair $\langle EX, k_X \rangle$ will be called the *absolute* of X. The isomorphism between the Boolean algebra $\mathcal{R}(X)$ and the clopen subsets of its Stone space EX defined in Theorem 6.22 will be denoted by $\lambda_X : \mathcal{R}(X) \to \mathcal{CO}(EX)$. We now state some of the properties of the absolute.

A function between topological spaces $f : X \to Y$ will be called *irreducible* if it is closed[7] and every time C is a closed subset of X, then $f[C] = Y$ if and only if $C = X$. So irreducible functions are in particular onto.

Proposition 6.37 Let X be a Tychonoff space. Then EX is a dense subspace of $\mathbf{G}(X)$. Thus, EX is Tychonoff and ED. Also, $k_X : EX \to X$ is a perfect and irreducible continuous function.

Proof. For $p \in X$, the collection $\{A \in \mathcal{R}(X) : p \in \mathrm{int}_X(A)\}$ can be easily seen to be a filter in the Boolean algebra $\mathcal{R}(X)$ so using the Axiom of Choice we may choose some ultrafilter $\mathcal{U}_p \in \mathbf{G}(X)$ that contains it.

Consider any basic open subset $\hat{A}$ of $\mathbf{G}(X)$, where A is an regular closed non-empty subset of X and let $p \in \mathrm{int}_X(A)$. Notice that it follows that $\mathcal{U}_p \in \widehat{A}$ and $k_X(\mathcal{U}_p) = p$. This shows that EX is dense in $\mathbf{G}(X)$ and k_X is onto. The fact that EX is ED follows from Proposition 6.30.

We will need the following two facts

($*$) If U is open in X, then $k_X^{\leftarrow}[U] = \bigcup \Big\{\lambda_X(A) \cap EX : A \in \mathcal{R}(X), A \subset U\Big\}$.

($\star$) If $B \in \mathcal{R}(X)$, then $k_x[\lambda_X(B)] = B$.

The right side of ($*$) is clearly contained in the left side. Now, let $\mathcal{U} \in EX$ be such that $p = k_X[\mathcal{U}] \in U$. By the regularity of X, there is an open set V of X such that $p \in V$ and $\mathrm{cl}_X(V) \subset U$. Let $A = \mathrm{cl}_X(V)$, since $\mathcal{U}$ is an ultrafilter, either A or its Boolean complement $X \setminus V$ is in $\mathcal{U}$. Since $p \in V$ we obtain that $A \in \mathcal{U}$ so $\mathcal{U} \in \lambda_X(A)$. This proves ($*$).

The left side of ($\star$) is clearly contained in the right side. To see the other inclusion, let $p \in B$. Let $\mathcal{V}$ be any ultrafilter extending $\{A \in \mathcal{R}(X) : p \in \mathrm{int}_X(A)\} \cup \{B\}$. Then it follows that $\mathcal{V} \in \lambda_X(B)$ and $k_X(\mathcal{V}) = p$. Thus, ($\star$) holds.

[7] Other authors do not require irreducible functions to be closed. Our definition is chosen to avoid repetition.

By $(*)$, k_X is continuous. Next, we see that k_X is irreducible. Let $F \subset EX$ be a closed, proper subset. By the definition of the topology of $\mathbf{st}(\mathcal{R}(X))$, there is a non-empty $A \in \mathcal{R}(X)$ such that $F \cap \lambda_X(A) = \emptyset$. By $(\star)$, $k_X[\lambda_X(\mathrm{cl}_X(X \setminus A))] = \mathrm{cl}_X(X \setminus A)$ which is a proper subset. Since $F \subset \lambda_X(\mathrm{cl}_X(X \setminus A))$, $k_X[F]$ is a proper subset. Now, let $p \in X \setminus k_X[F]$. Since $\mathcal{U}_p$ is an ultrafilter that is not in F, there is a non-empty regular closed set $A \subset X$ such that $F \cap \lambda_X(A) = \emptyset$ and $\mathcal{U}_p \in \lambda_X(A)$. Then $k_X[F] \subset k_X[\lambda_X(\mathrm{cl}_X(X \setminus A))] = \mathrm{cl}_X(X \setminus A)$ and $p \in k_X[A] \setminus k_X[F] = A \setminus \mathrm{cl}_X(X \setminus A) = \mathrm{int}_X(A)$ by $(\star)$. Thus, $p \in \mathrm{int}_X(A)$ and $\mathrm{int}_X(A) \cap k_X[F] = \emptyset$, this proves that k_X is closed.

Finally, we prove that k_X is perfect, we only need to prove that it has compact fibers. Since $\mathbf{G}(X)$ is compact, it is enough to prove that each fiber is a closed subset of $\mathbf{G}(X)$. Let $p \in X$ and assume that $\mathcal{U} \in \mathbf{G}(X)$ is such that $k_X[\mathcal{U}] \neq p$. Notice that $\mathcal{U}$ does not extend $\{A \in \mathcal{R}(X) : p \in \mathrm{int}_X(A)\}$, otherwise $\bigcap \mathcal{U} = \{p\}$ and $\mathcal{U}$ would map to p under k_X. So there is some $B \in \mathcal{R}(X)$ such that $p \in \mathrm{int}_X(B)$ but $B \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, $C = \mathrm{cl}_X(X \setminus B) \in \mathcal{U}$ so $\mathcal{U} \in \lambda_X(C)$. If $V = \mathrm{int}_X(B)$, then $k_X^{\leftarrow}(p) \subset k_X^{\leftarrow}[V]$. By $(*)$ and the fact that $\lambda_X(C) \cap \lambda_X(A) = \lambda_X(C \wedge A) = \emptyset$ for each $A \in \mathcal{R}(X)$ with $A \subset V$, we obtain that $\lambda_X(C) \cap k_X^{\leftarrow}(p) = \emptyset$. Thus, $k_X^{\leftarrow}(p)$ is closed in $\mathbf{G}(X)$ so it is compact. $\square$

Also, we have the following immediate consequence of Theorem 6.37.

Corollary 6.38 For every Tychonoff space X, the following are equivalent:

(a) X is compact,

(b) EX is compact,

(c) $\mathbf{G}(X) = EX$.

Notice also that k_X is a homeomorphism if and only if X is ED.

With this we can notice the following essential difference between the construction of βX with ultrafilters and the construction of EX. In Theorem 6.23 it was shown that for every point p of a discrete space X there is an ultrafilter $e(p)$ and in some sense $e(p)$ is the *only* ultrafilter that converges to p (since e is an embedding). In other constructions of βX for an arbitrary topological space X using ultrafilters, one gives an injective correspondence from points of X to ultrafilters in some sense (see for example, [161, 1.37]). However, in the case of the absolute, for one point $p \in X$ we may have several ultrafilters that converge to p, $\mathcal{U}_p$ in the proof of Proposition 6.37 was only one of them. More precisely, $|k_X^{\leftarrow}(p)| > 1$ in general.

Example 6.39 For every $r \in \mathbb{R}$, $|k_{\mathbb{R}}^{\leftarrow}(r)| \geq |\omega^*|$.

We simplify the notation by assuming, without loss of generality, that $r = 0$. For every $\mathcal{U} \in \omega^*$ we shall construct $\tilde{\mathcal{U}} \in E\mathbb{R}$ such that $k_{\mathbb{R}}[\tilde{\mathcal{U}}] = 0$ and if $\mathcal{U}_0, \mathcal{U}_1 \in \omega^*$ are such that $\tilde{\mathcal{U}_0} = \tilde{\mathcal{U}_1}$, then $\mathcal{U}_0 = \mathcal{U}_1$. For each $n < \omega$, let $x_n = \frac{1}{n+1}$ and let I_n be the middle-third closed interval between x_{n+1} and x_n, more specifically:

$$I_n = \left[\frac{2}{3} \cdot x_{n+1} + \frac{1}{3} \cdot x_n, \frac{1}{3} \cdot x_{n+1} + \frac{2}{3} \cdot x_n\right].$$

For each $A \subset \omega$, let $I_A = (\bigcup\{I_n : n < \omega\}) \cup \{0\}$. Notice that for each infinite $A \subset \omega$, $I_A \in \mathcal{R}(\mathbb{R})$. Using this, for each $\mathcal{U} \in \omega^*$, it is possible to find an ultrafilter $\tilde{\mathcal{U}}$ in the Boolean algebra $\mathcal{R}(\mathbb{R})$ that extends the collection

$$\{B \in \mathcal{R}(\mathbb{R}) : 0 \in \mathrm{int}_X(B)\} \cup \{I_A : A \in \mathcal{U}\}.$$

The fact that these ultrafilters are as requested is left to the reader. □

Figure 6.2: For $r \in \mathbb{R}$, $|k_{\mathbb{R}}^{\leftarrow}(r)| \geq |\omega^*|$.

In Theorem 6.47 we will see that $|\omega^*| = 2^{\mathfrak{c}}$ so each fiber of $k_{\mathbb{R}}$ is of size $2^{\mathfrak{c}}$. Later we will be interested in cases when k_X has fibers of cardinality 1 to obtain homeomorphisms, Proposition 7.12 is analogous to this fact. For the time being, we prove the following.

Lemma 6.40 Let X be a Tychonoff space and $p \in X$. Then $|k_X^{\leftarrow}(p)| = 1$ if and only if X is ED at p.

Proof. Consider the neighborhood filter of regular closed neighborhoods of p, $\mathcal{F} = \{A \in \mathcal{R}(X) : p \in \mathrm{int}_X(A)\}$. If X is ED at p, it is not hard to see that $\mathcal{F}$ is in fact a ultrafilter in the Boolean algebra $\mathcal{R}(X)$ so in this case $k_X^{\leftarrow}(p) = \{\mathcal{F}\}$.

If X is not ED at p, there are disjoint open subsets U and V such that $p \in \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V)$. Define $B_0 = \mathrm{cl}_X(U)$ and $B_1 = \mathrm{cl}_X(V)$. Then $B_0, B_1 \in \mathcal{R}(X)$ and $p \in B_0 \cap B_1$ but $B_0 \wedge B_1 = \mathrm{cl}_X(\mathrm{int}_X(B_0 \cap B_1)) = \emptyset$. For each $i \in 2$, the collection

$$\{B \in \mathcal{R}(X) : \text{ there is } A \in \mathcal{F} \text{ such that } A \cap B_i \subset B\}$$

is a filter in the Boolean algebra $\mathcal{R}(X)$ so it is contained in an ultrafilter $\mathcal{U}_i$. Then $\mathcal{U}_0, \mathcal{U}_1 \in EX$, $\mathcal{U}_0 \neq \mathcal{U}_1$ and $k_X[\mathcal{U}_i] = \{p\}$ for $i \in 2$. □

Now we know the existence of the absolute, we will state its topological characterization as we did with the Čech-Stone compactification.

Theorem 6.41 Let X be a Tychonoff space. Let $f_0 : Y_0 \to X$, $f_1 : Y_1 \to X$ be perfect and irreducible continuous functions where Y_0 and Y_1 are ED. Then there exists a homeomorphism $h : Y_0 \to Y_1$ such that $f_0 = f_1 \circ h$.

The proof of Theorem 6.41 is technical and long. We will not include a detailed proof as this would require too much space and background results. However, the idea of the proof follows one simple idea: every point of Y_0 is an ultrafilter in $\mathcal{R}(X)$ and the same goes for points in Y_1, one just has to pair them so that they have the same combinatorial properties.

$$\begin{tikzcd} Y_0 \arrow[rr, "h"] \arrow[dr, "f_0"'] & & Y_1 \arrow[dl, "f_1"] \\ & X & \end{tikzcd}$$

We will give a brief sketch of this idea in hope that the reader gets at least an intuitive feeling of what the proof entails. For a detailed proof see [135, 6.7].

Sketch of proof. For each $y \in Y_0$, let

$$\mathcal{F}_y = \{\mathrm{cl}_{Y_1}(f_1^{\leftarrow}[X \setminus f_0[Y_0 \setminus U]]) : U \text{ is open in } Y_0 \text{ and } y \in U\}$$

and define $h(y) \in Y_1$ as the only point in the intersection $\bigcap \mathcal{F}_y$. With some work, it is possible to prove that h is a well-defined continuous function. Moreover, h defined in this way satisfies $f_0 = f_1 \circ h$. To show that h is a homeomorphism, it is possible to define a function $h' : Y_1 \to Y_0$ in an analogous way to h. That is, for each $y \in Y_1$, let

$$\mathcal{G}_y = \{\mathrm{cl}_{Y_0}(f_0^{\leftarrow}[X \setminus f_1[Y_1 \setminus U]]) : U \text{ is open in } Y_1 \text{ and } y \in U\},$$

and define $h'(y)$ to be the only point in the intersection $\bigcap \mathcal{G}_y$. Then h' is also continuous and h' is the inverse function of h. This proves that h is a homeomorphism. $\square$

Thus, if Y is a Tychonoff ED space and $f : Y \to X$ is a perfect and irreducible continuous function where X is Tychonoff, we can say that $\langle Y, f\rangle$ is the absolute of X. Two Tychonoff spaces X and Y will be called *coabsolute* if they have the same absolute, more formally, if there is a ED Tychonoff space E and two perfect and irreducible continuous functions $f : E \to X$, $g : E \to Y$.

Corollary 6.42 If $f : X \to Y$ is a perfect and irreducible continous function between Tychonoff spaces, then X and Y are coabsolute.

Proof. If we consider the absolute $k_X : EX \to Y$ and take the composition $f \circ k_X : EX \to Y$, by Theorem 6.41 we notice that $\circ EX, f \circ k_X$ is the absolute of Y. Thus, X and Y are coabsolute. $\square$

We give our first application of the characterization by relating the Čech-Stone compactification with the absolute. More generally, the absolute is "preserved" under dense subsets as the following shows. Its proof follows easily from Proposition 6.30 and Theorem 6.41.

Proposition 6.43 Let X be a Tychonoff space and $D \subset X$ a dense subspace. If $f : Y \to X$ is the absolute of X, then $f{\restriction}_{f^{\leftarrow}[D]}: f^{\leftarrow}[D] \to D$ is the absolute of D.

Being an ED space, the absolute is a very complicated space in general. We give an example of the absolute, relating it to the Čech-Stone compactification.

Example 6.44 The absolute of a convergent sequence.

The convergent sequence $\omega + 1 = \omega \cup \{\omega\}$ is perhaps the simpliest compact Hausdorff space. Define the function $f : \beta\omega \to \omega + 1$ by $f(n) = n$ for $n < \omega$ and $f(p) = \omega$ for $p \in \omega^*$. It is not hard to prove that $\langle \beta\omega, f \rangle$ is in fact the absolute of $\omega + 1$. Notice that the point $\{\omega\}$ was "blown up" to the whole remainder ω^*.

In fact, if X is a discrete space and K is a compact space containing X densely, by Theorem 6.4 there is a continuous function $f : \beta X \to K$ such that $f{\restriction}_X$ is the identity function. Then it is easy to see that $\langle \beta X, f \rangle$ is the absolute of K. $\square$

A question that has been asked by several authors is whether given some Tychonoff space X, one can find all Tychonoff spaces Y such that X and Y are coabsolute. This question is related to our results as we will see in Corollary 7.13.

For the sake of completeness, we present some properties of irreducible functions that we will use later.

Lemma 6.45 Let $f : X \to Y$ be an irreducible continuous function between Tychonoff spaces.

(a) If $D \subset Y$ is dense in Y, then $f^{\leftarrow}[D]$ is dense in X.

(b) If $C \subset X$ is closed and nowhere dense in X, then $f[C]$ is closed and nowhere dense in Y.

(c) If $\beta f : \beta X \to \beta Y$ is the unique continuous extension of f, then βf is irreducible.

Proof. For (a), notice that if U is an open set of X with $U \cap f^{\leftarrow}[D] = \emptyset$, then $f^{\sharp}[U]$ is a non-empty open subset of Y disjoint from Y.

For (b), notice that $f[C]$ is closed since f is closed so we only have to prove that it is nowhere dense. Assume this is not the case and let $U \subset f[C]$ be a non-empty open set. Since C is closed and nowhere dense, then $V = f^{\leftarrow}[U] \setminus C$ is a non-empty open subset of X. Then $f[X \setminus U] = Y$ which contradicts the fact that f is irreducible. This proves (b).

Now we prove (c), notice that βf is immediately closed and onto. Let $F \subset \beta X$ be a closed proper subset of βX. Then there exists a non-empty open subset $U \subset X$ such that $\mathrm{Ex}_X(U) \cap F = \emptyset$ by Proposition 6.11. Let $G = X \setminus U$, then $F \subset \beta X \setminus \mathrm{Ex}_X(U) = \mathrm{cl}_{\beta X}(G)$. Since f is irreducible, then $V = Y \setminus f[G]$ is open and non-empty. Then $\beta f[F] \subset \beta f[\mathrm{cl}_{\beta X}(G)] \subset \mathrm{cl}_{\beta Y}(f[G]) = \mathrm{cl}_{\beta Y}(Y \setminus V)$, which is disjoint from V. Thus, $\beta f[F]$ is a proper subset of βY, this proves (c). $\square$

6.4 The importance of $\beta\omega$

In this section, we give some properties of $\beta\omega$ and applications to non pseudocompact spaces. Recall that by Theorem 6.23, we have the following algebraic model of $\beta\omega$: $\beta\omega = \mathbf{st}(\mathcal{P}(\omega))$, each $n \in \omega$ is identified to the principal ultrafilter $\{A \subset \omega : n \in A\}$ and the family $\{\widehat{A} : A \subset \omega\}$ is the family of all clopen subsets of $\beta\omega$ that moreover is a base.

Theorem 6.46 $w(\beta\omega) = \mathfrak{c}$.

Proof. By Lemma 0.15, there is a set $N \subset \mathcal{P}(\omega)$ of cardinality $w(\beta\omega)$ such that $\mathcal{B} = \{\widehat{A} : A \in N\}$ is a base. We may assume that $\mathcal{B}$ is closed under finite unions. If $A \subset \omega$, by compactness there is $m < \omega$ and $A_0, \ldots, A_m \in N$ such that $\widehat{A_0} \cup \ldots \cup \widehat{A_m} = \widehat{A}$. This implies that $A = A_0 \cup \ldots \cup A_m$ and since $\mathcal{B}$ is closed under finite unions, $A \in N$. This shows that $N = \mathcal{P}(\omega)$ so in fact $w(\beta\omega) = |\mathcal{P}(\omega)| = \mathfrak{c}$. $\square$

Now we calculate the cardinality of $\beta\omega$.

Theorem 6.47 $|\beta\omega| = 2^{\mathfrak{c}}$.

Notice that since $\mathbf{st}(\mathcal{P}(\omega)) \subset \mathcal{P}(\mathcal{P}(\omega))$, we have that $|\beta\omega| \leq 2^{\mathfrak{c}}$. So the difficult part is to prove that $|\beta\omega| \geq 2^{\mathfrak{c}}$. We will give two proofs. Pospíšil's original proof ([136]) consists in constructing a compactification of ω with remainder of power $2^{\mathfrak{c}}$. Later, S. Mrówka ([123]) gave a simplier proof in which it is shown that it was not necessary to construct such compactification. Our first proof is perhaps a summary of both proofs, notice it is of topological nature.

First Proof. Let X be the Cantor set with its usual topology (or any other compact crowded metrizable space) and fix some countable base $\mathcal{B}$ of X. Let $K = {}^X X$, that is, the set of functions $f : X \to X$ with the product topology. Clearly, K is a compact Hausdorff space and $|K| = 2^{\mathfrak{c}}$.

Let $\mathcal{G}$ be the set such that if $F \in \mathcal{G}$, then F is a finite collection of pairs $\langle B, C\rangle$ where $B, C \in \mathcal{B}$ and if $\langle B_0, C_0\rangle, \langle B_1, C_1\rangle \in F$ are such that $B_0 \cap B_1 \neq \emptyset$, then $\langle B_0, C_0\rangle = \langle B_1, C_1\rangle$. For each $F \in \mathcal{G}$, choose a function $f_F \in K$ such that if $\langle B, C\rangle \in F$, then $f_F[B] \subset C$. Notice that such f_F can be even chosen to be a continuous function, although this is not essential to the proof.

Let $D = \{f_F : F \in \mathcal{G}\}$. Then D is a countable subset of K and it is not hard to see that D is dense in K. Let $e : \omega \to D$ be a bijection, clearly e is a continuous function. By (2) in Theorem 6.4, there is a continuous function $f : \beta\omega \to K$ such that $f\restriction_\omega = e$. Since D is dense and $\beta\omega$ is compact, f is onto. Thus, $|\beta\omega| \geq |K| = 2^{\mathfrak{c}}$. $\square$

Our second proof is by Hausdorff [76] and is set-theoretic in nature. It is interesting to notice that Hausdorff's proof was given before the definition of the Čech-Stone compactification (and was therefore stated in another language).

Second Proof. According to Theorem 6.23, it is enough to prove that there are $2^{\mathfrak{c}}$ ultrafilters on ω. Let $\mathcal{F}_\omega = \{A \subset \omega : \omega \setminus A \text{ is finite}\}$ be the Frechet filter. Notice that every ultrafilter on ω that extends $\mathcal{F}_\omega$ is non-principal so it is contained in ω^*. Then we need to find $2^{\mathfrak{c}}$ different ultrafilters that extend $\mathcal{F}_\omega$. For this, we will construct an independent family.

Assume that $\mathcal{A} \subset \mathcal{P}(\omega)$ is an independent family. Then for every function $\phi : \mathcal{A} \to 2$ define

$$\mathcal{F}_\phi = \mathcal{F}_\omega \cup \{A : A \in \mathcal{A}, \phi(A) = 0\} \cup \{\omega \setminus A : A \in \mathcal{A}, \phi(A) = 1\}.$$

Then $\mathcal{F}_\phi$ has the finite intersection property since $\mathcal{A}$ is an independent family so it is contained in an ultrafilter $\mathcal{U}_\phi$. If $\phi \neq \psi$, let $A \in \mathcal{A}$ be such that $\phi(A) = 0$ and $\psi(A) = 1$, without loss of generality. Then $A \in \mathcal{U}_\phi$ and $A \notin \mathcal{U}_\psi$ so $\mathcal{U}_\phi \neq \mathcal{U}_\psi$. This proves that $\{\mathcal{U}_\phi : \phi \in {}^{\mathcal{A}}2\}$ is a family of $2^{|\mathcal{A}|}$ ultrafilters on ω. Thus, it is enough to find an independent family of size $\mathfrak{c}$ consisting of subsets of ω in order to show that $|\beta\omega| \geq 2^{\mathfrak{c}}$.

Let N be the set of pairs $\langle F, \mathcal{S}\rangle$ where $F \subset \omega$ is finite and $\mathcal{S} \subset \mathcal{P}(F)$. Then N is a countable infinite set. We will define an independent family $\mathcal{A} \subset \mathcal{P}(N)$ of size $\mathfrak{c}$, clearly this is enough to finish the proof.

For each $A \subset \omega$, define $\tilde{A} = \{\langle F, \mathcal{S}\rangle \in N : A \cap F \in \mathcal{S}\}$. We now argue that $\mathcal{A} = \{\tilde{A} : A \subset \omega\}$ is an independent family of size $\mathfrak{c}$.

First, let $A, B \subset \omega$ such that $A \neq B$. So there exists, without loss of generality, $x \in A \setminus B$. So $\langle \{x\}, \{\{s\}\}\rangle \in \tilde{A} \setminus \tilde{B}$. This proves that $|\mathcal{A}| = \mathfrak{c}$. To prove that $\mathcal{A}$ is independent, let $A_0, \ldots, A_m, B_0, \ldots, B_n$ be distinct subsets of ω, we must prove that

$$J = \tilde{A_0} \cap \ldots \tilde{A_m} \cap (N \setminus \tilde{B_0}) \cap \ldots \cap (N \setminus \tilde{B_n}) \neq \emptyset.$$

For each pair $\langle i, j\rangle \in m \times n$, $A_i \neq B_j$ so there is some $x(i,j) \in A_i \bigtriangleup B_j$. Let $G = \{x(i,j) : \langle i, j\rangle \in m \times n\}$. Then it is not hard to see that

$$\langle G, \{A_0 \cap G, \ldots, A_m \cap G\}\rangle \in \tilde{A_0} \cap \ldots \tilde{A_m} \cap (N \setminus \tilde{B_0}) \cap \ldots \cap (N \setminus \tilde{B_n}).$$

This shows that $\mathcal{A}$ is indeed independent and finishes the proof. $\square$

However, we can do more. Any infinite Hausdorff space has a countable discrete subset. So by the following Proposition 6.48, every infinite closed subset of $\beta\omega$ is of weight $\mathfrak{c}$ and cardinality $2^{\mathfrak{c}}$.

Proposition 6.48 If N is a countable discrete subset of $\beta\omega$, then $\mathrm{cl}_{\beta\omega}(N)$ is homeomorphic to $\beta\omega$.

Proof. Notice that $\mathrm{cl}_{\beta\omega}(N)$ is a compact space in which N is densely embedded. Since N is homeomorphic to ω, by Theorem 6.4 it is enough to prove that N is C^*-embedded in $\beta\omega$. By the Urysohn Extension Theorem 0.9, the fact that N is discrete and $\beta\omega$ is normal, it is enough to prove that if A and B are disjoint subsets of N, then $\mathrm{cl}_{\beta\omega}(A) \cap \mathrm{cl}_{\beta\omega}(B) = \emptyset$.

We may assume that $N = A \cup B$. Let $A = \{x_n : n < \omega\}$ and $B_n = \{y_n : n < \omega\}$ be enumerations. Recursively, it is possible to construct clopen sets $\{U_n : n < \omega\}$ and $\{V_n : n < \omega\}$ of $\beta\omega$ such that

- $(\cdot)$ if $n < \omega$, then $x_n \in U_n$, $y_n \in V_n$ and
- $(\cdot)$ if $m, n < \omega$, then $U_m \cap V_n = \emptyset$.

Let $U = \bigcup\{U_n : n < \omega\}$ and $V = \bigcup\{V_n : n < \omega\}$. Then U and V are disjoint open subsets of $\beta\omega$ with $A \subset U$ and $B \subset V$. By Corollary 6.35, $\beta\omega$ is ED so it follows that $\mathrm{cl}_{\beta\omega}(U) \cap \mathrm{cl}_{\beta\omega}(V) = \emptyset$. Thus, $\mathrm{cl}_{\beta\omega}(A) \cap \mathrm{cl}_{\beta\omega}(B) = \emptyset$. This finishes the proof. $\square$

As we saw on Example 6.5, there are spaces whose Čech-Stone compactification is obtained by adding just one point. This is not true for non-pseudocompact spaces because there are copies of $\beta\omega$, as the following result shows.

Proposition 6.49 Let X be a Tychonoff space and G a non-empty subset of βX of type G_δ. If $G \subset X^*$, then G contains a topological copy of ω^*. In particular, if X is non-pseudocompact, X^* contains a topological copy of ω^* and $|X^*| \geq 2^{\mathfrak{c}}$.

Proof. Assume that G is a subset of βX of type G_δ such that $G \subset X^*$. It is not hard to find a continuous function $f : \beta X \to [0,1]$ such that $f^{\leftarrow}(0) \subset G$. Since X is dense in βX, there is a countable discrete $N = \{x_n : n < \omega\} \subset X$ such that $f(x_n) > f(x_{n+1}) > 0$ for all $n < \omega$ and $\lim_{n\to\infty} f(x_n) = 0$.

Notice that $\mathrm{cl}_{\beta X}(N) \setminus N \subset X^*$ so if we prove that N is C^*-embedded in X, then by Theorem 6.4, there is a topological copy of ω^* in X^*. By the Urysohn Extension Theorem 0.9 and Lemma 0.8, it is enough to prove that every time $N = A_0 \cup A_1$ is a partition, then A_0 and B_0 are contained in disjoint zero sets.

To see this, let $\{J_n : n < \omega\}$ be a collection of pairwise disjoint closed intervals of $(0,1]$ such that $x_n \in J_n$. Then

$$Z_i = f^{\leftarrow}\big[(\bigcup\{J_n : n \in A_i\}) \cup \{0\}\big] \cap X$$

is a zero set of X that contains A_i for $i \in 2$ and $Z_0 \cap Z_1 = \emptyset$ since $f^{\leftarrow}(0) \cap X = \emptyset$. This shows that N is C^*-embedded in X and completes the proof. $\square$

In particular, we obtain the following.

Corollary 6.50 If X is any non-compact space and $p \in X^*$, then $\{p\}$ is not of type G_δ.

6.5 Non-homogeneity of ω^*

In this section, we will give a summary of some results related to the non-homogeneity of ω^*. Recall that a space X is *homogeneous* if every time $x, y \in X$ there exists a homeomorphism $h : X \to X$ such that $h(x) = y$. So in some way, this means that you cannot distinguish one point from another in a homogeneous space.

A discrete space is homogeneous and at first sight any two non-principal ultrafilters “look the same”. So a natural question is whether ω^* is homogeneous. In [142], Walter Rudin showed that, assuming **CH**, some points of ω^* are P-points, but there are also non-P-points. We shall only prove this result and give a summary of the further results on non-homogeneity of ω^*.

If X is a topological space and $p \in X$, then p is called a *P-point of X* if every time $\{U_n : n < \omega\}$ are open subsets of X with $p \in \bigcap\{U_n : n < \omega\}$, then there is an open set U such that $p \in U \subset \bigcap\{U_n : n < \omega\}$ (see Definition 2.21).

From Theorem 6.23, ω^* is the space of non-principal ultrafilters on ω. If $\mathcal{U}$ is an ultrafilter in ω, then its neighborhoods in ω^* are of the form $\widehat{A} \cap \omega^*$ for $A \in \mathcal{U}$. In $\beta\omega$, by the Stone Representation Theorem 6.22, clearly $\widehat{A} \subset \widehat{B}$ if and only if $A \subset B$. However, in ω^* the following holds.

Lemma 6.51 Let $A, B \subset \omega$. Then $\widehat{A} \cap \omega^* \subset \widehat{B} \cap \omega^*$ if and only if $A \setminus B$ is finite.

Proof. The result follows easily from the fact that is $A \setminus B$ is infinite, every ultrafilter extending $\mathcal{F}_\omega \cup \{A \setminus B\}$ (where $\mathcal{F}_\omega$ is the Fréchet filter) is a non-principal ultrafilter in $(\widehat{A} \cap \omega^*) \setminus (\widehat{B} \cap \omega^*)$. □

If $A, B \subset \omega$, we define the *almost inclusion* $A \subset^* B$ if $A \setminus B$ is finite. If $\mathcal{A}$ is a family of sets, then a *pseudointersection* of $\mathcal{A}$ is a set B such that $B \subset^* A$ for all $A \in \mathcal{A}$.

Theorem 6.52 **CH** implies that ω^* has P-points.

Proof. By the discussion above, we must find an ultrafilter $\mathcal{U}$ such that every time $\{A_n : n < \omega\} \subset \mathcal{U}$, then there is $A \in \mathcal{U}$ such that $A \subset^* A_n$ for all $n < \omega$. We will construct $\mathcal{U}$ by a recursion of length $\omega_1 = \mathfrak{c}$.

Let $\{A_\alpha : \alpha < \omega_1\}$ be an enumeration of $\mathcal{P}(\omega)$ and let $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ be an enumeration of $[[\omega]^\omega]^\omega$ such that every set repeats cofinally often (this is possible by Lemma 0.1). We will construct a sequence of filters $\{\mathcal{F}_\alpha : \alpha < \omega_1\}$ on ω such that

(a) if $\alpha < \beta < \omega_1$, then $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$;

(b) if $\alpha < \omega_1$, then $\mathcal{F}_{\alpha+1} \cap \{A_\alpha, \omega \setminus A_\alpha\} \neq \emptyset$;

(c) if $\alpha < \omega_1$ and $\mathcal{A}_\alpha \subset \mathcal{F}_\alpha$, then there is $B \in \mathcal{F}_{\alpha+1}$ such that $B \subset^* A$ for all $A \in \mathcal{A}_\alpha$,

(d) if $\alpha < \omega_1$, then there is a countable set $N_\alpha \subset \mathcal{F}_\alpha$ such that $\mathcal{F}_\alpha$ is the filter generated by N_α.

Define $\mathcal{F}_0 = \mathcal{F}_\omega$, the Fréchet filter. If $\alpha < \omega_1$ is a limit ordinal, let $F_\alpha = \bigcup\{F_\beta : \beta < \alpha\}$, this is a filter by (a). To see (d) when $\alpha < \omega_1$ is a limit ordinal, let $\{\alpha(n) : n < \omega\} \subset \alpha$ be such that $\sup\{\alpha(n) : n < \omega\} = \alpha$ and notice that the set $\bigcup\{N_{\alpha(n)} : n < \omega\}$ is countable and generates $\mathcal{F}_\alpha$ so let N_α be this set.

Once the construction is finished, let $\mathcal{U} = \bigcup\{\mathcal{F}_\alpha : \alpha < \omega_1\}$. Then $\mathcal{U}$ is an ultrafilter by (a) and (b). Since $\mathcal{F}_\omega \subset \mathcal{U}$, then $\mathcal{U}$ is non-principal. If $\{A_n : n < \omega\} \subset \mathcal{U}$, let $\alpha < \omega_1$ be such that $\{A_n : n < \omega\} \subset \mathcal{F}_\alpha$. Since sequences of infinite subsets are listed cofinally often, there is a $\beta \in (\alpha, \omega_1)$ such that $\mathcal{A}_\beta = \{A_n : n < \omega\}$ and by (c) there is a pseudointersection $B \in \mathcal{F}_{\beta+1} \subset \mathcal{U}$ of $\mathcal{A}$. This proves that $\mathcal{U}$ is a P-point.

Now let us show that this construction is indeed possible. We are only left to show that it is possible to carry out the succesor step. So assume that $\mathcal{F}_\beta$ is defined. If $\mathcal{A}_\beta \not\subset \mathcal{F}_\beta$, let $\mathcal{G} = \mathcal{F}_\beta$ and $M = N_\beta$.

Otherwise, by (d), let $N_\beta = \{B_k : k < \omega\}$ be an enumeration and let $\mathcal{A}_\beta = \{C_k : k < \omega\}$ be an enumeration. We must find $B \subset \omega$ that is a pseudointersection of $\mathcal{A}_\beta$ and such that $N_\beta \cup \{B\}$ has the finite intersection property. In this way, $N_\beta \cup \{B\}$ generates a filter that extends $\mathcal{F}_\beta$ and has a pseudointersection of A_β.

Let $[\omega]^{<\omega} = \{F_n : n < \omega\}$ be an enumeration such that every set repeats cofinally often (Lemma 0.1). Recursively, for each $n < \omega$, choose

$$x_n \in \big(\big(\bigcup\{B_k : k \in F_n\}\big) \cap (C_0 \cup \ldots \cup C_n)\big) \setminus \{x_0, \ldots, x_{n-1}\},$$

this is possible because we are considering finitely many sets contained in $\mathcal{F}_\beta$ and $\mathcal{F}_\beta$ contains the Fréchet filter. Finally, let $B = \{x_k : k < \omega\}$. It is not hard to see that B has the properties requested. Thus, define $\mathcal{G}$ to be the filter generated by $N_\beta \cup \{B\}$, or equivalently, the filter generated by $\mathcal{F}_\beta \cup \{B\}$. Also, define $M = N_\beta \cup \{B\}$.

Finally, consider $\mathcal{G}$ as defined in any of the two cases. There must be some $C \in \{A_\beta, \omega \setminus A_\beta\}$ such that $\mathcal{G} \cup \{C\}$ generates a filter. Let $\mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{G} \cup \{C\}$ and let $N_{\beta+1} = M \cup \{C\}$. Then all conditions of the construction hold. This finishes the construction and the proof of the Theorem. □

Corollary 6.53 **CH** implies that ω^* has a dense set of P-points.

Proof. Every clopen subset of ω^* is of the form $\widehat{A} \cap \omega^* = \mathrm{cl}_{\beta\omega}(A) \setminus A$ so it is homeomorphic to ω^* by Proposition 6.48. The result follows from Theorem 6.52 and the fact that ω^* is 0-dimensional. □

Corollary 6.54 **CH** implies that ω^* is not homogeneous.

Proof. Clearly, being a P-point is a topological property. By Theorem 6.52, ω^* has P-points. However, not every point of ω^* (or of any infinite compact Hausdorff space) is a P-point: let N be a countable discrete subset of ω^*, then no point of $\mathrm{cl}_{\omega^*}(N)$ is a P-point. Thus, there are at least two types of points: P-points and non-P-points. □

Of course, it is desirable to have a ZFC proof of the non-homogeneity of ω^*. The first such proof was given in Zdeněk Frolík.

Theorem 6.55 [64] There is a set $S \subset \omega^*$ such that (a) $|S| = 2^{\mathfrak{c}}$; (b) if $x, y \in S$ are such that there is a homeomorphism $h : \omega^* \to \omega^*$ with $h(x) = y$, then $x = y$; and (c) if $x \in \omega^*$ then there exists $y \in S$ and a homeomorphism $h : \omega^* \to \omega^*$ such that $h(x) = y$.

A proof of Theorem 6.55 can be found in [161, 3.40 to 3.46]. Other proofs of the non-homogeneity of ω^* using Frolík's arguments can be found in [161, 6.31] (Frolik's fixed point theorem) and [113] (which gives a proof in modern terminology). From Frolik's results we can also obtain the following whose proof can be found in [161, 4.12].

Theorem 6.56 [65] Let X be a non-pseudocompact space. Then X^* is not homogeneous.

Frolík's arguments prove that ω^* is not homogeneous. However, in some sense, those arguments do not prove "why" ω^* is not homogeneous. More precisely, the proof does not show two different points of ω^* with different topological properties. It was desirable to find a "honest" proof (in the words of van Douwen) of the non-homogeneity of ω^*.

One possible solution to this question was the existence of P-points of ω^* in ZFC. However, Saharon Shelah showed that there is a model of ZFC where ω^* has no P-points. The proof of this is a very profund result that relies on iteration of proper forcing. We refer the reader to Chapter VI, Section 4 of [154] for a proof of this fact. Also, recently a Diploma thesis with a very clear and detailed proof of Shelah's theorem has been written, see [169].

Even though it is not possible to find P-points of ω^* in ZFC, Kenneth Kunen found another special type of point. A point p in a topological space X is called a *weak P-point* if for every countable set $N \subset X \setminus \{p\}$ it follows that $p \notin \mathrm{cl}_X(N)$. As shown in Corollary 6.54, in any compact Hausdorff space there must be points that are not weak P-points.

Theorem 6.57 [100] There are weak P-points in ω^*. Thus, ω^* is not homogeneous because there are points with different topological properties.

The proof of Theorem 6.57 can be also found in [62, Chapter 6].

One of the strongest promoters of "honest" proofs of non-homogeneity was Eric K. van Douwen. In the next Chapter, we will start by giving a proof of existence of a special kind of points called *remote points* and its applications to the theory of non-homogeneity of Čech-Stone compactifications by van Douwen. These remote points will become our object of study in this Part.

Chapter 7

Remote Points and their Density

Let X be a Tychonoff space. A point $p \in X^*$ is called a *remote point* of X if $p \notin \mathrm{cl}_{\beta X}(A)$ for every nowhere dense subset A of X. Following van Douwen [36], we will denote the set of remote points of X by $\varrho(X)$. In some informal sense, points of X^* are "infinite points" of X and points in $\varrho(X)$ are "more infinite than [all] others" ([36, 1.3]).

As discussed in the end of Chapter 6, even though Theorem 6.56 had been proved, van Douwen wanted to find points with different topological properties that showed "why" some spaces of the form X^* were not homogeneous. In [35], van Douwen finally found an answer to his question in the following way.

Theorem 7.1 [35] Let X be a nowhere locally compact metrizable space. Then some but not all points of X^* are far from X^{**} in βX^*, and so X^* is not homogeneous.

A point $p \in Y$ is *far* from $A \subset Y$ if for there is no discrete $D \subset A$ closed in A such that $p \in \mathrm{cl}_Y(D)$. So every remote point of X is far from X^* in βX. Notice that however, Theorem 7.1 is not quite a "honest" proof of non-homogeneity because one has to see X^* from "outside" (from βX^*) to know why it is not homogeneous.

According to [114], van Douwen had proved the existence of remote points in some remainders using Martin's axiom and proved the non-homogeneity of these spaces using those remote points (this paper was published posthumosly as [40]). However, the proof was not published until finally in [36], van Douwen gave the same results in ZFC, that is, without additional hypothesis.

Our main interest is the set $\varrho(X)$ as a topological space. Roughly speaking, our main theorems in this Part (Theorem 9.13, Corollary 9.22 and Corollary 9.27) give a classification of homeomorphism type of $\varrho(X)$ when X is a metrizable space. In this Chapter, we will give some results on the homeomorphism type of $\varrho(X)$ when X is a locally compact metrizable space. These results were given in the 70s and 80s by R. Grant Woods and Catherine Gates (Theorems 7.20 and 7.12, respectively).

7.1 Existence of remote points

The first step is of course, proving that $\varrho(X) \neq \emptyset$ when X is a metrizable space. In [36], van Douwen proved that a non-pseudocompact space of countable π-weight has remote points. As there are metrizable spaces with arbitrarily large π-weight, we will present the argument given by Soo Bong Chae and Jeffrey H. Smith in [26].

Lemma 7.2 [26, Theorem 3] Let X be a regular space with a σ-locally finite π-base. Then for every non-empty open subset U of X and every $n < \omega$ there is a family $\mathcal{G}_n(U)$ of closed subsets of X contained in U such that every dense open subset of U contains an element of $\mathcal{G}_n(U)$ and any $n+1$ elements of $\mathcal{G}_n(U)$ have non-empty intersection.

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n < \omega\}$ be a π-base of X such that $\mathcal{B}_n$ is locally finite for all $n < \omega$. We may assume that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ for all $n < \omega$. If V is a non-empty open subset of X, define $\Xi(V)$ to be the smallest $m < \omega$ such that there is $W \in \mathcal{B}_m$ with $\mathrm{cl}_X(W) \subset V$.

For any non-empty open subset U of X and $n < \omega$, let us define a non-empty closed subset $G_n(U) \subset U$ by recursion. First, let

$$G_0(U) = \bigcup\{\mathrm{cl}_X(V) : V \in \mathcal{B}_{\Xi(U)}, \mathrm{cl}_X(V) \subset U\}.$$

By the definition of $\Xi(U)$ and the fact that $\mathcal{B}_n$ is locally finite for all $n < \omega$, we obtain that $G_0(U)$ is indeed a non-empty and closed subset of U (Lemma 0.20). For all $n < \omega$, define

$$G_{n+1}(U) = \left(\bigcup\{G_n(U \cap V) : V \in \mathcal{B}_{\Xi(U)}, U \cap V \neq \emptyset\}\right) \cup G_n(U).$$

Again, by similar arguments, $G_{n+1}(U)$ is a non-empty and closed subset of U. Finally, we define for $n < \omega$,

$$\mathcal{G}_n(U) = \{G_n(W) : W \text{ is an open dense subset of } U\}.$$

Clearly, $\mathcal{G}_n(U)$ is a collection of closed subsets of U. If W is dense open in U, then $G_n(W)$ is an element of $\mathcal{G}_n(U)$ contained in W. To show that this family is as required, we prove the following claim by induction.

Claim. If $U \subset X$ is a non-empty open subset then any $n+1$ elements of $\mathcal{G}_n(U)$ have non-empty intersection.

For $n = 0$, we just have to notice that if W is dense open in U, then $G_n(W) \neq \emptyset$, this has already been shown above. Now assume inductively that we have proved the claim for $n = k$. Let $W_0, \ldots, W_{k+1}$ be dense open subsets of U. We may assume that $\Xi(W_0) \leq \Xi(W_i)$ for all $1 \leq i \leq k+1$. Notice that this implies that $\mathcal{B}_{\Xi(W_0)} \subset \mathcal{B}_{\Xi(W_i)}$ for $1 \leq i \leq k+1$. By the definition of $\Xi(W_0)$, there is a non-empty $V \in \mathcal{B}_{\Xi(W_0)}$ such that $\mathrm{cl}_X(V) \subset W_0$. In particular, V is an non-empty open subset of U so $V \cap W_i$ is dense in V for all $1 \leq i \leq k+1$. From the inductive hypothesis appplied to the open set V, we obtain that the set

$$L = G_k(V \cap W_1) \cap G_k(V \cap W_2) \cap \ldots \cap G_k(V \cap W_{k+1})$$

is non-empty. For $1 \leq i \leq k+1$, by the recursive definition of $G_{k+1}(W_i)$ and the fact that $V \in \mathcal{B}_{\Xi(W_0)} \subset \mathcal{B}_{\Xi(W_i)}$, $L \subset G_k(V \cap W_i) \subset G_{k+1}(W_i)$. Notice that moreover, since $L \subset V$, then $L \subset G_0(W_0)$. By the definition of $G_{k+1}(W_0)$, we obtain that $L \subset G_0(W_0) \subset G_{k+1}(W_0)$. Then

$$L \subset G_{k+1}(W_0) \cap G_{k+1}(W_1) \cap \ldots \cap G_{k+1}(W_{k+1})$$

so this intersection is non-empty. This proves the claim and concludes the proof of this Lemma. $\square$

Theorem 7.3 [26, Theorem 1] Let X be a normal space with a σ-locally finite π-base. If G is a non-empty subset of βX of type G_δ such that $G \subset X^*$, then G contains $2^{\mathfrak{c}}$ remote points.

Proof. It is not hard to find a continuous function $f : \beta X \to [0,1]$ such that $f^{\leftarrow}(0) \subset G$. For each $n < \omega$, let $U_n = f^{\leftarrow}[(\frac{1}{n+1}, \frac{1}{n+2})]$ and choose a non-empty open subset V_n of X such that $\mathrm{cl}_X(V_n) \subset U_n$. Then $\{V_n : n < \omega\}$ is a locally finite family of non-empty open subsets of X whose closures are pairwise disjoint (this is a refined version of Lemma 0.19).

By Lemma 7.2, for each $n < \omega$, there is a family $\mathcal{G}_n(V_n)$ of closed subsets of V_n such that (a) every open dense subset of V_n contains an element of $\mathcal{G}_n(V_n)$ and (b) every $n+1$ elements of $\mathcal{G}_n(V_n)$ have non-empty intersection. For every $\mathcal{U} \in \omega^*$, consider the following family of closed subsets of X:

$$\mathcal{G}_{\mathcal{U}} = \big\{ \bigcup\{F_n : n \in A\} : A \in \mathcal{U}, \forall n \in A(F_n \in \mathcal{G}_n(V_n))\big\}.$$

For each $\mathcal{U} \in \omega^*$, let $R(\mathcal{U}) = \bigcap\{\mathrm{cl}_{\beta X}(G) : G \in \mathcal{G}_{\mathcal{U}}\}$. We will now prove that $\bigcup\{R(\mathcal{U}) : \mathcal{U} \in \omega^*\}$ consists of remote points of X contained in G.

Claim 1. For every $\mathcal{U} \in \omega^*$, $\emptyset \neq R(\mathcal{U}) \subset \varrho(X) \cap G$.

Fix some $\mathcal{U} \in \omega^*$. First, we show that $R(\mathcal{U})$ is a subset of G.

Let $m < \omega$. Since $\mathcal{U}$ is non-principal, there is $A \in \mathcal{U}$ such that $A \subset \omega \setminus m$. For each $n \in A$, let $F_n \in \mathcal{G}_n(U_n)$. Then $\bigcup\{F_n : n \in A\} \in \mathcal{G}_{\mathcal{U}} \cap (\bigcup\{V_n : n > m\})$. By the definition of the family $\{V_n : n < \omega\}$, this implies that

$$R(\mathcal{U}) \in \mathrm{cl}_{\beta X}(\{V_n : m < n < \omega\}) \subset G \cup (\bigcup\{\mathrm{cl}_{\beta X}(V_n) : m < n < \omega\})$$

But this holds for every $m < \omega$, so we obtain that $R(\mathcal{U}) \subset G$.

Since βX is a compact space, to prove that $R(\mathcal{U}) \neq \emptyset$, it is enough to prove that the family $\mathcal{G}_{\mathcal{U}}$ has the finite intersection property. Let $m < \omega$ and $\{G_k : k < m+1\} \subset \mathcal{G}_{\mathcal{U}}$. Then there exist $A_0, \ldots, A_m \in \mathcal{U}$ and for each $k < m+1$ a collection $\{F_n(k) : n \in A_k\}$ such that $F_n(k) \in \mathcal{G}_n(V_n)$ for all $n \in A_k$ and $G_k = \bigcup\{F_n(k) : n \in A_k\}$. Let $r \in A_0 \cap \ldots \cap A_m$ be such that $m < r$. Then by property (b) of the definition of $\mathcal{G}_r(V_r)$, $F_r(0) \cap \ldots \cap F_r(m) \neq \emptyset$. Thus, $G_0 \cap \ldots \cap G_m \neq \emptyset$.

Now we prove that $R(\mathcal{U}) \subset \varrho(X)$ so let F be a nowhere dense subset of X. Then for each $n < \omega$, $V_n \setminus F$ is a dense open subset of V_n. So by property (a) in the definition of $\mathcal{G}_n(V_n)$, there is $H_n \in \mathcal{G}_n(V_n)$ such that $H_n \cap F = \emptyset$. Notice that $H = \bigcup\{H_n : n < \omega\} \in \mathcal{G}_{\mathcal{U}}$. Since $F \cap H = \emptyset$ and X is normal, F and H can be separated by zero sets of X. From (3) in Theorem 6.4, we obtain that $\mathrm{cl}_{\beta X}(F) \cap \mathrm{cl}_{\beta X}(H) = \emptyset$. Thus, $R(\mathcal{U}) \cap \mathrm{cl}_{\beta X}(F) = \emptyset$. This implies that $R(\mathcal{U}) \subset \varrho(X)$ and concludes the proof of Claim 1.

Claim 2. If $\mathcal{U}_0, \mathcal{U}_1 \in \omega^*$ are such that $\mathcal{U}_0 \neq \mathcal{U}_1$, then $R(\mathcal{U}_0) \cap R(\mathcal{U}_1) = \emptyset$.

Since $\mathcal{U}_0 \neq \mathcal{U}_1$, there is $A \subset \omega$ such that $A \in \mathcal{U}_0$ and $\omega \setminus A \in \mathcal{U}_1$. For each $n < \omega$, let $F_n \in \mathcal{G}_n(V_n)$. Let $H_0 = \bigcup\{F_n : n \in A\}$ and $H_1 = \bigcup\{F_n : n \in \omega \setminus A\}$. Then $H_0 \in \mathcal{G}_{\mathcal{U}_0}$, $H_1 \in \mathcal{G}_{\mathcal{U}_1}$ and $H_0 \cap H_1 = \emptyset$. Since X is normal, by (3) in Theorem 6.4, we obtain that $\mathrm{cl}_{\beta X}(H_0) \cap \mathrm{cl}_{\beta X}(H_1) = \emptyset$. This implies that $R(\mathcal{U}_0) \cap R(\mathcal{U}_1) = \emptyset$ so Claim 2 has been proved.

By Claims 1, 2 and Theorem 6.47, we have proved the statement of the Theorem. $\square$

Notice that Theorem 7.3 is a kind of dual to Proposition 6.49: while Proposition 6.49 gives $2^{\mathfrak{c}}$ points that are in the closure of a discrete subset, Theorem 7.3 gives $2^{\mathfrak{c}}$ points that are not in the closure of any discrete subset.

The author of this dissertation was not able to remove the normality assumptions from Theorem 7.3. Nevertheless, Theorem 7.3 is stong enough for our main purpose: a metrizable space is clearly normal and by the Nagata-Smirnov metrization theorem (Theorem 0.22), it has a σ-locally finite base. Thus, Theorem 7.3 applies to metrizable spaces. Moreover, metrizable spaces are nearly realcompact by Proposition 6.14 so we obtain the following.

Corollary 7.4 If X is a metrizable non-compact space, then $\varrho(X)$ is dense in X^*.

Also, van Douwen's Theorem [36, Theorem 4.2] on existence of remote points can be easily obtained by a careful manipulation of Theorem 7.3 as follows.

Corollary 7.5 [36, Theorem 4.2] Let X be a Tychonoff space with countable π-weight. If G is a non-empty subset of βX of type G_δ such that $G \subset X^*$, then G contains $2^{\mathfrak{c}}$ remote points.

Proof. Let $Y = \beta X \setminus G$. Then $X \subset Y \subset \beta X$ so the equivalence of (0) and (1) in Theorem 6.4 (or Corollary 6.15) easily implies that $\beta Y = \beta X$. Also, notice that Y is σ-compact so Y is normal (this follows from Lemma 1.5.15 and Theorem 3.1.9 in [50]). If F is a nowhere dense subset of X, then $\mathrm{cl}_Y(F)$ is a nowhere dense subset of Y. From this, $\varrho(Y) \subset \varrho(X) \cap G$. Since X has a countable π-base and X is dense in Y, it is not hard to construct a countable π-base for Y. Obviously any countable π-base of Y is σ-locally finite. The result then follows by applying Theorem 7.3 to Y. $\square$

Now we will give some properties of remote points studied by van Douwen in [36]. First, we will notice that we have extremal disconnecteness at remote points and then use this to give van Douwen's "honest" proofs of non-homogeneity.

Proposition 7.6 [36, Corollary 5.2] Let X be a Tychonoff space and $p \in \varrho(X)$. Then βX is ED at p.

Proof. Let $p \in \varrho(X)$. First we prove a preliminary claim.

Claim: If U is an open subset of βX such that $p \in \mathrm{cl}_{\beta X}(U)$, then $p \in \mathrm{Ex}_X(U \cap X)$.

To prove the claim, notice that $\mathrm{cl}_{\beta X}(U) = \mathrm{cl}_{\beta X}(\mathrm{Ex}_X(U \cap X))$ by Proposition 6.11. Notice that $\mathrm{bd}_X(U \cap X)$ is a nowhere dense subset of X so $p \notin \mathrm{cl}_{\beta X}(\mathrm{bd}_X(U \cap X))$. By Lemma 6.12, we obtain that $p \in \mathrm{Ex}_X(U \cap X)$. This proves the claim.

Thus, assume that U and V are disjoint open subsets of βX such that $p \in \mathrm{cl}_{\beta X}(U) \cap \mathrm{cl}_{\beta X}(V)$. By the claim, p is in the open set $W = \mathrm{Ex}_X(U \cap X) \cap \mathrm{Ex}_X(V \cap X)$. However, by Proposition 6.10, $W = \mathrm{Ex}_X(U \cap V \cap X) = \emptyset$ since $U \cap V = \emptyset$. This contradiction proves that βX is ED at p. $\square$

In view of Proposition 7.6, we can give examples of spaces without remote points.

Example 7.7 Some spaces with no remote points.

In Example 6.5, we saw that if κ is an ordinal of uncountable cofinality, then $\beta\kappa = \kappa + 1$. So the Čech-Stone remainder of κ is just one point $\{\kappa\}$. However, if A is the set of limit ordinals of κ, then A is nowhere dense and $\kappa \in \mathrm{cl}_\kappa(A)$ so κ is not a remote point of κ. Thus, $\varrho(\kappa) = \emptyset$.

In Example 6.6 we considered a dense subspace $\mathbf{\Sigma}$ of the power ${}^\kappa 2$, where κ is an uncountable cardinal. We next argue that ${}^\kappa 2$ is not ED at any point. By Proposition 7.6 and the fact that $\beta\mathbf{\Sigma} = {}^\kappa 2$, we obtain that $\mathbf{\Sigma}$ has no remote points.

We will prove that ${}^\kappa 2$ is not ED at the constant 0 function $\overline{0}$, for other points the argument is similar. For each $\alpha < \kappa$, let $\pi_\alpha : {}^\kappa 2 \to 2$ be the projection onto the α-th coordinate. Let $U_0 = \pi_0^{\leftarrow}(1)$ and if $n < \omega$, let

$$U_{n+1} = (\bigcap\{\pi_k^{\leftarrow}(0) : k \leq n\}) \cap \pi_{n+1}^{\leftarrow}(1).$$

Then $\{U_n : n < \omega\}$ is a pairwise disjoint collection of clopen subsets of ${}^\kappa 2$. If $U = \bigcup\{U_{2n} : n < \omega\}$ and $V = \bigcup\{U_{2n+1} : n < \omega\}$, then U and V are disjoint open sets with $p \in \mathrm{cl}_{{}^\kappa 2}(U) \cap \mathrm{cl}_{{}^\kappa 2}(V)$. So ${}^\kappa 2$ is not ED at $\overline{0}$. $\square$

Notice that both spaces from Example 7.7 are pseudocompact. This is not a coincidence. In [44], Alan Dow proved that no pseudocompact space has remote points.

We would like to have a topological property opposite to extremal disconnectedness in non-remote points. Of course, we have to ask for more conditions on X since βX is ED at all its points when X is ED (Theorem 6.36). A point p in a space X is called a *κ-point* of X, where κ is a cardinal, if there is a collection $\mathcal{S}$ of pairwise disjoint open subsets of X such that $|\mathcal{S}| \geq \kappa$ and $p \in \mathrm{cl}_X(U)$ for each $U \in \mathcal{S}$. Clearly, $p \in X$ is a 2-point of X if and only if X is not ED at p.

Lemma 7.8 [36, Lemma 6.5] Let κ be a cardinal and let X be a Tychonoff space with a dense set of κ-points. If G is a non-empty subset of βX of type G_δ with $G \subset X^*$, then G contains at least $2^{\mathfrak{c}}$ points that are κ-points of βX.

Proof. We make the following assertions that can be shown analogously as in the proofs of Theorem 7.3 and Corollary 7.5. First, we may assume without loss of generality that G is a zero set of βX, call $Y = \beta X \setminus G$ so that $\beta X = \beta Y$ and $Y^* = G$. Also it is not hard to find a locally finite collection of non-empty open sets $\{V_n : n < \omega\}$ of βY with pairwise disjoint closures so that $\mathrm{cl}_{\beta Y}(V_n) \subset Y$ for all $n < \omega$ (again, this is a refined version of Lemma 0.19).

For each $n < \omega$, let $x_n \in V_n$ be a κ-point of X. It is not hard to see that for each $n < \omega$ there exists a family of pairwise disjoint open subsets $\{U(n, \alpha) : \alpha < \kappa\}$ of Y such that $x_n \in \bigcap\{\mathrm{cl}_Y(U(n, \alpha)) : \alpha < \kappa\}$. For every $n < \omega$ and $\alpha < \kappa$ we may further assume that $U(n, \alpha) \subset V_n$ so that $U(n, \alpha)$ is an open subset of βY as well. Let $U_\alpha = \bigcup\{U(n, \alpha) : n < \omega\}$ for each $\alpha < \kappa$ and let $F = \mathrm{cl}_{\beta Y}(\{x_n : n < \omega\})$. Notice that $\{U_\alpha : \alpha < \omega\}$ is a family of non-empty open subsets of βY such that $F \subset \bigcap\{\mathrm{cl}_{\beta Y}(U_\alpha) : \alpha < \omega\}$. Thus, every point of F is a κ-point of $\beta Y = \beta X$.

Since $\{V_n : n < \omega\}$ is a locally finite family, it is not hard to use the Urysohn Extension Theorem 0.9 to prove that $\{x_n : n < \omega\}$ is C^*-embedded in Y. Thus, by Corollary 6.15, F is homeomorphic to $\beta\omega$ and $|F| = 2^{\mathfrak{c}}$ by Theorem 6.47. Notice that this argument is basically the same one used in the proof of Proposition 6.49.

Then $F \cap X^*$ is a set of $2^{\mathfrak{c}}$ points that are κ-points of βX contained in G. This concludes the proof of this Lemma. □

With these tools we can provide proofs of non-homogeneity in certain cases.

Theorem 7.9 Let X be a nowhere locally compact Tychonoff space. If $\varrho(X)$ is dense in X and X fails to be ED at a dense set of points, then X^* is not homogeneous because it is ED at some points but not at others.

Proof. Notice that X^* is dense in βX since X is nowhere locally compact. Thus, by Proposition 7.6 and Lemma 6.32, $\varrho(X)$ is ED, so in particular X^* is ED at each point of $\varrho(X)$. The rest of the result follows from Lemma 7.8. □

For example, by Corollary 7.4 we have the following.

Corollary 7.10 If X is a nowhere locally compact metrizable space, then X^* is not homogeneous because it is ED at some points but not at others.

We remark that van Douwen gave many more applications of remote points in his paper [36]. As stated in the Introduction to this Part, further work has been done to investigate the existence of remote points in wider classes of spaces.

After van Douwen's results, Johannes Vermeer and Evert Wattel proved the following, which shows that in some cases, being a remote point is in fact a topological property.

Theorem 7.11 [160] Let X be a nowhere locally compact Tychonoff space. If $h : X^* \to X^*$ is a homeomorphism, then $h[\varrho(X)] = \varrho(X)$.

Proof. It is enough to prove that if $p \notin \varrho(X)$, then $h(p) \notin \varrho(X)$. Notice that since X is nowhere locally compact, X^* is dense in βX and thus, is nowhere locally compact as well. By (2) in Theorem 6.4, there exists a continuous function $f : \beta X^* \to \beta X$ such that $f\restriction_{X^*}: X^* \to X^*$ is the identity function. Also by (2) in Theorem 6.4 there is a unique function $\beta h : \beta X^* \to \beta X^*$ such that $\beta h\restriction_{X^*}= h$. Since h is a homeomorphism, βh can be easily see to be a homeomorphism as well.

It is not hard to see that $f[X^{**}] = X$ and $f\restriction_{X^{**}}: X^{**} \to X$ is an irreducible function from the fact that both X^* and X^{**} are dense in βX^*.

Thus, if $F \subset X$ is a closed and nowhere dense in X such that $p \in \mathrm{cl}_{\beta X}(F)$, then $G_0 = f^{\leftarrow}[F]$ is a closed and nowhere dense subset of X^{**}. We claim that $p \in \mathrm{cl}_{\beta X^*}(G_0)$. Assume that this is not the case and let U be an open subset of βX^* such that $p \in U$ and $U \cap G_0 = \emptyset$. Then $f^\sharp[U] = \beta X \setminus f[\beta X^* \setminus U]$ is an open subset of βX^* that contains p and misses F. This is a contradiction.

Let $G_1 = \beta h[G_0]$ and $H = f[G_1]$. Then G_1 is a closed nowhere dense subset of X^* such that $h(p) \in \mathrm{cl}_{\beta X^*}(G_1)$. Since $f\restriction_{X^{**}}$ is irreducible, by (b) in Lemma 6.45 it follows that H is a closed and nowhere dense subset of X and $h(p) \in \mathrm{cl}_{\beta X}(H)$. Thus, $h(p) \notin \varrho(X)$. $\square$

However, Theorem 7.11 cannot be generalized to locally compact spaces. It was shown by Joseph Yi-Chung Yu ([170]) that it is consistent that there is a autohomeomorphism of $\mathbb{R}^*$ that maps a remote point to a non remote point. A proof of this fact can also be found in [75].

7.2 Coabsolute spaces and remote points

As it has been mentioned before, our main problem is the following: if X and Y are two metrizable spaces, when are $\varrho(X)$ and $\varrho(Y)$ homeomorphic? In [66], Catherine Gates gave some tools that relate the absolute to this problem. More precisely, $\varrho(X)$ is homeomorphic to $\varrho(EX)$, see Corollary 7.13. In this section we will give Gate's proof of this result.

Proposition 7.12 [66, Theorem 2.4] Let X be a Tychonoff space, let Y be a normal space and let $f : X \to Y$ be an irreducible continuous function. Denote by $\beta f : \beta X \to \beta Y$ the unique continuous extension of f. Then $\varrho(X) = \beta f^{\leftarrow}[\varrho(Y)]$ and $\beta f\restriction_{\varrho(X)}: \varrho(X) \to \varrho(Y)$ is a homeomorphism.

Proof. We first show that

$$(*)\ \varrho(X) = \beta f^{\leftarrow}[\varrho(Y)].$$

First, let $p \in \varrho(X)$ and $q = \beta f(p)$, we want to show that $q \in \varrho(Y)$.

Claim. For every closed nowhere dense subset B of Y, $q \notin \mathrm{cl}_{\beta Y}(B)$.

Assume that this is not the case and let B be a witness closed nowhere dense set. Let $A = f^{\leftarrow}[B]$, then A is closed and nowhere dense since f is irreducible. Since $p \in \varrho(X)$, $p \notin \mathrm{cl}_X(A)$. From Proposition 6.11, there is a non-empty open set U of X such that $p \in \mathrm{Ex}_X(U)$ and $\mathrm{cl}_{\beta X}(\mathrm{Ex}_X(U)) \cap \mathrm{cl}_X(A) = \emptyset$. From the fact that f is closed it follows that B and $f[\mathrm{cl}_X(U)]$ are disjoint closed subsets of Y. Notice that (b) in Proposition 6.10 implies that $p \in \mathrm{cl}_{\beta X}(\mathrm{cl}_X(U))$. Thus, $q \in \mathrm{cl}_{\beta Y}(B) \cap \mathrm{cl}_{\beta Y}(f[\mathrm{cl}_X(U)])$, but this contadicts the normality of Y. So the Claim has been established.

We still have to show that $q \in X^*$ to prove that $q \in \varrho(Y)$. Assume that this is not the case, then $q \in Y$. If q is not an isolated point, then take $B = \{q\}$, this is a closed nowhere dense subset of Y so by the Claim, $q \notin B$, a contradiction. Thus, the only chance left is that q is an isolated point of Y. Then $f^{\leftarrow}(q)$ is clopen and by the irreducibility of f, $f^{\leftarrow}(q) = \{x\}$ for some isolated point $x \in X$. Since $p \in \mathrm{cl}_{\beta X}(X \setminus \{x\})$, then $q \in \mathrm{cl}_{\beta Y}(Y \setminus \{q\}) \subset \beta Y \setminus \{q\}$ since q is also an isolated point of βY. This is a contradiction. Thus, we have proved that $q \in \varrho(Y)$.

Now let $p \in \beta X$ be such that $q = \beta f(p) \in \varrho(Y)$. Since $f[X] = Y$, we obtain that $p \in X^*$. Assume that there is a closed nowhere dense subset A of X such that $p \in \mathrm{cl}_{\beta X}(A)$. By (b) in Lemma 6.45, $B = f[A]$ is closed and nowhere dense in Y. Moreover, $q \in \mathrm{cl}_{\beta Y}(B)$, this is a contradiction. Thus, $p \in \varrho(X)$. This argument completes the proof of $(*)$.

From $(*)$, it is not hard to see that $\beta f{\restriction}_{\varrho(X)} \colon \varrho(X) \to \varrho(Y)$ is continuous and closed, we just have to prove that it is one to one. So assume that there exists $p \in \varrho(Y)$ such that $|\beta f^{\leftarrow}(p)| > 1$. Then there are disjoint open subsets U_0 and U_1 of βX such that $U_i \cap \beta f^{\leftarrow}(p) \neq \emptyset$ for $i \in 2$. Let $V_i = \beta f^{\sharp}[U_i]$ for $i \in 2$. By (c) in Lemma 6.45, V_0 and V_1 are non-empty disjoint open subsets of βY. Moreover, it is not hard to see that $p \in \mathrm{cl}_{\beta Y}(V_0) \cap \mathrm{cl}_{\beta Y}(V_1)$. But this contradicts Proposition 7.6. This proves that in fact, $|\beta f^{\leftarrow}(p)| = 1$. Thus, $\beta f{\restriction}_{\varrho(X)} \colon \varrho(X) \to \varrho(Y)$ is a homeomorphism and the proof is complete. $\square$

Proposition 7.12 and Corollary 6.42 immediately show the following.

Corollary 7.13 If two normal spaces X and Y are coabsolute, then $\varrho(X)$ is homeomorphic to $\varrho(Y)$.

However, Proposition 7.12 gives us more information than Corollary 7.13. For one part, we don't just obtain some arbitrary homeomorphism but we know how to obtain it. Also, we do not need to "see" the absolute of X and the absolute of Y to compare $\varrho(X)$ and $\varrho(Y)$; we just need a function between X and Y.

We state two more results that will be useful for our results.

Lemma 7.14 Let X be a normal space and $U \subset X$ be open and dense in X. Then $\varrho(X)$ is homeomorphic to $\varrho(U)$.

Proof. Consider the absolute $k_X : EX \to X$. By Proposition 6.43 $k_X \restriction_{k_X^{\leftarrow}[U]}\colon k_X^{\leftarrow}[U] \to U$ is the absolute of U. Identify $k_X^{\leftarrow}[U]$ with EU and $k_X \restriction_{k_X^{\leftarrow}[U]}$ with k_U. Also, since k_X is irreducible and U is dense, by (a) in Lemma 6.45, EU is dense in EX. By Proposition 6.33, EU is a C^*-embedded subset of EX. This implies that $\mathrm{cl}_{\beta EX}(EU)$ can be identified with βEU. No remote point of EX can lie in $\beta EX \setminus \beta EU$ because this set is contained in the closure of the nowhere dense subset $EX \setminus EU$ of EX. Thus, it can be shown that $\varrho(EX) = \varrho(EU)$. By applying Corollary 7.13 it follows that $\varrho(U)$ and $\varrho(X)$ are homeomorphic. $\square$

Proposition 7.15 Let X be a normal space. Assume that Y is a regular closed subset of X and identify $\mathrm{cl}_{\beta X}(Y)$ with βY. Then $\varrho(Y) = \varrho(X) \cap \mathrm{cl}_{\beta X}(Y)$. Moreover, since $\mathrm{cl}_X(X - Y)$ is also a regular closed subset, we can write $\varrho(X) = \varrho(Y) \cup \varrho(\mathrm{cl}_X(X - Y))$ and $\varrho(Y), \varrho(\mathrm{cl}_X(X - Y))$ are disjoint clopen subsets of $\varrho(X)$.

Proof. Define Z to be the direct sum of Y and $\mathrm{cl}_X(X \setminus Y)$, formally, $Z = A_0 \cup A_1$, where $A_0 = Y \times \{0\}$ and $A_1 = \mathrm{cl}_X(X \setminus Y) \times \{1\}$. Let $\phi : Z \to X$ be the natural projection to the first coordinate. It is easy to see that ϕ is a perfect and irreducible continuous function so we may apply Proposition 7.12 to obtain that $\beta\phi \restriction_{\varrho(Z)}\colon \varrho(Z) \to \varrho(X)$ is a homeomorphism. Notice that A_0 and A_1 are complementary clopen subsets of Z. Thus, $\mathrm{cl}_{\beta Z}(A_i)$ can be identified with βA_i for $i \in 2$ and $\beta A_0 \cap \beta A_1 = \emptyset$. From this, it is straightforward that $\varrho(A_i) = \varrho(X) \cap \mathrm{cl}_{\beta Z}(A_i)$ for $i \in 2$, $\varrho(Z) = \varrho(A) \cup \varrho(B)$ and $\varrho(A) \cap \varrho(B) = \emptyset$. From the fact that both $\phi\restriction_{A_0}\colon A_0 \to Y$ and $\phi\restriction_{A_1}\colon A_1 \to \mathrm{cl}_X(X \setminus Y)$ are homeomorphisms, it is not hard to prove that $\beta\phi[\varrho(A_0)] = \varrho(Y)$ and $\beta\phi[\varrho(A_1)] = \varrho(\mathrm{cl}_X(X \setminus Y))$. The result follows from these observations. $\square$

7.3 Locally compact spaces

In this Section we will prove a result by R. Grant Woods that completely characterizes remote points of locally compact crowded metrizable spaces, see Theorem

7.20. We start by mapping the Cantor set to any compact metrizable space in an irreducible way.

Proposition 7.16 [167, Lemma 2.1] Let X be a compact crowded metrizable space. Then there is a irreducible continuous function $f : {}^{\omega}2 \to X$.

Proof. Fix some metric d for X. By recursion, we will construct a sequence of covers $\{\mathcal{U}_n : n < \omega\}$ of X and an increasing sequence $\{k_n : n < \omega\} \subset \omega$ such that for all $n < \omega$ the following hold

(a) $\mathcal{U}_n \subset \mathcal{R}(X)$,

(b) if $A \in \mathcal{U}_n$, then the diameter of A is $\leq \frac{1}{n+1}$,

(c) if $A, B \in \mathcal{U}_n$ and $A \neq B$, then $A \cap \mathrm{int}_X(B) = \emptyset$,

(d) $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$,

(e) if $A \in \mathcal{U}_n$, then $\{B \in \mathcal{U}_{n+1} : B \subset A\}$ covers A,

(f) $|\mathcal{U}_n| = 2^{k_n}$,

(g) if $A \in \mathcal{U}_n$, then $|\{B \in \mathcal{U}_{n+1} : B \subset A\}| = 2^{k_{n+1}-k_n}$.

Assume that we have constructed $\{\mathcal{U}_0, \ldots, \mathcal{U}_m\}$ and $\{k_0, \ldots, k_m\}$. Let $\mathcal{U}_m = \{A_i : i \leq 2^{k_m}\}$.

Fix $i \leq 2^{k_m}$. Since A_i is a compact metrizable space, there exists $p(i) < \omega$ and a cover $\mathcal{V}_i = \{V_j : j \leq p(i)\}$ of A_i by open sets in A_i of diameter $\leq \frac{1}{m+2}$. Recursively, for $j \leq p(i)$ we define an open set V'_j of A_i in the following way. First, $V'_0 = V_0$. If $j \leq p(i)$, let

$$V'_{j+1} = V_{j+1} \setminus \mathrm{cl}_{A_i}(V'_0 \cup \ldots V'_j).$$

Let $q(i) < \omega$ and let $\{W_j : j \leq q(i)\} = \{V'_j : j \leq p(i)\} \setminus \{\emptyset\}$ be a precise enumeration. Notice that for each $j \leq p(i)$, $\mathrm{cl}_{A_i}(W_j) \in \mathcal{R}(A_i)$. Moreover, since $A_i \in \mathcal{R}(X)$, it is not hard to see that $\mathrm{cl}_{A_i}(W_j) = \mathrm{cl}_X(W_j) \in \mathcal{R}(X)$. Finally, let $\mathcal{W}_i = \{\mathrm{cl}_{A_i}(W_j) : i \leq q(i)\}$.

Then it is easy to see that $\mathcal{W}_i$ is a family of $q(i)$ regular closed subsets of X contained in A_i, each with diameter $\leq \frac{1}{m+2}$ and with pairwise disjoint interiors (with respect to X). Notice that if we define $\mathcal{U}_{m+1} = \bigcup\{\mathcal{W}_i : i \leq 2^{k_m}\}$, then conditions (a) to (e) hold. However, conditions (f) and (g) still may not hold, we have to do additional work.

Let us make a general observation: in any crowded regular space Y for any $A \in \mathcal{R}(Y)$, there are $B, C \in \mathcal{R}(Y)$ such that $A = B \cup C$ and $B \cap C$ is nowhere dense. Let r be the smallest $j < \omega$ such that $2^j \geq \max\{q(i) : i \leq k_{m+1}\}$ and define $k_{m+1} = k_m + r$. From this observation, it is possible to modify each family $\mathcal{W}_i$ to obtain a family $\mathcal{W}_i'$ of precisely 2^r regular closed subsets of X contained in A_i, each one of diameter $\leq \frac{1}{m+2}$ and with pairwise disjoint interiors with respect to X, for each $i \leq k_{m+1}$. Then defining $\mathcal{U}_{m+1} = \bigcup\{\mathcal{W}_i' : i \leq 2^{k_m}\}$ conditions (a) to (g) hold. This completes the construction.

Now we will do a similar construction for the Cantor set. For every $s \in {}^{<\omega}2$, let $U_s = \{x \in {}^{\omega}2 : x(i) = s(i) \text{ for all } i \in \mathit{dom}(s)\}$. Then $\{U_s : s \in {}^{<\omega}2\}$ is a basis of clopen subsets of ${}^{\omega}2$. Using properties (d), (e) and (f), it is not hard to construct by recursion a bijection $\phi : \bigcup\{{}^{k_n}2 : n < \omega\} \to \bigcup\{\mathcal{U}_n : n < \omega\}$ such that $\phi[{}^{k_n}2] = \mathcal{U}_n$ for all $n < \omega$ and $\phi(s) \subset \phi(t)$ if and only if $t \subset s$.

Finally, define $f : {}^{\omega}2 \to X$ in the following way: if $s \in {}^{\omega}2$, let $f(s)$ be the only point in the intersection $\bigcap\{\phi(s\restriction_{k_n}) : n < \omega\}$. Clearly this function is well-defined. Notice that

$(*)$ $f[U_s] \subset \phi(U_s)$ for every $s \in \bigcup\{{}^{k_n}2 : n < \omega\}$.

To prove that f is continuous, let $t \in {}^{\omega}2$ and let V an open set in X such that $f(t) \in V$, we must find an open set U in ${}^{\omega}2$ such that $t \in U$ and $f[U] \subset V$. There exists $m < \omega$ such that $\{y \in X : d(f(t), y) \leq \frac{1}{m+1}\} \subset V$. Let $r = t\restriction_{k_m}$, then by the definition of m, property (b) and $(*)$, we have that $\phi(U_r) \subset V$. Let $U = U_r$, then U is an open set of ${}^{\omega}2$ such that $t \in U$ and $f[U_r] \subset \phi(U_r) \subset V$ by $(*)$. This proves the continuity of f and since f is a function between compact spaces, it is also closed.

Finally, we prove that f is irreducible. To see that f is onto, let $x \in X$. By property (e) recursively choose $A_n \in \mathcal{U}_n$ such that $x \in A_n$ and $A_{n+1} \subset A_n$ if $n < \omega$. Then $\{\phi^{-1}(A_n) : n < \omega\}$ is a strictly increasing sequence of functions and $\bigcup\{\phi^{-1}(A_n) : n < \omega\} = y \in {}^{\omega}2$ is such that $f(y) = x$.

Now, let $F \subset {}^{\omega}2$ be a closed proper subset, we must prove that $f[F]$ is proper. Then there is $t \in \bigcup\{{}^{k_n}2 : n < \omega\}$ such that $F \cap U_t \neq \emptyset$. Let $m = \mathit{dom}(t)$ and let $C = \phi(U_t)$. By $(*)$,

$$f[F] \subset \bigcup\{f[U_s] : \mathit{dom}(s) = m, s \neq t\} \subset \bigcup\{\phi(U_s) : \mathit{dom}(s) = m, s \neq t\}$$

so $f[F] \subset X \setminus \mathrm{int}_X(C)$ by property (c). Thus, $f[F] \neq {}^{\omega}2$ and f is then irreducible. $\square$

From this we obtain the following well-known result.

Corollary 7.17 If K is a compact crowded metrizable space, then K is coabsolute with the Cantor set.

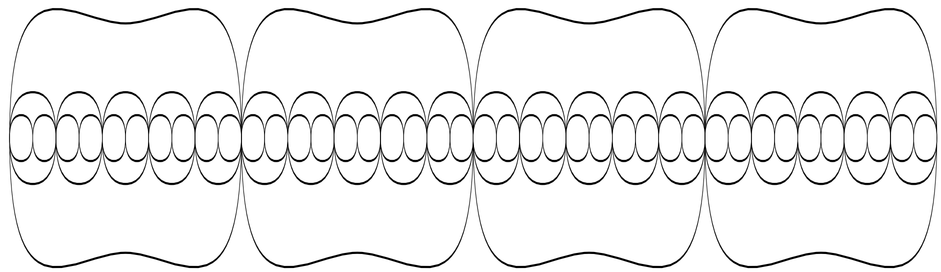

Figure 7.1: Refining covers in Proposition 7.16.

Next we give two results about the structure of locally compact metrizable spaces.

Proposition 7.18 [167, Theorem 1.1] Let X be a locally compact metrizable space of weight κ.

(a) If $\kappa = \omega$, then X is σ-compact.

(b) If $\kappa > \omega$, then X is the free union of precisely κ locally compact, σ-compact and non-compact metrizable spaces.

Proof. For each $x \in X$, let U_x be an open subset of X such that $x \in U_x$ and $\mathrm{cl}_X(U_x)$ is compact. Since X is paracompact (Theorem 0.21), there exists a locally finite open cover $\mathcal{U}$ that refines $\{U_x : x \in X\}$. Thus, every member of $\mathcal{U}$ has compact closure.

We define an equivalence relation in $\mathcal{U}$ in the following way. We will say that $U \sim V$ for $U, V \in \mathcal{U}$ if there exists $m < \omega$ and $\{U_n : n \leq m\} \subset \mathcal{U}$ such that $U_0 = U$, $U_m = V$ and $U_n \cap U_{n+1} \neq \emptyset$ for all $n < m$. For every $U \in \mathcal{U}$, let $\lfloor U \rfloor$ be the equivalence class of U defined by $\sim$. It is not hard to see that if $U, V \in \mathcal{U}$, then either $(\bigcup \lfloor U \rfloor) \cap (\bigcup \lfloor V \rfloor) = \emptyset$ or $(\bigcup \lfloor U \rfloor) = (\bigcup \lfloor V \rfloor)$. Then $\mathcal{V} = \{(\bigcup \lfloor U \rfloor) : U \in \mathcal{U}\}$ is a partition of X into clopen subsets.

Claim. $\lfloor U \rfloor$ is countable for each $U \in \mathcal{U}$.

Fix $U \in \mathcal{U}$. To prove the Claim, we define a function $\rho : \lfloor U \rfloor \to \omega$. Let $\rho(V)$ to be the smallest number $m < \omega$ such that there is $\{U_n : n \leq m\} \subset \mathcal{U}$ such that $U_0 = U$, $U_m = V$ and $U_n \cap U_{n+1} \neq \emptyset$ for all $n < m$. Clearly, ρ is well-defined by the definition of $\lfloor U \rfloor$. Inductively, let us prove that $S_n = \{V \in \lfloor U \rfloor : \rho(V) = n\}$ is finite for all $n < \omega$. Clearly, $S_0 = \{U\}$. Assume that S_k is finite, then

$$(*)\ S_{k+1} \subset \bigcup \{\{W \in \mathcal{U} : V \cap W \neq \emptyset\} : V \in S_k\}.$$

Since $\mathcal{U}$ is locally finite, $\{W \in \mathcal{U} : V \cap W \neq \emptyset\}$ is finite for all $V \in \mathcal{U}$ so equation $(*)$ implies that S_{k+1} is finite. Since $\lfloor U \rfloor = \bigcup\{S_n : n < \omega\}$, we have proved the Claim.

By the Claim and the fact that each element of $\mathcal{U}$ has compact closure, we easily obtain that $\mathcal{V}$ is a partition of X into countably many clopen, locally compact and σ-compact subsets of X. If $w(X) = \omega$, then $\mathcal{V}$ must be countable and X is σ-compact.

Moreover, since compact metrizable spaces have countable weight (Theorem 0.16), the weight of X is precisely $|\mathcal{V}|$. So in the case that $w(X) = \kappa > \omega$, we can give a precise enumeration $\mathcal{V} = \{V(\alpha, n) : \alpha < \kappa, n < \omega\}$. Let $V_\alpha = \bigcup\{V(\alpha, n) : n < \omega\}$ for each $\alpha < \kappa$. Then $\{V_\alpha : \alpha < \kappa\}$ is a partition of precisely κ locally compact, σ-compact and non-compact metrizable spaces. □

Proposition 7.19 Let X be a locally compact, σ-compact and non-compact Hausdorff space. Then there is a sequence $\{K_n : n < \omega\}$ of compact, regular closed subsets of X such that

(a) $X = \bigcup\{K_n : n < \omega\}$;

(b) if $m, n < \omega$ and $|m - n| > 1$, then $K_m \cap K_n = \emptyset$ and

(c) if $m, n < \omega$ and $m \neq n$, then $K_n \cap \mathrm{int}_X(K_m) = \emptyset$.

Moreover, if X is crowded then we may ask that K_n is crowded for every $n < \omega$.

Proof. Since X is a countable union of compact sets, there is a countable open cover $\mathcal{U} = \{U_n : n < \omega\}$ of X such that $\mathrm{cl}_X(U_n)$ is compact and non-empty for all $n < \omega$. We first define a sequence of compact sets $\{T_n : n < \omega\}$ such that $T_n \subsetneq \mathrm{int}_X(T_{n+1})$ for all $n < \omega$ and $X = \bigcup\{T_n : n < \omega\}$.

Let $T_0 = \mathrm{cl}_X(U_0)$. Recursively, if we have defined $\{T_n : n \leq m\}$, let $t(m) = \min\{n < \omega : U_n \not\subset T_m\}$ and let $\mathcal{U}_m \subset \mathcal{U}$ be a finite collection that covers T_m. Define $T_{m+1} = \mathrm{cl}_X(U_{t(m)} \cup (\bigcup \mathcal{U}_n))$. It is easy to see that the family $\{T_n : n < \omega\}$ constructed is as requested.

Finally, define $K_0 = T_0$ and $K_{n+1} = \mathrm{cl}_X(T_{n+1} \setminus K_n)$ for $n < \omega$. It is not hard to see that the collection $\{K_n : n < \omega\}$ is as requested. □

Finally, we can prove the following.

Theorem 7.20 [167] Let X be a non-compact, locally compact and crowded metrizable space of weight κ. Then $\varrho(X)$ is homeomorphic to $\varrho(\kappa \times {}^{\omega}2)$.

Proof. If $\kappa > \omega$, by Proposition 7.18, $X = \bigoplus\{X_\alpha : \alpha < \kappa\}$ where X_α is non-compact, locally compact, crowded and σ-compact for each $\alpha < \kappa$. Assume that

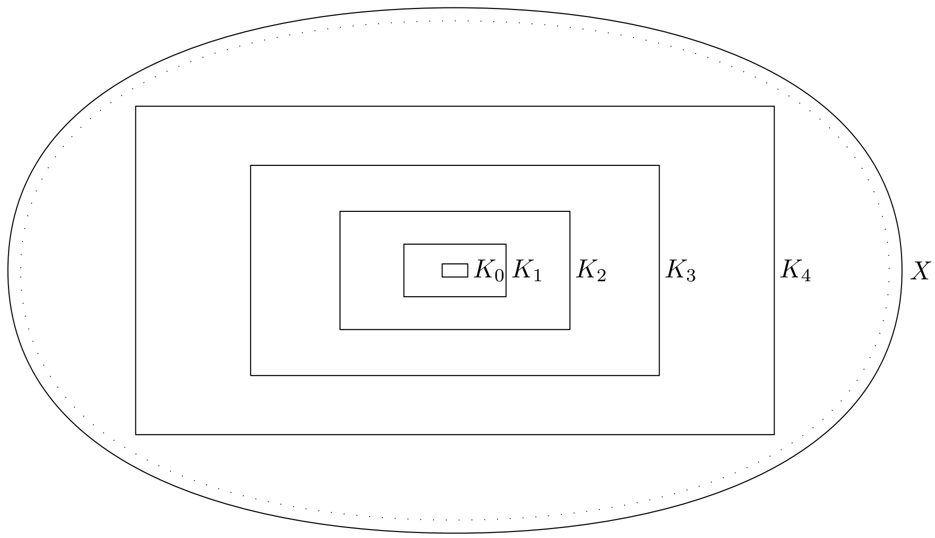

Figure 7.2: The regular closed sets in Proposition 7.19.

for each $\alpha < \kappa$ we are able to define a perfect, continuous and irreducible continuous function $f_\alpha : \omega \times {}^\omega 2 \to X_\alpha$. Then the function $f : \kappa \times \omega \times {}^\omega 2 \to X$ given by $f(\alpha, n, x) = f_\alpha(n, x)$ for all $(\alpha, n, x) \in \kappa \times \omega \times {}^\omega 2$ is also perfect, continuous and irreducible. By Proposition 7.12, f witnesses that $\varrho(X)$ is homeomorphic to $\varrho(\kappa \times \omega \times {}^\omega 2)$ and this set is homeomorphic to $\varrho(\kappa \times {}^\omega 2)$. If $\kappa = \omega$, then X is σ-compact by Proposition 7.18. If $f : \omega \times {}^\omega 2 \to X$ is a perfect, continuous and irreducible function, we have that $\varrho(X)$ is homeomorphic to $\varrho(\omega \times {}^\omega 2)$ by Proposition 7.12.

So for both cases it is enough to assume that X is σ-compact and find a perfect, continuous and irreducible function $f : \omega \times {}^\omega 2 \to X$. Let $\{K_n : n < \omega\}$ be a family of sets for X as in Proposition 7.19. For each $n < \omega$, by Proposition 7.16 there is a perfect, continuous and irreducible function $f_n : {}^\omega 2 \to K_n$. Define $f : \omega \times {}^\omega 2$ by $f(\langle n, x\rangle) = f_n(x)$ for all $\langle n, x\rangle \in \omega \times {}^\omega 2$. Clearly, f is continuous and onto.

To see that f is closed, let $F \subset \omega \times {}^\omega 2$ be closed. Then $f[F] = \bigcup\{f_n[F \cap (\{n\} \times {}^\omega 2)] : n < \omega\}$. Notice that $f_n[F \cap (\{n\} \times {}^\omega 2)]$ is closed in X because it is a closed subset of K_n for each $n < \omega$. By properties (b) and (c) of the family $\{K_n : n < \omega\}$, $\{f_n[F \cap (\{n\} \times {}^\omega 2)] : n < \omega\}$ is locally finite so its union $f[F]$ is then closed (Lemma 0.20).

Now we see that f is pefect, let $p \in X$. By property (b) of the sequence $\{K_n : n < \omega\}$, there are $m_0, m_1 < \omega$ such that $p \in K_n$ implies $n \in \{m_0, m_1\}$

(m_0 might be equal to m_1). Thus, $f^{\leftarrow}(p) = (\{m_0\} \times f^{\leftarrow}_{m_0}(p)) \cup (\{m_1\} \times f^{\leftarrow}_{m_1}(p))$. Thus, every fiber is compact and f is closed so f is thus perfect.

We only have to see that f is irreducible. Notice that f is onto because $f_n[{}^{\omega}2] = K_n$ for all $n < \omega$. Let $F \subset \omega \times {}^{\omega}2$ be a proper closed subset. Then there is $m < \omega$ such that $(\{m\} \times {}^{\omega}2) \setminus F$ is non-empty. Then

$$f[F] \subset (\bigcup\{K_n : n < \omega, n \neq m\}) \cup f_m[F \cap (\{m\} \times {}^{\omega}2)].$$

But f_m is an irreducible function so $U = K_m \setminus f_m[F \cap (\{m\} \times {}^{\omega}2)]$ is an non-empty open subset of K_m. From properties (b) and (c) of the collection $\{K_n : n < \omega\}$, we obtain that $\mathrm{int}_X(U) \neq \emptyset$ and $f[F] \cap \mathrm{int}_X(U) = \emptyset$. Thus, f is irreducible and the proof has been completed. □

One may also wonder what happens with locally compact spaces with isolated points. Recall from Example 6.44 that for every compact Hausdorff space X with a countable dense set of isolated points there is a perfect, continuous and irreducible function $f : \beta\omega \to X$. Also, notice that $\varrho(\omega) = \omega^*$. So for example, if $X = (\omega \times {}^{\omega}2) \oplus \omega$, then $\varrho(X) = \varrho(\omega \times {}^{\omega}2) \oplus \varrho(\omega)$ by Proposition 7.15. In general, the problem of finding a common model for $\varrho(X)$ in some larger class of locally compact metrizable spaces X is not obvious. However, in this dissertation we will not deal with this problem.

We finally remark that the following was later shown.

Theorem 7.21 [66, Corollary 5.8] Let X be a non-compact, separable and crowded metrizable space whose set of non-locally compact points is compact and non-empty. Then $\varrho(X)$ is homeomorphic to $\omega \times \varrho(\omega \times {}^{\omega}2)$.

Let us give an example of an space as in Theorem 7.21.

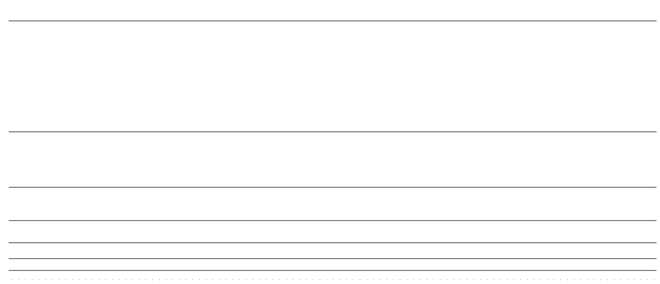

Figure 7.3: Example 7.22.

Example 7.22 A metrizable X space with $\varrho(X)$ homeomorphic to $\omega\times\varrho(\omega\times{}^{\omega}2)$.

For each $n<\omega$, let $I_n=\{\frac{1}{n+1}\}\times(0,1)$ and let C be the Cantor middle-third set in the interval $[0,1]$. Define

$$X=(\bigcup\{I_n : n<\omega\})\cup(\{0\}\times C)$$

as a subspace of the Euclidean plane ${}^2\mathbb{R}$. Then X is locally compact in the dense set $\bigcup\{I_n : n<\omega\}$ but no point in $\{0\}\times C$ has a relatively compact neighborhood. Thus, by Theorem 7.21, $\varrho(X)$ is homeomorphic to $\omega\times\varrho(\omega\times{}^{\omega}2)$. $\square$

Chapter 8

Special Tools

In this Chapter, we will develop four topics that at first sight has nothing to do with our problems on remote points but will nevertheless provide some key ideas for our results in Chapter 9. Since our main problem (Question 9.1) concerns metrizable spaces, the topics developed in this chapter will be mainly in the context of metrizable spaces.

8.1 Strongly 0-dimensional spaces

It turns out that in order to study spaces of remote points of metrizable spaces, it is enough to consider a special class of metrizable spaces (Proposition 9.6 below). We will give a brief summary of the background we need to prove this result.

It is not hard to convince oneself that a space X is 0-dimensional if for every pair $\langle p, F\rangle$ where $p \in X$, $F \subset X$ is closed and $p \notin F$, there is $C \in \mathcal{CO}(X)$ such that C separates p from F; more precisely, $p \in C$ and $C \cap F = \emptyset$. We can obtain another property if instead of separating a point from a closed set, we can separate two disjoint closed subsets.

A Tychonoff space X is *strongly* 0-*dimensional* if for every two disjoint zero sets $F, G \subset X$ there is $C \in \mathcal{CO}(X)$ such that $C \subset F$ and $C \cap G = \emptyset$. Clearly, if X is normal we may change "zero sets" in this definition by "closed sets".

We remark that for each $n \in \omega \cup \{\infty\}$, there are notions of n-dimensional which generalize 0-dimensional and strongly 0-dimensionality (see Chapter 7 of [50] or the specialized book [51]). There is a famous example of a metrizable space which is 0-dimensional but not strongly 0-dimensional, this space is due to Prabir Roy ([140] and [141]). We also refer the reader to [131, Chapter 7, Section 4] where this example is given in detail. We have, however, that these two notions coincide sometimes.

Proposition 8.1 If X is any 0-dimensional separable metrizable space, then X is strongly 0-dimensional.

Proof. Let F and G be disjoint closed subsets of X. By Lemma 0.15, there is a countable base $\mathcal{B} \subset \mathcal{CO}(X)$. For every $x \in X$, let $B_x \in \mathcal{B}$ such that either $B_x \cap F = \emptyset$ or $B_x \cap G = \emptyset$. Since $\mathcal{B}$ is countable, we can give an enumeration $\{B_x : x \in X\} = \{A_n : n < \omega\}$. For $n < \omega$, define $C_n = A_n \setminus (\bigcup\{A_m : m < n\})$. Then $\{C_n : n < \omega\}$ is a partition of X into clopen sets (some which may be empty). Moreover, for each $n < \omega$, either $C_n \cap F = \emptyset$ or $C_n \cap G = \emptyset$. Let $C = \bigcup\{C_n : n < \omega, C_n \cap F \neq \emptyset\}$ and $D = \bigcup\{C_n : n < \omega, C_n \cap F = \emptyset\}$. Then $C = X \setminus D$, $F \subset C$ and $G \subset D$. Thus, C and D are clopen subsets that separate F and G. This proves that X is strongly 0-dimensional. $\square$

As Corollary 9.6 shows, the "correct" notion of dimension 0 for us will be that of strongly 0-dimensional spaces. Let us give a characterization of strong dimension 0 for metrizable spaces.

Theorem 8.2 Let X be a metrizable space. Then the following conditions are equivalent.

(0) X is strongly 0-dimensional.

(1) For every metric on X there is a sequence $\{\mathcal{U}_n : n < \omega\}$ of partitions of X into clopen subsets such that (a) for all $n < \omega$, $\mathcal{U}_n$ has mesh $\leq \frac{1}{n+1}$ and (b) for each $n < \omega$, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.

(2) There is a metric on X and a sequence $\{\mathcal{U}_n : n < \omega\}$ of partitions of X into clopen subsets such that (a) for all $n < \omega$, $\mathcal{U}_n$ has mesh $\leq \frac{1}{n+1}$ and (b) for each $n < \omega$, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$.

Proof. Clearly, (1) implies (2). Assume (2) and let d be the metric in the hypothesis. Let F, G be two disjoint closed subsets of X. Define

$$V_0 = \bigcup\{U \in \mathcal{U}_0 : U \cap F \neq \emptyset, U \cap G = \emptyset\}$$

which is clopen since $\mathcal{U}_0$ is a partition and it is disjoint from G. Recursively, for each $n < \omega$, let

$$V_{n+1} = \bigcup\{U \in \mathcal{U}_{n+1} : U \cap (F \setminus (V_0 \cup \ldots \cup V_n)) \neq \emptyset, U \cap G = \emptyset\},$$

which is also clopen and disjoint from G.

Let $V = \bigcup\{V_n : n < \omega\}$, then V is an open set and clearly $V \cap G = \emptyset$. We claim that $F \subset V$ and $V \in \mathcal{CO}(X)$, this will show (0).

To see that $F \subset V$, assume this is not the case. Then there is $x \in F \setminus V$, let $m < \omega$ be such that $d(x, G) = \inf\{d(x, y) : y \in G\} < \frac{1}{m+1}$. Let $W \in \mathcal{U}_m$ be such that $x \in W$, notice that $W \cap G = \emptyset$. Since $x \notin V_m$, we obtain that $W \subset V_{m+1}$ by the definition of V_{m+1}. But this implies that $x \in V_{m+1}$, a contradiction. This proves that $F \subset V$.

Now we see that V is closed, since X is metrizable, we can check this using convergent sequences. Let $\{x_n : n < \omega\} \subset V$ have limit x. If there is $m < \omega$ such that $\{n < \omega : x_n \in V_m\}$ is infinite, then $x \in V_m \subset V$ as V_m is closed. If this is not the case, taking a subsequence if necessary, we may assume that there is a strictly increasing $s : \omega \to \omega$ such that $x_n \in V_{s(n)}$ for all $n < \omega$. For each $n < \omega$, let $y_n \in V_{s(n)} \cap F$ be such that $d(x_n, y_n) < \frac{1}{s(n)+1}$. Then clearly $\{y_n : n < \omega\}$ also converges to x. But F is closed so $x \in F$ and we have already proved that $F \subset V$ so $x \in V$. Thus V is closed. This completes the proof of the fact that (2) implies (0).

Now we assume (0) and prove (1). Let d be a metric on X. We will construct the partitions $\{\mathcal{U}_n : n < \omega\}$ recursively. Assume that we have constructed $\{\mathcal{U}_n : n \leq m\}$ for some $m < \omega$, now we have to construct $\mathcal{U}_{m+1}$. Since X is paracompact (Theorem 0.21), there is a locally finite cover $\mathcal{V}_0$ of X such that $\mathcal{V}_0$ refines $\mathcal{U}_{m+1}$. By taking more refinements, we may assume that $\mathcal{V}_0$ has mesh $\leq \frac{1}{m+2}$. Let $\mathcal{V}_1$ be a locally finite refinement of $\mathcal{V}_0$. By the regularity of X, we may further ask that for every $V \in \mathcal{V}_1$ there is $U \in V_0$ such that $\mathrm{cl}_X(V) \subset U$.

Let $\mathcal{V}_0 = \{V_\alpha : \alpha < \kappa\}$ be a precise enumeration, where κ is some cardinal. For each $\alpha < \kappa$, let $F_\alpha = \bigcup\{\mathrm{cl}_X(V) : V \in \mathcal{V}_1, \mathrm{cl}_X(V) \subset V_\alpha\}$, this is a closed subset of X since $\mathcal{V}_1$ is locally finite (Lemma 0.20). Define $\mathcal{G} = \{F_\alpha : \alpha < \kappa\}$, this is a closed cover of X that refines $\mathcal{V}_0$ with the additional property that this refinement is a shrinking: $F_\alpha \subset V_\alpha$ for all $\alpha < \kappa$.

Since X is strongly 0-dimensional, there is $W_\alpha \in \mathcal{CO}(X)$ such that $F_\alpha \subset W_\alpha \subset V_\alpha$ for all $\alpha < \kappa$. Let $\mathcal{W} = \{W_\alpha : \alpha < \kappa\}$. Then $\mathcal{W}$ is a cover since $\mathcal{G}$ is a cover. Let $x \in X$, since $\mathcal{V}_0$ is locally finite, let U be an open set such that $\{\alpha : V_\alpha \cap U \neq \emptyset\}$ is finite. If $\beta < \kappa$ is such that $U \cap W_\beta \neq \emptyset$, then $U \cap V_\beta \neq \emptyset$. This proves that $\{\alpha : W_\alpha \cap U \neq \emptyset\} \subset \{\alpha : V_\alpha \cap U \neq \emptyset\}$. Thus, $\mathcal{W}$ is a locally finite clopen cover of X.

Recursively, define $U_0 = W_0$ and $U_\alpha = W_\alpha \setminus (\bigcup\{W_\beta : \beta < \alpha\})$ if $0 < \alpha < \kappa$. Then by Lemma 0.20, $U_\alpha \in \mathcal{CO}(X)$ for all $\alpha < \kappa$. Define $\mathcal{U}_{m+1} = \{U_\alpha : \alpha < \kappa\} \setminus \{\emptyset\}$. Notice that $\mathcal{U}_{m+1}$ is a partition of X into clopen sets. Clearly, $\mathcal{U}_{m+1}$ refines $\mathcal{V}_0$ so the mesh of $\mathcal{U}_{m+1}$ is $\leq \frac{1}{m+2}$ and $\mathcal{U}_{m+1}$ refines $\mathcal{U}_m$. This proves (1) and concludes the proof of the Theorem. $\square$

We obtain the following Corollary. We remark that it is not true in general that a subspace of a strongly 0-dimensional space is also strongly 0-dimensional,

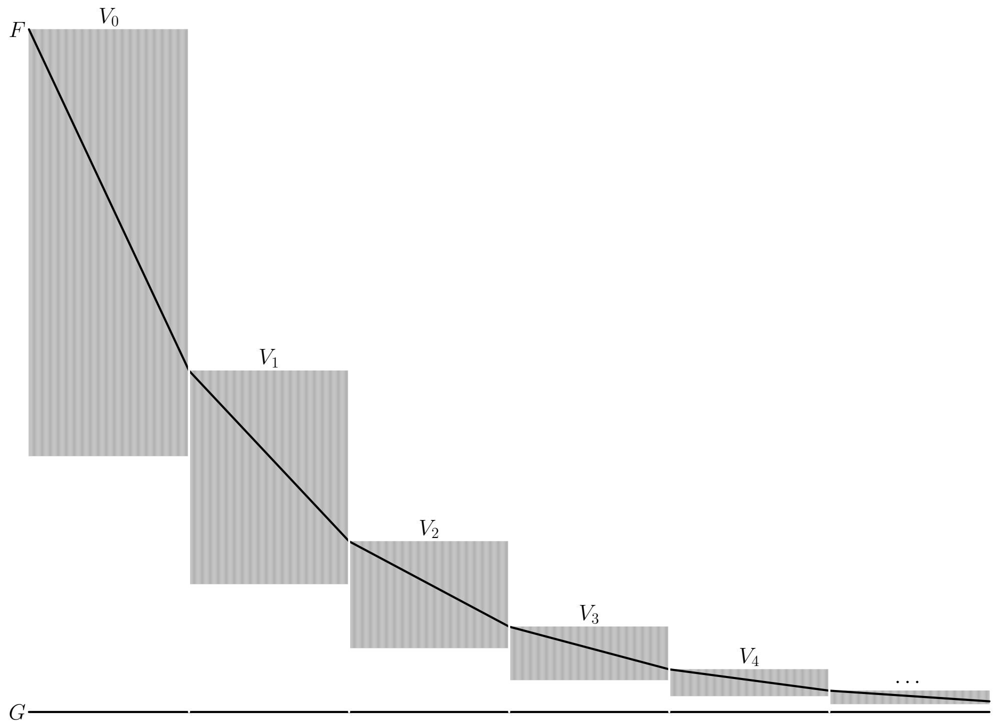

Figure 8.1: (2) implies (0) in Theorem 8.2.

as an example by Dowker [50, 6.2.20] shows.

Corollary 8.3 If X is a strongly 0-dimensional metrizable space and $Y \subset X$, then Y is strongly 0-dimensional as well.

Proof. Fix some metric for X, notice that this metric restricts to a metric for Y. From (1) in Theorem 8.2, we obtain a sequence of clopen partitions $\{\mathcal{U}_n : n < \omega\}$ for X. For all $n < \omega$, let $\mathcal{V}_n = \{U \cap Y : U \in \mathcal{U}_n\}$. Then $\{V_n : n < \omega\}$ witnesses (2) in Theorem 8.2 for Y. Thus, Y is also strongly 0-dimensional. $\square$

We now introduce the space which is the archetype for strongly 0-dimensional spaces.

Definition 8.4 If κ is an infinite cardinal, the product space ${}^\omega\kappa$ is called the *κ-Baire space*. There exists a very natural metric d in ${}^\omega\kappa$: if $x, y \in {}^\omega\kappa$ and $x \neq y$, let $d(x,y) = \frac{1}{\Delta(x,y)+1}$, where $\Delta(x,y) = \min\{n < \omega : x(n) \neq y(n)\}$. For each

$f \in {}^{<\omega}\kappa$, let $B_f = \{x \in {}^{\omega}\kappa : f \subset x\}$. For each $n < \omega$, let $\mathcal{B}_n = \{B_f : f \in {}^{n+1}\kappa\}$ and define $\mathcal{B} = \bigcup\{\mathcal{B}_n : n < \omega\}$.

Lemma 8.5 For every infinite cardinal κ, ${}^{\omega}\kappa$ is a nowhere locally compact, strongly 0-dimensional completely metrizable space of weight κ. Moreover, $\mathcal{B}$ from Definition 8.4 is a base of clopen sets all homeomorphic to ${}^{\omega}\kappa$ and d from Definition 8.4 is a complete metric that generates the topology of ${}^{\omega}\kappa$.

Proof. It is not hard to see that $\mathcal{B}$ is a base of clopen sets of ${}^{\omega}\kappa$ with the product topology. Moreover, if $f \in {}^{n+1}\kappa$ for $n < \omega$, then $B_f = \{f(0), \ldots, f(n)\} \times {}^{\omega\setminus(n+1)}\kappa$, which is homeomorphic to ${}^{\omega}\kappa$. Thus, $\mathcal{B}$ is a base of precisely κ clopen sets all homeomorphic to the whole space ${}^{\omega}\kappa$.

If $x \in {}^{\omega}\kappa$, $n < \omega$ and $f = x\restriction_{n+1}$, then $B_f = \{y \in {}^{\omega}\kappa : d(x,y) < \frac{1}{n+1}\}$. This proves that d generates the product topology on ${}^{\omega}\kappa$. Now we see that d is a complete metric. Let $\{x_n : n < \omega\} \subset {}^{\omega}\kappa$ be a Cauchy sequence, in particular, such that $d(x_m, x_n) \leq \frac{1}{m+1}$ if $m \leq n < \omega$. Let $t \in {}^{\omega}\omega$ be such that $t(n) = x_n(n)$ for all $n < \omega$. Inductively it follows that $x_n(k) = t(k)$ for all $k \leq n < \omega$. Then $\{x_n : n < \omega\}$ converges to t and d is thus complete.

Notice that $\mathcal{B}_n$ is a partition into clopen sets with mesh $\frac{1}{n+1}$. Let $f, g \in {}^{<\omega}\kappa$. If f and g are compatible, say $f \subset g$, then $B_g \subset B_f$. Otherwise, $B_f \cap B_g = \emptyset$. Thus, $\{\mathcal{B}_n : n < \omega\}$ satisfies condition (2) in Theorem 8.2. This implies that ${}^{\omega}\kappa$ is strongly 0-dimensional.

Since $|\mathcal{B}| = \kappa$, we obtain that ${}^{\omega}\kappa$ has weight $\leq \kappa$. Moreover, $\mathcal{B}_0$ is a partition of ${}^{\omega}\kappa$ into exactly κ disjoint non-empty open sets, this easily implies that $w({}^{\omega}\kappa) \geq \kappa$. Thus, $w({}^{\omega}\kappa) = \kappa$.

If ${}^{\omega}\kappa$ were locally compact at some point, then there would be an element of the base $\mathcal{B}$ that is compact. This and the fact that all elements of $\mathcal{B}$ are homeomorphic to ${}^{\omega}\kappa$ imply that ${}^{\omega}\kappa$ is compact. However, $\mathcal{B}_0$ is a partition of ${}^{\omega}\kappa$ into exactly κ disjoint non-empty open sets, which contradicts the compactness. Thus, ${}^{\omega}\kappa$ is nowhere locally compact. $\square$

In the rest of this Section we shall give properties of the κ-Baire space we will use later. The first property is that κ-Baire space is a universal space, see Example 2.2 in Part I for a related result.

Proposition 8.6 If X is a strongly 0-dimensional metrizable space of weight κ, then X can be embedded in ${}^{\omega}\kappa$.

Proof. Fix some metric on X. There is a family of clopen partitions $\{\mathcal{U}_n : n < \omega\}$ of X as in (1) of Theorem 8.2. Define $\mathcal{U} = \bigcup\{\mathcal{U}_n : n < \omega\}$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n <$

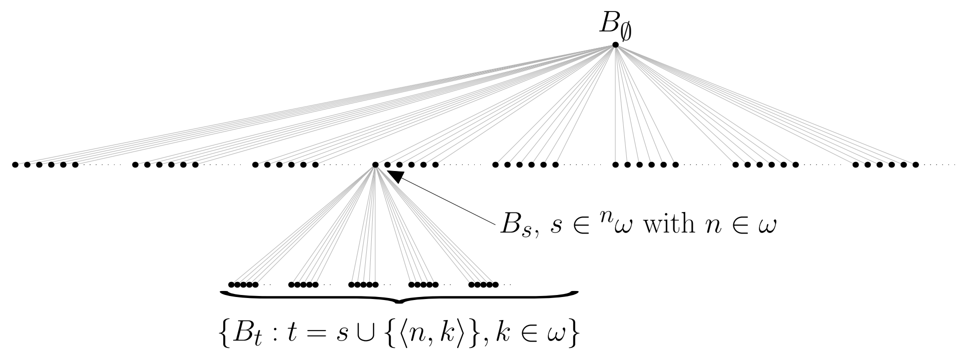

Figure 8.2: Base $\mathcal{B}$ from Definition 8.4 forms a tree.

$\omega\}$ be as in Definition 8.4, from Lemma 8.5 we know that $\mathcal{B}$ is a base of clopen subsets of ${}^\omega\kappa$.

Notice that $|\mathcal{B}_n| = \kappa$ for each $n < \omega$. Since X has weight κ, the partition $\mathcal{U}_n$ must have cardinality $\leq \kappa$ for each $n < \omega$. From this, it is not hard to use recursion to define a one-to-one function $\phi : \mathcal{U} \to {}^{<\omega}\kappa$ such that $\phi[\mathcal{U}_n] \subset {}^{n+1}\kappa$ for each $n < \omega$ and $\phi(V) \subset \phi(U)$ whenever $U, V \in \mathcal{U}$ and $U \subset V$.

Now we define $e : X \to {}^\omega\kappa$. If $x \in X$, the set $\{U \in \mathcal{U} : x \in U\}$ is linearly ordered by inclusion and intersects $\mathcal{U}_n$ in exactly one element for each $n < \omega$. Thus, $\{\phi(U) : x \in U \in \mathcal{U}\}$ is also linearly ordered by inclusion and intersects ${}^{n+1}\kappa$ in exactly one element for each $n < \omega$. This implies that $\{\phi(U) : x \in U \in \mathcal{U}\} = \{f_n : n < \omega\}$, where $f_n \in {}^{n+1}\kappa$ and $f_n \subset f_{n+1}$ for $n < \omega$. Define $e(x) = \bigcup\{\phi(U) : x \in U \in \mathcal{U}\} = \bigcup\{f_n : n < \omega\} \in {}^\omega\kappa$.

To see that e is continuous, notice that $e^{\leftarrow}[B_f] = \emptyset$ if $f \notin \phi[\mathcal{U}]$ and otherwise $e^{\leftarrow}[B_f] = \phi^{-1}(f)$. To see that e is open, notice that $e[U] = B_{\phi(U)} \cap e[X]$ for every $U \in \mathcal{U}$. Finally, let $x, y \in X$ with $x \neq y$. Then there is $n < \omega$ and $U, V \in \mathcal{U}_n$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. From this, $e(x) \in B_{\phi(U)}$, $e(y) \in B_{\phi(V)}$ and $B_{\phi(U)} \cap B_{\phi(V)} = \emptyset$ so $e(x) \neq e(y)$. This shows that e is an embedding. $\square$

One of the properties of the κ-Baire space is that it can be topologically characterized by simple topological properties. We divide our characterization in two parts as the special case when $\kappa = \omega$ uses other hypothesis.

Theorem 8.7 [3] Let X be a 0-dimensional, separable and metrizable space. Then X is homeomorphic to ${}^\omega\omega$ if and only if X is completely metrizable and nowhere locally compact.

Proof. We have already seen that ${}^\omega\omega$ is 0-dimensional, separable, nowhere locally compact and completely metrizable in Lemma 8.5. Let X be a space with the described properties and fix some complete metric.

Claim: For each non-empty clopen set $V \subset X$ and every $m < \omega$ there exists a partition $\{V(n) : n < \omega\}$ of U into exactly ω clopen subsets of diameter $\leq \frac{1}{m+1}$.

Let us prove the Claim. Clearly, V is also 0-dimensional. Since V is non-compact, by Lemma 0.19, there is a countable infinite closed and discrete set D. By Lemma 0.15, there is a countable base $\mathcal{V} \subset \mathcal{CO}(V)$. For each $x \in V$, let $V_x \in \mathcal{V}$ of diameter $\leq \frac{1}{m+1}$ such that $|V_x \cap D| \leq 1$. Since $\mathcal{V}$ is countable, there is an enumeration $\{W_n : n < \omega\} = \{V_x : x \in V\}$. Define $V_0 = W_0$ and $V_{n+1} = W_{n+1} \setminus (W_0 \cup \ldots \cup W_n)$ for $n < \omega$. Then $\{V_n : n < \omega\}$ is a partition of V into clopen subsets of diameter $\leq \frac{1}{m+1}$. Notice that $|V_n \cap D| \leq 1$ for each $n < \omega$. Since D is infinite, we must have that $\{V_n : n < \omega\}$ is infinite as well. Then $\{V_n : n < \omega\} \setminus \{\emptyset\}$ is the partition requested by the Claim.

Using the Claim recursively, it is not hard to construct a family of partitions $\{\mathcal{U}_n : n < \omega\}$ of X into clopen subsets such that $|\mathcal{U}_n| = \omega$, $\mathcal{U}_n$ has mesh $\leq \frac{1}{n+1}$ and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for each $n < \omega$. Let $\mathcal{U} = \bigcup\{\mathcal{U}_n : n < \omega\}$. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n < \omega\}$ be the collection from Definition 8.4, recall that $\mathcal{B}$ is a base by Lemma 8.5. Then it is not hard to construct a bijection $\phi : \mathcal{B} \to \mathcal{U}$ such that $\phi[\mathcal{B}_n] = \mathcal{U}_n$ for each $n < \omega$ and $\phi(U) \subset \phi(V)$ if and only if $U \subset V$ whenever $U, V \in \mathcal{B}$.

We now define a function $h : {}^\omega\omega \to X$. For each $x \in {}^\omega\omega$, the set $\{B \in \mathcal{B} : x \in B\}$ is decreasing and intersects $\mathcal{B}_n$ in exactly one element for each $n < \omega$ so $\{\phi(B) : x \in B \in \mathcal{B}\}$ is a decreasing chain of non-empty clopen subsets of X of diameter converging to 0. Since the metric we chose is complete, $\bigcap\{\phi(B) : x \in B \in \mathcal{B}\}$ is non-empty (Theorem 0.26) and consists of only one point, let $h(x)$ be this point. Notice that $h[B] = \phi(B)$ for each $B \in \mathcal{B}$ and $h^{\leftarrow}[U] = \phi^{-1}(U)$ for each $U \in \mathcal{U}$ so h is continuous and open. It is also not hard to see that h is one-to-one and onto so in fact h is a homeomorphism. Thus completes the proof of the Theorem. $\square$

One interesting consequence of this characterization is the following.

Corollary 8.8 The space of irrational numbers is homeomorphic to ${}^\omega\omega$.

We also obtain an embedding result as follows. An alternative proof is given in Example 2.2.

Corollary 8.9 If X is a 0-dimensional separable metrizable space, then X can be embedded in ${}^\omega 2$.

Proof. Let $Q = \{x \in {}^\omega 2 : \exists n < \omega\ \forall m \geq n\ (x(n) = x(m))\}$. Then it is not hard to see that Q is countable and dense in ${}^{\omega}2$. Thus, ${}^{\omega}2 \setminus Q$ is easily seen to be nowhere locally compact. By Theorem 8.7, ${}^{\omega}2 \setminus Q$ is homeomorphic to ${}^{\omega}\omega$. By Proposition 8.6, X may be embedded in ${}^{\omega}2 \setminus Q$. In particular, we have embedded X in ${}^{\omega}2$. □

Theorem 8.10 [148] Let X be a strongly 0-dimensional metrizable space and κ an uncountable cardinal. Then X is homeomorphic to ${}^{\omega}\kappa$ if and only if every non-empty open subset of X has weight κ.

Proof. Again, the κ-Baire space has these properties so we assume that X is some space with the properties and construct a homeomorphism $h : {}^{\omega}\kappa \to X$. Fix some complete metric on X.

Claim: For each non-empty clopen set $V \subset X$ and every $m < \omega$ there exists a partition $\{V(\alpha) : n < \kappa\}$ of U into exactly κ clopen subsets of diameter $\leq \frac{1}{m+1}$.

We start by proving the Claim. By (1) in Theorem 8.2, there is a sequence of clopen partitions $\{\mathcal{V}_n : n < \omega\}$ of X such that $\mathcal{V}_n$ has mesh $\leq \frac{1}{m+1}$ and $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$ for every $n < \omega$. It is not hard to see that $\mathcal{V} = \bigcup\{\mathcal{V}_n : n < \omega\}$ is a base of X so $|\mathcal{V}| \geq \kappa$. By discarding some members of the sequence $\{\mathcal{V}_n : n < \omega\}$, we may assume that $|\mathcal{V}_0| \geq \omega$ and $\mathcal{V}_0$ has mesh $\leq \frac{1}{m+1}$. Let $\{W_n : n < \omega\} \subset \mathcal{V}_0$ be countably infinite and indexed faithfully.

If $\mathbf{cof}(\kappa) = \omega$, let $\{\kappa_n : n < \omega\} \subset \kappa$ such that $\kappa = \sup\{\kappa_n : n < \omega\}$. Otherwise, let $\kappa_n = \kappa$ for every $n < \omega$.

Fix $n < \omega$. Notice that $\{V \in \mathcal{V} : V \subset W_n\}$ is a base of W_n and W_n is of weight κ by hypothesis. Notice that $|\{V \in \mathcal{V}_k : V \subset W_n\}| \leq |\{V \in \mathcal{V}_{k+1} : V \subset W_n\}|$ for every $k < \omega$. Thus, there exists $k(n) < \omega$ such that $|\{V \in \mathcal{V}_{k(n)} : V \subset W_n\}| \geq \kappa_n$.

We may choose $\{k(n) : n < \omega\}$ recursively such that $k(n) < k(n+1)$ for every $n < \omega$. Finally, let

$$\mathcal{W} = (\mathcal{V}_0 \setminus \{W_n : n < \omega\}) \cup \{V : \exists n < \omega(V \in \mathcal{V}_{k(n)}, V \subset W_n)\}.$$

Then $\mathcal{W}$ is a partition of V into $\geq \kappa$ clopen subsets of diameter $\leq \frac{1}{m+1}$. Since $w(V) = \kappa$, then $|\mathcal{W}| = \kappa$. This completes the proof of the Claim.

Using the Claim, we continue as in Theorem 8.7. That is, we recursively construct a sequence of partions $\{\mathcal{U}_n : n < \omega\}$ into clopen subsets such that $|\mathcal{U}_n| = \kappa$, $\mathcal{U}_n$ has mesh $\leq \frac{1}{n+1}$ and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for all $n < \omega$. Proceeding as in Theorem 8.7, it is not hard to construct an order isomorphism $\phi : \mathcal{U} \to \mathcal{B}$, where $\mathcal{B}$ is the cannonical base of ${}^{\omega}\kappa$ (Definition 8.4 and Lemma 8.5) and $\mathcal{U} = \bigcup\{\mathcal{U}_n : n < \omega\}$. Finally, using ϕ, it is not hard to construct the homeomorphism $h : {}^{\omega}\kappa \to X$ in a way completely analogous to Theorem 8.7. □

An interesting example we obtain is the following surprising fact.

Corollary 8.11 For each $n < \omega$, let X_n be a discrete space of cardinality ω_n. Then $\prod\{X_n : n < \omega\}$ is homeomorphic to ${}^{\omega}(\omega_\omega)$.

Proof. Let $S = \bigcup\{\prod\{X_n : n \leq k\} : k \in \omega\}$ and $X = \prod\{X_n : n < \omega\}$. For each $f \in S$, let $B_f = \{x \in X : f \subset x\}$. Then $\{B_f : f \in S\}$ is a base of X of cardinality ω_ω. This shows that $w(X) \leq \omega_\omega$.

Let $f \in S$ and $m \geq n = dom(f)$. The set $\prod\{X_k : n \leq k \leq m\}$ is a discrete set of cardinality ω_m. Thus, the set

$$\{B_{f\cup g} : g \in \prod\{X_k : n \leq k \leq m\}\}$$

is a partition of B_f into ω_m clopen subsets. This shows that $\omega_m \leq w(B_f) \leq w(X) \leq \omega_\omega$. Since this inequality is true when $n \leq m < \omega$, we obtain that $w(B_f) = w(X) = \omega_\omega$.

Thus, we have proved that X has a base of open sets of weight $\aleph_\omega$. Thus, every non-empty open subset of X has weight ω_ω.

Clearly, X is metrizable since it is a product of metrizable (discrete) spaces. Moreover, it is easy to define a metric d for X: let $d(x,y) = \frac{1}{n+1}$ if $n = \min\{k < \omega : x(k) \neq y(k)\}$. If $\mathcal{U}_n = \{B_f : f \in \prod\{X_k : k \leq n\}\}$, then $\{\mathcal{U}_n : n < \omega\}$ with the metric d witnesses (2) in Theorem 8.2 so X is strongly 0-dimensional.

Finally, by Theorem 8.10, we obtain that X is homeomorphic to ${}^{\omega}(\omega_\omega)$. $\square$

8.2 Games on topological spaces

The Baire Category Theorem 0.25 is one of the most important General Topology theorems. It is not only important in General Topology but also in other areas of mathematics. For example, one may infer the existence of continuous nowhere differentiable functions ([130, Chapter 11]) or Liouville Numbers ([130, Chapter 2]) from the Baire Category Theorem. Another interesting example is that the Baire Category Theorem shows that most subcontinua (a dense G_δ in the hyperspace of continua) of the plane are pseudoarcs (see [125, Exercise 12.70] or [105, Theorem 3.15]). Also, besides this direct applications, the Forcing Method and Martin's Axiom in Set Theory is a generalization of the Baire Category Theorem (this was mentioned in the introduction, see page xii).

A space X is a *Baire space* if it satisfies the Baire Category Theorem; that is, if each intersection of countably many open dense sets of X is dense. Here we explore *topological games*[1] that test this property in a metrizable space. In

[1]See Chapter 6 of [130] for a historical introduction.

general, a game will be played between two players, commonly named I and II, who choose open subsets of some fixed metrizable space in turns. These games have ω turns and at the end of the game, some rule on the chosen open sets determines the winner. The games we consider here test the Baire property in various degrees of strength according to whether one of the players has a winning strategy. For other types of games, see [94, Sections 20, 21]. We will define two games: the Choquet game and the strong Choquet game.

Let X be a topological space. The *Choquet game* of X is defined as follows:

($\cdot$) In turn 0, I chooses a non-empty open set $U_0 \subset X$; after this, II chooses a non-empty open set $V_0 \subset U_0$,

($\cdot$) In turn $n+1$, where $n < \omega$, I chooses a non-empty open set $U_{n+1} \subset V_n$; after this, II chooses a non-empty open set $V_{n+1} \subset U_{n+1}$.

We will say that II wins this round in the Choquet game if $\bigcap\{U_n : n < \omega\} = \bigcap\{V_n : n < \omega\} \neq \emptyset$; otherwise, I wins.

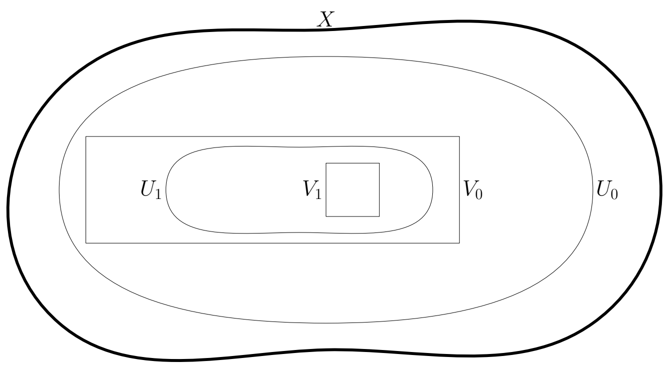

Figure 8.3: Playing a topological game in space X using open sets.

I	U_0		$\ldots$	U_n		U_{n+1}		$\ldots$	$\bigcap_{n<\omega} U_n = \emptyset$
II		V_0	$\ldots$		V_n		V_{n+1}	$\ldots$	$\bigcap_{n<\omega} V_n \neq \emptyset$

The *strong Choquet game* of X is defined as follows:

($\cdot$) In turn 0, I chooses a non-empty open set $U_0 \subset X$ and a point $x_0 \in U_0$; after this, II chooses an open set $V_0 \subset U_0$ such that $x_0 \in V_0$,

($\cdot$) In turn $n+1$, where $n < \omega$, I chooses a non-empty open set $U_{n+1} \subset V_n$ and a point $x_{n+1} \in U_{n+1}$; after this, II chooses a non-empty open set $V_{n+1} \subset U_{n+1}$ such that $x_{n+1} \in V_{n+1}$.

We will say that II wins this round in the strong Choquet game if $\bigcap\{U_n : n < \omega\} = \bigcap\{V_n : n < \omega\} \neq \emptyset$; otherwise, I wins.

I	U_0, x_0		$\dots$	U_n, x_n		$\dots$	$\bigcap_{n<\omega} U_n = \emptyset$
II		$x_0 \in V_0$	$\dots$		$x_n \in V_n$	$\dots$	$\bigcap_{n<\omega} V_n \neq \emptyset$

In both games, we will use the expression "draw U" when the player in turn chooses the open set U as part of his turn.

Notice that in some sense, the strong Choquet game gives player I an additional advantage over player II than in the Choquet game, since I can choose "where" II must draw his next open set V_n, by requiring that $x_n \in V_n$. We will formalize this below.

First, we must define what a *winning strategy* is. Informally, a winning strategy for one of the players A (where A is either I or II) is a rule that tells A what to draw in each of A's turns, according to the sets drawn by both I and II in previous turns and such that if A follows the rule of the strategy, then A wins.

Formally, we would have to give four definitions of winning strategy, one for each player in each of the two games. We will only give the definition of winning strategy for player II in the Choquet game, the other definitions can be given in a similar fashion and are left to the reader. Let X be a topological space and let

$$\begin{aligned} T &= \{\langle W_0, \dots, W_n\rangle : n < \omega,\ W_n \subset W_{n+1} \subset \dots \subset W_1 \subset W_0, \\ &\qquad \forall i \leq n (W_i \text{ is open in } X \text{ and non-empty })\}, \end{aligned}$$

that is, T is the tree of all *legal positions* in the Choquet game with the order defined as $\langle W_0, \dots, W_n\rangle \leq \langle W_0', \dots W_m'\rangle$ if and only if $n \leq m$ and $W_i = W_i'$ for all $i \leq n$. Notice that all branches of T are of order type ω. A *strategy* for II in the Choquet game is a subtree $T_0 \subset T$ such that the following conditions hold:

(a) if $\langle U_0, V_0, \dots, U_n\rangle \in T_0$ for some $n < \omega$, then there is a non-empty open subset V of X such that $\langle U_0, V_0, \dots, U_n, W\rangle \in T_0$ if and only if $W = V$,

(b) $\langle W\rangle \in T_0$ for every non-empty open subset W of X,

(c) if $\langle U_0, V_0, \dots, U_n, V_n\rangle \in T_0$ for some $n < \omega$, then for every non-empty open subset W of V_n it follows that $\langle U_0, V_0, \dots, U_n, V_n, W\rangle \in T_0$.

Informally, condition (a) gives player II a unique open set to draw in turn $n < \omega$ and conditions (b) and (c) include every possible draw of player I in turns 0 and $n + 1$, respectively. Finally, a *winning strategy* for II (in the Choquet game) is a strategy $T_0 \subset T$ such that (*d*) if $\{s_n : n < \omega\}$ is a branch in T_0, where $s_n = \langle U_0, \dots, U_n\rangle$, then $\bigcap\{U_n : n < \omega\} \neq \emptyset$.

Clearly, if player II plays the sets given in a winning strategy for the Choquet game, then II wins the Choquet game.

As the reader will notice in the proofs below, games give a nice and rather entertaining way to test properties. Our proofs using games will be quite informal, but in this way it will be easier to understand the usefulness of proofs using games. Our first result proves that the property of being a Baire space can be tested using games.

Lemma 8.12 Let X be a second countable space. Then X is a Baire space if and only if player I does not have a winning strategy in the Choquet game.

Proof. First assume that X is not a Baire space. Let $\{W_n : n < \omega\}$ be a sequence of dense open subsets of X and W an open subset of X such that W is disjoint from $\bigcap\{W_n : n < \omega\}$. We will now informally describe a winning strategy for player I. In turn 0, player I draws $U_0 = W$. Given $n < \omega$, assume that player II drew the open set V_n in turn n. Then player I draws $U_{n+1} = V_n \cap W_n$ in turn $n+1$; this set is clearly non-empty. Then $\bigcap\{U_n : n < \omega\} \subset W \cap (\bigcap\{W_n : n < \omega\}) = \emptyset$ so player I wins this round of the game. Thus, we have produced a winning strategy for player I.

Now let us assume that X is a Baire space and player I uses some strategy. We will play as player II and produce a round in which I does not win, so that the strategy is not a winning strategy. As player I is using a fixed strategy, all possible outcomes of the game will be decided by the sets drawn by player II. Notice that since player I's first move is to draw U_0, the rest of the game will be played inside U_0. Also, every open subspace of a Baire space is easily seen to be a Baire space. So we may assume that $U_0 = X$ without loss of generality for the rest of the proof.

Choose some countable base $\mathcal{B}$ of X such that $\emptyset \notin \mathcal{B}$. Let T be the tree of all $V \in {}^{<\omega}\mathcal{B}$ such that either $V = \emptyset$ or $0 \neq dom(V) = n+1$ for some $n < \omega$ and there is an instance of the game in which I uses the strategy and II draws $V(i)$ in step i for all $i \leq n$. Notice that T is countable. Let ϕ be the function defined in T such that $\phi(\emptyset) = X$ and whenever $V \in T$ and $0 \neq dom(V) = n+1$, $\phi(V)$ is the open set given by the strategy of player I in turn $n+1$ for games where the first n turns of II are given by $\langle V(0), \dots, V(n)\rangle$. Notice that $\phi(V) \subset V(n)$ by the definition of the Choquet game. We may think that ϕ is the strategy used by player I.

Let $V \in T$ be such that $dom(V) = n$ and let $B \in \mathcal{B}$ such that $B \subset \phi(V)$.

Define

$$\lfloor V, B \rfloor = \bigcup \{\phi(V \cup \{\langle n, W\rangle\}) : V \cup \{\langle n, W\rangle\} \in T, W \subset B, W \in \mathcal{B}\}.$$

Clearly, $\lfloor V, B\rfloor$ is an open subset of B and it is not hard to see that it is dense in B. Thus, the set

$$D = \bigcap\{\lfloor V, B\rfloor \cup (X \setminus \mathrm{cl}_X(B)) : V \in T \text{ and } B \in \mathcal{B} \text{ is such that } B \subset \phi(V)\}$$

is non-empty, being a countable intersection of dense open subsets of the Baire space X.

Let $x \in D$, we will now recursively choose open sets $\{V_n : n < \omega\} \subset \mathcal{B}$ and show that they can be drawn by player II in a round of the game in such a way that $x \in \bigcap\{V_n : n < \omega\}$. In this way, we will have a prove that the strategy used by player I is not a winning strategy.

In turn 0, player I draws $\phi(\emptyset) = U_0 = X$. Choose $B \in \mathcal{B}$ such that $x \in B$. Notice that $x \in \lfloor\emptyset, B\rfloor$ so by the definition of $\lfloor\emptyset, B\rfloor$ there exists $W \in \mathcal{B}$ such that $x \in \phi(\langle 0, W\rangle)$. Define $V_0 = W$, in this way $x \in V_0$. Now assume that $k < \omega$, we have defined $\{V_n : n \leq k\} \subset \mathcal{B}$ and that $x \in \phi(V)$, where $V \in {}^{k+1}\mathcal{B}$ is such that $V(n) = V_n$ for $n \leq k$ and $V \in T$. Let $B \in \mathcal{B}$ be such that $x \in B \subset \phi(V)$. Then by the definition of D, $x \in \lfloor V, B\rfloor$. By the definition of $\lfloor V, B\rfloor$, we have that there is some $W \in \mathcal{B}$ such that $x \in \phi(V \cup \{\langle k+1, W\rangle\})$ and $V \cup \{\langle k+1, W\rangle\} \in T$. Then it is possible to define $V_{k+1} = W$.

In this way, it is possible to continue the recursion to construct $\{V_n : n < \omega\}$ with $x \in \bigcap\{V_n : n < \omega\}$. As discussed before, this is a contradiction to the fact that player I was playing with a winning strategy. Thus, player I has no winning strategy. $\square$

A strengthening of the property of being a Baire space is the following. We will say that a topological space X is a *Choquet space* if player II has a winning strategy in the Choquet game for X. Further, X is a *strong Choquet space* if player II has a winning strategy in the strong Choquet game for X. So Choquet and strong Choquet are stronger properties than being a Baire space. In particular, it is not hard to see that the following implications hold for separable metrizable spaces. Recall that a *Polish space* is a completely metrizable separable space.

$$\text{Polish} \longrightarrow \text{strong Choquet} \longrightarrow \text{Choquet} \longrightarrow \text{Baire}$$

Our final result in this section gives a translation of the Choquet and strong Choquet spaces in terms of well-known topological propeties. A set A in a topological space X is said to be *comeager* if its complement $X \setminus A$ is meager; equivalently, if it contains a countable intersection of dense open subsets of X.

Theorem 8.13 Let X be a separable metrizable space and let Y be any Polish space such that X is dense in Y.

(1) X is a Choquet space if and only if X is comeager in Y.

(2) X is a strong Choquet space if and only if X is of type G_δ in Y if and only if X is Polish.

Proof. We begin with statement (1). First assume that X is comeager in the complete metric space (Y, d) so that $Y \setminus X = \bigcup\{F_n : n < \omega\}$, where F_n is nowhere dense in Y for each $n < \omega$. We now describe a strategy for player II in the Choquet game: in turn $n < \omega$, if player I draws U_n, player II chooses an open set W_n of Y such that $W_n \cap F_n = \emptyset$, $\mathrm{cl}_X(W_n \cap X) \subset U_n$ and W_n has diameter $\leq \frac{1}{n+1}$; then player II draws $V_n = W_n \cap X$. By the completeness of d, $\bigcap\{W_n : n < \omega\}$ is non-empty (using Theorem 0.26) and misses $X \setminus Y$, so $\bigcap\{V_n : n < \omega\} \neq \emptyset$.

Now assume that X is embedded in the complete metric space $\langle Y, d\rangle$, and II has a winning strategy ϕ for the Choquet game in X. We have to prove that there is a dense subset of Y of type G_δ contained in X. We will recursively construct a family $\{\mathcal{U}_n : n < \omega\}$, where such that

(a) $\mathcal{U}_n$ is a pairwise disjoint family of open subsets of Y for all $n < \omega$,

(b) if $U \in \mathcal{U}_n$, then U has diameter $\leq \frac{1}{n+1}$ for every $n < \omega$,

(c) $\bigcup \mathcal{U}_n$ is dense in X for all $n < \omega$,

(d) $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for all $n < \omega$ and

(e) for every $k < \omega$, if $\{V_n : n < k\}$ is such that $V_n \in \mathcal{U}_n$ and $V_{n+1} \subset V_n$ for each $n < k$, then there is an instance of the Choquet game such that II uses strategy ϕ and draws $V_n \cap X$ in turn n for each $n < k$.

For $n = 0$, consider the set $\mathcal{W}_0$ of all possible draws of player II in turn 0 using strategy ϕ and let $\mathcal{U}_0$ some maximal pairwise disjoint family of elements of $\{W : W \text{ is open in } Y, W \cap X \in \mathcal{W}_0\}$. We now show how to construct $\mathcal{U}_{k+1}$ once we have $\{\mathcal{U}_n : n \leq k\}$. Let $\mathcal{W}$ be the family of non-empty open sets contained in exactly one element of $\mathcal{U}_k$. Fix some $W \in \mathcal{W}$ and let W_n be the only element of $\mathcal{U}_n$ such that $W \subset W_n$ for $n \leq k$. By property (e), there is an instance of the Choquet game where II uses ϕ and draws $W_0 \cap X, \ldots, W_k \cap X$ succesively. Let player I draw set $W \cap X$ in turn $k+1$ and let $\phi(W)$ be the set given by ϕ in this situation for turn $k+1$ of player II. Define $\mathcal{U}_{k+1}$ to be a maximal pairwise disjoint family of elements of $\{U : U$ is open in Y and $U \cap X = \phi(W)$ for some $W \in \mathcal{W}\}$. Property (c) can be easily proved by induction.

For each $n < \omega$, let $U_n = \bigcup \mathcal{U}_n$, this is a dense open subset of Y. Notice that if $\{V_n : n < \omega\}$ is such that $V_n \in \mathcal{U}_n$ and $V_{n+1} \subset V_n$ for each $n < \omega$, from properties (b), (e) and the assumption that ϕ is a winning strategy we obtain that $\bigcap\{V_n : n < \omega\} = \{x\}$ for some $x \in X$. From properties (a) and (b) it is not hard to show that (compare with [146, Lemma 1.12.3])

$$\bigcap\{U_n : n < \omega\} = \bigcup\{\bigcap\{V_n : n < \omega\} : V_n \in \mathcal{U}_n, V_{n+1} \subset V_n \text{ for all } n < \omega\}.$$

So we obtain that $G = \bigcap\{U_n : n < \omega\}$ is a set of type G_δ of Y contained in X. Also, U_n is an open dense subset of the Polish, hence Baire space Y so G is dense. This proves that X is comeager in Y.

Now we prove the equivalence of (2). The fact that a metrizable space is Polish if and only if it is a G_δ set of every metrizable space in which it is embedded is well-known, see [50, 4.3.23 and 4.3.24]. If X is Polish we can give a winning strategy for player II in the strong Choquet game as follows. Let d a complete metric for X. In turn n, if player I draws a non-empty open set U_n and a point $p_n \in U_n$, let player II draw an open set V_n such that $p \in V_n$, $\mathrm{cl}_X(V_n) \subset U_n$ and V_n has diameter $\leq \frac{1}{n+1}$. From the completeness of the metric we obtain that $\bigcap\{V_n : n < \omega\} \neq \emptyset$ (Theorem 0.26) so this gives a winning strategy for player II.

Finally, assume that X is a dense subset of the Polish space Y and player II has a winning strategy for the strong Choquet game in X. Fix some complete metric for Y. We will show that X is a G_δ set of Y by constructing a family $\{W_n : n < \omega\}$ of open subsets of Y with intersection X.

The fact that player II has a winning strategy can be expressed in the following way. When player II uses a strategy, an instance of the game is completely defined by the pairs $\langle U_n, x_n\rangle$ drawn by player I. Every time $k < \omega$ and $\langle\langle U_0, x_0\rangle, \ldots, \langle U_k, x_k\rangle\rangle$ represents the first k draws of player I in a game where player II uses the winning strategy, the set drawn by player II in turn k will be denoted by $\psi\langle\langle U_0, x_0\rangle, \ldots, \langle U_k, x_k\rangle\rangle$.

We will define a collection of locally finite covers $\{\mathcal{U}_n : n < \omega\}$ of X by open subsets of Y and then W_n will be defined equal to $\bigcup \mathcal{U}_n$ for all $n < \omega$. The definition of $\{\mathcal{U}_n : n < \omega\}$ will be recursive and we will need to make auxiliar definitions. We need a collection $\mathcal{V}_n = \{V(n,x) : x \in X\}$ of open sets of Y such that $x \in V(n,x)$ for each $n < \omega$. For each $n < \omega$ and $U \in \mathcal{U}_{n+1}$, we will also define a finite set $\phi(n,U) \subset X$. The following conditions shall be satisfied:

(i) if $x \in X$, $n < \omega$ and $U \in \mathcal{U}_n$, then $x \in V(n,x) \subset U$, (*ii*) if $x \in X$ and $m \leq n < \omega$ then $\{U \in \mathcal{U}_m : V(n,x) \subset U\}$ is non-empty and finite,

(*iii*) if $n < \omega$, $U \in \mathcal{U}_{n+1}$ and $x \in \phi(n,U)$ then $U \subset V(n,x)$,

(*iv*) if $n < \omega$, $U \in \mathcal{U}_{n+1}$ and $x \in X$ is such that there is $W \in \mathcal{U}_n$ with $U \subset V(n,x) \subset W$, then there is $y \in \phi(n,U)$ with $U \subset V(n,y) \subset W$.

For $n < \omega$, let $\mathcal{U}_0$ be a locally finite open cover of X with open subsets of Y of diameter ≤ 1. For each $x \in X$, define $V(0,x)$ to be an open set of Y such that

$$(*)_0 \; V(0,x) \cap X = \bigcap \{\psi\langle\langle U, x\rangle\rangle : x \in U \in \mathcal{U}_0\},$$

this is possible as $\mathcal{U}_0$ is locally finite so the set on the right of equation $(*)_0$ is a finite intersection of open sets. Clearly (*i*) and (*ii*) hold in this step.

Assume that we have defined $\{\mathcal{U}_n : n \leq k\}$ and $\{\mathcal{V}_n : n \leq k\}$ and conditions (*i*), (*ii*), (*iii*) and (*iv*) hold in the current domain of definition. Let $\mathcal{U}_{k+1}$ be a locally finite refinement of $\mathcal{V}_k$ with mesh $\leq \frac{1}{k+2}$, as $\mathcal{V}_k$ may not cover Y, this is carried out in the space $\bigcup \mathcal{V}_k$, which is open in Y and contains X. We may also ask that every element of $\mathcal{U}_{k+1}$ intersects finitely many elements of $\mathcal{U}_k$ since $\mathcal{U}_k$ is locally finite. If $U \in \mathcal{U}_{k+1}$, there are only finitely many elements of $\mathcal{U}_k$ that contain U; using this it is not hard to define $\phi(n,U)$ in such a way that (*iii*) and (*iv*) hold.

For every $x \in X$, we choose an open set $V(k+1,x)$ of $\bigcup \mathcal{V}_k$ such that

$$\begin{aligned}(*)_{k+1} \; V(k+1,x) \cap X = \bigcap \{&\psi\langle\langle U_0, x_0\rangle, \ldots, \langle U_k, x_k\rangle, \langle U_{k+1}, x\rangle\rangle : \\ &x \in U_{k+1} \in \mathcal{U}_{k+1}, \forall i \leq k \; (U_i \in \mathcal{U}_i, x_i \in \phi(i, U_{i+1}) \cap U_i)\}\end{aligned}$$

To prove that this is possible, we have to prove that the right side of the equation in $(*)_{k+1}$ is a finite intersection. To do this, we have to show that the set of all such possible $\langle\langle U_0, x_0\rangle, \ldots, \langle U_k, x_k\rangle, \langle U_{k+1}, x\rangle\rangle$ is finite. Notice that since $\mathcal{U}_{k+1}$ is locally finite, there are only finitely many options for U_{k+1}. Once $\langle U_{i+1}, \ldots, U_k + 1\rangle$ and $\langle x_{i+1}, \ldots, x_{k+1}\rangle$ have been chosen for some $i \leq k$, we must choose U_i. Clearly, x_i is in the finite set $\phi(i, U_{i+1})$ so there are only finitely many possibilities for the choice of x_i. By property (*iv*), we must choose U_i such that $U_{i+1} \subset V(i, x_i) \subset U_i$ and there is only finitely many options by the definition of $\mathcal{U}_{k+1}$. This proves that there are only finitely many options for $\langle\langle U_0, x_0\rangle, \ldots, \langle U_k, x_k\rangle, \langle U_{k+1}, x\rangle\rangle$ so it is indeed possible to define $V(k+1,x)$.

It is not hard to prove that the inductive assumptions (*i*), (*ii*), (*iii*) and (*iv*) hold. This completes the definitions of the covers $\{\mathcal{U}_n : n < \omega\}$ and thus, of $W_n = \bigcup \mathcal{U}_n$ for all $n < \omega$. By definition, clearly $X \subset \bigcap \{W_n : n < \omega\}$, we must then prove the other inclusion. Let $p \in \bigcap\{W_n : n < \omega\}$. Recall that $\mathcal{U}_n$ has mesh $\leq \frac{1}{n+1}$. If we prove the following claim, we obtain that $p \in X$, which concludes the proof of this Lemma.

Claim. There is $\mathcal{S} = \{\langle U_n, x_n\rangle : n < \omega\}$ such that there is an instance of the game where player II uses the winning strategy, player I draws $\langle U_n, x_n\rangle$ in turn n, and $p \in U_n \in \mathcal{U}_n$ for all $n < \omega$.

We will construct a tree of all possibilities and from it, we will extract $\mathcal{S}$. Let T be the defined such that $s \in T$ if and only if $s = \emptyset$ or there is $n < \omega$ such that s is a function, $dom(s) = n + 1$ and $s(i) = \langle U_i, x_i\rangle$ for $i \leq n$, where

(v) if $i \leq n$, $p \in U_i \in \mathcal{U}_i$,

(vi) if $i < n$, $x_i \in \phi(i, U_{i+1}) \cap U_i$ and

(vii) there is $W \in \mathcal{U}_{n+1}$ such that $p \in W$ and $x_n \in \phi(n, W) \cap U_n$.

It is not hard to prove that if $s \in T$ and $t \subset s$, $t \in T$ so T is indeed a tree. For every $n < \omega$, let $T_n = \{s \in T : dom(s) = n\}$.

Let us argue that T_n is finite and non-empty for every $n < \omega$. This is clear for $n = 0$, let $0 < k < \omega$. Since $\mathcal{U}_{k+1}$ is locally finite, there are only finitely many $W \in \mathcal{U}_{k+1}$ with $p \in W$. By the recursive construction of $\mathcal{U}_{k+1}$ and properties (ii) and (iv), there exist $W \in \mathcal{U}_{k+1}$, $U_k \in \mathcal{U}_k$ and $x_k \in \phi(n, W)$ such that condition (vii) holds. Moreover, there are only finitely many such sets by the local finiteness of the covers and the fact that $\phi(n, W)$ is finite. Once $\langle U_{i+1}, \ldots, U_{k+1}\rangle$ and $\langle x_{i+1}, \ldots, x_{k+1}\rangle$ have been chosen so that conditions (v) and (vi) are satisfied for some $i \leq k$, it is possible to choose U_i and x_i and there are finitely many such choices. The argument is completely analogous to the previous one and we will omitt it.

So it has been proved that T_n is finite and non-empty for every $n < \omega$. Then it is possible to recursively choose $s_n \in T_n$ such that $\{t \in T : s_n \subset t\}$ is infinite and $s_n \subset s_{n+1}$ for all $n < \omega$ (this argument is known as König's lemma, see [94, 4.12]). Let $\mathcal{S}' = \bigcup\{s_n : n < \omega\}$ and define $\mathcal{S} = \{\mathcal{S}'(n) : n < \omega\}$.

We now prove the property of $\mathcal{S}$ given in the claim, let $\mathcal{S}(n) = \langle U_n, x_n\rangle$ for each $n < \omega$. From properties (iii), (v) and (vi), we obtain that $p \in U_{n+1} \subset V(n, x_n) \subset U_n$ for all $n < \omega$. Play the strong Choquet game as player I drawing $\langle U_n, x_n\rangle$ in turn n. Then player II responds with an open set V_n and by the definition of $V(n, x_n)$ it is possible to see that $V(n, x_n) \subset V_n \subset U_n$. So it is indeed possible to us, taking the role of player I, to draw $\langle U_{n+1}, x_{n+1}\rangle$ in turn $n + 1$. This completes the proof of the Claim and thus the proof of the Theorem. $\square$

8.3 Paracompact M-spaces

In our main problem for this Part (Question 9.1, p. 148), we are dealing with metrizable spaces only. However, by Corollary 7.13, our results will always extend to spaces coabsolute with the spaces we are considering, even if they are not metrizable. In this section we will mention a result by Ponomarev that characterizes spaces coabsolute with metrizable spaces and make an additional remark.

A space X is an *M-space* if there exists a sequence $\{\mathcal{C}_n : n < \omega\}$ of covers of X such that

(i) if $x_n \in St(x, \mathcal{C}_n)$ for each $n < \omega$, then $\{x_n : n < \omega\}$ has a cluster point,

(ii) for each $n < \omega$, $\mathcal{C}_{n+1}$ star-refines $\mathcal{C}_n$.

The following result was proved by Ponomarev. Its proof is long and requires many steps. By this reason, we will not give the details of the proof.

Proposition 8.14 [134] Let X be a Tychonoff space. Then the following are equivalent

(a) X is coabsolute with a metrizable space,

(b) there exists a metrizable space that is a perfect and irreducible continuous image of X,

(c) X is a paracompact M-space with a σ-locally finite π-base.

Actually, in [134], (c) says "paracompact p-space" (p-space in the sense of Arkhangel'skiĭ) but this is equivalent to the formulation we have given (see [72, Corollary 3.20]). So according to Propositions 7.12 and 8.14, we will be able to obtain results about remote points of paracompact M-spaces. However, our results will be stated in terms of metrizable spaces only. The main reason for doing this is that for our arguments it is enough to consider metrizable spaces. Moreover, the corresponding results for paracompact M-spaces can be easily obtained by using Proposition 8.14.

Remark 8.15 By Proposition 8.14, it is not hard to prove that in Proposition 9.12, Theorems 9.13, 9.26 and Corollary 9.27 we can change "(completely) metrizable" by "(Čech-complete) paracompact M-space with a σ-locally finite π-base".

We will now address another technical matter. Notice that Proposition 7.12 talks about irreducible closed mappings, while talking about coabsolutes we asked that the function be perfect. We now show that there is nothing else we can obtain using Proposition 7.12 inside the class of paracompact M-spaces.

Proposition 8.16 Let $f : X \to Y$ be an irreducible continuous function between paracompact M-spaces. Then f is perfect.

Proof. By (b) in Proposition 8.14, there exists a metrizable space M and a perfect and irreducible continuous function $g : Y \to M$. We will follow the proof of Vaĭnšteĭn's Lemma and the Hanai-Morita-Stone Theorem from [50, 4.4.16]. Let $h = g \circ f$.

Claim: For every $p \in M$, $\mathrm{bd}_X(h^{\leftarrow}(p))$ is countably compact.

To prove the Claim, let $\{x_n : n < \omega\} \subset \mathrm{bd}_X(h^{\leftarrow}(p))$. Since M is metrizable let $\{U_n : n < \omega\}$ be a local open basis of p such that $\mathrm{cl}_M(U_{n+1}) \subset U_n$ for each $n < \omega$. Let $\{\mathcal{C}_n : n < \omega\}$ be the sequence of open covers for X given by the definition of M-space. For each $n < \omega$, let $y_n \in (h^{\leftarrow}[U_n] \setminus h^{\leftarrow}(p)) \cap St(x_n, \mathcal{C}_n)$. Then it is easy to see that $\{h(y_n) : n < \omega\}$ is a sequence converging to p. Since h is a closed function, it follows that there exists a cluster point $q \in h^{\leftarrow}(p)$ of $\{y_n : n < \omega\}$.

We construct a strictly increasing function $\phi : \omega \to \omega$ with $\phi(n) \geq n+1$ for all $n < \omega$ as follows: for each $n < \omega$ let $\phi(n)$ be such that $y_{\phi(n)} \in St(q, \mathcal{C}_{n+1})$. Since $y_{\phi(n)} \in St(x_{\phi(n)}, \mathcal{C}_{\phi(n)})$ and $\phi(n) \geq n+1$, by condition (ii) in the definition of an M-space we obtain that $x_{\phi(n)} \in St(q, \mathcal{C}_n)$. Thus, by condition (i) in the definition of M-space we obtain that $\{x_{\phi(n)} : n < \omega\}$ has a cluster point. Such cluster point must be in $\mathrm{bd}_X(h^{\leftarrow}(p))$ so the Claim follows.

Now, by the Claim and the fact that X is paracompact we have that the set $\mathrm{bd}_X(h^{\leftarrow}(p))$ is compact for each $p \in M$. Since h is irreducible, either $\mathrm{bd}_X(h^{\leftarrow}(p)) = h^{\leftarrow}(p)$ or $h^{\leftarrow}(p)$ is a singleton. Thus, h is perfect and so is f. $\square$

We now present some examples of non-metrizable paracompact M-spaces so that the reader has an idea of what kind of generalization we have. Recall that the Sorgenfrey line $\mathbb{S}$ is the set $\mathbb{R}$ with $\{[x, y) : x, y \in \mathbb{R}, x < y\}$ as a base of open sets.

Example 8.17 The Alexandroff-Urysohn double arrow space $\mathbb{A}$.

Let $\mathbb{A}_0 = (0, 1] \times \{0\}$, $\mathbb{A}_1 = [0, 1) \times \{1\}$ and $\mathbb{A} = \mathbb{A}_0 \cup \mathbb{A}_1$. Define the lexicographic strict order on $\mathbb{A}$ as $\langle x, t\rangle < \langle y, s\rangle$ if $x < y$ or both $x = y$ and $t < s$. Then $\mathbb{A}$ is given the order topology. Notice that both $\mathbb{A}_0$ and $\mathbb{A}_1$ have the Sorgenfrey line topology as subspaces of $\mathbb{A}$ and both are dense in $\mathbb{A}$. It is easy to see that $\mathbb{A}$ is separable, first countable, compact, 0-dimensional and of weight $\mathfrak{c}$ (see Proposition 12.7 in Part III). The function $\pi : \mathbb{A} \to [0, 1]$ defined by $\pi(\langle x, t\rangle) = x$ is a ≤ 2-to-1 continuous function. Also, it is not hard to see that π is irreducible, so in fact π witnesses that $\mathbb{A}$ and $[0, 1]$ are coabsolute (Corollary 6.42). Since $\mathbb{A}$ is compact and of weight $\mathfrak{c}$, it is not metrizable (Theorem 0.16). $\square$

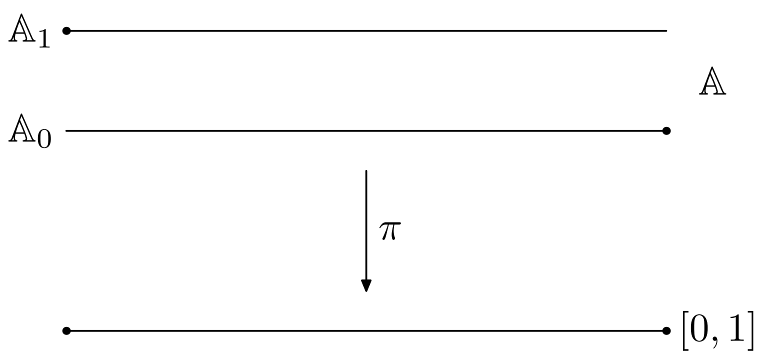

Figure 8.4: The double arrow space.

Being coabsolute to a metrizable space is not hereditary. As Example 8.17 shows, $\mathbb{A}$ is coabsolute with $[0, 1]$, and thus with any compact crowded metrizable space (Corollary 7.17). The Sorgenfrey line $\mathbb{S}$ is a subspace of $\mathbb{A}$ but it is not coabsolute with a metrizable space.

Proposition 8.18 The Sorgenfrey line $\mathbb{S}$ is not coabsolute with a metrizable space.

Proof. The Sorgenfrey line is paracompact (since it is Lindelöf) and has countable π-weight. However, we show next that $\mathbb{S}$ is not an M-space. Let $\{\mathcal{C}_n : n < \omega\}$ be the sequence of covers of $\mathbb{S}$.

For each $n < \omega$, let

$$E_n = \{p \in \mathbb{S} : \text{ there is } y \in \mathbb{S} \text{ such that } y < p \text{ and } (y, p) \cap St(p, \mathcal{C}_n) = \emptyset\}.$$

If $p \in E_n$, there is $U \in \mathcal{C}_n$ such that $p \in U$ and $x \in \mathbb{S}$ such that $[p, x) \subset U$ so $E_n \cap [p, x) = \{p\}$. This shows that E_n is discrete so it is countable.

Choose $p \in \mathbb{S} \setminus \bigcup\{E_n : n < \omega\}$. For each $n < \omega$, let $x_n \in \mathbb{S}$ be such that $x_n < p$ and $x_n \in St(p, \mathcal{C}_n)$. We can further ask that $x_n < x_{n+1}$ for every $n < \omega$.

Thus, $\{x_n : n < \omega\}$ is a strictly increasing sequence in $\mathbb{S}$, so it is discrete. This shows that condition (a) in the definition of an M-space does not hold. Thus, $\mathbb{S}$ is not an M-space. $\square$

Another general method that can be used to construct spaces coabsolute to metrizable spaces that are not metrizable are resolutions, see Chapter 3 in [162]. Of course, if X is a non-discrete metrizable space, then EX is not metrizable (since it is ED) and coabsolute to X. But EX is too "big" and it is sometimes hard to see inside the structure of this space. Using resolutions, one can obtain spaces coabsolute with X with some control of their properties.

8.4 $\mathfrak{c}$-points in some spaces

As seen in Proposition 7.6, βX is ED at each remote point for every Tychonoff space X. Also, this implies that when X is nowhere locally compact, then X^* is ED at each remote point as well (this follows from the proof of Theorem 7.9). However, if X is locally compact and metrizable, we will see that we have the opposite situation (Theorem 8.20).

Recall that a point p in a topological space X is a κ-point if there is a family of exactly κ disjoint open subsets of X that have p in their closure. It is not hard to show that, for example, in a metrizable space any non-isolated point is an ω-point. In [12], Balcar and Vojtáš proved that any point of ω^* is a $\mathfrak{c}$-point of ω^*, a fact that strongly contrasts with the fact that $\beta\omega$ is ED so no point of ω^* is a $\mathfrak{c}$-point of $\beta\omega$.

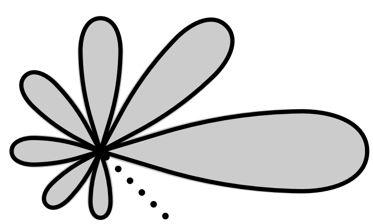

Figure 8.5: Every point in the plane is an ω-point.

Theorem 8.19 [12] Every point of ω^* is a $\mathfrak{c}$-point of ω^*.

The proof of Theorem 8.19 is hard and is outside the objectives of this thesis. A proof can be found in [62, Chapter 5, Theorem 3.5]. In [41], van Douwen gave a technique to transfer some properties of ω^* to spaces of the form X^* for some Tychonoff spaces X. We will present van Douwen's proof of the following.

Theorem 8.20 [41, Theorem 5.2] If X is a locally compact, non-compact, realcompact Tychonoff space, then every point of X^* is a $\mathfrak{c}$-point of X^*.

Proof. Fix $p \in X^*$. We will find a regular closed subset F of X^* such that $p \in F$ and a open continuous function $f : F \to \omega^*$. From this it follows that p is a $\mathfrak{c}$-point in the following way. By Theorem 8.19, there is a collection $\mathcal{V}$ of precisely $\mathfrak{c}$ pairwise disjoint open subsets of ω^* such that $f(p) \in \mathrm{cl}_{\omega^*}(V)$ for each $V \in \mathcal{V}$. Let $\mathcal{U} = \{f^{\leftarrow}[V] : V \in \mathcal{V}\}$, by the continuity of f we obtain that $\mathcal{U}$ is a collection of $\mathfrak{c}$ pairwise disjoint sets. Assume that $p \notin \mathrm{cl}_{\omega^*}(f^{\leftarrow}[V])$ for some $V \in \mathcal{V}$. Then $W = \omega^* \setminus \mathrm{cl}_{\omega^*}(f^{\leftarrow}[V])$ is an open set of ω^* so $f[W]$ is open, $f(p) \in f[W]$ and $f[W] \cap V = \emptyset$; thus, $f(p) \notin \mathrm{cl}_{\omega^*}(V)$, a contradiction. Thus, $\mathcal{U}$ witnesses that p is a $\mathfrak{c}$-point.

Let us describe how to construct such F and g. From the definition of real-compactness it is not hard to construct a continuous function $f : \beta X \to [0, 1]$ such that $f(p) = 0$ and $G = f^{\leftarrow}(0) \subset X^*$. Define $Y = \beta X \setminus f^{\leftarrow}(0) = f^{\leftarrow}[(0, 1]]$. It is not hard to see that Y is normal since it is locally compact and σ-compact (this follows from Lemma 1.5.15 and Theorem 3.1.9 in [50]). Notice that that $\beta X = \beta Y$ and $Y^* = G$.

Claim 1. G is a regular closed set of X^*.

To prove Claim 1, let U be open in X such that $G \cap \mathrm{Ex}_X(U) \neq \emptyset$. According to Proposition 6.11, it is enough to find an open set V of X such that $\emptyset \neq \mathrm{Ex}_X(V) \cap X^* \subset \mathrm{Ex}_X(U) \cap G$.

Since $\mathrm{Ex}_X(U) \cap X^* \neq \emptyset$ and U is dense in $\mathrm{Ex}_X(U)$ (Proposition 6.10), there are $\{y_n : n < \omega\} \subset U$ such that $y_{n+1} < y_n$ and $f(x_n) < \frac{1}{n+1}$ for $n < \omega$. Let $z_0 = 0$ and $z_{n+1} = \frac{1}{2}(x_{n+1} + x_{n+2})$ for $n < \omega$. For each $n < \omega$, let V_n be an open set of X such that $y_n \in V_n$, $f[V_n] \subset (z_{n+1}, z_n)$, $\mathrm{cl}_X(V_n) \subset U$ and $\mathrm{cl}_X(V_n)$ is compact. Notice that $\{\mathrm{cl}_X(V_n) : n < \omega\}$ is pairwise disjoint and locally finite. Let $V = \bigcup\{V_n : n < \omega\}$. Notice that $\mathrm{cl}_X(V) = \bigcup\{\mathrm{cl}_X(V_n) : n < \omega\}$ (Lemma 0.20) and $\mathrm{cl}_X(V) \subset U$.

From the definition of $\{V_n : n < \omega\}$ it follows that $f[\mathrm{cl}_{\beta X}(V) \cap X^*] \subset \{0\}$ so $\mathrm{cl}_{\beta X}(V) \cap X^* \subset G$. Then $\mathrm{Ex}_X(V) \cap X^* \subset G$ by (b) in Proposition 6.10 and $\mathrm{Ex}_X(V) \subset \mathrm{Ex}_X(U)$ since $V \subset U$. To complete the proof of the claim, we just need to show that $\mathrm{Ex}_X(V) \cap X^* \neq \emptyset$. Notice that $D = \{y_n : n < \omega\}$ is a closed and discrete subset of V so there is $y \in \mathrm{cl}_X(D) \cap X^*$. Clearly, $y \in \mathrm{cl}_{\beta X}(V) = \mathrm{cl}_{\beta X}(\mathrm{Ex}_X(V))$ (Proposition 6.10). By Proposition 6.12, it is enough to prove that $y \notin \mathrm{cl}_{\beta X}(\mathrm{bd}_X(V))$. Notice that $\mathrm{bd}_X(V) = \bigcup\{\mathrm{bd}_X(V_n) : n < \omega\}$. Also, $\mathrm{bd}_X(V)$

and D are disjoint closed subsets of the normal space Y so they are completely separated in Y. Thus, $\mathrm{cl}_{\beta Y}(\mathrm{bd}_X(V)) \cap \mathrm{cl}_{\beta Y}(D) = \emptyset$ by (3) in Theorem 6.4. But $\beta X = \beta Y$ so in fact $p \notin \mathrm{cl}_{\beta X}(\mathrm{bd}_X(V))$, which is what we wanted to prove. This completes the proof of Claim 1.

From the fact that Y is locally compact and σ-compact, we may apply Proposition 7.19 and find a collection of regular closed, non-empty compact subsets $\{K_n : n < \omega\}$ of Y that cover Y and such that for $m, n < \omega$, $K_m \cap K_n = \emptyset$ if $|m - n| > 1$ and $K_n \cap \mathrm{int}_X(K_m) = \emptyset$ if $m \neq n$. Let $W_0 = \bigcup\{\mathrm{int}_X(K_{2n}) : n < \omega\}$ and $W_1 = \bigcup\{\mathrm{int}_X(K_{2n+1}) : n < \omega\}$. Then $Y = \mathrm{cl}_Y(W_0) \cup \mathrm{cl}_Y(W_1)$ so $\beta Y = \mathrm{cl}_{\beta Y}(W_0) \cup \mathrm{cl}_{\beta Y}(W_1)$. We may assume without loss of generality that $p \in \mathrm{cl}_{\beta Y}(W_0)$. Let $Z = \mathrm{cl}_Y(W_0)$ and $F = \mathrm{cl}_{\beta Y}(W_0) \cap Y^*$.

Claim 2. F is a regular closed set of G that contains p.

Clearly, F is closed. Recall that $\mathrm{cl}_{\beta Y}(Z) = \mathrm{cl}_{\beta Y}(\mathrm{Ex}_Y(W_0))$ (Proposition 6.10). To prove Claim 2 it is enough to show that $\mathrm{Ex}_Y(W_0) \cap Y^*$ is dense in F. The argument is similar to the one in the proof of Claim 1. Let U be a non-empty subset of Y such that $F \cap \mathrm{Ex}_Y(U) \neq \emptyset$. Then there exists an infinite set $A \subset \{2n : n < \omega\}$ such that $U \cap K_m \neq \emptyset$ for each $m \in A$. For each $m \in A$, let $y_m \in U \cap \mathrm{int}_X(K_m)$. Then $D = \{y_m : m \in A\}$ is a closed and discrete set of W_0 so there is a point $y \in \mathrm{cl}_Y(D) \setminus D$. Since $D \cap \mathrm{bd}_Y(W_0) = \emptyset$ and Y is normal, $y \notin \mathrm{bd}_{\beta Y}(\mathrm{Ex}_Y(W_0))$ by (3) in Theorem 6.4. By Proposition 6.12, $y \in \mathrm{Ex}_Y(W_0)$. Thus, $\mathrm{Ex}_X(W_0) \cap Y^*$ is dense in F.

From Claims 1 and 2 we obtain that F is a regular closed subset of X^* with $p \in F$. We are only left with the task of defining an open continuous function $f : F \to \omega^*$.

Define a function $f_0 : Z \to \omega$ by $f_0(x) = n$ if $x \in K_{2n}$. Notice that K_{2n} is clopen in Z for each $n < \omega$ so it follows that f_0 is continuous. Since Y is normal, Z is C^*-embedded in βY. Thus, there is a continuous function $f_1 : \mathrm{cl}_{\beta Y}(Z) \to \beta\omega$ such that $f_1{\restriction_Z} = f_0$.

Notice that $f_1[F] = \omega^*$. By Lemma 0.11, $f_1[F] \subset \omega^*$. Also, $f_1[\mathrm{cl}_Y(W_0)] = f_0[\mathrm{cl}_Y(W_0)]$ is dense in $\beta\omega$ and $\mathrm{cl}_{\beta Y}(Z)$ is compact so in fact $f_1[F] = \omega^*$.

Finally, define $f : F \to \omega^*$ as $f = f_1{\restriction_F}$. We are only left to prove that f is open. Let U be an open set of Z, it is enough to prove that f maps $\mathrm{Ex}_Z(U) \cap F$ to an open set by Proposition 6.11. Let $A = \{n < \omega : U \cap K_{2n} \neq \emptyset\}$. Recall that $\beta\omega$ can be taken as the space of ultrafilters on ${}^{\omega}2$ (Theorem 6.23) and recall that $\widehat{B} = \{p \in \beta\omega : B \in p\}$ is clopen in $\beta\omega$ for each $B \subset \omega$. It is not hard to see that $f_1[\mathrm{Ex}_Z(U)] = \widehat{A}$ so $f[\mathrm{Ex}_Z(U) \cap F] = \widehat{A} \cap \omega^*$. Thus, f is indeed open and this completes the proof of this theorem. $\square$

Chapter 9

Homeomorphic Spaces of Remote Points

In this Chapter we will define our main problem and show the results we were able to obtain. In a general form, the problem we are interested in is the following.

> $(*)$ Given a Tychonoff space X, find all Y such that $\varrho(X)$ is homeomorphic to $\varrho(Y)$.

Notice that problems of type $(*)$ can be formulated every time we can construct a space in terms of some other in a topological way. Due to the results of Woods and Gates presented in Theorem 7.20 and 7.21, we already know that some classes of metrizable spaces have homeomorphic sets of remote points. The question we will focus on is the following.

Question 9.1 Let X be a metrizable non-compact space. Find some simple or known topological property $\mathbf{P}$ such that if Y is metrizable then Y has $\mathbf{P}$ if and only if $\varrho(X)$ is homeomorphic to $\varrho(Y)$.

A reason to restrict X to be metrizable is that we already know that we have a rich collection of remote points (Corollary 7.4). This will allow us to transfer some properties of X to $\varrho(X)$. However, by Proposition 8.14, we may also consider paracompact M-spaces, see Remark 8.15.

Some of our results are of the following type: if X has topological property $\mathbf{P}$ and $\varrho(X)$ is homeomorphic to $\varrho(Y)$, then Y also has $\mathbf{P}$. In particular we study properties such as dimension, local compactness, topological completeness and σ-compactness. For the other implication, our main results are perhaps Theorem 9.13 and Corollary 9.27 that characterize remote points of the irrationals and rationals up to some restrictions. These results generalize the following result of Eric K. van Douwen.

Theorem 9.2 [36, Theorem 16.2] $\varrho(\mathbb{Q})$ and $\varrho({}^\omega\omega)$ are not homeomorphic because $\varrho(\mathbb{Q})$ is a Baire space and $\varrho({}^\omega\omega)$ is meager.

However the main question that remains unanswered is the following.

Question 9.3 Find all metrizable X such that $\varrho(X)$ is homeomorphic to either $\varrho(\mathbb{Q})$ or $\varrho({}^\omega\omega)$.

See Questions 9.15 and 9.33 for reformulations of Question 9.3. In Section 9.2 we also give a classification of nowhere locally compact, completely metrizable spaces by a sort of cardinal invariant we call cellular type. It turns out that cellular type almost characterizes remote points for this class of spaces, see Corollary 9.22 and Example 9.24.

Finally, we remark that we know nothing about non-definable sets (of ${}^\omega 2$). Thus, we finish the discussion with the following question.

Question 9.4 Do there exist two Bernstein sets X and Y such that $\varrho(X)$ is not homeomorphic to $\varrho(Y)$?

9.1 Dimension and Local Compactness

We start by showing that it is enough to consider strongly 0-dimensional spaces. The following result is original but its proof resembles one given by Morita in [121, Part 5].

Proposition 9.5 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) For every metrizable space X of weight κ there exists a strongly 0-dimensional space Y of weight κ and a perfect and irreducible continuous function $f : Y \to X$.

Proof. Let d be the metric for ${}^\omega\kappa$ defined in Definition 8.4 and let d' be some compatible metric for X. In this proof we shall use some notation given in Definition 8.4. We shall construct construct locally finite covers $\{\mathcal{U}_n : n < \omega\}$ of non-empty regular closed sets of X in such a way that the following hold.

(a) $\mathcal{U}_n$ has mesh $\leq \frac{1}{n+1}$ for every $n < \omega$,

(b) $|\mathcal{U}_n| \leq \kappa$ for each $n < \omega$,

(c) if $n < \omega$ and $A, B \in \mathcal{U}_n$ are such that $A \neq B$, then $A \cap \operatorname{int}_X(B) = \emptyset$, (*d*) $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ for all $n < \omega$,

(*e*) if $n < \omega$ and $A \in \mathcal{U}_n$, then $\{B \in \mathcal{U}_{n+1} : B \subset A\}$ covers A.

The existence of this family of covers can be easily deduced from the following.

Claim 1. Let $A \in \mathcal{R}(X)$ and $m < \omega$. Then there exists a locally finite cover $\mathcal{V}$ of mesh $\leq \frac{1}{m+1}$ consisting of $\leq \kappa$ non-empty regular closed sets of A such that if $B, C \in \mathcal{V}$ and $B \neq C$, then $B \cap \mathrm{int}_A(C) = \emptyset$.

To prove Claim 1, let $\mathcal{W}$ be a locally finite open cover of A with open sets of diameter $\leq \frac{1}{m+1}$, this is possible by Stone's Theorem 0.21. Choose some well-order $\mathcal{W} = \{W_\alpha : \alpha < \lambda\}$ for some ordinal λ. Recursively, let $V_0 = W_0$ and $V_\alpha = W_\alpha \setminus \bigcup\{\mathrm{cl}_A(V_\beta) : \beta < \alpha\}$, these are pairwise disjoint open sets (Lemma 0.20). Since A has weight $\leq \kappa$, it follows that $\{V_\alpha : \alpha < \lambda\} \setminus \{\emptyset\}$ is a collection of $\leq \kappa$ pairwise disjoint open subsets of A. Let $\mathcal{V} = \{\mathrm{cl}_A(V_\alpha) : \alpha < \lambda\} \setminus \{\emptyset\}$. It is easy to see that $\mathcal{V}$ is as required by Claim 1.

Let $\mathcal{U} = \bigcup\{\mathcal{U}_n : n < \omega\}$. Recursively on $n < \omega$, it is not hard to choose $S_n \subset {}^{n+1}\kappa$ and a bijection $\phi_n : S_n \to \mathcal{U}_n$ in such a way that

- for every $g \in S_n$ we have $g{\restriction}_{m+1} \in S_m$ for all $m \leq n < \omega$ and
- if $g \in S_n$ then $\phi_n(g) \subset \phi_m(g{\restriction}_{m+1})$ for all $m \leq n < \omega$.

Define $S = \bigcup\{S_n : n < \omega\} \subset {}^{<\omega}\kappa$ and $\phi = \bigcup\{\phi_n : n < \omega\}$. Then $\phi : S \to \mathcal{U}$ is a bijection such that $\phi(h) \subset \phi(g)$ if and only if $g \subset h$ whenever $g, h \in S$. We will use ϕ to define f and Y. Let

$$Y = \big\{x \in {}^{\omega}\kappa : \forall n < \omega\ (x{\restriction}_{n+1} \in S_n) \text{ and } \bigcap\{\phi(x{\restriction}_{n+1}) : n < \omega\} \neq \emptyset\big\},$$

and define $f(x)$ to be the only point in $\bigcap\{\phi(x{\restriction}_{n+1}) : n < \omega\}$ for each $x \in Y$, this is well-defined as the diameter of $\phi(x{\restriction}_{n+1})$ is $\leq \frac{1}{n+1}$ for all $n < \omega$.

Notice that Y is strongly 0-dimensional with $w(Y) \leq \kappa$ by Lemma 8.5 and Corollary 8.3. Moreover, since every dense set of Y is mapped to a dense set of X, $\kappa = w(X) = d(X) \leq d(Y) = w(Y)$ so $w(Y) = \kappa$. We now prove that f is perfect, continuous and irreducible. We will use the following claim.

Claim 2. If $n < \omega$ and $g \in S_n$, then $f[B_g] = \phi(B_g)$.

We now prove Claim 2. The definition of f easily shows that $f[B_g] \subset \phi(B_g)$. Now let $x \in \phi(B_g)$. By property (*e*) it is possible to choose $A_m \in \mathcal{U}_m$ such that $x \in A_m$ and $A_{m+1} \subset A_m$ for every $m < \omega$ and $A_n = \phi(B_g)$. Then $\bigcup\{\phi^{-1}(A_m) : m < \omega\} = y \in Y$ is such that $y \in B_g$ and $f(y) = x$. This completes the proof of Claim 2.

To see that f is continuous, let $y \in Y$ and let U be an open set of X such that $f(x) \in U$. Let $m < \omega$ be such that $\{x \in X : d'(x, y) < \frac{1}{m+1}\} \subset U$. Thus, $A \subset U$ for every $A \in \mathcal{U}_m$ with $f(x) \in A$. Let $g = x{\restriction}_{m+2} \in {}^{m+2}\kappa$, then $V = B_g \cap Y = \{z \in Y : g \subset z\}$ is an open set (see Lemma 8.5) of Y such that $y \in V$ and $f[V] \subset U$. This shows that f is continuous.

Next, let us prove that f is perfect. We start by showing that if $p \in X$, then $f^{\leftarrow}(p)$ is compact. For each $n < \omega$, from the fact that $\mathcal{U}_n$ is locally finite, the set

$$F_n = \{\alpha < \kappa : \exists g \in S_n \ (p \in \phi_n(g),\ g(n) = \alpha)\}$$

is finite. Then $f^{\leftarrow}(p)$ is a closed subset of the compact set $\prod\{F_n : n < \omega\} \subset {}^{\omega}\kappa$ so it is compact as well.

Let $F \subset Y$ be closed, we now prove that $f[F]$ is closed. Let $x \in X \setminus f[F]$, since $f^{\leftarrow}(x)$ is compact, there is $m < \omega$ such that $\{d(y, z) : x = f(y), z \in F\} \leq \frac{1}{m+1}$. Let $T = \{g \in {}^{m+1}\kappa : B_g \cap F \neq \emptyset\}$ and $V = \bigcup\{B_g : g \in T\}$. Then $F \subset V \cap Y$ and $f^{\leftarrow}(x) \cap V \cap Y = \emptyset$. Thus, $f[F] \subset f[V \cap Y]$, $x \notin f[V \cap Y]$. By Claim 2, $f[V \cap Y] = \bigcup\{\phi(g) : g \in T \cap S_m\}$, this set is closed by Lemma 0.20. This proves that f is closed.

Notice that Claim 2 implies that f is onto. It is only left to prove that f is irreducible. Let $F \subset Y$ be closed and $F \neq Y$. As in the last paragraph, it is not hard to find $m < \omega$ and $T \subset {}^{m+1}\kappa$ such that $T \neq S_m$ and if $V = \bigcup\{B_g : g \in T\}$, then $F \subset V \cap Y$ and $f[V \cap Y] = \bigcup\{\phi(g) : g \in T \cap S_m\}$ by Claim 2. Let $h \in S_m \setminus T$. Then by condition (c), if $g \in T \cap S_m$, then $\phi(g) \cap \mathrm{int}_X(\phi(h)) = \emptyset$. Thus, $\mathrm{int}_X(\phi(h)) \setminus \bigcup\{\phi(g) : g \in T \cap S_m\}$ is non-empty. Thus, $f[V \cap Y]$ is a proper subset of X. This proves that f is irreducible. □

Thus, by Proposition 9.5 and Corollary 6.42 we obtain the following.

Corollary 9.6 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Each metrizable space is coabsolute with a strongly 0-dimensional metrizable space.

Thus, to study the remote points of a metrizable space X it is enough to assume that X is strongly 0-dimensional by Proposition 7.12 and Corollary 9.6.

The next step is to see that local compactness is distinguished by remote points. For any space X, let LX be the points where X is locally compact and $NX = X \setminus \mathrm{cl}_X(LX)$. Gates [66] has already noticed the following Corollary of Proposition 7.15.

Lemma 9.7 Let X be a normal space. Then $\varrho(X)$ is homeomorphic to the direct sum $\varrho(\mathrm{cl}_X(LX)) \oplus \varrho(\mathrm{cl}_X(NX))$.

Now we must find a topological way to distinguish between $\varrho(\mathrm{cl}_X(LX))$ and $\varrho(\mathrm{cl}_X(NX))$.

Theorem 9.8 Let X be a metrizable space. Then $\varrho(\mathrm{cl}_X(LX))$, if non-empty, contains a dense set of $\mathfrak{c}$-points and $\varrho(NX)$ is extremally disconnected.

Proof. Let us start with NX. By Proposition 7.6, $\beta(NX)$ is extremally disconnected at each point of $\varrho(NX)$. Notice that $(NX)^*$ is dense in $\beta(NX)$ so by Corollary 7.4, $\varrho(NX)$ is dense in $\beta(NX)$. It follows from Lemma 6.32 that $\varrho(NX)$ is extremally disconnected.

Let $Y = \mathrm{cl}_X(LX)$. Assume that Y is not compact so that $\varrho(Y) \neq \emptyset$. Let $\mathrm{Ex}_X(U)$ be a basic open subset of βY that intersects Y^*. By (b) in Proposition 6.10, $\mathrm{cl}_Y(U)$ is not compact so by Lemma 0.19 we may find a discrete collection $\{U_n : n < \omega\}$ of open sets of U. Since U is locally compact, we may assume that $\mathrm{cl}_Y(U_n)$ is compact for each $n < \omega$.

Let $A = \bigcup\{\mathrm{cl}_Y(U_n) : n < \omega\}$. Then A is a regular closed subset of Y (Lemma 0.20) and since Y is a normal space A and $X \setminus U$ are completely separated so $\mathrm{cl}_{\beta Y}(A) \subset \mathrm{Ex}_X(U)$ by (3) in Theorem 6.4. By Proposition 7.15 we have that $\varrho(A)$ is a clopen subspace of $\varrho(Y) \cap \mathrm{Ex}_X(U)$. Notice that A is realcompact by Proposition 6.9, as each $\mathrm{cl}_X(U_n)$ has countable weight by Theorem 0.16. By Proposition 8.20 each point of $\varrho(A)$ is a $\mathfrak{c}$-point of $\mathrm{cl}_Y(A) \setminus A$. By Corollary 7.4 it follows that each point of $\varrho(A)$ is a $\mathfrak{c}$-point of $\varrho(A)$. Thus, any point of $\varrho(A)$ is a $\mathfrak{c}$-point of $\varrho(Y)$ contained in $\mathrm{Ex}_X(U)$. $\square$

The following follows immediately from Lemma 7.14, Lemma 9.7 and Theorem 9.8.

Corollary 9.9 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X and Y be metrizable spaces. If $h : \varrho(X) \to \varrho(Y)$ is a homeomorphism then $h[\varrho(\mathrm{cl}_X(LX))] = \varrho(\mathrm{cl}_X(LY))$ and $h[\varrho(NX)] = \varrho(NY)$.

Results for LX were mentioned in Theorems 7.20 and 7.21. We will now direct our efforts towards nowhere locally compact spaces (that is, spaces where $X = NX$). If X is nowhere locally compact (and metrizable) and $\varrho(X)$ is homeomorphic to $\varrho(Y)$ then it follows from Corollary 9.9 that $\mathrm{cl}_X(LX)$ is compact. Since $\varrho(Y) = \varrho(NY)$ in this case, we may restrict to the case when both X and Y are nowhere locally compact.

9.2 Completely Metrizable Spaces

We would like to give a classification of the set of remote points of nowhere locally compact, completely metrizable spaces in the spirit of Theorem 7.20. Recall that a metrizable space is completely metrizable if and only if it is Čech-complete ([50, Theorem 4.3.26]).

We shall start by characterizing spaces coabsolute to the Baire space ${}^{\omega}\kappa$, where κ is an infinite cardinal, see Proposition 9.12.

Lemma 9.10 Let $f : X \to Y$ be a perfect and irreducible continuous function between Tychonoff spaces. Then

(a) X is a Čech-complete space if and only if Y is,

(b) X is nowhere locally compact if and only if Y is and

(c) X is σ-compact if and only if Y is.

Proof. Let $\beta f : \beta X \to \beta Y$ be the unique continuous extension of f (Theorem 6.4). By Lemma 0.11, $\beta f[X^*] = Y^*$.

Using βf it is easy to see that X^* is σ-compact if and only if Y^* is also σ-compact. This implies that X is Čech-complete if and only if Y is Čech complete.

If Y has some compact set K with non-empty interior, then $f^{\leftarrow}[K]$ is also a compact set because $f^{\leftarrow}[K] = \beta f^{\leftarrow}[K]$ is closed in βX and $f^{\leftarrow}[K]$ has non-empty interior in X. If $T \subset X$ is compact with non-empty interior, then $f[T]$ is a compact subset of Y and $f^{\sharp}[\mathrm{int}_X(T)]$ is a non-empty open subset of $f[T]$ since f is irreducible. These two observations prove (b).

A continuous image of a σ-compact space is clearly also σ-compact. As mentioned in the paragraph above, the preimage of a compact set under f is compact. Thus, if Y is σ-compact then X is σ-compact. Thus, (c) holds. $\square$

Recall that for metrizable spaces cellularity, weight and density all coincide (Theorem 0.17); we shall use this in what follows. We will say that X is *of uniform cellularity* (κ) if $c(X) = c(U)(= \kappa)$ for each non-empty open subset $U \subset X$. We are interested in spaces of uniform cellularity because ${}^{\omega}\kappa$ is such a space and this property is preserved as the following result shows.

Lemma 9.11

(i) Let $f : Y \to X$ be an irreducible continuous function. Then $c(X) = c(Y)$ and X is of uniform cellularity if and only if Y is.

(*ii*) Let X be a space and $D \subset X$ a dense subset. Then $c(X) = c(D)$ and X is of uniform cellularity if and only if D is.

Proof. We start with (*i*). That $c(X) = c(Y)$ is easy to prove. Assume that X is of uniform cellularity and let $U \subset Y$ be a non-empty open subset. Let $V = f^\sharp[U]$. Then it is easy to see that $f\restriction_{f^\leftarrow[V]}: f^\leftarrow[V] \to V$ is an irreducible and continuous function. Thus, $c(V) = c(f^\leftarrow[V])$. Since $f^\leftarrow[V] \subset U$ we obtain that $c(X) = c(V) = c(f^\leftarrow[V]) \leq c(U) \leq c(X)$ so $c(U) = c(X)$. The rest of the argument for (*i*) is similar and the proof of (*ii*) is not hard. □

Proposition 9.12 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a metrizable space and κ an infinite cardinal. Then X is coabsolute with ${}^\omega\kappa$ if and only if X is a nowhere locally compact, completely metrizable space of uniform cellularity κ.

Proof. If X is coabsolute with ${}^\omega\kappa$ then the result follows from Lemmas 9.10 and 9.11. Now assume that X has the properties given in the Proposition. By Corollary 9.6 we may assume that X is coabsolute with a strongly 0-dimensional metrizable space. The result now follows either from Theorem 8.7 or Theorem 8.10. □

Notice that in Proposition 9.12, nowhere locally compact can be omitted when $\kappa > \omega$ and of uniform cellularity ω simply means being separable.

Theorem 9.13 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a nowhere locally compact, completely metrizable space and $\kappa > \omega$. Then $\varrho(X)$ is homeomorphic to $\varrho({}^\omega\kappa)$ if and only if X is of uniform cellularity κ.

Proof. If X is of uniform cellularity κ, use Proposition 9.12 and Corollary 7.13. Now, assume that $\varrho(X)$ is homeomorphic to $\varrho({}^\omega\kappa)$. Both X and $\varrho(X)$ are dense in βX. Also, $\varrho({}^\omega\kappa)$ and ${}^\omega\kappa$ are dense in $\beta({}^\omega\kappa)$. By Lemma 9.11 we obtain that X is of uniform cellularity κ. □

Wondering if the hypothesis that X is complete is necessary in Theorem 9.13 we observe the following.

Example 9.14 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) For each completely metrizable, realcompact and non-compact space X there exists a Tychonoff space Y that is neither Čech complete nor an M-space such that $\varrho(X)$ is homeomorphic to $\varrho(Y)$.

Let $D = \{x_n : n < \omega\}$ be a countable closed discrete and infinite subset of X (Lemma 0.19). Let $p \in \mathrm{cl}_{\beta X}(D) \setminus X$ and let $Y = X \cup \{p\}$ as a subspace of βX. Notice that $\beta X = \beta Y$. We now show that Y is not Čech-complete and it is not an M-space.

Assume that Y is Čech-complete and let $\{U_n : n < \omega\}$ be a family of open subsets of βX whose intersection is Y. Thus, $\{U_n \cap X^* : n < \omega\}$ witnesses that $\{p\}$ is a G_δ set of X^*. But X is realcompact so there exists a subset of type G_δ of βX that contains p and misses X. It easily follows that $\{p\}$ is a G_δ set of βX. But this contradicts Corollary 6.50.

Now assume that Y is an M-space, we will reach a contradiction. Let $\{\mathcal{C}_n : n < \omega\}$ be a sequence of covers as in the definition of an M-space. Let d be a metric for X. As in the proof of Proposition 8.16, we consider a strictly increasing $\phi : \omega \to \omega$ as follows: let $\phi(n)$ be such that $x_{\phi(n)} \in St(p, \mathcal{C}_n)$. For each $n < \omega$, let $y_n \in St(p, \mathcal{C}_n) \setminus D$ be such that $d(x_{\phi(n)}, y_n) < \frac{1}{n+1}$. Then it can be proved that $\{y_n : n < \omega\}$ is a closed discrete subset of X. Since X is normal, $\{y_n : n < \omega\}$ is also closed in Y. But this contradicts (i) in the definition of an M-space.

Finally, $p \notin \varrho(X)$ because D is nowhere dense. Thus, $\varrho(X) = \varrho(Y)$. $\square$

However, since the space in Example 9.14 is not an M-space, we can ask the following more specific question.

Question 9.15 Let X and Y be metrizable spaces such that $\varrho(X)$ is homeomorphic to $\varrho(Y)$. If X is completely metrizable, must Y be also completely metrizable?

We have obtained a characterization of spaces with remote points homeomorphic to the remote points of some specific completely metrizable spaces in Theorem 9.13. In an effort to classify them all we make the following definition.

Definition 9.16 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a space, let $\mathcal{S}$ be a set of infinite cardinals and ϕ a function with domain $\mathcal{S}$ such that for each $\kappa \in \mathcal{S}$, $\phi(\kappa)$ is a cardinal $\geq \kappa$. We will say that X has *cellular type* $\langle \mathcal{S}, \phi \rangle$ if there exists a pairwise disjoint family of open subsets $\mathcal{B} = \{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ of X whose union is dense in X and such that if $\kappa \in \mathcal{S}$ and $\alpha < \phi(\kappa)$ then $V(\kappa, \alpha)$ is of uniform cellularity κ. In this case we say that $\mathcal{B}$ is a witness to the cellular type of X.

Lemma 9.17 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Every crowded metrizable space has a cellular type.

Proof. Let X be a crowded metrizable space. By the Kuratowski-Zorn Lemma, find a maximal pairwise disjoint family $\mathcal{U}$ of open, non-empty subsets of X of uniform cellularity. Notice that $\bigcup\mathcal{U}$ is dense, otherwise let V be an open subset of $X \setminus \mathrm{cl}_X(\bigcup\mathcal{U})$ of minimal cellularity; $\mathcal{U} \cup \{V\}$ contradicts the maximality of $\mathcal{U}$. We define

$$\mathcal{S} = \{\kappa : \kappa \text{ is a cardinal and there is } U \in \mathcal{U} \text{ such that } c(U) = \kappa\}.$$

Clearly $\mathcal{S}$ is a set of infinite cardinals because X is crowded. For each $\kappa \in \mathcal{S}$ let

$$\mathcal{U}(\kappa) = \{U \in \mathcal{U} : c(U) = \kappa\}.$$

We may assume that $|\mathcal{U}(\kappa)| \geq \kappa$ by the following argument. If the size of this family is smaller than κ, using the fact that the cellularity is attained in metrizable spaces (Theorem 0.18), replace $\mathcal{U}(\kappa)$ by a family of κ pairwise disjoint open subsets of $\bigcup\mathcal{U}(\kappa)$ that is dense in $\bigcup\mathcal{U}(\kappa)$. We finally define ϕ: for each $\kappa \in \mathcal{S}$ we let $\phi(\kappa) = |\mathcal{U}(\kappa)|$. Clearly $\mathcal{U}$ witnesses that X has cellular type $\langle\mathcal{S}, \phi\rangle$. $\square$

Lemma 9.18 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) If a Tychonoff space X has cellular types $\langle\mathcal{S}, \phi\rangle$ and $\langle\mathcal{T}, \psi\rangle$, then $\langle\mathcal{S}, \phi\rangle = \langle\mathcal{T}, \psi\rangle$.

Proof. Let $\{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ witness type $\langle\mathcal{S}, \phi\rangle$ and $\{W(\kappa, \alpha) : \kappa \in \mathcal{T}, \alpha < \psi(\kappa)\}$ witness type $\langle\mathcal{T}, \psi\rangle$. For each $\kappa \in \mathcal{S}$, $V(\kappa, 0)$ is a non-empty open subset of X so it must intersect some $W(\tau, \alpha)$, with $\tau \in \mathcal{T}$ and $\alpha < \psi(\tau)$. By the definition of uniform cellularity it follows that $\kappa = \tau$ so $\mathcal{S} \subset \mathcal{T}$. By a similar argument $\mathcal{S} = \mathcal{T}$. Notice that $V(\kappa, \alpha) \cap W(\tau, \beta) \neq \emptyset$ implies $\kappa = \tau$. Assume $\phi(\kappa) < \psi(\kappa)$ for some $\kappa \in \mathcal{S}$. For each $\alpha < \phi(\kappa)$ let $J(\alpha) = \{\beta < \psi(\tau) : V(\kappa, \alpha) \cap W(\kappa, \beta) \neq \emptyset\}$. Then $|J(\alpha)| \leq \kappa$ because $c(V(\kappa, \alpha)) = \kappa$. Let $\gamma \in \psi(\tau) \setminus \bigcup\{J(\alpha) : \alpha < \phi(\kappa)\}$, then it follows that $W(\kappa, \gamma)$ does not intersect any element of $\{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$, which contradicts the density of $\bigcup\{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$. This completes the proof. $\square$

So at least metrizable spaces have a cellular type and cellular type is unique. Sometimes cellular type can be transfered from one space to another. For example we have the following easy transfer result. Its proof is easy and will be left to the reader.

Lemma 9.19 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be any space and D a dense subset of X. Then X has cellular type $\langle\mathcal{S}, \phi\rangle$ if and only if D has cellular type $\langle\mathcal{S}, \phi\rangle$.

In the case of nowhere locally compact, completely metrizable spaces, cellular type can be transfered to the remote points. For this nice witnesses are needed.

Lemma 9.20 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a regular space with cellular type $\langle \mathcal{S}, \phi \rangle$. Then there exists a witness family $\mathcal{B}$ of the cellular type of X such that any two different members of $\mathcal{B}$ have disjoint closures.

Proof. Let $\{W(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ witness the cellular type of X. For each $\kappa \in \mathcal{S}$ and $\alpha < \phi(\kappa)$ let $\mathcal{B}(\kappa, \alpha)$ be a maximal family of open subsets of $W(\kappa, \alpha)$ whose closures are pairwise disjoint and contained in $W(\kappa, \alpha)$. Clearly $|\mathcal{B}(\kappa, \alpha)| \leq \kappa$. Give an enumeration $\{V(\kappa, \alpha) : \alpha < \phi(\kappa)\}$ of $\bigcup\{\mathcal{B}(\kappa, \alpha) : \alpha < \phi(\kappa)\}$. Clearly $\{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ is the witness we were looking for. $\square$

Theorem 9.21 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a nowhere locally compact and completely metrizable space of cellular type $\langle \mathcal{S}, \phi \rangle$. Then there exists a family $\{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ consisting of clopen subsets of $\varrho(X)$ that witnesses that $\varrho(X)$ has cellular type $\langle \mathcal{S}, \phi \rangle$ and has the additional property that for each $\kappa \in \mathcal{S}$ and $\alpha < \phi(\kappa)$, $V(\kappa, \alpha)$ is homeomorphic to $\varrho({}^{\omega}\kappa)$.

Proof. Let $\{W(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ witness the cellular type of X. By Lemma 9.20, we may assume that every two subsets of this family have disjoint closures. Let $D(\kappa, \alpha) = \mathrm{cl}_X(W(\kappa, \alpha))$ for each $\kappa \in \mathcal{S}$ and $\alpha < \phi(\kappa)$, also define

$$V(\kappa, \alpha) = \varrho(X) \cap \mathrm{cl}_{\beta X}(D(\kappa, \alpha)).$$

It easily follows from Proposition 7.15 that $V(\kappa, \alpha) = \varrho(D(\kappa, \alpha))$ for each $\kappa \in \mathcal{S}$, $\alpha < \phi(\kappa)$ and $\mathcal{B} = \{V(\kappa, \alpha) : \kappa \in \mathcal{S}, \alpha < \phi(\kappa)\}$ is a pairwise disjoint family of clopen subsets of $\varrho(X)$. Since $\varrho(X)$ is dense in βX (Corollary 7.4), $V(\kappa, \alpha) \neq \emptyset$ for each $\kappa \in \mathcal{S}$, $\alpha < \phi(\kappa)$ and $\bigcup \mathcal{B}$ is dense.

Finally, fix $\kappa \in \mathcal{S}$ and $\alpha < \phi(\kappa)$, we now prove that $V(\kappa, \alpha)$ is homeomorphic to $\varrho({}^{\omega}\kappa)$, which will complete the proof. Notice that $D(\kappa, \alpha)$ is nowhere locally compact, completely metrizable and of uniform cellularity κ. The result now follows from Theorem 9.13. $\square$

Corollary 9.22 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) If X and Y are nowhere locally compact, completely metrizable spaces with the same cellular type then $\varrho(X)$ and $\varrho(Y)$ have open dense homeomorphic subspaces.

The following is true but maybe not worth proving in detail. We leave it as an exercise to the reader.

Proposition 9.23 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80])

- Let X and Y be coabsolute Tychonoff spaces. Then X has cellular type $\langle \mathcal{S}, \phi \rangle$ if and only if Y has cellular type $\langle \mathcal{S}, \phi \rangle$.

- Every paracompact p-space has a cellular type.

So the remaining question is if cellular type completely characterizes remote points of nowhere locally compact and completely metrizable spaces. We end this section showing that this is not the case by means of an example.

Example 9.24 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) There exist two nowhere locally compact and completely metrizable spaces X and Y that have the same cellular type but such that $\varrho(X)$ is not homeomorphic to $\varrho(Y)$.

For each $n < \omega$, let X_n be homeomorphic to ${}^{\omega}\omega_n$ such that $\{X_n : n < \omega\}$ are pairwise disjoint. Let us define $X = \bigoplus\{X_n : n < \omega\}$, clearly this is a nowhere locally compact and completely metrizable space. For each $n < \omega$ let $K_n = \mathrm{cl}_{\beta X}(X_n)$ and let $P = \beta X \setminus \bigcup\{K_n : n < \omega\}$.

We now define T as the quotient space of βX obtained by identifying P to a point and let $\rho : \beta X \to T$ be this identification. Let $p \in T$ be such that $\{p\} = \rho[P]$ and define $Y = \{p\} \cup (\bigcup\{X_n : n < \omega\})$ as a subspace of T.

Notice X and Y have the same cellular type, simply because X is dense in Y (Lemma 9.19).

To see that Y is metrizable, we may use the Bing-Nagata-Smirnov Theorem (Theorem 0.22), since X is already metrizable it is enough to notice that Y is first-countable at p. Since Y is a G_δ in T, Y is completely metrizable.

We claim that Y is C^*-embedded in T. Let $f : Y \to [0,1]$ be a continuous function. Let $g = f{\restriction}_X$, we now prove that $\beta g : \beta X \to [0,1]$ is constant restricted to P. Assume this is not the case and let $x, y \in P$ be such that $\beta g(x) < \beta g(y)$. Let $\epsilon = \frac{1}{3}(\beta g(y) - \beta g(x))$ and define $U = \beta g^{\leftarrow}[(-\infty, \beta g(x) + \epsilon)]$ and $V = \beta g^{\leftarrow}[(\beta g(y) - \epsilon, \infty)]$. Then there exist two closed countable discrete subsets D_0, D_1 of X such that $D_0 \subset U$ and $D_1 \subset V$. Clearly, both D_0 and D_1 converge to p in Y, this contradicts the definition of U and V. Thus the function $F : T \to [0,1]$ given by $F(x) = \beta g(x)$ if $x \neq p$ and $\{F(p)\} = \beta g[P]$ is a continuous extension of f. Thus, $T = \beta Y$.

Since P is a non-empty G_δ subset of βX, by Theorem 7.3 we have that $\varrho(X) \cap P \neq \emptyset$. Also notice that

$$\begin{aligned} \varrho(X) = & \quad \varrho(Y) \cup (\varrho(X) \cap P), \quad \text{and} \\ \varrho(Y) = & \quad \bigcup\{\varrho(X_n) : n < \omega\}. \end{aligned}$$

To prove that $\varrho(X)$ is not homeomorphic to $\varrho(Y)$ it is enough to notice the following two facts which show different topological properties of points in $\varrho(X) \cap P$ to those in $\varrho(Y)$.

(a) if $n < \omega$, $c(\varrho(X_n)) = \omega_n$,

(b) if $q \in \varrho(X) \cap P$ then for every open set $U \subset \varrho(X)$ such that $q \in U$, $c(U) \geq \omega_\omega$.

Statement (a) is clear. For statement (b) let $q \in \varrho(X) \cap P$ and U be an open subset of $\varrho(X)$ with $q \in U$. Let $A \subset \omega$ be an infinite set such that $U \cap K_n \neq \emptyset$ for all $n \in A$. For each $n \in A$ we may choose a pairwise disjoint family of open sets $\{V(\alpha, n) : n < \omega_n\}$ of $\varrho(X_n) \cap U$. Then $\{V(\alpha, n) : n < \omega, \alpha < \omega_n\}$ is a collection of ω_ω pairwise open sets contained in U. Thus, $c(U) \geq \omega_\omega$. $\square$

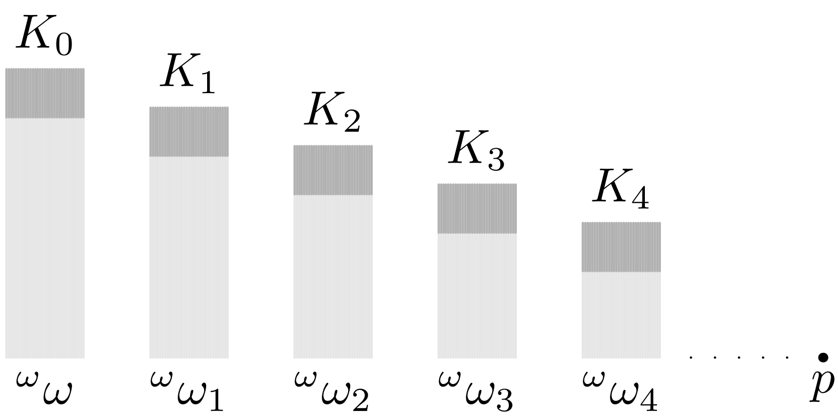

Figure 9.1: Space βY from Example 9.24. Space X is in lighter grey.

9.3 Meager vs Comeager

In this section we consider separable metrizable spaces. We already know some spaces X with $\varrho(X)$ homeomorphic to $\varrho({}^\omega\omega)$ (Theorem 9.13). We start by considering the problem for $\mathbb{Q}$.

Lemma 9.25 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Any two remainders of a nowhere locally compact Tychonoff space are coabsolute.

Proof. Let X be nowhere locally compact Tychonoff space. Consider any compactification T of X with remainder Y and let $f : \beta X \to T$ be the continuous extension of the identity function (Theorem 6.4). By Proposition 0.11, $X^* = f^{\leftarrow}[Y]$. Then $g = f\restriction_X\colon X^* \to Y$ is a perfect and irreducible continuous function. So Y is coabsolute with X^*. $\square$

Theorem 9.26 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a metrizable space. Then X is coabsolute with $\mathbb{Q}$ if and only if X is nowhere locally compact and σ-compact.

Proof. If X is coabsolute with $\mathbb{Q}$, then X is nowhere locally compact and σ-compact by Lemma 9.10. Assume now that X is nowhere locally compact and σ-compact. By Corollary 9.6 we may assume that X is strongly 0-dimensional. Since X is σ-compact, it is separable (Theorems 0.16 and 0.17) so it can be embedded in ${}^{\omega}2$ (Corollary 8.9). Moreover X is crowded so the closure of X in ${}^{\omega}2$ is also crowded. So we may assume that X is dense in ${}^{\omega}2$, let $Y = {}^{\omega}2 \setminus X$. But then Y is a separable, nowhere locally compact and completely metrizable 0-dimensional space. Then Y is homeomorphic to ${}^{\omega}\omega$ by Theorem 8.7. Notice that ${}^{\omega}2$ is a compactification of Y. Also, if $Q = \{x \in {}^{\omega}2 : \exists n < \omega\ \forall m \geq n\ (x(n) = x(m))\}$, then ${}^{\omega}2 \setminus Q$ is a dense set of ${}^{\omega}2$ homeomorphic to ${}^{\omega}\omega$ (Theorem 8.7). Thus, ${}^{\omega}2$ is a compactification of ${}^{\omega}\omega$ with remainder Q, that is homeomorphic to $\mathbb{Q}$ (see Theorem 10.3 in Part III). By Lemma 9.25 we obtain X is coabsolute with $\mathbb{Q}$. $\square$

Corollary 9.27 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) If X is a σ-compact, nowhere locally compact, metrizable space then $\varrho(X)$ is homeomorphic to $\varrho(\mathbb{Q})$.

Another proof of Theorem 9.26 can be given by considering the following result of van Mill and Woods.

Theorem 9.28 [119, Theorem 3.1] Let X be a σ-compact, nowhere locally compact metrizable space. Then there exists a perfect and irreducible continuous function $f : \mathbb{Q}\times{}^{\omega}2 \to X$ such that $f^{\leftarrow}(p)$ is homeomorphic to ${}^{\omega}2$ for every $p \in X$.

Notice we have the following situation: For two specific spaces ${}^{\omega}\omega$ and $\mathbb{Q}$ we have found non-trivial classes of metrizable spaces that have the same set of remote points as these spaces. Now we want to know if these classes of spaces are the best possible. We were not able to solve this problem but we will prove Proposition 9.31 that is in the spirit of van Douwen's Theorem 9.2. We will use the Choquet game for this purpose. Recall that every separable completely metrizable space is a Choquet space by Theorem 8.13. Also notice that a σ-compact nowhere locally compact metrizable space is meager. So in some sense ${}^{\omega}\omega$ and $\mathbb{Q}$ are dual.

Lemma 9.29 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let $f : X \to Y$ be an irreducible continuous function between crowded regular spaces. Then

(*a*) X is meager if and only if Y is meager, and

(*b*) X is a Choquet space if and only if Y is a Choquet space.

Proof. Let us start with (*a*). A meager space can be written as the union of ω closed nowhere dense subsets. Since f is closed irreducible, the image of a closed nowhere dense subset of X is also closed nowhere dense by Lemma 6.45. The other implication is easier.

Now we prove (*b*). First assume that X is a Choquet space. We will now use player II's strategy on X to produce one on Y. Every time player I draws an open set $U_n \subset Y$, let $W_n = f^{\leftarrow}[U_n]$. Using the strategy of player II on X, we obtain an open subset $V_n \subset W_n$ of X. Since f is irreducible, player II draws the non-empty open subset $f^{\sharp}[V_n]$ of Y. Since II wins in X, there exists $p \in \bigcap\{V_n : n < \omega\} = \bigcap\{W_n : n < \omega\}$ so $f(p) \in \bigcap\{U_n : n < \omega\} = \bigcap\{f^{\sharp}[V_n] : n < \omega\}$. Thus, II also wins in Y.

Now assume that Y is a Choquet space, again we transfer II's strategy to X. If I draws an open set $U_n \subset X$, consider $W_n = f^{\sharp}[U_n]$ which is open and non-empty in Y. Using II's strategy, we obtain an open subset $V_n \subset W_n$. Then player II draws $f^{\leftarrow}[V_n] \subset U_n$ on X. We know that II wins on Y so there exists $p \in \bigcap\{V_n : n < \omega\}$, clearly $f^{\leftarrow}(p) \subset \bigcap\{f^{\leftarrow}[V_n] : n < \omega\}$ so II wins on X as well. This completes the proof of (*b*). $\square$

Lemma 9.30 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X be a space and $Y \subset X$ be G_δ-dense in X. Then

(*a*) X is meager if and only if Y is meager, and

(*b*) X is a Choquet space if and only if Y is a Choquet space.

Proof. Start with (*a*). If X is meager, then Y is also meager because it is dense in X. Assume that Y is meager, so $Y = \bigcup\{Y_n : n < \omega\}$ where Y_n is nowhere dense for each $n < \omega$. Let $X_n = \mathrm{cl}_X(Y_n)$, this is a closed and nowhere dense subset of X for each $n < \omega$. Since $X \setminus \bigcup\{X_n : n < \omega\}$ is a subset of type G_δ of X that does not intersect Y, it must be empty. Thus, X is meager.

Now we prove (b). First assume that X is a Choquet space. As in the proof of Lemma 9.29, we transfer strategies. If player I draws an open subset U_n of Y, let U_n' be an open subset of X such that $U_n' \cap Y = U_n$. Player II's strategy gives an open subset $V_n \subset U_n'$ of X so player II draws $V_n \cap Y \neq \emptyset$. By player II's strategy in X, we know that $G = \bigcap\{V_n : n < \omega\}$ is a non-empty subset of X of type G_δ. Thus, $G \cap Y \neq \emptyset$, which implies that the described strategy for II is a winning strategy in Y.

Now assume that Y is a Choquet space. We again transfer II's strategy in Y to X. Every time I draws an open subset U_n of X, consider the open subset $W_n = U_n \cap Y$ of Y. The strategy in Y for player II gives an open subset $V_n \subset W_n$. Choose an open subset V_n' of X such that $V_n' \cap Y = V_n$. So II draws V_n'. Since $\bigcap\{V_n : n < \omega\}$ is non-empty, we obtain that $\bigcap\{V_n' : n < \omega\}$ is non-empty as well. This shows that II has a winning strategy so X is a Choquet space. We have finished the proof. □

Proposition 9.31 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) Let X and Y be two separable, nowhere locally compact metrizable spaces such that $\varrho(X)$ is homeomorphic to $\varrho(Y)$. Then

(i) X is a Choquet space if and only if Y is a Choquet space, and

(ii) X is meager if and only if Y is meager.

Proof. Let K be a metrizable compactification of X and T be a metrizable compactification of Y, and assume that $X \subset K$ and $Y \subset T$. Let $f : \beta X \to K$ and $g : \beta Y \to T$ continuously extend the identity function. Then $f{\restriction}_{X^*} \colon X^* \to K \setminus X$ and $g{\restriction}_{Y^*} \colon Y^* \to T \setminus Y$ are easily seen to be irreducible continuous functions. Notice that if we give T and K some metric, these spaces are the completion of X and Y, respectively, with respect to appropriate restrictions of these metrics. Thus, we can use Theorem 8.13.

Let us prove (i). Assume that X is a Choquet space. Then, by Theorem 8.13 X is comeager in K. Thus, $K \setminus X$ is meager. By Lemma 9.29 applied to $f{\restriction}_{X^*}$, X^* is meager. Recall $\varrho(X)$ is G_δ-dense in X^* (Theorem 7.3) so by Lemma 9.30, $\varrho(X)$ is meager. Thus, $\varrho(Y)$ is meager. Therefore, we can again use Lemmas 9.29 and 9.30 to obtain that Y^* and $T \setminus Y$ are meager as well. Again, Theorem 8.13 proves that Y is a Choquet space. Using a similar argument it is easy to prove (ii). □

One good hope to extend the results given above is to consider the strong Choquet game because the fact that player II has a winning strategy provides a way to prove that a space is completely metrizable (Theorem 8.13). However, it was not possible for the author to prove a transfer result analogous to Proposition 9.31. We could only prove the following observation, that is only half the way to the proof.

Proposition 9.32 (Hernández-Gutiérrez, Hrušák and Tamariz-Mascarúa, [80]) $\varrho(\mathbb{Q})$ is a strong Choquet space.

Proof. Fix some enumeration $\mathbb{Q} = \{q_n : n < \omega\}$. If player I draws the open set U_n of $\varrho(\mathbb{Q})$ and $x_n \in U_n$, then player II draws $V_n \cap \varrho(\mathbb{Q})$, where V_n is an open subset of $\beta\mathbb{Q}$ such that $x_n \in V_n$, $\mathrm{cl}_{\beta\mathbb{Q}}(V_n) \cap \varrho(\mathbb{Q}) \subset U_n$ and $q_n \notin V_n$. Then $G = \bigcap\{V_n : n < \omega\} = \bigcap\{\mathrm{cl}_{\beta\mathbb{Q}}(V_n) : n < \omega\} \neq \emptyset$ is a non-empty G_δ set of $\beta\mathbb{Q}$ contained in $\mathbb{Q}^*$. Recall that $\varrho(\mathbb{Q})$ intersects every G_δ subset of $\beta\mathbb{Q}$ contained in $\mathbb{Q}^*$ (Theorem 7.3). Thus, $G \cap \varrho(\mathbb{Q}) \neq \emptyset$. This implies that $\bigcap\{V_n \cap \varrho(\mathbb{Q}) : n < \omega\} \neq \emptyset$. Thus, the strategy described is a winning strategy for player II in the strong Choquet game for $\varrho(\mathbb{Q})$. $\square$

So the following remains unanswered.

Question 9.33 Let X be a metrizable space such that $\varrho(X)$ is homeomorphic to $\varrho(\mathbb{Q})$. Is X σ-compact?

Now we make some comments about the use of Proposition 9.31. Let X and Y be separable, nowhere locally compact and metrizable. If, for example, X has some open subset that is meager and Y is comeager then we can say that $\varrho(X)$ and $\varrho(Y)$ are not homeomorphic using Propositions 7.15 and 9.31. However we are not able to distinguish between, for example, $\varrho(\mathbb{Q})$ and $\varrho({}^\omega\omega \times \mathbb{Q})$ or between $\varrho({}^\omega\omega)$ and $\varrho({}^2P \cup {}^2\mathbb{Q})$ where $P = \mathbb{R} \setminus \mathbb{Q}$.

Part III

Countable Dense Homogeneous Spaces

Introduction

In [25], Cantor showed that any countable linearly ordered set with no least or greatest element that is densely ordered is order-isomorphic to the rational numbers. Cantor's method of proof is now known as back-and-forth induction (see [29, §4] for applications of back-and-forth induction to algebraic constructions). Although Cantor's result is order-theoretic, it can be shown to have topological consequences. In particular, for every two countable dense subsets $D, E \subset \mathbb{R}$ it is possible to find a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h[D] = E$ (Theorem 10.2).

Thus, it is natural to take the abstract property as a topological definition. A separable space X will be called *countable dense homogeneous* if every time $D, E \subset X$ are countable dense subsets of X, there is a homeomorphism $h : X \to X$ such that $h[D] = E$. Which spaces are countable dense homogeneous?

At least for connected spaces, countable dense homogeneity implies homogeneity (Corollary 10.10). However, homogeneity does not imply countable dense homogeneity (see examples in Section 10.1). It can be shown that many familiar spaces are countable dense homogeneous, such as manifolds, the Cantor set, separable Hilbert space and the space of irrational numbers (see Section 10.2). In fact, there are some characterizations of countable dense homogeneous spaces in the classes of locally compact spaces (see Theorem 10.25).

However, we know little outside the class of Polish spaces. In the Open Problems in Topology book, there was a paper [57] in which the authors asked whether there are separable metrizable spaces that are countable dense homogeneous but not complete. There are consistent examples of non-definable countable dense homogeneous spaces such as Bernstein sets (Theorem 10.30). Michael Hrušák and Beatriz Zamora-Áviles proved that such spaces cannot be Borel (Theorem 10.32). Later, Ilijas Farah, Michael Hrušák and Carlos Martínez-Ranero showed that there is a ZFC example of a non-Polish countable dense homogeneous metrizable space of cardinality ω_1 (Theorem 10.34).

One of the latest papers on the subject was due to Medini and Milovich ([110]). In this paper, ultrafilters with the Cantor ser topology are studied. In particular, some CDH and non-CDH ultrafilters are constructed under Martin's axiom for countable posets. The author and Professor Michael Hrušák were able to extend part of these results, in particular, we showed that non-meager P-filters are countable dense homogeneous (Theorem 11.12). Our results are in ZFC except for the fact that non-meager P-filters are known to exist only under special hypothesis (Theorem 11.3).

Another insteresting problem posed by Jan van Mill is to find compact CDH spaces of uncountable weight (Question 12.4). We were not able to solve this problem, but due to another result of the author (Theorem 12.25), we rule out many possible examples of such spaces (see Example 12.29).

Chapter 10 gives an introduction to countable dense homogeneity, Chapter 11 gives our results on filters and Chapter 12 gives the collection of results obtain related to the problem of compact CDH spaces. The research of this Part was done during the author's stay in the Centro de Ciencias Matemáticas, UNAM in Morelia city.

Chapter 10

Structure of Countable Dense Homogeneous Spaces

A space X is called *countable dense homogeneous* if it is Hausdorff, separable and every time D and E are countable dense subsets of X there is a homeomorphism $h : X \to X$ such that $h[D] = E$. We will abbreviate "countable dense homogeneous" as *CDH* from this point on. In this Chapter we give a summary of some general results about CDH spaces.

10.1 Examples and simple properties

Historically, in [25] Cantor gave the first result related to CDH spaces. The technique invented by Cantor is now called "back and forth induction".

Theorem 10.1 [25] Let $\langle Q, \leq \rangle$ be a countable linearly ordered set that has neither least nor greatest element and for every $a, b \in Q$ with $a < b$ there exists $c \in Q$ such that $a < c < b$. Then $\langle Q, \leq \rangle$ is order-isomorphic to $\langle \mathbb{Q}, \leq \rangle$.

Proof. Let $\mathbb{Q} = \{q_n : n < \omega\}$ and $Q = \{r_n : n < \omega\}$ be one-to-one enumerations. Recursively, we define finite bijections h_n for $n < \omega$ such that:

(a) if $n < \omega$, $\{q_0, \ldots, q_n\} \subset dom(h_n)$ and $\{r_0, \ldots, r_n\} \subset im(h_n)$;

(b) if $n < \omega$, $a, b \in dom(h_n)$ and $a < b$, then $h_n(a) < h_n(b)$;

(c) if $n < \omega$ then $h_n \subset h_{n+1}$.

Assume that we have defined h_m for some $m < \omega$, let us define h_{m+1}. We will define h_{m+1} in two steps, first we describe the construction of a bijection g

that extends h_m. If $q_{m+1} \in dom(h_m)$, let $g = h_m$. Otherwise, we must do some work. Let $dom(h_m) = \{a_0, \ldots, a_k\}$ and $im(h_m) = \{b_0, \ldots, b_k\}$ where $a_i < a_j$ and $b_i < b_j$ whenever $0 \leq i < j \leq k$. Consider the following intervals of $\mathbb{Q}$: $I_0 = (\leftarrow, a_0)$, $I_i = (a_i, a_{i+1})$ for $0 < i < k$ and $I_k = (a_k, \rightarrow)$. Also consider the following intervals of Q: $J_0 = (\leftarrow, b_0)$, $J_i = (b_i, b_{i+1})$ for $0 < i < k$ and $J_k = (b_k, \rightarrow)$. There exists $i \leq k$ such that $q_{k+1} \in I_i$. By the properties of Q, $J_i \neq \emptyset$ so we may choose $r \in J_i$. Let $g = h_m \cup \{\langle q_{m+1}, r\rangle\}$. Thus, we have defined g in both cases. If $r_{m+1} \in im(g)$, let $h_{m+1} = g$. Otherwise, we may use an argument as above to choose $q \in \mathbb{Q}$ such that $h_{m+1} = g \cup \{\langle q, r_{m+1}\rangle\}$ has properties (a) and (b).

This completes the construction of $\{h_n : n < \omega\}$. Define $h = \bigcup\{h_n : n < \omega\}$, then it is easy to check that $h : \mathbb{Q} \to Q$ is an order isomorphism. □

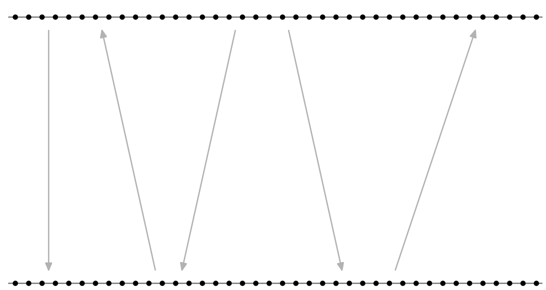

Figure 10.1: Using back-and-forth induction to prove Theorem 10.1.

From this result it is possible to obtain the first example of a CDH space.

Theorem 10.2 The real line $\mathbb{R}$ is CDH.

Proof. Let $D, E \subset \mathbb{R}$ two countable dense subsets of $\mathbb{R}$. From Theorem 10.1 it is easy to see that both D and E are order-isomorphic to $\mathbb{Q}$ with the order of $\mathbb{R}$. Thus, there is a order isomorphism $h : D \to E$. Define $H : \mathbb{R} \to \mathbb{R}$ by $H(x) = \inf\{h(d) : x \leq d, d \in D\}$. It is not hard to see that H is a well-defined order isomorphism. Thus, H is a autohomeomorphism of $\mathbb{R}$ as a topological space and $H[D] = E$. □

We also obtain the following characterization of spaces homeomorphic to $\mathbb{Q}$. This result shows the power of Cantor's technique.

Theorem 10.3 (Sierpiński) Every countable, first countable, crowded, regular space is homeomorphic to $\mathbb{Q}$.

Proof. Let X be countable, crowded and first countable. Since metrizable spaces are first countable and X is itself countable, X has a countable base so X is second countable. Then X is metrizable by Theorem 0.23.

Moreover, X is 0-dimensional by the following argument. Choose some compatible metric d for X. Let $x \in X$ and F be a closed set of X such that $x \notin F$. Let $f : X \to \mathbb{R}$ be the function defined as $f(y) = d(x, y)$ for all $y \in X$, this is a continuous function so $f[F] \subset [\epsilon, \infty)$ for some $\epsilon > 0$. Since X is countable, $f[X]$ is countable as well. Thus, there exists $\delta \in (0, \epsilon) \setminus f[X]$. Let $U = f^{\leftarrow}[[0, \delta)]$ and $V = f^{\leftarrow}[(\delta, \infty)]$. Then U and V are disjoint open subsets of X whose union is X and separate x from F.

Since X is second countable and 0-dimensional, we may assume that X is a subspace of ${}^{\omega}\omega$ (Propositions 8.1 and 8.6). Let $Y = \mathrm{cl}_{{}^{\omega}\omega}(X)$. Notice that Y is also a completely metrizable space (Proposition 0.24). Every open subspace of Y is also completely metrizable (again by Proposition 0.24) and no countable space is completely metrizable (more generally, it cannot be a Baire space by Theorem 0.25) so $Y \setminus X$ is dense in Y. Then there exists a countable dense subset $E \subset Y$ such that $E \cap X = \emptyset$. Define $Z = Y \setminus E$. It easily follows that Z is nowhere locally compact and by Theorem 8.7 and Corollary 8.8, Z is homeomorphic to the space of irrational numbers.

Thus, we have shown that X is homeomorphic to a countable dense subset of the irrational numbers. In particular X is a countable dense subset of $\mathbb{R}$. By Theorem 10.2, there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h[X] = \mathbb{Q}$. Thus, X is homeomorphic to $\mathbb{Q}$. $\square$

Corollary 10.4 Every countable, first countable, regular space can be embedded in $\mathbb{Q}$.

Proof. Let X be countable and first countable. By arguments similar to those in the proof of Theorem 10.3, we may assume that $X \subset {}^{\omega}\omega$. Let Q be a countable dense subset of ${}^{\omega}\omega$. Then X is a subspace of $X \cup Q$, which is homeomorphic to $\mathbb{Q}$ by Theorem 10.3. $\square$

Naturally, being CDH looks like being homogeneous. But the concepts are independent, as the following examples show.

Example 10.5 $\mathbb{Q}$ is not CDH although it is homogeneous.

Both $\mathbb{Q}$ and $\mathbb{Q} \setminus \{0\}$ are countable dense subsets of $\mathbb{Q}$ but there is no homeomorphism $h : \mathbb{Q} \to \mathbb{Q}$ that takes the whole space $\mathbb{Q}$ to the proper space $\mathbb{Q} \setminus \{0\}$. However, $\mathbb{Q}$ is homogeneous, as the homeomorphism $h : \mathbb{Q} \to \mathbb{Q}$ defined by $h(x) = x + q$ for some fixed $q \in \mathbb{Q}$ takes 0 to q. $\square$

We need the following to give our second example.

Lemma 10.6 If X_0 and X_1 are CDH spaces, then $X_0 \oplus X_1$ is CDH.

Proof. Let $D_0, D_1 \subset X_0 \oplus X_1$ be countable dense subsets. Let $D(i,j) = D_i \cap X_j$ for $i, j \in 2$. Let $h_0 : X_0 \to X_0$ and $h_1 : X_1 \to X_1$ be homeomorphisms such that $h_j[D(0,j)] = D(1,j)$ for $j \in 2$. Define $h : X_0 \oplus X_1 \to X_0 \oplus X_1$ by $h(x) = h_i(x)$ if $x \in X_i$ for $i \in 2$. Then h is a homeomorphism and $h[D_0] = D_1$. $\square$

Example 10.7 Let $\mathbb{S}^1 = \{\langle x, y\rangle \in^2 \mathbb{R} : x^2 + y^2 = 1\}$ be the unit circle. Then $\mathbb{S}^1$ is CDH. Thus, $\mathbb{R} \oplus \mathbb{S}^1$ is CDH but not homogeneous.

First, we need to argue that $\mathbb{S}^1$ is CDH. Let $D_0, D_1 \subset \mathbb{S}^1$ be countable dense subsets, choose $d_i, e_i \in D_i$ two different points for each $i \in 2$. Let $J(i,+)$ and $J(i,-)$ be two arcs whose union is $\mathbb{S}^1$ and their intersection is $\{d_i, e_i\}$, for $i \in 2$. Using the methods of Theorems 10.1 and 10.2, it is possible to find homeomorphisms $h_+ : J(0,+) \to J(1,+)$ and $h_- : J(0,-) \to J(1,-)$ such that $h_+[D_0 \cap J(0,+)] = D_1 \cap J(1,+)$ and $h_-[D_0 \cap J(0,-)] = D_1 \cap J(1,-)$. Then $h = h_+ \cup h_-$ is a homeomorphism $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h[D] = E$. The rest follows by Lemma 10.6. $\square$

Even though being CDH does not imply homogeneity, a CDH space must be a free sum of homogeneous spaces.

For each space X, $\mathcal{H}(X)$ denotes the set of all autohomeomorphisms of X. For every space X and $x \in X$, let $\mathcal{O}(x, X) = \{h(x) : h \in \mathcal{H}(X)\}$, we call this set the *orbit* of x (under homeomorphisms).

Theorem 10.8 Let X be a CDH space. Then $\mathcal{O}(x, X)$ is clopen and CDH for every $x \in X$.

Proof. We begin by proving the following.

Claim 1. For every $x \in X$, $\mathrm{cl}_X(\mathcal{O}(x, X))$ is a clopen subset of X.

Fix $x \in X$ and denote $O = \mathcal{O}(x, X)$. Notice that O is fixed under autohomeomorphisms of X.

If O were nowhere dense, there would be a countable dense set D of X that misses O. Notice that there is no homeomorphism $h : X \to X$ such that $h[D] = D \cup \{x\}$ so we obtain a contradiction. Thus, $\mathrm{int}_X(\mathrm{cl}_X(O)) \neq \emptyset$.

The next step to prove Claim 1 is showing that $\mathrm{int}_X(\mathrm{cl}_X(O))$ is dense in $\mathrm{cl}_X(O)$. Let U be an open subset of $\mathrm{cl}_X(O)$, choose $y \in U \cap O$ and $z \in \mathrm{int}_X(\mathrm{cl}_X(O))$. By the definition of orbit, there is a homeomorphism $h : X \to X$

such that $h(z) = y$. Since $h[\mathrm{int}_X(\mathrm{cl}_X(O))] = \mathrm{int}_X(\mathrm{cl}_X(O))$, it follows that $y \in \mathrm{int}_X(\mathrm{cl}_X(O))$. From this, $\mathrm{int}_X(\mathrm{cl}_X(O))$ intersects U so it is dense in $\mathrm{cl}_X(O)$.

Let $y \in \mathrm{cl}_X(O)$ be arbitrary, we now prove that $y \in \mathrm{int}_X(\mathrm{cl}_X(O))$. Since every open subset of X is separable, we may choose a countable dense subset D of $\mathrm{int}_X(\mathrm{cl}_X(O))$ and a countable dense subset E of $X \setminus \mathrm{cl}_X(O)$. Then $D \cup E$ is a countable dense subset of X. There exists a homeomorphism $h : X \to X$ such that $h[D \cup E] = D \cup E \cup \{y\}$. Since $h[\mathrm{cl}_X(O)] = \mathrm{cl}_X(O)$ we obtain that $h(y) \in D$. Then $h(y) \in \mathrm{int}_X(\mathrm{cl}_X(O))$. Moreover, $h^{\leftarrow}[\mathrm{int}_X(\mathrm{cl}_X(O))] = \mathrm{int}_X(\mathrm{cl}_X(O))$ so $x \in \mathrm{int}_X(\mathrm{cl}_X(O))$. Since y was arbitrary, we have completed the proof of Claim 1.

Claim 2. If $x, y \in X$ are such that $\mathrm{cl}_X(\mathcal{O}(x, X)) \cap \mathrm{cl}_X(\mathcal{O}(y, X)) \neq \emptyset$, then $\mathrm{cl}_X(\mathcal{O}(x, X)) = \mathrm{cl}_X(\mathcal{O}(y, X))$.

To prove Claim 2, assume that $x, y \in X$ are such that $\mathrm{cl}_X(\mathcal{O}(x, X)) \cap \mathrm{cl}_X(\mathcal{O}(y, X)) \neq \emptyset$ but $\mathrm{cl}_X(\mathcal{O}(x, X)) \setminus \mathrm{cl}_X(\mathcal{O}(y, X)) \neq \emptyset$. Let $p, q \in \mathcal{O}(x, X)$ be such that $p \in \mathrm{cl}_X(\mathcal{O}(y, X))$ and $q \notin \mathrm{cl}_X(\mathcal{O}(y, X))$, this is possible by Claim 1. Then there is a homeomorphism $h : X \to X$ such that $h(p) = q$. Since $h[\mathrm{cl}_X(\mathcal{O}(y, X))] = \mathrm{cl}_X(\mathcal{O}(y, X))$, then $q \in \mathrm{cl}_X(\mathcal{O}(y, X))$, a contradiction. With this we have proved Claim 2.

Now fix $x \in X$ for the rest of the proof, let $Y = \mathrm{cl}_X(\mathcal{O}(x, X))$. Notice that Y is CDH by the following argument. Let D and E be countable dense sets of Y. Choose F a countable dense in $X \setminus Y$, then there is a homeomorphism $h_0 : X \to X$ with $h_0[D \cup F] = E \cup F$. Since $h_0[Y] = Y$, $h_1 = h_0{\restriction_Y} : Y \to Y$ is a homeomorphism such that $h_1[D] = E$.

We must prove that $\mathcal{O}(x, X) = Y$ so assume this is not the case. By Claims 1 and 2, $\mathcal{O}(y, X)$ is dense in Y for every $y \in Y$. Then both $\mathcal{O}(x, X)$ and $Y \setminus \mathcal{O}(x, X)$ are dense in Y. Choose some countable dense set D of Y. Let $D_0 = D \cap \mathcal{O}(x, X)$ and $D_1 = D \setminus \mathcal{O}(x, X)$.

Case 1: $\mathrm{cl}_Y(D_0) = X$. Let $p \in Y \setminus \mathcal{O}(x, X)$ be arbitrary. Then there exists a homeomorphism $h : Y \to Y$ such that $h[D_0] = D_0 \cup \{p\}$. Then $p \in \mathcal{O}(x, X)$, which is a contradiction.

Case 2: $\mathrm{cl}_Y(D_1) = X$. There is a homemorphism $h : Y \to Y$ such that $h[D_1] = D_1 \cup \{x\}$. However, $D_1 \cap \mathcal{O}(x, X) = \emptyset$ so we obtain a contradiction again.

Case 3: Not Case 1 or 2. Let $U = Y \setminus \mathrm{cl}_Y(D_1)$, then U is a non-empty open set of Y and $D_0 \cap U$ is a countable dense set of U. Since $\mathrm{cl}_Y(D_0) \neq Y$, there is an non-empty open set V of Y such that $V \cap D_0 = \emptyset$. Notice that then $V \cap D_1$ is dense in V and $V \cap U = \emptyset$. Let $p \in V \cap \mathcal{O}(x, X)$. Define $E = (D_0 \cap U) \cup D_1$, then E is a countable dense subset of Y. Let $h : Y \to Y$ be a homeomorphism such that

$h[E] = E \cup \{p\}$. Notice that since both $\mathcal{O}(x, X)$ and $Y \setminus \mathcal{O}(x, X)$ are dense in Y, then $E \cap \mathcal{O}(x, X)$ has no isolated points. Notice that $V \cap (E \cup \{p\}) \cap \mathcal{O}(x, X) = \{p\}$ so p is an isolated point of $(E \cup \{p\}) \cap \mathcal{O}(x, X)$. But this is a contradiction to the fact that $h[E \cap \mathcal{O}(x, X)] = (E \cup \{p\}) \cap \mathcal{O}(x, X)$. Thus, this case is not possible.

Thus, we have obtain a contradiction on each of the three possibilities. This shows that contrary to our assumption, $\mathcal{O}(x, X) = Y$ and this finishes the proof of the Theorem. □

Thus, in particular we have the following.

Corollary 10.9 Every CDH space can be decomposed as a pairwise disjoint union of clopen CDH homogeneous spaces.

Corollary 10.10 [55, Theorem, p. 20] Every connected CDH space is homogeneous.

Let us give some other examples of spaces and whether they are CDH or not.

Example 10.11 The Sorgenfrey line $\mathbb{S}$ is CDH.

The Sorgenfrey line is the set $\mathbb{R}$ with the topology that has $\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$ as a base for the open sets (see [50, Example 1.2.2]). When $\mathbb{R}$ is given the Sorgenfrey line topology, it will be denoted by $\mathbb{S}$. The Sorgenfrey line is a very famous example in General Topology. Let us mention some properties of $\mathbb{S}$.

It is easy to see that every set from $\mathcal{B}$ is clopen so $\mathbb{S}$ is Hausdorff and 0-dimensional. As the set of rational numbers is dense in $\mathbb{S}$, it is separable. We now argue that $\mathbb{S}$ has weight $\mathfrak{c}$.

Since $|\mathcal{B}| = \mathfrak{c}$, $w(\mathbb{S}) \leq \mathfrak{c}$. Assume that $w(\mathbb{S}) < \mathfrak{c}$, we shall arrive to a contradiction. By Lemma 0.15, there exists $\mathcal{B}' = \{(x_\alpha, y_\alpha] : \alpha < w(\mathbb{S})\} \subset \mathcal{B}$ that is a base. However, if $y \in (0, 1] \setminus \{y_\alpha : \alpha < w(\mathbb{S})\}$, then there is no $B \in \mathcal{B}'$ such that $y \in B \subset (0, y]$. This is a contradiction so in fact $w(\mathbb{S}) = \mathfrak{c}$.

Since the Euclidean topology (that is, the topology in $\mathbb{R}$) is contained in the Sorgenfrey line topology, every dense set of $\mathbb{S}$ is also a dense subset of $\mathbb{R}$. Let D, E be two countable dense subsets of $\mathbb{S}$. By the argument given in Theorem 10.2, there is an order isomorphism $h : \mathbb{S} \to \mathbb{S}$ such that $h[D] = E$. It is not hard to see that any order isomorphism of $\mathbb{S}$ is a homeomorphism so indeed $\mathbb{S}$ is CDH. □

The following two examples will illustrate an interesting phenomenon: if a space can be decomposed into subsets such that any homeomorphism gives a permutation of such subsets, there is a big chance that the space is not CDH even if it is homogeneous. Such subsets are components in the first example and composants in the second one.

Example 10.12 $\mathbb{Q} \times \mathbb{R}$ is homogeneous but not CDH

Notice that the components of $\mathbb{Q}\times\mathbb{R}$ are precisely $\{\{q\}\times\mathbb{R} : q \in \mathbb{Q}\}$ and each one is nowhere dense in $\mathbb{Q} \times \mathbb{R}$. Thus it is possible to define two essentially diferent countable dense subsets D and E in the following way.

To define D, choose a countable dense subset $D_q \subset \{q\} \times \mathbb{R}$ for each $q \in \mathbb{Q}$ and define $D = \bigcup\{D_q : q \in \mathbb{Q}\}$. To construct E, let $\{U_n : n < \omega\}$ be an enumeration of a base of $\mathbb{Q} \times \mathbb{R}$. Recursively choose $\{q_n : n < \omega\} \subset \mathbb{Q}$ and $E = \{x_n : n < \omega\} \subset \mathbb{Q}\times\mathbb{R}$ such that $x_n \in U_n \cap (\{q_n\} \times \mathbb{R})$ and $q_n \notin \{q_k : k < n\}$ for all $n < \omega$. Then E intersects each component of the space in at most one point, while D intersects all components of the space in infinitely many points. Thus, there is no homeomorphism of the space that takes D to E. $\square$

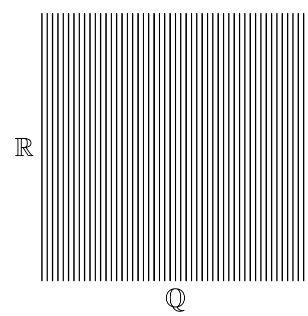

Figure 10.2: The components of $\mathbb{Q} \times \mathbb{R}$ are vertical lines.

Example 10.13 Homogeneous, metrizable continua that are not CDH; one of them a topological group.

A *continuum* is a compact, connected and Hausdorff space. A continuum X is *indecomposable* if every time $X = Y \cup Z$ where Y and Z are both continua, then $Y = X$ or $Z = X$. Two well-known examples of metrizable, homogeneous and indecomposable continua are the *dyadic solenoid* and the *pseudoarc*. The dyadic solenoid is the space

$$\left\{x \in {}^{\omega}(\mathbb{S}^1) : \forall n < \omega\ (x(n) = x(n+1) \cdot x(n+1))\right\},$$

where $z \cdot w$ denotes multiplication in the complex plane. A geometric construction of the solenoid can be described in the following way (also, see the discussion after 2.8 in [125]). Let T_0 be a solid torus in Euclidean space and for each $n<\omega$, let T_{n+1} be a solid torus contained in the solid torus T_n in such a way that T_{n+1} turns around T_n twice and the width of T_{n+1} is half the width of T_n (see Figure 10.1); then the dyadic solenoid is homeomorphic to $\bigcap\{T_n : n<\omega\}$. Since $\mathbb{S}^1$ is a topological group, it can be easily proved that a structure of topological group can be given to the dyadic solenoid. For a description of the pseudoarc, see [105] or [125, Exercise 1.23].

It turns out, however, that an indecomposable continuum is never CDH. If X is an indecomposable continuum, let $x \sim y$ mean that there is a continuum $Y \subsetneq X$ such that $x, y \in Y$; it can be easily proved that this is an equivalence relation. The equivalence classes of such equivalence relation are called *composants*. It can be proved that any metrizable indecomposable continuum X has precisely $\mathfrak{c}$ composants ([108]) and each of one is dense in X ([125, Exercise 5.20(a)]).

Let X be any metrizable indecomposable continuum. It is possible to construct two countable dense sets D and E of X in the following way. Let D be contained in exactly one composant of X and let $E = D \cup \{p\}$ where p is in a composant different to the one that contains D. Then there is no autohomeomorphism of X that sends D to E. This proves that X is not CDH. $\square$

We remark that the following properties are known of CDH metrizable continua.

Theorem 10.14 [17, Theorem 2] A CDH metrizable continuum is not irreducible between any two of its points

Theorem 10.15 [54] Any CDH metrizable continuum is locally connected.

We end this section with the following result that has been obtained recently.

Theorem 10.16 [9] If X is a countable dense homogeneous space, then $|X| \leq \mathfrak{c}$.

Proof. Assume that $|X| > \mathfrak{c}$, we will arrive to a contradiction. Let Q be some fixed countable dense subset of X. For each $x \in X$, there is a homeomorphism $h_x : Q \to Q \cup \{x\}$. Let $y(x) = h_x^{-1}(x) \in Q$. Since there are $> \mathfrak{c}$ homeomorphisms of the form h_x, there is $Y \subset X$ with $|Y| > \mathfrak{c}$ and $y \in Q$ such that $y(x) = y$ for all $x \in Y$. Moreover, there are at most $\mathfrak{c}$ functions from $Q \setminus \{y\}$ to Q so there is $Z \subset Y$ with $|Z| > \mathfrak{c}$ such that $h_p\restriction_{Q\setminus\{y\}} = h_q\restriction_{Q\setminus\{y\}}$ for all $p, q \in Z$.

Let $p, q \in Z$ with $p \neq q$. Then $h_p : Q \to X$ and $h_q : Q \to X$ are continuous functions that concide in the dense subset $Q \setminus \{y\}$. By Lemma 0.4, $h_p = h_q$ so $p = h_p(y) = h_q(y) = q$, a contradiction. This proves that $|X| \leq \mathfrak{c}$. $\square$

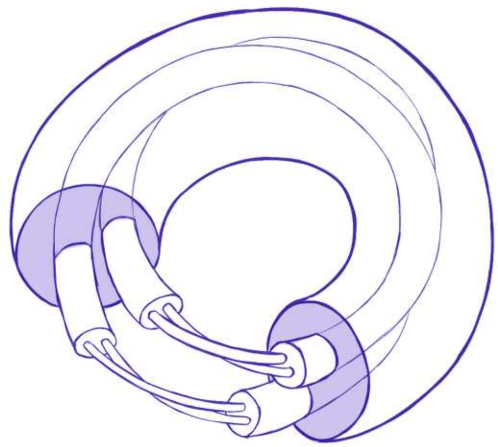

Figure 10.3: Construction of the dyadic solenoid.

10.2 Strongly locally homogeneous spaces

In the previous section, we showed that the real line $\mathbb{R}$ and the circle $\mathbb{S}^1$ are CDH. We would like to find more examples of CDH spaces. It turns out that many natural examples of homogeneous spaces turn out to be CDH. As in the proof of Theorem 10.2, the natural candidate for a proof of CDHness is defining a homeomorphism recursively between the dense sets and then extending it to the whole space. In this section we give a property that will give this result in a precise way. We shall restrict to separable metrizable spaces in this section. Recall that a space is *Polish* if it is separable and completely metrizable.

A space X will be called *strongly locally homogeneous*[1], *SLH* for short, if for every $p \in X$ and every open subset U if X such that $p \in U$ there is an open set V with $p \in V \subset U$ such that if $q \in V$ there is a homeomorphism $h : X \to X$ such that $h(p) = q$ and $h(x) = x$ for all $x \in X \setminus V$.

Theorem 10.17 [18] A SLH Polish space is CDH.

Theorem 10.17 was originally proved by Bennett in [17] for locally compact metrizable spaces. For a proof of Theorem 10.17, see [10, Theorem 14.1]. We will now give some examples of this result.

[1]This property is also called *representable* by some authors.

Example 10.18 The space of irrational numbers and the Cantor set are SLH Polish spaces. Thus, they are both CDH.

Recall that a *manifold* is a Hausdorff space such that every point has an open neighborhood homeomorphic to ${}^{n}\mathbb{R}$ for some $n \in \mathbb{N}$. Since Euclidean spaces are clearly SLH, manifolds are also SLH. Moreover, second countalbe manifolds are locally compact so they are Polish.

Corollary 10.19 If X is a second countable manifold, then X is CDH.

Separable Hilbert space is any space homeomorphic to ${}^{\omega}\mathbb{R}$. The *Hilbert cube* is ${}^{\omega}[0, 1]$. Both of these spaces are SLH (see [60]) and Polish so they provide infinite-dimensional examples of CDH spaces; separable Hilbert space is nowhere locally compact and the Hilbert cube is compact.

Example 10.20 Separable Hilbert space and the Hilbert cube are CDH.

Another famous example of SLH space is the *Menger universal curve* M, it is a locally connected metrizable continuum of dimension 1 that contains homeomorphic copies of all 1-dimensional separable metrizable spaces (see Figure 10.2). It was shown in [5] that the Menger universal curve is SLH.

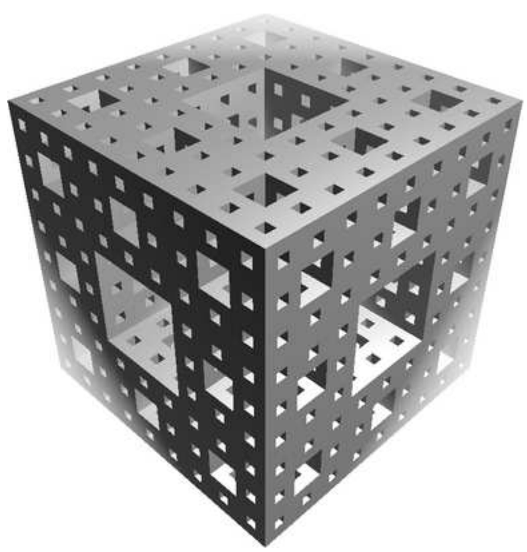

Figure 10.4: A step in the construction of the Menger universal curve.

Example 10.21 The Menger universal curve is SLH and CDH.

We remark that the following is known.

Theorem 10.22 Let X be a 1-dimensional metrizable continuum. Then the following are equivalent.

(a) X is CDH.

(b) X is homogeneous and locally connected.

(c) X is homeomorphic to either the circle or the Menger universal curve.

Proof. Both the circle and the Menger universal curve are SLH metrizable continua (see [5]) so they are CDH by Theorem 10.17. If X is a CDH metrizable continuum, then X is locally connected by Theorem 10.15. Then X is homogeneous by Corollary 10.10. A result of Anderson says that any homogeneous 1-dimensional locally connected metrizable continuum is either the circle or the Menger universal curve ([5] and [6]). □

Our final example is complete Erdős space. This space has already been mentioned in Example 2.3. Complete Erdős space is a 1-dimensional, totally disconnected Polish space. See [32] for more information on this space. It turns out that Complete Erdős is CDH but not CDH.

Example 10.23 Complete Erdős space is CDH ([93, Theorem 12]) but not SLH ([31, Proposition 6.9]).

Obviously SLH does not imply CDH in general, as $\mathbb{Q}$ is SLH but not CDH. However, the question of how much one can relax the hypothesis of completeness in Theorem 10.17 and still obtain CDH is interesting.

Example 10.24 [112] There is a subspace of the Euclidean plane that is a Baire space, connected, locally connected, SLH but not CDH.

10.3 Ungar's theorem and generalizations

By Corollary 10.10 it follows that in the presence of connectedness, CDH is stronger than homogeneity. One natural question is how strong it is. That is, is it possible that CDH spaces have stronger homogeneity properties? In the other direction, we would like to know if there is some charaterization of

CDH connected spaces in terms of their homogeneity. Ungar has shown such a characterization, we present it in Theorem 10.25.

Let X be a topological space and $n \in \mathbb{N}$. We will say that X is *n-homogeneous* if for every $F, G \subset X$ with $|F| = |G| = n$ there is a authomeomorphism $h : X \to X$ such that $h[F] = G$. We will say that X is *strongly n-homogeneous* if for every $F, G \subset X$ with $|F| = |G| = n$ and every bijection $\phi : F \to G$ there is a autohomeomorphism $h : X \to X$ such that $h(x) = \phi(x)$ whenever $x \in F$.

Theorem 10.25 [157] Let X be a locally compact, separable and metrizable space such that cannot be separated by any finite set. Then the following are equivalent.

(a) X is CDH.

(b) X is n-homogeneous for every $n \in \mathbb{N}$.

(c) X is strongly n-homogeneous for every $n \in \mathbb{N}$.

Notice that the hypothesis from Theorem 10.25 that the space X cannot be separated by finite subsets cannot be removed: the real line $\mathbb{R}$ is CDH (Theorem 10.2) and in fact it is n-homogeneous for every $n \in \mathbb{N}$ (follow the method of back-and-forth from Theorem 10.1 in finitely many steps) but it is not strongly 3-homogeneous since $\langle 0, 1, 2\rangle$ cannot be sent to $\langle 0, 2, 1\rangle$ by a homeomorphism. It has also been proved by van Mill that the hypothesis of local compactness cannot be removed.

Example 10.26 [116] There is a Polish space that is n-homogeneous for every $n \in \mathbb{N}$ but not CDH.

Very recently, Ungar's result has been strengthened by Michael Hrušák and Jan van Mill in the following way.

Theorem 10.27 [87] Let X be a locally compact separable metrizable space. The following are equivalent.

(a) X is CDH.

(b) For every finite subset F of X there is a partition $\mathcal{U}$ of $X \setminus F$ into relatively clopen sets such that for every $U \in \mathcal{U}$ and every $p, q \in U$ there is a homeomorphism $h : X \to X$ such that $h(p) = q$ and $f[F] = F$.

(b) For every finite subset F of X there is a partition $\mathcal{U}$ of $X \setminus F$ into relatively clopen sets such that for every $U \in \mathcal{U}$ and every $p, q \in U$ there is a homeomorphism $h : X \to X$ such that $h(p) = q$ and $f(x) = x$ for every $x \in F$.

10.4 Questions on Definability

Notice that all our results on CDH spaces so far are for completely metrizable spaces, so it is natural to wonder whether there are non-complete CDH spaces. In their Open Problems in Topology paper [57], Fitzpatrick and Zhou precisely asked that question.

Question 10.28 Does there exist a CDH metrizable space that is not completely metrizable?

Question 10.29 For which 0-dimensional subsets X of $\mathbb{R}$ is ${}^{\omega}X$ CDH?

The first results in this direction were given in the presence of some hypothesis that goes beyond ZFC set theory.

If $X \subset \mathbb{R}$, we say that X is a *Bernstein set* if every compact crowded subspace of $\mathbb{R}$ (equivalently, every Cantor set contained in $\mathbb{R}$) intersects both X and $\mathbb{R} \setminus X$. It is not hard to construct a Bernstein set: let $\{C_\alpha : \alpha < \mathfrak{c}\}$ be the collection of all Cantor sets contained in X and recursively choose $x_\alpha, y_\alpha \in C_\alpha$ such that $x_\alpha \notin \{x_\beta : \beta < \alpha\} \cup \{y_\beta : \beta < \alpha\}$ and $y_\alpha \notin \{x_\beta : \beta \leq \alpha\} \cup \{y_\beta : \beta < \alpha\}$ for all $\alpha < \mathfrak{c}$, then $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ is a Bernstein set.

A Bernstein set is not completely metrizable, as it is known that every uncountable Polish space contains a copy of the Cantor set $\mathfrak{c}$ ([94, 6.5]). Even though it is easy to prove the existence of Bernstein sets in ZFC alone, it has not been yet possible to show the existence of a CDH Bersnstein set without further hypothesis. The only result known so far about this is the following one.

Theorem 10.30 [13] If $\mathbf{MA}(\text{countable})$ holds, then there exists a CDH Bernstein set.

Thus, the following remains unanswered.

Question 10.31 Can the existence of a CDH Bernstein set be proved in ZFC?

Let us recall some facts about Borel sets. Let X be a Polish space. By recursion on an ordinal α, we define the following sets.

$$\begin{aligned}
\mathbf{\Sigma}^0_0(X) &= \Big\{U \subset X : U \text{ is open}\Big\}, \\
\mathbf{\Sigma}^0_\alpha(X) &= \Big\{\bigcup \mathcal{A} : \mathcal{A} \subset \bigcup\{\mathbf{\Pi}^0_\beta(X) : \beta < \alpha\}, |\mathcal{A}| \leq \omega\Big\}, \text{ for each } \alpha > 0. \\
\mathbf{\Pi}^0_\alpha(X) &= \Big\{X \setminus A : A \in \Sigma^\alpha_0\Big\}, \text{ for each } \alpha.
\end{aligned}$$

So $\mathbf{\Sigma}^0_0(X)$ are open sets, $\mathbf{\Pi}^0_0(X)$ are closed, $\mathbf{\Sigma}^0_1(X)$ are F_σ and $\mathbf{\Pi}^0_1(X)$ are G_δ. By induction it is possible to prove that for $\alpha < \beta < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) \cup \mathbf{\Pi}^0_\alpha(X) \subset \mathbf{\Sigma}^0_\beta(X)$. Let $\mathcal{BOR}(X) = \bigcup\{\mathbf{\Sigma}^0_\alpha(X) : \alpha < \omega_1\}$. Sets from $\mathcal{BOR}(X)$ are called *Borel* sets of X.

In some sense, Borel sets are "definable". For example, in $\mathbb{R}$, open sets are just countable unions of intervals, which can be easily defined; taking the σ-algebra generated by such a collection should also be "definable". It can also be proved that being a Borel set of some Polish space is a topological property: if X is a Borel subset of Y and Z is a Polish space in which X is densely embedded, then X is also a Borel set of Z. (see [146, 3.3.7]). Thus, we can define a separable metrizable space to be a *Borel space* if it is a Borel set of some (equivalently, each) Polish space in which it is (densely) embedded. Notice that Polish spaces are Borel themselves.

One of the reasons Borel spaces are thought to be definable is that they have many properties in common to closed sets. In particular, a Borel space X has the *perfect set property*, that is, there is a Cantor set contained in X ([94, 13.6]). This is a big contrast with Bernstein sets, which do not contain Cantor sets.

It turns out that CDH Borel spaces can only be Polish, this was shown by Michael Hrušák and Beatriz Zamora-Avilés.

Theorem 10.32 [86] Let X be a separable metrizable space.

(a) If X is a CDH Borel space, then X is completely metrizable.

(b) If ${}^\omega X$ is CDH, then X is a Baire space.

In fact, it is possible to give a list of Borel 0-dimensional CDH spaces that are ω-powers in the spirit of Question 10.29.

Theorem 10.33 [86] Let $X \subset {}^\omega 2$ be Borel. Then the following are equivalent.

(a) ${}^\omega X$ is CDH.

(b) X is a subset of type G_δ in ${}^\omega 2$.

(c) ${}^{\omega}X$ is homeomorphic to a point, ${}^{\omega}2$ or ${}^{\omega}\omega$.

A space X is a λ-*set* if every countable subset of X is a subset of type G_δ in X. By the Baire Category Theorem (Theorem 0.25), a Polish space cannot be a λ-set. A CDH λ-set has been shown to exist in ZFC, this answers Question 10.28.

Theorem 10.34 [53] There is a CDH set of reals X of size ω_1 that is a λ-set and thus, not completely metrizable.

The techniques used for the proof of Theorem 10.34 produce spaces that cannot be Baire spaces, so by (b) in Theorem 10.32 they cannot answer Question 10.29.

We finally show some results about the Baire property of CDH spaces. The following is a straighforward generalization of [56, Lemma 3.2]. Recall that a space X is *meager* if it is a countable union of nowhere dense subsets of X.

Lemma 10.35 Let X be a crowded, first countable, separable, meager space. Then X has a countable dense set that is a set of type G_δ of X.

Proof. Using first countability and separability, it is possible to show that there is a countable π-base $\mathcal{B} = \{B_n : n < \omega\}$ of X. Let $X = \bigcup\{F_n : n < \omega\}$ be such that F_n is closed and nowhere dense in X for every $n < \omega$. We may assume that $F_m \subset F_n$ for all $m \leq n < \omega$. Recursively choose $x_n \in B_n \setminus (F_n \cup \{x_m : m < n\})$ for all $n < \omega$.

Let $G_n = (X \setminus F_n) \cup \{x_m : m \leq n\}$ for all $n < \omega$ and define $D = \{x_n : n < \omega\}$. Since X is first countable, G_n is a set of type G_δ of X. Also, $G_n \subset G_m$ if $m < n$ and $\bigcap\{G_n : n < \omega\} = D$. Thus, D is dense, countable and of type G_δ of X. $\square$

Proposition 10.36 [56, Lemma 3.1] If X is a homogeneous space, then X is either meager or a Baire space.

Proof. If X is not a Baire space, there is a countable collection of dense open sets $\{D_n : n < \omega\}$ and a non-empty open set $U \subset X$ such that $U \cap (\bigcap\{D_n : n < \omega\}) = \emptyset$. Then U is a meager open subset of X. Since X is homogeneous, every point of X has a meager neighbordhood.

Let $\mathcal{U}$ be a maximal collection of open meager sets whose closures do not intersect pairwise and let $F = X \setminus (\bigcup \mathcal{U})$. It is not hard to prove that F is closed and nowhere dense.

For each $U \in \mathcal{U}$, let $\{F(n, U) : n < \omega\}$ be a collection of nowhere dense closed subsets whose union is $\mathrm{cl}_X(U)$. For each $n < \omega$, let

$$F_n = F \cup (\bigcup\{F(n,U) : U \in \mathcal{U}\}).$$

It is not hard to see that F_n is a nowhere dense closed set of X. Further, $X = \bigcup\{F_n : n < \omega\}$ so X is meager. □

From Lemma 10.35 and Proposition 10.36 we obtain another structure theorem about CDH spaces that extends Corollary 10.9.

Corollary 10.37 Any first countable CDH space is a disjoint union of clopen CDH λ-sets and CDH Baire spaces.

Proof. Let X be CDH. By Corollary 10.9, X is a direct sum of CDH homogeneous spaces. Take one of this summands Y, it is enough to prove that Y is a λ-set or a Baire space. If Y is not a Baire space, then it is meager by Proposition 10.36. If Y has an isolated point, then it is discrete because it is homogeneous and homogeneous spaces are Baire spaces so we obtain a contradiction. Thus, Y is crowded. We conclude the proof by using Lemma 10.35 and noting that any countable subset of Y is contained in a countable dense subset. □

Chapter 11

Countable Dense Homogeneous Filters

The purpose of this Chapter is to give the proofs of the results we obtained on CDH filters, which consistently answer Question 10.28 and give a non-definable Example for Question 10.29. We will first give the theoretical background and mention previous results given by Medini and Milovich. After this, we will present the results obtain. Such results have been submitted for their publication in [81].

11.1 $\mathcal{P}(\omega)$ with the Cantor set topology and filters

The Cantor set is usually defined as the set ${}^{\omega}2$ of functions $f : \omega \to 2$. Recall that there is a natural bijection $\Xi : \mathcal{P}(\omega) \to {}^{\omega}2$ such that $\Xi(A)$ is the characteristic function of A. Using this bijection, it is natural to identify the Cantor set with the power set of the natural numbers $\mathcal{P}(\omega)$. In this way, any subset of $\mathcal{P}(\omega)$ can be considered as a separable metrizable space.

The subsets of $\mathcal{P}(\omega)$ we will be considering are filters. We have already seen that ultrafilters are points in some spaces (see the Stone Representation Theorem 6.22). Our point of view here is different, as a single filter will be considered a topological space. We will assume that all filters under the discussion in this Chapter are filters on ω and they all contain the Frechet filter $\mathcal{F}_\omega = \{A \subset \omega : \omega \setminus A \text{ is finite}\}$.

Filters are combinatorial objects; by this we mean that their topological properties are equivalent to properties that can be stated in combinatorial terms, without refering to the topology. As all our filters contain the Frechet filter, which is a countable dense subset of the Cantor set, all filters under consideration are dense subsets of $\mathcal{P}(\omega)$.

If $\mathcal{X} \subset {}^{\omega}2$, define $\mathcal{X}^* = \{\omega \setminus A : A \in \mathcal{X}\}$. Notice that $\mathcal{X}^{**} = \mathcal{X}$ for all $\mathcal{X} \subset {}^{\omega}2$. An *ideal* on ${}^{\omega}2$ is a set $\mathcal{I} \subset {}^{\omega}2$ such that $\mathcal{I}^*$ is a filter and $\mathcal{F}$ and $\mathcal{I}$ are called *dual*. The function $A \mapsto \omega \setminus A$ is easily seen to be a autohomeomorphism of $\mathcal{P}(\omega)$ so it follows that any set $\mathcal{X} \subset \mathcal{P}(\omega)$ is homeomorphic to its dual $\mathcal{X}^*$. Sometimes it is easier to give results in the language of ideals than in the language of filters (for example, Lemma 11.11 below).

The sum modulo 2 in $2 = \{0, 1\}$, defined as $0 + 0 = 0 = 1 + 1$ and $0 + 1 = 1 = 1 + 0$, makes 2 a topological group. Thus, coordinate-wise sum in ${}^{\omega}2$ also gives an structure of topological group. The corresponding operation in ${}^{\omega}2$ is the symmetric difference, that is $A \bigtriangleup B = (A \setminus B) \cup (B \setminus A)$. Thus, $(\mathcal{P}(\omega), \bigtriangleup, \emptyset)$ is a topological group. Notice that the symmetric difference is *nilpotent*, that is, $x \bigtriangleup x = \emptyset$ for all $x \in \mathcal{P}(\omega)$. We will now see that filters are homogeneous, using the topological group structure of $(\mathcal{P}(\omega), \bigtriangleup, \emptyset)$.

As discussed before, it is enough to prove that every ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is homogeneous. By the definition of ideal, $\mathcal{I} \bigtriangleup y = \{x \bigtriangleup y : x \in \mathcal{I}\}$ is contained in $\mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. Thus, $(\mathcal{I} \bigtriangleup y) \bigtriangleup y \subset \mathcal{I} \bigtriangleup y \subset \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. From nilpotency, $(\mathcal{I} \bigtriangleup y) \bigtriangleup y = \mathcal{I}$ so in fact $\mathcal{I} \bigtriangleup y = \mathcal{I}$ for every $y \in {}^{\omega}2$. With this observation, given $p, q \in \mathcal{I}$, the function $h : \mathcal{I} \to \mathcal{I}$ defined by $h(x) = x \bigtriangleup (p \bigtriangleup q)$ is a homeomorphism such that $h(p) = q$. This shows that $\mathcal{I}$ is homogeneous.

Proposition 11.1 Every filter $\mathcal{F}$ on ω that extends the Fréchet filter is a homogeneous and dense subset of the Cantor set $\mathcal{P}(\omega)$. The dual ideal $\mathcal{I} = \mathcal{F}^*$ is homeomorphic to $\mathcal{F}$ via the homeomorphism $A \mapsto \omega \setminus A$. Moreover, $\mathcal{I}$ is closed under the symmetric diference of $\mathcal{P}(\omega)$.

Now we would like to start considering properties that some filters have but others do not. Recall that the Ultrafilter Theorem UFT in page xi is not provable from ZFC. But clearly $\mathcal{F}_\omega$ is definable. So ultrafilters must be also topologically distinct from the Fréchet filter in some way.

Let X be a Polish space. Recall that a Borel set of X is any subset in the σ-algebra generated by open subsets of X. A set $A \subset X$ is said to have the *Baire property*[1] if there is a meager set $M \subset X$ such that $A \bigtriangleup M$ is open (perhaps empty). It can be shown that the sets with the Baire property are precisely those in the smallest σ-algebra generated by open sets and meager sets ([94, Proposition 8.22]). In particular, all Borel sets have the Baire property. It turns out that the Baire property is the right notion to topologically distinguish ultrafilters from filters like the Fréchet filter.

Lemma 11.2 [14, Theorem 4.1.2] Let $\mathcal{F}$ be a filter on ω. Then the following

[1]Do not confuse with the concept of a Baire space.

are equivalent.

(a) $\mathcal{F}$ has the Baire property.

(b) $\mathcal{F}$ is meager in itself.

(c) There is a partition of ω into finite sets $\{J_n : n < \omega\}$ such that for each $F \in \mathcal{F}$ there is $m < \omega$ such that $X \cap J_n \neq \emptyset$ whenever $m \leq n < \omega$.

So meager filters are, in some sense, definable and small. It is not hard to see that an ultrafilter is in fact non-meager (use (c) in Lemma 11.2).

An important class of filters are P-filters. A filter $\mathcal{F}$ on ω will be called a P-filter if every time $\{F_n : n < \omega\} \subset \mathcal{F}$ there exists $F \in \mathcal{F}$ such that $F \setminus F_n$ is finite for all $n < \omega$. We have considered ultrafilters that are P-filters in Section 6.5, we called them "P-points in ω^*". If $A, B \subset \omega$, we define the *almost inclusion* $A \subset^* B$ if $A \setminus B$ is finite. If $\mathcal{A}$ is a family of sets, then a *pseudointersection* of $\mathcal{A}$ is a set B such that $B \subset^* A$ for all $A \in \mathcal{A}$. Thus, a P-filter is a filter in which every countable family has a pseudointersection.

Clearly, $\mathcal{F}_\omega$ is a P-filter. But as discussed in Section 6.5, the existence of P-points is independent of ZFC. So a natural question is whether there are non-meager P-filters in ZFC. This is a major question in Set Theory and has not been solved, see [14, 4.4.C] for more details. Basically, the following is everything that is known.

Theorem 11.3 [91] If all P-filters are meager then $\mathfrak{t} < \mathfrak{b} = \mathfrak{d} < \mathbf{cof}[\mathfrak{d}]^\omega$ and there are inner models with large cardinals.

Non-meager P-filters surprisingly have the most interesting combinatorial properties. As seen in [169], they play an important role in the proof of the independence of existence of P-points in ω^*. It turns out that these filters will be just what we need for CDHness (Theorem 11.12). We will need a combinatorial characterization of non-meager P-filters. Let us give some notation: if $s \in {}^{<\omega}A$ for some set A and $a \in A$, $s^\frown a$ denotes $s \cup \{(n, a)\}$ where $n = dom(s)$.

Definition 11.4 Let $\mathcal{X} \subset [\omega]^\omega$. A tree $T \subset {}^{<\omega}([\omega]^{<\omega})$ is called a *$\mathcal{X}$-tree of finite sets* if for each $s \in T$ there is $X_s \in \mathcal{X}$ such that for every $a \in [X_s]^{<\omega}$ we have $s^\frown a \in T$.

Lemma 11.5 [104, Lemma 1.3] Let $\mathcal{F}$ be a filter on $\mathcal{P}(\omega)$ that extends the Fréchet filter. Then $\mathcal{F}$ is a non-meager P-filter if and only if every $\mathcal{F}$-tree of finite sets has a branch whose union is in $\mathcal{F}$.

Finally, we mention that the ω-power of a non-meager P-filter is topologically homeomorphic to a non-meager P-filter. This result was originally proved by Shelah (see [143, Fact 4.3, p. 327] or [169, Lemma 3.10]).

Lemma 11.6 If $\mathcal{F}$ is a non-meager P-filter, then ${}^{\omega}\mathcal{F}$ is homeomorphic to a non-meager P-filter.

Proof. Let

$$\mathcal{G} = \{A \subset \omega \times \omega : \forall n < \omega (A \cap (\{n\} \times \omega) \in \mathcal{F})\}.$$

Notice that $\mathcal{G}$ is homeomorphic to ${}^{\omega}\mathcal{F}$. It is easy to see that G is a filter on $\omega \times \omega$. We next prove that $\mathcal{G}$ is a non-meager P-filter.

Let $\{A_k : k < \omega\} \subset \mathcal{G}$. For each $\{k, n\} \subset \omega$, let $A_k^n = \{x \in \omega : (n, x) \in A_k\} \in \mathcal{F}$. Since $\mathcal{F}$ is a P-filter, there is $A \in \mathcal{F}$ such that $A \subset^* A_k^n$ for all $\{k, n\} \subset \omega$. Let $f : \omega \to \omega$ be such that $A \setminus f(n) \subset A_k^n$ for all $k \leq n$. Let

$$B = \bigcup \{\{n\} \times (A \setminus f(n)) : n < \omega\}.$$

Then it is easy to see that $B \in \mathcal{G}$ and B is a pseudointersection of $\{A_n : n < \omega\}$. So $\mathcal{G}$ is a P-filter.

Let $\{J_k : k < \omega\}$ a partition of $\omega \times \omega$ into finite subsets. Recursively, we define a sequence $\{F_n : n < \omega\} \subset \mathcal{F}$ and a sequence $\{A_n : n < \omega\} \subset [\omega]^{\omega}$ such that $A_{n+1} \subset A_n$ and $A_n \subset \{k < \omega : J_k \cap (\{n\} \times F_n) = \emptyset\}$ for all $n < \omega$.

For $n = 0$, since $\mathcal{F}$ is non-meager, by Lemma 11.2 there is $F_0 \in \mathcal{F}$ such that $\{k < \omega : J_k \cap (\{0\} \times F_0) = \emptyset\}$ is infinite, call this last set A_0. Assume that we have the construction up to $m < \omega$, then $\mathcal{B} = \{J_k \cap (\{m+1\} \times \omega) : k \in A_m\}$ is a family of pairwise disjoint finite subsets of $\{m+1\} \times \omega$. If $\bigcup \mathcal{B}$ is finite, let $F_{m+1} \in \mathcal{F}$ be such that $F_{m+1} \cap (\bigcup \mathcal{B}) = \emptyset$ and let $A_{m+1} = A_m$. If $\bigcup \mathcal{B}$ is infinite, let $\{B_k : k \in A_m\}$ be any partition of $(\{m+1\} \times \omega) \setminus \bigcup \mathcal{B}$ into finite subsets (some possibly empty). For each $k \in A_m$, let $C_k = (J_k \cap (\{m+1\} \times \omega)) \cup B_k$. Then $\{C_k : k \in A_m\}$ is a partition of $\{m+1\} \times \omega$ into finite sets so by Lemma 11.2, there is $F_{m+1} \in \mathcal{F}$ such that $\{k \in A_m : C_k \cap (\{m+1\} \times F_{m+1}) = \emptyset\}$ is infinite, call this set A_{m+1}. This completes the recursion.

Define an increasing function $s : \omega \to \omega$ such that $s(0) = \min A_0$ and $s(k+1) = \min (A_{k+1} \setminus \{s(0), \ldots, s(k)\})$ for $k < \omega$. Also, define $t : \omega \to \omega$ such that $t(0) = 0$ and

$$t(n+1) = \min \{m < \omega : (J_{s(0)} \cup \ldots \cup J_{s(n)}) \cap (\{n+1\} \times \omega) \subset \{n+1\} \times m\}.$$

Finally, let

$$G = \bigcup \{\{n\} \times (F_n \setminus t(n)) : n < \omega\}.$$

Then $G \in \mathcal{G}$ and for all $k < \omega$, $G \cap J_{s(k)} = \emptyset$. Thus, $\mathcal{G}$ is non-meager by Lemma 11.2. $\square$

11.2 CDH Ultrafilters

In [110], Andrea Medini and David Milovich studied some topological properties of ultrafilters. In particular, they study CDHness of ultrafilters and prove the following result.

Theorem 11.7 [110, Theorems 15, 21, 24, 41, 43 and 44] Assume $\mathbf{MA}(\text{countable})$. Then there exists an ultrafilter $\mathcal{U} \subset \mathcal{P}(\omega)$ with any of the following properties:

(a) $\mathcal{U}$ is CDH and a P-point,

(b) $\mathcal{U}$ is CDH and not a P-point,

(c) $\mathcal{U}$ is not CDH and not a P-point, or

(d) ${}^{\omega}\mathcal{U}$ is CDH.

The construction of CDH ultrafilters from Theorem 11.7 roughly follows the following. Give an enumeration of all pairs of countable dense subsets of $\mathcal{P}(\omega)$, $\{\langle D_\alpha, E_\alpha\rangle : \alpha < \mathfrak{c}\}$, and construct the ultrafilter $\mathcal{U}$ in a recursion of length $\mathfrak{c}$. In step $\alpha < \mathfrak{c}$, if $D_\alpha \cup E_\alpha$ is contained in the filter we have constructed so far one finds a homeomorphism $h_\alpha : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $h_\alpha[D_\alpha] = E_\alpha$ and $h_\alpha[\mathcal{U}] = \mathcal{U}$. Of course the hard part in this argument is to find such a homeomorphism h_α that restricts to the yet to construct ultrafilter $\mathcal{U}$. Medini and Milovich found a way to characterize such homeomorphisms in a simple way ([110, Lemma 20]). We will now prove this result in the form that will be useful for us. Notice that the following result is stated in terms of ideals since the characterization is more naturally expressed in this way.

Lemma 11.8 Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal, $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ a continuous function and D a countable dense subset of $\mathcal{I}$. If there exists $x \in \mathcal{I}$ such that $\{d \triangle f(d) : d \in D\} \subset \mathcal{P}(x)$, then $f[\mathcal{I}] = \mathcal{I}$.

Proof. Since D is dense in $\mathcal{P}(\omega)$ and $d \triangle f(d) \subset x$ for all $d \in D$, by continuity it follows that $y \triangle f(y) \subset x$ for all $y \in \mathcal{P}(\omega)$. Then $y \triangle f(y) \in \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. Since $\mathcal{I}$ is closed under $\triangle$ and $\triangle$ is nilpotent, it is easy to see that $y \in \mathcal{I}$ if and only if $f(y) \in \mathcal{I}$ for all $y \in \mathcal{P}(\omega)$. $\square$

We make two final comments. If one looks at Theorem 11.7, a natural question is whether all P-points are CDH ([110, Question 11]). As our Theorem 11.12 shows, the answer is in the affirmative.

Also, Medini and Milovich ask whether an ultrafilter is a P-point if and only if it is *hereditarily Baire* (that is, every subspace is a Baire space). We remark that this was already known as the following result shows.

Proposition 11.9 [107, Theorem 1.2] Let $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter. Then $\mathcal{F}$ is hereditarily Baire if and only if $\mathcal{F}$ is a non-meager P-filter.

What completeness properties must CDH filters have? First, they must be non-definable.

Proposition 11.10 (Hernández-Gutiérrez and Hrušák, [81]) Let $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter. If one of $\mathcal{F}$ or ${}^{\omega}\mathcal{F}$ is CDH, then $\mathcal{F}$ is non-meager.

Proof. If ${}^{\omega}\mathcal{F}$ is CDH, then $\mathcal{F}$ is non-meager by (b) in Theorem 10.32. Assume that $\mathcal{F}$ is CDH. If $\mathcal{F}$ is the Fréchet filter, then $\mathcal{F}$ is homeomorphic to $\mathbb{Q}$ (Theorem 10.3) so it is not CDH (Example 10.5). If $\mathcal{F}$ is not the Fréchet filter, there exists $x \in \mathcal{F}$ such that $\omega \setminus x$ is infinite. Thus, $C = \{y : x \subset y \subset \omega\}$ is a copy of the Cantor set contained in $\mathcal{F}$. Assume that $\mathcal{F}$ is meager, let us arrive to a contradiction.

Let $D \subset \mathcal{F}$ be a countable dense subset of $\mathcal{F}$ such that $D \cap C$ is dense in C. Since $\mathcal{F}$ is meager in itself, by Lemma 10.35, it is possible to find a countable dense subset E of $\mathcal{F}$ that is a G_δ set relative to $\mathcal{F}$. Let $h : \mathcal{F} \to \mathcal{F}$ be a homeomorphism such that $h[D] = E$. Then $h[D \cap C]$ is a countable dense subset of the Cantor set $h[C]$ that is a relative G_δ subset of $h[C]$, this is impossible by the Baire Category Theorem 0.25. This contradiction shows that $\mathcal{F}$ is non-meager and completes the proof. $\square$

Thus, CDH filters must be non-meager (Proposition 11.10) and there are consistent examples of CDH filters that are hereditarily Baire (P-points by (a) in Theorem 11.7) and some that are Baire but not hereditarily Baire ((c) in Theorem 11.7 and Proposition 11.9).

11.3 Non-meager P-filters are Countable Dense Homogeneous

In this Section we will prove our main result. The proof can be naturally divided in two parts: first a Lemma that gives a combinatorial property of non-meager

P-filters and then the proof that this combinatorial property implies CDHness. Recall that $\Xi(x) \in {}^{\omega}2$ denotes the characteristic function of $x \in \mathcal{P}(\omega)$.

Lemma 11.11 (Hernández-Gutiérrez and Hrušák, [81]) Let $\mathcal{I}$ be a non-meager P-ideal and D_0, D_1 be two countable dense subsets of $\mathcal{I}$. Then there exists $x \in \mathcal{I}$ such that

(i) for each $d \in D_0 \cup D_1$, $d \subset^* x$ and

(ii) for each $i \in 2$, $d \in D_i$, $n < \omega$ and $t \in {}^{n \cap x}2$, there exists $e \in D_{1-i}$ such that $d \setminus x = e \setminus x$ and $\Xi(e){\restriction_{n \cap x}} = t$.

Proof. Let $\mathcal{F} = \mathcal{I}^*$. We will construct an $\mathcal{F}$-tree of finite sets T and use Lemma 11.5 to find $x \in \mathcal{I}$ with the properties listed. Let us give an enumeration $(D_0 \cup D_1) \times {}^{<\omega}2 = \{\langle d_n, t_n \rangle : n < \omega\}$ such that $\{d_n : n \equiv i \ (mod\ 2)\} = D_i$ for $i \in 2$.

The definition of T will be by recursion. For each $s \in T$ we also define $n(s) < \omega$, $F_s \in \mathcal{F}$ and $\phi_s : dom(s) \to D_0 \cup D_1$ so that the following properties hold.

(1) $\forall s, t \in T\ (s \subsetneq t \Rightarrow n(s) < n(t))$,

(2) $\forall s \in T\ \forall k < dom(s)\ (s(k) \subset n(s{\restriction_{k+1}}) \setminus n(s{\restriction_k}))$,

(3) $\forall s, t \in T\ (s \subset t \Rightarrow F_t \subset F_s)$,

(4) $\forall s \in T\ (F_s \subset \omega \setminus n(s))$,

(5) $\forall s, t \in T\ (s \subset t \Rightarrow \phi_s \subset \phi_t)$,

(6) $\forall s \in T,$ if $k = dom(s)$ $((d_{k-1} \cup \phi_s(k-1)) \setminus n(s) \subset \omega \setminus F_s)$.

Since $\emptyset \in T$, let $n(\emptyset) = 0$ and $F_\emptyset = \omega$. Assume we have $s \in T$ and $a \in F_s$, we have to define everything for $s^\frown a$. Let $k = dom(s)$. We start by defining $n(s^\frown a) = \max\{k, \max(a), dom(t_k)\} + 1$. Next we define $\phi_{s^\frown a}$. We only have to do it at k because of (5). We have two cases.

Case 1. There exists $m < dom(t_k)$ with $t_k(m) = 1$ and $m \in s(0) \cup \ldots \cup s(k-1)$. We simply declare $\phi_{s^\frown a}(k) = d_k$.

Case 2. Not Case 1. We define $r_{s^\frown a} \in {}^{n(s^\frown a)}2$ in the following way.

$$r_{s^\frown a}(m) = \begin{cases} d_k(m), & \text{if } m \in s(0) \cup \ldots \cup s(k-1) \cup a, \\ t_k(m), & \text{if } m \in dom(t_k) \setminus (s(0) \cup \ldots \cup s(k-1) \cup a), \\ 1, & \text{in any other case.} \end{cases}$$

Let $i \in 2$ be such that $i \equiv k$ (mod 2). So $d_k \in D_i$, let $\phi_{s^\frown a}(k) \in D_{1-i}$ be such that $\phi_{s^\frown a}(k) \cap n(s^\frown a) = (r_{s^\frown a})^{\leftarrow}(1)$, this is possible because D_{1-i} is dense in $\mathcal{P}(\omega)$. Finaly, define

$$F_{s^\frown a} = (F_s \cap (\omega \setminus d_{k-1}) \cap (\omega \setminus \phi_{s^\frown a}(k-1))) \setminus n(s^\frown a).$$

Clearly, $F_{s^\frown a} \in \mathcal{F}$ and it is easy to see that conditions (1) – (6) hold.

By Lemma 11.5, there exists a branch $\{\langle y_0, \ldots, y_n \rangle : n < \omega\}$ of T whose union $y = \bigcup\{y_n : n < \omega\}$ is in $\mathcal{F}$. Let $x = \omega \setminus y \in \mathcal{I}$. We prove that x is the element we were looking for. It is easy to prove that (6) implies (i).

We next prove that (ii) holds. Let $i \in 2$, $n < \omega$, $t \in {}^{n \cap x}2$ and $d \in D_i$. Let $k < \omega$ be such that $\langle d_k, t_k \rangle = \langle d, t' \rangle$, where $t' \in {}^{n}2$ is such that $t' \restriction_{n \cap x} = t$ and $t' \restriction_{n - x} = \underline{0}$. Consider step $k+1$ in the construction of y, that is, the step when $y(k+1)$ was defined. Notice that we are in Case 2 of the construction and $r_{y \restriction_{k+1}}$ is defined. Then $\phi_{y \restriction_{k+1}}(k) = e$ is an element of D_{1-i}. It is not very hard to see that $d \setminus x = e \setminus x$ and $\Xi(e) \restriction_{n \cap x} = t$. This completes the proof of the Lemma. $\square$

Theorem 11.12 (Hernández-Gutiérrez and Hrušák, [81]) Let $\mathcal{F}$ be a non-meager P-filter on $\mathcal{P}(\omega)$ extending the Fréchet filter. Then both $\mathcal{F}$ and ${}^{\omega}\mathcal{F}$ are CDH.

Proof. By Lemma 11.6, it is enough to prove that $\mathcal{F}$ is CDH. Let $\mathcal{I} = \mathcal{F}^*$, it is enough to prove that $\mathcal{I}$ is CDH. Let D_0 and D_1 be two countable dense subsets of $\mathcal{I}$. Let $x \in \mathcal{I}$ be given by Lemma 11.11.

We will construct a homeomorphism $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $h[D_0] = D_1$ and

$$\forall d \in D \ (d \bigtriangleup h(d) \subset x). \tag{$\star$}$$

By Lemma 11.8, $h[\mathcal{I}] = \mathcal{I}$ and we will have finished.

We shall define h by approximations. By this we mean the following. We will give a strictly increasing sequence $\{n(k) : k < \omega\} \subset \omega$ and in step $k < \omega$ a homeomorphism (permutation) $h_k : \mathcal{P}(n(k)) \to \mathcal{P}(n(k))$ such that

$$\forall j < k < \omega \ \forall a \in \mathcal{P}(n(k)) \ (h_k(a) \cap n(j) = h_j(a \cap n(k))). \tag{$*$}$$

By ($*$), we can define $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ to be the inverse limit of the h_k, which is a homeomorphism.

Let $D_0 \cup D_1 = \{d_n : n < \omega\}$ in such a way that $\{d_n : n \equiv i \ (mod\ 2)\} = D_i$ for $i \in 2$. To make sure that $h[D_0] = D_1$, in step k we have to decide the value of $h(d_k)$ when k is even and the value of $h^{-1}(d_k)$ when k is odd. We do this by approximating a bijection $\pi : D_0 \to D_1$ in ω steps by a chain of finite bijections $\{\pi_k : k < \omega\}$ and letting $\pi = \bigcup\{\pi_k : k < \omega\}$. In step $k < \omega$, we would like to

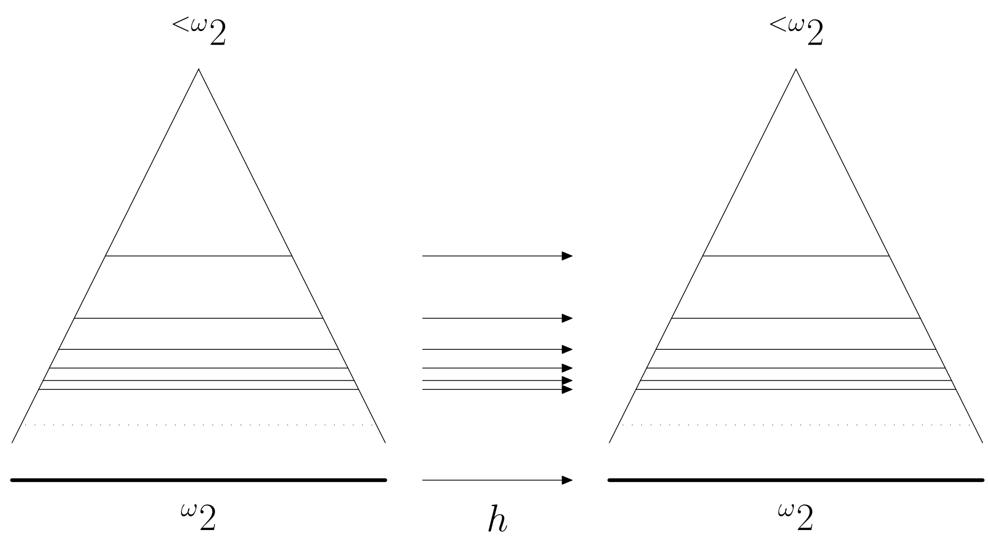

Figure 11.1: Constructing a homeomorphism on ${}^{\omega}2$ as the inverse limit of homeomorphisms.

have π_k defined on some finite set so that the following conditions hold whenever $\pi_k \subset \pi$:

$(a)_k$ if $j < k$ is even, then $h_k(d_j \cap n(k)) = \pi(d_j) \cap n(k)$, and

$(b)_k$ if $j < k$ is odd, then $h_k(d_j \cap n(k)) = \pi^{-1}(d_j) \cap n(k)$.

Notice that once π is completely defined, if $(a)_k$ and $(b)_k$ hold for all $k < \omega$, then $h[D] = E$. As we do the construction, we need to make sure that the following two conditions hold.

$$\begin{array}{ll} (c)_k & \forall i \in n(k) \setminus x \ \ \forall a \in \mathcal{P}(n(k)) \ (i \in a \Leftrightarrow i \in h_k(a)) \\ (d)_k & \forall d \in dom(\pi_k) \ (d \setminus x = \pi_k(d) \setminus x) \end{array}$$

Condition $(c)_k$ is a technical condition that will help us carry out the recursion. Notice that if we have condition $(d)_k$ for all $k < \omega$, then $(\star)$ will hold.

Assume that we have defined $n(0) < \ldots < n(s-1)$, $h_0, \ldots, h_{s-1}$ and a finite bijection $\pi_s \subset D_0 \times D_1$ with $\{d_r : r < s\} \subset dom(\pi_s) \cup dom(\pi_s^{-1})$ in such a way that if $\pi \supset \pi_s$, then $(a)_{s-1}$, $(b)_{s-1}$, $(c)_{s-1}$ and $(d)_{s-1}$ hold. Let us consider the case when s is even, the other case can be treated in a similar fashion.

If $d_s = \pi_s^{-1}(d_r)$ for some odd $r < s$, let $n(s) = n(s-1) + 1$. If we let $\pi_{s+1} = \pi_s$, it is easy to define h_s so that it is compatible with h_{s-1} in the sense of $(*)$, in such a way that $(a)_s$, $(b)_s$, $(c)_s$ and $(d)_s$ hold for any $\pi \supset \pi_{s+1}$. So we may assume this is not the case.

Notice that the set $S = \{d_r : r < s+1\} \cup \{\pi_s(d_r) : r < s, r \equiv 0\ (mod\ 2)\} \cup \{\pi_s^{-1}(d_r) : r < s, r \equiv 1\ (mod\ 2)\}$ is finite. Choose $p < \omega$ so that $d_s \setminus p \subset x$. Let $r_0 = h_{s-1}(d_s \cap n(s-1)) \in \mathcal{P}(n(s-1))$. Choose $n(s-1) < m < \omega$ and $t \in {}^{m\cap x}2$ in such a way that $t^{-1}(1) \cap n(s-1) = r_0 \cap n(s-1) \cap x$ and t is not extended by any element of $\{\chi(a) : a \in S\}$. By Lemma 11.11, there exists $e \in E$ such that $d_s \setminus x = e \setminus x$ and $\chi(e) \restriction_{m\cap x} = t$. Notice that $e \notin S$ and $\chi(e) \restriction_{n(s-1)} = r_0$. We define $\pi_{s+1} = \pi_s \cup \{(d_s, e)\}$. Notice that $(d)_s$ holds in this way.

Now that we have decided where π will send d_s, let $n(s) > \max\{p, m\}$ be such that there are no two distinct $a, b \in S \cup \{\pi_{s+1}(d_s)\}$ with $a \cap n(s) = b \cap n(s)$. Topologically, all elements of $S \cup \{\pi_{s+1}(d_s)\}$ are contained in distinct basic open sets of measure $1/(n(s)+1)$.

Finally, we define the bijection $h_s : \mathcal{P}(n(s)) \to \mathcal{P}(n(s))$. For this part of the proof we will use characteristic functions instead of subsets of ω (otherwise the notation would become cumbersome). Therefore, we may say $h_r : {}^{n(r)}2 \to {}^{n(r)}2$ is a homeomorphism for $r < s$.

Let $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)\setminus x}2$ be a pair of compatible functions. Notice that $(h_{s-1}(q), q')$ are also compatible by $(c)_{s-1}$. Consider the following condition.

$$\nabla(q, q') : \quad \forall a \in {}^{n(s)}2\ (q \cup q' \subset a \Leftrightarrow h_{s-1}(q) \cup q' \subset h_s(a))$$

Notice that if we define h_s so that $\nabla(q, q')$ holds for each pair $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)\setminus x}2$ of compatible functions, then $(*)$ and $(c)_s$ hold as well.

Then for each pair $(q, q') \in {}^{n(s-1)}2 \times {}^{n(s)\setminus x}2$ of compatible functions we only have to find a bijection $g : {}^T2 \to {}^T2$, where $T = (n(s) \cap x) \setminus n(s-1)$ (this bijection will depend on such pair) and define $h_s : {}^{n(s)}2 \to {}^{n(s)}2$ as

$$h_s(a) = h_{s-1}(q) \cup q' \cup g(f \restriction_T),$$

whenever $a \in {}^{n(s)}2$ and $q \cup q' \subset a$. There is only one restriction in the definition of g and it is imposed by conditions $(a)_s$ and $(b)_s$; namely that g is compatible with the bijection π_{s+1} already defined. However by the choice of $n(s)$ this is not hard to do. This finishes the inductive step and the proof. □

Notice that Theorem 11.12 consistently answers Question 10.28 and gives consistent examples for Question 10.29. We finish this Section and Chapter with the natural questions of what else can it be done?

Question 11.13 Is there a combinatorial characterization of CDH filters?

Question 11.14 Is there a CDH filter (ultrafilter) in ZFC? Is there a non-CDH and non-meager filter (ultrafilter) in ZFC?

Chapter 12

Compact CDH spaces of uncountable weight

In this Chapter we will consider another natural question regarding the existence of CDH spaces. This question is whether there are compact CDH spaces of uncountable weight. We will present some consistent examples from the literature in the first section. After this, we will explore a specific space that was a natural candidate for a counterexample: the Alexandroff-Urysohn double arrow space $\mathbb{A}$. Inspired on $\mathbb{A}$, we shall show some restrictions on CDH spaces that are products, see Theorem 12.25. The results presented in this Chapter have been submitted for publication in the paper [78].

12.1 Some consistent examples

The first candidates for non-metrizable CDH spaces are of course "generalizations" of metrizable spaces that are known to be CDH. For example, we have seen that the Sorgenfrey line is CDH (Example 10.11). Other candidates for such spaces are manifolds. The following result is of Steprāns and Zhou.

Theorem 12.1 [147]

- Every separable manifold of weight $< \mathfrak{b}$ is CDH.
- There is a separable manifold of weight $\mathfrak{c}$ that is not CDH.

On one hand, Theorem 12.1 generalizes Corollary 10.19 but it also restricts Theorem 10.17, as manifolds are SLH. So, for example, it is independent of ZFC whether there is a separable non-CDH manifold of weight ω_1. The following question of Steprāns and Zhou is still open.

Question 12.2 Let

$$\tau = \min\{\kappa : \text{ there is a separable manifold of weight } \kappa \text{ that is not CDH}\}.$$

Is $\tau = \mathfrak{c}$?

The next spaces that we may think of are infinite products of CDH spaces, in particular the powers ${}^{\kappa}2$, ${}^{\kappa}\mathbb{R}$ and ${}^{\kappa}[0,1]$.

Theorem 12.3 ([147] and [86]) Let X be one of 2, $\mathbb{R}$ or $[0,1]$. Then

$$\mathfrak{p} = \min\{\kappa : {}^{\kappa}X \text{ is not CDH}\}.$$

Thus, taking any model in which $\mathfrak{p} > \omega_1$, for example any model of $\mathbf{MA} + \mathfrak{c} > \omega_1$, ${}^{\omega_1}2$ is a compact CDH space of uncountable weight. This motivates the following question.

Question 12.4 [10, Problem 15.6] Does there exist a compact CDH space of countable weight in ZFC?

Question 12.4 remains unsolved. Jan van Mill and Alexander Arhangel'skiĭ have recently obtained a consistent result using **CH**.

Theorem 12.5 [9] **CH** implies that there is a compact CDH space of uncountable weight that is hereditarily separable and hereditarily Lindelöf.

12.2 The double arrow space

A natural candidate for a ZFC example of a compact CDH space of uncountable weight is the Alexandroff-Urysohn double arrow space $\mathbb{A}$.

Definition 12.6 Let $\mathbb{A}_0 = (0,1] \times \{0\}$, $\mathbb{A}_1 = [0,1) \times \{1\}$ and $\mathbb{A} = \mathbb{A}_0 \cup \mathbb{A}_1$. Define the lexicographic strict order on $\mathbb{A}$ as $\langle x, t\rangle < \langle y, s\rangle$ if $x < y$ or both $x = y$ and $t < s$. Then $\mathbb{A}$ is given the order topology. Define the function $\pi : \mathbb{A} \to [0,1]$ by $\pi(\langle x, t\rangle) = x$.

Notice that both $\mathbb{A}_0$ and $\mathbb{A}_1$ have the Sorgenfrey line topology as subspaces of $\mathbb{A}$ and both are dense in $\mathbb{A}$ (see Example 10.11).

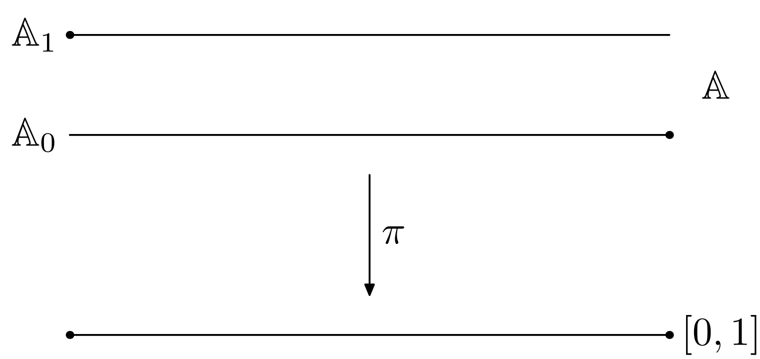

Figure 12.1: The double arrow space.

Proposition 12.7 $\mathbb{A}$ is separable, first countable, compact, 0-dimensional and of weight $\mathfrak{c}$.

Proof. The set $\mathbb{Q} \times \{0,1\}$ is clearly a countable dense subset of $\mathbb{A}$. If $p \in \mathbb{A} \setminus \{\langle 0,1\rangle, \langle 1,0\rangle\}$ let $\{x_n : n < \omega\} \subset (0,1)$ be strictly increasing and $\{y_n : n < \omega\} \subset (0,1)$ be strictly decreasing such that $\pi(p) = \lim x_n = \lim y_n$; then $\{[\langle x_n, 1\rangle, \langle y_n, 0\rangle] : n < \omega\}$ is a countable local base of clopen sets at p. In a similar manner it is possible to find such countable local bases of clopen sets at $\langle 0,1\rangle$ and $\langle 1,0\rangle$. This shows that $\mathbb{A}$ is first countable and 0-dimensional.

To see that $\mathbb{A}$ is compact, it is enough to prove it is sequentially compact. Let $\{x_n : n < \omega\} \subset \mathbb{A}$. Notice that any sequence in $\mathbb{A}$ contains either an increasing sequence or a decreasing sequence. So we may assume that $\{x_n : n < \omega\}$ is strictly increasing, the other case is analogous. Since $\{\pi(x_n) : n < \omega\} \subset [0,1]$, let $x = \sup\{\pi(x_n) : n < \omega\}$. Then $x \in (0,1]$ and $\langle x, 0\rangle = \sup\{x_n : n < \omega\}$.

Notice that $\{(\langle 0,1\rangle, \langle x,0\rangle] : x \in (0,1]\} \cup \{[\langle y,1\rangle, \langle 1,0\rangle) : y \in [0,1)\}$ is a subbase of the topology of $\mathbb{A}$ of cardinality $\mathfrak{c}$ so $w(\mathbb{A}) \leq \mathfrak{c}$. Moreover, $\mathbb{A}_0$ is a subspace of $\mathbb{A}$ that is homeomorphic to $\mathbb{S}$. In Example 10.11 we saw that $w(\mathbb{S}) = \mathfrak{c}$ so $w(\mathbb{A}) \geq \mathfrak{c}$. Thus, $w(\mathbb{A}) = \mathfrak{c}$. $\square$

The function $\pi : \mathbb{A} \to [0,1]$ is a $\leq$ 2-to-1 continuous function. Also, it is not hard to see that π is irreducible, so in fact π witnesses that $\mathbb{A}$ and $[0,1]$ are coabsolute (Corollary 6.42). Since $\mathbb{A}$ is compact and of weight $\mathfrak{c}$, it is not metrizable (Theorem 0.16).

So $\mathbb{A}$ is one possible candidate for a ZFC answer to Question 12.4. However, recently, Arhangel'skiĭ and van Mill have proved that this is not the case.

Theorem 12.8 [9, Theorem 3.2] $\mathbb{A}$ is not CDH.

So the most natural candidate does not work. Arhangel'skiĭ, van Mill and Michael Hrušák wondered if it was possible to obtain a CDH space using some modification of $\mathbb{A}$.

Question 12.9 [9, Question 3.3] Is $\mathbb{A} \times {}^{\omega}2$ CDH?

Question 12.10 (van Mill and Hrušák) Is ${}^{\omega}\mathbb{A}$ CDH?

Recently, in [87], M. Hrušák and J. van Mill have studied the topological types of countable dense subsets of separable metrizable spaces. If X is a separable space and κ a cardinal number, we will say that X has κ types of countable dense subsets if κ is the minimum cardinal number such that there is a collection $\mathcal{D}$ of countable dense subsets of X such that $|\mathcal{D}| \leq \kappa$ and for any countable dense subset $E \subset X$ there is $D \in \mathcal{D}$ and a homeomorphism $h : X \to X$ such that $h[D] = E$. Among other results, they were able to obtain the following result.

Theorem 12.11 [87] A locally compact separable metrizable space that is homogeneous and not CDH has uncountably many types of countable dense subsets.

So another natural question was the following.

Question 12.12 (Hrušák) How many types of countable dense subsets does $\mathbb{A}$ have?

For the rest of this section we will focus on proving properties of $\mathbb{A}$ that will help us answer Questions 12.9, 12.10 and 12.12. First we will answer Question 12.12, as it only requires an analysis of the topology of $\mathbb{A}$. The answer to Questions 12.9 and 12.10 will be presented in Section 12.3, where we give more general results on products, see Corollary 12.26 and Example 12.29.

So let us start with the proof of Theorem 12.16 below. We need two previous properties of $\mathbb{A}$. The first one says that autohomeomorphisms of $\mathbb{A}$ behave in a very simple manner. We need a preliminary result about the Sorgenfrey line.

Lemma 12.13 The Sorgenfrey line $\mathbb{S}$ is a Baire space.

Proof. To prove Claim 1, let $\{U_n : n < \omega\}$ be a collection of dense open subsets of $\mathbb{S}$. Then, using the definition of the Sorgenfrey line topology, it is easy to see that the Euclidean interior $V_n = \mathrm{int}_{\mathbb{R}}(U_n)$ is dense in U_n for all $n < \omega$. In particular, $\{V_n : n < \omega\}$ is a collection of dense open subsets of $\mathbb{R}$. Since $\mathbb{R}$ is a Baire space (it is completely metrizable), $\bigcap\{V_n : n < \omega\} \neq \emptyset$ so $\bigcap\{U_n : n < \omega\} \neq \emptyset$. □

The following result is a more explicit version of [9, Lemma 3.1]. According to [9], the main idea comes from van Douwen.

Proposition 12.14 Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism. Then there exists a collection $\mathcal{U}$ of pairwise disjoint clopen intervals of $\mathbb{A}$ such that $\bigcup\mathcal{U}$ is dense in $\mathbb{A}$ and for every $J \in \mathcal{U}$, $h\restriction_J \colon J \to \mathbb{A}$ is either increasing or decreasing.

Proof. Notice that the statement of the Proposition follows from the following statement.

$(*)$ For every non-empty open set $U \subset \mathbb{A}$ there are $p, q \in (0,1)$ such that $p < q$, $[\langle p,1\rangle, \langle q,0\rangle] \subset U$ and h is either increasing or decrasing on $[\langle p,1\rangle, \langle q,0\rangle]$.

Fix a non-empty open set $U \subset \mathbb{A}$. To prove $(*)$, we will use a Baire category type argument. Notice that both $\mathbb{A}_0$ and $\mathbb{A}_1$ are homeomorphic to the Sorgenfrey line. Let $X = \mathbb{A}_1 \cap U \cap h^{\leftarrow}[\mathbb{A}_1]$ and $Y = \mathbb{A}_1 \cap U \cap h^{\leftarrow}[\mathbb{A}_0]$. Then $X \cup Y = \mathbb{A}_1 \cap U$, which is a Baire space by Lemma 12.13. So it is impossible that both X and Y are meager in $\mathbb{A}_1 \cap U$. Let us first assume that X is non-meager in $\mathbb{A}_1 \cap U$.

For each $n < \omega$, let $X_n = \{x \in X : h[[\pi(x), \pi(x) + 1/(n+1)) \times \{1\}] \subset [h(x), \rightarrow)\}$. Then by the continuity of h, $X = \bigcup\{X_n : n < \omega\}$. Then again, since X is non-meager in $\mathbb{A}_1 \cap U$, then there is an $m < \omega$ such that X_m is non-meager in $\mathbb{A}_1 \cap U$. In particular, $\mathrm{cl}_{\mathbb{A}_1}(X_m)$ has non-empty interior so let $V \subset U$ be a non-empty open set of $\mathbb{A}_1$ such that X_m is dense in V. Let $p, q \in V$ be such that $0 < q - p < 1/(m+1)$ and $(p,q) \subset V$. We claim that h is increasing in $(p,q) \subset \mathbb{A}_1$.

Assume that there exist $x, y \in (p,q) \cap X_m$ such that $x < y$ but $h(y) < h(x)$. By the continuity of h and the fact that X_m is dense in V, there are $a \in (x,y) \cap X_m$ and $b \in (y,q) \cap X_m$ such that $h(a) \in [h(x), \rightarrow)$ and $h(b) \in [h(y), h(x))$. Then $h(y) < h(b) < h(x) < h(a)$. But $a \in X_m$ and $b \in [\pi(a), \pi(a) + 1/(m+1)) \cap \mathbb{A}_1$ so $h(b) \in [h(a), \rightarrow)$. This means that $h(b) < h(a)$ and $h(a) < h(b)$, a contradiction.

Thus, h is increasing in $(p,q) \subset \mathbb{A}_1$. Since $(p,q) \subset \mathbb{A}$ is a dense subset of $[\langle p,1\rangle, \langle q,0\rangle]$, we obtain that h is also increasing in this interval.

If Y is non-meager in $\mathbb{A}_1 \cap U$ the argument is analogous with the exception that h is decreasing in the interval obtained in this way. This completes the proof of $(*)$ and the Proposition. $\square$

We also need $\mathfrak{c}$ pairwise non-homeomorphic countable metrizable spaces. The first proof of this fact was apparently given by Stefan Mazurkiewicz and Wacław Sierpiński in [109]. A different proof was given by Brian, van Mill and Sabedissen in [21].

Lemma 12.15 [87, Lemma 2.4] The number of distinct homeomorphism classes of countable metrizable spaces is $\mathfrak{c}$.

The proof of Theorem 12.8 consisted in exhibiting two different countable dense subsets of $\mathbb{A}$. We will manipulate those two dense subsets and get $\mathfrak{c}$ different ones. If $(X,<)$ is a linearly ordered set, an interval $[a,b] \subset X$ with $a<b$ and $X \cap (a,b) = \emptyset$ will be called a *jump*.

Theorem 12.16 (Hernández-Gutiérrez, [78]) $\mathbb{A}$ has $\mathfrak{c}$ types of countable dense subsets.

Proof. Since $|\mathbb{A}| = \mathfrak{c}$, it is enough to find $\mathfrak{c}$ countable dense subsets of $\mathbb{A}$ that are different with respect to autohomeomorphisms of $\mathbb{A}$.

The classical middle-thirds Cantor set in $[0,1]$ is the complement of a union of countably many open intervals $\{(x_n, y_n) : n < \omega\}$. For each $n < \omega$, let $J_n = [\langle x_n, 1\rangle, \langle y_n, 0\rangle]$, this is a clopen subinterval of $\mathbb{A}$. Define

$$X = \mathbb{A} \setminus \bigcup\{J_n : n < \omega\}.$$

Then X is a closed, crowded and nowhere dense subset of X. Since X is first countable and regular, every countable dense subset of X is homeomorphic to the space of rational numbers $\mathbb{Q}$ (Theorem 10.3). Since $\mathbb{Q}$ is universal for countable metrizable spaces (Corollary 10.4), by Lemma 12.15 there is a collection $\{C_\alpha : \alpha < \mathfrak{c}\}$ of countable subsets of X that are pairwise non-homeomorphic.

Let Q_0, Q_1 be two disjoint countable dense subsets of $\bigcup\{(x_n, y_n) : n < \omega\}$ (with the Euclidean topology). For each $n < \omega$, choose $z_n \in (x_n, y_n)$ and define the following sets: $K_n^0 = (x_n, z_n)$, $K_n^1 = (z_n, y_n)$, $J_n^0 = [\langle x_n, 1\rangle, \langle z_n, 0\rangle]$ and $J_n^1 = [\langle z_n, 1\rangle, \langle y_n, 0\rangle]$. Let D be the subset of $\mathbb{A} \setminus X$ defined such that

(∗) if $n < \omega$ and $i, j \in 2$, then $\pi[J_n^i \cap \mathbb{A}_j \cap D] = Q_{i\cdot j} \cap K_n^j$.

Notice that D is then a countable dense subset of $\mathbb{A} \setminus X$ and thus, of $\mathbb{A}$. For each $\alpha < \mathfrak{c}$, let $D_\alpha = D \cup C_\alpha$. Then D_α is a countable dense subset of $\mathbb{A}$. Assume that there are $\beta < \gamma < \mathfrak{c}$ and a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ such that $h[D_\beta] = D_\gamma$, we shall arrive to a contradiction.

Claim: $h[X] = X$.

We shall prove the claim proceeding by contradiction. Then there are $k < \omega$, $i \in 2$ and $x \in J_k^i$ such that $h(x) \in X$. By the definition of the middle-thirds Cantor set, it is easy to see that every neighborhood of a point of X contains some interval from $\{J_n : n < \omega\}$. Thus, by continuity there exist $m < \omega$ and clopen intervals I_0 and I_1 such that $I_0 \subset J_k^i$, $I_1 \subset J_m^{1-i}$ and $h[I_0] \subset I_1$.

By Proposition 12.14, we may assume that h is increasing or decreasing on I_0. Let us consider the case when h is increasing on I_0, the other case is treated

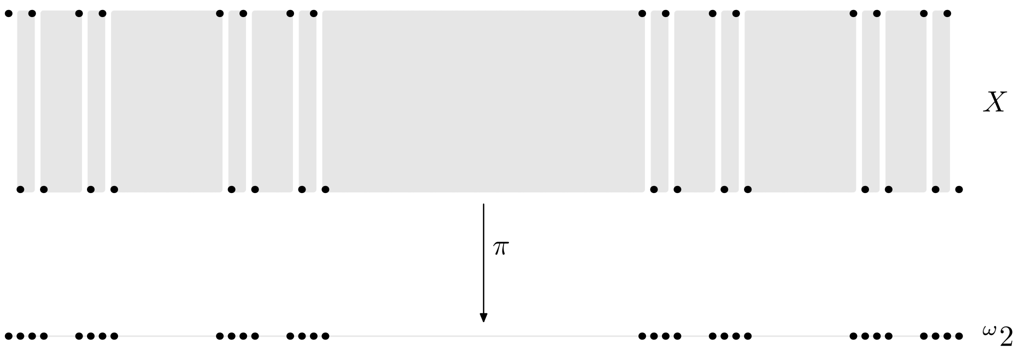

Figure 12.2: Set X from Theorem 12.16, its complement is a countable union of clopen segments (in lighter grey). Notice that X maps onto the usual Cantor set under π and π is one-to-one in the endpoints of the Cantor set.

similarly. Compactness implies that every clopen subset of $\mathbb{A}$ is a finite union of clopen intervals. Thus, $h[I_0]$ can be written as the union of finitely many clopen intervals $[a_0, b_0] \cup \ldots \cup [a_t, b_t]$. Since h is increasing in I_0, $[h^{-1}(a_0), h^{-1}(b_0)]$ is a subinterval of I_0. Thus, we may assume that $h[I_0] = I_1$. In particular, $h{\restriction}_{I_0} \colon I_0 \to I_1$ is an order isomorphism.

Assume that $i = 0$, the other case is entirely symmetric. Since $I_0 \subset J_k^0$, by $(*)$, there are $p, q \in I_0 \cap D$ such that $[p, q]$ is a jump in $\mathbb{A}$. Thus, $[h(p), h(q)]$ is a jump in $\mathbb{A}$ such that $h(p), h(q) \in I_1 \cap D$. But then $h(p), h(q) \in J_m^1 \cap D$ and $\pi(h(p)) = \pi(h(q))$. This contradicts $(*)$ so we have proved the claim.

Thus, given that $D_\alpha \cap X = C_\alpha$ for all $\alpha < \mathfrak{c}$, we obtain that $h[C_\beta] = C_\gamma$ from the claim. But this contradicts the choice of the family $\{C_\alpha : \alpha < \mathfrak{c}\}$. Thus, there can be no such homeomorphism h and we have finished the proof. $\square$

We will next give a characterization of those subsets of $\mathbb{A}$ that are homeomorphic to $\mathbb{A}$. Compare this with [23, Theorem 4.6] in which a classification of subsets of the Sorgenfrey line homeomorphic to the Sorgenfrey line itself are given.

Proposition 12.17 (Hernández-Gutiérrez, [78]) Every closed and crowded subset of $\mathbb{A}$ is homeomorphic to $\mathbb{A}$.

Proof. Let $X \subset \mathbb{A}$ be closed and crowded. We will construct a homeomorphism $h : X \to \mathbb{A}$.

Since X is compact, $T = \pi[X]$ is a compact subspace of $[0, 1]$. Notice that since X is crowded, T is crowded as well. View T as an ordered subspace of

$[0,1]$ and consider the equivalence relation $\sim$ on T obtained by defining $x \sim y$ if either $x = y$ or $[\min\{x,y\}, \max\{x,y\}]$ is a jump in T. Since T is crowded, each equivalence class of $\sim$ consists of at most two points. Since the equivalence classes are convex, the quotient $T/\sim$ can be linearly ordered in a natural way. Then it is not hard to see that the resulting order topology coincides with the quotient space topology and is homeomorphic to $[0,1]$. Thus, there exists a continuous function $f : T \to [0,1]$ that is order preserving and the following property holds.

> $(*)$ Fix $t \in [0,1]$. Then $|f^{\leftarrow}(t)| \neq 1$ if and only if $f^{\leftarrow}(t) = \{x,y\}$, where $0 < x < y < 1$ and $[x,y] \cap T = \{x,y\}$. Moreover, if $f^{\leftarrow}(t) = \{x\}$, then x is not in a jump of T.

The function f works as the classical continuous function collapsing jumps in the Cantor middle-third set to points in $[0,1]$. The following is not hard to prove from the fact that X is crowded.

> $(\star)$ If $[x,y]$ is a jump in T, then $X \cap \{\langle x,0\rangle, \langle x,1\rangle\} = \{\langle x,0\rangle\}$ and $X \cap \{\langle y,0\rangle, \langle y,1\rangle\} = \{\langle y,1\rangle\}$.

Define $h : X \to \mathbb{A}$ by $h(\langle x,t\rangle) = \langle f(x), t\rangle$. We will prove that h is a homeomorphism. Since both X and $\mathbb{A}$ are linearly ordered topological spaces, to prove that h is continuous and one-to-one, it is enough to prove that the strict order is preserved by h.

Let $\langle x,t\rangle < \langle y,s\rangle$ be two points in X. If $x = y$, then $0 = t < s = 1$ so $h(\langle x,t\rangle) = \langle f(x),t\rangle < \langle f(x),s\rangle = h(\langle y,s\rangle)$. If $f(x) < f(y)$, then also $h(\langle x,t\rangle) < h(\langle y,s\rangle)$. So assume that $x < y$ and $f(x) = f(y)$. By $(*)$, $[x,y]$ is a jump in T. Using $(\star)$ it is not hard to see that $t = 0$ and $s = 1$. Thus, $h(\langle x,t\rangle) = \langle f(x),0\rangle < \langle f(x),1\rangle = h(\langle y,s\rangle)$.

Finally, we show that h is onto. Let $\langle x,t\rangle \in \mathbb{A}$. By $(*)$, there are two cases.

Case 1: There is a unique $y \in T$ with $x = f(y)$.

By $(*)$, y is not in a jump. If $y \in \{0,1\}$, then clearly $\langle y, 1-y\rangle \in T$ and $h(\langle y,1-y\rangle) = \langle x,t\rangle$. Assume that $y \notin \{0,1\}$. Then for all $n < \omega$ there are $z_n^0, z_n^1 \in T$ such that

$$y - \frac{1}{n+1} < z_n^0 < y < z_n^1 < y + \frac{1}{n+1}.$$

For each $n < \omega$ and $i \in 2$, let $t_n^i \in 2$ be such that $\langle z_n^i, t_n^i\rangle \in X$. Notice that $\lim_{n\to\infty} \langle z_n^0, t_n^0\rangle = \langle y,0\rangle$ and $\lim_{n\to\infty} \langle z_n^0, t_n^0\rangle = \langle y,1\rangle$. By the compactness of X, $\{\langle y,0\rangle, \langle y,1\rangle\} \subset X$. Thus, $\langle y,t\rangle \in X$ and $h(\langle y,t\rangle) = \langle x,t\rangle$.

Case 2: There are $y_0, y_1 \in T$ with $y_0 < y_1$ and $x = f(y_0) = f(y_1)$.

By $(*)$, $[y_0, y_1]$ is a jump in T. By $(\star)$, we obtain that $\langle y_i, i\rangle \in X$ for $i \in 2$. Thus, $\langle y_t, t\rangle \in X$ and $h(\langle y_t, t\rangle) = \langle x, t\rangle$.

This concludes the proof that h is the homeomorphism that we were looking for. $\square$

Recall that a space X is *scattered* if every subspace of X has an isolated point.

Corollary 12.18 Any compact and metrizable subset of $\mathbb{A}$ is scattered. In particular, $\mathbb{A}$ does not contain topological copies of ${}^{\omega}2$.

We finally show some interesting results on homeomorphisms of $\mathbb{A}$ that have interesting properties. These results were discovered while trying to prove Theorem 12.16.

For example, in the proof of Theorem 12.16 we used a crowded subspace X of $\mathbb{A}$ to construct $\mathfrak{c}$ different dense subsets of $\mathbb{A}$. The author's first idea was to use a scattered space, but this does not work by Proposition 12.19.

For any space Y, we recursively define

- $Y^{(0)} = Y$,
- $Y^{(1)} = Y \setminus \{y \in Y : y \text{ is isolated}\}$,
- $Y^{(\alpha+1)} = (Y^{(\alpha)})^{(1)}$ for every ordinal α, and
- $Y^{(\beta)} = \bigcap\{Y^{(\alpha)} : \alpha < \beta\}$ for every limit ordinal β.

These operations are called *Cantor-Bendixon derivatives*. Define

$$\mathbf{rk}(Y) = \min\{\alpha : Y^{(\alpha)} = Y^{(\alpha+1)}\},$$

we will call this the *Cantor-Bendixon rank* of Y. If X is scattered then it follows that $\mathbf{rk}(X) = \min\{\alpha : X^{(\alpha)} = \emptyset\}$.

Proposition 12.19 (Hernández-Gutiérrez, [78]) Let $C \subset \mathbb{A}_0$ be compact and scattered. Then there is a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ such that $h[\mathbb{A}_0 \setminus C] = \mathbb{A}_0$.

Proof. We will prove this result by induction on $\mathbf{rk}(C) = \kappa$. Since any Cantor-Bendixon derivative of C is closed in C and C is compact, κ is a successor ordinal $\tau + 1$ and $C^{(\tau)}$ is a finite non-empty set. There are pairwise disjoint clopen intervals $J_0, \ldots, J_m$ of $\mathbb{A}$ such that $|C^{(\tau)} \cap J_i| \leq 1$ for each $i \leq m$ and $\mathbb{A} = J_0 \cup \ldots \cup J_m$. Every clopen interval of $\mathbb{A}$ is of the form $[\langle p, 0\rangle, \langle q, 1\rangle]$ where $0 \leq p < q \leq 1$. So J_i is order-isomorphic (in particular, homeomorphic) to $\mathbb{A}$ for $i \leq m$. Thus, it is enough to define the homeomorphism in each of the intervals J_i for $i \leq m$. For the rest of the proof let us assume without loss of generality that $C^{(\tau)} = \{\langle p, 0\rangle\}$ for some $p \in (0, 1]$.

First assume that $p \neq 1$, let $A_0 = [\langle 0, 1\rangle, \langle p, 0\rangle]$ and $A_1 = [\langle p, 1\rangle, \langle 1, 0\rangle]$. Let $f_0 : [0, p] \to [1/2, 1]$ and $f_1 : [p, 1] \to [0, 1/2]$ be order isomorphisms and define $f : \mathbb{A} \to \mathbb{A}$ by

$$f(\langle q, t\rangle) = \begin{cases} \langle f_0(q), t\rangle & \text{if } \langle q, t\rangle \in A_0, \\ \langle f_1(q), t\rangle & \text{if } \langle q, t\rangle \in A_1. \end{cases}$$

Then f is a homeomorphism, $f[C] \subset \mathbb{A}_0$ and $f(\langle p, 0\rangle) = \langle 1, 0\rangle$. This shows that we may assume that $p = 1$ for the rest of the proof.

Let $\{x_n : n < \omega\} \subset (0, 1)$ be increasing such that $\sup\{x_n : n < \omega\} = 1$. Let $I_0 = [\langle 0, 1\rangle, \langle x_n, 0\rangle]$ and $I_{n+1} = [\langle x_n, 1\rangle, \langle x_{n+1}, 0\rangle]$ for $n < \omega$. Notice that

$$\mathbb{A} = \big(\bigcup\{I_n : n < \omega\}\big) \cup \{\langle 1, 0\rangle\}$$

and I_n is a clopen subset order-isomorphic to $\mathbb{A}$ for each $n < \omega$. Since $C^{(\tau)} = \{\langle 1, 0\rangle\}$, $\mathbf{rk}(C \cap I_n) < \tau$ for each $n < \omega$. Thus, by the inductive hypothesis, there exists a homeomorphism $h_n : I_n \to [\langle 1/(n+2), 1\rangle, \langle 1/(n+1), 0\rangle]$ such that $h_n[(\mathbb{A}_0 \cap I_n) \setminus (C \cap I_n)] = [1/(n+2), 1/(n+1)) \times \{0\}$ for each $n < \omega$. We define $h : \mathbb{A}_0 \to \mathbb{A}_0$ by

$$h = \langle\langle 1, 0\rangle, \langle 0, 1\rangle\rangle \cup \big(\bigcup\{h_n : n < \omega\}\big).$$

It is not hard to see that h is a homeomorphism and $h[\mathbb{A}_0 \setminus C] = \mathbb{A}_0$. This completes the proof. $\square$

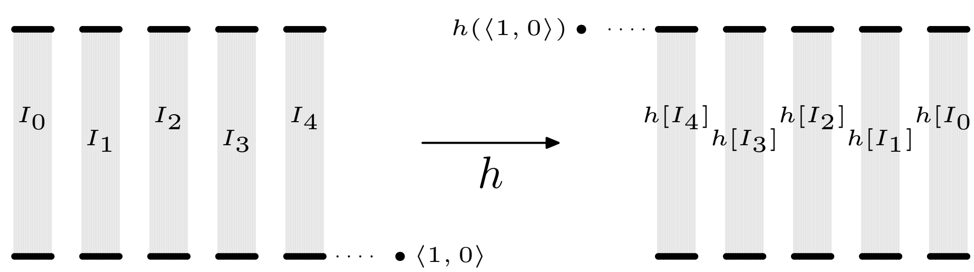

Figure 12.3: Proving Proposition 12.19.

Even if $\mathbb{A}$ is not "compatible" with the Cantor set (Corollary 12.18), it seems from Proposition 12.19 that it may be somehow compatible with compact scattered spaces. The product of the Cantor set with a compact, metrizable and scattered space is homeomorphic to the Cantor set (this follows from [50, 6.2.A(c)]). However, there is not an equivalent theorem with $\mathbb{A}$.

Proposition 12.20 (Hernández-Gutiérrez, [78]) $\mathbb{A} \times (\omega + 1)$ is not homeomorphic to $\mathbb{A}$.

Proof. Assume that there is a homeomorphism $h : \mathbb{A} \times (\omega+1) \to \mathbb{A}$, we will reach a contradiction. It is not hard to prove that every clopen subset of $\mathbb{A}$ is the finite union of clopen intervals. Then for all $n < \omega$ there is $m_n < \omega$ and a collection of disjoint clopen subsets $J(n, 0), \ldots, J(n, m_n)$ of $\mathbb{A}$ such that $h[J(n, i) \times \{n\}]$ is a clopen interval for all $i \leq m_n$. The family

$$\mathcal{U} = \{J(n, i) : n < \omega, i \leq m_n\}$$

is a countable collection of clopen subsets of $\mathbb{A}$. Since $\mathbb{A}$ has weight $\mathfrak{c}$ (Proposition 12.7), $\mathcal{U}$ is not a base of $\mathbb{A}$. From the compactness of $\mathbb{A}$ it is possible to find $x, y \in \mathbb{A}$ with $x \neq y$ and such that $x \in W$ if and only if $y \in W$ for all $W \in \mathcal{U}$.

By the definition of the product topology, $\{\langle x, n\rangle : n < \omega\}$ converges to $\langle x, \omega\rangle$ and $\{\langle y, n\rangle : n < \omega\}$ converges to $\langle y, \omega\rangle$. Let $p = h(\langle x, \omega\rangle)$, we will show that $\{h(\langle y, n\rangle) : n < \omega\}$ converges to p, this contradicts the injectiveness of h and we will have finished.

Assume that $p \in \mathbb{A}_0$, the other case is entirely analogous. It is enough to prove that for every $q \in \mathbb{A}$ with $q < p$ there is an $k < \omega$ with $h(\langle y, k\rangle) \in (q, p]$. Since $h[\mathbb{A} \times \{\omega\}]$ is crowded, there is $r \in h[\mathbb{A} \times \{\omega\}] \cap (q, p)$. By the continuity of h, there exists $k < \omega$ such that $h(\langle x, k\rangle) \in (q, p)$. Let $j \leq m_k$ be such that $\langle x, k\rangle \in J(k, j)$. Since $h[J(k, j) \times \{k\}]$ is an interval that intersects (q, p) and does not contain its endpoints, $h[J(k, j) \times \{k\}] \subset (q, p)$. Then $h(\langle y, k\rangle) \in (q, p)$. This proves that $\{h(\langle y, n\rangle) : n < \omega\}$ converges to p and as discussed above, finishes the proof. $\square$

Since every compact, metrizable and scattered space contains a clopen convergent sequence, we obtain the following.

Corollary 12.21 (Hernández-Gutiérrez, [78]) Let C be compact Hausdorff and countable. Then $\mathbb{A} \times C$ is neither homogeneous nor CDH.

Question 12.22 Let C, D be compact Hausdorff and countable. Is it true that $\mathbb{A} \times C$ is homeomorphic to $\mathbb{A} \times D$ if and only if $\mathbf{rk}(C) = \mathbf{rk}(D)$?

No countable power of $\mathbb{A}$ is CDH (Example 12.29) so one might ask about the finite powers. It turns out that no finite power of $\mathbb{A}$ is CDH, as Proposition 12.24 below shows. The technique we will use is entirely analogous to the original proof of Theorem 12.8. We need a preliminary lemma which can be proved in an analogous way to the proof of Proposition 12.14.

Proposition 12.23 (Hernández-Gutiérrez, [78]) Let $f : \mathbb{A} \to \mathbb{A}$ be a continuous function. Then there exists a collection $\mathcal{U}$ of pairwise disjoint clopen intervals of $\mathbb{A}$ such that $\bigcup \mathcal{U}$ is dense in $\mathbb{A}$ and for every $J \in \mathcal{U}$, $f\restriction_J : J \to \mathbb{A}$ is either non-increasing or non-decreasing.

Proposition 12.24 (Hernández-Gutiérrez, [78]) If $1 \leq n < \omega$, then ${}^n\mathbb{A}$ is not CDH.

Proof. Assume that $n \geq 2$, since the case $n = 1$ is Theorem 12.8. Let $Q = \mathbb{Q} \cap (0,1)$, $D = Q \times \{0\}$ and $E = Q \times \{0,1\}$. Then D and E are countable dense subsets of $\mathbb{A}$ so ${}^nD$ and ${}^nE$ are countable dense subsets of ${}^n\mathbb{A}$. Assume that there is a homeomorphism $h : {}^n\mathbb{A} \to {}^n\mathbb{A}$ such that $h[{}^nE] = {}^nD$, we will reach a contradiction.

Let $d \in D$ and define $X = \{x \in {}^n\mathbb{A} : x(i) = d \text{ for all } 1 \leq i \leq n-1\}$, this is a topological copy of $\mathbb{A}$ and ${}^nE \cap X$ is dense in X. For $i \leq n-1$, let $\pi_i : X \to \mathbb{A}$ be the projection to the i-th coordinate and let $f_i = \pi_i \circ h\restriction_X : X \to \mathbb{A}$, this is a continuous function.

By Proposition 12.23 there is some non-empty interval $J \subset X$ such that $f_i\restriction_J : J \to \mathbb{A}$ is either non-decreasing or non-increasing for each $i \leq n-1$. Notice that it is impossible that f_i is constant on an interval for all $i \leq n-1$ because this would contradict the injectivity of h. Thus, we may assume that $f_j\restriction_J : J \to \mathbb{A}$ is one-to-one for some $j \leq n$.

Let $p, q \in J$ be such that $\pi(p) = \pi(q) \in Q$ so that $p, q \in {}^nE$ and q is the immediate succesor of p in the order of X. By the fact that f_j is strictly increasing, it is not hard to prove that $f_j(p) \in \mathbb{A}_0$ and $f_j(q) \in \mathbb{A}_1$. But $h(q) \in {}^nD$ by the choice of h and $\pi_j[{}^nD] \subset D$ is disjoint from $\mathbb{A}_1$. This is a contradiction. This shows that such a homeomorphism h cannot exist. $\square$

12.3 Results on products

In this Section we will prove that neither $\mathbb{A} \times {}^\omega 2$ nor ${}^\omega\mathbb{A}$ are CDH. Our results are nevertheless more general and place restrictions on products of spaces that are CDH.

Theorem 12.25 (Hernández-Gutiérrez, [78]) Let X and Y be two crowded spaces of countable π-weight. If $X \times Y$ is CDH, then X contains a subset homeomorphic to ${}^{\omega}2$ if and only if Y contains a subset homeomorphic to ${}^{\omega}2$.

Proof. Assume that X contains a subspace homeomorphic to the Cantor set and Y does not, we shall arrive to a contradiction. Since $X \times Y$ contains a Cantor set, there is a countable dense subset $D \subset X \times Y$ and $Q \subset D$ such that $\mathrm{cl}_{X\times Y}(Q) \approx {}^{\omega}2$. We shall construct a countable dense subset $E \subset X \times Y$ that does not have this property.

Let $\mathcal{B} = \{U_n \times V_n : n < \omega\}$ be a π-base of the product $X \times Y$, where U_n is open in X and V_n is open in Y for each $n < \omega$. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections. Recursively, choose $\{e_n : n < \omega\} \subset X \times Y$ such that $\pi_X(e_n) \in U_n$ and

$$\pi_Y(e_n) \in V_n \setminus \{\pi_Y(e_0), \ldots, \pi_Y(e_{n-1})\}$$

for all $n < \omega$. Let $E = \{e_n : n < \omega\}$. Thus, $\pi_Y{\restriction}_E \colon E \to Y$ is one-to-one.

Assume that there is some autohomeomorphism of $X \times Y$ that takes D to E. Then there exists $R \subset E$ such that $K = \mathrm{cl}_{X\times Y}(R)$ is homeomorphic to ${}^{\omega}2$. Notice that $T = \pi_Y[K]$ is a compact subset of Y of countable weight. Since Y does not contain topological copies of the Cantor set, T is scattered. Then T contains an isolated point p. Since $(X \times \{p\}) \cap K$ is a clopen subset of K, $\pi_Y{\restriction}_R \colon R \to Y$ is one to one and R is dense in K, we obtain that $(X \times \{p\}) \cap K$ is a singleton. But K is crowded so this is a contradiction. Thus, the theorem follows. $\square$

We immediately obtain the following from Theorem 12.25 and Proposition 12.17. This answers Question 12.9.

Corollary 12.26 (Hernández-Gutiérrez, [78]) $\mathbb{A} \times {}^{\omega}2$ is not CDH.

Corollary 12.28 gives a necessary condition on X for ${}^{\omega}X$ to be CDH. It is the first criterion of this kind that works for non-metrizable spaces. See also the discussion in Example 12.29(c). This contrasts with the following result of Alan Dow and Elliott Pearl.

Theorem 12.27 [47] If X is regular, first countable and 0-dimensional, then ${}^{\omega}X$ is homogeneous.

Corollary 12.28 (Hernández-Gutiérrez, [78]) Let Z be a crowded space of countable π-weight. If ${}^{\omega}Z$ is CDH, then Z contains a subspace homeomorphic to ${}^{\omega}2$.

Proof. Assume that Z contains no subspace homeomorphic to ${}^{\omega}2$. Let $X = Z$ and $Y = {}^{\omega}Z$. Then both X and Y are crowded spaces of countable π-weight. Notice that Z has at least two points so Y contains a topological copy of ${}^{\omega}2$. By Theorem 12.25, we obtain a contradiction. □

We finally present some examples.

Example 12.29 (Hernández-Gutiérrez, [78])

(a) If $\mathbb{Q}$ is the space of rational numbers, ${}^{\omega}\mathbb{Q}$ is not CDH, this was first shown by Fitzpatrick and Zhou ([56]) and obviously follows from Theorem 10.32.

(b) If $\mathbb{S}$ is the Sorgenfrey line, $\mathbb{S}$ is CDH by Example 10.11. However, ${}^{\omega}\mathbb{S}$ is not CDH.

(c) By Theorem 10.32, any separable and metrizable space X such that ${}^{\omega}X$ must be a Baire space. Thus, a Bernstein set is a natural candidate for such a space. However, by Corollary 12.28, the ω-power of a Bernstein set is not CDH. Thus, we have obtained additional conditions for spaces to have their ω-power CDH. In fact, the only (consistently) known non-Borel spaces X such that ${}^{\omega}X$ is CDH are filters (Theorem 11.7 of Medini and Milovich and Theorem 11.12) and filters always contain Cantor sets (see the proof of Proposition 11.10).

(d) Finally, by Proposition 12.17, ${}^{\omega}\mathbb{A}$ is not CDH, as announced. □

Notice that Theorem 12.25 applies to first countable separable spaces. This observation is important since any absolute example of a CDH compact space of uncountable weight must be first countable by the following result of Arhangel'skiĭ and van Mill.

Theorem 12.30 [9] Under $2^{\omega} < 2^{\omega_1}$, every CDH compact space is first countable.

So the existence of compact CDH spaces of uncountable weight is still open. The presence of Cantor sets in spaces whose ω-power is CDH may shed some light on this problem. We end the discussion with a related problem.

Question 12.31 Let X be a compact CDH space. Does it follow that X must contain topological copies of the Cantor set?

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