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THE RUIN PROBABILITY AND THE EXPECTED UTILITY FOR INSURANCE COMPANIES IN THE PRESENCE OF STOCHASTIC VOLATILITY ON INVESTMENTS

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Dedicado a

mis padres y a mis hermanos.

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Introducción

La teoría de riesgo en general y la probabilidad de ruina en particular son partes de las matemáticas de los seguros que tratan con modelos estocásticos de una compañía de seguros. Para más detalles en teoría de riesgo, buenas referencias incluyen a Asmussen (2000) y Rolski et al. (1999). En años recientes se ha autorizado a la comapñías aseguradoras a invertir parte de su reserva en activos con riesgo, de donde surge el problema de encontrar una estrategia *óptima* de inversión. Siguiendo la teoría clásica de riesgo actuarial el problema que se plantea es el encontrar la estrategia (*si existe*) que minimiza la probabilidad de ruina. Por otro lado, tenemos la teoría económica que considera a las llamadas funciones de utilidad como un enfoque adecuado para este problema, en este caso, lo que se pretende es encontrar la estrategia (*si existe*) que maximiza la utilidad esperada. Ambos puntos de vista son parte de la teoría del riesgo y del control estocástico.

El estudio de la probabilidad de ruina tiene una historia larga, la cual inició con los artículos clásicos de Cramér y Lundberg, quienes fueron los primeros en considerar el problema de la ruina de una compañía de seguros. Cramér y Lundberg encontraron que la teoría de procesos estocásticos proporciona el enfoque más apropiado para modelar reclamos en una compañía de seguros. En 1903, Lundberg introdujo un modelo simple capaz de describir la dinámica básica del portafolio de una aseguradora, desde entonces se ha llevado a cabo mucho trabajo en esta área. Uno de los resultados más conocidos para el proceso de riesgo afirma que la probabilidad de ruina como función de la riqueza inicial $\psi(x)$ esta acotada como sigue (véase [1]):

$$Ce^{-\nu x} \leq \psi(x) \leq e^{-\nu x},$$

Este resultado significa que la probabilidad de ruina decrece de forma exponencial con respecto a la riqueza inicial. Hipp y Plum (2000), consideraron el proceso clásico de riesgo con la oportunidad de invertir en un activo con riesgo modelado por el movimiento Browniano geométrico. La probabilidad de ruina en este caso se minimiza escogiendo una estrategia de inversión adecuada la cual se obtiene utilizando la ecuación de Hamilton-Jacobi-Bellman. Gaier et al. (2003) obtuvieron una estimación de la probabilidad de ruina de tipo exponencial con una tasa que mejora el parámetro clásico de Lundberg. Esta tasa fue obtenida proponiendo una estrategia que consiste en invertir una cantidad constante de dinero en el activo con riesgo. Hipp y Schmidli (2004) demostraron que esta estrategia es asintóticamente óptima cuando la riqueza inicial tiende a infinito. Por otro lado, el estudio de la función de utilidad esperada ha sido muy importante tanto en finanzas como en riesgo. Ferguson (1965) conjeturó que maximizar la utilidad exponencial de una riqueza terminal está estrictamente relacionado con minimizar la probabilidad de la ruina. Ferguson estudio el problema del valor esperado de la utilidad de la riqueza para un modelo discreto para el inversionista. Browne (1965) verificó la conjetura de Ferguson para un proceso de riesgo modelado por el movimiento Browniano con deriva y con la posibilidad de inversión en un activo con riesgo modelado por el movimiento Browniano geométrico, sin considerar la cuenta de banco. Browne concluyó que la estrategia óptima que minimiza la probabilidad de ruina es también óptima en el problema de maximizar la utilidad exponencial de una riqueza terminal. En la presencia de la cuenta de banco esta equivalencia no se satisface. Yang y Zhang (2005) consideraron el modelo clásico de riesgo perturbado por el movimiento Browniano. Al asegurador se le permite invertir en la cuenta de banco y en el activo con riesgo. Yang y Zhang obtuvieron una expresión cerrada de la estrategia óptima cuando la función de utilidad es exponencial. Fernández el al. (2008) consideraron el modelo de riesgo con la posibilidad de inversión en la cuenta de banco y en el activo con riesgo modelado por el movimiento Browniano geométrico. Utilizando el enfoque Hamilton- Jacobi-Bellman, encontraron la estrategia óptima cuando las preferencias son exponenciales y obtuvieron una forma cerrada de la solución. El caso en el que el proceso de reclamaciones está modelado por un proceso de saltos puros y el asegurador tiene la posibilidad de invertir en múltiples activos con riesgo sin la posibilidad de una cuenta de banco fue estudiado por Wang (2008). Wang encontró que la estrategia óptima al maximizar la utilidad exponencial consiste en invertir una cantidad constante de dinero en cada activo con riesgo. Mientras que para obtener la estrategia óptima de reaseguro de la cedente, Guerra y Centeno (2008) estudiaron la relación entre maximizar el coeficiente de ajuste y maximizar el valor esperado de la utilidad exponencial de la riqueza.

El presente trabajo se encuentra organizado como sigue. El capítulo 1 se dedica al repaso de los resultados clásicos de la teoría de riesgo. Se introduce el modelo clásico de riesgo y se muestra la importancia de la teoría de ecuaciones diferenciales en modelar la probabilidad de ruina para un asegurador con la posibilidad de incluir pagos de reaseguro y dividendos. Finalmente, en la última sección del capítulo 1 introducimos el problema de la probabilidad de ruina cuando al asegurador se le permite invertir en un activo con riesgo. Se demuestran los resultados más conocidos tanto para la probabilidad de ruina como para la función de utilidad esperada. En capítulo 2 se introduce el problema de la presente tesis. Se presenta el primer resultado importante relacionado con nuestro trabajo de investigación, el cual fue obtenido por Rubio (2010). Este resultado consiste en un teorema de verificación, el cual relaciona la probabilidad de ruina con la ecuación de Hamilton-Jacobo-Bellman. En el capítulo 3 obtenemos una cota superior una cota inferior para la probabilidad de ruina vía expresiones generales de martingalas exponenciales. Para la cota superior demostramos un teorema con condiciones generales para la existencia de una función θ, el cual afirma que la cota superior es de la forma $e^{-\theta(z)x}$. Ilustramos este teorema por el modelo truncado de Scott y consideramos dos casos de funciones θ, para las cuales la estrategia K y la cota superior son obtenidas de forma explícita. Finalmente, obtenemos la cota inferior bajo las hipótesis usuales de reclamaciones con distribuciones con momento exponencial uniforme (véase [13]). Los resultados de este capítulo están publicados en Badaoui y Fernández (véase [2]). En el capítulo 4, proporcionamos un teorema de verificación para el problema de optimización que relaciona la función de costo con la ecuación de Hamilton-Jacobi-Bellman, el cual se demuestra utilizando la teoría de martingalas y la formula de Itô. También se demuestra un teorema de existencia y unicidad cuando las preferencias del asegurador son de tipo exponencial. Se obtiene una solución explícita para la ecuación diferencial parcial. Como consecuencia se obtiene una forma explícita de la estrategia óptima, la cual depende únicamente del factor externo y el tiempo. Para desarrollar algunos resultados numéricos demostramos la consistencia y la estabilidad del esquema numérico. Se demuestra que el problema de Cauchy está bien planteado para completar las condiciones del teorema Lax de la convergencia. Presentamos resultados para el modelo de Scott cuando las reclamaciones son de tipo exponencial. Finalmente, estudiamos la reserva de la compañía de seguros bajo la estrategia óptima y se prueba la propiedad de supermartingala del proceso de riesgo para obtener una cota superior de la probabilidad de ruina en horizonte finito.

En el capítulo 5, se presentan algunas conclusiones y líneas de investigación futuras. Finalmente incluimos en el Apéndice A algunos resultados sobre ecuaciones diferenciales parciales y ecuaciones diferenciales estocásticas y en el Apéndice B algunos resultados sobre el método de Diferencias Finitas.

Introduction

Risk theory in general and ruin probabilities in particular are the part of insurance mathematics that deal with the stochastic models of an insurance business. For more details on risk theory, good references include Assmussen (2000) and Rolski et al. (1999). In recent years, insurance companies are allowed to invest part of their wealth in risky assets, this gives rise to the problem of looking for an *optimal* strategy of investment. Following the classical theory of actuarial risk the problem is looking for a strategy (*if it exists*) that minimize the ruin probability. On the other hand, we have the economic theory which consider the so called utility functions as an appropriate approach for this problem, in this case the intention is to find the optimal strategy (*if it exists*) which maximize the expected utility. Both views are parts of risk theory and stochastic control theory.

The study of ruin probabilities has a long history that started with the classical papers of Cramér and Lundberg, which first considered the ruin problem of an insurance company. They realized that the theory of stochastic processes provides the most appropriate framework for modeling claims in an insurance business [27]. In 1903, Lundberg introduced a simple model which is capable of describing the basic dynamic of an insurance portfolio, and ever since, much work has been carried out in this area. A well-known fact in risk theory is that the probability of ruin as a function of initial wealth $\psi(x)$ is bounded for the classical risk process as follows (see [1]):

$$Ce^{-\nu x} \leq \psi(x) \leq e^{-\nu x},$$

which means that the probability of ruin decreases exponentially with respect to the initial wealth.
Hipp and Plum (2000) consider the classical risk process with the opportunity to invest in a risky asset modeled by a geometric Brownian motion. The probability of ruin in this case is minimized by the choice of a suitable investment strategy, which is determined by using the Hamilton-Jacobi-Bellman equation. Gaier et al. (2003) obtained an estimate for the ruin probability of exponential type with a rate that improves the classical Lundberg parameter, by proposing a strategy that consists in investing a constant amount of money in the risky asset. Hipp and Schmidli (2004) showed that this strategy is asymptotically optimal as the initial wealth tends to infinity.
On the other hand, the study of the expected utility function has been very important in both finance and insurance. Ferguson (1965) conjectured that maximizing exponential utility from terminal wealth is strictly related to minimizing the probability of ruin. Ferguson studied the problem of expected utility of wealth under a discrete model for the investor. Browne (1995) verified the Ferguson conjecture for a risk process modeled by a Brownian motion with drift, with the possibility of investment in a risky asset which follows a geometric Brownian motion, but without a risk-free interest rate. He concluded that the optimal strategy that minimizes the probability of ruin is also optimal in maximizing the exponential utility of terminal wealth. In the presence of a positive interest rate, this equivalence does not hold. Yang and Zhang (2005) considered the classical risk model perturbed by a standard Brownian motion. The insurer is allowed to invest in the money market and a risky asset. They obtained a closed form expression of the optimal strategy when the utility function is exponential. Fernández et al.

(2008) considered the risk model with the possibility of investment in the money market and a risky asset modeled by a geometric Brownian motion. Via the Hamilton-Jacobi-Bellman approach, they found the optimal strategy when the insurer's preferences are exponential. In this case as well, a closed form solution is given. The optimal strategy is then used to get an estimate of the ruin probability. The case in which the claim process is a pure jump process and the insurer has the option of investing in multiple risky assets without the risk-free option was studied by Wang (2007). Wang found that the optimal strategy of maximizing the exponential utility of terminal wealth consists in putting a fixed amount of money in each risky asset, while to get the optimal reinsurance from the ceding company, Guerra and Centeno (2008) studied the relationship between maximizing the adjustment coefficient and maximizing the expected utility of wealth for the exponential utility function.

The organization of this dissertation is as follows. Chapter 1 is dedicated to recalling the classical results of risk theory. We introduce the classical risk model and we show the importance of differential equations theory in modeling the probability of ruin for the insurer with the possibility of including reinsurance and dividend payments. Finally, in the last section of Chapter 1 we introduce the problem of ruin probabilities when the insurer is allowed to invest in a risky asset. We show the most well-known results obtained for both the ruin probability and the expected utility function. Chapter 2 introduces our model. We present the first important result related to our research problem, which was obtained by Rubio (2010). This result consists in a verification theorem which relates the ruin probability with the Hamilton-Jacobi-Bellman equation. In Chapter 3 we obtain upper and lower bounds for the ruin probability by giving general expressions for exponential martingales. For the upper bound we prove a theorem with general and abstract conditions for the existence of a function θ that guarantees that the upper bound is of the form $e^{-\theta(z)x}$. We illustrate this theorem with a truncated Scott models, and we consider two cases of functions θ, for which a strategy K and the bound are explicitly obtained. Finally, we obtain the lower bound under the usual hypothesis of uniform exponential moment of the tail distribution of the claims (see [13]). The results of this chapter are published in Badaoui and Fernández [2]. In chapter 4, we provide a verification theorem for the optimization problem which relates the value function with the Hamilton-Jacobi-Bellman equation, which is proven by using martingale theory and Itô's formula. We also prove an existence and uniqueness theorem when the insurer's preferences are exponential, and we obtain an explicit solution for the partial differential equation (PDE). Consequently, an explicit form for the optimal strategy is obtained, which depends only on the external factor and time. To develop some numerical results, we prove consistency and stability of the explicit scheme. The well-posedness of the Cauchy problem is proven to complete the conditions of the Lax theorem for convergence. We present results for the Scott model when claim-size is exponentially distributed. Finally, we study the reserve of the insurance company under the optimal strategy, and we prove a supermartingale property of the risk process to get an upper bound for the ruin probability in finite horizon.

In Chapter 5, we present some conclusions and further research lines. Finally we include in Appendix A some results about Partial Differential Equations and Stochastic Differential Equations and in Appendix B some results about the Finite Difference Method.

Chapter 1

Non-Life Insurance

1.1 The Classical Risk Model

The traditional approach in ***risk theory*** is to consider a model of the risk business of an insurance company and study the probability of ruin, i.e., the probability that the risk business will ever be below some specified (negative) value. We start by formulating the usual risk model. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. The Cramér-Lundberg process or classical risk process X, is defined as

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i, \tag{1.1}$$

where x is the initial capital, c is the premium rate, $\{N_t\}$ is a Poisson process with rate λ, and the claims $\{Y_i\}$ are i.i.d. and independent of $\{N_t\}$, having a common distribution function G with $G(0) = 0$, and $\mu = \mathbb{E}[Y_i]$.
We denote the moment generating function by $M_Y(r) = \mathbb{E}[e^{rY_i}]$. The claim times are denoted by $T_1 < T_2 < \dots$, and $\forall\, i \geq 1$, T_i is $\Gamma(i, \lambda)$ distributed. The inter occurrence times (i.e., the time periods between claims) are exponentially distributed with parameter $\lambda > 0$. The following figure shows the sample path of X_t. A classical risk process $\{X_t\}$ as defined above, is a model

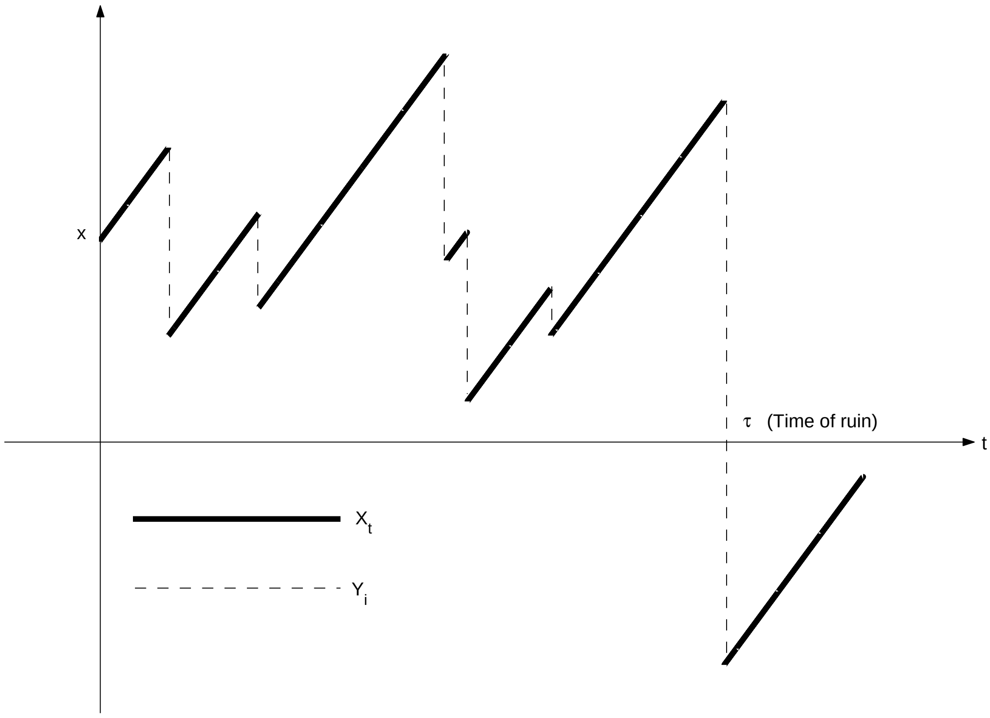

Figure 1.1: The classical risk process and claims

for the time evolution of the reserves of an insurance company. A possible measure of risk is the probability of ruin. Let

$$\tau = \begin{cases} \inf\{t > 0, X_t < 0\}, & \text{if } \{t > 0, X_t < 0\} \neq \emptyset, \\ \infty, & \text{in other case.} \end{cases}$$

Then the ruin probability is defined as $\psi(x) = \mathbb{P}[\tau < \infty]$. Sometimes it is convenient to use the survival probability $\delta(x) = 1 - \psi(x) = \mathbb{P}[\tau = \infty]$. Another important condition is $c > \lambda\mu$, called the net profit condition, which means that per unit of time the premium income exceeds the expected aggregate claim amount given by

$$\mathbb{E}\left[\sum_{i=1}^{N_t} Y_i\right] = \lambda\mu t.$$

It can also be shown that if the net profit condition does not hold, then $\psi(x) = 1$ for all $x \geq 0$.

Observation 1.1 *The reason to choose a classical risk process as a model of insurance risk is mathematical, owing to the nice properties of the Compound Poisson process from which many results often follow easily. However, the model also admits a natural interpretation, by assuming that the insurer starts with an initial amount of money, collects premium and pays claims. If a problem cannot be solved for the Cramér-Lundberg model, then there is no chance to do so for more realistic models. In the following theorem, we prove that $\delta(x)$ satisfies an integro-differential equation which can be found in Grandell (1991).*

Theorem 1.1 *The survival probability $\delta(x)$ is continuous and differentiable everywhere except for a countable set, where G is not continuous. Moreover $\delta(x)$ satisfies the following integro-differential equation:*

$$c\delta'(x) = \lambda \left[\delta(x) - \int_0^x \delta(x-y)\, dG(y) \right]. \tag{1.2}$$

Proof. Since the Poisson process is a renewal process, let h be small. Then:

$$\begin{aligned}
\delta(x) &= \mathbb{E}[\mathbb{P}[\tau = \infty \mid X_{T_1 \wedge h}]] \\
&= \mathbb{E}[\mathbb{I}_{T_1 \geq h}\mathbb{P}[\tau = \infty \mid X_{T_1 \wedge h}]] + \mathbb{E}[\mathbb{I}_{T_1 < h}\mathbb{P}[\tau = \infty \mid X_{T_1 \wedge h}]] \\
&= \mathbb{E}[\mathbb{I}_{T_1 \geq h}\delta(x+ch)] + \mathbb{E}[\mathbb{I}_{T_1 < h}\delta(x + cT_1 - Y_1)] \\
&= e^{-\lambda h}\delta(x+ch) + \int_0^\infty \int_0^{x+ct} \delta(x+ct-y)\lambda e^{-\lambda t}\, dG(y)\, dt
\end{aligned}$$

then

$$\begin{aligned}
\frac{\delta(x) - \delta(x)e^{-\lambda h}}{h} &= ce^{-\lambda h}\frac{\delta(x+ch) - \delta(x)}{ch} + \\
&\quad \frac{1}{h}\int_0^\infty \int_0^{x+ct} \delta(x+ct-y)\lambda e^{-\lambda t}\, dG(y)\, dt.
\end{aligned}$$

By letting $h \to 0$, we obtain

$$\lambda\delta(x) = c\delta'(x) + \lambda \int_0^x \delta(x-y)\, dG(y),$$

which leads to (1.2). □

Observation 1.2 *In general* (1.2) *cannot be solved analytically, but we can compute the survival probability in* (1.2) *numerically. A solution to* (1.2) *can be found using the Laplace transform (see [31]), but it is extremely difficult to find a closed form solution for* $\delta(x)$, *because the inverse of the Laplace transform cannot be found in the general case.*

Now by using (1.2), we get a closed form for $\psi(0)$ and $\delta(0)$. We have:

$$\begin{aligned}
\frac{c}{\lambda}(\delta(x) - \delta(0)) &= \frac{1}{\lambda}\int_0^x c\delta'(s)\, ds \\
&= \int_0^x \delta(s)\, ds - \int_0^x \int_0^s \delta(s-y)\, dG(y)\, ds \\
&= \int_0^x \delta(s)\, ds - \int_0^x \int_y^x \delta(s-y)\, ds\, dG(y) \\
&= \int_0^x \delta(s)\, ds - \int_0^x \int_0^{x-y} dG(y)\, \delta(x) dx \\
&= \int_0^x \delta(x-s)(1-G(s))\, ds.
\end{aligned}$$

Then

$$\frac{c}{\lambda}(\delta(x) - \delta(0)) = \int_0^x \delta(x-s)(1-G(s))\, ds.$$

Letting $x \to \infty$, then by the bounded convergence theorem we have:

$$\frac{c}{\lambda}(1 - \delta(0)) = \int_0^\infty (1-G(s))\, ds = \mu,$$

then it is easy to obtain:

$$\delta(0) = 1 - \frac{\lambda\mu}{c}, \quad \psi(0) = \frac{\lambda\mu}{c}.$$

In the following example, we produce an explicit formula for the ruin probability when the claim sizes Y_i are exponentially distributed. This result was first established by Lundberg (1903).

EXAMPLE 1.1 Let the claims be Exp(α) distributed. Then by equation (1.2), we have:

$$c\delta'(x) = \lambda \left[\delta(x) - e^{-\alpha x} \int_0^x \delta(y) \alpha e^{\alpha y}\, dy \right]. \tag{1.3}$$

Differentiating (1.3) yields

$$c\delta''(x) = \lambda \left[\delta'(x) + \alpha e^{-\alpha x} \int_0^x \delta(y) \alpha e^{\alpha y}\, dy - \alpha \delta(x) \right] = \lambda \delta'(x) - \alpha c \delta'(x),$$

thus producing the following ordinary differential equation:

$$\delta''(x) = \left(\frac{\lambda}{c} - \alpha \right) \delta'(x).$$

The solution is given by:

$$\delta(x) = A + Be^{(\frac{\lambda}{c} - \alpha)x}.$$

Since $\delta(0) = 1 - \dfrac{\lambda}{\alpha c}$, the solution is given by:

$$\delta(x) = 1 - \frac{\lambda}{\alpha c} e^{(\frac{\lambda}{c} - \alpha)x} \text{ and } \psi(x) = \frac{\lambda}{\alpha c} e^{(\frac{\lambda}{c} - \alpha)x},$$

which gives a closed form for the survival probability and the ruin probability.

We know that in general (1.2) does not have an explicit solution, but an upper bound for the ruin probability can be obtained. We begin by introducing a number $R > 0$, called ***the adjustment coefficient or the Lundberg exponent***. For the classical risk process, R is defined as the unique positive root of

$$\theta(r) = 0,$$

where

$$\theta(r) = \lambda \left(M_Y(r) - 1 \right) - cr.$$

Provided that for some quantity γ, $0 < \gamma \leq \infty$, $M_Y(r)$ is finite for all $r < \gamma$ with $\lim_{r \to \gamma -} M_Y(r) = \infty$ (see [16]), then this is a technical condition which we require for a turning point to exist, because θ is a decreasing function at zero and convex.

Observation 1.3 *If the claims are exponentially distributed, we can produce an explicit value for R, but in other cases we must solve $\theta(r) = 0$ numerically.*

The following theorem is considered a connection between the adjustment coefficient and the ruin probability, which gives an upper bound for the latter. This upper bound is attributed to Lundberg.

Theorem 1.2

$$\psi(x) < e^{-Rx}, \forall x \geq 0, \tag{1.4}$$

where R is the adjustment coefficient.

The proof of Theorem 1.2 will be a consequence of the next lemma.

Lemma 1.1 *Let $r \in \mathbb{R}$ such that $M_Y(r) < \infty$, then the stochastic process*

$$\left\{e^{-rX_t - \theta(r)t}\right\}_{t \geq 0}$$

is a martingale.

Proof. By the independent increments property of the compound Poisson process we get:

$$\begin{aligned} \mathbb{E}[e^{-rX_t-\theta(r)t} \mid \mathcal{F}_s] &= \mathbb{E}[e^{-r(X_t-X_s)}]e^{-rX_s-(\lambda M_Y(r)-1)t+crs} \\ &= \mathbb{E}[e^{r\sum_{i=N_s+1}^{N_t} Y_i}]e^{-rX_s-\lambda(M_Y(r)-1)t+crs} \\ &= e^{-rX_s-\lambda(M_Y(r)-1)s+crs} = e^{-rX_s-\theta(r)s}. \end{aligned}$$

Therefore:
$\mathbb{E}[e^{-rX_t-\theta(r)t} \mid \mathcal{F}_s] = X_s$, which finishes the proof of the lemma. □

Proof. (**Theorem 1.2**) Using Lemma 1.1 for $r = R$, then by the stopping time theorem we have:

$$e^{-Rx} = e^{-RX_0} = \mathbb{E}[e^{-RX_{\tau\wedge t}}] \geq \mathbb{E}[e^{-RX_{\tau\wedge t}}; \tau \leq t] = \mathbb{E}[e^{-RX_\tau}; \tau \leq t].$$

By the monotone convergence theorem and letting t tend to infinity, we get:

$$e^{-Rx} \geq \mathbb{E}[e^{-RX_\tau}; \tau < \infty] > \mathbb{P}[\tau < \infty] = \psi(x).$$

□

OBSERVATION 1.4 *The lower bound for the ruin probability is given by*

$$\psi(x) \leq Ce^{-Rx},$$

for more details, see [1, 31].

In the next theorem, we present an asymptotic behavior for the ruin probability.

THEOREM 1.3 *(The Cramér-Lundberg approximation) Assume that the adjustment coefficient exists and that* $\frac{\lambda}{c}\int_0^\infty xe^{Rx}(1-G(x))\,dx < \infty$. *Then:*

$$\lim_{x\to\infty} \psi(x)e^{Rx} = \frac{c-\lambda\mu}{\lambda M_Y'(R)-c} \tag{1.5}$$

Proof. Let R be the adjustment coefficient. Then by (1.2), we have

$$\psi(x)e^{Rx} = e^{Rx}\int_x^\infty \frac{\lambda}{c}(1-G(y))\,du + \int_0^x \psi(x-y)e^{R(x-y)}e^{Rx}\frac{\lambda}{c}(1-G(y))\,dy,$$

which is a renewal equation. Then by the renewal theorem, [11] we get:

$$\lim_{x\to\infty} \psi(x)e^{Rx} = \kappa\int_0^\infty e^{Rx}\int_x^\infty \frac{\lambda}{c}(1-G(y))\,dy\,dx,$$

where

$$\begin{aligned}
\kappa &= \int_0^\infty xe^{Rx}\frac{\lambda}{c}(1-G(x))\,dx \\
&= \frac{\lambda}{c}\int_0^\infty\int_x^\infty xe^{Rx}\,dG(y)\,dx \\
&= \frac{\lambda}{c}\int_0^\infty\int_0^y xe^{Rx}\,dx\,dG(y) \\
&= \frac{\lambda}{cR^2}\int_0^\infty (Rye^{Ry}-e^{Ry}+1)\,dG(y) \\
&= \frac{\lambda RM_Y'(R)-cR}{cR^2} = \frac{\lambda M_Y'(R)-cR}{c},
\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty e^{Rx}\int_x^\infty \frac{\lambda}{c}(1-G(y))\,dy\,dx &= \frac{\lambda}{c}\int_0^\infty\int_0^y e^{Rx}dx(1-G(y))\,dy\\ &= \frac{\lambda}{cR}\int_0^\infty (e^{Ry}-1)(1-G(y))\,dy\\ &= \frac{1}{cR}(c-\lambda\mu).\end{aligned}$$

Then the limit value follows easily from this expression. □

Thus for large values of x, we obtain an approximation to $\psi(x)$. This approximation is called the ***Cramér-Lundberg approximation.***

1.2 Reinsurance and Ruin

In this section, we study the ruin problem for the classical risk model that involves reinsurance. For most insurance companies, the premium volume is not big enough to carry the risk, especially in the case of subexponential claims (see Appendix A). Therefore the insurers try to share a part of the risk with other companies. Assume that the company has the possibility of buying proportional reinsurance. Then let b_t be the retention level in force at time t. This means that for a claim Y happening at time t the insurer pays b_tY and the reinsurer covers $(1-b_t)Y$. As compensation for the risk, the insurer has to pay a premium at a rate of $c - c(b)$, and thus the premium rate left for the insurer is $c(b)$. We assume that $c(b)$ is increasing, continuous, $c(1) = c$, and that the insurer cannot insure the whole portfolio ($c(0) < 0$). We restrict b_t to be smaller than one because $b_t > 1$ would not be realistic. Let the filtration $\mathcal{F}_t$ generated by X_t^1 and X_t^1 be the Cramér-Lundberg process. Schmidli (2001), among other authors (see [20, 22, 23]), included the reinsurance program in the classical risk model. Schmidli (2001) considered the classical risk process with reinsurance as follows:

Definition 1.1 *A* ***reinsurance strategy*** *is an adapted process b_t with val-*

ues in $[0,1]$*. The risk process then becomes:*

$$X_t^b = x + \int_0^t c(b_s)\, ds - \sum_{i=1}^{N_t} b_{T_i} Y_i \tag{1.6}$$

For the risk process given by (1.6), ruin time is:

$$\tau = \tau_b = \inf\left\{t \geq 0 : X_t^b < 0\right\}$$

and the survival probability is

$$\delta_b(x) = \mathbb{P}[\tau_b = \infty \mid X_0^b = x].$$

Our aim is to minimize the ruin probability or to find the optimal value

$$\delta(x) = \sup_{b_t} \delta_b(x),$$

and finally to prove that $\delta(x)$ satisfies the following equation:

$$f'(x) = \inf_{b\in(\bar{b},1]} \frac{\lambda}{c(b)} \left[f(x) - \int_0^{\frac{x}{b}} f(x-by)\, dG(y) \right], \tag{1.7}$$

where

$$\bar{b} = \inf\left\{b \in [0,1] : c(b) > 0\right\}.$$

Equation (1.7) is called the Hamilton-Jacobi-Bellman (HJB) equation. Now we are going to give a motivation to study the HJB equation (1.7). For $b \in [0,1]$ and small h, we have:

$$\begin{aligned}
\delta(x) &= \mathbb{E}[\mathbb{P}[\tau = \infty \mid X_{T_1\wedge h}]] \\
&= \mathbb{E}[\mathbb{I}_{T_1\geq h}\mathbb{P}[\tau = \infty \mid X_{T_1\wedge h}]] + \mathbb{E}[\mathbb{I}_{T_1<h}\mathbb{P}[\tau = \infty \mid X_{T_1\wedge h}]] \\
&= \mathbb{E}[\mathbb{I}_{T_1\geq h}\delta(x+ch)] + \mathbb{E}[\mathbb{I}_{T_1<h}\delta(x+cT_1-Y_1)] \\
&= e^{-\lambda h}\delta(x+ch) + \int_0^\infty \int_0^{x+c(b)t/b} \delta(x+c(b)t-y)\lambda e^{-\lambda t}\, dG(y)\, dt,
\end{aligned}$$

then

$$\begin{aligned}\frac{\delta(x)-\delta(x)e^{-\lambda h}}{h} &= c(b)e^{-\lambda h}\frac{\delta(x+ch)-\delta(x)}{c(b)h}+ \\ & \frac{1}{h}\int_0^\infty \int_0^{x+c(b)t/b} \delta(x+c(b)t-y)\lambda e^{-\lambda t}\,dG(y)\,dt.\end{aligned}$$

By letting $h \to 0$, we obtain

$$\lambda\delta(x) = c(b)\delta'(x) + \lambda\int_0^x \delta(x-by)\,dG(y)$$

which leads to the HJB equation (1.7).

The main result in this section is the following theorem.

THEOREM 1.4 *Let $f(x)$ be the unique solution to equation* (1.7) *with* $f(0) = 1$*. Then* $\delta(x) = \dfrac{f(x)}{f(\infty)}$*. The strategy* $(b^*(X_t))$ *is optimal, where* $b^*(x)$ *is an argument that minimizes the right hand side of* (1.7)*.*

THEOREM 1.5 *There exists a unique solution for* (1.7) *with f(0)=1. The solution is bounded, strictly increasing and continuously differentiable.*

Proof. It is easy to show that

$$f(x) - \int_0^{\frac{x}{b}} f(x-by)\,dG(y) = 1 - G(x/b) + \int_0^x f'(z)(1-G((x-z)/b)\,dz,$$

then let $\mathscr{V}$ be the operator acting on strictly positive functions g via

$$\mathscr{V}g(x) = \inf_{b\in(\bar{b},1]} \frac{\lambda}{c(b)}\left[1-G(x/b)+\int_0^x g(z)(1-G((x-z)/b)\,dz\right].$$

Now we show the existence of a solution. Let $\delta_1(x)$ be the survival probability without reinsurance, then by (1.2) we have

$$\begin{aligned}\delta_1'(x) &= \frac{\lambda}{c}\left[\delta_1(x) - \int_0^x \delta_1(x-y)\,dG(y)\right] \\ &= \frac{\lambda}{c}\left[1-G(x)\delta(0)+\int_0^x \delta_1'(z)(1-G(x-z))\,dz\right].\end{aligned}$$

Let $g_0(x) = \dfrac{c\delta_1'(x)}{\lambda\mu}$ and define $g_n(x) = \mathscr{V} g_{n-1}(x)$ recursively, then

$$g_1(x) \le g_0(x).$$

To show that $g_n(x)$ is decreasing in n, we suppose that $g_{n-1}(x) \ge g_n(x)$ and let b_n be the point for which $\mathscr{V} g_{n-1}(x)$ attains the minimum. Then

$$\begin{aligned} g_n(x) - g_{n-1}(x) &= \mathscr{V} g_{n-1}(x) - \mathscr{V} g_n(x) \\ &\ge \frac{\lambda}{c(b_n)} \int_0^x g_{n-1}(z)(1 - G((x-z)/b_n)\,dz \\ &\quad - \frac{\lambda}{c(b_n)} \int_0^x g_n(z)(1 - G((x-z)/b_n)\,dz \\ &= \frac{\lambda}{c(b_n)} \int_0^x (g_{n-1}(z) - g_n(z))(1 - G((x-z)/b_n)\,dz \ge 0. \end{aligned}$$

Then $g_n(x) \ge g_{n+1}(x)$, therefore $\lim_{n\to\infty} g_n(x) = g(x)$ exists, and by the bounded convergence theorem we get

$$\mathscr{V} g(x) = g(x).$$

Now we define

$$f(x) = 1 + \int_0^x g(z)\,dz, \tag{1.8}$$

then by the bounded convergence theorem $f(x)$ satisfies (1.7). Furthermore, by (1.8) it follows that $f(x)$ is increasing and continuously differentiable.

$$\begin{aligned} f(x) &= 1 + \int_0^x g(z)\,dz \\ &\le 1 + \frac{c}{\lambda\mu} \int_0^x \delta_(' z)\,dz \\ &\le 2, \end{aligned}$$

then $f(x)$ is bounded.

From (1.7)

$$f'(0) = \lambda \inf_{b\in(\bar{b},1]} \frac{1}{c(b)} > 0.$$

Since $f(x-by) \leq f(x)$ because f is increasing then:

$$\frac{\lambda}{c(b)}\left[f(x)-\int_0^{\frac{x}{b}} f(x-by)\,dG(y)\right] \geq f(x)(1-G(x/b))\frac{\lambda}{c(b)},$$

therefore

$$f'(x) \geq f(x)\inf_{b\in(\bar{b},1]}\frac{\lambda}{c(b)}(1-G(x/b)) > 0.$$

Thus f is strictly increasing.
To show that the solution to (1.7) is unique, let $f_1(x)$ and $f_2(x)$ be two solutions to (1.7) with $f_1(0) = f_2(0) = 1$. Denote by $g_i(x) = f_i(x)$ the derivatives and by $b_i(x)$ the point where the minimum is taken. Fix $\bar{x} \geq 0$ and let $x_1 = \inf\{\min_i c(b_i(x)) : 0 \leq x \leq \bar{x}\}/(2\lambda)$. Then we define $x_n = nx_1 \wedge \bar{x}$, and thus we have $f_1(x) = f_2(x)$ for $n = 0$ on $[0, x_0]$. Now we suppose that $f_1(x) = f_2(x)$ on $[0, x_n]$ and let $m = \sup_{x_n\leq x\leq x_{n+1}} \mid g_1(x) - g_2(x) \mid$, then for $x \in [x_n, x_{n+1}]$:

$$\begin{aligned}
g_1(x)-g_2(x) &= \mathscr{V}g_1(x)-\mathscr{V}g_2(x)\\
&\leq \frac{\lambda}{c(b_2)(x)}\int_0^x g_1(z)(1-G((x-z)/b_2(x))\,dz\\
&\quad -\frac{\lambda}{c(b_2(x))}\int_0^x g_2(z)(1-G((x-z)/b_2(x))\,dz\\
&\leq \frac{\lambda}{c(b_2(x))}\int_0^x (g_1(z)-g_2(z))(1-G((x-z)/b_2(x))\,dz\\
&\leq \frac{\lambda}{c(b_2(x))}\int_{x_n}^x (g_1(z)-g_2(z))(1-G((x-z)/b_2(x))\,dz\\
&\leq \frac{\lambda m x_1}{c(b_2(x))} \leq \frac{m}{2}.
\end{aligned}$$

Similarly, we get $g_2(x) - g_1(x) \leq \frac{m}{2}$. Then $\mid g_1(x) - g_2(x) \mid \leq \frac{m}{2}$, which is only possible if $m = 0$. This shows that:
$f_1(x) = f_2(x)$ on $[0, x_{n+1}]$ and $f_1(x) = f_2(x)$ on $[0, \bar{x}]$. And because $\bar{x}$ was arbitrary, any solution must be unique. □

LEMMA 1.2 *Let $b(x)$ be the value that minimizes (1.7), then $x \longmapsto b(x)$ is measurable.*

Proof. The right hand side of (1.7) is continuous in both b and x. We choose $a \in (\bar{b}, 1]$ and let

$$m_a(x) = \inf_{b \leq a} \frac{\lambda}{c(b)} \left[f(x) - \int_0^{\frac{x}{b}} f(x - by) \right] dG(y).$$

We then have $\{b(x) > a\} = \{x : f'(x) < m_a(x)\}$. This shows that $b(x)$ is measurable because $f'(x)$ is continuous. □

Lemma 1.3 *Let b_t be an arbitrary strategy. Then with probability one either ruin occurs or X_t^b diverges to infinity as $t \to \infty$*

Proof. See Lemma 1.10. □

Lemma 1.4 *Let $g : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that*

$$\mathbb{E} \sum_{i=1}^{N_t} \mid g(X_{T_i}) - g(X_{T_i -}) \mid < \infty$$

for all $t \geq 0$ $n \in \mathbb{N}$. Then g is in the domain of the extended generator of $\mathbb{A}$ X_t, where

$$\mathbb{A} g(x) = c g'(x) + \lambda (\int_0^\infty g(x - y) \, dG(y) - g(x)).$$

Proof. First we have to show that the process

$$M_t = g(X_t) - g(x) - \int_0^t \mathbb{A} g(X_s) \, ds$$

is a martingale. By Itô's formula we obtain:

$$M_t = \sum_{i=1}^{N_t} g(X_{T_i}) - g(X_{T_i -}) - \lambda \int_0^t \int_0^\infty g(X_s - y) - g(X_s) \, dG(y) \, ds$$

We observe that:

$$\sum_{i=1}^{N_t} g(X_{T_i}) - g(X_{T_i -}) = \int_0^t g(X_s) - g(X_s -) \, dN_s,$$

as in the proof of Lemma 1.9 we get that M_t is a martingale. □

The results obtained above enable us to prove the main result.

Proof. (**Theorem 1.4**) Let b_t be an arbitrary strategy with corresponding process X^b. Then from Lemma 1.4 the process

$$Y_t = f(X_t^b) - \int_0^t \left(c(b_s)f'(X_s^b) + \lambda \left[\int_0^{X_s^b/b_s} f(X_s^b - b_s y)\, dG(y) - f(X_s^b) \right] \right) ds$$

is a martingale. Since τ^b is a stopping time, then $Y_{\tau^b \wedge t}$ is also a martingale. By the stopping time theorem, we get

$$\begin{aligned} f(x) &= \mathbb{E}[f(X_{\tau\wedge t})] - \mathbb{E}\Bigg[\int_0^{t\wedge\tau} c(b_s) \\ &\quad \times \left(f'(X_s^b) - \frac{\lambda}{c(b_s)}(f(X_s^b) - \int_0^{X_s^b/b_s} f(X_s^b - b_s y)\, dG(y)) \right) \Bigg] ds \\ &\geq \mathbb{E}[f(X_{\tau\wedge t})]. \end{aligned}$$

because $f(x)$ satisfies (1.7). Furthermore

$$\mathbb{E}[f(X_t^b)\mathbb{I}_{\tau>t}] = \mathbb{E}[f(X_{\tau\wedge t}^b)] \leq f(x).$$

Letting $t \to \infty$ yields $f(x) \geq f(\infty)\mathbb{P}[\tau = \infty]$, then $f(x)/f(\infty) \geq \delta(x)$.

Redoing the calculation with the strategy b^* produces $f(x) = \mathbb{E}[f(X_t^b)\mathbb{I}_{\tau>t}]$, because $\{f(X_{\tau\wedge t}^*)\}$ is a martingale. Then $f(x) = f(\infty)\delta^*(x)$. This shows that $\delta(x) = f(x)/f(\infty)$. □

We show in the next lemma that under reasonable conditions reinsurance should not be taken.

LEMMA 1.5 *We assume that* $\liminf_{b\uparrow 1}(1-b)^{-1}(c - c(b)) > 0$ *and that* $G(x)$ *has a bounded density close to zero. Then there exists* $\varepsilon > 0$ *such that* $b(x) = 1$ *for* $x \leq \varepsilon$.

Proof. Let $H(x,b) = c(b)f'(x) + \lambda(\mathbb{E}(f(x - bY) - f(x))$ then we get

$$\frac{H(x,1) - H(x,b)}{1-b} = \frac{c - c(b)}{1-b} f'(x) + \lambda \mathbb{E}\left[\frac{f(x-Y) - f(x-bY)}{1-b} \mathbb{I}_{Y \leq x/b} \right]$$

By Taylor's theorem, there exists $\zeta(Y) \in (x-Y, x-bY)$ such that

$$f(x-bY) = f(x-Y) + Y(1-b)f'(\zeta(Y)),$$

then $\mathbb{E}\left[\frac{f(x-Y)-f(x-bY)}{1-b}\mathbb{I}_{Y\le x/b}\right] = -\mathbb{E}\left[Yf'(\zeta(Y))\mathbb{I}_{Y\le x/b}\right].$

Since $G(x)$ has a bounded density, then by the bounded convergence theorem we get

$$\lim_{b\to 1}\mathbb{E}\left[\frac{f(x-Y)-f(x-bY)}{1-b}\mathbb{I}_{Y\le x/b}\right] = -\mathbb{E}\left[Yf'(\zeta(Y))\mathbb{I}_{Y\le x}\right].$$

Then for small enough values of x, the right hand side can be made small. Thus $H(x,b)$ is strictly increasing in b at $b=1$.

Now we assume that $b(x) \neq 1$ close to zero, then $\limsup_{x\downarrow 0} b(x) < 1$, so that $b(x)$ jumps at zero because $b(0)=1$. Since $H(x,b)$ is continuous in both x and b then $H(0,b) = H(x,b)$ for some $b<1$, but $H(0,b) = c(b)f'(x) - \lambda f(x)$ is strictly increasing in b, which is a contradiction. Thus $b(x)=1$ for low levels of capital. □

1.3 Optimal Dividends

In this section, we extend the Cramér-Lundberg theorem so that a dividend is paid out. A dividend process is an adapted increasing process $\{D_t\}$ with $D_{0-}=0$. The surplus process then becomes

$$X_t^D = x + ct - \sum_{i=1}^{N_t} Y_i - D_t.$$

The time of ruin is defined as $\tau^D = \inf\left\{t : X_t^D < 0\right\}$. The value of a dividend strategy is defined as

$$V^D(x) = \mathbb{E}\left[\int_{0-}^{\tau^D-} e^{\delta t}\, dD_t\right].$$

The value function is $V(x) = \sup_D V^D(x)$. Then our goal is to find an optimal value where the supremum is taken over all increasing adapted processes.

LEMMA 1.6 *The function $V(x)$ is bounded by*

$$x + \frac{c}{\lambda + \delta} \leq V(x) \leq x + \frac{c}{\delta}.$$

Proof. We consider the dividend strategy $D_t = x + ct$, which means that all of the surplus is paid out as dividends. This strategy has the value

$$\begin{aligned} V^D(x) &= \mathbb{E}\left[\int_{0-}^{\tau^{D-}} e^{-\delta t}\, dD_t\right] \\ &= x + \int_0^\infty \int_0^t c e^{-\delta s}\, ds \lambda e^{-\lambda t}\, dt \quad \textit{because} \quad T_1 = \tau^D \\ &= x + \frac{c}{\lambda + \delta} \end{aligned}$$

Then $V(x) = \sup_D V^D(x) \geq x + \frac{c}{\lambda + \delta}$ yields the lower bound. For the upper bound, we have $D_t \leq x + ct$, otherwise ruin occurs. Then

$$V^D(x) \leq \int_0^\infty D_t \delta e^{-\delta t}\, dt,$$

and

$$V^D(x) < \int_0^\infty (x + ct)\delta e^{-\delta t}\, dt = x + \frac{c}{\delta}.$$

This yields the upper bound. □

LEMMA 1.7 *The function $V(x)$ is locally Lipschitz continuous on $[0, \infty)$. Moreover, for $y \leq x$*

$$x - y \leq V(x) - V(y) \leq V(x)\frac{\lambda + \delta}{c}(x - y).$$

Proof. Let $x > y$ and D_t be a dividend strategy for initial capital y. We consider the following strategy $\tilde{D}_t = x - y + D_t$ then:

$$V(x) \geq V^{\tilde{D}}(x) = V^D(y) + x - y,$$

then

$$V(x) - V^D(x) \geq x - y.$$

Taking the supremum over $V^D(y)$ we get:

$$V(x) - V(y) \geq x - y.$$

Now let D_t be a strategy for initial capital x and let $h = (x-y)/c$ denote the time needed to reach x from y if no claims occur. For the second inequality, we use the following strategy:

$$\tilde{D}_t = \begin{cases} 0 & \text{if } \ t < h \quad or \quad T_1 \leq h \\ D_{t-h} & \text{if } \ T_1 \wedge t > h \\ D_0 & \text{if } \ t = h \quad and \quad T_1 > h \end{cases}$$

Taking the supremum over D, we get

$$\begin{aligned} V(y) \ &\geq \ V^{\tilde{D}} = \mathbb{E}\left[\int_{0_-}^{\tau^{\tilde{D}_-}} e^{-\delta t}\, dD_t\right] \\ &\geq \ \mathbb{E}\left[\int_{0_-}^{T_1 \wedge h} e^{-\delta t}\, d\tilde{D}_t\right] + \mathbb{E}\left[\int_{T_1 \wedge h}^{\tau^{\tilde{D}_-}} e^{-\delta t}\, d\tilde{D}_t\right] \\ &\geq \ \mathbb{E}\left[\mathbb{I}_{T_1 \geq h} \int_{h}^{\tau^{\tilde{D}_-}} e^{-\delta t}\, d\tilde{D}_t\right] \\ &\geq \ \mathbb{E}\left[\mathbb{I}_{T_1 \geq h} \int_{h}^{\tau^{\tilde{D}_-}} e^{-\delta t}\, dD_{t-h}\right] \\ &\geq \ \mathbb{E}\left[\mathbb{I}_{T_1 \geq h} \int_{0_-}^{\tau^{D_-}} e^{-\delta h} e^{-\delta t}\, dD_t\right] \\ &\geq \ \mathbb{E}\left[\mathbb{I}_{T_1 \geq h} \mathbb{E}\left[\int_{0_-}^{\tau^{D_-}} e^{-\delta h} e^{-\delta t}\, dD_t \mid T_1\right]\right] \\ &\geq \ \int_{h}^{\infty} e^{-\delta h} \mathbb{E}\left[\int_{0_-}^{\tau^{D_-}} e^{-\delta s}\, dD_s \mid T_1 = t\right] \lambda e^{-\lambda t}\, dt \\ &\geq \ e^{-(\lambda+\delta)h} V(x), \end{aligned}$$

then we get $V(x) - V(y) \leq V(x)(1 - e^{-(\lambda+\delta)h} \leq V(x)\frac{\lambda+\delta}{c}(x-y)$. □

1.3.1 The Hamilton-Jacobi-Bellman Equation

As a motivation for the Hamilton-Jacobi-Bellman equation, we restrict to dividend processes of the form $D_t = \int_0^t u_s\, ds$, where $0 \leq u_s \leq d_0$ is a bounded rate. We let $d_0 > c$, $h > 0$ and fix $u \in [0, d_0]$. If $x = 0$, we assume $u \leq c$; and if $x > 0$, we take h small enough such that $x + (c-u)h \geq 0$ (i.e., ruin does not occur because of the dividend payments). Now we choose $\varepsilon > 0$, and consider the following strategy:

$$u_t = \begin{cases} u & \text{if } \ 0 \leq t \leq T_1 \wedge h \\ u^{\varepsilon}_{t-T_1\wedge h} & \text{if } \ T_1 \wedge t > h \end{cases}$$

where u^{ε}_t is a strategy for initial capital $X_{T_1\wedge h}$ given by $V^{\varepsilon}(X_{T_1\wedge h}) > V(X_{T_1\wedge h}) - \varepsilon$. Because $V(x) = \sup_D V^D(x)$, then we get:

$$\begin{aligned}
V(x) \;\geq\;& \mathbb{E}\left[\mathbb{I}_{T_1\leq h}\int_0^{T_1} e^{-\delta s}\, dD_s\right] + \mathbb{E}\left[\mathbb{I}_{T_1\geq h}\int_0^{h} e^{-\delta s}\, dD_s\right] + \\
& \mathbb{E}\left[\mathbb{I}_{T_1\leq h}\int_{T_1}^{\tau_D} e^{-\delta s}\, dD_s\right] + \mathbb{E}\left[\mathbb{I}_{T_1\geq h}\int_{h}^{\tau_D} e^{-\delta s}\, dD_s\right] \\
\geq\;& \int_0^h \int_0^t e^{-\delta s} u\, ds \lambda e^{-\lambda t} + \int_h^\infty \int_0^h e^{-\delta s} u\, ds \lambda e^{-\lambda t} + \\
& \mathbb{E}\left[\mathbb{I}_{T_1\leq h}\int_{T_1}^{\tau_D} e^{-\delta s} u_{s-T_1}\, ds\right] + \mathbb{E}\left[\mathbb{I}_{T_1\geq h}\int_{h}^{\tau_D} e^{-\delta s} u_{s-h}\, ds\right] \\
\geq\;& \int_0^h \int_0^t e^{-\delta s} u\, ds \lambda e^{-\lambda t} + \int_h^\infty \int_0^h e^{-\delta s} u\, ds \lambda e^{-\lambda t} + \\
& \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{I}_{T_1\leq h}\int_{T_1}^{\tau_D} e^{-\delta s} u_{s-T_1}\, ds \mid T_1, Y_1\Big]\Big] + \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{I}_{T_1\geq h}\int_{h}^{\tau_D} e^{-\delta s} u_{s-h}\, ds \mid T_1\Big]\Big] \\
\geq\;& e^{-\lambda h}\Big[\int_0^h u e^{-\delta t}\, dt + e^{-\delta h} V^{\varepsilon}(x + (c-u)h)\Big] + \int_0^h \Big[\int_0^t u e^{-\delta s}\, ds \\
& + e^{-\delta t}\int_0^{x+(c-u)t} V^{\varepsilon}(x + (c-u)t - y)\, dG(y)\Big] \lambda e^{-\lambda t}\, dt \\
\geq\;& e^{-\lambda h}\Big[\int_0^h u e^{-\delta t}\, dt + e^{-\delta h} V(x + (c-u)h)\Big] + \int_0^h \Big[\int_0^t u e^{-\delta s}\, ds \\
& + e^{-\delta t}\int_0^{x+(c-u)t} V(x + (c-u)t - y)\, dG(y)\Big] \lambda e^{-\lambda t}\, dt - \varepsilon.
\end{aligned}$$

Since the constant ε is arbitrary, we can let it tend to zero. Then dividing by h yields

$$\begin{aligned} 0 \geq & \frac{V(x+(c-u)h)-V(x)}{h} - \frac{1-e^{-(\delta+\lambda)h}}{h} V(x+(c-u)h) \\ & + e^{-\lambda h} \frac{1}{h} \int_0^h u e^{-\delta t}\, dt + \frac{1}{h} \int_0^h \Big[\int_0^t u e^{-\delta s}\, ds \\ & + e^{-\delta t} \int_0^{x+(c-u)t} V(x+(c-u)t-y)\, dG(y) \Big] \lambda e^{-\lambda t}\, dt. \end{aligned}$$

Letting $h \to 0$, we get

$$(c-u)V'(x) + \lambda \int_0^x V(x-y)\, dG(y) - (\lambda+\delta)V(x) + u \leq 0.$$

Therefore we consider the following Hamilton-Jacobi-Bellman equation:

$$\max \left\{ cV'(x) + \lambda \int_0^x V(x-y)\, dG(y) - (\lambda+\delta)V(x), 1 - V'(x) \right\} = 0. \quad (1.9)$$

The optimal dividend is expected to be constructed the following way. Let

$$\mathcal{B}_0 = \{x : V'(x- > 1)\}$$

$$\mathcal{B}_c = \{x \notin \mathcal{B}_0 : \exists x_n, x_n \uparrow x\} \cup \tilde{\mathcal{B}}_c,$$

where $\tilde{\mathcal{B}}_c = 0$ if $V(0) = \dfrac{c}{\lambda+\delta}$ and the empty set otherwise. Let

$$\mathcal{B}_\infty = [0,\infty) \setminus (\mathcal{B}_0 \cup \mathcal{B}_c).$$

Now we define the following strategy D_t^*. If $x \in \mathcal{B}_0$, we do not pay out dividends. If $x \in \mathcal{B}_c$, we pay out a dividend $dD_t^* = cdt$ until the next jump.

THEOREM 1.6 *(Schmidli, [37]) The strategy $\{D_t^*\}$ is an optimal strategy, that is*

$$V^{D^*}(x) = V(x),$$

and $V(x)$ is the solution to (1.9).

1.4 Optimal Investment

Browne (1995) considered a risk process modeled by a Brownian motion with drift given by:

$$X_t = x + \alpha dt + \beta W_t^{(1)},$$

with the possibility of investment in a risky asset which follows a geometric Brownian motion given by

$$dS_t = S_t(\mu dt + \sigma dW_t). \tag{1.10}$$

W_t^1 is a Brownian motion such that $\mathbb{E}[W_t W_t^{(1)}] = \rho t$, where $\rho^2 \neq 1$ is the correlation coefficient. Let $\mathcal{K}$ be the set of all admissible adapted strategies, i.e., K_t is a non-anticipative function (see [30], p.40) and satisfies, for any T,

$$\int_0^T K_t^2 \, dt < \infty, a.s. \tag{1.11}$$

OBSERVATION 1.5 *The set $\mathcal{K}$ of all admissible strategies depends on the filtration generated by X_t and S_t. In all of the cases studied in this chapter, we will conserve the same notation for the set of all admissible strategies $\mathcal{K}$. Only the filtration will depend on the risk processes and the risky assets.*

If the company is allowed to invest in the risky asset (1.10) at time t under an investment strategy $K \in \mathcal{K}$, then the wealth of the company X_t^K at time t is given by:

$$X_t = x + \int_0^t (\alpha + \mu K_s) \, ds + \int_0^t \sigma K_t \, dW_t + \int_0^t \beta \, dW_t^{(1)}. \tag{1.12}$$

From standard arguments in stochastic control, Browne showed that the value function $V(x) = \sup_{K \in \mathcal{K}} \mathbb{P}[X_\tau^K \geq b \mid X_0 = x]$, where $\tau^K = \min\left\{\tau_a^K, \tau_b^K\right\}$ and $\tau_x^K = \inf\left\{t > 0 : X_t^K = x\right\}$ satisfies the following HJB equation:

$$\max_{K \in \mathbb{R}} \left[(\alpha + \mu K)V_x + \frac{1}{2}(\sigma^2 K^2 + \beta^2 + 2\rho\sigma\beta K)V_{xx}\right] = 0, \text{ for } a \leq x \leq b. \tag{1.13}$$

When $b \uparrow \infty$ and $a \downarrow 0$, he arrived at a surprising result: the optimal strategy that minimizes the ruin probability is constant (see [3], p.10).
Hipp and Plum (2000) considered the classical risk process (1.1) with opportunity to invest in a risky asset modeled by a geometric Brownian motion (1.10). The objective in this case is to find how to invest in the risky asset in order to minimize the probability of ruin. More precisely, let X_t be the classical Cramér-Lundberg model

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i. \tag{1.14}$$

Now the insurer has the possibility to invest in a risky asset described by a geometric Brownian motion

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{1.15}$$

where $\mu, \sigma \in \mathbb{R}$. The standard Brownian motion $\{W_t\}$ is assumed to be independent of the process X_t.
Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by the processes X and S, and $\mathbb{E}_t[\cdot]$ be the notation for the conditional expectation $\mathbb{E}[\ \cdot \mid \mathcal{F}_t]$. We will denote by $K = \{K_t,\ t \geq 0\}$ the investment strategy of the insurer in each time period t in the risky asset, and by $\mathcal{K}$ the same set of non-anticipative strategies considered above. Let $X_t^K := X(t, x, K)$ be the wealth of the insurer at time t if he chooses the admissible strategy K to invest in the risky asset, then the process X^K satisfies

$$X_t^K = x + \int_0^t (c + \mu K_s)\, ds + \int_0^t \sigma K_s \, dW_s - \sum_{i=1}^{N_t} Y_i. \tag{1.16}$$

The time of ruin is defined as

$$\tau(x, K) = \inf \left\{ t : X^K < 0 \right\},$$

and the survival probability is given by

$$\phi(x, K) = \mathbb{P}\left[\tau(x, K) = \infty\right].$$

The problem consists in maximizing the survival probability, that is solving

$$\phi^*(x) = \sup_{K \in \mathcal{K}} \phi(x, K),$$

and finding an optimal strategy $K^* \in \mathcal{K}$ such that

$$\phi^*(x) = \phi(x, K^*).$$

Using Itô's lemma, Hipp and Plum (2000) suggested the following HJB equation for the optimal survival probability:

$$\sup_{K \in \mathbb{R}} \left\{ \frac{1}{2}\sigma^2 K^2 \phi''(x) + (c + \mu K)\phi'(x) + \lambda \int_0^\infty \phi(x-y) - \phi(x)\, dG(y) \right\} = 0. \quad (1.17)$$

The following theorem relates the value function with the survival probability:

THEOREM 1.7 *Assume that there exits a solution $\phi^*(x)$ to the HJB equation* (1.17) *with the maximizing function K^* with the following properties:*

- $\phi^*(0), \phi^*(0) > 0$ *and* $\phi^*(x) = 0$ *for* $x < 0$.
- $\lim_{x\to\infty} \phi^*(x) = 1$.
- $\phi^*(x)$ *is twice continuously differentiable on* $\{x > 0\}$.

Then ${\phi^*}'(x) > 0$ *for* $x > 0$, *and if* $K(t)$ *is an arbitrary admissible strategy for which the reserve process* X^K *is defined on* $0 \le t \le \infty$, *then the corresponding survival probability* $\phi(x)$ *satisfies:*

$$\phi(x) \le \phi^*(x)$$

with equality for K^*.

OBSERVATION 1.6 *The main idea of the proof of the verification theorem in risk theory and finance is now classic and many authors have used it [18, 35, 7]. This idea can be seen in theorem 4.1 of the present thesis. However, to prove theorem 1.7, Hipp and Plum used a new element which consists in proving the asymptotic behavior of the risk process on the set* $\{\tau = \infty\}$. *This property of the risk process was obtained by constructing a new strategy of investment.*

THEOREM 1.8 *Suppose that* $G(x)$ *has a bounded density, then*

$$\phi(x) = \sup_{K \in \mathcal{K}} \phi(x, K),$$

solves the HJB equation (1.17) *and the optimal strategy of investment is given by:*

$$K^*_{t-} = K^*(X_{t-}) = -\frac{\mu}{\sigma^2}\frac{\phi'(X_{t-})}{\phi''(X_{t-})}. \tag{1.18}$$

Proof. We will give a sketch of the proof, for more details see [18].

Step 1. The first step consists in substituting the strategy K^*_{t-} given by (1.18) in the HJB equation (1.17), which leads to:

$$\lambda \int_0^\infty \phi(x-y) - \phi(x)\, dG(y) + c\phi'(x) = \frac{\mu}{2\sigma^2}\frac{(\phi'(x))^2}{\phi''(x)}. \tag{1.19}$$

Let $H = 1 - G$, then (1.19) can be transformed to:

$$\psi''(x)\left(\lambda \int_0^x \phi'(x-y)H(y)\, dy + c(\phi'(x) - H(x))\right) = \frac{1}{2}(\phi'(x))^2, \tag{1.20}$$

which can be written for $v = \phi'$ as:

$$\begin{aligned} v'(x)\left(-\lambda \int_0^x v(x-y)H(y)\, dy + c(v(x) - H(x))\right) &= \frac{1}{2}(v(x))^2 \\ v(0) &= 1. \end{aligned} \tag{1.21}$$

Let $w(x) = v(x^2)$, then (1.21) is transformed to:

$$w'(x)\mathbb{L}[w](x) = (w(x))^2, w(0) = 1, \tag{1.22}$$

where

$$\mathbb{L}[w](x) = -2\lambda x \int_0^1 tw(xt)H(x^2(1-x^2))\,dt + c\frac{w(x)-H(x^2)}{x}.$$

Step 2. We define $Q_\epsilon := \sup_{0<x\leq\epsilon} \frac{1}{x} \mid w'(x) - w'(0) \mid$ for $\epsilon > 0$ and $w \in C^1[0,\epsilon]$. Let $R_\epsilon := \{w \in C^1[0,\epsilon], Q_\epsilon[w] < \infty\}$ endowed with the norm

$$\| w \|_\epsilon = \max\{\| w \|_\infty, \mid w'(0) \mid, \epsilon Q_\epsilon\}$$

be a Banach space, and

$$D_{\epsilon,M} := \left\{ w \in R_\epsilon : w(0) = 1, w'(0) = -\frac{1}{\sqrt{c}}, \| w - 1 \|_\infty \leq \frac{1}{3}, Q_\epsilon[w]. \right\},$$

be a closed subset.

Step 3. The operator T defined by:

$$T(w)(x) := 1 + \int_0^x \frac{w(y)^2}{\mathbb{L}[w](y)}\,dy, \quad w \in D_{\epsilon,M}, x \in [0,\epsilon],$$

maps $D_{\epsilon,M}$ into itself and is a contraction. By Banach's fixed point theorem there exists a fixed point $w \in D_{\epsilon,M}$ which is a solution to (1.21) in $[0,\epsilon]$.

Step 4. This step consists in extending the solution to $[0,\infty]$. □

OBSERVATION 1.7 *The strategy given by* (1.18) *is different from the one obtained by Browne, because at time t it is a function of the wealth of the insurance company. In the case of exponentially distributed claim size and special parameter values, Hipp and Plum (2000) gave an explicit solution for the HJB equation* (1.17).

A follow-up paper by Schmidli (2002) considered the classical risk process and allowed investment in a risky asset modeled by a geometric Brownian motion and proportional reinsurance. Here the ruin probability is

$$\psi(x) = \inf_{K\in\mathcal{K}} \psi(x,K),$$

where $\psi(x,K)=\mathbb{P}[\tau^K<\infty]$ satisfies the following HJB equation:

$$\inf_{b\in[0,1]}\Big[\inf_{K\in\mathbb{R}}\frac{1}{2}\sigma^2K^2\psi''(x)+(c-c(b)+\mu K)\psi'(x) \\ +\lambda\int_0^x \psi(x-y)-\psi(x)\,dG(y)\Big]=0. \quad (1.23)$$

Via the HJB equation (1.23), Schmidli found an optimal strategy $K^*(X_{t-})$ and $b^*(X_{t-})$, where $K^*(x)=-\dfrac{\mu}{\sigma^2}\dfrac{\psi'(x)}{\psi''(x)}$ and $b^*(x)$ is the argument minimizing the HJB equation (1.23) after substituting $K^*(x)$. Numerical procedures to solve the HJB equation (1.23) were further developed by Schmidli.

Gaier et al. (2003) obtained an estimate for the ruin probability of exponential type with a rate that improves the classical Lundberg parameter, by proposing a strategy that consists in investing a constant amount of money in the risky asset. Let $h:\mathbb{R}_+\to\mathbb{R}_+$ be the moment generating function of Y such that $h(0)=0$, $h(r)=M_Y(r)-1$ and let:

$$\theta(r)=\lambda h(r)-cr-\frac{\mu^2}{2\sigma^2}.$$

We assume as in the case of the Cramér-Lundberg model without investment that there exists $r_\infty\in(0,\infty]$ such that $h(r)<\infty$ for $r<r_\infty$ and $\lim_{r\to r_\infty}h(r)=\infty$. Hence, the ruin probability is estimated as follows:

THEOREM 1.9 *The minimal ruin probability $\psi^*(x)$ of an insurer investing in a risky asset can be bounded from above by*

$$\psi^*(x)\le e^{-\hat{r}x},$$

where $0<\hat{r}<r_\infty$ is the positive solution of:

$$\lambda\,(M_Y(r)-1)=cr+\frac{\mu^2}{2\sigma^2}.$$

The proof of this theorem will be a consequence of the following lemma:

LEMMA 1.8 *Let* $M(t,x,K,r) = e^{-rX_t^K}$*, then the process* $M(t,x,\hat{K},\hat{r})$ *is a martingale, where*

$$\hat{K} = \frac{\mu}{\hat{r}\sigma^2}, \tag{1.24}$$

and $\hat{r}$ *is the unique solution of:*

$$\theta(r) = 0.$$

Proof. We define $f : \mathbb{R} \times [0, r_\infty) \to \mathbb{R}$

$$f(K,r) = \lambda h(r) - (\mu K + c)r + \frac{1}{2}\sigma^2 K^2 r^2 \tag{1.25}$$

then $f(\hat{K},\hat{r}) = 0$. Then we have

$$\begin{aligned} \mathbb{E}[M(t,0,\hat{K},\hat{r})] &= \mathbb{E}[e^{-\hat{r}(ct - \sum_{i=1}^{N_t} Y_i + \mu t\hat{K} + \sigma\hat{K}W_t)}] \\ &= e^{-\hat{r}(c+\mu\hat{K})t}\mathbb{E}[e^{\hat{r}\sum_{i=1}^{N_t} Y_i}]\mathbb{E}[e^{-\hat{r}\sigma\hat{K}W_t}] \\ &= e^{-\hat{r}(c+\mu\hat{K})t}e^{h(\hat{r})\lambda t}e^{(\sigma^2\hat{K}^2\hat{r}^2/2)t} \\ &= e^{f(\hat{K},\hat{r})} \\ &= 1. \end{aligned}$$

Thus, $M(t,0,\hat{K},\hat{r})$ has a finite moment. By using the stationary and independent increments of the process $X_t^{\hat{K}}$, we obtain for $0 \le t \le T$

$$\begin{aligned} \mathbb{E}_t[M(T,x,\hat{K},\hat{r})] &= \mathbb{E}_t[e^{-\hat{r}X_T^{\hat{K}}}] \\ &= e^{-\hat{r}X_t^{\hat{K}}}\mathbb{E}_t[e^{-\hat{r}(X_{T-t}^{\hat{K}} - X_t^{\hat{K}})}] \\ &= e^{-\hat{r}X^{\hat{K}}}\mathbb{E}_t[e^{-\hat{r}(X_{T-t}^{\hat{K}} - X_0^{\hat{K}})}] \\ &= e^{-\hat{r}X^{\hat{K}}}\mathbb{E}_t[e^{-\hat{r}Y(T-t,0,\hat{K})}] \\ &= e^{-\hat{r}X^{\hat{K}}}\mathbb{E}[M(T-t,0,\hat{K},\hat{r})] \\ &= e^{-\hat{r}X_t^{\hat{K}}} \\ &= M(t,x,\hat{K},\hat{r}), \end{aligned}$$

therefore $M(t,x,\hat{K},\hat{r})$ is a martingale. □

***Proof.*(Theorem 1.9)** We know that $M(t,x,\hat{K},\hat{r})$ is a martingale with respect to the filtration $\mathbb{F}$, then the stopped process $\tilde{M}(t,x,\hat{K},\hat{r}) = M(t\wedge \tau,x,\hat{K},\hat{r})$ is also a martingale, so that

$$\begin{aligned} e^{-\hat{r}x} &= \tilde{M}(0,x,\hat{K},\hat{r}) \\ &= \mathbb{E}[(\tilde{M}(t,x,\hat{K},\hat{r})] \\ &= \mathbb{E}[M(\tau,x,\hat{K},\hat{r})\mathbb{1}_{\tau<t}] + \mathbb{E}[M(t,x,\hat{K},\hat{r})\mathbb{1}_{t\leq\tau}] \\ &\geq \mathbb{E}[M(\tau,x,\hat{K},\hat{r})\mathbb{1}_{\tau<t}]. \end{aligned}$$

By using monotone convergence, we get

$$\lim_{t\to\infty} \mathbb{E}[M(\tau,x,\hat{K},\hat{r})\mathbb{1}_{\tau<t}] = \mathbb{E}[M(\tau,x,\hat{K},\hat{r})\mathbb{1}_{\tau<\infty}],$$

then:

$$e^{-\hat{r}x} \geq \mathbb{E}[M(\tau,x,\hat{K},\hat{r}) \mid \tau<\infty]\mathbb{P}[\tau<\infty],$$

therefore:

$$\begin{aligned} \psi(x,\hat{K}) &= \mathbb{P}[\tau<\infty] \\ &\leq \frac{e^{-\hat{r}x}}{\mathbb{E}[M(\tau,x,\hat{K},\hat{r}) \mid \tau<\infty]}. \end{aligned}$$

Then

$$\psi^*(x) \leq e^{-\hat{r}x},$$

because $M(\tau,x,\hat{K},\hat{r}) \geq 1$. □

Observation 1.8 *1. We know that the classical Lundberg exponent R is the positive solution to*

$$h(r) = \frac{c}{\lambda}r,$$

and $\hat{r}$ is the positive solution of

$$h(r) = \frac{c}{\lambda}r + \frac{\mu^2}{2\lambda\sigma^2}.$$

The figure below shows that $\hat{r} > R$, which means that including investment in the classical risk model improves the bound given by Lundberg.

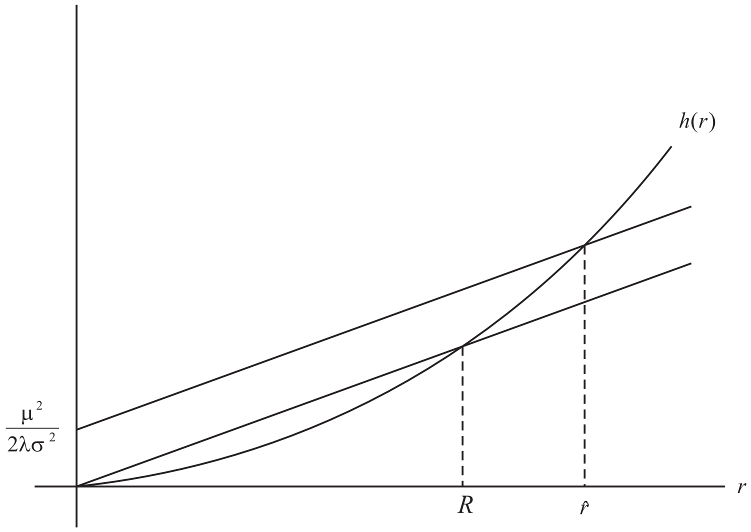

Figure 1.2: The adjustment coefficient R.

2. *The case in which the bond yields interest was considered in [13], where by the same argument as in the case of zero interest, the authors produce an upper bound for the ruin probability.*

Hipp and Schmidli (2004) showed in the following theorem that the strategy proposed by Gaier et al. (2003) is asymptotically optimal as initial wealth tends to infinity.

THEOREM 1.10 *(Hipp and Schmidli, [21]) In the small claim case:*

$$\lim_{x\to\infty} K^*(x) = \hat{K}.$$

Liu and Yang (2004) considered the model by Hipp and Plum (2000) incorporating a risk-free interest rate. In this case, a closed form solution cannot be obtained. Hence, they provided numerical results for the behavior of investment under different claim-size distributions.

Definition 1.2 *Let* $< r < r_\infty$ *be given. We say that* Y *has a uniform moment in the tail distribution for* r *if the following condition holds true*

$$\sup_{z\geq 0} \mathbb{E}[e^{-r(z-Y)} \mid Y > z] < \infty.$$

Lemma 1.9 *Let* $0 < r < r_\infty$ *and* $K \in \mathcal{K}$*. The difference of the processes*

$$\int_0^{t\wedge\tau} M(s-, x, K, \hat{r})(e^{rY_{N_s}} - 1)\, dN_s$$

and

$$\mathbb{E}[e^{rY} - 1] \int_0^{t\wedge\tau} M(s-, x, K, r)\lambda\, ds,$$

is a martingale with respect to the filtration $\mathbb{F}$*.*

Proof. We know that $N = (N_t)_{t\geq 0}$ is a finite variation process, then the integral with respect to N makes sense as a pathwise Lebesgue-Stieltjes integral. Then

$$\begin{aligned}
&\int_0^{t\wedge\tau} M(s-, x, K, \hat{r})(e^{rY_{N_s}} - 1)\, dN_s \\
&= \sum_{n=1}^{\infty} M(T_n-, x, K, r)(e^{rY_n} - 1)\mathbb{1}_{\{t\wedge\tau \geq T_n\}}
\end{aligned}$$

Now by taking the expectation for $t \leq T$:

$$\begin{aligned}
& \mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,\hat{r})(e^{rY_{N_s}}-1)\,dN_s] \\
= \quad & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} M(T_n-,x,K,r)(e^{rY_n}-1)\mathbb{1}_{\{T\wedge\tau\geq T_n\geq t\wedge\tau\}}] \\
= \quad & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} \mathbb{E}_{T_n-}[M(T_n-,x,K,r)(e^{rY_n}-1)\mathbb{1}_{\{T\wedge\tau\geq T_n\geq t\wedge\tau\}}]] \\
= \quad & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} \mathbb{E}_{T_n-}[e^{rY_n}-1]M(T_n-,x,K,r)\mathbb{1}_{\{T\wedge\tau\geq T_n\geq t\wedge\tau\}}] \\
= \quad & \mathbb{E}_{t\wedge\tau}[\sum_{n=1}^{\infty} \mathbb{E}[e^{rY}-1]M(T_n-,x,K,r)\mathbb{1}_{\{T\wedge\tau\geq T_n\geq t\wedge\tau\}}] \\
= \quad & \mathbb{E}[e^{rY}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)\,dN_s)] \\
= \quad & \mathbb{E}[e^{rY}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)\,d(N_s-\lambda s)+ \\
& \quad \int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)\lambda\,ds] \\
= \quad & \mathbb{E}[e^{rY}-1]\mathbb{E}_{t\wedge\tau}[\int_{t\wedge\tau}^{T\wedge\tau} M(s-,x,K,r)\lambda\,ds].
\end{aligned}$$

In the last line, we used that $N_t - \lambda t$ is a martingale and that the integral of any measurable and bounded process with respect to $N_t - \lambda t$ is also a martingale, thus completing the proof. □

Theorem 1.11 *Suppose that Y has a uniform exponential moment in the tail distribution for $\hat{r}$. Then for each $K \in \mathcal{K}$, the process $(\tilde{M}(t,x,K,\hat{r}))_{t\geq 0}$ is a uniformly integrable submartingale.*

Proof. Using Itô's lemma with jumps for arbitrary $K \in \mathcal{K}$, we get:

$$\begin{aligned}
& \tilde{M}(t,x,K,\hat{r}) - \tilde{M}(0,x,K,\hat{r}) \\
& = \int_0^{t\wedge\tau} M(s-,x,K,\hat{r})\,ds - rb\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})K_s dW_s \\
& + \int_0^{t\wedge\tau} M(s-,x,K,\hat{r})(e^{\hat{r}Y_{N_s}})\,dN_s - \mathbb{E}[e^{\hat{r}Y}-1]\int_0^{t\wedge\tau} M(s-,x,K,\hat{r})\lambda\,ds
\end{aligned}$$

The stochastic integral with respect to the Brownian motion is integrable because $K \in \mathcal{K}$ is integrable and $M(s-, x, K, \hat{r}) \leq 1$. It was shown by the lemma above that the difference of the two processes

$$\int_0^{t\wedge\tau} M(s-, x, K, \hat{r})(e^{\hat{r}Y_{N_s}})\, dN_s - \mathbb{E}[e^{\hat{r}Y} - 1] \int_0^{t\wedge\tau} M(s-, x, K, \hat{r})\lambda\, ds,$$

is a martingale.

Finally it is easy to obtain:

$$f(K, \hat{r}) = \frac{1}{2}\hat{r}^2\sigma^2(K - \hat{K}) \geq 0.$$

So we deduce that for $0 \leq t \leq T$, $\int_{t\wedge\tau}^{T\wedge\tau} M(s-, x, K, \hat{r}) f(K, \hat{r})\, ds$ is a submartingale. Then $(\tilde{M}(t, x, K, \hat{r}))$ is a local submartingale. Now to show that $(\tilde{M}(t, x, K, \hat{r}))_{t\geq 0}$ is a submartingale, we use the following process:

$$\tilde{M}^* = \sup_{t\geq 0} \mid \tilde{M}(t) \mid,$$

then,

$$\begin{aligned}
\mathbb{E}[M^*] &\leq \mathbb{E}[\tilde{M}(t, x, K, \hat{r}) \mid \tau < \infty] \\
&\leq \mathbb{E}[\tilde{M}(t, x, K, \hat{r}) \mid \tau < \infty, X(\tau-) > 0].
\end{aligned}$$

Let $H(dt, dy)$ be the joint probability distribution of τ and $X^K(\tau-)$, then

$$\begin{aligned}
\mathbb{E}[M^*] &\leq \mathbb{E}[\tilde{M}(t, x, K, \hat{r}) \mid \tau < \infty, R(\tau-) > 0] \\
&= \int_0^\infty \int_0^\infty H(dt, dy) \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \\
&\leq \sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \int_0^\infty \int_0^\infty H(dt, dy) \\
&= \sup_{y\geq 0} \int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)} \\
&< \infty.
\end{aligned}$$

Since each

$$\mid \tilde{M}(t) \mid \leq \tilde{M}^*,$$

then by the dominated convergence theorem we get that $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale. □

LEMMA 1.10 *If Y has a uniform exponential distribution moment in the tail distribution for $\hat{r}$, then for an arbitrary $K \in \mathcal{K}$ and $x \in \mathbb{R}_+$, the stopped wealth process $X^K_{t\wedge\tau}$ converges almost surely on $\tau = \infty$ to ∞ for $t \to \infty$.*

Proof. From Theorem 1.11 and by applying Doob's convergence theorem, the stopped wealth process $X^K_{t\wedge\tau}$ converges a.s. for $t \to \infty$. There exists $d > 0$ such that $\mathbb{P}[Y > d] > 0$, then we define $E_n := \{Y_n > d\}$. Therefore $\mathbb{P}[E_n^c] < 1$, and $\{E_j\}_{j=1}^{\infty}$. Then :

$$\mathbb{P}[\bigcup_{i=1}^{\infty} \bigcup_{n\geq k} E_n^c] = \lim_{k\to\infty} \mathbb{P}[\bigcup_{n\geq k} E_n^c] = \lim_{k\to\infty} \prod_{n\geq k} \mathbb{P}[E_n^c] = 0,$$

which means that a jump of size greater than d occurs infinitely often, and cannot be compensated by the continuous stochastic integral ${}_0^t\sigma K_s\, dW_s$. Then the wealth stopped at the time of ruin cannot converge to a nonzero value with positive probability. □

THEOREM 1.12 *Assume that Y has a uniform exponential in the tail distribution for $\hat{r}$. Then for every $K \in \mathcal{K}$:*

$$\psi(x,K) \geq Ce^{-\hat{r}x},$$

where

$$C = \inf_{y\geq 0} \frac{\displaystyle\int_y^\infty dF(u)}{\displaystyle\int_y^\infty e^{-\hat{r}(y-z)} dF(z)}.$$

Proof. We know that $\tilde{M}(t,x,K,\hat{r})$ is a uniformly integrable submartingale, then from Doob's optional sampling theorem we obtain:

$$\begin{aligned} \tilde{M}(0,x,K,\hat{r}) &= e^{-\hat{r}x} \\ &\leq \mathbb{E}[\tilde{M}(\tau,x,K,\hat{r})] \end{aligned}$$

and

$$\mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})] =$$
$$\mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) \mid \tau < \infty]\mathbb{P}[\tau < \infty] + \mathbb{E}[\lim_{t\to\infty} \tilde{M}(t, x, K, \hat{r}) \mid \tau = \infty]\mathbb{P}[\tau = \infty]$$

Then by Lemma 1.10, we obtain:

$$e^{-\hat{r}x} \le \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r})] = \mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) \mid \tau < \infty]\mathbb{P}[\tau < \infty],$$

then proceeding as in the proof of Theorem 1.11, we see that:

$$\begin{aligned} \psi(x, K) = \mathbb{P}[\tau < \infty] &\ge \frac{e^{-\hat{r}x}}{\mathbb{E}[\tilde{M}(\tau, x, K, \hat{r}) \mid \tau < \infty]} \\ &\ge \inf_{y\ge 0} \frac{1}{\int_y^\infty e^{-\hat{r}(y-z)} \frac{dF(z)}{\int_y^\infty dF(u)}} \\ &\ge Ce^{-\hat{r}x}, \end{aligned}$$

where

$$C = \inf_{y\ge 0} \frac{\displaystyle\int_y^\infty dF(u)}{\displaystyle\int_y^\infty e^{-\hat{r}(y-z)} dF(z)}.$$

□

Finally, Hipp and Schmidli (2004) obtain an equivalent asymptotic behavior for the ruin probability as in the case of the Cramér-Lundberg model without investment.

Theorem 1.13 *(Hipp and Schmidli, [21]) There exists a constant* $\xi \in (0, \infty)$ *such that*

$$\lim_{x\to\infty} \psi(x)e^{\hat{r}x} = \xi.$$

1.5 Optimization and Utility Functions

The study of the optimization of the expected utility function is very important in both finance and insurance. Ferguson (1965) conjectured that maximizing exponential utility from terminal wealth is strictly related to minimizing the probability of ruin. Ferguson studied the problem of expected utility of wealth under a discrete time-sapce model for the investor. Browne (1995) verified the conjecture in the same model mentioned above, but without the risk-free interest rate, by assuming that the investor has an exponential utility function:

$$u(x) = \nu - \frac{\gamma}{\theta} e^{-\theta x}, \tag{1.26}$$

where $\gamma > 0$ and $\theta > 0$. Since $-\dfrac{u''(x)}{u'(x)} = \theta > 0$, then this utility function features risk aversion. According to Browne:

Such utility functions play a prominent role in insurance mathematics and actuarial practice, since they are the only functions under which the principle zero utility gives a fair premium that is independent of the level of reserves of an insurance company (see [14], p. 68).

Let $\mathcal{K}$ be the set of non-anticipative functions satisfying (1.11). The value function considered in [3] is given by: $V(t,x) = \sup_{K \in \mathcal{K}} \mathbb{E}[u(X_T^K) \mid X_t^K = x]$, which consists in maximizing utility from terminal wealth at a fixed terminal time. $V(t,x)$ satisfies the following HJB equation:

$$\begin{cases} V_t + \max_{K \in \mathbb{R}} \left[(\alpha + \mu K) V_x + \frac{1}{2}(\sigma^2 K^2 + \beta^2 + 2\rho\sigma\beta K) V_{xx} \right] = 0, \; t < T. \\ V(T,x) = 1 \text{ for } x \in [0, \infty[\end{cases} \tag{1.27}$$

Browne arrived at the result that the optimal strategy that minimizes the ruin probability is also optimal in maximizing exponential utility from terminal wealth. In the presence of a positive interest rate, this equivalence does not hold. Yang and Zhang (2005) considered the classical risk process perturbed by a standard Brownian motion W_t^1 such that $\mathbb{E}[W_t W_t^{(1)}] = \rho t$, where $\rho^2 \neq 1$. The insurer is allowed to invest in the money market and a risky asset given by (1.10). The money market is modeled by:

$$dS_t^0 = rS_t^0 dt. \tag{1.28}$$

If K_t is the amount invested at time t in the risky asset and the rest of the wealth $(X_t - K_t)_{t\geq 0}$ is invested in the money market, then the wealth process evolves as:

$$X_t^K = x + \int_0^t c + \Big((\mu - r)K_s + rX_s\Big)\, ds + \int_0^t \sigma K_s \, dW_s + \beta \int_0^t dW_t^1 - \sum_{i=1}^{N_t} Y_i.$$

Let u be the exponential utility function given by (1.26), and let $\mathcal{K}$ be the same set considered by Browne (1995), then the value function

$$V(t,x) = \sup_{K\in\mathcal{K}} \mathbb{E}[u(X_T^K) \mid X_t^K = x]$$

satisfies the following HJB equation:

$$\begin{cases} V_t + \sup\limits_{K\in\mathbb{R}} \left[(c + rx + (\mu - r)K)V_x + \dfrac{1}{2}(\sigma^2 K^2 + \beta^2 + 2\rho\sigma\beta K)V_{xx}\right] \\ +\lambda \displaystyle\int_0^x V(t, x-y) - V(t,x)\, dG(y) = 0, \; t < T, \\ V(T,x) = 1 \text{ for } x \in [0,\infty[. \end{cases} \tag{1.29}$$

By assuming that the HJB equation (1.29) has a classical solution V, which satisfies $V_x > 0$ and $V_{xx} < 0$, then differentiating with respect to K, they obtained a closed form expression of the optimal strategy:

$$K^*(t) = -\frac{\mu - r}{\sigma^2}\frac{V_x}{V_{xx}} - \frac{\rho\beta}{\sigma}. \tag{1.30}$$

By inserting (1.30) into the HJB equation (1.29) and after simplifying, we obtain the following nonlinear PDE:

$$\begin{cases} V_t + \left[\alpha + rx - \dfrac{\rho\beta(\mu - r)}{\sigma}\right] V_x - \dfrac{1}{2}\left(\dfrac{\mu - r}{\sigma}\right)^2 \dfrac{V_x^2}{V_{xx}} + \dfrac{1}{2}\beta^2(1-\rho^2)V_{xx} \\ +\lambda \displaystyle\int_0^x V(t, x-y) - V(t,x)\, dG(y) = 0, \; t < T, \\ V(T,x) = 1 \text{ for } x \in [0,\infty[. \end{cases} \tag{1.31}$$

To solve (1.31), Yang and Zhang – inspired by Browne (2005) – tried to find a solution of the form:

$$V(t,x) = c - \frac{\gamma}{\theta} \exp\left\{ \theta x e^{r(T-t)} - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (T-t) + w(T-t) \right\} \quad (1.32)$$

Plugging (1.32) into equation (1.31), $w(t)$ satisfies the following ordinary differential equation:

$$\begin{cases} w'(t) &= -\theta\left[\alpha - \frac{\rho\beta(\mu - r)}{\sigma}\right] e^{r(T-t)} + \frac{1}{2}\theta^2\beta^2(1-\rho^2)e^{2r(T-t)} \\ & + \ \lambda \int_0^\infty \left[\exp\left\{\theta y e^{r(T-t)}\right\} - 1\right] dG(y), \ t < T, \\ w(0) &= 0. \end{cases} \quad (1.33)$$

It is clear that a closed form solution of $w(\cdot)$ depends on the distribution of Y. Finally, the optimal strategy of investment is given by:

$$K^*(t) = \frac{\mu - r}{\theta\sigma^2} e^{-r(T-t)} - \frac{\rho\beta}{\sigma}.$$

Fernández et al. (2008) considered the same model as Yang and Zhang (2005) but without perturbation ($\beta = 0$). The wealth process can then be written as:

$$X_t^K = x + \int_0^t c + \Big((\mu - r)K_s + rX_s\Big)\, ds + \int_0^t \sigma K_s \, dW_{1s} - \sum_{i=1}^{N_t} Y_i.$$

In this case, the set of admissible strategies $\mathcal{K}$ is defined as follows:

Definition 1.3 *We say that $K = (K_t)_{t \geq 0}$ is an admissible strategy if it is an $\mathcal{F}_t$-progressively measurable process such that:*

$$\mathbb{P}[| K_t | \leq C_K, 0 \leq t \leq T] = 1,$$

where C_K is a constant which may depend on the strategy K. We denote the set of admissible strategies as $\mathcal{K}$.

Let u be a utility function, then the value function to maximize is:

$$\psi(t,x) = \sup_{K \in \mathcal{K}} \mathbb{E}[u(X_T^K) \mid X_t^K = x],$$

which consists in maximizing the expected utility of wealth with finite horizon $T > 0$. Via the Hamilton-Jacobi-Bellman approach, under suitable conditions of integrability

$$\psi(t,x) = \sup_{K \in \mathcal{K}} \mathbb{E}[u(X_T^K) \mid X_t^K = x]$$

satisfies the following Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \psi_t + \sup_K \left[(c + rx + (\mu - r)K)\psi_x + \frac{1}{2}\sigma^2 K^2 \psi_{xx} \right] \\ +\lambda \int_0^x \psi(t, x-y) - \psi(t,x)\, dG(y) = 0, \;\; t < T, \\ \psi(T,x) = 1 \text{ for } x \in [0, \infty[. \end{cases} \tag{1.34}$$

To gain precision in the study of the utility function, Fernández et al. (2008) considered the case in which the insurer's preferences are exponential, i.e., $u(x) = -e^{-\alpha x}$ and $\alpha > 0$. In this case, the HJB equation given by (1.34) has an explicit solution which can be written in closed form as

$$\psi(t,x) = -\xi(t) \exp\left\{-\alpha x e^{r(T-t)}\right\},$$

where $\xi(t)$ is the solution to the following ordinary differential equation:

$$\xi_t - \left[\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} - \lambda \theta_s + c\alpha e^{r(T-t)} \right] \xi = 0,$$

and $\theta_t = \int_0^\infty \left(\exp\left\{\alpha y e^{r(T-t)}\right\} - 1 \right) dG(y)$. The optimal expected utility function is given by:

$$\begin{aligned} \psi(t,x) &= \exp\left\{ -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2}(T-t) + \frac{c\alpha}{r}(1 - e^{r(T-t)}) + \lambda \int_t^T \theta_s \, ds \right\} \\ &\quad . \exp\left\{-\alpha x e^{r(T-t)}\right\}. \end{aligned} \tag{1.35}$$

A candidate for the optimal strategy is:

$$K^*(t) = \frac{\mu - r}{\alpha\sigma^2} e^{-r(T-t)}. \tag{1.36}$$

Finally, Fernández et al. (2008) considered the wealth process $X_t^{K^*}$ under the optimal strategy (1.36). They obtained the following estimation for the ruin probability:

$$\psi(x) \leq e^{-\nu^* x},$$

where ν^* is the positive solution to the equation:

$$h_r(\nu) = -\nu \left(c + \frac{(\mu - r)^2}{\alpha\sigma^2} \right) e^{-rT} + \frac{\nu^2}{2} \frac{(\mu - r)^2}{\alpha^2\sigma^2} e^{-2rT} + \lambda(M_Y(\nu) - 1) = 0.$$

OBSERVATION 1.9 *If the interest rate of the money market $r = 0$, the approach by Fernández et al. (2008) can be used to recover some known results in risk theory (the upper bound by Gaier et al. (2003) and the optimal strategy by Hipp and Schmidli (2004)). The idea consists in considering for each $\alpha > 0$, the root $\nu(\alpha)$ of $h_0(\nu)$. By using the implicit function theorem, they showed that $\nu(\alpha)$ reaches its maximum when $\nu(\alpha) = \alpha$, and this is precisely the value for which the upper bound obtained by Gaier et al. (2003), and the asymptotically optimal strategy by Hipp and Plum (2004) can be recovered.*

The case in which the claim process is a pure jump process and the insurer has the option of investing in multiple risky assets without a risk-free option was studied by Wang (2007). Wang found that the optimal strategy of maximizing the exponential utility of terminal wealth consists in investing a fixed amount of money in each risky asset, while to get the optimal form of reinsurance from the ceding company, Guerra and Centeno (2008) studied the relationship between maximizing the adjustment coefficient and maximizing the expected utility of wealth for the exponential utility function.

Chapter 2

The Stochastic Volatility Model

This chapter is devoted to formulating the problem of our research, which consists in a model of an insurance company allowed to invest in a risky asset and a bank account in the presence of stochastic volatility. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which carries the following independent stochastic processes:

- A Poisson process $\{N_t\}_{t\geq 0}$ with intensity $\lambda > 0$ and jump times $\{T_i\}_{i\geq 1}$.
- A sequence $\{Y_i\}_{i\geq 1}$ of i.i.d. positive random variables with common distribution G.
- W_{1t} and W_{2t} are independent standard Brownian motions.
- The filtration $\mathcal{F}_t$ is defined by

  $$\mathcal{F}_t = \sigma \left\{ W_{1s}, W_{2s}, Y_i \mathbf{1}_{[i \leq N_s]}, 0 \leq s \leq t, i \geq 1 \right\}$$

  with the usual conditions.

In previous articles ([7, 13, 18, 20, 42]), the asset price was modeled by a geometric Brownian motion given by

$$dS_t = S_t(\mu dt + \sigma dW_t). \tag{2.1}$$

Empirical observations of financial markets show that some indicators of market volatility behave in a highly erratic manner, which makes it unrealistic to assume μ, σ and r are constant over long periods of time. This fact has motivated several authors to study the so-called stochastic volatility models (see among others [8, 38, 43]). Here, we consider an extension of this model. If the parameters in (2.1) are stochastic (see [4, 9]), then the asset price satisfies the following stochastic differential equation:

$$dS_t = S_t(\mu(Z_t)dt + \sigma(Z_t)dW_{1t}) \quad \text{with } \; S_0 = 1 \tag{2.2}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are respectively the return rate and volatility functions. Z is an external factor modeled as a diffusion process solving

$$dZ_t = g(Z_t)dt + \beta(\rho dW_{1t} + \varepsilon dW_{2t}) \quad \text{with } Z_0 = z \in \mathbb{R} \tag{2.3}$$

where $|\ \rho\ | \leq 1, \varepsilon = \sqrt{1-\rho^2}$ and $\beta \neq 0$. The parameter ρ is the correlation coefficient between W_{1t} and $\tilde{W} = \rho W_{1t} + \varepsilon W_{2t}$, thus the external factor can be written as

$$Z_t = z + \int_0^t g(Z_s)ds + \beta \int_0^t d\tilde{W}_s. \tag{2.4}$$

Our model also contains a bank account given by the equation

$$dS_t^0 = S_t^0 r(Z_t)dt, \tag{2.5}$$

where $r(\cdot)$ is the interest rate function. The process Z_t can be interpreted as the behavior of some economic factor that has an impact on the dynamics of the risky asset and the bank account (see for example [2, 4, 9, 30]). For instance, the external factor can be modeled by the mean reverting Ornstein-Uhlenbeck (O-U) process:

$$dZ_t = \delta(\kappa - Z_t)\,dt + \beta\,d\tilde{W}_t, \quad Z_0 = z$$

where δ and κ are constant and the risky asset price can be given by the Scott model [30]:

$$dS_t = S_t(\mu_0 dt + e^{Z_t} dW_{1t}) \quad \text{with } \; S_0 = 1. \tag{2.6}$$

Here, we assume that μ_0 is constant.
Let $\mathcal{K}$ be the set of all admissible adapted strategies, i.e., K_t is a non-anticipative function (see [30], p.40) and satisfies, for any T,

$$\int_0^T K_t^2 \sigma^2(Z_t)\, dt < \infty, a.s. \tag{2.7}$$

Then if X_t is the insurer's wealth, and he invests an amount $K_t \in \mathcal{K}$ in the risky asset and the remaining reserve $X_t - K_t$ in the bank account, then the wealth process $X_t^K := X(t,x,z,K)$ can be written as:

$$X_t^K = x + \int_0^t c + \Big((\mu(Z_s) - r(Z_s))K_s + r(Z_s)X_s \Big) ds + \int_0^t K_s \sigma(Z_s)\, dW_{1s} - \sum_{i=1}^{N_t} Y_i, \tag{2.8}$$

where $x \geq 0$ is the initial reserve of the insurance company and c is the constant premium rate. The time of ruin for the risk process given by (2.8) is defined as:

$$\tau^K := \tau(x,z,K) = \inf \left\{ t > 0, X_t^K < 0 \right\}.$$

The ruin probability and the survival probability are given, respectively, by:

$$\psi(x,z,K) = \mathbb{P}[\tau^K < \infty],$$

$$\delta(x,z,K) = \mathbb{P}[\tau^K = \infty].$$

The main purpose of risk theory is the study of the following value functions:

$$\psi(x,z) = \inf_{K \in \mathcal{K}} \psi(x,z,K), \tag{2.9}$$

and

$$\delta(x,z) = \sup_{K \in \mathcal{K}} \delta(x,z,K). \tag{2.10}$$

Rubio (2010) recently studied a control problem in his Ph.D. dissertation, which consists in the optimization of the ruin probability for an insurer allowed to invest in a risky asset defined by the stochastic volatility model given above. He obtained a verification theorem which relates the optimization problem with the HJB equation. The proof of the verification theorem is inspired in the works of Hipp and Plum (2000) and Schmidli (2002). Since volatility problems are very difficult to solve using traditional theory, he applied a technique based on stopping times to get the convergence property of the risk process, which was the key to solving the asymptotic behavior of wealth on $\{\tau = \infty\}$. Because of the importance of the results of Rubio (2010) and their strong connection with our work, we will recall his main result with a sketch of the proof he provided. First we start by introducing some hypotheses and lemmas which will be useful in this chapter.

HYPOTHESIS 2.1 *1.* $r = 0$.

2. *g is Lipschitz continuous with linear growth and satisfies:*

$$\int_0^x \exp\left\{-\int_0^z g(u)\right\} dz \underset{x\to\pm\infty}{\longrightarrow} \infty .$$

3. $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ *are bounded Lipschitz continuous and satisfy:*

$$0 < \sigma_0 < \sigma(\cdot) < \sigma_1, 0 < \mu_0 \le \mu(\cdot),$$

for some constants $\mu_0, \sigma_0, \sigma_1$.

HYPOTHESIS 2.2

g is Lipschitz continuous with linear growth and satisfies:

$$\int_0^x \exp\left\{-\int_0^z g(u)\right\} dz \underset{x\to\pm\infty}{\to} \infty .$$

OBSERVATION 2.1 *Hypothesis 2.1 implies that the process Z_t is recurrent, i.e., for all $a, b \in \mathbb{R}$*

$$\mathbb{P}_a\left[Z_t = b \text{ } i.o\right] = 1.$$

Since its diffusion coefficient never vanishes, the process satisfies

$$\mathbb{P}\left[\sup_{t>0} Z_t = \infty\right] = \mathbb{P}\left[\inf_{t>0} Z_t = -\infty\right] = 1.$$

The main purpose is to characterize the value function $\delta(x,z)$. Following the same arguments of classical control theory, Rubio (2010) related the value function given by (2.10) with the following HJB equation:

$$\sup_{K\in\mathbb{R}}\left\{\mathscr{L}^K(f)+\lambda\int_0^\infty\left(f(t,x-y,z)-f(t,x,z)\right)\,dG(y)\right\} \tag{2.11}$$

where

$$\begin{aligned}\mathscr{L}^K(f) &= f_t+\tfrac{1}{2}\sigma^2(z)K^2f_{xx}+\tfrac{1}{2}\beta^2f_{zz}+\rho\beta K\sigma(z)f_{xz}\\ &\quad +(c+(\mu(z)-r(z))K+r(z)x)f_x+g(z)f_z.\end{aligned}$$

If $f_{xx}<0$, the supremum is attained at

$$K^*(x,z)=\frac{\mu(z)f_z+\rho\beta\sigma(z)f_{xz}}{\sigma^2(z)f_{xx}}. \tag{2.12}$$

More details about the derivation of this equation are given in Chapter 4.

THEOREM 2.1 *(Verification Theorem, Rubio 2010) Assume Hypothesis 2.1 or 2.2. Assume also that there exists a solution $f(x,z)$ of the HJB given by* (2.11) *with a maximizing function given by $K^*(x,z)$, that is locally Lipschitz continuous, with the following properties:*

1. *$f(x,z)=0$ for $(x,z)\in(-\infty,0)\times\mathbb{R}$.*
2. *$f\in C^2[(0,\infty)\times\mathbb{R}]\cap C[[\,0,\infty\,)\times\mathbb{R}]$.*
3. *$f_x>0$ and $f_{xx}<0$.*
4. *$K^*(0,\cdot)=0$.*

Then f is bounded. Furthermore, $f(\infty,z)$ is constant and for any admissible strategy K, the following inequality is satisfied:

$$\delta(x,z,K)\le\frac{f(x,z)}{f(\infty,z)}\le\delta(x,z,K^*),$$

where $K^*_t=K^*(X_{t-},Z_t)$, and hence we obtain the equality for this strategy.

To prove this theorem, we need the following lemmas.

LEMMA 2.1 *We consider the stochastic processes γ_t, μ_t and σ_t such that $0 < \mu_0 \leq \mu_t$, $0 < \sigma_0 \leq \sigma_t \leq \sigma_1$ for some constants $\mu_0, \sigma_0, \sigma_1$. We also assume that μ_t and σ_t are continuous processes. For $a, b, c > 0$ we define the process:*

$$\pi_t = \gamma_t + a + b\int_0^t \mu_s\, ds + c\int_0^t \sigma_s\, dW_s.$$

If $\gamma_t \geq 0$ for some $K \in \mathcal{F}$, then:

$$\pi_t \underset{t\to\infty}{\rightarrow} \infty$$

over the set $\mathcal{K}$.

LEMMA 2.2 *Let*

$$\pi_t = x + \alpha\int_0^t \mu_s\, ds + \beta\int_0^t \sigma_s\, dW_s,$$

with $0 < \sigma < \sigma_1$ and $0 \leq \mu_0 \leq \mu$, then:

$$\mathbb{P}[for\ some\ t; \pi_t < 0] \leq \exp\left\{\frac{-2\alpha\mu_0}{\beta^2\sigma_1^2}x\right\}.$$

Proof.(Verification Theorem)
Case 1: Assume Hypothesis 2.1.
Step 1. The lower bound:
Let $K \in \mathcal{K}$ be any admissible strategy and X_t^K be the risk process given by (2.8). To prove that f is bounded over $\{\tau^K = \infty\}$, the idea consists in showing that the process $X_t^K \underset{t\to\infty}{\rightarrow} \infty$. Hence, we consider a family of risk processes asymptotically close to the process X_t^ε, with the property $X_t^\varepsilon \underset{t\to\infty}{\rightarrow} \infty$ over the set $\{\tau^K = \infty\}$. This idea was motivated by Hipp and Plum in [18] and [19].
Define K^ε as $K_t^\varepsilon = K_t + \varepsilon$, and $X^\varepsilon(0) = x + \varepsilon$ with ruin time τ^ε. Now we

analyze the process X_t^ε given by:

$$\begin{aligned} X_t^\varepsilon &= x+\varepsilon+\int_0^t c+\Big((\mu(Z_s)-r(Z_s))(K_s+\varepsilon^2)+r(Z_s)X_s\Big)\,ds \\ &+ \int_0^t (K_s+\varepsilon^2)\sigma(Z_s)\,dW_{1s}-\sum_{i=1}^{N_t} Y_i. \end{aligned} \tag{2.13}$$

Since

$$\left\{\tau^\varepsilon<\tau^K\right\}\subset\left\{\exists\, t, \varepsilon+\varepsilon^2\int_0^t \mu(Z_s)\,ds+\varepsilon^2\int_0^t \sigma(Z_s)\,dW_{1s}<0\right\},$$

by Lemma 2.2 we have:

$$\mathbb{P}[\tau^\varepsilon<\tau^K]\le \exp\left\{\frac{-2\mu_0}{\sigma_1^2}\right\}.$$

Then, thanks to Lemma 2.1 we obtain $X_t^\varepsilon \underset{t\to\infty}{\to} \infty$ over the set $\left\{\tau^K=\infty\right\}$. To prove the boundness of f we follow the approach by Schmidli (2002). We consider the following stopping times. Let $a\in\mathbb{R}$, and $n,m,M\in\mathbb{N}$. Let

$$\alpha_n^\varepsilon := \inf\left\{t>0, X_t^\varepsilon\notin[0,n]\right\},$$

$$\gamma_M := \inf\left\{t>0, Z_t\notin[-M,M]\right\},$$

$$\nu_m^a := \inf\left\{t>m, Z_t=a\right\}.$$

From the non-explosion in finite time of the process Z_t and Observation 2.1, we have the following properties:

$$\gamma_M \underset{M\to\infty}{\to} \infty,\ a.s,$$

$$\nu_m^a<\infty, \nu_m^a \underset{m\to\infty}{\to} \infty,\ a.s,$$

and

$$\alpha_n^\varepsilon \underset{n\to\infty}{\to} \tau^\varepsilon,\ a.s.$$

For ease of notation, $X_t^K := X^K(t)$. The process

$$\left\{f(X^\varepsilon,Z)(t\wedge\alpha_n^\varepsilon\wedge\gamma_M\wedge\nu_m^a)-\int_0^{t\wedge\alpha_n^\varepsilon\wedge\gamma_M\wedge\nu_m^a}\mathscr{L}^{K_s^\varepsilon}(X^\varepsilon(s-),Z_s)\,ds\right\}_{t\ge0}.$$

is a local martingale. More details about this fact can be seen in Chapters 3 and 4.
Let $\{\pi_k\}_{k=1}^{\infty}$ be a sequence of localization times such that $\lim_{k\to\infty} \pi_k = \infty$, a.s. Since for $0 < s < t \wedge \alpha_n^\varepsilon \wedge \gamma_M \wedge \nu_m^a \wedge \pi_k$:

$$\mathscr{L}^{K_s^\varepsilon} \leq 0,$$

then for all $t \geq 0, n, m, M, k \in \mathbb{N}$

$$\mathbb{E}[f(X^\varepsilon, Z)(t \wedge \alpha_n^\varepsilon \wedge \gamma_M \wedge \nu_m^a \wedge \pi_k)] \leq f(x+\varepsilon, z).$$

Over the set $]-\infty, n] \times [-M, M]$, f is bounded, so by the dominated convergence theorem:

$$f(x+\varepsilon, z) \geq \mathbb{E}[f(X^\varepsilon, Z)(t \wedge \alpha_n^\varepsilon \wedge \gamma_M \wedge \nu_m^a)].$$

Since $f \geq 0$, Fatou's lemma for $n \to \infty$ and $M \to \infty$ implies:

$$f(x+\varepsilon, z) \geq \mathbb{E}[f(X^\varepsilon, Z)(t \wedge \tau^\varepsilon \wedge \nu_m^a)].$$

Letting $t \to \infty$, and since $\nu_m^a < \infty, a.s.$

$$f(x+\varepsilon, z) \geq \mathbb{E}[f(X^\varepsilon(\nu_m^a), a)\mathbb{1}_{\{\tau^\varepsilon=\infty\}}].$$

Then

$$f(x+\varepsilon, z) \geq \mathbb{E}[f(X^\varepsilon(\nu_m^a), a)\mathbb{1}_{\{\tau^\varepsilon=\infty, \tau^K=\infty\}}].$$

Letting $m \to \infty$, we get:

$$f(x+\varepsilon, z) \geq f(\infty, a)\mathbb{P}[\tau^\varepsilon = \infty, \tau^K] \geq f(\infty, a)\left(\mathbb{P}[\tau^K = \infty] - \exp\left(\frac{-2\mu_0}{\sigma_1\varepsilon}\right)\right).$$

Finally:

$$\begin{aligned}\mathbb{P}[\tau^K = \infty] &= \mathbb{P}[\tau^\varepsilon = \infty, \tau^K = \infty] + \mathbb{P}[\tau^\varepsilon = \infty^K, \tau^\varepsilon < \tau^K] \\ &\leq \mathbb{P}[\tau^\varepsilon = \infty, \tau^K = \infty] + \mathbb{P}[\tau^\varepsilon < \tau^K].\end{aligned}$$

Letting $\varepsilon \downarrow 0$:

$$f(x,z) \geq f(\infty,a)\mathbb{P}[\tau^K = \infty].$$

For the strategy $K \equiv 0$, $\mathbb{P}[\tau^0 = \infty] > 0$, since $f(x,z)$ is finite, for all $a \in \mathbb{R}$, $f(\infty,z)$ is finite and:

$$0 < \mathbb{P}[\tau^0 = \infty] \leq \frac{f(x,z)}{f(\infty,a)}$$

which implies that $\limsup_{a\to\pm\infty} f(\infty,a) < \infty$, so $f(\infty,a)$ is a bounded function. Since f is increasing in x $f(x,z) \leq f(\infty,a)$, then f is bounded. Finally, for any admissible strategy K and $a \in \mathbb{R}$:

$$\delta(x,z,K) \leq \frac{f(x,z)}{f(\infty,a)}.$$

Step 2. The upper bound:
Let X^* be the risk process with investment strategy K^*. For this process, we define the following stopping times:

$$\alpha_n^* := \inf\left\{t > 0, X_t^* \notin [0,n]\right\},$$

$$\gamma_M := \inf\left\{t > 0, Z_t \notin [-M,M]\right\},$$

$$\nu_m^a := \inf\left\{t > m, Z_t = a\right\}.$$

Proceeding as in step 1, for $0 < s < t \wedge \alpha_n^* \wedge \gamma_M \wedge \nu_m^a$ we have that

$$\left\{f(X^*,Z)(t \wedge \alpha_n^* \wedge \gamma_M \wedge \nu_m^a)\right\}_{t\geq 0}.$$

is a martingale. Then, for all $t \geq 0, n, m, M \in \mathbb{N}$,

$$f(x,z) = \mathbb{E}[f(X^*,Z)(t \wedge \alpha_n^* \wedge \gamma_M \wedge \nu_m^a)].$$

By letting $n \to \infty, M \to \infty$, we get

$$f(x,z) = \mathbb{E}[f(X^*,Z)(t \wedge \tau^* \wedge \nu_m^a)].$$

Letting $t \to \infty$, we have:

$$f(x,z) = \mathbb{E}[f(X^*, Z)(\tau^* \wedge \nu_m^a)].$$

$$\begin{aligned} f(x,z) &= \mathbb{E}[f(X^*, Z)(\tau^*, \nu_m^a)\left(\mathbb{1}_{\{\tau^*=\infty\}} + \mathbb{1}_{\{\tau^*<\infty\}}\right)] \\ &\leq f(\infty, a)\mathbb{P}[\tau^* = \infty] + \mathbb{E}[f(X^*, Z)(\tau^* \wedge \nu_m^a)\mathbb{1}_{\{\tau^*<\infty\}}]. \end{aligned}$$

Since $\nu_m^a \underset{m\to\infty}{\to} \infty \, a.s$, and letting $m \to \infty$:

$$f(x,z) \leq f(\infty, a)\mathbb{P}[\tau^* = \infty] = \mathbb{E}[f(0, Z_{\tau^*})\mathbb{1}_{\{\tau^*<\infty, X^*(\tau^*)=0\}}],$$

thus $\mathbb{P}[\tau^* < \infty, X^*(\tau^*) = 0]$, which is true since the process (X^*, Z) is a strong Markov process. Finally, we obtain:

$$\delta(x,z,K) \leq \frac{f(x,z)}{f(\infty,z)} = \delta(x,z,K^*).$$

Case 2 Assume Hypothesis 2.2.
In this case, $0 < r_0 \leq r(\cdot)$. Again, we follow the idea set forth by Hipp and Plum in [18] and [19]. Define X_t^ε with initial wealth $x + \varepsilon$, ruin time τ^ε and strategy K_t. Thus, we have:

$$X_t^\varepsilon - X_t^K = \varepsilon + \int_0^t r(Z_s)(X_s^\varepsilon - X_s)\, ds,$$

then:

$$X_t^\varepsilon = X_t^K + \varepsilon \exp\left\{\int_0^t r(Z_s)\, ds\right\}.$$

Over the set $\left\{\tau^K = \infty\right\}$, it must be that $X_t^K \geq 0$, then:

$$\begin{aligned} X_t^\varepsilon &\geq \varepsilon \exp\left\{\int_0^t r(Z_s)\, ds\right\} \\ &\geq \varepsilon \exp\{r_0 t\} \underset{t\to\infty}{\to} \infty \end{aligned}$$

We have $X_t^\varepsilon \geq X_t^K$ a.s., and then:

$$\mathbb{P}[\tau^\varepsilon < \tau^K] = 0.$$

Since the proof is exactly the same as when $r = 0$, we only prove that there exists a strategy K such that $\mathbb{P}[\tau^K = \infty] > 0$. We consider the following processes:

$$X_t^0 = x - \varepsilon + ct - \sum_{i=1}^{N_t} Y_i$$

and

$$X_t^1 = x + \int_0^t (c + r(Z_s))X_{s-}^K \, ds - \sum_{i=1}^{N_t} Y_i.$$

Then $\{X_t^0 - X_t^1\}_{t\geq 0}$ is a continuous process. Over the set $\{\tau_0 = \infty\}$, we obtain that:

$$\begin{aligned} X_t^0 - X_t^1 &= -\varepsilon - \int_0^t r(Z_s)X_{s-}^1 \, ds \\ &\leq +\varepsilon - \int_0^t r(Z_s)(X_s^0 - X_{s-}^1) \, ds. \end{aligned}$$

An application of Gronwall's lemma (see Appendix A), over the set $\{\tau_0 = \infty\}$ leads to:

$$X_t^0 - X_t^1 \leq -\varepsilon \exp\left\{ - \int_0^t r(Z_s) \, ds \right\},$$

thus:

$$X_t^1 \geq X_t^0.$$

Finally, we conclude that:

$$\mathbb{P}[\tau_1 = \infty] \geq \mathbb{P}[\tau_0 = \infty].$$

□

Chapter 3

The Ruin Probability

In this chapter, we consider a model for an insurance company where the insurer has to face a claims process, which follows a compound Poisson process with finite exponential moments. The insurer is allowed to invest in a bank account and in a risky asset described by a geometric Brownian motion with stochastic volatility that depends on an external factor modeled as a diffusion process. The main purpose is to obtain upper and lower bounds for the ruin probabilities that recover the known bounds for constant volatility models. Finally, we apply the results to a truncated Scott model.

3.1 Introduction

Let X_t^K be the wealth of an insurer given by (2.8). Our aim in this chapter is to obtain bounds for the expression

$$\psi(x,z) = \inf_{K \in \mathcal{K}} \psi(x,z,K),$$

defined in chapter 2, under the following hypothesis:

HYPOTHESIS 3.1 *1. The functions $\mu(\cdot)$, $\sigma(\cdot)$ and $g(\cdot)$ are such that there exists a strong solution for equations* (2.2) *and* (2.4). *The function $r(\cdot)$*

is continuous, positive, and

$$r(z) < \mu(z), \text{ for all } z \in \mathbb{R}.$$

2. *Let Y be a random variable with the common distribution G of the claims. There exists $\theta_\infty \in (0, \infty]$ such that $M_Y(\theta) = \mathbb{E}[e^{\theta Y}] < \infty$ for all $\theta \in [0, \theta_\infty)$, and $h(\theta) = M_Y(\theta) - 1$ satisfies*

$$\lim_{\theta \to \theta_\infty} h(\theta) = \infty.$$

Hypothesis 3.1 will be implicitly assumed in what follows.

3.2 Decomposition of $e^{-\theta(Z_t)X_t^K}$

In this section, we will introduce some notation that will be used throughout the rest of this chapter. We will denote by

$$\alpha_t := \int_0^t r(Z_s)\, ds.$$

Let $\theta : \mathbb{R} \to]0, \infty[$ in $C_b^2(\mathbb{R})$ (twice differentiable functions with bounded derivatives), and K be an admissible strategy, then Itô's lemma implies that:

$$\begin{aligned} e^{-\alpha_t}\theta(Z_t)X_t^K \quad = \quad & \theta(z)x + \int_0^t e^{-\alpha_s}\theta(Z_s)(c + (\mu(Z_s) - r(Z_s))K_s)\, ds + \\ & \int_0^t e^{-\alpha_s}(\theta(Z_s)K_s\sigma(Z_s) + \rho\beta\theta'(Z_s)X_s)\, dW_{1s} + \\ & \int_0^t \varepsilon\beta e^{-\alpha_s}\theta'(Z_s)X_s\, dW_{2s} + \int_0^t e^{-\alpha_s}\theta'(Z_s)X_s g(Z_s)\, ds + \\ & \frac{1}{2}\int_0^t \beta^2 e^{-\alpha_s}\theta''(Z_s)X_s\, ds + \int_0^t \rho\beta\sigma(Z_s)K_s e^{-\alpha_s}\theta'(Z_s)\, ds - \\ & \int_0^t \int_0^\infty y\theta(Z_s)e^{-\alpha_s}\, \bar{N}(dy, ds). \end{aligned}$$

where $\bar{N}$ is the Poisson random measure on $\mathbb{R}_+ \times [0, \infty[$ defined by

$$\bar{N} = \sum_{n \geq 1} \delta_{(T_n, Y_n)}.$$

PROPOSITION 3.1 *For each function* $\theta \in C_b^2((\mathbb{R})$ *and* $K \in \mathcal{K}$, *let*

$$H_\theta^{K,r}(t,x,z) = \exp\left\{-e^{-\alpha_t}\theta(Z_t)X_t^K\right\}.$$

Then

$$H_\theta^{K,r}(t,x,z) = e^{-\theta(z)x} e^{\mathcal{D}_\theta^K(t)+\mathcal{E}_\theta(t)} e^{\int_0^t f_{\theta,r}(e^{-\alpha_s}, X_s, Z_s, K_s)\, ds},$$

where

$$f_{\theta,r} : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

is given by

$$\begin{aligned} f_{\theta,r}(u,x,z,K) \;&=\; \lambda h(\theta(z)u) - (c + (\mu(z) - r(z))K)\theta(z)u - xu\theta'(z)g(z) \\ &\quad -\tfrac{1}{2}\beta^2 xu\theta''(z) + \tfrac{1}{2}(Ku\sigma(z)\theta(z) + \rho\beta u\theta'(z)x)^2 \\ &\quad +\tfrac{1}{2}\varepsilon^2\beta^2u^2x^2\theta'^2(z) - \rho\beta uK\sigma(z)\theta'(z). \end{aligned} \tag{3.1}$$

$\mathcal{D}_\theta^K(t)$ *and* $\mathcal{E}_\theta(t)$ *are the local martingales given by*

$$\begin{aligned} \mathcal{D}_\theta^K(t) = &-\int_0^t e^{-\alpha_s}(\theta(Z_s)K_s\sigma(Z_s) + \rho\beta\theta'(Z_s)X_s)dW_{1s} - \int_0^t \varepsilon\beta e^{-\alpha_s}\theta'(Z_s)X_s dW_{2s} \\ &-\frac{1}{2}\int_0^t e^{-2\alpha_s}(\theta(Z_s)K_s\sigma(Z_s) + \rho\beta\theta'(Z_s)X_s)^2 ds - \frac{1}{2}\int_0^t \varepsilon^2\beta^2 e^{-2\alpha_s}\theta'^2(Z_s)X_s^2\, ds, \end{aligned} \tag{3.2}$$

$$\mathcal{E}_\theta(t) = \int_0^t\int_0^\infty y\theta(Z_s)e^{-\alpha_s}\,\bar{N}(dy,ds) - \lambda\int_0^t\int_0^\infty (e^{\theta(Z_s)ye^{-\alpha_s}} - 1)\, dG(y)\, ds. \tag{3.3}$$

The proof of Proposition 3.1 is not shown, as it is straightforward.
When the rate $r = 0$, we have the following corollary that will be used in Example 3.1 and in the estimation of the lower bound.

COROLLARY 3.1 *If* $r = 0$, *then:*

$$H_\theta^{K,0}(t,x,z) = e^{-\theta(z)x} e^{\mathcal{D}_\theta^K(t)+\mathcal{E}_\theta(t)} e^{\int_0^t f_\theta^*(X_s, Z_s, K_s)\, ds},$$

with

$$f_\theta^*(x,z,K) := f_{\theta,0}(1,x,z,K).$$

3.3 The Upper Bound

Our aim in this section is to get an upper bound for the ruin probabilities of the form $e^{-\theta(z)x}$, for some function θ.

THEOREM 3.1 *If there exists an admissible strategy* $K \in \mathcal{K}$ *and* $\theta : \mathbb{R} \to]0, \infty[$ *in* $C_b^2(\mathbb{R})$*, such that* $H_\theta^{K,r}(t, x, z)$ *is a supermartingale with respect to the filtration* $(\mathcal{F}_t)_{t \geq 0}$*, then:*

$$\psi(x, z) \leq C_r e^{-\theta(z)x},$$

with

$$0 \leq C_r = \inf_{K \in \mathcal{K}} \frac{1}{\mathbb{E}[H_\theta^{K,r}(\tau^K, x, z) \mid \tau^K < \infty]} \leq 1.$$

Proof. Since for each strategy $K \in \mathcal{K}$,

$$X_t^K < 0 \text{ if and only if } e^{-\alpha_t}\theta(Z_t)X_t^K < 0,$$

it is equivalent to study the ruin probability for this process. By hypothesis, $H_\theta^{K,r}(t, x, z)$ is a supermartingale. Then by the optional sampling theorem, we obtain:

$$\begin{aligned} e^{-\theta(z)x} &\geq \mathbb{E}[H_\theta^{K,r}(t \wedge \tau^K, x, z)] \\ &\geq \mathbb{E}[H_\theta^{K,r}(\tau^K, x, z)\mathbf{1}_{\{\tau^K < t\}}]. \end{aligned}$$

Now, letting $t \to \infty$,

$$e^{-\theta(z)x} \geq \mathbb{E}[H_\theta^{K,r}(\tau^K, x, z) \mid \tau^K < \infty]\mathbb{P}[\tau^K < \infty],$$

therefore

$$\psi(x, z, K) = \mathbb{P}[\tau^K < \infty] \leq \frac{e^{-\theta(z)x}}{\mathbb{E}[H_\theta^{K,r}(\tau^K, x, z) \mid \tau^K < \infty]},$$

and

$$\psi(x, z) = \inf_{K \in \mathcal{K}} \psi(x, z, K) \leq \inf_{K \in \mathcal{K}} \frac{e^{-\theta(z)x}}{\mathbb{E}[H_\theta^{K,r}(\tau^K, x, z) \mid \tau^K < \infty]},$$

then

$$\psi(x,z) \leq C_r e^{-\theta(z)x},$$

with

$$C_r = \inf_{K\in\mathcal{K}} \frac{1}{\mathbb{E}[H_\theta^{K,r}(\tau^K,x,z) \mid \tau^K < \infty]}.$$

□

OBSERVATION 3.1 *1. Observe that given a function θ, for each $x, z \in \mathbb{R}$ fixed, the function $f_{\theta,r}(u,x,z,K)$ given by expression (3.1) is a quadratic form in K. This suggests taking K as a root of this equation. The point here is to emphasize the existence of this root. Example 3.1 is a particular case of this result.*

2. If we can take K as a root, then from Proposition 3.1:

$$H_\theta^{K,r}(t,x,z) = e^{-\theta(z)x} e^{\mathcal{D}_\theta^K(t)+\mathcal{E}_\theta(t)}.$$

Then we have that $H_\theta^{K,r}(t,x,z)$ is a local martingale, since it is the product of a continuous and a pure jump local martingale. Furthermore, because it is positive, it is a supermartingale.

In the following example, we consider a function θ for a truncated Scott model. We could consider a truncation via the stochastic volatility in such a way that it belongs to $C_b^2([-m,m])$ (see [4]). However, to avoid technicalities we stop the process Z.

EXAMPLE 3.1 *We assume that the claims are exponentially distributed with parameter $\eta > 0$, and also that $\rho = 1$ and $r = 0$. We consider a truncated Scott model in the following sense:*

Let $Z_t,\ t \geq 0$ be given by

$$dZ_t = \gamma(\delta - Z_t)dt + \beta dW_{1t}, \quad Z_0 = z$$

and for each $m > 0$, define the stopping time τ_m as follows

$$\tau_m = \inf\{t > 0, \mid Z_t \mid > m\}.$$

Let $Z_t^m = Z_{t\wedge\tau_m}$, and

$$dS_t^m = S_t^m(\mu_0 dt + e^{Z_t^m} dW_{1t}), \quad S_0^m = 1.$$

From the convex property of h and Hypothesis (3.1) ($\lim_{\theta\to\theta_\infty} h(\theta) = \infty$), *we have that for each*
$-m \leq z \leq m$, *$\theta(z)$ is defined as the positive solution of:*

$$\lambda h(\theta(z)) = c\theta(z) + \frac{\mu_0^2}{2e^{2z}}, \quad \textit{with } \theta(z) < \eta. \tag{3.4}$$

Straightforward calculations show that $\theta(z)$ satisfies:

$$\theta^2(z) + \frac{1}{c}\left(\lambda - \eta c + \frac{\mu_0^2}{2}e^{-2z}\right)\theta(z) - \frac{\eta\mu_0^2}{2c}e^{-2z} = 0, \tag{3.5}$$

and is given by:

$$\theta(z) = \frac{1}{2}\left[-\frac{1}{c}\left(\lambda - \eta c + \frac{\mu_0^2}{2}e^{-2z}\right) + \sqrt{\frac{1}{c^2}\left(\lambda - \eta c + \frac{\mu_0^2}{2}e^{-2z}\right)^2 + \frac{2\eta\mu_0^2}{c}e^{-2z}}\right]. \tag{3.6}$$

It is clear that $\theta(z) \in C_b^2$, for $z \in [-m, m]$ and $\theta(z) \leq \eta$. Then from the exponential decomposition given in Corollary 3.1, we define the following equation:

$$f_\theta^*(x, z, K) = \frac{1}{2}\theta^2(z)\sigma^2(z)K^2 + b(x, z)K + a(x, z) = 0, \tag{3.7}$$

where

$$a(x, z) = \frac{1}{2}\beta^2\theta'^2(z)x^2 - \Big(g(z)\theta'(z) + \frac{1}{2}\beta^2\theta''(z)\Big)x + \frac{\mu_0^2}{2}e^{-2z},$$

and

$$b(x, z) = \Big(-\mu_0\theta(z) - \beta\sigma(z)\theta'(z) + \beta x\sigma(z)\theta(z)\theta'(z)\Big).$$

Then from Observation 3.1, we can define $K_*(x,z)$ *as the solution of* (3.7) *given by*

$$K_*(x,z) = \frac{-b(x,z) + \sqrt{b^2(x,z) - 2a(x,z)\theta^2(z)\sigma^2(z)}}{\theta^2(z)\sigma^2(z)},$$

if this root exists. We do not have general conditions on the coefficients for its existence; however this indeed holds in some particular cases. For example, if we take

$$\mu_0 = 0.01,\ c = 1,\ \lambda = 1,\ \beta = 10,\ g(z) = 0.01(1-z), \eta = 1.1,$$

with the help of Mathematica v.7 it can be shown that for $m = -\frac{1}{2}\log(2\mu_0 c\eta + 2\mu_0\lambda)$, *equation* (3.7) *admits a positive solution. The following figures show the behavior of the function* $\theta(z)$ *and the strategy* $K_*(x,z)$ *for different values of* z.

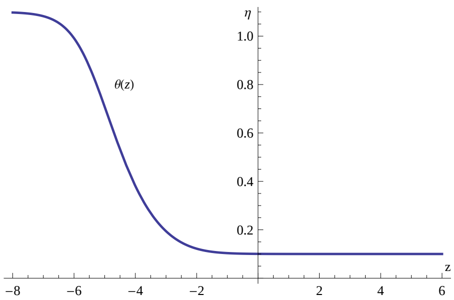

Figure 3.1: The function $\theta(z)$

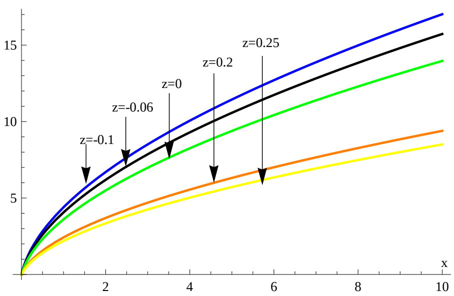

Figure 3.2: The admissible strategy $K_*(x,z)$

The following corollary gives an upper bound for the ruin probability when the return from the bound and the risky asset is bounded from below.

COROLLARY 3.2 *We assume that there exists a constant $R_1 > 0$ such that*

$$0 < R_1 \leq \frac{\mu(z) - r(z)}{\sigma(z)} \quad \forall z \in \mathbb{R}. \tag{3.8}$$

Then the ruin probability $\psi(x, z)$ of an insurer investing in a risky asset can be bounded from above by

$$\psi(x, z) \leq e^{-\hat{\theta}x},$$

where $0 < \hat{\theta} < \theta_\infty$ is the unique positive solution of

$$\lambda h(\theta) = c\theta + \frac{1}{2}R_1^2. \tag{3.9}$$

Proof. The existence of $\hat{\theta}$ is a consequence of the convex property of $h(\theta)$ and the second part of **Hypothesis** (3.1). Using Theorem 3.1, we only need to prove that the process $H_{\hat{\theta}}^{\hat{K},r}(t, x, z)$ is a supermartingale, where

$$\hat{K}_t = \frac{\mu(Z_t) - r(Z_t)}{\hat{\theta}\sigma^2(Z_t)}.$$

Since $\hat{\theta}$ is constant, we can see that the function $f_{\hat{\theta},r}$ given in Proposition 3.1 is reduced to

$$\begin{aligned}
f_{\hat{\theta},r}(e^{-\alpha_s}, x, z, \hat{K}) &= \lambda h(\hat{\theta}e^{-\alpha_s}) - (c + (\mu(z) - r(z))K)\hat{\theta}e^{-\alpha_s} + \frac{1}{2}\left(\hat{K}e^{-\alpha_s}\hat{\theta}\sigma(z)\right)^2 \\
&= \lambda h(\hat{\theta}e^{-\alpha_s}) - \left(c\hat{\theta} + \frac{(\mu(z) - r(z))^2}{\sigma^2(z)}\right)e^{-\alpha_s} + \frac{1}{2}e^{-2\alpha_s}\frac{(\mu(z) - r(z))^2}{\sigma^2(z)} \\
&\leq \lambda h(\hat{\theta}e^{-\alpha_s}) - \left(c\hat{\theta} + \frac{1}{2}\frac{(\mu(z) - r(z))^2}{\sigma^2(z)}\right)e^{-\alpha_s} \\
&\leq e^{-\alpha_s}\left(\lambda h(\hat{\theta}) - c\hat{\theta} - \frac{1}{2}\frac{(\mu(z) - r(z))^2}{\sigma^2(z)}\right).
\end{aligned}$$

The last inequality follows from the fact that the function $q(x) = e^{px} - pe^{x}$, with $x \geq 0$ and $0 \leq p \leq 1$ fixed, is decreasing, and by taking $x = \hat{\theta} y$, $p = e^{-\alpha_s}$. Finally, from the definition of $\hat{\theta}$ we have

$$f_{\hat{\theta},r}(e^{-\alpha_s}, X_s, Z_s, \hat{K}_s) \leq 0,$$

and then we get that $H_{\hat{\theta}}^{\hat{K},r}(t,x,z)$ is a supermartingale. □

OBSERVATION 3.2 *1. If $c > \lambda\mu$, then the Lundberg coefficient $\nu > 0$ exists and $\hat{\theta} > \nu$.*

2. If $\inf_z \dfrac{\mu(z) - r(z)}{\sigma(z)} = 0$, following a similar procedure as in Corollary 3.2, we can obtain that:
$\psi(x,z) \leq e^{-\nu x}$, under the assumption that $c > \lambda\mu$.

3. Observe that $\mu(z) - r(z)$ represents the premium return from investing in the risky asset. Therefore, small values of $\dfrac{\mu(z) - r(z)}{\sigma(z)}$ correspond to very large volatilities. Thus R_1 can be viewed in some sense as a measure of the risk aversion of the investor.

4. When $\mu(z)$ and $\sigma(z)$ are constants and $r = 0$, we obtain the same bound as in [13]:

$$\psi(x,z) \leq e^{-\hat{\theta}x},$$

where $\hat{\theta}$ is the unique positive solution of:

$$\lambda h(\theta) = c\theta + \frac{\mu^2}{2\sigma^2}.$$

5. The case in which μ, σ and r are constant was studied in [13] under the restriction that the interest force is equal to the inflation force, and only the prime and the claims are affected by inflation. In this case, *the wealth process becomes*

$$X_t^K = x + \int_0^t (ce^{rs} + K_s\mu + (X_s - K_s)r)\,ds + \int_0^t K_s\sigma\,dW_{1s} - \sum_{i=1}^{N_t} e^{rT_i}Y_i.$$

This assumption is not so clear to us, as it seems that it is used only for technical reasons. In our case, we can deal without it .

EXAMPLE 3.2 *As an application of Corollary 3.2, we consider another truncated Scott model. Assume*
$0 < r < \mu_0$ *is constant, and let* $Z_t,\ S_t^{1,m}\ t \geq 0$ *be given by*

$$dZ_t = \gamma(\delta - Z_t)dt + \beta dW_{1t}, \quad Z_0 = z,$$

and for each $m > 0$,

$$dS_t^{1,m} = S_t^{1,m}(\mu_0 dt + \sigma(Z_t)dW_{1t}), \quad S_0^{1,m} = 1,$$

with

$$\sigma(z) = \begin{cases} e^z & \text{if} \quad z \in [-m, m], \\ e^m & \text{if} \quad z \in]m, \infty[, \ \text{for some } m > 0. \\ e^{-m} & \text{if} \quad z \in]-\infty, -m[, \end{cases}$$

Then $R_1 = (\mu_0 - r)e^{-m}$. *In particular, if the claims are exponentially distributed with parameter* η, *we have that* (3.9) *becomes:*

$$\lambda\left(\frac{\eta}{\eta - \theta} - 1\right) = c\theta + \frac{(\mu_0 - r)^2}{2}e^{-2m}$$

which leads to:

$$\theta^2 + \frac{1}{c}\left(\lambda - \eta c + \frac{(\mu_0 - r)^2}{2}e^{-2m}\right) - \frac{\eta(\mu_0 - r)^2}{2c}e^{-2m} = 0.$$

Then

$$\hat{\theta} = \frac{1}{2}\left[-\frac{1}{c}\left(\lambda - \eta c + \frac{(\mu_0 - r)^2}{2}e^{-2m}\right) + \sqrt{\frac{1}{c^2}\left(\lambda - \eta c + \frac{(\mu_0 - r)^2}{2}e^{-2m}\right)^2 + \frac{2\eta(\mu_0 - r)^2}{c}e^{-2m}}\right].$$

The admissible strategy is given by:

$$\hat{K}_t = \frac{\mu_0 - r}{\hat{\theta}\sigma^2(Z_t)}.$$

Then by Corollary 3.2, we get that for all $(x, z) \in [0, \infty[\times\mathbb{R}$

$$\psi(x, z) \leq e^{-\hat{\theta}x}.$$

3.4 The Lower Bound

In this section, we assume $r = 0$, i.e., the bank account is not taken in consideration. Then, in order to get a lower bound for the ruin probabilities, we assume the following:

HYPOTHESIS 3.2 *There exists a constant* $R_2 > 0$ *such that:*

$$0 < \frac{\mu(z)}{\sigma(z)} \leq R_2 \;\; \forall z \in \mathbb{R}. \tag{3.10}$$

DEFINITION 3.1 *Let* $0 < \theta < \theta_\infty$ *be given. We say that* Y *has a uniform exponential moment in the tail distribution for* θ *if the following condition holds:*

$$\sup_{z \geq 0} \mathbb{E}[e^{-\theta(z-Y)} \mid Y > z] < \infty.$$

THEOREM 3.2 *Assume that* Y *has a uniform exponential moment in the tail distribution for* θ^*. *Then:*

$$\psi(x, z) \geq C^* e^{-\theta^* x} \quad \forall z \in \mathbb{R},$$

with

$$0 < C^* = \inf_{y \geq 0} \frac{\displaystyle\int_y^\infty dG(u)}{\displaystyle\int_y^\infty e^{-\theta^*(y-z)} dG(z)} \leq 1,$$

and θ^* *is the unique positive solution of:*

$$\lambda h(\theta) = c\theta + \frac{1}{2} R_2^2. \tag{3.11}$$

To prove Theorem 3.2, we need the following lemma. For ease of notation, we will denote the ruin time as $\tau := \tau^K$.

LEMMA 3.1 *Suppose that* Y *has a uniform exponential moment in the tail distribution for* θ^**. Then for each* $K \in \mathcal{K}$*, the process* $H_{\theta^*}^{K,0}(t \wedge \tau, x, z)$ *is a uniformly integrable submartingale.*

Proof. The proof will be given in two steps. In Step 1, we will prove that $H_*^K = \sup_{t \geq 0} H_{\theta^*}^{K,0}(t \wedge \tau, x, z)$ has a finite first moment, and in Step 2, that $H_{\theta^*}^{K,0}(t \wedge \tau, x, z)$ is a local submartingale.
Step 1. The existence of θ^* is a consequence of the convex property of $h(\theta)$ and **Hypothesis** 3.1. Our aim is to prove that H_*^K has a first finite moment. We observe that:

$$H_*^K = \begin{cases} H_{\theta^*}^{K,0}(\tau, x, z) > 1 & \text{on} \quad [\tau < \infty] \cap [X(\tau^-, x, z) > 0], \\ H_{\theta^*}^{K,0}(\tau, x, z) = 1 & \text{on} \quad [\tau < \infty] \cap [X(\tau^-, x, z) = 0], \\ \sup_{t \geq 0} H_{\theta^*}^{K,0}(t, x, z) \leq 1 & \text{on} \quad [\tau = \infty]. \end{cases}$$

Then

$$\begin{aligned} \mathbb{E}[H_*^K] &= \mathbb{E}[H_*^K \mathbb{1}_{[\tau=\infty]} + H_*^K \mathbb{1}_{\{[\tau<\infty] \cap [X(\tau^-,x,z)=0]\}}] \\ &\quad + \mathbb{E}[H_*^K \mathbb{1}_{\{[\tau<\infty] \cap [X(\tau^-,x,z)>0]\}}] \\ &\leq 2 + \mathbb{E}[H_*^K \mathbb{1}_{\{[\tau<\infty] \cap [X(\tau^-,x,z)>0]\}}] \end{aligned} \tag{3.12}$$

On the other hand, given that ruin occurs at a jump in time, then the jump is conditioned to be greater or equal than $X^K(\tau^-, x, z) > 0$. More precisely, let Y be an independent copy of $(Y_i)_{i\geq 1}$, then

$$\begin{aligned}
&\mathbb{E}[H^{K,0}_{\theta^*}(\tau, x, z) \mid \tau = t, X(\tau-, x, z) = v] \\
&= \mathbb{E}[e^{-\theta^*(X(\tau^-,x,z)-Y)} \mid \tau = t, X(\tau-, x, z) = v] \\
&= \mathbb{E}[e^{-\theta^*(v-Y)} \mid \tau = t, X(\tau-, x, z) = v] \\
&= \mathbb{E}[e^{-\theta^*(v-Y)} \mid Y > v] = \int_v^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_0^\infty dG(s)}. \qquad (3.13)
\end{aligned}$$

Now let $M(dt, dv)$ be the joint distribution of $(\tau, X(\tau-, x, z))$, then from (3.12) and (3.13) we have

$$\begin{aligned}
&\mathbb{E}[H^K_*] \leq 2 + \mathbb{E}[\mathbb{E}[H^K_* \mathbb{1}_{[\tau<\infty]\cap[X^K(\tau-,x,z)>0]} \mid \tau, X^K(\tau-, x, z)]] \\
&= 2 + \int_0^\infty \int_0^\infty M(dt, dv) \int_v^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_v^\infty dG(s)} \\
&\leq 2 + \mathbb{P}[\tau < \infty, X^K(\tau-, x, z) > 0] \sup_{v\geq 0} \int_y^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_v^\infty dG(s)} \\
&\leq 2 + \sup_{v\geq 0} \int_v^\infty e^{-\theta^*(v-u)} \frac{dG(u)}{\int_v^\infty dG(s)} < \infty,
\end{aligned}$$

where the last inequality follows from the hypothesis that Y has a uniform exponential moment in the tail distribution.

Step 2. We know that

$$H^{K,0}_{\theta^*}(t, x, z) = e^{-\theta^* x} e^{\mathcal{D}^K_{\theta^*}(t)+\mathcal{E}_{\theta^*}(t)} e^{\int_0^t f^*_{\theta^*}(X_s, Z_s, K_s)\, ds}.$$

By using **Hypothesis** 3.2 and (3.11), we get that for all $T \geq 0$ such that $t \leq T$:

$$f^*_{\theta^*}(X_t, Z_t, K_t) \geq \frac{1}{2}\theta^{*2}\sigma^2(Z_t) \left(K_t - \frac{\mu(Z_t)}{\theta^*\sigma^2(Z_t)}\right)^2 \geq 0.$$

Following a similar procedure as in the proof of Corollary 3.2, we obtain that $H^{K,0}_{\theta^*}(t\wedge\tau,x,z)$ is a local submartingale. By using Step 1 and the dominated convergence theorem, we get that $H^{K,0}_{\theta^*}(t\wedge\tau,x,z)$ is a uniformly integrable submartingale (see [29], Theorem I.51). □

LEMMA 3.2 *If Y has a uniform exponential distribution moment in the tail distribution for θ^*, then for an arbitrary $K\in\mathcal{K}$ and $(x,z)\in\mathbb{R}_+\times\mathbb{R}$, the stopped wealth process $X^K(t\wedge\tau,x,z)$ converges almost surely on $\tau=\infty$ to ∞ for $t\to\infty$.*

Proof. Lemma 3.1 implies that $H^{K,0}_{\theta^*}(t\wedge\tau,x,z)$ is a uniformly integrable submartingale. By Doob's convergence theorem (see [32],Theorem II.69.1),

$$\lim_{t\to\infty} H^{K,0}_{\theta^*}(t\wedge\tau,x,z) \text{ exists a.s.}$$

Then, $X_{t\wedge\tau}$ converges when $t\to\infty$. Now to prove that

$$\mathbb{P}[\lim_{t\to\infty} X_{t\wedge\tau}=\infty \mid \tau=\infty]=1,$$

we work toward a contradiction. We assume that

$$\mathbb{P}[\lim_{t\to\infty} X_{t\wedge\tau}<\infty,\tau=\infty]>0,$$

then there exists $m_0>0$ such that

$\mathbb{P}[\sup_{t\geq 0} X_t\leq m_0,\tau=\infty]>0$, because it is a downward jump. We consider the following event $[\sum_{i=1}^{N(1)} Y_i>2m_0+c]$. Since the compound Poisson process has an unbounded support, then:

$$\mathbb{P}[\sum_{i=1}^{N(1)} Y_i>2m_0+c]=p_0>0.$$

Let $(t_k)_{k\geq 0}$ be a sequence of points given by:

$$t_1<t_1+1<t_2<t_2+1<\dots$$

Because the compound Poisson process has stationary and independent increments, then $(\sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i)_{k\geq 0}$ is a sequence of i.i.d. random variables and represents the number of claims in an interval of length one. Now we consider the following sequence of r.v. $(D_k)_{k\geq 0}$ defined by:

$$D_k = \mathbf{1}_{\left\{\sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0\right\}}$$

By the strong law of large numbers, we get that:

$$\frac{1}{n}\sum_{k=1}^{n} D_k \xrightarrow{a.s} p_0,$$

then

$$\mathbb{P}\left[\sum_{k=1}^{\infty} \mathbf{1}_{\left\{\sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0\right\}} = \infty\right] = 1,$$

which implies that

$$\mathbb{P}\left[\bigcap_n \bigcup_{k\geq n} \sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0\right] = 1.$$

Now let

$$A = [\sup_{t\geq 0} X_t \leq m_0, \tau = \infty],$$

and

$$B = [\bigcap_n \bigcup_{k\geq n} \sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i > 2m_0].$$

Observe that on A, we have that for $T_i \leq t < T_{i+1}$

$$X_t = X_{T_i} + c + \int_{T_i}^{t} \mu(Z_s)K_s\,ds + \int_{t_k}^{t} \sigma(Z_s)K_s\,dW_{1s} < m_0, \tag{3.14}$$

and then

$$\int_{T_i}^{t} \mu(Z_s)K_s\,ds + \int_{T_i}^{t} \sigma(Z_s)K_s\,dW_{1s} \leq m_0 - c \quad \text{on} \quad A.$$

On the other hand, the following occurs on B infinitely often:

$$\begin{aligned} X_{t_k+1} &= X_{t_k} + c + \int_{t_k}^{t_k+1} \mu(Z_s)K_s\,ds + \int_{t_k}^{t_k+1} \sigma(Z_s)K_s\,dW_{1s} - \sum_{i=N_{t_k}+1}^{N_{t_k+1}} Y_i \\ &\leq X_{t_k} + c + \int_{t_k}^{t_k+1} \mu(Z_s)K_s\,ds + \int_{t_k}^{t_k+1} \sigma(Z_s)K_s\,dW_{1s} - 2m_0. \quad (3.15) \end{aligned}$$

Then from (3.14) and (3.15), we get that $P[A \cap B] = 0$. Since $P[B] = 1$, then $P[A] = 0$. □

Proof. **(Theorem 3.2.)** Since $H^{K,0}_{\theta^*}(t \wedge \tau, x, z)$ is a submartingale, then

$$e^{-\theta^* x} \leq \mathbb{E}[H^{K,0}_{\theta^*}(t \wedge \tau, x, z)]$$

and

$$\mathbb{E}[H^{K,0}_{\theta^*}(t \wedge \tau, x, z)] = \mathbb{E}[H^{K,0}_{\theta^*}(\tau, x, z), \tau < t] + \mathbb{E}[H^{K,0}_{\theta^*}(t, x, z), \tau > t].$$

By letting $t \to \infty$

$$e^{-\theta^* x} \leq \mathbb{E}[H^{K,0}_{\theta^*}(\tau, x, z) \mid \tau < \infty]\mathbb{P}[\tau < \infty] + \mathbb{E}[\lim_{t\to\infty} H^{K,0}_{\theta^*}(t, x, z) \mid \tau = \infty]\mathbb{P}[\tau = \infty].$$

By Lemma 3.2, we get that:

$$e^{-\theta^* x} \leq \mathbb{E}[H^{K,0}_{\theta^*}(\tau, x, z) \mid \tau < \infty]\mathbb{P}[\tau < \infty],$$

and

$$\begin{aligned} \psi(x, z, K) = \mathbb{P}[\tau < \infty] &\geq \frac{e^{-\theta^* x}}{\mathbb{E}[H^{K,0}_{\theta^*}(\tau, x, z) \mid \tau < \infty]} \\ &\geq \inf_{y \geq 0} \frac{1}{\displaystyle\int_y^\infty e^{-\theta^*(y-z)} \frac{dG(z)}{\int_y^\infty dG(u)}} \\ &\geq C^* e^{-\theta^* x}. \end{aligned}$$

Finally,

$$\psi(x, z) \geq C^* e^{-\theta^* x},$$

with

$$C^* = \inf_{y \geq 0} \frac{\displaystyle\int_y^\infty dG(u)}{\displaystyle\int_y^\infty e^{-\theta^*(y-z)} dG(z)}.$$

□

As a consequence of Corollary 3.2 and Theorem 3.2, we get the following estimations for the ruin probability when initial capital tends to infinity.

COROLLARY 3.3 *If θ^+ is the positive solution of (3.9) when $r = 0$, then:*

$$C^* e^{-\theta^* x} \leq \psi(x, z) \leq e^{-\theta^+ x}.$$

Furthermore, for all $z \in \mathbb{R}$:

$$-\theta^* \leq \liminf_{x \to \infty} \frac{1}{x} \ln(\psi(x, z)) \leq \limsup_{x \to \infty} \frac{1}{x} \ln(\psi(x, z)) \leq -\theta^+.$$

Chapter 4

Optimization and Utility Functions

The main purpose of this chapter is to extend the results obtained for a geometric Brownian motion in [7] to the stochastic volatility model introduced in Chapter 2. Following the same approach as [7], first we will establish the connection between the optimization problem and a Hamilton-Jacobi-Bellman equation, via a verification theorem.

When the utility function is of exponential type, we prove an existence theorem for he HJB equation, which is expressed as the product of an exponential term and the solution of a parabolic partial differential equation. We also prove an existence and uniqueness theorem when the insurer's preferences are exponential, and we obtain an explicit solution for the partial differential equation (PDE). Consequently, a closed form for the optimal strategy is obtained, which depends only on the external factor and time. To develop some numerical results, we prove consistency and stability of the explicit scheme. The well-posedness of the Cauchy problem is proven to complete the conditions of the Lax theorem for convergence. We present results for the Scott model when claim-size is exponentially distributed. Finally, we study the reserve of the insurance company under the optimal strategy, and prove a supermartingale property of the risk process to get an upper bound for the ruin probability in finite horizon.

4.1 The Optimization Problem

In this section we will introduce the problem that will be studied along this chapter.

DEFINITION 4.1 *We say that $K = (K_t)_{t\geq 0}$ is an admissible strategy if it is an $\mathcal{F}_t$-progressively measurable process such that:*

$$\mathbb{P}[\mid \sigma_t K_t \mid\leq C_K, 0 \leq t \leq T] = 1,$$

where C_K is a constant which may depend on the strategy K. We denote the set of admissible strategies as $\mathcal{K}$.

We consider the same problem described in chapter 2 of an insurer with wealth X_t^K that invests an amount $K_t \in \mathcal{K}$ in the risky asset and the remaining reserve $X_t - K_t$ in the bank account. If at time $s < T$ the wealth of the company is x and the external factor is z, then the wealth process satisfies:

$$\left\{\begin{array}{rcl} X_t^{s,x,z,K} & = & x + \int_s^t \Big(c + (\mu(Z_v) - r(Z_v))K_v + r(Z_v)X_v\Big)\, dv \\ & & + \int_s^t \sigma(Z_v) K_v \, dW_{1v} - \sum_{i=N_s+1}^{N_t} Y_i \\ Z_s & = & z, \end{array}\right. \tag{4.1}$$

with the convention that $\sum_{i=1}^{0} = 0$, and that when $s = 0$ we write X_t^K.

A utility function $U : \mathbb{R} \to \mathbb{R}$ is defined as a twice continuously differentiable function, with the property that $U(\cdot)$ is strictly increasing and strictly concave. Now we consider the optimization problem consisting in maximizing the expected utility wealth at time T, i.e., we are interested in the following value function:

$$V(s,x,z) = \sup_{K\in\mathcal{K}} \mathbb{E}\left[U(X_T^{s,x,z,K})\right]. \qquad (4.2)$$

We say that an admissible strategy K^* is optimal if

$$V(s,x,z) = \mathbb{E}\left[U(X_T^{s,x,z,K^*})\right].$$

4.2 Verification Theorem

Our aim is to relate the value function given in (4.2), which is associated to a stochastic control problem, to a well suited PDE. This allows for an explicit solution in good cases and a set of verification arguments. In view of the Markovian structure of the model given by (X_t^K, Z_t), the HJB associated with the control problem (4.2) is given by:

$$\lambda\int_0^\infty \left(f(t,x-y,z)-f(t,x,z)\right)\,dG(y) + \sup_{\pi\in\mathbb{R}}\mathscr{L}^\pi f(t,x,z) = 0, \qquad (4.3)$$

with terminal condition $f(T,x,z) = U(x)$ and

$$\begin{aligned}\mathscr{L}^\pi f(t,x,z) &= f_t + \tfrac{1}{2}\sigma^2(z)\pi^2 f_{xx} + \tfrac{1}{2}\beta^2 f_{zz} + \rho\beta\pi\sigma(z) f_{xz} \\ &\quad + (c + (\mu(z)-r(z))\pi + r(z)x) f_x + g(z) f_z.\end{aligned}$$

Now we establish a verification theorem, which relates the value function ψ with the HJB equation.

Theorem 4.1 *(The Verification Theorem)*
Assume that there exists a classical solution $f(t,x,z)\in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ to the HJB equation (4.3) *with terminal condition $f(T,x,z) = U(x)$. Assume also that for each $K\in\mathcal{K}$*

$$\int_0^T\int_0^\infty \mathbb{E}|f(s,X_{s-}^K - y, Z_s) - f(s,X_{s-}^K, Z_s)|^2\,dG(y)\,ds < \infty; \qquad (4.4)$$

$$\int_0^T \mathbb{E}|K_{s-} f_x(s, X_{s-}^K, Z_s)|^2\, ds < \infty; \tag{4.5}$$

$$\int_0^T \mathbb{E}|K_{s-} f_z(s, X_{s-}^K, Z_s)|^2\, ds < \infty. \tag{4.6}$$

Then for each $s \in [0, t]$, $(x, z) \in \mathbb{R}^2$

$$f(s, x, z) \geq V(s, x, z).$$

If, in addition, there exists a bounded measurable function $K^*(t, x, z)$ *such that:*

$$K^*(t, x, z) \in argmax_{\pi\in\mathbb{R}} \left\{ \frac{1}{2}\sigma^2(z)\pi^2 f_{xx} + \rho\beta\pi\sigma(z) f_{xz} + (\mu(z) - r(z))\pi f_x \right\},$$

then $K_t^* = K(t, X_t, Z_t)$ *defines an optimal strategy and*

$$f(s, x, z) = V(s, x, z) = \mathbb{E}[U(X_T^{s,x,z,K^*})].$$

Proof. Let $K \in \mathcal{K}$. Itô's formula implies that for any $v \in [s, T]$,

$$\begin{aligned}
f(v, X_v^{s,x,z,K}, Z_v) &= f(s, x, z) + \int_s^v f_t(t, X_{t-}^{s,x,z,K}, Z_t)\, dt \\
&+ \int_s^v \left(c + (\mu(Z_t) - r(Z_t))K_{t-} + r(Z_t)X_{t-}^{s,x,z,K}\right) f_x(t, X_{t-}^{s,x,z,K}, Z_t)\, dt \\
&+ \int_s^v g(Z_t) f_z(t, X_{t-}^{s,x,z,K}, Z_t)\, dZ_t + \int_s^v \sigma(Z_t)K_{t-} f_x(t, X_{t-}^{s,x,z,K})\, dW_{1t} \\
&+ \int_s^v \beta f_z(t, X_{t-}^{s,x,z,K}, Z_t)\, d\tilde{W}_t + \frac{1}{2}\int_s^v \sigma^2(Z_t)K_{t-}^2 f_{xx}(t, X_{t-}^{s,x,z,K})\, dt \\
&+ \frac{1}{2}\int_s^v \beta^2 f_{zz}(t, X_{t-}^{s,x,z,K})\, dt + \int_s^v \rho\beta\sigma(Z_t)K_{t-} f_{xz}(t, X_{t-}^{s,x,z,K}, Z_t)\, dt \\
&+ \int_s^v \int_0^\infty \left(f(t, X_{t-}^{s,x,z,K} - y, Z_t) - f(t, X_{t-}^{s,x,z,K}, Z_t)\right) \bar{N}(dy\, dt)
\end{aligned} \tag{4.7}$$

where $\bar{N}$ is the Poisson random measure on $\mathbb{R}_+ \times [0, \infty[$ defined by

$$\bar{N} = \sum_{n\geq 1} \delta_{(T_n, Y_n)}.$$

Compensating (4.7) by:

$$\lambda \int_s^v \int_0^\infty \left(f(t, x-y, z) - f(t,x,z)\, dG(y) \right) \, dt,$$

we obtain the following:

$$\begin{aligned} f(v, X_v^{s,x,z,K}, Z_v) &= f(s,x,z) + \int_s^v \mathscr{L}^{K_t} f(t, X_{t-}^{s,x,z,K}, Z_t)\, dt \\ &+ \int_s^v \sigma(Z_t) K_{t-} f_x \, dW_{1t} + \int_s^v \beta f_z(t, X_{t-}^{s,x,z,K}, Z_t)\, d\tilde{W}_t \\ &+ \int_s^v \int_0^\infty \Big(f(t, X_{t-}^{s,x,z,K} - y, Z_t) - f(t, X_{t-}^{s,x,z,K}, Z_t) \Big) \, \bar{N}(dy\, dt) \\ &+ \lambda \int_s^v \int_0^\infty \left(f(t, x-y, z) - f(t,x,z) \right) \, dG(y)\, dt \end{aligned} \tag{4.8}$$

The assumptions of the verification theorem ((4.5),(4.6)) imply that all the stochastic integrals with respect to the Brownian motion are matingales. By assumption (4.4) of the verification theorem:

$$\int_s^v \int_0^\infty f(t, X_{t-}^{s,x,z,K} - y, Z_t) - f(t, X_{t-}^{s,x,z,K}, Z_t) \bar{N}(dy\, dt)$$

$$-\lambda \int_s^v \int_0^\infty f(t, X_{t-}^{s,x,z,K} - y, Z_t) - f(t, X_{t-}^{s,x,z,K}, Z_t)\, dG(y)\, dt$$

is a martingale (see [24], p.63). Then, taking expectations in (4.8) yields:

$$\begin{aligned} \mathbb{E}\left[f(v, X_v^{s,x,z,K}, Z_v) \right] &= f(s,x,z) + \mathbb{E}\left[\int_s^v \mathscr{L}^{K_t} f(t, X_{t-}^{s,x,z,K}, Z_t)\, dt \right] \\ + \lambda \mathbb{E} &\left[\int_s^v \int_0^\infty \Big(f(t, X_{t-}^{s,x,z,K} - y, Z_t) - f(t, X_{t-}^{s,x,z,K}, Z_t) \Big) \, dG(y)\, dt \right]. \end{aligned}$$

Since f satisfies the HJB equation (4.3), we obtain that

$$\mathbb{E}[f(v, X_v^{s,x,z,K}, Z_v)] \leq f(s,x,z), \tag{4.9}$$

and letting $v = T$ in (4.9), we get that

$$f(s,x,z) \geq V(s,x,z).$$

To justify the second part of the theorem, we repeat the above calculations for the strategy given by K_t^*. Then we have

$$f(s,x,z) = \mathbb{E}[U(X_T^{s,x,z,K^*})] \leq V(s,x,z),$$

and with the first part of the proof we get that

$$f(s,x,z) = \mathbb{E}[U(X_T^{s,x,z,K^*})] = V(s,x,z).$$

□

In the next section, we present a closed representation of the solution to the HJB equation (4.3) when the insurer has exponential preferences. We also obtain a closed form of the optimal strategy of investment.

4.3 Existence of a Solution for the Exponential Utility Function

In this section, we prove that (4.3) has a unique solution when the the insurer's preferences are exponential, i.e., the utility function is given by:

$$U(x) = -e^{-\alpha x}, \qquad \alpha > 0.$$

In addition to parts 1 and 2 of **Hypothesis** (3.1), we will assume the following:

HYPOTHESIS 4.1 *1. $r(z) = r$ is constant and $\rho = 0$;*

2. g is uniformly Lipschitz and bounded;

3. $\dfrac{(\mu(z)-r)^2}{\sigma^2(z)}$ has a bounded first derivative.

In view of the form of the utility function, we conjecture the following function as a solution to the HJB equation (4.3) :

$$f(t,x,z) = -\xi(z,t)\exp\left\{-\alpha x e^{r(T-t)}\right\}, \tag{4.10}$$

where $\xi(z,t)$ will be defined below as a solution to a Cauchy problem. From the definition of $\phi(t,x,z)$, we have:

$$\begin{aligned}
f_t(t,x,z) &= (-\xi_t - \alpha x r\xi e^{r(T-t)})\exp\left\{-\alpha x e^{r(T-t)}\right\} && (4.11)\\
f_x(t,x,z) &= \alpha\xi e^{r(T-t)}\exp\left\{-\alpha x e^{r(T-t)}\right\} && (4.12)\\
f_{xx}(t,x,z) &= -\alpha^2\xi e^{2r(T-t)}\exp\left\{-\alpha x e^{r(T-t)}\right\} && (4.13)\\
f_z(t,x,z) &= -\xi_z\exp\left\{-\alpha x e^{r(T-t)}\right\} && (4.14)\\
f_{zz}(t,x,z) &= -\xi_{zz}\exp\left\{-\alpha x e^{r(T-t)}\right\} && (4.15)
\end{aligned}$$

Substituting the last expressions in (4.3), we get:

$$\begin{aligned}
0 &= -\xi_t - \frac{1}{2}\beta^2\xi_{zz} + c\alpha e^{r(T-t)}\xi - \lambda\xi\int_0^\infty \left(\exp\left\{\alpha y e^{r(T-t)}\right\} - 1\right) dG(y)\\
&- g(z)\xi + \sup_{\pi\in\mathbb{R}}\left\{-\frac{1}{2}\sigma^2(z)\pi^2\alpha^2 e^{2r(T-t)} + \alpha\xi e^{r(T-t)}(\mu(z)-r)\pi\right\} \quad (4.16)
\end{aligned}$$

and the maximum is achieved at:

$$K^*(t,z) = \frac{\mu(z)-r}{\alpha\sigma^2(z)}e^{-r(T-t)}$$

Now we substitute K^* in (4.16) to obtain the following Cauchy problem:

$$\begin{cases}
\xi_t + \frac{1}{2}\beta^2\xi_{zz} + g(z)\xi_z - \left(\dfrac{1}{2}\dfrac{(\mu(z)-r)^2}{\sigma^2(z)} + c\alpha e^{r(T-t)} - \lambda\theta_t\right)\xi = 0\\
\\
\xi(z,T) = 1
\end{cases} \tag{4.17}$$

where

$$\theta_t = \int_0^\infty \left(\exp\left\{\alpha y e^{r(T-t)}\right\} - 1\right) dG(y).$$

Theorem 4.2 *(Existence and Uniqueness Theorem)*
Assume that

$$\int_0^\infty \exp\left\{8\alpha y e^{rT}\right\} dG(y) < \infty \tag{4.18}$$

and

$$\int_0^\infty y \exp\left\{8\alpha y e^{rT}\right\} dG(y) < \infty. \tag{4.19}$$

Then the Cauchy problem given by (4.17) *has a unique solution, which satisfies the following conditions:*

$$|\xi(z,t)| \leq C_1(1+|z|) \tag{4.20}$$

$$|\xi_z(z,t)| \leq C_2(1+|z|) \tag{4.21}$$

where C_1 *and* C_2 *are constants.*

Proof. In order to prove this theorem, first we verify that the Cauchy problem given by (4.17) satisfies the conditions of Theorem A.1 (see Appendix A):
Since β is constant, then it is Lipschitz continuous, Hölder continuous, and the operator $\frac{1}{2}\beta^2\partial_{zz}$ is uniformly elliptic.
By **Hypothesis** 4.1, we know that $g(z)$ is bounded and uniformly Lipschitz continuous.
Now we prove that:

$$d(z,t) = -\frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)} - c\alpha e^{r(T-t)} + \lambda\theta_t$$

is bounded and uniformly Hölder continuous in compact subsets of $\mathbb{R}\times[0,T]$.
By **Hypothesis** 4.1, it is easy to check that the first term of $d(z,t)$ is bounded by $\frac{1}{2}\frac{(\mu_1-r)^2}{\sigma_0^2}$. The second term is bounded by $c\alpha e^{rT}$. To prove that θ_t is

bounded, we observe that

$$\begin{aligned} |\ \theta_t\ | &= |\int_0^\infty \left(\exp\left\{\alpha y e^{r(T-t)}\right\} - 1\right)\, dG(y)\ | \\ &\le \int_0^\infty \exp\left\{\alpha y e^{r(T-t)}\right\} dG(y) \\ &\le \int_0^\infty \exp\left\{\alpha y e^{rT}\right\} dG(y) < \infty. \end{aligned}$$

In the last line, we used (4.18), thus $d(z,t)$ is bounded. Now we prove that $d(z,t)$ is uniformly Hölder continuous in compact subsets of $\mathbb{R} \times [0,T]$. Let

$$l(z) = \frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)}.$$

Since by **Hypothesis** 4.1 $l'(z)$ is bounded, then by Lemma A.1 $l(z)$ is uniformly Hölder continuous with exponent $h = 1/2$, i.e., for all $(z, z_0) \in \mathbb{R}^2$

$$|\ l(z) - l(z_0)\ | \le C\ |\ z - z_0\ |^{\frac{1}{2}}\ .$$

For the second term of $d(z,t)$, we use the mean value theorem to obtain that for all $(t,t_0) \in [0,T]^2$:

$$|\ c\alpha e^{r(T-t)} - c\alpha e^{r(T-t)}\ | \le \alpha r c e^{rT}\ |\ t - t_0\ |,$$

then $c\alpha e^{r(T-t)}$ is uniformly Lipschitz in $[0,T]$. Therefore $c\alpha e^{r(T-t)}$ is uniformly Hölder continuous in the compact set $[0,T]$. For the third term of $d(z,t)$, the mean value theorem implies that there exists $t^* \in]t_0, t[$ such that:

$$\begin{aligned} |\ \theta_t - \theta_{t_0}\ | &= |\int_0^\infty \exp\left\{\alpha y e^{r(T-t)}\right\} - \exp\left\{\alpha y e^{r(T-t_0)}\right\} dG(y)\ | \\ &= \left(\int_0^\infty \alpha y e^{rT} \exp\left\{\alpha y e^{r(T-t^*)}\right\} dG(y)\right)\ |\ t - t_0\ | \\ &\le \alpha e^{rT}\left(\int_0^\infty y \exp\left\{\alpha y e^{rT}\right\} dG(y)\right)\ |\ t - t_0\ |\ . \end{aligned}$$

By (4.19), we get that θ_t is uniformly Lipschitz continuous in $[0,T]$, and then $d(z,t)$ is uniformly Hölder continuous in compact subsets of $\mathbb{R} \times [0,T]$.

Since the Cauchy problem (4.17) is homogeneous with a constant terminal condition, then the right hand side of (4.17) has the property of linear growth. Finally, the conditions of Theorem A.1 are satisfied, and thus the Cauchy problem (4.17) has a unique solution $\xi(z,t)$ which satisfies (4.20) and (4.21). □

The next theorem relates the value function with the HJB equation.

THEOREM 4.3 *(Main Theorem)*
If ((4.18),(4.19)) *are satisfied, then the value function defined by* (4.2) *has the form:*

$$V(t,x,z) = -\xi(z,t)\exp\left\{-\alpha x e^{r(T-t)}\right\},$$

where $\xi(t,z)$ *is the unique solution of* (4.17)*, and*

$$K^*(t,z) = \frac{\mu(z)-r}{\alpha\sigma^2(z)}e^{-r(T-t)}$$

is an optimal strategy. When $r=0$*, we have:*

$$V(t,x,z) = -\xi(z,t)\exp\{-\alpha x\},$$

and

$$K^*(t,z) = \frac{\mu(z)}{\alpha\sigma^2(z)}.$$

Proof. We have already checked that

$$f(t,x,z) = -\xi(z,t)\exp\left\{-\alpha x e^{r(T-t)}\right\},$$

solves the HJB equation (4.3). Then to prove that $f(t,x,z)$ is the true value function, we shall verify that assumptions ((4.4), (4.5) (4.6)) of Theorem 4.1 are satisfied by $f(t,x,z)$. First, we consider the case in which $r=0$. Let $K \in \mathcal{K}$ be an admissible strategy, then:

$$\begin{aligned}&\int_0^\infty \mathbb{E}|f(t,X_{t-}^K - y, Z_t) - f(t,X_{t-}^K,Z_t)|^2\,dG(y)\\ &= \left(\int_0^\infty (e^{\alpha y}-1)^2\,dG(y)\right)\mathbb{E}\left[\xi^2(Z_t,t)\exp\left\{-2\alpha X_{t-}^K\right\}\right].\end{aligned}$$

To get condition (4.4), we need only obtain an estimate of:

$$\mathbb{E}\left[\xi^2(Z_t, t)\exp\left\{-2\alpha X_{t-}^K\right\}\right].$$

We observe that

$$\begin{aligned}\mathbb{E}\left[\xi^2(Z_t, t)\exp\left\{-2\alpha X_{t-}^K\right\}\right] &\leq C_1^2\mathbb{E}\left[(1+|Z_t|)^2 e^{-2\alpha X_t^K}\right] \\ &\leq C_1^2\left[\mathbb{E}(1+|Z_t|)^4\right]^{1/2}\left[\mathbb{E}\, e^{-4\alpha X_{t-}^K}\right]^{1/2}.\end{aligned}$$

In the first line, we used (4.20). From the first line to the second, we used Hölder's inequality. By Theorem A.2 (see Appendix B), we know that

$$\mathbb{E}(\sup_{0\leq t\leq T} Z_t^4) \leq C(1+|z|^4).$$

Then we only have to estimate $\mathbb{E}\left[e^{-4\alpha X_{t-}^K}\right]$. Let

$$L_t = -8\alpha\int_0^t \sigma(Z_s)K_s\, dW_{1s} - \frac{64\alpha^2}{2}\int_0^t \sigma^2(Z_s)K_s^2\, ds.$$

Then

$$\begin{aligned}\mathbb{E}\left[e^{-4\alpha X_{t-}^K}\right] &\leq \mathbb{E}\left[\exp\left\{-4\alpha\int_0^t \sigma(Z_s)K_s\, dW_{1s} + 4\alpha\sum_{i=1}^{N_{t-}} Y_i\right\}\right] \\ &\leq \mathbb{E}\left[\exp\left\{-4\alpha\int_0^t \sigma(Z_s)K_s\, dW_{1s} + 4\alpha\sum_{i=1}^{N_{t-}} Y_i\right\}\right] \\ &\leq \mathbb{E}\left[\exp\left\{\frac{1}{2}L_t + 16\alpha^2\int_0^t \sigma^2(Z_s)K_s^2\, ds + 4\alpha\sum_{i=1}^{N_{t-}} Y_i\right\}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[e^{-4\alpha X_{t-}^K}\right] &\leq e^{16\alpha^2 C_K^2 T}\mathbb{E}\left[\exp\left\{\frac{1}{2}L_t + 4\alpha\sum_{i=1}^{N_{t-}} Y_i\right\}\right] \\ &\leq e^{16\alpha^2 C_K^2 T}\left[\mathbb{E}\left[\exp\{L_t\}\right]\right]^{1/2}\left[\mathbb{E}\left[\exp\{8\alpha\sum_{i=1}^{N_{t-}} Y_i\}\right]\right]^{1/2}.\end{aligned}$$

Since $\exp\{L_t\}$ is a martingale and using (4.18), we obtain:

$$\begin{aligned}\mathbb{E}\left[e^{-4\alpha X_{t-}^K}\right] &\leq e^{16\alpha^2 C_K^2 T}\left[\mathbb{E}\left[\exp\{8\alpha\sum_{i=1}^{N_{t-}} Y_i\}\right]\right]^{1/2}\\ &\leq e^{16\alpha^2 C_K^2 T}\exp\left\{\frac{\lambda t}{2}\left(\int_0^\infty (e^{8\alpha y}-1)\,dG(y)\right)\right\} < \infty,\end{aligned}$$

which proves (4.4). In order to prove conditions (4.5) and (4.6), we observe that:

$$\mathbb{E}|f_x(t, X_t^K, Z_t)|^2 \leq C_1\left[\mathbb{E}\left[1+|Z_t|\right]^4\right]^{1/2}\left[\mathbb{E}\left[e^{-4\alpha X_{t-}^K}\right]\right]^{1/2}$$

and

$$\mathbb{E}|f_z(t, X_t^K, Z_t)|^2 \leq C_2\left[\mathbb{E}\left[1+|Z_t|\right]^4\right]^{1/2}\left[\mathbb{E}\left[e^{-4\alpha X_{t-}^K}\right]\right]^{1/2}.$$

Then by the same arguments as above, we get conditions (4.5) and (4.6). For the case in which the interest rate $r \neq 0$, let $\tilde{X}_t^K = e^{r(T-t)}X_t^K$. An application of Itô's formula shows that $\tilde{X}$ satisfies the following SDE:

$$\begin{aligned}\tilde{X}_t^K &= xe^{rT} + \int_0^t e^{r(T-s)}(c + (\mu(Z_s) - r)K_s)\,ds\\ &\int_0^t e^{r(T-s)}K_s\,dW_{1s} - \int_0^t\int_0^\infty e^{r(T-s)}y\bar{N}(dsdy),\end{aligned} \tag{4.22}$$

which corresponds to the case in which the interest rate is zero with drift $e^{r(T-t)}(c + (\mu(Z_t) - r)K_t)$, and the result can be derived in a similar way to the first part of the proof. □

4.4 Exponential Claim Distribution

In this example, we solve the Cauchy problem (4.17) by using the finite-difference method. First, we assume that the claims are exponentially distributed with parameter b, then for $T < \frac{1}{r}\log(b/\alpha)$ we get that:

$$\theta_t = \frac{\alpha e^{r(T-t)}}{b - \alpha e^{r(T-t)}},$$

and (4.17) becomes:

$$\xi_t+\frac{1}{2}\beta^2\xi_{zz}+g(z)\xi_z-\left(\frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)}+c\alpha e^{r(T-t)}-\frac{\lambda\alpha e^{r(T-t)}}{b-\alpha e^{r(T-t)}}\right)\xi=0. \quad (4.23)$$

Since the numerical computations can only be performed on finite domains, the first step is to reduce the Cauchy problem (4.17) to a bounded domain, i.e., $\mathbb{R}$ is replaced by $[-a,a]$, and to add artificial boundary conditions. Then the Cauchy problem to solve is the following:

$$\begin{cases} \xi_t+\frac{1}{2}\beta^2\xi_{zz}+g(z)\xi_z-\left(\frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)}+c\alpha e^{r(T-t)}-\lambda\theta_t\right)\xi=0 \\ \\ \xi(z,T)=1, \quad \forall z\in]-a,a[, \\ \xi(z,t)=1, \quad \forall z\notin]-a,a[\times[0,T]. \end{cases} \quad (4.24)$$

From [12], we know that the solution of (4.24) exists and is unique. The imposed boundary conditions give a good error estimate for large values of a. This result was shown in [15] by using the Feynman-Kac formula for parabolic PDE.

The first step is to discretize (4.24) in the domain $D:=[-a,a]\times[0,T]$. A uniform grid on D is given by:

$$\begin{aligned} z_i &= -a+(i-1)h, \quad && i=1\dots N, \quad && h=2a/N-1, \\ t_j &= (j-1)k, && j=1\dots M, && k=T/M-1. \end{aligned}$$

The space and time derivatives are discretized using finite differences as follows:

$$\begin{aligned} \xi_t(z_i,t_j) &\simeq \frac{\xi(z_i,t_j)-\xi(z_i,t_j-k)}{k} \\ \xi_z(z_i,t_j) &\simeq \frac{\xi(z_i+h,t_j)-\xi(z_i-h,t_j)}{2h} \\ \xi_{zz}(z_i,t_j) &\simeq \frac{\xi(z_i+h,t_j)-2\xi(z_i,t_j)+\xi(z_i-h,t_j)}{h^2}. \end{aligned}$$

Since our Cauchy problem is given with a terminal condition, we follow the same procedure as [30], but backward in time. We denote by $\xi_i^j:=\xi(z_i,t_j)$

the solution on the discretized domain. Then by substituting the derivatives by the expressions given above, (4.24) becomes:

$$\frac{\xi_i^j - \xi_i^{j-1}}{k} + \frac{1}{2}\beta^2 \frac{\xi_{i+1}^j - 2\xi_i^j + \xi_{i-1}^j}{h^2} + g(z_i)\frac{\xi_{i+1}^j - \xi_{i-1}^j}{2h}$$
$$- \left(\frac{(\mu(z_i) - r)^2}{2\sigma^2(z_i)} + c\alpha e^{r(T-t_j)} - \frac{\lambda \alpha e^{r(T-t_j)}}{b - \alpha e^{r(T-t_j)}} \right) \xi_i^j = 0$$

then

$$-\frac{\xi_i^{j-1}}{k} + \left(\frac{1}{k} - \frac{\beta^2}{h^2} - \left(\frac{(\mu(z_i) - r)^2}{2\sigma^2(z_i)} + c\alpha e^{r(T-t_j)} - \frac{\lambda \alpha e^{r(T-t_j)}}{b - \alpha e^{r(T-t_j)}} \right) \right) \xi_i^j$$
$$+ \left(\frac{\beta^2}{2h^2} + \frac{1}{2h} g(z_i) \right) \xi_{i+1}^j + \left(\frac{\beta^2}{2h^2} - \frac{1}{2h} g(z_i) \right) \xi_{i-1}^j = 0$$

Then for $j = 2 \dots M$ and $i = 2 \dots N-1$, ξ_i^j satisfies the following explicit scheme:

$$\xi_i^{j-1} = \left(1 - \frac{k\beta^2}{h^2} - k \left(\frac{(\mu(z_i) - r)^2}{2\sigma^2(z_i)} + c\alpha e^{r(T-t_j)} - \frac{\lambda \alpha e^{r(T-t_j)}}{b - \alpha e^{r(T-t_j)}} \right) \right) \xi_i^j \quad (4.25)$$
$$+ \left(\frac{k\beta^2}{2h^2} + \frac{k}{2h} g(z_i) \right) \xi_{i+1}^j + \left(\frac{k\beta^2}{2h^2} - \frac{k}{2h} g(z_i) \right) \xi_{i-1}^j.$$

The final condition is given by:

$$\xi_i^M = 1, \qquad \text{for all } i = 1 \dots N.$$

The imposed boundary conditions will be given by:

$$\xi_1^j = 1 \text{ for all } j = 1 \dots M-1,$$
$$\xi_{N+1}^j = 1 \text{ for all } j = 1 \dots M-1.$$

Our algorithm given by the explicit scheme, final condition and the imposed boundary conditions is backward in time, forward in space, and hence, by the explicit scheme, the numerical solution can be computed.

Consistency: Let

$$\mathscr{L}\xi = \xi_t + \frac{1}{2}\beta^2 + g(z)\xi_z + GN(t,z)\xi,$$

where

$$GN(z,t) = -\left(\frac{(\mu(z)-r)^2}{2\sigma^2(z)} + c\alpha e^{r(T-t)} - \frac{\lambda\alpha e^{r(T-t)}}{b-\alpha e^{r(T-t)}}\right).$$

The difference operator is given by:

$$\mathscr{L}_{h,k}\xi = \frac{\xi_i^j - \xi_i^{j-1}}{k} + \frac{1}{2}\beta^2\frac{\xi_{i+1}^j - 2\xi_i^j + \xi_{i-1}^j}{h^2} + g(z_i)\frac{\xi_{i+1}^j - \xi_{i-1}^j}{2h} + GN(z_i,t_j)\xi_i^j.$$

By Taylor's series expansion:

$$\xi_i^j = \xi_i^{j-1} + k\xi_t + \frac{1}{2}k^2\xi_{tt} + O(k^3),$$

$$\xi_{i\pm1}^j = \xi_i^j \pm h\xi_z + \frac{1}{2}h^2\xi_{zz} \pm \frac{1}{6}h^3\xi_{zzz} + O(h^4).$$

The derivatives on the right hand side are all evaluated at (z_i, t_j). Then

$$\frac{\xi_i^j - \xi_i^{j-1}}{k} = \xi_t + \frac{1}{2}k\xi_{tt} + O(k^2),$$

$$\frac{1}{2}\beta^2\frac{\xi_{i+1}^j - 2\xi_i^j + \xi_{i-1}^j}{h^2} = \frac{1}{2}\beta^2\xi_{zz} + O(h^2)$$

and

$$g(z_i)\frac{\xi_{i+1}^j - \xi_{i-1}^j}{2h} = g(z_i)\xi_z + \frac{1}{6}h^2 g(z_i)\xi_{zz} + O(h^3).$$

Then:

$$\mathscr{L}_{h,k}\xi = \xi_t + \frac{1}{2}k\xi_{tt} + \frac{1}{2}\beta^2\xi_{zz} + g(z)\xi_z + \frac{1}{6}h^2 g(z_i)\xi_{zz} + GN(z_i,t_j) + O(h^2+h^3+k^2),$$

and

$$\begin{aligned}\mathscr{L}_{h,k}\xi - \mathscr{L}\xi &= \frac{1}{2}k\xi_{tt} + \frac{1}{6}h^2 g(z_i)\xi_{zz} + O(h^2+h^3+k^2)\\ &\longrightarrow 0 \ as \ (h,k) \to 0.\end{aligned}$$

Therefore the explicit scheme given by (4.25) is consistent.

Stability: Now we explore the stability of the explicit scheme, which is an important property that ensures that the numerical solution at a given point does not blow up when $(\triangle z, \triangle t) \to 0$. From von Neumann (see [39, 40]), a numerical scheme is stable if the module of the gain (amplification factor) is less than the unit. Now, to determine if the explicit scheme given by (4.25) is stable for the parameters given above, we use an idea adapted from the Fourier method. Hence, since our PDE involves variable coefficients, then the stability condition obtained for the constant coefficients scheme can be used to give stability conditions for the same scheme applied to parabolic PDE with variable coefficients (see [39]). The partial differential equation given by (4.24) is linear, thus we only need to consider one Fourier mode (see [40]), i.e., we admit that the solution of the explicit scheme has the form:

$$\xi_i^n = A^n e^{q\nu i \triangle z}, \tag{4.26}$$

where ν is the wave number and $q = \sqrt{-1}$. The function A will be specified below. Let $\omega = \nu \triangle z$, then for each z_i, t_n fixed, the amplification factor is a function of ω defined as:

$$P(\omega) = \frac{\xi_i^{n-1}}{\xi_i^n}.$$

Then substituting (4.26) into (4.25) gives:

$$\begin{aligned}
P(\omega) &= \frac{\xi_i^{n-1}}{\xi_i^n} = \frac{1}{A(\omega)} \\
&= \left(1 - \frac{k\beta^2}{h^2} + kGN(z_i, t_n)\right) \frac{\xi_i^n}{\xi_i^n} + \left(\frac{k\beta^2}{2h^2} + \frac{k}{2h} g(z_i)\right) \frac{\xi_{i+1}^n}{\xi_i^n} \\
&+ \left(\frac{k\beta^2}{2h^2} - \frac{k}{2h} g(z_i)\right) \frac{\xi_{i-1}^n}{\xi_i^n} \\
&= \left(1 - \frac{k\beta^2}{h^2} + kGN(z_i, t_n)\right) + \left(\frac{k\beta^2}{2h^2} + \frac{k}{2h} g(z_i)\right) e^{q\omega} \\
&\quad + \left(\frac{k\beta^2}{2h^2} - \frac{k}{2h} g(z_i)\right) e^{-q\omega}
\end{aligned}$$

Hence, the module of P is such that:

$$\mid P(\omega) \mid^2 = \left(1 - \frac{k\beta^2}{h^2} + kGN + \frac{k\beta^2}{h^2} \cos(\omega)\right)^2 + \frac{k^2}{h^2} g^2(z_i) \sin^2(\omega) \tag{4.27}$$

The stability condition $\mid P(\omega) \mid^2 \leq 1$ must hold for every ω. Now to get an explicit condition for which $\mid P(\omega) \mid^2 \leq 1$ we proceed as follows. Let

1. $M_1 = \min_{(z,t)\in D} \frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)}, M_2 = \max_{(z,t)\in D} \frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)}$.
2. $T < \frac{1}{r}\ln\left(\frac{b}{\alpha}\left(\frac{M_1+c\alpha}{\lambda+M_1+c\alpha}\right)\right)$.

Notation

$$M_4 = M_1 + c\alpha - \frac{\lambda\alpha e^{rT}}{b-\alpha e^{rT}}$$
$$M_5 = M_2 + c\alpha - \frac{\lambda\alpha}{b-\alpha}$$

Main purpose

If:

$$k^2 < \min\left(\left(\frac{2M_4h^2}{h^2M_4^2+M_3}\right)^2, \frac{h^4}{M_5^2h^4+(M_3+4\beta^2M_5)h^2+4\beta^4}\right)$$

then:

$$\mid P \mid^2 \leq 1$$

Step 1. First we prove that

$$-M_5 < GN < -M_4 < 0.$$

Since the function

$$\frac{\lambda\alpha e^{r(T-t)}}{b-\alpha e^{r(T-t)}}$$

is decreasing because:

$$\frac{d}{dt}\left(\frac{\lambda\alpha e^{r(T-t)}}{b-\alpha e^{r(T-t)}}\right) = -\frac{\lambda\alpha e^{r(T-t)}}{(b-\alpha e^{r(T-t)})^2} < 0,$$

then the minimum is reached at $t = T$ and the maximum at $t = 0$. Hence:

$$GN \leq -M_1 - c\alpha + \frac{\lambda \alpha e^{rT}}{b - \alpha e^{rT}}$$

and

$$GN < -M_4.$$

To get $GN > -M_5$ we proceed similarly as above .

Step 2.

$$-\frac{2k\beta^2}{h^2} - kM_5 \leq \left(1 + \frac{k\beta^2}{h^2}(\cos(\omega) - 1) + kGN\right) \leq 1 + kGN < 1 - kM_4.$$

Then:

$$\left(1 + \frac{k\beta^2}{h^2}(\cos(\omega) - 1) + kGN\right)^2 \leq \max\left[(1 - kM_4)^2, \left(\frac{2k\beta^2}{h^2} + kM_5\right)^2\right],$$

furthermore:

$$\mid P \mid^2 \leq \max\left[(1 - kM_4)^2 + \frac{k^2}{h^2}M_3, \left(\frac{2k\beta^2}{h^2} + kM_5\right)^2 + \frac{k^2}{h^2}M_3\right].$$

Then

$$(1 - kM_4)^2 + \frac{k^2}{h^2}M_3 < 1,$$

implies

$$k < \frac{2M_4h^2}{k^2M_4^2 + M_3}.$$

Now

$$\left(\frac{2k\beta^2}{h^2} + kM_5\right)^2 + \frac{k^2}{h^2}M_3 < 1,$$

leads to

$$k^2 < \frac{h^2}{4\beta^4h^{-2} + 4\beta^2M_5 + h^2M_5^2 + M_3}$$

Finally, a sufficient condition for stability is given by:

$$k^2 < \min\left(\left(\frac{2M_4h^2}{h^2M_4^2 + M_3}\right)^2, \frac{h^4}{M_5^2h^4 + (M_3 + 4\beta^2M_5)h^2 + 4\beta^4}\right) \tag{4.28}$$

Observation 4.1 *The stability condition given by* (4.28) *can be written as:*

$$k^2 < Ch^4$$

where:

$$C = \max\left(\left(\frac{2M_4}{M_3}\right)^2, \frac{1}{4\beta^2}\right)$$

Well-posedness: The well-posedness condition is given for the initial parabolic PDE (see [39]). First we make the following substitution:

$$\omega(z,t) = \omega(z, T-t),$$

then the parabolic PDE given by (4.17) can be written as:

$$\begin{cases} -\omega_t + \frac{1}{2}\beta^2\omega_{zz} + g(z)\omega_z - \left(\frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)} + c\alpha e^{rt} - \frac{\lambda\alpha e^{rt}}{b-\alpha e^{rt}}\right)\omega = 0, \\ \\ \omega(z,0) = 1, \quad \forall z \in \mathbb{R}. \end{cases} \tag{4.29}$$

Let

$$HN(z,t) = -\left(\frac{1}{2}\frac{(\mu(z)-r)^2}{\sigma^2(z)} + c\alpha e^{rt} - \frac{\lambda\alpha e^{rt}}{b-\alpha e^{rt}}\right),$$

then by the Feynman-Kac formula, we have:

$$\begin{aligned} \omega(z,t) &= \mathbb{E}\left[\omega(Z_t,0)\exp\left(\int_0^t HN(Z_s,s)\right)ds \mid Z_0 = z\right] \\ &\leq e^{\lambda\alpha\int_0^t \frac{e^{rs}}{b-\alpha e^{rs}}ds}\,\mathbb{E}\left[\omega(Z_t,0)e^{-\frac{1}{2}\int_0^t \frac{(\mu(Z_s)-r)^2}{\sigma^2(Z_s)}}\Big| Z_0 = z\right] \\ &\leq e^{\lambda\alpha\int_0^t \frac{e^{rs}}{b-\alpha e^{rs}}ds}\,\mathbb{E}\left[\omega(Z_t,0)\right] \\ &\leq C_T = e^{\lambda\alpha\int_0^T \frac{e^{rs}}{b-\alpha e^{rs}}ds} = e^{-\frac{\lambda}{r}\log\left(\frac{b-\alpha e^{rT}}{b-\alpha}\right)} = \left(\frac{b-\alpha}{b-\alpha e^{rT}}\right)^{\frac{\lambda}{r}}. \end{aligned}$$

From the first line to the second line, we used $e^{-\int_0^t c\alpha e^{rs}\, ds} \leq 1$. From the second line to the last line, we used $\omega(Z_t, 0) = 1$. Now to complete the proof of well-posedness, we have:

$$\begin{aligned} \omega(z,t) &\leq C_T \omega(z,0) \\ \omega^2(z,t) &\leq C_T^2 \omega^2(z,0) \end{aligned}$$

and then

$$\int_{-\infty}^{\infty} \omega^2(z,t)\, dz \leq C_T^2 \int_{-\infty}^{\infty} \omega^2(z,0)\, dz, \ \forall\ 0 \leq t \leq T,$$

which proves the well-posedness of the problem given by (4.17) (see [39]). Since the explicit scheme given by (4.25) is consistent, stable and well-posed, then from the Lax theorem, it is also convergent (see [40]).

Example 4.1 In this example, we consider the case in which μ, σ and r are constant i.e., the external factor does not affect the insurer preferences. This model was studied in [7]. The utility function in the absence of the external factor will be denoted as $V(t,x)$ and is given by:

$$\begin{aligned} V(t,x) &= -\exp\left\{-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t) + \frac{c\alpha}{r}(1-e^{r(T-t)}) - \int_t^T \frac{\lambda\alpha e^{r(T-s)}}{b-\alpha e^{r(T-s)}}\, ds\right\} \\ &\quad .\exp\left\{-\alpha x e^{r(T-t)}\right\}. \end{aligned} \tag{4.30}$$

Let $\mu = 0.3, \sigma = e^{-1},\ r = 0.04,\ b = 2,\ c = 5,\ \lambda = 3, \alpha = 0.02$ and $T = 5$. The following figure shows the behavior of the analytic solution:

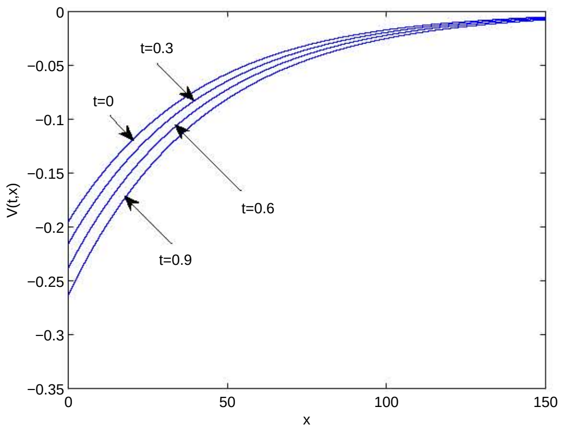

Figure 4.1: The analytic solution $V(t,x)$

To make sure that our algorithm given by the explicit scheme recovers the solution given by (4.30), we take $\delta = 0,\ \kappa = 0, \beta = 0\,,\ \mu = 0.3, \sigma = e^z$, $r = 0.04,\ b = 2,\ c = 5,\ \lambda = 3, \alpha = 0.02, T = 5,\ \ a = 2, h = 0.01$ and $k = 0.0001$. Then the numerical solution is obtained by substituting $z = -1$ in the explicit scheme.

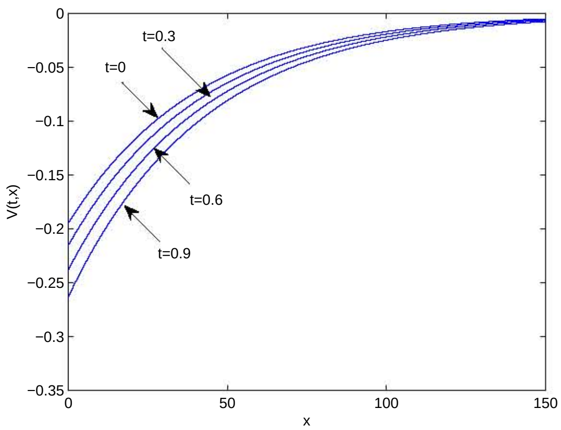

Figure 4.2: The numerical solution $V(t,x)$

OBSERVATION 4.2 *From Figures 4.1 and 4.2, we conclude the following:*

1. *The analytic solution given by* (4.30) *and the numerical solution given by the explicit scheme* (4.25) *have the same behavior.*

2. *The utility function for an insurer investing in a risky asset following a geometric Brownian motion is increasing in wealth and decreasing in time.*

3. *If in addition to* **Hypothesis** *4.1, we assume that μ, σ are constant, $g(z) = 0$, and $\beta = 0$, then the Cauchy problem* (4.17) *is reduced to an ordinary differential equation, which can be solved explicitly. Furthermore, we recover the results obtained in [7].*

EXAMPLE 4.2 In this example, we consider the stochastic volatility model described by the Scott model with

$$\begin{aligned} &\delta = 0.1, \kappa = 1, \mu = 0.3, \sigma(z) = e^z, r = 0.04, \beta = 0.3, b = 2, c = 5, \\ &\lambda = 3, \alpha = 0.02, T = 5\, a = 2, h = 0.01 \,\text{and}\, k = 0.0001. \end{aligned} \tag{4.31}$$

The main purpose is to study the behavior of the insurer's utility function and his optimal investment strategy by checking how it is affected by the external factor. To obtain a general conclusion, we make several simulations, which are illustrated by the following figures for different fixed values of x and t:

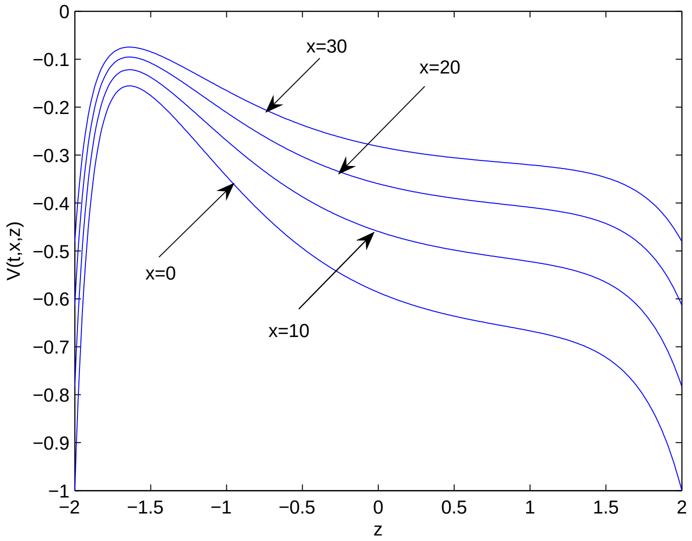

Figure 4.3: The numerical solution at $t = 0$

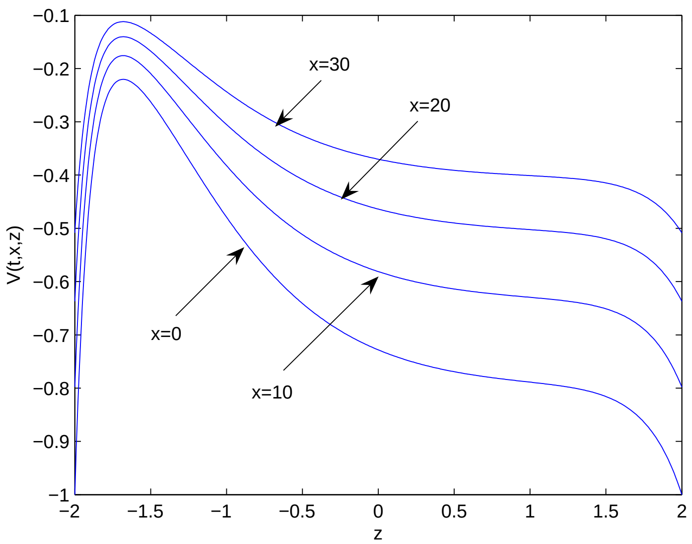

Figure 4.4: The numerical solution at $t = 2$

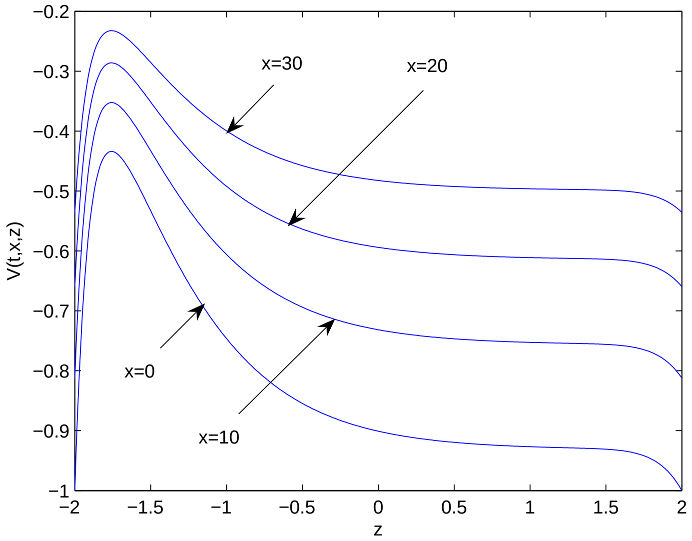

Figure 4.5: The numerical solution at $t = 4$

From Figures 4.3, 4.4 and 4.5, we conclude the following observations:

Observation 4.3 *1. The utility function as a function of x is increasing.*

2. The utility function as a function of t is decreasing.

3. For x and t fixed, the utility function is increasing for small values of z. When the impact of the external factor impact is small the utility function is increasing. However as the external factor increases, the utility function decreases because of the higher risk involved.

The following figure shows the behavior of the optimal strategy.

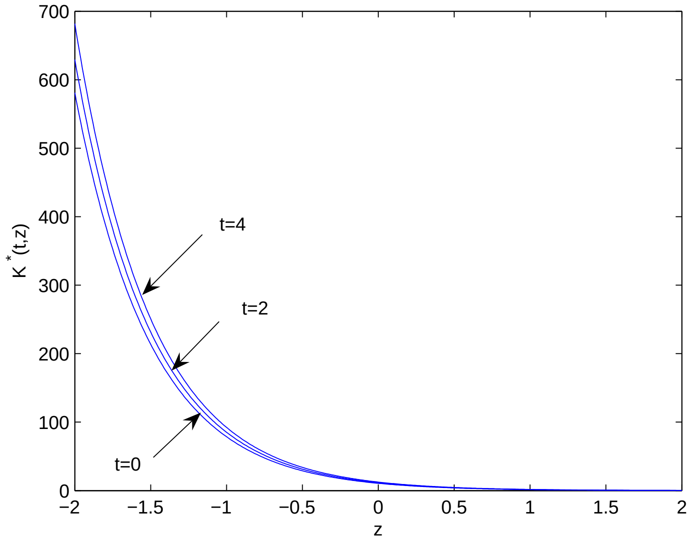

Figure 4.6: The optimal strategy $K^*(t, z)$

OBSERVATION 4.4 *1. The optimal strategy is decreasing as a function of the external factor, as we can see from Figure 4.6 and the the insurer will invest less in the risky asset due to the higher risk involved. The utility function as a function of time is increasing.*

2. The optimal strategy depends only on time and the external factor.

3. The insurer's wealth does not affect his decision.

4.5 Ruin Probability

In this section, we obtain an upper bound for the ruin probability when the insurer follows the optimal strategy outlined in the previous section. First, in addition to the previous hypothesis, we will make the following assumptions:

HYPOTHESIS 4.2 *1.* $\sigma_0 \leq \sigma(\cdot) \leq \sigma_1$ *for some constants* $\sigma_1 > \sigma_0 > 0$*;*

2. $r < \mu_0 \leq \mu(\cdot) \leq \mu_1$*;*

The risk process associated with the optimal strategy K^* is given by:

$$\begin{aligned} X_t^{K^*} &= x + ct + \int_0^t \left(\frac{(\mu(Z_s) - r)^2}{\alpha \sigma^2(Z_s)} e^{-r(T-s)} + r X_s^{K^*} \right) ds \\ &\quad + \int_0^t \frac{(\mu(Z_s) - r)}{\alpha \sigma(Z_s)} e^{-r(T-s)} \, dW_{1s} - \sum_{i=1}^{N_t} Y_i. \end{aligned} \tag{4.32}$$

For this problem, the time of ruin in finite horizon is defined as

$$\tau^*(x,z) = \begin{cases} \inf\left\{0 \leq t \leq T : X_t^{K^*} < 0\right\}, & \text{if } \left\{0 \leq t \leq T : X_t^{K^*} < 0\right\} \neq \emptyset \\ \infty, & \text{in other case.} \end{cases}$$

and the ruin probability is given by:

$$\mathbb{P}[\tau^*(x,z) < T].$$

Let

$$\bar{X}_t^* = e^{-rt} X_t^{K^*}.$$

An application of Itô's lemma leads to

$$\begin{aligned} \bar{X}_t^* &= x + \int_0^t e^{-rT} \left(c + \frac{(\mu(Z_s) - r)^2}{\alpha \sigma^2(Z_s)} \right) ds \\ &\quad + \int_0^t e^{-rT} \frac{(\mu(Z_s) - r)}{\alpha \sigma(Z_s)} \, dW_{1s} - \sum_{i=1}^{N_t} e^{-rT_i} Y_i. \end{aligned} \tag{4.33}$$

Before stating the main theorem of this section, we assume that:
The law of random variables $(Y_i)_{i \geq 1}$ admits a Laplace transform $M_Y(\gamma)$ for $\gamma \in (0, \gamma_\infty]$ and

$$\lim_{\gamma \to \gamma_\infty} M_Y(\gamma) = \infty, \tag{4.34}$$

with the following safety loading condition:

$$\left(c + \frac{(\mu_0 - r)^2}{\alpha \sigma_1^2} \right) e^{-rT} - \lambda \mathbb{E}[Y_1] > 0.$$

Now we define the function $\mathscr{G} : [0, \gamma_\infty) \to \mathbb{R}$ by:

$$\mathscr{G}(\gamma) = -\gamma \left(c + \frac{(\mu_0 - r)^2}{\alpha \sigma_1^2} \right) e^{-rT} + \frac{\gamma^2}{2} \frac{(\mu_1 - r)^2}{\alpha^2 \sigma_0^2} e^{-2rT} + \lambda(M_Y(\gamma) - 1).$$

The following theorem gives an upper bound for the ruin probabilities in finite horizon.

THEOREM 4.4 *(The Upper Bound)*
The ruin probability can be bounded from above by

$$\mathbb{P}[\tau^*(x, z) < T] \leq e^{-\gamma^* x},$$

where γ^* *is the unique positive root of:*

$$\mathscr{G}(\gamma) = 0.$$

To get the proof of the theorem, we need the following lemma.

LEMMA 4.1 *The exponential process*

$$H_t = \exp\left\{-\gamma^* \bar{X}_t^*\right\}$$

is a supermartingale.

Proof. The existence of γ^* is a consequence of the convex property of $\mathscr{G}(\gamma)$ and (4.34). In order to prove that H_t is a supermartingale, first we prove that H_t has a first finite moment. We observe that

$$\begin{aligned} -\gamma^* \bar{X}_t^* &= -\gamma^* x - \gamma^* \int_0^t \left(c + \frac{(\mu(Z_s) - r)^2}{\alpha \sigma^2(Z_s)} \right) e^{-rT} \, ds \\ &\quad -\gamma^* \int_0^t \frac{(\mu(Z_s) - r)}{\alpha \sigma(Z_s)} e^{-rT} \, dW_{1s} + \gamma^* \sum_{i=1}^{N_t} e^{-rT_i} Y_i \end{aligned}$$

Compensating the last equation by:

$$\frac{\gamma^{*2}}{2} \int_0^t \frac{(\mu(Z_s) - r)^2}{\alpha^2 \sigma^2(Z_s)} e^{-2rT} \, ds,$$

we get that:

$$\begin{aligned}
-\gamma^* \bar{X}_t^* &= -\gamma^* x - \gamma^* \int_0^t \left(c + \frac{(\mu(Z_s) - r)^2}{\alpha \sigma^2(Z_s)} \right) e^{-rT}\, ds \\
&\quad -\gamma^* \int_0^t \frac{(\mu(Z_s) - r)}{\alpha \sigma(Z_s)} e^{-rT}\, dW_{1s} - \frac{\gamma^{*2}}{2} \int_0^t \frac{(\mu(Z_s) - r)^2}{\alpha^2 \sigma^2(Z_s)} e^{-2rT}\, ds \\
&\quad + \frac{\gamma^{*2}}{2} \int_0^t \frac{(\mu(Z_s) - r)^2}{\alpha^2 \sigma^2(Z_s)} e^{-2rT}\, ds + \gamma^* \sum_{i=1}^{N_t} e^{-rT_i} Y_i.
\end{aligned}$$

Using **Hypothesis** (4.2) and observing that

$$\exp\left\{ -\gamma^* \int_0^t \frac{(\mu(Z_s) - r)}{\alpha \sigma(Z_s)} e^{-rT}\, dW_{1s} - \frac{\gamma^{*2}}{2} \int_0^t \frac{(\mu(Z_s) - r)^2}{\alpha^2 \sigma^2(Z_s)} e^{-2rT}\, ds \right\} \tag{4.35}$$

is a martingale, and that the compound Poisson process has stationary independent increments, we get that

$$\begin{aligned}
\mathbb{E}[H_t] &= \mathbb{E}[\exp\{-\gamma^* \bar{X}_t^*\}] \\
&\leq e^{-\gamma^* x} \mathbb{E}\Big[\exp\left\{ \int_0^t -\gamma^* \left(c + \frac{(\mu_0 - r)^2}{\alpha \sigma_1^2} \right) e^{-rT} + \frac{\gamma^{*2}}{2} \frac{(\mu_1 - r)^2}{\alpha^2 \sigma_0^2} e^{-2rT}\, ds \right\} \\
&\quad \times \exp\{\lambda t (M_Y(\gamma^*) - 1)\} \Big] \\
&\leq e^{-\gamma^* x} \mathbb{E}\left[e^{\int_0^t \mathscr{G}(\gamma^*)\, ds} \right] \\
&\leq e^{-\gamma^* x} < \infty.
\end{aligned}$$

Therefore, H_t has a finite first moment. Denote the conditional expectation $\mathbb{E}[. \mid \mathcal{F}_t]$ by $\mathbb{E}_t[\,\cdot\,]$. Let $0 \leq s \leq t$. Then proceeding as above, we get:

$$\begin{aligned}
\mathbb{E}_s[H_t] &= \mathbb{E}_s[e^{-\gamma^*(\bar{X}_t^* - \bar{X}_s^*)}] e^{-\gamma^* \bar{X}_s^*} \\
&\leq \mathbb{E}\left[e^{\int_s^t \mathscr{G}(\gamma^*)\, dv} \right] H_s \\
&\leq H_s.
\end{aligned}$$

From the first line to the second, we used (4.35) and the fact that the compound Poisson process has independent increments. Therefore H_t is a supermartingale. □

Proof. **(Theorem 4.4)** We know that H_t is a supermartingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then by the optional sampling theorem for supermartingales, we get:

$$\begin{aligned} e^{-\gamma^* x} &\geq \mathbb{E}[H_{\tau^*\wedge T}] \\ &\geq \mathbb{E}[H_{\tau^*}\mathbb{I}_{\tau^*<T}] \\ &\geq \mathbb{E}[H_{\tau^*} \mid \tau^* < T]\mathbb{P}[\tau^* < T]. \end{aligned}$$

Then

$$e^{-\gamma^* x} \geq \mathbb{E}[H_{\tau^*} \mid \tau^* < T]\mathbb{P}[\tau^* < T],$$

therefore

$$\mathbb{P}[\tau^*(x,z) < T] \leq \frac{e^{-\gamma^* x}}{\mathbb{E}[H_{\tau^*} \mid \tau^* < T]},$$

and then

$$\mathbb{P}[\tau^*(x,z) < T] \leq e^{-\gamma^* x},$$

because $H_{\tau^*} \geq 1$ on $[\tau^* < T]$. □

Chapter 5

Conclusion And Further Research

The risk model with investment described in this article is more general in the sense that the coefficients μ and σ depend on an external factor modeled as a diffusion process. Using the same approach as Gaier et al. (2003), we get an upper bound and a lower bound for the ruin probabilities. When the risky asset price is given by the Scott model, we obtain an upper bound which depends on the external factor. We make some observations concerning how our model allows to recover the known bounds for ruin probabilities.
The problem of the expected utility of the insurer was successfully solved by using stochastic control techniques. When the insurer's preferences are exponential, the value function is related to a parabolic PDE. We develop an explicit numerical scheme to solve the parabolic PDE, and we observe that the utility function depends on the external factor. When the impact of the external factor is small the utility function is increasing. However as the external factor increases, the utility function decreases because of the higher risk involved. Finally, we use the optimal strategy of investment to produce an upper bound for the ruin probability.

We conclude by outlining further research directions from the present work:

- The inclusion of dividend payments in the risk model with investment.
- The study of the ruin problem by using the PDE approach.
- The asymptotic behavior of the ruin probability.
- The study of the expected utility under more complex utility functions, for example Hara and Logarithmic.
- The implementation of an implicit scheme, the finite elements method.
- The implementation of Monte Carlo methods.

Appendix A

Parabolic PDE

A.1 Parabolic Partial Differential Equations

To clarify the statement of our research problem, we shall devote this section to a short introduction to some concepts of parabolic PDE's. The goal is to avoid technicalities and communicate only the basic terminology, definitions and important results of existence and uniqueness for parabolic PDE's, which will be useful in chapter 4.

DEFINITION A.1 *Let* $E = \sum_{i,j=1}^{n} a_{ij}(x,t)\partial_{x_i}\partial_{x_j}$.

1. *We say that* E *is uniformly elliptic, if there exist* $\lambda_0, \lambda_1 > 0$ *such that*

$$\lambda_0 \mid y \mid^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t) y_i y_j \leq \lambda_1 \mid y \mid^2$$

for all $y \in \mathbb{R}^n$ *and all* $(x,t)\mathbb{R}^n \times [0,T]$.

2. *A function* f *on* $\mathbb{R}^n \times [0,T]$ *is called Hölder continuous in* x *with exponent* $0 < h \leq 1$*, uniformly with respect to* t *in compact subsets of* $\mathbb{R}^n \times [0,T]$*, if for each compact set* $D \subset \mathbb{R}^n$ *there is a constant* c_D *such*

that

$$| f(x,t) - f(y,t) | \leq c_D | x - y |^h, \forall x, y \in D\,, \forall t \in [0,T].$$

3. *f is said to be uniformly Hölder continuous in (t,x) in compact subsets of $\mathbb{R}^n \times [0,T]$ if for each compact set $D \subset \mathbb{R}^n \times [0,T]$ there is a constant C such that*

$$| f(x,t) - f(y,s) | \leq C(| x - y |^h + | t - s |^{h/2}), \forall (x,t), (y,s) \in D.$$

Theorem A.1 *(Friedman 1975, [12])*
We consider the following Cauchy problem:

$$\begin{cases} u_t(x,t) + Lu(x,t) = f(x,t) & in\ \mathbb{R}^n \times [0,T) \\ u(x,T) = h(x) & in\ \mathbb{R}^n \end{cases} \tag{A.1}$$

where L is given by:

$$Lu = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u.$$

If the Cauchy problem (A.1) *satisfies the following conditions:*

1. *the coefficients of L are uniformly elliptic;*

2. *the functions a_{ij}, b_i are bounded in $\mathbb{R}^n \times [0,T]$ and uniformly Lipschitz continuous in (x,t) in compact subsets of $\mathbb{R}^n \times [0,T]$;*

3. *the functions a_{ij} are Hölder continuous in x, uniformly with respect to (x,t) in $\mathbb{R}^n \times [0,T]$;*

4. *the function $c(x,t)$ is bounded in $\mathbb{R}^n \times [0,T]$ and uniformly Hölder continuous in (x,t) in compact subsets of $\mathbb{R}^n \times [0,T]$.*

5. *$f(x,t)$ is continuous in $\mathbb{R}^n \times [0,T]$, uniformly Hölder continuous in x with respect to (x,t) and*

$$|f(x,t)| \leq B(1+|x|^\gamma);$$

6. *$h(x)$ is continuous in $\mathbb{R}^n$ and*

$$|h(x)| \leq B(1+|x|^\gamma),$$

with $\gamma > 0$;

then there exists a unique solution u of the Cauchy problem (A.1) *satisfying:*

$$|u(x,t)| \leq const(1+|x|^\gamma) \qquad and \qquad |u_x(x,t)| \leq const(1+|x|^\gamma).$$

LEMMA A.1 *Let f be a real positive bounded function with bounded derivative, then f is uniformly Hölder continuous with exponent $h = \frac{1}{2}$ i.e.,*

$$| f(x) - f(y) | \leq C \mid x - y \mid^{1/2} .$$

Proof. By the mean value theorem and using that $f'(x)$ is bounded, we have:

$$| f^2(x) - f^2(y) | \leq K \mid x - y \mid,$$

where K is a constant. Then

$$| f(x) - f(y) | \leq K \mid x - y \mid^{1/2},$$

because f is positive. □

THEOREM A.2 *(Pham, [28])*
Let X_t be a stochastic processes defined by the following SDE:

$$X_t = x + \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dW_s,$$

with a standard Brownian motion W_t. We assume that for some $L > 0$, the coefficients satisfy:

$$| f(x) - f(y) | + | g(x) - g(y) | \leq L | x - y |$$

$$| f(x) | + | g(x) | \leq L(1 + | x |),$$

for all $(x, y) \in \mathbb{R}^2$. Let $T > 0$ and $p \geq 2$. Then, there exists $C_p > 0$ such that for all $(t, x) \in [0, T) \times \mathbb{R}$ we have:

$$\mathbb{E}[\sup_{0 \leq t \leq T} | X_t |^p] \leq C_p(1 + | x |^p).$$

A.1.1 Itô's Formula For Jump Diffusion Processes

Itô's formula is an important tool to derive the HJB equation. Since the processes involved in this research paper are diffusions with jumps, we recall the following result in stochastic calculus.

PROPOSITION A.1 *Let $X_t, t \geq 0$ be a diffusion process with jumps defined as the sum of a drift term, a Brownian stochastic integral and a compound Poisson process:*

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s + \sum_{i=1}^{N_t} \triangle X_i, \tag{A.2}$$

where b_t and σ_t are continuous non-anticipating processes with:

$$\mathbb{E}\left[\int_0^T \sigma_t^2 \, dt\right] < \infty.$$

Then, for any $C^{1,2}$ function $F : [0, T] \times \mathbb{R} \to \mathbb{R}$, the process $Y_t = F(t, X_t)$ can be represented as:

$$\begin{aligned} F(t, X_t) - F(0, X_0) &= \int_0^t \left[\frac{\partial F}{\partial s}(s, X_s) + b_s \frac{\partial F}{\partial x}(s, X_s)\right] ds \\ &+ \frac{1}{2} \int_0^t \sigma_s^2 \frac{\partial^2 F}{\partial^2 s}(s, X_s) \, ds + \int_0^t \sigma_s \frac{\partial F}{\partial x}(s, X_s) \, dW_s \\ &+ \sum_{i \geq 1, T_i \leq t} \left[F(T_i, X_{T_i^-} + \triangle X_i) - F(T_i, X_{T_i^-})\right]. \end{aligned} \tag{A.3}$$

Lemma A.2 *(Gronwall's Lemma) Let* $\alpha \in \mathbb{R}$, $k(t) \geq 0$ *continuous and* $\gamma \in C(\mathbb{R})$. *If*

$$\gamma(t) \leq \alpha + \int_0^t k(s)\gamma(s)\,ds,$$

then

$$\gamma(t) \leq \alpha \exp\left\{\int_0^t k(s)\,ds\right\}.$$

Appendix B

The Finite Difference Method

It is a well-known fact that in general a closed smooth solution of parabolic PDE's does not exist, especially when the coefficients are not constant. Therefore, one is forced to use a numerical method. In this section, we recall some basic notions about numerical methods such as consistency, stability and convergence of numerical schemes, and how to estimate the error between the analytic solution and the numerical solution. We present the basic material necessary to do scientific computation. The difference scheme is an approximation of the parabolic PDE, which can be written as a linear system to be solved at each time step starting at the initial condition. The basic idea of finite difference schemes is to replace derivatives by finite differences in the parabolic PDE.

Now we introduce some notation which will be useful in understanding the basic theory of numerical analysis. To avoid the complexity of operators depending on n variables, the following results will be announced only for an operator depending on two variables, time t and space $x \in \mathbb{R}$.

Notation B.1 *1. Let F and G be functions of the same parameter x.*

We write

$$F = O(G) \qquad as \quad x \to 0,$$

if

$$\mid \frac{F}{G} \mid \leq D,$$

for some constant D and x sufficiently small.

2.

$$Qu(x,t) = u_t + \beta u_{xx} + g(x)u_x + \Lambda(t,x)u. \tag{B.1}$$

3. $u_i^j := u(i\triangle x, j\triangle t)$.

4. $u^{j+1} := (\cdots, u_{-1}^{j+1}, u_0^{j+1}, u_1^{j+1}, \cdots)^T$.

5. $\hat{u}(\omega,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-i\omega x} u(x,t)\, dx$ *is the Fourier transform of* $u(x,t)$.

6. $\hat{v}(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\sum_{m=-\infty}^{\infty} e^{-im\omega} v_m$ *is the discrete Fourier transform of* $v = (\cdots, v_{-1}, v_0, v_1, \cdots)^T$, *for* $\omega \in [-\pi, \pi]$.

7. $L_2(\mathbb{R}) = \left\{ v : \mathbb{R} \to \mathbb{C} : \displaystyle\int_{\mathbb{R}} \mid v(x) \mid^2 \; dx < \infty \right\}$ *with the norm:*

$$\|v\|_2 = \sqrt{\int_{\mathbb{R}} \mid v(x) \mid^2 \; dx}.$$

The domain of functions in L_2 can be $\mathbb{R}$ or $[-\pi, \pi]$.

8. $l_2 = \left\{ v = (\cdots, v_{-1}, v_0, v_1, \cdots)^T : \sum_{k=0}^{\infty} |v_k|^2 < \infty \right\}$ *with the norm*

$$\|u\|_2 = \sqrt{\sum_{k=-\infty}^{\infty} |u_k|^2}.$$

$l_{2,\triangle x}$ *vectors are the solution to our difference scheme at time step* j. *The* $l_{2,\triangle x}$ *norm is given by:*

$$\|u\|_{2,\triangle x} = \sqrt{\sum_{k=-\infty}^{\infty} |u_k|^2 \triangle x}.$$

To clarify the conditions that must be met for the implementation of the finite difference method, we start by working with the known heat equation:

Example B.1

$$\begin{cases} u_t(x,t) &= \nu u_{xx} \quad \text{in } \mathbb{R} \times [0,T) \\ u(x,0) &= h(x) \quad \text{in } \mathbb{R} \end{cases} \tag{B.2}$$

The first step is to discretize the domain. A uniform grid on the domain is given by:

$$x(i) = i\triangle x = ih \qquad i \geq 0$$
$$t(j) = j\triangle t = jk \qquad j \geq 0.$$

The basic idea of replacing derivatives by finite differences can be motivated as follows:

$$u_x(x,t) = \lim_{\epsilon \to 0} \frac{u(x+\epsilon,t) - u(x,t)}{\epsilon}.$$

Then some basic formulas that can be used are the following:

$$u_x(x,t) \approx \frac{u(x+h,t) - u(x,t)}{h} \tag{B.3}$$

$$u_x(x,t) \approx \frac{u(x+h,t) - u(x-h,t)}{2h} \tag{B.4}$$

$$u_t(x,t) \approx \frac{u(x,t+k) - u(x,t)}{k} \tag{B.5}$$

$$u_{xx}(x,t) \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}. \tag{B.6}$$

Naturally there are more formulas, but each one just gives a different degree of precision. Now we substitute equations (B.5) and (B.6) in problem (B.2). Thus we obtain the following discrete problem:

$$\frac{u(x,t+k) - u(x,t)}{k} = \nu \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}.$$

To simplify the notation, we take $x = x(i), t = t(j)$ and $u(x(i), t(j)) := u_i^j$. The result is then given by:

$$\frac{u_i^{j+1} - u_i^j}{k} = \nu \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}, \tag{B.7}$$

which can be rewritten as:

$$u_i^{j+1} = \nu \frac{k}{h^2}(u_{i+1}^j - 2u_i^j + u_{i-1}^j) + u_i^j. \tag{B.8}$$

The initial condition is discretized as:

$$u_i^0 = h(x(i)).$$

Through equation (B.8), the numerical solution can be computed by advancing in the direction of t.

However, the problem that still remains is to determine how well the solution to the difference equation approximates the solution to the parabolic PDE.

Definition B.1 *The finite difference scheme $Q_{k,h}u_i^j = 0$ is pointwise consistent with the partial differential equation $Qu = 0$ at point (x,t) if for any smooth function $\psi(x,t)$,*

$$Q\psi \mid_i^j - Q_{k,h}\psi\,(i\triangle x, j\triangle t) \to 0$$

as $\triangle x, \triangle t \to 0$ *and* $(i\triangle x, j\triangle t) \to (x,t)$, *where* $Q_{k,h}$ *is the difference operator obtained after substituting the derivatives by their corresponding Taylor expansion in* (B.1).

Observation B.1 *Consistency means that the discrete equation approximates the continuous equation, i.e., the difference between the exact solution of the numerical and mathematical models should vanish as the grid spacing approaches zero. The principal tool used to sketch the proof of consistency is based on Taylor's series expansion.*

Example B.2 In this example, we will investigate the consistency of (B.2). Let

$$\mathscr{L}\psi = \psi_t - \nu\psi_{xx},$$

then the difference operator is given by:

$$\mathscr{L}_{k,h}\psi = \frac{\psi_i^{j+1} - \psi_i^j}{k} - \nu\frac{\psi_{i+1}^j - 2\psi_i^j + \psi_{i-1}^j}{h^2}.$$

By Taylor's series we have:

$$\psi_i^{j+1} = \psi_i^j + k\psi_t + \frac{1}{2}k^2\psi_{tt} + O(k^3),$$

$$\psi_{i\pm1}^j = \psi_i^j \pm h\psi_x + \frac{1}{2}h^2\psi_{xx} \pm \frac{1}{6}h^3\psi_{xxx} + O(h^4).$$

The derivatives on the right hand side are all evaluated at $x(i), t(j)$. Then:

$$\mathscr{L}_{k,h}\psi = \psi_t + \frac{1}{2}k\psi_{tt} - \nu\psi_{xx} + O(h^2) + O(k^2).$$

Thus

$$\begin{aligned}\mathscr{L}\psi - \mathscr{L}_{k,h} &= \frac{1}{2}k\psi_{tt} + O(h^2) + O(k^2)\\ &\longrightarrow 0 \ as \ (h,k) \to 0.\end{aligned}$$

Therefore the scheme is consistent.

Definition B.2 *The difference scheme* $u^{j+1} = Mu^j$, *where* M *is an infinite matrix, is said to be stable with respect to the norm* $\|\cdot\|_{2,\triangle x}$ *if there exist positive constants* $\triangle x^*$ *and* $\triangle t^*$, *and non-negative constants* N *and* ϑ *such that:*

$$\| u^{j+1} \|_{2,\triangle x} \leq N e^{\vartheta (j+1)\triangle t} \| u^0 \|_{2,\triangle x}, \tag{B.9}$$

for $0 < \triangle x \leq \triangle x^*$ *and* $0 < \triangle t \leq \triangle t^*$.

Observation B.2 *Since it can be shown that* $\| u \|_{2,\triangle x} = \sqrt{\triangle x} \, \| \hat{u} \|_2$, *(see [40], p. 99), an equivalent definition of stability is given by:*

Definition B.3 *The difference scheme* $u^{j+1} = Mu^j$, *where* M *is an infinite matrix, is said to be stable with respect to the norm* $\|\cdot\|_2$ *if there exist positive constants* $\triangle x^*$ *and* $\triangle t^*$, *and non-negative constants* N *and* ϑ *such that:*

$$\| \hat{u}^{j+1} \|_2 \leq N e^{\vartheta (j+1)\triangle t} \| \hat{u}^0 \|_2, \tag{B.10}$$

$0 < \triangle x \leq \triangle x^*$ *and* $0 < \triangle t \leq \triangle t^*$.

Proposition B.1 *The sequence* $\{u^n\}$ *is stable in* $l_{2,\triangle x}$ *if and only if* $\{\hat{u}^n\}$ *is stable in* $L_2([-\pi, \pi])$.

In the next example, we analyze the stability of the difference scheme given by (B.8).

$$u_i^{j+1} = \nu \frac{k}{h^2}(u_{i+1}^j - 2u_i^j + u_{i-1}^j) + u_i^j.$$

Example B.3 First we start by taking the discrete Fourier transform on both sides of (B.8).

$$\begin{aligned}
\hat{u}^{j+1}(\omega) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\omega} u_m^{j+1} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\omega} \left(\nu \frac{k}{h^2} (u_{m+1}^j - 2u_m^j + u_{m-1}^j) + u_m^j \right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{\nu k}{h^2} \sum_{m=-\infty}^{\infty} e^{-im\omega} u_{m+1}^j - \frac{1}{\sqrt{2\pi}} \frac{2\nu k}{h^2} \sum_{m=-\infty}^{\infty} e^{-im\omega} u_m^j \\
&\quad + \frac{1}{\sqrt{2\pi}} \frac{\nu k}{h^2} \sum_{m=-\infty}^{\infty} e^{-im\omega} u_{m-1}^j + \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\omega} u_m^j,
\end{aligned}$$

then:

$$\begin{aligned}
\hat{u}^{j+1}(\omega) &= \frac{\nu k}{h^2} e^{i\omega} \hat{u}^j(\omega) - \frac{2\nu k}{h^2} \hat{u}^j(\omega) + \frac{\nu k}{h^2} e^{-i\omega} \hat{u}^j(\omega) + \hat{u}^j(\omega) \\
&= \left(\frac{2\nu k}{h^2} \cos(\omega) + 1 - \frac{2\nu k}{h^2} \right) \hat{u}^j(\omega) \\
&= \left(1 - \frac{2\nu k}{h^2} (1 - \cos(\omega)) \right) \hat{u}^j(\omega) \\
&= \left(1 - \frac{4\nu k}{h^2} \sin^2(\omega/2) \right)^{j+1} \hat{u}^j(\omega).
\end{aligned}$$

Note that if we restrict $\nu \frac{k}{h^2}$ such that:

$$\left| 1 - \frac{4\nu k}{h^2} \sin^2(\omega/2) \right| \leq 1, \tag{B.11}$$

then we can choose $N = 1$ and $\vartheta = 0$ and thus satisfy inequality (B.10):

$$\left| 1 - \frac{4\nu k}{h^2} \sin^2(\omega/2) \right| \leq 1.$$

Condition (B.11) is equivalent to

$$-1 \leq 1 - \frac{4\nu k}{h^2} \sin^2(\omega/2) \leq 1.$$

We observe that $1 - \frac{4\nu k}{h^2} \sin^2(\omega/2) \leq 1$ is always true, and

$$1 - \frac{4\nu k}{h^2} \sin^2(\omega/2) \geq 1,$$

or

$$\frac{4\nu k}{h^2}\sin^2(\omega/2) \geq 2,$$

which is true when $\nu\dfrac{k}{h^2} \leq 1/2$. Then $\nu\dfrac{k}{h^2} \leq 1$ is a sufficient condition for stability.

If

$$\left|1 - \frac{4\nu k}{h^2}\sin^2(\omega/2)\right| > 1,$$

then

$$\left|1 - \frac{4\nu k}{h^2}\sin^2(\omega/2)\right|^{j+1}$$

is greater than $Ne^{\vartheta(j+1)\triangle t}$ for any N and ϑ, because for large values of j, $Ne^{\vartheta(j+1)\triangle t}$ is bounded, while $\left|1 - \dfrac{4\nu k}{h^2}\sin^2(\omega/2)\right|^{j+1}$ is not. Finally, $\nu\dfrac{k}{h^2} \leq 1$ is both a necessary and sufficient condition for stability.

Observation B.3 *Instead of studying stability as we did above, the approach for linear PDE's is to consider a discrete Fourier mode, i.e., the solution of the explicit scheme has the form (see [40]):*

$$u_i^j = A^j(\omega)e^{i\sqrt{-1}\omega\triangle x}.$$

Definition B.4 *The initial value problem for the first-order partial differential equation $Qu = 0$ is well-posed if for any time $T \geq 0$, there is a constant C_T such that any solution $u(t,x)$ satisfies:*

$$\int_{-\infty}^{\infty} \mid u(t,x) \mid^2 \, dx \leq C_T \int_{-\infty}^{\infty} \mid u(0,x) \mid^2 \, dx \tag{B.12}$$

for $0 \leq t \leq T$.

Observation B.4 *After applying the Fourier transform in space any equation of first order in the time derivative can be expressed in the following form:*

$$\hat{u}_t = q(\omega)\hat{u}(t,\omega), \tag{B.13}$$

then the initial value problem for this equation has the form

$$\hat{u}(t,\omega) = e^{q(\omega)t}\hat{u}(0,\omega),$$

where $\hat{u}(0,\omega)$ *is the Fourier transform of the initial condition.*

Theorem B.1 *The necessary and sufficient condition for equation* (B.13) *to be well-posed, i.e., to satisfy the estimate* (B.12)*, is that there exist a constant* $\bar{q}$ *such that:*

$$Re(q(\omega)) \leq \bar{q}, \text{ for all values of } \omega. \tag{B.14}$$

Example B.4 In this example, we investigate the well-posedness of the problem given by (B.2). First, we apply the Fourier transform in space to (B.2), then we get:

$$\int_{-\infty}^{\infty} u_t(t,x)e^{-i\omega x}\,dx = \nu\int_{-\infty}^{\infty} u_{xx}(t,x)e^{-i\omega x}\,dx.$$

Integration by parts implies

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(t,x)e^{-\omega x}\,dx = \nu u_x e^{-i\omega x}\Big]_{-\infty}^{\infty} + i\nu\omega\int_{-\infty}^{\infty} u_x(t,x)e^{-i\omega x}\,dx.$$

Assuming that $u(-\infty,x) = u(\infty,x) = 0$ for all $x \in \mathbb{R}$, and by repeating the process of integration by parts we obtain:

$$\hat{u}_t(t,\omega) = -\nu\omega^2\hat{u}(t,\omega),$$

which is an ordinary first order differential equation with the following solution:

$$\hat{u}(t,\omega) = e^{-\nu\omega^2 t}\hat{u}(0,\omega).$$

Then by taking $q(\omega) = -\nu\omega^2$ and $\bar{q} = 0$, the conditions of theorem (B.1) are satisfied. Finally, problem (B.2) is well-posed.

The following theorem shows how convergence, consistency and stability are connected.

THEOREM B.2 *(The Lax Equivalence Theorem, [40])*
A consistent, two level difference scheme (a scheme involving only the nth and the (n+1)-th time levels) for a well-posed linear initial-value problem is convergent if and only if it is stable.

OBSERVATION B.5 *Since a Cauchy problem can only be solved numerically in a bounded domain, we first study the approximation of the solution of the corresponding localized problem. Since this approximation implies a certain error, the theorem below will give an estimation between the solution of the Cauchy problem and the localized problem.*

We consider the following Cauchy problem and assume that the conditions of Theorem A.1 are satisfied:

$$\begin{cases} u_t(x,t) + Qu(x,t) &= 0 \quad \text{in } \mathbb{R} \times [0,T) \\ u(x,T) &= 1 \quad \text{in } \mathbb{R} \end{cases} \tag{B.15}$$

The localization of the Cauchy problem (B.15) is done by imposing the Dirichlet boundary conditions as follows. Let $a > 0$, then the localized problem is given by:

$$\begin{cases} u_t(x,t) + Qu(x,t) &= 0 \quad \text{in }]-2a, 2a[\times [0,T) \\ u(x,T) &= 1 \quad \text{in }]-2a, 2a[\\ u(x,t) &= 1 \quad \text{in } \{-2a, 2a\} \times [0,T] \end{cases} \tag{B.16}$$

The following theorem gives an estimation for the localization error.

THEOREM B.3 *(Gonçalves, [15])*
Let u be the unique solution of problem (B.15) *in* $C^{2,1}(]-2a,2a[)\times[0,T]$ *and* u_a *the unique solution of problem* (B.16) *in* $C^{2,1}(]-2a,2a[\times[0,T])$. *Then, for all* $q \geq 1$, $t \in [0,T]$ *and* $x \in [-2a,2a]$,

$$|u_a(x,t)-u(x,t)| \leq N(1+|x|^{q+\gamma}+|x|^q a^\gamma)a^{-q},$$

where N is a constant depending on T, q, and γ is the growth condition imposed over both functions f and h in Theorem A.1.

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