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FACULTAD DE CIENCIAS

UNA ESTRATIFICACIÓN, A TRAVÉS DE VARIEDADES DE STIEFEL COMPLEJAS ORTOGONALES GENERALIZADAS, DEL ESPACIO TWISTORIAL DE LA 2n-ESFERA CONFORME PARA n $\geq$ 3

QUE PARA OBTENER EL GRADO ACADÉMICO DE

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P R E S E N T A

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Resumen

Para $n \geq 1$, consideremos a la esfera unitaria $\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ dotada de la métrica estándar (la cual es de curvatura seccional positiva constante igual a 1) y elijamos una orientación en dicha esfera. Reflejamos este hecho diciendo que $\mathbb{S}^{2n}$ es la $2n$-esfera estándar. Llamamos $2n$-esfera conforme a toda esfera Σ de dimensión $2n$ que sea conformemente equivalente a la esfera estándar $\mathbb{S}^{2n}$. Consideremos al *espacio twistorial*

$$\mathfrak{Z}(\mathbb{S}^{2n}) := SO(2n+1)/U(n)$$

de la $2n$-esfera estándar, el cual parametriza al conjunto de todas las estructuras casi-complejas ortogonales definidas en $\mathbb{S}^{2n}$ que son compatibles con la métrica y la orientación (cf. Introducción de este trabajo).

Definimos $\mathbf{q}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ como

$$(x_1, ..., x_{n+1}, y_1, ..., y_{n+1}) \longmapsto 2 \sum_{j=1}^{n+1} x_j y_j$$

El conjunto de *espinores puros* correspondiente a la forma cuadrática $\mathbf{q}_{2n+2}$ consiste de dos componentes irreducibles, cada una de las cuales es biholomorfa al espacio twistorial $\mathfrak{Z}(\mathbb{S}^{2n})$ (cf. definiciones 1.3 y 1.4, Proposición 1.11 en el Capítulo 1 de esta tesis).

Denotemos por $\mathcal{S}_{n+1} \simeq \mathbb{C}^{\frac{n(n+1)}{2}}$ al espacio lineal formado por todas las matrices complejas antisímetricas de tamaño $(n+1)\times(n+1)$, y por $\mathbf{G}(n+1, 2n+2)$ a la variedad grassmanniana de subespacios lineales complejos $(n+1)$-dimensionales contenidos en $\mathbb{C}^{2n+2}$. Consideremos al conjunto

$$\mathbf{\Gamma}_{n+1} := \left\{ \big(Z, M(Z)\big) \in \mathbb{C}^{2n+2} \mid Z \in \mathbb{C}^{n+1}, M \in \mathcal{S}_{n+1} \right\} \subset \mathbf{G}(n+1, 2n+2)$$

de todas las gráficas (horizontales) de endomorfismos lineales antisimétricos de $\mathbb{C}^{n+1}$ y a su cerradura de Zariski

$$\overline{\mathbf{\Gamma}}_{n+1} \subset \mathbf{G}(n+1, 2n+2)$$

Un resultado clásico de Geometría Algebraica permite establecer que $\overline{\mathbf{\Gamma}}_{n+1}$ es biholomorfa a $\mathfrak{Z}(\mathbb{S}^{2n})$ (cf. Capítulo 6 de [**23**], Proposición 1.12 en el Capítulo 1 de esta tesis). Utilizamos este hecho para describir una estratificación natural del espacio twistorial $\mathfrak{Z}(\mathbb{S}^{2n})$, a través de lo que hemos llamado el *tipo* de un espinor puro correspondiente a $\mathbf{q}_{2n+2}$ y *variedades de Stiefel complejas ortogonales generalizadas* de $\mathbb{C}^{n+1}$ (cf. definiciones 1.7 y 1.13, Teorema 1.14 en el Capítulo 1 de esta tesis). Los resultados que describen a dicha estratificación se encuentran en la última sección del tercer capítulo de este trabajo. Es de esperar que dicha estratificación resulte de utilidad para adquirir información geométrica y algebraica relevante sobre el espacio twistorial $\mathfrak{Z}(\mathbb{S}^{2n})$ en el caso en el que $n \geq 3$, por ejemplo, para calcular los grupos de cohomología de dicho espacio twistorial.

En el caso particular del espacio twistorial $\mathfrak{Z}(\mathbb{S}^6)$ de la 6-esfera estándar, es muy conocido que dicho espacio es biholomorfo a una hipersuperficie cuádrica compleja y no singular contenida en $\mathbb{P}^7_{\mathbb{C}}$ (cf. Capítulo 8 de [**28**]). Motivados por las secciones 3 y 5 de [**45**], describimos explícitamente la construcción de una foliación real-analítica, por variedades proyectivas complejas isomorfas al espacio proyectivo complejo tridimensional $\mathbb{P}^3_{\mathbb{C}}$, de la hipersuperficie $\mathfrak{Z}(\mathbb{S}^6)$. El espacio cociente de esta foliación es una 6-esfera conforme y dicha foliación resulta ser *riemanniana* respecto a la métrica de Fubini-Study e isométricamente equivalente a la *fibración twistorial*

$$\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$$

de $\mathbb{S}^6$, cuyas fibras son isomorfas a $\mathbb{P}^3_{\mathbb{C}}$ (cf. Introducción, §4 del segundo capítulo de esta tesis).

Aclaración: Con el objetivo de que este trabajo pudiese ser accesible a un mayor número de personas interesadas en los temas que se tratan en él, hemos optado por redactarlo, en lo subsecuente, en Lengua Inglesa.

Abstract

A stratification, in terms of generalised complex orthogonal Stiefel manifolds, of the twistor space of the conformal $2n$-sphere with $n \geq 3$

Given $n \geq 1$, let us consider the unit sphere $\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ endowed with the standard metric (which is of constant positive sectional curvature equal to 1) and let us choose an orientation on this sphere. We shall denote this fact by saying that $\mathbb{S}^{2n}$ is the standard $2n$-sphere. We shall call any $2n$-dimensional sphere Σ which is conformally equivalent to the standard sphere $\mathbb{S}^{2n}$ a conformal $2n$-sphere. Let us consider the *twistor space*

$$\mathfrak{Z}(\mathbb{S}^{2n}) := SO(2n+1)/U(n)$$

of the standard $2n$-sphere, which parametrises the set of all orthogonal almost-complex structures defined on $\mathbb{S}^{2n}$ which are compatible with the metric and the orientation (cf. Introduction to this thesis).

Let us define $\mathbf{q}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ by

$$(x_1, ..., x_{n+1}, y_1, ..., y_{n+1}) \longmapsto 2\sum_{j=1}^{n+1} x_j y_j$$

The set of *pure spinors* corresponding to the quadratic form $\mathbf{q}_{2n+2}$ consists of two irreducible components, each of them biholomorphic to the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ (cf. definitions 1.3 and 1.4, Proposition 1.11 in Chapter 1 of this thesis).

Let us denote by $\mathcal{S}_{n+1} \simeq \mathbb{C}^{\frac{n(n+1)}{2}}$ the linear space formed by all complex skew-symmetric $(n+1) \times (n+1)$ matrices, and by $\mathbf{G}(n+1, 2n+2)$ the Grassmannian manifold of linear $(n+1)$-dimensional complex subspaces contained in $\mathbb{C}^{2n+2}$. Let us consider the set

$$\mathbf{\Gamma}_{n+1} := \left\{ \big(Z, M(Z)\big) \in \mathbb{C}^{2n+2} \mid Z \in \mathbb{C}^{n+1}, M \in \mathcal{S}_{n+1} \right\} \subset \mathbf{G}(n+1, 2n+2)$$

of all the (horizontal) graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^{n+1}$ and its Zariski closure

$$\overline{\mathbf{\Gamma}}_{n+1} \subset \mathbf{G}(n+1, 2n+2)$$

A well-known result in Algebraic Geometry enables one to establish that $\overline{\mathbf{\Gamma}}_{n+1}$ is biholomorphic to the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ (cf. Chapter 6 of [**23**], Proposition 1.12 in Chapter 1 of this thesis). We make use of this fact to describe a natural stratification of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$, in terms of what we have called the *type* of a pure spinor corresponding to $\mathbf{q}_{2n+2}$ and *generalised complex orthogonal Stiefel manifolds* of $\mathbb{C}^{n+1}$ (cf. definitions 1.7 and 1.13, Theorem 1.14 in Chapter 1 of this thesis). The statements describing the afore-mentioned stratification are to be found in the last section of Chapter 3 of this work. We would expect this stratification to become useful for obtaining relevant geometric and algebraic information about the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ in the case when $n \geq 3$. An instance of this would be to calculate the cohomology groups of this twistor space.

In the particular case of the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere, it is widely known that this twistor space is biholomorphic to a non-singular complex quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$ (cf. Chapter 8 of [**28**]). Motivated by §3 and §5 of [**45**], we explicitly construct a real-analytic foliation of this quadric hypersurface, by complex projective linear 3-folds, such that its quotient space is a conformal 6-sphere. This foliation turns out to be *Riemannian* with respect to the Fubini-Study metric and isometrically equivalent to the *twistor fibration*

$$\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$$

of $\mathbb{S}^6$, the fibres of which are isomorphic to the complex projective 3-dimensional space $\mathbb{P}^3_{\mathbb{C}}$ (cf. Introduction, §4 of the second chapter of this thesis).

Introduction

One of the most fruitful and beautiful contributions to Mathematics is the idea of the *geometric realisation* of a set of objects as a manifold endowed with some structure (differentiable, Riemannian, real-analytic, complex, algebraic, etc.). For instance:

(1) Julius Plücker parametrised the set of complex projective lines contained in $\mathbb{P}^3_{\mathbb{C}}$ as a non-singular complex quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$ (cf. [**39**], [**27**], §2 and §3 in Chapter 8 of [**28**]).

(2) In [**20**], Hermann Grassmann generalised some of Plücker's ideas to obtain the non-singular, compact, complex algebraic *Grassmannian manifolds* which parametrise the sets of k-dimensional complex linear subspaces in $\mathbb{C}^m$ with $0 \leq k \leq m$.

(3) In his doctoral dissertation [**32**], Sophus Lie established a bijective correspondence between the set of all oriented and conformal hyperspheres Σ_{k-1} contained in the unit sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ with $k \geq 2$ (also called *Lie spheres*), and the set of all points on a real quadric hypersurface $Q_{k+1} \subset \mathbb{P}^{k+2}_{\mathbb{R}}$ given by the equation $\langle x, x \rangle = 0$ where $\langle\, , \rangle$ is an indefinite bilinear form on $\mathbb{R}^{k+3}$ with signature $(k+1, 2)$ (cf. [**15**]).

Another important instance is found in Élie Cartan's book [**14**] where, given a complex linear space V furnished with a non-singular quadratic form q, he describes a parametrisation of the set of *maximal totally null subspaces* of the quadratic space (V, q) in terms of the *pure spinors* for q. If V is odd-dimensional, then the set of pure spinors for q consists of a single irreducible component. When V is even-dimensional, this set of pure spinors consists of two irreducible and not canonically isomorphic components. As it is thoroughly explained by Claude Chevalley in his book [**16**], Cartan's work on pure spinors and their relationship to maximal totally null subspaces may be elegantly generalised to the case of a non-singular quadratic form defined on a linear space V over a field $\mathbb{K}$ of characteristic other than 2, and some of Cartan's developments on this subject may even be dealt with when $\mathbb{K}$ is of characteristic 2 (cf. the review written by Jean Dieudonné contained in [**16**]). In this thesis, we shall be concerned with the complex linear space

$\mathbb{C}^{2n+2}$ with $n \geq 1$, furnished with the quadratic form $\mathbf{q}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ given by

$$(x_1, ..., x_{n+1}, y_1, ..., y_{n+1}) \longmapsto 2 \sum_{j=1}^{n+1} x_j y_j$$

(cf. Chapter 1). One significant fact is that each irreducible component of the set of pure spinors corresponding to the quadratic form $\mathbf{q}_{2n+2}$ is isomorphic to the Riemannian symmetric space

$$SO(2n+2)/U(n+1)$$

of complex structures defined on $\mathbb{R}^{2n+2}$ which preserve both the orientation and the standard inner product of $\mathbb{R}^{2n+2}$ (cf. [**9**], [**28**]).

Moreover, for $n \geq 1$ one has the celebrated twistor fibration

$$\mathbf{p}_{2n} : \mathfrak{Z}(\mathbb{S}^{2n}) \longrightarrow \mathbb{S}^{2n}$$

where the twistor space

$$\mathfrak{Z}(\mathbb{S}^{2n}) := SO(2n+1)/U(n)$$

parametrises the set of orthogonal almost-complex structures defined on the standard $2n$-sphere $\mathbb{S}^{2n} = SO(2n+1)/SO(2n)$ which are compatible with the orientation and the metric of this conformal $2n$-sphere. The fibres of $\mathbf{p}_{2n}$, called twistor fibres, are isomorphic to $SO(2n)/U(n)$ (cf. [**38**], [**37**], [**11**], [**12**], [**43**]). Given that there exists a diffeomorphism of Riemannian symmetric spaces

$$\mathfrak{Z}(\mathbb{S}^{2n}) = SO(2n+1)/U(n) \simeq SO(2n+2)/U(n+1)$$

we have that the fibres of $\mathbf{p}_{2n}$ are isomorphic to the twistor space $\mathfrak{Z}(\mathbb{S}^{2(n-1)})$ of the standard $2(n-1)$-sphere, and, also, we get that the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ parametrises each of the components of the set of pure spinors corresponding to the quadratic form $\mathbf{q}_{2n+2}$. That is to say, the twistor space of the standard $2n$-sphere is a geometric realisation of each of the two connected components of the set of maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Furthermore, the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ is *always* a non-singular complex projective variety of dimension $\frac{n(n+1)}{2}$, and it has a canonical projective embedding, by means of the *spin representation*, in $\mathbb{P}_{\mathbb{C}}^{2^n-1}$ (cf. [**25**], Chapter 1 of this thesis). As examples, for $n = 1, 2, 3$ we have that each of the components of the set of pure spinors corresponding to $\mathbf{q}_{2n+2}$ is, respectively, isomorphic to $\mathfrak{Z}(\mathbb{S}^2) \simeq \mathbb{P}_{\mathbb{C}}^1$, to $\mathfrak{Z}(\mathbb{S}^4) \simeq \mathbb{P}_{\mathbb{C}}^3$, and to $\mathfrak{Z}(\mathbb{S}^6) \simeq Q$ where $Q \subset \mathbb{P}_{\mathbb{C}}^7$ is a non-singular complex quadric hypersurface (cf. chapters 1 and 2 of this thesis).

Amongst these twistor fibrations, the most studied and understood is the *Calabi-Penrose fibration*

$$\mathbf{p}_4 : \mathfrak{Z}(\mathbb{S}^4) := SO(5)/U(2) \simeq \mathbb{P}_{\mathbb{C}}^3 \longrightarrow \mathbb{S}^4$$

with fibre isomorphic to $\mathbb{P}_{\mathbb{C}}^1$ (cf. [**13**], [**2**], [**40**]). This fibration implies that $\mathbb{P}_{\mathbb{C}}^3$ has a C^∞-ruling by complex projective lines. This ruling is not holomorphic since $\mathbb{S}^4$ does not admit a complex structure (indeed, it does not even admit an almost-complex structure).

In the unceasing spirit of Plücker, Grassmann, Klein and Lie, the Calabi-Penrose fibration also allows us to parametrise the set $\mathbf{\Sigma}^{2,4}$ formed by all oriented and conformal 2-spheres contained in $\mathbb{S}^4$ (that is to say, 2-dimensional spheres contained in $\mathbb{S}^4$ furnished with a metric of constant positive sectional curvature and a chosen orientation) together with all the points in $\mathbb{S}^4$, which are the non-oriented and trivial spheres in $\mathbb{S}^4$. If $\Sigma \subset \mathbb{S}^4$ is any oriented and conformal 2-sphere, then Σ has a *canonical* horizontal lift to a complex projective line $\ell_+ \subset \mathbb{P}_{\mathbb{C}}^3$. Denoting by Σ^- the sphere Σ with the opposite orientation, we have that Σ^- lifts to another complex projective line $\ell_- \subset \mathbb{P}_{\mathbb{C}}^3$ such that $\ell_\pm$ is the set of focal points of $\ell_\mp$ under the normal exponential map for the Fubini-Study metric in $\mathbb{P}_{\mathbb{C}}^3$ (cf. [**30**]). We know that each point in $\mathbb{S}^4$ lifts to a fibre of $\mathbf{p}_4$. On the other hand, the image of any complex projective line under $\mathbf{p}_4$ is either an oriented conformal 2-sphere or a point (cf. [**13**], [**17**], [**8**], [**5**]). Thus, the afore-mentioned Plücker quadric hypersurface in $\mathbb{P}_{\mathbb{C}}^5$ is a geometric realisation of the set $\mathbf{\Sigma}^{2,4}$.

The twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$ is quite interesting also. It is well-known that the twistor space

$$\mathfrak{Z}(\mathbb{S}^6) := SO(7)/U(3) \simeq SO(8)/U(4)$$

is biholomorphic to a non-singular complex quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$ (cf. §5 in Chapter 8 of [**28**]). We have that the fibres of $\mathbf{p}_6$ are isomorphic to $\mathfrak{Z}(\mathbb{S}^4) \simeq \mathbb{P}^3_{\mathbb{C}}$, and that the quadric hypersurface $\mathfrak{Z}(\mathbb{S}^6)$ parametrises each of the following sets:

(a) The Grassmannian manifold $\mathbf{G}^+(2, \mathbb{R}^8)$ of oriented 2-planes in $\mathbb{R}^8$ (cf. [**36**] and [**30**], or §1 of Chapter 2 of this thesis).

(b) Either of the two irreducible components of the *Fano variety* $\mathbf{F}_{3,6}$ of linear 3-folds contained in a smooth complex quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$ (cf. [**23**] and [**24**], or Theorem 1.10 in the first chapter of this thesis).

In complete analogy with the case of the Calabi-Penrose fibration, let us consider the set $\mathbf{\Sigma}^{4,6}$ consisting of the oriented and conformal 4-spheres contained in $\mathbb{S}^6$ and all the points in $\mathbb{S}^6$. The inverse image under $\mathbf{p}_6$ of every point in $\mathbb{S}^6$ is isomorphic to $\mathbb{P}^3_{\mathbb{C}}$. If $\Sigma \subset \mathbb{S}^6$ is an oriented and conformal 4-sphere, then its inverse image $\mathbf{p}_6^{-1}(\Sigma)$ is a linear 3-fold $\Omega_+ \simeq \mathbb{P}^3_{\mathbb{C}}$, and the restriction of $\mathbf{p}_6$ to Ω_+ is isometrically equivalent to $\mathbf{p}_4$. Denoting by Σ^- the 4-sphere Σ with the opposite orientation, we have that the inverse image of Σ^- under $\mathbf{p}_6$ is another linear 3-fold Ω_-. The restriction of $\mathbf{p}_6$ to Ω_- is, again, isometrically equivalent to $\mathbf{p}_4$ (cf. [**45**]). We get that $\Omega_\pm$ is the set of focal points of $\Omega_\mp$ with respect to the Fubini-Study metric in $\mathfrak{Z}(\mathbb{S}^6)$. Thus, the quadric hypersurface $\mathfrak{Z}(\mathbb{S}^6)$ is a geometric realisation of the set $\mathbf{\Sigma}^{4,6}$ (cf. [**45**], §3 of [**41**]).

When $n \geq 4$, we have that the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ is *never* a quadric hypersurface since $2^n - 1$ is, precisely, the minimal dimension of a projective space in which the $\frac{n(n+1)}{2}$-dimensional complex variety $\mathfrak{Z}(\mathbb{S}^{2n})$ can be embedded (cf. [**25**]). However, this twistor space is an intersection of a finite number of *quadric* hypersurfaces in $\mathbb{P}^{2^n-1}_{\mathbb{C}}$ (cf. [**14**], [**16**]). Thus, one may come up to a better understanding of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ with $n \geq 4$, by profiting from the study of the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere.

Let us briefly describe the contents of this work:

► In Chapter 1 we give the fundamental definitions, fix the notation, and state some relevant known results. This chapter represents an overview of several ways to approach the study of twistor spaces of standard even-dimensional spheres and we give the main ideas that shall help the reader to understand some of the afore-made claims about the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere.

► In Chapter 2 we present an extended discussion of the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere. The second section of this chapter contains some results that allow us to retrieve some relevant information regarding the pure spinors for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$, for instance, the *type* of such a spinor (cf. Definition 1.7 in the first chapter) from the coordinates of these spinors. This section also contains the descriptions of two stratifications of the twistor space $\mathfrak{Z}(\mathbb{S}^6)$, the first of which is given in terms of graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^4$. While dealing with the second stratification, we shall come across a particular *generalised complex orthogonal Stiefel manifold* of $\mathbb{C}^4$ (cf. Definition 1.13 in the first chapter) and the manner in which this manifold provides information about the hypersurface $\mathfrak{Z}(\mathbb{S}^6)$. In the third section we introduce a special kind of 4×4 complex matices which we have called *Hamilton matrices.* The study of the skew-symmetric Hamilton matrices, together with sections 3 and 5 of [**45**], give rise to our revisiting the twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$ in the fourth section of this chapter.

► In Chapter 3 we generalise the ideas of §2 of the previous chapter to the case of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere for $n \geq 1$. Our main contribution is contained in the last section of this chapter, and it consists of the description of natural stratifications of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ in terms of the *type* of the pure spinors corresponding to $\mathbf{q}_{2n+2}$ and *generalised complex orthogonal Stiefel manifolds* of $\mathbb{C}^{n+1}$. When $n+1$ is even, we have two stratifications of $\mathfrak{Z}(\mathbb{S}^{2n})$ while, in the case when $n+1$ is odd, it suffices with one such stratification (cf. Proposition 1.8 in the first chapter). We would like to aknowledge our gratitude to Dr. Gregor Weingart for having kindly pointed out to our attention that these stratifications are reminiscent to Wilhelm Wirtinger's classification of the orbits of the canonical action of the unitary group on the Grassmannian manifolds of *real* linear subspaces in a complex linear space.

Contents

CHAPTER 1

Definitions and notation

Let us begin by defining some geometric and algebraic objects which will turn out to be fundamental in the ensuing constructions. In what follows, unless otherwise stated, we shall define all transformations between linear spaces using the corresponding standard bases, and we shall denote by the same symbol a transformation between linear spaces and the corresponding matrix representation of such transformation. We shall use standard projective homogeneous coordinates to define points in projective spaces. Our main references regarding Clifford algebras and spinors are [**14**], [**16**], [**29**], [**19**], and [**28**]. For the case of complex varieties, we shall refer the reader to [**4**], [**46**], [**44**], [**23**], [**21**], [**24**], and [**34**].

Let $n \geq 1$. In the standard base $\left\{ e_j \mid 1 \leq j \leq 2n+2 \right\}$ of the complex linear space $\mathbb{C}^{2n+2}$, we set once and for all

$$\mathcal{H} := \bigoplus_{j=1}^{n+1} \mathbb{C}e_j \quad \text{and} \quad \mathcal{V} := \bigoplus_{j=n+2}^{2n+2} \mathbb{C}e_j$$

We call $\mathcal{H}$ and $\mathcal{V}$, respectively, the horizontal space and the vertical space of $\mathbb{C}^{2n+2}$. Since $\mathbb{C}^{2n+2} = \mathcal{H} \oplus \mathcal{V}$, every point of $\mathbb{C}^{2n+2}$ may be uniquely written as a pair (x, y) where

$$x = \sum_{j=1}^{n+1} x_j e_j \in \mathcal{H} \ \text{ and } \ y = \sum_{j=1}^{n+1} y_j e_{j+n+1} \in \mathcal{V}$$

For every nonzero $(x, y) \in \mathbb{C}^{2n+2}$, we will denote by $[x : y]$ the point $[x_1 : \ldots : x_{n+1} : y_1 : \ldots : y_{n+1}] \in \mathbb{P}^{2n+1}_{\mathbb{C}}$ defined by (x, y).

Let us consider the $\mathbb{C}$-linear canonical involution $\mathcal{I}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ given by

$$(x, y) \longmapsto (y, x)$$

We shall also denote by $\mathcal{I}_{2n+2}$ the induced involution of $\mathbb{P}^{2n+1}_{\mathbb{C}}$. We have that the diagonal $\Delta := \{ (x,x) \in \mathbb{C}^{2n+2} \mid x \in \mathcal{H} \} \simeq \mathbb{C}^{n+1}$ is the fixed-point set of $\mathcal{I}_{2n+2}$.

Let us consider the quadratic form $\mathbf{q}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ given by

$$(x,y) \longmapsto 2\sum_{j=1}^{n+1} x_j y_j$$

Let us denote by $\mathbf{0}_{n+1}$ and $\mathbf{Id}_{n+1}$, respectively, the zero and the identity $(n+1)\times(n+1)$ complex matrices. Then, the matrix corresponding to the bilinear form $\mathbf{B}_{2n+2} : \mathbb{C}^{2n+2} \times \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ associated to $\mathbf{q}_{2n+2}$ is given as

$$\mathbf{B}_{2n+2} = \begin{pmatrix} \mathbf{0}_{n+1} & \mathbf{Id}_{n+1} \\ \mathbf{Id}_{n+1} & \mathbf{0}_{n+1} \end{pmatrix}$$

and we have that $\mathbf{q}_{2n+2}$ is of maximal rank. Therefore,

$$Q_{2n+1} := \{ (x,y) \in \mathbb{C}^{2n+2} \mid \mathbf{q}_{2n+2}(x,y) = 0 \}$$

is a complex affine quadric hypersurface with an isolated singularity at the origin, and

$$\mathbf{Q}_{2n} := \{ [x:y] \in \mathbb{P}^{2n+1}_{\mathbb{C}} \mid \mathbf{q}_{2n+2}(x,y) = 0 \}$$

is a non-singular (or smooth) complex projective quadric hypersurface.

Let us consider the orthogonal group

$$O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2}) := \\ \{ L : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2} \mid L \text{ is a linear automorphism, } \mathbf{q}_{2n+2} \circ L = \mathbf{q}_{2n+2} \}$$

of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, and the special orthogonal group

$$SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2}) := \{ L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2}) \mid \det(L) = 1 \}$$

Given that $\mathcal{I}_{2n+2} \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, we have that the quadric hypersurfaces Q_{2n+1} and $\mathbf{Q}_{2n}$ are preserved by $\mathcal{I}_{2n+2}$.

Definition 1.1. *We say that a complex linear subspace $U \subset \mathbb{C}^{2n+2}$ is* totally null *for the quadratic form $\mathbf{q}_{2n+2}$ if $\mathbf{q}_{2n+2}(x,y) = 0$ for all $(x,y) \in U$.*

Given that the $(n+1)$-dimensional complex linear subspaces $\mathcal{H}, \mathcal{V} \subset \mathbb{C}^{2n+2}$ are totally null for $\mathbf{q}_{2n+2}$, we have that *any* other totally null subspace for $\mathbf{q}_{2n+2}$ has complex dimension at most $n+1$ (cf. I.3.4 in [**16**]). Then, we say that the quadratic form $\mathbf{q}_{2n+2}$ is of maximal index $n+1$ and we shall denote this fact by $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$.

Definition 1.2. *If $U \subset \mathbb{C}^{2n+2}$ is a complex linear subspace which is totally null for $\mathbf{q}_{2n+2}$ and $dim(U) = \mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$, then we call U a* maximal totally null subspace *of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.*

The *spinors* for the quadratic form $\mathbf{q}_{2n+2}$ are certain elements of the Clifford algebra associated to the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. We shall denote this algebra by $\mathcal{C}\ell_{2n+2}$, its product by $\cdot$, and we would like to state a few facts about it. By definition, $\mathcal{C}\ell_{2n+2}$ is an associative complex algebra with unit $\mathbf{1} \in \mathbb{C}$, which contains as a linear subspace and is generated by $\mathbb{C}^{2n+2}$ in such a way that, for each $(x,y) \in \mathbb{C}^{2n+2}$, it holds that

$$(x,y) \cdot (x,y) = \mathbf{q}_{2n+2}(x,y) \cdot \mathbf{1}$$

Moreover, $\mathcal{C}\ell_{2n+2}$ satisfies the following universal property: Let $\mathcal{A}$ be an associative complex algebra with unit 1, which product we denote by juxtaposition. If there exists a linear transformation $L : \mathbb{C}^{2n+2} \longrightarrow \mathcal{A}$ such that

$$L(x,y)L(x,y) = \mathbf{q}_{2n+2}(x,y)\, 1, \text{ for each } (x,y) \in \mathbb{C}^{2n+2},$$

then there exists a unique algebra homomorphism $\Psi : \mathcal{C}\ell_{2n+2} \longrightarrow \mathcal{A}$ which extends L.

The complex dimension of $\mathcal{C}\ell_{2n+2}$ is $2^{dim(\mathbb{C}^{2n+2})} = 2^{2n+2}$. Since $dim(\mathbb{C}^{2n+2})$ is even, we have that $\mathcal{C}\ell_{2n+2}$ is a central simple algebra and it is isomorphic to the full algebra of $2^{n+1} \times 2^{n+1}$ complex matrices.

Let us denote by $\mathcal{C}\ell^{+}_{2n+2}$ and by $\mathcal{C}\ell^{-}_{2n+2}$ the complex linear subspaces of $\mathcal{C}\ell_{2n+2}$ generated by the products of, respectively, an even number and an odd number of elements of $\mathbb{C}^{2n+2}$. We have that

$$\mathcal{C}\ell^{\pm}_{2n+2} \cdot \mathcal{C}\ell^{\pm}_{2n+2} \subset \mathcal{C}\ell^{+}_{2n+2} \text{ and that } \mathcal{C}\ell^{\pm}_{2n+2} \cdot \mathcal{C}\ell^{\mp}_{2n+2} \subset \mathcal{C}\ell^{-}_{2n+2}$$

Then, the $\mathbb{Z}_2$-graduation of $\mathcal{C}\ell_{2n+2}$ is given as $\mathcal{C}\ell_{2n+2} = \mathcal{C}\ell^{+}_{2n+2} \oplus \mathcal{C}\ell^{-}_{2n+2}$. Since $\mathbb{C} \cdot \mathbf{1} \subset \mathcal{C}\ell^{+}_{2n+2}$, we get that $\mathcal{C}\ell^{+}_{2n+2}$ is a subalgebra of $\mathcal{C}\ell_{2n+2}$.

Let us set

$$\mathcal{G}_{2n+2} := \left\{ \eta \in \mathcal{C}\ell_{2n+2} \mid \eta \text{ is invertible and } \eta \cdot \mathbb{C}^{2n+2} \cdot \eta^{-1} = \mathbb{C}^{2n+2} \right\}$$

It is clear that $\mathcal{G}_{2n+2}$ forms a group under the product in $\mathcal{C}\ell_{2n+2}$ and we call it the Clifford group of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. For each $\eta \in \mathcal{G}_{2n+2}$, let us define a linear automorphism $\boldsymbol{\chi}_{\eta} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ by

$$(x, y) \longmapsto \eta \cdot (x, y) \cdot \eta^{-1}$$

Thus, for every $\eta \in \mathcal{G}_{2n+2}$ and $(x, y) \in \mathbb{C}^{2n+2}$, it holds that

$$\mathbf{q}_{2n+2}\big(\boldsymbol{\chi}_{\eta}(x, y)\big) \cdot \mathbf{1} = (\eta \cdot (x, y) \cdot \eta^{-1}) \cdot (\eta \cdot (x, y) \cdot \eta^{-1}) = \mathbf{q}_{2n+2}(x, y) \cdot \mathbf{1},$$

and we have that $\boldsymbol{\chi}_{\eta} \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. The group representation $\boldsymbol{\chi} : \mathcal{G}_{2n+2} \longrightarrow O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ given by

$$\eta \longmapsto \boldsymbol{\chi}_{\eta}$$

is called the vector representation of $\mathcal{G}_{2n+2}$. Again, since $dim(\mathbb{C}^{2n+2})$ is even, we have that:

(a) The vector representation $\boldsymbol{\chi}$ is surjective.

(b) The kernel of $\boldsymbol{\chi}$ coincides with $\mathbb{C}^{*} \cdot \mathbf{1}$.

(c) $\boldsymbol{\chi}(\mathcal{G}_{2n+2} \cap \mathcal{C}\ell^{+}_{2n+2}) = SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

(d) The set $\mathbb{C}^{*} \cdot \mathbf{1} \ \cup \ (\mathcal{G}_{2n+2} \cap \mathbb{C}^{2n+2})$ generates $\mathcal{G}_{2n+2}$.

(e) Furthermore, the Clifford group $\mathcal{G}_{2n+2}$ is a set of generators for the Clifford algebra $\mathcal{C}\ell_{2n+2}$ (cf. II.4.1 in [**16**]).

We shall now proceed to specifying the *spinors* for $\mathbf{q}_{2n+2}$. For every complex linear subspace $U \subset \mathbb{C}^{2n+2}$, let us denote by $\mathcal{C}\ell_U$ the subalgebra of $\mathcal{C}\ell_{2n+2}$ generated by U. If U is totally null for $\mathbf{q}_{2n+2}$, then the complex algebras $\mathcal{C}\ell_U$ and the full Grassmann algebra $\bigwedge U$ of U are isomorphic and, thus, may be identified. In particular, we have

Definition 1.3. *Let us consider the horizontal space $\mathcal{H} \subset \mathbb{C}^{2n+2}$, and we set*

$$\mathbf{S}_{2n+2} := \bigwedge \mathcal{H} = \mathcal{C}\ell_{\mathcal{H}}$$

We say that the 2^{n+1}-dimensional complex algebra $\mathbf{S}_{2n+2}$ is the space of spinors *for the quadratic form $\mathbf{q}_{2n+2}$.*

We shall regard the set

$$\left\{ e_{j_0 \cdots j_r} := e_{j_0} \wedge ... \wedge e_{j_r} \mid 1 \leq j_0 < ... < j_r \leq n+1,\ 0 \leq r \leq n+1 \right\}$$

as the standard base for $\mathbf{S}_{2n+2}$. The empty product is equal to $\mathbf{1} \in \mathbb{C}$ and we set

$$\boldsymbol{\omega} := e_{1\ldots(n+1)} = e_1 \wedge ... \wedge e_{n+1} = e_1 \cdot ... \cdot e_{n+1}$$

as the generator of $\bigwedge^{n+1} \mathcal{H} \simeq \mathbb{C}$.

If $s \in \mathbf{S}_{2n+2}$ is nonzero, we shall denote by $[s] \in \mathbb{P}(\mathbf{S}_{2n+2}) \simeq \mathbb{P}_{\mathbb{C}}^{2^{n+1}-1}$ the point defined by s. The $\mathbb{Z}_2$-graduation of the space of spinors is given as $\mathbf{S}_{2n+2} = \mathbf{S}^+_{2n+2} \oplus \mathbf{S}^-_{2n+2}$, where

$$\mathbf{S}^+_{2n+2} := \bigoplus_{\substack{0 \leq k \leq n+1 \\ k \text{ even}}} \bigwedge{}^k \mathcal{H} \quad \text{and} \quad \mathbf{S}^-_{2n+2} := \bigoplus_{\substack{0 \leq k \leq n+1 \\ k \text{ odd}}} \bigwedge{}^k \mathcal{H}.$$

Then, $\mathbf{S}^{\pm}_{2n+2} \subset \mathcal{C}\ell^{\pm}_{2n+2}$, and we say that the 2^n-dimensional complex linear space $\mathbf{S}^+_{2n+2}$ (respectively, $\mathbf{S}^-_{2n+2}$) is the $\mathsf{space\ of\ even\ half-spinors}$ (respectively, $\mathsf{odd\ half-spinors}$) for the quadratic form $\mathbf{q}_{2n+2}$.

Let us describe the *pure spinors* for the quadratic form $\mathbf{q}_{2n+2}$. For the vertical space $\mathcal{V}$, we set

$$\mathbf{v} := e_{n+2} \cdot \ldots \cdot e_{2n+2} = e_{n+2} \wedge \ldots \wedge e_{2n+2} \in \bigwedge \mathcal{V} = \mathcal{C}\ell_{\mathcal{V}}$$

If U is any other maximal totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and $\{ u_j \mid 1 \leq j \leq n+1 \}$ is any base of U, we set

$$\mathbf{u} := u_1 \cdot \ldots \cdot u_{n+1} = u_1 \wedge \ldots \wedge u_{n+1} \in \bigwedge U = \mathcal{C}\ell_U$$

Then, up to nonzero complex multiples, $\mathbf{u}$ is uniquely determined by U. Furthermore (cf. II.2.2 in [**16**]), we have that

$$\mathcal{C}\ell_{2n+2} \cdot \mathbf{u} \quad \text{and} \quad \mathbf{u} \cdot \mathcal{C}\ell_{2n+2}$$

are, respectively, a minimal left ideal and a minimal right ideal of $\mathcal{C}\ell_{2n+2}$ and, in particular, it holds that

$$\mathcal{C}\ell_{2n+2} \cdot \mathbf{v} = \mathcal{C}\ell_{\mathcal{H}} \cdot \mathbf{v} \quad \text{and} \quad \mathbf{v} \cdot \mathcal{C}\ell_{2n+2} = \mathbf{v} \cdot \mathcal{C}\ell_{\mathcal{H}}$$

For every maximal totally null subspace U we have that

$$\mathcal{C}\ell_{\mathcal{H}} \cdot \mathbf{v} \,\cap\, \mathbf{u} \cdot \mathcal{C}\ell_{2n+2}$$

is a 1-dimensional complex linear subspace of $\mathcal{C}\ell_{2n+2}$ (cf. III.1.1 in [**16**]) and, given that $\mathcal{C}\ell_{\mathcal{H}} = \mathbf{S}_{2n+2}$, this intersection may be written in the form

$$\mathcal{C}\ell_{\mathcal{H}} \cdot \mathbf{v} \,\cap\, \mathbf{u} \cdot \mathcal{C}\ell_{2n+2} = S_U \cdot \mathbf{v}$$

for a unique 1-dimensional complex linear subspace $S_U \simeq \mathbb{C}$ of $\mathbf{S}_{2n+2}$. With this notation, we have

Definition 1.4. *We say that any generator of the complex linear space* S_U represents *the maximal totally null subspace* U*. Any nonzero spinor* $s \in \mathbf{S}_{2n+2}$ *which represents some maximal totally null subspace of* $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ *is called a* pure spinor *for the quadratic form* $\mathbf{q}_{2n+2}$.

Now, we shall elaborate on the relationship that exists between the pure spinors for the quadratic form $\mathbf{q}_{2n+2}$ and the maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. Let us denote by $End(\mathbf{S}_{2n+2})$ the complex $(2^{n+1})^2$-dimensional associative algebra, with unit $\mathbf{Id}$, of linear endomorphisms of the space of spinors $\mathbf{S}_{2n+2}$. As complex linear spaces, $\mathcal{C}\ell_{2n+2}$ and $End(\mathbf{S}_{2n+2})$ are isomorphic (since they are of the same dimension). Furthermore, there exists an irreducible (or simple) algebra representation

$$\boldsymbol{\rho} : \mathcal{C}\ell_{2n+2} \longrightarrow End(\mathbf{S}_{2n+2})$$

called the spin representation of $\mathcal{C}\ell_{2n+2}$, which also turns out to be an algebra isomorphism (cf. [**16**], [**29**], [**19**], [**28**]), and which is obtained as follows:

(1) For each

$$x = \sum_{j=1}^{n+1} x_j e_j \in \mathcal{H},$$

the transformation $L_x : \mathbf{S}_{2n+2} \longrightarrow \mathbf{S}_{2n+2}$ given by

$$s \longmapsto x \wedge s$$

belongs to $End(\mathbf{S}_{2n+2})$. For the linear transformation $\mathbf{L} : \mathcal{H} \longrightarrow End(\mathbf{S}_{2n+2})$ given as $x \longmapsto L_x$, it holds that $\mathbf{L}(x) \circ \mathbf{L}(x)$ vanishes identically on $\mathbf{S}_{2n+2}$, for all $x \in \mathcal{H}$.

(2) For each

$$y = \sum_{j=1}^{n+1} y_j e_{j+n+1} \in \mathcal{V},$$

let us consider the linear form $f_y : \mathcal{H} \longrightarrow \mathbb{C}$ given by $x \longmapsto 2\mathbf{B}_{2n+2}(y, x)$. Then (cf. Lemma 3.3 on page 45 of [**16**]), there exists a *unique* degree -1 homogeneous derivation

$$D_y : \mathbf{S}_{2n+2} \longrightarrow \mathbf{S}_{2n+2}$$

that extends f_y. Thus, $D_y \in End(\mathbf{S}_{2n+2})$, $D_y(\mathbf{1}) = 0$ and $D_y \circ D_y$ vanishes identically on $\mathbf{S}_{2n+2}$. It is clear that the transformation $\mathbf{D} : \mathcal{V} \longrightarrow End(\mathbf{S}_{2n+2})$ given as $y \longmapsto D_y$ is linear.

(3) Let us define a linear transformation $R : \mathbb{C}^{2n+2} \longrightarrow End(\mathbf{S}_{2n+2})$ by

$$(x, y) \longmapsto R_{(x,y)} := \mathbf{L}(x) + \mathbf{D}(y) = L_x + D_y$$

For all $(x, y), (x', y') \in \mathbb{C}^{2n+2}$ and $\xi \in \mathcal{H}$, we get that

$$\begin{gathered}
R_{(x,y)}\big(R_{(x',y')}(\xi)\big) + R_{(x',y')}\big(R_{(x,y)}(\xi)\big) = \\
L_x\big(L_{x'}(\xi)\big) + L_x\big(D_{y'}(\xi)\big) + D_y\big(L_{x'}(\xi)\big) + D_y\big(D_{y'}(\xi)\big) + \\
L_{x'}\big(L_x(\xi)\big) + L_{x'}\big(D_y(\xi)\big) + D_{y'}\big(L_x(\xi)\big) + D_{y'}\big(D_y(\xi)\big) = \\
(x \wedge x' \wedge \xi) + 2\mathbf{B}_{2n+2}(y', \xi)x + D_y(x' \wedge \xi) + \\
(x' \wedge x \wedge \xi) + 2\mathbf{B}_{2n+2}(y, \xi)x' + D_{y'}(x \wedge \xi) = \\
2\mathbf{B}_{2n+2}(y', \xi)x + 2\mathbf{B}_{2n+2}(y, x')\xi - 2\mathbf{B}_{2n+2}(y, \xi)x' + \\
2\mathbf{B}_{2n+2}(y, \xi)x' + 2\mathbf{B}_{2n+2}(y', x)\xi - 2\mathbf{B}_{2n+2}(y', \xi)x = \\
2\mathbf{B}_{2n+2}\big((x, y), (x', y')\big)\mathbf{Id}(\xi)
\end{gathered}$$

Since the space of spinors $\mathbf{S}_{2n+2}$ is generated by $\mathcal{H}$, for all $(x, y), (x', y') \in \mathbb{C}^{2n+2}$ and each $s \in \mathbf{S}_{2n+2}$, it holds that

$$R_{(x,y)}\big(R_{(x',y')}(s)\big) + R_{(x',y')}\big(R_{(x,y)}(s)\big) = 2\mathbf{B}_{2n+2}\big((x, y), (x', y')\big)\mathbf{Id}(s)$$

and, thus, $R_{(x,y)} \circ R_{(x,y)} = \mathbf{q}_{2n+2}(x, y)\mathbf{Id}$ for all $(x, y) \in \mathbb{C}^{2n+2}$. By the universal property of $\mathcal{C}\ell_{2n+2}$, we get that R may be uniquely extended to an algebra homomorphism

$$\boldsymbol{\rho} : \mathcal{C}\ell_{2n+2} \longrightarrow End(\mathbf{S}_{2n+2})$$

For each $\eta \in \mathcal{C}\ell_{2n+2}$, we shall denote by $\boldsymbol{\rho}_\eta$ the linear endomorphism $\boldsymbol{\rho}(\eta)$. In particular, for every $(x, y) \in \mathbb{C}^{2n+2}$, we have that the linear endomorphism $\boldsymbol{\rho}_{(x,y)}$ interchanges the spaces of half-spinors for $\mathbf{q}_{2n+2}$.

The following result is a consequence of III.1.4 in [**16**], and it describes the parametrisation, by means of the pure spinors for $\mathbf{q}_{2n+2}$, of the set of maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ that we mentioned in the Introduction.

THEOREM 1.5. *Let U be a maximal totally null subspace of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and let $s \in \mathbf{S}_{2n+2}$ be a pure spinor for $\mathbf{q}_{2n+2}$ which represents U. Let $\boldsymbol{\rho} : \mathcal{C}\ell_{2n+2} \longrightarrow End(\mathbf{S}_{2n+2})$ be the spin representation of $\mathcal{C}\ell_{2n+2}$. Then, the following statements hold:*

(i) $U = \{ (x,y) \in \mathbb{C}^{2n+2} \mid \boldsymbol{\rho}_{(x,y)}(s) = 0 \}$.

(ii) If $\varsigma \in \mathbf{S}_{2n+2}$ is such that $\boldsymbol{\rho}_{(x,y)}(\varsigma) = 0$ for all $(x,y) \in U$, then $\varsigma = cs$ for some $c \in \mathbb{C}$.

By *(i)* of Theorem 1.5, the spin representation of $\mathcal{C}\ell_{2n+2}$ allows us to completely determine the points contained in a maximal totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ once we know a pure spinor which represents such subspace (and, conversely, a pure spinor for $\mathbf{q}_{2n+2}$ is determined by the maximal totally null subspace it represents).

It is important to remark that *not* every nonzero spinor for the quadratic form $\mathbf{q}_{2n+2}$ is a pure spinor. Indeed, it may be proven (cf. III.1.5 in [**16**]) that a pure spinor for $\mathbf{q}_{2n+2}$ is *always* a half-spinor (this is the reason for calling it "pure") and that there exist even and odd pure spinors for $\mathbf{q}_{2n+2}$. Moreover, if $n \geq 3$, then *not* all nonzero half-spinors are pure spinors for $\mathbf{q}_{2n+2}$ (cf. [**14**], [**16**], Chapter 2 of this thesis).

By *(ii)* of Theorem 1.5, we have that all pure spinors which represent a fixed maximal totally null subspace U have the same parity. We shall say, then, that U is an even or an odd maximal totally null subspace according to the parity of its representative spinors. This allows us to sort out all the maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ in two families, and we would like to understand these families. As a consequence of III.1.10 in [**16**], we have

LEMMA 1.6. *Let U and U' be two maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. Then, $dim(U \cap U') \equiv \mathbf{idx}(\mathbf{q}_{2n+2}) = n + 1 \pmod 2$ if and only if U and U' have the same parity.*

For each pure spinor $s \in \mathbf{S}_{2n+2}$, we shall denote by $\mathcal{K}_s$ the maximal totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ represented by s. That is to say,

$$\mathcal{K}_s := \{ (x,y) \in \mathbb{C}^{2n+2} \mid \boldsymbol{\rho}_{(x,y)}(s) = 0 \}$$

By Theorem 1.5, if $c \in \mathbb{C}^*$, then $\mathcal{K}_{cs}$ is equal to $\mathcal{K}_s$. In particular, given that for all $x \in \mathcal{H}$ and $y \in \mathcal{V}$ it holds that

$$\boldsymbol{\rho}_x(\boldsymbol{\omega}) = x \wedge \boldsymbol{\omega} = 0 \text{ and } \boldsymbol{\rho}_y(\mathbf{1}) = D_y(\mathbf{1}) = 0,$$

we get that $\mathcal{V} = \mathcal{K}_{\mathbf{1}}$ is *always* an even maximal totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. By Lemma 1.6, we have that $\mathcal{H} = \mathcal{K}_{\boldsymbol{\omega}}$ is an even maximal totally null subspace if and only if $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is even.

The action of the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on the set of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ is transitive (cf. I.4.3 in [**16**]), and two such subspaces U and U' belong to the same family if and only if there exists $A \in SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $A(U) = U'$. Since the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ parametrises each of the two families of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, we have that $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ acts transitively on $\mathfrak{Z}(\mathbb{S}^{2n})$. The orbit of the vertical space $\mathcal{V}$ under the action of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ consists of all the even maximal totally null subspaces (cf. [**9**], [**19**]) and, hence, we may also think of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ as the homogeneous space

$$SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/Isot_{\mathcal{V}}$$

where $Isot_{\mathcal{V}}$ denotes the isotropy group of $\mathcal{V}$. In Chapter 3 we shall go back to these actions of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and, in particular, we shall describe the isotropy group $Isot_{\mathcal{V}}$ in terms of elements of the general linear group $GL(n+1, \mathbb{C})$ and certain $(n+1) \times (n+1)$ skew-symmetric complex matrices.

Next, we would like to elaborate on the two families of maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, and on some of the claims made in the Introduction regarding the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere $\mathbb{S}^{2n}$.

The following construction is motivated by Hermann Schubert's work (cf. [**42**], [**26**]), and it is crucial to the description of the stratifications of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ that we shall undertake in chapters 2 and 3 (see Remark 1.9). We would like to define certain sets of pure spinors for $\mathbf{q}_{2n+2}$ in terms of the dimension of the incidence of the maximal totally null subspaces with the horizontal space $\mathcal{H}$ and the vertical space $\mathcal{V}$. For every pure spinor $s \in \mathbf{S}_{2n+2}$, let us set

$$r_{\mathcal{H}}(s) := dim(\mathcal{H} \cap \mathcal{K}_s) \quad \text{and} \quad r_{\mathcal{V}}(s) := dim(\mathcal{V} \cap \mathcal{K}_s)$$

Definition 1.7. *For $n \geq 1$, let $k, k' \in \{0, ..., n+1\}$ be such that $0 \leq k + k' \leq n+1$. Let $s \in \mathbf{S}_{2n+2}$ be a pure spinor for the quadratic form $\mathbf{q}_{2n+2}$. We shall say that s is* of type (k, k') *if $r_{\mathcal{H}}(s) = k$ and $r_{\mathcal{V}}(s) = k'$. We shall call the nonempty set*

$$\mathbf{T}_{n+1}(k, k') := \{\, s \in \mathbf{S}_{2n+2} \mid s \text{ is pure of type } (k, k') \,\}$$

the $(k, k')-$ type set *of* $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

If $s \in \mathbf{S}_{2n+2}$ is pure of type (k, k'), we shall say that the maximal totally null subspace $\mathcal{K}_s = \mathcal{K}_{[s]} \subset \mathbb{C}^{2n+2}$ and that the point $[s] \in \mathbb{P}(\mathbf{S}_{2n+2}) \simeq \mathbb{P}_{\mathbb{C}}^{2^{n+1}-1}$ are of type (k, k') as well.

By Theorem 1.5 we get that

$$\mathbf{T}_{n+1}(0, n+1) = \{\, c \cdot \mathbf{1} \in \mathbf{S}^{+}_{2n+2} \mid c \in \mathbb{C}^* \,\} \simeq \{\, [\mathbf{1}] \,\} \simeq \{\, \mathcal{V} \,\}$$

and that

$$\mathbf{T}_{n+1}(n+1, 0) = \{\, c \cdot \boldsymbol{\omega} \in \mathbf{S}_{2n+2} \mid c \in \mathbb{C}^* \,\} \simeq \{\, [\boldsymbol{\omega}] \,\} \simeq \{\, \mathcal{H} \,\}$$

Furthermore, as a consequence of Lemma 1.6, we have

Proposition 1.8. *Let $n \geq 1$. Then, the following statements hold:*

(i) If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ is even, then the disjoint unions

$$\bigsqcup_{\substack{0 \leq k, k', k+k' \leq n+1 \\ k, k' \text{ even}}} \mathbf{T}_{n+1}(k, k') \quad \text{and} \quad \bigsqcup_{\substack{1 \leq k, k', k+k' \leq n+1 \\ k, k' \text{ odd}}} \mathbf{T}_{n+1}(k, k')$$

parametrise, respectively, the family of even maximal totally null subspaces and the family of odd maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

(ii) If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ *is odd, then the disjoint unions*

$$\bigsqcup_{\substack{0\le k,k',k+k'\le n+1 \\ k \text{ even, } k' \text{ odd}}} \mathbf{T}_{n+1}(k,k') \quad \textit{and} \quad \bigsqcup_{\substack{0\le k,k',k+k'\le n+1 \\ k \text{ even, } k' \text{ odd}}} \mathbf{T}_{n+1}(k',k)$$

parametrise, respectively, the family of even maximal totally null subspaces and the family of odd maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Remark 1.9. *In chapters 2 and 3 we shall study the geometry of the* (k,k')*-type set* $\mathbf{T}_{n+1}(k,k')$ *for all* $n \geq 1$ *and* $k, k' \in \{0,...,n+1\}$ *such that* $0 \leq k+k' \leq n+1$*. These sets will turn out to be the building-blocks for the stratifications of the twistor space* $\mathfrak{Z}(\mathbb{S}^{2n})$ *that we have already mentioned in the Introduction.*

From Proposition 1.8, it follows that:

(1) If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ is even, then the canonical involution $\mathcal{I}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ *preserves* each of the two families of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

(2) If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ is odd, then the canonical involution $\mathcal{I}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ *interchanges* the two families of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

To contrast with this behaviour of the canonical involution, let us consider the $\mathbb{C}$-linear involution $\mathcal{J}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ given by

$$(x,y) = (x_1,...,x_n,x_{n+1},y_1,...,y_n,y_{n+1}) \longmapsto (y_1,...,y_n,x_{n+1},x_1,...,x_n,y_{n+1})$$

We shall also denote by $\mathcal{J}_{2n+2}$ the induced involution of $\mathbb{P}^{2n+1}_{\mathbb{C}}$. Given that $q_{2n+2} \circ \mathcal{J}_{2n+2} = \mathbf{q}_{2n+2}$, we have that $\mathcal{J}_{2n+2} \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. Furthermore, if $n+1$ is even, then $\det(\mathcal{J}_{2n+2}) = -1$ and, hence, $\mathcal{J}_{2n+2} \notin SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. If $n+1$ is odd, then $\det(\mathcal{J}_{2n+2}) = 1$ and $\mathcal{J}_{2n+2} \in SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. Therefore:

(1') If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ is even, then the involution $\mathcal{J}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ *interchanges* the two families of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

(2') If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ is odd, then the involution $\mathcal{J}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ *preserves* each of the two families of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Thus, we may use either the canonical involution $\mathcal{I}_{2n+2}$ or the involution $\mathcal{J}_{2n+2}$ depending on whether we want to change the parity of the maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ or not.

On another matter, if $1 \leq j \leq n+1$ and U is a complex j-dimensional totally null subspace of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, then the complex projective space $\mathbb{P}(U) \simeq \mathbb{P}^{j-1}_{\mathbb{C}}$ is a linear subspace (or a point) of the quadric hypersurface $\mathbf{Q}_{2n} \subset \mathbb{P}^{2n+1}_{\mathbb{C}}$ defined by $\mathbf{q}_{2n+2}$. Therefore, understanding the maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ (that is to say, understanding the pure spinors for the quadratic form $\mathbf{q}_{2n+2}$) is equivalent to understanding the linear n-folds contained in $\mathbf{Q}_{2n}$, and we shall proceed to describe one way to achieve so.

For $1 \leq j \leq m$, we shall denote by $\mathbf{G}(j{+}1, m{+}1)$ the Grassmannian manifold of complex $(j+1)$-dimensional linear subspaces of $\mathbb{C}^{m+1}$. Equivalently, $\mathbb{G}(j,m)$ shall denote the Grassmannian manifold of linear j-folds contained in $\mathbb{P}^m_{\mathbb{C}}$. For any complex projective variety $X \subset \mathbb{P}^m_{\mathbb{C}}$, let us define

$$\mathbf{F}_j(X) := \big\{\, \Lambda \in \mathbb{G}(j,m) \mid \Lambda \subset X \,\big\}$$

We have (cf. [**24**]) that the set $\mathbf{F}_j(X)$ is a subvariety of $\mathbb{G}(j,m)$, and we shall call it the Fano variety of linear $j-$folds contained in X. It holds that

$$\mathbf{F}_j(X) = \bigcap_H \mathbf{F}_j(H)$$

where this intersection in $\mathbb{G}(j,m)$ is taken over all the hypersurfaces $H \subset \mathbb{P}^m_{\mathbb{C}}$ such that $X \subset H$. If X is itself a hypersurface of degree $d \geq 1$, then

$$dim\big(\mathbf{F}_j(X)\big) = (j+1)(m-j) - \binom{j+d}{d}$$

If, furthermore, we consider the case when X is a non-singular *quadric* hypersurface, then we can completely understand the behaviour of the linear subspaces contained in X as is shown in the next result (cf. p.735 in [**23**], Theorem 22-13 and Theorem 22-14 in [**24**]).

THEOREM 1.10. *For $j, m \geq 1$, let $Q \subset \mathbb{P}_{\mathbb{C}}^{m+1}$ be a non-singular complex quadric hypersurface. Then, the following statements hold:*

(i) The Fano variety $\mathbf{F}_{j,m} := \mathbf{F}_j(Q)$ of linear j-folds contained in Q is non-singular.

(ii) If $j > \frac{m}{2}$, then $\mathbf{F}_{j,m}$ is empty.

(iii) If $j < \frac{m}{2}$, then $\mathbf{F}_{j,m}$ is irreducible of complex dimension $(j+1)\left(\frac{2m-3j}{2}\right)$.

(iv) If $j = \frac{m}{2}$, then $\mathbf{F}_{j,m}$ is of complex dimension $\frac{j(j+1)}{2}$ and it consists of two complex $\frac{j(j+1)}{2}$-dimensional irreducible components. Moreover:

(a) For any two linear j-folds $\Lambda, \Lambda' \subset Q$, we have that $dim(\Lambda \cap \Lambda') \equiv j \,(\mathrm{mod}\, 2)$ if and only if Λ and Λ' belong to the same irreducible component of $\mathbf{F}_{j,m}$ (compare to Lemma 1.6 above).

(b) For every linear $(j-1)$-fold contained in Q, there exist exactly two linear j-folds in Q containing it, and such j-folds belong to opposite irreducible components (compare to III.1.11 in [**16**]*) .*

(c) Each irreducible component of $\mathbf{F}_{j,m}$ is isomorphic to the Fano variety $\mathbf{F}_{j-1}(Q')$ of linear $(j-1)$-folds contained in a non-singular complex $(m-1)$-dimensional quadric hypersurface $Q' \subset \mathbb{P}_{\mathbb{C}}^m$.

As a consequence of this theorem, we have that the Fano variety

$$\mathbf{F}_{n,2n} := \mathbf{F}_n(\mathbf{Q}_{2n})$$

of linear n-folds contained in the non-singular complex quadric hypersurface $\mathbf{Q}_{2n} \subset \mathbb{P}_{\mathbb{C}}^{2n+1}$, parametrises the set of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+1})$, that is to say, it parametrises the set of pure spinors for $\mathbf{q}_{2n+2}$.

Let us denote by $\mathbf{F}^{+}_{n,2n}$ (respectively, by $\mathbf{F}^{-}_{n,2n}$) the component of $\mathbf{F}_{n,2n}$ which contains the projective spaces associated to the even (respectively, odd) maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. By Proposition 1.8, the canonical involution $\mathcal{I}_{2n+2} : \mathbb{P}_{\mathbb{C}}^{2n+1} \longrightarrow \mathbb{P}_{\mathbb{C}}^{2n+1}$ preserves the irreducible components $\mathbf{F}^{\pm}_{n,2n}$ when $n+1$ is even, and it interchanges these irreducible components when $n+1$ is odd. In contrast, the involution $\mathcal{J}_{2n+2} : \mathbb{P}_{\mathbb{C}}^{2n+1} \longrightarrow \mathbb{P}_{\mathbb{C}}^{2n+1}$ interchanges the irreducible components $\mathbf{F}^{\pm}$ when $n+1$ is even, and it preserves them when $n+1$ is odd.

On a first point of view (as was mentioned in the Introduction), the twistor space $\mathfrak{Z}(\mathbb{S}^{2n}) \simeq SO(2n+2)/U(n+1)$ of the standard $2n$-sphere is biholomorphic to each component of the space of pure spinors for the quadratic form $\mathbf{q}_{2n+2}$. Thus, on a second point of view, we have

Proposition 1.11. *For $n \geq 1$, the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere is biholomorphic to (therefore, it parametrises) each of the irreducible components of the Fano variety $\mathbf{F}_{n,2n} := \mathbf{F}_{n}(\mathbf{Q}_{2n})$ of linear n-folds contained in the non-singular complex quadric hypersurface $\mathbf{Q}_{2n} \subset \mathbb{P}^{2n+1}_{\mathbb{C}}$ defined by the quadratic form $\mathbf{q}_{2n+2}$.*

In order to consider a third point of view, we would like to study the relationship between twistor spaces of standard even-dimensional spheres and complex skew-symmetric matrices. If $m \geq 1$, we shall denote by $\mathcal{M}(m,\mathbb{C})$ the algebra of all $m \times m$ complex matrices and the complex $\frac{m(m-1)}{2}$-dimensional subspace of skew-symmetric matrices shall be denoted by $\mathcal{S}_{m}$.

For $n \geq 1$, let us define $\mathbf{\Gamma} : \mathcal{M}(n+1,\mathbb{C}) \longrightarrow \mathbf{G}(n+1, 2n+2)$ by

$$M \longmapsto \mathbf{\Gamma}(M) := \left\{ \big(Z, M(Z)\big) \in \mathbb{C}^{2n+2} \mid Z \in \mathbb{C}^{n+1} \right\}$$

That is to say, $\mathbf{\Gamma}$ is simply the transformation that associates to each $M \in \mathcal{M}(n+1,\mathbb{C})$ the (horizontal) graph $\mathbf{\Gamma}(M)$ of the linear endomorphism $M : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$. We have (cf. Chapter 6 in [**23**]) that the $(n+1)$-dimensional subspace $\mathbf{\Gamma}(M)$ (therefore, also its image $\mathcal{I}_{2n+2}\big(\mathbf{\Gamma}(M)\big)$ under the canonical involution) is totally null for $\mathbf{q}_{2n+2}$ if and only if $M \in \mathcal{S}_{n+1}$. This is the reason for being concerned, in this thesis, only with the set

$$\mathbf{\Gamma}_{n+1} := \mathbf{\Gamma}(\mathcal{S}_{n+1}) \simeq \mathbb{C}^{\frac{n(n+1)}{2}}$$

of graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^{n+1}$. We would like to remark that, since $\mathbf{Id}_{n+1} \notin \mathcal{S}_{n+1}$, we have that the fixed-point set $\Delta = \left\{ (x,x) \in \mathbb{C}^{2n+2} \mid x \in \mathcal{H} \right\} \simeq \mathbb{C}^{n+1}$ of $\mathcal{I}_{2n+2}$ *does not* belong to $\mathbf{\Gamma}_{n+1}$.

For each $M \in \mathcal{S}_{n+1}$, we have that $dim(\mathbf{\Gamma}(M) \cap \mathcal{V}) = 0$. By Proposition 1.8, we get that all the maximal totally null subspaces contained in $\mathbf{\Gamma}_{n+1}$ have the same parity (they are even maximal totally null subspaces if and only if $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is even). That is to say, the projective spaces associated to the elements of $\mathbf{\Gamma}_{n+1}$ belong to one and the same irreducible component of the Fano variety $\mathbf{F}_{n,2n}$.

Let us denote the Zariski closure of $\mathbf{\Gamma}_{n+1}$, taken in the Grassmannian manifold $\mathbf{G}(n+1, 2n+2)$, by $\overline{\mathbf{\Gamma}}_{n+1}$. In order to obtain this Zariski closure one adds to $\mathbf{\Gamma}_{n+1}$ a complex projective variety of codimension at least 2 (cf. pages 18 and 19 below, Proposition 2.7 in Chapter 2, §3.2 in Chapter 3). Therefore (cf. [**4**], [**46**], [**44**], [**23**], [**21**], [**34**]), under the corresponding *Plücker embedding*

$$\mathbf{G}(n+1, 2n+2) \longrightarrow \mathbb{P}_{\mathbb{C}}^{\binom{2n+2}{n+1}-1}$$

the Zariski closure $\overline{\mathbf{\Gamma}}_{n+1}$ coincides with the closure of $\mathbf{\Gamma}_{n+1}$ *induced by the Fubini-Study metric* in $\mathbb{C}^{2n+2}$. Furthermore, it may be proven (cf. Chapter 6 in [**23**]) that the component of the Fano variety $\mathbf{F}_{n,2n}$ which parametrises $\mathbf{\Gamma}_{n+1}$, coincides with the Zariski closure $\overline{\mathbf{\Gamma}}_{n+1}$. By Proposition 1.8, we have that $\mathbf{F}^{+}_{n,2n} \simeq \overline{\mathbf{\Gamma}}_{n+1}$ if and only if $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is even.

In order to construct the complex projective variety $\mathcal{E} := \overline{\mathbf{\Gamma}}_{n+1} - \mathbf{\Gamma}_{n+1}$, we recur to the following tool. Let us denote by $\boldsymbol{\lambda}_{n+1} : \mathcal{S}_{n+1} \to \mathbb{C}$ the transformation which assigns to each $M \in \mathcal{S}_{n+1}$ the *unique* complex number $\boldsymbol{\lambda}_{n+1}(M)$ such that

$$\det(M) = \big(\boldsymbol{\lambda}_{n+1}(M)\big)^2$$

We shall say that the complex number $\boldsymbol{\lambda}_{n+1}(M)$ is the Pfaffian of M. If $n+1$ is even, then $\boldsymbol{\lambda}_{n+1}(M)$ is a complex polynomial (in the $\frac{n(n+1)}{2}$ coordinates of M) of degree $\frac{n+1}{2}$ for all $M \in \mathcal{S}_{n+1}$. If $n+1$ is odd, then $\boldsymbol{\lambda}_{n+1}(\mathcal{S}_{n+1}) = \{\,0\,\}$.

Let us set

$$\mathcal{S}^{*}_{n+1} := \{\, M \in \mathcal{S}_{n+1} \mid \boldsymbol{\lambda}_{n+1}(M) \neq 0 \,\},$$

$$\mathcal{S}^{0}_{n+1} := \mathcal{S}_{n+1} - \mathcal{S}^{*}_{n+1} = \{\, M \in \mathcal{S}_{n+1} \mid \boldsymbol{\lambda}_{n+1}(M) = 0 \,\},$$

$$\mathbf{\Gamma}^{*}_{n+1} := \mathbf{\Gamma}(\mathcal{S}^{*}_{n+1}) \quad \text{and} \quad \mathbf{\Gamma}^{0}_{n+1} := \mathbf{\Gamma}(\mathcal{S}^{0}_{n+1})$$

That is to say, $M \in \mathcal{S}_{n+1}$ is an invertible matrix if and only if $M \in \mathcal{S}^{*}_{n+1}$, and $\mathcal{S}^{*}_{n+1} = \varnothing$ when $n+1$ is odd.

Moreover, for $n+1$ even, we have that $\mathbf{\Gamma}_{n+1} = \mathbf{\Gamma}^{*}_{n+1} \sqcup \mathbf{\Gamma}^{0}_{n+1}$ and

$$\overline{\mathbf{\Gamma}}_{n+1} = \mathbf{\Gamma}_{n+1} \sqcup \mathcal{I}_{2n+2}(\mathbf{\Gamma}^0_{n+1}) \sqcup \{\text{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1}\}$$

where $\mathcal{I}_{2n+2}$ is the canonical involution of $\mathbb{C}^{2n+2}$. In this case, we get that the set $\mathbf{\Gamma}^*_{n+1}$ is isomorphic to the type set $\mathbf{T}_{n+1}(0,0)$.

For $n+1$ odd, we have that $\mathbf{\Gamma}_{n+1} = \mathbf{\Gamma}^0_{n+1}$ and

$$\overline{\mathbf{\Gamma}}_{n+1} = \mathbf{\Gamma}^0_{n+1} \sqcup \{\text{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1}\}$$

For $n \geq 1$, we have that $\mathbf{\Gamma}^0_{n+1}$ is isomorphic to the complex affine variety $\mathcal{S}^0_{n+1}$ (which is of dimension $\frac{n(n+1)-2}{2}$ and has an isolated singularity at the zero matrix $\mathbf{0}_{n+1}$). In order to obtain the set of limits at infinity of the elements in $\mathbf{\Gamma}^0_{n+1}$, we projectivise and compactify $\mathcal{S}^0_{n+1}$. By the *Fundamental Theorem of Projective Geometry*, every 1-dimensional complex linear subspace of $\mathcal{S}^0_{n+1}$ intersects the hyperplane at infinity in $\mathbb{P}_{\mathbb{C}}^{\frac{n(n+1)-2}{2}}$ in one point, and, therefore, the locus of such points is the afore-mentioned set of limits.

We would like to remark that an easy way to determine the set of limits at infinity of the elements in $\mathbf{\Gamma}^0_{n+1}$, is to keep in mind Lemma 1.6, Proposition 1.8, and the fact that the values of (k, k'), for the type sets contained in this set of limits, cannot be smaller than those for the type sets contained in $\mathbf{\Gamma}^0_{n+1}$ (cf. theorems 3.15 and 3.16).

The fact that $\overline{\mathbf{\Gamma}}_{n+1}$ coincides with one of the irreducible components of the Fano variety $\mathbf{F}_{n,2n}$, together with Proposition 1.11 and [**1**], imply the following

PROPOSITION 1.12. *Let $n \geq 1$. Then, the following statements hold:*

(i) The twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere is biholomorphic (indeed, algebraically isomorphic) to the Zariski closure $\overline{\mathbf{\Gamma}}_{n+1} \subset \mathbf{G}(n+1, 2n+2)$ of the set $\mathbf{\Gamma}_{n+1}$ of graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^{n+1}$.

(ii) The twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere is birationally equivalent to the complex projective space $\mathbb{P}_{\mathbb{C}}^{\frac{n(n+1)}{2}}$.

When $n \geq 3$, it is a remarkable consequence of the above proposition that the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$, being an intersection of a finite number of quadric hypersurfaces in $\mathbb{P}_{\mathbb{C}}^{2^n-1}$ (see the Introduction), is a rational complex projective variety. We would also like to remark that, for $n \geq 2$, this result has proven to be a fundamental tool in the study of the moduli space of superminimal immersions of $\mathbb{S}^2$ in $\mathbb{S}^{2n}$ (cf. [**6**], [**17**], [**33**], [**5**]).

Let us briefly illustrate some of the arguments given above for the case when $n = 1, 2$.

► $n = 1$ By Proposition 1.11, we have that the twistor space

$$\mathfrak{Z}(\mathbb{S}^2) := SO(3)/\mathbb{C}^*$$

parametrises each of the components of the Fano variety $\mathbf{F}_{1,2}$ of complex projective lines contained in the non-singular complex quadric hypersurface $\mathbf{Q}_2 \subset \mathbb{P}_{\mathbb{C}}^3$ defined by the quadratic form $\mathbf{q}_4 : \mathbb{C}^4 \longrightarrow \mathbb{C}$. Since $\mathbf{idx}(\mathbf{q}_4) = 2$ is even, we have that the canonical involution $\mathcal{I}_4 : \mathbb{P}_{\mathbb{C}}^3 \longrightarrow \mathbb{P}_{\mathbb{C}}^3$ preserves each of these components and the involution $\mathcal{J}_4 : \mathbb{P}_{\mathbb{C}}^3 \longrightarrow \mathbb{P}_{\mathbb{C}}^3$ interchanges them.

Given that $\mathbf{\Gamma}_2^* = \mathbf{T}_2(0,0)$ and $\mathbf{\Gamma}_2^0 = \mathbf{T}_2(2,0)$, we get that

$$\mathbf{\Gamma}_2 = \mathbf{T}_2(0,0) \sqcup \mathbf{T}_2(2,0) \simeq \mathcal{S}_2 \simeq \mathbb{C} \quad \text{and}$$

$$\left\{ \text{limits at infinity of the elements in } \mathbf{\Gamma}_2^0 \right\} = \varnothing$$

Then, $\mathcal{I}_4(\mathbf{\Gamma}_2^0) = \mathbf{T}_2(0,2) \simeq \{ [\mathbf{1}] \}$ and we get that

$$\mathbf{F}_{1,2}^+ \quad \simeq \quad \overline{\mathbf{\Gamma}}_2 = \mathbf{\Gamma}_2 \sqcup \mathcal{I}_4(\mathbf{\Gamma}_2^0) \quad \simeq \quad \mathbb{P}_{\mathbb{C}}^1$$

On the other hand, since all nonzero half-spinors for $\mathbf{q}_4$ are pure (cf. III.4.4 in [**16**]) and the spaces of half-spinors $\mathbf{S}_2^{\pm}$ for $\mathbf{q}_4$ are isomorphic to $\mathbb{C}^2$, we get that

$$\mathbf{F}_{1,2}^- \quad \simeq \quad \mathbf{T}_2(1,1) = \mathbb{P}(\mathbf{S}_2^-) \quad \simeq \quad \mathbb{P}_{\mathbb{C}}^1$$

► $n = 2$ Using Proposition 1.11 again, we get that that the twistor space

$$\mathfrak{Z}(\mathbb{S}^4) := SO(5)/U(2) \simeq SO(6)/U(3)$$

is biholomorphic to each of the components of the Fano variety $\mathbf{F}_{2,4}$ of linear 2-folds contained in the non-singular quadric hypersurface $\mathbf{Q}_4 \subset \mathbb{P}^5_{\mathbb{C}}$ defined by the quadratic form $\mathbf{q}_6 : \mathbb{C}^6 \longrightarrow \mathbb{C}$. Since $\mathbf{idx}(\mathbf{q}_6) = 3$ is odd, we get that the canonical involution $\mathcal{I}_6 : \mathbb{P}^5_{\mathbb{C}} \longrightarrow \mathbb{P}^5_{\mathbb{C}}$ interchanges the irreducible components $\mathbf{F}^{\pm}_{2,4}$ and the involution $\mathcal{J}_6 : \mathbb{P}^5_{\mathbb{C}} \longrightarrow \mathbb{P}^5_{\mathbb{C}}$ preserves them. By Theorem 1.10, we know that $\mathbf{F}^{\pm}_{2,4}$ is isomorphic to the Fano variety $\mathbf{F}_1(Q')$ of complex projective lines contained in a non-singular complex quadric hypersurface $Q' \subset \mathbb{P}^4_{\mathbb{C}}$. In turn, $\mathbf{F}_1(Q')$ is isomorphic to $\mathbb{P}^3_{\mathbb{C}}$ (cf. [**24**]).

Explicitly, we have that

$$\mathbf{F}^{-}_{2,4} \simeq \overline{\mathbf{\Gamma}}_3 = \mathbf{\Gamma}^0_3 \sqcup \left\{ \text{limits at infinity of the elements in } \mathbf{\Gamma}^0_3 \right\}$$

Moreover,

$$\mathbf{\Gamma}^0_3 = \mathbf{T}_3(1,0) \sqcup \mathbf{T}_3(3,0) \simeq \mathcal{S}_3 \simeq \mathbb{C}^3$$

and the set $\mathbf{T}_3(1,2)$ consists of the limits at infinity of all the elements in this $\mathbb{C}^3$. Therefore, $\mathbf{T}_3(1,2) \simeq \mathbb{P}^2_{\mathbb{C}}$.

To finish this chapter, let us consider $1 \leq j \leq m$ and a non-singular quadratic form $q : \mathbb{C}^m \longrightarrow \mathbb{C}$. Let us denote by $B : \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$ the bilinear form associated to q. Given that B is non-degenerate, if $U \subset \mathbb{C}^m$ is a complex j-dimensional subspace, then we have that the orthogonal complement of U with respect to B

$$U^{\perp_B} := \left\{ z \in \mathbb{C}^m \mid B(z,u) = 0 \text{ for all } u \in U \right\}$$

is a complex subspace of $\mathbb{C}^m$ of dimension $m - j \geq 0$.

Definition 1.13. *For $m \geq 1$, let $0 \leq k, k' \leq m$ be such that $0 \leq k + k' \leq m$. Let $q : \mathbb{C}^m \longrightarrow \mathbb{C}$ be a non-singular quadratic form and let $B : \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$ be the associated bilinear form. Then, the nonempty set*

$$\mathbf{V}^B_{\mathbb{C}}(k,k';m) := \left\{ (U,W) \in \mathbf{G}(k,m) \times \mathbf{G}(k',m) \mid U \perp_B W \right\}$$

is a smooth, compact, complex subvariety of $\mathbf{G}(k,m)\times\mathbf{G}(k',m)$ *of dimension*

$$m(k+k')-kk'-k^2-(k')^2,$$

and we shall call it a generalised complex orthogonal Stiefel manifold *of the quadratic space* $(\mathbb{C}^m,q)$.

Let us denote by $\pi_1 : \mathbf{V}^B_{\mathbb{C}}(k,k';m) \longrightarrow \mathbf{G}(k,m)$ and $\pi_2 : \mathbf{V}^B_{\mathbb{C}}(k,k';m) \longrightarrow \mathbf{G}(k',m)$ the canonical projections into the first and second coordinates, respectively. For all $(U,W)\in \mathbf{V}^B_{\mathbb{C}}(k,k';m)$ we have that

$$\pi_1^{-1}(U) = \left\{\, (U,V)\in \mathbf{V}^B_{\mathbb{C}}(k,k';m) \mid V\in U^{\perp_B}\simeq \mathbb{C}^{m-k}, dim(V)=k' \,\right\} \text{ and}$$

$$\pi_2^{-1}(W) = \left\{\, (V,W)\in \mathbf{V}^B_{\mathbb{C}}(k,k';m) \mid V\in W^{\perp_B}\simeq \mathbb{C}^{m-k'}, dim(V)=k \,\right\}$$

One can verify that both π_1 and π_2 are holomorphic submersions and, given that $\mathbf{V}^B_{\mathbb{C}}(k,k';m)$ is compact, we obtain

THEOREM 1.14. *For* $m\geq 1$, *let* $0\leq k,k'\leq m$ *be such that* $0\leq k+k'\leq m$. *Let* $q:\mathbb{C}^m\longrightarrow\mathbb{C}$ *be a non-singular quadratic form and let* $B:\mathbb{C}^m\times\mathbb{C}^m\longrightarrow\mathbb{C}$ *be the associated bilinear form. Let us consider the generalised complex orthogonal Stiefel manifold* $\mathbf{V}^B_{\mathbb{C}}(k,k';m)$ *of the quadratic space* $(\mathbb{C}^m,q)$. *Then, we have two locally trivial and holomorphic fibrations*

$$\begin{tikzcd} & \mathbf{V}^B_{\mathbb{C}}(k,k';m) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbf{G}(k,m) & & \mathbf{G}(k',m) \end{tikzcd}$$

The fibre of π_1 *is isomorphic to the complex Grassmannian manifold* $\mathbf{G}(k',m-k)$, *and the fibre of* π_2 *is isomorphic to the complex Grassmannian manifold* $\mathbf{G}(k,m-k')$.

REMARK 1.15. *Every generalised complex orthogonal Stiefel manifold is, in fact, a standard complex partial flag manifold (cf.* [**35**]*). We hope that, from our context in the following two chapters, it will be clear why we have chosen to make and use Definition* 1.13.

CHAPTER 2

The twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere

Let us recall that the twistor space

$$\mathfrak{Z}(\mathbb{S}^6) := SO(7)/U(3) \simeq SO(8)/U(4)$$

of the standard 6-sphere is a 6-dimensional complex projective variety, which is biholomorphic to a non-singular quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$ (cf. §5 in Chapter 8 of [**28**]). This twistor space may be discussed from several points of view, since it is a geometric realisation of each of the following sets:

(a) The set $\mathbf{\Sigma}^{4,6}$ of conformal and oriented 4-spheres contained in $\mathbb{S}^6$ together with the points in $\mathbb{S}^6$.

(b) The Grassmannian manifold $\mathbf{G}^+(2, \mathbb{R}^8)$ of oriented 2-planes in $\mathbb{R}^8$.

(c) Either of the two irreducible components of the Fano variety $\mathbf{F}_{3,6}$ of linear 3-folds contained in a non-singular complex quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$.

We have already described parametrisation (a) in the Introduction, and we refer the reader to [**25**] for a generalisation of it. In §2.1 we shall explain parametrisation (b).

The second section is devoted to elaborating on the parametrisations (c) (cf. Proposition 1.11). We use these parametrisations to describe two stratifications of $\mathfrak{Z}(\mathbb{S}^6)$. The first of these stratifications, given in terms of graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^4$, represents an alternative proof of the fact that the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ is biholomorphic to a non-singular complex quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$ (cf. Proposition 2.8).

In §2.3 we study a special kind of elements of $\mathcal{M}(4, \mathbb{C})$, which we have called *Hamilton matrices*. In particular we show that, in the Grassmannian manifold $\mathbf{G}(4, 8)$, the Zariski closure of the set of graphs of linear endomorphisms of $\mathbb{C}^4$ defined by *skew-symmetric* Hamilton matrices is isomorphic to a non-singular complex quadric hypersurface in $\mathbb{P}^4_{\mathbb{C}}$. The last section is motivated by our previous study of skew-symmetric Hamilton matrices and sections 3 and 5 of [**45**], and it consists of an explicit construction of a real-analytic foliation of $\mathfrak{Z}(\mathbb{S}^6)$, by linear 3-folds, the quotient space of which is a conformal 6-sphere. We also prove that this foliation is *Riemannian* with respect to the Fubini-Study metric in $\mathfrak{Z}(\mathbb{S}^6)$, and isometrically equivalent to the twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$.

1. The Grassmannian manifold $\mathbf{G}^+(2, \mathbb{R}^{m+1})$

For $m \geq 1$, let us consider the Fermat polynomial $\mathcal{F}_{m+1} : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}$ given by

$$(z_1, ..., z_{m+1}) \longmapsto \sum_{j=1}^{m+1} z_j^2$$

The quadratic form $\mathcal{F}_{m+1}$ has maximal rank. Then, the complex affine quadric hypersurface

$$Q_m^{\mathrm{aff}} := \left\{ (z_1, ..., z_{m+1}) \in \mathbb{C}^{m+1} \mid \mathcal{F}_{m+1}(z_1, ..., z_{m+1}) = 0 \right\}$$

has an isolated singularity at the origin, and the complex projective quadric hypersurface

$$Q_{m-1} := \left\{ [z_1 : \dots : z_{m+1}] \in \mathbb{P}^m_{\mathbb{C}} \mid \mathcal{F}_{m+1}(z_1, ..., z_{m+1}) = 0 \right\}$$

is non-singular. Let $\mathbb{S}^{2m+1} \subset \mathbb{C}^{m+1}$ be the unit sphere and let us consider the link

$$\mathcal{L} := \mathbb{S}^{2m+1} \cap Q_m^{\mathrm{aff}}$$

For each $Z = (z_1, ..., z_{m+1}) \in \mathbb{C}^{m+1}$ and $1 \leq j \leq m+1$, we set

$$x_j(Z) := \Re(z_j), y_j(Z) := \Im(z_j) \in \mathbb{R} \text{ and}$$

$$X(Z) := \big(x_1(Z), ..., x_{m+1}(Z)\big), Y(Z) := \big(y_1(Z), ..., y_{m+1}(Z)\big) \in \mathbb{R}^{m+1}$$

For each $Z = (z_1, ..., z_{m+1}) \in Q_m^{\mathrm{aff}}$, we have that

$$\|X(Z)\|^2 - \|Y(Z)\|^2 = 0 \text{ and } 2\langle X(Z), Y(Z)\rangle = 0$$

where $\langle\, , \rangle$ denotes the standard inner product in $\mathbb{R}^{m+1}$. If, furthermore, $Z = (z_1, ..., z_{m+1}) \in \mathcal{L}$, we have that $\|X(Z)\|^2 + \|Y(Z)\|^2 = 1$. Therefore,

$$\mathcal{L} = \Big\{ \big(X(Z), Y(Z)\big) \in \mathbb{R}^{m+1}\times\mathbb{R}^{m+1} \mid \|X(Z)\| = \|Y(Z)\| = \frac{1}{\sqrt{2}}, X(Z) \perp Y(Z) \Big\}$$

and $\mathcal{L}$ is diffeomorphic to the *Stiefel manifold* $\mathbf{V}(2, m+1)$ of oriented orthonormal 2-frames in $\mathbb{R}^{m+1}$.

We would like to understand the behaviour of the points in $\mathcal{L} \simeq \mathbf{V}(2, m+1)$ as we projectivise the affine quadric hypersurface Q_m^{aff}. In order to do this, we need to consider the natural action of the circle $\mathbb{S}^1$ on the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ (that is to say, coordinate-wise multiplcation by complex numbers of modulus 1). If $\Pi \in \mathbf{V}(2, m+1)$, then the whole orbit of Π under this action is contained in the *oriented* 2-dimensional subspace of $\mathbb{R}^{m+1}$ spanned by Π. Therefore, such an orbit may be identified with this oriented 2-plane, and we get that the complex projective quadric hypersurface Q_{m-1} is diffeomorphic to the corresponding Grassmannian manifold $\mathbf{G}^+(2,\mathbb{R}^{m+1})$ of oriented 2-planes in $\mathbb{R}^{m+1}$. In particular, for $m = 7$, we have that $\mathfrak{Z}(\mathbb{S}^6) \simeq Q_6$ is diffeomorphic to the Grassmannian manifold $\mathbf{G}^+(2,\mathbb{R}^8)$ of oriented 2-planes in $\mathbb{R}^8$.

We would like to remark that some of the implications of the existence of these diffeomorphisms

$$Q_{m-1} \simeq \mathbf{G}^+(2,\mathbb{R}^{m+1})$$

are to be found in the study of minimal surfaces (cf. [**17**], [**10**], [**8**]).

2. Two stratifications of the twistor space $\mathfrak{Z}(\mathbb{S}^6)$

We recall that the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$ given by $(x,y) \longmapsto 2\sum_{j=1}^4 x_j y_j$ is of maximal rank 8 and maximal index 4. Therefore, we have that

$$\mathbf{Q}_6 := \left\{ [x:y] \in \mathbb{P}^7_{\mathbb{C}} \mid \mathbf{q}_8(x,y) = 0 \right\}$$

is a non-singular complex quadric hypersurface.

The space of spinors

$$\mathbf{S}_8 = \mathbf{S}_8^+ \oplus \mathbf{S}_8^- \subset \mathcal{C}\ell_8$$

for the quadratic form $\mathbf{q}_8$ (cf. Definition 1.3) has complex dimension 16. Then, for each $s \in \mathbf{S}_8$, there exist unique $a_1, ..., a_{16} \in \mathbb{C}$ such that $s = \varsigma + \varkappa$ where

$$\varsigma := a_1 \cdot \mathbf{1} + a_6 e_{12} + a_7 e_{13} + a_8 e_{14} + a_9 e_{23} + a_{10} e_{24} + a_{11} e_{34} + a_{16} \cdot \boldsymbol{\omega} \in \mathbf{S}_8^+,$$

$$\varkappa := a_2 e_1 + a_3 e_2 + a_4 e_3 + a_5 e_4 + a_{12} e_{123} + a_{13} e_{124} + a_{14} e_{134} + a_{15} e_{234} \in \mathbf{S}_8^-,$$

and the form $\boldsymbol{\omega} = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is considered as the generator of $\bigwedge^4 \mathcal{H} \simeq \mathbb{C}$. We shall denote these facts by

$$\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \in \mathbf{S}_8^+,$$

$$\varkappa = (a_2, a_3, a_4, a_5, a_{12}, a_{13}, a_{14}, a_{15}) \in \mathbf{S}_8^-,$$

$$\text{and } s = (a_1, ..., a_{16}) \in \mathbf{S}_8$$

For every $(x,y) = (x_1, ..., x_4, y_1, ..., y_4) \in \mathbb{C}^8$ and $s = (a_1, ..., a_{16}) \in \mathbf{S}_8$, we have that the spin representation $\boldsymbol{\rho} : \mathcal{C}\ell_8 \longrightarrow End(\mathbf{S}_8)$ of the Clifford algebra $\mathcal{C}\ell_8$ of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ is given by

$$\boldsymbol{\rho}_{(x,y)}(s) = 2(a_2 y_1 + a_3 y_2 + a_4 y_3 + a_5 y_4)\cdot \mathbf{1}$$

$$+ (a_1 x_1 - 2a_6 y_2 - 2a_7 y_3 - 2a_8 y_4)e_1 + (a_1 x_2 + 2a_6 y_1 - 2a_9 y_3 - 2a_{10} y_4)e_2$$

$$+ (a_1 x_3 + 2a_7 y_1 + 2a_9 y_2 - 2a_{11} y_4)e_3 + (a_1 x_4 + 2a_8 y_1 + 2a_{10} y_2 + 2a_{11} y_3)e_4$$

$$+ (a_3 x_1 - a_2 x_2 + 2a_{12} y_3 + 2a_{13} y_4)e_{12} + (a_4 x_1 - a_2 x_3 - 2a_{12} y_2 + 2a_{14} y_4)e_{13}$$

$$+ (a_5 x_1 - a_2 x_4 - 2a_{13} y_2 - 2a_{14} y_3)e_{14} + (a_4 x_2 - a_3 x_3 + 2a_{12} y_1 + 2a_{15} y_4)e_{23}$$

$$+ (a_5 x_2 - a_3 x_4 + 2a_{13} y_1 - 2a_{15} y_3)e_{24} + (a_5 x_3 - a_4 x_4 + 2a_{14} y_1 + 2a_{15} y_2)e_{34}$$

$$+ (a_9 x_1 - a_7 x_2 + a_6 x_3 - 2a_{16} y_4)e_{123} + (a_{10} x_1 - a_8 x_2 + a_6 x_4 + 2a_{16} y_3)e_{124}$$

$$+ (a_{11} x_1 - a_8 x_3 + a_7 x_4 - 2a_{16} y_2)e_{134} + (a_{11} x_2 - a_{10} x_3 + a_9 x_4 + 2a_{16} y_1)e_{234}$$

$$+ (a_{15} x_1 - a_{14} x_2 + a_{13} x_3 - a_{12} x_4)\boldsymbol{\omega}$$

(cf. [**14**], [**16**], [**29**], [**19**], [**28**], Chapter 1 of this thesis).

Not all nonzero half-spinors for $\mathbf{q}_8$ are pure spinors but, one of the remarkable consequences of the *Principle of Triality* (cf. Chapter IV of [**16**], Lecture 20 of [**19**], Appendix B of [**28**]) is that, when dealing with an 8-dimensional complex linear space endowed with a quadratic form which is of maximal rank 8 and maximal index 4, we can determine all pure spinors for such a quadratic form in a nice geometric way as follows.

Let us define $\boldsymbol{\gamma} : \mathbf{S}_8 \longrightarrow \mathbb{C}$ by

$$(a_1, ..., a_{16}) \longmapsto 2(a_1a_{16} + a_2a_{15} - a_3a_{14} + a_4a_{13} - a_5a_{12} - a_6a_{11} + a_7a_{10} - a_8a_9)$$

The quadratic form $\boldsymbol{\gamma}$ has maximal rank 16 and maximal index 8. In order to get the parametrisations (c), we need to consider the restrictions $\boldsymbol{\gamma}^{\pm}$ of the quadratic form $\boldsymbol{\gamma}$ to each of the spaces of half-spinors for $\mathbf{q}_8$. Given that the quadratic form $\boldsymbol{\gamma}^{\pm}$ has rank 8 and $\mathbf{idx}(\boldsymbol{\gamma}^{\pm}) = 4$, IV.1.1 in [**16**] implies

PROPOSITION 2.1. *Let $s \in \mathbf{S}_8 = \mathbf{S}_8^+ \oplus \mathbf{S}_8^-$ be any spinor for the quadratic form $\mathbf{q}_8$. Then, s is a pure spinor for $\mathbf{q}_8$ if and only if s is a nonzero half-spinor and $\boldsymbol{\gamma}(s) = 0$.*

Let us consider the non-singular complex quadric hypersurfaces

$$\mathbf{Q}_6^+ := \left\{ [\varsigma] \in \mathbb{P}(\mathbf{S}_8^+) \simeq \mathbb{P}^7_{\mathbb{C}} \mid \boldsymbol{\gamma}^+(\varsigma) = 0 \right\}$$

$$\text{and } \mathbf{Q}_6^- := \left\{ [\varkappa] \in \mathbb{P}(\mathbf{S}_8^-) \simeq \mathbb{P}^7_{\mathbb{C}} \mid \boldsymbol{\gamma}^-(\varkappa) = 0 \right\}$$

By Proposition 2.1, we get that $\mathbf{Q}_6^+$ (respectively, $\mathbf{Q}_6^-$) parametrises the set of all even (respectively, odd) half-spinors for $\mathbf{q}_8$ which are pure. Therefore (cf. Theorem 1.10 and Proposition 1.11), the irreducible component $\mathbf{F}^{\pm}_{3,6}$ of the non-singular 6-dimensional complex projective variety $\mathbf{F}_{3,6} = \mathbf{F}_3(\mathbf{Q}_6)$ is biholomorphic to $\mathbf{Q}_6^{\pm}$. That is to say,

$$\mathbf{F}^{\pm}_{3,6} \text{ is biholomorphic to } \mathbf{Q}_6$$

(the kind of algebraic variety which we started with) which, in turn, is biholomorphic to the twistor space $\mathfrak{Z}(\mathbb{S}^6)$.

Since $\mathbf{idx}(\mathbf{q}_8) = 4$ is even, we know (cf. Chapter 1) that the involution $\mathcal{J}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}^8$ given by

$$(x, y) = (x_1, ..., x_4, y_1, ..., y_4) \longmapsto (y_1, y_2, y_3, x_4, x_1, x_2, x_3, y_4)$$

changes the parity of the maximal totally null subspaces of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$. Hence, the involution of $\mathbb{P}^7_{\mathbb{C}}$ induced by $\mathcal{J}_8$ interchanges the irreducible components $\mathbf{F}^{\pm}_{3,6} \simeq \mathbf{Q}_6^{\pm}$, and, in this sense, we say that the complex quadric hypersurfaces $\mathbf{Q}_6^{\pm}$ are *indistinguishable*. In the next two subsections we will recur to Proposition 1.8 and the type sets $\mathbf{T}_4(k,k') \subset \mathbf{S}_8$, with $k,k' \in \{0,1,2,3,4\}$, (cf. Definition 1.7) to *distinguish* $\mathbf{Q}_6^{+}$ from $\mathbf{Q}_6^{-}$, and to obtain the two afore-mentioned stratifications of $\mathfrak{Z}(\mathbb{S}^6)$.

Let us go back to the involution $\mathcal{J}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}^8$. Given a maximal totally null subspace U of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ and a pure spinor $s \in \mathbf{S}_8$ which represents U, we would like to determine a pure spinor which represents the maximal totally null subspace $\mathcal{J}_8(U)$.

Let us define $\mathbb{C}$-linear involutions $\boldsymbol{\varphi}^{\pm} : \mathbf{S}_8^{\pm} \longrightarrow \mathbf{S}_8^{\mp}$ by

$$(a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \xmapsto{\boldsymbol{\varphi}^+} (-2a_9, 2a_7, -2a_6, 4a_{16}, \frac{a_1}{2}, -a_{11}, a_{10}, -a_8)$$

$$(a_2, a_3, a_4, a_5, a_{12}, a_{13}, a_{14}, a_{15}) \xmapsto{\boldsymbol{\varphi}^-} \big(2a_{12}, \frac{-a_4}{2}, \frac{a_3}{2}, -a_{15}, \frac{-a_2}{2}, a_{14}, -a_{13}, \frac{a_5}{4}\big)$$

Then, $\boldsymbol{\varphi} : \mathbf{S}_8 = \mathbf{S}_8^{+} \oplus \mathbf{S}_8^{-} \longrightarrow \mathbf{S}_8^{-} \oplus \mathbf{S}_8^{+} = \mathbf{S}_8$ given by

$$s = \varsigma + \varkappa \quad \xmapsto{\boldsymbol{\varphi}} \quad \boldsymbol{\varphi}^+(\varsigma) + \boldsymbol{\varphi}^-(\varkappa)$$

is a $\mathbb{C}$-linear involution such that

$$-\gamma\big(\boldsymbol{\varphi}(s)\big) = 2\gamma^+(\varsigma) + \frac{\gamma^-(\varkappa)}{2}$$

for all $s = (a_1, ..., a_{16}) = \varsigma + \varkappa \in \mathbf{S}_8$. By Proposition 2.1, we have that $\boldsymbol{\varphi}$ transforms pure spinors into pure spinors. Since $\boldsymbol{\varphi}$ interchanges the spaces of half-spinors for the quadratic form $\mathbf{q}_8$, we have that no pure spinor for $\mathbf{q}_8$ is fixed by $\boldsymbol{\varphi}$. For each $(x,y) \in \mathbb{C}^8$, we get that

$$\boldsymbol{\rho}_{(x,y)}(s) = 0 \text{ if and only if } \boldsymbol{\rho}_{\mathcal{J}_8(x,y)}\big(\boldsymbol{\varphi}(s)\big) = 0$$

where $\boldsymbol{\rho} : \mathcal{C}\ell_8 \longrightarrow End(\mathbf{S}_8)$ is the spin representation. By Theorem 1.5, we have the following

Proposition 2.2. *Let U be a maximal totally null subspace of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ and let $s \in \mathbf{S}_8^{\pm}$ be a pure spinor for $\mathbf{q}_8$ which represents*

U. Then, the pure spinor $\boldsymbol{\varphi}(s) \in \mathbf{S}_8^{\mp}$ represents the maximal totally null subspace $\mathcal{J}_8(U)$.

COROLLARY 2.3. *If $U \subset \mathbb{C}^8$ is a 4-dimensional complex linear subspace which is invariant under the involution $\mathcal{J}_8$, then U is not totally null for $\mathbf{q}_8$.*

On the other side, the canonical involution $\mathcal{I}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}^8$ preserves the parity of the maximal totally null subspaces of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ (since $\mathbf{idx}(\mathbf{q}_8) = 4$ is even). Given a maximal totally null subspace U of $(\mathbb{C}^8, \mathbf{q}_8)$ and a pure spinor $s \in \mathbf{S}_8$ which represents U, we would like to determine a representative spinor for $\mathcal{I}_8(U)$. Let us define $\mathbb{C}$-linear involutions $\daleth^{\pm} : \mathbf{S}_8^{\pm} \longrightarrow \mathbf{S}_8^{\pm}$ by

$$(a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \xmapsto{\daleth^+} \left(4a_{16}, -a_{11}, a_{10}, -a_9, -a_8, a_7, -a_6, \frac{a_1}{4}\right)$$

$$(a_2, a_3, a_4, a_5, a_{12}, a_{13}, a_{14}, a_{15}) \xmapsto{\daleth^-} \left(2a_{15}, -2a_{14}, 2a_{13}, -2a_{12}, \frac{-a_5}{2}, \frac{a_4}{2}, \frac{-a_3}{2}, \frac{a_2}{2}\right)$$

For the $\mathbb{C}$-linear involution $\daleth : \mathbf{S}_8 = \mathbf{S}_8^+ \oplus \mathbf{S}_8^- \longrightarrow \mathbf{S}_8^+ \oplus \mathbf{S}_8^- = \mathbf{S}_8$ given by $s = \varsigma + \varkappa \xmapsto{\daleth} \daleth^+(\varsigma) + \daleth^-(\varkappa)$, it holds that

$$\boldsymbol{\gamma}\big(\daleth(s)\big) = \boldsymbol{\gamma}^+(\varsigma) + \boldsymbol{\gamma}^-(\varkappa)$$

for each $s = \varsigma + \varkappa \in \mathbf{S}_8$. By Proposition 2.1, we get that $\daleth$ transforms pure spinors into pure spinors and, since $\daleth$ preserves $\mathbf{S}_8^{\pm}$, we have that $\daleth^{\pm} \in SO(\mathbf{S}_8^{\pm}, \boldsymbol{\gamma}^{\pm}) \simeq SO(\mathbb{C}^8, \mathbf{q}_8)$. For each $(x, y) \in \mathbb{C}^8$, we get that

$$\boldsymbol{\rho}_{(x,y)}(s) = 0 \text{ if and only if } \boldsymbol{\rho}_{\mathcal{I}_8(x,y)}\big(\daleth(s)\big) = 0$$

Hence, Theorem 1.5 implies the following

PROPOSITION 2.4. *Let U be a maximal totally null subspace of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ and let $s \in \mathbf{S}_8^{\pm}$ be a pure spinor for $\mathbf{q}_8$ which represents U. Then, the pure spinor $\daleth(s) \in \mathbf{S}_8^{\pm}$ represents the maximal totally null subspace $\mathcal{I}_8(U)$.*

To finish this preamble, let us consider two distinct maximal totally null subspaces U and U' of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$. By Lemma 1.6, we know that $dim(U \cap U') \in \{0, 2\}$ if and only if U and U' have the same parity. The (non-degenerate) bilinear form

$$\boldsymbol{\beta} : \mathbf{S}_8 \times \mathbf{S}_8 \longrightarrow \mathbb{C}$$

associated to the quadratic form $\boldsymbol{\gamma} : \mathbf{S}_8 \longrightarrow \mathbb{C}$, also allows us to determine the value of $dim(U \cap U')$, as is shown in the next result (cf. III.2.3, III.2.4 and III.1.12 in [**16**]).

Proposition 2.5. *Let U and U' be distinct maximal totally null subspaces of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ and let $s, s' \in \mathbf{S}_8$ be pure spinors for $\mathbf{q}_8$ which represent U and U', respectively. Then, the following statements hold:*

(i) If U and U' are of opposite parity, then $\boldsymbol{\beta}(s, s') = 0$.

(ii) We have that $dim(U \cap U') \neq 0$ if and only if $\boldsymbol{\beta}(s, s') = 0$. Hence, if U and U' have the same parity, then $dim(U \cap U') = 2$ if and only if $\boldsymbol{\beta}(s, s') = 0$.

(iii) We have that $dim(U \cap U') = \mathbf{idx}(\mathbf{q}_8) - 2$ if and only if $s + s'$ is a pure spinor for $\mathbf{q}_8$. Furthermore, in this case we have that the nonzero complex linear combinations $cs + c's'$ are pure spinors for $\mathbf{q}_8$ which represent all the maximal totally null subspaces U'' of $(\mathbb{C}^8, \mathbf{q}_8)$ such that either $U'' \cap U = U' \cap U$ or $U'' = U$.

We shall denote by $\boldsymbol{\beta}^{\pm}$ the restrictions of $\boldsymbol{\beta}$ to the spaces $\mathbf{S}_8^{\pm}$ of half-spinors for $\mathbf{q}_8$.

2.1. The first stratification of $\mathfrak{Z}(\mathbb{S}^6)$.

By Proposition 1.11, we know that the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere is algebraically isomorphic to each of the irreducible components $\mathbf{F}_{3,6}^{\pm}$ of the Fano variety

$$\mathbf{F}_{3,6} = \{ \Omega \in \mathbb{G}(3,7) \mid \Omega \subset \mathbf{Q}_6 \}$$

where $\mathbf{Q}_6 \subset \mathbb{P}^7_{\mathbb{C}}$ is the non-singular quadric hypersurface defined by the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$. In this subsection, we shall only consider the even component $\mathbf{F}^+_{3,6}$. Since $\mathbf{idx}(\mathbf{q}_8) = 4$, Proposition 1.8 implies that

$$\mathbf{F}^+_{3,6} \quad \simeq \quad \bigsqcup_{k,k',k+k' \in \{\,0,2,4\,\}} \mathbf{T}_4(k,k')$$

We know that $\mathbf{T}_4(0,4) \simeq \{\,[\mathbf{1}]\,\}$ and that $\mathbf{T}_4(4,0) \simeq \{\,[\boldsymbol{\omega}]\,\}$. Regarding $\mathbf{T}_4(k,k')$ with $k,k' \in \{\,0,2\,\}$, we have

PROPOSITION 2.6. *Let $\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \in \mathbf{S}^+_8$ be a pure spinor for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$ such that $\varsigma \notin \mathbf{T}_4(0,4) \sqcup \mathbf{T}_4(4,0)$. Then, the following statements hold:*

(i) $\varsigma \in \mathbf{T}_4(0,0)$ if and only if $a_1, a_{16} \neq 0$.

(ii) $\varsigma \in \mathbf{T}_4(0,2)$ if and only if $a_1 \neq 0$ and $a_{16} = 0$.

(iii) $\varsigma \in \mathbf{T}_4(2,0)$ if and only if $a_1 = 0$ and $a_{16} \neq 0$.

(iv) $\varsigma \in \mathbf{T}_4(2,2)$ if and only if $a_1 = a_{16} = 0$.

PROOF. Since the maximal totally null subspace $\mathcal{K}_\varsigma \subset \mathbb{C}^8$ represented by ς is different from the horizontal space $\mathcal{H}$ and the vertical space $\mathcal{V}$, we have that $r_{\mathcal{H}}(\varsigma) = dim(\mathcal{K}_\varsigma \cap \mathcal{H}), r_{\mathcal{V}}(\varsigma) = dim(\mathcal{K}_\varsigma \cap \mathcal{V}) \in \{\,0,2\,\}$. Given that ς is pure, Proposition 2.1 implies that $a_1, a_6, ..., a_{11}, a_{16} \in \mathbb{C}$ are not all zero, and that $a_1 a_{16} - a_6 a_{11} + a_7 a_{10} - a_8 a_9 = 0$. Hence, if $a_1 a_{16} = 0$, then we have that $a_6, ..., a_{11} \in \mathbb{C}$ are not all equal to zero. Since

$$\boldsymbol{\beta}^+(\varsigma, \boldsymbol{\omega}) = a_1 \quad \text{and} \quad \boldsymbol{\beta}^+(\varsigma, \mathbf{1}) = a_{16}$$

Proposition 2.5 implies that $r_{\mathcal{H}}(\varsigma) = 2$ if and only if $a_1 = 0$, and that $r_{\mathcal{V}}(\varsigma) = 2$ if and only if $a_{16} = 0$. $\square$

On the other hand, given that $\mathbf{idx}(\mathbf{q}_8) = 4$, propositions 1.11 and 1.12 imply that $\mathbf{F}^+_{3,6}$ coincides with the Zariski closure $\overline{\mathbf{\Gamma}}_4 \subset \mathbf{G}(4,8)$ of the set

$$\mathbf{\Gamma}_4 = \{\,\mathbf{\Gamma}(M) \mid M \in \mathcal{S}_4\,\}$$

of (horizontal) graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^4$. For every $1 \leq i < j \leq 4$, let us define $\mathbf{M}_{ij} \in \mathcal{S}_4$ to be the matrix with ij-coordinate equal to 1, therefore, its ji-coordinate equals -1, and the other coordinates are all equal to 0. We shall regard

$$\left\{ \mathbf{M}_{ij} \mid 1 \leq i < j \leq 4 \right\}$$

as the standard base of $\mathcal{S}_4$. Then, for each $M \in \mathcal{S}_4$, there exist unique $m_{ij} \in \mathbb{C}$ with $1 \leq i < j \leq 4$, such that $M = \sum_{1 \leq i < j \leq 4} m_{ij} \mathbf{M}_{ij}$. We shall denote this fact by

$$M = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathcal{S}_4$$

and, if $M \neq \mathbf{0}_4$, we shall denote by $[M] \in \mathbb{P}(\mathcal{S}_4) \simeq \mathbb{P}^5_{\mathbb{C}}$ the point defined by M.

The Pfaffian transformation $\boldsymbol{\lambda}_4 : \mathcal{S}_4 \longrightarrow \mathbb{C}$ is given by

$$(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \longmapsto m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}$$

The quadratic form $\boldsymbol{\lambda}_4$ is of maximal rank 6 and maximal index 3. Thus, the complex affine quadric hypersurface

$$\mathcal{Q}_5^{\mathrm{aff}} := \mathcal{S}_4^0 = \left\{ M \in \mathcal{S}_4 \mid \boldsymbol{\lambda}_4(M) = 0 \right\}$$

has an isolated singularity at the zero matrix $\mathbf{0}_4$, and the non-singular complex quadric hypersurface

$$\mathcal{Q}_4 := \left\{ [M] \in \mathbb{P}(\mathcal{S}_4) \mid \boldsymbol{\lambda}_4(M) = 0 \right\}$$

is projectively equivalent to the Plücker quadric hypersurface $\mathbf{F}_1(\mathbb{P}^3_{\mathbb{C}}) \subset \mathbb{P}^5_{\mathbb{C}}$ (cf. Introduction and Chapter 1).

Recalling that $\mathcal{S}_4^* = \left\{ M \in \mathcal{S}_4 \mid \boldsymbol{\lambda}_4(M) \neq 0 \right\}$, that $\boldsymbol{\Gamma}_4^* = \boldsymbol{\Gamma}(\mathcal{S}_4^*)$ and that $\boldsymbol{\Gamma}_4^0 = \boldsymbol{\Gamma}(\mathcal{S}_4^0)$, we have that

$$\overline{\mathbf{\Gamma}}_4 = \mathbf{\Gamma}_4^* \sqcup \mathbf{\Gamma}_4^0 \sqcup \mathcal{I}_8(\mathbf{\Gamma}_4^0) \sqcup \left\{ \text{limits at infinity of the elements in } \mathbf{\Gamma}_4^0 \right\}$$

where $\mathcal{I}_8$ is the canonical involution of $\mathbb{C}^8$. Thus,

$$\begin{aligned} \mathbf{T}_4(0,0) &\simeq \mathbf{\Gamma}_4^*, \\ \mathbf{T}_4(2,0) &\simeq \mathbf{\Gamma}_4^0 - \left\{ \mathbf{\Gamma}(\mathbf{0}_4) = \mathcal{H} \right\}, \\ \mathbf{T}_4(0,2) &\simeq \mathcal{I}_8\big(\mathbf{\Gamma}_4^0 - \left\{ \mathbf{\Gamma}(\mathbf{0}_4) \right\}\big) \text{ and} \\ \mathbf{T}_4(2,2) &\simeq \left\{ \text{limits at infinity of the elements in } \mathbf{\Gamma}_4^0 \right\} \end{aligned}$$

Hence, we have

Proposition 2.7. *The following statements hold:*

(i) The type set $\mathbf{T}_4(0,0)$ is a complex affine variety isomorphic to complex 6-dimensional affine space $\mathbb{A}^6_{\mathbb{C}}$. Therefore, $\mathbf{T}_4(0,0) \subset \mathfrak{Z}(\mathbb{S}^6)$ is a Zariski open and dense subset.

(ii) The type sets $\mathbf{T}_4(2,0), \mathbf{T}_4(0,2) \subset \mathfrak{Z}(\mathbb{S}^6)$ are complex affine varieties biholomorphic to the non-singular and non-compact quadric hypersurface $\mathcal{Q}_5^{\mathit{aff}} - \left\{ \mathbf{0}_4 \right\}$.

(iii) The type set $\mathbf{T}_4(2,2)$ is a complex projective variety isomorphic to the non-singular quadric hypersurface $\mathcal{Q}_4 \subset \mathbb{P}^5_{\mathbb{C}}$. Thus, $\mathbf{T}_4(2,2)$ is biholomorphic to the Plücker quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.

Proposition 2.8. *The twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere is biholomorphic to a non-singular quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$.*

Proof. The facts that $\mathbf{T}_4(0,4) \simeq \left\{ [\mathbf{1}] \right\}$ and that $\mathbf{T}_4(4,0) \simeq \left\{ [\boldsymbol{\omega}] \right\}$ together with Proposition 2.7 imply that

$$\mathfrak{Z}(\mathbb{S}^6) \simeq \overline{\mathbf{\Gamma}}_4 \simeq \bigsqcup_{k,k',k+k' \in \{0,2,4\}} \mathbf{T}_4(k,k')$$

is algebraically equivalent to

$$\mathbb{A}^6_{\mathbb{C}} \sqcup \mathcal{Q}_5^{\mathrm{aff}} \sqcup \mathcal{Q}_4$$

That is to say, the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ is biholomorphic to the disjoint union of $\mathbb{A}^6_{\mathbb{C}}$ and the cone over the Plücker quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$. In turn, this disjoint union is biholomorphic to a non-singular complex quadric hypersurface in $\mathbb{P}^7_{\mathbb{C}}$. □

To continue our analysis of the type sets $\mathbf{T}_4(k,k')$ with $k,k' \in \{0,2\}$, we would like to explicitly describe the pure spinors for the quadratic form $\mathbf{q}_8$ which belong to each of these sets (cf. §2.4).

By Definition 1.7, Proposition 2.1 and (iv) of Proposition 2.6, we have that $\mathbf{T}_4(2,2)$ is given as

$$\left\{ \eta = (0, ca_6, ..., ca_{11}, 0) \in \mathbf{S}^+_8 \mid c \in \mathbb{C}^*, a_6, ..., a_{11} \in \mathbb{C} \text{ not all zero, } \boldsymbol{\gamma}^+(\eta) = 0 \right\}$$

Recalling that the involution $\mathfrak{I}^+ : \mathbf{S}^+_8 \longrightarrow \mathbf{S}^+_8$ is given by

$$(a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \longmapsto \big(4a_{16}, -a_{11}, a_{10}, -a_9, -a_8, a_7, -a_6, \frac{a_1}{4}\big),$$

we have that $\mathbf{T}_4(2,2)$ remains invariant under $\mathfrak{I}^+$. The set of points in $\mathbf{T}_4(2,2)$ which are fixed by $\mathfrak{I}^+$ is of the form

$$\left\{ (0, a, b, c, -c, b, -a, 0) \in \mathbf{S}^+_8 \mid a, b, c \in \mathbb{C} \text{ not all zero }, a^2 + b^2 + c^2 = 0 \right\}$$

That is to say, this set of fixed points is parametrised by the non-singular conic $Q_1 \subset \mathbb{P}^2_{\mathbb{C}}$ defined by the Fermat polynomial $\mathcal{F}_3 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ (cf. §2.1 and §2.3).

To describe the pure spinors contained in the type sets $\mathbf{T}_4(0,0)$, $\mathbf{T}_4(0,2)$ and $\mathbf{T}_4(2,0)$ (and to give a specific biholomorphism between $\mathbf{T}_4(2,2)$ and the Plücker quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$), we proceed to determine a pure spinor, for the quadratic form $\mathbf{q}_8$, which represents the graph $\boldsymbol{\Gamma}(M) \in \boldsymbol{\Gamma}_4 - \{\mathcal{H}, \mathcal{V}\}$.

We begin by associating to each even pure spinor $\varsigma \in \mathbf{S}^+_8$ a couple of elements in $\mathcal{S}_4$ as follows. Let us define $\hat{\mathsf{E}}, \check{\mathsf{E}} : \mathbf{S}^+_8 \longrightarrow \mathcal{S}_4$ by

$$\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \longmapsto \hat{\mathsf{E}}_\varsigma := \begin{pmatrix} 0 & -a_{11} & a_{10} & -a_9 \\ a_{11} & 0 & -a_8 & a_7 \\ -a_{10} & a_8 & 0 & -a_6 \\ a_9 & -a_7 & a_6 & 0 \end{pmatrix} \text{ and}$$

$$\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \longmapsto \check{\mathsf{E}}_\varsigma := \begin{pmatrix} 0 & a_6 & a_7 & a_8 \\ -a_6 & 0 & a_9 & a_{10} \\ -a_7 & -a_9 & 0 & a_{11} \\ -a_8 & -a_{10} & -a_{11} & 0 \end{pmatrix}$$

Then, $\hat{\mathsf{E}}$ and $\check{\mathsf{E}}$ are $\mathbb{C}$-linear and holomorphic transformations. For each even pure spinor $\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16})$, we have that

$$\boldsymbol{\lambda}_4(\hat{\mathsf{E}}_\varsigma) = \boldsymbol{\lambda}_4(\check{\mathsf{E}}_\varsigma) = a_1 a_{16} - \frac{\gamma^+(\varsigma)}{2}$$

and that $\hat{\mathsf{E}}_\varsigma \check{\mathsf{E}}_\varsigma = \boldsymbol{\lambda}_4(\hat{\mathsf{E}}_\varsigma)\mathbf{Id}_4 = \check{\mathsf{E}}_\varsigma \hat{\mathsf{E}}_\varsigma$. For a pure spinor $\varsigma \in \mathbf{S}_8^+$, propositions 2.1 and 2.6 imply that

$$\hat{\mathsf{E}}_\varsigma, \check{\mathsf{E}}_\varsigma \in \mathcal{S}_4^* \quad \text{if and only if} \quad \varsigma \in \mathbf{T}_4(0,0)$$

Given that the $\mathbb{C}$-linear transformation $\check{\mathsf{E}} : \mathbf{S}_8^+ \longrightarrow \mathcal{S}_4$ is holomorphic and that

$$[\check{\mathsf{E}}_\eta] \in \mathcal{Q}_4 \quad \text{for all} \quad \eta \in \mathbf{T}_4(2,2),$$

we have a holomorphic transformation $\boldsymbol{\varepsilon} : \mathbf{T}_4(2,2) \longrightarrow \mathcal{Q}_4$ given by

$$\eta \longmapsto [\check{\mathsf{E}}_\eta]$$

Conversely, let $M = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathcal{S}_4$ be nonzero and such that $[M] \in \mathcal{Q}_4$. Then,

$$\eta_M := (0, m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}, 0) \in \mathbf{S}_8^+$$

is a nonzero half-spinor for $\mathbf{q}_8$. Since

$$\boldsymbol{\gamma}^+(\eta_M) = -\boldsymbol{\lambda}_4(M) = 0,$$

Proposition 2.1 implies that η_M is a pure spinor for $\mathbf{q}_8$. By (iv) of Proposition 2.6, we have that

$$\eta_{_M} \in \mathbf{T}_{_4}(2,2) \quad \text{for all nonzero} \quad M \in Sk_{_4}^0$$

If $M' \in \mathcal{S}_{_4}$ is another representative of $[M] \in \mathcal{Q}_{_4}$, then $\eta_{_{M'}} = c\eta_{_M}$ for some $c \in \mathbb{C}^*$, and we have that $\eta_{_{M'}}$ and $\eta_{_M}$ define the same point in $\mathbf{T}_{_4}(2,2)$ (cf. Theorem 1.5). With this notation, let us define $\boldsymbol{\eta} : \mathcal{Q}_{_4} \longrightarrow \mathbf{T}_{_4}(2,2)$ by

$$[M] \longmapsto \eta_{_M}$$

Then, $\boldsymbol{\eta}$ is a well-defined holomorphic transformation. For all $\eta \in \mathbf{T}_{_4}(2,2)$ and $[M] \in \mathcal{Q}_{_4}$, we have that

$$\boldsymbol{\eta}\big(\boldsymbol{\varepsilon}(\eta)\big) = \eta \quad \text{and} \quad \boldsymbol{\varepsilon}\big(\boldsymbol{\eta}([M])\big) = [M]$$

Thus, $\boldsymbol{\varepsilon} : \mathbf{T}_{_4}(2,2) \longrightarrow \mathcal{Q}_{_4}$ is a biholomorphism.

Now, let us define $\hat{\mathsf{s}} : \mathcal{S}_{_4} \longrightarrow \mathbf{S}_{_8}^+$ by

$$\begin{gathered}\hat{\mathsf{s}}(M) := \hat{\mathsf{s}}(m_{_{12}}, m_{_{13}}, m_{_{14}}, m_{_{23}}, m_{_{24}}, m_{_{34}}) := \\ \big(2\boldsymbol{\lambda}_{_4}(M), -m_{_{34}}, m_{_{24}}, -m_{_{23}}, -m_{_{14}}, m_{_{13}}, -m_{_{12}}, \frac{1}{2}\big)\end{gathered}$$

Then, for each $M \in \mathcal{S}_{_4}$, we have that $\hat{\mathsf{s}}(M) \in \mathbf{S}_{_8}^+$ is nonzero and that $\boldsymbol{\gamma}^+\big(\hat{\mathsf{s}}(M)\big) = 0$. By Proposition 2.1, we have that $\hat{\mathsf{s}}(M)$ is a pure spinor for $\mathbf{q}_{_8}$. Moreover, if $M \in \mathcal{S}_{_4}$ and $Z \in \mathbb{C}^4$, then the point $\big(Z, M(Z)\big) \in \mathbb{C}^8$ satisfies that

$$\boldsymbol{\rho}_{_{(Z,M(Z))}}\big(\hat{\mathsf{s}}(M)\big) = 0$$

where $\boldsymbol{\rho} : \mathcal{C}\ell_{_8} \longrightarrow End(\mathbf{S}_{_8})$ is the spin representation. By Theorem 1.5, we get that $\hat{\mathsf{s}}(M)$ represents the horizontal graph $\boldsymbol{\Gamma}(M)$. By Proposition 2.4, for each $M \in \mathcal{S}_{_4}$, we have that

$$\check{\mathsf{s}}(M) := \daleth^+\big(\hat{\mathsf{s}}(M)\big) = \big(2, m_{_{12}}, m_{_{13}}, m_{_{14}}, m_{_{23}}, m_{_{24}}, m_{_{34}}, \frac{\boldsymbol{\lambda}_{_4}(M)}{2}\big) \in \mathbf{S}_{_8}^+$$

is a pure spinor for $\mathbf{q}_{_8}$ which represents the vertical graph $\mathcal{I}_{_8}\big(\boldsymbol{\Gamma}(M)\big)$. Thus, we have

LEMMA 2.9. *Let $M \in \mathcal{S}_4$. Then, $\hat{\mathsf{s}}(M)$ and $\check{\mathsf{s}}(M)$ are even pure spinors for the quadratic form $\mathbf{q}_8$ which, respectively, represent the maximal totally null subspace $\mathbf{\Gamma}(M)$ and its image $\mathcal{I}_8\big(\mathbf{\Gamma}(M)\big)$ under the canonical involution of $\mathbb{C}^8$.*

If $M = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathcal{S}_4$, then it holds that

$$\hat{\mathsf{E}}_{\hat{\mathsf{s}}(M)} = \check{\mathsf{E}}_{\check{\mathsf{s}}(M)} = M \text{ and } \hat{\mathsf{E}}_{\check{\mathsf{s}}(M)} = \check{\mathsf{E}}_{\hat{\mathsf{s}}(M)} = \begin{pmatrix} 0 & -m_{34} & m_{24} & -m_{23} \\ m_{34} & 0 & -m_{14} & m_{13} \\ -m_{24} & m_{14} & 0 & -m_{12} \\ m_{23} & -m_{13} & m_{12} & 0 \end{pmatrix}$$

Thus, if $M \in \mathcal{S}_4^*$, then its inverse matrix is given as

$$M^{-1} = \frac{1}{\boldsymbol{\lambda}_4(M)} \hat{\mathsf{E}}_{\check{\mathsf{s}}(M)}$$

and, in this case, the maximal totally null subspace $\mathcal{I}_8\big(\mathbf{\Gamma}(M)\big)$ coincides with $\mathbf{\Gamma}(M^{-1})$.

REMARK 2.10. *Let us consider a sequence $\left\{ M_n \in \mathcal{S}_4^* \mid n \in N \right\}$ such that $\lim\limits_{n \longrightarrow \infty} |\boldsymbol{\lambda}_4(M_n)| = \infty$. Since*

$$\lim_{n \longrightarrow \infty} M_n^{-1} = \lim_{n \longrightarrow \infty} \frac{1}{\boldsymbol{\lambda}_4(M_n)} \hat{\mathsf{E}}_{\check{\mathsf{s}}(M_n)} = \mathbf{0}_4,$$

we have that $\lim\limits_{n \longrightarrow \infty} \mathcal{I}_8\big(\mathbf{\Gamma}(M_n)\big) = \mathbf{\Gamma}(\mathbf{0}_4) = \mathcal{H}$. Hence, $\lim\limits_{n \longrightarrow \infty} \mathbf{\Gamma}(M_n) = \mathcal{I}_8(\mathcal{H}) = \mathcal{V}$.

By Proposition 2.6, Lemma 2.9 and the above discussion, we have

PROPOSITION 2.11. *The following statements hold:*

(i) Every pure spinor for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$ which belongs to the type set $\mathbf{T}_4(2,2)$ is of the form

$$\eta = (0, ca_6, ..., ca_{11}, 0)$$

where $c \in \mathbb{C}^*$, $a_6, ..., a_{11} \in \mathbb{C}$ are not all zero, and $\boldsymbol{\gamma}^+(\eta) = -a_6 a_{11} + a_7 a_{10} - a_8 a_9 = 0$.

(ii) $\mathbf{T}_4(0,0) = \{\, \hat{\mathsf{s}}(M) \in \mathbf{S}_8^+ \mid M \in \mathcal{S}_4^* \,\}$

Equivalently, every pure spinor for $\mathbf{q}_8$ *which belongs to the type set* $\mathbf{T}_4(0,0)$ *is of the form*

$$\big(2\boldsymbol{\lambda}_4(M), -m_{34}, m_{24}, -m_{23}, -m_{14}, m_{13}, -m_{12}, \frac{1}{2}\big)$$

for some $M = \sum_{1 \le i < j \le 4} m_{ij}\mathbf{M}_{ij} \in \mathcal{S}_4^*$.

(iii) $\mathbf{T}_4(2,0) = \{\, \hat{\mathsf{s}}(M) \in \mathbf{S}_8^+ \mid M \in \mathcal{S}_4^0$ *is nonzero* $\}$

Therefore, every element of the type set $\mathbf{T}_4(2,0)$ *is of the form*

$$\big(0, -m_{34}, m_{24}, -m_{23}, -m_{14}, m_{13}, -m_{12}, \frac{1}{2}\big)$$

where $(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathcal{S}_4^0 - \{\, \mathbf{0}_4 \,\}$.

(iv) $\mathbf{T}_4(0,2) = \beth^+\big(\mathbf{T}_4(2,0)\big) = \{\, \check{\mathsf{s}}(M) \in \mathbf{S}_8^+ \mid M \in \mathcal{S}_4^0$ *is nonzero* $\}$

Thus, every pure spinor for $\mathbf{q}_8$ *which belongs to the type set* $\mathbf{T}_4(0,2)$ *is of the form*

$$(2, m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}, 0)$$

where $(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathcal{S}_4^0 - \{\, \mathbf{0}_4 \,\}$.

Let $\hat{\mathsf{s}}(M) \in \mathbf{T}_4(0,0)$. Since $M \in \mathcal{S}_4^*$, the pure spinor

$$\check{\mathsf{s}}(M) = \beth^+\big(\hat{\mathsf{s}}(M)\big) = \boldsymbol{\lambda}_4(M)\hat{\mathsf{s}}(M^{-1})$$

belongs to $\mathbf{T}_4(0,0)$ and we have that the type set $\mathbf{T}_4(0,0)$ is invariant under the involution $\beth^+ : \mathbf{S}_8^+ \longrightarrow \mathbf{S}_8^+$. The elements of $\mathbf{T}_4(0,0)$ which are fixed by $\beth^+$ are of the form (cf. §2.3)

$$\big(2, a, b, c, -c, b, -a, \frac{1}{2}\big) \text{ with } a, b, c \in \mathbb{C} \text{ such that } a^2 + b^2 + c^2 = -1$$

To round up this subsection, we would like to describe the manner in which any two even maximal totally null subspaces of $(\mathbb{C}^8, \mathbf{q}_8)$, distinct from $\mathcal{H}$ and $\mathcal{V}$, intersect each other. Let us consider the bilinear form

$$\mathsf{B}_4 : \mathcal{S}_4 \times \mathcal{S}_4 \longrightarrow \mathbb{C}$$

associated to the Pfaffian transformation $\boldsymbol{\lambda}_4 : \mathcal{S}_4 \longrightarrow \mathbb{C}$. We have that the Fermat polynomial $\mathcal{F}_6 : \mathcal{S}_4 \simeq \mathbb{C}^6 \longrightarrow \mathbb{C}$ is given by

$$(m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \longmapsto \sum_{1 \leq i < j \leq 4} m_{ij}^2$$

Then, the bilinear form

$$\mho : \mathcal{S}_4 \times \mathcal{S}_4 \longrightarrow \mathbb{C}$$

associated to the quadratic form $\mathcal{F}_6$ is given as

$$\Big((m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}), (n_{12}, n_{13}, n_{14}, n_{23}, n_{24}, n_{34})\Big) \longmapsto \sum_{1 \leq i < j \leq 4} m_{ij} n_{ij}$$

For $M, N \in \mathcal{S}_4$, we get that

$$\mathsf{B}_4(M, N) = \frac{1}{2} \mho(M, \hat{\mathsf{E}}_{\check{\mathsf{s}}(N)})$$

Therefore, we have that

$$\boldsymbol{\beta}^+\big(\hat{\mathsf{s}}(M), \hat{\mathsf{s}}(N)\big) = \boldsymbol{\beta}^+\big(\check{\mathsf{s}}(M), \check{\mathsf{s}}(N)\big) = \boldsymbol{\lambda}_4(M - N) = \boldsymbol{\lambda}_4(M) + \boldsymbol{\lambda}_4(N) - 2\mathsf{B}_4(M, N),$$

$$\text{and} \quad \boldsymbol{\beta}^+\big(\hat{\mathsf{s}}(M), \check{\mathsf{s}}(N)\big) = 1 + \boldsymbol{\lambda}_4(M)\boldsymbol{\lambda}_4(N) + \mho(M, N)$$

For $\eta, \eta' \in \mathbf{T}_4(2,2)$, we have that

$$\boldsymbol{\beta}^+(\hat{\mathsf{s}}(M), \eta) = \mho(M, \check{\mathsf{E}}_\eta), \quad \boldsymbol{\beta}^+(\check{\mathsf{s}}(M), \eta) = \mho(M, \hat{\mathsf{E}}_\eta),$$
$$\text{and} \quad \boldsymbol{\beta}^+(\eta, \eta') = \mho(\hat{\mathsf{E}}_\eta, \check{\mathsf{E}}_{\eta'})$$

These equations and Proposition 2.5 imply

LEMMA 2.12. *Let $M, N \in \mathcal{S}_4$ be distinct matrices and let $\eta, \eta' \in \mathbf{T}_4(2,2)$ be distinct pure spinors. Then, the following statements hold:*

(i) $dim\big(\mathbf{\Gamma}(M) \cap \mathbf{\Gamma}(N)\big) = dim\big(\mathcal{I}_8\big(\mathbf{\Gamma}(M)\big) \cap \mathcal{I}_8\big(\mathbf{\Gamma}(N)\big)\big) = 2$ *if and only if* $M - N \in \mathcal{S}_4^0$.

(ii) If $\check{\mathsf{s}}(N) \in \mathbf{T}_4(0,2)$, *then* $dim\big(\mathcal{I}_8\big(\mathbf{\Gamma}(N)\big) \cap \mathbf{\Gamma}(M)\big) = 2$ *if and only if* $\mho(M,N) = -1$.

(iii) $dim(\mathbf{\Gamma}(M) \cap \mathcal{K}_\eta) = 2$ *if and only if M and $\check{\mathsf{E}}_\eta$ are mutually orthogonal with respect to $\mho$.*

(iv) $dim(\mathcal{I}\big(\mathbf{\Gamma}(M)\big) \cap \mathcal{K}_\eta) = 2$ *if and only if M and $\hat{\mathsf{E}}_\eta$ are mutually orthogonal with respect to $\mho$.*

(v) $dim(\mathcal{K}_\eta \cap \mathcal{K}_{\eta'}) = 2$ *if and only if $\hat{\mathsf{E}}_\eta$ and $\check{\mathsf{E}}_{\eta'}$ are mutually orthogonal with respect $\mho$.*

2.2. The second stratification of $\mathfrak{Z}(\mathbb{S}^6)$.

In this subsection we shall study the irreducible component

$$\mathbf{F}^-_{3,6} \quad \simeq \quad \mathbf{T}_4(1,1) \sqcup \mathbf{T}_4(1,3) \sqcup \mathbf{T}_4(3,1)$$

of the Fano variety $\mathbf{F}_{3,6}$, which parametrises the odd maximal totally null subspaces of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ (cf. Proposition 1.8).

Since

$$\mathbf{F}^+_{3,6} \cap \mathbf{F}^-_{3,6} = \varnothing \quad \text{ and } \quad \mathbf{F}^+_{3,6} \simeq \overline{\mathbf{\Gamma}}_4,$$

we have that *none* odd maximal totally null subspace of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ is either a graph of a skew-symmetric linear endomorphism of $\mathbb{C}^4$ or a limit (in the complex Grassmannian variety $\mathbf{G}(4,8)$) of such graphs. Hence, in comparison to the nice geometric description we found in §2.2.1, for the even maximal totally null subspaces of $(\mathbb{C}^8, \mathbf{q}_8)$, it could be harder to describe the odd maximal totally null subspaces.

By Theorem 1.5, given an odd pure spinor $\varkappa \in \mathbf{S}_8^-$, the spin representation $\boldsymbol{\rho} : \mathcal{C}\ell_8 \longrightarrow End(\mathbf{S}_8)$ is a very useful tool to determine the coordinates of the points $(x, y) \in \mathbb{C}^8$ which belong to the maximal totally null subspace $\mathcal{K}_\varkappa$ represented by $\varkappa$. Once such coordinates were made explicit, we could, amongst other things, determine the type of the odd pure spinor $\varkappa$. Nonetheless, there is an easier way to determine such type (cf. Proposition 2.14) which, in turn, will help us to describe the sets $\mathbf{T}_4(k, k')$ with $k, k' \in \{1, 3\}$.

Let us define $\mathbf{x}, \mathbf{y} : \mathbf{S}_8^- \longrightarrow \mathbb{C}^4$, respectively, by

$$\varkappa = (b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \longmapsto \mathbf{x}_\varkappa := (b_2, b_3, b_4, b_5) \text{ and}$$

$$\varkappa = (b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \longmapsto \mathbf{y}_\varkappa := (b_{15}, -b_{14}, b_{13}, -b_{12})$$

Then, $\mathbf{x}$ and $\mathbf{y}$ are $\mathbb{C}$-linear and holomorphic transformations. Given that the horizontal and vertical spaces $\mathcal{H}, \mathcal{V} \subset \mathbb{C}^8$ are isomorphic to $\mathbb{C}^4$, for each $\varkappa \in \mathbf{S}_8^-$, we have that $(\mathbf{x}_\varkappa, \mathbf{y}_\varkappa) \in \mathbb{C}^8$ and it holds that

$$\mathbf{q}_8(\mathbf{x}_\varkappa, \mathbf{y}_\varkappa) = \boldsymbol{\gamma}^-(\varkappa)$$

Thus, by Proposition 2.1, we have that $\varkappa$ is a pure spinor for $\mathbf{q}_8$ if and only if $[\mathbf{x}_\varkappa : \mathbf{y}_\varkappa] \in \mathbf{Q}_6$. Furthermore, in this case, we get that

$$\boldsymbol{\rho}_{(\mathbf{x}_\varkappa, \mathbf{y}_\varkappa)}(\varkappa) = 0$$

and Theorem 1.5 implies that $(\mathbf{x}_\varkappa, \mathbf{y}_\varkappa)$ belongs to the maximal totally null subspace $\mathcal{K}_\varkappa$ represented by $\varkappa$.

On the other hand, let us define a bilinear transformation $\bullet : \mathbf{S}_8^- \times \mathbf{S}_8^+ \longrightarrow \mathbb{C}^8$ as follows (cf. Chapter IV of [**16**], Appendix B of [**28**]). For each $\varkappa = (b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \in \mathbf{S}_8^-$ and each $\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \in \mathbf{S}_8^+$, we set

$$\varkappa \bullet \varsigma := \big(2a_{16}\mathbf{x}_\varkappa + 2\check{\mathsf{E}}_\varsigma(\mathbf{y}_\varkappa), \hat{\mathsf{E}}_\varsigma(\mathbf{x}_\varkappa) + a_1\mathbf{y}_\varkappa\big) \in \mathbb{C}^8$$

where $\hat{\mathsf{E}}_\varsigma, \check{\mathsf{E}}_\varsigma \in \mathcal{S}_4$ have been defined in §2.2.1. For all $\varkappa \in \mathbf{S}_8^-$ and $\varsigma \in \mathbf{S}_8^+$ it holds that

$$\mathbf{q}_8(\varkappa \bullet \varsigma) = \boldsymbol{\gamma}^-(\varkappa)\boldsymbol{\gamma}^+(\varsigma)$$

Thus, by Proposition 2.1, we have that $\varkappa$ and ς are pure spinors for $\mathbf{q}_8$ if and only if $\varkappa \bullet \varsigma \in \mathbb{C}^8$ is nonzero and the point $[\varkappa \bullet \varsigma] \in \mathbb{P}^7_{\mathbb{C}}$, defined by $\varkappa \bullet \varsigma$, belongs to $\mathbf{Q}_6$. In this case, it holds that

$$\boldsymbol{\rho}_{\varkappa \bullet \varsigma}(\varkappa) = (\varkappa \bullet \varsigma) \bullet \varkappa = 0 \quad \text{and} \quad \boldsymbol{\rho}_{\varkappa \bullet \varsigma}(\varsigma) = (\varkappa \bullet \varsigma) \bullet \varsigma = 0$$

(cf. IV.2.2 and IV.2.3 in [**16**]). By Theorem 1.5, we have that

$$\varkappa \bullet \varsigma \in \mathcal{K}_\varkappa \cap \mathcal{K}_\varsigma$$

Moreover, since the pure spinors $\varkappa$ and ς have opposite parity, Lemma 1.6 implies that $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_\varsigma) \in \{1, 3\}$. As a consequence of (i) of Proposition 2.5, IV.4.2 and IV.4.3 in [**16**], we have

Proposition 2.13. *Let $\varkappa \in \mathbf{S}_8^-$ and $\varsigma \in \mathbf{S}_8^+$ be pure spinors for the quadratic form $\mathbf{q}_8$. Then, $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_\varsigma) = 1$ if and only if $\varkappa \bullet \varsigma \in \mathbb{C}^8$ is nonzero. Furthermore, in this case we have that $\varkappa \bullet \varsigma$ is a generator of $\mathcal{K}_\varkappa \cap \mathcal{K}_\varsigma$.*

Proposition 2.13 allows us to easily describe the elements belonging to the type sets $\mathbf{T}_4(k, k')$ with $k, k' \in \{1, 3\}$, as in shown in the next result.

Proposition 2.14. *Let $\varkappa = (b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \in \mathbf{S}_8^-$ be a pure spinor for the quadratic form $\mathbf{q}_8$. Then, the following statements hold:*

(i) *$\varkappa \in \mathbf{T}_4(1, 1)$ if and only if $\mathbf{x}_\varkappa, \mathbf{y}_\varkappa \in \mathbb{C}^4$ are both nonzero.*

(ii) *$\varkappa \in \mathbf{T}_4(1, 3)$ if and only if $\mathbf{x}_\varkappa \in \mathbb{C}^4$ is nonzero and $\mathbf{y}_\varkappa \in \mathbb{C}^4$ is the origin.*

(iii) *$\varkappa \in \mathbf{T}_3(3, 1)$ if and only if $\mathbf{x}_\varkappa \in \mathbb{C}^4$ is the origin and $\mathbf{y}_\varkappa \in \mathbb{C}^4$ is nonzero.*

Proof. We know that $\mathbf{1}, \boldsymbol{\omega} \in \mathbf{S}_8^+$ are pure spinors for the quadratic form $\mathbf{q}_8$. Since $\hat{\mathsf{E}}_{\boldsymbol{\omega}} = \check{\mathsf{E}}_{\boldsymbol{\omega}} = \mathbf{0}_4 = \hat{\mathsf{E}}_{\mathbf{1}} = \check{\mathsf{E}}_{\mathbf{1}}$, we get that

$$\varkappa \bullet \boldsymbol{\omega} = (2b_2, 2b_3, 2b_4, 2b_5, 0, 0, 0, 0) := (2\mathbf{x}_\varkappa, 0, 0, 0, 0) \text{ and}$$

$$\varkappa \bullet \mathbf{1} = (0, 0, 0, 0, b_{15}, -b_{14}, b_{13}, -b_{12}) := (0, 0, 0, 0, \mathbf{y}_\varkappa)$$

By Proposition 2.13, we have that

$$r_{\mathcal{H}}(\varkappa) = dim(\mathcal{K}_\varkappa \cap \mathcal{H}) = 1 \quad \text{if and only if} \quad \mathbf{x}_\varkappa \in \mathbb{C}^4 \text{ is nonzero, and that}$$
$$r_{\mathcal{V}}(\varkappa) = dim(\mathcal{K}_\varkappa \cap \mathcal{V}) = 1 \quad \text{if and only if} \quad \mathbf{y}_\varkappa \in \mathbb{C}^4 \text{ is nonzero}$$

Since $r_{\mathcal{H}}(\varkappa), r_{\mathcal{V}}(\varkappa) \in \{1, 3\}$, we have the result. □

Proposition 2.14 implies that

$$\mathbf{T}_4(1,3) \ni \varkappa = (b_2, b_3, b_4, b_5, 0, 0, 0, 0) \longmapsto [b_2 : b_3 : b_4 : b_5] := [\mathbf{x}_\varkappa] \in \mathbb{P}^3_{\mathbb{C}} \text{ and}$$

$$\mathbf{T}_4(3,1) \ni \varkappa' = (0, 0, 0, 0, b'_{12}, b'_{13}, b'_{14}, b'_{15}) \longmapsto [b'_{15} : -b'_{14} : b'_{13} : -b'_{12}] := [\mathbf{y}_{\varkappa'}] \in \mathbb{P}^3_{\mathbb{C}}$$

define biholomorphisms between $\mathbb{P}^3_{\mathbb{C}}$ and, respectively, $\mathbf{T}_4(1,3)$ and $\mathbf{T}_4(3,1)$.

We recall that the involution $\beth^- : \mathbf{S}^-_8 \longrightarrow \mathbf{S}^-_8$ is given by

$$(b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \longmapsto \left(2b_{15}, -2b_{14}, 2b_{13}, -2b_{12}, \frac{-b_5}{2}, \frac{b_4}{2}, \frac{-b_3}{2}, \frac{b_2}{2}\right)$$

Hence, besides Proposition 2.4, we have in Proposition 2.14 another means to show that $\beth^-$ interchanges $\mathbf{T}_4(1,3)$ with $\mathbf{T}_4(3,1)$, and that the type set $\mathbf{T}_4(1,1)$ remains invariant under $\beth^-$. The points in $\mathbf{T}_4(1,1)$ which are fixed by $\beth^-$ are of the form

$$\left(b_2, b_3, b_4, b_5, \frac{-b_5}{2}, \frac{b_4}{2}, \frac{-b_3}{2}, \frac{b_2}{2}\right) \text{ with } b_2, ..., b_5 \in \mathbb{C} \text{ not all zero, } \sum_{j=2}^{5} b_j^2 = 0$$

Therefore, this fixed-point set is parametrised by the non-singular quadric hypersurface $Q_2 \subset \mathbb{P}^3_{\mathbb{C}}$ defined by the Fermat polynomial $\mathcal{F}_4 : \mathbb{C}^4 \longrightarrow \mathbb{C}$.

Regarding our description of the type set $\mathbf{T}_4(1,1)$, a second step is

PROPOSITION 2.15. *$\mathbf{T}_4(1,1)$ fibres differentiably over $\mathbb{C}^3 - \{(0,0,0)\}$ with fibre isomorphic to $\mathbb{P}^3_{\mathbb{C}}$.*

PROOF. By Proposition 1.11, the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere is biholomorphic to $\mathbf{F}^-_{3,6}$. We know that the twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$ is differentiable and its fibres are isomorphic to $\mathbb{P}^3_{\mathbb{C}}$. Given that the orthogonal group $O(\mathbb{C}^8, \mathbf{q}_8)$ acts transitively on the set of maximal totally null subspaces, we may assume that the linear 3-folds $\mathbf{T}_4(1,3)$ and $\mathbf{T}_4(3,1)$ are (distinct) fibres of $\mathbf{p}_6$ (cf. III.1.7, III.3.3 and Lemma 1 on p. 152 in [**16**]). Then,

$$\mathbf{T}_4(1,1) = \mathfrak{Z}(\mathbb{S}^6) - \big(\mathbf{T}_4(1,3) \sqcup \mathbf{T}_4(3,1)\big)$$

and we have that the restriction of $\mathbf{p}_6$ to $\mathbf{T}_4(1,1)$ establishes a diffeomorphism between $\mathbf{T}_4(1,1)$ and $\mathbb{S}^6 - \{\mathcal{N}, \mathcal{S}\}$, where $\mathcal{N}$ and $\mathcal{S}$ are the North and South poles of $\mathbb{S}^6$, respectively.

Since $\mathbb{S}^6 - \{\mathcal{N}, \mathcal{S}\}$ is diffeomorphic to $\mathbb{S}^5 \times \mathbb{R}$ which, in turn, is diffeomorphic to $\mathbb{C}^3 - \{(0,0,0)\}$, we get that the restriction of $\mathbf{p}_6$ to $\mathbf{T}_4(1,1)$ is a differentiable fibration over $\mathbb{C}^3 - \{(0,0,0)\}$, with fibre isomorphic to $\mathbb{P}^3_{\mathbb{C}}$. $\square$

In order to continue our description of $\mathbf{T}_4(1,1)$, let us denote by

$$\mathcal{B}_4 : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$$

the bilinear form associated to the Fermat polynomial $\mathcal{F}_4 : \mathbb{C}^4 \longrightarrow \mathbb{C}$. Since the quadratic form $\mathcal{F}_4$ has maximal rank 4, we have that $\mathcal{B}_4$ is non-degenerate. Let us consider the generalised complex orthogonal Stiefel manifold

$$\mathbf{V}^{\mathcal{B}_4}_{\mathbb{C}}(1,1;4) = \{(L_1, L_2) \in \mathbf{G}(1,4) \times \mathbf{G}(1,4) \mid L_1 \perp_{\mathcal{B}_4} L_2\}$$

of the quadratic space $(\mathbb{C}^4, \mathcal{F}_4)$ (cf. Definition 1.13). We have that $\mathbf{V}^{\mathcal{B}_4}_{\mathbb{C}}(1,1;4)$ is a complex 5-dimensional, smooth and compact subvariety of $\mathbf{G}(1,4) \times \mathbf{G}(1,4) \simeq \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^3_{\mathbb{C}}$. By Theorem 1.14, we get

Proposition 2.16. *For $j = 1, 2$, let us denote by $\pi_j : \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_4}(1,1;4) \longrightarrow \mathbf{G}(1,4) \simeq \mathbb{P}^3_{\mathbb{C}}$ the canonical projection*

$$(L_1, L_2) \longmapsto L_j$$

into the jth-coordinate. Then, we have two locally trivial and holomorphic fibrations

$$\begin{tikzcd} & \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_4}(1,1;4) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^3_{\mathbb{C}} & & \mathbb{P}^3_{\mathbb{C}} \end{tikzcd}$$

with fibres isomorphic to $\mathbb{P}^2_{\mathbb{C}}$.

Let $\varkappa = (b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \in \mathbf{T}_4(1,1)$. By Proposition 2.14, we know that $\mathbf{x}_\varkappa = (b_2, b_3, b_4, b_5)$ and $\mathbf{y}_\varkappa = (b_{15}, -b_{14}, b_{13}, -b_{12})$ are both nonzero in $\mathbb{C}^4$. Therefore

$$L_1^\varkappa := \mathbb{C}\mathbf{x}_\varkappa \quad \text{and} \quad L_2^\varkappa := \mathbb{C}\mathbf{y}_\varkappa$$

belong to $\mathbf{G}(1,4) \simeq \mathbb{P}^3_{\mathbb{C}}$. By Proposition 2.1 we get that

$$\mathcal{B}_4(\mathbf{x}_\varkappa, \mathbf{y}_\varkappa) = \frac{\gamma^-(\varkappa)}{2} = 0$$

and, thus, it holds that

$$(L_1^\varkappa, L_2^\varkappa) \in \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_4}(1,1;4)$$

With this notation, let us define $\Psi : \mathbf{T}_4(1,1) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_4}(1,1;4)$ by

$$\varkappa \longmapsto (L_1^\varkappa, L_2^\varkappa)$$

Let $P = (L_1, L_2) \in \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_4}(1,1;4)$ be any point and let

$$\ell^1_{_P} := (\ell^1_{_1}, \ell^1_{_2}, \ell^1_{_3}, \ell^1_{_4}),\ \ell^2_{_P} := (\ell^2_{_1}, \ell^2_{_2}, \ell^2_{_3}, \ell^2_{_4}) \in \mathbb{C}^4$$

be generators for $L_{_1}$ and $L_{_2}$, respectively. Let us set

$$\varkappa_{_P} := (\ell^1_{_1}, \ell^1_{_2}, \ell^1_{_3}, \ell^1_{_4}, -\ell^2_{_4}, \ell^2_{_3}, -\ell^2_{_2}, \ell^2_{_1}) \in \mathbf{S}^-_{_8}$$

Then, $\varkappa_{_P} \in \mathbf{S}^-_{_8}$ is nonzero and we have that

$$\gamma^-(\varkappa_{_P}) = 2\mathcal{B}_{_4}(\ell^1_{_P}, \ell^2_{_P}) = 0,$$

$$\mathbf{x}_{\varkappa_P} = \ell^1_{_P} \quad \text{and} \quad \mathbf{y}_{\varkappa_P} = \ell^2_{_P}$$

Again, by propositions 2.1 and 2.14, we have that $\varkappa_{_P} \in \mathbf{T}_{_4}(1,1)$. Since it holds that $\varkappa_{_P}$ and $c\varkappa_{_P}$ define the same point in $\mathbf{T}_{_4}(1,1)$ for all $c \in \mathbb{C}^*$ (cf. Definition 1.7 and Theorem 1.5), and given that

$$\Psi(c\varkappa_{_P}) = \Psi(\varkappa_{_P}) = (L_{_1}, L_{_2}) = P,$$

we get that the fibre $\Psi^{-1}(P)$ is isomorphic to $\mathbb{C}^*$. Recalling that

$$\mathcal{S}^*_{_2} = \left\{ \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \in \mathcal{M}(2,\mathbb{C}) \mid \mu \in \mathbb{C}^* \right\} \simeq \mathbb{C}^*$$

we have proven

Proposition 2.17. $\Psi : \mathbf{T}_{_4}(1,1) \longrightarrow \mathbf{V}^{\mathcal{B}_4}_{\mathbb{C}}(1,1;4)$ *is a locally trivial and holomorphic fibration with fibre isomorphic to* $\mathbb{C}^* \simeq \mathcal{S}^*_{_2}$.

As a consequence of propositions 2.16 and 2.17 we have

Proposition 2.18. *There exist two locally trivial and holomorphic fibrations*

$$
\begin{tikzcd}
& \mathbf{T}_4(1,1) \arrow[d, "\Psi"] & \\
& \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_4}(1,1;4) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^3_{\mathbb{C}} & & \mathbb{P}^3_{\mathbb{C}}
\end{tikzcd}
$$

with fibres isomorphic to $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{C}^*$.

On another matter, let $\varkappa, \varkappa' \in \mathbf{S}^-_8$ be distinct pure spinors for $\mathbf{q}_8$ and let $\mathcal{K}_\varkappa, \mathcal{K}_{\varkappa'} \subset \mathbb{C}^8$ be the odd maximal totally null subspaces represented by $\varkappa$ and $\varkappa'$, respectively. By Lemma 1.6, we know that $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_{\varkappa'}) \in \{0, 2\}$ and we would like to show how to determine the value of this dimension in terms of the bilinear form $\mathcal{B}_4$. Since

$$
\boldsymbol{\beta}^-(\varkappa, \varkappa') = \mathcal{B}_4\big(\mathbf{x}(\varkappa + \varkappa'), \mathbf{y}(\varkappa + \varkappa')\big) = \mathcal{B}_4(\mathbf{x}_\varkappa + \mathbf{x}_{\varkappa'}, \mathbf{y}_\varkappa + \mathbf{y}_{\varkappa'}),
$$

Proposition 2.5 implies the following result.

LEMMA 2.19. *Let $\varkappa, \varkappa' \in \mathbf{S}^-_8$ be distinct pure spinors for $\mathbf{q}_8$. Then, the following statements hold:*

(i) If $\varkappa, \varkappa' \in \mathbf{T}_4(1,1)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_{\varkappa'}) = 2$ if and only if $\mathbf{x}(\varkappa + \varkappa')$ and $\mathbf{y}(\varkappa + \varkappa')$ are mutually orthogonal with respect to $\mathcal{B}_4$.

(ii) If $\varkappa, \varkappa' \in \mathbf{T}_4(1,3)$ or $\varkappa, \varkappa' \in \mathbf{T}_4(3,1)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_{\varkappa'}) = 2$.

(iii) If $\varkappa \in \mathbf{T}_4(1,1)$ and $\varkappa' \in \mathbf{T}_4(1,3)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_{\varkappa'}) = 2$ if and only if $\mathbf{x}_{\varkappa'}$ and $\mathbf{y}_\varkappa$ are mutually orthogonal with respect to $\mathcal{B}_4$.

(iv) If $\varkappa \in \mathbf{T}_4(1,1) \sqcup \mathbf{T}_4(1,3)$ and $\varkappa' \in \mathbf{T}_4(3,1)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_{\varkappa'}) = 2$ if and only if $\mathbf{x}_\varkappa$ and $\mathbf{y}_{\varkappa'}$ are mutually orthogonal with respect to $\mathcal{B}_4$.

To finish this section, let us briefly go back to the type sets $\mathbf{T}_4(k, k')$ with $k, k' \in \{0, 1, 2, 3\}$. We would like to determine, in terms of the relevant elements of $\mathcal{S}_4$, the dimension of the intersection of any two opposite-parity maximal totally null subspaces of $(\mathbb{C}^8, \mathbf{q}_8)$ which are different from the horizontal space $\mathcal{H}$ and the vertical space $\mathcal{V}$.

If $\eta \in \mathbf{T}_4(2,2)$, $\varkappa \in \mathbf{S}_8^-$ and $M \in \mathcal{S}_4$ is nonzero, then we have that

$$\varkappa \bullet \hat{\mathsf{s}}(M) = (\mathbf{x}_\varkappa + 2\check{\mathsf{E}}_{\hat{\mathsf{s}}(M)}(\mathbf{y}_\varkappa), M(\mathbf{x}_\varkappa) + 2\boldsymbol{\lambda}_4(M)\mathbf{y}_\varkappa),$$

$$\varkappa \bullet \check{\mathsf{s}}(M) = (\boldsymbol{\lambda}_4(M)\mathbf{x}_\varkappa + 2M(\mathbf{y}_\varkappa), \hat{\mathsf{E}}_{\check{\mathsf{s}}(M)}(\mathbf{x}_\varkappa) + 2\mathbf{y}_\varkappa),$$

$$\text{and } \varkappa \bullet \eta = \big(2\check{\mathsf{E}}_\eta(\mathbf{y}_\varkappa), \hat{\mathsf{E}}_\eta(\mathbf{x}_\varkappa)\big)$$

These equations and Proposition 2.13 imply the following

LEMMA 2.20. *Let $M \in \mathcal{S}_4$ be nonzero. Let $\eta \in \mathbf{T}_4(2,2)$ and let $\varkappa \in \mathbf{S}_8^-$ be a pure spinor for $\mathbf{q}_8$. Then, the following statements hold:*

(i) If $\varkappa \in \mathbf{T}_4(1,1)$ and $M \in \mathcal{S}_4^*$, then $dim(\boldsymbol{\Gamma}(M) \cap \mathcal{K}_\varkappa) = 3$ if and only if $M(\mathbf{x}_\varkappa) = -2\boldsymbol{\lambda}_4(M)\mathbf{y}_\varkappa$.

(ii) If $\varkappa \in \mathbf{T}_4(1,3) \sqcup \mathbf{T}_4(3,1)$ and $M \in \mathcal{S}_4^*$, then $dim(\boldsymbol{\Gamma}(M) \cap \mathcal{K}_\varkappa) = 1$.

(iii) If $\varkappa \in \mathbf{T}_4(1,1)$ and $M \in \mathcal{S}_4^0$, then $dim(\boldsymbol{\Gamma}(M) \cap \mathcal{K}_\varkappa) = 3$ if and only if $\mathbf{x}_\varkappa = -2\check{\mathsf{E}}_{\hat{\mathsf{s}}(M)}(\mathbf{y}_\varkappa)$.

(iv) If $\varkappa \in \mathbf{T}_4(1,3)$ and $M \in \mathcal{S}_4^*$, then $dim(\boldsymbol{\Gamma}(M) \cap \mathcal{K}_\varkappa) = 1$.

(v) If $\varkappa \in \mathbf{T}_4(3,1)$ and $M \in \mathcal{S}_4^0$, then $dim(\boldsymbol{\Gamma}(M) \cap \mathcal{K}_\varkappa) = 3$ if and only if $\mathbf{y}_\varkappa \in Ker(\check{\mathsf{E}}_{\hat{\mathsf{s}}(M)})$.

(vi) If $\varkappa \in \mathbf{T}_4(1,1)$ and $M \in \mathcal{S}_4^0$, then $dim(\mathcal{I}_8\big(\boldsymbol{\Gamma}(M)\big) \cap \mathcal{K}_\varkappa) = 3$ if and only if $\hat{\mathsf{E}}_{\check{\mathsf{s}}(M)}(\mathbf{x}_\varkappa) = -2\mathbf{y}_\varkappa$.

(vii) If $\varkappa \in \mathbf{T}_4(1,3)$ and $M \in \mathcal{S}_4^0$, then $dim(\mathcal{I}_8\big(\boldsymbol{\Gamma}(M)\big) \cap \mathcal{K}_\varkappa) = 3$ if and only if $\mathbf{x}_\varkappa \in Ker(\hat{\mathsf{E}}_{\check{\mathsf{s}}(M)})$.

(viii) If $\varkappa \in \mathbf{T}_4(3,1)$ and $M \in \mathcal{S}_4^0$, then $dim(\mathcal{I}_8\big(\boldsymbol{\Gamma}(M)\big) \cap \mathcal{K}_\varkappa) = 1$.

(ix) If $\varkappa \in \mathbf{T}_4(1,1)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_\eta) = 3$ if and only if $\check{\mathsf{E}}_\eta(\mathbf{y}_\varkappa) = \hat{\mathsf{E}}_\eta(\mathbf{x}_\varkappa)$ is the zero vector.

(x) If $\varkappa \in \mathbf{T}_4(1,3)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_\eta) = 3$ if and only if $\mathbf{x}_\varkappa \in Ker(\hat{\mathsf{E}}_\eta)$.

(xi) If $\varkappa \in \mathbf{T}_4(3,1)$, then $dim(\mathcal{K}_\varkappa \cap \mathcal{K}_\eta) = 3$ if and only if $\mathbf{y}_\varkappa \in Ker(\check{\mathsf{E}}_\eta)$.

3. The complex algebra of Hamilton matrices

In the standard base of $\mathcal{S}_4$, let us define $\mathbf{I} := -(\mathbf{M}_{12} + \mathbf{M}_{34})$, $\mathbf{J} := -(\mathbf{M}_{13} - \mathbf{M}_{24})$, and $\mathbf{K} := -(\mathbf{M}_{14} + \mathbf{M}_{23})$. That is to say,

$$\mathbf{I} := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{J} := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{K} := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We have that $\boldsymbol{\lambda}_4(\mathbf{I}) = \boldsymbol{\lambda}_4(\mathbf{J}) = \boldsymbol{\lambda}_4(\mathbf{K}) = 1$ and that $\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -\mathbf{Id}_4$. Therefore, the linear automorphisms of $\mathbb{C}^4$ defined by $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ induce (orthogonal) complex structures on $\mathbb{R}^4 \subset \mathbb{C}^4$.

Let us consider the complex linear space

$$\mathfrak{H} := \left\{ r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K} \in \mathcal{M}(4, \mathbb{C}) \,|\, r, a, b, c \in \mathbb{C} \right\}$$

generated by $\left\{ \mathbf{Id}_4, \mathbf{I}, \mathbf{J}, \mathbf{K} \right\}$. We have that

$*$	$\mathbf{Id}_4$	$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$
$\mathbf{Id}_4$	Id_4	I	J	K
$\mathbf{I}$	I	$-\mathrm{Id}_4$	K	-J
$\mathbf{J}$	J	-K	$-\mathrm{Id}_4$	I
$\mathbf{K}$	K	J	-I	$-\mathrm{Id}_4$

where $*$ denotes matrix multiplication and we multiply the corresponding element on the first column by the corresponding element on the first row.

Thus, $\mathfrak{H}$ is a 4-dimensional complex, noncommutative, associative algebra with unit $\mathbf{Id}_4$. Given that the multiplication table for the generators of $\mathfrak{H}$ is reminiscent of that for the generators $\left\{ 1, i, j, k \right\}$ of the Hamilton quaternion algebra $\mathbb{H}$, we shall call $\mathfrak{H}$ the complex algebra of Hamilton matrices.

For each $(r, a, b, c) \in \mathbb{C}^4$, we have that

$$\det(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) = (r^2 + a^2 + b^2 + c^2)^2$$

and it holds that

$$(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) * (r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) =$$
$$(r^2 - a^2 - b^2 - c^2)\mathbf{Id}_4 + 2r(a\mathbf{I} + b\mathbf{J} + c\mathbf{K})$$

The analogue in $\mathfrak{H}$ to quaternion conjugation is the $\mathbb{C}$-linear involution $\boldsymbol{\kappa}$: $\mathfrak{H} \longrightarrow \mathfrak{H}$ given by

$$r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K} \quad \longmapsto \quad r\mathbf{Id}_4 - a\mathbf{I} - b\mathbf{J} - c\mathbf{K}$$

Indeed, for all $(r, a, b, c), (r', a', b', c') \in \mathbb{C}^4$ it holds that

$$(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) * \boldsymbol{\kappa}(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) = (r^2 + a^2 + b^2 + c^2)\mathbf{Id}_4 \text{ and}$$

$$\boldsymbol{\kappa}\big((r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K}) * (r'\mathbf{Id}_4 + a'\mathbf{I} + b'\mathbf{J} + c'\mathbf{K})\big) =$$
$$\boldsymbol{\kappa}(r'\mathbf{Id}_4 + a'\mathbf{I} + b'\mathbf{J} + c'\mathbf{K}) * \boldsymbol{\kappa}(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K})$$

Thus, $r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K} \in \mathfrak{H}$ is an invertible matrix if and only if $r^2 + a^2 + b^2 + c^2 \neq 0$ and, in this case, the inverse matrix is given as

$$(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K})^{-1} = \frac{1}{r^2 + a^2 + b^2 + c^2}\boldsymbol{\kappa}(r\mathbf{Id}_4 + a\mathbf{I} + b\mathbf{J} + c\mathbf{K})$$

We are mainly interested on discussing the set $\mathfrak{H} \cap \mathcal{S}_4$ of all skew-symmetric Hamilton matrices. We would like to remark that this interest springs from the fact that the skew-symmetric Hamilton matrices were, precisely, the first ones we studied and this study allowed us to discern the relationship between skew-symmetric matrices and twistor spaces in higher dimensions.

Let us define $\mathbf{H} : \mathbb{C}^3 \longrightarrow \mathcal{S}_4$ by

$$(a, b, c) \longmapsto \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -c & b \\ b & c & 0 & -a \\ c & -b & a & 0 \end{pmatrix}$$

Then, $\mathfrak{H} \cap \mathcal{S}_4 = \mathbf{H}(\mathbb{C}^3)$ and we shall say that the matrix $\mathbf{H}(a, b, c) \in \mathbf{H}(\mathbb{C}^3)$ is the Hamilton matrix for the point $(a, b, c) \in \mathbb{C}^3$.

For each $(a, b, c) \in \mathbb{C}^3$, it holds that $\boldsymbol{\lambda}_4\big(\mathbf{H}(a, b, c)\big) = a^2 + b^2 + c^2$, therefore,

$$\boldsymbol{\lambda}_4\big(\mathbf{H}(a,b,c)\big) = \mathcal{F}_3(a,b,c)$$

where $\mathcal{F}_3 : \mathbb{C}^3 \longrightarrow \mathbb{C}$ is the Fermat polynomial defined in §2.1.

For each $(a,b,c) \in \mathbb{C}^3$, we also have that

$$\mathbf{H}(a,b,c)\mathbf{H}(a,b,c) = -\boldsymbol{\lambda}_4\big(\mathbf{H}(a,b,c)\big)\mathbf{Id}_4 = -\mathcal{F}_3(a,b,c)\mathbf{Id}_4$$

Hence, $\mathbf{H}(a,b,c) \in \mathcal{S}_4^0$ if and only if $\mathbf{H}(a,b,c)$ is a step-2 nilpotent matrix in $\mathcal{S}_4$. That is to say, $\mathbf{H}(a,b,c) \in \mathcal{S}_4^0$ if and only if $\mathrm{Im}\big(\mathbf{H}(a,b,c)\big) = \mathrm{Ker}\big(\mathbf{H}(a,b,c)\big)$.

For all $(a,b,c), (a',b',c') \in \mathbb{C}^3$ it holds that

$$\begin{aligned} &\mathbf{H}(a,b,c)\mathbf{H}(a',b',c') \in \mathbf{H}(\mathbb{C}^3) \quad \text{ if and only if} \\ &0 = \mho\big(\mathbf{H}(a,b,c), \mathbf{H}(a',b',c')\big) = aa' + bb' + cc' \end{aligned}$$

where $\mho : \mathcal{S}_4 \times \mathcal{S}_4 \longrightarrow \mathbb{C}$ is the bilinear form associated to the Fermat polynomial $\mathcal{F}_6 : \mathcal{S}_4 \simeq \mathbb{C}^6 \longrightarrow \mathbb{C}$. In this case, we have that

$$\mathbf{H}(a,b,c)\mathbf{H}(a',b',c') = \mathbf{H}(bc' - cb', ca' - ac', ab' - ba')$$

Now, we would like to describe the Zariski closure

$$\overline{\mathbf{\Gamma}}_{\mathbf{H}} \subset \mathbf{G}(4,8)$$

of the set

$$\mathbf{\Gamma}_{\mathbf{H}} := \big\{\, \mathbf{\Gamma}\big(\mathbf{H}(a,b,c)\big) \in \mathbf{\Gamma}_4 \mid (a,b,c) \in \mathbb{C}^3 \,\big\}$$

of all horizontal graphs of linear endomorphisms of $\mathbb{C}^4$ defined by skew-symmetric Hamilton matrices. Let us set

$$\mathbf{\Gamma}_{\mathbf{H}}^0 := \big\{\, \mathbf{\Gamma}\big(\mathbf{H}(a,b,c)\big) \in \mathbf{\Gamma}_{\mathbf{H}} \mid \mathcal{F}_3(a,b,c) = 0 \,\big\}$$

Then (cf. §2.2.1), we have that

$$\overline{\mathbf{\Gamma}}_{\mathbf{H}} = \mathbf{\Gamma}_{\mathbf{H}} \sqcup \mathcal{I}_8(\mathbf{\Gamma}^0_{\mathbf{H}}) \sqcup \left\{ \text{limits in } \mathbf{T}_4(2,2) \text{ of the elements in } \mathbf{\Gamma}^0_{\mathbf{H}} \right\}$$

We have that $\mathbf{\Gamma}_{\mathbf{H}} \simeq \mathbb{C}^3$. Since $\mathcal{I}_8(\mathbf{\Gamma}^0_{\mathbf{H}})$ is isomorphic to the complex affine quadric hypersurface $Q_2^{\mathrm{aff}} \subset \mathbb{C}^3$ defined by $\mathcal{F}_3$, we get (cf. Proposition 2.7) that the set of limits in

$$\mathbf{T}_4(2,2) \simeq \mathcal{Q}_4 \subset \mathbb{P}^5_{\mathbb{C}}$$

of the elements in $\mathcal{I}_8(\mathbf{\Gamma}^0_{\mathbf{H}})$ is isomorphic to the non-singular conic $Q_1 \subset \mathbb{P}^2_{\mathbb{C}}$ defined by $\mathcal{F}_3$.

Therefore, $\overline{\mathbf{\Gamma}}_{\mathbf{H}}$ is obtained by adding to the complex affine 3-dimensional space $\mathbb{A}^3_{\mathbb{C}}$ a complex quadric hypersurface, which is compact with an isolated singularity, in $\mathbb{P}^3_{\mathbb{C}}$. Thus, we have proven

PROPOSITION 2.21. *The Zariski closure*

$$\overline{\mathbf{\Gamma}}_{\mathbf{H}} \subset \mathbf{G}(4,8)$$

of the set $\mathbf{\Gamma}_{\mathbf{H}}$ *of all horizontal graphs of linear endomorphisms of* $\mathbb{C}^4$ *defined by skew-symmetric Hamilton matrices is biholomorphic to a non-singular complex quadric hypersurface in* $\mathbb{P}^4_{\mathbb{C}}$.

4. The twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$ revisited

Let us begin by stating

DEFINITION 2.22. *Let* $(\mathcal{M}, g)$ *be a complete Riemannian manifold and let* $\mathcal{F}$ *be a foliation of* $(\mathcal{M}, g)$. *We say that* $\mathcal{F}$ *is* Riemannian *if any two leaves of* $\mathcal{F}$ *are locally equidistant.*

Equivalently, $\mathcal{F}$ *is a Riemannian foliation if and only if every geodesic of* $(\mathcal{M}, g)$ *orthogonal to one leaf, is orthogonal to every leaf it meets (cf.* [**18**]*).*

Let us consider the non-singular complex quadric hypersurfaces

$$\mathbf{Q}_{6} \subset \mathbb{P}^7_{\mathbb{C}} \quad \text{and} \quad \mathbf{Q}^+_{6} \subset \mathbb{P}(\mathbf{S}^+_{8}) \simeq \mathbb{P}^7_{\mathbb{C}}$$

respectively defined by the (maximal rank 8 and maximal index 4) quadratic forms $\mathbf{q}_{8} : \mathbb{C}^8 \longrightarrow \mathbb{C}$ and $\boldsymbol{\gamma}^+ : \mathbf{S}^+_{8} \longrightarrow \mathbb{C}$ (cf. §2.2). We endow $\mathbf{Q}_{6} \simeq \mathbf{Q}^+_{6}$ with the Fubini-Study metric, which we shall denote by g_{F-S}.

In this section, inspired by §3 and §5 of [**45**], we will explicitly construct a real-analytic fibration

$$\mathbf{\Phi} : \mathbf{Q}_{6} \longrightarrow \mathbf{\Sigma}^6$$

with fibres isomorphic to $\mathbb{P}^3_{\mathbb{C}}$, where $\mathbf{\Sigma}^6 \subset \mathbf{Q}^+_{6}$ is a conformal 6-sphere. The fibration $\mathbf{\Phi}$ will turn out to be isometrically equivalent to the twistor fibration $\mathbf{p}_{6} : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$ of the standard 6-sphere. In this way, we will produce a Riemannian foliation of $(\mathbf{Q}_{6}, g_{F-S})$.

Let us consider the $\mathbb{R}$-linear transformation $\mathbf{A} : \mathbb{C}^3 \longrightarrow \mathcal{S}_{4}$ given by

$$(a,b,c) \longmapsto \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -\overline{c} & \overline{b} \\ b & \overline{c} & 0 & -\overline{a} \\ c & -\overline{b} & \overline{a} & 0 \end{pmatrix}$$

REMARK 2.23. *If $(a,b,c) \in \mathbb{C}^3$, then $\mathbf{A}(a,b,c) \in \mathcal{S}_{4}$ coincides with the Hamilton matrix $\mathbf{H}(a,b,c)$ for the point (a,b,c) if and only if $(a,b,c) \in \mathbb{R}^3$. This is the motivation for defining the transformation $\mathbf{A}$ as we have done.*

For all $(a,b,c) \in \mathbb{C}^3$ it holds that

$$\boldsymbol{\lambda}_{4}(a,b,c) := \boldsymbol{\lambda}_{4}\big(\mathbf{A}(a,b,c)\big) = |a|^2 + |b|^2 + |c|^2 \in \mathbb{R}$$

Hence, $\mathbf{A}(a,b,c) \in \mathcal{S}^*_{4}$ if and only if $(a,b,c) \neq (0,0,0)$. In this case, the inverse matrix is given as

$$\mathbf{A}^{-1}(a,b,c) = \frac{-1}{|a|^2 + |b|^2 + |c|^2}\mathbf{A}(\overline{a}, \overline{b}, \overline{c})$$

For each $(a, b, c) \in \mathbb{C}^3$, we shall denote the graph $\mathbf{\Gamma}\big(\mathbf{A}(a, b, c)\big)$ of the linear endomorphism $\mathbf{A}(a, b, c) : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$ by $\mathbf{\Gamma}(a, b, c)$. By Lemma 2.9, we have that

$$\hat{\mathsf{s}}(a, b, c) := \hat{\mathsf{s}}\big(\mathbf{A}(a, b, c)\big) = \big(2(|a|^2 + |b|^2 + |c|^2), \overline{a}, \overline{b}, \overline{c}, c, -b, a, \frac{1}{2}\big) \text{ and}$$

$$\check{\mathsf{s}}(a, b, c) := \check{\mathsf{s}}\big(\mathbf{A}(a, b, c)\big) = \big(2, -a, -b, -c, -\overline{c}, \overline{b}, -\overline{a}, \frac{|a|^2 + |b|^2 + |c|^2}{2}\big)$$

are even pure spinors for $\mathbf{q}_8$ which represent, respectively, the graph $\mathbf{\Gamma}(a, b, c)$ and its image $\mathcal{I}_8\big(\mathbf{\Gamma}(a, b, c)\big)$ under the canonical involution $\mathcal{I}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}^8$.

If $(a, b, c) \neq (0, 0, 0)$, then we have that $\mathcal{I}_8\big(\mathbf{\Gamma}(a, b, c)\big) = \mathbf{\Gamma}\big(\mathbf{A}^{-1}(a, b, c)\big)$ and, hence,

$$\check{\mathsf{s}}(a, b, c) = \hat{\mathsf{s}}\big(\mathbf{A}^{-1}(a, b, c)\big)$$

Let $(a, b, c), (a', b', c') \in \mathbb{C}^3$ be distinct points. Since $a - a', b - b', c - c' \neq 0$, we have that $\mathbf{A}(a - a', b - b', c - c') \in \mathcal{S}_4^*$. Therefore,

$$dim\big(\mathbf{\Gamma}(a, b, c) \cap \mathbf{\Gamma}(a', b', c')\big) = 0$$

Let us consider a sequence $\big\{ (a_n, b_n, c_n) \in \mathbb{C}^3 \mid n \in \mathbb{N} \big\}$ such that $\lim\limits_{n \longrightarrow \infty} \boldsymbol{\lambda}_4(a_n, b_n, c_n) = \infty$. Then, by Remark 2.10, we have that

$$\lim_{n \longrightarrow \infty} \mathbf{\Gamma}(a_n, b_n, c_n) = \mathcal{V}$$

where $\mathcal{V} \subset \mathbb{C}^8$ is the vertical space.

Since $\mathbb{C}^3 \cup \big\{ \infty \big\}$ is diffeomorphic to the standard 6-sphere $\mathbb{S}^6$, we get that the transformation $\mathbf{\Gamma} \circ \mathbf{A} : \mathbb{C}^3 \cup \big\{ \infty \big\} \longrightarrow \mathbf{G}(4, 8)$ given by

$$(a, b, c) \longmapsto \mathbf{\Gamma}(a, b, c) \text{ for } (a, b, c) \in \mathbb{C}^3 \quad \text{and} \quad \infty \longmapsto \mathcal{V}$$

defines a diffeomorphism between $\mathbb{S}^6 \simeq \mathbb{C}^3 \cup \big\{ \infty \big\}$ and its image under $\mathbf{\Gamma} \circ \mathbf{A}$.

Therefore, we have that $\mathbb{S}^6$ is a geometric realisation of each of the following sets:

(1) The family of graphs

$$\mathbf{G}(4,8) \supset \left\{ \mathbf{\Gamma}(a,b,c) \mid (a,b,c) \in \mathbb{C}^3 \right\} \sqcup \mathcal{V}$$

Every member of this family is an even maximal totally null subspace of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$, and the dimension of the intersection of any two distinct members of this family is zero.

(2) The family of projectivised graphs

$$\mathbb{G}(3,7) \supset \left\{ \mathbb{P}\big(\mathbf{\Gamma}(a,b,c)\big) \mid (a,b,c) \in \mathbb{C}^3 \right\} \sqcup \mathbb{P}(\mathcal{V})$$

Every member of this family is a linear 3-fold contained in $\mathbf{Q}_6$, and any two distinct members of this family are disjoint. **The members of this family will be the leaves of the foliation we are constructing.**

(3) The family of representative spinors

$$\mathbf{Q}_6^+ \supset \left\{ [\hat{\mathsf{s}}(a,b,c)] \mid (a,b,c) \in \mathbb{C}^3 \right\} \sqcup \left\{ [\mathbf{1}] \right\}$$

The members of this family are pairwise distinct even pure spinors for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$. **Certain members of this family will be the images of the points in the quadric hypersurface $\mathbf{Q}_6 \subset \mathbb{P}^7_{\mathbb{C}}$ under the fibration $\mathbf{\Phi}$.**

REMARK 2.24. *Considering parametrisations* (1) *and* (2) *above (or through direct calculation), it can be proven that if* $(x,y) \in \mathbb{C}^8$ *is such that* $x \in \mathcal{H}$ *is nonzero and* $\mathbf{q}_8(x,y) = 0$*, then there exists a* **unique** *point*

$$\big(a(x,y), b(x,y), c(x,y)\big) \in \mathbb{C}^3$$

such that

$$(x,y) \in \mathbf{\Gamma}\big(a(x,y), b(x,y), c(x,y)\big) \subset \mathbb{C}^8$$

Equivalently, $[x : y] \in \mathbb{P}\big(\mathbf{\Gamma}\big(a(x,y), b(x,y), c(x,y)\big)\big) \subset \mathbf{Q}_6$.

REMARK 2.25. *Considering parametrisation* (3) *above, let us define* $\phi_{\mathcal{H}}, \phi_{\mathcal{V}} : \mathbb{C}^3 \cup \{\infty\} \longrightarrow \mathbf{Q}_6^+$ *by*

$$\phi_{\mathcal{H}}(a,b,c) = [\hat{\mathsf{s}}(a,b,c)] \text{ for all } (a,b,c) \in \mathbb{C}^3, \text{ and } \phi_{\mathcal{H}}(\infty) = [\mathbf{1}];$$

$$\phi_{\mathcal{V}}(a,b,c) = [\check{\mathsf{s}}(a,b,c)] \text{ for all } (a,b,c) \in \mathbb{C}^3, \text{ and } \phi_{\mathcal{V}}(\infty) = [\boldsymbol{\omega}]$$

Then, we have that $\phi_{\mathcal{H}}$ *and* $\phi_{\mathcal{V}}$ *are real-analytic embbedings. Given that*

$$\phi_{\mathcal{H}}(0,0,0) = [\boldsymbol{\omega}] = \phi_{\mathcal{V}}(\infty) \quad \text{ and } \quad \phi_{\mathcal{V}}(0,0,0) = [\mathbf{1}] = \phi_{\mathcal{H}}(\infty),$$

we get that the images of $\mathbb{C}^3 - \{(0,0,0)\}$ *under* $\phi_{\mathcal{H}}$ *and* $\phi_{\mathcal{V}}$ *coincide. In particular, it holds that* $\phi_{\mathcal{H}}$ *and* $\phi_{\mathcal{V}}$ *are homeomorphisms from* $\mathbb{C}^3 \cup \{\infty\} \simeq \mathbb{S}^6$ *into its respective images.*

Next, let us describe the **base space of the fibration** $\mathbf{\Phi}$, by considering the conjugate $\mathbb{C}$-linear involution $\mathfrak{f} : \mathbf{S}_8^+ \longrightarrow \mathbf{S}_8^+$ given by

$$\varsigma = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \longmapsto (\overline{a_1}, \overline{a_{11}}, -\overline{a_{10}}, \overline{a_9}, \overline{a_8}, -\overline{a_7}, \overline{a_6}, \overline{a_{16}})$$

For each $\varsigma \in \mathbf{S}_8^+$, we have that

$$\boldsymbol{\gamma}^+\big(\mathfrak{f}(\varsigma)\big) = \overline{\boldsymbol{\gamma}^+(\varsigma)}$$

and, hence, $\mathfrak{f}$ transforms even pure spinors into even pure spinors (cf. Proposition 2.1). Let us consider the bilinear form $\boldsymbol{\beta}^+ : \mathbf{S}_8^+ \times \mathbf{S}_8^+ \longrightarrow \mathbb{C}$ associated to $\boldsymbol{\gamma}^+$. Given that for all $\varsigma, \varsigma' \in \mathbf{S}_8^+$ it holds that

$$\boldsymbol{\beta}^+\big(\mathfrak{f}(\varsigma), \mathfrak{f}(\varsigma')\big) = \overline{\boldsymbol{\beta}^+(\varsigma, \varsigma')},$$

we have that $\mathfrak{f}$ is a real form of the quadratic space $(\mathbf{S}_8^+, \boldsymbol{\gamma}^+)$. Moreover, the fixed-point set

$$Fix(\mathfrak{f}) := \left\{\, \varsigma \in \mathbf{S}_8^+ \mid \mathfrak{f}(\varsigma) = \varsigma \,\right\}$$

is a *real* vector space isomorphic to $\mathbb{R}^8$, and the restriction of $\boldsymbol{\beta}^+$ to $Fix(\mathfrak{f})$ has signature (or is of *Lorentz type*) $(1,7)$.

Let us set

$$\mathbf{\Sigma}^6 := \left\{\, [\varsigma] \in \mathbf{Q}_6^+ \mid \varsigma \in Fix(\mathfrak{f}) \simeq \mathbb{R}^8 \,\right\}$$

It is immediate that

$$\mathbf{T}_4(0,4) \sqcup \mathbf{T}_4(4,0) = \left\{\, [\mathbf{1}], [\boldsymbol{\omega}] \,\right\} \subsetneq \mathbf{\Sigma}^6$$

By Proposition 2.11, we get that

$$\big(\mathbf{T}_4(0,2) \sqcup \mathbf{T}_4(2,0) \sqcup \mathbf{T}_4(2,2)\big) \cap \mathbf{\Sigma}^6 = \varnothing \text{ and}$$

$$\mathbf{T}_4(0,0) \cap \mathbf{\Sigma}^6 = \left\{\, [\hat{\mathfrak{s}}(a,b,c)] \in \mathbf{Q}_6^+ \mid (a,b,c) \in \mathbb{C}^3 \text{ is nonzero} \,\right\}$$

By remarks 2.24 and 2.25, we get a diffeomorphism

$$\mathbb{S}^6 \simeq \mathbb{C}^3 \cup \left\{\, \infty \,\right\} \simeq \mathbf{\Sigma}^6,$$

and also that the embeddings $\phi_{\mathcal{H}}$ and $\phi_{\mathcal{V}}$ are *real-analytic* coordinate charts of $\mathbf{\Sigma}^6$.

Since the real form $\mathfrak{f}$ has signature $(1,7)$, we have that the group formed by all those isometries of $(\mathbf{Q}_6^+, g_{F-S})$ which preserve the 6-dimensional sphere $\mathbf{\Sigma}^6$ is isomorphic to the real spin group $Spin(7,\mathbb{R})$. Hence, $\mathbf{\Sigma}^6$ is of constant Gaussian curvature.

With the above notation, let us define the fibration $\mathbf{\Phi} : \mathbf{Q}_6 \longrightarrow \mathbf{\Sigma}^6$ by

$$\mathbf{\Phi}([x:y]) = \begin{cases} [\mathbf{1}], & \text{if } x \in \mathcal{H} \text{ is zero;} \\ [\hat{\mathsf{s}}(a(x,y), b(x,y), c(x,y))], & \text{if } x \in \mathcal{H} \text{ is nonzero} \end{cases}$$

Let $\mathcal{F}$ be the foliation of $(\mathbf{Q}_6, g_{F-S})$ the leaves of which are the linear 3-folds

$$\left\{ \mathbb{P}\big(\mathbf{\Gamma}(a,b,c)\big) \mid (a,b,c) \in \mathbb{C}^3 \right\} \cup \mathbb{P}(\mathcal{V})$$

(cf. Remark 2.24). Moreover, since there exists (cf. [**22**], [**7**], [**36**]) an irreducible group representation

$$\varrho : Spin(7,\mathbb{R}) \longrightarrow Aut(\mathbb{R}^8)$$

of the real spin group $Spin(7,\mathbb{R})$ (where $Aut(\mathbb{R}^8)$ denotes the group of linear automorphisms of $\mathbb{R}^8$), we have that the representation ϱ acts on the Grassmannian manifold $\mathbf{G}^+(2,\mathbb{R}^8)$ of oriented 2-planes in $\mathbb{R}^8$. By §2.1, we know that

$$\mathbf{G}^+(2,\mathbb{R}^8) \simeq \mathfrak{Z}(\mathbb{S}^6)$$

Thus, ϱ acts on the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ of the standard 6-sphere. This action is transitive with isotropy group isomorphic to the unitary group $U(3)$. Therefore,

$$\mathfrak{Z}(\mathbb{S}^6) \simeq Spin(7,\mathbb{R})/U(3)$$

Since the non-singular complex quadric hypersurface $\mathbf{Q}_6 \subset \mathbb{P}^7_{\mathbb{C}}$ is biholomorphic to $\mathfrak{Z}(\mathbb{S}^6)$, we have that $Spin(7,\mathbb{R})$ acts on $\mathbf{Q}_6$, and this action is transitive on the set of leaves of the foliation $\mathcal{F}$. The isotropy group of a given leaf is isomorphic to the special unitary group $SU(4)$. Hence, the transformation

$$\mathfrak{Z}(\mathbb{S}^6) \simeq Spin(7,\mathbb{R})/U(3) \quad \longrightarrow \quad Spin(7,\mathbb{R})/SU(4) \simeq \mathbb{S}^6$$

coincides with the twistor fibration $\mathbf{p}_6$ of the standard 6-sphere.

With this notation, we summarise the above discussion in the form of

THEOREM 2.26. *Let $\mathbf{Q}_6 \subset \mathbb{P}^7_{\mathbb{C}}$ be the non-singular complex quadric hypersurface defined by the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$. We endow $\mathbf{Q}_6$ with the Fubini-Study metric g_{F-S}. Let $\mathcal{F}$ be the foliation of $\mathbf{Q}_6$ the leaves of which are the linear 3-folds*

$$\mathbb{G}(3,7) \supset \left\{ \mathbb{P}\big(\mathbf{\Gamma}(a,b,c)\big) \mid (a,b,c) \in \mathbb{C}^3 \right\} \cup \mathbb{P}(\mathcal{V})$$

Then, the following statements hold:

(i) The foliation $\mathcal{F}$ is real-analytic.

*(ii) The foliation $\mathcal{F}$ is invariant under the action of the real spin group $Spin(7,\mathbb{R})$ on $\mathbf{Q}_6$. This action is by isometries of $(\mathbf{Q}_6, g_{F-S})$ and, furthermore, it is transitive on the space of leaves of $\mathcal{F}$ (cf. [**22**], [**7**], [**36**]).*

(iii) The space of leaves of $\mathcal{F}$ may be identified with the 6-dimensional sphere

$$\mathbf{\Sigma}^6 := \left\{ [\varsigma] \in \mathbf{Q}_6^+ \mid \varsigma \in Fix(\mathfrak{f}) \simeq \mathbb{R}^8 \right\} =$$

$$\left\{ [\mathbf{1}], [\boldsymbol{\omega}] \right\} \sqcup \left\{ [\hat{\mathfrak{s}}(a,b,c)] \mid (a,b,c) \in \mathbb{C}^3 \text{ is nonzero} \right\}$$

Therefore, we may endow $\mathbf{\Sigma}^6$ with the standard metric (of constant positive sectional curvature) in such a way that the canonical projection (that is to say, the twistor fibration)

$$\mathbf{p}_6 : \mathbf{Q}_6 \simeq \mathfrak{Z}(\mathbb{S}^6) \quad \longrightarrow \quad \mathbb{S}^6 \simeq \mathbf{\Sigma}^6$$

is a Riemannian submersion with fibre isomorphic to a linear 3-fold.

(iv) The fibration

$$\mathbf{\Phi} : \mathbf{Q}_6 \longrightarrow \mathbf{\Sigma}^6$$

is Riemannian with respect to the Fubini-Study metric g_{F-S} and, up to scaling, it is isometrically equivalent to the twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$.

4.1. The set of "non-fibres" of the fibration $\mathbf{\Phi} : \mathbf{Q}_6 \longrightarrow \mathbf{\Sigma}^6$.

Let us recall that the conjugate $\mathbb{C}$-linear involution $\mathfrak{f} : \mathbf{S}_8^+ \longrightarrow \mathbf{S}_8^+$ is given by

$$(a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \longmapsto (\overline{a_1}, \overline{a_{11}}, -\overline{a_{10}}, \overline{a_9}, \overline{a_8}, -\overline{a_7}, \overline{a_6}, \overline{a_{16}})$$

Let $s = (a_1, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{16}) \in \mathbf{S}_8^+$ be a pure spinor for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$ such that $s \neq \mathfrak{f}(s)$. Therefore,

$$s \in \mathbf{T}_4(0,2) \sqcup \mathbf{T}_4(2,0) \sqcup \mathbf{T}_4(2,2) \quad \text{ or}$$

$$s \in \mathbf{T}_4(0,0) - \left\{ \hat{\mathsf{s}}\big(\mathbf{\Gamma}(a,b,c)\big) \mid (a,b,c) \in \mathbb{C}^3 \right\}$$

We know that $\mathfrak{f}(s) \in \mathbf{S}_8^+$ is also a pure spinor for $\mathbf{q}_8$. Let $\mathcal{K}_s$ and $\mathcal{K}_{\mathfrak{f}(s)}$ be the (even) maximal totally null subspaces of $(\mathbb{C}^8, \mathbf{q}_8)$ represented by s and $\mathfrak{f}(s)$, respectively. Given that

$$[s], [\mathfrak{f}(s)] \in \mathbf{Q}_6^+ - \mathbf{\Sigma}^6,$$

we have that the linear 3-folds

$$\mathbb{P}(\mathcal{K}_s), \mathbb{P}(\mathcal{K}_{\mathfrak{f}(s)}) \subset \mathbf{Q}_6$$

are *not* leaves of the foliation $\mathcal{F}$ described in Theorem 2.26. Equivalently, $\mathbb{P}(\mathcal{K}_s)$ and $\mathbb{P}(\mathcal{K}_{\mathfrak{f}(s)})$ are *not* fibres of the fibration $\mathbf{\Phi} : \mathbf{Q}_6 \longrightarrow \mathbf{\Sigma}^6$.

Since $\mathcal{K}_s$ and $\mathcal{K}_{\mathfrak{f}(s)}$ are *distinct* even maximal totally null subspaces of $(\mathbb{C}^8, \mathbf{q}_8)$, Lemma 1.6 implies that

$$dim(\mathcal{K}_s \cap \mathcal{K}_{\mathfrak{f}(s)}) \in \{ 0, 2 \}$$

By Proposition 2.11, we get that:

► If $s \in \mathbf{T}_4(0,2) \sqcup \mathbf{T}_4(2,0) \sqcup \mathbf{T}_4(2,2)$, then

$$\boldsymbol{\beta}^+(\mathfrak{f}(s), s) = -\sum_{j=6}^{11} |a_j|^2$$

▶ If $s \in \mathbf{T}_4(0,0) - \{\, \hat{\mathsf{s}}\big(\boldsymbol{\Gamma}(a,b,c)\big) \mid (a,b,c) \in \mathbb{C}^3 \,\}$, then there exists a unique

$$M = (m_{12}, m_{13}, m_{14}, m_{23}, m_{24}, m_{34}) \in \mathcal{S}_4^*$$

such that $s = \hat{\mathsf{s}}(M)$. Hence,

$$\boldsymbol{\beta}^+(\mathfrak{f}(\varsigma), \varsigma) = -\big(|m_{12} - \overline{m_{34}}|^2 + |m_{13} + \overline{m_{24}}|^2 + |m_{14} - \overline{m_{23}}|^2\big)$$

In both cases we have that

$$\boldsymbol{\beta}^+(\mathfrak{f}(\varsigma), \varsigma) \neq 0$$

which, by Proposition 2.5, implies that $dim(\mathcal{K}_s \cap \mathcal{K}_{\mathfrak{f}(s)}) = 0$. Thus,

$$\mathbb{P}(\mathcal{K}_s) \cap \mathbb{P}(\mathcal{K}_{\mathfrak{f}(s)}) = \varnothing$$

and we have proven

PROPOSITION 2.27. *Let $s \in \mathbf{S}_8^+$ be a pure spinor for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$ such that $s \neq \mathfrak{f}(s)$. Let $\mathcal{K}_s$ and $\mathcal{K}_{\mathfrak{f}(s)}$ be the even maximal totally null subspaces of the quadratic space $(\mathbb{C}^8, \mathbf{q}_8)$ which are represented by s and $\mathfrak{f}(s)$, respectively. Then, the linear 3-folds*

$$\mathbb{P}(\mathcal{K}_s), \mathbb{P}(\mathcal{K}_{\mathfrak{f}(s)}) \subset \mathbf{Q}_6$$

are not fibres of the fibration $\boldsymbol{\Phi} : \mathbf{Q}_6 \longrightarrow \boldsymbol{\Sigma}^6$, and they are disjoint.

4.2. Odd pure spinors for $\mathbf{q}_8$ and the foliation $\mathcal{F}$.

Let us consider an odd pure spinor

$$\varkappa = (b_2, b_3, b_4, b_5, b_{12}, b_{13}, b_{14}, b_{15}) \in \mathbf{S}_8^-$$

for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$. Let $\mathcal{K}_\varkappa \subset \mathbb{C}^8$ be the maximal totally null subspace represented by $\varkappa$. Inspired by §5 of [**45**], we would like to show how to assign a *canonical* linear 2-fold

$$\Pi_\varkappa \simeq \mathbb{P}^2_{\mathbb{C}}$$

to the linear 3-fold $\mathbb{P}(\mathcal{K}_\varkappa) \subset \mathbf{Q}_6$. We remark that this linear 3-fold is not a leaf of the foliation $\mathcal{F}$ of $(\mathbf{Q}_6, g_{F-S})$ described in Theorem 2.26.

► If $\varkappa \in \mathbf{T}_4(1,3)$, then $\mathcal{K}_\varkappa \cap \mathcal{V} \simeq \mathbb{C}^3$. By propositions 2.13 and 2.14, we know that

$$\mathbf{x}_\varkappa = (b_2, b_3, b_4, b_5) \in \mathbb{C}^4$$

is nonzero and it is a generator of $\mathcal{K}_\varkappa \cap \mathcal{H}$. Using the spin representation $\boldsymbol{\rho} : \mathcal{C}\ell_8 \longrightarrow End(\mathbf{S}_8)$ for the Clifford algebra $\mathcal{C}\ell_8$ (cf. §2.2) and Theorem 1.5, we get that

$$\Pi_\varkappa := \mathbb{P}(\mathcal{K}_\varkappa \cap \mathcal{V}) =$$
$$\Big\{ [-ab_3 - bb_4 - cb_5 : ab_2 : bb_2 : cb_2] \in \mathbb{P}^3_{\mathbb{C}} \mid (a,b,c) \in \mathbb{C}^3 \text{ is nonzero} \Big\}$$

is a linear 2-fold contained in $\mathbb{P}(\mathcal{K}_\varkappa) \simeq \mathbb{P}^3_{\mathbb{C}}$.

► If $\varkappa \in \mathbf{T}_4(3,1)$, then $\mathcal{K}_\varkappa \cap \mathcal{H} \simeq \mathbb{C}^3$ and

$$\mathbf{y}_\varkappa = (b_{15}, -b_{14}, b_{13}, -b_{12}) \in \mathbb{C}^4$$

is a generator of $\mathcal{K}_\varkappa \cap \mathcal{V}$ (by propositions 2.13 and 2.14). In this case, a calculation using the spin representation for $\mathcal{C}\ell_8$ and Theorem 1.5 give us that

$$\Pi_\varkappa := \mathbb{P}(\mathcal{K}_\varkappa \cap \mathcal{H}) =$$
$$\Big\{ [ab_{14} - bb_{13} + cb_{12} : ab_{15} : bb_{15} : cb_{15}] \in \mathbb{P}^3_{\mathbb{C}} \mid (a,b,c) \in \mathbb{C}^3 \text{ is nonzero} \Big\}$$

is a linear 2-fold contained in $\mathbb{P}(\mathcal{K}_\varkappa) \simeq \mathbb{P}^3_{\mathbb{C}}$.

▶ If $\varkappa \in \mathbf{T}_4(1,1)$, let us assume that for each nonzero $(a,b,c) \in \mathbb{C}^3$ it holds that

$$dim(\mathbf{\Gamma}(a,b,c) \cap \mathcal{K}_\varkappa) = 1$$

where $\mathbf{\Gamma}(a,b,c) := \mathbf{\Gamma}\big(\mathbf{A}(a,b,c)\big)$. Since $\mathcal{H} = \mathbf{\Gamma}(0,0,0)$,

$$\infty \quad \overset{\mathbf{\Gamma}\circ\mathbf{A}}{\longmapsto} \quad \mathcal{V} \quad \text{and}$$

$$dim(\mathcal{K}_\varkappa \cap \mathcal{H}) = dim(\mathcal{K}_\varkappa \cap \mathcal{V}) = 1,$$

we conclude that the diffeomorphism

$$(\mathbf{\Gamma} \circ \mathbf{A})\big(\mathbb{C}^3 \cup \{\,\infty\,\}\big) \simeq \mathbb{S}^6$$

establishes a *homeomorphism* between $\mathbb{S}^6$ and $\mathbb{P}(\mathcal{K}_\varkappa) \simeq \mathbb{P}^3_{\mathbb{C}}$, which is a contradiction.

Hence, by Lemma 1.6, there exists a nonzero $(a,b,c) \in \mathbb{C}^3$ such that

$$dim(\mathbf{\Gamma}(a,b,c) \cap \mathcal{K}_\varkappa) = 3$$

Let us assume that there exists *another* nonzero $(a',b',c') \in \mathbb{C}^3$ such that

$$dim(\mathbf{\Gamma}(a',b',c') \cap \mathcal{K}_\varkappa) = 3$$

Therefore,

$$dim\big(\mathbf{\Gamma}(a,b,c) \cap \mathbf{\Gamma}(a',b',c')\big) \geq 2$$

which is also a contradiction. Thus, this nonzero $(a, b, c) \in \mathbb{C}^3$ is *unique.*

Let $(x, y) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{C}^8$ be any point. Using the spin representation for $\mathcal{C}\ell_8$ and Theorem 1.5, we get that (x, y) belongs to $\mathbf{\Gamma}(a, b, c) \cap \mathcal{K}_\varkappa$ if and only if $y = \mathbf{A}(a, b, c)(x)$ and

$$(ab_3+bb_4+cb_5)x_1-(ab_2-\overline{c}b_4+\overline{b}b_5)x_2-(bb_2+\overline{c}b_3-\overline{a}b_5)x_3-(cb_2-\overline{b}b_3+\overline{a}b_4)x_4 = 0$$

All the coefficients in this last equation are zero if and only if $\mathbf{x}_\varkappa = (b_2, b_3, b_4, b_5) \in \mathbb{C}^4$ is the origin. Given that $\varkappa \in \mathbf{T}_4(1, 1)$, Proposition 2.14 implies that $\mathbf{x}_\varkappa$ is nonzero. Therefore, in this case we have that

$$\Pi_\varkappa := \mathbb{P}(\mathbf{\Gamma}(a, b, c) \cap \mathcal{K}_\varkappa)$$

is a linear 2-fold contained in $\mathbb{P}(\mathcal{K}_\varkappa)$. We may summarise the above discussion in the following

Proposition 2.28. *Let $\varkappa \in \mathbf{S}_8^-$ be an odd pure spinor for the quadratic form $\mathbf{q}_8 : \mathbb{C}^8 \longrightarrow \mathbb{C}$. Let $\mathcal{K}_\varkappa \subset \mathbb{C}^8$ be the maximal totally null subspace represented by $\varkappa$. Then, the following statements hold:*

(i) The linear 3-fold

$$\mathbb{P}(\mathcal{K}_\varkappa) \subset \mathbf{Q}_6 \simeq \mathfrak{Z}(\mathbb{S}^6)$$

*is not a fibre of the fibration $\mathbf{\Phi} : \mathbf{Q}_6 \longrightarrow \mathbf{\Sigma}^6$ described in Theorem 2.26. Moreover, under the twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$, $\mathbb{P}(\mathcal{K}_\varkappa)$ projects surjectively onto $\mathbb{S}^6$ (cf. Remark 5.7 in [**45**]).*

(ii) $\mathbb{P}(\mathcal{K}_\varkappa) \simeq \mathbb{P}^3_{\mathbb{C}}$ contains a **canonical** *linear 2-fold*

$$\Pi_\varkappa \simeq \mathbb{P}^2_{\mathbb{C}}$$

such that $\Pi_\varkappa$ is contained in a fibre of $\mathbf{p}_6$ and, thus, collapses to a point. The restriction of $\mathbf{p}_6$ to the complement, in this twistor fibre, of $\Pi_\varkappa$ is injective. Hence, this restriction is a **differentiable blow-down** *$\mathbb{P}^3_{\mathbb{C}} \longrightarrow \mathbb{S}^6$.*

(iii) *Furthermore, the construction of* $\Pi_{\varkappa}$ *gives us a real-analytic subbundle of the twistor fibration* $\mathbf{p}_6$ *with typical fibre isomorphic to* $\mathbb{P}^2_{\mathbb{C}}$.

To finish this section, let us recall that the standard 6-sphere $\mathbb{S}^6$ admits a *natural* orthogonal almost-complex structure induced by the product in the algebra of octonions $\mathbb{O} \simeq \mathbb{R}^8$ (cf. [**3**], [**31**]). The above construction of

$$\Pi_{\varkappa} \simeq \mathbb{P}^2_{\mathbb{C}}$$

for every odd pure spinor $\varkappa \in \mathbf{S}_8^-$, gives us a continuous selection of a linear 2-fold in each fibre of the twistor fibration $\mathbf{p}_6 : \mathfrak{Z}(\mathbb{S}^6) \longrightarrow \mathbb{S}^6$. For each $\zeta \in \mathbb{S}^6$, let us denote this canonical linear 2-fold, contained in the fibre $\mathbf{p}_6^{-1}(\zeta)$, by $\mathbb{P}^2_{\zeta}$. With respect to the Fubini-Study metric d_{F-S} on $\mathbb{P}^3_{\mathbb{C}} \simeq \mathbf{p}_6^{-1}(\zeta)$, there is a *unique* focal point

$$F(\zeta) \in \mathbf{p}_6^{-1}(\zeta)$$

with the property that $d_{F-S}(F(\zeta), \mathbb{P}^2_{\zeta})$ is maximal. This, in turn, gives a section of the twistor space $\mathfrak{Z}(\mathbb{S}^6)$ and, therefore, determines an orthogonal almost-complex structure on the standard 6-sphere $\mathbb{S}^6$. We would like to remark that the construction of this section is achieved without recuring to the product in $\mathbb{O}$.

CHAPTER 3

The twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ with $n \geq 1$

For $n \geq 1$, let us recall that the horizontal and vertical spaces $\mathcal{H}, \mathcal{V} \subset \mathbb{C}^{2n+2}$ are defined as

$$\mathcal{H} := \bigoplus_{j=1}^{n+1} \mathbb{C}e_j \quad \text{and} \quad \mathcal{V} := \bigoplus_{j=1}^{n+1} \mathbb{C}e_{n+1+j}$$

where $\{ e_j \mid 1 \leq j \leq 2n+2 \}$ is the standard base of $\mathbb{C}^{2n+2}$. Let us denote the standard base of $\mathbb{C}^{n+1}$ by $\{ \widetilde{e}_j \mid 1 \leq j \leq n+1 \}$. As complex linear spaces, $\mathcal{H}$, $\mathcal{V}$ and $\mathbb{C}^{n+1}$ are isomorphic and, thus, may be identified. From now on, we shall assume that this has been done through the linear transformations defined by

$$\mathcal{H} \ni e_j \quad \xmapsto{\varphi_1} \quad \widetilde{e}_j \in \mathbb{C}^{n+1}, \text{ and}$$

$$\mathbb{C}^{n+1} \ni \widetilde{e}_j \quad \xmapsto{\varphi_2} \quad e_{n+1+j} \in \mathcal{V}$$

Let us recall (cf. Chapter 1 and §2.1) that the maximal rank quadratic forms $\mathbf{q}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ and (the Fermat polynomial) $\mathcal{F}_{n+1} : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ are given by

$$(x, y) = (x_1, ..., x_{n+1}, y_1, ..., y_{n+1}) \quad \xmapsto{\mathbf{q}_{2n+2}} \quad 2\sum_{j=1}^{n+1} x_j y_j$$

$$\text{and } z = (z_1, ..., z_{n+1}) \quad \xmapsto{\mathcal{F}_{n+1}} \quad \sum_{j=1}^{n+1} z_j^2$$

Thus, the (non-degenerate) bilinear forms $\mathbf{B}_{2n+2} : \mathbb{C}^{2n+2} \times \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ and $\mathcal{B}_{n+1} : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ associated, respectively, to $\mathbf{q}_{2n+2}$ and $\mathcal{F}_{n+1}$ are given as

$$\mathbf{B}_{2n+2}\big((x,y),(x',y')\big) = \sum_{j=1}^{n+1}(x_j y'_j + x'_j y_j)$$

$$\text{and} \qquad \mathcal{B}_{n+1}(z,z') = \sum_{j=1}^{n+1} z_j z'_j$$

Let us consider the canonical involution $\mathcal{I}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$, and let

$$x = \sum_{j=1}^{n+1} x_j e_j \in \mathcal{H} \qquad \text{and} \qquad y = \sum_{j=1}^{n+1} y_j e_{n+1+j} \in \mathcal{V}$$

be any points. We have that

$$x \quad \overset{\mathcal{I}_{2n+2}}{\longmapsto} \quad \breve{x} := \sum_{j=1}^{n+1} x_j e_{n+1+j} \in \mathcal{V} \simeq \mathbb{C}^{n+1}$$

$$\text{and} \quad y \quad \overset{\mathcal{I}_{2n+2}}{\longmapsto} \quad \widehat{y} := \sum_{j=1}^{n+1} y_j e_j \in \mathcal{H} \simeq \mathbb{C}^{n+1}$$

Therefore,

$$\mathcal{B}_{n+1}(x,\widehat{y}) = \mathcal{B}_{n+1}(\breve{x},y) = \frac{1}{2}\mathbf{q}_{2n+2}(x,y)$$

which, in turn, implies that

$$x \perp_{\mathcal{B}_{n+1}} \widehat{y} \quad \text{if and only if} \quad \mathbf{q}_{2n+2}(x,y) = 0$$

Let $k, k' \in \big\{ 0, ..., n+1 \big\}$ be such that $0 \leq k + k' \leq n+1$, and let us consider a pure spinor $s \in \mathbf{S}_{2n+2}$ for the quadratic form $\mathbf{q}_{2n+2}$ such that s is of type (k,k') (cf. definitions 1.4 and 1.7). Let $\mathcal{K}_s$ be the maximal totally null subspace, of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, represented by s. We set

$$U_s := \mathcal{K}_s \cap \mathcal{H} \qquad \text{and} \qquad W_s := \mathcal{I}_{2n+2}(\mathcal{K}_s \cap \mathcal{V})$$

Since $s \in \mathbf{T}_{n+1}(k,k')$ and $U_s, W_s \subset \mathcal{H} \simeq \mathbb{C}^{n+1}$, we have that

$$U_s \in \mathbf{G}(k, n+1) \quad \text{and} \quad W_s \in \mathbf{G}(k', n+1)$$

For all $u \in U_s$ and $w \in W_s$, it holds that

$$(u, \breve{w}) \in \mathcal{K}_s$$

and, since $\mathcal{K}_s$ is totally null for $\mathbf{q}_{2n+2}$, we get that

$$U_s \perp_{\mathcal{B}_{n+1}} W_s$$

Thus (cf. Definition 1.13),

$$(U_s, W_s) \in \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, k'; n+1)$$

for all $s \in \mathbf{T}_{n+1}(k, k')$.

By Proposition 1.8, we know that the type sets

$$\mathbf{T}_{n+1}(k, k') \subset \mathbf{S}_{2n+2} \quad \text{with } 0 \leq k, k', k+k' \leq n+1$$

are the bulding-blocks of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere.

Hence, in order to get a better understanding of $\mathfrak{Z}(\mathbb{S}^{2n})$, it would be fruitful to discuss the relationship between the (k, k')-type set $\mathbf{T}_{n+1}(k, k')$ of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and the generalised complex orthogonal Stiefel manifold $\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, k'; n+1)$ of the quadratic space $(\mathbb{C}^{n+1}, \mathcal{F}_{n+1})$. We shall do so in §3.2, after we have taken a closer look at the actions of the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and the special orthogonal group $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on the set of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

1. The action of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$

Let us recall that, for $n \geq 1$, the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ is the group formed (under the usual composition of linear transformations or, equivalently, matrix multiplication) by those linear automorphisms

$$L : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$$

which preserve the quadratic form $\mathbf{q}_{2n+2}$ (therefore, they preserve the bilinear form $\mathbf{B}_{2n+2}$). The special orthogonal group $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ consists of those $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $\det(L) = 1$.

We know (cf. I.4.3 and III.1.6 in [**16**]) that the action of the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on the set of maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ is transitive, and that two such subspaces U and W have the same parity (that is to say, they belong to one and the same family of maximal totally null subspaces) if and only if there exists $A \in SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $A(U) = W$.

Let U and W be two isomorphic complex linear subspaces of $\mathbb{C}^{2n+2}$. We say that a linear isomorphism $\phi : U \longrightarrow W$ is a $\mathbf{q}_{2n+2}$-isomorphism if

$$\mathbf{q}_{2n+2}\big(\phi(x, y)\big) = \mathbf{q}_{2n+2}(x, y)$$

for all $(x, y) \in U$. Therefore, if $L : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ is an orthogonal automorphism, then its restriction

$$L_U : U \longrightarrow L(U)$$

to U is a $\mathbf{q}_{2n+2}$-isomorphism. Conversely, every $\mathbf{q}_{2n+2}$-isomorphism $\phi : U \longrightarrow W$ may be extended to an element of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ (cf. I.4.1 in [**16**]). If, furthermore, we assume that U and W are totally null for $\mathbf{q}_{2n+2}$, then every isomorphism between them is a $\mathbf{q}_{2n+2}$-isomorphism. In particular, every automorphism of a maximal totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ may be extended to an element of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Definition 3.1. *For $n \geq 1$ and $1 \leq m \leq n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$, let us consider an m-dimensional complex linear subspace $U \subset \mathbb{C}^{2n+2}$ which is totally null for the quadratic form $\mathbf{q}_{2n+2}$. If $1 \leq k \leq m$ and $\big\{ u_j \mid 1 \leq j \leq k \big\} \subset U$ is linearly independent over $\mathbb{C}$, then we say that the finite sequence*

$$(u_1, \ldots, u_k) \in \underbrace{U \times \ldots \times U}_{\vdash k\, factors \dashv}$$

is a null $k-$ framing *of* $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

For $n \geq 1$ and $1 \leq k \leq m \leq n{+}1$, let us consider two complex m-dimensional subspaces $U, W \subset \mathbb{C}^{2n+2}$ which are totally null for the quadratic form $\mathbf{q}_{2n+2}$, and let

$$\left\{\, u_j \mid 1 \leq j \leq k \,\right\} \subset U \quad \text{ and } \quad \left\{\, w_j \mid 1 \leq j \leq k \,\right\} \subset W$$

be linearly independent sets over $\mathbb{C}$. Hence,

$$(u_1, ..., u_k) \in \underset{\vdash k\, factors \dashv}{U \times ... \times U} \quad \text{ and } \quad (w_1, ..., w_k) \in \underset{\vdash k\, factors \dashv}{W \times ... \times W}$$

are two null k-framings of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. We complete these linearly independent sets to bases

$$\left\{\, u_j \mid 1 \leq j \leq m \,\right\} \quad \text{ and } \quad \left\{\, w_j \mid 1 \leq j \leq m \,\right\}$$

of U and W, respectively. Let $L : U \longrightarrow W$ be the linear space isomorphism defined by

$$L(u_j) \longmapsto w_j \qquad \text{for all } 1 \leq j \leq m$$

Since L is a $\mathbf{q}_{2n+2}$-isomorphism, we have that there exists $\widehat{L} \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $\widehat{L}$ extends L. In particular, $\widehat{L}(u_j) = w_j$ for all $1 \leq j \leq m$. Thus,

$$(w_1, ..., w_k) = \big(\widehat{L}(u_1), ..., \widehat{L}(u_k)\big)$$

and we have proven the following two results.

LEMMA 3.2. *Let $n \geq 1$ and let $1 \leq k \leq n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$. Then, the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ acts transitively on the set of null k-framings of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.*

LEMMA 3.3. *Let $n \geq 1$ and $1 \leq k \leq n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$. Let U be a k-dimensional complex linear subspace of the horizontal space $\mathcal{H}$. Let W be* *a maximal totally null subspace of* $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ *and let* $N \subset W$ *be a* k-*dimensional complex linear subspace. Then, there exists* $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ *such that* $L(W) = \mathcal{H}$ *and* $L(N) = U$*.*

The following result is a direct consequence of Lemma 1 on page 152 of [**16**].

LEMMA 3.4. *For* $n \geq 1$ *and* $0 \leq r \leq n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$*, let* U*,* U'*,* W *and* W' *be maximal totally null subspaces of the quadratic space* $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ *such that* $dim(U \cap W) = dim(U' \cap W') = r$*. Then, there exists* $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ *such that* $L(U) = U'$ *and* $L(W) = W'$*.*

Recalling Definition 1.7, as a consequence of Lemma 3.4, we have

COROLLARY 3.5. *For* $n \geq 1$*, let* $k, k' \in \{0, ..., n+1\}$ *be such that* $0 \leq k + k' \leq n+1$*. Then, the orthogonal group* $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ *acts transitively on the* (k, k')*-type set* $\mathbf{T}_{n+1}(k, k')$*. Hence,* $\mathbf{T}_{n+1}(k, k')$ *may be thought of as the homogeneous space*

$$O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/Isot_U$$

where $Isot_U$ *denotes the isotropy group of* $U \in \mathbf{T}_{n+1}(k, k')$*.*

PROOF. Let U and W be two maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ both of type (k, k'). Since

$$dim(U \cap \mathcal{V}) = k' = dim(W \cap \mathcal{V})$$

Lemma 3.4 implies that there exists $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $L(\mathcal{V}) = \mathcal{V}$ and $L(U) = W$. $\square$

1.1. The action of $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$.

In this subsection, we would like to study the action of the special orthogonal group $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on each of the two families of maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Let us recall (cf. Chapter 1) that the vector representation

$$\chi : \mathcal{G}_{2n+2} \longrightarrow O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$$

of the Clifford group $\mathcal{G}_{2n+2} \subset \mathcal{C}\ell_{2n+2}$ of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ is such that

$$\boldsymbol{\chi}(\mathcal{G}_{2n+2} \cap \mathcal{C}\ell^{+}_{2n+2}) = SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$$

Let U be a maximal totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, and let $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ be such that $L(U) = U$. By III.2.6 in [**16**], we get that

$$L = \boldsymbol{\chi}(\zeta) \text{ for some } \zeta \in \mathcal{G}_{2n+2} \cap \mathcal{C}\ell^{+}_{2n+2}$$

Thus, $L \in SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and we get that the isotropy group $Isot_{U}$ of U, under the action of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, is a subgroup of $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Therefore, as a consequence of Corollary 3.5 we have

Corollary 3.6. *For $n \geq 1$, let $k, k' \in \{0, ..., n+1\}$ be such that $0 \leq k + k' \leq n+1$. Then, the special orthogonal group $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ acts transitively on the (k, k')-type set $\mathbf{T}_{n+1}(k, k')$.*

Proof. Given $U, W \in \mathbf{T}_{n+1}(k, k')$, Corollary 3.5 implies the existence of $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $L(\mathcal{V}) = \mathcal{V}$ and $L(U) = W$. Since $L \in Isot_{\mathcal{V}}$, we get that $L \in SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. $\square$

On the other hand, let us recall (cf. Chapter 1) that, for $m \geq 1$, $\mathcal{M}(m, \mathbb{C})$ denotes the algebra of complex $m \times m$ matrices and that $\mathcal{S}_{m} \subset \mathcal{M}(m, \mathbb{C})$ denotes the $\frac{m(m-1)}{2}$-dimensional complex linear subspace of skew-symmetric matrices. The Pfaffian transformation $\boldsymbol{\lambda}_{m} : \mathcal{S}_{m} \longrightarrow \mathbb{C}$ satisfies that

$$\det(E) = \big(\boldsymbol{\lambda}_{m}(E)\big)^2$$

for all $E \in \mathcal{S}_{m}$. We have that

$$\mathcal{S}^{*}_{m} := \{\, E \in \mathcal{S}_{m} \mid \boldsymbol{\lambda}_{m}(E) \neq 0 \,\},\ \mathcal{S}^{0}_{m} := \{\, E \in \mathcal{S}_{m} \mid \boldsymbol{\lambda}_{m}(E) = 0 \,\},$$
$$\boldsymbol{\Gamma}^{*}_{m} := \boldsymbol{\Gamma}(\mathcal{S}^{*}_{m}) \quad \text{and} \quad \boldsymbol{\Gamma}^{0}_{m} := \boldsymbol{\Gamma}(\mathcal{S}^{0}_{m})$$

where $\boldsymbol{\Gamma} : \mathcal{S}_{m} \longrightarrow \mathbf{G}(m, 2m)$ is the transformation which assigns to each skew-symmetric matrix E the graph

$$\mathbf{\Gamma}(E) := \left\{ \left(Z, E(Z)\right) \in \mathbb{C}^{2m} \mid Z \in \mathbb{C}^m \right\}$$

of the linear endomorphism $E : \mathbb{C}^m \longrightarrow \mathbb{C}^m$.

We also recall that $\mathbf{0}_m, \mathbf{Id}_m \in \mathcal{M}(m, \mathbb{C})$ denote, respectively, the zero and the identity matrices, and that the general linear group

$$GL(m, \mathbb{C}) = \left\{ A \in \mathcal{M}(m, \mathbb{C}) \mid \det(A) \neq 0 \right\}$$

consists of all the linear automorphisms of $\mathbb{C}^m$. In what follows we shall denote matrix transposition by ^{t}.

Let $n \geq 1$. For each $C, C' \in \mathcal{M}(n+1, \mathbb{C})$, let us define

$$\mathsf{diag}(C, C') := \begin{pmatrix} C & \mathbf{0}_{n+1} \\ \mathbf{0}_{n+1} & C' \end{pmatrix} \in \mathcal{M}(2n+2, \mathbb{C})$$

If $A \in GL(n+1, \mathbb{C})$, let us set

$$\widetilde{A} := (A^{-1})^t = (A^t)^{-1} \in GL(n+1, \mathbb{C}) \quad \text{ and } \quad D_A := \mathsf{diag}(A, \widetilde{A})$$

Then, $\det(D_A) = 1$ and we have that $D_A \in GL(2n+2, \mathbb{C})$. In particular, $\mathbf{Id}_{n+2} = D_{\mathbf{Id}_{n+1}}$. For all $A, A' \in GL(n+1, \mathbb{C})$, we get that:

(1) $\widetilde{AA'} = \widetilde{A}\,\widetilde{A'}$ and, thus, $D_A D_{A'} = D_{AA'}$.

(2) $D_A^{-1} = D_{A^{-1}}$ and $D_A^t = D_{A^t}$.

(3) $D_A \mathbf{B}_{2n+2} D_A^t = \mathbf{B}_{2n+2}$.

Therefore,

$$\mathcal{D}_{2n+2} := \left\{ D_A := \mathsf{diag}(A, \widetilde{A}) \mid A \in GL(n+1, \mathbb{C}) \right\}$$

is a subgroup of $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, and we have a group monomorphism

$$\psi : GL(n+1, \mathbb{C}) \hookrightarrow SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2}) \quad \text{ given by } \quad A \longmapsto D_A$$

Let $A \in SO(n+1) \subset GL(n+1,\mathbb{C})$. Then, $AA^t = \mathbf{Id}_{n+1}$ and we have that $\widetilde{A} = A$. Hence, $D_A = \mathsf{diag}(A, A) \in SO(2n+2)$ and the restriction of ψ to $SO(n+1)$ defines a group monomorphism $SO(n+1) \hookrightarrow SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

More generally, let $A \in U(n+1) \subset GL(n+1,\mathbb{C})$. Then, $AA^* = A^*A = \mathbf{Id}_{n+1}$ where $A^* := \overline{A}^t$ denotes the conjugate transpose matrix. We have that $\widetilde{A} = \overline{A}$. Thus, $D_A = \mathsf{diag}(A, \overline{A})$ and the restriction of ψ to $U(n+1)$ defines a group monomorphism $U(n+1) \hookrightarrow SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

Let us get on to describing the isotropy groups $Isot_{\mathcal{H}}, Isot_{\mathcal{V}} \subset SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ corresponding to the horizontal and vertical spaces $\mathcal{H}, \mathcal{V} \subset \mathbb{C}^{2n+2}$, under the action of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. If $L \in SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, then we may write

$$L = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where $A_j \in \mathcal{M}(n+1,\mathbb{C})$ for all $1 \leq j \leq 4$. Given that L preserves the bilineal form $\mathbf{B}_{2n+2} : \mathbb{C}^{2n+2} \times \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}$ associated to $\mathbf{q}_{2n+2}$, we have that

$$A_2 A_1^t,\ A_3 A_4^t \in \mathcal{S}_{n+1} \qquad \text{and} \qquad A_2 A_3^t + A_1 A_4^t = \mathbf{Id}_{n+1}$$

Therefore:

► L preserves the horizontal space $\mathcal{H} \subset \mathbb{C}^{2n+2}$ if and only if $A_3 = \mathbf{0}_{n+1}$, $A_1 \in GL(n+1,\mathbb{C})$, $A_4 = (A_1^{-1})^t$, $A_2 \in \mathcal{M}(n+1,\mathbb{C})$ and $A_2 A_1^t \in \mathcal{S}_{n+1}$. That is to say,

$$Isot_{\mathcal{H}} = \left\{ \begin{pmatrix} A & M \\ \mathbf{0}_{n+1} & \widetilde{A} \end{pmatrix} \mid A, M \in \mathcal{M}(n+1, C), \det(A) \neq 0, MA^t \in \mathcal{S}_{n+1} \right\}$$

► L preserves the vertical space $\mathcal{V} \subset \mathbb{C}^{2n+2}$ if and only if $A_2 = \mathbf{0}_{n+1}$, $A_1 \in GL(n+1,\mathbb{C})$, $A_4 = (A_1^{-1})^t$, $A_3 \in \mathcal{M}(n+1,\mathbb{C})$ and $A_3 A_1^{-1} \in \mathcal{S}_{n+1}$. Thus,

$$Isot_{\mathcal{V}} = \left\{ \begin{pmatrix} C & \mathbf{0}_{n+1} \\ N & \widetilde{C} \end{pmatrix} \mid C, N \in \mathcal{M}(n+1, C), \det(C) \neq 0, NC^{-1} \in \mathcal{S}_{n+1} \right\}$$

► L preserves both $\mathcal{H}$ and $\mathcal{V}$ if and only if $A_2 = A_3 = \mathbf{0}_{n+1}$, $A_1 \in GL(n+1,\mathbb{C})$ and $A_4 = (A_1^{-1})^t$. Hence, we get that

$$\mathcal{D}_{2n+2} \subset Isot_{\mathcal{H}} \cap Isot_{\mathcal{V}}$$

Therefore, $\mathcal{D}_{2n+2}$ acts transitively on the type sets $\mathbf{T}_{n+1}(0, n+1) \simeq \{\mathcal{V}\}$ and $\mathbf{T}_{n+1}(n+1, 0) \simeq \{\mathcal{H}\}$.

If $(x, y) \in \mathbb{C}^{2n+2}$ is such that $x \in \mathcal{H}$ is nonzero and $y \in \mathcal{V}$ is nonzero, and $A \in GL(n+1, \mathbb{C})$, then

$$D_A(x, y) = \begin{pmatrix} A & \mathbf{0}_{n+1} \\ \mathbf{0}_{n+1} & \widetilde{A} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x) \\ \widetilde{A}(y) \end{pmatrix} \notin \mathcal{H}, \mathcal{V}$$

Therefore, we have

PROPOSITION 3.7. *Let $n \geq 1$. Then, the following statements hold:*

(i) The group $\mathcal{D}_{2n+2}$ is the subgroup of $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ of those elements which preserve, precisely, the horizontal space $\mathcal{H}$ and the vertical space $\mathcal{V}$.

(ii) The group $\mathcal{D}_{2n+2}$ preserves the type of every maximal totally null subspace of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

For $n \geq 1$, by Theorem 1.10, Proposition 1.11 and the above discussion, we may think of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere as either of the homogeneous spaces

$$SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/Isot_{\mathcal{H}} \quad \text{or} \quad SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/Isot_{\mathcal{V}}$$

By Corollary 3.6, Proposition 1.11 and (i) of Proposition 1.12, we have (cf. Chapter 7 of [**28**])

PROPOSITION 3.8. *For $n \geq 1$, the special orthogonal group $SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ acts transitively on each of the two irreducible components of the Fano variety $\mathbf{F}_{n,2n}$ of linear n-folds contained in the non-singular quadric hypersurface $\mathbf{Q}_{2n} \subset \mathbb{P}_{\mathbb{C}}^{2n+1}$ defined by the quadratic form $\mathbf{q}_{2n+2}$. Moreover:*

(i) If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ is even, then

$$SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/Isot_{\mathcal{V}} \quad \simeq \quad \mathbf{F}^{+}_{n,2n} \quad \simeq \quad \overline{\mathbf{\Gamma}}_{n+1}$$
$$= \mathbf{\Gamma}_{n+1} \sqcup \mathcal{I}_{2n+2}(\mathbf{\Gamma}^0_{n+1}) \sqcup \left\{ \textit{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \right\}$$

(ii) If $\mathbf{idx}(\mathbf{q}_{2n+2}) = n+1$ *is odd, then*

$$SO(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/Isot_{\mathcal{H}} \quad \simeq \quad \mathbf{F}^{-}_{n,2n} \quad \simeq \quad \overline{\mathbf{\Gamma}}_{n+1}$$
$$= \mathbf{\Gamma}^0_{n+1} \sqcup \left\{ \textit{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \right\}$$

To continue our study of the groups $Isot_{\mathcal{H}}$ and $Isot_{\mathcal{V}}$, let us define

$$\mathcal{N}_{\mathcal{H}} := \left\{ \begin{pmatrix} \mathbf{Id}_{n+1} & M \\ \mathbf{0}_{n+1} & \mathbf{Id}_{n+1} \end{pmatrix} \in GL(2n+2, \mathbb{C}) \mid M \in \mathcal{S}_{n+1} \right\}$$

and

$$\mathcal{N}_{\mathcal{V}} := \left\{ \begin{pmatrix} \mathbf{Id}_{n+1} & \mathbf{0}_{n+1} \\ N & \mathbf{Id}_{n+1} \end{pmatrix} \in GL(2n+2, \mathbb{C}) \mid N \in \mathcal{S}_{n+1} \right\}$$

Then, $\mathcal{N}_{\mathcal{H}} \subset Isot_{\mathcal{H}}$ and $\mathcal{N}_{\mathcal{V}} \subset Isot_{\mathcal{V}}$. It is clear that

$$\mathcal{N}_{\mathcal{H}} \cap \mathcal{D}_{2n+2} = \left\{ \mathbf{Id}_{2n+2} \right\} = \mathcal{N}_{\mathcal{V}} \cap \mathcal{D}_{2n+2}$$

We have that both $\mathcal{N}_{\mathcal{H}}$ and $\mathcal{N}_{\mathcal{V}}$ are Abelian groups isomorphic to the additive group of complex $(n+1) \times (n+1)$ skew-symmetric matrices $\mathcal{S}_{n+1} \simeq \mathbb{C}^{\frac{n(n+1)}{2}}$. Furthermore, $\mathcal{N}_{\mathcal{H}}$ is a normal subgroup of $Isot_{\mathcal{H}}$ and $\mathcal{N}_{\mathcal{V}}$ is a normal subgroup of $Isot_{\mathcal{V}}$.

Let $A, C, M, N \in \mathcal{M}(n+1, \mathbb{C})$ be such that $\det(A), \det(C) \neq 0$ and $MA^t, NC^{-1} \in \mathcal{S}_{n+1}$. Then, we have that

$$Isot_{\mathcal{H}} \ni \begin{pmatrix} A & M \\ \mathbf{0}_{n+1} & \widetilde{A} \end{pmatrix} = \begin{pmatrix} \mathbf{Id}_{n+1} & MA^t \\ \mathbf{0}_{n+1} & \mathbf{Id}_{n+1} \end{pmatrix} \begin{pmatrix} A & \mathbf{0}_{n+1} \\ \mathbf{0}_{n+1} & \widetilde{A} \end{pmatrix} \in \mathcal{N}_{\mathcal{H}} \mathcal{D}_{2n+2}$$

and

$$Isot_{\mathcal{V}} \ni \begin{pmatrix} C & \mathbf{0}_{n+1} \\ N & \widetilde{C} \end{pmatrix} = \begin{pmatrix} \mathbf{Id}_{n+1} & \mathbf{0}_{n+1} \\ NC^{-1} & \mathbf{Id}_{n+1} \end{pmatrix} \begin{pmatrix} C & \mathbf{0}_{n+1} \\ \mathbf{0}_{n+1} & \widetilde{C} \end{pmatrix} \in \mathcal{N}_{\mathcal{V}} \mathcal{D}_{2n+2}$$

Therefore, we have proven

PROPOSITION 3.9. *Let $n \geq 1$. Then, the following statements hold:*

(i) The isotropy group $Isot_{\mathcal{H}}$ of the horizontal space under the action of the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, on the set of maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, is the semidirect product of $\mathcal{N}_{\mathcal{H}}$ and $\mathcal{D}_{2n+2}$.

(ii) The isotropy group $Isot_{\mathcal{V}}$ of the vertical space under the action of the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, on the set of maximal totally null subspaces of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, is the semidirect product of $\mathcal{N}_{\mathcal{V}}$ and $\mathcal{D}_{2n+2}$.

DEFINITION 3.10. *For $n \geq 1$, let U and W be two maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $dim(U \cap W) = 0$. Then, we say that U and W are* linearly independent subspaces *of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.*

For instance, if $M \in \mathcal{S}_{n+1}$, then the vertical space $\mathcal{V}$ and the horizontal graph $\mathbf{\Gamma}(M)$ are linearly independent subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$. Thus, the vertical graph $\mathcal{I}_{2n+2}\big(\mathbf{\Gamma}(M)\big)$ and the horizontal space $\mathcal{H}$ are also linearly independent subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$.

We would like to remark that if U and W are linearly independent subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, then Lemma 1.6 implies that U and W are of the same parity if and only if $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is even.

PROPOSITION 3.11. *Let $n \geq 1$. Then, the following statements hold:*

(i) The group $\mathcal{N}_{\mathcal{V}}$ acts transitively on the set $\mathbf{\Gamma}_{n+1}$ of graphs of skew-symmetric linear endomorphisms of $\mathbb{C}^{n+1}$.

(ii) The group $\mathcal{N}_{\mathcal{H}}$ acts transitively on the image $\mathcal{I}_{2n+2}(\mathbf{\Gamma}_{n+1})$ of $\mathbf{\Gamma}_{n+1}$ under the canonical involution $\mathcal{I}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$.

PROOF. For each $M \in \mathcal{S}_{n+1}$ we have that

$$\begin{pmatrix} \mathbf{Id}_{n+1} & \mathbf{0}_{n+1} \\ -M & \mathbf{Id}_{n+1} \end{pmatrix} \in \mathcal{N}_{\mathcal{V}} \quad \text{and} \quad \begin{pmatrix} \mathbf{Id}_{n+1} & -M \\ \mathbf{0}_{n+1} & \mathbf{Id}_{n+1} \end{pmatrix} \in \mathcal{N}_{\mathcal{H}}$$

where $-M \in \mathcal{S}_{n+1}$ denotes the additive inverse matrix of M. For each $Z \in \mathbb{C}^{n+1}$, we get that

$$\begin{pmatrix} \mathbf{Id}_{n+1} & \mathbf{0}_{n+1} \\ -M & \mathbf{Id}_{n+1} \end{pmatrix} \begin{pmatrix} Z \\ M(Z) \end{pmatrix} = \begin{pmatrix} Z \\ \hat{0}_{n+1} \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} \mathbf{Id}_{n+1} & -M \\ \mathbf{0}_{n+1} & \mathbf{Id}_{n+1} \end{pmatrix} \begin{pmatrix} M(Z) \\ Z \end{pmatrix} = \begin{pmatrix} \hat{0}_{n+1} \\ Z \end{pmatrix}$$

where $\hat{0}_{n+1} \in \mathbb{C}^{n+1}$ denotes the origin. Hence, every element of $\mathbf{\Gamma}_{n+1}$ may be taken to $\mathcal{H}$ by a matrix in $\mathcal{N}_{\mathcal{V}}$, and every element of $\mathcal{I}_{2n+2}(\mathbf{\Gamma}_{n+1})$ may be taken to $\mathcal{V}$ by a matrix in $\mathcal{N}_{\mathcal{H}}$. □

COROLLARY 3.12. *Let $n \geq 1$. Then, the following statements hold:*

(i) The coordinate-wise action of the orthogonal group $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ on the set

$$\mathcal{P} := \big\{ (U, W) \mid U \textit{ and } W \textit{ are linearly independent subspaces of } (\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2}) \big\}$$

is transitive.

(ii) The isotropy group $Isot_{(U,W)}$ of $(U, W) \in \mathcal{P}$ is isomorphic to $\mathcal{D}_{2n+2}$. Therefore (cf. Theorem 1.10), the Fano variety $\mathbf{F}_{n,2n}$ may be thought of as the homogeneous space $O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})/\mathcal{D}_{2n+2}$.

PROOF. Let $(U, W) \in \mathcal{P}$. Then, there exists $L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ such that $L(U) = \mathcal{V}$. Thus, $dim(L(W) \cap \mathcal{V}) = 0$ and $L(W) \in \mathbf{\Gamma}_{n+1}$. By Proposition 3.11, there exists $L' \in \mathcal{N}_{\mathcal{V}} \subset Isot_{\mathcal{V}}$ such that $L'\big(L(W)\big) = \mathcal{H}$. Hence, $L' \circ L \in O(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ is such that

$$(U, W) \quad \xmapsto{L' \circ L} \quad (\mathcal{V}, \mathcal{H})$$

and we have (i). Since $(\mathcal{V},\mathcal{H}) \in \mathcal{P}$, by (i) of Proposition 3.7, we have that $Isot_{(\mathcal{V},\mathcal{H})} = \mathcal{D}_{2n+2}$, which proves (ii). $\square$

2. The stratifications of $\mathfrak{Z}(\mathbb{S}^{2n})$ with $n \geq 1$

Let $n \geq 1$. Recalling Definition 1.7, by Proposition 1.8, we know that the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere may be decomposed (or stratified) as

$$\bigsqcup_{\substack{0\leq k,k',k+k'\leq n+1\\ k,k'\ even}} \mathbf{T}_{n+1}(k,k') \quad \text{or} \quad \bigsqcup_{\substack{1\leq k,k',k+k'\leq n+1\\ k,k'\ odd}} \mathbf{T}_{n+1}(k,k')$$

when $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is even, and as

$$\bigsqcup_{\substack{0\leq k,k',k+k'\leq n+1\\ k\ even,\ k'\ odd}} \mathbf{T}_{n+1}(k,k') \quad \text{or} \quad \bigsqcup_{\substack{0\leq k,k',k+k'\leq n+1\\ k\ even,\ k'\ odd}} \mathbf{T}_{n+1}(k',k)$$

when $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is odd.

To finish our present discussion of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere, we would like to describe each type set $\mathbf{T}_{n+1}(k,k')$ for the quadratic space $(\mathbb{C}^{2n+2},\mathbf{q}_{2n+2})$ (that is to say, each of the building-blocks of the strata in the above decompositions) in terms of the corresponding generalised complex orthogonal Stiefel manifold of the quadratic space $(\mathbb{C}^{n+1},\mathcal{F}_{n+1})$, where $\mathcal{F}_{n+1} : \mathbb{C}^m \longrightarrow \mathbb{C}$ is the Fermat polynomial introduced in §2.1 (cf. Definition 1.13).

Given that the bilinear form $\mathcal{B}_{n+1} : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$, associated to $\mathcal{F}_{n+1}$, is non-degenerate, we have that

$$\begin{aligned}\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(0,n{+}1;n{+}1) &= \big\{\, (U,W) \in \mathbf{G}(0,n{+}1)\times\mathbf{G}(n{+}1,n{+}1) \mid U \perp_{\mathcal{B}_{n+1}} W \,\big\}\\ &\simeq \mathbb{C}^{n+1} \simeq \mathcal{V} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(n{+}1,0;n{+}1) &= \big\{\, (U,W) \in \mathbf{G}(n{+}1,n{+}1)\times\mathbf{G}(0,n{+}1) \mid U \perp_{\mathcal{B}_{n+1}} W \,\big\}\\ &\simeq \mathbb{C}^{n+1} \simeq \mathcal{H}\end{aligned}$$

► Therefore, for all $n \geq 1$, there exists a locally trivial and holomorphic fibration $\Theta : \mathbf{T}_{n+1}(0, n+1) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(0, n+1; n+1)$ with fibre isomorphic to a point.

There also exists a locally trivial and holomorphic fribration $\Theta' : \mathbf{T}_{n+1}(n+1, 0) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(n+1, 0; n+1)$ with fibre isomorphic to a point. ◄

More generally, let $n \geq 1$ and let $k, k' \in \{ 0, ..., n+1 \}$ be such that

$$0 < k + k' = n + 1 = \mathbf{idx}(\mathbf{q}_{2n+2})$$

Then $k' = n + 1 - k$ and let us consider the generalised complex orthogonal Stiefel manifold

$$\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, n+1-k; n+1) = \{ (U, W) \in \mathbf{G}(k, n+1) \times \mathbf{G}(n+1-k, n+1) \mid U \perp_{\mathcal{B}_{n+1}} W \}$$

of the quadratic space $(\mathbb{C}^{n+1}, \mathcal{F}_{n+1})$. If $(U, W) \in \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, n+1-k; n+1)$, then there exists a **unique** $s \in \mathbf{T}_{n+1}(k, k') = \mathbf{T}_{n+1}(k, n+1-k)$ such that

$$U = U_s := \mathcal{K}_s \cap \mathcal{H} \quad \text{ and } \quad W = W_s := \mathcal{K}_s \cap \mathcal{V}$$

where $\mathcal{K}_s$ is the maximal totally null subspace of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ represented by s.

► Thus, for all $n \geq 1$ and $0 \leq k \leq n+1$, there exists a locally trivial and holomorphic fibration

$$\Phi : \mathbf{T}_{n+1}(k, n+1-k) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, n+1-k; n+1)$$

with fibre isomorphic to a point. ◄

Now let us consider $n \geq 1$ and $k, k' \in \{ 0, ..., n+1 \}$ such that

$$0 \leq k + k' < n + 1$$

and let us set

$$r := (n+1) \,-\, (k+k')$$

Then, $0 < r \leq n+1$. Furthermore, by Proposition 1.8, we have:

(1) If $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is even, then k and k' are **both even** or **both odd**. Hence, $0 < r$ is an **even** integer.

(2) If $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is odd, then k and k' are of **opposite parity**. Hence, $0 < r$ is an **even** integer.

By (1) and (2) above (cf. page 16 above), we have that

$$\mathcal{S}_r^* := \left\{\, M \in \mathcal{S}_r \mid \boldsymbol{\lambda}_r(M) \neq 0 \,\right\} \neq \varnothing$$

Let us consider the generalised complex orthogonal Stiefel manifold $\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, k'; n+1)$ of $(\mathbb{C}^{n+1}, \mathcal{F}_{n+1})$, and a point (U, W) in this manifold. Since $k + k' < n+1$, we have that there exist, at least, two **distinct** pure spinors $s, s' \in \mathbf{T}_{n+1}(k, k')$ such that

$$\mathcal{K}_s \cap \mathcal{H} := U_s = U = U_{s'} := \mathcal{K}_{s'} \cap \mathcal{H}$$

and

$$\mathcal{K}_s \cap \mathcal{V} := W_s = W = W_{s'} := \mathcal{K}_{s'} \cap \mathcal{V}$$

where $\mathcal{K}_s$ and $\mathcal{K}_{s'}$ are the maximal totally null subspaces of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ represented, respectively, by s and s'. In order to recover the maximal totally null subspace $\mathcal{K}_s$ from the $(k+k')$-dimensional complex linear and totally null subspace of $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ generated by $U_s = U$ and $W_s = W$, we have to specify an r-dimensional complex linear subspace $N \subset \mathbb{C}^{2n+2}$ such that:

(a) N is totally null for the quadratic form $\mathbf{q}_{2n+2}$.

(b) It holds that

$$\boldsymbol{\rho}_{(x,y)}(s) = 0 \quad \text{ for all } (x, y) \in N$$

where $\boldsymbol{\rho} : \mathcal{C}\ell_{2n+2} \longrightarrow End(\mathbf{S}_{2n+2})$ is the spin representation of the Clifford algebra $\mathcal{C}\ell_{2n+2}$ (cf. Theorem 1.5).

(c) It holds that $dim(N \cap \mathcal{H}) = 0 = dim(N \cap \mathcal{V})$.

Therefore, N is isomorphic to the graph

$$\mathbf{\Gamma}(M) = \left\{ \big(Z, M(Z)\big) \in \mathbb{C}^{2r} \mid Z \in \mathbb{C}^r \right\}$$

for a **unique** $M \in \mathcal{S}^*_r$.

► Therefore, for all $n \geq 1$ and $k, k' \in \left\{ 0, ..., n+1 \right\}$ such that $0 \leq k + k' < n+1$, there exists a locally trivial and holomorphic fibration

$$\Psi : \mathbf{T}_{n+1}(k, k') = \mathbf{T}_{n+1}(k, n+1-k-r) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, n+1-k-r; n+1)$$

with fibre isomorphic to $\mathcal{S}^*_r$. ◄

Recalling (cf. Definition 1.13) that $dim\big(\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, k'; n+1)\big)$ is equal to

$$(n+1)(k+k') - kk' - k^2 - (k')^2,$$

we may summarise the above argumentation in the following result (cf. Proposition 2.17).

PROPOSITION 3.13. *For $n \geq 1$, let $k, k' \in \left\{ 0, ..., n+1 \right\}$ be such that $0 \leq k + k' \leq n+1$. We set $r := (n+1) - (k+k')$. Thus, $0 \leq r$ is an even integer. Let us consider the type set $\mathbf{T}_{n+1}(k, k')$, for the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$, and the generalised complex orthogonal Stiefel manifold $\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, k'; n+1)$, of the quadratic space $(\mathbb{C}^{n+1}, \mathcal{F}_{n+1})$. Then, the following statements hold:*

(i) If $k + k' = n+1$, then there exists a locally trivial and holomorphic fibration $\Phi : \mathbf{T}_{n+1}(k, n+1-k) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, n+1-k; n+1)$ with fibre isomorphic to a point.

Hence, $\mathbf{T}_{n+1}(k, n+1-k)$ coincides with the $k(n+1-k)$-dimensional complex manifold $\mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, n+1-k; n+1)$.

(ii) If $k + k' < n+1$, then there exists a locally trivial and holomorphic fibration $\Psi : \mathbf{T}_{n+1}(k, k') \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(k, k'; n+1)$ with fibre isomorphic to

$\mathcal{S}^*_r$. Hence, this fibre is a Zariski open and dense subset in $\mathcal{S}_r \simeq \mathbb{C}^{\frac{r(r-1)}{2}}$, and we have that $dim\big(\mathbf{T}_{n+1}(k,k')\big) = \frac{n(n+1)+k+k'-k^2-(k')^2}{2}$.

By (iii) of Proposition 2.7 and (i) of Proposition 3.13 we have

Corollary 3.14. *The generalised complex orthogonal Stiefel manifold $\mathbf{V}^{\mathcal{B}_4}_{\mathbb{C}}(2,2;4)$ of the quadratic space $(\mathbb{C}^4, \mathcal{F}_4)$ is biholomorphic to the Plücker quadric hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.*

Taking into account the discussion on page 17 above, regarding the set

$$\big\{ \text{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \big\},$$

propositions 1.8, 1.11, 3.13, (i) of Proposition 1.12 and definitions 1.7 and 1.13, we may summarise our description of the stratifications of the twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$, given in terms of the type sets $\mathbf{T}_{n+1}(k,k')$ of the quadratic space $(\mathbb{C}^{2n+2}, \mathbf{q}_{2n+2})$ and the corresponding generalised complex orthogonal Stiefel manifolds $\mathbf{V}^{\mathcal{B}_{n+1}}_{\mathbb{C}}(k,k';n+1)$ of the quadratic space $(\mathbb{C}^{n+1}, \mathcal{F}_{n+1})$, in the following two theorems.

Theorem 3.15. *Let $n \geq 1$ be such that $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ is odd. Then, the following statements hold:*

(i) The twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ of the standard $2n$-sphere is biholomorphic to

$$\mathbf{F}^-_{n,2n} \quad \simeq \quad \bigsqcup_{\substack{0 \leq k,k',k+k' \leq n+1 \\ k \text{ even, } k' \text{ odd}}} \mathbf{T}_{n+1}(k',k) \quad \simeq$$

$$\overline{\mathbf{\Gamma}}_{n+1} = \mathbf{\Gamma}^0_{n+1} \sqcup \big\{ \textit{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \big\}$$

(ii) $\mathbf{F}^+_{n,2n} = \mathcal{I}_{2n+2}(\mathbf{F}^-_{n,2n})$ where $\mathcal{I}_{2n+2} : \mathbb{P}^{2n+1}_{\mathbb{C}} \longrightarrow \mathbb{P}^{2n+1}_{\mathbb{C}}$ is the canonical involution.

(iii) There exists a locally trivial and holomorphic fibration $\Phi : \mathbf{T}_{n+1}(1,n) \longrightarrow \mathbf{V}^{\mathcal{B}_{n+1}}_{\mathbb{C}}(1,n;n+1)$ with fibre isomorphic to a point. Thus, $\mathbf{T}_{n+1}(1,n)$ is an n-dimensional complex manifold. Furthermore, $\mathbf{T}_{n+1}(1,n)$ is a Zariski open and dense subset of

$$\Big\{ \textit{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \Big\} = \overline{\mathbf{\Gamma}}_{n+1} - \mathbf{\Gamma}^0_{n+1}$$

and we have that $\mathbf{T}_{n+1}(1,n)$ *is the stratum of biggest dimension in* $\mathbf{F}^-_{n,2n}$.

THEOREM 3.16. *Let* $n \geq 1$ *be such that* $n+1 = \mathbf{idx}(\mathbf{q}_{2n+2})$ *is even. Then, the following statements hold:*

(i) The twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ *of the standard* $2n$*-sphere is biholomorphic to*

$$\mathbf{F}^-_{n,2n} \quad \simeq \quad \bigsqcup_{\substack{0 \leq k,k',k+k' \leq n+1 \\ k,k' \text{ odd,}}} \mathbf{T}_{n+1}(k,k')$$

Moreover, there exists a locally trivial and holomorphic fibration $\Psi : \mathbf{T}_{n+1}(1,1) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(1,1;n+1)$ *with fibre isomorphic to* $\mathcal{S}^*_{n-1}$ *(cf. Proposition* 2.17*). Thus,* $\dim\big(\mathbf{T}_{n+1}(1,1)\big) = \frac{n(n+1)}{2}$*. We get that* $\mathbf{T}_{n+1}(1,1)$ *is a Zariski open and dense subset of* $\mathbf{F}^-_{n,2n}$*, and it is the stratum of biggest dimension.*

(ii) The twistor space $\mathfrak{Z}(\mathbb{S}^{2n})$ *of the standard* $2n$*-sphere is biholomorphic to*

$$\mathbf{F}^+_{n,2n} \quad \simeq \quad \bigsqcup_{\substack{0 \leq k,k',k+k' \leq n+1 \\ k,k' \text{ even,}}} \mathbf{T}_{n+1}(k,k') \quad \simeq$$

$$\overline{\mathbf{\Gamma}}_{n+1} = \mathbf{\Gamma}_{n+1} \sqcup \mathcal{I}_{2n+2}(\mathbf{\Gamma}^0_{n+1}) \sqcup \Big\{ \textit{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \Big\}$$

where $\mathcal{I}_{2n+2} : \mathbb{C}^{2n+2} \longrightarrow \mathbb{C}^{2n+2}$ *is the canonical involution. Moreover:*

(a) $\mathbf{T}_{n+1}(0,0)$ *is isomorphic to* $\mathcal{S}^*_{n+1}$*. Therefore,* $\mathbf{T}_{n+1}(0,0)$ *is a Zariski open and dense subset of* $\mathbf{F}^+_{n,2n}$ *(cf. (i) of Proposition* 2.7*).*

(b) The set $\mathcal{I}_{2n+2}(\mathbf{\Gamma}^0_{n+1})$ *is a Zariski open and dense subset of* $\overline{\mathbf{\Gamma}}_{n+1} - \mathbf{\Gamma}_{n+1}$ *(cf. (ii) of Proposition* 2.7*).*

(c) The set $\Lambda := \Big\{ \textit{limits at infinity of the elements in } \mathbf{\Gamma}^0_{n+1} \Big\}$ *is biholomorphic to the complex projective variety defined by the equation* $\boldsymbol{\lambda}_{n+1}(M) = 0$.

That is to say, Λ is obtained when we projectivise and compactify the complex $\frac{n(n+1)-2}{2}$-dimensional affine variety $\mathcal{S}^0_{n+1} := \big\{ M \in \mathcal{S}_{n+1} \mid \boldsymbol{\lambda}_{n+1}(M) = 0 \big\}$.

(d) *There exists a locally trivial and holomorphic fibration $\Phi : \mathbf{T}_{n+1}(2,2) \longrightarrow \mathbf{V}_{\mathbb{C}}^{\mathcal{B}_{n+1}}(2,2;n+1)$ with fibre isomorphic to $\mathcal{S}^*_{n-3}$.*

Thus, dim$\big(\mathbf{T}_{n+1}(2,2)\big) = \frac{n(n+1)-4}{2}$ and $\mathbf{T}_{n+1}(2,2)$ is a dense subset of Λ (cf. (iii) of Proposition 2.7).

Bibliography

[1] D. Akhiezer, *Dense orbits with two ends*, Math. USSR Izvestija 11 (2) (1977) 293-307.

[2] M. Atiyah, N. Hitchin, I. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London A362 (1978) 425-461.

[3] J. Baez, *The octonions*, Bull. Amer. Math. Soc. 39 (2) (2002) 145-205.

[4] E. Bishop, *Conditions for the analyticity of certain sets*, Mich. Math. Jour. 11 (1964) 289-304.

[5] J. Bolton, L. Fernández, J. Wood, *Some recent developments in the study of minimal 2-spheres in spheres*, Geometry related to the theory of integrable systems, Re. Ins. Math. Sci., Kyoto Univ. (2008) 151-169.

[6] R. Bryant, *Conformal and minimal immersions of compact surfaces into the* 4-*sphere*, J. Diff. Geom. 17 (1982) 455-473.

[7] R. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. 126 (1987) 525-576.

[8] P. Budinich, M. Rigoli, *Cartan spinors, minimal surfaces and strings*, I.S.A.S., Trieste, 1987.

[9] P. Budinich, A. Trautman, *The spinorial chessboard*, Trieste Notes in Physics, Springer-Verlag, Berlin, 1988.

[10] F. Burstall, *A twistor description of harmonic maps of a 2-sphere into a Grassmannian*, Math. Ann. 274 (1986) 61-74.

[11] F. Burstall, S. Gutt, J. Rawnsley, *Twistor spaces for Riemannian symmetric spaces*, Math. Ann 295 (1993) 729-743.

[12] F. Burstall, O. Muškarov, G. Grantcharov, J. Rawnsley, *Hermitian structures on Hermitian symmetric spaces*, J. Geom. Phys. 10 (1993) 245-249.

[13] E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Diff. Geom. 1 (1967) 111-125.

[14] É. Cartan, *The Theory of Spinors*, Dover Publications Inc., New York, 1981.

[15] T. Cecil, *Lie sphere geometry with applications to submanifolds*, Universitext, Springer, 2008.

[16] C. Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras*, Collected Works Volume 2, P. Cartier, C. Chevalley (Eds.), Springer-Verlag, Berlin, 1997.

[17] J. Eells, S. Salamon, *Twistorial construction of harmonic maps of surfaces into four-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 12 (4) (1985) 589-640.

[18] J. Eells, A. Verjovsky, *Harmonic and Riemannian foliations*, Bol.Soc. Mat. Mexicana 4 (3) (1998) 1-12.

[19] W. Fulton, J. Harris, *Representation Theory. A First Course*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1991.

[20] H. Grassmann, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, Verlag, Leipzig, 1844.

[21] H. Grauert, *Set theoretic complex equivalence relations*, Math. Ann. 265 (1983) 137-148.

[22] A. Gray, *Vector cross products on manifolds*, Trans. AMS 141 (1969) 465-504.

[23] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[24] J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics 133, Springer-Verlag, New York, 1992.

[25] Y. Inoue, *Twistor spaces of even dimensional Riemannian manifolds*, J. Math. Kyoto Univ. 32 (1992) 101-134.

[26] S. Kleiman, D. Laksov, *Schubert Calculus*, The American Mathematical Monthly, 79 (1972) 1061-1082.

[27] F. Klein, S. Lie, *Sur une certaine famille de courbes et de surfaces*, Comptes Rendus 70 (1870) 1222-1226.

[28] S. Klein, *The complex quadric from the standpoint of Riemannian geometry*, Ph.D. Thesis, Universität zu Köln, Köln, 2005.

[29] H. Lawson, M. Michelsohn, *Spin Geometry*, Princeton University Press, New Jersey, 1989.

[30] D. Lê, J. Seade, A. Verjovsky, *Quadrics, orthogonal actions and involutions in complex projective spaces*, Ens. Math. 49 (1-2) (2003) 173-203.

[31] C. LeBrun, *Orthogonal Complex Structures on* S^6, Proc. AMS 101 (1987) 136-138.

[32] S. Lie, *Ueber Complexe, insbesondere Linien-und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differential-Gleichungen*, Math. Ann. 5 (1872) 145-208.

[33] B. Loo, *Irreducibility of moduli space of harmonic* 2*-spheres in* S^4, Bol. Soc. Bras. Mat. 30 (2) (1999) 163-181.

[34] B. McKay, *Extension phenomena for holomorphic geometric structures*, SIGMA 5 (058) (2009).

[35] D. Monk, *The geometry of flag manifolds*, Proc. London Math.Soc 9 (3) (1959) 253-286.

[36] E. Musso, *On the twistor space of the six-sphere*, Bull. Austral. Math. Soc. 39 (1989) 119-127.

[37] N. O'Brian, J. Rawnsley, *Twistor spaces*, Ann. Global Anal. Geom. 3 (1) (1985) 29-58.

[38] R. Penrose, *The twistor programme*, Rep. Math. Phys. 12 (1977) 65-76.

[39] J. Plücker, *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, Verlag, Leipzig, 1868-1869.

[40] M. Pontecorvo, *Complex structures on Riemannian four-manifolds*, Math. Ann. 309 (1997) 159-177.

[41] E. Puente, A. Verjovsky, *Skew-symmetric complex matrices, pure spinors, the twistor space of the conformal* $2n$*-sphere, and the Fano variety of linear* n*-folds of a nonsingular complex quadric hypersurface in* $\mathbb{P}^{2n+1}_{\mathbb{C}}$, manuscript, 2011.

[42] H. Schubert, *Kalkül der abzählenden Geometrie*, Teubner, Leipzig, 1874.

[43] J. Seade, A. Verjovsky, *Higher Dimensional Kleinian groups*, Math. Ann. 322 (2002) 279-300.

[44] Y. Siu, *Techniques of extension of analytic objects*, Lecture Notes in Pure and Appl. Math., 8, M. Dekker Inc., New York, 1974.

[45] M. Slupinski, *The twistor space of the conformal six sphere and vector bundles on quadrics*, J. of Geom. and Phys. 19 (1996) 246-266.

[46] G. Stolzenberg, *Volumes, limits, and extensions of analytic varieties*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1966.