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A Contribution to the Theory of (k; l)-kernels in Digraphs

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A Contribution to the Theory of (k, l)-kernels in Digraphs

César Hernández Cruz

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To my family:
Gilberto,
Rebeca,
Rodrigo,
Daniela.

With love.

Preface

The study of digraphs still represent a small part of Graph Theory. Nonetheless, it has grown a lot in the past twenty years. It suffices to consider Graph Theory texts, past and present. When those texts started to appear, digraphs were developed in a single chapter, sometimes only lightly related with the rest of the material. Nowadays, some of the most popular Graph Theory books consider digraphs as an interwoven subject through all the text. And, as in every fast growing subject inside Mathematics, there are a lot of simple but clever things that can be proved within. The dissertation you hold in your hand is an exploration of a subject within digraphs that has not received a lot of attention due to its difficulty: (k, l)-kernels in digraphs.

Five years ago, Professor Hortensia Galeana-Sánchez imparted two courses on kernels in digraphs in the Mathematics Graduate Program at the Autonomous National University of México. During those courses we studied not only kernels in digraphs, but some of its generalizations, mainly kernels by monochromatic paths and (k, l)-kernels. At that time, I was surprised to notice that, although (k, l)-kernels seem to be a more "natural" generalization of the concept of kernel than kernels by monochromatic paths, only a few very general results related to this concept were known, in contrast with the number of results known, involving large families of digraphs, for kernels and kernels by monochromatic paths.

As a matter of fact, after completing the courses, I felt that finding general sufficient conditions for a digraph to have a (k, l)-kernel was a very difficult problem, so I began working on kernels by monochromatic paths under the supervision of Professor Galeana-Sánchez. Luckily, I did not get any results working with kernels by monochromatic paths. But more than that, I managed to prove a result, if not about (k, l)-kernels, resembling a classic result about k-kernels. Shortly afterwards, I obtained my first result about (k, l)-kernels, proving the existence of (k, l)-kernels in quasi-transitive digraphs.

The aforementioned results, together with all the work that I developed under the supervision of Professor Galeana-Sánchez during the following four years, is now contained in the present work.

Of course, the material in the present dissertation has received a logical order, different from the chronological order in which it was obtained. At its early stages, this work grew erratically, guided more by chance than by a deep understanding of the subject. Chronologically, our first result is the main theorem of Section 7.2, followed by the main theorem of Section 3.3, and followed by the main results of Sections 2.3, 4.3, 3.2, 6.2 and 5.5, in that order. Once these results were obtained (as well as a deeper understanding of the subject), we began to organize the material in articles that were submitted to specialized journals for their publication. Most of the results of this work were obtained during this stage. Since the mathematical context for every result was included in the respective article, those articles were the base for the chapters of the present thesis. In fact, to the present day articles corresponding to Chapters 2, 3 and 7 are published. Also, and I am very thankful for that, Professor Galeana-Sánchez encouraged me to include Section 5.6, which is based in an accepted paper that I developed independently. Except for Chapter 7, the rest of the chapters are in the chronological order of the articles.

Except from some very powerful results due to A. Włoch and I. Włoch, most of the results about (k,l)-kernels were obtained without the aid of previous results. In fact, when we started this work, most of the existing results about (k,l)-kernels in digraphs were related to operations in digraphs and how the (k,l)-kernels are preserved, but we focused on finding large families of digraphs with (k,l)-kernel. Also, we managed to generalize some results valid for kernels to (k,l)-kernels. In some cases, more than a result, we obtained proof techniques that may be (and were) used to obtain further results. Also, the structure of some families of digraphs were studied and described, *e.g*, unilateral cyclically k-partite digraphs or 3-transitive digraphs. Also, an infinite family of families of digraphs is introduced here and some of its basic properties studied, k-transitive digraphs and k-quasi-transitive digraphs, with $k \geq 2$ an integer.

In Chapter 1, a brief introduction to the basic concepts of Digraph Theory is given, along with the definition and classic results about kernels and (k,l)-kernels in digraphs. In Chapters 2 to 6, different families of digraphs are studied. In some cases the whole family is proven to have (k,l)-kernel for some pairs of integers k and l, and in other cases sufficient conditions are given for the family to have a (k,l)-kernel. In Chapter 7, the notion of (k,l)-kernel is generalized for weighted digraphs. In Chapter 8 (k,l)-kernels in infinite digraphs are considered, obviously, a summary of the results of the whole dissertation is made for the infinite case. Many open problems and conjectures are proposed.

We expect the present work to begin to fill the gap that has formed through the past twenty years, since the introduction of the concept of (k,l)-kernel in 1991, between the study of kernels and (k,l)-kernels. Principally, we tried to contribute in the study of general sufficient conditions for the existence of (k,l)-kernels in digraphs, and the possible analogies between kernels and (k,l)-kernels.

Acknowledgments

I should like to thank to all who made this project possible. To my parents Rebeca and Gilberto who trusted and supported me through all this years, and always made my life easier so I could concentrate in my studies. To my brother Rodrigo, who is always there for me, to help me and cheer me up. To Daniela, for her love, her wonderful company, her support in every ambit of my life and the daily motivation to be a better human being. To my tutor Hortensia, who guided me through my graduate studies unharmed and taught me how to do mathematical research. To Francisco Raggi, for kindly letting me use his office, for offering his friendship to me, and for teaching me with his example how to be a remarkable member of our university. To Juan José Montellano for showing me a different face of Graph Theory (and the power of combinatorics). To Rafael Rojas, who is a strong influence in the way I perceive and think Mathematics through formalism and Logic. To my fellow students during graduate studies, principally to Martín and Natalia. To my students and assistants, principally to the wonderful group that I taught for two years and from which I learned a lot: Ely, Jesús, Natalia Valdez, Ariana, Patricio, Carolina, Natalia Martínez and other not so constant members. To all the members of the jury who kindly gave their time to review this work: Hortensia Galeana, Isidoro Gitler, Juan José Montellano, Eduardo Rivera and specially to Iwona Włoch, whom I have not met yet, but is always as kind as anyone can be. To the members of my Tutoral Committee, Hortensia, Juancho and Hugo Rincón. To Juan Orendain, for being so chido. And last but not least, to the Autonomous National University of México, UNAM,

which has been a home to me since I was fifteen years old, for more than twelve years, and has gave me all the opportunities that a student and a human being can receive to grow in every aspect of his life.

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Abstract

Despite the fact that the contents of this work are specialized, we have managed to present them in a self contained fashion, starting from almost nothing but the definition of digraph and building and defining every necessary notion. Obviously, by doing this it was intended that anyone with basic mathematical skills could read and understand the whole material. Nonetheless, it is clear that also specialists of the field will read it and, since they know by heart all the basics, we understand that they would like to get to the point. This brief section is a quick guide through the material presented here, indicating the highlights of the work.

Let us recall that our principal aim is to find large families of digraphs with (k, l)-kernel, for different integers of k and l. So the structure of each chapter (except Chapter 1) is simple. Every chapter has a short introductory section where a class of digraphs is defined and the principal properties and known results are mentioned. Afterwards, a section is devoted to prove some results that will be used in the final section to prove sufficient conditions for the existence of (k, l)-kernels in the family of digraphs considered. In Chapter 1, basic notions of (Di)Graph Theory can be found in the first sections. Later, in Section 1.7, the concept of kernel is introduced, alongside with a brief overview of the the role of kernels in Graph Theory and some of the main results related to the existence of kernels in digraphs. In Section 1.8 the notion of (k, l)-kernel is defined. It is easy to give an overview of the existing results about (k, l)-kernels, since the concept was introduced in 1991 and the number of articles devoted to the subject is limited.

The original results of the work begin in Chapter 2, where cyclically k-partite digraphs are considered. A digraph $D = (V, A)$ is cyclically k-partite if a partition $\{V_0, V_1, \ldots, V_{k-1}\}$ of V exists, such that every arc of D is a V_iV_{i+1}-arc (mod k). In Section 2.2, a structural characterization is given for unilateral cyclically k-partite digraphs. In Section 2.3 the characterization is

used to prove that every unilateral digraph such that every directed cycle has length $\equiv 0 \pmod k$ and every directed cycle with precisely one obstruction has length $\equiv 2 \pmod k$ has a k-kernel.

A digraph $D = (V, A)$ is right-(left-)pretransitive if the existence of the directed path (u, v, w) in D impliest that $(u, w) \in A(D)$ or $(w, v) \in A(D)$ ($(v, u) \in A(D)$). A digraph $D = (V, A)$ is quasi-transitive if the existence of the directed path (u, v, w) in D implies that $(u, w) \in A$ or $(w, u) \in A$. In Chapter 3 both generalizations of transitive digraphs are studied. A new definition of (k, l)-semikernel (an original contribution of this work) is given in section 3.1. Also it is proved that if every vertex of D is a (k, l)-semikernel, then D has a (k, l)-kernel. In Section 3.2 it is proved that every right-pretransitive digraph such that every directed triangle is symmetrical has a k-kernel for every integer $k \geq 2$. An analogous result is proved for left-pretransitive digraphs. In Section 3.3 it is proved that if D is a quasi-transitive digraph, then D has a (k, l)-kernel for every pair of integers k, l such that $k \geq 4$ and $3 \leq l \leq k - 1$ or $k = 3$ and $l = 2$.

Semicomplete multipartite digraphs are studied in Chapter 4. In section 4.2, the k-transitive closure $C_k(D)$ of a digraph $D = (V, A)$ is considered. The vertex set of $C_k(D)$ is V, and $(u, v) \in A(D)$ if and only if $d_D(u, v) \leq k$. It is direct to observe that D has a k-kernel if and only if $C_{k-1}(D)$ has a kernel. In section 4.3 the k-transitive closure is used to prove that every semicomplete multipartite digraph T has a k-kernel for every $m \geq 2$, $k \geq 4$. If every directed cycle of length 4 in T intersects 4 different classes of T, then T has a 3-kernel for every $m \geq 2$. Also, two distinct characterizations for semicomplete multipartite digraphs with a 3-kernel are given.

In Chapter 5 we introduce three new families of digraphs (also an original contribution of this work), two of them generalizing transitive and quasi-transitive digraphs respectively; a digraph D is k-transitive if whenever $(x_0, x_1, \ldots, x_k)$ is a directed path of length k in D, then $(x_0, x_k) \in A(D)$; k-quasi-transitive digraphs are analogously defined. In Section 5.3 some structural results about k-kernels are proved and used to prove that a k-transitive digraph has an n-kernel for every $n \geq k$. Also, we prove that a k-transitive digraph has a k-king if and only if it has a unique initial strong component. The study of k-quasi-transitive digraphs is divided between Section 5.4, where some basic structural results are proved; and Section 5.5, where it is proved that for even $k \geq 2$, every k-quasi-transitive digraph has an n-kernel for every $n \geq k+2$; and that every 3-quasi-transitive digraph has k-kernel for every $k \geq 4$. The fact that, for even $k \in \mathbb{Z}$, a k-quasi-transitive digraph has a $(k+1)$-king if and only if it has a unique initial strong component is also proved. Similar results are proved for k-quasi-transitive digraphs where $k \in \mathbb{Z}^+$ is odd, but it was not possible to get results as good as the obtained for the even case.

The following conjecture is proposed in Chapter 6: If D is a digraph with circumference l, then D has a l-kernel. This conjecture is proved for two families of digraphs and a partial result is obtained for a third family. In Section 6.2 it is proved that if D is a σ-strong digraph with circumference l, then D has a $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-kernel for every $k \geq 2$. A digraph D is locally in(out)-semicomplete if whenever $(v,u),(w,u) \in A(D)$ $((u,v),(u,w) \in A(D))$ and $v \neq w$, then $(v,w) \in A(D)$ or $(w,v) \in A(D)$. In Section 6.3 it is proved that if D is a locally in/out-semicomplete digraph such that, for a fixed integer $l \geq 1$, $(u,v) \in A(D)$ implies $d(v,u) \leq l$, then D has a (k,l)-kernel for every $k \geq 2$. As a consequence of this theorems we have that every $(l-1)$-strong digraph with circumference l and every locally out-semicomplete digraph with circumference l have an l-kernel, and every locally in-semicomplete digraph with circumference l has an l-solution. Also, in Section 6.4 we prove that every k-quasi-transitive digraph with circumference $l \leq k$ has a n-kernel for every $n \geq k$.

Chapter 7 is different from the rest. We propose an extension of the definition of (k,l)-kernel to (arc-)weighted digraphs, verifying which of the existing results for k-kernels are valid in this extension. If D is a digraph and $\mathsf{w} : A(D) \to \mathbb{Z}$ is a weight function for the arcs of D, we can restate the problem of finding a k-kernel in the following way. If $\mathscr{C}$ is a walk in D, the weight of $\mathscr{C}$ is defined as $\mathsf{w}(\mathscr{C}) := \sum_{f \in A(\mathscr{C})} \mathsf{w}(f)$. A subset $S \subseteq V(D)$ is (k,w)-independent if, for every $u,v \in S$ there does not exist an uv-directed path of weight less than k. A subset $S \subseteq V(D)$ will be (l,w)-absorbent if, for every $u \in V(D) \setminus S$, there exists an uS-directed path of weight less than or equal to l. A subset $N \subseteq V(D)$ is a (k,l,w)-kernel if it is (k,w)-independent and (l,w)-absorbent. In Section 7.1 it is proved, among other results, that every transitive digraph has a $(k,k-1,\mathsf{w})$-kernel for every k, that if T is a tournament and $\mathsf{w}(a) \leq \frac{k-1}{2}$ for every $a \in A(T)$, then T has a (k,w)-kernel and that if every directed cycle in a quasi-transitive digraph D has weight $\leq \frac{k-1}{2}+1$, then D has a (k,w)-kernel. In Section 7.2 weighted digraphs where the weight function $\mathsf{w} : A(D) \to G$ has an arbitrary group as codomain is considered.

In Chapter 8, Infinite digraphs are studied. In Section 8.1, a brief survey

about (k,l)-kernels in infinite digraphs is given. The rest of the chapter is devoted to consider the results of previous chapters and to see which of them can be stated for infinite digraphs. The main tool for generalizing those results is a lemma, stating that if D is an infinite digraph such that every vertex of D is a (k,l)-semikernel, then D has a (k,l)-kernel.

Contents

Chapter 1

Introduction

1.1 Digraphs and Subdigraphs

Our most basic objects are digraphs (or directed graphs). A **digraph** consists of a non-empty set $V(D)$ of elements called **vertices**, together with a finite set $A = A(D)$ of ordered pairs of distinct vertices of V called **arcs**. We call $V(D)$ the vertex set and $A(D)$ the arc set of D. We will often write $D = (V, A)$, which means that V and A are the vertex set and arc set of D, respectively. The **order** (**size**) of D is the cardinality of $V(D)$ $(A(D))$. In the present work all digraphs will be considered to have finite order and size, unless stated otherwise.

For an arc (u, v) the first vertex u is its **tail** and the second vertex v is its **head**. We also say that the arc (u, v) leaves u and enters v. The head and tail of an arc are its end-vertices; we say that the end-vertices are adjacent, *i.e.* u is adjacent to v and v is adjacent to u. If (u, v) is an arc, we also say that v absorbs u (or u is absorbed by v) and denote it by $u \to v$. As a dual notion, if (u, v) is an arc, we say that u dominates v. We say that a vertex u is incident to an arc a if u is the head or tail of a. For subsets X and Y of $V(D)$, an XY-arc is an arc with tail in X and head in Y. For disjoint subsets X and Y of $V(D)$, $X \to Y$ means that every vertex of X dominates every vertex of Y, and $X \mapsto Y$ means that $X \to Y$ and there are no YX-arcs. In the digraph D of Figure 1.1, $\{v_4, v_5\} \mapsto \{v_8, v_9\}$.

The above definition of a digraph implies that we allow a digraph to have arcs with the same end-vertices, that is to say (u, v) and (v, u), but we do not allow it to have **parallel** (also called multiple) arcs, that is, pairs of arcs with the same tail and the same head, or **loops** (*i.e.* arcs whose head and tail coincide). When parallel arcs and loops are admissible we speak of **pseudodigraphs**; directed pseudographs without loops are **multidigraphs**. Nonetheless, in the present work we will always consider digraphs, that is, digraphs without loops and without parallel arcs, unless stated otherwise. We will give, in most of the cases, terminology and notation for digraphs only, but their extension to pseudodigraphs can be given in the natural way. Figure 1.1 depicts a digraph D and a pseudodigraph D'.

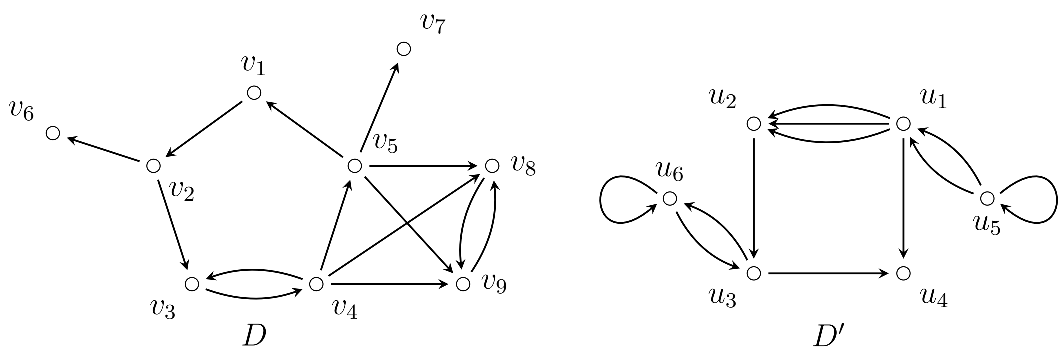


Figure 1.1: A digraph D with symmetrical arcs $(v_3, v_4), (v_8, v_9)$ and a pseudodigraph D' with loops at $(u_5, u_5), (u_6, u_6)$ and parallel arcs at $(u_1, u_2), (u_5, u_1)$.

If v is a vertex of the digraph D, the sets $N_D^+(v)$, $N_D^-(v)$ and $N(v) = N_D^+(v) \cup N_D^-(v)$ are called the **out-neighborhood**, **in-neighborhood** and **neighborhood** of v respectively. We call the vertices in $N_D^+(v)$, $N_D^-(v)$ and $N(v) = N_D^+(v) \cup N_D^-(v)$ the **out-neighbors**, **in-neighbors** and **neighbors** of v, respectively. In Figure 1.1, $N_D^+(v_5) = \{v_1, v_7, v_8, v_9\}$ and $N_D^-(v_5) = \{v_4\}$. For a vertex v, the **out-degree** of v, denoted $d^+(v)$, is the number of arcs with tail v. Let us remark that if D is a digraph (*i.e.*, without loops or parallel arcs), then the out-degree of a vertex equals the number of out-neighbors of this vertex. This is not the case for pseudodigraphs, we may observe that, for example in Figure 1.1, $N_{D'}^+(u_1) = \{u_2, u_4\}$, but $d_{D'}^+(u_1) = 4$. The **in-degree** and **degree** of a vertex v are analogously defined and denoted $d^-(v)$ and $d(v)$, respectively.

A digraph H is a **subdigraph** of a digraph D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and every arc in $A(H)$ has both end vertices in $V(H)$. If $V(H) = V(D)$, we say that H is a **spanning subdigraph** of D. If every arc of $A(D)$ with both end-vertices in $V(H)$ is in $A(H)$, we say that H is induced by $X = V(D)$ (we write $H = D[X]$) and call H an **induced** of D. If H is a subdigraph of D, then we say that D is a **superdigraph** of H.

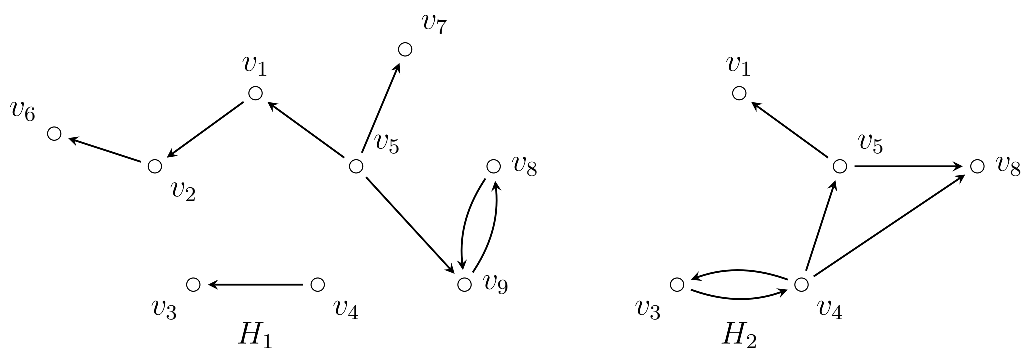


Figure 1.2: Both digraphs H_1 and H_2 are subdigraphs of the digraph D of Figure 1.1. The digraph H_1 is a spanning non-induced subdigraph of D and H_2 is the subdigraph of D induced by the set $\{v_1, v_3, v_4, v_5, v_8\}$.

1.2 Isomorphism and Basic Operations on Digraphs

Suppose $D = (V, A)$ is a digraph and let xy be an arc of D. By **reversing the arc** (x, y), we mean that we replace the arc (x, y) by the arc (y, x).

A pair of directed digraphs D and H are **isomorphic** (denoted by $D \cong H$) if there exists a bijection $\varphi : V(D) \to V(H)$ such that $(x, y) \in A(D)$ if and only if $(\varphi(x), \varphi(y)) \in A(H)$ for every ordered pair x, y of vertices in D. The function φ is an **isomorphism**. As is usual in mathematics, we will often not distinguish between isomorphic digraphs. For example, we may say that there is only one digraph on a single vertex and there are exactly three digraphs with two vertices. For a set of digraphs Ψ we say that a digraph D belongs to Ψ or is a member of Ψ (denoted $D \in \Psi$) if D is isomorphic to a digraph in Ψ. Since we usually do not distinguish between isomorphic digraphs, we will often write $D = H$ instead of $D \cong H$ for isomorphic D and H.

The **converse** or **dual** of a digraph D is the digraph H which one obtains from D by reversing all arcs. It is easy to verify, using only the definitions of isomorphism and converse, that a pair of digraphs are isomorphic if and only if their converses are isomorphic. To obtain subdigraphs we use the following operations of **deletion**. For a digraph D and a set $B \subseteq A(D)$, the digraph $D-B$ is the spanning subdigraph of D with arc set $A(D) \setminus B$. If $X \subseteq V(D)$, the digraph $D-X$ is the digraph induced by $V(D) \setminus X$, *i.e.* $D-X = D[V(D) \setminus X]$. For a subdigraph H of D, we define $D-H = D-V(H)$. Since we do not distinguish between a singleton $\{x\}$ and the element x itself, we will often write $D-x$ rather than $D-\{x\}$. If H is a non-induced subdigraph of D, then there is an arc $a=(x,y)$ such that $x, y \in V(H)$ and $(x,y) \in A(D) \setminus A(H)$. We can construct another subdigraph H' of D by adding a of H; $H' = H + a$.

Let G be a subdigraph of a digraph D. The **contraction** of G in D is a digraph D/G with $V(D/G) = \{g\} \cup (V(D) \setminus V(G))$, where g is a 'new' vertex not in D and $A(D/G) = A(D-G) \cup \big\{(v,g) \big| (v,y) \in A(D), y \in V(G)\big\} \cup \big\{(g,v) \big| (x,v) \in A(D), x \in V(G)\big\}$. Note that D has no parallel arcs and, if D is loopless, then D/G is also loopless. Let $G_1, G_2, \ldots, G_t$ be vertex-disjoint subdigraphs of D. Then

$$D/\{G_1, G_2, \ldots, G_t\} = (\ldots((D/G_1)/G_2)\ldots)/G_t.$$

Clearly, the resulting digraph $D/\{G_1, G_2, \ldots, G_t\}$ does not depend on the order of $G_1, G_2, \ldots, G_t$. Contraction can be defined for sets of vertices, rather than subdigraphs. It suffices to view a set of vertices X as a subdigraph with vertex set X and no arcs.

To construct 'bigger' digraphs from 'smaller' ones, we will often use the following operation called **composition**. Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $G_1, G_2, \ldots, G_n$ be digraphs which are pairwise vertex-disjoint. The composition $D[G_1, G_2, \ldots, G_n]$ is the digraph L with vertex set $\bigcup_{i=1}^n V(G_i)$ and arc set $(\bigcup_{i=1}^n A(G_i)) \cup \big\{g_i g_j \big| g_i \in V(G_i), g_j \in V(G_j), (v_i, v_j) \in A(D)\big\}$.

The **Cartesian product** of a family of digraphs $D_1, D_2, \ldots D_n$, denoted by $D_1 \Box D_2 \Box \ldots \Box D_n$ or $\prod_{i=1}^n D_i$, where $n \geq 2$, is the digraph D having $V(D) = V(D_1) \times V(D_2) \times \cdots \times V(D_n)$ (the Cartesian product) and a vertex $(u_1, u_2, \ldots, u_n)$ dominates a vertex $(v_1, v_2, \ldots, v_n)$ of D if and only if there exists an $r \in \{1, 2, \ldots, n\}$ such that $(u_r, v_r) \in A(D_r)$ and $u_i = v_i$ for all $i \in \{1, 2, \ldots, n\} \setminus \{r\}$. It is easy to observe that if x_j is a fixed vertex in $V(G_j)$

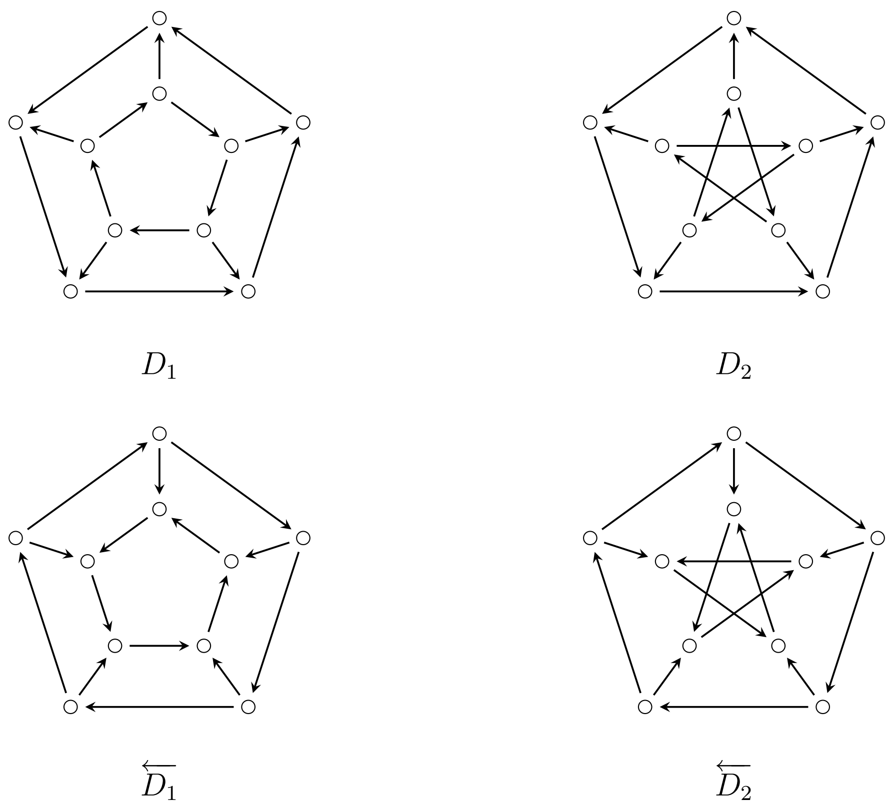


Figure 1.3: Two digraphs D_1 and D_2 and their duals. Also, D_1 is isomorphic to its dual but but D_2 is not.

for $j \neq i$, then the subdigraph induced by $\big\{(x_1, \ldots, y_i, \ldots, x_n)\big|y_i \in V(G_i)\big\}$ is isomorphic to G_i. Examples of contraction, composition and cartesian product are shown in Figure 1.3, but first we will introduce some new concepts.

1.3 Walks, Trails, Paths and Cycles

Let D be a multidigraph. A (directed) **walk** in D is an alternating sequence $\mathscr{C} = x_1a_1x_2a_2x_3 \ldots x_{k-1}a_{k-1}x_k$ of vertices x_i and arcs a_j from D such that the tail of a_i is x_i and the head of a_i is x_{i+1} for every $i \in \{1, 2, \ldots, k-1\}$. The **length** of the walk $\mathscr{C}$ is $k - 1$. A walk is **even** (**odd**) if it has even (odd) length. A walk $\mathscr{C}$ is **closed** if $x_1 = x_k$, and open otherwise. The set of vertices $\{x_1, x_2, \dots x_k\}$ is denoted by $V(\mathscr{C})$; the set of arcs $\{a_1, a_2, \dots, a_{k-1}\}$ is denoted by $A(\mathscr{C})$. We say that $\mathscr{C}$ is a walk **from** x_1 **to** x_k or an x_1x_k-walk. If $\mathscr{C}$ is open, then we say that the vertex x_1 is the **initial** vertex of $\mathscr{C}$, the vertex x_k is the **terminal** vertex of $\mathscr{C}$ and x_1 and x_k are the **end-vertices** of $\mathscr{C}$. When the arcs of $\mathscr{C}$ are defined from the context or simply unimportant (as in digraphs without parallel arcs), we will denote $\mathscr{C}$ by $(x_1, x_2, \dots, x_k)$.

A **trail** is a walk in which all arcs are distinct. Sometimes, we identify a trail $\mathscr{C}$ with the digraph $(V(\mathscr{C}), A(\mathscr{C}))$ which is a subdigraph of D. If the vertices of $\mathscr{C}$ are distinct, $\mathscr{C}$ is a **path**. If the vertices $x_1, x_2, \dots, x_{k-1}$ are distinct, $k \geq 3$ and $x_1 = x_k$, $\mathscr{C}$ is a **cycle**. Since paths and cycles are special cases of walks, the length of a path and a cycle is already defined. The same remark is valid for other parameters and notions, *e.g.* an xy-path. A **longest** path (cycle) in D is a path (cycle) of maximal length in D.

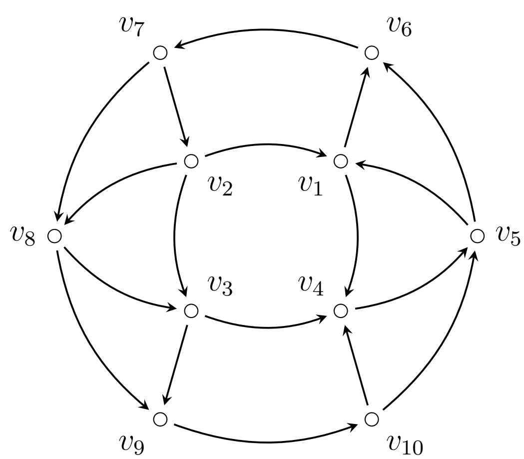
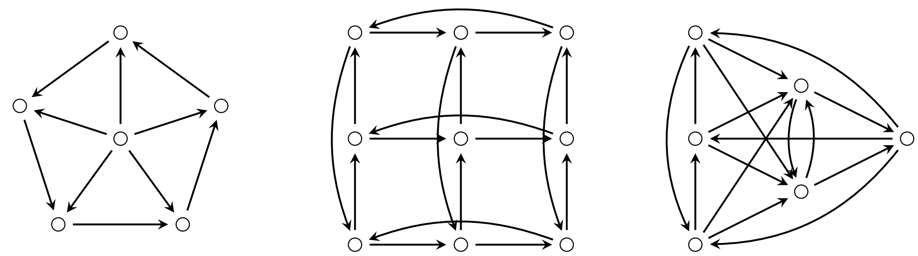


Figure 1.4: Examples of directed walks, trails, paths and cycles: $(v_3, v_9, v_{10}, v_4, v_5, v_1, v_6, v_7, v_2, v_3, v_4, v_5, v_6, v_7, v_2, v_3)$ is a closed walk but not a trail, $(v_2, v_8, v_3, v_9, v_{10}, v_5, v_1, v_6, v_7, v_2, v_3, v_4, v_5, v_6)$ is a trail but not a path, $\mathscr{C} = (v_1, v_6, v_7, v_2, v_8, v_3, v_9, v_{10}, v_4, v_5, v_1)$ is an spanning cycle and $(v_5, v_1, v_6, v_7, v_2, v_8)$ is a subpath of $\mathscr{C}$.

When $\mathscr{C}$ is a cycle and x is a vertex of $\mathscr{C}$, we say that $\mathscr{C}$ is a cycle **through** x. In a digraph D, a loop is also considered a cycle (of length one). A **k-cycle** is a cycle of length k and it is ussually denoted by C_k. An

$|V|$-cycle (spanning cycle) is called Hamiltonian. The minimum integer k for which d has a k-cycle is the girth of D; denoted by $g(D)$. If D does not have a cycle, we define $g(D) = \infty$. If $g(D)$ is finite then we call a cycle of length $g(D)$ a shortest cycle in D. The digraph in Figure 1.4 has girth 3, since (v_1, v_4, v_5, v_1) is a 3-cycle and it has no 2-cycles since it is asymmetrical.



H_1 $\quad$ $C_3 \square C_3$ $\quad$ $C_3[C_3, C_2, K_1]$

Figure 1.5: Examples of contraction, cartesian product and composition. The digraph H_1 is obtained from D_1 (or D_2) of Figure 1.2 contracting the inner 5-cycle.

For subsets X, Y of $V(D)$, an xy-path $\mathscr{C}$ is an XY-path if $x \in X$ and $y \in Y$. Note that if $X \cap Y \neq \varnothing$ then a vertex $x \in X \cap Y$ forms an XY-path by itself. Sometimes we will talk about an HH'-path when H and H' are subdigraphs of D. By this we mean a $V(H)V(H')$-path in D.

Let $\mathscr{C} = (x_1, x_2, \ldots, x_k)$, $\mathscr{D} = (y_1, y_2, \ldots, y_t)$ be a pair of walks in a digraph D. The walks $\mathscr{C}$ and $\mathscr{D}$ are **disjoint** if $V(\mathscr{C}) \cap V(\mathscr{D}) = \varnothing$ and **arc-disjoint** if $A(\mathscr{C}) \cap A(\mathscr{D}) = \varnothing$. If $\mathscr{C}$ and $\mathscr{D}$ are open walks, they are called **internally disjoint** if $\{x_2, x_3, \ldots, x_{k-1}\} \cap V(\mathscr{D}) = \varnothing$ and $V(\mathscr{C}) \cap \{y_2, y_3, \ldots, y_{t-1}\} = \varnothing$. In the digraph of Figure 1.4, the v_7v_4-paths (v_7, v_8, v_3, v_4) and (v_7, v_2, v_1, v_4) are internally disjoint.

We will use the following notation for a path or a cycle $\mathscr{C} = (x_1, x_2, \ldots, x_k)$ (recall that $x_1 = x_k$ if $\mathscr{C}$ is a cycle):

$$x_i \mathscr{C} x_j = (x_i, x_{i+1}, \ldots, x_j).$$

It is easy to see that $x_i \mathscr{C} x_j$ is a path; we call it the **subpath** of $\mathscr{C}$ from x_i to x_j. If $1 < i \leq k$ then the **predecessor** of x_i on $\mathscr{C}$ is the vertex x_{i-1}. If $1 \leq i < k$ then the **successor** of x_i on $\mathscr{C}$ is the vertex x_{i+1}.

Now, we present a very basic and very important result about walks, paths and cycles in digraphs.

Proposition 1.3.1. *Let D be a digraph and let x, y be a pair of distinct vertices in D. If D has an xy-walk $\mathscr{C}$, then D contains an xy-path $\mathscr{D}$ such that $A(\mathscr{D}) \subseteq A(\mathscr{C})$. If D has a closed xx-walk $\mathscr{C}$, then D contains a cycle $\mathscr{D}$ through x such that $A(\mathscr{D}) \subseteq A(\mathscr{C})$.*

A digraph D is **acyclic** if it has no cycle. Acyclic digraphs form a well-studied family of digraphs, in particular, due to the following important properties.

Proposition 1.3.2. *Every acyclic digraph has a vertex of in-degree zero as well as a vertex of out-degree zero.*

If D is a digraph, a vertex v in $V(D)$ will be called an **initial vertex** if $d^-(v) = 0$ and **terminal vertex** if $d^+(v) = 0$. Thus, Proposition 1.3.2 affirms that every acyclic digraph has at least one initial and one terminal vertex.

Let D be a digraph and let $x_1, x_2, \ldots, x_n$ be an ordering of its vertices. We call this ordering an **acyclic ordering** if, for every arc x_ix_j in D, we have $i < j$. Clearly an acyclic ordering of D induces an acyclic ordering of every subdigraph H of D. Since no cycle has an acyclic ordering, no digraph with a cycle has an acyclic ordering. On the other hand, the following holds.

Proposition 1.3.3. *Every acyclic digraph has an acyclic ordering of its vertices.*

The following proposition, although rather obvious, is very useful for our purposes.

Proposition 1.3.4. *Let D be an acyclic digraph and $v \in V(D)$ a non-initial, non-terminal vertex. Then, there are an initial vertex x and a terminal vertex y such that an xv-path and a vy-path exist.*

1.4 Strong and Unilateral Connectivity

In a digraph D a vertex v is **reachable** from a vertex u if D has a uv-walk. In particular, a vertex is reachable from itself. By Proposition 1.3.1, v is reachable from u if and only if D contains a uv-path. A digraph D

is **strongly connected** (or just **strong**) if, for every pair u, v of distinct vertices in D, there exists a uv-walk and a vu-walk. In other words, D is strong if every vertex of D is reachable from every other vertex of D. We define a digraph with one vertex to be strongly connected.

A digraph D is **complete** if, for every pair u, v of distinct vertices of D, both (u, v) and (v, u) are arcs of D. For a strong digraph $D = (V, A)$, a set $S \subset V$ is a **separator** (or a **separating set**) if $D - S$ is not strong. A digraph D is **k-strongly connected** (or **k-strong**) if $|V| \geq k+1$ and D has no separator with less than k vertices. It follows from the definition of strong connectivity that a complete digraph with n vertices is $(n-1)$-strong, but is not n-strong. The largest integer k such that D is k-strongly connected is the **vertex-strong connectivity** of D (denoted by $\kappa(D)$). If a digraph D is not strong, we set $\kappa(D) = 0$.

A **strong component** of a digraph D is a maximal induced subdigraph of D which is strong. If $D_1, \ldots, D_t$ are the strong components of D, then clearly $\bigcup_{i=1}^{t} V(D_i) = V(D)$ (recall that a digraph with only one vertex is strong). Moreover, we must have $V(D_i) \cap V(D_j) = \varnothing$ for every $i \neq j$; thus, $\{V(D_i)\}_{i=1}^{n}$ is a partition of $V(D)$. Let D be a digraph with strong components $\{V(D_i)\}_{i=1}^{n}$, then the **condensation** $D^\star$ of D, is a digraph such that $V(D^\star) = \{D_i\}_{i=1}^{n}$ and $(D_i, D_j) \in A(D^\star)$ if and only if there is a D_iD_j-arc in D. The subdigraph of D induced by the vertices of a cycle in D is strong, *i.e.*, is contained in a strong component of D. Thus, $D^\star$ is acyclic. The strong components of D which are vertices of $D^\star$ of in-degree (out-degree) zero are the **initial** (**terminal**) **strong components** of D. The remaining strong components of D are called **intermediate strong components** of D.

A digraph D is **unilateral** if, for every pair u, v of vertices of D, either u is reachable from v or v is reachable from u. Clearly, every strong digraph is unilateral. Figure 1.6 depicts an unilateral digraph D and its condensation $D^\star$.

The following are characterizations of strong and unilateral digraphs.

Proposition 1.4.1. *A digraph D is strong if and only if there exists a spanning closed walk in D. A digraph D is unilateral if and only if there exists a spanning walk in D.*

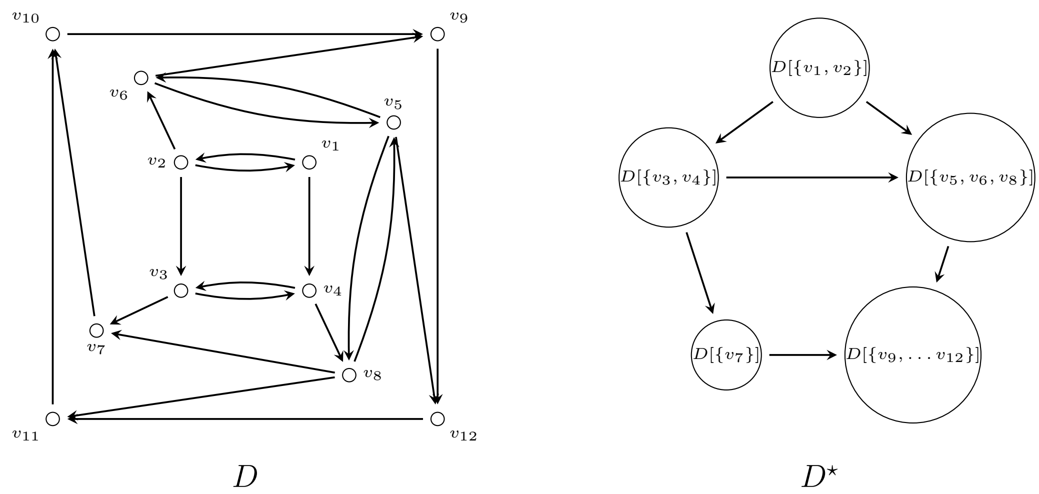


Figure 1.6: An unilateral digraph D with spanning directed walk $(v_3, v_4, v_1, v_2, v_6, v_5, v_8, v_7, v_{10}, v_9, v_{12}, v_{11})$ and its condensation $D^\star$.

1.5 Undirected Graphs, Biorientations and Orientations

An **undirected graph** (or a **graph**) $G = (V, E)$ consists of a non-empty finite set $V = V(G)$ of elements called vertices and a finite set $E = E(G)$ of unordered pairs of vertices called edges. We call $V(G)$ the **vertex set** and $E(G)$ the **edge set** of G. In other words, an edge $\{x, y\}$ is a 2-element subset of $V(G)$. We will often denote $\{x, y\}$ just by xy. If $xy \in E(G)$, we say that the vertices x and y are **adjacent**. Notice that, in the above definition of a graph, we do not allow loops or parallel edges. The **complement** $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

When parallel edges and loops are admissible we speak of **pseudographs**; pseudographs with no loops are **multigraphs**. For a pseudograph G, a pseudodigraph D is called a **biorientation** of G if D is obtained from G by replacing each edge $\{x, y\}$ of G by either (x, y) or (y, x) or the pair (x, y) and (y, x) (except for a loop (x, x) which is replaced by a (directed) loop at x). Note that different copies of the edge xy in G may be replaced by different arcs in D. An **orientation** of a graph G is a biorientation of G having no

2-cycle and no loops. An orientation of a graph is often called an **oriented graph**. Clearly, every digraph is a biorientation and every oriented graph an orientation of some undirected graph. The **underlying graph** $UG(D)$ of a digraph D is the unique graph G such that D is a biorientation of G. For a graph G, the complete biorientation of G (Denoted by $\overleftrightarrow{G}$) is a biorientation D of G such that $(x,y) \in A(D)$ implies $(y,x) \in A(D)$. An arc (x,y) in a digraph $D = (V,A)$ is **symmetrical** if $(y,x) \in A$ and **asymmetrical** if $(y,x) \notin A(D)$. A digraph D is **symmetrical** (**asymmetrical**) if every arc of D is symmetrical (asymmetrical). Clearly, D is symmetrical if and only if D is the complete biorientation of some graph. An **oriented path (cycle)** is an orientation of a path (cycle).

Under the previous notions, an oriented graph is just an asymmetrical digraph. A **tournament** is a loopless oriented graph where every pair of distinct vertices are adjacent. In other words, a digraph T with vertex set $\{v_1, v_2, \ldots, v_n\}$ is a tournament if exactly one of the arcs (v_i, v_j) and (v_j, v_i) is in T for every $i \neq j \in \{1, 2, \ldots, n\}$.

A graph G is **connected** if its complete biorientation $\overleftrightarrow{G}$ is strongly connected. Similarly, G is k-connected if $\overleftrightarrow{G}$ is k-strong. Strong components in $\overleftrightarrow{G}$ are **connected components**, or just **components** in G. A **bridge** in a connected graph G is an edge whose deletion from G makes G disconnected. A graph G is ***k*-edge-connected** if the graph obtained from G after deletion of at most $k-1$ edges is connected. Clearly, a connected pseudograph is bridgeless if and only if it is 2-edge-connected. The **neighborhood** $N_G(x)$ of a vertex x in G is the set of vertices adjacent to x. The **degree** $d(x)$ of a vertex x is the number of edges, except loops, having x as an end-vertex. A pair of graphs G and H are isomorphic if $\overleftrightarrow{G}$ and $\overleftrightarrow{H}$ are isomorphic.

A digraph is *connected* if its underlying graph is connected. The notions of walks, trails, paths and cycles in undirected pseudographs are analogous to those for pseudodigraphs (we merely disregard orientations). An ***xy*-path** in an undirected pseudograph is a path whose end-vertices are x and y. When we consider a digraph and its underlying graph $UG(D)$, we will often call walks of D **directed** (to distinguish between them and those in $UG(D)$). In particular, we will speak of directed paths, cycles and trails. An undirected graph is a **forest** if it has no cycle. A connected forest is a **tree**. It is easy to see that every connected undirected graph has a **spanning tree**, *i.e.* a spaning subgraph, which is a tree. A digraph D is an **oriented forest** (**tree**) if D is an orientation of a forest (tree).

The following well-known theorem is due to Robbins.

Theorem 1.5.1. *A connected graph G has a strongly connected orientation if and only if G has no bridge.*

A set Q of vertices in a graph or a digraph H is **independent** if the graph (digraph) $H[Q]$ has no edges (arcs). A **(proper) colouring** of a digraph or graph H is a partition of $V(H)$ into (disjoint) independent sets. The minimum number $\chi(H)$, of independent sets in a proper colouring of H is the **chromatic number** of H.

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. A graph G is called *Berge* if it does not contain an induced odd hole (induced odd cycle of length ≥ 5) nor odd-antihole (complement of an odd hole). Claude Berge conjectured [13, 14] that a graph is perfect if and only if it is Berge, this result, known as The Strong Perfect Graph Theorem, was proved by Chudnovsky, et al. [26].

1.6 Classes of Digraphs and Graphs

In this section, we define certain families of digraphs and graphs which will be used in various chapters of this work.

A graph G is **complete** if every pair of distinct vertices in G are adjacent. We will denote the complete graph on n vertices (which is unique up to isomorphism) by K_n. Its complement $\overline{K}_n$ has no edge. Clearly, a tournament of order n is an orientation of K_n.

A graph G is ***p*-partite** if there exists a partition $V_1, V_2, \ldots, V_p$ of $V(G)$ into p independent sets. The special case of a p-partite graph when $p = 2$ is called a **bipartite graph**. We often denote a p-partite graph G with partition $V_1, V_2, \ldots, V_p$ by $G = (V_1, V_2, \ldots, V_p)$. A p-partite graph G is **complete *p*-partite** if, for every pair $x \in V_i$, $y \in V_j$ $(i \neq j)$, the edge xy is in G. A complete graph on n vertices is clearly a complete n-partite graph for which every partite set is a singleton. We denote the complete p-partite graph with partite sets of cardinalities $n_1, n_2, \ldots, n_p$ by $K_{n_1,n_2,\ldots,n_p}$. Complete p-partite graphs for $p \geq 2$ are also called **complete multipartite graphs**.

A ***p*-partite digraph** is a biorientation of a p-partite graph. Bipartite digraphs are of special interest. It is well-known that an undirected graph is bipartite if and only if it has no cycle of odd length. The obvious extension of this statement to cycles in digraphs is not valid. However, the obvious extension does hold for strong bipartite digraphs.

Theorem 1.6.1. *A strongly connected digraph is bipartite if and only if it has no cycle of odd length.*

An extension of this result to digraphs whose cycles are all of length $\equiv 0$ modulo $k \geq 2$ will be given in section 2.

Recall that tournaments are orientations of complete graphs. A **semicomplete digraph** is a biorientation of a complete graph. The complete biorientation of a complete graph is a **complete digraph** (denoted by $\overleftrightarrow{K}_n$). The notion of semicomplete digraphs and their special subclass, tournaments, can be extended in various ways. A biorientation of a complete p-partite (multipartite) graph is a **semicomplete *p*-partite (multipartite) digraph**. A **multipartite tournament** is an orientation of a complete multipartite graph. A semicomplete 2-partite digraph is also called a **semicomplete bipartite digraph**. A **bipartite tournament** is a semicomplete bipartite digraph with no 2-cycles.

Recall that a digraph D is acyclic if D has no (directed) cycle. Obviously, every acyclic digraph is an oriented graph. A digraph D is **transitive** if, for every pair of arcs $(x, y), (y, z) \in A(D)$, such that $x \neq z$, the arc (x, z) is also in D. It is easy to show that a tournament is transitive if and only if it is acyclic. Sometimes, we will deal with transitive oriented graphs, *i.e.* transitive digraphs with no cycle of length two. A digraph D is **quasi-transitive** if, for every triple x, y, z of distinct vertices of D such that $(x, y), (y, z) \in A(D)$, there is at least one arc between x and z. Clearly, a semicomplete digraph is quasi-transitive. Note that, if there is only one arc between x and z, it can have any direction; hence quasi-transitive digraphs are generally not transitive.

1.7 Kernels and Semikernels

Given a digraph $D = (V, A)$, a set $N \subseteq V$ is a **kernel** of D if it is independent and absorbent. Morgenstern and von Neumann introduced the concept of a kernel in a digraph when describing winning positions in 2-person games. Since their introduction in [92] in 1953, kernels have found various applications in many mathematical fields. For instance in Game Theory, where a kernel represents a set of winning positions [23, 34, 92]; in the study of combinatorial games, where the existence of a kernel is closely related to the existence of a winning strategy [15, 16, 34]; in Mathematical Logic, where kernels in graphical representations of finite theories represent minimal sets of counterexamples [19]; in list edge-colourings of graphs (see for example [21]). But also, kernels have been related to very important conjectures in Graph Theory, like Berge's Strong Perfect Graph Conjecture or the Laborde-Payan-Xuong Conjecture.

Also, kernels have an obvious interest by themselves as they are "optimal" sets in a relation. Let us consider a simple example: Imagine that you want to assemble a team of experts for intergalactic exploration and colonization. Obviously, the spacecraft has only limited space and resources, thus, the team of experts must be as small as possible, but it must cover all branches of human knowledge. To choose the team, you can model the problem by considering one vertex for each candidate and if u and v are candidates, the arc (u, v) will represent that candidate v is better than candidate u (*i.e.*, candidate v can effectively replace candidate u in the spacecraft). If two candidates are incomparable, then no arc should be placed between them. A kernel in the resulting digraph will represent an optimal solution: Since the kernel is absorbent, for every person not chosen, there will be someone in the kernel that is a better choice for the mission. Also, since the kernel is independent, for every couple u, v chosen, u and v will be incomparable (and thus, no one better than the other). Although this example is fictional and simplistic, surely the reader can think of a lot of real life problems where a kernel can model an optimal solution. So, one may ask, why kernels are not widely used to solve all this problems? The answer is simple. First of all, not every digraph has a kernel, consider for instance a directed triangle. Also, in [27], Chvátal proved that the problem of determining if a digraph has a kernel is NP-complete. Thus, there are not efficient algorithms to determine if an arbitrary digraph has a kernel. In [33], Fraenkel proved that the problem remains NP-complete even for planar digraphs D with degree constraints $d^+(x) \leq 2, d^-(x) \leq 2$ and $d(x) \leq 3$ for all vertices x.

Since we cannot decide effectively whether an arbitrary digraph has a kernel or not, one of our best possible strategies is to find sufficient conditions to guarantee the existence of a kernel in a digraph. Obviously, we would like to find sufficient conditions that are polynomial time verifiable, for the sake of applications, but this is not always possible. So, we are also interested in sufficient conditions that imply that large families of digraphs have a kernel. The first result of this kind is due to Morgenstern and von Neumann. In [92] it is proved the following theorem.

Theorem 1.7.1. *Every acyclic digraph has a unique kernel.*

A lot of work has been done around this theorem, one of the first generalizations is due to Richardson, [83].

Theorem 1.7.2 (Richardson)**.** *Every digraph without directed odd cycles has a kernel.*

The original proof of Richardson's Theorem was very extense and complicated. In the search of a shorter proof, Víctor Neumann-Lara introduced two concepts. If D is a digraph, a set $S \subseteq V(D)$ is a **semikernel** (local kernel) of D if S is independent and for every vertex $v \in N^+(S)$ there exists a vertex $u \in S$ such that $(v, u) \in A(D)$. A digraph D is **kernel-perfect** if every induced subdigraph of D has a kernel. In [80], Neumann-Lara proved the following sufficient condition for a digraph to be kernel-perfect (and thus, to have a kernel).

Theorem 1.7.3. *Let D be a (possibly infinite) digraph. If every induced subdigraph of D has a semikernel, then D is kernel-perfect.*

In [52], Galeana-Sánchez and Neumann-Lara, using the notion of semikernel, gave sufficient conditions for a digraph to be a kernel-perfect digraph. If $\mathscr{C}$ is a cycle in a digraph D, a pseudodiagonal of $\mathscr{C}$ is an arc with both end-vertices in $V(\mathscr{C})$ but not in $E(\mathscr{C})$.

Theorem 1.7.4. *If every directed cycle of odd length in D has two pseudodiagonals with consecutive terminal end-vertices, then D is kernel-perfect.*

Another classical sufficient condition for a digraph to be kernel-perfect is due to Berge and Duchet, [17].

Theorem 1.7.5. *If every directed cycle of D has at least one symmetrical arc, then D is kernel-perfect.*

Two classical results describing large families of digraphs with a kernel can be found in the book of Claude Berge [11]. First, that every maximal independent set in a symmetrical digraph is a kernel. Also, that in a transitive digraph D, a set $N \subseteq V(D)$ is a kernel if and only if N is a minimal absorbant set. Moreover, in a transitive digraph D, a kernel is obtained by choosing one vertex in each terminal component of D, thus all the kernels have the same cardinality.

The list is extense. We could go on listing sufficient conditions for a digraph to have a kernel or to be kernel-perfect, or families of digraphs that have a kernel, but hopefully, this few examples will give the reader an idea of the work done in this direction. To emphasize the importance of kernels, we would like to go deeper in the relation between kernels and two important problems in Graph Theory.

The first problem is a solved one: The Strong Perfect Graph Conjecture (now the Strong Perfect Graph Theorem). Before stating the theorem, we need some definitions and terminology. A **clique** in a graph G is a subset $C \subseteq V(G)$ such that $G[C]$ is a complete graph. A maximum clique is a clique of the largest possible size in G. The **clique number** of G, $\omega(G)$ is the number of vertices in a maximum clique in G. A graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. An odd hole in a graph is an induced odd cycle, and an odd anti-hole in a graph is an induced subgraph isomporphic to the complement of an odd cycle. The following theorem was conjectured by Berge in 1963 and finally proved in 2006 by Chudnovsky, Robertson, Seymour and Thomas in [26].

Theorem 1.7.6 (Strong Perfect Graph Theorem)**.** *A graph D is perfect if and only if it has no odd holes nor odd anti-holes of length greater than or equal to 5.*

During the time that Theorem 1.7.6 was still a conjecture, a lot of new theory was developed in the attempts to solve it. An alternative characterization of perfect graph was conjectured by Berge and Duchet. A biorientation of a graph G is called **clique-acyclic** if every clique has a kernel. An undirected graph G is called **kernel-solvable** if every clique-acyclic orientation of it has a kernel. Kernel-solvable graphs were introduced by Berge and Duchet [18], who conjectured that they are the same as perfect graphs. The fact that kernel-solvable graphs are perfect follows from Theorem 1.7.6. This alternative characterization of perfect graphs is completed by the following theorem of Boros and Gurvich, [22].

Theorem 1.7.7. *Every perfect graph G is kernel-solvable.*

An excellent survey on kernels, perfect graphs and cores of cooperative games is due to Boros and Gurvich, [23].

The second problem where kernels are related remains open. A longest path in a digraph D is a directed path of maximum length in D. In [78], Laborde, Payan and Xuong conjectured that every digraph has an independent set intersecting every longest path. This conjecture has been open since 1982. As a special case, they proved that their conjecture is true for every digraph with a kernel.

1.8 (k,l)-kernels

We have already mentioned that not every digraph has a kernel. An obvious alternative was outlined in the previus section, to find sufficient conditions for the existence of kernels in digraphs. Another usual way to proceed in Mathematics is to consider a weakening of the concept that preserves some of the central properties of the original concept, without being as restrictive as the original. One weakening of the concept of kernel has been already introduced in the previous section, a semikernel. Nonetheless, the basic example of a digraph that does not have a kernel, the directed triangle, neither has a semikernel. Chvátal and Lovász introduced in [28] the concept of quasi-kernel of a digraph (although they originally called it a semi-kernel). A set $S \subseteq V(D)$ is a **quasi-kernel** of the digraph D if it is independent and for every vertex $u \notin S$ there exists $v \in S$ such that $d(u,v) \leq 2$. This notion generalizes the notion of kernel, as it is clear that every kernel is a quasi-kernel. Besides, a quasi-kernel is also an independent and 'absorbing' structure, but instead of the usual notion of absorbence, it 'absorbs' every vertex outside the quasi-kernel at distance 2. Quasi-kernels were introduced to generalize the famous result due to Landau, [79], asserting that in every tournament T there exists a vertex $u \in V(T)$ such that $d(u,v) \leq 2$ for every $v \in V(T)$. This simple weakening, *i.e*, letting a quasi-kernel absorb the remaining vertices of the digraph at distance 2, gives place to the following surprising result.

Theorem 1.8.1 (Chvátal, Lovász)**.** *Every digraph has a quasi-kernel.*

With this excellent result and the idea of changing the (distance 1) absorbence of a kernel for the 'distance 2 absorbence' in a quasi-kernel, it immediately comes to mind a further generalization of the notion of kernel. It is true that a quasi-kernel can be thought as a 'not so good' kernel, and it turned out that it was easier to determine if a digraph has a quasi-kernel than determining if a digraph has a kernel. But, although mathematicians like short and elegant solutions, we do not like very easy problems. Then, it comes to mind, if we let a quasi-kernel absorb the remaining vertices of the digraph at distance 2 and the problem became very easy, we can put an aditional restriction on our absorbing set of vertices to get an interesting problem once again. One obvious restriction is to ask the vertices in our set to be at a greater distance. With this, we do not only get an interesting (and more general) problem again, we also get the possibility to retrieve more information when we use this structure to model a situation. For example, if we want to bring a new service to the population of a city, and this service will accesible to the population through a set of service centers that will be located across the city, we can optimize the number and location of service centers using the following reasoning. We want that, from every point of the city, a service center can be reached in at most distance $l \in \mathbb{Z}^+$, so all the population have an easy acces to the service we are offering. But building and mantaining a lot of service centers is expensive, so we do not want to have redudant centers, that is two center covering almost the same area, so, service centers should be away enough from each other, let us say, at distance $k \in \mathbb{Z}^+$. A digraph can easily model the map of the city, and a solution to our problem would be a set N of vertices such that, the distance between vertices of N is at least k and the distance from a vertex not in N to a vertex in N is at most l.

In [74], Kwaśnik and Borowiecki introduced the concept of (k, l)-kernel, generalizing both, kernels and quasi-kernels. If D is a digraph and $S \subseteq V(D)$, we say that S is ***k*-independent** if $d(u, v) \geq k$ for every pair of distinct vertices $u, v \in S$; and we call S ***l*-absorbent** if for every $u \in V \setminus S$, there exists $v \in S$ such that $d(u, v) \leq l$. A **(*k*,*l*)-kernel** in a digraph D is a set $N \subseteq V(D)$ that is k-independent and l-absorbent. Thus, a kernel is a $(2, 1)$-kernel and a quasi-kernel is a $(2, 2)$-kernel. A dual to the notion of (k, l)-kernel can be defined in the following way. We call a set $S \subseteq V$ ***l*-dominating** if for every $u \in V \setminus S$, there exists $v \in S$ such that $d(v, u) \leq l$. A **(*k*,*l*)-solution** in a digraph D is a set $N \subseteq V(D)$ that is k-independent and l-dominating. Many results concerning (k, l)-kernels can be dualized to (k, l)-solutions using the dual digraph $\overleftarrow{D}$.

A structure related to (k, l)-solutions has received a lot of attention in the last couple of decades. An ***l*-king** is an l-dominating set consisting of a single vertex. Thus, clearly an l-king is a (k, l)-solution for every integer $k \geq 2$. An ***l*-serf** is an l-absorbing set consisting in a single vertex. The notion of l-serf dualizes that of l-king, hence, an l-serf is a (k,l)-kernel for every integer $k \geq 2$.

Now, once again, we have that for an arbitrary choice of a pair of positive integers k, l, it does not necessarily exists a (k,l)-kernel. One would expect that most (or at least some) of the existing sufficient conditions for the existence of kernels in digraphs could be generalized to (k,l)-kernels. Nonetheless, for some choices of k and l, the most general sufficient condition for the existence of kernels in digraphs fails for (k,l)-kernels. There are acyclic digraphs without a $(k, k-2)$ for every integer $k \geq 3$, moreover, it can be observed that the directed path of length $k-1$ fails to have a $(k, k-2)$-kernel.

Clearly, a (k,l)-kernel is also an (n,m)-kernel for every $2 \leq n \leq k$ and every $m \geq l$. So, for a (k,l)-kernel, an (n,m)-kernel with $2 \leq n \leq k$ and $m \geq l$ can be think as a relaxation of the (k,l)-kernel, and an (n,m)-kernel with $n \geq k$ and $m \leq l$ a strengthening of the (k,l)-kernel. For $l \leq k-2$ it can be easily observed that there exist acyclic digraphs (not only acyclic digraphs, but directed paths) that does not have a (k,l)-kernel. Also, we will prove in Chapter 2 that every acyclic digraph has a unique $(k, k-1)$-kernel for every integer $k \geq 2$. So, every acyclic digraph has a (k,l)-kernel for every pair of integers k, l such that $l \geq k-1$. From this observations we can conclude that $(k, k-1)$-kernels are of particular interest since they are good candidates to generalize properties of kernels. We define a ***k*-kernel** to be a $(k, k-1)$-kernel. Under this definition a kernel in the usual sense is a 2-kernel.

Despite the fact that (k,l)-kernels and, in particular, k-kernels seem to be a good generalization of the notion of kernel, the study of both concepts have been very limited, in comparison to the vast aumont of articles and applications that exist related to kernels. When the development of this work began, there were only a few articles devoted to prove the existence of general sufficient conditions for the existence of (k,l)-kernels in digraphs or to show that large families of digraphs have a (k,l)-kernel. One of the cornerstones in the study of sufficient conditions for the existence of k-kernels in digraphs, and therefore of the present work, is the following result due to Kwaśnik [74].

Theorem 1.8.2. *Let D be a strong digraph. If every directed cycle of D has length $\equiv 0$ (mod k), then D has a k-kernel.*

Theorem 1.8.2 is very important because it is a generalization of Richardson's Theorem (Theorem 1.7.2). It is also the first generalization of a major theorem about kernels to k-kernels. Later, Galeana-Sánchez gave a slight generalization of this result in [35]. Other papers devoted to finding sufficient conditions for the existence of (k, l)-kernels in digraphs, generalizing known sufficient conditions for the existence of kernels, include those of Galeana-Sánchez [36], Galeana Sánchez with Rincón-Mejía [60], of Bród, Włoch and Włoch [24] and the work of Kwaśnik, Włoch and Włoch. In [77], Kwaśnik, Włoch and Włoch prove that the problem of finding a k-kernel in a digraph D is equivalent to the problem of finding a kernel in a digraph D^{k-1} obtained from D.

The work dealing with the existence of (k, l)-kernels in some families of digraphs include the article Kucharska and Kwaśnik [73], where the existence of (k, l)-kernels in special superdigraphs of a directed path and a directed cycle are considered; and also the articles of Galeana-Sánchez and Pastrana-Ramírez [57, 58], where sufficient conditions for the existence of k-kernels in the orientation of the line graph and path graph are given. In [97], Włoch and Włoch construct families of digraphs having (k, l)-kernels for distinct values of k and l.

A subject that has received a lot of attention is the relation between (k, l)-kernels and some operations on digraphs. The existence and structure of (k, l)-kernels in distinct products of digraphs have been largely studied by many authors, *e.g.*, by Kwaśnik in [74, 76], by Kwaśnik, Włoch and Włoch in [77], by Włoch and Włoch in [95, 96, 98] and by Szumny, Włoch and Włoch in [86, 87]. In some cases, results characterizing the structure of (k, l)-kernels in a product of digraphs is given in terms of the (k, l)-kernels in the factors of the product. Also, the number of (k, l)-kernels, the number of k-independent sets and the number of l-absorbing sets in products of digraphs are studied.

Other operations have been also considered. Galeana-Sánchez and Gómez give in [42] sufficient conditions for a state splitting to preserve kernels, a (k, l)-kernels and (k, l)-semikernels. In [55], Galeana-Sánchez and Pastrana-Ramírez construct, for any given digraph D, a digraph $s(S)$ such that D has a k-kernel if and only if $s(S)$ has a k-kernel. In [59] they also prove that the number of k-kernels in $s(S)$ is the same as in D. New operations are also defined and considered in [56] by Galeana-Sánchez and Pastrana-Ramírez.

The aforementioned articles, along with the articles derived from the present work [45, 46, 47, 48, 49, 50, 51, 67], seem to be the only existing publications on the subject of (k, l)-kernels in digraphs.

Chapter 2

Cyclically k-partite digraphs

2.1 Introduction

Several classes of k-partite graphs and digraphs have been extensively studied as they are a natural generalization of bipartite graphs and digraphs; k-partite tournaments (*e.g.* [4]), which have been studied for hamiltonicity and pancyclism, and cyclically k-partite digraphs stand out for their multiple properties.

In this section we introduce another well known class of k-partite digraphs. A digraph D is **cyclically k-partite** if there exists a k-partition of $V(D)$, $V_0, V_1, \dots V_{k-1}$ such V_i that every arc of D is a V_iV_{i+1}-arc (mod k). It is clear from the definition that cyclically k-partite digraphs are k-partite digraphs, since V_i is independent for every $i \in \{1, 2, \dots, k\}$. Cyclically k-partite digraphs have received attention for their connection with matrix theory (*e.g* [25]) in the study of the properties of cyclic matrices and some special cases of diagonal matrices since the digraph associated with an irreducible matrix with imprimitivity index k is exactly a k-partite digraph. The aim of this chapter is to find structural properties of cyclically k-partite graphs and digraphs which are of general interest and that we can use to state sufficient conditions for the existence of k-kernels in some families of digraphs.

As we have already mentioned in Chapter 1, M. Kwasnik stated the following generalization of Richardson's Theorem for k-kernels. If D is a strongly connected digraph such that every directed cycle in D has length $\equiv 0$ (mod k) then D has a k-kernel. It has been noticed that the hypothesis of being strongly connected cannot be dropped, and, altough diverse counterexamples have been considered for the non strongly conected case (e.g. [85]), all of these examples are non unilateral, so the question arises. Can the strong connectedness be substituted for unilaterality? The answer is no, and the following digraph is a counterexample, showing that the hypothesis in Theorem 1.8.2 is sharp.

If the digraph in Figure 2.1 had a 3-kernel, since vertex 10 has outdegree 0 (and thus can not be absorbed by any other vertex) it should be in the 3-kernel, hence vertices $2, 7, 4, 8, 6$ and 9 would be 2-absorbed. The only vertices that could 2-absorb vertex 1 are $2, 3$ and 7, but the distance from vertex 7 to vertex 10 is one, and distance from vertex 2 to vertex 10 is two, so they can not be in the 3-kernel and the only remaining possibilities are that vertex 1 is in the 3-kernel or vertex 3 is in the 3-kernel. We will show that vertex 3 can not be in the 3-kernel and by symmetry vertex 1 neither can be in the 3-kernel. Let us assume that 3 is in the 3-kernel. Now, vertex 5 can be 2-absorbed by vertices $1, 6$ or 9 but $d(1, 3) = 2$, $d(6, 10) = 2$ and $d(9, 10) = 1$ and hence none of them can be added to the 3-kernel but neither can vertex 5, since vertex 3 is at distance two from vertex 5. Consequently, digraph in Figure 2.1 does not have a 3-kernel, its only directed cycle, $(1, 2, 3, 4, 5, 6, 1)$ has length $\equiv 0 \pmod 3$, and is unilaterally connected.

From the previous example it is clear that Theorem 1.8.2 is as best as possible with respect to connectedness. But, strong connectedness is not the only hypothesis of the theorem. The connection between cyclically k-partite digraphs and k-kernels was first noticed in [35], where Galeana-Sánchez proves Theorem 1.8.2 showing that any strongly connected digraph D such that every directed cycle has length $\equiv 0 \pmod k$ is cyclically k-partite. Thanks to the strong connectedness, it is easy to observe that, if $D = (V_1, V_2, \ldots, V_k)$, then V_i is a k-absorbent set for every $i \in \{0, 1, \ldots, k-1\}$, *i.e.*, that for every vertex $u \in V(D) \setminus V_i$, there is a vertex $v \in V_i$ such that $d(u, v) \leq k$. So, our strategy in this chapter is to propose new sufficient conditions for a digraph D to be cyclically k-partite and to find a k-absorbing set in this partition, without asking D to be strong.

2.2 Cyclically k-partite digraphs.

As we mentioned in the previous section, a strong digraph D such that every directed cycle has length $\equiv 0 \pmod k$ is cyclically k-partite. But it is easy

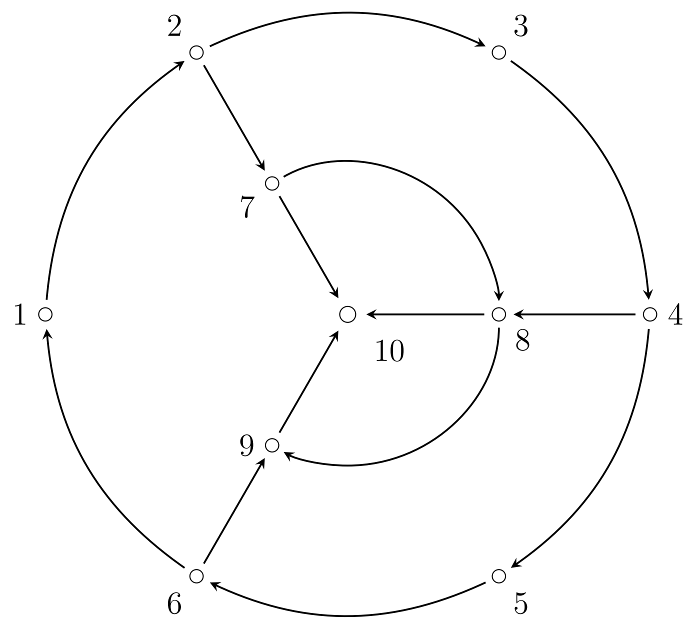


Figure 2.1: Counterexample to a version of Theorem 1.8.2 with weaker hypothesis.

to observe that the converse is also true, so a characterization of strong cyclically k-partite digraphs is obtained. We would like to characterize unilateral cyclically k-partite digraphs. Obviously, if we replace strong by unilateral connectedness, we need to ask for an additional restriction on the structure of a digraph D if we wish that D results cyclically k-partite. The cycles of the digraph are the obvious starting point to look for this additional restriction.

Definition 2.2.1. A closed walk $\mathscr{C} = (x_0, x_1, \ldots x_n, x_{n+1} = x_0)$ is directed with an obstruction at vertex x_n if there exists a directed walk $\mathscr{C}' = (x_0, x_1, \ldots, x_n)$ and an arc $(x_0, x_n) \in A(D) \setminus A(\mathscr{C}')$ such that $\mathscr{C} = \mathscr{C}' \cup (x_0, x_n)$.

Figure 2.2 shows a cycle $\mathscr{C} = (0, 1, 2, 3, 4, 5, 6, 7, 0)$ with an obstruction at vertex 7, where $\mathscr{C}' = (0, 1, 2, 3, 4, 5, 6, 7)$ and $\mathscr{C} = \mathscr{C}' \cup (0, 7)$. If we reverse the arc $(0, 7)$, the sequence $\mathscr{C}$ will denote a directed cycle.

In Definition 2.2.1 it is important to notice that $(x_0, x_n) \notin A(\mathscr{C}')$ so its reversal turns $\mathscr{C}$ into a closed directed walk. Figure 2.3 digraph (i) shows a digraph with a closed walk $\mathscr{C} = (0, 1, 2, 0, 3, 4, 2, 3, 0)$ such that there exist a

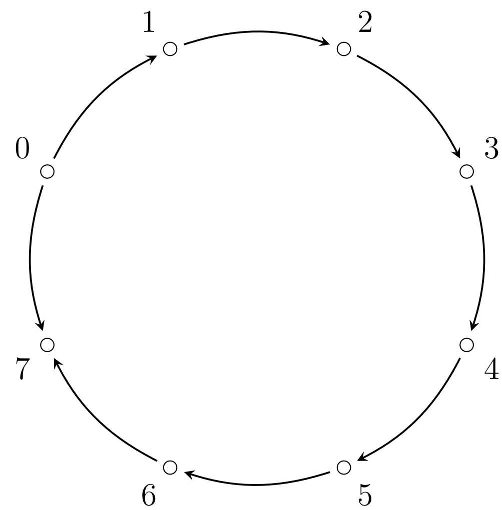


Figure 2.2: A cycle with an obstruction at vertex 7.

directed walk $\mathscr{C}' = (0, 1, 2, 0, 3, 4, 2, 3)$ and an arc $(0, 3) \in A(\mathscr{C}')$ such that $\mathscr{C} = \mathscr{C}' \cup (0, 3)$, but as it can be observed in digraph (ii), the reversal of (x_0, x_n) does not turn $\mathscr{C}$ into a directed walk.

With this definition we state some lemmas leading to a characterization of unilateral cyclically k-partite digraphs.

Lemma 2.2.2. *If $\mathscr{C}$ is a directed closed walk with one obstruction then $\mathscr{C}$ contains a cycle with at most one obstruction.*

Proof. Since $\mathscr{C}$ is a directed closed walk with an obstruction, then $\mathscr{C} = (x_0, x_1, \ldots, x_n, x_{n+1} = x_0)$ where $\mathscr{C}' = (x_0, \ldots, x_n)$ is a directed walk and (x_0, x_n) is an arc not in $A(\mathscr{C}')$. If $\mathscr{D}$ is the directed closed walk obtained from $\mathscr{C}$ by reversing the arc (x_0, x_n), then it is a well known result[1] that $\mathscr{D}$ contains a directed cycle $\mathscr{D}_1$. If $\mathscr{D}_1$ contains the arc (x_n, x_0), then $\mathscr{C}$ contains the directed cycle with one obstruction at vertex x_n $\mathscr{C}_1$, obtained by the reversing of the arc (x_n, x_0) in $\mathscr{D}_1$; if $\mathscr{D}_1$ does not contain the arc (x_n, x_0), then $\mathscr{C}$ contains the directed cycle $\mathscr{D}_1$. ■

Lemma 2.2.3. *If D is a digraph such that every directed cycle in D has length $\equiv 0$ (mod k), then every directed closed walk has length $\equiv 0$ (mod k).*

[1] This result can be found in [11].

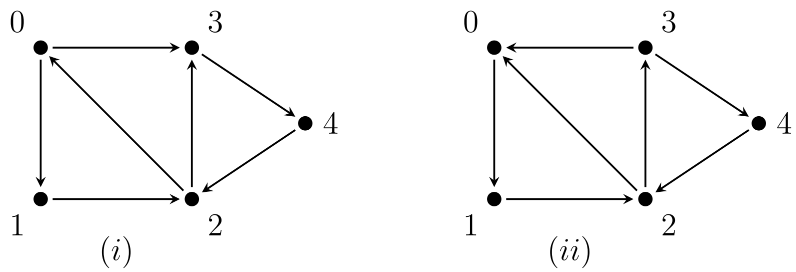


Figure 2.3: Digraph (ii) is obtained from digraph (i) by the reversal of the arc $(0,3)$. The closed walk $(0,1,2,0,3,4,2,3,0)$ is not directed in digraph (i) nor in digraph (ii).

Proof. By induction on $\ell(\mathscr{C}) = n$, where $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ is the directed closed walk. If $n \leq k$, since every directed closed walk contains a directed cycle and every directed cycle in D has length $\equiv 0 \pmod{k}$, then $n = k$. If $n > k$, then $\mathscr{C}$ contains a directed cycle $\mathscr{C}_1 = x_i\mathscr{C}(x_j = x_i)$, where $j > i$. It is clear that if $\mathscr{C}_2 = x_0\mathscr{C}x_i \cup x_j\mathscr{C}x_n$, then $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ and $\ell(\mathscr{C}) = \ell(\mathscr{C}_1) + \ell(\mathscr{C}_2)$. By induction hypothesis $\ell(\mathscr{C}_2) \equiv 0 \pmod{k}$ and $\ell(\mathscr{C}_1) \equiv 0 \pmod{k}$ because $\mathscr{C}_1$ is a directed cycle. Hence, $\ell(\mathscr{C}) \equiv 0$ (mod k). ∎

Lemma 2.2.4. *Let D be a digraph. If every directed cycle has length $\equiv 0$ (mod k) and every directed cycle with one obstruction has length $\equiv r$ (mod k) then every directed closed walk $\mathscr{C}$ with one obstruction fulfills that $\ell(\mathscr{C}) \equiv r$ (mod k).*

Proof. By induction on $\ell(\mathscr{C})$. If $\ell(\mathscr{C}) \leq k$ then $\mathscr{C}$ cannot repeat interior vertices, because it wolud contain a directed closed walk and then a directed cycle, but every directed cycle has length $\equiv 0 \pmod{k}$, thus, $\mathscr{C}$ is a directed cycle with one obstruction and by hypothesis has length $\equiv r \pmod{k}$. If $\ell(\mathscr{C}) > k$ and $\mathscr{C}$ does not repeat interior vertices, then again $\mathscr{C}$ is a directed cycle with one obstruction. Otherwise, there exist an interior vertex x_i such that $x_i\mathscr{C}(x_j = x_i) = \mathscr{C}_1$ is a directed closed walk and as in the proof of Lemma 2.2.3, $\mathscr{C}_2 = x_0\mathscr{C}x_i \cup x_j\mathscr{C}x_n$ is such that $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2$ and $\ell(\mathscr{C}) = \ell(\mathscr{C}_1) + \ell(\mathscr{C}_2)$. But, in virtue of Lemma 2.2.3, $\ell(\mathscr{C}_1) \equiv 0 \pmod{k}$ and by induction hypothesis $\ell(\mathscr{C}_2) \equiv r \pmod{k}$. Thence, $\ell(\mathscr{C}) \equiv r \pmod{k}$. ∎

The desired sufficient condition for an unilateral digraph to be cyclically k-partite can be proved now.

Lemma 2.2.5. *If D is an unilateral digraph such that every directed cycle has length $\equiv 0 \pmod k$ and every directed cycle with one obstruction has length $\equiv 2 \pmod k$, then D is cyclically k-partite.*

Proof. First observe that to have a k-partition of D we need at least k vertices, so we will suppose that $\left|V(D)\right| \geq k$. Since D is unilateral, there exists a spanning directed walk $\mathscr{C} = (v_0, v_1, \ldots, v_n)$ and we can consider the subsets $V_i = \left\{v_r \middle| r \equiv i \pmod{k}\right\}$, $0 \leq i \leq k-1$ of $V(D)$. The set $\{V_i\}_{i=0}^{k-1}$ is a partition of $V(D)$. To prove that $\bigcup_{i=0}^{k-1} V_i = V(D)$ and $V_i \neq \varnothing$ for $0 \leq i \leq k-1$ it suffices to observe that $\mathscr{C}$ is a spanning directed walk and thus has length greater than or equal to k, it follows that $v_i \in V_i$ for $0 \leq i \leq k-1$, then $V_i \neq \varnothing$. Also, if $v \in V(D)$ then $v = v_r$ for some $0 \leq r \leq n$, but $\{0, 1, \ldots, k-1\}$ is a complete system of distinct representatives $(\mathrm{mod}\ k)$ hence $r \equiv i \pmod k$ for some $0 \leq i \leq k-1$ and $v \in V_i$. Finally, to prove that $V_j \cap V_k = \varnothing$, let v_r be a vertex in $V(D)$, if v_r appears only once in $\mathscr{C}$ then $r \equiv i \pmod k$ for a unique $i \in \{0, 1, \ldots, k-1\}$ and consequently v_r belongs to V_i for a unique $i \in \{0, 1, \ldots, k-1\}$; if v_r appears more than once in $\mathscr{C}$ we can suppose without loss of generality that $v_r = v_s$ with $r < s$ and then $v_r \mathscr{C} v_s$ is a directed closed walk which, in virtue of Lemma 2.2.3, has length $\equiv 0 \pmod k$ so $r \equiv s \pmod k$ and $v_r \in V_i$ for a unique i.

Let $(x, y) \in A(D)$, then $x = v_r$, $y = v_s$ for some $r, s \in \{0, 1, \ldots n\}$. If $s < r$, then $y\mathscr{C}x \cup (x, y)$ is a directed closed walk and it follows from Lemma 2.2.3 that $\ell\left(y\mathscr{C}x \cup (x, y)\right) \equiv 0 \pmod k$ and since $\ell(y\mathscr{C}x) = r - s$, then $r - s + 1 \equiv 0$ and hence $s \equiv r + 1 \pmod k$ therefore (x, y) is a V_iV_{i+1}-arc for some $i \in \{0, 1, \ldots, k-1\}$. If $r < s$, then $s = r + 1$ when $(x, y) \in A(D)$ or $x\mathscr{C}y \cup (x, y)$ is a directed closed walk with one obstruction in y and in virtue of Lemma 2.2.4 $\ell\left(x\mathscr{C}y \cup (x, y)\right) \equiv 2 \pmod k$, but $\ell(x\mathscr{C}y) = s - r$, thus $s - r + 1 \equiv 2 \pmod k$ and $s \equiv r + 1 \pmod k$; in either case (x, y) is a V_iV_{i+1}-arc for some $i \in \{0, 1, \ldots, k-1\}$ and we can conclude that D is a cyclically k-partite digraph. ■

Well, we have half of the job done, we found sufficient conditions for an unilateral digraph to be cyclically k-partite, so the natural question arises. Are these sufficient condition also necessary? The answer to this question is yes, not only for unilateral digraphs, but for every cyclically k-partite digraph as well.

Lemma 2.2.6. *If D is a cyclically k-partite digraph, then every directed cycle has length $\equiv 0 \pmod{k}$ and every directed cycle with one obstruction has length $\equiv 2 \pmod{k}$.*

Proof. Let D be a cyclically k-partite digraph. Is clear that every directed cycle has length $\equiv 0 \pmod{k}$, so let $\mathscr{C} = (x_0, x_1, \ldots, x_n, x_{n+1} = x_0)$ be a directed cycle with one obstruction at vertex x_n (and hence $(x_0, x_n) \in A(D)$). Without loss of generality let us assume that $\{V_i\}_{i=0}^{k-1}$ is the cyclical k-partition and that $x_0 \in V_0$, then $x_n \in V_1$. Since $x_1 \in V_1$, $\ell(x_1 \ldots x_n) \equiv 0 \pmod{k}$, but $\mathscr{C} = (x_0, x_1) \cup x_1 \mathscr{C} x_n \cup (x_0, x_n)$, so $\ell(\mathscr{C}) \equiv 2 \pmod{k}$. ■

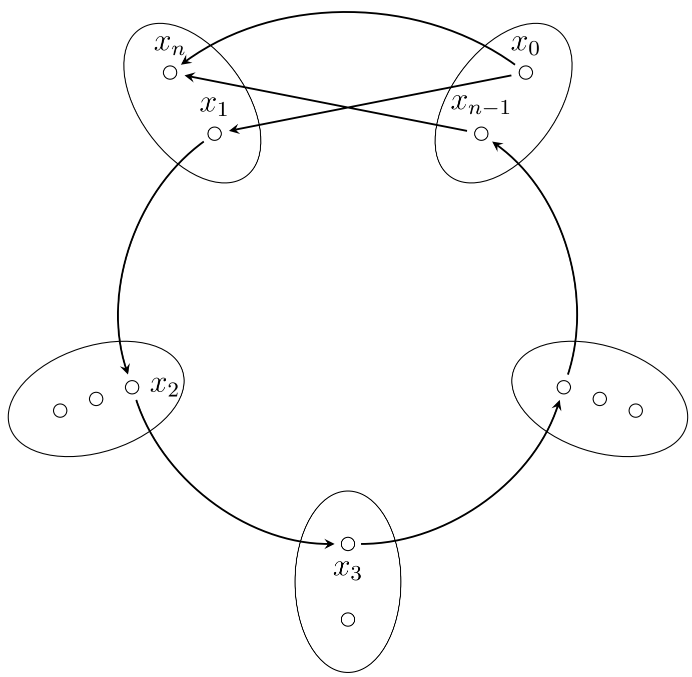


Figure 2.4: An illustration for the proof of Lemma 2.2.6, a directed cycle with one obstruction in a cyclically k-partite digraph.

The characterization is then obtained.

Theorem 2.2.7. *If D is an unilateral digraph then D is cyclically k-partite if and only if every directed cycle has length $\equiv 0 \pmod{k}$ and every directed cycle with one obstruction has length $\equiv 2 \pmod{k}$.*

Proof. It follows from Lemma 2.2.5 and Lemma 2.2.6. ■

And as an immediate consequence of these theorem, we have the following corollary that generalizes a classical characterization of bipartite digraphs. It is known that a strongly connected digraph is bipartite if and only if every directed cycle has even length. We have the following characterization for unilateral digraphs.

Corollary 2.2.8. *Let D be an unilateral digraph, then D is bipartite if and only if every directed cycle with at most one obstruction has even length.*

Proof. The sufficiency is trivial as every cycle (directed or not) of the digraph is of even length. For the necessity set $k = 2$, then $2 \equiv 0 \pmod{k}$ and the hypothesis of Theorem 2.2.7 are fullfiled, so D is cyclically 2-partite and then bipartite. ■

Thus, we have characterizations for strongly connected and unilateral cyclically k-partite digraphs. Thus, in terms of connectedness the following step would be connected digraphs. Unfortunately, the method used in the proof of the existing characterizations makes extensive use of the existence of a directed spanning walk, which we do not have in merely connected digraphs. The following theorem gives a sufficient condition for a graph to have a cyclically k-partite orientation. This theorem is of great interest on its own because it generalizes a classic result in Graph Theory, and also, its contrapositive form gives some information on the structural properties of non cyclically k-partite digraphs (and graphs).

Besides, we introduce the bridge graph of a given graph, a new tool that we found very useful in the proof of the theorem. If G is a graph, the **bridge graph of $\boldsymbol{G}$** is the graph $Br(G)$ with vertex set $\{H \subseteq G \big| H \text{ is a maximal bridgeless subgraph of } G\}$ and such that $H_1H_2 \in E\left(Br(G)\right)$ if and only if there is a bridge between H_1 and H_2 in G. It is clear from the definition that every edge of $Br(G)$ is a bridge, and thus, $Br(G)$ is a tree. Moreover, there is a bijection between edges in $Br(G)$ and bridges in G.

Theorem 2.2.9. *Let G be a graph such that every cycle has length $\equiv 0 \pmod{k}$, then G admits a cyclically k-partite orientation.*

Proof. By induction on $n = \left|V\left(Br(G)\right)\right|$. If $n = 1$, then G is a bridgeless graph, so it admits a strongly connected orientation $\mathcal{O}(G)$. Since every cycle of G has length $\equiv 0 \pmod{k}$, then $\mathcal{O}(G)$ is strongly connected and every directed cycle has length $\equiv 0 \pmod{k}$, thus $\mathcal{O}(G)$ is cyclically k-partite. Assume the result valid for every graph G with $|V(Br(G))| < n$ and let G be a graph such that $|V(Br(G))| = n$. If H is a leaf in $Br(G)$, then $G - H$ is a connected graph with $|V(Br(G))| = n - 1$, and by induction hypothesis it is cyclically k-partite with k-partition $\mathsf{P} = \{V_0, V_1, \ldots, V_{k-1}\}$. Since H is bridgless, it is also cyclically k-partite with k-partition $\mathsf{Q} = \{W_0, W_1, \ldots W_{k-1}\}$and there is only one edge $e \in E(G)$ between H and $G - H$. If we orient e so it has tail in H and head in $G - H$, and we rename the elements of Q to obtain Q' such that the arc obtained by the orientation of e has tail in W_i and head in $V_{i+1} \pmod{k}$, as this is the only arc between the orientations of H and $G - H$, $\mathsf{R} = \{V_0 \cup W_0, V_1 \cup W_1, \ldots, V_{k-1} \cup W_{k-1}\}$ is a cyclical k-partition of G. ■

This condition is suficient, but not necessary as the example in Figure 2.5 shows.

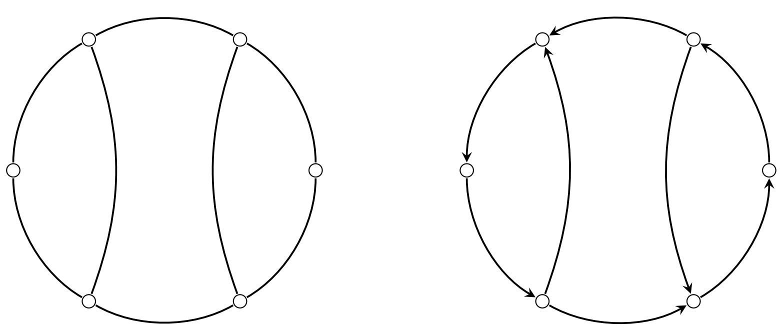


Figure 2.5: A graph with cycles of length 3, 4 and 5 and a cyclically 3-partite orientation of the same graph.

However Theorem 2.2.9 has interesting consequences.

Theorem 2.2.10. *If G is a graph such that every cycle has length $\equiv 0$ (mod k), then G is cyclically k-partite.*

Proof. It suffices to consider a cyclically k-partite orientation of G, it result obvious that G is itself cyclically k-partite. ■

For $k = 2$ this is a classical Graph Theory theorem, asserting that if every cycle of G is even, then G is bipartite. For the $k = 2$ (bipartite) case the necessity is also true, but Figure 2.5 demonstrates that it is not true for every k, as a matter of fact, for every other k we can find a cyclically k-partite graph with a 4-cycle as Figure 2.6 shows. The idea of this construction can be extended to find cyclically k-partite graphs with cycles of every even length.

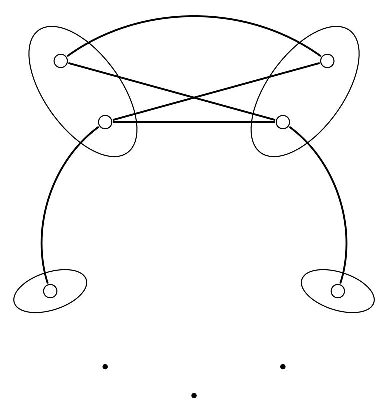


Figure 2.6: Example of a cyclically k-partite digraph with a 4-cycle.

As a final consequence of Theorem 2.2.9 in this section, we give the following corollary.

Corollary 2.2.11. *Let G be graph such that every cycle has length $\equiv 0$ (mod k) with $k = 2n - 1$, $n \in \mathbb{N}$, then $\chi(G) \leq 3$.*

Proof. Let us recall that for any graph G, $\chi(G) < 3$ if and only if G has no cycles of odd length, so if we assume that G has at least one cycle, since k is odd the equality $\chi(G) = 3$ must hold. It follows from Theorem 2.2.10 that G is cyclically k-partite with partition $\{V_1, V_2, \ldots, V_k\}$ and the elements of the partition form an odd cycle. As every element of the partition is an independent set, it suffices to give a 3-colouring for the k-cycle $(V_1, V_2, \ldots V_k, V_0)$ and assign the same color as V_i to each vertex in V_i for every $i \in \{1, 2, \ldots k\}$. ■

2.3 Cyclically k-partite digraphs and k-kernels.

From the proof of definition of a cyclically k-partite digraph we can observe the following.

Proposition 2.3.1. *If D is a cyclically k-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$, then V_i is k-independent in D for every $i \in \{0, 1, \ldots, k-1\}$.*

Proof. Since every arc of D is a V_iV_{i+1}-arc (mod k) for some $i \in \{0, 1, \ldots, k-1\}$ then for each $i \in \{0, 1, \ldots, k-1\}$, every V_iV_i-walk must pass throug each V_j, $j \neq i$ before getting back to V_i. ■

Before proving the main theorem of this section, we need to state a simple result that generalizes Theorem 1.7.1. A very elegant proof of the following theorem can be found in [24].

Theorem 2.3.2. *Every acyclic digraph has a unique k-kernel for every $k \geq 2$.*

Theorem 2.3.3. *If D is a unilateral digraph such that every directed cycle has length $\equiv 0$ (mod k) and every directed cycle with one obstruction has length $\equiv 2$ (mod k), then D has a k-kernel.*

Proof. If D has less than k vertices, then D cannot contain directed cycles (since every directed cycle has at least k vertices), using Theorem 2.3.2 we can conclude that D has a k-kernel. So, we can suppose without loss of generality that $|V(D)| \geq k$.

In virtue of Theorem 2.2.5, D is a cyclically k-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$ and as a consequence of the unilaterality, there exists a directed spanning walk $\mathscr{C} = (v_0, v_1, \ldots, v_n)$ in D. Let be V_j such that $v_n \in V_j$. It is a direct observation that V_j is $(k-1)$-absorbent; for every $u \in V(D) \setminus V_j$, $u = x_r$, $r \equiv i$ (mod k) for some $0 \leq i \leq k-1$, $i \neq j$ and $r \neq n$, as $u \notin V_j$. We have two cases:

Case 1. If $j < i$ it suffices to consider the directed walk $(x_r, x_{r+1}, \ldots, x_{r+(k-i+j)})$. In the virtue that $x_r \in V_i$, it is the case that $x_{r+(k-i+j)} \in V_{i+(k-i+j)}$ (mod k), but $i + (k-i+j) \equiv k+j \equiv j$ (mod k), and as $j - i < 0$ it follows that $k - i + j \leq k - 1$ therefore $\ell(x_r, x_{r+1} \ldots, x_{r+(k-i+j)}) \leq k-1$ and x_r results to be $(k-1)$-absorbed by V_j.

Case 2. If $i < j$, then $0 \leq i < j \leq k-1$ and thence $j - i \leq k-1$. Considering the directed walk $(x_r, x_{r+1}, \ldots, x_{r+(j-i)})$, analogously to Case

1, $x_{r+(j-i)} \in V_{i+(j-i)} \pmod{k}$, but $i + (j - i) = j$ so $x_{r+(j-i)} \in V_j$ and $\ell\left(x_r, x_{r+1}, \ldots, x_{r+(j-i)}\right) = j - i \leq k - 1$, finally x_r results $(k-1)$-absorbed by V_j.

Besides, it follows from Proposition 2.3.1 that V_j is a k-independent set.

V_j is then k-independent and $(k-1)$-absorbent and is therefore the desired k-kernel. ∎

Also from the observation of the proof of Theorem 2.3.3, we have good prospects for k-kernels in cyclically k-partite digraphs, we just have to find an absorbing element of the k-partition. It is also clear that unilateral cyclically k-partite "like" structures have k-kernel.

Let us make further observations of the proof of Theorem 2.3.3. The absorbence in the proposed k-kernel, V_0 (without loss of generality), is granted due to the existence of the spanning directed walk, for any vertex it suffices to "follow" this walk to get eventually k-absorbed. The independence follows from the disposition of the arcs between the elements of the k-partition, but this disposition guarantee independence for every element of the partition, not only the one we did choose as our k-kernel, therefore we can reverse any number of arcs as long as we do not reverse arcs in the directed spanning walk (conserving the absorption) and as long as we do not create any V_0V_0-paths of length $< k$. We can also add any number of V_iV_j-arcs as long as $j < i \neq 1$, since these arcs will not affect independence.

Corollary 2.3.4. *Let D be an unilateral cyclically k-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$, k-kernel V_0 and spanning directed walk $\mathscr{C}$. If D' is obtained from D by reversing any number of arcs not in $A(\mathscr{C})$ nor of the form V_0V_1 or $V_{k-1}V_0$, or adding any number of V_iV_j-arcs with $j < i \neq 1$, then V_0 is a k-kernel for D'.*

Proof. The absorption is a consequence of the existence of $\mathscr{C}$ in D'. For the independence, observe that every arc with tail in V_0 has head in V_1, and every arc with head in V_0 has tail in V_{k-1}, thus, every V_0V_0-walk must pass through every element of the k-partition of D', and consequently has length greater or equal than k. All the added arcs go "backwards" in the k-partition, so the V_0V_0 distance cannot be shortened. ∎

But unilateral digraphs are not the only cyclically k-partite digraphs with kernel, directed trees are also cyclically k-partite and have k-kernel since they are acyclical. Despite this fact, cyclically k-partite digraphs with a k-kernel are not easy to find. To finalize this chapter, we will explore some sufficient conditions for a cyclically k-partite digraph to have a k-kernel.

Our following corollary continues analyzing the relation between k-kernels and cyclically k-partite digraphs. Let us introduce a definition before the corollary. If D is a digraph, $N \subseteq V(D)$ will be called **independent by directed paths** if for every $u, v \in N$ there are not uv-paths in D. Analogously N will be called **absorbent by directed paths** if for every $u \in V(D) \setminus N$ there exists $v \in N$ such that $d(u, v) \in \mathbb{N}$. If a set is independent by directed paths and absorbent by directed paths it will be called a **kernel by directed paths**.

Corollary 2.3.5. *Let $D = (V_0, V_1, \dots V_{k-1})$ be a cyclically k-partite digraph. If there exists $N \subset V_i$ for some $i \in \{0, 1, \dots, k-1\}$ such that N is absorbent by directed paths, then V_i is a k-kernel of D.*

Proof. Let $N \subseteq V_i$ be the set absorbent by directed paths in D. We affirm that V_i is the desired k-kernel. Clearly, V_i is independent. For the absortion we have that for every vertex $u \in V(D) \setminus V_i$ there exists a uV_i-directed path $\mathscr{C}$ because $N \subseteq V_i$. The digraph $D[V(\mathscr{C})]$ induced by the set of vertices of $\mathscr{C}$ is a unilateral cyclically k-partite digraph with spanning walk $\mathscr{C}$, so, by Theorem 2.3.3, u is $(k-1)$-absorbed by V_i in $D[V(\mathscr{C})]$ and thence is $(k-1)$-absorbed by V_i in D. ■

As an easy consequence of Corollary 2.3.5 we have the following. If $D = (V_0, V_1, \dots, V_k)$ is a cyclically k-partite digraph such that every terminal vertex of D is contained in V_0, then V_0 is a k-kernel of D. This follows immediatly from the observation that the set $N = \left\{v \in V(D) \middle| d^+(v) = 0\right\}$ is a kernel by directed paths in D. Let us observe that the case where there is only one terminal vertex is equivalent, k-kernel-wise, to the case where every terminal vertex is contained in the same class, say V_0. The following problem was proposed by Professor Gitler-Goldwain and it seems to be very difficult, in contrast to the aforementioned case.

Problem 2.3.6. Let $k \geq 2$ be an integer. Is it true that a cyclically k-partite digraph with exactly 2 terminal vertices, u and v, such that u and v belong to distinct classes of D, has a k-kernel? If the answer if affirmative, how does that k-kernel look like?

Although Problem 2.3.6 is still open, from the observation of distinct extamples of cyclically k-partite digraphs with and without a k-kernel, we propose the following conjecture.

Conjecture 2.3.7. *Let $k \geq 2$ be an integer. If D is a cyclically k-partite digraph such that there are terminal vertices of D in at most $k-1$ distinct classes of D, then D has a k-kernel.*

We finalize the section, and the chapter, with another consequence of Corollary 2.3.5.

Corollary 2.3.8. *Let $\mathcal{D} = \{D_i\}_{i=1}^{n}$ be a family of disjoint unilateral cyclically k-partite digraphs, $\mathcal{W} = \{\mathscr{W}_i\}_{i=0}^{n}$ a family of directed walks such that $\mathscr{W}_i$ is a directed spanning walk for D_i and v_i is the end vertex of $\mathscr{W}_i$ for every $i \in \{0, 1, \ldots, k-1\}$. If D_0 is a cyclically k-partite digraph with partition $\{V_i\}_{i=0}^{k-1}$ and k-kernel V_0 such that $v_i \in V_0$ for every $i \in \{1, 2, \ldots, n\}$, then $\bigcup_0^n \{D_i\}$ has a kernel.*

Proof. This is a direct aplication of Corollary 2.3.5. Just observe that V_0 is a kernel by directed paths for $\bigcup_0^n \{D_i\}$. ■

This last corolary was one of the first generalizations we found for non-unilateral cyclically k-partite digraphs, it is a star shaped digraph where each point of the star is a unilateral cyclically k-partite digraph, and all these digraphs converge at the k-kernel of another cyclically k-partite digraph.

Chapter 3

Classic generalizations of transitive digraphs

3.1 Introduction

As we have already mentioned, the existence of (k,l)-kernels have been proved only for a few large families of digraphs. In our search for large families of digraphs with (k,l)-kernel, or at least k-kernel as in the case of acyclic digraphs, we turn our attention to some of the best existing results about the existence of kernels in families of digraphs. Just as Chapter 2 is inspired in Theorem 1.8.2, Theorem 3.1.1 has been a motivation for the results we present in this chapter. In [11], the following very general result can be found:

Theorem 3.1.1. *If D is a transitive digraph, then D has a kernel. Moreover, every kernel consists in one vertex from every terminal strong component of D, so all kernels of D have the same cardinality.*

We will focus on three families of digraphs which generalize transitive digraphs: quasi-transitive digraphs and right-/left-pretransitive digraphs.

Let us recall that a digraph D is quasi-transitive if $(u,v),(v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(w,u) \in A(D)$. We define a digraph D to be **right-(left-)pretransitive** if $(u,v),(v,w) \in A(D)$ implies $(u,w) \in A(D)$ or $(w,v) \in A(D)$ $((v,u) \in A(D))$.

It is clear from the definition that every transitive digraph is quasi-transitive and right-/left-pretransitive. Thus, these families of digraphs effectively generalize transitive digraphs.

Related to right- and left-pretransitive digraphs is also Theorem 3.1.2 below (proved by Galeana-Sánchez and Rojas-Monroy in [61]), which generalizes a result of Duchet ([30]).

Theorem 3.1.2 (Galeana-Sánchez, Rojas-Monroy)**.** *Let D be a (possibly infinite) digraph. Suppose that there exist two subdigraphs of D say D_1 and D_2 such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \varnothing$), where D_1 is a right-pretransitive digraph, D_2 is a left-pretransitive digraph, and D_i contains no infinite outward path for $i \in \{1, 2\}$. Then D is a kernel-perfect digraph.*

Corollary 3.1.3 (Duchet)**.** *If D is a right (resp. left) pretransitive digraph, then D is kernel-perfect.*

So we have great motivation for studying these families of digraphs. Also, Bang-Jensen and Huang have studied quasi-transitive digraphs. Among other results related to our research they have relevant results concerning 3-kings in quasi-transitive digraphs [9] and a structural characterization of quasi-transitive digraphs [8].

We conclude the present section introducing a new definition and presenting a lemma ressembling Theorem 1.7.3, which will be very useful not only in this chapter, but throughout the whole work.

We have already mentioned in the introduction that the notion of semikernel of a digraph has proved to be very useful in the study of the kernel problem, principally because of Theorem 1.7.3, due to Neumann-Lara. In [45], Galeana-Sánchez and the author proposed a generalization of this concept, the k-semikernel of a digraph, but a more general concept can be defined and can be used to prove an analogous version of the aforementioned result due to Neumann-Lara. Let D be a digraph. A subset $S \subseteq V(D)$ will be a ***(k,l)*-semikernel** of D if (i) S is k-independent; and (ii) for every vertex $v \in V(D) \setminus S$, $d(S, v) \leq k - 1$ implies $d(v, S) \leq l$.

The condition (ii) will be often referred as "the second (k, l)-semikernel condition". A $(k, k - 1)$-semikernel will be simply called a k-semikernel. There was a previous attempt to define a (k, l)-semikernel of a digraph due to Kucharska and Kwaśnik in [73] which was also used by Galeana-Sánchez and Gómez-Aiza in [42]. Despite the fact that the definition they proposed worked well in the context they were using it, it was not possible to find an analogous result to the one of Neumann-Lara with that definition, but is worth observing that under either definition of (k, l)-semikernel, the k-semikernels remain the same.

Lemma 3.1.4. *Let D be a digraph such that $\{v\}$ is a (k,l)-semikernel for every vertex $v \in V(D)$, then D has a (k,l)-kernel.*

Proof. Since every vertex in D is a (k,l)-semikernel, then D has at least one non-empty (k,l)-semikernel and thus we can consider a $(\subseteq)$maximal (k,l)-semikernel of D, namely $S \subseteq V(D)$. If S is l-absorbent then S is a (k,l)-kernel of D, so let us assume that S is not l-absorbent, therefore there must exist a vertex $v \in V(D) \setminus S$ such that $d(v,S) > l$. Let us observe that $d(S,v) > k-1$ because, by the second condition of (k,l)-semikernel, $d(S,v) \leq k-1$ implies that $d(v,S) \leq l$ but v is not l-absorbed by S. We will consider two cases.

Case 1. If $k-1 \leq l$, then $k-1 \leq l < d(v,S)$, so, in view that $d(S,v) > k-1$, we have that $S' = S \cup \{v\}$ is a k-independent set. Moreover, if $u \in V(D)$ is such that there exists an $S'u$-directed path $\mathscr{C}$ of length less than or equal to $k-1$ then, since S is a (k,l)-semikernel, if $\mathscr{C}$ is a Su-directed path, then there exists an uS-directed path of length less than or equal to $k-1$, but this path is also a uS'-directed path; and since $\{v\}$ is also a (k,l)-semikernel, then if $\mathscr{C}$ is a vu-directed path, this implies that there exists a uv-directed path of length less than or equal to $k-1$, which is also a uS'-directed path, and then S' is a (k,l)-semikernel properly containing S which contradicts the election of S as a maximal (k,l)-semikernel.

Case 2. If $l < k-1$, then we can assume that $d(v,S) \leq k-1$, otherwise $S \cup \{v\}$ would be k-independent and we can proceed as in Case 1. So, since $\{v\}$ is a (k,l)-semikernel, then $d(S,v) \leq l < k-1$ which results in a contradiction.

In both cases a contradiction arises from the assumption that S is not l-absorbent, so S must be l-absorbent and hence the desired (k,l)-kernel. ■

In section 3.2 we study some properties of right-(left-)pretransitive digraphs as a set up to use Lemma 3.1.4 to prove that if D is a right-(left-) pretransitive strong digraph such that every directed triangle of D is symmetrical, then D has a k-kernel for every integer $k \geq 3$. This result will be used along with a brief structural analysis of non-strong right-pretransitive digraphs to prove that the result is also valid for non-strong digraphs in the right-pretransitive case. A conjecture and an open problem are proposed on the matter. In section 3.3 a structural characterization of quasi-transitive digraphs is used along with a previous result about (k,l)-kernels in the composition of digraphs to prove that every quasi-transitive digraph has (k,l)-kernel for every integers $k > l \geq 3$ or $k = 3$ and $l = 2$. An analysis of the

(2-)kernels in quasi-transitive digraphs is made from the point of view of the Strong Perfect Graph Theorem. At the end of both sections, results about (k, l)-solutions in digraphs are obtained by means of dualization.

3.2 Pretransitive Digraphs

In the particular families of digraphs we will be studying in this work the existence of k-solutions will be very close to the existence of k-kernels, so we find it useful to state the following remark.

Remark 3.2.1. If N is a (k, l)-kernel of D then N is a (k, l)-solution of $\overleftarrow{D}$.

There is a notorious duality in the definitions of right and left-pretransitive digraphs and as there is also a duality in the definitions of k-kernels and k-solutions. In view of both definitions the next remark will prove to be very useful once we have the appropriate tools.

Remark 3.2.2. Let D be a digraph. D is a right-pretransitive digraph if and only if $\overleftarrow{D}$ is a left-pretransitive digraph.

We will prove two lemmas about the structure of right-pretransitive digraphs; the second one will be dualized using Remark 3.2.2 to obtain an analogous result about left-pretransitive digraphs.

Lemma 3.2.3. *If D is a right-pretransitive digraph and $(x_0, x_1, \ldots, x_n)$ is an asymmetrical directed path in D then $(x_0, x_i) \in A(D)$ for every $i \in \{2, 3, \ldots, n\}$.*

Proof. Straightforward, by induction on n. ■

Lemma 3.2.4. *If D is a right-pretransitive digraph then $Asym(D)$ is acyclic. Moreover, every directed triangle in D has at least two symmetrical arcs.*

Proof. We will prove the second part first. Let $C_3 = (x, y, z, x)$ be a directed triangle in D. Since D is right-pretransitive and $(x, y), (y, z) \in A(D)$ we can conclude that $(x, z) \in A(D)$ or $(z, y) \in A(D)$. In either case the result is a directed triangle with a symmetrical arc, so let us suppose without loss of generality that $(x, z) \in A(D)$. Then we can consider the arcs $(z, x) \in A(D)$ and $(x, y) \in A(D)$, for the right-pretransitivity of D we know that $(z, y) \in A(D)$ or $(y, x) \in A(D)$. In either case C_3 has at least two symmetrical arcs.

For the first part, suppose by contradiction that $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ is a directed cycle in Asym(D), where D is a right-pretransitive digraph. If we consider the asymmetrical directed path $(x_0, x_1, \ldots, x_{n-1})$, it follows from Lemma 3.2.3 that $(x_0, x_{n-1}) \in A(D)$; we don't know whether (x_0, x_{n-1}) is an asymmetrical arc or not, but together with (x_{n-1}, x_n) and as a consequence of the right-pretransitivity of D we have that $(x_0, x_n) \in A(D)$ or $(x_n, x_{n-1}) \in A(D)$, but (x_n, x_0) and (x_{n-1}, x_n) are both asymmetrical arcs of D, thus we obtain the desired contradiction. Since the contradiction arises from the assumption that there is a directed cycle in Asym(D), then Asym(D) is acyclic. ■

Lemma 3.2.5. *If D is a left-pretransitive digraph then Asym(D) is acyclic. Moreover, every directed triangle in D has at least two symmetrical arcs.*

Proof. The result follows straightforward from Remark 3.2.2 and Lemma 3.2.4. ■

Our next result was part of our first attempt to implement a classic proof method in kernel theory to k-kernels. For kernels (2-kernels), once it is proved that digraphs in a certain family have nonempty semikernels it suffices to consider a ($\subseteq$-)maximal semikernel S for a digraph D. If the set of vertices not absorbed by S is not empty, then we can find a nonempty semikernel S' for $D \setminus (S \cup N^-(S))$. From here is easy to prove that $S \cup S'$ is a semikernel of D, contradicting the choice of S. When working with k-kernels we have a problem: suppose that we have proved that a certain family of digraphs have nonempty semikernel and consider a digraph D in such family. Then we can find a maximal k-semikernel S of D and, if S is $(k-1)$-absorbent, S is the desired k-kernel. But if not, we consider a k-semikernel S' for the subdigraph T of D induced by the vertices not $(k-1)$-absorbed by S; it remains clear that $S \cup S'$ is k-independent and that every vertex reached from S must reach $S \cup S'$ in D but, suppose that there is a vertex $v \in V(T)$ such that the only $S'v$-directed path of length less than or equal to $k-1$ in D is $(x_0, x_1, x_2, \ldots, x_n = v)$, where $x_0 \in S' \subseteq V(T)$, but $x_i \in V(D \setminus T)$ for some $1 \leq i \leq n-1$, then v is not reached by S' in T and then v may not reach S' in D, and as v is in T, v does not reach S in D, so $S \cup S'$ may not be a k-semikernel in D. It is in view of this problem that we proposed Lemma 3.1.4, were we prove that if **every** vertex $v \in V(D)$ is a (k, l)-semikernel **of D**, then D has a (k, l)-kernel. Nevertheless, this result is interesting by itself as a local property of the class of right-pretransitive digraphs is found.

Theorem 3.2.6. *If D is a right-pretransitive digraph, then D has a k-semikernel consisting of a single vertex for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. If D has no asymmetrical arcs, then D is a symmetrical digraph and each vertex is trivially a k-semikernel of D for every $k \geq 2$.

So, let us assume that $\mathrm{Asym}(D) \neq \varnothing$. In virtue of Lemma 3.2.4 $\mathrm{Asym}(D)$ is acyclic, so we can choose a vertex v with exdegree 0 in $\mathrm{Asym}(D)$. We claim that $\{v\}$ is a k-semikernel of D for every $k \geq 2$. As $\{v\}$ is k-independent for every $k \in \mathbb{N}$, it suffices to prove that for every $k \geq 2$ if a vw-directed path of length less than or equal to $k-1$ exists, then a wv-directed path of length less than or equal to $k-1$ exists.

Since v has exdegree 0 in $\mathrm{Asym}(D)$, if $(v, w) \in A(D)$ for some $w \in V(D)$, then such arc must be symmetrical, so $(w, v) \in A(D)$ and the second condition of k-semikernel is fulfilled for $k = 2$. Let k be greater than 2. We will prove by induction on n that if a vw-directed path of length $n \leq k-1$ exists, then there exists a wv-directed path of length less than or equal to $k-1$. The case $n = 1$ has been already proved, is the same as case $k = 2$. Let us assume the result valid for every vw-directed path of length $m < n$ and let $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = w)$ be a vw-directed path of length $n \leq k-1$. For the choice of v we know that (v_0, v_1) is a symmetrical arc of D. If every arc in $\mathscr{C}$ is symmetrical, then the directed path $\mathscr{C}^{-1}$ is the one we have been looking for. Otherwise, there must be a first asymmetrical arc in $A(\mathscr{C})$, let us say (v_i, v_{i+1}), $1 \leq i$. So we can consider the arcs $(v_{i-1}, v_i), (v_i, v_{i+1}) \in A(D)$ and, since D is right-pretransitive and $(v_{i+1}, v_i) \notin A(D)$, necessarily $(v_{i-1}, v_{i+1}) \in A(D)$, and hence $v_0 \mathscr{C} v_{i-1} \cup (v_{i-1}, v_{i+1}) \cup v_{i+1} \mathscr{C} w$ is a vw-directed path of length $n-1$. Inductive hypothesis assures the existence of a wv-directed path of length less than or equal to $k-1$, which concludes the proof. The desired result follows from the induction principle.

■

We have already proved that right/left-pretransitive digraphs have at least two symmetrical arcs in every directed triangle. In view of this property, it is not very restrictive to ask for a right/left-pretransitive digraph to have only symmetrical directed triangles. As the next lemma shows (only after a little technical lemma), this is a sufficient condition along with strong connectedness to prove that every right/left-pretransitive digraph have a k-kernel.

Lemma 3.2.7. *If D is a right-pretransitive digraph such that every directed triangle is symmetrical and $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a directed path such that (x_i, x_{i+1}) is a symmetrical arc for every $i \in \{0, 1, \ldots, n-2\}$ and (x_{n-1}, x_n) is an asymmetrical arc of D, then $(x_i, x_n) \in A(D)$ for every $i \in \{0, 1, \ldots, n-1\}$. Moreover, every such arc is asymmetrical.*

Proof. By induction on n. For $n = 2$, $(x_0, x_1), (x_1, x_2) \in A(D)$ so, since D is right-pretransitive then $(x_2, x_1) \in A(D)$ or $(x_0, x_2) \in A(D)$, but (x_1, x_2) is an asymmetrical arc, hence $(x_0, x_2) \in A(D)$. Besides, if $(x_2, x_0) \in A(D)$, then (x_0, x_1, x_2, x_0) would be a directed triangle and it should be symmetrical by hypothesis, but (x_1, x_2) is an asymmetrical arc; it follows that (x_0, x_2) is also an asymmetrical arc. So, let us assume the result valid for every path with the required conditions and length less than n. If $\mathscr{C} = (x_0, x_1, \ldots, x_n)$ is a directed path with the required conditions and length n, clearly $(x_1, x_2, \ldots, x_n)$ is a directed path with the required conditions and length $n - 1 < n$, and from the inductive hypothesis we have the existence of the asymmetrical arcs (x_i, x_n) for every $i \in \{1, 2, \ldots, n\}$. To finish the inductive step we have to prove that $(x_0, x_n) \in A(D)$ and it is an asymmetrical arc. But $(x_0, x_1), (x_1, x_n)$ is a directed path of length 2 where the first arc is symmetrical and the second arc is asymmetrical, so it follows from the case $n = 2$ that $(x_0, x_n) \in A(D)$ is an asymmetrical arc. The desired result follows from the principle of mathematical induction. ■

Lemma 3.2.8. *Let $k \geq 2$ be an integer. If D is a right-pretransitive strong digraph such that every directed triangle is symmetrical, then every vertex of D is a k-semikernel of D.*

Proof. Let $k \geq 2$ be an integer. Let $v \in V(D)$ be any vertex, consider $w \in V(D)$ such that there exists a vw-directed path of length less than or equal to $k - 1$ and let $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = w)$ be a vw-directed path of minimum length. Then $n \leq k - 1$. For every pair of arcs $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}) \in A(\mathscr{C})$, the arc (v_i, v_{i+2}) can not exist in $A(D)$, because it would contradict the choice of $\mathscr{C}$ as a vw-directed path of minimum length, so, for the right pretransitve hypothesis, for every $0 \leq i \leq n - 2$ the arc $(v_{i+2}, v_{i+1}) \in A(D)$ must exist. If $(v_1, v_0) \in A(D)$, the directed path $(v_n, v_{n-1}, \ldots, v_0)$ would be a wv-directed path of length $n \leq k - 1$. If $(v_1, v_0) \notin A(D)$, as D is strong, there exists a v_1v-directed path in D, say $\mathscr{D} = (v_1 = z_0, z_1, \ldots, z_m = v)$. We can suppose without loss of generality

that $\mathscr{D}$ is of minimum length and its length is greater than 1. So, we can consider the arcs $(z_i, z_{i+1}), (z_{i+1}, z_{i+2}) \in A(D)$, and since $\mathscr{D}$ has minimum length, once again we have the existence of the arcs $(z_{i+2}, z_{i+1}) \in A(D)$ for every $0 \leq i \leq m-2$. Also, we have the arcs $(z_{m-1}, v), (v, z_0) \in A(D)$, and by hypothesis we know that $(v, z_0 = v_1)$ is not a symmetrical arc, thence it follows from right pretransitivity the existence of the arc $(z_{m-1}, z_0) \in A(D)$. But (z_{m-1}, z_0) must be an asymmetrical arc of D, in other case, (z_0, z_{m-1}, z_m, z_0) would be a directed triangle and all of its arcs would be symmetrical for hypothesis, in particular the arc $(z_m, z_0) = (v, v_1)$ would be symmetrical, contrary to our assumption. So the directed path $(z_1, z_2, \ldots, z_{m-1}, z_0)$ fulfills the hypothesis of Lemma 3.2.7 and as a consequence the arcs (z_i, z_0) are asymmetrical arcs of D for every $i \in \{1, 2, \ldots m-1\}$; in particular $(z_1, z_0) \in A(D)$ and it should be an asymmetrical arc, but $(z_0, z_1) \in A(D)$, which turns out to be a contradiction. Since the contradiction arises from the assumption $(v_1, v_0) \notin A(D)$, we can conclude that $(v_1, v_0) \in A(D)$ and thence there exists a wv-directed path of length less than or equal to $k-1$.
■

Theorem 3.2.9. *If D is a right-pretransitive strong digraph such that every directed triangle is symmetrical, then D has k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. It follows from Lemmas 3.1.4 and 3.2.8. ■

Lemma 3.2.10. *If D is a left-pretransitive strong digraph such that every directed triangle is symmetrical, then $\{v\}$ is a k-semikernel of D for every $v \in V(D)$.*

Proof. Let D be a left-pretransitive strong digraph such that every directed triangle is symmetrical. In virtue of Remark 3.2.2 $\overleftarrow{D}$ is a right-pretransitive digraph such that every directed triangle is symmetrical, so it follows from Lemma 3.2.8 that $\{v\}$ is a k-semikernel of $\overleftarrow{D}$ for every $v \in V(D) = V(\overleftarrow{D})$. Let v be a vertex in $V(D)$ and $k \geq 2$ an integer. It is clear that $\{v\}$ is k-independent for every k, so let us consider a vw-directed path of length less than or equal to $k-1$ $\mathscr{C}$. It is also obvious that $\mathscr{C}^{-1}$ is a wv-directed path of length less than or equal to $k-1$ in $\overleftarrow{D}$, and since $\{w\}$ is a k-semikernel of $\overleftarrow{D}$, then there exists a vw-directed path of length less than or equal to $k-1$ in $\overleftarrow{D}$, say $\mathscr{D}$. But $\mathscr{D}^{-1}$ is hence a wv-directed path of length $\leq k-1$ in D, consequently $\{v\}$ fulfills both k-semikernel conditions and the result follows.
■

Theorem 3.2.11. *If D is a left-pretransitive strong digraph such that every directed triangle is symmetrical, then D has k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. The result follows from Lemmas 3.1.4 and 3.2.10. ■

The following corollary is obtained directly by dualization.

Corollary 3.2.12. *If D is a right-(left-)pretransitive strong digraph such that every directed triangle is symmetrical, then D has a k-solution for every $k \in \mathbb{N}$, $k \geq 2$.*

For right-pretransitive digraphs we can improve our results. Let us state a lemma about the structure of non-strong right-pretransitive digraphs, but first we will need some notation. Let A and B be non-empty subsets of $V(D)$, if for every $a \in A$ and every $b \in B$ we have that $(a, b) \in A(D)$, we will write $A \mapsto B$. When $A = \{v\}$ for some $v \in V(D)$, we will simply write $v \mapsto B$, and analogously if $B = \{v\}$. If S and T are subdigraphs of D (*e.g.*, strong components) we may abuse of notation to write $S \mapsto T$ instead of $V(S) \mapsto V(T)$.

Lemma 3.2.13. *Let be D a right-pretransitive digraph, S and T strong components of D. If there exist $s \in S$ and $t \in T$ such that $(s, t) \in A(D)$, then $S \mapsto t$.*

Proof. Let v be a vertex in $V(S) \setminus \{s\}$. Since S is a strong component of D, we have that $d(v, s) \in \mathbb{N}$. We will prove by induction on $n = d(v, s)$ that $(v, t) \in A(D)$. If $d(v, s) = 1$, then $(v, s), (s, t) \in A(D)$. By the right-pretransitivity of D we have that $(v, t) \in A(D)$ or $(t, s) \in A(D)$. But S and T are distinct strong components of D and $(s, t) \in A(D)$. Since $D^\star$ is an acyclic digraph, $(t, s) \notin A(D)$, thus $(v, t) \in A(D)$.

For the inductive step it suffices to observe that, if $d(v, s) = n$, then there exists a vs-directed path in S, $(v = x_0, x_1, \ldots, x_n = s)$, realizing the distance from v to s. It is clear that $d(x_1, s) = n - 1$, so by induction hypothesis, $(x_1, t) \in A(D)$. Together with $(v, x_1) \in A(D)$, we may use the same argument that we used in the base case. ■

Theorem 3.2.14. *Let D be a right-pretransitive digraph such that every directed triangle is symmetrical, then D has k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. Let $k \geq 2$ be a fixed integer. We will proceed by induction on $n = |V(D^\star)|$. If $n = 1$, then D is a strong digraph and the result follows from Theorem 3.2.9. So let us assume that $n \geq 2$.

Let D be a digraph such that $|V(D^\star)| = n$ and S an initial strong component of D. Clearly $(D \setminus S)^\star = D^\star \setminus S$, so $|V((D \setminus S)^\star)| = n-1$. By induction hypothesis, $D \setminus S$ has a k-kernel, say N'. If $d_D(S, N') \geq k$, then, by Theorem 3.2.9 we can choose a k-kernel N'' of S. We know that $d_D(N', S) = \infty$ because S is an initial component of D, so $N = N' \cup N''$ is k-independent. Also it follows from the fact that N' is $k-1$-absorbent in $D \setminus S$ and N'' is $k-1$-absorbent in S that N is $k-1$-absorbent in D. Thus, N is the desired k-kernel.

If $d_D(S, N') \leq k-1$, then there is a vertex $s \in S$ and a vertex $t \in N'$ such that there exist a directed path $(s = x_0, x_1, \ldots, x_r = t)$ of length $r \leq k-1$. We can choose s and t in such way that $x_1 \in V(D) \setminus S$. Since s and x_1 are in distinct strong components, in virtue of Lemma 3.2.13 we can conclude that $S \mapsto x_1$, which implies that $d(v, t) \leq k-1$ for every $v \in V(S)$. Thus, N' is a $k-1$-absorbent set in D. Also, since S is an initial component, there are no $N'S$-directed paths, so N' is k-independent in D. Hence, N' is the desired k-kernel. ■

Dualization does not work as good as we would like for Lemma 3.2.13 and Theorem 3.2.14. The next results have straightforward proofs by means of dualization.

Lemma 3.2.15. *Let be D a left-pretransitive digraph, S and T strong components of D. If there exist $s \in S$ and $t \in T$ such that $(s, t) \in A(D)$, then $s \mapsto T$.*

Theorem 3.2.16. *Let D be a left-pretransitive digraph such that every directed triangle is symmetrical, then D has k-solution for every $k \in \mathbb{N}$, $k \geq 2$.*

So, two obvious problems arise.

Problem 3.2.17. Is it true that every right-pretransitive digraph such that every directed triangle is symmetrical has a k-solution for every integer $k \geq 2$?

A positive answer for the question proposed in Problem 3.2.17 would imply that every left-pretransitive digraph such that every directed triangle is symmetrical has a k-kernel for every integer $k \geq 2$. The remaining question about existence of k-kernels in right/left-pretransitive digraphs would be the following.

Problem 3.2.18. Are the hypotheses in Theorems 3.2.9 and 3.2.11 on the directed triangles sharp?

In virtue of Lemmas 3.2.4 and 3.2.5, Problem 3.2.18 is equivalent to asking if it is true that every right/left-pretransitive strong digraph has a k-kernel for every integer $k \geq 3$ or if there is a right/left-pretransitive strong digraph without a k-kernel for some integer $k \geq 3$.

3.3 Quasi-transitive Digraphs

Results about the existence of kernels in digraphs include the work of Galeana-Sánchez and Rojas-Monroy [61], where they proved that if D is a quasi-transitive digraph such that every directed triangle has at least two symmetrical arcs, then D has a kernel. Surprisingly, although quasi-transitive digraphs need an additional condition to have a kernel, in the case of k-kernels with $k \geq 3$, no additional condition is required.

As we have already mentioned, in [8] Bang-Jensen and Huang prove a structural characterization of quasi-transitive digraphs. This characterization theorem is now stated.

Theorem 3.3.1 (Bang-Jensen and Huang [8])**.** *Let D be a digraph which is quasi-transitive.*

1. *If D is not strong, then there exists an acyclic, transitive oriented graph T with vertices $\{u_1, u_2, \dots u_t\}$ and quasi-transitive strong digraphs H_1, $H_2, \dots H_t$ such that $D = T[H_1, H_2, \dots H_t]$, where H_i is substituted u_i, $i = 1, 2, \dots, t$.*

2. *If D is strong, then there exists a strong semicomplete digraph S with vertices $\{v_1, v_2, \dots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \dots, Q_s$ such that each Q_i is either a vertex or is non-strong and $D = S[Q_1, Q_2, \dots, Q_s]$, where Q_i is substituted for v_i, $i = 1, 2, \dots, s$.*

Using this characterization and a result due to Szumny, Włoch and Włoch about (k, l)-kernels in digraph compositions we are able to derive easily that every quasi-transitive digraph has a (k, l)-kernel for every $k \geq 4$, $k-1 \geq l \geq 3$

or $k = 3$ and $l = 2$, in particular, every quasi-transitive digraph has a k-kernel for $k \geq 3$. We include this proof since is a very direct and easy consequence of Theorems 3.3.1 and 3.3.2, but also, the authors have developed another proof of this fact using local properties of the quasi-transitive digraphs rather than global arguments (like those from Theorems 3.3.1 and 3.3.2) that will not be included in this chapter. Nonetheless, the other proof can be used when working with infinite digraph, so it will be presented eventually. For the case $k = 2$ we simply mention the existing results about kernels in digraphs.

To state the next result we need some new notation. If D is a digraph with vertex set $V(D) = \{v_1, v_2, \dots v_n\}$, we denote by $\mathcal{C}_D^{\mu}(v_i)$ the family of all circuits in D containing the vertex v_i and of length at most μ. If $D[G_1, G_2, \dots G_n]$ is a digraph composition, we denote by G_i^c the copy of G_i as an induced subdigraph of $D[G_1, G_2, \dots G_n]$.

Theorem 3.3.2 (Szumny, Włoch, Włoch [86])**.** *Let $k \geq 2$, $1 \leq l \leq k-1$ be integers. A subset $J^* \subseteq V(D[G_1, G_2, \dots G_n])$ is a (k,l)-kernel of the composition $D[G_1, G_2, \dots G_n]$ if and only if there exists a (k,l)-kernel $J \subseteq V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i \big| v_i \in J\}$, $J_i \subseteq V(G_i^c)$ and for every $i \in \mathcal{I}$*

1. *J_i is a (k,l)-kernel of G_i^c if $\mathcal{C}_D^{k-1}(v_i) = \varnothing$ or*

2. *J_i is 1-element set containing an arbitrary vertex of $V(G_i^c)$ if $\mathcal{C}_D^{l}(v_i) \neq \varnothing$ or*

3. *J_i is 1-element set containing an l-absorbent vertex of G_i^c, otherwise.*

To make an adequate use of this theorem we need to prove the following lemma.

Lemma 3.3.3. *If D is a strong semicomplete digraph and $v \in V(D)$, then v is contained in a directed cycle of length 2 or 3.*

Proof. Let D be a semicomplete digraph and $v \in V(D)$ a vertex. If any arc incident to or from v is symmetrical, then v is contained in a directed cycle of length 2. If every arc indicent to and from v is asymmetrical, we can consider the in-neighbourhood and out-neighbourhood of v, $N^-(v)$ and $N^+(v)$. Since D is a strong semicomplete digraph, there must exist a $N^+(v)N^-(v)$-arc in D, say uw, and thus (v, u, w, v) is a directed cycle of length 3. ■

To finish the setup to prove the main theorem of this section, we need to observe that Theorem 3.1.1 has a very nice generalization for (k,l)-kernels. Let D be a digraph and let $x_1, x_2, \ldots, x_n$ be an ordering of its vertices. We call this ordering an acyclic ordering if, for every arc $(x_i, x_j) \in A(D)$, we have $i < j$. In [4], the following characterization of transitive digraphs is left as an excersise.

Proposition 3.3.4. *Let D be a digraph with an acyclic ordering $D_1, D_2, \ldots,$ D_p of its strong components. The digraph D is transitive if and only if each of D_i is complete, $D^\star$ is a transitive oriented graph, and $D = D^\star[D_1, D_2, \ldots, D_p]$, where $D^\star$ is the condensation of*

Using Proposition 3.3.4, Theorem 3.1.1 can be generalized as follows.

Theorem 3.3.5. *If D is a transitive digraph, then D has a (k,l)-kernel for every $k \geq 2$ and every $l \geq 1$. Moreover, every (k,l)-kernel consists in one vertex from every terminal strong component of D, so all (k,l)-kernels of D have the same cardinality.*

Proof. Let D be a transitive digraph with an acyclic ordering $D_1, D_2, \ldots, D_p$ of its strong components. From Proposition 3.3.4 we have that $D^\star$ is a transitive acyclic digraph and $D = D^\star[D_1, D_2, \ldots, D_p]$, so, if v is a vertex of D that does not belong to a terminal strong component of D, then there exists a terminal strong component of D, say S, such that $d(v,s) = 1$ for every $s \in S$. Besides, D_i is a complete digraph for every $i \in \{1, 2, \ldots, p\}$, so every vertex in D_i is absorbed by every other vertex in D_i. From these observations we can conclude that if we choose one vertex in every terminal strong component, then we obtain an (1-)absorbent set, say N. Also, for every vertex $v \in N$, since v is in a terminal strong component of D, there are no directed paths from v to any other strong component of D, so N is k-independent for every $k \geq 2$. The set N is the desired (k,l)-kernel, we have already observed that it is k-independent, and every for every vertex $u \in V(D) \setminus N$, there exists a vertex $v \in N$ such that $d(u,v) = 1 \leq l$, for each $l \geq 1$. ■

Previous theorem can be generalized in the following way for quasi-transitive digraphs.

Theorem 3.3.6. *If D is a quasi-transitive digraph, then D has a (k,l)-kernel for every pair of integers k, l such that $k \geq 4$ and $3 \leq l \leq k-1$ or $k = 3$ and $l = 2$.*

Proof. Let $k \geq 4$ and $3 \leq l \leq k-1$ or $k = 3$ and $l = 2$ be a fixed pair of integers. The proof is by mathematical induction on the order of D. If $|V(D)| = 1$ the result follows trivially, so let us assume the result valid for every quasi-transitive digraph with fewer than m vertices and let D be a digraph with exactly m vertices. We have two cases, when D is strong and when D is non-strong.

Case 1 If D is non-strong, as a consequence of Theorem 3.3.1 there exists an acyclic, transitive oriented graph T with vertices $\{u_1, u_2, \ldots u_t\}$ and quasi-transitive strong digraphs $H_1, H_2, \ldots H_t$ such that $D = T[H_1, H_2, \ldots H_t]$. Theorem 3.3.5 assures the existence of a (k,l)-kernel with $k \geq 3$ and $2 \leq l \leq k-1$ for every transitive digraph, so we can consider a (k,l)-kernel of T, say J, and since $H_1, H_2, \ldots H_t$ are quasi-transitive digraphs of order strictly smaller than m, it follows from the inductive hypothesis that every H_i has (k,l)-kernel J_i. Since T is acyclic, we just have to consider the first case of Theorem 3.3.2, which asks H_i to have a (k,l)-kernel for every $u_i \in J$ such that $\mathcal{C}_D^{k-1}(u_i) = \varnothing$. It follows from Theorem 3.3.2 that D has a (k,l)-kernel.

Case 2 If D is strong, as a consequence of Theorem 3.3.1 there exists a strong semicomplete digraph S with vertex set $\{v_1, v_2, \ldots, v_s\}$ and quasi-transitive digraphs $Q_1, Q_2, \ldots, Q_s$ such that Q_i is a single vertex or is non-strong, and $D = S[Q_1, Q_2, \ldots, Q_s]$. Since S is a semicomplete digraph, it follows from a well known result[1] that S has a 1-vertex quasi-kernel, which without loss of generality can be chosen as $\{v_1\}$. So $\{v_1\}$ is k-independent for every k and 2-absorbent, which implies that is also l-absorbent for every $2 \leq l \leq k-1$. Being S a strong semicomplete digraph, and as a consequence of Lemma 3.3.3, for every vertex $v \in V(S)$ there exists a directed cycle of length 2 or 3 containing v. Therefore, if $l \geq 3$, $\mathcal{C}_S^l(v_1) \neq \varnothing$ and in such case, applying Theorem 3.3.2, it suffices to consider $J = \{v_1\}$ and $J_1 = \{u\}$, where $u \in V(Q_1^c)$ is an arbitrary vertex. If $k = 3$, $l = 2$ and $\mathcal{C}_S^l(v_1) \neq \varnothing$, it also suffices to consider $J = \{v_1\}$ and $J_1 = \{u\}$, where $u \in V(Q_1^c)$ is an arbitrary vertex. If $k = 3$, $l = 2$ and $\mathcal{C}_S^l(v_1) = \varnothing$, then, as $k-1 = 2$, also $\mathcal{C}_S^{k-1}(x_1) = \varnothing$, and for the first case of Theorem 3.3.2, we only need to choose J_1 as a $(3,2)$-kernel for Q_1 that exists for the inductive hypothesis. It follows from Theorem 3.3.2 that $J^* = J_1$ is a (k,l)-kernel of D.

The result now follows from the principle of mathematical induction.

■

As mentioned above, case $k = 2$ is not covered by Theorem 3.3.6, but in

[1]Every tournament has a $(2,2)$-kernel.

this case a k-kernel is a kernel in the classical sense of Berge. For kernels in quasi-transitive digraphs we have a powerful sufficient condition given by the Strong Perfect Graph Theorem. Let us recall that Theorem 1.7.7 states that A graph G is perfect if and only if it is kernel solvable. Hence, applying Theorem 1.7.7 and remembering that underlying graphs of asymmetrical quasi-transitive digraphs are comparability graphs[2] and that comparability graphs are perfect[3], we obtain a sufficient condition for an asymmetrical quasi-transitive digraph D to have a kernel (in fact, to be kernel perfect).

Theorem 3.3.7. *If D is an asymmetrical quasi-transitive digraph such that every maximal semicomplete subdigraph of D has a kernel, then D is kernel perfect.*

Also, Galeana-Sánchez and Rojas-Monroy proved in [61] the following sufficient condition for a quasi-transitive digraph to have a kernel.

Theorem 3.3.8. *Let D be a (possibly infinite) digraph such that $D = D_1 \cup D_2$ (possibly $A(D_1) \cap A(D_2) \neq \varnothing$), where D_i is a quasi-transitive subdigraph of D which contains no asymmetrical (in D) infinite outward path. If every triangle contained in D has at least two symmetrical arcs, then D is a kernel perfect digraph.*

Corollary 3.3.9. *If D is a quasi-transitive digraph such that every triangle contained in D has at least two symmetrical arcs, then D is kernel-perfect.*

To finalize this section, we present a dualization of the results obtained.

Remark 3.3.10. Let D be a digraph. Then D is a quasi-transitive digraph if and only if $\overleftarrow{D}$ is a quasi-transitive digraph.

Corollary 3.3.11. *If D is a quasi-transitive digraph, then D has (k,l)-solution for every pair of integers k,l such that $k \geq 4$ and $3 \leq l \leq k-1$ or $k = 3$ and $l = 2$.*

[2] Proved by Ghouila-Houri in [63].
[3] Proved by Berge in [12].

Chapter 4

Multipartite tournaments

4.1 Introduction

At this point of the work, four families of digraphs have been studied with some detail. The main strategy in the previous chapters was to consider an existing (very promising) result, valid for a family $\mathcal{F}$ of digraphs, and try to prove analogous results for families containing $\mathcal{F}$. In Chapter 2, our base theorem was valid for strong cyclically k-partite digraphs and we prove an analogous theorem for unilateral cyclically k-partite digraphs. In Chapter 3, our starting point was a result for transitive digraphs and we proved results for quasi-transitive and right-(left-)pretransitive digraphs. So, following the same strategy once again, the turn of tournaments has come.

Let us recall that Theorem 1.8.1 states that every digraph has a $(2,2)$-kernel. When a $(2,2)$-kernel N is considered for a tournament T, since an independent set of T has size at most 1, it becomes clear that $N = \{v\}$ for some $v \in V(T)$. So, N is k-independent for every integer $k \geq 2$ and l-absorbent for every integer $l \geq 2$. Thus, tournaments have (k,l)-kernel for every pair of integers $k, l \geq 2$. Generalizations of tournaments are great candidates to have (k,l)-kernel.

Multipartite tournaments are among the most widely studied families of digraphs, as the survey of Volkmann [91] shows. This family has been studied in diverse contexts, such as hamiltonicity, pancyclicity, properties of cyles and paths, etc. But a subject that has received a lot of attention is the existence and number of 3-kings and 4-kings.

An obvious necessary condition for a digraph to have a k-king (k-serf) is that it has at most one initial (terminal) vertex. Clearly that restriction is not necessary for the case of k-solutions (k-kernels). Gutin [64] and independently Petrovic and Thomassen [81] proved that every multipartite tournament with at most one initial vertex has a 4-king, and there are infinitely many examples of multipartite tournaments without 3-kings. So, the two directions that have been studied since then are to find the number and distribution of 4-kings in multipartite tournaments without a 3-king and sufficient conditions (or characterizations) for a multipartite tournament to have 3-kings [65, 71, 82, 88].

It is clear that a k-serf (k-king) is also a k-kernel (k-solution), but the converse is not true, so although sufficient conditions for the existence of k-kings can be transformed in sufficient conditions for the existence of k-serfs and thus for the existence of k-kernels, characterization theorems can be extended in neither way. As we have seen k-kernels and k-solutions are generalizations of kernels and solutions, but they are also generalizations of k-serfs and k-kings, so a characterization theorem for multipartite tournaments having 3-kernel is very valuable. In this Chapter we give a theorem with two distinct such characterizations.

4.2 k-transitive closure

Sometimes, in Graph Theory the problem of determining if a graph G possesses a given property, can be reduced to the problem of determining if another graph, obtained from G by means of some operation, also posses the property. As an excellent example we can recall the famous result due to Bondy and Chvátal stating that a graph is hamiltonian if and only if its closure is hamiltonian.

If we apply this idea to the problem of finding a k-kernel in a digraph we may follow the following reasoning. Let D be a digraph, we wish to build a new digraph $\mathcal{C}_k(D)$ such that, $(k+1)$-independence in D is related with independence in $\mathcal{C}_k(D)$ as well as k-absorbency in D is related with absorbency in $\mathcal{C}_k(D)$. Moreover, this relation must be given in such a way that we can reduce the problem of finding a k-kernel in D to find a kernel in $\mathcal{C}_k(D)$. If this is possible, we could use the known sufficient conditions for the existence of kernels in digraphs to derive existence of k-kernels in families of digraphs.

Such digraph was introduced by Kwaśnik, Włoch and Włoch in [77].

If D is a digraph and $k \in \mathbb{N}$, the ***k*-transitive closure** of D is the digraph $\mathcal{C}_k(D)$ such that $V(\mathcal{C}_k(D)) = V(D)$ and $A(\mathcal{C}_k(D)) = \{(u, v)|$ there is a uv-directed walk of length $\leq k$ in $D\}$. We clearly have from the definition that $\mathcal{C}_1(D) = D$. In [77], the following result is proved.

Lemma 4.2.1. *If D is a digraph then $\mathcal{C}_k(D)$ has a kernel if and only if D has a $(k+1)$-kernel.*

Lemma 4.2.1 can be combined with all the known results about sufficient conditions for the existence of kernels in digraphs in a way that, if we can asseverate the existence of a kernel in $\mathcal{C}_k(D)$, then we will have as an immediate consequence the existence of a $(k+1)$-kernel in D.

The well known result we will use within this chapter is the following theorem due to Berge and Duchet [17].

Theorem 4.2.2. *If every directed cycle of D has at least one symmetrical arc, then D is kernel-perfect.*

There are other results imposing conditions on the directed cycles of a digraph as a sufficient condition for the existence of a kernel, so directed cycles in the closure of a digraph are important structure to be considered. We may ask under what conditions the closure of a digraph does not have directed cycles of odd length, or every directed cycle has at least one symmetrical arc, or every directed cycle has at least two pseudo-diagonals, or whatever fits for the closure to have a kernel. As the first example using this technique, in the Section 4.3 we present the aforementioned result for multipartite tournaments.

Another result that has an interesting corollary is due to Chvátal [27].

Theorem 4.2.3. *It is NP-complete to recognize whether a directed graph has a kernel, or not.*

From which we can derive a corollary.

Corollary 4.2.4. *For every $k \geq 2$ it is NP-complete to recognize whether a directed graph has a k-kernel, or not.*

So, as anyone can imagine, finding a k-kernel in a digraph, or recognizing if there is none, is a problem as difficult as the analogous for kernels.

4.3 k-kernels

Although the name of the chapter is multipartite tournaments, we will work with a larger class of digraphs, semicomplete multipartite digraphs. Another trivial remark about duality, which be useful for the aims of this work is now given.

Remark 4.3.1. If T is a semicomplete m-partite digraph, then $\overleftarrow{T}$ is also a semicomplete m-partite digraph.

When we are working with multipartite tournaments, sometimes is very important that between any two vertices in distinct classes there is one and only one arc. For the existence of k-kernels this is not the case. It is sufficient for us to assume that between any two vertices in distinct classes there is at least one arc, thus, we prefer to work with semicomplete multipartite digraphs rather than multipartite tournaments. Let us make an additional rermark before proving the main result of the section. If we were working with multipartite tournaments, we would be able to derive the existence of a 5-kernel in every multipartite tournament from the fact that every m-partite tournament has a 4-king [64, 81]. Nonetheless, the existence of 4-kernels and the sufficient condition for an m-partite tournament to have a 3-kernel cannot be obtained in this way.

Theorem 4.3.2. *Let $T = (V_1, V_2, \ldots, V_m)$ be a semicomplete m-partite digraph, then T has a k-kernel for every $m \geq 2$, $k \geq 4$. If every directed cycle of length 4 in T intersects 4 different classes of T, then T has a 3-kernel for every $m \geq 2$.*

Proof. Let T be a semicomplete m-partite digraph with $m \geq 2$. Let $k \geq 3$ be an integer and consider the $(k-1)$-transitive closure of T. In virtue of Lemma 4.2.1 to prove the result it suffices to show that $\mathcal{C}_{k-1}(T)$ has a kernel. With that in mind, we will prove that every directed cycle in $\mathcal{C}_{k-1}(T)$ has at least one symmetrical arc. This will be done by induction on the length of the cycle. For the base case let us consider a directed cycle $\mathscr{C}$ of length 3 in $\mathcal{C}_{k-1}(T)$. We have three possible cases, that the three vertices of $\mathscr{C}$ are in distinct classes of T, that two vertices are in the same class and the third one is in a different class, and that the three vertices are in the same class of T.

Case 1. If the three vertices of the cycle are in distinct classes of T, let us say, $\mathscr{C} = (x, y, z, x)$ with $x \in X$, $y \in Y$, $z \in Z$ and $X \neq Y$, $X \neq Z$, $Y \neq Z$.

If any of the arcs in $\mathscr{C}$ is not an arc of T, for instance $(x, y) \notin A(T)$, since T is a semicomplete multipartite digraph and x, y are in distinct classes of T, it must exist the arc $(y, x) \in A(T)$ and therefore also $(y, x) \in A(\mathcal{C}_{k-1}(T))$, and thus this is the symmetrical arc we want. So, we can assume without loss of generality that the three arcs of $\mathscr{C}$ are arcs of T, in particular we have that $(x, y), (y, z) \in A(T)$, such arcs conform a xz-directed walk of length $2 \leq k-1$ and thence $(x, z) \in A(\mathcal{C}_{k-1}(T))$. We can conclude that the arc $(z, x) \in A(\mathscr{C})$ is symmetrical.

Case 2. If $\mathscr{C} = (x_1, y, x_2, x_1)$ with $x_1, x_2 \in X$, $y \in Y$, $Y \neq X$, once again we can assume without loss of generality that $(x_1, y), (y, x_2) \in A(T)$, and thus $(x_1, x_2) \in A(\mathcal{C}_{k-1})(T)$ so the arc $(x_2, x_1) \in A(\mathscr{C})$ is a symmetrical arc.

Case 3. If $\mathscr{C} = (x_1, x_2, x_3, x_1)$ with $x_i \in X$ $i \in \{1, 2, 3\}$, following from the fact that T is a semicomplete m-partite digraph and x_1, x_2 are in the same class of T, we have that $(x_1, x_2) \notin A(T)$, but as $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$, then there exists a x_1x_2-directed path of length less than or equal to $k-1$ in T, namely $\mathscr{D} = (u_0 = x_1, u_1, \ldots, u_{p-1}, u_p = x_2)$, where $p \leq k-1$ and $u_1, u_{p-1} \notin X$ (the length of $\mathscr{D}$ can be 2, in that case $u_1 = u_{p-1}$). Since $u_1 \notin X$, then $(u_1, x_3) \in A(T)$ or $(x_3, u_1) \in A(T)$. If $(u_1, x_3) \in A(T)$, then $d_T(x_1, x_3) = 2$ and hence $(x_1, x_3) \in A(\mathcal{C}_{k-1}(T))$, which turns out to be the wanted symmetrical arc. In the latter case $(x_3, u_1) \cup (u_1 \mathscr{D} x_2)$ is a x_3x_2-directed path of length $p \leq k-1$ and in this way $(x_3, x_2) \in A(\mathcal{C}_{k-1}(T))$ resulting in the symmetrical arc we have been looking for.

Let us assume for the inductive step that every directed cycle of $\mathcal{C}_{k-1}(D)$ of length less than or equal to n has at least one symmetrical arc and let $\mathscr{C}$ be a directed cycle of length $n \geq 4$ in $\mathcal{C}_{k-1}(T)$. Let us observe three (arbitrarily chosen) consecutive vertices of $\mathscr{C}$, we have 5 cases.

Case 1. The considered segment of the directed cycle is (x, y, z) with $x \in X$, $y \in Y$, $z \in Z$ and $X \neq Y$, $X \neq Z$, $Y \neq Z$. Once again we can assume without loss of generality that $(x, y), (y, z) \in A(T)$ and thence $(x, z) \in A(\mathcal{C}_{k-1}(T))$. If $(x, z) \notin A(T)$, then $(z, x) \in A(T)$ and together with the arc (x, y) we can deduce the existence of the arc $(z, y) \in A(\mathcal{C}_{k-1}(T))$, which would be a symmetrical arc of $\mathscr{C}$. Thus, we can suppose that $(x, z) \in A(T)$. We have that $(x, z) \cup (z \mathscr{C} x)$ is a directed cycle of length $n-1$ which has a symmetrical arc for the induction hypothesis. If the symmetrical arc that exists for the induction hypothesis is in $z \mathscr{C} x$, then it is a symmetrical arc of $\mathscr{C}$. Let us assume then that the symmetrical arc is (x, z). Since $(z, x) \notin A(T)$, but $(z, x) \in A(\mathcal{C}_{k-1}(T))$, a zx-directed path of length greater than or equal to 2 but less than or equal to $k-1$ exists in T, namely $\mathscr{D} = (z = u_0, u_1, \ldots, u_{p-1}, u_p = x)$ with $p \leq k-1$. If $u_1 \notin Y$, then $(u_1, y) \in A(T)$ or $(y, u_1) \in A(T)$. In the former case, together with the arc (z, u_1) it follows the existence of the arc $(z, y) \in A(\mathcal{C}_{k-1}(T))$, that would be a symmetrical arc in $\mathscr{C}$. When $(y, u_1) \in A(T)$, $(y, u_1) \cup (u_1 \mathscr{D} x)$ is a yx-directed path of length $p \leq k-1$, which implies that $(y, x) \in A(\mathcal{C}_{k-1}(T))$ and this would be the symmetrical arc of $\mathscr{C}$ we have been looking for. If $u_1 \in Y$ and $p > 2$, then $k \geq 4$ and necessarily $u_2 \neq x$ and $u_2 \notin Y$, so $(u_2, y) \in A(T)$ or $(y, u_2) \in A(T)$. For the former case we can consider the directed path (z, u_1, u_2, y) in T and, since $k \geq 4$, the existence of this path implies the existence of the arc $(z, y) \in A(\mathcal{C}_{k-1}(T))$; for the latter case the directed path $(y, u_2) \cup (u_2 \mathscr{D} x)$ is in T and has length $p-1 < k-1$ so the existence of the arc $(y, x) \in A(\mathcal{C}_{k-1}(T))$ can be deduced. Finally, if $u_1 \in Y$ and $p = 2$, then $\mathscr{D} = (z, u_1, x)$ and it follows that $k \geq 4$, because for $k = 3$ we are assuming that every directed cycle of length 4 in T intersects 4 distinct classes of T but (x, y, z, u_1, x) is a directed cycle of length 4 in T that intersects only 3 different classes of T. So, $(x, y) \cup \mathscr{D}$ is a zy-directed path of length $3 \leq k-1$ and hence $(z, y) \in A(\mathcal{C}_{k-1}(T))$ is the desired symmetrical arc.

Case 2. The considered segment of $\mathscr{C}$ is (x_1, x_2, x_3) with $x_1, x_2, x_3 \in X$. Since $(x_1, x_2) \notin A(T)$, but $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$, there must exist a x_1x_2-directed path of length $2 \leq p \leq k-1$ in T, namely $\mathscr{D} = (x_1 = u_0, u_1, \ldots u_{p-1}, u_p = x_2)$. Let us observe that $(u_0, u_1) \in A(T)$, so $u_1 \notin X$ and hence $(x_3, u_1) \in A(T)$ or $(u_1, x_3) \in A(T)$. In the former case $(x_3, u_1) \cup (u_1 \mathscr{D} x_2)$ is a x_3x_2-directed path of length $p \leq k-1$ in T, and the existence of the arc $(x_3, x_2) \in A(\mathcal{C}_{k-1}(T))$ now follows, moreover, this arc is a symmetrical arc of $\mathscr{C}$. If $(u_1, x_3) \in A(T)$, together with the arc (x_1, u_1) we can deduce that $(x_1, x_3) \in A(\mathcal{C}_{k-1}(T))$ and, analogously to Case 1, $(x_1, x_3) \cup (x_3 \mathscr{C} x_1)$ is a directed cycle of length $n-1$ in $\mathcal{C}_{k-1}(T)$ which has at least one symmetrical arc by the induction hypothesis; if the symmetrical arc is other than (x_1, x_3) then $\mathscr{C}$ would have a symmetrical arc, so we can suppose that $(x_3, x_1) \in A(\mathcal{C}_{k-1}(T))$ from where we can deduce the existence of $\mathscr{E} = (x_3 = v_0, v_1, \ldots v_{q-1}, v_q = x_1)$, a x_3x_1-directed path of length $2 \leq q \leq k-1$ in T. Since $(x_3, v_1) \in A(T)$ it follows that $v_1 \notin X$ and thus $(v_1, x_2) \in A(T)$ or $(x_2, v_1) \in A(T)$. If $(v_1, x_2) \in A(T)$ then (x_3, v_1, x_2) directed path in T and therefore $(x_3, x_2) \in A(\mathcal{C}_{k-1}(T))$ is a symmetrical arc in $\mathscr{C}$. If $(x_2, v_1) \in A(T)$ we can consider the x_2x_1-directed path $(x_2, v_1) \cup (v_1 \mathscr{E} x_1)$ of length $q \leq k-1$ which implies the existence of the arc $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$, a symmetrical arc of $\mathscr{C}$.

Case 3. The considered segment of the directed cycle is (x_1, x_2, y) with $x_1, x_2 \in X$, $y \in Y$, $X \neq Y$. Let us assume without loss of generality that $(x_2, y) \in A(T)$, and since x_1 and y are in distinct classes of T, then one of the two possible arcs between them must exist in T. If $(y, x_1) \in A(T)$, then together with the arc (x_2, y) the existence of the arc $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$ can be derived, and this is the desired symmetrical arc. Let us suppose then that $(x_1, y) \in A(T)$, analogously to Case 1, $(x_1, y) \cup (y\mathscr{C}x_1)$ is a directed cycle of length $n-1$ which has a symmetrical arc by induction hypothesis, moreover, such arc must be (x_1, y) or we would have the symmetrical arc in $\mathscr{C}$ we have been looking for. Since $(y, x_1) \in \mathcal{C}_{k-1}(T)$ but $(y, x_1) \notin A(T)$, then a yx_1-directed path of length $2 \leq p \leq k-1$ exists in T, namely $\mathscr{D} = (y = u_0, u_1, \ldots, u_{p-1}, u_p = x_1)$. But, as $(u_{p-1}, x_1) \in A(T)$, then $u_{p-1} \notin X$ and therefore $(u_{p-1}, x_2) \in A(T)$ or $(x_2, u_{p-1}) \in A(T)$. In the former case $(y\mathscr{D}u_{p-1}) \cup (u_{p-1}, x_2)$ is a yx_2-directed path of length $p \leq k-1$ which implies the existence of $(y, x_2) \in A(\mathcal{C}_{k-1}(T))$ resulting a symmetrical arc of $\mathscr{C}$. In the latter case, together with the arc (u_{p-1}, x_1) the existence of the arc $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$ can be inferred, and this is the wanted symmetrical arc.

Case 4. The considered segment of $\mathscr{C}$ is (y, x_1, x_2) with $x_1, x_2 \in X$, $y \in Y$, $X \neq Y$. Since the length of $\mathscr{C}$ is greater than or equal to 4, then the cycle has at least another vertex, namely z, such that (x_1, x_2, z) is also a segment of $\mathscr{C}$. If $z \in Z \neq X$ then we have the same situation as in Case 3. If $z \in X$ then we have the same situation as in Case 2. In any case, we know that $\mathscr{C}$ has at least one symmetrical arc.

Case 5. The considered segment in the directed cycle is (x_1, y, x_2) with $x_1, x_2 \in X$, $y \in Y$, $X \neq Y$. We can assume once again without loss of generality that $(x_1, y), (y, x_2) \in A(T)$, from where we can infer that $(x_1, x_2) \in A(\mathcal{C}_{k-1}(T))$ and $(x_1, x_2) \cup (x_2\mathscr{C}x_1)$ is a directed cycle of length $n-1$ in $\mathcal{C}_{k-1}(T)$ which has at least one symmetrical arc by induction hypothesis. Such arc must be (x_1, x_2) or the existence of a symmetrical arc in $\mathscr{C}$ would be already proven. Therefore $(x_2, x_1) \in A(\mathcal{C}_{k-1}(T))$ and a x_2x_1-directed path $\mathscr{D} = (x_2 = u_0, u_1, \ldots u_{p-1}, u_p = x_1)$ must exist in T with $2 \leq p \leq k-1$. If $p = 2$ then $k \geq 4$ because (x_1, y, x_2, u_1, x_1) is a directed cycle of length 4 intersecting only 3 distinct classes of T, which can not occur for $k = 3$, so (x_2, u_1, x_1, y) is a x_2y-directed path of length $3 \leq k-1$ and then $(x_2, y) \in A(\mathcal{C}_{k-1}(T))$ is a symmetrical arc of $\mathscr{C}$. If $p > 2$ then there is at least one index $1 \leq i \leq p-1$ such that $u_i \notin Y$ and hence $(u_i, y) \in A(T)$ or $(y, u_i) \in A(T)$. In the former case, $(x_2\mathscr{D}u_i) \cup (u_i, y)$ is a x_2y-directed path of length less than or equal to $p \leq k-1$ and therefore $(x_2, y) \in A(\mathcal{C}_{k-1}(T))$ is the desired symmetrical arc in $\mathscr{C}$. For the case when $(y, u_i) \in A(T)$ we can take into account the yx_1-directed path of length less than or equal to $p \leq k-1$,$(y, u_i) \cup (u_i \mathscr{D} x_1)$ from which the existence of the arc $(y, x_1) \in A(\mathcal{C}_{k-1}(T))$ can be deduced and is, in fact, a symmetrical arc of $\mathscr{C}$.

Since the cases are exhaustive, the desired result follows from the principle of mathematical induction and an application of Theorem 4.2.2. ■

Corollary 4.3.3. *If T is a semicomplete m-partite digraph with $m \geq 2$ and no directed cycles of length 4, then T has a 3-kernel.*

Corollary 4.3.4. *If T is a semicomplete m-partite digraph with $m \geq 2$, then T has a k-solution for every $k \geq 4$.*

Proof. It follows from Remark 4.3.1 and Theorem 4.3.2. ■

Before making a full study of the case $k = 3$, let us observe that for case $k = 2$ we have the usual notion of kernel in Berge's sense and we know that every bipartite digraph has a kernel, so every semicomplete bipartite digraph has a kernel. For semicomplete m-partite digraphs with $m \geq 3$ we can always find a m-partite tournament without a kernel, that is to say, a tournament without vertices with exdegree equal to 0 on m vertices. As a matter of fact, since absorbency is 1-absorbency, in this case a semicomplete m-partite digraph T will have a kernel if and only if there is a class X of T such that for every vertex $v \in V(T) \setminus X$ there exists a vX-arc in T. In this case X will be the desired kernel. If we think a tournament of order m as an m-partite tournament this will happen if and only if there is a vertex of exdegree equal to 0 in the tournament.

For the case $k = 3$ we can also construct a m-partite tournament for every $m \geq 2$ without a 3-kernel. For $m = 2$ it suffices to consider the directed cycle of length 4, C_4, or the strong orientation of $K_{3,3}$ presented below.

In both digraphs depicted in Figure 4.1 the partition is given by the circular white vertices and the black starred ones. In both cases, every vertex 2-absorbs every other vertex except for its ex-neighborhood, so, the vertices in its ex-neighborhood are not 2-absorbed nor can be added to the 3-kernel.

For $m \geq 3$, we can define the m-partite tournament $T_{C_4,m}$ with vertex set $V(T_{C_4,m}) = \{1, 1', 2, 2'\} \cup \{3, \ldots m\}$ and arc set $A(T_{C_4,m}) = \{(1,2), (2,1'), (1',2'), (2',1)\} \cup \bigcup_{i=3}^{m} \{(i,1), (i,1'), (i,2), (i,2')\} \cup \bigcup_{i<j} \{(j,i)\}$ whit vertex partition given by $V(T_{C_4,m}) = \bigcup_{i=1}^{2} \{i, i'\} \cup \bigcup_{i=3}^{m} \{i\}$.

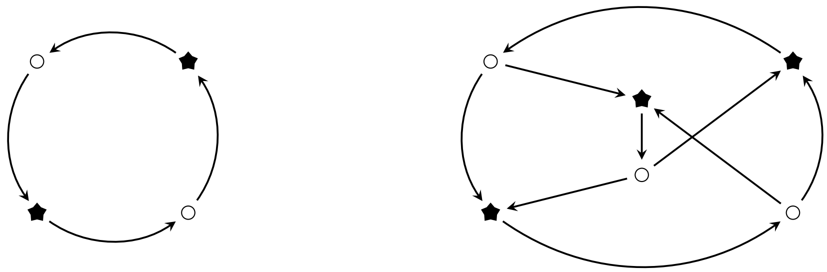


Figure 4.1: Examples of bipartite tournaments without a 3-kernel

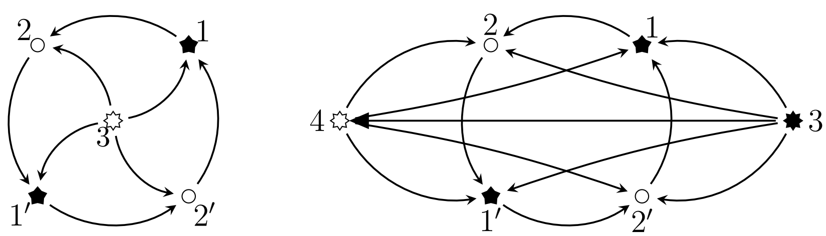


Figure 4.2: $T_{C_4,3}$ and $T_{C_4,4}$

As an obvious consequence of Theorem 4.3.2, every example of a m-partite tournament T without a 3-kernel must have a copy of C_4 as a subdigraph and this copy of C_4 must intersect 4 distinct classes of T. Unfortunately, the sufficient condition presented in Theorem 4.3.2 for a m-partite tournament to have a 3-kernel is not necessary. Figure 4.3.5 shows an example of a bipartite tournament with a copy of C_4 which clearly does not intersect 4 different classes of T and with a 3-kernel.

The central vertex in the bipartite tournament of Figure 4.3 is a 3-kernel for the tournament.

But, let us observe the structure of a 3-kernel in a m-partite tournament T. Obviously, since vertices in distinct classes are always adjacent, not only for $k = 3$ but for every k, a k-kernel must be contained in a single class of T. The next proposition explore a necessary condition for a m-partite tournament to have a 3-kernel.

Proposition 4.3.5. *If T is a semicomplete m-partite digraph with a 3-kernel N, then for every $v \in N$, the set $\{v\}$ is a 2-absorbent set of $T - (X \setminus \{v\})$*

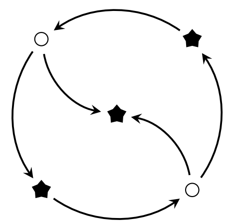


Figure 4.3: An example of a bipartite tournament with a copy of C_4 and a 3-kernel.

where X is the class of T which contains N.

Proof. Let T be a semicomplete m-partite digraph with 3-kernel N and X the class of T that contains N. Let $v \in N$ be an arbitrary vertex. Clearly the in-neighborhood of v is 2-absorbed by v, so let us think of a vertex $u \in V(T) \setminus (X \cup N^-(v))$, then since T is a semicomplete multipartite digraph, $u \in N^+(v)$, and since N is a 3-kernel of T, there must exist a vertex $w \in N$ such that w 2-absorbs u. If $w = v$ we are done.

For the bipartite case, every vertex in $V(T) \setminus X$ can only be 2-absorbed at distance 1 by N, so if $w \neq v$, then the existence of the directed path (v, u, w) would contradict the 3-independence of N. So for the bipartite case it follows that $u \in N^-(v)$.

For $m \geq 3$, let us assume that $v \neq w$; if $u \in N^-(w)$, then (v, u, w) would be a NN-directed path of length 2 in T and N would not be 3-independent, so $u \in N^+(w)$ and then there exists a vertex $z \notin X$ such that (u, z, w) is a directed path in T. But, as $z \notin X$, then $(v, z) \in A(T)$ or $(z, v) \in A(T)$; the former case can not occur because the directed path (v, z, w) would contradict the 3-independence of N. So, $(z, v) \in A(D)$ and (u, z, v) is a directed path of length 2 in T, so $\{v\}$ 2-absorbs u. Since u was chosen arbitrarily in $V(T) \setminus (X \cup N^-(v))$, then $\{v\}$ is an absorbent set for $T - (X \setminus \{v\})$. ■

Well, at this point the obvious question arise, is the converse of Proposition 4.3.5 also true? That is, if T is a semicomplete m-partite digraph with a vertex v such that $\{v\}$ is a 2-absorbent set of $T - (X \setminus \{v\})$ where X is the class of T that contains v, then T has a 3-kernel? In that case we would have a characterization of semicomplete multipartite digraphs with 3-kernel. Not only the answer to this question is yes, we also have a third equivalence inspired in the observations made about the directed cycles of length 4 when we were looking for a 3-kernel.

Theorem 4.3.6. *Let T be a semicomplete m-partite digraph with $m \geq 2$, then the following assertions are equivalent.*

1. *T has a 3-kernel.*
2. *There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v, $\{v\}$ is a 2-absorbent set of $T - (X \setminus \{v\})$.*
3. *There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v, $\{v\}$ 2-absorbs in T every $x \in \{v\} \cup (T \setminus X)$ such that x is in a directed cycle of length 4 of T.*

Proof. For (i) implies (ii) we just have to choose an arbitrary vertex in the 3-kernel, the result follows from Proposition 4.3.5.

Trivially (ii) implies (iii).

For (iii) implies (i), let v be a vertex that fulfills the property stated in (iii). Let $R = \{v\} \cup \{u \in V(T) \big| d_T(u, v) \leq 2\}$, that is, the set of vertices that are 2-absorbed by v. As a consequence of the choice of v and Corollary 4.3.3, if $S := T \setminus R$ is non empty, it has a 3-kernel, namely N (if it is empty, then $\{v\}$ is a 3-kernel of T). If $N \cup \{v\}$ is a 3-independent set in T, then since N is 2-absorbent in S and $\{v\}$ 2-absorbs every vertex in R, then $N \cup \{v\}$ is a 2-absorbing set and thus the desired 3-kernel of T. If $N \cup \{v\}$ is not 3-independent then there are two possibilities, that $N \subseteq X$ or that $N \subseteq Y$ for a class $Y \neq X$ of T. In either case, in virtue of Claim 4.3.7, we can assume without loss of generality that N is 3-independent in T. Our next claim will be proved after completing the proof of the theorem.

Claim 4.3.7. *If the 3-kernel N that exists for S is not 3-independent in T, we can choose $K \subseteq N$ such that K is 3-independent in T, every vertex 2-absorbed by N in S is also absorbed by K and every vertex in $N \setminus K$ is 2-absorbed by K.*

If $N \subseteq X$, since N is 3-independent, $N \cup \{v\}$ is not 3-independent but $v \in X$, there must be a vx-directed path of length two or a xv-directed path of length two in T for some $x \in N$. But the latter case can not occur, or x would be 2-absorbed by v and it would belong to R. So, the former case occurs and v is 2-absorbed by N in T. If we prove that $N^-(v) \subseteq N^-(x)$, then every vertex 2-absorbed by v will be also 2-absorbed by x and then N will be the desired 3-kernel. Let $y \in N^{-}(v)$ be chosen arbitrarily. Since T is a semicomplete multipartite digraph, $x \in X$ and $y \notin X$, then $(x, y) \in A(T)$ or $(y, x) \in A(T)$; but if $(x, y) \in A(T)$, then (x, y, v) would be a xv-directed path of length 2, contradicting that $x \notin R$, so $(y, x) \in A(T)$ and then $y \in N^{-}(x)$, so $N^{-}(v) \subseteq N^{-}(x)$.

If $N \subseteq Y$ for a class $Y \neq X$ of T, then for every $x \in N$, $(v, x) \in A(T)$, so v and $N^{-}(v)$ are 2-absorbed by N in T. Let u be a vertex in R such that $(v, u) \in A(T)$, then there is a vertex $w \in N^{-}(v)$ such that $(u, w) \in A(T)$. If $w \notin Y$ and $x \in N$, then $(w, x) \in A(T)$ or $(x, w) \in A(T)$. If $(x, w) \in A(T)$, since $w \in N^{-}(v)$, x would be 2-absorbed by v and thus an element of R, which is not the case. It follows that $(w, x) \in A(T)$, then (u, w, x) is a directed path in T and u is 2-absorbed by N in T. If $w \in Y$, then $u \notin Y$, and then, if $x \in N$, $(u, x) \in A(T)$ or $(x, u) \in A(T)$. If $(x, u) \in A(T)$, then (v, x, u, w, v) is a directed cycle of length 4 in T, and by the choice of v fulfilling the conditions in (iii), x would be 2-absorbed by v and $x \in R$, which would be a contradiction. Thus, $(u, x) \in A(T)$ and then N 2-absorbs every vertex in R and is the desired 3-kernel. ■

Proof of Claim 4.3.7 Let us recall that $S := T - R$. In virtue of Proposition 4.3.5, if $N \subseteq Z$ for a class Z of T, then every vertex in N 2-absorbs every vertex in $S \setminus Z$, so we just have to find a subset $K \subseteq N$ that is 3-independent and 2-absorbs every vertex in $Z \setminus (R \cup N)$ and every vertex in $N \setminus K$. Let $x \in Z \setminus (R \cup N)$ an arbitrarily chosen vertex, since N is a 3-kernel for S and $x \in Z \setminus N$, then $d_S(x, u) = 2$ for some vertex $u \in N$, so there is a vertex $y \in V(T) \setminus (Z \cup R)$ such that (x, y, u) is a directed path in T. Since $y \notin Z$, then for every other vertex $w \in N$, $(w, y) \in A(T)$ or $(y, w) \in A(T)$, but if $(w, y) \in A(T)$, then $d_S(w, u) = 2$, contradicting the 3-independence of N in S, so $(y, w) \in A(T)$ and then every vertex in $Z \setminus (R \cup N)$ is 2-absorbed by every vertex in N.

Let us prove by means of mathematical induction on the cardinality of N that we can always choose a subset $K \subseteq N$ such that K is 3-independent and 2-absorbs in T every vertex in $N \setminus K$. If $|N| = 1$, then $K = N$ will work. If $|N| = 2$ and N is 3-independent in T we are done; otherwise $N = \{u, w\}$ and it follows from the 3-dependence of N that $d_S(u, w) = 2$ or $d_S(w, u) = 2$, let us assume without loss of generality that $d_S(u, w) = 2$, then u is 2-absorbed by $\{w\}$ and hence $\{w\} = K$. Suppose for the induction hypothesis that if $|N| < n$ we can find $K \subseteq N$ with the desired property and let N be a 3-kernel for S with cardinality n. If N is 3-independent we are done, otherwise, there exists vertices $u, w \in N$ such that $d_S(u, w) = 2$, so u is 2-absorbed by N and $N' := N \setminus \{u\}$ is a 3-kernel of $S - u$, so by induction hypothesis there exists $K' \subseteq N'$ such that K' is 3-independent in T and absorbs every vertex in $N' \setminus K'$. If K' 2-absorbs u, then $K = K'$ is the desired set. If K' does not absorb u and $K' \cup \{u\}$ is 3-independent in T, then $K = K' \cup \{u\}$ is the subset we have been looking for. If K' does not 2-absorb u and $K' \cup \{u\}$ is not 3-independent, since N' 2-absorbs u, then $K' \subset N'$ and thus $|K' \cup \{u\}| \leq |N'| < |N|$. We also have that $K' \cup \{u\}$ is a 3-kernel of $S - (N \setminus (K' \cup \{u\}))$, so it follows from the induction hypothesis that there exists $K \subset K' \cup \{u\}$ such that K 2-absorbs in T every vertex in $(K' \cup \{u\}) \setminus K$. Finally, we need K to 2-absorb in T every vertex in $N \setminus (K' \cup \{u\})$ but, if $x \in N \setminus (K' \cup \{u\})$ for the choice of K' there exist $z \in K'$ and $y \in V(T) \setminus Z$ such that (x, y, z) is a directed path in T. If k is an arbitrary vertex in K, then $(k, y) \in A(T)$ or $(y, k) \in A(T)$. If $(k, y) \in A(T)$, then $d_T(k, z) = 2$, contradicting the 3-independence of K' in T, so $(y, k) \in A(T)$ and thence K 2-absorbs x in T. The result follows by the principle of mathematical induction. ■

Corollary 4.3.8. *Let T be a semicomplete m-partite digraph with $m \geq 2$, then the following assertions are equivalent.*

1. *T has a 3-solution.*

2. *There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v, $\{v\}$ is a 2-dominating set of $T - (X \setminus \{v\})$.*

3. *There is a vertex $v \in V(T)$ such that, if X is the class of T that contains v, $\{v\}$ 2-dominates in T every $x \in \{v\} \cup (T \setminus X)$ such that x is in a directed cycle of length 4 of T.*

Proof. It follows from Remark 4.3.1 and Theorem 4.3.6. ■

Since every m-partite tournament is a semicomplete m-partite digraph, every result stated in this section for semicomplete m-partite digraphs remain valid for m-partite tournaments.

Chapter 5

k-transitive and k-quasi-transitive digraphs

5.1 Introduction

Classic generalizations of transitive digraphs have been studied in Chapter 3. Since transitive digraphs are possibly the best behaved digraphs in the context of (k, l)-kernels, in the present chapter we introduce a further generalization of transitive digraphs. A brief structural analysis of this generalization is made and some results concerning k-kernels are derived.

We want to emphasize two aspects about this chapter, the generalization of transitive and quasi-transitive digraphs that we introduce and the results obtained about k-kings in this new families of digraphs. So, we will begin with a little expansion of our background on quasi-transitive digraphs, and k-kings.

A graph is a **comparability graph** if it admits a transitive orientation, Berge proved that comparability graphs are perfect. Quasi-transitive digraphs were introduced in [63] by Ghouila-Houri to characterize comparability graphs as those graphs that admit a quasi-transitive orientation, so, every asymmetrical quasi-transitive digraph can be reoriented into a asymmetrical transitive digraph. Later, quasi-transitive digraphs were studied as a generalization of semicomplete digraphs, where they were found to have very nice properties, among them, maybe the nicest is the recursive structural characterization stated in Theorem 3.3.1, which can be used to prove that, for instance, the longest path and cycle problems are polynomial time solvable in this family, or to characterize Hamiltonian and traceable quasi-transitive digraphs. So, the family of quasi-transitive digraphs is a good one to verify the behavior of difficult problems.

Also, a generalization of quasi-transitive in the same direction that we propose have been studied before, that is 3-quasi-transitive digraphs, which where introduced by Bang-Jensen in the context of arc-locally semicomplete digraphs. Arc locally semi-complete digraphs were thought as a generalization of semicomplete digraphs that could also contain semicomplete bipartite digraphs [3]. This family is characterized by two sets of forbidden substructures (families $\mathscr{H}_1$ and $\mathscr{H}_2$ in Figure 5.1), both of them arising from orientations of the path of length 3, but whit this logic, another two sets of forbidden substructures can be considered; 3-quasi-transitive digraphs are those digraphs which do not have any subdigraph of the family $\mathscr{H}_3$ of Figure 5.1 as an induced subdigraph. In [3], Bang-Jensen obtained a characterization of 3-quasi-transitive strong digraphs which turned out to be incomplete, but Galeana-Sánchez, Goldfeder and Urrutia completed this characterization in [43]. This class of digraphs also has a lot of structure. In [94], Wang and Wang study the structure of non-strong 3-quasi-transitive digraphs.

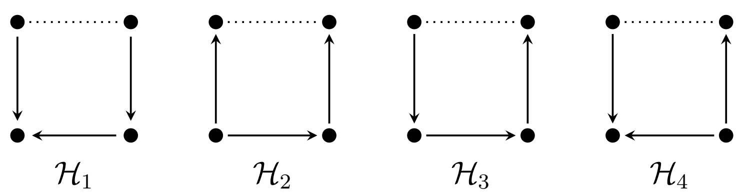


Figure 5.1: Each of the 4 digraphs shown denote a class of digraphs $\mathscr{H}_i$ with 4 vertices containing the 3 arcs shown and having no arc between the two vertices with a dotted edge between them. All other arcs not shown and not with the same end vertices as the dotted edge are possible in $\mathscr{H}_i$.

As we mentioned in Chapter 1, a particular case of k-kernels is considering a k-kernel consisting in only one vertex. Being a $(k-1)$-absorbing set, this kind of k-kernel can be found in the literature under the name of $(k-1)$-serf, but the most popular version of this problem is considering its dual, *i.e.*, the existence of k-kings. The problem of finding k-kings in digraphs have been largely studied for some classes of digraphs, the principal are multipartite tournaments and multipartite semicomplete digraphs [64, 65, 71, 81, 82, 88], but also Bang-Jensen and Huang explored this problem for quasi-transitive digraphs in [9], proving in particular that a quasi-transitive digraph has a 3-king if and only if it has one unique terminal strong component. In both these families, considering necessary restrictions like the number of vertex with out-degree equal to zero of the number of initial components, the existence was proved for small k (4 and 3 respectively), so the problem that has been approached since then is finding the number and configurations of k-kings in such digraphs.

5.2 k-path-transitive digraphs

We begin the development of this chapter introducing a quite simple family of digraphs that will be used as a tool to prove some results in the following sections. A digraph D is called ***k*-path-transitive** if whenever there are a uv-directed path of length less than or equal to k and a vw-directed path of length less than or equal to k, then there exists a uw-directed path of length less than or equal to k. Digraphs of this family have a very simple characterization.

Lemma 5.2.1. *A digraph D is k-path-transitive if and only if whenever $u, v \in V(D)$ and there exists a uv-directed path in D, then $d(u, v) \leq k$.*

Proof. First let D be a k-path-transitive digraph, $u, v \in V(D)$ two arbitrary distinct vertices and $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ a uv-directed path in D. We will prove by induction on n that $d(u, v) \leq k$. If $n \leq k$ then we are done. Let us assume that the result holds for every $m < n$ and consider the uv-directed path $\mathscr{C}$ of length $n \geq k+1$. Clearly (x_0, x_1) is a x_0x_1-directed path of length $\leq k$ and $x_1 \ldots x_{k+1}$ is a x_1x_{k+1}-directed path of length less than or equal to k, then, by the k-path-transitivity of D there must exist a x_0x_{k+1}-directed path of length $\leq k$, let us say, $\mathscr{C}'$. So $\mathscr{C}' \cup x_{k+1}\mathscr{C}x_n$ is a x_0x_n-directed path of length less than n and by induction hypothesis it follows that $d(u, v) \leq k$.

Now, let D be a digraph such that whenever $u, v \in V(D)$ and there exists a uv-directed path in D, then $d(u, v) \leq k$. Let $\mathscr{C}$ and $\mathscr{D}$ be uv and vw-directed paths of length less than or equal to k, then $\mathscr{C} \cup \mathscr{D}$ is a uw-directed path in D so $d(u, w) \leq k$ and a uw-directed path of length less than or equal to k exists. ■

In Chapter 2 we defined a kernel by directed paths. Berge proved that every digraph has a kernel by directed paths, a proof of this fact can be consulted in [11], and as an obvious dual result it can be derived that every digraph has a solution by directed paths.

Theorem 5.2.2. *If D is a k-path transitive digraph then D has an n-kernel for every $n \geq k+1$.*

Proof. It suffices to to choose a kernel by directed paths of D, let us say N, we affirm than N is also an n-kernel. It is clearly n-independent for every $n \geq k$ because N is independent by directed paths. Now, let $u \in V(D) \setminus N$ be an arbitrary vertex in the complement of N, then there is a uv-directed path for some $v \in N$, because N is absorbent by directed paths, but in virtue of Lemma 5.2.1, there is also a uv-directed path of length less than or equal to k, so N is $n-1$-absorbent for every $n \geq k+1$. Thence, N is an n-kernel for D. ■

The particular case of k-kings is considered in the next theorem.

Theorem 5.2.3. *Let D be a k-path transitive digraph, then D has a k-king if and only if D has a unique initial strong component. Moreover, every vertex in the unique initial strong component of D is a k-king.*

Proof. If D has a k-king v, then the component that contains v is clearly the unique initial strong component of the digraph. If D has a unique initial strong component, is suffices to choose any vertex in such component, this vertex is a solution by directed paths and hence a k-king in virtue of Lemma 5.2.1. ■

5.3 k-transitive digraphs

Our following definition generalizes the notion of transitive digraphs. A digraph D is ***k*-transitive** if whenever $\mathscr{C} = (x_0, x_1, \ldots, x_k)$ is a directed path of length k in D, then $(x_0, x_k) \in A(D)$.

As a rather simple consequence of the previous definition we have the following lemma.

Lemma 5.3.1. *Let $k \geq 2$ be an integer. If D is a k-transitive digraph, then D is $(k-1)$-path-transitive.*

Proof. Let $u, v \in V(D)$ be arbitrary distinct vertices and let $\mathscr{C} = (u = x_0, x_1,$
$\ldots, x_n = v)$ be a uv-directed path. We will prove by induction on n that $d(u,v) \leq k-1$. If $n \leq k-1$ then we are done. So let us assume that $n \geq k$, then, by the k-transitivity of D, since $x_0 \mathscr{C} x_k$ is a directed path of length k in D, $(x_0, x_k) \in A(D)$, so $(x_0, x_k) \cup x_k \mathscr{C} x_n$ is a uv-directed path of length strictly less than n, we can derive from the induction hypothesis that $d(u,v) \leq k-1$. The result follows from the principle of mathematical induction and Lemma 5.2.1. ■

Just like the transitive case, the k-transitive case is very simple to analyze, at least the obvious generalization of the theorem that affirm that if D is a (2-)transitive digraph, then D has a (2-)kernel, which can be found in [11].

Theorem 5.3.2. *Let $k \geq 2$ be an integer. If D is a k-transitive digraph, then D has an n-kernel for every $n \geq k$.*

Proof. It follows immediately from Lemma 5.3.1 and Theorem 5.2.2. ■

And once again, the particular case of k-kings.

Theorem 5.3.3. *Let D be a k-transitive digraph, then D has a $(k-1)$-king if and only if D has a unique initial strong component. Moreover, every vertex in the unique initial strong component of D is a $(k-1)$-king.*

Proof. It is clear from Lemma 5.3.1 and Theorem 5.2.3. ■

Let us make the rather obvious observation that a digraph D is k-transitive if and only if $\overleftarrow{D}$ is k-transitive, so every result for k-kernels has a dual for k-solutions, and the same is true for k-kings and k-serfs.

Thus, since our main interest is to find families of digraphs with k-kernel, we only present a simple exploration of both the k-path-transitive and k-transitive digraphs, but considering the rich structure of transitive digraphs, a lot of questions arise concerning the structure of both strong and non-strong k-transitive digraphs. It is clear that transitive strong digraphs are complete digraphs, and that the condensation of a transitive digraphs is again a transitive digraph. However, this is not true for k-transitive digraphs, k-transitive strong digraphs are not complete digraphs and the condensation of a k-transitive digraph is not k-transitive, but k-path-transitive. So is a natural question to ask if k-transitive digraphs have a nice structural characterization. At least is easy to observe that for every $k \geq 2$, a k-transitive strong digraph have diameter $\leq k-1$. Also, what happens to the n-kernels for $n \leq k$ in k-transitive digraphs? It is clear that a directed k-cycle is a k-transitive digraph that does not have a $(k-1)$-kernel. Can k-transitive digraphs possesing an n-kernel for $n \leq k$ be characterized? We think that these are two interesting problems. The case $k = 3$ will be analyzed at the final section of this chapter.

5.4 k-quasi-transitive digraphs: Preliminaries

Among the families we introduce in this work, k-quasi-transitive digraph seem to be the most interesting one. At least for us, the most intuition-defying results where obtained for this family. A digraph D is called ***k*-quasi-transitive** if, whenever $(x_0, x_1, \ldots, x_k)$ is a directed path of length k in D, then $(x_0, x_k) \in A(D)$ or $(x_k, x_0) \in A(D)$.

From the definition above it is clear that a quasi-transitive digraph in the usual sense is a 2-quasi-transitive digraph. Also, 3-quasi-transitive digraphs have been studied in [3] and strong 3-quasi-transitive digraphs characterized in [43].

Analogously to the previously studied families, we have a dualization remark.

Remark 5.4.1. Let D be a digraph, then D is a k-quasi-transitive digraph if and only if $\overleftarrow{D}$ is a k-quasi-transitive digraph.

Proceeding as Bang-Jensen in the study of quasi-transitive digraphs we propose the following lemmas.

Lemma 5.4.2. *Let $k \in \mathbb{N}$ be an even natural number, D a k-quasi-transitive digraph and $\mathscr{C} = (x_0, x_1, \ldots, x_{k+3})$ a directed path such that $d(x_0, x_{k+3}) = k+3$ and $(x_{k+3}, x_1) \in A(D)$, then $(x_{k+3}, x_{k-2i}) \in A(D)$ for every $0 \leq i \leq \frac{k}{2}$. In particular $(x_{k+3}, x_0) \in A(D)$.*

Proof. By induction on i. For the base case, let $i = 0$, then $(x_{k+3}, x_1) \cup x_1 \mathscr{C} x_k$ is clearly a $x_{k+3}x_k$-directed path of length k. Since D is k-quasi-transitive then $(x_{k+3}, x_k) \in A(D)$ or $(x_k, x_{k+3}) \in A(D)$, but $d(x_0, x_{k+3}) = k+3$, so $(x_k, x_{k+3}) \notin A(D)$ and therefore $(x_{k+3}, x_k) \in A(D)$.

For the inductive step, let us assume that $(x_{k+3}, x_{k-2i}) \in A(D)$ for every $0 \leq i < n \leq \frac{k}{2}$. Clearly $\mathscr{C}' = (x_{k+3}, x_{k-2(n-1)}) \cup x_{k-2(n-1)}\mathscr{C}x_k \cup (x_k, x_0) \cup x_0\mathscr{C}x_{k-2n}$ is a directed path and $\ell(\mathscr{C}') = 1 + \ell(x_{k-2(n-1)}\mathscr{C}x_k) + 1 + \ell(x_0\mathscr{C}x_{k-2n}) = 2 + 2(n-1) + k - 2n = k$ and therefore $(x_{k+3}, x_{k-2n}) \in A(D)$ or $(x_{k-2n}, x_{k+3}) \in A(D)$, but since $d(x_0, x_{k+3}) = k + 3$, then $(x_{k-2n}, x_{k+3}) \notin A(D)$ which implies that $(x_{k+3}, x_{k-2n}) \in A(D)$.

The desired result now follows from the principle of mathematical induction.

■

Lemma 5.4.3. *Let $k \in \mathbb{N}$ be an even natural number, D a k-quasi-transitive digraph and $u, v \in V(D)$ such that a uv-directed path exists. Then:*

1. *If $d(u,v) = k$, then $d(v,u) = 1$.*
2. *If $d(u,v) = k + 1$, then $d(v,u) \leq k + 1$.*
3. *If $d(u,v) \geq k + 2$, then $d(v,u) = 1$*

Proof.

1. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_k = v)$ be a directed path in D that realizes the distance from u to v. Since D is k-quasi-transitive, then $(u,v) \in A(D)$ or $(v,u) \in A(D)$, but $d(u,v) = k$, so $(u,v) \notin A(D)$, therefore $(v,u) \in A(D)$.

2. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_k, x_{k+1} = v)$ be a directed path in D that realizes the distance from u to v. From the k-quasi-transitivity and the fact that $d(u,v) = k+1$ it follows that $(x_{k+1}, x_1), (x_k, x_0) \in A(D)$. Observe that $(x_{k+1}, x_1) \cup x_1\mathscr{C}x_k \cup (x_k, x_0)$ is a vu-directed path of length $k + 1$.

3. By induction on $n = d(u,v)$. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ be a directed path in D. If $n = k + 2$, then by the k-quasi-transitivity of D we have that $(x_0, x_k) \in A(D)$ or $(x_k, x_0) \in A(D)$ and also $(x_2, x_n) \in A(D)$ or $(x_n, x_2) \in A(D)$. Since $d(x_0, x_n) = n$, in the former case we have that $(x_0, x_k) \notin A(D)$ and in the latter case we can deduce that $(x_2, x_n) \notin A(D)$, so $(x_k, x_0), (x_n, x_2) \in A(D)$. Now, let $\mathscr{C}' = (x_n, x_2) \cup x_2\mathscr{C}x_k \cup (x_k, x_0)$ be a directed path in D. It is clear that $\ell(\mathscr{C}') = 1 + \ell(x_2\mathscr{C}x_k) + 1 = 1 + (k-2) + 1 = k$, and by the k-quasi-transitivity of D and the fact that $d(x_0, x_n) = n$ it follows that $(x_n, x_0) \in A(D)$. So the base case holds.

If $n = k+3$, then by the base case and Lemma 5.4.2, we have that $d(v,u) = 1$. So we can assume that $n > k+3$ and by the inductive hypothesis and the fact that $\ell(x_2 \mathscr{C} x_n) \geq k+2$, we can deduce that $(x_n, x_2) \in A(D)$. It is clear that $\mathscr{C}' = (x_n, x_2) \cup x_2 \mathscr{C} x_k \cup (x_k, x_0)$ is a vu-directed path of length k. From the k-quasi-transitivity of D and the fact that $d(u,v) = n$ we can deduce that $(v,u) \in A(D)$.

The result now follows from the principle of mathematical induction. ■

Lemma 5.4.4. *Let $k \in \mathbb{N}$ be an odd natural number, D a k-quasi-transitive digraph and $u, v \in V(D)$ such that a uv-directed path exists. Then:*

1. *If $d(u,v) = k$, then $d(v,u) = 1$.*
2. *If $d(u,v) = k+1$, then $d(v,u) \leq k+1$.*
3. *If $d(u,v) = n \geq k+2$ with n odd, then $d(v,u) = 1$*
4. *If $d(u,v) = n \geq k+3$ with n even, then $d(v,u) \leq 2$*

Proof.

1. As in Lemma 5.4.3.
2. As in Lemma 5.4.3.
3. Will be proved along with (4.).
4. By induction on $n = d(u,v)$. For the case $n = k+2$ the proof is as in Lemma 5.4.3. So, to complete the base case let us consider the case $n = k+3$. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ be a directed path. By the case $n = k+2$ we know that $(x_n, x_1) \in A(D)$ and, clearly $(x_n, x_1) \cup x_1 \mathscr{C} x_k$ is a $x_n x_k$-directed path of length k, so by the k-quasi-transitivity of D, $(x_n, x_k) \in A(D)$ or $(x_k, x_n) \in A(D)$, but the latter case can not occur because $d(u,v) = n$, and then $(x_n, x_k) \in A(D)$. Also from (1.) we know that $(x_k, x_0) \in A(D)$, so (x_n, x_k, x_0) is a vu-directed path of length 2 and $d(v,u) \leq 2$.

   For the inductive step let us assume that $n > k+3$ and that $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ is a directed path. If n is odd, then by induction hypothesis $(x_n, x_2) \in A(D)$, also we know that $(x_k, x_0) \in A(D)$, so $(x_n, x_2) \cup x_2 \mathscr{C} x_k \cup (x_k, x_0)$ is a $x_n x_0$-directed path of length k. By the k-quasi-transitivity and the fact that $d(u,v) = n$ we can deduce that $(x_n, x_0) \in A(D)$. If n is even, then by induction hypothesis

$(x_n, x_1) \in A(D)$. So $(x_n, x_1) \cup x_1 \mathscr{C} x_k$ is a $x_n x_k$-directed path of length k and it follows from the k-quasi-transitivity of D and $d(u,v) = n$ that $(x_n, x_k) \in A(D)$, once again, (x_n, x_k, x_0) is a directed path of length 2 and therefore $d(v,u) \leq 2$. This completes the inductive step and the result follows from the principle of mathematical induction.

■

Our next lemma also resembles a result obtained by Bang-Jensen in the study of quasi-transitive digraphs, although we were unable to characterize k-quasi-transitive non-strong digraphs, a nice behavior is observed in the condensation of a k-quasi-transitive digraph. If D is a digraph, A and B are strong components of D, we denote by $A \overset{k}{\to} B$ the fact that every vertex of A k-dominates every vertex of B. If D is a digraph, the ***k*-condensation** of D is the digraph $D^\star_k$ such that $V(D^\star_k)$ is the set of strong components of D, and if A and B are strong components of D, then $(A,B) \in A(D^\star_k)$ if and only if there is a AB-directed path of length less than or equal to k in D.

Lemma 5.4.5. *Let D be a k-quasi-transitive digraph. If $A \neq B$ are strong components of D such that there exists a AB-directed path in D, then $A \overset{k-1}{\to} B$.*

Proof. Since there exists an AB-directed path in D, then for every $u \in V(A)$ and $v \in V(B)$ a uv-directed path exists. By Lemmas 5.4.3 and 5.4.4 it must be the case that $d(u,v) \leq k-1$ for if not, there would exist a vu-directed path, which can not happen because A and B are distinct strong components of D and a AB-directed path already exists. ■

Lemma 5.4.6. *Let D be a k-quasi-transitive digraph, then the condensation $D^\star$ of D is k-path-transitive. Also, the $(k-1)$-condensation $D^\star_{k-1}$ of D is transitive.*

Proof. Let D be a k-quasi-transitive digraph and $A, B \in V(D^\star)$ be two strong components of D such that there is a AB-directed path in D. Then, by Lemma 5.4.5, $A \overset{k-1}{\to} B$ and since $d_{D^\star}(A,B) \leq d_D(A,B)$ we have that $d_{D^\star}(A,B) \leq k-1$. It follows from Lemma 5.2.1 that $D^\star$ is k-path-transitive.

Now, let A, B and C be strong components of D such that $(A,B), (B,C) \in A(D^\star_{k-1})$, then by Lemma 5.4.5, there exists an AC-directed path in D, and again by the same lemma, $A \overset{k-1}{\to} C$, thus $(A,C) \in A(D^\star_{k-1})$. ■

Let us remark that not only the $(k-1)$-condensation of D is transitive, also we can think of D it in terms of some kind of "composition" over its $(k-1)$-condensation in the next way. In virtue of Lemmas 5.4.5 and 5.4.6, $(u,v) \in A(D^\star_{k-1})$ if and only if $u \overset{k-1}{\to} v$ in D. And clearly for $k = 2$, D is just a quasi-transitive digraph in the usual sense and Lemmas 5.4.5 and 5.4.6 are those obtained by Bang-Jensen stating that if $A \neq B$ are strong components of D such that there is a AB-arc then $A \to B$, and that any non-strong quasi-transitive digraph is a composition of strong quasi-transitive digraphs over a non-strong transitive digraph (its condensation).

The next few lemmas are oriented to prove that every k-quasi-transitive digraph has a $(k+2)$-kernel with even k. Also a sufficient condition will be stated for the same result to hold with odd k.

Lemma 5.4.7. *Let $k \geq 2$ be an integer and D be a k-quasi-transitive digraph. For every integer $n \geq 2$ there does not exist a directed cycle $\mathscr{C}$ of length n in D such that, with at most one exception, for every arc $(x,y) \in A(\mathscr{C})$ holds that $d(y,x) \geq k+1$.*

Proof. Let us proceed by induction on n and by contradiction in both the base case and the inductive step. If $n \leq k+1$, let $\mathscr{C}$ be a directed cycle with length n and the property stated in the hypothesis of the lemma, then we can choose an arc $(x,y) \in A(\mathscr{C})$ such that $d(y,x) \geq k+1$, but the directed path $y\mathscr{C}x$ has length $\ell(y\mathscr{C}x) = n-1 < k+1$ which results in a contradiction, so the result holds for every $n \leq k+1$.

For the inductive step let $n \geq k+2$ be an integer and $\mathscr{C} = (x_0, x_1, \ldots, x_n, x_0)$ a directed cycle of length n with the desired property. If there is an arc $(x,y) \in A(\mathscr{C})$ such that $d(y,x) \leq k$ we can assume without loss of generality that it is the arc (x_1, x_2), if there is no such arc the argumentation is the same. Since our only exception is the arc (x_1, x_2), then $d(x_1, x_0) \geq k+1$, but D is k-quasi-transitive and $(x_0, x_1, \ldots, x_k)$ is a directed path of length k; if $(x_k, x_0) \in A(D)$ we would have a contradiction because $(x_1, x_2, \ldots, x_k, x_0)$ would be a x_1x_0-directed path of length k, so $(x_0, x_k) \in A(D)$ and therefore $\mathscr{C}' = (x_0, x_k) \cup x_k\mathscr{C}x_0$ is a directed cycle in which every arc $(x,y) \in A(\mathscr{C}')$, with the possible exception of (x_0, x_k), fulfills that $d(y,x) \geq k+1$. But $\ell(\mathscr{C}') < \ell(\mathscr{C}) = n$ and, by induction hypothesis, there are no directed cycles with this property of length less than n, so a contradiction arises from the assumption of the existence of $\mathscr{C}$. We conclude that no such cycle of length n exists. ■

Lemma 5.4.8. *Let $k \geq 2$ be an integer and D be a k-quasi-transitive digraph, then there exists a vertex $v \in V(D)$ such that whenever $(v,u) \in A(D)$, then $d(u,v) \leq k$.*

Proof. We will proceed by contradiction. Let us assume that for every vertex $v \in V(D)$ there exists an arc $(v,u) \in V(D)$ such that $d(u,v) \geq k+1$. Then, since the subdigraph H of D induced by these arcs has $\delta^+(H) \geq 1$, then there exist a directed cycle $\mathscr{C}$ in D such that for every arc $(v,u) \in A(\mathscr{C})$, $d(u,v) \geq k+1$, which clearly results in a contradiction by Lemma 5.4.7. ■

Lemma 5.4.9. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive digraph, then D has a $(k+2)$-semikernel consisting in a single vertex.*

Proof. By Lemma 5.4.8 we can choose a vertex $v \in V(D)$ such that for every arc $(v,u) \in A(D)$, $d(u,v) \leq k$. So let $u \in V(D)$ be a vertex such that $2 \leq d(v,u) \leq k+1$. It can not happen that $d(u,v) \geq k+2$, because this would imply by Lemma 5.4.3 that $d(v,u) = 1$, but $2 \leq d(v,u)$, so $d(u,v) \leq k+1$ and thus $\{v\}$ is a $(k+2)$-semikernel of D. ■

A problem arose while working with the odd case since we could not find a good analog for Lemma 5.4.9 because, although almost the same proof can be done, we can not assure that once we have chosen a vertex v such that for every arc (v,u) it follows that $d(u,v) \leq k+1$, if we choose a vertex u such that $d(v,u) = 2$ then it will be the case that $d(u,v) \leq k+1$ like in the even case.

So a weaker analog of Lemma 5.4.9 will be proposed and proved.

Lemma 5.4.10. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive digraph such that at least one vertex $v \in S = \{u \in V(D) \big| (u,w) \in A(D)$ implies that $d(w,u) \leq k+1\}$ is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$ then $\{v\}$ is a $(k+2)$-semikernel for D.*

Proof. By Lemma 5.4.8 the set S is non empty and also there is a vertex $v \in S$ such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$. So let $u \in V(D)$ be a vertex such that $3 \leq d(v,u) \leq k+1$. It can not happen that $d(u,v) \geq k+2$, because this would imply by Lemma 5.4.3 that $d(v,u) \leq 2$, but $3 \leq d(v,u)$, so $d(u,v) \leq k+1$ and thus $\{v\}$ is a $(k+2)$-semikernel of D. ■

At this point we have two possible courses of action. The one we will not follow is to prove directly that, for even k, whenever a k-quasi-transitive digraph has a $(k+2)$-semikernel then it has a $(k+2)$-kernel; this can be achieved by considering a $\subseteq$-maximal $(k+2)$-semikernel and proving by means of contradiction that it is $(k+1)$-absorbent. But, even though it is a more efficient way to prove this fact, this would not give any information about the structure of the $(k+2)$-kernel. So, we will use a couple of lemmas (including Lemma 5.4.6) that will help us to know how a $(k+2)$-kernel look like, we will begin proving the strong case.

Lemma 5.4.11. *Let D be a k-quasi-transitive strong digraph. If D has a non-empty $(k+2)$-semikernel S, then S is a $(k+2)$-kernel of D.*

Proof. Let $S \subseteq V(D)$ be a $(k+2)$-semikernel for D and $N^-_{k+1}(S)$ the set of all vertices in D which are $(k+1)$-absorbed by S. Define $T := V(D) \setminus \big(S \cup N^-_{k+1}(S)\big)$. If $T = \varnothing$, then S is a $(k+2)$-kernel of D. If $T \neq \varnothing$, then we can consider a vertex $v \in T$ which, by the definition of T, is not $(k+1)$-absorbed by S, but since D is strong, there exists a vS-directed path. Let $u \in S$ be a vertex such that $d(v,u) = d(v,S)$, then $d(v,u) \geq k+2$ because $v \notin N^-_{k+1}(S)$, but from Lemmas 5.4.3 and 5.4.4 it can be derived that $d(u,v) \leq 2$. This fact, altogether with the second $(k+2)$-semikernel condition implies that $v \in N^-_{k+1}(S)$ which results in a contradiction. Since the contradiction arises from assuming that $T \neq \varnothing$, we can conclude that $T = \varnothing$ and then S is a $(k+2)$-kernel for D. ■

5.5 k-quasi-transitive digraphs: Main Results

Theorem 5.5.1. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive strong digraph, then D has an n-kernel for every $n \geq k+2$.*

Proof. By Lemma 5.4.9, D has a $(k+2)$-semikernel N consisting in a single vertex, but by Lemma 5.4.11, N is indeed a $(k+2)$-kernel of D. But since N has only one vertex, then N is n-independent for every $n \geq k+2$, and since it is $(k+1)$-absorbent, then it is $(n-1)$-absorbent for every $n \geq k+2$, so N is an n-kernel for every $n \geq k+2$. ■

Theorem 5.5.2. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive digraph, then D has an n-kernel for every $n \geq k+2$.*

Proof. In virtue of Lemmas 5.4.6 and 5.5.1, it suffices to choose a subset $N \subseteq V(D)$ consisting in an n-kernel for every terminal component of D, this set will be n-independent for every $n \in \mathbb{Z}^+$ because every such n-kernel consist in a single vertex and terminal components are path-independent. Also N will be $(k+1)$-absorbent because every n-kernel is inside its component and every vertex of D not in a terminal component is $(k-1)$-absorbed by every vertex in some terminal component. ■

Let us recall that the out(in)-radius of a digraph is defined as $\min\{d(x,V) \big| x \in V\}$ $(\min\{d(V,x) \big| x \in V\})$, and that a digraph D has a finite out(in)-radius if and only if it has a unique initial (terminal) strong component.

Corollary 5.5.3. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive digraph.*

1. *D has finite in-radius if and only if D has a $(k+1)$-serf.*
2. *D has finite out-radius if and only if D has a $(k+1)$-king.*

Proof. Let us prove the first assertion and the second one will follow immediately from Remark 5.4.1.

If D has finite out-radius, then D has a unique terminal strong component, so it suffices to pick a $(k+2)$-kernel there $\{v\}$. The result is clear from recalling that $\{v\}$ is a $(k+1)$-absorbing set, so v is the $(k+1)$-serf. ■

At this point we want to remark that for every $k \geq 2$ we can find a k-quasi-transitive digraph that does not have a k-kernel, that is to say, the directed cycle of length $k+1$, C_{k+1}, which also is an example of a k-quasi-transitive digraph with a k-king rather than a $(k+1)$-king. Nevertheless we have been unable to find a k-quasi-transitive digraph that does not have a $(k+1)$-kernel, so the question remain open, and since every quasi-transitive digraph has a 3-kernel we are inclined to state the next conjecture.

Conjecture 5.5.4. *If $k \geq 2$ is an even integer and D is a k-quasi-transitive strong digraph, then D has a $(k+1)$-kernel.*

It suffices to consider the strong case, in virtue of Lemma 5.4.6 this would imply that every k-quasi-transitive strong digraph has a $(k+1)$-kernel.

Next, we deal with the odd case again.

Theorem 5.5.5. *Let $k \geq 3$ be an odd integer and let D be a k-quasi-transitive strong digraph such that at least one vertex $v \in S = \{u \in V(D) \big| (u,w) \in A(D)$*
implies that $d(w,u) \leq k+1\}$ *is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$, then D has an n-kernel for every $n \geq k+2$.*

Proof. It is analog to Theorem 5.5.1. ∎

Theorem 5.5.6. *Let $k \geq 3$ be an odd integer and let D be a k-quasi-transitive digraph such that at least one vertex $v \in S = \{u \in V(D) \big| (u,w) \in A(D)$* implies
that $d(w,u) \leq k+1\}$ *is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$, then D has an n-kernel for every $n \geq k+2$.*

Proof. It is analog to Theorem 5.5.2. ∎

Despite this fact, we were actually able to work out an odd case. As we have mentioned before, Galeana-Sánchez and Goldfeder successfully characterized the 3-quasi-transitive strong digraphs, their theorem goes as follows.

Theorem 5.5.7 (Galeana-Sánchez, Goldfeder)**.** *If D is a 3-quasi-transitive strong digraph, then D is either a semicomplete digraph or a bipartite semicomplete digraph or a digraph of the family depicted below.*

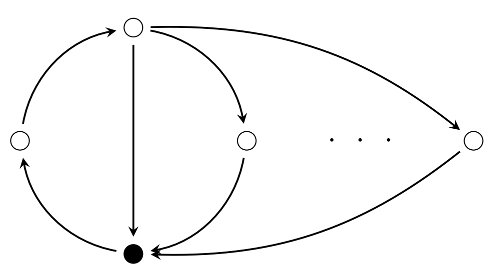


Figure 5.2: The third family of 3-quasi-transitive digraphs.

The dots in Figure 5.2 indicate that any number of intermediate vertices can be added respecting the direction of the arcs.

Also, in Chapter 4, we proved Theorem 4.3.2, stating that every m-partite semicomplete digraph has a k-kernel for every $m \geq 2$, $k \geq 4$. Recall that every semicomplete digraph has a k-kernel for every $k \geq 2$. In view of this results we can deduce the following.

Theorem 5.5.8. *If D is a 3-quasi-transitive strong digraph, then D has a k-kernel for every $k \geq 4$.*

Proof. In virtue of Theorem 5.5.7, D is either a semicomplete digraph, or a bipartite semicomplete digraph, or a digraph of the third family depicted in the theorem. If D is semicomplete, then D has a k-kernel for every $k \geq 2$. If D is a bipartite semicomplete digraph, then by Theorem 4.3.2 it has a k-kernel for every $k \geq 4$. If D is a digraph of the third family, then it suffices to pick the filled vertex in Figure 5.2; that vertex is clearly a k-kernel for every $k \geq 3$. ■

Theorem 5.5.9. *If D is a 3-quasi-transitive digraph, then D has a k-kernel for every $k \geq 4$.*

Proof. Let $k \geq 4$ be an integer. Let $\{S_i\}_{i=i}^{k}$ be the set of terminal strong components of D and $N_i \subseteq S_i$ a k-kernel for S_i, $1 \leq i \leq k$, which exists by Lemma 5.5.8. It is clear from Lemma 5.4.6 that $N = \bigcup_{i=1}^{k} N_i$ is a k-kernel for D. The set N is clearly k-independent since each N_i is, and they are contained in terminal components. Also, every vertex not in a terminal component is 2-absorbed by every vertex in some terminal component. ■

We can get again a corollary about kings and serfs.

Corollary 5.5.10. *Let D be a 3-quasi-transitive digraph and let $n \geq 2$ be an integer.*

- *D has an n-king if and only if D has finite in-radius and the terminal strong component of D has an n-king.*
- *D has an n-serf if and only if D has finite out-radius and the initial strong component of D has an n-serf.*

Proof. The proof is analog to the proof of Corollary 5.5.3. ■

We would like to point out that it follows from Theorem 5.5.7 and Corollary 5.5.10 that a 3-quasi-transitive digraph with finite out-radius (in-radius) which initial (terminal) strong component is not a bipartite semicomplete digraph always have a 2-king (2-serf). Sufficient conditions for the existence of n-kings in the case when the digraph does not have a 2-king (2-serf) can be obtained from the extensive bibliography (*e.g.* [64, 65, 71, 81, 82, 88]) about kings in multipartite semicomplete digraphs.

Recalling that the directed cycle of length 4 has no 3-kernel, the result of Theorem 5.5.9 is as good as it gets, resembling the case when $k = 2$. So, considering that from the case $k = 2$ we conjectured that for even k, every k-quasi-transitive digraph has a $(k+1)$-kernel, we have two conjectures on the matter for the odd case.

Conjecture 5.5.11. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive strong digraph, then D has a $(k+2)$-kernel.*

Conjecture 5.5.12. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive strong digraph, then D has a $(k+1)$-kernel.*

The former would match the results obtained for the even case in this work, while the latter would match the results obtained for the case $k = 3$ for every odd integer.

Before the last section of this chapter, where the particular case of 3-transitive digraphs is studied, we want to propose a seemingly interesting problem relating k-transitive and k-quasi-transitive digraphs.

Problem 5.5.13. Is it true that a graph G can receive a k-transitive orientation if and only if G can receive a k-quasi-transitive orientation?

5.6 3-transitive digraphs

In this section we wiil characterize strong 3-transitive digraphs and give a thorough description of the structure of non-strong 3-transitive digraphs. We will use these results to characterize 3-transitive digraphs with a kernel. This characterization and Theorem 5.3.2 completes the study of k-kernels for 3-transitive digraphs for every integer $k \geq 2$.

We begin this section with another very simple remark about duality.

Remark 5.6.1. A digraph D is a 3-transitive digraph if and only if $\overleftarrow{D}$ is 3-transitive.

The following is another simple, yet useful, property of k-transitive digraphs.

Proposition 5.6.2. *If D is a k-transitive digraph, then D is $k + n(k-1)$-transitive for every $n \in \mathbb{N}$.*

Proof. Let D be a k-transitive digraph. We will proceed by induction on n.

For $n = 1$, consider $(v_0, v_1, \ldots, v_{k+(k-1)})$, a directed path of length $k + (k-1)$. From the k-transitivity of D we have that $(v_0, v_k) \in A(D)$, so $(v_0, v_k, v_{k+1}, \ldots, v_{k+(k-1)})$ is a v_0v_k-directed path of length k, and by the k-transitivity of D, we have that $(v_0, v_{k+(k-1)}) \in A(D)$.

Let us assume the result valid for $n-1$ and let $(v_0, v_1, \ldots, v_{k+n(k-1)})$ be a directed path of length $k + n(k-1)$ in D. By the induction hypothesis $(v_0, v_{k+(n-1)(k-1)}) \in A(D)$, and clearly $(v_0, v_{k+(n-1)(k-1)}, \ldots, v_{k+n(k-1)})$ is a directed path of length k in D. It follows from the k-transitivity that $(v_0, v_{k+n(k-1)}) \in A(D)$.

The result is now obtained by the Principle of Mathematical Induction. ■

As a particular case of Proposition 5.6.2, we can observe that a 3-transitive digraph is n-transitive for every odd integer n. We can state this observation as the following corollary.

Corollary 5.6.3. *Let D be a 3-transitive digraph and $(v_0, v_1, \ldots, v_n)$ a directed path in D. Then $(v_0, v_i) \in A(D)$ for every odd integer $1 \le i \le n$.*

Proof. It is straightforward from Proposition 5.6.2. ■

In [96], Wang and Wang proved some results describing the structure of non-strong 3-quasi transitive digraphs. Since every 3-transitive digraph is also 3-quasi-transitive, the properties stated next hold also for 3-transitive digraphs.

Proposition 5.6.4 ([96])**.** *Let D' be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph D and let $s \in V(D) \setminus V(D')$ with at least one arc from D' to s and $D' \Rightarrow s$. Then each of the following holds:*

1. *If D is a bipartite digraph with bipartition (X, Y) and there exists a vertex of X which dominates s, then $X \mapsto s$.*

2. *If D' is a non-bipartite digraph, then $D' \mapsto s$.*

In the case of 3-transitive digraphs, the condition $D' \Rightarrow s$ in Proposition 5.6.4 not necessary. The following proposition is some kind of analogous of Proposition 5.6.4 for 3-transitive digraphs, emphasizing the behavior of certain strong subdigraphs.

Proposition 5.6.5. *Let D be a 3-transitive digraph and $v \in V(D)$. The following statements hold:*

1. *For every C_3 in D such that there is a C_3v-arc in D, then $C_3 \to v$.*
2. *For every C_3 in D such that there is a vC_3-arc in D, then $v \to C_3$.*
3. *For every $\overleftrightarrow{K_n}$ in D, $n \geq 3$, such that there is a $\overleftrightarrow{K_n}v$-arc in D, then $\overleftrightarrow{K_n} \to v$.*
4. *For every $\overleftrightarrow{K_n}$ in D, $n \geq 3$, such that there is a $v\overleftrightarrow{K_n}$-arc in D, then $v \to \overleftrightarrow{K_n}$.*
5. *For every $\overleftrightarrow{K_{n,m}} = (X, Y)$ in D such that there is a Xv-arc in D, then $X \to v$.*
6. *For every $\overleftrightarrow{K_{n,m}} = (X, Y)$ in D such that there is a vX-arc in D, then $v \to X$.*

Proof. For *1*. Let $C_3 = (x, y, z, x)$ be a cycle in D and $(x, v) \in A(D)$. The existence of the directed path (y, z, x, v) in D, implies that $(y, v) \in A(D)$. Finally, since (z, x, y, v) is a directed path of length 3 in D, $(z, v) \in A(D)$. Thus $C_3 \to v$.

For *2*. It suffices to dualize *1* using Remark 5.6.1.

For *3*. Let $D[S]$, with $S = \{1, 2, \ldots, n\}$, be a complete subdigraph of D and $(1, v) \in A(D)$. Let $i \in S \setminus 1$ be an arbitrary vertex. Remember that $n \geq 3$, so there exists a vertex $j \in S \setminus \{1, i\}$. Now, since $D[S] = \overleftrightarrow{K_n}$, we have the existence of the directed path $(i, j, 1, v)$, which implies that $(i, v) \in A(D)$. But i is an arbitrary vertex of $D[S]$, and then we can conclude that $D[S] \to v$.

For *4*. It suffices to dualize *3* using Remark 5.6.1.

For *5*. Let $\overleftrightarrow{K_{n,m}} = (X, Y)$ be a complete subdigraph of D and $x \in X$. If $|X| = 1$, then we are done. If not, let $z \in X$ be a vertex such that $z \neq x$. Since $Y \neq \varnothing$, there is a vertex $y \in Y$. Also, $(z, y), (y, x) \in A(D)$, because $D[X \cup Y]$ is a complete bipartite digraph. So (z, y, x, v) is a directed path of length 3 in D and hence, $(z, v) \in A(D)$. Thus, $X \to v$.

For *6*. It suffices to dualize *5* using Remark 5.6.1. ■

The following proposition is also due to Wang and Wang.

Proposition 5.6.6 ([96])**.** *Let D' be a non-trivial strong subdigraph of a* 3*-transitive digraph D. For any $s \in V(D) \setminus V(D')$, if there exists a directed path between s and D', then s and D' are adjacent.*

In the case of 3-transitive digraphs we can be a little more specific. The proof of the following proposition will be omitted since it is almost the same as the one given by Wang and Wang in [96].

Proposition 5.6.7. *Let D' be a non-trivial strong subdigraph of a* 3*-transitive digraph D and $s \in V(D) \setminus V(D')$. Then each of the following holds:*

1. *If there exists an sD'-directed path in D, then an sD'-arc exists.*
2. *If there exists a $D's$-directed path in D, then a $D's$-arc exists.*

The following couple of propositions will be used later to characterize strong 3-transitive digraphs.

Proposition 5.6.8. *Let D be a strong* 3*-transitive digraph of order $n \geq 4$. If D is semicomplete, then D is complete.*

Proof. For any $(x, y) \in A(D)$, let $P = (y_0, y_1, \ldots, y_s)$ be a shortest path from y to x. If $s \geq 3$, then by Corollary 5.6.3 we can find a shorter path than P from y to x. Suppose that $s = 2$, then (x, y, y_1, x) is a C_3 in D. Let $D' = D[\{x, y, y_1\}]$. Since the order of D is $n \geq 4$, there exists $v \in V(D) \setminus V(D')$. Also, D is strong, so a $D's$-directed path and an sD'-directed path exist in D. It follows from Propositions 5.6.5 (*1* and *2*) and 5.6.7 that $(y_1, v), (v, x) \in A(D)$. So (y, y_1, v, x) is a directed path of length 3 in D and hence, $(y, x) \in A(D)$. This contradicts that $s = 2$. Thus, $(y, x) \in A(D)$. ■

Proposition 5.6.9. *Let D be a strong* 3*-transitive digraph. If D is semicomplete bipartite, then D is complete bipartite.*

Proof. Let (X, Y) be the bipartition of D. It suffices to prove that for any $(v, u) \in A(D)$, $(u, v) \in A(D)$. Since D is strong, there exists a path P from u to v of length n. Again, since D is bipartite and u and v belong to the different partite, n must be odd. By Corollary 5.6.3, $(u, v) \in A(D)$. ■

We are ready now for the characterization theorem. We just need to define two digraphs. Let C_3^* and C_3^{**} be directed triangles with one and two symmetrical arcs, respectively. Digraphs C_3, C_3^* and C_3^{**} are shown in Figure 5.3

The characterization of strong 3-transitive digraphs is now proved.

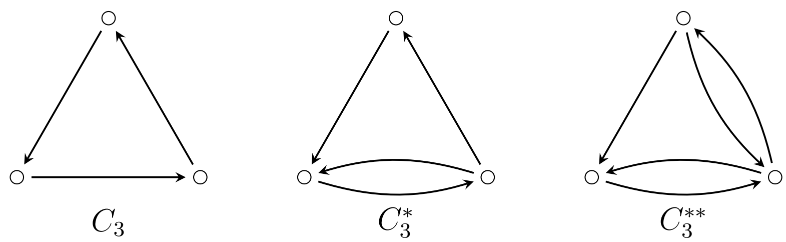


Figure 5.3: The digraphs C_3, C_3^* and C_3^{**}.

Proposition 5.6.10. *A strong digraph D is 3-transitive if and only if it is one of the following:*

1. *A complete digraph*
2. *A complete bipartite digraph*
3. *C_3, C_3^* or C_3^{**}.*

Proof. Since every 3-transitive digraph is 3-quasi-transitive, in virtue of Theorem 5.5.7, a strong 3-transitive digraph must be either semicomplete, semicomplete bipartite or isomorphic to F_n. But F_n is not 3-transitive, so a strong 3-transitive digraph must be either semicomplete or semicomplete bipartite. It is clear that every strong digraph of order less than or equal to 3 is either complete, complete bipartite or one of the digraphs C_3, C_3^* or C_3^{**}. If D has order greater than or equal to 4, and it is a semicomplete digraph, it follows from Proposition 5.6.8 that D is complete. Finally, if D is semicomplete bipartite, it follows from Proposition 5.6.9 that D is complete bipartite. ■

Corollary 5.6.11. *Let D be a 3-transitive digraph. Then D is Hamiltonian if and only D is strong and it is not bipartite or it is regular.*

Let us recall that Proposition 5.6.5 describes the interaction of a single vertex with some subdigraphs of a 3-transitive digraph D. This covers the case when a strong component of D consists of a single vertex. In [96], the following proposition is proved.

Proposition 5.6.12. *Let S_1 and S_2 be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one D_1D_2-arc. Then either*

$D_1 \mapsto D_2$ or the digraph induced by $D_1 \cup D_2$ is a semicomplete bipartite digraph.

As it was noted before, every 3-transitive digraph is a 3-quasi-transitive digraph, so Proposition 5.6.12 is also valid for 3-transitive digraphs. In an attempt to be more explicit with the interaction between non-trivial strong components of a 3-transitive digraph, we state the following proposition. Nonetheless, we omit the proof, since it is very similar to the proof of Proposition 5.6.12.

Proposition 5.6.13. *Let D be a 3-transitive digraph and S_1, S_2 be distinct strong components of D such that there exists an S_1S_2-arc. The following statements hold:*

1. *If S_1 contains a subdigraph isomorphic to C_3, then $S_1 \to S_2$.*
2. *If S_2 contains a subdigraph isomorphic to C_3, then $S_1 \to S_2$.*
3. *If S_i is a complete bipartite digraph with bipartition (X_i, Y_i) for $i \in \{1,2\}$ and if the S_1S_2-arc is an X_1X_2-arc, then $X_1 \to X_2$.*
4. *If S_i is a complete bipartite digraph with bipartition (X_i, Y_i) for $i \in \{1,2\}$ and there exist an X_1X_2-arc and a Y_1X_2-arc, then $S_1 \to S_2$.*
5. *If S_i is a complete bipartite digraph with bipartition (X_i, Y_i) for $i \in \{1,2\}$ and there exist an X_1X_2-arc and a X_1Y_2-arc, then $S_1 \to S_2$.*

As a direct consequence of Propositions 5.6.7 and 5.6.13, we have the following corollary.

Corollary 5.6.14. *Let D be a 3-transitive digraph D and S_1 a strong component of D which contains a subdigraph isomorphic to C_3. If $S_1 \to v$ for some vertex $v \in V$, then $S_1 \to u$ for every vertex $u \in V$ that can be reached from v. Dually, if $v \to S_1$ for some vertex $v \in V$, then $u \to S_1$ for every vertex $u \in V$ that reaches v.*

We have already proved that the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. The following results are devoted to a deeper exploration of the similarities between these families of digraphs. A structural characterization of 3-transitive digraphs that are transitive is given.

Theorem 5.6.15. *Let D be a non-strong 3-transitive digraph with strong components $S_1, S_2, \ldots, S_p$. Then $D = D^\star[S_1, S_2, \ldots, S_p]$ if and only if, for every pair of strong components S_i, S_j of D, such that an S_iS_j-arc exists in D, then:*

1. *If S_i, S_j are complete bipartite digraphs, then $D[S_i \cup S_j]$ is not bipartite.*

2. *If S_i is a complete bipartite digraph and S_2 consists of a single vertex v, then $D[S_1 \cup \{v\}]$ is not bipartite.*

3. *If S_i consists of a single vertex v and S_2 is a complete bipartite digraph, then $D[\{v\} \cup S_2]$ is not bipartite.*

Proof. Clearly, if $D = D^\star[S_1, S_2, \ldots, S_p]$, then for every pair of strong components S_i, S_j of D such that there is an S_iS_j-arc in D, then $S_i \to S_j$. Thus, the three conditions are fulfilled.

Now, for the 'if' implication, let S_i, S_j be distinct strong components of D such that there is an S_iS_j-arc in D. In virtue of Proposition 5.6.10, S_i and S_j are either complete, complete bipartite, C_3, C_3^* or C_3^{**}. If S_i or S_j is neither complete bipartite nor consists of a single vertex, then by Propositions 5.6.5 and 5.6.13 we can conclude that $S_i \to S_j$.

So, let us assume that S_i and S_j are complete bipartite digraphs with bipartitions (X_i, Y_i) and (X_j, Y_j), respectively. Clearly, every arc between S_i and S_j is an S_iS_j-arc, because they are distinct strong components of D. If every S_iS_j-arc were an X_iX_j-arc or a Y_iY_j-arc, then $D[S_i \cup S_j]$ would be a bipartite digraph with bipartition $(X_i \cup Y_j, Y_i \cup X_j)$, contradiction our hypothesis. Analogously, if every S_iS_j-arc were a X_iY_j-arc or a Y_iX_j-arc, then $D[S_i \cup S_j]$ would be a bipartite digraph with bipartition $(X_i \cup X_j, Y_i \cup Y_j)$, contradicting our hypothesis. Then an X_iX_j-arc and a Y_iX_j-arc exist or an X_iY_j-arc and a Y_iY_j-arc exist. It follows from *4* and *5*, respectively, of Proposition 5.6.13, that $S_i \to S_j$.

If S_i is a complete bipartite digraph with bipartition (X, Y) and S_j consists of a single vertex v, then it is not possible that every S_iS_j-arc is an Xv-arc, because $D[S_i \cup \{v\}]$ would be a bipartite digraph with bipartition $(X, Y \cup \{v\})$, contradicting hypothesis *2*. Analogously, if every S_iS_j-arc is a Yv-arc, then $D[S_i \cup \{v\}]$ is a bipartite digraph with bipartition $(X \cup \{v\}, Y)$, which results in a contradiction. So, there must exist a Xv-arc and a Yv-arc in D. By *5* of Proposition 5.6.5, we have that $S_i \to v$.

The case in which S_i consists of a single vertex v and S_j is a complete bipartite digraph can be obtained analogously to the previous case, using *6* of Proposition 5.6.5. ■

Theorem 5.6.16. *Let D be a 3-transitive digraph. Then $D^\star$ is a transitive digraph if and only if for every triplet of strong components S_1, S_2, S_3 of D, such that: S_i consists of a single vertex v_i, $i \in \{1, 3\}$; S_2 is either a single vertex v_2 or a complete bipartite digraph with bipartition (X, Y) and $v_1 \to v_2 \to v_3$ or $v_1 \to X \to v_3$ but there are neither v_1Y-arcs nor Yv_3-arcs in D, respectively, then $(v_1, v_3) \in A(D)$.*

Proof. Let D be a 3-transitive digraph. If $D^\star$ is a transitive digraph, then for every triplet of strong components S_1, S_2 and S_3 of D, such that there is an S_1S_2-arc in D and an S_2S_3-arc in D, then there is an S_1S_3-arc in D. In particular, if S_1 and S_3 consist of single vertices v_1 and v_3 respectively, then $(v_1, v_3) \in A(D)$.

For the converse, let D be a 3-transitive digraph and S_1, S_2 and S_3 strong components of D, such that there is an S_1S_2-arc in D and an S_2S_3-arc in D. We will prove that there is an S_1S_3-arc in D. If S_1 contains an isomorphic copy of C_3, then, by Corollary 5.6.14, we have that $S_1 \to S_3$ in$D$. If S_3 contains an isomorphic copy of C_3, again, by Corollary 5.6.14, we have that $S_1 \to S_3$. So, let us assume that neither S_1 nor S_3 contains an isomorphic copy of C_3.

It follows from Proposition 5.6.10 that S_1 and S_3 are either a single vertex or complete bipartite digraphs. If S_1 is not a single vertex, then it is a complete bipartite digraph with bipartition (X_1, Y_1). Let us assume without loss of generality that the S_1S_2-arc is an X_1S_2-arc. Let (x_1, u) be the S_1S_2-arc in D. Since S_2 is a strong component of D, we have, by Propositions 5.6.10 and 5.6.13, two cases. The first case is that a vertex $s_3 \in V(S_3)$ exists, such that $(u, s_3) \in A(D)$. In this case is clear that, for any vertex $y_1 \in Y_1$ (recall that $Y_1 \neq \varnothing$), (y_1, x_1, u, s_3) is a directed path of length 3 in D. By the 3-transitivity of D, we have that $(y_1, s_3) \in A(D)$, the desired S_1S_3-arc. The second case is that vertices $v \in V(S_2)$ and $s_3 \in V(S_3)$ exist, such that $(u, v), (v, s_3) \in A(D)$. Again, it is clear that (x_1, u, v, s_3) is a directed path of length 3 and thus, $(x_1, s_3) \in A(D)$, the desired S_1S_3-arc. The case when S_3 is a complete bipartite digraph can be obtained dualizing the previous argument using Remark 5.6.1.

So, the remaining cases are when S_1 and S_3 consist of single vertices. We have again two cases. First, when S_2 contains a subdigraph isomorphic to

C_3, then $S_2 \to S_3$. So, there exist vertices $s_1 \in V(S_1), u, v \in V(S_2), s_3 \in V(S_3)$ such that $(s_1, u), (u, v), (v, s_3) \in A(D)$. Thus, (s_1, u, v, s_3) is a directed path of length 3 in D. By the 3-transitivity of D, $(s_1, s_3) \in A(D)$ is the desiredS_1S_3-arc. If S_2 does not contain a subdigraph isomorphic to C_3, then S_2 is a single vertex or complete bipartite. If S_2 is a single vertex v_2 or a complete bipartite digraph with bipartition (X, Y) such that $v_1 \to v_2 \to v_3$ or $v_1 \to X \to v_3$ but there are neither v_1Y-arcs nor Yv_3-arcs in D, respectively, then, by hypothesis $(v_1, v_3) \in A(D)$. Hence, we have the existence of an S_1S_3-arc. The remaining case is that S_2 is a complete bipartite digraph with bipartition (X, Y) such that $v_1 \to X \to v_3$, and either a v_1Y-arc or a Yv_3-arc exists. In the first case we have by Proposition 5.6.13 that $v_1 \to S_2$, and thus, vertices $u \in X$, $v \in Y$ exist such that $(v_1, v), (u, v_3) \in A(D)$. So, (v_1, v, u, v_3) is a directed path of length 3 in D. For the second case, again by Proposition 5.6.13, it follows that $S_2 \to v_3$. Then, vertices $u \in X$ and $v \in Y$ exist such that $(v_1, u), (v, v_3) \in A(D)$. Therefore, (v_1, u, v, v_3) is a directed path of length 3 in D. In either case, it follows by the 3-transitivity of D that $(v_1, v_3) \in A(D)$. So an S_1S_3-arc exists.

Since the cases are exhaustive, we have that $D^\star$ is transitive. ■

Corollary 5.6.17. *Let D be a 3-transitive digraph. Then D is a transitive digraph if and only if every strong component of D is a complete digraph and, for every triplet of strong components S_1, S_2, S_3 of D, such that: S_i consists of a single vertex v_i, $i \in \{1, 3\}$; S_2 is either a single vertex v_2 or a symmetrical arc $(v_2, v_2') \in A(D)$ and $v_1 \to v_2 \to v_3$ but $(v_1, v_2'), (v_2', v_3) \notin A(D)$, then $(v_1, v_3) \in A(D)$.*

Proof. It is clear from Proposition 3.3.4 and Theorems 5.6.15 and 5.6.16. ■

Corollary 5.6.18. *Let D be a 3-transitive digraph. If every strong component of D is a complete digraph of order greater than or equal to 3, then D is transitive.*

Proof. Let D be a 3-transitive digraph such that every strong component of D is a complete digraph of order greater than or equal to 3. Then, by Theorem 5.6.16, it is clear that $D^\star$ is transitive. Also, in virtue of Theorem 5.6.13, we can observe that $S_i \to S_j$ for every pair of strong components S_i, S_j of D such that there exists an S_iS_j-arc in D. Thus, $D = D^\star[S_1, S_2, \ldots, S_n]$, where $\{S_1, S_2, \ldots, S_n\}$ is the set of strong components of D and $D^\star$ is transitive. So, by Theorem 3.3.4, D is transitive. ■

As we have already shown, the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. We know that the condensation of a transitive digraph is again transitive. A characterization of 3-transitive digraphs with a transitive condensation has been already given, but a natural question arises. Is the condensation of a 3-transitive digraph 3-transitive again? Sadly, the answer is no, Figure 5.4 shows a counterexample to this fact.

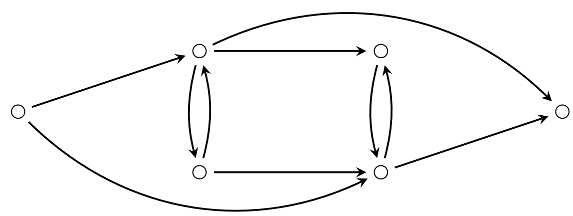


Figure 5.4: A 3-transitive digraph without 3-transitive condensation.

Following similar ideas to those used to characterize the 3-transitive digraphs with a transitive condensation in Theorem 5.6.16, we can characterize 3-transitive digraphs with a 3-transitive condensation. The 'bad' configurations, preventing the condensation of a 3-transitive digraph to be 3-transitive, are pointed out in the following theorem.

Theorem 5.6.19. *Let D be a 3-transitive digraph. Then $D^\star$ is a 3-transitive digraph if and only if for every 4-set, $\{S_1, S_2, S_3, S_4\}$, of strong components of D such that: S_i consists of a single vertex v_i, $i \in \{1, 4\}$ and one of the following conditions is fulfilled:*

1. *S_2 consists of single vertex v_2 and S_3 is a complete bipartite digraph with bipartition (X, Y), such that $v_1 \to v_2 \to X$ and $Y \to v_4$, but there are neither v_2Y-arcs nor Xv_4-arcs in D;*

2. *S_2 is a complete bipartite digraph with bipartition (X, Y) and S_3 consists of single vertex v_3, such that $v_1 \to X$ and $Y \to v_3 \to v_4$, but there are neither v_1Y-arcs nor Xv_3-arcs in D;*

3. *S_j is a complete bipartite digraph with bipartition (X_j, Y_j), $j \in \{2, 3\}$, such that $v_1 \to X_2 \to X_3$ and $Y_3 \to v_4$, but there are neither v_1Y_2-*

arcs, v_1X_3-arcs, Y_2v_4-arcs, nor X_3v_4-arcs, and $D[V(S_2) \cup V(S_3)]$ is a semicomplete bipartite digraph,

then $(v_1, v_4) \in A(D)$.

Proof. Let D be a 3-transitive digraph. If $D^\star$ is a 3-transitive digraph, then for every 4-set of strong components $\{S_1, S_2, S_3, S_4\}$ of D, such that there is an S_iS_{i+1}-arc in D, $i \in \{1, 2, 3\}$, then there is an S_1S_3-arc in D. In particular, if S_1 and S_4 consist of single vertices v_1 and v_4 respectively, then $(v_1, v_4) \in A(D)$.

Conversely, let $\{S_1, S_2, S_3, S_4\}$ be a 4-set of strong components of D such that there is an S_iS_{i+1}-arc in D, $i \in \{1, 2, 3\}$. If S_1 or S_4 are non-trivial, then by Proposition 5.6.7, there exists an S_1S_4-arc in D. So let us assume without loss of generality that S_i consists of a single vertex S_i, $i \in \{1, 4\}$. Suppose that S_2 or S_3 contains C_3 as a subdigraph. It can be easily derived from Corollary 5.6.14 the existence of an S_1S_4-arc in D. So, we have 3 cases.

Before the analysis of the cases, let us recall that, by Proposition 5.6.5, if $S = (X, Y)$ is a bipartite strong component of D and $v \in V(D) \setminus V(S)$ such that a vX-arc exists, then $v \to X$; and if an Xv-arc exists, then $X \to v$.

The first case is when S_2 consists of single vertex v_2 and S_3 is a complete bipartite digraph with bipartition (X, Y). Clearly, if a v_2X-arc, and an Xv_4-arc exist, then $v_2 \to X \to v_4$. Thus, a v_1v_4-directed path of length 3 exists and $(v_1, v_4) \in A(D)$ by the 3-transitivity of D. Analogously, if a v_2Y-arc and a Yv_4-arc exist in D, clearly $(v_1, v_4) \in A(D)$. So, we can assume without loss of generality that $v_2 \to X, Y \to v_4$ and there are neither v_2Y-arcs nor Xv_4-arcs in D. Then, by hypothesis, $(v_1, v_4) \in A(D)$.

The second case is when S_2 is a complete bipartite digraph with bipartition (X, Y) and S_3 consists of single vertex v_3. But this case is just the dual of the first case, so, using Remark 5.6.1, it can be easily shown that $(v_1, v_4) \in A(D)$.

The third case is when S_j is a complete bipartite digraph with bipartition (X_j, Y_j), $j \in \{2, 3\}$. Let us assume without loss of generality that $v_1 \to X_2$ and $Y_3 \to v_4$. If $X_2 \to Y_3$, then $v_1 \to X_2 \to Y_3 \to v_4$ and clearly $(v_1, v_4) \in A(D)$. If $Y_2 \to X_3$, it is easy to observe that $X_2 \to Y_3$. So, we can suppose that $X_2 \to X_3$ (thus $Y_2 \to Y_3$) and that there are neither X_2Y_3-arcs nor Y_2X_3-arcs. Thus, $D[V(S_2) \cup V(S_3)]$ is semicomplete bipartite. If $v_1 \to Y_2$, then $v_1 \to Y_2 \to Y_3 \to v_4$ and we are done. If $v_1 \to X_3$, then $v_1 \to X_3 \to Y_3 \to v_4$ and $(v_1, v_4) \in A(D)$. Symmetrically, if $Y_2 \to v_4$ or $X_3 \to v_4$ we can conclude that $(v_1, v_4) \in A(D)$. Hence, we can suppose that there are neither v_1Y_2-arcs, v_1X_3-arcs, Y_2v_4-arcs, nor X_3v_4-arcs in D. By hypothesis $(v_1, v_4) \in A(D)$.

Since the cases are exhaustive, we have that $D^\star$ is 3-transitive. ■

Now 3-transitive digraphs have been studied in detail, we are ready to talk about kernels again.

Theorem 5.6.20. *Let D be a 3-transitive digraph. Then D has a kernel if and only if it has no terminal strong component isomorphic to C_3.*

Proof. The 'only if' part will be proved by contrapositive. Let D be a 3-transitive digraph such that a terminal strong component S is isomorphic to C_3. Let $V(S) = \{v_0, v_1, v_2\}$ and $A(S) = \{(v_i, v_{i+1})\}_{i=0}^{2}$ (mod 3). Since S is terminal, we have that $d^+(v) = 1$ for every $v \in V(S)$. Thus, the only out-neighbor of v_i is v_{i+1} (mod 3). It is clear that S has no kernel and vertices in S cannot be absorbed by any other vertex in D, thus, D has no kernel.

The 'if' implication will be proved by induction on the number of strong components of D. Let us assume that D is strong. It can be directly verified that the digraphs mentioned in Proposition 5.6.10, except for C_3 have a kernel. So, let us assume that every 3-transitive digraph such that no terminal strong component isomorphic to C_3 and with n strong components has a kernel. Let D be a 3-transitive digraph such that no terminal strong component isomorphic to C_3 and with $n + 1$ strong components. Let us recall that $D^\star$ is an acyclic digraph, so, we can consider an initial strong component S of D. By induction hypothesis, $D - S$ has a kernel N. If S is not a complete bipartite digraph, then, either S consists of a single vertex or contains a subdigraph isomorphic to C_3. If S consists of a single vertex v, and v is absorbed by N, we are done. If v is not absorbed by N, since S is initial, $N \cup \{v\}$ is independent and thus a kernel of D. If D contains a subdigraph isomorphic to C_3, we can use Corollary 5.6.14 to prove that $S \mapsto S_t$ for some terminal strong component S_t of D. But since S_t is terminal, at least one vertex of S_t must belong to N, and thus S is absorbed by N. So, N is a kernel of D. If S is a complete bipartite digraph, we must consider three cases. Let (X, Y) be the bipartition of S. If neither X nor Y is absorbed by N, then we consider $N \cup X$. Since S is an initial component, every arc between X and N must be an XN-arc. But if such arc exists, we would have by Proposition 5.6.5.*5* that $X \to n \in N$, contradicting our assumption. So $N \cup X$ is an independent set, and $Y \to X$ because S is a complete bipartite digraph. Thus, $N \cup X$ is a kernel for D. If some vertex of X is absorbed by N, then by Proposition

5.6.5.*5* X is absorbed by N. So let us assume that Y is not absorbed by N. Once again, since S is an initial component, every arc between N and Y must be a YN-arc, but no such arc can exist. So, $N \cup Y$ is an independent absorbent set of D, and hence a kernel of D. The case when Y is absorbed but X is not is analogous. Finally, if S is absorbed by N, we have that N is the desired kernel of D.

Since in every case D has a kernel, the result follows from the Principle of Mathematical Induction. ■

As we mentioned earlier in this section, Theorem 5.6.20 completes the study of k-kernels in 3-transitive digraphs for $k \geq 2$. This very simple characterization makes us wonder if the family of k-transitive digraphs with $k \geq 4$ and n-kernel for $n \leq k$ will have a simple characterization.

Chapter 6

Digraphs with a given circumference

6.1 Introduction

In [83], Richardson proved that every digraph without odd cycles has a 2-kernel, as a particular case of this result we can observe that if a digraph D has circumference 2, then D has a 2-kernel. As a matter of fact, it is easy to prove that if a digraph D has circumference 2, then D has a k-kernel for every $k \geq 2$. A short proof of this fact is given in the following section. From this, we conjecture that if D is a digraph with circumference l, then D has a l-kernel. Also, a stronger version of this conjecture is proposed, if D is a digraph with circumference l, then D has a k-kernel for every $k \geq l$. As we noted before, the strong version of the conjecture is true for $l = 2$. The aim of this chapter is to prove this conjecture true for some families of digraphs, including σ-strong digraphs and locally in/out-semicomplete digraphs.

Maybe this chapter is somewhat technical, and also not as general as the results in the previous chapters, but the conjecture introduced here is by no means simple. In Section 6.2, we prove that if D is a σ-strong digraph with circumference l then $\sigma \leq l-1$ and D has a $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-kernel for every integer $k \geq 2$. In Section 6.3 we prove that if D is a locally out(in)-semicomplete digraph with circumference $l+1$, then D has a (k, l)-kernel(solution) for every integer $k \geq 2$. In Section 6.4, we prove that if D is a k-quasi-transitive digraph with circumference $l \leq k$, then D has a n-kernel for every $n \geq k$.

Before starting to work with specific families of digraphs, we want to make some general observations.

The following lemma is a very simple one, but it will help us to prove some results to propose a conjecture about the relation between the circumference of a digraph and the existence of certain k-kernels. It also inspired the results of the next section.

Lemma 6.1.1. *Let D be a strong digraph with circumference l. If $(u,v) \in A(D)$, then $d(v,u) \leq l-1$.*

Proof. Let $(u,v) \in A(D)$ and $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = u)$ a vu-directed path (which exists because D is strong). It is clear that $\mathscr{C} \cup (u,v)$ is a directed cycle in D and therefore $\ell(\mathscr{C} \cup (u,v)) = n+1 \leq l$, this implies that $n \leq l-1$. ■

The following theorem, as it was mentioned earlier, is the inspiration for the conjecture proposed in this work.

Theorem 6.1.2. *Let D be a digraph with circumference 2, then D has a k-kernel for every $k \geq 2$.*

Proof. It can be easily observed that if D has circumference 2, then every strong component of D is a symmetrical digraph. Otherwise, let $(u,v) \in A(D)$ be an asymmetrical arc in a strong component of D. Since u and v are in the same component, then there is a vu-directed path $\mathscr{C}$ of length at least 2 in D. But $\mathscr{C} \cup (u,v)$ is a directed cycle of length greater than or equal to 3. It follows from this observation that the underlying graph of every strong component of D is a tree.

The existence of a k-kernel can now be proved by induction on $|V(D)|$. If $|V(D)| = 1$, the result is obtained trivially. If not, let v be a leaf in the underlying graph of an initial strong component S of D. So, v has only one (in and out) neighbor in D. By the Induction Hypothesis, $D - v$ has a k-kernel N. If v is $(k-1)$-absorbed by N in D, then N is a kernel for D. Otherwise, we have two cases. If $N \cap V(S) = \varnothing$, then, since S is an initial component, there are not NS-directed paths in D. Thus, $N \cup \{v\}$ is a k-independent (because v is a leaf of the underlying graph of S), $(k-1)$-absorbent set in D, and thus a k-kernel. If $N \cap V(S) \neq \varnothing$, then we affirm that there are not Nv-directed path of length less than or equal to $(k-1)$ in D. This is because, since S is an initial component, the only vertices of

N that can reach v are also in S, but S is a symmetrical digraph. Hence, the existence of a Nv-directed path of length less than or equal to $(k-1)$ in D impies the existence of a vN-directed path of length less than or equal to $(k-1)$ in D, but v is not $(k-1)$-absorbed by N. Once again, v is a leaf in the underlying graph of S, thus $N \cup \{v\}$ is a k-independent, $(k-1)$-absorbent set of D. In both cases the existence of a k-kernel is proved.

The result follows from the principle of mathematical induction. ■

6.2 σ-strongly connected digraphs

Recall that, in Section 1.4, we define σ-strong connectivity. As a first observation, let us notice that if D is a σ-strong digraph with circumference l, then $l \geq \sigma + 1$. To prove this, let $\mathscr{C}$ be a longest cycle in D. If $|V(\mathscr{C})| \leq \sigma$, then fix an arc (x, y) in $\mathscr{C}$ and delete all vertices of $\mathscr{C} - \{x, y\}$ and the arc (x, y). The resulting digraph is strongly connected (since $|V(\mathscr{C})| \leq \sigma$), so there is an xy-path of length at least 2 in D. Thus, a cycle longer than $\mathscr{C}$ can be constructed in D, contradicting the choice of $\mathscr{C}$.

The degree of strong connectivity of a digraph has consequences on the distances between its vertices. The next couple of lemmas show this relation.

Lemma 6.2.1. *Let D be a σ-strong digraph with circumference l, $k \geq 2$ a fixed integer and $\mathscr{C} = (x_0, x_1 \ldots, x_m)$ a directed path of length m. If $m = q\sigma + r$ where q and r are given by the division algorithm, then:*

1. *If $r = 0$, then $d(x_m, x_0) \leq (l - \sigma)q$.*
2. *If $r > 0$, then $d(x_m, x_0) \leq (l - r) + (l - \sigma) \left\lfloor \frac{m-1}{\sigma} \right\rfloor$.*

Proof. For (i) we have that $m = q\sigma$ and we will proceed by induction on q. If $q = 0$, then $\mathscr{C} = (x_0)$ and there is nothing to prove, so let us suppose that $q \geq 1$. Let us consider the set $S = \{x_{(q-1)\sigma+1}, x_{(q-1)\sigma+2}, \ldots, x_{q\sigma-1}\}$, it is clear that $|S| = \sigma - 1$ and hence $D \setminus S$ is strong, so there exists an $x_m x_{(q-1)\sigma}$-directed path in $D \setminus S$, namely $\mathscr{D}$ which is clearly internally disjoint with $\mathscr{E} = (x_{(q-1)\sigma}, x_{(q-1)\sigma+1}, \ldots, x_m)$, so $\mathscr{D} \cup \mathscr{E}$ is a directed cycle in D. But recalling that D has circumference l, we have that $\ell(\mathscr{D} \cup \mathscr{E}) = \ell(\mathscr{D}) + \ell(\mathscr{E}) = \ell(\mathscr{D}) + \sigma \leq l$, and thence $\ell(\mathscr{D}) \leq l - \sigma$. By induction hypothesis there exists an $x_{(q-1)\sigma}x_0$-directed path $\mathscr{D}'$ of length less than or equal to $(l - \sigma)(q - 1)$, so $\mathscr{D} \cup \mathscr{D}'$ is an $x_m x_0$-directed path of length less than or equal to $(l - \sigma)q$. The desired result now follows from the Principle of Mathematical Induction.

For (ii) we have that $m = q\sigma + r$ with $0 < r < k$, so $q = \left\lfloor \frac{m-1}{\sigma} \right\rfloor$. If we consider the set $S = \{x_{q\sigma+1}, x_{q\sigma+2}, \ldots, x_{q\sigma+(r-1)}\}$ with cardinality $|S| = r - 1 \leq \sigma - 1$ we can observe that there is an $x_m x_{q\sigma}$-directed path $\mathscr{D}$ in $D \setminus S$ because D is σ-strong. The directed path $\mathscr{D}$ is internally disjoint with the directed path $\mathscr{E} = (x_{q\sigma}, x_{q\sigma+1}, \ldots, x_m)$, therefore $\mathscr{D} \cup \mathscr{E}$ is a directed cycle in D and thus, $\ell(\mathscr{D} \cup \mathscr{E}) = \ell(\mathscr{D}) + \ell(\mathscr{E}) = \ell(\mathscr{D}) + r \leq l$, so $\ell(\mathscr{D}) \leq l - r$. By (i) we know that $d(x_{q\sigma}, x_0) \leq (l - \sigma)q$ and we have just proved that $d(x_m, x_{q\sigma}) \leq l - r$, by the triangle inequality we can conclude that $d(x_m, x_0) \leq (l - r) + (l - \sigma)q = (l - r) + (l - \sigma) \left\lfloor \frac{m-1}{\sigma} \right\rfloor$. ■

Lemma 6.2.2. *Let D be a σ-strong digraph with circumference l, then for every $v \in V(D)$, $\{v\}$ is a $(k, (l-1) + (l-\sigma) \left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-semikernel for every integer $k \geq 2$.*

Proof. Let us recall that $\sigma \leq l - 1$. Let $k \geq 2$ and $v \in V(D)$ be fixed and let $\mathscr{C} = (v = x_0, x_1, \ldots, x_m)$ be a vx_m-directed path of length $m \leq k - 1$. In virtue of Lemma 6.2.1, $d(x_m, v) \leq (l-1)+(l-\sigma) \left\lfloor \frac{m-1}{\sigma} \right\rfloor \leq (l-1)+(l-\sigma) \left\lfloor \frac{k-2}{\sigma} \right\rfloor$ and then $\{v\}$ fulfills the second $(k, (l-1)+(l-\sigma) \left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-semikernel condition. ■

The principal theorem of the section is now proved. It explores what kind of (k, l)-kernels exists with given values of circumference and strong connectivity.

Theorem 6.2.3. *Let D be a σ-strong digraph with circumference l. Then D has a $(k, (l-1) + (l-\sigma) \left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 3.1.4 and 6.2.2. ■

We would like to point out a special case of Theorem 6.2.3, concerning the conjecture proposed above.

Corollary 6.2.4. *Let D be a $(l-1)$-strong digraph with circumference l, then D has an l-kernel.*

Proof. In virtue of Theorem 6.2.3, D has an $(l, (l-1) + (l-(l-1)) \left\lfloor \frac{l-2}{l-1} \right\rfloor)$-kernel. But for every integer $l \geq 2$, we have that $\left\lfloor \frac{l-2}{l-1} \right\rfloor = 0$. Thus D has an $(l, l-1)$-kernel. ■

6.3 Locally in/out-semicomplete digraphs

These families of digraphs have been largely studied by Bang-Jensen, *et al.* A very surprising theorem, that is also particularly useful in the study of k-kernels, will be stated after the formal definition of these classes of digraphs. A digraph D is **locally in-semicomplete** if whenever $(v, u), (w, u) \in A(D)$, then $(v, w) \in A(D)$ or $(w, v) \in A(D)$. Dually, a digraph D is **locally out-semicomplete** if whenever $(u, v), (u, w) \in A(D)$, then $(v, w) \in A(D)$ or $(w, v) \in A(D)$. Finally, D is **locally semicomplete** if it is both, locally out-semicomplete and locally in-semicomplete.

The theorem due to Bang-Jensen, Huang and Prisner is now stated.

Theorem 6.3.1 ([10])**.** *A locally in-semicomplete digraph D of order $n \geq 2$ is Hamiltonian if and only if D is strong.*

Let us observe that the definition of a locally in-semicomplete digraph is equivalent to the fact that for every $v \in V(D)$, $D[N^-(v)]$ is a semicomplete digraph; analogously for the locally out-semicomplete an locally semicomplete digraphs. Also, we may observe that every directed cycle is a locally in/out-semicomplete digraph. Let us recall that a directed cycle of length ℓ has k-kernel if and only if $\ell \equiv 0 \pmod{k}$, so there is an infinite subfamily of locally in/out-semicomplete strong digraphs that does not have a k-kernel for a fixed k, so it is not surprising that heavy restrictions have to be considered in order to guarantee the existence of k-kernels.

The following remark will be useful to dualize some results from locally out-semicomplete digraphs to locally in-semicomplete digraphs.

Remark 6.3.2. A digraph D is locally in-semicomplete if and only if $\overleftarrow{D}$ is locally out-semicomplete. As a consequence, a digraph D is locally semicomplete if and only if $\overleftarrow{D}$ is locally semicomplete.

The previous remark extends Theorem 6.3.1 to locally out-semicomplete digraphs, so a locally out-semicomplete digraph D of order $n \geq 2$ is Hamiltonian if and only of D is strong. Since we are studying classes of digraphs in which we can find a k-kernel, where k depends on the circumference of the digraph, any condition that we find on this family of digraphs can be easily verified thanks to Theorem 6.3.1.

Our next lemma is just a technical one that will be used to prove the one after it.

Lemma 6.3.3. *Let $l \geq 1$ be an integer, D a locally out-semicomplete digraph and $(x_0, x_1, \ldots, x_n)$ is a x_0x_n-directed path of length $n \leq l$. If $(x_0, v_0) \in A(D)$ and $(x_n, v_0) \notin A(D)$, then $d(v_0, x_n) \leq l$.*

Proof. If $v_0 = x_i$ for some $1 \leq i \leq n-1$, then $(v_0, x_{i+1}, \ldots, x_n)$ is a v_0x_n-directed path of length less than or equal to $n \leq l$, so let us take for granted that $v_0 \neq x_i$ for all $0 \leq i \leq n$. For each $0 \leq i < n$, if $(x_i, v_0) \in A(D)$ then $(x_{i+1}, v_0) \in A(D)$ or $(v_0, x_{i+1}) \in A(D)$, because $(x_i, x_{i+1}) \in A(D)$ and D is locally out-semicomplete. So, since $(x_0, v_0) \in A(D)$, let us consider the greatest $0 \leq i \leq n$ such that $(x_i, v_0) \in A(D)$. Clearly $i \neq n$, because $(x_n, v_0) \notin A(D)$, and by the choice of i, $(x_{i+1}, v_0) \notin A(D)$, thus, $(v_0, x_{i+1}) \in A(D)$ and $(v_0, x_{i+1}, \ldots, x_n)$ is a v_0x_n-directed path of length less than or equal to $n \leq l$. ■

The hypothesis of the following lemma may look a bit odd, but we observed that, to use Lemma 3.1.4, only the vertices reached at distance one failed to fulfill the second (k, l)-semikernel condition. It is easy to observe that for a connected locally out-semicomplete digraph D such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$ then $d(u, v) \leq l$, the digraph D results strongly connected, and thus Hamiltonian by Theorem 6.3.1, so it will always have a one vertex $|V(D)|$-kernel, but a better result can be proved with this hypothesis.

Lemma 6.3.4. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$ then $d(v, u) \leq l$. Then $\{v\}$ is a (k, l)-semikernel for every integer $k \geq 2$ and every $v \in V(D)$.*

Proof. Let $(v = v_0, v_1, \ldots, v_m)$ be a vv_m-directed path of length m. We will prove by induction on m that $d(v_m, v) \leq l$. If $m = 1$, then by hypothesis $d(v_m, v) \leq l$. So let us consider the result valid for $m - 1$ and let $(v = v_0, v_1, \ldots, v_m)$ be a vv_m-directed path of length m. By induction hypothesis there exists a $v_{m-1}v$-directed path of length less than or equal to l, besides $(v_{m-1}, v_m) \in A(D)$ and since $d(v, v_m) \geq 2$, $(v, v_m) \notin A(D)$, it follows from Lemma 6.3.3 that $d(v_m, v) \leq l$. ■

Theorem 6.3.5. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 3.1.4 and 6.3.4. ■

In the following corollary let us point out a special case.

Corollary 6.3.6. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a k-kernel for every integer $k \geq l + 1$.*

The following proposition can be derived from Lemma 6.1.1 and Theorem 6.3.5, but it is also a trivial consequence of Theorem 6.3.1.

Proposition 6.3.7. *Let D be a locally out-semicompete strong digraph with circumference $l + 1$, then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. Since every locally out-semicomplete strong digraph is Hamiltonian, then $l + 1 = |V(D)|$. Trivially, for every vertex $v \in V(D)$, $\{v\}$ is a k-independent, l-absorbent set. ■

Now we will obtain analogous results for locally in-semicomplete digraphs by means of dualization.

Lemma 6.3.8. *Let D be a locally in-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then $\{v\}$ is a (k, l)-semikernel for every integer $k \geq 2$ and every $v \in V(D)$.*

Proof. By Remark 6.3.2, $\overleftarrow{D}$ is a locally out-semicomplete digraph, and is straightforward to verify that the hypothesis of Lemma 6.3.4 hold and hence for every vertex $u \in V(\overleftarrow{D}) = V(D)$, $\{u\}$ is a (k, l)-semikernel for $\overleftarrow{D}$ for every $k \geq 2$. Now, let $v \in V(D)$ be an arbitrary vertex and $\mathscr{C}$ be a vw-directed path in D, then $\mathscr{C}^{-1}$ is a wv-directed path in $\overleftarrow{D}$ and since $\{w\}$ is a (k, l)-semikernel of $\overleftarrow{D}$ for every $k \geq 2$, then there exists a vw-directed path of length less than or equal to l in $\overleftarrow{D}$, namely $\mathscr{D}$. Thus $\mathscr{D}^{-1}$ is a wv-directed path of length less than or equal to l in D. Thence, $\{v\}$ is a (k, l)-semikernel of D for every $k \geq 2$. ■

Theorem 6.3.9. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 3.1.4 and 6.3.8. ■

Corollaries analogous to those of the locally out-semicomplete case can be also obtained.

Corollary 6.3.10. *Let D be a locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a k-kernel for every integer $k \geq l + 1$.*

Proposition 6.3.11. *Let D be a locally out-semicompete strong digraph with circumference $l + 1$, then D has a (k, l)-kernel for every integer $k \geq 2$.*

It is our desire to find families of digraphs such that circumference k implies the existence of a k-kernel, Propositions 6.3.7 and 6.3.11 tell us that locally out/in-semicomplete strong digraphs have this property. But these results follow easily from Theorem 6.3.1, so we will prove that the result can be improved at least in the locally out-semicomplete case for non-strong digraphs, however, the dualization method in this case is not enough to prove the corresponding locally in-semicomplete case, but a dual result about (k, l)-solutions can be obtained.

The next theorems are due to Bang Jensen and Gutin [5].

Theorem 6.3.12. *Let D be a locally out-semicomplete digraph and S, T distinct strong components of D. If a vertex $b \in T$ absorbs some vertex in S, then $S \mapsto b$.*

Theorem 6.3.13. *Let D be a locally in-semicomplete digraph and S, T distinct strong components of D. If a vertex $a \in S$ dominates some vertex in T, then $a \mapsto T$.*

And can be easily generalized as follows.

Lemma 6.3.14. *Let D be a locally out-semicomplete digraph and S, T distinct strong components of D. If some vertex of S is l-absorbed by a vertex $b \in T$, then $S \overset{l}{\mapsto} b$.*

Proof. By induction on l. Case $l = 1$ is Theorem 6.3.12, so let us consider S and T distinct strong components of D such that a vertex $b \in T$ l-absorbs some vertex in $a \in S$, then there must exist an ab-directed path $\mathscr{C} = (a = v_0, v_1, \ldots, v_n = b)$ of length $n \leq l$. If $n < l$ by induction hypothesis we are done, so let us consider that $n = l$. Let v_j be the first vertex of $\mathscr{C}$ not in A, then for every $i < j$, $v_i \in S$ and v_j absorbs v_{j-1}, and by Theorem 6.3.12, $A \mapsto v_j$; thus, for every $v \in S$, $(v, v_j, \ldots, v_n = b)$ is a vb-directed path of length less than or equal to l, therefore $S \overset{l}{\mapsto} b$. ∎

Lemma 6.3.15. *Let D be a locally in-semicomplete digraph and S, T distinct strong components of D. If some vertex in T is l-dominated by a vertex $a \in S$, then $a \overset{l}{\mapsto} T$.*

Proof. The proof is analogous to the one of the previous Lemma. ■

Lemma 6.3.16. *Let D be a locally out-semicomplete digraph, $(y_0, y_1, \ldots, y_s)$ a y_0y_s-directed path in D and $x \in V(D)$ a vertex such that $(y_0, x) \in V(D)$ but $(x, y_j) \notin A(D)$ for every $1 \leq j \leq s$, then $(y_j, x) \in A(D)$ for every $0 \leq j \leq s$.*

Proof. By induction on j. For $j = 0$, $(y_0, x) \in A(D)$ by hypothesis. If $(y_j, x) \in A(D)$, since $(y_j, y_{j+1}) \in A(D)$ also, by the locally out-semicomplete hypothesis $(x, y_{j+1}) \in A(D)$ or $(y_{j+1}, x) \in A(D)$, but by hypothesis $(x, y_{j+1}) \notin A(D)$, so $(y_{j+1}, x) \in A(D)$. ■

Theorem 6.3.17. *Let D be a locally out-semicomplete digraph with circumference $l + 1$, then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. By induction on $|V(D)|$. If $|V(D)| = 1$ the result is obvious, so let us consider the result valid for every digraph with $|V(D)| < n$ and let D be a digraph with $|V(D)| = n$. By Proposition 6.3.7 we may choose a (k, l)-kernel for every terminal strong component of D; let N_1 be the union of all such (k, l)-kernels. If M is the set of all vertices l-absorbed by N_1, and for some strong component S of D, a vertex $u \in S$ is also in M, then in virtue of Lemma 6.3.14 $S \subseteq M$, so every strong component of D is either contained in M or in $V(D) \setminus M$. By induction hypothesis there exists a (k, l)-kernel for $D \setminus M$, namely N_2. If N_2 is k-independent in D then clearly $N_1 \cup N_2$ is a k-independent set in D.

Let us assume that N_2 is not k-independent in D to reach a contradiction. We know that N_2 is k-independent in $D \setminus M$, so if $u, v \in N_2$ are such that $d_D(u, v) \leq k - 1$, then every uv-directed path in D must have at least one vertex in M. Let $\mathscr{C} = (u = x_0, x_1, \ldots, x_r = v)$ be a uv-directed path in D and let w be the last vertex of $\mathscr{C}$ that is in M; it is clear that $u \neq w \neq v$, so $w = x_i$ for some $1 \leq i \leq n - 1$ and $x_{i+1} \in V(D) \setminus M$. But $w \in M$ implies that w is l-absorbed by N_1, thus there exists a wN_1-directed path $\mathscr{D} = (w = y_0, y_1, \ldots, y_s)$ of length $s \leq l$. Let us observe that $(w, y_1), (w, x_{i+1}) \in A(D)$, and that $(x_{i+1}, y_j) \notin A(D)$ for every $1 \leq j \leq s$, for this would imply that $(x_{i+1}, y_j, y_{j+1}, \ldots, y_s)$ is a directed path of length less than or equal to l and hence x_{i+1} would be in M which can not occur; therefore by Lemma 6.3.16 we have that $(y_s, x_{i+1}) \in A(D)$, which results in a contradiction because $y_s \in N_1$ is a vertex in a terminal component of D. Thence, N_2 is k-independent in D.

As we observed earlier, $N_1 \cup N_2 = N$ is a k-independent set of D, also N_1 is l-absorbent in M and N_2 is l-absorbent in $D \setminus M$, so N is l-absorbent in D, and then is the desired (k,l)-kernel. ■

Corollary 6.3.18. *Let D be a locally out-semicompete digraph with circumference l, then D has a k-kernel for every integer $k \geq l$.*

Although we can not prove analogous results to those of Theorem 6.3.17 and Corollary 6.3.18 for locally in-semicomplete digraphs, we can dualize these results by means of Remark 6.3.2 to (k,l)-solutions and k-solutions.

Theorem 6.3.19. *Let D be a locally in-semicomplete digraph with circumference $l+1$, then D has a (k,l)-solution for every integer $k \geq 2$.*

Proof. Let D be a locally in-semicomplete digraph with circumference $l+1$. In virtue of Remark 6.3.2 $\overleftarrow{D}$ is a locally out-semicomplete digraphs with circumference $l+1$, and by Theorem 6.3.18, $\overleftarrow{D}$ has a (k,l)-kernel, namely N. Clearly N is k-independent and l-absorbent in D and thus a (k,l)-solution. ■

Corollary 6.3.20. *Let D be a locally in-semicompete digraph with circumference l, then D has a k-solution for every integer $k \geq l$.*

Let us recall that the problem of determining if a digraph has a k-kernel or not is NP-complete, also the problem of finding a longest cycle in a digraph is NP-complete, so Theorems 6.3.17 and 6.3.19 become more valuable if stated in the next way.

Corollary 6.3.21. *Let D be a locally out-semicompete digraph with set of strong components $\mathcal{C}$ and $l+1 = \max_{H\in\mathcal{C}} \left|V(H)\right|$, then D has a (k,l)-kernel for every integer $k \geq 2$.*

Proof. In virtue of Theorem 6.3.1 and the fact that every directed cycle is contained in a single strong component of D, the circumference of D is equal to the greatest of the orders of the strong components of D, so this is merely a different statement for Theorem 6.3.17. ■

Corollary 6.3.22. *Let D be a locally in-semicompete digraph with set of strong components $\mathcal{C}$ and $l+1 = \max_{H\in\mathcal{C}} \left|V(H)\right|$, then D has a (k,l)-solution for every integer $k \geq 2$.*

In view of the theorems that we have proved, these results are not as best as possible, because if D is a locally out-semicomplete digraph we may have as an hypothesis that whenever $(u,v) \in A(H)$ then $d_H(v,u) \leq l$ for every strong component H of D and we would get a strengthening of Theorems 6.3.17 and 6.3.19, but in the form of the two previous corollaries we may decide the existence of a (k,l)-kernel in polynomial time, this is because the only thing we have to do is to find the condensation and calculate the order of the strong components of D, which can be done in polynomial time.

To finish this section, let us observe that some of this results can be extended to k-kings and k-serfs. Considering Lemma 6.3.4, we may conclude that, for $k \geq l$, any (k,l)-kernel in a locally out-semicomplete digraph such that, whenever $(u,v) \in A(D)$ we have that $d(u,v) \leq l$, consists in a single vertex. Assume the contrary, and for a fixed integer $k \geq 2$, let N be a (k,l)-kernel with more than one vertex in a locally out-semicomplete digraph that fulfills the aforementioned hypothesis, and let $u,v \in N$. It also follows from Lemma 6.3.4 that $\{u\}$ is a $(d(u,v)+1,l)$-semikernel, and then $d(v,u) \leq l \leq k$, contradicting the k-independence of k. Thus, the next result is obtained.

Theorem 6.3.23. *Let D be a locally in/out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u,v) \in A(D)$, then $d(v,u) \leq l$, then D has a $(l+1)$-serf and a $(l+1)$-king.*

6.4 k-quasi-transitive digraphs

Let us recall from Chapter 5 that a digraph D is k-quasi-transitive if the existence of a directed path $(v_0, v_1, \ldots, v_k)$ of length k in D implies that $(v_0, v_k) \in A(D)$ or $(v_k, v_0) \in A(D)$. In this section we will prove a very simple proposition concerning k-quasi-transitive digraphs with an additional circumference restriction.

Proposition 6.4.1. *Let D be a k-quasi-transitive digraph. If D does not contain directed cycles of length $k+1$, then D is k-transitive.*

Proof. Let $(v_0, v_1, \ldots, v_k)$ be a directed path in D. If $(v_k, v_0) \in A(D)$, then $(v_0, v_1, \ldots, v_k, v_0)$ is a directed path of lenght $k+1$ in D, which cannot occur. Thus, $(v_0, v_k) \in A(D)$ and D is k-transitive. ■

Corollary 6.4.2. *Let D be a k-quasi-transitive digraph. If D does not contain directed cycles of length $k+1$, then D has an n-kernel for every $n \geq k$.*

As a particular case of this result, we have that if D is a k-quasi-transitive digraph with circumference $l \leq k$, then D has an n-kernel for every $n \geq k$.

Chapter 7

Weighted digraphs

As it was mentioned in Chapter 1, we think that (k,l)-kernels have potential to be used in real life models, *e.g.*, if a digraph represents the map of a city, a (k,l)-kernel is an optimum distribution of a service or good someone may offer to the population, according to the parameters k and l this distribution could be done choosing an appropiate distance between the service centers (k-independence) to avoid saturation in one zone and also an appropiate distance so all the population in the city have an easy access to the service (l-absorbence). In addition to this point of view, we want to add further information, not only the distance matters, but transportation use to have aditional costs, it may be time, or some toll, this information may be added by means of a weight function for the arcs of the digraph, so every arc would represent a distance unit and its weight the cost to cross it. So, our next aim is to generalize the concept of k-kernel adding weights to the arcs of the digraph and study the possible generalization for well known results on k-kernels.

In Chapter 2, we explored variants of Theorem 1.8.2, relaxing the strong connectedness to unilaterality but restricting the length of certain cycles besides the directed ones to have a specific congruence modulo k. The idea came from the observation that the congruence to 0 (mod k) together with the strong connectedness is a very strong combination, yielding a cyclically k-partite structure. The work developed in the current chapter arises from the attempt to obtain further resemblings of Theorem 1.8.2. Our first try, reflected in Section 7.1, did not worked so well in the direction of Theorem 1.8.2, but we won some insight of the weighted case. In the development of Section 7.2 we emphasized the $\equiv 0 \pmod k$ condition and realized that given a normal subgroup H of a group G it makes sense to think in the congruence modulo H, so an analog of Theorem 1.8.2 can be positively obtained.

7.1 Weighted digraphs

If D is a digraph and $\mathsf{w} : A(D) \to \mathbb{Z}$ is a weight function for the arcs of D, we can restate the problem of finding a k-kernel in the next way. If $\mathscr{C}$ is a walk in D, the weight of $\mathscr{C}$ is defined as $\mathsf{w}(\mathscr{C}) := \sum \mathsf{w}(f)_{f \in A(\mathscr{C})}$. A subset $S \subseteq V(D)$ is (k, w)-independent if, for every $u, v \in S$ it does not exist an uv-directed path of weight less than k. A subset $S \subseteq V(D)$ will be (l, w)-absorbent if, for every $u \in V(D) - S$ exists an uS-directed path of weight less or equal than l. A subset $N \subseteq V(D)$ is a (k, l, w)-kernel if it is (k, w)-independent and (l, w)-absorbent.

Let us observe that if we want to find a (k, w)-kernel, an arc with weight greater or equal than k will not contribute in any walk between vertices for independence nor absorbence; to avoid the case when there is an arc between two vertices and they remain (k, w)-independent or when a vertex can not $(k-1, \mathsf{w})$-absorb some of its in-neighbours, we will consider only weights between 1 and $k-1$ for the arcs. When w is the constant function equal to 1, a (k, l, w)-kernel is a (k, l)-kernel in Kwasnik's sense, and as usual, a $(k, k-1, \mathsf{w})$-kernel will be simply called (k, w)-kernel. Despite the fact that this definition generalizes efectively the notion of (k, l)-kernel, and thus the notion of k-kernel, many results does not remain true when w is not identically the consant 1.

For convenience we will say that the weighted distance from vertex u to vertex v respect to the weight function w is the minimum weight of all the uv-directed paths, no matter the length. We will denote this as $d^{\mathsf{w}}(u, v)$, as $d(u, v)$ will denote the usual distance.

Proposition 7.1.1. *Theorem 1.8.2 is false for (k, w)-kernels.*

Proof. In the digraph at the right of Figure 7.1 there is only one cycle of length $6 \equiv 0 \pmod 3$, nonetheless has no 3-kernel. For every two vertices in this digraph one of the two weighted distances between them is two, then every maximal 3-independent set consist of exactly one vertex. But for every two vertices one of the two weighted distances between them is four, so for each maximal 3-independent set there exists a vertex that can not be 2-absorbed. ∎

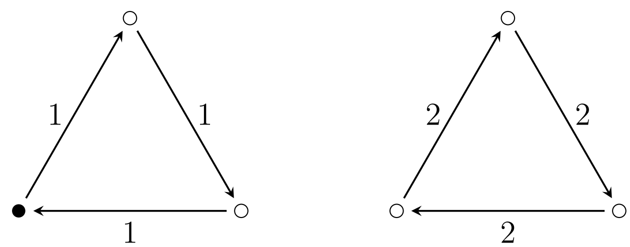


Figure 7.1: Figure of the left is C_3 which have a 3-kernel (filled vertex) in the usual sense. On the right, a counterexample to the weighted version of Theorem 1.8.2.

The following theorem, a generalization of a result due to Berge [11], about symmetrical digraphs is another example of a result that does not remain valid if a weight funcion is considered.

Theorem 7.1.2. *If D is a symmetrical digraph and $k \geq 2$ then every maximal k-independent susbset of vertices is a k-kernel.*

Proposition 7.1.3. *Theorem 7.1.2 is false for (k, w)-kernels.*

Proof. The digraph on Figure 7.2 is a counterexample.

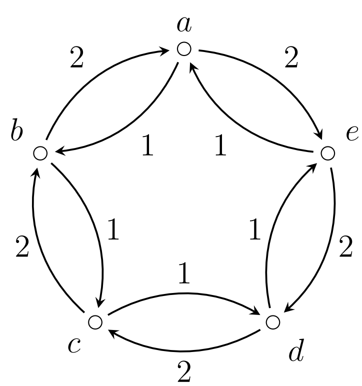


Figure 7.2: Counterexample to the weighted version of Theorem 7.1.2.

The set $\{a\}$ is a maximal 3-independent subset of vertices since vertices b and c are at weighted distance one and two, respectively, from a and vertices

e and d are at weighted distance one and two, respectively, to a. However, vertex a is at weighted distance three from vertex c, thus c can not be $(2, \mathsf{w})$-absorbed by a.

■

Note that the digraph in Figure 7.2 does not have a 3-kernel, so we go beyond the statement of Theorem 7.1.2, it is not only that maximal k-independent sets in symmetrical digraphs are not k-kernels, there are symmetrical digraphs without a k-kernel. However, there is a simple case when a symmetrical digraph has a kernel for every $k \in \mathbb{N}$. Also, and just as a curious observation, the weight assignment for the arcs of the digraph in Figure 7.2 conform a nowhere-zero 3-flow for the given digraph.

Theorem 7.1.4. *If D is a symmetrical digraph and $\mathsf{w} : A(D) \to \mathbb{Z}$ is a constant weight function for the arcs of D, then every maximal (k, w)-independent subset of $V(D)$ is a (k, w)-kernel for every $k \geq 2$.*

Proof. First, let us observe that for $k = 2$, since every arc have weight $\leq k - 1$, the result is the classical result for kernels in symmetrical non weighted digraphs. The proof is analogous to the proof of the original result. Let N be a maximal (k, w)-independent subset of $V(D)$. If N is $(k-1, \mathsf{w})$-absorbent, then N is the desired (k, w)-kernel. So, let us assume that there exists a vertex $v \in V(D)$ such that it is not $(k-1, \mathsf{w})$-absorbed by N. Then $d^{\mathsf{w}}(v, N) \geq k$, but D is symmetrical and w is a constant function, so $d^{\mathsf{w}}(N, v) \geq k$ and $N \cup \{v\}$ is (k, w)-independent, contradicting the choice of N as a maximal (k, w)-independent set. ■

Figure 7.1 shows that result of Theorem 7.1.4 is not as obvious as it may seem, there are other theorems that become invalid even in the constant weights case. Also, the hypothesis of Theorem 7.1.4 are not tight, there are symmetrical digraphs with non constant weight functions and k-kernel as the example in Figure 7.3 shows; sets $\{b\}$ and $\{c\}$ are both 3-kernels for the digraph.

Theorem 7.1.5. *If D is an acyclic digraph and $\mathsf{w} : A(D) \to \mathbb{Z}$ is a weight function for the arcs of D, then D has a unique (k, w)-kernel for each $k \geq 2$.*

Proof. Let us proceed by induction on $|V(D)|$ with fixed $k \geq 2$. If $|V(D)| = 1$, the only vertex of D is the desired (k, w)-kernel. Assuming the result valid

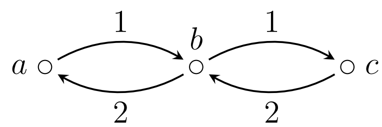


Figure 7.3: Symmetrical digraph with 3-kernel and non constant weight function.

for every acyclic digraph D such that $|V(D)| < n$, let D be an acyclic digraph with $|V(D)| = n$. Since D is an acyclic digraph, there exists $v \in V(D)$ such that $d^-(v) = 0$. Now, $D - v$ is an acyclic digraph on $n-1$ vertices and by induction hypothesis has a unique (k, w)-kernel N'. There are two cases:

Case 1. If v is $(k-1, \mathsf{w})$-absorbed by N' in D, then N' is the $(k-1, \mathsf{w})$-kernel we have been looking for.

Case 2. If v is not $(k-1, \mathsf{w})$-absorbed by N' in D, then there are not vN'-directed paths of weight less or equal than $k-1$ and, as v has indegree 0 there are not $N'v$-directed paths in D, in particular there are not $N'v$-directed paths of weight less or equal than $k-1$ and hence $N = N' \cup \{v\}$ is (k, w)-independent and $(k-1, \mathsf{w})$-absorbent in D. We have found in N the desired (k, w)-kernel.

Finally, observe that in either case N' is unique by inductive hypothesis. If M is a (k, w)-kernel for D, $M \setminus \{v\}$ is (k, w)-independent in $D - v$ and, as $d_D^-(v) = 0$, v can not $(k-1, \mathsf{w})$ absorb any other vertex, therefore $M \setminus \{v\}$ is $(k-1, \mathsf{w})$-absorbent in $D - v$ and a (k, w)-kernel of $D - v$. It follows than $M \setminus \{v\} = N \setminus \{v\}$ and hence $M = N$, the unique (k, w)-kernel of D. ■

If we extend the notion of diammeter to match with our new weighted distance, we can derive some results, altough rather simple, we have seen that other results become invalid in the weighted versions.

Lemma 7.1.6. *If D is a digraph, $\mathsf{w} : A(D) \to \mathbb{Z}$ a weight function for $A(D)$ and under this function D has weighted diammeter less or equal than $k-1$, then every vertex of D is a (h, w)-kernel for each $k \leq h$.*

Proof. Let $v \in V(D)$, $h \geq k$ and $\mathsf{w} : A(D) \to \mathbb{Z}$ a weight function for the arcs of D. Clearly, as $\{v\}$ has only one vertex, is a (h, w)-independent set. Since D has diammeter less or equal than $k-1$ then, for every $u \in V(D)$ if $\mathscr{C}$ is a uv-directed path, $\sum \mathsf{w}(f)_{f \in A(\mathscr{C})} \leq k-1 \leq h-1$, thus $\{v\}$ is a $(h-1, \mathsf{w})$-absorbent set and consequently a (h, w)-kernel. ■

We will introduce a definition that has proved to be very useful in kernel theory, the weighted case is no exception and we will use it in many proofs in the rest of this work.

Definition 7.1.7. If D is a digraph, the *condensation digraph of* D or simply the condensation of D is the digraph $D^\star$ which vertices are the strong components of D, $\{D_1, D_2, \ldots, D_p\}$ and $(D_i, D_j) \in A(D^\star)$ if and only if there exist a D_iD_j-arc in D.

It is direct to observe that $D^\star$ has no directed cycles, and so, for every digraph D, the condensation $D^\star$ has at least one vertex of indegree 0 and one vertex of exdegree 0, these will be called initial and terminal components of D (or vertices of $D^\star$) respectively.

Theorem 7.1.8. *If D is a digraph, $\mathsf{w} : A(D) \to \mathbb{Z}$ is a weight function for the arcs of D and every strong component of D has diameter less or equal than $k - 1$ under the weight function, then D has a (k, w)-kernel.*

Proof. By induction on the number of strong components of D. If D has a unique strong component, then D is strongly conected and has diammeter less or equal than $k - 1$. In virtue of Lemma 7.1.6 any vertex of D is a (k, w)-kernel for D.

Let D be a digraph with φ strong components of diammeter less or equal than $k - 1$ and suppose for inductive hypothesis the theorem valid for every digraph with less than φ strong components. Since the condensation digraph $D^\star$ of D has not directed cycles we can consider an initial strong component of D, say C_0. The digraph $D - C_0$ has $\varphi - 1$ strong components and all of its components have diammeter less or equal than $k - 1$. As a consequence of the inductive hypothesis $D - C_0$ has (k, w)-kernel N'. It is easy to observe that N' is (k, w)-independent not only in $D - C_0$ but in all D because C_0 is an initial component and there are no $(D - C_0)C_0$-paths in D and therefore there are no new $N'N'$-paths in D, so if N' is $(k - 1, \mathsf{w})$-absorbent in D, then N' is a (k, w)-kernel of D, if not, there exists a vertex $v \in V(C_0)$ not $(k - 1, \mathsf{w})$-absorbed by N' and hence, there are not vN'-directed paths with weight less or equal than $k - 1$ in D. Being C_0 an initial component there exist neither $N'v$-directed paths in D, so $N = N' \cup \{v\}$ is a (k, w)-independent set in D. Is easy to observe that N is $(k - 1, \mathsf{w})$-absorbent set for D, this is because N' $(k - 1, \mathsf{w})$-absorbs all vertices in $D - C_0$ and, as a result of Lemma 7.1.6, $\{v\}$ $(k - 1, \mathsf{w})$-absorbs all vertices in C_0. Consequently, N is a (k, w)-kernel for D and the desired result follows from the induction principle. ■

The following Lemma and its consequences take advantage of the existing results for (k,l)-kernels, in some cases those results can be adapted with hypothesis which are not very restrictive to fit in the weighted case.

Lemma 7.1.9. *Let D is a digraph and $\mathsf{w}: A(D) \to \mathbb{Z}$ a weight function for the arcs of D. If N is a (k,l)-kernel for D, and for every directed path $\mathscr{C}$, of length less or equal than l, the condition $\mathsf{w}(\mathscr{C}) \leq k-1$ holds, then N is a (k,w)-kernel for D.*

Proof. Since N is k-independent, the weighted distance between vertices of N is greater or equal than k, and thus N is (k,w)-independent. Also, N is l-absorbent, so for a vertex $v \in V(D) \setminus N$ there exists a vN-directed path $\mathscr{C}$ of length $\leq l$, but for hypothesis, $\mathsf{w}(\mathscr{C}) \leq k-1$, therefore v is $(k-1,\mathsf{w})$-absorbed by N. ■

We have two direct applications of Lemma 7.1.9 for cases $l=1$ and $l=2$.

Theorem 7.1.10. *If D is a transitive digraph and $\mathsf{w}: A(D) \to \mathbb{Z}$ is a weight function for the arcs of D, then D has a (k,w)-kernel.*

Proof. We just have to observe the structure of a k-kernel in a transitive digraph. Since every transitive digraph is the composition of an acyclic transitive digraph $D^\star$ (the condensation of D) and complete digraphs $D_1, D_2, \ldots, D_p$, a k-kernel in a transitive digraph can be constructed choosing one vertex in every terminal strong component of D. In virtue of Proposition 3.3.4, such k-kernel will be not only k-independent but independent by directed paths, and not only $(k-1)$-absorbent, but 1-absorbent. So if N is a k-kernel for a transitive digraph D, then N is a $(k,1)$-kernel in D and the result follows from Lemma 7.1.9. ■

Let us recall that as a direct consequence of Theorem 1.8.1, we can obtain that every tournament has a 2-absorbent vertex. Also, as we already mentioned in Chapter 4, an immediate consequence of this observation is that every tournament has a $(k,2)$-kernel for every $k \geq 2$, the set containing the 2-absorbent vertex will work.

Theorem 7.1.11. *If T is a tournament and $\mathsf{w}: A(T) \to \mathbb{Z}$ is a weight function for the arcs of T such that every directed path of length 2 has weight less or equal than $k-1$, then T has a vertex v such that $\{v\}$ is a (k,w)-kernel.*

Proof. It follows directly from the previous observation and Lemma 7.1.9. ■

Corollary 7.1.12. *If T is a tournament and $\mathsf{w} : A(T) \to \mathbb{Z}$ is a weight function for the arcs of T such that $\mathsf{w}(a) \leq \frac{k-1}{2}$ for all $a \in A(T)$, then T has a (k, w)-kernel.*

Proof. Let (x, y, z) be a directed path, then $\mathsf{w}(x, y, z) = \mathsf{w}(x, y) + \mathsf{w}(y, z) \leq \frac{k-1}{2} + \frac{k-1}{2} = k - 1$. The result now follows from theorem 7.1.11. ■

To prove our next result, we will need an structural result for quasi-transitive digraphs. The following proposition, due to Bang-Jensen and Huang, can be consulted in [8].

Proposition 7.1.13. *Let D be a quasi-transitive digraph. Suppose that $P = (x_1, x_2, \ldots, x_k)$ is a minimal x_1x_k-path. Then the subdigraph induced by $V(P)$ is a semicomplete digraph and $(x_j, x_i) \in A(D)$ for every $2 \leq i + 1 < j \leq k$, unless $k = 4$, in which case the arc between x_1 and x_k may be absent.*

In the proof of the following theorem the technique is, again, to choose a k-kernel for each of the terminal strong components of the given digraph.

Theorem 7.1.14. *If $k \geq 3$, D is a quasi-transitive digraph and $\mathsf{w} : A(D) \to \mathbb{Z}$ is a weight function for the arcs of D such that every directed cycle has weight less or equal than $\left\lfloor \frac{k-1}{2} \right\rfloor + 1$, then D has a k-kernel.*

Proof. We know that every quasi-transitive digraph has a k-kernel for every $k \geq 3$. Let N be a k-kernel for D, we will prove that N is the (k, w)-kernel we are looking for. It is clear that N is (k, w)-independent in D because is k-independent and the weighted distance is greater or equal than the distance. For the $(k-1, \mathsf{w})$-absorbence let $v \in V(D) \setminus N$ be a vertex not in N. If v is not in a terminal strong component of D, then, by Theorem 3.3.1 and the observation to Theorem 3.3.6, v is absorbed at distance 1 by N, so it is $(k-1, \mathsf{w})$-absorbed by N. If v is in a terminal strong component S of D, then there is at least one vertex $u \in V(S)$ such that $u \in N$ and u $(k-1)$-absorbs v, so we have three posibilities, that $d(u, v) = 1$, $d(u, v) \notin \{1, 3\}$ and $d(u, v) = 3$.

Case 1 If the non weighted distance from u to v is 1, then, as S is strong, there must be a vu-directed path $\mathscr{C}$ and together with the arc (u, v) it forms a directed cycle, wich by hypothesis must have total weight $\leq \left\lfloor \frac{k-1}{2} \right\rfloor + 1$, so $\mathsf{w}(\mathscr{C}) \leq \frac{k-1}{2}$, and thence v is $(k-1, \mathsf{w})$-absorbed by N.

Case 2 If the non weighted distance from u to v is not 1 nor 3, then in virtue of Proposition 7.1.13, v is 1-absorbed by N, and then, since the arc weights are bounded by $k-1$, v is $(k-1, \mathsf{w})$-absorbed by N.

Case 3 If the non weighted distance from u to v is 3, then there exists a uv-directed path $\mathscr{C} = (u, x, y, v)$ and we have several subcases. Observe that $d(v, u) \in \{1, 3\}$, otherwise, as D is quasi-transitive, Proposition 7.1.13 would imply that $d(u, v) = 1$ which contradicts the assumption for this case.

Case 3.1 If $(v, u) \in A(D)$, then v is $(k-1, \mathsf{w})$-absorbed by N because $\mathsf{w}(v, u) \leq k - 1$.

Case 3.2 If $d(v, u) = 3$ then one of the subcases depicted in Figure 7.5 occurs.

Case 3.2.a Because of the weight restriction for the directed cycles, the two cycles $C_1 = (u, x, y, u)$ and $C_2 = (x, y, v, x)$ fulfill $\mathsf{w}(C_1), \mathsf{w}(C_2) \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 1$, but $\mathsf{w}(C_1) = p_1 + p_2 + q_2$ and $\mathsf{w}(C_2) = p_2 + p_3 + q_1$, adding this inequalities we have.

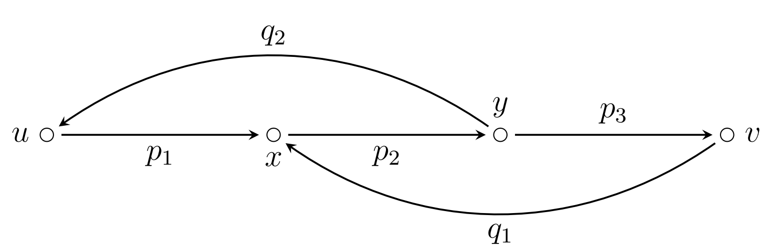


Figure 7.4: Case 3.2.a.

$$\begin{aligned}
p_1 + 2p_2 + p_3 + q_1 + q_2 &\leq 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2 \\
q_1 + p_2 + q_2 &\leq 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2 - (p_1 + p_2 + p_3) \\
&\leq 2\left\lfloor \frac{k-1}{2} \right\rfloor + 2 - 3 \\
&= 2\left\lfloor \frac{k-1}{2} \right\rfloor - 1 \\
&< k - 1
\end{aligned}$$

So, $\mathsf{w}(v, x, y, u) = q_1 + p_2 + q_2 < k - 1$ and thus, v is $(k-1, \mathsf{w})$-absorbed by N.

Case 3.2.b Since $d(u, v) = 3$, the quasi-transitivity and the existence of the arcs $(u, x), (x, y), (y, v)$ implies that $(y, u), (v, x) \in A(D)$, which reduces this case to case 3.2.a.

Case 3.2.c This case can not occur, since $(u, x), (x, y) \in A(D)$, by the quasi-transitive hypothesis, then (u, y) or (y, u) must be in $A(D)$, but $d(u, v) = d(v, u) = 3$, and (u, y) together with (y, v) would imply that $d(u, v) = 2$, also (y, u) together with (v, y) would imlpy that $d(v, u) = 2$, as a contradiction arise in both cases, this case is impossible.

Case 3.2.d The same argument used in Case 3.2.c shows that this case can not happen.

Case 3.2.e Since $(x, y), (y, v) \in A(D)$, by the quasi-transitive hypothesis, then (x, v) or (v, x) must be in $A(D)$, but analogous to Case 3.2.c, (u, x) together with (x, v) would imply that $d(u, v) = 2$, also (v, x) together with (x, u) would imlpy that $d(v, u) = 2$, as a contradiction arise in both cases, this case is impossible.

Case 3.2.f This configuration can be reduced to the one in Case 3.2.a, observe that $(u, x), (x, y) \in A(D)$ and, as $d(u, v) = 3$, $(y, u) \in A(D)$, therefore we have the same configuration as in the mentioned case.

Case 3.2.g This case can also be reduced to Case 3.2.a, we just need the existence of the arc (v, x), which is justified by the existence of the arcs $(x, y), (y, v)$, the quasi-transitive hypothesis and the fact that $d(u, v) = 3$, and thence $(x, v) \notin A(D)$.

Since the cases are exhaustive, we can conclude that every vertex in $V(D)$

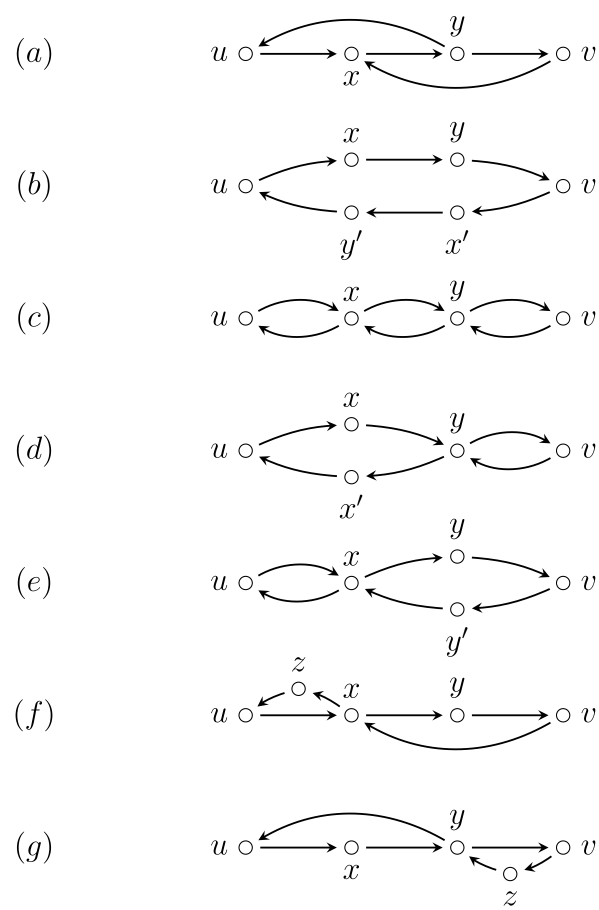


Figure 7.5: Subcases for Case 3.2 in the proof of Theorem 7.1.14.

is $(k-1, \mathsf{w})$-absorbed by N. Thus, as N is (k, w)-independent, it is the (k, w)-kernel we have been looking for. ■

The hypothesis of Theorem 7.1.14 are sufficient but not necessary. Figure 7.6 shows a digraph with directed cycles of weight greater than $\left\lfloor \frac{5-1}{3} \right\rfloor + 1 = 2$ and a 5-kernel. Cycle (a, d, b, c, a) has weight 8 and cycle (a, b, c, a) has weight 6; however, sets $\{b\}, \{c\}$ and $\{d\}$ are 5-kernels for the digraph. Also, we do not know if the bound for the weight of the cycles is tight, we were unable to find an example in which the equality is necessary, fact that make us think that the bound could be improved.

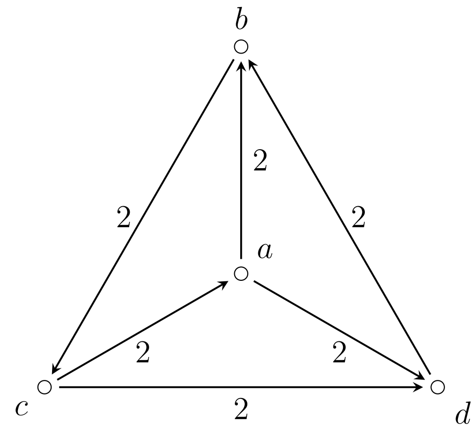


Figure 7.6: A digraph with directed cycles of weight greater than $\left\lfloor \frac{5-1}{3} \right\rfloor + 1 = 2$ and a 5-kernel.

Finally, let us observe that Figure 7.6 is also a counterexample for the assertion that every quasi-transitive digraph has a (k, w)-kernel for every integer $k \geq 3$ because it does not have a $(4, \mathsf{w})$-kernel.

7.2 Digraphs with group weights

As many results that are valid in the non weighted case does not remain valid in the weighted case, we have that the problem of finding a (k, w)-kernel in a given digraph is vastly more complicated that the non weighted one, so inspired in Theorem 1.8.2 and in view of the great difficulty that the integer (or even natural)-valued functions represent, we considered the weight function to have an arbitrary group for codomain since a group is the simplest algebraic structure where a definition of congruence exists and we think that the key to theorem 1.8.2 is the congruence modulo 0 condition. Let us recall that if G is a group and H is a subgroup of G, then if $g, h \in G$, $g \equiv h \pmod{H}$ if and only if $gh^{-1} \in H$. Nonetheless, as an arbitrary group has no order, we can not state a direct analogy between a walk's weight and its length, thus, former results cannot be formally generalized, but an interesting result can be stated ressembling Theorem 1.8.2.

Let D be a digraph, G a group and $\mathsf{w} : A(D) \to G$ a weight function for the arcs of D. If H is a subgroup of G we will say that a walk $\mathscr{P} =$

$(v_0, v_1, \dots v_n)$ has weight in H, if $\sum_{i=0}^{n-1} \mathsf{w}(v_i v_{i+1}) = \mathsf{w}(\mathscr{P}) \in H$. Let H be a subgroup of G, a subset $S \subseteq V(D)$, is ***H*-independent** if for all $u, v \in S$ does not exists an uv-path of weight in H nor a vu-path of weight in H. We will say that S is ***H*-absorbent** if for each $u \in V(D) - S$ exists a uS-path of weight in H. A H-independent, H-absorbent set will be a ***H*-kernel**.

Example 7.2.1. If D is a digraph, and we let $G = \mathbb{Z}$, $H = n\mathbb{Z}$ and $\mathsf{w} \equiv 1$, then a $(n\mathbb{Z})$-kernel will be a subset $N \subseteq V(D)$ such that between every pair of vertices of N there are no walks of length a multiple of n and from every vertex in $V(D) \setminus N$ there is a walk in D of length a multiple of n.

Lemma 7.2.2. *If D is a digraph, G a group, H a normal subgroup of G and $\mathsf{w} : A(D) \to G$ a weight function for $A(D)$ such that every directed cycle $\mathscr{C}$ fulfills $\mathsf{w}(\mathscr{C}) \in H$, then every uv-directed walk with weight in H contains a uv-directed path with weight in H.*

Proof. Before beginning the proof we want to point out that, being H a normal subgroup of G it does not matter in which vertex we begin summing the weight of a cycle (or, in other words, where we begin crossing the cycle), for if the arcs of the cycle are $a_1, a_2, \dots, a_k$ with weights $b_1, b_2, \dots b_k$ respectively then, for the normality of H, if $b_1 + b_2 + \cdots + g_k \in H$ also $-g_1 + g_1 + g_2 + \cdots + g_k + g_1 \in H$, hence for every cyclic permutation of the indexes the sum of the weights of the cycle is in H. We proceed with the proof by induction on the length of the uv-directed walk. If the length is 1, then the walk is a path and the base case of the induction follows. Let's assume the result valid for every uv-directed walk of length strictly less than n and let $\mathscr{R} = (u = x_0, x_1, \dots, x_{n-1}, x_n = v)$ an uv-directed walk of length n. If $x_i \neq x_j$ for each $i \neq j$ then $\mathscr{R}$ is a path. If not, then exists a cycle $\mathscr{C} = (x_i, x_{i+1}, \dots, x_{k-1}, x_k = x_i)$ contained as a subsequence of $\mathscr{R}$. Let $\mathscr{R}' = (x_0, \dots, x_i)$ and $\mathscr{R}''(x_{k+1}, \dots, x_n)$, then $\mathscr{R} = \mathscr{R}'\mathscr{C}\mathscr{R}''$ therefore $\mathsf{w}(\mathscr{R}) = \mathsf{w}(\mathscr{R}') + \mathsf{w}(\mathscr{C}) + \mathsf{w}(\mathscr{R}'')$. For the theorem hypothesis $\mathsf{w}(\mathscr{C}) \in H$ and $\mathsf{w}(\mathscr{R}) \in H$, since H is normal in G, it follows that $-\mathsf{w}(\mathscr{R}') + \mathsf{w}(\mathscr{R}) + \mathsf{w}(\mathscr{R}') = \mathsf{w}(\mathscr{C}) + \mathsf{w}(\mathscr{R}'') + \mathsf{w}(\mathscr{R}) \in H$ and after a few simple calculations $\mathsf{w}(\mathscr{R}'') + \mathsf{w}(\mathscr{R}') \in H$; using the normality of H once again we obtain $-\mathsf{w}(\mathscr{R}'') + \mathsf{w}(\mathscr{R}'') + \mathsf{w}(\mathscr{R}') + \mathsf{w}(\mathscr{R}'') = \mathsf{w}(\mathscr{R}') + \mathsf{w}(\mathscr{R}'') \in H$. Therefore $\mathscr{R}'\mathscr{R}''$ is a uv-directed walk with weight in H of length strictly less than n, the desired result follows from the induction hypothesis. ■

Our next theorem is inspired in Theorem 1.8.2.

Theorem 7.2.3. *If D is a digraph, G a group, H a normal subgroup of G and $\mathsf{w} : A(D) \to G$ is a weight function for the arcs of D such that every directed cycle of D has weight in H, then D has H-kernel.*

Proof. Let N_0 be a maximal H-independent subset of $V(D)$ such that if $\text{Abs}(N_0)$ is the set of vertices H-absorbed by N_0 then $|\text{Abs}(N_0) \cup N_0|$ is maximum. If $V(D) - (N_0 \cup \text{Abs}(N_0)) = \varnothing$, then N_0 is a H-kernel for D. If $V(D) - (N_0 \cup \text{Abs}(N_0)) \neq \varnothing$, then exists a vertex v_0 that is not in N_0 nor H-absorbed by it. Since N_0 is maximal H-independent and v_0 is not H-absorbed by N_0, it must exist in D a N_0v_0-directed path of weight in H. Let $A \subseteq N_0$ be the set of vertices in N_0 that are the initial vertex of a directed path with final vertex v_0 and weight in H and let $N_1 = (N_0 - A) \cup \{v_0\}$. Clearly N_1 is H-independent because N_0 was, and as we added v_0 we removed from N_0 all the vertices that reached v_0 with a directed path with weight in H, besides it does not exist any v_0N_0-directed path with weight in H in D. Also, $\text{Abs}N_0 \subseteq \text{Abs}N_1$; for the election of A N_1 H-absorbs every vertex in A and, if x is a vertex in en $\text{Abs}(N_0)$ absorbed by A, then an xa-directed path of weight in H existed for some $a \in A$, but we have the existence of an av_0-directed path of weight in H so, there exists a xv_0-directed walk with weight in H and, for Lemma 7.2.2, there is a xv_0-directed path with weight H in D, which finally results in the H-absortion of x by N_1. Now, $|\text{Abs}(N_0) \cup N_0| < |\text{Abs}(N_1) \cup N_1|$, since N_1 H-absorbs at least one more vertex than N_0, that is to say v_0, and $\text{Abs}(N_0) \subseteq \text{Abs}(N_1)$. Nonetheless this results in a contradiction to the election of N_0 as a set that maximizes $|\text{Abs}(N_0) \cup N_0|$. So, N_0 is the H-kernel we have been looking for. ■

Chapter 8

Infinite digraphs

8.1 Introduction

Theorem 3.3.5 in Chapter 3 asserts that every transitive digraph has a (k, l)-kernel for every pair of integers k, l such that $k \geq 2$, $l \geq 1$. Also, Theorem 2.3.2 in Chapter 2 states that acyclic digraphs have a unique k-kernel for every integer $k \geq 2$. It has been also mentioned that every semicomplete digraph has a k-kernel for every integer $k \geq 3$. In view of this results, it comes as a surprise the existence of a semicomplete transitive acyclic infinite digraph without k-kernel for every integer $k \geq 2$. The digraph $D^\sharp$ is the digraph with $V(D^\sharp) = \mathbb{N}$ and such that $(n, m) \in A(D^\sharp)$ if and only if $n < m$. Clearly, $D^\sharp$ is a tournament and thus, a maximal independent set of $D^\sharp$ consists in a single vertex. It is also clear that $D^\sharp$ is transitive and acyclic. But it is also clear that for every vertex $n \in V(D^\sharp)$ and for every $m > n$, m is not k-absorbed by n for every integer $k \geq 2$. In particular, for every $n \in V(D^\sharp)$, $n + 1$ is not k-absorbed by n, and thus, $\{n\}$ is not a k-kernel of D for every $n \in V(D^\sharp)$. The digraph $D^\sharp$ was introduced and first studied by Rojas-Monroy and Villarreal-Valdés in [84], where they also give sufficient conditions for distinct families of infinite digraphs to have a kernel.

It is clear that the behavior of infinite digraphs respect to (k, l)-kernels is different from the finite case. But, how different it is? In this chapter we will explore similarities and differences between the finite and the infinite cases of certain families of digraphs. Surprisingly (again) some results are generalizable straightforward from the finite case, and in some other cases, adding a few new hypothesis will get the work done. Some results are not generalizable, but in some cases a weak analog of some results were obtained.

In previous years, some work has been done in the direction of finding sufficient condition for the existence of kernels in infinite digraphs, let us mention the most remarkable results in this direction. The first results about the existence of kernels in an infinite digraph can be found in [80], where Neumann-Lara proved that every semikernel is contained in a maximal semikernel. Also he proved a very powerful result stating that if every induced subdigraph of D has a nonempty semikernel, then D is a kernel-perfect digraph (a digraph such that every induced subdigraph has a kernel). A digraph such that for every vertex $v \in V(D)$ we have that $N^+(v)$ is a finite set is called *outwardly finite*. In [31], Duchet and Meyniel proved that an outwardly finite digraph D is kernel-perfect if and only if every **finite** induced subdigraph of D has a kernel. As a corollary of this result, they proved that if D is an outwardly finite digraph such that every odd directed cycle $\mathscr{C}$ has the following property: if all arcs of $\mathscr{C}$ are incident to a subset T of vertices of $\mathscr{C}$, then some chord of $\mathscr{C}$ has its head in T. Then D is kernel-perfect. Let us recall that an arc of D is a *chord* of a directed cycle $\mathscr{C}$ if it has its endpoints in $\mathscr{C}$ but is not an arc of $\mathscr{C}$.

In [84] Rojas-Monroy and Villarreal Valdés proved that if every cycle and every infinite outward path of D has a symmetrical arc, then D is a kernel-perfect digraph. An immediate consequence of this result is that every symmetrical digraph is kernel-perfect. A digraph is right (left) pretransitive if $(u, v), (v, w) \in A(D)$ implies that $(u, w) \in A(D)$ or $(w, v) \in A(D)$ $((v, u) \in A(D))$. In the same paper is also proved that if D is an infinite right/left pretransitive digraph, such that every infinite outward path has a symmetrical arc, then D is a kernel-perfect digraph. As an easy consequence it is proved that every infinite transitive digraph such that every infinite outward path has a symmetrical arc is kernel-perfect. A very general theorem generalizing the very first result of Kernel Theory proved by von Neumann and Morgenstern states that every acyclic digraph without infinite outward paths is a kernel-perfect digraph. Another generalization of a classical result, this time due to Richardson, states that if D is an infinite digraph such that D contains no infinite outward path and contains no odd cycle, then D is a kernel-perfect digraph. Also, the aforementioned digraph $D^\sharp$ is introduced in this paper and its following properties proved: $D^\sharp$ contains no terminal strong component, no absorbing set distinct from $V(D^\sharp)$, no vertex with null out-degree and no kernel.

In [44] Galeana-Sánchez and Guevara work with semikernels modulo F, where $F \subseteq A(D)$. A set $S \subseteq V(D)$ is a *semikernel modulo* F of D if S is an independent set of vertices such that, for every $z \in V(D) \setminus S$ for which there exists an (S, z)-arc of $D - F$, there also exists a (z, S)-arc. We will also need the notion of asymmetrically transitive. A digraph D is asymmetrically transitive if whenever $(u, v), (v, w) \in \mathrm{Asym}(D)$, then $(u, w) \in \mathrm{Asym}(D)$. The following result is proved: Let D be a digraph and D_1 an asymmetrically subdigraph of D. If D has no infinite outward path contained in $\mathrm{Asym}(D_1)$, D is Γ_{D_1} free and every induced subdigraph of D has a nonempty semikernel modulo $A(D_1)$, then D is a kernel perfect digraph. In the previous statement, Γ_{D_1} is a set of 16 digraphs, that are forbidden to appear as induced subdigraphs of D. As a consequence of this result, they proved that every infinite quasi-transitive digraph such that every directed triangle contained in D is symmetrical and D has no asymmetrical infinite outward path is a kernel-perfect digraph. Another consequence is that every infinite bipartite digraph has a kernel.

Not exactly about sufficient conditions for the existence of (k, l)-kernels in digraphs, but also related, there is a paper by Erdös and Soukup. It is a well known result of Chvátal and Lovász [28] that every finite digraph has a $(2, 2)$-kernel. Once again, the digraph $D^\sharp$ is a counterexample for the infinite version of this result. In [32], Erdös and Soukup work in a very interesting conjecture stating that for any digraph D, there exist a partition (V_0, V_1) of $V(D)$ such that $D[V_0]$ has a $(2, 2)$-kernel and $D[V_1]$ a $(2, 2)$-solution. They focus on infinite tournaments. Again, not exactly about sufficient conditions for the existence of (k, l)-kernels in digraphs, but related, we can mention the work of Fraenkel [34], where he uses Game Theory tools to analyze the structure of the kernels of a digraph, and some of his results are valid in infinite digraphs.

The rest of the chapter is structured as follows. In Section 8.2 we prove a very important (and general) sufficient condition for a digraph to have (k, l)-kernel, that will be used later on. In Section 8.3 we prove that every symmetrical digraph has a k-kernel for every integer $k \geq 2$ and propose a sufficient condition for acyclic infinite digraphs to have a k-kernel for every integer $k \geq 2$. In Section 8.4 we find a sufficient condition for infinite transitive digraphs to have (k, l)-kernel for every pair of integers k, l such that $k \geq 2$ and $l \geq 1$; the condition is that every infinite outward path $(x_i)_{i \in \mathbb{N}}$ has an arc of the form (x_j, x_i) with $i < j$. In Section 8.5 we prove that every infinite cyclically k-partite strong digraph has at least k different k-kernels; also, we prove that if D is a digraph with a (possibly bi-infinite) spanning directed walk such that every directed cycle has length $\equiv 0 \mod k$, and every directed cycle with exactly one obstruction has length $\equiv 2 \mod k$, then D is cyclically k-partite and thus has a k-kernel. The main result of Section 8.6 is that every infinite quasi-transitive digraph without infinite outward paths has a (k, l)-kernel for every pair of integers k, l such that $k \geq 4$ and $l \geq 3$ or $k = 3$ and $l = 2$. Section 8.7 is about pretransitive digraphs, we prove that if D is an infinite right/left pretransitive strong digraph such that every directed triangle is symmetrical, then D has a k-kernel for every $k \in \mathbb{N}$, $k \geq 2$. The results of Sections 8.8 and 8.9 are straightforward generalizations from the finite case, we give sufficient conditions for the existence of (k, l)-kernels in terms of the circumference of a digraph in κ-strong digraphs and in locally in/out-semicomplete digraphs. Section 8.10 is the final section of the article, we work with k-transitive and k-quasi-transitive digraphs, the main results are: If $k \geq 2$ be an even integer and D is an infinite k-quasi-transitive digraph such that for every infinite outward path $(x_i)_{i\in\mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$, then D has an (n, m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k+1$. And, if D is an infinite k-transitive digraph such that every infinite outward path $(x_i)_{i\in\mathbb{N}}$ has an arc of the form (x_j, x_i) with $i < j$, then D has a (n, m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k - 1$. Moreover, every (n, m)-kernel of D consists in choosing one vertex from every terminal component of D.

8.2 Two useful lemmas in infinite digraphs

We begin with a very simple result which is vastly used in its finite form in Kernel Theory. Nonetheless, in the infinite case we depend on the Axiom of Choice to prove it.

Lemma 8.2.1. *Let D be a digraph. If D has a non-empty (k, l)-semikernel, then D has a maximal (k, l)-semikernel.*

Proof. Since D has at least one non-empty (k, l)-semikernel and thus, the set $\mathcal{S}$ of all (k, l)-semikernels of D is non-empty. Clearly $(\mathcal{S}, \subseteq)$ is a POSET. Let $\{C_i\}_{i\in I} = \mathcal{C} \subseteq \mathcal{S}$ be a chain in $(\mathcal{S}, \subseteq)$. We will prove that $\mathcal{C}$ has an upper bound, and by Zorn's Lemma we will have the existence of a maximal (k, l)-semikernel in D.

If $C = \bigcup_{i\in I} C_i$, then it is easy to observe that $C \in \mathcal{S}$. To prove that C is independent, let $u, v \in C$, then $u \in C_i$ and $v \in C_j$ for some $i, j \in I$.

Since $\mathcal{C}$ is a chain, we can assume without loss of generality that $C_i \subseteq C_j$, and then $u, v \in C_j$, but C_j is a (k,l)-semikernel and hence is k-independent, so $d(u,v), d(v,u) \geq k$, and then C is k-independent. To prove the second (k,l)-semikernel condition, let $v \in C$ and $u \in V(D)$ such that $d(v,u) \leq k-1$. Recall that $v \in C$ implies that $v \in C_i$ for some $i \in I$. But C_i is a (k,l)-semikernel, thus $d(v,u) \leq k-1$ implies $d(u,v) \leq l$. Then, $C \in \mathcal{S}$ and clearly $C_i \subseteq C$ for every $i \in I$. ■

The following lemma is the infinite version of Lemma 3.1.4

Lemma 8.2.2. *Let D be a digraph. If $\{v\}$ is a (k,l)-semikernel of D for every $v \in V(D)$, then D has a (k,l)-kernel.*

Proof. By Lemma 8.2.1, we can consider a $(\subseteq)$maximal (k,l)-semikernel of D, namely $S \subseteq V(D)$. If S is l-absorbent then S is a (k,l)-kernel of D, so let us assume that S is not l-absorbent, therefore there must exist a vertex $v \in V(D) \setminus S$ such that $d(v,S) > l$. Let us observe that $d(S,v) > k-1$ because, by the second condition of (k,l)-semikernel, $d(S,v) \leq k-1$ implies that $d(v,S) \leq l$ but v is not l-absorbed by S. We will consider two cases.

Case 1. If $k-1 \leq l$, then $k-1 \leq l < d(v,S)$, so, in view that $d(S,v) > k-1$, we have that $S' = S \cup \{v\}$ is a k-independent set. Moreover, if $u \in V(D)$ is such that there exists an $S'u$-directed path $\mathscr{C}$ of length less than or equal to $k-1$ then, since S is a (k,l)-semikernel, if $\mathscr{C}$ is a Su-directed path, then there exists an uS-directed path of length less than or equal to $k-1$, but this path is also a uS'-directed path; and since $\{v\}$ is also a (k,l)-semikernel, then if $\mathscr{C}$ is a vu-directed path, this implies that there exists a uv-directed path of length less than or equal to $k-1$, which is also a uS'-directed path, and then S' is a (k,l)-semikernel properly containing S which contradicts the choice of S as a maximal (k,l)-semikernel.

Case 2. If $l < k-1$, then we can assume that $d(v,S) \leq k-1$, otherwise $S \cup \{v\}$ would be k-independent and we can proceed as in Case 1. So, since $\{v\}$ is a (k,l)-semikernel, then $d(S,v) \leq l < k-1$ which results in a contradiction.

In both cases a contradiction arises from the assumption that S is not l-absorbent, so S must be l-absorbent and hence the desired (k,l)-kernel. ■

8.3 Two simple families of infinite digraphs with k-kernel

Let us recall that Theorem 2.3.2 affirms that every acyclic digraph has a unique k-kernel for every integer $k \geq 2$. As we have seen with the digraph $D^\sharp$, the direct generalization for infinite digraphs is not valid, so we propose a sufficient condition for an infinite acyclic digraph to have k-kernel for every integer $k \geq 2$. To simplify the proof we will use the k-transitive closure defined in Section 4.2. As we mentioned before, the finite case of the following lemma was first stated by Bród, Włoch and Włoch in [24]. We omit the proof of the infinite case since it is basically the same as the finite case.

Lemma 8.3.1. *If D is a (possibly infinite) digraph then $\mathcal{C}_k(D)$ has a kernel if and only if D has a $(k+1)$-kernel.*

Now we use Lemma 8.3.1 to prove the following theorem.

Theorem 8.3.2. *If D is an infinite acyclic digraph without infinite outward paths, then D has a k-kernel for every integer $k \geq 2$.*

Proof. Since D acyclic implies that $\mathcal{C}_k(D)$ is acyclic for every $k \in \mathbb{N}$. In virtue of Lemma 8.3.1 it suffices to prove that if D is acyclic and does not have infinite outward paths, then D has a kernel.

Let us observe that if D is an acyclic digraph without infinite outward paths, then for every vertex $v \in V(D)$, there exists a vertex $w \in V(D)$ such that $d(v,w) \in \mathbb{N}$ and $d^+(w) = 0$. It suffices to consider the terminal vertex in a directed path of maximum length with initial vertex v. So, let us consider the following recursive sequence of subsets of $V(D)$.

- $S_0 = \left\{ v \in V(D) \middle| d^+_D(v) = 0 \right\}$.
- $S_{n+1} = \left\{ v \in V(D) \middle| d^+_{D_{n+1}}(v) = 0 \right\}$.

Where $\{D_n\}$ is a sequence of subdigraphs of D defined recursively.

- $D_0 = D$.
- $D_{n+1} = D_n \setminus (N^-(S_n) \cup S_n)$.

By the previous observation, there exists an $n_0 \in \mathbb{N}$ such that $S_n = D_n = \varnothing$ for every $n > n_0$. Let n_0 be the least natural number with such property. We affirm that $N = \bigcup_{i=0}^{n_0} S_i$ is a kernel for D. First, let us observe that N is absorbent. Let $v \in V(D) \setminus N$ be arbitrarily chosen. Let i be the greatest integer such that $v \in V(D_i)$, clearly $0 \leq i \leq n_0$. Since $v \notin V(D_{i+1})$, by the definition of D_{i+1}, $v \in N^-(S_n) \cup S_n$. But $v \notin N$, so $v \notin S_n$. This implies that $v \in N^-(S_n) \subseteq N$, and thus v is absorbed by N. To observe that N is independent, let $u, v \in N$ and $i, j \in \mathbb{N}$ such that $u \in S_i$ and $v \in S_j$. Let us assume without loss of generality that $i \leq j$. If $i = j$, then $u, v \in S_i$ and by the definition of S_i, $d^+_{D_i}(u) = d^+_{D_i}(v) = 0$. Since D_i is an induced subdigraph of D and $u, v \in V(D_i)$, then $(u, v), (v, u) \notin A(D)$. If $i < j$, then $(v, u) \notin A(D)$; otherwise $v \in N^-(S_i)$ which would imply that $v \notin V(D_{i+1})$ contrary to our assumption. Also $(u, v) \notin A(D)$ since $u, v \in V(D_i)$, $d^+_{D_i}(u) = 0$ and D_i is an induced subdigraph of D. So, N is a kernel for D and the result follows. ■

Although asking for a digraph not to have infinite outward paths is not very restrictive in the cardinality sense, because a digraph without infinite outward paths can has a vertex set of arbitrary cardinality, we think that is somewhat restrictive in a structural sense. In the development of this result we worked with another hypotheses, but we were unable to get the desired result nor a counterexample for our conjectures. So we propose the following conjectures.

Conjecture 8.3.3. *If D is an infinite acyclic digraph such that every vertex has a finite out-neighborhood, then D has a k-kernel for every integer $k \geq 2$.*

Conjecture 8.3.4. *If D is an infinite digraph without cycles (directed or undirected), then D has a k-kernel for every $k \geq 2$.*

The second conjecture has been proved true for $k = 2$ in [44], since a digraph without directed and undirected cycles is a bipartite digraph and thus has a kernel.

Another of the first sufficient conditions that were found for a digraph to have a kernel is to ask for symmetry. Berge proved the finite version of the following theorem for $k = 2$, a proof can be found in [11]. We want to emphasize that, although simple, we could not find a proof without using the Axiom of Choice.

Theorem 8.3.5. *If D is a symmetrical digraph, then D has a k-kernel for every integer $k \geq 2$. Moreover, every maximal k-independent subset of D is a k-kernel.*

Proof. By Zorn's Lemma we can choose a maximal k-independent subset of $V(D)$, say N. We affirm that N is the desired k-kernel. By our choice N is k-independent. Let $v \in V(D) \setminus N$ be an arbitrary vertex. Since N is a maximal k-independent subset, then $N \cup \{v\}$ is not k-independent. If $d(v, N) \leq k-1$, then v is $(k-1)$-absorbed by N. So $d(N, v) \leq k-1$, and then, there exists $u \in N$ such that there exists an uv-directed path of length less than or equal to $k-1$, but since D is symmetrical there also exists a vu-directed path of length less than or equal to $k-1$, and thus, v is $(k-1)$-absorbed by N. ■

8.4 Transitive digraphs

As we have already mentioned in previous sections of this chapter, there are infinite transitive digraphs (even transitive tournaments) that do not have k-kernel for any $k \geq 2$. In [84] and independently in [44] it is proved that if D is a transitive digraph such that every infinite outward path has at least one symmetrical arc, then D has a kernel. We will generalize this result, weakening the condition of the existence of a symmetrical arc in every infinite outward path and proving the existence of (k,l)-kernels with $k \geq 2$, $l \geq 1$ (and thus, the aforementioned result is the case $k = 2$ and $l = 1$). But before stating and proving our generalization we need to define a relation.

Let D be a digraph with set of strong components $\mathfrak{C}$. We define the relation $\preccurlyeq$ on $\mathfrak{C}$ in the next way. For every $\mathsf{C}_1, \mathsf{C}_2 \in \mathfrak{C}$ we have that $\mathsf{C}_1 \preccurlyeq \mathsf{C}_2$ if and only if there exists a $\mathsf{C}_1\mathsf{C}_2$-directed path in D. In [84] it is observed that $\langle \mathfrak{C}, \preccurlyeq \rangle$ is a reflexive partial order whose maximal elements (in the case that there exist) are the terminal strong components of D.

Remark 8.4.1. It is direct to observe that if D is a transitive digraph, $\mathsf{C}_1, \mathsf{C}_2 \in \mathfrak{C}$ and $\mathsf{C}_1 \preccurlyeq \mathsf{C}_2$, then for every $u \in V(\mathsf{C}_1)$ and every $v \in V(\mathsf{C}_2)$, we have that $(u, v) \in F(D)$.

Theorem 8.4.2. *Let D be an infinite transitive digraph such that every infinite outward path $(x_i)_{i \in \mathbb{N}}$ has an arc of the form (x_j, x_i) with $i < j$, then D has a (k,l)-kernel for every pair of integers k, l such that $k \geq 2$, $l \geq 1$.*

Moreover, every (k,l)-kernel of D consists in choosing one vertex from every terminal component of D.

Proof. It suffices to prove that if C_0 is a strong component of D then there exists a terminal component T of D such that $\mathsf{C}_0 \preccurlyeq \mathsf{T}$. This is because, since D is a transitive digraph, by Remark 8.4.1 every vertex in C_0 will be absorbed by every vertex in T. Also, if we choose one vertex in every terminal strong component of D, the set of the chosen vertices will be k-independent for every integer $k \geq 2$, because every vertex is in a distinct terminal component. So, every set consisting of one vertex from every terminal component of D will be k-independent and absorbent, and thus, l-absorbent, for every pair of integers k, l such that $k \geq 2$, $l \geq 1$.

We will proceed by contradiction. Assume that for every $\mathsf{C} \in \mathfrak{C}$ such that $\mathsf{C}_0 \preccurlyeq \mathsf{C}$ there exists $\mathsf{C}' \in \mathfrak{C}$ such that $\mathsf{C}' \neq \mathsf{C}$ and $\mathsf{C} \preccurlyeq \mathsf{C}'$. In virtue of the Axiom of Choice we can build a sequence $(\mathsf{C}_i)_{i\in\mathbb{N}}$ satisfying $\mathsf{C}_0 \preccurlyeq \mathsf{C}_1$ and, for every $i < j$, $\mathsf{C}_i \neq \mathsf{C}_{i+1}$ and $\mathsf{C}_i \preccurlyeq \mathsf{C}_j$. Appealing again to the Axiom of Choice, let us choose a vertex $v_i \in V(\mathsf{C}_i)$ for every $i \in \mathbb{N}$. Since D is transitive, by Remark 8.4.1, $(v_i)_{i\in\mathbb{N}}$ is an infinite outward path in D. Moreover, if $i < j$, $(x_j, x_i) \notin F(D)$. In the contrary case we would have that (x_j, x_i) is a $\mathsf{C}_j\mathsf{C}_i$-arc and thus a $\mathsf{C}_j\mathsf{C}_i$ directed path, which by the definition of $\preccurlyeq$ implies that $\mathsf{C}_j \preccurlyeq \mathsf{C}_i$. Since $\mathsf{C}_i \preccurlyeq \mathsf{C}_j$ by the construction of $(\mathsf{C}_n)_{n\in\mathbb{N}}$ it would follow from the antisymmetry of $\preccurlyeq$ that $\mathsf{C}_i = \mathsf{C}_j$. Again, by the construction of $(\mathsf{C}_n)_{n\in\mathbb{N}}$, we know that $j \neq i+1$, but this implies the existence of a directed cycle $(\mathsf{C}_i, \mathsf{C}_{i+1}, \ldots, \mathsf{C}_j, \mathsf{C}_i)$ in $D^\star$, which results in a contradiction because $D^\star$ is acyclic. Therefore $(x_i)_{i\in\mathbb{N}}$ is an infinite outward path in D such that $(x_j, x_i) \notin F(D)$ for each $i < j$, which results in a contradiction. Since the contradiction arises from assuming that for every $\mathsf{C} \in \mathfrak{C}$ such that $\mathsf{C}_0 \preccurlyeq \mathsf{C}$ there exists $\mathsf{C}' \in \mathfrak{C}$ such that $\mathsf{C}' \neq \mathsf{C}$ and $\mathsf{C} \preccurlyeq \mathsf{C}'$, there must exists a strong component T, such that $\mathsf{C}_0 \preccurlyeq \mathsf{T}$ and for every $\mathsf{C}' \in \mathfrak{C}$, $\mathsf{C}' = \mathsf{T}$ or $\mathsf{T} \not\preccurlyeq \mathsf{C}'$. Thence, T is a $\preccurlyeq$-maximal element of $\mathfrak{C}$, and thus a terminal component of D.

As noted before, if $N \subseteq V(D)$ has one vertex from every terminal component of D, then N is a (k,l)-kernel for every pair of integers k, l such that $k \geq 2$, $l \geq 1$. ∎

Theorem 8.4.2 gives us a sufficient structural condition for an infinite transitive digraph to have a (k,l)-kernel. It is easy to observe that this condition is not necessary. As an example take the digraph $D^\sharp$ and add one new vertex v along with the arc (n, v) for every $n \in \mathbb{N}$. This digraph clearly has the (k, l)-kernel $\{v\}$ for every pair of integers k, l such that $k \geq 2$ and $l \geq 1$, but does not fulfill the condition given in Theorem 8.4.2. Nonetheless, if we take a closer look at the proof of Theorem 8.4.2, we can observe that the required condition implies that every strong component of the digraph is absorbed by some terminal component. This is the condition that characterize the transitive (infinite) digraphs that have a kernel.

Theorem 8.4.3. *Let D be a possibly infinite transitive digraph. Then D has a (k, l)-kernel for every pair of integers k, l such that $k \geq 2$ and $l \geq 1$ if and only if for every strong component S of D, there exists a terminal component T of D such that $\mathsf{S} \mapsto \mathsf{T}$.*

Proof. The theorem is obviously valid for finite digraphs, let us prove the infinite case.

The "if" implication follows directly from the proof of Theorem 8.4.2. For the "only if" part, let D be a transitive digraph that has a (k, l)-kernel for every pair of integers such that $k \geq 2$ and $l \geq 1$, and N be a $(2, 1)$-kernel of D. Let us assume that there exists a strong component S of D such that no terminal component of D can be reached from S by a directed path. Since N is a kernel of D, and by the transitivity of D it follows that there must exists a vertex $v \in N$ such that $\mathsf{S} \mapsto v$. Moreover, if R is the strong component of D containing v, then $\mathsf{S} \mapsto \mathsf{R}$. By our assumption, R is not a terminal component. If R' is a strong component reached by R, then $\mathsf{R} \mapsto \mathsf{R}'$, in particular $v \mapsto \mathsf{R}'$. Now, R and R' are different strong components, thus, vertices in R' cannot be absorbed by v. But N is a kernel of D, and then, there must exist another vertex $u \in N$ contained in a strong component R_1 such that $\mathsf{R}' \mapsto u$. But D is transitive, which implies that $\mathsf{R} \mapsto \mathsf{R}_1$, in particular, $(v, u) \in A(D)$, contradicting the independence of N. Thus, for every strong component S of D, there exists a terminal component T of D such that $\mathsf{S} \mapsto \mathsf{T}$. ■

Although Theorem 8.4.3 characterize transitive digraphs with (k, l)-kernel for every pair of integers k, l such that $k \geq 2$ and $l \geq 1$, we do not have a characterization in terms of structural properties of the digraphs like in Theorem 8.4.2. So, we propose the following problem.

Problem 8.4.4. Find a structural characterization of transitive digraphs with (k, l)-kernel for every pair of integers k, l such that $k \geq 2$, $l \geq 1$.

8.5 Cyclically k-partite digraphs

One of the most general results relating cyclically k-partite digraphs and k-kernels remains valid for infinite digraphs.

Theorem 8.5.1. *Let D be a (possibly infinite) cyclically k-partite digraph. If, with at most one exception, for every $v \in V(D)$ we have that $d^+(v) \geq 1$, then D has a k-kernel.*

Proof. Let D be a cyclically k-partite digraph with vertex partition $\{V_0, V_1, \ldots, V_{k-1}\}$. We may assume without loss of generality that there is a vertex $v \in V(D)$ such that $d^+(v) = 0$, and $v \in V_0$. Since every arc of D is a V_iV_{i+1}-arc (mod k), it is clear that V_i is k-independent for each $i \in \{0, 1, \ldots, k-1\}$. If $u \in V(D) \setminus V_0$, then $u \in V_i$ for some $i \in \{1, 2, \ldots, k-1\}$, and since D is cyclically k-partite and every $w \in D - v$ has $d^+(w) \geq 1$, there exists a uV_0-directed path of length less than or equal to $k-1$. Hence, V_0 is a k-kernel for D. ■

In Chapter 2, unilateral cyclically k-partite digraphs are characterized[1] as those digraphs such that every directed cycle has length $\equiv 0 \pmod k$ and every directed cycle with one obstruction[2] has length $\equiv 2 \pmod k$. This characterization was obtained while trying to generalize Theorem 1.8.2, which asserts that a strong digraph such that every directed cycle has length $\equiv 0 \pmod k$ has a k-kernel. As we mentioned earlier, the best way to prove Theorem 1.8.2 is to prove that, under such hypotheses, a digraph is cyclically k-partite. The generalization of Theorem 1.8.2 to the inifinte case worked just fine. Nonetheless the unilateral case presented some complications that will be mentioned shortly. The following lemma will be stated without proof since the proof is exactly the same as for finite digraphs.

Lemma 8.5.2. *Let D be an infinite digraph. If every directed cycle of D has length $\equiv 0 \pmod k$, then every directed closed walk of D has length $\equiv 0 \pmod k$.*

Theorem 8.5.3. *Let D be an infinite strong digraph. If every directed cycle in D has length $\equiv 0 \pmod k$, then D is cyclically k-partite.*

[1] Theorem 2.2.7.
[2] Definition 2.2.1.

Proof. Since D is strong, for every $u, v \in V(D)$, $d(u,v) \in \mathbb{N}$. Let $v \in V(D)$ be a fixed vertex and $V_i = \{u \in V(D) \big| d(v,u) \equiv i \pmod{k}\}$ for each $0 \leq i \leq k-1$. We affirm that $\{V_i\}_{i=0}^{k-1}$ is a cyclic partition of $V(D)$. First we will prove that $\{V_i\}_{i=0}^{k-1}$ is a partition. Clearly $V_i \cap V_j = \varnothing$ if and only if $i \neq j$, because $d(v,u)$ is uniquely determined for every $u \in V(D)$. Also, it follows from the first observation of the proof that $\bigcup_{i=1}^{k-1} V_i = V(D)$. It remains to prove that $V_i \neq \varnothing$ for every $0 \leq i \leq k-1$. So, it follows from the fact that D is strong that $d^-(v) \geq 1$. Thus, let $u \in N^-(v)$ be an in-neighbor of v. Again by the strength of D, there exists a vu-directed path, say $\mathscr{C} = (v = x_0, x_1, \ldots, x_n = u)$, and then $\mathscr{C}' = \mathscr{C} \cup (u,v)$ is a directed cycle in D. Without loss of generality can choose $\mathscr{C}$ to realize the distance from v to u. It follows by the main hypothesis of the theorem that $n \equiv 0 \pmod{k}$. Since $\mathscr{C}$ is a uv-directed path of minimum length, we have that $d(v, x_i) \equiv i \pmod{k}$ for $0 \leq i \leq k-1$. Thus $V_i \neq \varnothing$ for every $0 \leq i \leq k-1$. We have already seen that $\{V_i\}_{i=0}^{k-1}$ is a partition, let us prove that it is cyclic.

Let $(u,w) \in A(D)$ be an arbitrary arc and let us assume that $u \in V_i$, $w \in V_j$ for some $0 \leq i, j \leq k-1$. We will prove that $j \equiv i+1 \pmod{k}$. Let $\mathscr{C}$ and $\mathscr{D}$ be vu and vw-directed paths of minimum length, respectively. By the strength of D it also exists a wv-directed path, say $\mathscr{D}'$. Clearly $\mathscr{D} \cup \mathscr{D}'$ and $\mathscr{C} \cup (u,w) \cup \mathscr{D}'$ are closed directed walks in D. Hence, by Lemma 8.5.2, we have that $\ell(\mathscr{D} \cup \mathscr{D}') \equiv \ell(\mathscr{C} \cup (u,w) \cup \mathscr{D}') \equiv 0 \pmod{k}$. But $\ell(\mathscr{D} \cup \mathscr{D}') = \ell(\mathscr{D}) + \ell(\mathscr{D}')$ and $\ell(\mathscr{C} \cup (u,w) \cup \mathscr{D}') = \ell(\mathscr{C}) + 1 + \ell(\mathscr{D}')$. Therefore $\ell(\mathscr{D}) \equiv \ell(\mathscr{C}) + 1$. Since $\mathscr{C}$ and $\mathscr{D}$ realize the distances from v to u and from v to w respectively, we have that $d(u,v) \equiv i \pmod{k}$ and $d(w,v) \equiv j \equiv i+1$.

Thus, every arc of D is a V_iV_{i+1}-arc $\pmod{k}$ and hence $\{V_i\}_{i=0}^{k-1}$ is a cyclical partition. ■

Corollary 8.5.4. *Let D be an infinite strong digraph, then D is bipartite if and only if every directed cycle has even length.*

Theorem 8.5.5. *Let D be an infinite strong digraph. If every directed cycle of D has length $\equiv 0 \pmod{k}$, then D has at least k distinct k-kernels.*

Proof. By Theorem 8.5.3, D is cyclically k-partite with partition $\{V_i\}_{i=0}^{k-1}$. By Theorem 8.5.1, every V_i is a k-kernel for D. ■

As we have already seen in Theorem 8.5.5, the generalization of Richardson's Theorem remain valid in the infinite case. However, we were not able

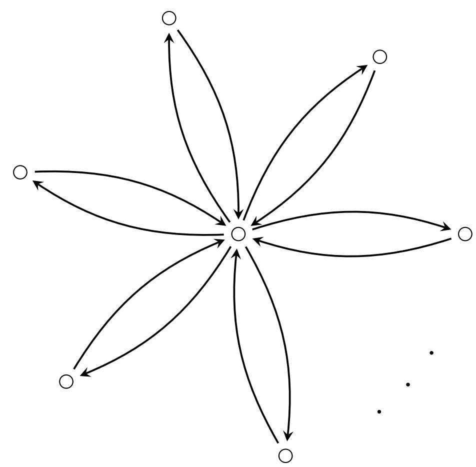


Figure 8.1: A flower with an infinite number of petals, example of an infinite strong digraph without a closed directed spanning walk.

to prove the infinite version of Theorem 2.2.7. The reason is that the widely known characterization of unilateral digraphs does not work for infinite digraphs. There are infinite unilateral digraphs without directed spanning walk. Less important to our concern, but also worth of mention, the characterization of strong digraphs neither works in the infinite case. There are infinite strong digraphs without a closed directed spanning walk. An examples of this fact is shown in Figure 8.1.

So, we state the following conjecture.

Conjecture 8.5.6. *Let D be an infinite unilateral digraph. If every directed cycle of D has length $\equiv 0 \pmod k$ and every directed cycle of D with one obstruction has length $\equiv 2 \pmod k$, then D is cyclically k-partite.*

Since the proof of this result for the finite case relies heavily in the fact that every unilateral digraph has a directed spanning walk, we may consider this condition as a hypothesis to prove a similar result. Nevertheless, this hypothesis is very restrictive in the cardinality sense. The existence of a spanning directed walk implies that $|V(D)| \leq \aleph_0$. The digraphs considered for Conjecture 8.5.6 can have an arbitrarily large set of vertices.

The the following theorem ressembles Theorem 2.2.7. Since the proof is very similar we will omit some details. A **bi-infinite directed walk** in a digraph D is an integer-indexed sequence of vertices of D, $\{v_i\}_{i\in\mathbb{Z}}$ such that $v_iv_{i+1} \in A(D)$ for every $i \in \mathbb{Z}$.

Theorem 8.5.7. *Let D be an infinite digraph with a (possibly bi-infinite) directed spanning walk. If every directed cycle in D has length $\equiv 0 \pmod k$ and every directed cycle with one obstruction has length $\equiv 2 \pmod k$ then D is cyclically k-partite and thus has a k-kernel.*

Sketch of Proof By hypothesis there exists a spanning directed walk indexed with the set of integers $\mathscr{C} = (\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots)$ and we can consider the subsets $V_i = \left\{v_r \middle| r \equiv i \ (\mathrm{mod} k)\right\}$, $0 \leq i \leq k-1$ of $V(D)$. The set $\{V_i\}_{i=0}^{k-1}$ is a partition of $V(D)$. To prove that $\bigcup_{i=0}^{k-1} V_i = V(D)$ and $V_i \neq \varnothing$ for $0 \leq i \leq k-1$ it suffices to observe that $\mathscr{C}$ is a spanning directed walk. To prove that $V_j \cap V_k = \varnothing$, let v_r be a vertex in $V(D)$, if v_r appears only once in $\mathscr{C}$ then $r \equiv i \pmod k$ for a unique $i \in \{0, 1, \ldots, k-1\}$ and consequently v_r belongs to V_i for a unique $i \in \{0, 1, \ldots, k-1\}$; if v_r appears more than once in $\mathscr{C}$ we can assume without loss of generality that $v_r = v_s$ with $r < s$ and then $v_r\mathscr{C}v_s$ is a directed closed walk of finite length which, in virtue of Lemma 8.5.2, has length $\equiv 0 \pmod k$ so $r \equiv s \pmod k$ and $v_r \in V_i$ for a unique i. We have to observe that this partition is cyclic.

Let $(x, y) \in A(D)$, then $x = v_r$, $y = v_s$ for some $r, s \in \mathbb{Z}$. If $s < r$, then $y\mathscr{C}x \cup (x, y)$ is a directed closed walk of finite length and the same argument used in the proof of Theorem 2.2.7 can be used to prove that $s \equiv r + 1 \pmod k$. If $r < s$, then $s = r + 1$ when $(x, y) \in A(D)$ or $x\mathscr{C}y \cup (x, y)$ is a directed closed walk of finite length with one obstruction and the same argument of the proof of Theorem 2.2.7 can be used to prove that $s \equiv r + 1 \pmod k$.

Therefore D is cyclically k-partite and by Theorem 8.5.1 it has a k-kernel.
■

8.6 Quasi-transitive digraphs

Let us recall that a recursive structural characterization of the family of quasi-transitive digraphs was given by Bang-Jensen and Huang[3]. This structural characterization was a central part of the proof given in Chapter 3 for the finite version of the principal result of this section. Nonetheless, the aforementioned characterization only works for finite digraphs, and since a

[3] Theorem 3.3.1.

fundamental part of the proof is done by induction, we were unable to find an analogous characterization theorem for infinite digraphs. In the present section, a different approach is considered. We will prove that every quasi-transitive strong digraph has a (k,l)-semikernel and use local properties of quasi-transitive digraphs to prove that this (k,l)-semikernel is also a (k,l)-kernel.

We begin with some technical results.

Lemma 8.6.1. *Let D be an infinite quasi-transitive digraph and $u, v \in V(D)$. If $d(v,u) = 2$ or $4 \leq d(v,u) \in \mathbb{N}$, then $d(u,v) = 1$. If $d(v,u) = 3$, then $d(u,v) \leq 3$.*

Proof. The proof of the finite case remains valid since it is proved by induction on $d(v,u) \in \mathbb{N}$. ∎

Lemma 8.6.2. *Let D be an infinite quasi-transitive digraph. If A and B are strong components of D such that there is an AB-arc in D, then $A \mapsto B$. Hence, the condensation of D, $D^\star$, is transitive.*

Proof. Since there is an AB-arc in D, for every $a \in V(A)$ and every $b \in V(B)$, there is an ab-directed path in D. The proof of the finite case of the first part of the lemma remains valid since it is proved by induction on $d(a,b) \in \mathbb{N}$ that $(a,b) \in A(D)$. To prove that $D^\star$ is transitive, let (A,B) and (B,C) be arcs of $D^\star$. Then, by the first part of the lemma, there are vertices $a \in V(A)$, $b \in V(B)$ and $c \in V(C)$, such that $(a,b),(b,c) \in A(D)$. Since D is quasi-transitive, $(a,c) \in A(D)$ or $(c,a) \in A(D)$. But since $D^\star$ is acyclic, $(c,a) \notin A(D)$. Thus, $(a,c) \in A(D)$ and this implies that $(A,C) \in A(D^\star)$. ∎

The following lemma is "one half" of Bang-Jensen and Huang's characterization theorem. The strong case is the one that could not be obtained for infinite digraphs.

Lemma 8.6.3. *Let D be an infinite non-strong digraph. Then D is quasi-transitive if and only if there exist an acyclic transitive digraph T with vertex set $V(T) = \{v_i\}_{i \in I}$ and a family of strong quasi-transitive digraphs $\{Q_i\}_{i \in I}$ such that $D = T[Q_i]_{i \in I}$.*

Proof. The sufficiency is straightforward to verify. For the necessity, let D be a quasi-transitive digraph. Let T be the condensation of D, *i.e.*, $T = D^\star$. Hence, T is acyclic. Recall that $V(T) = \{C_i\}_{i\in I}$ is the set of strong components of D. Since D is quasi-transitive, C_i is a quasi-transitive strong digraph for every $i \in I$. So, let $Q_i = C_i$ for every $i \in I$. It follows from the definition of $D^\star$ and Lemma 8.6.2 that $D = T[Q_i]_{i\in I}$ and T is transitive. ■

From this point on, our results aim to prove that every quasi-transitive digraph has a (k,l)-semikernel. Maybe some lemmas like the following one may look a bit odd.

Lemma 8.6.4. *Let D be an infinite quasi-transitive digraph. Then, for every directed cycle $\mathscr{C}$ of D, there are at least two arcs of $\mathscr{C}$, say $(u_1,v_1),(u_2,v_2) \in A(\mathscr{C})$ such that $d(v_i,u_i) \leq 2$, $i \in \{1,2\}$.*

Proof. By induction on $\ell(\mathscr{C})$. If $\ell(\mathscr{C}) = 2$ or $\ell(\mathscr{C}) = 3$, the result is clear. Let $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ be a directed cycle of length $n \geq 4$ in D. Since D is quasi-transitive and $(x_0,x_1),(x_1,x_2) \in A(D)$, then $(x_0,x_2) \in A(D)$ or $(x_2,x_0) \in A(D)$. In the latter case it is clear that $d(x_1,x_0), d(x_2,x_1) \leq 2$ and we are done. In the former case, let us apply the induction hypothesis to the cycle $\mathscr{C}' = (x_0,x_2) \cup (x_2\mathscr{C}x_0)$, which has length $n-1$, to obtain two arcs with the desired condition in $A(\mathscr{C}')$. Since $A\big(\mathscr{C}' - (x_0,x_2)\big) \subset A(\mathscr{C})$, if the two arcs obtained from the induction hypothesis are different from (x_0,x_2) we are done. Let us assume that one of the arcs is (x_0,x_2). Hence, $d(x_2,x_0) \leq 2$. If $d(x_2,x_0) = 1$, it is the case we have already analyzed. So $d(x_2,x_0) = 2$. Let $v \in V(D)$ be a vertex such that $(x_2,v),(v,x_0) \in A(D)$. If $v = x_1$, then the arcs $(x_0,x_1),(x_1,x_2) \in A(D)$ are symmetrical and we are done. If not, we have that $(x_1,x_2),(x_2,v) \in A(D)$. Since D is quasi-transitive, $(x_1,v) \in A(D)$, which implies that $d(x_1,x_0) \leq 2$; or $(v,x_1) \in A(D)$, which implies that $d(x_2,x_1) \leq 2$. In either case we reach de desired conclusion. ■

Lemma 8.6.5. *If D is an infinite quasi-transitive digraph without infinite outward paths, then for every $S \subseteq V(D)$ there exists a vertex $v \in S$ such that, if $u \in S$ and $(v,u) \in A(D)$, then $d_D(u,v) \leq 2$.*

Proof. Let S be a subset of $V(D)$ and suppose that there is no vertex in S with the desired property. Then for every $x \in S$ there exists a vertex $y \in S$ such that $(x,y) \in A(D)$ and $d_D(y,x) > 2$. Since every vertex in S has at least one neighbor in S with this property and there are not infinite outward paths in D, then there must exist a directed cycle $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ such that $V(\mathscr{C}) \subseteq S$ and $d_D(x_i, x_{i-1}) > 2$ with $1 \leq i \leq n$. This contradicts Lemma 8.6.4. Since the contradiction arose from the assumption that there is no vertex in S with the desired property, then there must exist at least one such vertex. ■

Lemma 8.6.6. *Let D be an infinite quasi-transitive digraph without infinite outward paths. If D has non-empty 3-semikernel, then D has 3-kernel.*

Proof. Since D has a non-empty 3-semikernel, by Lemma 8.2.1 we can consider S be a maximal 3-semikernel of D. If S is 2-absorbent, then S is the desired 3-kernel. If S is not 2-absorbent, we can consider $T \subseteq V(D)$, the set of vertices not 2-absorbed by S. It follows from Lemma 8.6.5 the existence of a vertex $v \in T$ such that if $u \in T$ and $(v, u) \in A(D)$ then $d(u, v) \leq 2$. As a consequence of Lemma 8.6.1, whenever $u \in T$ and $d(v, u) = 2$ it follows that $(u, v) \in A(D)$. Besides, $d(v, S) \geq 3$ and $d(S, v) \geq 3$, as a matter of fact, in virtue of Lemma 8.6.1 we have that $d(v, S) = 3 = d(S, v)$. So $S \cup \{v\}$ is a 3-independent subset of $V(D)$. By the choice of v and since S is a 3-semikernel, $S \cup \{v\}$ fulfills the second property of 3-semikernel, so it is a 3-semikernel properly containing S, contradicting the maximality of S. Therefore, S is 2-absorbent and thus a 3-kernel. ■

Lemma 8.6.7. *Let D be an infinite quasi-transitive strong digraph and k, l be a pair of integers such that $k \geq 4$, $3 \leq l \leq k-1$. If D has a (k, l)-semikernel, then D has a (k, l)-kernel.*

Proof. Since D has a non-empty (k, l)-semikernel, by Lemma 8.2.1 we can consider S to be a maximal (k, l)-semikernel of D. If S is l-absorbent, then S is the (k, l)-kernel we have been looking for. If it is not l-absorbent, let $v \in V(D)$ be a vertex not l-absorbed by S. Since D is strong, there must exist a vS-directed path of length greater than or equal to $l+1 \geq 4$. Let $s \in S$ be the final vertex in such directed path. In virtue of Lemma 8.6.1 we have that $(s, v) \in A(D)$, and since S is a (k, l)-semikernel of D, a vS-directed path of length less than or equal to l must exist in D, which results in a contradiction because we choose v as a vertex not l-absorbed by S. Hence S is l-absorbent and it turns out to be the desired (k, l)-kernel. ■

Lemma 8.6.8. *Let D be an infinite quasi-transitive digraph without infinite outward paths and k, l be a pair of integers such that $k \geq 4$ and $3 \leq l \leq k-1$*

or $k = 3$ and $l = 2$. Then there is a vertex $v \in V(D)$ such that $\{v\}$ is a (k, l)-semikernel of D.

Proof. It suffices to consider $S = V(D)$ and an application of Lemma 8.6.5 to find a vertex $v \in V(D)$ such that whenever $u \in V(D)$ and $(v, u) \in A(D)$ then $d(u, v) \leq 2$. It follows from Lemma 8.6.1 that the set $\{v\}$ is a (k, l)-semikernel. ■

We finish this section with our two principal results.

Theorem 8.6.9. *If D is an infinite quasi-transitive strong digraph without infinite outward paths, then D has a k-kernel for every $k \in \mathbb{N}$, $k \geq 3$.*

Proof. It is an immediate consequence of Lemmas 8.6.6, 8.6.7 and 8.6.8. ■

Theorem 8.6.10. *If D is an infinite quasi-transitive digraph without infinite outward paths, then D has a $(k.l)$-kernel for every pair of integers k, l such that $k \geq 4$ and $3 \leq l \leq k - 1$ or $k = 3$ and $l = 2$.*

Proof. In virtue of Theorem 8.6.9 every terminal strong component of D has a (k, l)-kernel. If D has $\{D_i\}_{i \in I}$ as set of strong terminal components (where I can be an infinite set), it suffices to choose a (k, l)-kernel N_i for every D_i. The union $N = \bigcup_{i \in I} N_i$ is a (k, l)-kernel of D. Since $N_i \subseteq D_i$ and D_i is a terminal component for every $i \in I$ it is clear that N is k-independent. In addition, for each $i \in I$, if $v \in V(D_i)$, then v is l-absorbed by N_i. So it suffices to prove that if v is in a non-terminal strong component then it is l-absorbed by some vertex in N. But by Lemma 8.6.2 $D^\star$ is an acyclic transitive digraph. Since D does not have infinite outward paths, $D^\star$ does not have infinite outward paths. Thus, by Lemma 8.6.3 every non-terminal strong component of D is 1-absorbed by a terminal strong component of D in $D^\star$. Therefore, by Lemma 8.6.2 every vertex in a non-terminal strong component is 1-absorbed by every vertex in at least one terminal strong component of D, and then is l-absorbed by N. So N is a k-independent, l-absorbent set, and thus the desired (k, l)-kernel. ■

8.7 Pretransitive digraphs

The main result of this section is proved with the aid of Lemma 8.2.2 in a very similar way that the finite version[4] is proved.

[4]Lemma 3.2.8.

Lemma 8.7.1. *Let $k \geq 2$ be an integer. If D is a right pretransitive infinite strong digraph such that every directed triangle is symmetrical, then every vertex of D is a k-semikernel of D.*

Idea of Proof Let $k \geq 2$ be an integer. Let $v \in V(D)$ be any vertex, consider $w \in V(D)$ such that there exists a vw-directed path of length less than or equal to $k-1$ and let $\mathscr{C} = (v = v_0, v_1, \ldots, v_n = w)$ be a vw-directed path of minimum length. For every such w, $\mathscr{C}$ is a directed path of finite length. Also, since D is strong, $d(w, v) \in \mathbb{N}$. So, the same argument used in the finite version of this theorem can be used to prove that $d(w, v) \leq k-1$. ■

Theorem 8.7.2. *If D is an infinite right pretransitive strong digraph such that every directed triangle is symmetrical, then D has a k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. It follows from Lemmas 8.2.2 and 8.7.1. ■

The analogous results for left pretransitive digraphs can be easily obtained like in finite digraphs by means of dualization.

Lemma 8.7.3. *If D is an infinite left pretransitive strong digraph such that every directed triangle is symmetrical, then $\{v\}$ is a k-semikernel of D for every $v \in V(D)$.*

Proof. Exactly like the finite case[5]. ■

Theorem 8.7.4. *If D is an infinite left pretransitive strong digraph such that every directed triangle is symmetrical, then D has a k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Proof. The result follows from Lemmas 8.2.2 and 8.7.3. ■

In Section 3.2, it is also proved that if D is a right pretransitive digraph such that every directed triangle is symmetrical, then D has a k-kernel for every $k \in \mathbb{N}$, $k \geq 2$. However, the proof of this fact was done by induction on $|V(D^\star)|$, which can be uncountable if D is infinite. So, since we were unable to find a proof for infinite digraphs, we state the following conjecture.

[5] Lemma 3.2.10.

Conjecture 8.7.5. *If D is a right pretransitive infinite digraph such that every directed triangle is symmetrical and does not contain infinite outward paths, then D has a k-kernel for every $k \in \mathbb{N}$, $k \geq 2$.*

Obviously D must have a restriction on its infinite outward paths because, since the digraph $D^\sharp$ is transitive, is both right/left pretransitive and every directed triangle is symmetrical (provided that it has none) but we already have observed that it does not have k-kernel for any $k \in \mathbb{N}$. One remarkable observation is that, in Theorems 8.7.2 and 8.7.4 this condition is not necessary, the digraphs may contain infinite outward paths and the results remain valid.

8.8 κ-strongly connected digraphs

The results of this section have somewhat technical proofs and are direct generalizations of the respective finite versions. Once again, the main tool to change from finite to infinite digraphs is Lemma 8.2.2.

The proofs of some results will be omitted for the sake of brevity since, as many of them are local properties, the proof is just like in finite digraphs and can be consulted in Section 6.2.

Lemma 8.8.1. *Let D be a σ-strong digraph with circumference l, $k \geq 2$ a fixed integer and $\mathscr{C} = (x_0, x_1 \ldots, x_m)$ a directed path of length m. If $m = q\sigma + r$ where q and r are given by the division algorithm, then:*

1. *If $r = 0$, then $d(x_m, x_0) \leq (l - \sigma)q$.*
2. *If $r > 0$, then $d(x_m, x_0) \leq (l - r) + (l - \sigma)\left\lfloor \frac{m-1}{\sigma} \right\rfloor$.*

Proof. The proof is by induction on q, which is finite. So the proof for the finite case remains valid. ■

The proof of the following theorem is also like the one of the finite version, we will reproduce it to emphasize that it is also valid for infinite digraphs.

Lemma 8.8.2. *Let D be a σ-strong digraph with circumference l, then for every $v \in V(D)$, $\{v\}$ is a $(k, (l-1) + (l-\sigma)\left\lfloor \frac{k-2}{\sigma} \right\rfloor)$-semikernel for every integer $k \geq 2$.*

Proof. Let $k \geq 2$ and $v \in V(D)$ be fixed and let $\mathscr{C} = (v = x_0, x_1, \ldots, x_m)$ be a vx_m-directed path of length $m \leq k-1$. In virtue of Lemma 8.8.1 $d(x_m, v) \leq (l-1)+(l-\kappa)\lfloor\frac{m-1}{\kappa}\rfloor \leq (l-1)+(l-\kappa)\lfloor\frac{k-2}{\kappa}\rfloor$ and then $\{v\}$ fulfills the second $(k, (l-1)+(l-\kappa)\lfloor\frac{k-2}{\kappa}\rfloor)$-semikernel condition. ■

Just as in the finite case, we are ready to prove the main theorem of the section.

Theorem 8.8.3. *Let D be a σ-strong digraph with circumference l. Then D has a $(k, (l-1)+(l-\sigma)\left\lfloor\frac{k-2}{\sigma}\right\rfloor)$-kernel for every integer $k \geq 2$.*

Proof. It follows immediately from Lemmas 8.2.2 and 8.8.2. ■

At this point, it comes as no surprise that the same consequences of Theorem 6.2.3 can be derived from Theorem 8.8.3.

8.9 Locally in/out-semicomplete digraphs

This section is very brief. We only remark which of the existing results for locally in/out-semicomplete digraphs remain valid in the infinite case. Once again in this section, Lemma 8.2.2 makes the proofs for the infinite case possible.

The following couple of lemmas remain valid from the finite case.

Lemma 8.9.1. *Let $l \geq 1$ be an integer, D a locally out-semicomplete infinite digraph and $(x_0, x_1, \ldots, x_n)$ is a x_0x_n-directed path of length $n \leq l$. If $v_0 \in V(D)$ is such that $(x_0, v_0) \in A(D)$ and $(x_n, v_0) \notin A(D)$, then $d(v_0, x_n) \leq l$.*

Proof. The proof of the finite case remains valid since we only work with the vertices of the x_0x_n-directed path. ■

Lemma 8.9.2. *Let D be a locally out-semicomplete infinite digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$ then $d(v, u) \leq l$. Then $\{u\}$ is a (k, l)-semikernel for every integer $k \geq 2$ and every $u \in V(D)$.*

Proof. Consider a vertex $w \in V(D)$ such that a vw-directed path exists in D. It is proved by induction on $d(v, w)$ that $d(w, v) \leq l$. The proof of the finite case remains valid. ■

And have the following theorem as a direct consequence.

Theorem 8.9.3. *Let D be a locally out-semicomplete infinite digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a (k, l)-kernel for every integer $k \geq 2$.*

Proof. Is a direct consequence of Lemmas 8.2.2 and 8.9.2. ■

The two following corollaries are straightforward derived from Theorem 8.9.3.

Corollary 8.9.4. *Let D be a locally out-semicomplete infinite digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a k-kernel for every integer $k \geq l + 1$.*

Corollary 8.9.5. *Let D be a locally out-semicompete infinite strong digraph with circumference $l + 1$, then D has a (k, l)-kernel for every integer $k \geq 2$.*

The results of Lemma 8.9.2 and Theorem 8.9.3 can also be dualized by means of Remark 6.3.2. The principal result obtained by means of dualization is now stated.

Theorem 8.9.6. *Let D be an infinite locally out-semicomplete digraph such that, for a fixed integer $l \geq 1$, whenever $(u, v) \in A(D)$, then $d(v, u) \leq l$. Then D has a (k, l)-kernel for every integer $k \geq 2$.*

One of the main results obtained in Section 6.3 states that every locally out-semicomplete digraph with circumference $l+1$ has a (k, l)-kernel for every $k \geq 2$. This is, in the finite case the condition of strong connectivity can be dropped in Corollary 8.9.5. Nonetheless, the proof of the finite case was done by induction on $|V(D)|$, and we were unable to find a proof that works for the infinite case. In the same section, we conjectured that if D is a digraph with circumference l, then D has a k-kernel for every $k \leq l$. For the infinite case it seems that there should also be a condition on the infinite outward paths. Proving the result for some families such as locally out-semicomplete digraphs would be a good start point.

8.10 k-transitive and k-quasi-transitive digraphs

Since we are less interested in the structure of infinite k-transitive and k-quasi-transitive digraphs than in the results that can be generalized from Chapter 5, we will omit the family of k-path-transitive digraphs. Thus, we begin our results with a simple technical lemma used to work directly with k-transitive digraphs.

Lemma 8.10.1. *Let $k \geq 2$ be an integer, D a k-transitive infinite digraph and $u, v \in V(D)$. If there exists a uv-directed path in D, then $d(u,v) \leq k-1$.*

Proof. Let $u, v \in V(D)$ be arbitrary distinct vertices and let $\mathscr{C} = (u = x_0, x_1, \ldots, x_n = v)$ be a uv-directed path. We will prove by induction on n that $d(u,v) \leq k-1$. If $n \leq k-1$ then we are done. So let us assume that $n \geq k$, then, by the k-transitivity of D, since $x_0 \mathscr{C} x_k$ is a directed path of length k in D, $(x_0, x_k) \in A(D)$, so $(x_0, x_k) \cup x_k \mathscr{C} x_n$ is a uv-directed path of length strictly less than n, we can derive from the induction hypothesis that $d(u,v) \leq k-1$. The result follows from the principle of mathematical induction. ■

The following theorem generalize the result of Theorem 8.4.2.

Theorem 8.10.2. *Let D be an infinite k-transitive digraph such that every infinite outward path $(x_i)_{i \in \mathbb{N}}$ has an arc of the form (x_j, x_i) with $i < j$, then D has a (n,m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k-1$. Moreover, every (n,m)-kernel of D consists in choosing one vertex from every terminal component of D.*

Proof. This proof is very similar to the proof of Theorem 8.4.2, so we will use the same notation for the relation $\preccurlyeq$. It suffices to prove that if C_0 is a strong component of D then there exists a terminal component T of D such that $\mathsf{C}_0 \preccurlyeq \mathsf{T}$. This is because, since D is a k-transitive digraph, by Lemma 8.10.1 every vertex in C_0 will be $(k-1)$-absorbed by every vertex in T. Also, if we choose one vertex in every terminal strong component of D, the set of the chosen vertices will be k-independent for every integer $k \geq 2$, because every vertex is in a distinct terminal component. So, every set consisting of one vertex from every terminal component of D will be n-independent and $(k-1)$-absorbent, and thus, m-absorbent, for every pair of integers n, m such that $n \geq 2$, $m \geq k-1$.

We will proceed by contradiction. Assume that for every $\mathsf{C} \in \mathfrak{C}$ such that $\mathsf{C}_0 \preccurlyeq \mathsf{C}$ there exists $\mathsf{C}' \in \mathfrak{C}$ such that $\mathsf{C}' \neq \mathsf{C}$ and $\mathsf{C} \preccurlyeq \mathsf{C}'$. In virtue of the Axiom of Choice we can build a sequence $(\mathsf{C}_i)_{i\in\mathbb{N}}$ satisfying $\mathsf{C}_0 \preccurlyeq \mathsf{C}_1$ and, for every $i < j$, $\mathsf{C}_i \neq \mathsf{C}_{i+1}$ and $\mathsf{C}_i \preccurlyeq \mathsf{C}_j$. As in the proof of Theorem 8.4.2, we want to obtain an infinite outward in D from this sequence. Let us choose a vertex $x_1 \in V(\mathsf{C}_1)$. Since D is k-transitive, using the Axiom of Choice we can recursively construct an infinite outward path in D in the following way: The first vertex is x_1; If x_n has been chosen in $V(\mathsf{C}_i)$ for some $i \in \mathbb{N}$, choose x_{n+1} as any vertex such that $(x_n, x_{n+1}) \in A(D)$ and $x_{n+1} \in V(\mathsf{C}_j)$ with $i < j$. We affirm that such vertex exists because D is a k-transitive digraph and $(\mathsf{C}_i)_{i\in\mathbb{N}}$ is an infinite chain in the partial order $\preccurlyeq$. Clearly, x_n can reach (at a finite distance) C_{i+r} for every $r \in \mathbb{N}$, so we can choose a vertex $x_{n+1} \in V(\mathsf{C}_j)$ for $i < j$ such that a $x_n x_{n+1}$-directed path of length $\equiv 0 \mod k^2$ exists. Thus, the k-transitivity of D implies that $(x_n, x_{n+1}) \in A(D)$.

Now, we have an infinite outward path $\{x_i\}_{i\in\mathbb{N}}$. Moreover, if $i < j$, $(x_j, x_i) \notin F(D)$. This is because, by the construction of $(x_i)_{i\in\mathbb{N}}$, if $i < j$, then $x_i \in V(\mathsf{C}_r)$ and $x_j \in V(\mathsf{C}_s)$ for some $r < s$. Then we would have that $\mathsf{C}_s \preccurlyeq \mathsf{C}_r$ for some $r < s$ and it would follow from the antisymmetry of $\preccurlyeq$ that $\mathsf{C}_r = \mathsf{C}_s$. By the construction of $(\mathsf{C}_i)_{i\in\mathbb{N}}$, we know that $s \neq r+1$, but this implies the existence of a directed cycle $(\mathsf{C}_r, \mathsf{C}_{r+1}, \dots, \mathsf{C}_s, \mathsf{C}_r)$ in $D^\star$, which results in a contradiction because $D^\star$ is acyclic. Therefore $(x_i)_{i\in\mathbb{N}}$ is an infinite outward path in D such that $(x_j, x_i) \notin F(D)$ for each $i < j$, which results in a contradiction. From this point we can conclude as in the proof of Theorem 8.4.2. ■

To work with k-quasi-transitive digraphs, we will use a very similar technique that the one we used to work with quasi-transitive digraphs, using local properties to prove the existence of (n,m)-semikernels in strong k-quasi-transitive digraphs. Then we prove that such (n,m)-semikernels are indeed (n,m)-kernels. The three following lemmas originally stated in Section 5.4 for finite digraphs, trivially remain valid for the infinite case. Although they are only a tool for our immediate concern of finding (n,m)-kernels, we think that they are very interesting on their own.

Lemma 8.10.3. *Let $k \in \mathbb{N}$ be an even natural number, D a k-quasi-transitive infinite digraph and $u, v \in V(D)$ such that a uv-directed path exists. Then:*

1. *If $d(u,v) = k$, then $d(v,u) = 1$.*

2. *If $d(u,v) = k+1$, then $d(v,u) \leq k+1$.*

3. *If $d(u,v) \geq k+2$, then $d(v,u) = 1$*

Lemma 8.10.4. *Let $k \in \mathbb{N}$ be an odd natural number, D a k-quasi-transitive infinite digraph and $u, v \in V(D)$ such that a uv-directed path exists. Then:*

1. *If $d(u,v) = k$, then $d(v,u) = 1$.*

2. *If $d(u,v) = k+1$, then $d(v,u) \leq k+1$.*

3. *If $d(u,v) = n \geq k+2$ with n odd, then $d(v,u) = 1$*

4. *If $d(u,v) = n \geq k+3$ with n even, then $d(v,u) \leq 2$*

It can be observed that k-quasi-transitive digraphs have a "better" behavior when k is an even integer. This fact will have important consequences to our concern.

Lemma 8.10.5. *Let D be a k-quasi-transitive digraph. If $A \neq B$ are strong components of D such that there exists an AB-directed path in D, then $A \overset{k-1}{\rightarrow} B$.*

The following lemma resembles Lemma 8.6.4.

Lemma 8.10.6. *Let D be a (possibly infinite) k-quasi-transitive digraph. Then, for every directed cycle $\mathscr{C}$ of D, there are at least r arcs of $\mathscr{C}$, say $(u_i, v_i) \in A(\mathscr{C})$ such that $d(v_i, u_i) \leq k$, $i \in \{1, 2, \ldots, r\}$, $r = \min\{k, \ell(\mathscr{C})\}$.*

Proof. By induction on $\ell(\mathscr{C})$. If $\ell(\mathscr{C}) \leq k+1$, the result is clear. Let $\mathscr{C} = (x_0, x_1, \ldots, x_n = x_0)$ be a directed cycle of length $n \geq k+2$ in D. Since D is k-quasi-transitive and $(x_0, x_1, \ldots, x_k) \in A(D)$, then $(x_0, x_k) \in A(D)$ or $(x_k, x_0) \in A(D)$. In the latter case it is clear that $d(x_i, x_{i-1}) \leq k$ for $1 \leq i \leq k$ and we are done. In the former case, let us apply the induction hypothesis to the cycle $\mathscr{C}' = (x_0, x_k) \cup (x_k \mathscr{C} x_0)$, which has length $n - k + 1 < n$, to obtain k arcs with the desired condition in $A(\mathscr{C}')$. Since $A\big(\mathscr{C}' - (x_0, x_k)\big) \subset A(\mathscr{C})$, if the k arcs obtained from the induction hypothesis are different from (x_0, x_k) we are done. Let us assume that one of the arcs is (x_0, x_k). Hence, $d(x_k, x_0) \leq k$. If $d(x_k, x_0) = 1$, it is the case we have already analyzed. So $d(x_k, x_0) > 1$. Let $\mathscr{D} = (x_k = y_0, y_1, \ldots, y_s = x_0)$ be a $x_k x_0$-directed path with $s \leq k$. If $y_1 = x_1$, then $d(x_k, x_{k-1}) \leq k$ and we are done. Let us assume that $y_1 \neq x_1$. Since D is k-quasi-transitive and $x_1 \mathscr{C} x_k \cup (x_k, y_1)$ is a $x_1 y_1$-directed path of length k, it follows that $(y_1, x_1) \in A(D)$, which implies that $d(x_k, x_{k-1}) \leq k$ because $(x_k, y_1, x_1) \cup (x_1 \mathscr{C} x_{k-1})$ is a $x_k x_{k-1}$ directed path of length k; or $(x_1, y_1) \in A(D)$, which implies that $d(x_1, x_0) \leq k$ because $(x_1, y_1) \cup (y_1 \mathscr{D} x_0)$ is a $x_k x_{k-1}$ directed path of length less than or equal to k. In either case we reach de desired conclusion. ■

Lemma 8.10.7. *Let $k \geq 2$ be an integer and D be an infinite k-quasi-transitive digraph such that for every infinite outward path $(x_i)_{i \in \mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$. Then there exists a vertex $v \in V(D)$ such that whenever $(v, u) \in A(D)$, then $d(u, v) \leq k$.*

Proof. We will proceed by contradiction. Let us assume that for every vertex $v \in V(D)$ there exists an arc $(v, u) \in V(D)$ such that $d(u, v) \geq k+1$. Then, since the subdigraph H of D induced by these arcs has $\delta^+(H) \geq 1$, then we have two possibilities. There exist a directed cycle $\mathscr{C}$ in D such that for every arc $(v, u) \in A(\mathscr{C})$, $d(u, v) \geq k+1$, which clearly results in a contradiction by Lemma 8.10.6. Or there exists an infinite outward path $\mathscr{C} = (x_i)_{i \in \mathbb{N}}$ such that for every arc $(x_i, x_{i+1}) \in A(\mathscr{C})$, $d(x_{i+1}, x_i) \geq k+1$. But by hypothesis there is an arc $(x_j, x_i) \in A(D)$ with $i < j$. So, $(x_i \mathscr{C} x_j) \cup (x_j, x_i)$ is a directed cycle in D. By Lemma 8.10.6, at least one arc $(x_i, x_{i+1}) \in A(\mathscr{C})$ is such that $d(x_{i+1}, x_i) \leq k$, which results in a contradiction. ■

Lemma 8.10.8. *Let $k \geq 2$ be an even integer and let D be an infinite k-quasi-transitive digraph such that for every infinite outward path $(x_i)_{i \in \mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$. Then D has a $(k+2)$-semikernel consisting in a single vertex.*

Proof. By Lemma 8.10.7 we can choose a vertex $v \in V(D)$ such that for every arc $(v, u) \in A(D)$, $d(u, v) \leq k$. So let $u \in V(D)$ be a vertex such that $2 \leq d(v, u) \leq k+1$. It can not happen that $d(u, v) \geq k+2$, because this would imply by Lemma 8.10.3 that $d(v, u) = 1$, but $2 \leq d(v, u)$, so $d(u, v) \leq k+1$ and thus $\{v\}$ is a $(k+2)$-semikernel of D. ■

Lemma 8.10.9. *Let $k \geq 3$ be an odd integer and D be an infinite k-quasi-transitive digraph such that for every infinite outward path $(x_i)_{i \in \mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$ and such that at least one vertex $v \in S = \{u \in V(D) \mid (u, w) \in A(D)$ implies that $d(w, u) \leq k+1\}$ is such that whenever $d(v, x) = 2$ then $d(x, v) \leq k+1$. Then $\{v\}$ is a $(k+2)$-semikernel for D.*

Proof. By Lemma 8.10.7 the set S is non empty and also there is a vertex $v \in S$ such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$. So let $u \in V(D)$ be a vertex such that $3 \leq d(v,u) \leq k+1$. It can not happen that $d(u,v) \geq k+2$, because this would imply, by Lemma 8.10.3, that $d(v,u) \leq 2$, but $3 \leq d(v,u)$, so $d(u,v) \leq k+1$ and thus $\{v\}$ is a $(k+2)$-semikernel of D. ■

Lemma 8.10.10. *Let D be an infinite k-quasi-transitive strong digraph. If D has a non-empty $(k+2)$-semikernel S, then S is a $(k+2)$-kernel of D.*

Proof. Let $S \subseteq V(D)$ be a $(k+2)$-semikernel for D and $N^-_{k+1}(S)$ the set of all vertices in D which are $(k+1)$-absorbed by S. Define $T := V(D) \setminus \big(S \cup N^-_{k+1}(S)\big)$. If $T = \varnothing$, then S is a $(k+2)$-kernel of D. If $T \neq \varnothing$, then we can consider a vertex $v \in T$ which, by the definition of T, is not $(k+1)$-absorbed by S, but since D is strong, there exists a vS-directed path. Let $u \in S$ be a vertex such that $d(v,u) = d(v,S)$, then $d(v,u) \geq k+2$ because $v \notin N^-_{k+1}(S)$, but from Lemmas 8.10.3 and 8.10.4 it can be derived that $d(u,v) \leq 2$. This fact, altogether with the second $(k+2)$-semikernel condition implies that $v \in N^-_{k+1}(S)$ which results in a contradiction. Since the contradiction arises from assuming that $T \neq \varnothing$, we can conclude that $T = \varnothing$ and then S is a $(k+2)$-kernel for D. ■

Theorem 8.10.11. *Let $k \geq 2$ be an even integer and let D be a k-quasi-transitive strong digraph such that for every infinite outward path $(x_i)_{i\in\mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$. Then D has an (n,m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k+1$.*

Proof. By Lemma 8.10.8, D has a $(k+2)$-semikernel N consisting in a single vertex, but by Lemma 8.10.10, N is indeed a $(k+2)$-kernel of D. But since N has only one vertex, then N is n-independent for every $n \geq 2$, and since it is $(k+1)$-absorbent, then it is m-absorbent for every $m \geq k+1$, so N is an (n,m)-kernel for every pair of integers n, m such that $n \geq 2$, ■

Theorem 8.10.12. *Let $k \geq 2$ be an even integer and let D be an infinite k-quasi-transitive digraph such that for every infinite outward path $(x_i)_{i\in\mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$. Then D has an (n,m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k+1$.*

Proof. Once again it suffices to prove that if C_0 is a strong component of D then there exists a terminal component T of D such that $\mathsf{C}_0 \preccurlyeq \mathsf{T}$. This is because, in virtue of Lemmas 8.10.5 and 8.10.11, if we choose a subset $N \subseteq V(D)$ consisting in an (n,m)-kernel for every terminal component of D, this set will be n-independent for every $n \in \mathbb{Z}^+$ because every such (n,m)-kernel consist in a single vertex and terminal components are path-independent. Also N will be $(k+1)$-absorbent because every (n,m)-kernel is inside its component and every vertex of D not in a terminal component is $(k-1)$-absorbed by every vertex in some terminal component.

Let us observe that the same proof of Theorem 8.10.2 works for k-quasi-transitive digraphs. The only part of the proof where the hypothesis of being k-transitive is used is when we recursively construct an infinite outward path in D (using the Axiom of Choice). When $x_n \in V(\mathsf{C}_i)$ has been chosen and we choose x_{n+1} as any vertex such that $(x_n, x_{n+1}) \in A(D)$ and $x_{n+1} \in V(\mathsf{C}_j)$ with $i < j$.

Using k-quasi-transitivity instead of k-transitivity, we affirm that such vertex exists because $(\mathsf{C}_i)_{i\in\mathbb{N}}$ is an infinite chain in the partial order $\preccurlyeq$. Clearly, x_n can reach (at a finite distance) C_{i+r} for every $r \in \mathbb{N}$. Moreover, for every C_j such that $i < j$ and $x \in V(\mathsf{C}_j)$ we have that $d(x_n, x) \leq k-1$. Otherwise, by Lemmas 8.10.3 and 8.10.4 we would have the existence of a xx_n-directed path, but there are not $\mathsf{C}_j\mathsf{C}_i$-directed paths for $i < j$ because we would have a cycle in $D^\star$. Now, let l be an integer greater than i and $y \in V(\mathsf{C}_l)$ an arbitrary vertex. Then $d(x_n, y) \leq k-1$. If $d(x_n, y) = 1$, then $y = x_{n+1}$. If not, let $\mathscr{C}$ be an $x_n y$-directed path of minimum length and consider an arbitrary vertex $z \in V(\mathsf{C}_j)$ for some $j \geq l$ such that a yz-directed path $\mathscr{D}$, internally disjoint with $\mathscr{C}$ and of length $k - d(x_n, y)$, exists (such vertex always exists since $(\mathsf{C}_n)_{n\in\mathbb{N}}$ is an infinite sequence). Clearly $\mathscr{C} \cup \mathscr{D}$ is a $x_n z$-directed path of length k. By the k-quasi-transitivity of D, $(x_n, z) \in A(D)$ or $(z, x_n) \in A(D)$, but since $z \in V(\mathsf{C}_j)$ with $i < j$, it follows that $(x_n, z) \in A(D)$, and thus we can choose $x_{n+1} = z$. So, the infinite outward path can be constructed and the rest of the proof is just like the aforementioned proof. ∎

Theorem 8.10.13. *Let $k \geq 3$ be an odd integer and let D be an infinite k-quasi-transitive strong digraph such that for every infinite outward path $(x_i)_{i\in\mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$ and such that at least one vertex $v \in S = \{u \in V(D) \big| (u,w) \in A(D) \text{ implies that } d(w,u) \leq k+1\}$ is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$. Then D has an (n,m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k+1$.*

Proof. It is analogous to Theorem 8.10.11. ∎

Theorem 8.10.14. *Let $k \geq 3$ be an odd integer and let D be an infinite k-quasi-transitive digraph such that for every infinite outward path $(x_i)_{i \in \mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$ and such that at least one vertex $v \in S = \{u \in V(D) \big| (u,w) \in A(D)$ implies that $d(w,u) \leq k+1\}$ is such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$. Then D has an n-kernel for every $n \geq k+2$.*

Proof. It is analogous to Theorem 8.10.12. ■

As a final comment, we would like to point out again that, in virtue of Lemma 8.10.7, the set $S = \{u \in V(D) \big| (u,w) \in A(D)$ implies that $d(w,u) \leq k+1\}$ in Lemma 8.10.9 is always non-empty. So, it would suffice to prove that there is a vertex $v \in S$ such that whenever $d(v,x) = 2$ then $d(x,v) \leq k+1$ for every $x \in V(D)$ to have as a consequence that, for every integer $k \geq 2$, every k-quasi-transitive digraph has a (n,m)-kernel for every pair of integers n, m such that $n \geq 2$, $m \geq k+1$.

From the various properties that k-quasi-transitive digraphs have proved to have, we state the following conjecture.

Conjecture 8.10.15. *If $k \geq 3$ is an odd integer and D is a k-quasi-transitive strong digraph such that for every infinite outward path $(x_i)_{i \in \mathbb{N}}$ there exists an arc (x_j, x_i) with $i < j$, then D has a non-empty $(k+2)$-kernel.*

Recall that in the finite case in Section 5.5 we proved, by means of a structural characterization of 3-quasi-transitive digraphs, that every 3-quasitransitive digraph has a 4-kernel. Since the structural characterization works only for finite digraphs, an analogous for infinite digraphs could not be obtained. Nonetheless, we believe that the existence of a $(k+2)$-kernel can be replaced by the existence of a $(k+1)$-kernel in Conjecture 8.10.15. Once again, a good starting point would be to prove the result for infinite 3-quasi-transitive digraphs.

Summary

When the development of the present work began, few general families of digraphs where known to possess a (k,l)-kernel. Acyclic digraphs, strong cyclically k-partite digraphs, symmetrical digraphs, transitive digraphs, semicomplete digraphs and some superdigraphs of the directed cycle and the directed path were almost all of those families.

In the present work, eight new familes where studied. Maybe it is not the case that all the digraphs in the eight families have a (k,l)-kernel for arbitrary values of k and l. But sufficient conditions were given for all these families to have a (k,l)-kernel for many values of k and l. Moreover, new proof techniques that may be used to obtain further results were introduced. Also, various open problems and conjectures were proposed thrhough the text. Most of them concerning (k,l)-kernels, but also some of them are about the structure of new families of digraphs. In fact, new families of digraphs were defined and studied and some new definitions were given.

In the Preface, we mentioned that this work was intended to fill the gap between kernels and (k,l)-kernels, since there were many results about kernels that seemed generalizable to (k,l)-kernels but no such generalizations were known. Sadly, the gap has not been filled. Nonetheless, a narrow bridge has been built, based on the few results that we were able to obtain and the open problems and conjectures that we proposed. Hopefully, this narrow bridge will be solid enough to be widened by future research on the field, and maybe one day our knowledge on (k,l)-kernels will be as broad as our knowledge on kernels.

As a final thought, (k,l)-kernels seem to have a lot of potential for real-life applications. The main reasons that prevent us from using them is the difficulty to determine if a given digraph has a (k,l)-kernel and to find the (k,l)-kernel. Maybe this work can be inscribed within Pure Mathematics, but the intention of most of the results were to find an easy to verify sufficient condition for a digraph to have a (k,l)-kernel and to describe as much as possible the structure of such (k,l)-kernel. Maybe, hopefully sooner than later, some of the results presented in this work will be helpful to implement optimal solutions to some of the many problems that humankind faces everyday.

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