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STOCHASTIC OPTIMAL CONTROL, HJB EQUATIONS AND SOME APPLICATIONS TO RISK THEORY

T E S I S

QUE PARA OBTENER EL GRADO ACADÉMICO DE

DOCTOR EN CIENCIAS

P R E S E N T A

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Por fin llegó... primero quiero dar las gracias a las personas que nunca, nunca me han faltado: mis padres y mi hermano. Gracias de todo corazón, sin ustedes esto no hubiera sucedido, de verdad.

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Stochastic optimal control, HJB equations and some applications to risk theory.

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Introduction

The Theory of Control has presented a great development in the last sixty years thanks to its application to a wide variety of interesting problems: risk theory, consumption problems, production control, investment's portfolios, exit problems, among others. See [?], [?], [?], [?], [?] and the reference therein.

In a broad sense, control is the action responsible for evolution of a process applied in order to achieve a desired goal. Control theory has many approaches depending on the dynamic of the controlled process: The deterministic case next to the calculus of variations, has a long history starting with the brachistochrone problem solved by Johann Bernoulli nearly 300 years ago (see [?] for a general presentation). The stochastic case has its beginning in the late 1950's and early 1960's, however its development has been very intensive since then (see [?], [?] or [?]). Two types of problems are generated depending on the considerations made on time: discrete or continuous. For a survey on the discrete theory see [?], [?], [?] or [?].

In this work we deal with stochastic control problems in continuous case. In the deterministic case, the evolution of a system is in general modeled with a differential equation of the form

$$\frac{dX(s)}{ds} = b(X(s))$$

Many interesting problems present some features which are random or simply unknown for the observer. One way of modeling this is by considering an Ito's jump diffusion in $\mathbb{R}^d$,

$$dX(s) = b(X(s))ds + \sigma(X(s))dW(s) + \int_{\mathbb{R}^n} \gamma(X(s), z)M(ds, dz)$$

where W is a Brownian motion and $M(\cdot)$ represents a Poisson random measure generated by the jumps of a Levy process (in this work we only consider the special case of a compound Poisson process).

The control theory assumes that the dynamic of the system can be changed via a *control process* $A = \{A(s)\}$. Generally, the control process are selected over a class of predictable stochastic processes with some integrability properties (denoted by $\mathcal{A}$). In this case, we assume that the state process with control A has the following dynamic

$$\begin{aligned} X(s) =& x + \int_0^s b(t-s, X(s-), A(s))ds + \int_0^s \sigma(t-s, X(s-), A(s))dW(s) \\ &+ \int_{(0,s]} \int_{\mathbb{R}^n} \gamma(t-s, X(s-), A(s-), z) M(ds, dz). \end{aligned}$$

An stochastic control problem consists in optimize a given goal, in particular, we are interested in the case when the goal is given by means of an utility function, $V(\cdot; A)$. In this work we consider two types of problems: first, we consider the problem in a finite horizon t and in this case we only work with continuous diffusion processes, that is, we assume that the jump part does not exists. Second, we consider the problem in an infinite horizon. In this case we focus in a jump risk process and study the probability of ruin.

Finite horizon.

We are interested in determine and analyze the *value function* V defined as

$$V(t, x) := \sup_{A \in \mathcal{A}} \{V(t, x; A)\} \tag{1}$$

where x represents the initial state of the system, t is the final horizon and

$$\begin{aligned} V(t, x; A) :=& \mathbb{E}_x \left[\int_0^t e^{\int_0^s c(t-r, X_A(r), A(r))dr} f(t-s, X_A(s), A(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^t c(t-r, X_A(r), A(r))dr} h(X_A(t)) \right] \end{aligned}$$

with c representing a discounting factor, f a running utility function and h the final utility function.

Two main questions arise: What can we say about the value function V and in case of existence of an optimal control A^*, i.e. a control that satisfies $V(\cdot; A^*) = V(\cdot)$, what can we say about it?

A well-known approach to these problems is the Dynamic Programming Principle and the Hamilton-Jacobi-Bellman equations (HJB equations). By considering constant control processes acting over very short times (see section **??** for a more detailed explanation), it can be proposed, by an heuristic argument, that the value function V satisfies the following HJB equation

$$\begin{aligned} -u_t(t,x) + \sup_{\alpha\in\Lambda}\{\mathcal{D}^\alpha[u](t,x) + f(t,x,\alpha)\} =& 0, \qquad (t,x)\in(0,\infty)\times\mathbb{R}^d \\ u(0,x) =& h(x), \quad x\in\mathbb{R}^d \end{aligned} \tag{2}$$

where

$$\begin{aligned} \mathcal{D}^\alpha[u](r,x) := & \sum_{ij} a_{ij}(r,x,\alpha)D_{ij}u(r,x) + \sum_i b_i(r,x,\alpha)D_iu(r,x) \\ & + c(r,x,\alpha)u(r,x) \end{aligned}$$

with $\{a_{ij}\} = a = \sigma\sigma'$ and $\Lambda\subset\mathbb{R}^m$ is the set of control values.

Since the derivation of the HJB equation is heuristic only, further results are needed to guarantee that the value function is in fact a solution to the HJB equation: a *Verification Theorem* and an *Existence Theorem*. The Verification Theorem states that if a solution to the HJB equation exists, then it has to be the value function and so the solution is unique. More precisely, it is important to notice that the supremum in equation (1) is taken over a class of stochastic process while the supremum in equation (2) is taken over a set of real numbers. In general, the optimal control policy is given in a feedback form $A^*(s) = \alpha^*(t-s, X(s))$, where

$$\alpha^*(t,x) := \operatorname{argmax}_{\alpha\in\Lambda}\{\mathcal{D}^\alpha[u](t,x) + f(t,x,\alpha)\}$$

In that case, there exists an optimal utility function and the equality $V(\cdot;A^*) = V(\cdot)$ is fulfilled.

The remaining result is an Existence Theorem for the solution to the HJB equation. In general, the existence of a classical solution to equation (2) is not an easy task to solve due the non linearity in the second order derivatives. There exists some general results, however, they need some restrictive hypotheses like the boundedness of all the coefficients and their derivatives (see [**?**] Chapter 6 or [**?**] Chapter IV).

In case σ does not depend on the control, the HJB equation becomes a semilinear equation of the form

$$\begin{aligned} -u_t(t,x) + \sum_{ij} a_{ij}(t,x)D_{ij}(t,x) + \sup_{\alpha\in\Lambda}\{\mathcal{L}^\alpha[u](t,x) + f^\alpha(t,x)\} =& 0, \quad \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) =& h(x), \qquad x\in\mathbb{R}^d, \end{aligned} \tag{3}$$

where

$$\mathcal{L}^\alpha := \sum_i b_i(t,x,\alpha) D_i u(t,x) + c(t,x,\alpha)u(t,x)$$

and

$$f^\alpha(\cdot) := f(\cdot,\alpha).$$

There exist many interesting problems in which the HJB equation can be reduced into a equation of the form of equation (3) (see e.g. [?], [?], [?], [?], [?], [?] and [?]). However, there are no general procedures to prove the existence of a classical solution and each equation is treated in a particular way. Despite this, in all the papers mentioned above, the basis for the solution to the HJB equation is a result proved by Fleming (see [?] Theorem VI.6.2). It is assumed that the control set Λ is compact, $c \equiv 0$, the functions $b, \sigma \in C^{1,2}$ with σ, σ_x and b_x bounded. In this case, the boundedness is relaxed for the data functions f and h which are assumed to have a polynomial growth and C^2 regularity. The main idea used to prove this theorem is a linearization technique (see [?] Appendix E), that is, approximate the solution of equation (3) by equations of the form

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x) \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= h(x) \quad \text{for } x \in \mathbb{R}^d. \end{aligned} \tag{4}$$

where

$$\mathcal{L}[u](t,x) = \sum_{i,j} a_{ij}(t,x) D_{ij} u(t,x) + \sum_i b_i(t,x) D_i u(t,x).$$

In Chapter **??** we study the existence and uniqueness of a classical solution to equation (3) when the coefficients σ, b, c and f are locally Hölder in t and locally Lipschitz in (x,α), not necessarily differentiable, σ and b have linear growth, c is bounded from above and f has a polynomial growth of any order. h is a continuous function with polynomial growth and $\Lambda \subset \mathbb{R}^m$ is a connected compact set. We assume the ellipticity condition to be local, that is, for any $[0,T] \times A \subset [0,\infty) \times \mathbb{R}^d$ there exists $\lambda(T,A)$ such that $\sum a_{ij}(t,x)\xi_i\xi_j \geq \lambda(T,A)\|\xi\|^2$ for all $x, \xi \in A$ and $t \in [0,T]$. These hypotheses were considered due the combination of the unboundedness and the continuity of the coefficients. As we present in section **??**, there exist some stochastic control problems in which these hypotheses appear naturally.

We construct a solution by approximation with linear parabolic equations. Despite the approximation technique is standard, the linear equations involved can not be solved with the traditional results. Therefore, we study the existence of a classical solution to the Cauchy problem for a second order linear parabolic equation when the coefficients fulfil the same hypotheses of the ones of the semilinear problem (3).

In Chapter **??** we study the existence and uniqueness of a classical solution to a more general problem, the Cauchy-Dirichlet problem, for a linear parabolic differential equation in a general unbounded domain. Let $\mathcal{L}$ be the differential operator

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x) D_{ij} u(t,x) + \sum_{i=1}^{d} b_i(t,x) D_i u(t,x)$$

where $\{a_{ij}\} = a = \sigma\sigma'$, $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The Cauchy-Dirichlet problem is

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x), \qquad (t,x) \in (0,\infty) \times D, \\ u(0,x) &= h(x), \qquad x \in D, \\ u(t,x) &= g(t,x), \qquad (t,x) \in (0,\infty) \times \partial D \end{aligned} \tag{5}$$

where $D \subset \mathbb{R}^d$ is an unbounded, open, connected set with regular boundary.

In the case of bounded domains, the Cauchy-Dirichlet problem is well understood (see [**?**] and [**?**] for a detailed description of this problem). Moreover, when the domain is unbounded and the coefficients are bounded, the existence of a classical solution to equation (5) is well known. For a survey of this theory see [**?**] and [**?**] where the problem is studied with analytical methods and [**?**] for a probabilistic approach.

In the last years, parabolic equations with unbounded coefficients in unbounded domains have been studied in great detail. For the particular case when $D = \mathbb{R}^d$, there exists many papers in which the existence, uniqueness and regularity of the solution is studied under different hypotheses on the coefficients; see e.g. [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] and [**?**].

In the case of general unbounded domains, Fornaro, Metafune and Priola in [**?**] studied the homogeneous, autonomous Cauchy-Dirichlet problem. They proved using analytical methods in semigroups, the existence and uniqueness of a solution to the Cauchy-Dirichlet problem when the coefficients are C^1 locally Hölder continuous, with a_{ij} bounded, b and c functions with a Lyapunov type growth and D has a C^2 boundary. Schauder type estimates were obtained for the gradient of the solution in terms of the data. Bertoldi and Fornaro in [?] obtained analogous results for the Cauchy-Neumann problem for an unbounded convex domain. Later, in [?] Bertoldi, Fornaro and Lorenzi, generalized the method to non-convex sets with C^2 boundary. They studied the existence, uniqueness and gradient estimates for the Cauchy-Neumann problem. For a survey of this results see [?].

In the paper of Hieber, Lorenzi and Rhandi [?], the existence and uniqueness of a classical solution for the autonomous, non-homogeneous Cauchy-Dirichlet and Cauchy-Neumann problems is proved. The domain is consider to be an exterior domain with C^3 boundary. The coefficients are assumed to be C^3-Hölder continuous functions with Lyapunov type growth. The continuity properties of the semigroup generated by the solution of the parabolic problem are studied in the spaces $C_b(\overline{D})$ and $L^p(D)$.

In all the papers cited above, the uniformly elliptic condition is assumed, that is, there exists $\lambda > 0$ such that $\sum a_{ij}(t,x)\xi_i\xi_j \geq \lambda\|\xi\|^2$ for all $t \geq 0$ and $x, \xi \in \overline{D}$.

In this work we prove the existence and uniqueness of a classical solution to equation (5), when the coefficients are locally Lipschitz continuous in x and locally Hölder continuous in t, a_{ij} has a quadratic growth, b_i has linear growth and c is bounded from above. We allow f, g and h to have a polynomial growth of any order. We also consider the elliptic condition to be local, that is, for any $[0,T] \times A \subset [0,\infty) \times \overline{D}$ there exists $\lambda(T,A)$ such that $\sum a_{ij}(t,x)\xi_i\xi_j \geq \lambda(T,A)\|\xi\|^2$ for all $t \in [0,T]$ and $x, \xi \in A$. We assume that D is an unbounded, connected, open set with regular boundary (see [?] Chapter III Section 4, for a definition of regular boundary). Furthermore, we prove that the solution is locally Hölder continuous up to the second space derivative and the first time derivative.

Our approach is using stochastic differential equations and parabolic differential equations in bounded domains. For proving existence, many analytical methods construct the solution by solving the problem in nested bounded domains that approximate the domain D. The problem here lies in proving the convergence of the approximating solutions to the global one. This presents some drawbacks depending on the geometry of the domain and the regularity of the functions involved. Unlike these methods, first we propose as a solution to equation (5), a functional of the solution to a SDE,

$$\begin{aligned} v(t,x) =& \mathbb{E}_x \left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^t c(t-r,X(r))dr} h(X(t)) \mathbb{1}_{\tau_D \geq t} \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^{\tau_D} c(t-r,X(r))dr} g(t-\tau_D, X(\tau_D)) \mathbb{1}_{\tau_D < t} \right] \end{aligned}$$

where

$$dX(s) = b(t-s,X(s))ds + \sigma(t-s,X(s))dW(s), \quad X(0) = x,$$

and

$$\tau_D := \inf\{s > 0 | X(s) \notin \overline{D}\}$$

Using the continuity of the paths of the SDE we prove that this function is continuous in $[0,\infty) \times \overline{D}$. Then, using the theory of parabolic equations in bounded domains, we study locally the regularity of the function v and prove that it is a $C^{1,2}$ function. Finally, with some standard arguments we prove that it solves the Cauchy-Dirichlet problem. This kind of idea has been used for several partial differential problems (see [?], [?] and [?]).

Infinite horizon.

In Chapter ?? we work with some risk processes and present some of the partial results of this investigation. The notation in this part corresponds with the usual one of this theory. A natural field of the application of control techniques is insurance mathematics. Since 1903, when Lundberg proposed a collective risk model based on a Poisson claim process, the theory of non-life insurance has presented a great development. The Cramér-Lundberg model is

$$R(t) = x + ct - S(t),$$

where $x \geq 0$ is the initial capital, $c > 0$ stands for the premium income rate and $S(t) = \sum_{n=1}^{N(t)} \xi_n$, where $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ, $\{\xi_n\}_{n=1}^{\infty}$ are i.i.d. positive random variables independent of the Poisson process, corresponding to the incoming claims, with common distribution Q and mean $\mu < \infty$.

One of the main concerns for an insurance company is to analyze the possibility of a default, that is, study the probability of ruin or survival

$$\psi(x) := \mathbb{P}\left[\tau < \infty | R(0) = x\right],$$
$$\delta(x) := \mathbb{P}\left[\tau = \infty | R(0) = x\right].$$

where

$$\tau := \inf\{t > 0 | R(t) < 0\}$$

is the ruin time.

Since then, more complicated models have been proposed in order to reflect more accurately real aspects of the insurance's field. There are many authors for this problem, e.g. [?], [?], [?], [?], [?], [?], [?] and the references therein. It has been considered problems as investment, reinsurance, payment of dividends, severity of the ruin and combinations of them. See [?] for a very nice survey of this theory.

In all the problems mentioned above, the control theory plays an essential role in order to find optimal strategies (for investment, reinsurance, etc.). In this chapter, we focus in the problem when the insurer company puts its capital in some investment instruments: a non-risk bonus and a risky asset. Here is the extra problem of finding an optimal investment strategy that maximizes the survival probability. This problem was solved by Hipp and Plum in [?] and [?] when the non-risk rate is constant and the risky asset is a Geometric Brownian motion, that is,

$$\begin{aligned} dZ_0(t) =& Z_0(t) r dt, \\ dZ(t) =& Z(t)\mu dt + Z(t)\sigma dW(t), \end{aligned}$$

where $\mu > r \geq 0$, $\sigma > 0$ and $\{W(t)\}_{t\geq 0}$ is an standard Brownian motion independent of the claim process $\{S(t)\}_{t\geq 0}$.

The reserve process with investment strategy A is

$$\begin{aligned} X(t; A) =& R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s) \frac{dZ_0(s)}{Z_0(s)} \\ =& x + \int_0^t [c + bA_s + rX(s-)] ds + \int_0^t \sigma A_s dW(s) - S(t), \end{aligned}$$

where $b := \mu - r$ and the survival probability is defined as

$$\delta(x; A) := \mathbb{P}\left[\tau(A) = \infty | X(0; A) = x\right].$$

with

$$\tau(A) := \inf\{t > 0 | X(t; A) < 0\}.$$

Hipp and Plum proved in [?] and [?], via a Verification and an Existence Theorem, that the optimal survival probability is the unique classical solution to the HJB equation

$$\sup_{\alpha \in \mathbb{R}} \{\mathcal{M}^{\alpha}[f](x)\} = 0,$$

with

$$\mathcal{M}^{\alpha}[f](x) := \frac{1}{2}\sigma^2\alpha^2 f''(x) + (c + b\alpha + rx)f'(x) + \lambda \int_0^{\infty} [f(x-z) - f(x)]dQ(z)$$

and boundary conditions

- $\lim_{x\to\infty} f(x) = 1$ and
- $\alpha^*(0) = 0$

where

$$\alpha^*(x) = -\frac{bf'(x)}{\sigma^2 f''(x)}.$$

It is important to notice that the boundary conditions are not usual, this happens because the risk process can not reach the zero boundary continuously. This imposes some additional considerations to prove the existence of a classical solution. Even for the Verification Theorem, despite the proof follows the traditional arguments, it presents some additional problems due the behavior of the risk process when $t \to \infty$ and the degeneration of the optimal process at $x = 0$.

We are interested in generalize the results of Hipp and Plum when the rates of the investment instruments are stochastic and depend on an external factor. We assume that the external factor has the following dynamic

$$Y(t) = y + \int_0^t g(Y(s))ds + \beta(\rho W_1(t) + \epsilon W_2(t)),$$

with $0 \le \rho \le 1$, $\epsilon = \sqrt{1-\rho^2}$ and the investment instruments fulfil

$$\begin{aligned} dZ_0(t) =& Z_0(t)r(Y(t))dt, \\ dZ(t) =& Z(t)\mu(Y(t))dt + Z(t)\sigma(Y(t))dW_1(t), \end{aligned}$$

The reserve process with investment strategy $\{A_t\}_{t\geq 0}$ is

$$\begin{aligned} X(t;A) =& R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s)\frac{dZ_0(s)}{Z_0(s)} \\ =& x + \int_0^t [c + b(Y(s))A_s + r(Y(s))X(s-)]ds \\ &+ \int_0^t \sigma(Y(s))A_s dW_1(s) - S(t), \end{aligned}$$

where $b(\cdot) := (\mu - r)(\cdot)$.

The value function is defined as

$$\delta(x,y) := \sup_A\{\delta(x,y;A)\}$$

with

$$\delta(x,y;A) := \mathbb{P}\left[\tau(A) = \infty | X(0;A) = x, Y(0) = y\right].$$

In this case we propose as a HJB equation for the optimal survival probability

$$\sup_{\alpha\in\mathbb{R}}\{\mathcal{L}^{\alpha}[f](x,y)\} = 0$$

where

$$\begin{aligned} \mathcal{L}^{\alpha}[f](x,y) :=& g(y)f_y(x,y) + (c + b(y)\alpha + xr(y))f_x(x,y) \\ &+ \frac{1}{2}\beta^2 f_{yy}(x,y) + \frac{1}{2}\sigma(y)^2\alpha^2 f_{xx}(x,y) + \beta\rho\sigma(y)f_{xy}(x,y) \\ &+ \lambda\int_0^{\infty}(f(x-z,y) - f(x,y))dQ(z). \end{aligned}$$

with $\alpha^*(0,\cdot) = 0$ where

$$\alpha^*(x,y) = -\frac{b(y)f_x(x,y) + \beta\rho\sigma(y)f_{xy}(x,y)}{\sigma^2(y)f_{xx}(x,y)},$$

We prove a Verification Theorem. The Existence Theorem is a work in progress.

The work is divided as follows: In Chapter **??** we present the preliminaries and many of the main ideas used throughout the rest of the work. We first present a general overview for the theory of stochastic optimal control. Next we discuss some of the general ideas used to prove the existence of solutions to semilinear parabolic differential equations. Finally, we look through the relation between the linear partial differential equations and the stochastic differential equations. Chapter **??** presents the main result and the proofs for the existence of a classical solution to the Cauchy-Dirichlet problem for class of linear parabolic differential equations with unbounded coefficients in a unbounded domain. Chapter **??** is devoted to prove the results for the semilinear problem. In section **??** of this chapter, we apply the results proved to an optimal consumption problem. This problem shows situations in which the hypotheses appear naturally. In Chapter **??** we present the problem of optimal investment for an insurance company in an incomplete market when the coefficients of the investment instruments are stochastic. We present the partial results of this investigation. Finally, in Appendix **??** the reader will find some of the results used in the proofs of the theorems.

Introducción.

La Teoría del Control ha tenido un gran desarrollo en los últimos sesenta años gracias a la gran variedad de aplicaciones que presenta: teoría del riesgo, problemas de consumo, control de producción, portafolios de inversión, problemas de salida, entre otros. Véase [?], [?], [?], [?], [?] y las referencias en ellos.

En un sentido amplio, el control es la acción ejercida en la evolución de un proceso con la finalidad de obtener una meta deseada. La teoría del control tiene varias clasificaciones de acuerdo a la dinámica del proceso controlado: El caso determinista junto con el cálculo de variations, tiene una larga historia que comienza con el problema de la braquistocrona resuelto por Johann Bernoulli hace aproximadamente 300 años (véase [?] para una presentación general). El caso estocástico tuvo sus principios a finales de los 1950's y principios de los 1960's, sin embargo su desarrollo ha sido muy intenso desde entonces (véase [?], [?] o [?]). Se generan dos tipos de problemas dependiendo de las consideraciones sobre la evolución temporal: discreto y continuo. Véase [?], [?], [?] o [?] para un panorama general sobre esta teoría en el caso discreto.

En este trabajo trataremos problemas de control estocástico en tiempo continuo. Desde la perspectiva determinista, la evolución de un sistema se modela de manera general mediante una ecuación diferencial de la forma

$$\frac{dX(s)}{ds} = b(X(s))$$

Muchos problemas interesantes presentan características que son aleatorias o simplemente desconocidas para el observador. Una manera de modelar esto es considerando una difusión de Ito con saltos en $\mathbb{R}^d$

$$dX(s) = b(X(s))ds + \sigma(X(s))dW(s) + \int_{\mathbb{R}^n} \gamma(X(s), z)M(ds, dz)$$

donde W es un movimiento Browniano y $M(\cdot)$ representa una medida aleatoria Poisson generada por los saltos de un proceso de Levy (en este trabajo consideraremos solamente el caso especial de un proceso Poisson compuesto).

La teoría del control supone que la dinámica del sistema puede ser modificada mediante un *proceso de control* $A = \{A(s)\}$. Generalmente, los procesos de control son seleccionados dentro de una clase de procesos estocásticos predecibles con ciertas condiciones de integrabilidad (denotados por $\mathcal{A}$). En este caso, suponemos que la dinámica del proceso de estado con control A está dada por

$$\begin{aligned} X(s) =& x + \int_0^s b(t-s, X(s-), A(s))ds + \int_0^s \sigma(t-s, X(s-), A(s))dW(s) \\ &+ \int_{(0,s]} \int_{\mathbb{R}^n} \gamma(t-s, X(s-), A(s-), z) M(ds, dz). \end{aligned}$$

Un problema de control estocástico consiste en optimizar cierta meta dada, en particular, estamos interesados en el caso que dicha meta está dada por una función de utilidad, $V(\cdot; A)$. En este trabajo consideramos dos tipos de problemas: primero, consideramos el problema en un horizonte finito t y en este caso trabajamos únicamente con procesos de difusión continuos, es decir, suponemos que la parte de saltos no existe. Segundo, consideramos el problema en horizonte infinito. En este caso nos enfocamos en un proceso de riesgo con saltos y estudiamos la probabilidad de ruina.

Horizonte finito.

Estamos interesados en derminar y analizar la *función de valor* V definida por

$$V(t,x) := \sup_{A \in \mathcal{A}} \{V(t,x;A)\} \tag{6}$$

donde x representa el estado inicial del sistema, t es el horizonte final y

$$\begin{aligned} V(t,x;A) :=& \mathbb{E}_x \left[\int_0^t e^{\int_0^s c(t-r, X_A(r), A(r))dr} f(t-s, X_A(s), A(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^t c(t-r, X_A(r), A(r))dr} h(X_A(t)) \right] \end{aligned}$$

con c representando un factor de descuento, f la función de utilidad corriente y h la función de utilidad final.

De esto surgen dos preguntas: ¿Qué podemos decir acerca de la función V y en caso de que existe un control óptimo A^*, i.e., un control que satisfaga $V(\cdot; A^*) = V(\cdot)$, qué podemos decir de éste?

Un método usual para este tipo de problemas es el Principio de la Programación Dinámica y las ecuaciones de Hamilton-Jacobi-Bellman (ecuaciones de HJB). Considerando procesos de control constantes actuando sobre intervalos de tiempo muy pequeños (véase la sección **??** para una explicación detallada), se puede proponer mediante un argumento heurístico, que la función de valor V satisface la siguiente ecuación de HJB

$$\begin{aligned} -u_t(t,x) + \sup_{\alpha\in\Lambda}\{\mathcal{D}^\alpha[u](t,x) + f(t,x,\alpha)\} =& 0, \qquad (t,x) \in (0,\infty)\times\mathbb{R}^d \\ u(0,x) =& h(x), \quad x \in \mathbb{R}^d \end{aligned} \tag{7}$$

donde

$$\begin{aligned} \mathcal{D}^\alpha[u](r,x) :=& \sum_{ij} a_{ij}(r,x,\alpha)D_{ij}u(r,x) + \sum_i b_i(r,x,\alpha)D_iu(r,x) \\ &+ c(r,x,\alpha)u(r,x) \end{aligned}$$

con $\{a_{ij}\} = a = \sigma\sigma'$ y $\Lambda \subset \mathbb{R}^m$ representa el conjunto de valores del control.

Debido a que la derivación de la ecuación de HJB es heurística, se requiere de ciertos resultados adicionales para garantizar que la función de valor es de hecho la solución para la ecuación de HJB: un *Teorema de Verificación* y un *Teorema de Existencia.* El Teorema de Verificación establece que el en caso que exista una solución de la ecuación de HJB, ésta tiene que ser la función de valor y por lo tanto la solución es única. Es importante notar que el supremo en la ecuación (6) se toma sobre una clase de procesos estocásticos mientras que el supremo en la ecuación (7) se toma sobre un conjunto de reales. En general, la estrategia de control óptima está dada de forma retroactiva $A^*(s) = \alpha^*(t-s, X(s))$, donde

$$\alpha^*(t,x) := \operatorname{argmax}_{\alpha\in\Lambda}\{\mathcal{D}^\alpha[u](t,x) + f(t,x,\alpha)\}$$

En este caso, se tiene una función de utilidad óptima y se cumple la igualdad $V(\cdot; A^*) = V(\cdot)$.

El resultado restante es un Teorema de Existencia para la solución de la ecuación de HJB. En general, probar la existencia de una solución clásica para la ecuación (7) no es una tarea sencilla debido a la no linealidad en las derivadas de segundo orden. Existen algunos resultados generales, sin embargo, requieren de hipótesis restrictivas sobre los coeficientes, como el acotamiento de éstos y sus derivadas (véase [?] Capítulo 6 o [?] Capítulo IV).

En el caso que σ no dependa del control, la ecuación de HJB se convierte en una ecuación semilineal de la forma

$$\begin{aligned} -u_t(t,x) + \sum_{ij} a_{ij}(t,x)D_{ij}(t,x) + \sup_{\alpha\in\Lambda}\{\mathcal{L}^\alpha[u](t,x) + f^\alpha(t,x)\} =&0, \quad \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) =&h(x), \qquad x\in\mathbb{R}^d, \end{aligned} \tag{8}$$

donde

$$\mathcal{L}^\alpha := \sum_i b_i(t,x,\alpha)D_i u(t,x) + c(t,x,\alpha)u(t,x)$$

y

$$f^\alpha(\cdot) := f(\cdot,\alpha).$$

Existen muchos problemas interesantes para los cuales la ecuación de HJB se reduce a una ecuación de esta forma (véase e.g. [?], [?], [?], [?], [?], [?] y [?]). Sin embargo, no existen procedimientos generales para probar la existencia de soluciones clásicas y por lo tanto cada ecuación es tratada de manera particular. A pesar de esto, en todos los artículos mencionados, la base para la existencia de una solución para la ecuación de HJB es un resultado probado por Fleming (véase [?] Teorema VI.6.2). En éste se trabaja en un espacio de control Λ acotado, $c\equiv 0$, las funciones $b,\sigma\in C^{1,2}$ con σ, σ_x y b_x acotadas. En este caso, se relaja la hipótesis de acotamiento para los datos f y h a los cuales se les permite tener un crecimiento polinomial y regularidad C^2. La principal idea detrás de la prueba de este teorema es una técnica de linealización (véase [?] Apéndice E), estos es, se aproxima la solución de la ecuación (8) por ecuaciones de la forma

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) = -f(t,x) \quad (t,x)\in(0,\infty)\times\mathbb{R}^d, \\ u(0,x) = h(x) \quad \text{para } x\in\mathbb{R}^d. \end{aligned} \tag{9}$$

donde

$$\mathcal{L}[u](t,x) = \sum_{i,j} a_{ij}(t,x)D_{ij}u(t,x) + \sum_i b_i(t,x)D_i u(t,x).$$

En el Capítulo **??** estudiamos la existencia y unicidad de una solución clásica para la ecuación (8) cuando los coeficientes σ, b, c y f son localmente Hölder en t y localmente Lipschitz en (x, α), no necesariamente diferenciables, σ y b tienen crecimiento lineal, c es acotado por arriba y f presente un crecimiento polinomial de cualquier orden. h es una función continua con crecimiento polinomial y $\Lambda \subset \mathbb{R}^m$ es un conjunto conexo y compacto. Suponemos la condición de elipticidad localmente, esto es, para todo $[0,T] \times A \subset [0, \infty) \times \mathbb{R}^d$ existe $\lambda(T, A)$ tal que $\sum a_{ij}(t,x)\xi_i\xi_j \geq \lambda(T,A)\|\xi\|^2$ para todo $x, \xi \in A$ y $t \in [0,T]$. Estas hipótesis son consideradas debido a la combinación entre el no acotamiento y la continuidad de los coeficientes. En la sección **??** presentamos un problema de control estocástico en donde estos supuestos aparecen de manera natural.

La solución la construimos mediante una aproximación por ecuaciones parabólicas lineales. A pesar de que esta técnica es estándar, las ecuaciones lineales no cumplen las hipótesis de los resultados tradicionales. Por lo tanto, estudiamos la solución clásica al problema de Cauchy de una ecuación parabólica lineal de segundo orden cuando los coeficientes satisfacen las mismos supuestos que los del problema semilineal.

En el Capítulo **??** estudiamos la existencia y unicidad de una solución general para un problema más general, el problema de Cauchy-Dirichlet, para una ecuación diferencial parabólica lineal en un dominio no acotado general. Sea $\mathcal{L}$ el operador diferencial

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x) D_{ij} u(t,x) + \sum_{i=1}^{d} b_i(t,x) D_i u(t,x)$$

donde $\{a_{ij}\} = a = \sigma\sigma'$, $D_i = \frac{\partial}{\partial x_i}$ y $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. El problem de Cauchy-Dirichlet es

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x), \qquad (t,x) \in (0,\infty) \times D, \\ u(0,x) &= h(x), \qquad x \in D, \\ u(t,x) &= g(t,x), \qquad (t,x) \in (0,\infty) \times \partial D \end{aligned} \tag{10}$$

donde $D \subset \mathbb{R}^d$ es un conjunto no acotado, abierto, conexo con frontera regular.

En el caso de dominios acotados, el problema de Cauchy-Dirichlet está muy bien entendido (véase [**?**] y [**?**] para una descripción detallada de este problema). Es más, cuando el dominio es no acotado y los coeficientes son acotados, el problema de la existencia de soluciones clásicas para la ecuación (10) está resuelto. Para un estudio general de esta teoría véase [**?**] y [**?**]

donde el problema es tratado mediante argumentos de análisis y [?] para un tratamiento probabilístico.

En los últimos años, se han estudiado en gran detalle las ecuaciones parabólicas con coeficientes no acotados en dominios no acotados. En el caso particular $D = \mathbb{R}^d$, existen muchos artículos en los que se estudia la existencia, unicidad y regularidad de las soluciones bajo una gran variedad de hipótesis sobre los coeficientes; véase e.g. [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?] y [?].

En el caso de dominios no acotados generales, Fornaro, Metafune y Priola en [?] estudiaron el problema de Cauchy-Dirichlet homogéneo y autónomo. Probaron mediante argumentos de semigrupos, la existencia y unicidad de una solución clásica del problema de Cauchy-Dirichlet cuando los coeficientes son C^1 localmente Hölder continuos, con a_{ij} acotado, b y c funciones con un crecimiento del tipo Lyapunov y D con regularidad C^2. Se obtuvieron estimaciones del tipo Schauder para el gradiente de la solución en términos de los datos del problema. Bertoldi y Fornaro en [?] obtuvieron resultados análogos para el problema de Cauchy-Neumann para un dominios convexo, no acotado. Posteriormente, en [?] Bertoldi, Fornaro y Lorenzi, generalizaron el método para conjuntos no-convexos con frontera C^2. Estudiaron la existencia, unicidad y estimaciones del gradiente para el problema de Cauchy-Neumann. Para un compendio de esta teoría véase [?].

En el artículo de Hieber, Lorenzi y Rhandi [?], se probó la existencia y unicidad de una solución clásica del problema autónomo, no homogéneo de Cauchy-Dirichlet y de Cauchy-Neumann. El dominio se considera que es un dominio exterior con frontera C^3. Los coeficientes son C^3-Hölder continuos con un crecimiento del tipo Lyapunov. Se estudiaron las propiedades de continuidad del semigrupo generado por la solución del problema parabólico en los espacios $C_b(\overline{D})$ and $L^p(D)$.

En todos los trabajos mencionados, se supone la condición de elipticidad uniforme, esto es, existe $\lambda > 0$ tal que $\sum a_{ij}(t,x)\xi_i\xi_j \geq \lambda\|\xi\|^2$ para todo $t \geq 0$ y $x, \xi \in \overline{D}$.

En este trabajo probamos la existencia y unicidad de una solución clásica de la ecuación (10), cuando los coeficientes son localmente Lipschitz continuos en x y localmente Hölder continuos en t, a_{ij} tiene un crecimiento cuadrático, b_i tiene un crecimiento lineal y c es acotado por encima. Las función f, g y h tienen un crecimiento lineal de cualquier orden. También consideramos que la condición de elipticidad es local, esto es, para todo $[0,T] \times A \subset [0,\infty) \times \overline{D}$ existe $\lambda(T,A)$ tal que $\sum a_{ij}(t,x)\xi_i\xi_j \geq \lambda(T,A)\|\xi\|^2$ para todo

$t \in [0, T]$ y $x, \xi \in A$. Suponemos que D es un conjunto no acotado, conexo con frontera regular (véase [?] Capítulo III Sección 4, para una defición de frontera regular). Aún más, probamos que la solución es localmente Hölder continua hasta la segunda derivada espacial y la primera derivada temporal.

Nuestro tratamiento se centra en ecuaciones diferenciales estocásticas y ecuaciones diferenciales parabólicas en dominios acotados. Para probar existencia, muchos métodos de análisis construyen la solución resolviendo el problema en una sucesión anidada de dominios acotados que aproximan el dominio D. El problema en este caso radica en probar la convergencia de las soluciones aproximanentes a la global, lo que presenta varios inconvenientes dependiendo de la geometría del dominio y la regularidad de las funciones involucradas. A diferencia de estos métodos, primero proponemos como una solución para la ecuación (10) un funcional de la solución de una ecuación diferencial estocástica,

$$\begin{aligned} v(t,x) =& \mathbb{E}_x \left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^t c(t-r,X(r))dr} h(X(t)) \mathbb{1}_{\tau_D \geq t} \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^{\tau_D} c(t-r,X(r))dr} g(t-\tau_D, X(\tau_D)) \mathbb{1}_{\tau_D < t} \right] \end{aligned}$$

donde

$$dX(s) = b(t-s,X(s))ds + \sigma(t-s,X(s))dW(s), \quad X(0) = x,$$

y

$$\tau_D := \inf\{s > 0 | X(s) \notin \overline{D}\}$$

Usando la continuidad de las trayectorias de la ecuación diferencial estocástica probamos que esta función es continua en $[0,\infty) \times \overline{D}$. Luego, utilizando la teoría de las ecuaciones diferenciales parabólicas en dominios acotados, estudiamos localmente la regularidad de la función v y probamos que es una función $C^{1,2}$. Finalmente, mediante algunos argumentos estándares probamos que ésta resuelve el problema de Cauchy-Dirichlet. Este tipo de ideas han sido utilizados en varios problemas de ecuaciones diferenciales parciales (véase [?], [?] y [?]).

Horizonte infinito.

En el Capítulo **??** trabajamos con algunos procesos de riesgo y presentamos algunos de los resultados parciales de esta investigación. La notación en esta parte del trabajo corresponde a la notación usual en este teoría. Un campo natural para la aplicación de la teoría del control el la matemática actuarial o del seguro. Desde 1903 cuando Lundberg propuso un modelo de riesgo colectivo basado en un proceso de reclamaciones Poisson, la teoría del seguro de no-vida ha presentado un gran desarrollo. El modelo de Cramér-Lundberg es

$$R(t) = x + ct - S(t),$$

donde $x \geq 0$ es el capital inicial, $c > 0$ representa la tasa de primas y $S(t) = \sum_{n=1}^{N(t)} \xi_n$, con $\{N(t)\}_{t\geq 0}$ un proceso Poisson con intensidad λ, $\{\xi_n\}_{n=1}^{\infty}$ variables aleatorias i.i.d. positivas independientes del proceso Poisson, que representan las reclamaciones, con función de distribución común Q y media $\mu < \infty$.

Una de las principales preocupaciones de una compañía de seguros en analizar la posibilidad de un incumplimiento, es decir, estudiar la probabilidad de ruina o supervivencia

$$\psi(x) := \mathbb{P}\left[\tau < \infty | R(0) = x\right],$$
$$\delta(x) := \mathbb{P}\left[\tau = \infty | R(0) = x\right].$$

donde

$$\tau := \inf\{t > 0 | R(t) < 0\}$$

es el tiempo de ruina.

Desde entonces, se han propuesto modelos más complejos que reflejan de manera más precisa distintos aspectos del seguro. Existen muchos autores para este problema, e.g. [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**] y las referencias en ellos. Se han considerado aspectos como la inversión, el reaseguro, el pago de dividendos, la severidad de la ruina y combinaciones entre ellos. Véase [**?**] para un muy buen resumen de esta teoría.

En todos los problemas mencionados arriba, la teoría del control juega un rol esencial para la obtención de estrategias óptimas (para inversión, reaseguro, etc.). En este capítulo, nos enfocamos en el problema donde la compañía de seguros coloca su capital en algunos instrumentos de inversión: un cuenta sin riesgo y un activo con riesgo. En este caso se tiene el problema extra de encontrar una estrategia de inversión óptima que maximice su probabilidad de supervivencia. Este problema fue resuelto por Hipp y Plum en [?] y [?] cuando la tasa libre de riesgo es constante y el activo con riesgo es un movimiento Browniano geométrico, esto es

$$\begin{aligned} dZ_0(t) =& Z_0(t) r dt, \\ dZ(t) =& Z(t)\mu dt + Z(t)\sigma dW(t), \end{aligned}$$

donde $\mu > r \geq 0$, $\sigma > 0$ y $\{W(t)\}_{t\geq 0}$ es un movimiento Browniano estándar independiente del proceso de reclamaciones $\{S(t)\}_{t\geq 0}$.

El proceso de reserva con estrategia de inversión A es

$$\begin{aligned} X(t;A) =& R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s) \frac{dZ_0(s)}{Z_0(s)} \\ =& x + \int_0^t [c + bA_s + rX(s-)] ds + \int_0^t \sigma A_s dW(s) - S(t), \end{aligned}$$

donde $b := \mu - r$ y la probabilidad de supervivencia se define como

$$\delta(x;A) := \mathbb{P}\left[\tau(A) = \infty | X(0;A) = x\right].$$

con

$$\tau(A) := \inf\{t > 0 | X(t;A) < 0\}.$$

Hipp y Plum probaron en [?] y [?], mediante un Teorema de Verificación y uno de Existencia, que la probabilidad de supervivencia óptima es la única solución clásica de la ecuación de HJB

$$\sup_{\alpha\in\mathbb{R}} \{\mathcal{M}^\alpha[f](x)\} = 0,$$

con

$$\mathcal{M}^\alpha[f](x) := \frac{1}{2}\sigma^2\alpha^2 f''(x) + (c + b\alpha + rx)f'(x) + \lambda \int_0^\infty [f(x-z) - f(x)] dQ(z)$$

y condiciones de frontera

- $\lim_{x\to\infty} f(x) = 1$ y
- $\alpha^*(0) = 0$

donde

$$\alpha^*(x) = -\frac{bf'(x)}{\sigma^2 f''(x)}.$$

Es importante notar que las condiciones de frontera no son comunes a las condiciones de frontera tradicionales. Esto se debe a que el proceso de riesgo no puede alcanzar la frontera cero de manera continua. Esto impone consideraciones adicionales para probar la existencia de una solución clásica. Incluso para el Teorema de Verificación, a pesar de que la prueba sigue los argumentos tradicionales, ésta presenta algunos problemas adicionales debido al comportamiento del proceso de riesgo cuando $t \to \infty$ y la degeneración del proceso óptimo en $x = 0$.

Estamos interesados en generalizar los resultados de Hipp y Plum al caso en que las tasas de los instrumentos de inversión son estocásticas y dependen de un factor externo. Suponemos que el factor externo tiene la siguiente dinámica

$$Y(t) = y + \int_0^t g(Y(s))ds + \beta(\rho W_1(t) + \epsilon W_2(t)),$$

con $0 \leq \rho \leq 1$, $\epsilon = \sqrt{1-\rho^2}$ y los instrumentos de inversión satisfacen

$$\begin{aligned} dZ_0(t) =& Z_0(t)r(Y(t))dt, \\ dZ(t) =& Z(t)\mu(Y(t))dt + Z(t)\sigma(Y(t))dW_1(t), \end{aligned}$$

El proceso de reserva con estrategia de inversión $\{A_t\}_{t\geq 0}$ es

$$\begin{aligned} X(t;A) =& R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s)\frac{dZ_0(s)}{Z_0(s)} \\ =& x + \int_0^t [c + b(Y(s))A_s + r(Y(s))X(s-)]ds \\ &+ \int_0^t \sigma(Y(s))A_s dW_1(s) - S(t), \end{aligned}$$

donde $b(\cdot) := (\mu - r)(\cdot)$.

La función de valor está definido por

$$\delta(x,y) := \sup_A \{\delta(x,y;A)\}$$

con

$$\delta(x,y;A) := \mathbb{P}\left[\tau(A) = \infty | X(0;A) = x, Y(0) = y\right].$$

En este caso proponemos la siguiente ecuación de HJB para la probabilidad de supervivencia óptima

$$\sup_{\alpha\in\mathbb{R}}\{\mathcal{L}^{\alpha}[f](x,y)\}=0$$

donde

$$\begin{aligned}\mathcal{L}^{\alpha}[f](x,y) &:= g(y)f_y(x,y)+(c+b(y)\alpha+xr(y))f_x(x,y)\\ &+\frac{1}{2}\beta^2 f_{yy}(x,y)+\frac{1}{2}\sigma(y)^2\alpha^2 f_{xx}(x,y)+\beta\rho\sigma(y)f_{xy}(x,y)\\ &+\lambda\int_0^{\infty}(f(x-z,y)-f(x,y))dQ(z).\end{aligned}$$

con $\alpha^*(0,\cdot)=0$ y

$$\alpha^*(x,y)=-\frac{b(y)f_x(x,y)+\beta\rho\sigma(y)f_{xy}(x,y)}{\sigma^2(y)f_{xx}(x,y)},$$

Probamos un Teorema de Verificación. El Teorema de Existencia se encuentra aún en desarrollo.

EL trabajo se encuentra dividido de la siguiente forma: En el Capítulo **??** presentamos los preliminares y varias de las principales ideas usadas a lo largo del trabajo. Primero presentamos un panorama general sobre la teoría de control estocástico óptimo. Después discutimos algunas de las ideas generales usadas para probar la existencia de soluciones para las ecuaciones diferenciales parabólicas semilineales. Finalmente, revisamos la relación entre las ecuaciones diferenciales parciales lineales y las ecuaciones diferenciales estocásticas. El Capítulo **??** presenta el resultado principal y las pruebas para la existencia de una solución clásica del problema de Cauchy-Dirichlet para una clase de ecuaciones parabólicas lineales con coeficientes no acotados en un dominio no acotado. El Capítulo **??** es dedicado a probar los resultados del problema semilineal. En la sección **??** de este capítulo, aplicamos los resultados probados a un problema de consumo óptimo. Este problema muestra situaciones en las que los supuestos de este trabajo aparecen naturalmente. En el Capítulo **??** presentamos el problema de inversión óptima de una compañía de seguros en un mercado incompleto cuando los coeficientes de los instrumentos de inversión son estocásticos. Presentamos los resultados parciales de esta investigación. Finalmente, en el Apéndice **??** el lector encontrará algunos de los resultados utilizados a lo largo del trabajo.

Preliminaries.

In this chapter we present an introduction to the problem of stochastic optimal control and some of the problems and techniques involved with it. We are interested in providing a general view about this subject, for that, in this chapter we present some general results without hypotheses and for the proofs we only sketch the main ideas. In the following chapters we present some particular problems with all the hypotheses needed and all the details of the proofs.

0.1 Preliminaries.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{s\geq 0})$ be a complete filtered probability space and let $\{W\} = \{W_i\}_{i=1}^d$ be a d-dimensional brownian motion defined in it.

Let $\{\theta(s)\}_{s\geq 0}$, with $\theta(0) = 0$ be a cadlag Levy process and define M as the associated Poisson random measure defined as

$$M(t,U) = \sum_{s\in(0,t]} \mathbb{1}_U(\Delta\theta(s))$$

for $t \geq 0$ and $U \in \mathcal{B}(\mathbb{R})$, where $\Delta\theta(s) = \theta(s) - \theta(s-)$. Define the intensity measure ν as

$$\nu(U) := \mathbb{E}\left[M(1,U)\right]$$

0.2 Stochastic optimal control problem.

Stochastic control is a relatively young branch of the mathematics. However, thanks to the wide amount of applications that it present, this field has seen a great development in the last sixty years.

The concept of control can be described as a process that is influencing the behavior of a dynamical system with the objective to get a desired goal. If the goal is to optimize a given "utility" function, the problem is referred as *optimal control*.

There exist many approaches to the optimal control problem depending on the model considered for the dynamic of the system. We will focus in the problem when the system is modeled by Ito's diffusions and Stochastic Differential Equations.

A *stochastic control problem* has the following elements:

- Let $X = \{X(s)\}_{s\in\mathbf{T}}$ be a stochastic process representing the state of the controlled system that takes values in $\mathbb{R}^d$. We assume that the process is controlled in some set $D \subset \mathbb{R}^d$.

- Let $A = \{A(s)\}_{s\in\mathbf{T}}$ be the control applied to the system. We assume that the control takes values in the set $\Lambda \subset \mathbb{R}^m$.

- Let $\mathcal{A}$ be the set of "admissible" controls.

We consider the problem in two main settings: finite and infinite horizon. In the following we explain the main characteristic for both problems.

0.2.1 Finite time horizon.

In general, for a finite time horizon problem we assume that the system X with control A, has the following dynamic

$$\begin{aligned} X(s) =& x + \int_0^s b(t-s, X(s-), A(s))ds + \int_0^s \sigma(t-s, X(s-), A(s))dW(s) \\ &+ \int_{(0,s]}\int_{\mathbb{R}^n} \gamma(t-s, X(s-), A(s-), z)M(ds, dz). \end{aligned} \tag{1}$$

Define the exit time of the controlled region as

$$\tau_D := \inf\{s \geq 0 | X_A(s) \notin \overline{D}\}.$$

We consider a performance criteria V defined as

$$\begin{aligned} V(t,x;A) :=& \mathbb{E}_x \left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X_A(r),A(r))dr} f(t-s,X_A(s),A(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^t c(t-r,X_A(r),A(r))dr} h(X_A(t)) \mathbb{1}_{\tau_D\geq t} \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^{\tau_D} c(t-r,X_A(r),A(r))dr} g(t-\tau_D,X_A(\tau_D)) \mathbb{1}_{\tau_D< t} \right] \end{aligned} \tag{2}$$

where

- c represents a discounting factor,
- f represents a running utility function and
- g and h represent the final utility functions. For simplicity of notation, let G denote both final utility functions.

In general, it is assumed that an utility function $U : (0,\infty) \to \mathbb{R}$ is strictly increasing, strictly concave, continuous differentiable and satisfies the Inada's condition ($U'(0) = \infty$ and $U'(\infty) = 0$). These properties reflects some economic aspects about the subjective preferences of an agent. The introduction to the utility functions go back to Daniel Bernoulli and the Saint Petersburg's paradox where a naive person would bet a large amount of money in order to obtain a small profit. Despite these properties are rational in an economical sense, there exists some "utility" functions which fails to fulfil them and nevertheless, are very interesting in other branches, like the probability of ruin for the insurance mathematics.

Back to the optimal problem, the objective is to maximize the expected utility function, that is, study the function V defined as

$$V(t,x) := \sup_{A\in\mathcal{A}} \{V(t,x;A)\}. \tag{3}$$

There are many aspects of the function V that we are interested in study: the regularity of the function, an analytic expression, the existence of an optimal strategy, among others. One of the main techniques used to answer this concerns is the Dynamic Programming Principle and the Hamilton-Jacobi-Bellman equations (HJB equations). This consists in finding a differential equation whose solution is the value function V.

Next, we present an "intuitive" argument to propose a HJB equation for the value function. Consider an strategy A such that $A(s) \equiv \alpha$ for some

$\alpha \in \Lambda$, denote by X_α the respective controlled process. Thanks to the Markov property, for $0 < h \ll 1$ we can argue that,

$$\begin{aligned} V(t,x) \geq & \mathbb{E}_x \left[\int_0^h e^{\int_0^s c(t-r,X_\alpha,\alpha)dr} f(t-s,X_\alpha,\alpha) ds \right] \\ & + \mathbb{E}_x \left[e^{\int_0^h c(t-r,X_\alpha(r),\alpha)dr} V(t-h,X_\alpha(h)) \right]. \end{aligned} \tag{4}$$

If we assume that $V \in C^{1,2}$, applying Itô's formula to $e^{\int_0^h c}V$ we get

$$\begin{aligned} & e^{\int_0^h c(t-r,X_\alpha(r),\alpha)dr} V(t-h,X_\alpha(h)) - V(t,x) \\ &= \int_0^h e^{\int_0^s c(t-r,X_\alpha(r),\alpha)dr} (-V_t + \mathcal{D}^\alpha[V])(t-s,X_\alpha(s-))ds \\ &\quad + \int_0^h e^{\int_0^s c(t-r,X_\alpha(r),\alpha)dr} \sigma(t-s,X_\alpha(s-),\alpha) DV(t-s,X_\alpha(s-)) \cdot dW(s) \\ &\quad + \iint_{(0,h]\times\mathbb{R}^n} [V(t-s,X_\alpha(s-)+\gamma(t-s,X_\alpha(s-),\alpha,z)) - V(t-s,X_\alpha(s-))] \tilde{M}(ds,dz) \end{aligned}$$

where $\tilde{M} := M - \nu$ is the compensated Poisson random measure and $\mathcal{D}$ is the integro-differential operator

$$\begin{aligned} \mathcal{D}^\alpha[u](r,x) :=& \sum_{ij} a_{ij}(r,x,\alpha) D_{ij} u(r,x) + \sum_i b_i(r,x,\alpha) D_i u(r,x) + c(r,x,\alpha) u(r,x) \\ & + \int_{\mathbb{R}^n} [u(r,x+\gamma(r,x,\alpha,z)) - u(r,x)] \nu(dz). \end{aligned} \tag{5}$$

with $\{a_{ij}\} = a = \sigma\sigma'$.

If we assume that the integrals with respect to the Brownian motion and the compensated Poisson Random Measure are martingales, taking expectation and substituting back in equation (4) we get

$$V(t,x) \geq V(t,x) + \mathbb{E}_x \left[\int_0^h e^{\int_0^s c(t-r,X_\alpha,\alpha)dr} \left[-V_t + \mathcal{D}^\alpha[V] + f \right] (t-s,X_\alpha,\alpha) ds \right]$$

Dividing by h, letting $h \downarrow 0$ and taking the supremum with respect to α, suggest that V satisfies the following equation

$$\begin{aligned} -u_t(t,x) + \sup_{\alpha\in\Lambda} \{ \mathcal{D}^\alpha[u](t,x) + f(t,x,\alpha) \} =& 0, \qquad (t,x) \in (0,\infty) \times D \\ u(t,x) =& G(t,x), \quad (t,x) \in \partial((0,\infty) \times D) \end{aligned} \tag{6}$$

where $\mathcal{D}$ is defined as in equation (5). The boundary condition follows by making $t = 0$ or $x \in \partial D$ in equation (2).

To prove that the value function V is the solution to the HJB equation (equation (6)), two main results are needed: a Verification Theorem and an Existence Theorem. The Verification Theorem states that if a solution to the HJB equation exists, then it has to be the optimal utility function and so the solution is unique. The Existence Theorem proves the existence of a solution to the HJB equation.

Next, we present a general formulation of a Verification Theorem and give a sketch of the main ideas used in the proof. This theorem asserts that in case of existing a classical solution to the HJB equation, this solution is the value function. The significance of this theorem lies in the fact that with it, the probabilistic problem is transformed into a deterministic one. The supremum in equation (3) is taken over the set of admissible strategies. In general these sets are made up of the class of predictable processes with some general integration properties. This make the direct analysis of the value function V a difficult task. Despite equation (6) is a nonlinear equation, and hence is not an easy one to work with, the supremum in this case is taken over a subset of $\mathbb{R}^m$ and in many cases is easier to work with the solution to equation (6) than working directly with the value function V.

Depending on the control problem different hypotheses are needed, here we only want to give a general formulation, so we omit them for this theorem.

Theorem 0.2.1 (Verification Theorem.)**.** *Assume there exists* $v \in C([0,\infty)\times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$*, solution to equation* (6) ***+ extra hypotheses****. Let*

$$\alpha^*(t,x) = argmax_{\alpha\in\Lambda} \left\{\mathcal{D}^\alpha[v](t,x) + f(t,x,\alpha)\right\}$$

and assume that $A^*(s) := \alpha^*(t-s, X(s))$ *is an "admissible" strategy. Then, for any* $A \in \mathcal{A}$ *we have that*

$$V(t,x;A) \le v(t,x) = V(t,x;A^*)$$

In particular, the solution is unique and

$$v(t,x) = \sup_{A\in\mathcal{A}} \left\{V(t,x;A)\right\}.$$

Sketch of the proof. The proof is divided in two main steps. In the first step we consider an arbitrary admissible strategy and we prove that its expected

utility function is a lower bound for the solution of the HJB equation. In the second step we work with the optimal strategy and we prove that its expected utility function is an upper bound for the solution of the HJB equation. In both cases the main tools used are Ito's formula and martingale arguments.
Step 1. Lower bound.
Let $A \in \mathcal{A}$ be any admissible process and denote by $X(s) = X(s; A)$ the controlled process with strategy A. Since $v \in C^{1,2}$, applying Ito's rule we get for $s \leq t \wedge \tau_D$

$$
\begin{aligned}
& e^{\int_0^s c(t-r,X(r),A(r))dr} v(t-s, X(s)) = v(t,x) \\
& + \int_0^s e^{\int_0^r c(t-y,X(y-),A(y))dy} (-v_t + \mathcal{D}^{A(s)}[v])(t-r, X(r-))dr \\
& + \int_0^s e^{\int_0^r c(t-y,X(y-),A(y))dy} Dv(t-r, X(r-))\sigma(t-r, X(r-))dW(r) \\
& + \iint_{(0,s]\times\mathbb{R}^n} [v(t-r, X(r-) + \gamma(t-r, X(r-), \alpha, z)) - v(t-r, X(r-))]\tilde{M}(dr, dz).
\end{aligned}
$$

From here we conclude that

$$
\begin{aligned}
& e^{\int_0^s c(t-r,X(r),A(r))dr} v(t-s, X(s)) \\
& - \int_0^s e^{\int_0^r c(t-y,X(y-),A(y))dy} (-v_t + \mathcal{D}^{A(s)}[v])(t-r, X(r-))dr
\end{aligned}
$$

is a martingale for $0 \leq s \leq t$.
Letting $s \uparrow t$

$$
\begin{aligned}
v(t,x) \geq & \mathbb{E}_x \left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r),A(r))dr} f(t-s, X(s), A(r))ds \right] \\
& + \mathbb{E}_x \left[e^{\int_0^t c(t-r,X(r),A(r))dr} h(X(t)) \mathbb{1}_{\tau_D \geq t} \right] \\
& + \mathbb{E}_x \left[e^{\int_0^{\tau_D} c(t-r,X(r),A(r))dr} g(t-\tau_D, X(\tau_D)) \mathbb{1}_{\tau_D < t} \right].
\end{aligned} \tag{7}
$$

And so $V(t, x; a) \leq v(t, x)$.
Step 2. Upper bound.
Repeating the same arguments with the optimal strategy A^*, we get an equality in equation (7) instead of an inequality. So we get that

$$V(t, x; A^*) = v(t, x)$$

and the proof is complete. □

0.2.2 Infinite time horizon.

For an infinite time horizon problem, we consider that the dynamic of the process X with control A is

$$\begin{aligned} X(s) =& x + \int_0^s b(X(s-), A(s))ds + \int_0^s \sigma(X(s-), A(s))dW(s) \\ &+ \int_{(0,s]} \int_{\mathbb{R}^n} \gamma(X(s-), A(s-), z)M(ds, dz). \end{aligned} \tag{8}$$

Again, the exit time is defined as

$$\tau_D := \inf\{s \geq 0 | X_A(s) \notin \overline{D}\}.$$

In this case the performance criteria is

$$\begin{aligned} V(x; A) :=& \mathbb{E}_x \left[\int_0^{\tau_D} e^{\int_0^s c(X_A(r), A(r))dr} f(X_A(s), A(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^{\tau_D} c(X_A(r), A(r))dr} g(X_A(\tau_D)) \right] \end{aligned} \tag{9}$$

where

- c represents a discounting factor,
- f represents a running utility function and
- g represents the final utility function.

As in the finite horizon, the objective is to maximize the expected utility function, that is, study the function V defined as

$$V(x) := \sup_{A \in \mathcal{A}} \{V(x; A)\}.$$

Following similar arguments as the ones of the finite time horizon, we propose as a HJB equation for this problem the following one

$$\begin{aligned} \sup_{\alpha \in \Lambda} \{\mathcal{E}^\alpha[u](x) + f(x, \alpha)\} =& 0, \qquad x \in D \\ u(x) =& g(x), \quad x \in \partial D \end{aligned} \tag{10}$$

where $\mathcal{E}$ is defined as

$$\begin{aligned}\mathcal{E}^\alpha[u](x) :=& \sum_{ij} a_{ij}(x,\alpha) D_{ij} u(x) + \sum_i b_i(x,\alpha) D_i u(x) + c(x,\alpha)u(x) \\ &+ \int_{\mathbb{R}^n} [u(x+\gamma(x,\alpha,z)) - u(x)]\nu(dz).\end{aligned} \tag{11}$$

with $\{a_{ij}\} = a = \sigma\sigma'$.

As in the finite time horizon, we need to prove both, a Verification and an Existence Theorem. A similar theorem as Theorem 0.2.1 can be stated in this case. The discussion made is also valid for this problem.

0.3 Existence to the HJB equation.

Once the Verification Theorem is proved an existence result is needed. At this stage, the original probabilistic problem has been transformed into a deterministic differential one. In general, the existence of a classical solution to equation (6) is a very difficult problem and so there exist many approaches depending on the control problem (see e.g. [?], [?], [?], [?], [?] [?], [?] and the references therein). In the next section we present a general method for a simplified problem when the HJB equation can be reduced into a semilinear differential equation.

0.3.1 Semilinear differential equation.

In many interesting problems (see e.g. [?], [?], [?], [?], [?], [?] and [?]) the HJB equation can be reduced to an equation of the form

$$\begin{aligned}-u_t(t,x) + \sum_{ij} a_{ij}(t,x) D_{ij}(t,x) + \sup_{\alpha\in\Lambda} \{\mathcal{L}^\alpha[u](t,x) + f^\alpha(t,x)\} =&0, \quad \text{in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) =&h(x), \qquad x\in\mathbb{R}^d,\end{aligned} \tag{12}$$

where

$$\mathcal{L}^\alpha := \sum_i b_i(t,x,\alpha) D_i u(t,x) + c(t,x,\alpha)u(t,x)$$

and

$$f^\alpha(\cdot) := f(\cdot,\alpha).$$

In a more general case, we are interested in studying the following semi-linear parabolic equation

$$
\begin{aligned}
-u_t(t,x) + \sum_{ij} a_{ij}(t,x)D_{ij}(t,x) + \sup_{\alpha\in\Lambda}\{\mathcal{L}^\alpha[u](t,x) + f^\alpha(t,x)\} = 0, \quad \text{in } (0,\infty)\times D, \\
u(0,x) = h(x), \qquad x \in D, \\
u(t,x) = g(t,x), \qquad (t,x) \in (0,\infty)\times\partial D.
\end{aligned} \tag{13}
$$

This equation is the HJB equation for the following control problem: the dynamic of the state process, X, with control A is given by

$$dX(s) = b(t-s, X(s), A(s))ds + \sigma(t-s, X(s))dW(s), \quad X(0) = x.$$

The utility function is defined as

$$
\begin{aligned}
V(t,x;A) :=& \mathbb{E}_x\left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X_A(r),A(r))dr} f(t-s, X_A(s), A(s))ds\right] \\
&+ \mathbb{E}_x\left[e^{\int_0^t c(t-r,X_A(r),A(r))dr} h(X_A(t))\mathbb{1}_{\tau_D\geq t}\right] \\
&+ \mathbb{E}_x\left[e^{\int_0^{\tau_D} c(t-r,X_A(r),A(r))dr} g(t-\tau_D, X_A(\tau_D))\mathbb{1}_{\tau_D<t}\right]
\end{aligned} \tag{14}
$$

and the value function is

$$V(t,x) := \sup_{A\in\mathcal{A}}\{V(t,x;A)\}.$$

We are interested in the existence of a classical solution to equation (13). Depending on the conditions of the coefficients and the domain, there exist different approaches to solve this problem. One of this approaches is by approximation with linear parabolic equations. Despite this technique is standard (see [**?**], Appendix E), depending on the coefficients and the domain, the existence of classical solutions to the linear problem is not trivial. For this reason we are interested in study the existence of classical solutions to linear parabolic equations under some broad assumptions on the coefficients and the domain. Next, we present a general formulation of an Existence Theorem and give a sketch of the ideas behind the linearization technique. As in the Verification Theorem, depending on the control problem different hypotheses are needed, since we only want to give a general formulation, we omit them.

Theorem 0.3.1 (Existence Theorem.). *Consider the following equation*

$$\begin{aligned} -u_t(t,x) + \sum_{ij} a_{ij}(t,x) D_{ij}(t,x) + \sup_{\alpha\in\Lambda} \{\mathcal{L}^\alpha[u](t,x) + f^\alpha(t,x)\} = 0, \quad & \text{in } (0,\infty)\times D, \\ u(0,x) = h(x), \qquad & x \in D, \\ u(t,x) = g(t,x), \qquad & (t,x) \in (0,\infty)\times\partial D. \end{aligned} \tag{15}$$

Then under ***hypotheses****, there exists a classical solution to equation* (15)*.*

Sketch of the proof. Let $\mathcal{L}_2[u]$ be defined as

$$\mathcal{L}_2[u] := -u_t + \sum_{ij} a_{ij} D_{ij} u.$$

Let $\alpha_0 \in \Lambda$ and $u^{(0)}$ be the solution to

$$\begin{aligned} \mathcal{L}_2[u^{(0)}] + \mathcal{L}_1^{\alpha_0}[u^{(0)}] + f^{\alpha_0} = 0, \quad & (0,\infty)\times D \\ u^{(0)}(t,x) = G(t,x), \quad & (t,x) \in \partial((0,\infty)\times D). \end{aligned}$$

For $n \geq 1$, let

$$A^{(n-1)} := \text{argmax}_{\alpha\in\Lambda} \left\{\mathcal{L}_1^\alpha[u^{(n-1)}] + f^\alpha\right\},$$

and $u^{(n)}$ be the solution to

$$\begin{aligned} \mathcal{L}_2[u^{(n)}] + \mathcal{L}_1^{A^{(n-1)}}[u^{(n)}] + f^{A^{(n-1)}} = 0, \quad & (0,\infty)\times D \\ u^{(n)}(t,x) = G(t,x), \quad & (t,x) \in \partial((0,\infty)\times D). \end{aligned}$$

Because

$$A^{(n)} \in \text{argmax} \left\{\mathcal{L}_1^\alpha[u^{(n)}] + f^\alpha\right\}$$

using a form of the maximum principle for parabolic equations, we can prove that

$$u^{(n)} \leq u^{(n+1)}.$$

The desired solution u of equation (15) is the limit of $u^{(n)}$ as $n \to \infty$. □

0.3.2 Linear differential equations.

We are interested in the existence and uniqueness of a classical solution to the Cauchy-Dirichlet problem for a linear parabolic differential equation. Let $\mathcal{L}$ be the differential operator

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x) D_{ij} u(t,x) + \sum_{i=1}^{d} b_i(t,x) D_i u(t,x)$$

where $\{a_{ij}\} = a = \sigma\sigma'$, $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The Cauchy-Dirichlet problem is

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) = -f(t,x), &\qquad (t,x) \in (0,\infty) \times D, \\ u(0,x) = h(x), &\qquad x \in D, \\ u(t,x) = g(t,x), &\qquad (t,x) \in (0,\infty) \times \partial D \end{aligned} \tag{16}$$

where $D \subset \mathbb{R}^d$ is an open, connected set.

In the case of bounded domains, the Cauchy-Dirichlet problem is well understood (see [?] and [?] for a detailed description of this problem). Moreover, when the domain is unbounded and the coefficients are bounded, the existence of a classical solution to equation (16) is well known. For a survey of this theory see [?] and [?] where the problem is studied with analytical methods and [?] for a probabilistic approach.

In the case of general unbounded domains, many authors have considered unbounded coefficients, (b and σ) satisfying a Lyapunov type growth assumption, and bounded data. Using the theory of semigroups, the existence of a classical solution to the Cauchy-Dirichlet problem has been studied in [?], [?], [?], [?], among others. The main technique used in this papers is to consider a sequence of nested bounded domains that approximate the domain D, solve the problem in the bounded domains and then prove that the sequence of solutions converges to the solution of the original problem. This construction impose some restrictions about the geometry of the domain and the regularity of the functions. In all these cases the coefficients are at least C^1.

Our motivation for studying linear equations came from stochastic control problems. In the proof of Theorem 0.3.1 the coefficients of the linear equations have some terms of the form

$$A^{(n-1)} := \text{argmax}_{\alpha\in\Lambda} \left\{ \mathcal{L}_1^{\alpha}[u^{(n-1)}] + f^{\alpha} \right\}.$$

In general, this coefficients only satisfies a continuity regularity property, hence we are interested in studying solutions to equation (16) with continuous unbounded coefficients, not necessarily differentiable. A suitable approach for this is a probabilistic one. In the following section we present some results concerning the existence of classical solutions to parabolic problems using probabilistic methods.

0.3.3 A probabilistic approach.

Consider the Cauchy problem for a linear parabolic differential equation, that is,

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x) \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= h(x) \quad \text{for } x \in \mathbb{R}^d. \end{aligned} \tag{17}$$

This is a special case of the Cauchy-Dirichlet problem (16) when $D = \mathbb{R}^d$. A very well-known result, the Feynman-Kac's Theorem, relates the solution to the Cauchy problem with a functional of the solution to a stochastic differential equation. We enounce it

Theorem 0.3.2 (Feynman-Kac's Formula.). *Let $\sigma : [0,\infty) \times \mathbb{R}^d$ and $b : [0,\infty) \times \mathbb{R}^d$ be continuous functions, locally Lipschitz continuous in x with the following growth condition: for all $T > 0$, exists $K_1(T)$ such that*

$$\|\sigma(t,x)\|^2 + \|b(t,x)\|^2 \leq K_1(T)^2(1 + \|x\|^2),$$

for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$.

Let $c : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$, $f(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$ be continuous functions such that for all $T > 0$, exists $K_2(T)$

$$|h(x)| + |f(t,x)| \leq K_3(T)(1 + \|x\|^{2k})$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$ for some $k \geq 1$.

Assume there exists a classical solution $u \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d)$ to equation (17) with the polynomial growth condition

$$\sup_{0 \leq t \leq T} |u(t,x)| \leq C(1 + \|x\|^{2\mu}), \quad x \in \mathbb{R}^d,$$

for some constants $C > 0$ and $\mu \geq 1$.

Then u has the representation

$$u(t,x) = \mathbb{E}_x \left[\int_0^t e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds + e^{\int_0^t c(t-r,X(r))dr} h(X(t)) \right], \tag{18}$$

where X is the solution to the stochastic differential equation

$$dX(s) = b(t-s,X(s))ds + \sigma(t-s,X(s))dW(s), \quad X(0) = x.$$

For a proof of this theorem see [**?**] section 6.3.

This theorem states that in case of existence of a classical solution u, then it has the representation given in equation (18). The natural question is if we can proceed in the other direction, that is, define $v : [0,\infty) \times \mathbb{R}^d$ as

$$\begin{aligned} v(t,x) =& \mathbb{E}_x \left[\int_0^t e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^t c(t-r,X(r))dr} h(X(t)) \right]. \end{aligned} \tag{19}$$

and then prove, that under some suitable assumptions about the coefficients, the function v satisfies the Cauchy problem.

In order to understand this approach, in the following part we will explain it considering only the Brownian motion and the Laplace operator.

0.3.4 Brownian motion and the Laplace operator.

Consider the Cauchy problem for the heat equation,

$$\begin{aligned} u_t(t,x) =& \frac{1}{2}\Delta u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) =& h(x), \qquad x \in \mathbb{R}^d. \end{aligned} \tag{20}$$

In this section, we present a general procedure to construct a solution to equation (20). In all the theorems we present only the main ideas used in the proofs (see [**?**] chapter 4 for a detailed discussion).

The relation between the Brownian motion and the Laplace operator has many approaches, one of them is the following: the Fundamental Theorem of Calculus states that if $g : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is a function such that $g \in C^{1,1}([0,\infty) \times \mathbb{R}^d)$ and $\gamma : [0,\infty) \to \mathbb{R}^d$ is a differentiable curve, then

$$g(s,\gamma(s)) = g(0,\gamma(0)) + \int_0^s g_t(r,\gamma(r))dr + \int_0^s Dg(r,\gamma(r)) \cdot d\gamma(r).$$

The Brownian motion has the characteristic that despite their sample paths are continuous, they are nowhere differentiable with unbounded variation. This requires a new form of integration (Ito's integral), and so we get that if $g \in C^{1,2}([0,\infty) \times \mathbb{R}^d)$ and $W(s)$ is a d-dimensional Brownian motion, then

$$\begin{aligned} g(s, W(s)) =& g(0, W(0)) + \int_0^s g_t(r, W(r))dr + \int_0^s Dg(r, W(r)) \cdot dW(r) \\ &+ \int_0^s \frac{1}{2}\Delta g(r, W(r))dr \\ =& g(0, W(0)) + \int_0^s \left(g_t(r, W(r)) + \frac{1}{2}\Delta g(r, W(r)) \right) dr \\ &+ \int_0^s Dg(r, W(r)) \cdot dW(r). \end{aligned}$$

The extra term (Δg) comes from the quadratic variation of the Brownian motion. Hence, Ito's formula and some martingale arguments are the fundamental tools needed in order to construct a solution to equation (20) with the Brownian motion.

For simplicity, assume that h is a bounded function. The first step is to find a martingale

Theorem 0.3.3. *Assume $u \in C^{1,2}((0,\infty) \times \mathbb{R}^d)$ satisfies equation (20). Then*

$$M(s) := u(t-s, W(s)),$$

is a local martingale.

Proof. It follows from Ito's formula

$$\begin{aligned} u(t-s, W(s)) =& u(t, W(0)) + \int_0^s \left(-u_t(t-r, W(r)) + \frac{1}{2}\Delta u(t-r, W(r)) \right) dr \\ &+ \int_0^s Dg(t-r, W(r)) \cdot dW(r) \\ =& \text{local martingale} \end{aligned}$$

since $-u_t + \frac{1}{2}\Delta u = 0$. □

Next we prove that the solution is unique

Theorem 0.3.4. *Assume $u \in C([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$ is a bounded solution to equation* (20)*. Then*

$$u(t,x) = \mathbb{E}_x\left[h(W(t))\right].$$

Proof. Since u is bounded, then $M(s)$ is an uniformly integrable martingale, and so is closed. Thanks to the martingale property

$$u(t,x) = \mathbb{E}_x\left[u(t-s, W(s))\right].$$

Letting $s \uparrow t$ and using the boundary condition the proof is complete. □

Let $v : [0,\infty)\times\mathbb{R}^d$ be defined as

$$v(t,x) := \mathbb{E}_x\left[h(W(t))\right]. \tag{21}$$

The next step is to prove that if $v \in C^{1,2}$ then it fulfils equation (20) and so we have existence.

Theorem 0.3.5. *Let v be defined as in equation* (21)*. Assume that $v \in C([0,\infty)\times\mathbb{R}^d)\cap C^{1,2}((0,\infty)\times\mathbb{R}^d)$. Then v fulfils equation* (20)*.*

Proof. The Markov property implies that for any $s < t$

$$\mathbb{E}_x\left[h(W(t))|\mathcal{F}_s\right] = \mathbb{E}_{W(s)}\left[h(W(t-s))\right] = v(t-s, W(s))$$

and so $M_s := v(t-s, W(s))$ is a martingale. Again, using Ito's formula we get

$$\begin{aligned} v(t-s, W(s)) = & v(t,x) + \int_0^s \left(v_s + \frac{1}{2}\Delta v\right)(t-r, W(r))dr \\ & + \text{local martingale.} \end{aligned}$$

Combining both equations we get that

$$\left\{\int_0^s \left(-v_t + \frac{1}{2}\Delta v\right)(t-r, W(r))dr\right\}_{0\le s\le t}$$

is a local martingale. Since it is continuous and locally of bounded variation, then it has to be identically 0, and so

$$-v_t(t,x) + \frac{1}{2}\Delta v(t,x) = 0$$

for all $(t,x) \in (0,\infty)\times\mathbb{R}^d$.

The boundary condition is proved by letting $t \downarrow 0$ and the Dominated Convergence Theorem. □

The final part is to prove the regularity of v. This follows from the continuity of the Brownian motion with respect to x and the integration with respect to the Gaussian kernel. We enounce the theorem (see [**?**] section 4.1 for a proof).

Theorem 0.3.6. *Assume that h is a bounded continuous function and let v be defined as in equation* (21)*. Then $v \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2}((0,\infty) \times \mathbb{R}^d)$*

In conclusion, we presented a method that constructs a solution to the Heat Equation by means of a Brownian motion. In the next chapters, we use similar ideas to construct a solution for some more general parabolic problems using, instead of the Brownian motion, a stochastic process that satisfies a stochastic differential equation.

0.4 Risk process.

0.4.1 Cramér-Lundberg model.

The theory of non-life risk has presented a great development since Lundberg in 1903 introduced a collective risk model based on a Poisson claim process. Lundberg proposed that the reserve capital of an insurance company has the following dynamic

$$R(t) = x + ct - S(t),$$

where

- $x \geq 0$ is the initial capital,
- $c > 0$ stands for the premium income rate and
- $S(t) = \sum_{n=1}^{N(t)} \xi_n$, where $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ and jump times $\{\eta_n\}_{n=1}^{\infty}$; $\{\xi_n\}_{n=1}^{\infty}$ are i.i.d. positive random variables independent of the Poisson process, corresponding to the incoming claims, with common distribution Q and mean $\mu < \infty$.

This process is one of the simplest models for the reserve capital. It only considers three fundamental aspects: the original reserve, an income corresponding to the charged premiums and an outcome corresponding to the paid claims. It assumes that the number of insured clients is large enough so that the premium rate is constant independent of the claims received by the company. The exponential interarrival times (because of the Poisson process), makes this process a suitable one for modeling non-life insurance due to the lack of memory. Despite its simplicity, this model is not completely understood and it is still studied in these days.

One of the main concerns for the company is to understand the possibility of a default, that is, the probability that the reserve capital becomes insufficient in order to face a claim. For that, define the ruin time as

$$\tau := \inf\{t > 0 | R(t) < 0\}$$

and the ruin and survival probability as

$$\psi(x) := \mathbb{P}\left[\tau < \infty | R(0) = x\right],$$
$$\delta(x) := \mathbb{P}\left[\tau = \infty | R(0) = x\right].$$

The probability of ruin is not a classical utility function as we discussed in Section 0.2.1, however it is a very important topic for the insurance mathematics. For a regulation agency it is important to study the probability of ruin as a function of the reserve capital in order to demand a minimum retention level for the insurance company to guarantee the non-default with its clients.

The dynamic of the Cramér-Lundberg model is an special case of the dynamic described in equation (8), when:

- $b(x, \alpha) \equiv c$,
- $\sigma(x, \alpha) \equiv 0$,
- $\gamma(x, \alpha, z) = -z$ and the Poisson measure is generated by the Levy process $\{S(t)\}_{t \geq 0}$.

The performance criteria (ruin probability) is an special case of function (9) when:

- $c(x, \alpha) \equiv 0$,
- $f(x, \alpha) \equiv 0$,
- $g(x) \equiv 1$.

As in the general case, one of the main tools for the study of the properties of the ruin probability is the HJB equations' approach. For the survival probability, we have the following theorem (for a proof see [**?**] section 5.3)

Theorem 0.4.1. *The survival probability $\delta(x)$ is continuous in $\mathbb{R}_+$. Let Q denote the distribution function of the claim ξ. If Q admits a density function then $\delta \in C^1$ and fulfils*

$$0 = c\delta'(x) + \lambda \int_0^\infty [\delta(x-z) - \delta(x)]dQ(z). \tag{22}$$

Thanks to equation (22), if the claims has an exponential distribution then

$$\psi(x) = \frac{\lambda\mu}{c} \exp\left\{ -\frac{c-\lambda\mu}{c\mu} x \right\}.$$

In the general case, thanks to equation (22), some bounds and the asymptotic behavior for the ruin probability are known (see [?] section 5.4).

0.4.2 Ruin model with investment.

Since the introduction of the Cramér-Lundberg process, more complex models have been studied taking into account aspects of the insurance like investment, reinsurance, payment of dividends among other possibilities. There are many authors for this problem (see [?], [?], [?], [?], [?], [?] and [?] and the references therein). In all these cases, finding the ruin probability is one of the main problems. In this section, we focus in the investment problem for an insurance company, in particular in [?] and [?].

Consider the Cramér-Lundberg process,

$$R(t) = x + ct - S(t),$$

and assume that additionally we let the insurance company to put their reserve capital into some investment instruments: a non-risk bonus and a risky asset, with the following dynamics

$$\begin{aligned} dZ_0(t) =& Z_0(t) r dt, \\ dZ(t) =& Z(t)\mu dt + Z(t)\sigma dW(t), \end{aligned}$$

where $\mu > r \geq 0$, $\sigma > 0$ and $\{W(t)\}_{t\geq 0}$ is an standard Brownian motion independent of the claim process $\{S(t)\}_{t\geq 0}$.

Let $\mathcal{A}$ denote the set of all admissible investment strategies and let $A = \{A_t\}_{t\geq 0} \in \mathcal{A}$ where A_t denotes the amount of money invested in the risky asset at time t. The reserve process with investment strategy A is

$$\begin{aligned} X(t;A) =& R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s) \frac{dZ_0(s)}{Z_0(s)} \\ =& x + \int_0^t [c + bA_s + rX(s-)]ds + \int_0^t \sigma A_s dW(s) - S(t), \end{aligned} \tag{23}$$

where $b := \mu - r$.

We define the ruin time as

$$\tau(A) := \inf\{t > 0 | X(t;A) < 0\}$$

and the respective probability of ruin and survival

$$\psi(x;A) := \mathbb{P}\left[\tau(A) < \infty | X(0;A) = x\right],$$

$$\delta(x;A) := \mathbb{P}\left[\tau(A) = \infty | X(0;A) = x\right].$$

Let

$$\delta(x) := \sup_{A \in \mathcal{A}} \{\delta(x;A)\}.$$

Our goal is to analyze the survival probability under an optimal investment strategy A_t^*, that is,

$$\delta(x;A^*) = \delta(x).$$

Again, this problem is an special case of the general problem presented in section 0.2.2, considering:

- $b(x,\alpha) = c + rx + (\mu - r)\alpha$,
- $\sigma(x,\alpha) = \sigma\alpha$,
- $\gamma(x,\alpha,z) = -z$ and the Poisson measure is generated by the Levy process $\{S(t)\}_{t \geq 0}$,
- $c(x,\alpha) \equiv 0$,
- $f(x,\alpha) \equiv 0$,
- $g(x) \equiv 1$.

Following a similar argument as the one used to propose equation (6), we propose the following HJB equation for this problem

$$\sup_{\alpha\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2\alpha^2 f''(x)+(c+b\alpha+rx)f'(x)+\lambda\int_0^\infty[\delta(x-z)-\delta(x)]dQ(z)\right\}=0. \tag{24}$$

If $f''(x)<0$ then the supremum exists, and in this case

$$\alpha^*(x)=-\frac{bf'(x)}{\sigma^2 f''(x)}. \tag{25}$$

0.4.3 Verification and Existence Theorems.

In this section, we enounce the Verification and the Existence Theorems for the Optimal Investment problem given in [**?**].

Theorem 0.4.2 (Verification Theorem.)**.** *Assume there exists a solution $f(x)$ of* (24) *with maximizing function $\alpha^*(x)$ with the following properties:*

- $f(0), f'(0)>0$,
- $f(x)=0$ *for* $x<0$,
- $\lim_{x\to\infty} f(x)=1$ *and*
- $f(x)\in C^2(0,\infty)$.
- $\alpha^*(0)=0$

Then $f'(x)>0$ for $x>0$, and if A_t is an arbitrary admissible investment strategy for which the reserve process $X(t;A), t\geq 0$ is defined, then the corresponding survival probability satisfies

$$\delta(x;A)\leq f(x)=\delta(x;A^*),\quad x\geq 0$$

where $A^*_t=\alpha^*(X(t-))$.

The main ideas are similar to the ones for Theorem 0.2.1, however there exist some particular features of this problem that are worth to be mentioned. The boundary conditions are not classical in a differential equation's sense. This happens because the risk process cannot reach the boundary in a continuous form ($\alpha^*(0)=0$). The other boundary condition is given in infinite, this leads to the extra problem of proving that the risk process defined in equation (23) tends to infinite for any admissible strategy. These problems modify the general procedure considered for a proof to a Verification Theorem. In Chapter **??** we give a detailed proof of this Theorem for a more general investment model.

Regarding the existence problem, substituting α^* in equation (24) we get that

$$\lambda \int_0^\infty f(x-z)dQ(z) - \lambda f(x) + (c+rx)f'(x) = \frac{1}{2}\frac{\mu^2 f'(x)^2}{\sigma^2 f''(x)}. \tag{26}$$

As we mentioned above there exist some difficulties with this equation: the boundary condition, the integral part and the nonlinear part. Hipp and Plum in [**?**] solve this problem by considering each problem separately. The solution to equation (26) is constructed by approximation. There are two main steps in the approximation: the first one approximates the integral part $\int f dQ$ and the second one approximates the nonlinearity $\frac{f'}{f''}$. Next, we present the Existence Theorem and give a brief sketch of the proof.

Theorem 0.4.3 (Existence Theorem.)**.** *Assume that Q has a density q. Then equation* (24) *has a solution $f(x)$ with the properties required by the Verification Theorem*

Sketch of the proof. We consider two main steps:
Step 1. Approximation of the integral part.
Consider the following equation

$$\lambda g(x) - \lambda f(x) + (c+rx)f'(x) = \frac{1}{2}\frac{\mu^2 f'(x)^2}{\sigma^2 f''(x)}, \tag{27}$$

and assume there exists a classical solution. Then let δ_0 be the solution to equation (6) and define f_n for $n \geq 1$ as the solution to

$$\lambda \int_0^x f_{n-1}(x-z)dQ(z) - \lambda f_n(x) + (c+rx)f_n'(x) = \frac{1}{2}\frac{\mu^2 f_n'(x)^2}{\sigma^2 f_n''(x)}.$$

Then it is proved that $f := \lim_{n\to\infty} f_n$ is the solution to equation (26).
Step 2. Approximation of the nonlinear part.
To find a solution to equation (27) the problem is transformed into the following equivalent system

$$\lambda(V(x) - u(x)) - c(x)V'(x) = \frac{1}{2}\sqrt{U(x)}V'(x), \tag{28}$$

and

$$\sqrt{U(x)}\left[\left(\lambda+\frac{1}{2}-c'(x)\right)V'(x)-\lambda g'(x)\right]+c(x)V'(x)=\frac{1}{4}U'(x)V'(x), \quad (29)$$

To find a solution to the system (28), (29) an approximation argument is used again, that is, define $V_0 \equiv 0$ and for $n \geq 0$ solve the following equations alternatively

$$\frac{1}{4}U'_{n+1}=c(x)+\sqrt{U_{n+1}(x)}\left[\lambda+\frac{1}{2}-c'(x)-\frac{g'(x)(c(x)+\frac{1}{2}\sqrt{U_{n+1}(x)})}{\lambda(V_n(x)-g(x))}\right]. \quad (30)$$

and then

$$V'_{n+1}(x)=\frac{\lambda(V_{n+1}(x)-g(x))}{c(x)+\frac{1}{2}\sqrt{U_{n+1}(x)}} \quad (31)$$

It can be proved that $(U,V)=\lim_{n\to\infty}(U_n,V_n)$ is the solution to the system (28), (29) and so there exists a solution to equation (27). □

0.4.4 Risk process with stochastic volatility.

In recent years, different generalizations of the classical Black-Scholes model for the dynamics of the asset prices have been studied. It has been considered that the parameters of the model are stochastic and depend on external factors. External factor can be: a leader interest rate, an exchange rate or another asset price that have a strong influence in the market. We work with generalized investment instruments.

We are interested in generalize the results of [**?**] and [**?**] when the market has the following components:

- Let $\{W_1(t),W_2(t)\}_{t\geq 0}$ be a two dimensional standard Brownian motion independent of the process $R(t)$.

- The external factor has the following dynamic

$$Y(t)=y+\int_0^t g(Y(s))ds+\beta(\rho W_1(t)+\epsilon W_2(t)),$$

  with $0\leq\rho\leq 1$, $\epsilon=\sqrt{1-\rho^2}$.

- For investment we have a non-risk bonus and a risky asset, both depending on the external factor

$$\begin{aligned}dZ_0(t) =&Z_0(t)r(Y(t))dt,\\ dZ(t) =&Z(t)\mu(Y(t))dt + Z(t)\sigma(Y(t))dW_1(t),\end{aligned}$$

The reserve process with investment strategy $\{A_t\}_{t\geq 0}$ is

$$\begin{aligned}X(t;A) =&R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s)\frac{dZ_0(s)}{Z_0(s)}\\ =&x + \int_0^t [c + b(Y(s))A_s + r(Y(s))X(s-)]ds\\ &+ \int_0^t \sigma(Y(s))A_s dW_1(s) - S(t),\end{aligned}$$

where $b(\cdot) := (\mu - r)(\cdot)$.

We define the ruin time as

$$\tau(A) := \inf\{t > 0 | X(t;A) < 0\}$$

and the respective probability of ruin and survival

$$\psi(x,y;A) := \mathbb{P}\left[\tau(A) < \infty | X(0;A) = x, Y(0) = y\right],$$

$$\delta(x,y;A) := \mathbb{P}\left[\tau(A) = \infty | X(0;A) = x, Y(0) = y\right].$$

Our goal is to analyze the survival probability under an optimal investment strategy A_t^* that maximize it over all admissible strategies, that is,

$$\delta(x,y;A^*) = \sup_{A\in\mathcal{A}}\{\delta(x,y;A)\}.$$

In Chapter **??** we present some of the partial results of this investigation.

Chapter 1

Linear parabolic differential equations.

In this chapter we study the existence and uniqueness of a classical solution to the Cauchy-Dirichlet problem for a linear parabolic differential equation in a general unbounded domain. Let $\mathcal{L}$ be the differential operator

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x) D_{ij} u(t,x) + \sum_{i=1}^{d} b_i(t,x) D_i u(t,x)$$

where $\{a_{ij}\} = a = \sigma\sigma'$, $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The Cauchy-Dirichlet problem is

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x), && (t,x) \in (0,\infty) \times D, \\ u(0,x) &= h(x), && x \in D, \\ u(t,x) &= g(t,x), && (t,x) \in (0,\infty) \times \partial D \end{aligned} \tag{1.1}$$

where $D \subset \mathbb{R}^d$ is an unbounded, open, connected set with regular boundary.

As we mentioned in the introduction, when the domain is bounded or the coefficients are bounded the problem is well understood (see [?], [?], [?], [?] and [?])

In the last years, parabolic equations with unbounded coefficients in unbounded domains have been studied in great detail. Many authors have considered unbounded coefficients, (b and σ) satisfying a Lyapunov type growth assumption, and bounded data. Using the theory of semigroups, the existence of a classical solution to the Cauchy-Dirichlet problem has been studied by Fornaro, Metafune and Priola (2004) in [?], Bertoldi and Fornaro (2005) in [?], Bertoldi, Fornaro and Lorenzi (2007) in [?], Hieber, Lorenzi and Rhandi (2007) in [?] among others.

We follow the same ideas presented in Section ??.

1.1 Preliminaries, hypotheses and notation.

In this section we present the hypotheses and the notation used in this chapter.

We will consider $D \subset \mathbb{R}^d$ an unbounded, open, connected set with boundary ∂D and closure $\overline{D}$. We assume that D has a regular boundary, that is, for any $x \in \partial D$, x is a regular point (see [?] Chapter III Section 4 or [?] Chapter 2 Section 4, for a detailed discussion of regular points). We denote the hypotheses on D as **H0**.

1.1.1 Stochastic differential equation.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{s\geq 0})$ be a complete filtered probability space and let $\{W\} = \{W_i\}_{i=1}^d$ be a d-dimensional brownian motion defined in it. For $t \geq 0$ and $x \in \overline{D}$ consider the stochastic differential equation

$$dX(s) = b(t-s, X(s))ds + \sigma(t-s, X(s))dW(s), \quad X(0) = x, \tag{1.2}$$

where $b = \{b_i\}_{i=1}^d$ and $\sigma = \{\sigma_{ij}\}_{i,j=1}^d$. Despite this process is the natural one for solving equation (1.1), it does not posses many good properties. The continuity of the flow process does not imply the continuity with respect to t. Furthermore, although this process is a strong Markov process, is not homogeneous in time, a very useful property for proving the results in this chapter.

To overcome these difficulties, we augment the dimension considering the following process

$$d\xi(s) = -ds, \quad \xi(0) = t. \tag{1.3}$$

Then the process $\{\xi(s), X(s)\}$ is solution to

$$\begin{aligned} d\xi(s) =& -ds, \\ dX(s) =& b(\xi(s), X(s))ds + \sigma(\xi(s), X(s))dW(s), \end{aligned} \tag{1.4}$$

with $(\xi(0), X(0)) = (t, x)$. Throughout this chapter we will use both processes, $X(s)$ and $(\xi(s), X(s))$, in order to simplify the exposition.
We need to define the following stopping times

$$\tau_D := \inf\{s > 0 | X(s) \notin \overline{D}\} \tag{1.5}$$

and

$$\tau := \tau_D \wedge t. \tag{1.6}$$

Remark 1.1.1. *Observe that τ is the exit time of the process $(\xi(s), X(s))$ from the set $[0, \infty) \times \overline{D}$, i.e.*

$$\tau = \inf\{s > 0 | (\xi(s), X(s)) \notin [0, \infty) \times \overline{D}\}.$$

We can not guarantee that the process $X(s)$ leaves the set D in a finite time, however the process $\xi(s)$ reaches the boundary $s = 0$ at time t. Thus, the joint process $(\xi(s), X(s))$ leaves the set $[0, \infty) \times \overline{D}$ in a bounded time.

We assume the following hypotheses on the coefficients b and σ. We denote them by **H1**. The matrix norm considered is $\|\sigma\|^2 := \mathrm{tr}\sigma\sigma' = \sum_{i,j} \sigma_{ij}^2$.

H1:

Let

$$\begin{aligned} \sigma(r, x) :& \mathbb{R} \times \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d) \\ b(r, x) :& \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \end{aligned}$$

be continuous functions such that

1. **(Continuity.)** Let $\lambda \in (0, 1)$. For all $T > 0$, $n \geq 1$ there exists $L_1(T, n)$ such that

   $$\|\sigma(r, x) - \sigma(s, y)\|^2 + \|b(r, x) - b(s, y)\|^2 \leq L_1(T, n)^2 (|r - s|^{2\lambda} + \|x - y\|^2),$$

   for all $|r|, |s| \leq T$, $\|x\| \leq n$, $\|y\| \leq n$.

2. **(Linear growth.)** For each $T > 0$, there exists a constant $K_1(T)$ such that

   $$\|\sigma(r, x)\|^2 + \|b(r, x)\|^2 \leq K_1(T)^2 (1 + \|x\|^2),$$

   for all $|r| \leq T$, $x \in \mathbb{R}^d$.

3. **(Local ellipticity.)** Let $A \subset \overline{D}$ be any bounded, open, connected set and $T > 0$. There exists $\lambda(T, A) > 0$ such that for all $(r, x) \in [0, T] \times \overline{A}$ and $\eta \in \overline{A}$
$$\sum_{i,j} a_{ij}(r, x)\eta_i\eta_j \geq \lambda(T, A)\|\eta\|^2.$$
where $\{a_{ij}\} = a = \sigma\sigma'$

Remark 1.1.2. *Observe that the local ellipticity is only assumed on* $[0, \infty) \times \overline{D}$. *This condition is used to prove the existence of a classical solution to equation* (1.1) *and so is only needed in that set. The local Lipschitz condition and the linear growth are assumed on* $\mathbb{R} \times \mathbb{R}^d$ *to ensure the existence of a strong solution to equation* (1.4) *for* $s \in [0, \infty)$.

Remark 1.1.3. *It follows from the non degeneracy (the local ellipticity) of the process* $X(s)$, *the regular boundary of the set* D *and Lemma 4.2, Chapter 2 in [**?**], that for any* $x \in D$
$$\mathbb{P}_{t,x}\left[\tau = \tau'\right] = 1.$$
where $\tau' := \inf\{s > 0 | (\xi(s), X(s)) \notin (0, \infty) \times D\}$ *(see Remark 1.1.1).*

The next proposition presents some of the properties of the process (ξ, X) required in this work.

Proposition 1.1.1. *As a consequence of* ***H1***, (ξ, X) *has the following properties:*

- *for all* $(t, x) \in [0, \infty) \times \mathbb{R}^d$, *there exists a unique strong solution to equation* (1.4),
- *the process* $\{\xi(s), X(s)\}_{s \geq 0}$ *is a strong homogeneous Markov process,*
- *the process* $\{\xi(s), X(s)\}_{s \geq 0}$ *does not explode in finite time a.s.,*
- *the flow process* $\{\xi(s, t), X(s, x)\}_{s \geq 0, (t,x) \in [0,\infty) \times \mathbb{R}^d}$ *is continuous a.s.,*
- *for all* $x \in \mathbb{R}^d$, $T > 0$ *and* $r \geq 1$
$$\mathbb{E}_x\left[\sup_{0 \leq s \leq T} \|X(s)\|^{2r}\right] \leq C(T, K_1, r)(1 + \|x\|^{2r}). \tag{1.7}$$

Proof. See [?] chapter 6 or [?] chapter V for a proof of these properties. □

Alternatively, we may assume these less restrictive assumptions. We denote them by **H1'**

H1':

Let

$$\sigma(r,x):[0,\infty)\times\mathbb{R}^d\to\mathcal{M}(\mathbb{R}^d\times\mathbb{R}^d)$$
$$b(r,x):[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d,$$

be continuous functions such that

1. **(Continuity.)** Let $\lambda\in(0,1)$. For all $T>0$, $n\geq 1$ there exists $L_1(T,n)$ such that

   $$\|\sigma(r,x)-\sigma(s,y)\|^2+\|b(r,x)-b(s,y)\|^2\leq L_1(T,n)^2(|r-s|^{2\lambda}+\|x-y\|^2),$$

   for all $0\leq r,s\leq T$, $\|x\|\leq n$, $\|y\|\leq n$.

2. **(Linear growth.)** For each $T>0$, there exists a constant $K_1(T)$ such that

   $$\|\sigma(r,x)\|^2+\|b(r,x)\|^2\leq K_1(T)^2(1+\|x\|^2),$$

   for all $0\leq r\leq T$, $x\in\mathbb{R}^d$.

3. **(Local ellipticity.)** Let $A\subset\overline{D}$ be any bounded, open, connected set and $T>0$. There exists $\lambda(T,A)>0$ such that for all $(r,x)\in[0,T]\times\overline{A}$ and $\eta\in\overline{A}$

   $$\sum_{i,j}a_{ij}(r,x)\eta_i\eta_j\geq\lambda(T,A)\|\eta\|^2.$$

   where $\{a_{ij}\}=a=\sigma\sigma'$

Remark 1.1.4. *If we assume **H1'** we can extend the functions $b(r,x)$ and $\sigma(r,x)$ to be defined for negative values of r as follows: let $\hat{b}$ and $\hat{\sigma}$ be defined as*

$$\hat{b}(r,x)=\begin{cases}b(r,x), & \text{if } r\geq 0,\\ b(0,x), & \text{if } r<0,\end{cases}$$

and

$$\hat{\sigma}(r,x)=\begin{cases}\sigma(r,x), & \text{if } r\geq 0,\\ \sigma(0,x), & \text{if } r<0.\end{cases}$$

*It is easy to see that these functions satisfy **H1** with the same constants L_1 and K_1.*

1.1.2 The Cauchy-Dirichlet problem.

Consider the following differential operator

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x)D_{ij}u(t,x) + \sum_{i=1}^{d} b_i(t,x)D_i u(t,x)$$

where $D_i = \frac{\partial}{\partial x_i}$, $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ and $\{a_{ij}\}_{i,j=1}^d = a = \sigma\sigma'$. For the rest of the chapter, we assume that the coefficients of $\mathcal{L}$ satisfies **H1**.
The Cauchy-Dirichlet problem for a linear parabolic equation is

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x), && (t,x) \in (0,\infty)\times D, \\ u(0,x) &= h(x), && x \in D, \\ u(t,x) &= g(t,x), && (t,x) \in (0,\infty)\times \partial D. \end{aligned} \tag{1.8}$$

We assume the following hypotheses for the functions c, f, h and g. We denote them by **H2**.

H2:

1. Let

$$\begin{aligned} c(r,x) &: [0,\infty)\times \overline{D} \to \mathbb{R} \\ f(r,x) &: [0,\infty)\times \overline{D} \to \mathbb{R}, \end{aligned}$$

   be continuous functions such that

   - **(Continuity.)** Let $\lambda \in (0,1)$. For all $T > 0$, $n \geq 1$ there exists a constant $L_2(T,n)$ such that

     $$\|f(r,x)-f(s,y)\|^2+\|c(r,x)-c(s,y)\|^2 \leq L_2(T,n)^2(|r-s|^{2\lambda}+\|x-y\|^2),$$

     for all $0 \leq s,r \leq T$, $x,y \in \overline{D}$ with $\|x\| \leq n$, $\|y\| \leq n$.
   - **(Growth.)** There exists $c_0 \geq 0$ such that

     $$c(r,x) \leq c_0 \quad \text{for all } (r,x) \in [0,\infty)\times \overline{D}.$$

     There exists $k > 0$, such that for all $T > 0$, a constant $K_2(T)$ exists such that

     $$|f(r,x)| \leq K_2(T)(1+\|x\|^k),$$

     for all $0 \leq r \leq T$, $x \in \overline{D}$

2. Let

$$\begin{aligned} h(x) :& D \to \mathbb{R} \\ g(r,x) :& (0,\infty) \times \partial D \to \mathbb{R}, \end{aligned}$$

be continuous functions such that

- **(Growth.)** There exists $k > 0$, such that for all $T > 0$, there exists a constant $K_3(T)$,

$$|h(x)| + |g(r,x)| \leq K_3(T)(1 + \|x\|^k)$$

  for all $(r,x) \in [0,T] \times \overline{D}$.

- **(Consistency.)** There exists consistency in the intersection of the space and the time boundaries, that is,

$$h(x) = g(0,x)$$

  for $x \in \partial D$.

1.1.3 Additional notation.

If μ is a locally Lipschitz function defined in some set R, then for any bounded open set A for which $\overline{A} \subset R$, we denote by $K_\mu(A)$ and $L_\mu(A)$, the constants

$$K_\mu(A) := \sup_{x \in A} \|\mu(x)\| < \infty,$$
$$L_\mu(A) := \sup_{x,y \in A, x \neq y} \frac{\|\mu(x) - \mu(y)\|}{\|x - y\|} < \infty.$$

If $\nu : [0,\infty) \to \mathbb{R}^d$, then for all $T > 0$

$$\|\nu\|_T := \sup_{0 \leq s \leq T} \|\nu(s)\|.$$

The space $C^{1,2,\lambda}_{\text{loc}}((0,\infty) \times D)$ is the space of all functions such that they and all their derivatives up to the second order in x and first order in t, are locally Hölder of order λ,

1.2 Main result.

In this section we present the main result of this chapter and some parts of the proof.

Theorem 1.2.1. *Assume **H0**, **H1** and **H2**. Then there exists a unique solution* $u \in C([0,\infty) \times \overline{D}) \cap C^{1,2,\lambda}_{loc}((0,\infty) \times D)$ *to equation* (1.8). *The solution has the representation*

$$\begin{aligned} u(t,x) =& \mathbb{E}_x\left[\int_0^\tau e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right] \\ &+ \mathbb{E}_x\left[e^{\int_0^t c(t-r,X(r))dr} h(X(t))\mathbb{1}_{\tau_D \geq t}\right] \\ &+ \mathbb{E}_x\left[e^{\int_0^{\tau_D} c(t-r,X(r))dr} g(t-\tau_D, X(\tau_D))\mathbb{1}_{\tau_D < t}\right] \end{aligned}$$

where X *is the solution to the stochastic differential equation*

$$dX(s) = b(t-s,X(s))ds + \sigma(t-s,X(s))dW(s), \quad X(0) = x,$$

and $\tau := \tau_D \wedge t$, *with*

$$\tau_D := \inf\{s > 0 | X(s) \notin \overline{D}\}.$$

Furthermore, for all $T > 0$

$$\sup_{0 \leq t \leq T} |u(t,x)| \leq C(T, c_0, K_1, K_2, K_3, k)(1 + \|x\|^k), \quad x \in \overline{D}, \tag{1.9}$$

where c_0, K_1, K_2, K_3 *and* k *are the constants defined in **H1** and **H2**.*

The proof of this Theorem is given by several Lemmas. The method we will use has the following steps: first we define a functional of the process X as a candidate solution. Let $v : [0,\infty) \times \overline{D} \to \mathbb{R}$ be defined as

$$\begin{aligned} v(t,x) :=& \mathbb{E}_x\left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right] \\ &+ \mathbb{E}_x\left[e^{\int_0^t c(t-r,X(r))dr} h(X(t))\mathbb{1}_{\tau_D \geq t}\right] \\ &+ \mathbb{E}_x\left[e^{\int_0^{\tau_D} c(t-r,X(r))dr} g(t-\tau_D, X(\tau_D))\mathbb{1}_{\tau_D < t}\right] \end{aligned} \tag{1.10}$$

If $v \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$, then there exists some standard arguments (see [**?**] chapter 4) to prove that v is the unique solution to equation

(1.8). The rest of this section is devoted to proving Theorem 1.2.1 in the case when v is a "regular" function. The proof is divided into two lemmas: the first one proves that if $v \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$, then v is a solution to equation (1.8) and hence we get existence. The second one proves that in case of existence of a classical solution, u, to equation (1.8), then it is unique and has the form given by v in equation (1.10). The regularity of v is proved in Section 1.3 below.

The next proposition gives an extension of the boundary data to all the space $[0,\infty) \times \mathbb{R}^d$. This extension is given to simplify the notation and is required in the proofs to Lemmas 1.3.1 and 1.3.2 below.

Proposition 1.2.1. *Assume* ***H2****. Then there exists a continuous function* $G : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\begin{aligned} G(t,x) =& g(t,x), \quad (t,x) \in (0,\infty) \times \partial D \\ G(0,x) =& h(x), \quad x \in D. \end{aligned}$$

Proof. Thanks to the consistency condition in **H2** and the continuity of g and h, we can extend by Tietze's Extension Theorem (see [**?**] section 2.6) the functions g, h from the closed set $\{0\} \times \overline{D} \cup [0,\infty) \times \partial D$ to a continuous function G defined in $[0,\infty) \times \mathbb{R}^d$. □

As a consequence of Proposition 1.2.1 we can write v in equation (1.10) as follows

$$\begin{aligned} v(t,x) =& \mathbb{E}_x \left[\int_0^\tau e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^\tau c(t-r,X(r))dr} G(t-\tau, X(\tau)) \right]. \end{aligned} \tag{1.11}$$

We are ready to prove the existence and uniqueness of a solution to equation (1.8) assuming $v \in C^{1,2}$.

Lemma 1.2.1. *Assume* ***H0****,* ***H1*** *and* ***H2****. Let v be defined as in equation* (1.11) *and assume that* $v \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$*. Then v fulfils the following equation*

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) =& -f(t,x) \quad (t,x) \in (0,\infty) \times D, \\ u(t,x) =& G(t,x) \quad (t,x) \in \partial((0,\infty) \times D). \end{aligned}$$

Furthermore, for all $T > 0$, there exists C such that

$$\sup_{0\le t\le T} |v(t,x)| \le C(T, c_0, K_1, K_2, K_3, k)(1 + \|x\|^k), \quad x \in \overline{D},$$

*where c_0, K_1, K_2, K_3 and k are the constants defined in **H1** and **H2**.*

Proof. Let $0 \le \alpha \le t$, then following the same argument used to prove equation (1.46) in the proof of Theorem 1.3.2 in section 1.3 below we have that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^\tau e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds + e^{\int_0^\tau c(t-r,X(r))dr} G(t-\tau, X(\tau)) \middle| \mathcal{F}_\alpha\right]\\
&= \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\\
&\quad + e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))} v(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)).
\end{aligned} \tag{1.12}$$

Because of **H1** and **H2** we have that the random variable inside the conditional expectation is integrable and so the lefthand side of equation (1.12) is a $\mathcal{F}_\alpha$-martingale, for $\alpha \in [0,t]$. Since $v \in C^{1,2}$ we can apply Ito's formula to $e^{\int_0^\alpha c dr} v()$ to get

$$\begin{aligned}
e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} v(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) &= v(t,x)\\
&+ \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} (-v_t + \mathcal{L}[v] + cv)(t-s,X(s))ds\\
&+ \int_0^{\alpha\wedge\tau_D} Dv(t-s,X(s)) \cdot \sigma(t-s,X(s))dW(s).
\end{aligned} \tag{1.13}$$

It follows from the continuity of Dv, σ and $X(\cdot)$ that

$$\sup_{0\le s\le\alpha} \|Dv(t-s,X(s))\|\|\sigma(t-s,X(s))\|$$

is a.s. finite and then

$$\int_0^{\alpha\wedge\tau_D} Dv(t-s,X(s)) \cdot \sigma(t-s,X(s))dW(s)$$

is a local martingale for $0 \leq \alpha \leq t$. So combining equations (1.12) and (1.13) we get that

$$M(\alpha) := \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr}(-v_t + \mathcal{L}[v] + cv + f)(t-s, X(s))ds$$

is a continuous local martingale for $\alpha \in [0,t]$. Since M is locally of bounded variation then $M(\alpha) \equiv 0$. This implies that $-v_t + \mathcal{L}[v] + cv + f = 0$ for all $(t,x) \in (0,\infty) \times D$. The boundary condition follows from the regularity of the set D, the local ellipticity condition and the continuity of v in $[0,\infty) \times \overline{D}$.

The second statement of the Theorem is proved with the same argument used to prove equations (1.21) and (1.34) in the proofs of Lemmas 1.3.1 and 1.3.2 in Section 1.3 below. □

The next Lemma proves the uniqueness of the solution.

Lemma 1.2.2. *Assume* ***H0****,* ***H1*** *and* ***H2****. Assume there exists a classical solution* $u \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \cap D)$ *to equation*

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) =& -f(t,x) \quad (t,x) \in (0,\infty) \times D, \\ u(t,x) =& G(t,x) \quad (t,x) \in \partial((0,\infty) \times D), \end{aligned} \tag{1.14}$$

such that for all $T > 0$*, exists* C *for which*

$$\sup_{0\leq t\leq T} |u(t,x)| \leq C(T)(1 + \|x\|^\mu) \tag{1.15}$$

for some $\mu > 0$*. Then* u *has the following representation*

$$\begin{aligned} u(t,x) =& \mathbb{E}_x\left[\int_0^\tau e^{\int_0^s c(t-r,X(r))dr} f(t-s, X(s))ds\right] \\ &+ \mathbb{E}_x\left[e^{\int_0^\tau c(t-r,X(r))dr} G(t-\tau, X(\tau))\right]. \end{aligned}$$

and hence the solution is unique.

Proof. Consider, for $\alpha \in [0,t]$, the process

$$e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)).$$

Applying Ito's rule, we get

$$\begin{aligned} e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) &= u(t,x) \\ &+ \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr}(-u_t+\mathcal{L}[u]+cu)(t-s,X(s))ds \\ &+ \int_0^{\alpha\wedge\tau_D} Du(t-s,X(s))\cdot\sigma(t-s,X(s))dW(s). \end{aligned}$$

A similar argument as the one used in the proof to Lemma 1.2.1 shows that

$$\int_0^{\alpha\wedge\tau_D} Du(t-s,X(s))\cdot\sigma(t-s,X(s))dW(s)$$

is a local martingale. Due to equation (1.14) we conclude that

$$\begin{aligned} M(\alpha) :=& e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) \\ &+ \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \end{aligned}$$

is a local martingale for $\alpha\in[0,t]$. Let $\{\theta_n\}_{n\geq 1}$ be a sequence of localization times for $M(\alpha)$, i.e., $\theta_n\uparrow\infty$ a.s. as $n\to\infty$ and $M(\alpha\wedge\theta_n)$ is a martingale for all $n\geq 1$. Then for all $n\geq 1$

$$\begin{aligned} u(t,x) =& \mathbb{E}_x\left[e^{\int_0^{\alpha\wedge\tau_D\wedge\theta_n} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D\wedge\theta_n, X(\alpha\wedge\tau_D\wedge\theta_n))\right] \\ &+ \mathbb{E}_x\left[\int_0^{\alpha\wedge\tau_D\wedge\theta_n} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right]. \end{aligned}$$

Since $0\leq\alpha\wedge\tau_D\wedge\theta_n\leq t$, using equation (1.15) we get

$$\begin{aligned} e^{\int_0^{\alpha\wedge\tau_D\wedge\theta_n} c(t-r,X(r))dr} & |u(t-\alpha\wedge\tau_D\wedge\theta_n, X(\alpha\wedge\tau_D\wedge\theta_n))| \\ &\leq e^{c_0 t}C(t)\left(1+\|X(\alpha\wedge\tau_D\wedge\theta_n)\|^\mu\right) \\ &\leq e^{c_0 t}C(t)\left(1+\sup_{0\leq s\leq t}\|X(s)\|^\mu\right). \end{aligned}$$

And

$$\begin{aligned} &\left|\int_0^{\alpha\wedge\tau_D\wedge\theta_n} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right| \\ &\qquad\leq \int_0^{\alpha\wedge\tau_D\wedge\theta_n} e^{c_0 s}K_2(t)\left(1+\|X(s)\|^k\right)ds \\ &\qquad\leq e^{c_0 t}tK_2(t)\left(1+\sup_{0\leq s\leq t}\|X(s)\|^k\right). \end{aligned}$$

By equation (1.7) and the Dominated Convergence Theorem, letting $n \to \infty$ we get

$$\begin{aligned}u(t,x) =&\mathbb{E}_x\left[e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr}u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D))\right]\\ &+\mathbb{E}_x\left[\int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right].\end{aligned}$$

Letting $\alpha \uparrow t$, a similar argument and the boundary condition proofs that

$$\begin{aligned}u(t,x) =&\mathbb{E}_x\left[e^{\int_0^{t\wedge\tau_D} c(t-r,X(r))dr}G(t-t\wedge\tau_D, X(t\wedge\tau_D))\right]\\ &+\mathbb{E}_x\left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right],\end{aligned}$$

and the proof is complete. □

1.3 Regularity of v.

In this section we prove that $v \in C([0,\infty)\times\overline{D}) \cap C^{1,2,\lambda}_{\text{loc}}((0,\infty)\times D)$. First, we prove using the continuity of the flow process X, that v is a continuous function in $[0,\infty)\times\overline{D}$. Since we are only assuming the continuity of the coefficients, then the flow is not necessarily differentiable and so we can not prove the regularity of v in terms of the regularity of the flow. To prove that $v \in C^{1,2}$, we show that v is the solution to a parabolic differential equation in a bounded domain, for which we have the existence of a classical solution and hence $v \in C^{1,2}$.

1.3.1 Continuity of v.

Let (ξ, X) denote the solution to equation (1.4) and G be defined as in Proposition 1.2.1, then v has the following form

$$\begin{aligned}v(t,x) =&\mathbb{E}_{t,x}\left[\int_0^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right]\\ &+\mathbb{E}_{t,x}\left[e^{\int_0^\tau c(\xi(r),X(r))dr}G(\xi(\tau),X(\tau))\right].\end{aligned} \tag{1.16}$$

For simplicity, we write $v = v_1 + v_2$, where

$$v_1(t,x) := \mathbb{E}_{t,x}\left[\int_0^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right] \tag{1.17}$$

and

$$v_2(t,x) := \mathbb{E}_{t,x}\left[e^{\int_0^\tau c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\right]. \tag{1.18}$$

Theorem 1.3.1. *Assume **H0**, **H1** and **H2**. Let v be defined as in equation* (1.16)*. Then v is continuous on $[0,\infty)\times\overline{D}$.*

The proof to this Theorem is divided into two lemmas.

Lemma 1.3.1. *Assume **H0**, **H1** and **H2**. Let v_1 be defined as in equation* (1.17)*. Then v_1 is continuous on $[0,\infty)\times\overline{D}$.*

Proof. First we prove the continuity on $(0,\infty)\times D$. For that, let

$$(t_n,x_n)\xrightarrow[n\to\infty]{}(t,x)$$

in $(0,\infty)\times D$ and $\epsilon>0$. We need to prove that there exists $N(\epsilon)\in\mathbb{N}$ such that for all $n\geq N$

$$|v_1(t_n,x_n)-v_1(t,x)|<\epsilon.$$

Denote by (ξ,X) and (ξ_n,X_n) the solutions to equation (1.4) with initial conditions (t,x) and (t_n,x_n) respectively. Let τ and τ_n be their corresponding exit times from $[0,\infty)\times\overline{D}$.
Let $\alpha>0$, then there exists $N_1\in\mathbb{N}$ such that for all $n\geq N_1$

$$|(t_n,x_n)-(t,x)|<\alpha. \tag{1.19}$$

Observe that for all $n\geq N_1$, we get

$$\begin{aligned}\tau_n \leq & t_n \leq t+\alpha,\\ \tau \leq & t \leq t+\alpha.\end{aligned} \tag{1.20}$$

Define the random variables Y_n as

$$\begin{aligned}Y_n := \Bigg| & \int_0^{\tau_n} e^{\int_0^s c(\xi_n(r),X_n(r))dr} f(\xi_n(s),X_n(s))ds\\ & -\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\Bigg|.\end{aligned}$$

The sequence $\{Y_n\}_{n\geq N_1}$ is uniformly integrable. To see that

$$\begin{aligned}\mathbb{E}\left[Y_n^2\right] \leq & 2\mathbb{E}\left[\left|\int_0^{\tau_n} e^{\int_0^s c(\xi_n(r),X_n(r))dr} f(\xi_n(s),X_n(s))ds\right|^2\right] \\ & + 2\mathbb{E}\left[\left|\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right|^2\right] \\ \leq & 2\mathbb{E}\left[\left|\int_0^{\tau_n} e^{c_0(t+\alpha)} K_2(t+\alpha)(1+\|X_n(s)\|^k)ds\right|^2\right] + C_{t,x} \\ \leq & 2e^{2c_0(t+\alpha)}K_2^2(t+\alpha)\mathbb{E}\left[\left|\int_0^{\tau_n}\left(1+\sup_{0\leq r\leq t+\alpha}\|X_n(r)\|^k\right)ds\right|^2\right] + C_{t,x} \\ \leq & 4e^{2c_0(t+\alpha)}K_2^2(t+\alpha)(t+\alpha)^2\left(1+\mathbb{E}\left[\sup_{0\leq r\leq t+\alpha}\|X_n(r)\|^{2k}\right]\right) + C_{t,x} \\ \leq & C(1+K(1+\|x_n\|^{2k})) + C_{t,x} \\ \leq & C(1+K(1+(\|x\|+\alpha)^{2k})) + C_{t,x} < \infty,\end{aligned} \tag{1.21}$$

where we use (1.19), (1.20), (1.7) and the polynomial growth of f.
Let $M > 0$, $0 < \eta < 1$ and $\beta > 0$ and define the set

$$E_{M,n,\eta,\beta} := \{\|X\|_{t+\alpha} \leq M\} \cap \{\|X_n - X\|_{t+\alpha} \leq \eta\} \cap \{|\tau_n - \tau| \leq \beta\}. \tag{1.22}$$

Then

$$\begin{aligned}|v_1(t_n,x_n) - v_1(t,x)| \leq & \int_\Omega Y_n d\mathbb{P} \\ = & \int_{E_{M,n,\eta,\beta}} Y_n d\mathbb{P} + \int_{\Omega\setminus E_{M,n,\eta,\beta}} Y_n d\mathbb{P}.\end{aligned}$$

Since the sequence $\{Y_n\}$ is uniformly integrable, there exists $\delta(\epsilon)$ such that for any $E \in \mathcal{F}$ that satisfies $\mathbb{P}[E] < \delta$, we have

$$\sup_{n\geq N_1}\int_E Y_n d\mathbb{P} < \frac{\epsilon}{2}. \tag{1.23}$$

It follows from Remark 1.1.3, Proposition 1.1.1 and Theorems **??**, **??** in Appendix **??**, the existence of M and N_2 such that

$$\mathbb{P}[\Omega \setminus E_{M,n,\eta,\beta}] < \delta(\epsilon)$$

for all $n \geq N_2$. Then for $n \geq N_1 \vee N_2$ we get that

$$|v_1(t_n, x_n) - v_1(t,x)| \leq \int_{E_{M,n,\eta,\beta}} Y_n d\mathbb{P} + \frac{\epsilon}{2}.$$

For simplicity of notation, we write the set $E_{M,n,\eta,\beta}$ as E and define

$$A := [0, t+\alpha] \times [-M-1, M+1]^d \tag{1.24}$$

and

$$D_t := [0, t+\alpha] \times \overline{D}. \tag{1.25}$$

On the set E, for all $n \geq N_1$ and $0 \leq s \leq t+\alpha$, it is satisfied that

$$(\xi_n(s), X_n(s)), (\xi(s), X(s)) \in A.$$

We have that

$$\begin{aligned}\int_E Y_n d\mathbb{P} \quad \leq \quad & \int_E \int_0^{\tau_n \wedge \tau} \Big| e^{\int_0^s c(\xi_n(r), X_n(r)dr} f(\xi_n(s), X_n(s)) \\ & -e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s), X(s)) \Big| ds d\mathbb{P} & (1.26) \\ + \quad & \int_E \int_{\tau_n \wedge \tau}^{\tau_n \vee \tau} \Big(e^{\int_0^s c(\xi_n(r), X_n(r))dr} |f(\xi_n(s), X_n(s))| \mathbb{1}_{\tau_n > \tau} \\ & + e^{\int_0^s c(\xi(r),X(r))dr} |f(\xi(s), X(s))| \mathbb{1}_{\tau_n \leq \tau} \Big) ds d\mathbb{P}. & (1.27)\end{aligned}$$

We first analyze (1.27).

$$\begin{aligned}(1.27) \leq & \int_E \int_{\tau_n \wedge \tau}^{\tau_n \vee \tau} e^{c_0(t+\alpha)} K_f(A \cap D_t)(\mathbb{1}_{\tau_n \leq \tau} + \mathbb{1}_{\tau_n > \tau}) ds d\mathbb{P} \\ = & e^{c_0(t+\alpha)} K_f(A \cap D_t) \int_E |\tau_n - \tau| d\mathbb{P} \\ \leq & e^{c_0(t+\alpha)} K_f(A \cap D_t) \beta.\end{aligned}$$

For (1.26) we get

$$\begin{aligned}(1.26) \quad \leq \quad & \int_E \int_0^{\tau_n \wedge \tau} e^{\int_0^s c(\xi_n(r), X_n(r))dr} \\ & \times |f(\xi_n(s), X_n(s)) - f(\xi(s), X(s))| ds d\mathbb{P} & (1.28) \\ + \quad & \int_E \int_0^{\tau_n \wedge \tau} f(\xi(s), X(s)) \\ & \times \left| e^{\int_0^s c(\xi_n(r), X_n(r))dr} - e^{\int_0^s c(\xi(r),X(r))dr} \right| ds d\mathbb{P}. & (1.29)\end{aligned}$$

Now

$$\begin{aligned}(1.28) \leq & \int_E e^{c_0(t+\alpha)} \int_0^{\tau_n \wedge \tau} L_f(A\cap D_t)(|t_n - t|^\lambda + \|X_n(s) - X(s)\|)dsd\mathbb{P} \\ \leq & e^{c_0(t+\alpha)}(t+\alpha)L_f(A\cap D_t)(|t_n-t|^\lambda + \eta).\end{aligned}$$

For (1.29) we need the following bound

$$\begin{aligned}\left|e^{\int_0^s c(\xi_n(r),X_n(r))dr} - e^{\int_0^s c(\xi(r),X(r))dr}\right| &= e^{\int_0^s c(\xi(r),X(r))dr} \\ &\times \left|\exp\left\{\int_0^s (c(\xi_n(r),X_n(r)) - c(\xi(r),X(r)))dr\right\} - 1\right| \\ &\leq e^{c_0 s}\left(\exp\left\{\int_0^s |c(\xi_n(r),X_n(r)) - c(\xi(r),X(r))|dr\right\} - 1\right) \\ &\leq e^{c_0 s}\left(\exp\left\{\int_0^s L_c(A\cap D_t)(|t_n-t|^\lambda + \|X_n(r) - X(r)\|)dr\right\} - 1\right) \\ &\leq e^{c_0 s}(\exp\{L_c(A\cap D_t)s(|t_n-t|^\lambda+\eta)\} - 1).\end{aligned} \tag{1.30}$$

since $|e^x - 1| \leq e^{|x|} - 1$. If we choose $N_3 \in \mathbb{N}$ such that $|t_n - t|^\lambda \leq \frac{1}{2L_c(A\cap D_t)(t+\alpha)}$ for all $n \geq N_3$ and $\eta \leq \frac{1}{2L_c(A\cap D_t)(t+\alpha)}$, we get by the Mean Value Theorem that

$$\left|e^{\int_0^s c(\xi_n(r),X_n(r))dr} - e^{\int_0^s c(\xi(r),X(r))dr}\right| \leq e^{c_0 s} e L_c(A\cap D_t)s(|t_n-t|^\lambda + \eta). \tag{1.31}$$

Then

$$(1.29) \leq K_f(A\cap D_t)e^{c_0(t+\alpha)} e L_c(A\cap D_t)(t+\alpha)^2(|t_n-t|^\lambda+\eta).$$

Hence, to prove continuity we proceed as follows

- Let $\epsilon > 0$ and $0 < \alpha \ll 1$.
- Let $N_1 \geq \mathbb{N}$ such that for all $n \geq N_1$

$$\|(t_n, x_n) - (t, x)\| < \alpha.$$

- Let $\delta(\epsilon) > 0$ that fulfils the uniformly integrable condition (1.23).
- Take $M > 0$ such that $\mathbb{P}\left[\|X\|_{t+\alpha} > M\right] < \frac{\delta(\epsilon)}{3}$.

- Define $A := [0, t+\alpha] \times [-M-1, M+1]^d$ and $D_t := [0, t+\alpha] \times \overline{D}$.
- Let
$$\eta < \min\left\{1, \frac{1}{2L_c(A\cap D_t)(t+\alpha)}, \frac{\epsilon}{16e^{c_0(t+\alpha)}(t+\alpha)L_f(A\cap D_t)}\right.$$
$$\left., \frac{\epsilon}{16K_f(A\cap D_t)e^{c_0(t+\alpha)}eL_c(A\cap D_t)(t+\alpha)}\right\}.$$
- Chose $N_2 \in \mathbb{N}$ such that $\mathbb{P}\left[\|X_n - X\|_{t+\alpha} > \eta\right] \leq \frac{\delta(\epsilon)}{3}$ for all $n \geq N_2$
- Let $N_3 \in \mathbb{N}$ such that
$$|t_n - t|^\lambda < \min\left\{\frac{1}{2L_c(A\cap D_t)(t+\alpha)}, \frac{\epsilon}{16e^{c_0(t+\alpha)}(t+\alpha)L_f(A\cap D_t)}\right.$$
$$\left., \frac{\epsilon}{16K_f(A\cap D_t)e^{c_0(t+\alpha)}eL_c(A\cap D_t)(t+\alpha)}\right\}.$$
for all $n \geq N_3$
- Let
$$\beta < \frac{\epsilon}{4e^{c_0(t+\alpha)}K_f(A\cap D_t)},$$
and chose $N_4 \in \mathbb{N}$ such that for all $n \geq N_4$, $\mathbb{P}\left[|\tau_n - \tau| > \beta\right] \leq \frac{\delta(\epsilon)}{3}$.

Thus if $N = N_1 \vee N_2 \vee N_3 \vee N_4$, then for all $n \geq N$
$$|v_1(t_n, x_n) - v_1(t, x)| < \epsilon.$$
Therefore v_1 is continuous in $(0, \infty) \times D$.

For the continuity at the boundary we make a similar argument. Let $(t_n, x_n) \xrightarrow[n\to\infty]{} (t, x)$, where $(t_n, x_n) \in (0, \infty) \times D$ and $(t, x) \in \partial((0, \infty) \times D)$, that is, either $t = 0$ or $x \in \partial D$. In both cases we get that $\tau = 0$ a.s. and so $v_1(t, x) = 0$. Then we need to prove that
$$|v_1(t_n, x_n)| \xrightarrow[n\to\infty]{} 0.$$
Let $0 < \alpha \ll 1$ and $N_1 \in \mathbb{N}$ such that
$$\|(t_n, x_n) - (t, x)\| < \alpha.$$

We get

$$\tau_n \leq t_n < t + \alpha$$

for all $n \geq N_1$. For the continuity we have

$$\begin{aligned}|v_1(t_n, x_n)| \leq & \mathbb{E}\left[\int_0^{\tau_n} e^{\int_0^s c(\xi_n(r), X_n(r))dr} |f(\xi_n(s), X_n(s))| ds\right] \\ \leq & \mathbb{E}\left[\int_0^{\tau_n} e^{c_0 s} K_2(t+\alpha)(1 + \|X_n(s)\|^k) ds\right] \\ \leq & e^{c_0(t+\alpha)} K_2(t+\alpha)\mathbb{E}\left[\tau_n \left(1 + \sup_{0\leq r\leq t+\alpha} \|X_n(r)\|^k\right)\right] \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

The convergence follows from the uniform integrability of

$$\tau_n \left(1 + \sup_{0\leq r\leq t+\alpha} \|X_n(r)\|^k\right)$$

for $n \geq N_1$, that $\tau_n \xrightarrow[n\to\infty]{a.s.} 0$ (see Theorem **??** in Appendix **??**) and Theorem 5.2 in Chapter 5 of [**?**]. This completes the proof. □

Lemma 1.3.2. *Assume **H0**, **H1** and **H2**. Let v_2 be defined as in equation* (1.18)*. Then v_2 is continuous on* $[0, \infty) \times \overline{D}$*.*

Proof. We use an analogous argument to the one in the proof of Lemma 1.3.1. First we prove the continuity in $(0, \infty) \times D$. Let

$$(t_n, x_n) \xrightarrow[n\to\infty]{} (t, x),$$

with $(t_n, x_n), (t, x) \in (0, \infty) \times D$. Denote by (ξ_n, X_n) and (ξ, X) the solutions to equation (1.4) with initial conditions (t_n, x_n) and (t, x) respectively, and let τ_n and τ be its corresponding exit times from $[0, \infty) \times \overline{D}$. Let $0 < \alpha \ll 1$ and N_1 such that for all $n \geq N_1$

$$\|(t_n, x_n) - (t, x)\| < \alpha. \tag{1.32}$$

This implies that

$$\begin{aligned}\tau_n &\leq t + \alpha \\ \tau &\leq t + \alpha.\end{aligned} \tag{1.33}$$

First we prove that the sequence of random variables

$$Y_n := \left| e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} G(\xi_n(\tau_n), X_n(\tau_n)) - e^{\int_0^{\tau} c(\xi(r), X(r))dr} G(\xi(\tau), X(\tau)) \right|$$

is uniformly integrable for all $n \geq N_1$.

$$\begin{aligned}
\mathbb{E}\left[Y_n^2\right] \leq & 2\mathbb{E}\left[\left| e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} G(\xi_n(\tau_n), X_n(\tau_n)) \right|^2\right] \\
& + 2\mathbb{E}\left[\left| e^{\int_0^{\tau} c(\xi(r), X(r))dr} G(\xi(\tau), X(\tau)) \right|^2\right] \\
\leq & 2\mathbb{E}\left[e^{2c_0(t+\alpha)} K_3^2(t+\alpha)(1 + \|X_n(\tau_n)\|^k)^2\right] + C_{t,x} \\
\leq & 2e^{2c_0(t+\alpha)} K_3^2(t+\alpha)\mathbb{E}\left[(1 + \sup_{0 \leq r \leq t+\alpha} \|X_n(r)\|^k)^2\right] + C_{t,x} \\
\leq & 4e^{2c_0(t+\alpha)} K_3^2(t+\alpha)\left(1 + \mathbb{E}\left[\sup_{0 \leq r \leq t+\alpha} \|X_n(r)\|^{2k}\right]\right) + C_{t,x} \\
\leq & C(1 + K(1 + \|x_n\|^{2k})) + C_{t,x} \\
\leq & C(1 + K(1 + (\|x\| + \alpha)^{2k})) + C_{t,x} < \infty,
\end{aligned} \tag{1.34}$$

where we use equations (1.32), (1.33), (1.7) and polynomial growth of G in $\partial((0,\infty) \times D)$. As in Lemma 1.3.1, let $\epsilon > 0$, then there exists $\delta(\epsilon) > 0$ such that

$$\sup_{n \geq N_1} \int_E Y_n d\mathbb{P} < \frac{\epsilon}{2} \tag{1.35}$$

for all $E \in \mathcal{F}$, with $\mathbb{P}[E] < \delta(\epsilon)$.

Let $E_{M,n,\eta,\beta}$ be defined as in equation (1.22), and chose $M > 0$ and $N_2 \in \mathbb{N}$ such that

$$\mathbb{P}[\Omega \setminus E_{M,n,\eta,\beta}] < \delta(\epsilon),$$

for all $n \geq N_2$.

For simplicity of notation, denote $E_{M,n,\eta,\beta}$ as E. Then

$$\begin{aligned}
|v_2(t_n, x_n) - v_2(t, x)| &\leq \int_E Y_n d\mathbb{P} + \int_{\Omega \setminus E} Y_n d\mathbb{P} \\
&\leq \int_E Y_n d\mathbb{P} + \frac{\epsilon}{2}.
\end{aligned}$$

Let A and D_t be defined as in Lemma 1.3.1 (equations (1.24) and (1.25)). Then on the set E we get that for all $n \geq N_1$ and $0 \leq s \leq t + \alpha$,

$$(\xi_n(s), X_n(s)), (\xi(s), X(s)) \in A.$$

So

$$\begin{aligned} \int_E Y_n d\mathbb{P} \quad &\le \quad \int_E e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} \\ &\quad \times |G(\xi_n(\tau_n), X_n(\tau_n)) - G(\xi(\tau), X(\tau))| \, d\mathbb{P} \qquad (1.36) \\ &+ \quad \int_E |G(\xi(\tau), X(\tau))| \\ &\quad \left| e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} - e^{\int_0^{\tau} c(\xi(r),X(r))dr} \right| d\mathbb{P}. \qquad (1.37) \end{aligned}$$

We study each addend of the righthand side separately.

$$\begin{aligned} (1.36) \quad \le \quad & e^{c_0(t+\alpha)} \int_E |G(t_n - \tau_n, X_n(\tau_n)) - G(t_n - \tau_n, X(\tau_n))| d\mathbb{P} \qquad (1.38) \\ & + e^{c_0(t+\alpha)} \int_E |G(t_n - \tau_n, X(\tau_n)) - G(t - \tau, X(\tau))| d\mathbb{P}. \qquad (1.39) \end{aligned}$$

First we get a bound for (1.38). Since G is continuous, then it is uniformly continuous on A. Then for $\epsilon > 0$, there exists $\gamma(c_0, t, \alpha, \epsilon, M)$ such that

$$|G((t_1, x_1)) - G(t_2, x_2))| < \frac{\epsilon}{8e^{c_0(t+\alpha)}}$$

for all $(t_1, x_1), (t_2, x_2) \in A$ with $\|(t_1, x_1) - (t_2, x_2)\| < \gamma(c_0, t, \alpha, \epsilon, M)$. On the set E, we have $(t_n - \tau_n, X_n(\tau_n)), (t_n - \tau_n, X(\tau_n)) \in A$ and

$$\|(t_n - \tau_n, X_n(\tau_n)) - (t_n - \tau_n, X(\tau_n))\| < \eta.$$

So if we choose $\eta < \gamma$, we get

$$(1.38) < \frac{\epsilon}{8e^{c_0(t+\alpha)}}.$$

Next we study (1.39). Thanks to Theorem **??** we know that $\tau_n \xrightarrow[n\to\infty]{a.s} \tau$. This and the continuity of $X(\cdot)$ and G implies that

$$G(t_n - \tau_n, X(\tau_n)) \xrightarrow[n\to\infty]{a.s.} G(t - \tau, X(\tau)).$$

On the set E we have that $(t_n - \tau_n, X(\tau_n)), (t - \tau, X(\tau)) \in A$ and so

$$|G(t_n - \tau_n, X(\tau_n)) - G(t - \tau, X(\tau))| \mathbb{1}_E \le 2K_G(A).$$

By the Dominated Convergence Theorem, there exists, $N_3 \in \mathbb{N}$ such that

$$(1.39) < \frac{\epsilon}{8e^{c_0(t+\alpha)}}$$

for all $n \geq N_3$.
To give a bound for (1.37) we observe that on the set E

$$\begin{aligned}
&\left|\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr - \int_0^{\tau} c(\xi(r), X(r))dr\right| \\
&\quad \leq \int_0^{\tau_n \wedge \tau} |c(\xi_n(r), X_n(r)) - c(\xi(r), X(r))|dr \\
&\quad\quad + \int_{\tau_n \wedge \tau}^{\tau_n \vee \tau} \left(|c(\xi_n(r), X_n(r))|\mathbb{1}_{\tau_n \geq \tau} + |c(\xi(r), X(r))|\mathbb{1}_{\tau_n < \tau}\right) dr \\
&\quad \leq \int_0^{\tau_n \wedge \tau} L_c(A \cap D_t)(|t_n - t|^\lambda + \|X_n(r) - X(r)\|)dr + K_c(A \cap D_t)|\tau_n - \tau| \\
&\quad \leq L_c(A \cap D_t)(t+\alpha)(|t_n - t|^\lambda + \eta) + K_c(A \cap D_t)\beta.
\end{aligned}$$

Making a similar argument as the one made in equations (1.30) and (1.31) we get

$$\begin{aligned}
&\left|e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} - e^{\int_0^{\tau} c(\xi(r), X(r))dr}\right| \\
&\qquad\qquad \leq e^{c_0(t+\alpha)} e\left[L_c(A \cap D_t)(t+\alpha)(|t_n - t|^\lambda + \eta) + K_c(A \cap D_t)\beta\right],
\end{aligned}$$

if $|t_n - t|^\lambda < \frac{1}{3L_c(A\cap D_t)(t+\alpha)}$, $\eta < \frac{1}{3L_c(A\cap D_t)(t+\alpha)}$ and $\beta < \frac{1}{3K_c(A\cap D_t)}$. Then

$$(1.37) \leq K_G(A) e^{c_0(t+\alpha)} e\left[L_c(A \cap D_t)(t+\alpha)(|t_n - t|^\lambda + \eta) + K_c(A \cap D_t)\beta\right].$$

Hence, to prove continuity we proceed as follows

- Let $\epsilon > 0$ and $0 < \alpha \ll 1$.
- Let $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$\|(t_n, x_n) - (t, x)\| < \alpha.$$

- Let $\delta(\epsilon) > 0$ that fulfils the uniformly integrable condition (1.35).
- Take $M > 0$ such that $\mathbb{P}\left[\|X\|_{t+\alpha} > M\right] < \frac{\delta(\epsilon)}{3}$.

- Define $A := [0, t+\alpha] \times [-M-1, M+1]^d$ and $D_t := [0, t+\alpha] \times \overline{D}$.
- Let

$$\eta < \min\left\{1, \gamma(c_0, t, \alpha, \epsilon, M), \frac{1}{3L_c(A \cap D_t)(t+\alpha)}\right.$$
$$\left., \frac{\epsilon}{12K_G(A)e^{c_0(t+\alpha)}eL_c(A \cap D_t)(t+\alpha)}\right\}.$$

- Chose $N_2 \in \mathbb{N}$ such that $\mathbb{P}\left[\|X_n - X\|_{t+\alpha} > \eta\right] < \frac{\delta(\epsilon)}{3}$ for all $n \geq N_2$
- Let

$$\beta < \min\left\{\frac{1}{3K_c(A \cap D_t)}, \frac{\epsilon}{12K_G(A)e^{c_0(t+\alpha)}eK_c(A \cap D_t)}\right\}$$

- Chose $N_3 \in \mathbb{N}$ such that $\mathbb{P}\left[|\tau_n - \tau| > \beta\right] < \frac{\delta(\epsilon)}{3}$ for all $n \geq N_3$.
- Let $N_4 \in \mathbb{N}$ such that

$$|t_n - t|^\lambda < \min\left\{\frac{1}{3(L_c(A \cap D_t)(t+\alpha)}\right.$$
$$\left., \frac{\epsilon}{12K_G(A \cap D_t)e^{c_0(t+\alpha)}eL_c(A \cap D_t)(t+\alpha)}\right\}.$$

  for all $n \geq N_4$.
- Let $N_5 \in \mathbb{N}$ to get

$$\int_E |G(t_n - \tau_n, X(\tau_n)) - G(t - \tau, X(\tau))| d\mathbb{P} < \frac{\epsilon}{8e^{c_0(t+\alpha)}},$$

  for all $n \geq N_4$.

So for $N = N_1 \vee N_2 \vee N_3 \vee N_4 \vee N_5$, we have that if $n \geq N$ then

$$|v_2(t_n, x_n) - v_2(t, x)| < \epsilon,$$

and we conclude that v_2 is continuous over $(0, \infty) \times D$.
Next we prove the continuity in the boundary. Let $(t_n, x_n) \xrightarrow[n\to\infty]{} (t, x)$,

where $(t_n, x_n) \in (0,\infty) \times D$ and $(t,x) \in \partial((0,\infty)\times D)$, that is, either $t = 0$ or $x \in \partial D$. In both cases we get that $\tau = 0$ a.s.. We need to prove that

$$|v_2(t_n, x_n) - G(t,x)| \xrightarrow[n\to\infty]{} 0.$$

Let $0 < \alpha \ll 1$ and $N_1 \in \mathbb{N}$ such that

$$\|(t_n, x_n) - (t,x)\| < \alpha.$$

So, for all $n \geq N_1$,

$$\tau_n \leq t_n < t + \alpha.$$

We have that

$$\begin{aligned} |v_2(t_n, x_n) - G(t,x)| \;\; &\leq \;\; \mathbb{E}\Big[e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} \\ &\qquad \times |G(t_n - \tau_n, X_n(\tau_n)) - G(t,x)|\Big] \qquad (1.40) \\ &\quad + \;\; \mathbb{E}\Big[|G(t,x)| \left| e^{\int_0^{\tau_n} c(\xi_n(r), X(r))dr} - 1 \right|\Big]. \qquad (1.41) \end{aligned}$$

Because $\left|e^{\int_0^{\tau_n} c(\xi_n, X_n)} - 1\right| \leq e^{c_0(t+\alpha)} + 1$ and $\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr \xrightarrow[n\to\infty]{a.s.} 0$ (due to Theorem **??**, that $\tau = 0$ a.s. and the continuity of c in $[0,\infty) \times \overline{D}$), by the Dominated Convergence Theorem we get for (1.41)

$$\mathbb{E}\Big[|G(t,x)| \left| e^{\int_0^{\tau_n} c(\xi_n(r), X(r))dr} - 1 \right|\Big] \xrightarrow[n\to\infty]{} 0.$$

Next we work with (1.40). As in equation (1.34) we can prove that the sequence

$$\left\{ e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} |G(t_n - \tau_n, X_n(\tau_n)) - G(t,x)| \right\}_{n \geq N_1}$$

is uniformly integrable. We have that

$$\begin{aligned} (1.40) \;\; &\leq \;\; e^{c_0(t+\alpha)} \mathbb{E}\left[|G(t_n - \tau_n, X_n(\tau_n)) - G(t_n - \tau_n, X(\tau_n))|\right] \qquad (1.42) \\ &\quad + e^{c_0(t+\alpha)} \mathbb{E}\left[|G(t_n - \tau_n, X(\tau_n)) - G(t,x))|\right]. \qquad (1.43) \end{aligned}$$

We repeat the same arguments made for the estimates to equations (1.38) and (1.39) with equation (1.42) and (1.43) respectively. Then we can prove that

$$\mathbb{E}\left[|G(t_n - \tau_n, X_n(\tau_n)) - G(t_n - \tau_n, X(\tau_n))|\right] \xrightarrow[n\to\infty]{} 0,$$

and

$$\mathbb{E}\left[|G(t_n - \tau_n, X(\tau_n)) - G(t,x))|\right] \xrightarrow[n\to\infty]{} 0.$$

So $v_2 \in C([0,\infty) \times \overline{D})$ and the proof is complete. □

1.3.2 Differentiability of v.

Let $0 \leq T_0 < T_1$ and $A \subset D$ be a bounded, open, connected set with C^2 boundary. Consider the following parabolic differential equation

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) = -f(t,x) \quad (t,x) \in (T_0, T_1] \times A, \\ u(T_0, x) = v(T_0, x) \quad \text{for } x \in A, \\ u(t,x) = v(t,x) \quad \text{for } (t,x) \in (T_0, T_1] \times \partial A. \end{aligned} \tag{1.44}$$

where the boundary data is v. If we assume **H0**, **H1** and **H2**, by the continuity of v (Theorem 1.3.1) and Theorem **??** we can guarantee the existence of a unique classical solution to equation (1.44). To prove the regularity of v, we show that it coincides with the solution to equation (1.44) in the set $(T_0, T_1) \times A$ and so $v \in C^{1,2}((T_0, T_1) \times A)$. Since T_0, T_1 and A are arbitrary, we get the desired regularity. We are ready to prove the next Theorem.

Theorem 1.3.2. *Assume* ***H0****,* ***H1*** *and* ***H2****. Let v be defined as in equation* (1.16)*. Then $v \in C^{1,2,\lambda}_{loc}((0,\infty) \times \mathbb{R}^d)$.*

Proof. Let w be the solution to equation (1.44). Define the following stopping times

$$\begin{aligned} \theta_T :=& \inf\{s > 0 | \xi(s) < T_0\} \\ \theta_A :=& \inf\{s > 0 | X(s) \notin \overline{A}\}, \\ \theta :=& \theta_T \wedge \theta_A. \end{aligned}$$

Following the same arguments of Section 5 in Chapter 6 of [**?**], we can prove that w has the following representation

$$\begin{aligned} w(t,x) =& \mathbb{E}_x \left[\int_0^\theta e^{\int_0^s c(t-r, X(r))dr} f(t-s, X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^\theta c(t-r, X(r))ds} v(t-\theta, X(\theta)) \right]. \end{aligned} \tag{1.45}$$

Next we prove that v satisfies the following equality

$$\begin{aligned} v(t,x) =& \mathbb{E}_{t,x} \left[\int_0^\theta e^{\int_0^s c(\xi(r), X(r))dr} f(\xi(s), X(s))ds \right] \\ &+ \mathbb{E}_{t,x} \left[e^{\int_0^\theta c(\xi(r), X(r))ds} v(\xi(\theta), X(\theta)) \right]. \end{aligned} \tag{1.46}$$

Let v_1 and v_2 be defined as in equations (1.17) and (1.18). We will use the following representation of v_1 and v_2,

$$v_1(t,x) = \mathbb{E}_{t,x}\left[\int_0^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right]$$

and

$$v_2(t,x) = \mathbb{E}_{t,x}\left[e^{\int_0^\tau c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\right].$$

First we work with v_1

$$\begin{aligned} v_1(t,x) &= \mathbb{E}_{t,x}\left[\mathbb{E}\left[\int_0^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\middle|\mathcal{F}_\theta\right]\right] \\ &= \mathbb{E}_{t,x}\left[\mathbb{E}\left[\int_0^\theta e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\middle|\mathcal{F}_\theta\right]\right] \quad (1.47)\\ &\quad + \mathbb{E}_{t,x}\left[\mathbb{E}\left[\int_\theta^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\middle|\mathcal{F}_\theta\right]\right] \quad (1.48) \end{aligned}$$

We study the addends of the righthand side separately

$$(1.47) = \mathbb{E}_{t,x}\left[\int_0^\theta e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right].$$

For (1.48) we make a couple of changes of variable to get

$$\begin{aligned} (1.48) =& \mathbb{E}_{t,x}\left[\mathbb{E}\left[\int_0^{\tau-\theta} e^{\int_0^{s+\theta} c(\xi(r),X(r))dr} f(\xi(s+\theta),X(s+\theta))ds\middle|\mathcal{F}_\theta\right]\right] \\ =& \mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr}\mathbb{E}\left[\int_0^{\tau-\theta} e^{\int_0^s c(\xi(r+\theta)),X(r+\theta))dr} f(\xi(s+\theta),X(s+\theta))ds\middle|\mathcal{F}_\theta\right]\right]. \end{aligned}$$

Since $\theta < \tau_D$ (see Remark 1.1.1) and θ is bounded, we get that

$$\begin{aligned} \tau =& \inf\{s>0|(\xi(s),X(s)) \notin [0,\infty)\times\overline{D}\} \\ =& \theta + \inf\{s>0|(\xi(s+\theta),X(s+\theta)) \notin [0,\infty)\times\overline{D}\}, \end{aligned}$$

so

$$\tau - \theta = \Theta_\theta \circ \tau \quad (1.49)$$

where Θ. denotes the shift operator. Since the process (ξ, X) is a homogeneous strong Markov process, we get that

$$\begin{aligned}&\mathbb{E}\left[\int_0^{\Theta_\theta\circ\tau} e^{\int_0^s c(\Theta_\theta\circ(\xi,X)(r))dr} f(\Theta_\theta\circ(\xi,X)(s))ds\middle|\mathcal{F}_\theta\right]\\ =&\mathbb{E}_{\xi(\theta),X(\theta)}\left[\int_0^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(r),X(s))ds\right]\\ =&v_1(\xi(\theta),X(\theta)).\end{aligned} \tag{1.50}$$

So

$$\begin{aligned}v_1(t,x)=&\mathbb{E}_{t,x}\left[\int_0^\theta e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))\right]\\ &+\mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr} v_1(\xi(\theta),X(\theta))\right].\end{aligned} \tag{1.51}$$

Next we study v_2. Again, for the integral we use a couple of changes of variables to get

$$\begin{aligned}v_2(t,x)=&\mathbb{E}_{t,x}\left[\mathbb{E}\left[e^{\int_0^\tau c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\middle|\mathcal{F}_\theta\right]\right]\\ =&\mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr}\mathbb{E}\left[e^{\int_0^{\tau-\theta} c(\xi(r+\theta),X(r+\theta))dr} G(\xi(\tau),X(\tau))\middle|\mathcal{F}_\theta\right]\right].\end{aligned}$$

We write

$$G(\xi(\tau),X(\tau))=G(\xi(\tau-\theta+\theta),X(\tau-\theta+\theta)).$$

Then, the argument of the conditional expectation can be written as

$$e^{\int_0^{\Theta_\theta\circ\tau} c(\Theta_\theta\circ(\xi,X)(r))dr} G(\Theta_\theta\circ(\xi,X)(\Theta_\theta\circ\tau))$$

Using a similar argument as the one in equations (1.49) and (1.50) we get

$$\begin{aligned}v_2(t,x)=&\mathbb{E}_{t,x}\Big[e^{\int_0^\theta c(\xi(r),X(r))dr}\\ &\times\mathbb{E}_{\xi(\theta),X(\theta)}\left[e^{\int_0^{\tau-\theta} c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\right]\Big]\\ =&\mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr} v_2(\xi(\theta),X(\theta))\right].\end{aligned} \tag{1.52}$$

Combining equations (1.51) and (1.52) we prove that (1.46) holds.
So due to equations (1.45) and (1.46) we have that $v = w$. Since $w \in C^{1,2,\lambda}((T_0,T_1)\times A)$ (see Theorem **??** below) and T_0 T_1 and A are arbitrary we get that $v \in C^{1,2}((0,\infty)\times\mathbb{R}^d)\cap C^{1,2,\lambda}_{\text{loc}}((0,\infty)\times\mathbb{R}^d)$ and the proof is complete. □

We are ready to proof the Main Theorem

Proof of Theorem 1.2.1. The proof follows from Theorems 1.3.1 and 1.3.2 and Lemmas 1.2.1 and 1.2.2. □

Chapter 2

Linear parabolic differential equations.

In this chapter we study the existence and uniqueness of a classical solution to the Cauchy-Dirichlet problem for a linear parabolic differential equation in a general unbounded domain. Let $\mathcal{L}$ be the differential operator

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x) D_{ij} u(t,x) + \sum_{i=1}^{d} b_i(t,x) D_i u(t,x)$$

where $\{a_{ij}\} = a = \sigma\sigma'$, $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$. The Cauchy-Dirichlet problem is

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x), && (t,x) \in (0,\infty) \times D, \\ u(0,x) &= h(x), && x \in D, \\ u(t,x) &= g(t,x), && (t,x) \in (0,\infty) \times \partial D \end{aligned} \tag{2.1}$$

where $D \subset \mathbb{R}^d$ is an unbounded, open, connected set with regular boundary.

As we mentioned in the introduction, when the domain is bounded or the coefficients are bounded the problem is well understood (see [41], [40], [26], [43] and [25])

In the last years, parabolic equations with unbounded coefficients in unbounded domains have been studied in great detail. Many authors have considered unbounded coefficients, (b and σ) satisfying a Lyapunov type growth assumption, and bounded data. Using the theory of semigroups, the existence of a classical solution to the Cauchy-Dirichlet problem has been studied by

Fornaro, Metafune and Priola (2004) in [24], Bertoldi and Fornaro (2005) in [2], Bertoldi, Fornaro and Lorenzi (2007) in [3], Hieber, Lorenzi and Rhandi (2007) in [32] among others.

We follow the same ideas presented in Section 1.3.4.

2.1 Preliminaries, hypotheses and notation.

In this section we present the hypotheses and the notation used in this chapter.

We will consider $D \subset \mathbb{R}^d$ an unbounded, open, connected set with boundary ∂D and closure $\overline{D}$. We assume that D has a regular boundary, that is, for any $x \in \partial D$, x is a regular point (see [41] Chapter III Section 4 or [17] Chapter 2 Section 4, for a detailed discussion of regular points). We denote the hypotheses on D as **H0**.

2.1.1 Stochastic differential equation.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{s\geq 0})$ be a complete filtered probability space and let $\{W\} = \{W_i\}_{i=1}^d$ be a d-dimensional brownian motion defined in it. For $t \geq 0$ and $x \in \overline{D}$ consider the stochastic differential equation

$$dX(s) = b(t-s, X(s))ds + \sigma(t-s, X(s))dW(s), \quad X(0) = x, \tag{2.2}$$

where $b = \{b_i\}_{i=1}^d$ and $\sigma = \{\sigma_{ij}\}_{i,j=1}^d$. Despite this process is the natural one for solving equation (2.1), it does not posses many good properties. The continuity of the flow process does not imply the continuity with respect to t. Furthermore, although this process is a strong Markov process, is not homogeneous in time, a very useful property for proving the results in this chapter.

To overcome these difficulties, we augment the dimension considering the following process

$$d\xi(s) = -ds, \quad \xi(0) = t. \tag{2.3}$$

Then the process $\{\xi(s), X(s)\}$ is solution to

$$\begin{aligned} d\xi(s) &= -\,ds, \\ dX(s) &= b(\xi(s), X(s))ds + \sigma(\xi(s), X(s))dW(s), \end{aligned} \tag{2.4}$$

with $(\xi(0), X(0)) = (t, x)$. Throughout this chapter we will use both processes, $X(s)$ and $(\xi(s), X(s))$, in order to simplify the exposition.
We need to define the following stopping times

$$\tau_D := \inf\{s > 0 | X(s) \notin \overline{D}\} \tag{2.5}$$

and

$$\tau := \tau_D \wedge t. \tag{2.6}$$

Remark 2.1.1. *Observe that τ is the exit time of the process $(\xi(s), X(s))$ from the set $[0, \infty) \times \overline{D}$, i.e.*

$$\tau = \inf\{s > 0 | (\xi(s), X(s)) \notin [0, \infty) \times \overline{D}\}.$$

We can not guarantee that the process $X(s)$ leaves the set D in a finite time, however the process $\xi(s)$ reaches the boundary $s = 0$ at time t. Thus, the joint process $(\xi(s), X(s))$ leaves the set $[0, \infty) \times \overline{D}$ in a bounded time.

We assume the following hypotheses on the coefficients b and σ. We denote them by **H1**. The matrix norm considered is $\|\sigma\|^2 := \mathrm{tr}\sigma\sigma' = \sum_{i,j} \sigma_{ij}^2$.

H1:

Let

$$\begin{aligned} \sigma(r, x) &: \mathbb{R} \times \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d) \\ b(r, x) &: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \end{aligned}$$

be continuous functions such that

1. **(Continuity.)** Let $\lambda \in (0, 1)$. For all $T > 0$, $n \geq 1$ there exists $L_1(T, n)$ such that

   $$\|\sigma(r, x) - \sigma(s, y)\|^2 + \|b(r, x) - b(s, y)\|^2 \leq L_1(T, n)^2(|r - s|^{2\lambda} + \|x - y\|^2),$$

   for all $|r|, |s| \leq T$, $\|x\| \leq n$, $\|y\| \leq n$.

2. **(Linear growth.)** For each $T > 0$, there exists a constant $K_1(T)$ such that

   $$\|\sigma(r, x)\|^2 + \|b(r, x)\|^2 \leq K_1(T)^2(1 + \|x\|^2),$$

   for all $|r| \leq T$, $x \in \mathbb{R}^d$.

3. **(Local ellipticity.)** Let $A \subset \overline{D}$ be any bounded, open, connected set and $T > 0$. There exists $\lambda(T, A) > 0$ such that for all $(r, x) \in [0, T] \times \overline{A}$ and $\eta \in \overline{A}$
$$\sum_{i,j} a_{ij}(r, x)\eta_i\eta_j \geq \lambda(T, A)\|\eta\|^2.$$
where $\{a_{ij}\} = a = \sigma\sigma'$

Remark 2.1.2: *Observe that the local ellipticity is only assumed on* $[0, \infty) \times \overline{D}$*. This condition is used to prove the existence of a classical solution to equation* (2.1) *and so is only needed in that set. The local Lipschitz condition and the linear growth are assumed on* $\mathbb{R} \times \mathbb{R}^d$ *to ensure the existence of a strong solution to equation* (2.4) *for* $s \in [0, \infty)$.

Remark 2.1.3. *It follows from the non degeneracy (the local ellipticity) of the process* $X(s)$*, the regular boundary of the set* D *and Lemma 4.2, Chapter 2 in [17], that for any* $x \in D$

$$\mathbb{P}_{t,x}[\tau = \tau'] = 1.$$

where $\tau' := \inf\{s > 0 | (\xi(s), X(s)) \notin (0, \infty) \times D\}$ *(see Remark 2.1.1).*

The next proposition presents some of the properties of the process (ξ, X) required in this work.

Proposition 2.1.1. *As a consequence of **H1**,* (ξ, X) *has the following properties:*

- *for all* $(t, x) \in [0, \infty) \times \mathbb{R}^d$*, there exists a unique strong solution to equation* (2.4)*,*
- *the process* $\{\xi(s), X(s)\}_{s \geq 0}$ *is a strong homogeneous Markov process,*
- *the process* $\{\xi(s), X(s)\}_{s \geq 0}$ *does not explode in finite time a.s.,*
- *the flow process* $\{\xi(s,t), X(s,x)\}_{s \geq 0, (t,x) \in [0,\infty) \times \mathbb{R}^d}$ *is continuous a.s.,*
- *for all* $x \in \mathbb{R}^d$*,* $T > 0$ *and* $r \geq 1$

$$\mathbb{E}_x\left[\sup_{0 \leq s \leq T} \|X(s)\|^{2r}\right] \leq C(T, K_1, r)(1 + \|x\|^{2r}). \tag{2.7}$$

Proof. See [57] chapter 6 or [39] chapter V for a proof of these properties. □

Alternatively, we may assume these less restrictive assumptions. We denote them by **H1'**

H1':

Let

$$\sigma(r,x) : [0,\infty) \times \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$$
$$b(r,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d,$$

be continuous functions such that

1. **(Continuity.)** Let $\lambda \in (0,1)$. For all $T > 0$, $n \geq 1$ there exists $L_1(T,n)$ such that

   $$\|\sigma(r,x)-\sigma(s,y)\|^2+\|b(r,x)-b(s,y)\|^2 \leq L_1(T,n)^2(|r-s|^{2\lambda}+\|x-y\|^2),$$

   for all $0 \leq r, s \leq T$, $\|x\| \leq n$, $\|y\| \leq n$.

2. **(Linear growth.)** For each $T > 0$, there exists a constant $K_1(T)$ such that

   $$\|\sigma(r,x)\|^2 + \|b(r,x)\|^2 \leq K_1(T)^2(1 + \|x\|^2),$$

   for all $0 \leq r \leq T$, $x \in \mathbb{R}^d$.

3. **(Local ellipticity.)** Let $A \subset \overline{D}$ be any bounded, open, connected set and $T > 0$. There exists $\lambda(T,A) > 0$ such that for all $(r,x) \in [0,T] \times \overline{A}$ and $\eta \in \overline{A}$

   $$\sum_{i,j} a_{ij}(r,x)\eta_i\eta_j \geq \lambda(T,A)\|\eta\|^2.$$

   where $\{a_{ij}\} = a = \sigma\sigma'$

Remark 2.1.4. *If we assume* ***H1'*** *we can extend the functions $b(r,x)$ and $\sigma(r,x)$ to be defined for negative values of r as follows: let $\hat{b}$ and $\hat{\sigma}$ be defined as*

$$\hat{b}(r,x) = \begin{cases} b(r,x), & \textit{if } r \geq 0, \\ b(0,x), & \textit{if } r < 0, \end{cases}$$

and

$$\hat{\sigma}(r,x) = \begin{cases} \sigma(r,x), & \textit{if } r \geq 0, \\ \sigma(0,x), & \textit{if } r < 0. \end{cases}$$

It is easy to see that these functions satisfy ***H1*** *with the same constants L_1 and K_1.*

2.1.2 The Cauchy-Dirichlet problem.

Consider the following differential operator

$$\mathcal{L}[u](t,x) := \sum_{i,j=1}^{d} a_{ij}(t,x) D_{ij} u(t,x) + \sum_{i=1}^{d} b_i(t,x) D_i u(t,x)$$

where $D_i = \frac{\partial}{\partial x_i}$, $D_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ and $\{a_{ij}\}_{i,j=1}^d = a = \sigma\sigma'$. For the rest of the chapter, we assume that the coefficients of $\mathcal{L}$ satisfies **H1**.
The Cauchy-Dirichlet problem for a linear parabolic equation is

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) &= -f(t,x), && (t,x) \in (0,\infty) \times D, \\ u(0,x) &= h(x), && x \in D, \\ u(t,x) &= g(t,x), && (t,x) \in (0,\infty) \times \partial D. \end{aligned} \tag{2.8}$$

We assume the following hypotheses for the functions c, f, h and g. We denote them by **H2**.

H2:

1. Let

$$c(r,x) : [0,\infty) \times \overline{D} \to \mathbb{R}$$
$$f(r,x) : [0,\infty) \times \overline{D} \to \mathbb{R},$$

be continuous functions such that

- **(Continuity.)** Let $\lambda \in (0,1)$. For all $T > 0$, $n \geq 1$ there exists a constant $L_2(T,n)$ such that

  $$\|f(r,x) - f(s,y)\|^2 + \|c(r,x) - c(s,y)\|^2 \leq L_2(T,n)^2(|r-s|^{2\lambda} + \|x-y\|^2),$$

  for all $0 \leq s, r \leq T$, $x, y \in \overline{D}$ with $\|x\| \leq n$, $\|y\| \leq n$.
- **(Growth.)** There exists $c_0 \geq 0$ such that

  $$c(r,x) \leq c_0 \quad \text{for all } (r,x) \in [0,\infty) \times \overline{D}.$$

  There exists $k > 0$, such that for all $T > 0$, a constant $K_2(T)$ exists such that

  $$|f(r,x)| \leq K_2(T)(1 + \|x\|^k),$$

  for all $0 \leq r \leq T$, $x \in \overline{D}$

2. Let

$$h(x) : D \to \mathbb{R}$$
$$g(r, x) : (0, \infty) \times \partial D \to \mathbb{R},$$

be continuous functions such that

- **(Growth.)** There exists $k > 0$, such that for all $T > 0$, there exists a constant $K_3(T)$,

$$|h(x)| + |g(r, x)| \leq K_3(T)(1 + \|x\|^k)$$

  for all $(r, x) \in [0, T] \times \overline{D}$.

- **(Consistency.)** There exists consistency in the intersection of the space and the time boundaries, that is,

$$h(x) = g(0, x)$$

  for $x \in \partial D$.

2.1.3 Additional notation.

If μ is a locally Lipschitz function defined in some set R, then for any bounded open set A for which $\overline{A} \subset R$, we denote by $K_\mu(A)$ and $L_\mu(A)$, the constants

$$K_\mu(A) := \sup_{x \in A} \|\mu(x)\| < \infty,$$
$$L_\mu(A) := \sup_{x, y \in A, x \neq y} \frac{\|\mu(x) - \mu(y)\|}{\|x - y\|} < \infty.$$

If $\nu : [0, \infty) \to \mathbb{R}^d$, then for all $T > 0$

$$\|\nu\|_T := \sup_{0 \leq s \leq T} \|\nu(s)\|.$$

The space $C^{1,2,\lambda}_{\text{loc}}((0, \infty) \times D)$ is the space of all functions such that they and all their derivatives up to the second order in x and first order in t, are locally Hölder of order λ,

2.2 Main result.

In this section we present the main result of this chapter and some parts of the proof.

Theorem 2.2.1. *Assume **H0**, **H1** and **H2**. Then there exists a unique solution $u \in C([0,\infty) \times \overline{D}) \cap C^{1,2,\lambda}_{loc}((0,\infty) \times D)$ to equation (2.8). The solution has the representation*

$$\begin{aligned} u(t,x) =& \mathbf{E}_x\left[\int_0^{\tau} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right] \\ &+ \mathbf{E}_x\left[e^{\int_0^t c(t-r,X(r))dr} h(X(t))\mathbf{1}_{\tau_D \geq t}\right] \\ &+ \mathbf{E}_x\left[e^{\int_0^{\tau_D} c(t-r,X(r))dr} g(t-\tau_D, X(\tau_D))\mathbf{1}_{\tau_D < t}\right] \end{aligned}$$

where X is the solution to the stochastic differential equation

$$dX(s) = b(t-s,X(s))ds + \sigma(t-s,X(s))dW(s), \quad X(0) = x,$$

and $\tau := \tau_D \wedge t$, with

$$\tau_D := \inf\{s > 0 | X(s) \notin \overline{D}\}.$$

Furthermore, for all $T > 0$

$$\sup_{0 \leq t \leq T} |u(t,x)| \leq C(T, c_0, K_1, K_2, K_3, k)(1 + \|x\|^k), \quad x \in \overline{D}, \tag{2.9}$$

*where c_0, K_1, K_2, K_3 and k are the constants defined in **H1** and **H2**.*

The proof of this Theorem is given by several Lemmas. The method we will use has the following steps: first we define a functional of the process X as a candidate solution. Let $v : [0,\infty) \times \overline{D} \to \mathbb{R}$ be defined as

$$\begin{aligned} v(t,x) :=& \mathbf{E}_x\left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right] \\ &+ \mathbf{E}_x\left[e^{\int_0^t c(t-r,X(r))dr} h(X(t))\mathbf{1}_{\tau_D \geq t}\right] \\ &+ \mathbf{E}_x\left[e^{\int_0^{\tau_D} c(t-r,X(r))dr} g(t-\tau_D, X(\tau_D))\mathbf{1}_{\tau_D < t}\right] \end{aligned} \tag{2.10}$$

If $v \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$, then there exists some standard arguments (see [16] chapter 4) to prove that v is the unique solution to equation (2.8). The rest of this section is devoted to proving Theorem 2.2.1 in the case when v is a "regular" function. The proof is divided into two lemmas: the first one proves that if $v \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$, then v is a solution to equation (2.8) and hence we get existence. The second one proves that in case of existence of a classical solution, u, to equation (2.8), then it is unique and has the form given by v in equation (2.10). The regularity of v is proved in Section 2.3 below.

The next proposition gives an extension of the boundary data to all the space $[0,\infty) \times \mathbb{R}^d$. This extension is given to simplify the notation and is required in the proofs to Lemmas 2.3.1 and 2.3.2 below.

Proposition 2.2.1. *Assume* **H2**. *Then there exists a continuous function* $G : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\begin{aligned} G(t,x) =& g(t,x), \quad (t,x) \in (0,\infty) \times \partial D \\ G(0,x) =& h(x), \quad x \in D. \end{aligned}$$

Proof. Thanks to the consistency condition in **H2** and the continuity of g and h, we can extend by Tietze's Extension Theorem (see [15] section 2.6) the functions g, h from the closed set $\{0\} \times \overline{D} \cup [0,\infty) \times \partial D$ to a continuous function G defined in $[0,\infty) \times \mathbb{R}^d$. □

As a consequence of Proposition 2.2.1 we can write v in equation (2.10) as follows

$$\begin{aligned} v(t,x) =& \mathbb{E}_x \left[\int_0^\tau e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^\tau c(t-r,X(r))dr} G(t-\tau, X(\tau)) \right]. \end{aligned} \tag{2.11}$$

We are ready to prove the existence and uniqueness of a solution to equation (2.8) assuming $v \in C^{1,2}$.

Lemma 2.2.1. *Assume* **H0**, **H1** *and* **H2**. *Let v be defined as in equation* (2.11) *and assume that $v \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \times D)$. Then v fulfils the following equation*

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) =& -f(t,x) \quad (t,x) \in (0,\infty) \times D, \\ u(t,x) =& G(t,x) \quad (t,x) \in \partial((0,\infty) \times D). \end{aligned}$$

Furthermore, for all $T > 0$, there exists C such that

$$\sup_{0\le t\le T} |v(t,x)| \le C(T, c_0, K_1, K_2, K_3, k)(1 + \|x\|^k), \quad x \in \overline{D},$$

*where c_0, K_1, K_2, K_3 and k are the constants defined in **H1** and **H2**.*

Proof. Let $0 \le \alpha \le t$, then following the same argument used to prove equation (2.46) in the proof of Theorem 2.3.2 in section 2.3 below we have that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^T e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds + e^{\int_0^\tau c(t-r,X(r))dr} G(t-\tau, X(\tau))\middle| \mathcal{F}_\alpha\right] \\
&= \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \\
&\quad + e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))} v(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)).
\end{aligned} \tag{2.12}$$

Because of **H1** and **H2** we have that the random variable inside the conditional expectation is integrable and so the lefthand side of equation (2.12) is a $\mathcal{F}_\alpha$-martingale, for $\alpha \in [0,t]$. Since $v \in C^{1,2}$ we can apply Ito's formula to $e^{\int_0^\alpha cdr} v()$ to get

$$\begin{aligned}
e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} v(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) &= v(t,x) \\
&+ \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr}(-v_t + \mathcal{L}[v] + cv)(t-s,X(s))ds \\
&+ \int_0^{\alpha\wedge\tau_D} Dv(t-s,X(s))\cdot\sigma(t-s,X(s))dW(s).
\end{aligned} \tag{2.13}$$

It follows from the continuity of Dv, σ and $X(\cdot)$ that

$$\sup_{0\le s\le\alpha} \|Dv(t-s,X(s))\|\|\sigma(t-s,X(s))\|$$

is a.s. finite and then

$$\int_0^{\alpha\wedge\tau_D} Dv(t-s,X(s))\cdot\sigma(t-s,X(s))dW(s)$$

is a local martingale for $0 \le \alpha \le t$. So combining equations (2.12) and (2.13) we get that

$$M(\alpha) := \int_0^{\alpha \wedge \tau_D} e^{\int_0^s c(t-r,X(r))dr}(-v_t + \mathcal{L}[v] + cv + f)(t-s, X(s))ds$$

is a continuous local martingale for $\alpha \in [0,t]$. Since M is locally of bounded variation then $M(\alpha) \equiv 0$. This implies that $-v_t + \mathcal{L}[v] + cv + f = 0$ for all $(t,x) \in (0,\infty) \times D$. The boundary condition follows from the regularity of the set D, the local ellipticity condition and the continuity of v in $[0,\infty) \times \overline{D}$.

The second statement of the Theorem is proved with the same argument used to prove equations (2.21) and (2.34) in the proofs of Lemmas 2.3.1 and 2.3.2 in Section 2.3 below. □

The next Lemma proves the uniqueness of the solution.

Lemma 2.2.2. *Assume* ***H0, H1*** *and* ***H2****. Assume there exists a classical solution* $u \in C([0,\infty) \times \overline{D}) \cap C^{1,2}((0,\infty) \cap D)$ *to equation*

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) =& -f(t,x) \quad (t,x) \in (0,\infty) \times D, \\ u(t,x) =& G(t,x) \quad (t,x) \in \partial((0,\infty) \times D), \end{aligned} \tag{2.14}$$

such that for all $T > 0$, *exists* C *for which*

$$\sup_{0 \le t \le T} |u(t,x)| \le C(T)(1 + \|x\|^\mu) \tag{2.15}$$

for some $\mu > 0$. *Then* u *has the following representation*

$$\begin{aligned} u(t,x) =& \mathbb{E}_x \left[\int_0^\tau e^{\int_0^s c(t-r,X(r))dr} f(t-s, X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^\tau c(t-r,X(r))dr} G(t-\tau, X(\tau)) \right]. \end{aligned}$$

and hence the solution is unique.

Proof. Consider, for $\alpha \in [0,t]$, the process

$$e^{\int_0^{\alpha \wedge \tau_D} c(t-r,X(r))dr} u(t - \alpha \wedge \tau_D, X(\alpha \wedge \tau_D)).$$

Applying Ito's rule, we get

$$\begin{aligned} e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) &= u(t,x) \\ &+ \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr}(-u_t+\mathcal{L}[u]+cu)(t-s,X(s))ds \\ &+ \int_0^{\alpha\wedge\tau_D} Du(t-s,X(s))\cdot\sigma(t-s,X(s))dW(s). \end{aligned}$$

A similar argument as the one used in the proof to Lemma 2.2.1 shows that

$$\int_0^{\alpha\wedge\tau_D} Du(t-s,X(s))\cdot\sigma(t-s,X(s))dW(s)$$

is a local martingale. Due to equation (2.14) we conclude that

$$\begin{aligned} M(\alpha) :=& e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) \\ &+ \int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \end{aligned}$$

is a local martingale for $\alpha \in [0,t]$. Let $\{\theta_n\}_{n\geq 1}$ be a sequence of localization times for $M(\alpha)$, i.e., $\theta_n \uparrow \infty$ a.s. as $n \to \infty$ and $M(\alpha\wedge\theta_n)$ is a martingale for all $n \geq 1$. Then for all $n \geq 1$

$$\begin{aligned} u(t,x) =& \mathbb{E}_x\left[e^{\int_0^{\alpha\wedge\tau_D\wedge\theta_n} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D\wedge\theta_n, X(\alpha\wedge\tau_D\wedge\theta_n))\right] \\ &+ \mathbb{E}_x\left[\int_0^{\alpha\wedge\tau_D\wedge\theta_n} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right]. \end{aligned}$$

Since $0 \leq \alpha\wedge\tau_D\wedge\theta_n \leq t$, using equation (2.15) we get

$$\begin{aligned} e^{\int_0^{\alpha\wedge\tau_D\wedge\theta_n} c(t-r,X(r))dr} &|u(t-\alpha\wedge\tau_D\wedge\theta_n, X(\alpha\wedge\tau_D\wedge\theta_n))| \\ &\leq e^{c_0 t}C(t)\left(1+\|X(\alpha\wedge\tau_D\wedge\theta_n)\|^\mu\right) \\ &\leq e^{c_0 t}C(t)\left(1+\sup_{0\leq s\leq t}\|X(s)\|^\mu\right). \end{aligned}$$

And

$$\begin{aligned} &\left|\int_0^{\alpha\wedge\tau_D\wedge\theta_n} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds\right| \\ &\leq \int_0^{\alpha\wedge\tau_D\wedge\theta_n} e^{c_0 s}K_2(t)\left(1+\|X(s)\|^k\right)ds \\ &\leq e^{c_0 t}tK_2(t)\left(1+\sup_{0\leq s\leq t}\|X(s)\|^k\right). \end{aligned}$$

By equation (2.7) and the Dominated Convergence Theorem, letting $n \to \infty$ we get

$$\begin{aligned} u(t,x) =& \mathbb{E}_x \left[e^{\int_0^{\alpha\wedge\tau_D} c(t-r,X(r))dr} u(t-\alpha\wedge\tau_D, X(\alpha\wedge\tau_D)) \right] \\ &+ \mathbb{E}_x \left[\int_0^{\alpha\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right]. \end{aligned}$$

Letting $\alpha \uparrow t$, a similar argument and the boundary condition proofs that

$$\begin{aligned} u(t,x) =& \mathbb{E}_x \left[e^{\int_0^{t\wedge\tau_D} c(t-r,X(r))dr} G(t-t\wedge\tau_D, X(t\wedge\tau_D)) \right] \\ &+ \mathbb{E}_x \left[\int_0^{t\wedge\tau_D} e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right], \end{aligned}$$

and the proof is complete. □

2.3 Regularity of v.

In this section we prove that $v \in C([0,\infty)\times\overline{D}) \cap C^{1,2,\lambda}_{\text{loc}}((0,\infty)\times D)$. First, we prove using the continuity of the flow process X, that v is a continuous function in $[0,\infty)\times\overline{D}$. Since we are only assuming the continuity of the coefficients, then the flow is not necessarily differentiable and so we can not prove the regularity of v in terms of the regularity of the flow. To prove that $v \in C^{1,2}$, we show that v is the solution to a parabolic differential equation in a bounded domain, for which we have the existence of a classical solution and hence $v \in C^{1,2}$.

2.3.1 Continuity of v.

Let (ξ, X) denote the solution to equation (2.4) and G be defined as in Proposition 2.2.1, then v has the following form

$$\begin{aligned} v(t,x) =& \mathbb{E}_{t,x} \left[\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds \right] \\ &+ \mathbb{E}_{t,x} \left[e^{\int_0^{\tau} c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau)) \right]. \end{aligned} \tag{2.16}$$

For simplicity, we write $v = v_1 + v_2$, where

$$v_1(t,x) := \mathbb{E}_{t,x} \left[\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds \right] \tag{2.17}$$

and

$$v_2(t,x) := \mathbb{E}_{t,x}\left[e^{\int_0^\tau c(\xi(r),X(r))dr}G(\xi(\tau),X(\tau))\right]. \tag{2.18}$$

Theorem 2.3.1. *Assume **H0**, **H1** and **H2**. Let v be defined as in equation* (2.16). *Then v is continuous on $[0,\infty)\times\overline{D}$.*

The proof to this Theorem is divided into two lemmas.

Lemma 2.3.1. *Assume **H0**, **H1** and **H2**. Let v_1 be defined as in equation* (2.17). *Then v_1 is continuous on $[0,\infty)\times\overline{D}$.*

Proof. First we prove the continuity on $(0,\infty)\times D$. For that, let

$$(t_n,x_n)\xrightarrow[n\to\infty]{}(t,x)$$

in $(0,\infty)\times D$ and $\epsilon>0$. We need to prove that there exists $N(\epsilon)\in\mathbb{N}$ such that for all $n\geq N$

$$|v_1(t_n,x_n)-v_1(t,x)|<\epsilon.$$

Denote by (ξ,X) and (ξ_n,X_n) the solutions to equation (2.4) with initial conditions (t,x) and (t_n,x_n) respectively. Let τ and τ_n be their corresponding exit times from $[0,\infty)\times\overline{D}$.
Let $\alpha>0$, then there exists $N_1\in\mathbb{N}$ such that for all $n\geq N_1$

$$|(t_n,x_n)-(t,x)|<\alpha. \tag{2.19}$$

Observe that for all $n\geq N_1$, we get

$$\begin{aligned}\tau_n &\leq t_n\leq t+\alpha,\\ \tau &\leq t\leq t+\alpha.\end{aligned} \tag{2.20}$$

Define the random variables Y_n as

$$\begin{aligned}Y_n := \Bigg|&\int_0^{\tau_n} e^{\int_0^s c(\xi_n(r),X_n(r))dr}f(\xi_n(s),X_n(s))ds\\ &-\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr}f(\xi(s),X(s))ds\Bigg|.\end{aligned}$$

The sequence $\{Y_n\}_{n\geq N_1}$ is uniformly integrable. To see that

$$
\begin{aligned}
\mathbb{E}\left[Y_n^2\right] \leq & 2\mathbb{E}\left[\left|\int_0^{\tau_n} e^{\int_0^s c(\xi_n(r),X_n(r))dr} f(\xi_n(s),X_n(s))ds\right|^2\right] \\
& + 2\mathbb{E}\left[\left|\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right|^2\right] \\
\leq & 2\mathbb{E}\left[\left|\int_0^{\tau_n} e^{c_0(t+\alpha)} K_2(t+\alpha)(1+\|X_n(s)\|^k)ds\right|^2\right] + C_{t,x} \\
\leq & 2e^{2c_0(t+\alpha)} K_2^2(t+\alpha)\mathbb{E}\left[\left|\int_0^{\tau_n}\left(1+\sup_{0\leq r\leq t+\alpha}\|X_n(r)\|^k\right)ds\right|^2\right] + C_{t,x} \\
\leq & 4e^{2c_0(t+\alpha)} K_2^2(t+\alpha)(t+\alpha)^2\left(1+\mathbb{E}\left[\sup_{0\leq r\leq t+\alpha}\|X_n(r)\|^{2k}\right]\right) + C_{t,x} \\
\leq & C(1+K(1+\|x_n\|^{2k})) + C_{t,x} \\
\leq & C(1+K(1+(\|x\|+\alpha)^{2k})) + C_{t,x} < \infty,
\end{aligned}
\tag{2.21}
$$

where we use (2.19), (2.20), (2.7) and the polynomial growth of f.
Let $M > 0$, $0 < \eta < 1$ and $\beta > 0$ and define the set

$$
E_{M,n,\eta,\beta} := \{\|X\|_{t+\alpha} \leq M\} \cap \{\|X_n - X\|_{t+\alpha} \leq \eta\} \cap \{|\tau_n - \tau| \leq \beta\}. \tag{2.22}
$$

Then

$$
\begin{aligned}
|v_1(t_n,x_n) - v_1(t,x)| &\leq \int_\Omega Y_n d\mathbb{P} \\
&= \int_{E_{M,n,\eta,\beta}} Y_n d\mathbb{P} + \int_{\Omega\setminus E_{M,n,\eta,\beta}} Y_n d\mathbb{P}.
\end{aligned}
$$

Since the sequence $\{Y_n\}$ is uniformly integrable, there exists $\delta(\epsilon)$ such that for any $E \in \mathcal{F}$ that satisfies $\mathbb{P}[E] < \delta$, we have

$$
\sup_{n\geq N_1} \int_E Y_n d\mathbb{P} < \frac{\epsilon}{2}. \tag{2.23}
$$

It follows from Remark 2.1.3, Proposition 2.1.1 and Theorems A.1.2, A.3.1 in Appendix A, the existence of M and N_2 such that

$$
\mathbb{P}[\Omega \setminus E_{M,n,\eta,\beta}] < \delta(\epsilon)
$$

for all $n \geq N_2$. Then for $n \geq N_1 \vee N_2$ we get that

$$|v_1(t_n, x_n) - v_1(t,x)| \leq \int_{E_{M,n,\eta,\beta}} Y_n d\mathbb{P} + \frac{\epsilon}{2}.$$

For simplicity of notation, we write the set $E_{M,n,\eta,\beta}$ as E and define

$$A := [0, t+\alpha] \times [-M-1, M+1]^d \tag{2.24}$$

and

$$D_t := [0, t+\alpha] \times \overline{D}. \tag{2.25}$$

On the set E, for all $n \geq N_1$ and $0 \leq s \leq t+\alpha$, it is satisfied that

$$(\xi_n(s), X_n(s)), (\xi(s), X(s)) \in A.$$

We have that

$$\begin{aligned}\int_E Y_n d\mathbb{P} \leq & \int_E \int_0^{\tau_n \wedge \tau} \Big| e^{\int_0^s c(\xi_n(r), X_n(r))dr} f(\xi_n(s), X_n(s)) \\ & - e^{\int_0^s c(\xi(r), X(r))dr} f(\xi(s), X(s)) \Big| ds d\mathbb{P} \qquad (2.26) \\ & + \int_E \int_{\tau_n \wedge \tau}^{\tau_n \vee \tau} \Big(e^{\int_0^s c(\xi_n(r), X_n(r))dr} |f(\xi_n(s), X_n(s))| \mathbb{1}_{\tau_n > \tau} \\ & + e^{\int_0^s c(\xi(r), X(r))dr} |f(\xi(s), X(s))| \mathbb{1}_{\tau_n \leq \tau} \Big) ds d\mathbb{P}. \qquad (2.27)\end{aligned}$$

We first analyze (2.27).

$$\begin{aligned}(2.27) \leq & \int_E \int_{\tau_n \wedge \tau}^{\tau_n \vee \tau} e^{c_0(t+\alpha)} K_f(A \cap D_t)(\mathbb{1}_{\tau_n \leq \tau} + \mathbb{1}_{\tau_n > \tau}) ds d\mathbb{P} \\ = & e^{c_0(t+\alpha)} K_f(A \cap D_t) \int_E |\tau_n - \tau| d\mathbb{P} \\ \leq & e^{c_0(t+\alpha)} K_f(A \cap D_t)\beta.\end{aligned}$$

For (2.26) we get

$$\begin{aligned}(2.26) \leq & \int_E \int_0^{\tau_n \wedge \tau} e^{\int_0^s c(\xi_n(r), X_n(r))dr} \\ & \times |f(\xi_n(s), X_n(s)) - f(\xi(s), X(s))| ds d\mathbb{P} \qquad (2.28) \\ & + \int_E \int_0^{\tau_n \wedge \tau} f(\xi(s), X(s)) \\ & \times \left| e^{\int_0^s c(\xi_n(r), X_n(r))dr} - e^{\int_0^s c(\xi(r), X(r))dr} \right| ds d\mathbb{P}. \qquad (2.29)\end{aligned}$$

Now

$$\begin{aligned}(2.28) \leq \int_E e^{c_0(t+\alpha)} \int_0^{\tau_n \wedge \tau} L_f(A \cap D_t)(|t_n - t|^\lambda + \|X_n(s) - X(s)\|) ds d\mathbb{P} \\ \leq e^{c_0(t+\alpha)}(t+\alpha) L_f(A \cap D_t)(|t_n - t|^\lambda + \eta).\end{aligned}$$

For (2.29) we need the following bound

$$\begin{aligned}\left| e^{\int_0^s c(\xi_n(r), X_n(r)) dr} - e^{\int_0^s c(\xi(r), X(r)) dr} \right| &= e^{\int_0^s c(\xi(r), X(r)) dr} \\ &\times \left| \exp\left\{ \int_0^s (c(\xi_n(r), X_n(r)) - c(\xi(r), X(r))) dr \right\} - 1 \right| \\ &\leq e^{c_0 s} \left(\exp\left\{ \int_0^s |c(\xi_n(r), X_n(r)) - c(\xi(r), X(r))| dr \right\} - 1 \right) \\ &\leq e^{c_0 s} \left(\exp\left\{ \int_0^s L_c(A \cap D_t)(|t_n - t|^\lambda + \|X_n(r) - X(r)\|) dr \right\} - 1 \right) \\ &\leq e^{c_0 s}(\exp\{L_c(A \cap D_t) s(|t_n - t|^\lambda + \eta)\} - 1).\end{aligned} \tag{2.30}$$

since $|e^x - 1| \leq e^{|x|} - 1$. If we choose $N_3 \in \mathbb{N}$ such that $|t_n - t|^\lambda \leq \frac{1}{2L_c(A\cap D_t)(t+\alpha)}$ for all $n \geq N_3$ and $\eta \leq \frac{1}{2L_c(A\cap D_t)(t+\alpha)}$, we get by the Mean Value Theorem that

$$\left| e^{\int_0^s c(\xi_n(r), X_n(r)) dr} - e^{\int_0^s c(\xi(r), X(r)) dr} \right| \leq e^{c_0 s} e L_c(A \cap D_t) s(|t_n - t|^\lambda + \eta). \tag{2.31}$$

Then

$$(2.29) \leq K_f(A \cap D_t) e^{c_0(t+\alpha)} e L_c(A \cap D_t)(t+\alpha)^2(|t_n - t|^\lambda + \eta).$$

Hence, to prove continuity we proceed as follows

- Let $\epsilon > 0$ and $0 < \alpha \ll 1$.
- Let $N_1 \geq \mathbb{N}$ such that for all $n \geq N_1$

$$\|(t_n, x_n) - (t, x)\| < \alpha.$$

- Let $\delta(\epsilon) > 0$ that fulfils the uniformly integrable condition (2.23).
- Take $M > 0$ such that $\mathbb{P}[\|X\|_{t+\alpha} > M] < \frac{\delta(\epsilon)}{3}$.

- Define $A := [0, t+\alpha] \times [-M-1, M+1]^d$ and $D_t := [0, t+\alpha] \times \overline{D}$.
- Let
$$\eta < \min\left\{1, \frac{1}{2L_c(A\cap D_t)(t+\alpha)}, \frac{\epsilon}{16e^{c_0(t+\alpha)}(t+\alpha)L_f(A\cap D_t)}\right.$$
$$\left., \frac{\epsilon}{16K_f(A\cap D_t)e^{c_0(t+\alpha)}eL_c(A\cap D_t)(t+\alpha)}\right\}.$$
- Chose $N_2 \in \mathbb{N}$ such that $\mathbb{P}\left[\|X_n - X\|_{t+\alpha} > \eta\right] \leq \frac{\delta(\epsilon)}{3}$ for all $n \geq N_2$
- Let $N_3 \in \mathbb{N}$ such that
$$|t_n - t|^\lambda < \min\left\{\frac{1}{2L_c(A\cap D_t)(t+\alpha)}, \frac{\epsilon}{16e^{c_0(t+\alpha)}(t+\alpha)L_f(A\cap D_t)}\right.$$
$$\left., \frac{\epsilon}{16K_f(A\cap D_t)e^{c_0(t+\alpha)}eL_c(A\cap D_t)(t+\alpha)}\right\}.$$
for all $n \geq N_3$
- Let
$$\beta < \frac{\epsilon}{4e^{c_0(t+\alpha)}K_f(A\cap D_t)},$$
and chose $N_4 \in \mathbb{N}$ such that for all $n \geq N_4$, $\mathbb{P}\left[|\tau_n - \tau| > \beta\right] \leq \frac{\delta(\epsilon)}{3}$.

Thus if $N = N_1 \vee N_2 \vee N_3 \vee N_4$, then for all $n \geq N$
$$|v_1(t_n, x_n) - v_1(t, x)| < \epsilon.$$
Therefore v_1 is continuous in $(0,\infty) \times D$.

For the continuity at the boundary we make a similar argument. Let $(t_n, x_n) \xrightarrow[n\to\infty]{} (t, x)$, where $(t_n, x_n) \in (0,\infty)\times D$ and $(t,x) \in \partial((0,\infty)\times D)$, that is, either $t = 0$ or $x \in \partial D$. In both cases we get that $\tau = 0$ a.s. and so $v_1(t,x) = 0$. Then we need to prove that
$$|v_1(t_n, x_n)| \xrightarrow[n\to\infty]{} 0.$$
Let $0 < \alpha \ll 1$ and $N_1 \in \mathbb{N}$ such that
$$\|(t_n, x_n) - (t, x)\| < \alpha.$$

We get

$$\tau_n \le t_n < t + \alpha$$

for all $n \ge N_1$. For the continuity we have

$$\begin{aligned}
|v_1(t_n, x_n)| \le & \mathbb{E}\left[\int_0^{\tau_n} e^{\int_0^s c(\xi_n(r), X_n(r))dr} |f(\xi_n(s), X_n(s))| ds\right] \\
\le & \mathbb{E}\left[\int_0^{\tau_n} e^{c_0 s} K_2(t+\alpha)(1 + \|X_n(s)\|^k) ds\right] \\
\le & e^{c_0(t+\alpha)} K_2(t+\alpha)\mathbb{E}\left[\tau_n\left(1 + \sup_{0\le r\le t+\alpha} \|X_n(r)\|^k\right)\right] \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

The convergence follows from the uniform integrability of

$$\tau_n\left(1 + \sup_{0\le r\le t+\alpha} \|X_n(r)\|^k\right)$$

for $n \ge N_1$, that $\tau_n \xrightarrow[n\to\infty]{a.s.} 0$ (see Theorem A.1.2 in Appendix A) and Theorem 5.2 in Chapter 5 of [29]. This completes the proof. □

Lemma 2.3.2. *Assume **H0**, **H1** and **H2**. Let v_2 be defined as in equation* (2.18). *Then v_2 is continuous on* $[0, \infty) \times \overline{D}$.

Proof. We use an analogous argument to the one in the proof of Lemma 2.3.1. First we prove the continuity in $(0, \infty) \times D$. Let

$$(t_n, x_n) \xrightarrow[n\to\infty]{} (t, x),$$

with $(t_n, x_n), (t, x) \in (0, \infty) \times D$. Denote by (ξ_n, X_n) and (ξ, X) the solutions to equation (2.4) with initial conditions (t_n, x_n) and (t, x) respectively, and let τ_n and τ be its corresponding exit times from $[0, \infty) \times \overline{D}$. Let $0 < \alpha \ll 1$ and N_1 such that for all $n \ge N_1$

$$\|(t_n, x_n) - (t, x)\| < \alpha. \tag{2.32}$$

This implies that

$$\begin{aligned}\tau_n &\le t + \alpha \\ \tau &\le t + \alpha.\end{aligned} \tag{2.33}$$

First we prove that the sequence of random variables

$$Y_n := \left| e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} G(\xi_n(\tau_n), X_n(\tau_n)) - e^{\int_0^{\tau} c(\xi(r),X(r))dr} G(\xi(\tau), X(\tau)) \right|$$

is uniformly integrable for all $n \geq N_1$.

$$\begin{aligned}
\mathbb{E}\left[Y_n^2\right] \leq & 2\mathbb{E}\left[\left|e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} G(\xi_n(\tau_n), X_n(\tau_n))\right|^2\right] \\
& + 2\mathbb{E}\left[\left|e^{\int_0^{\tau} c(\xi(r),X(r))dr} G(\xi(\tau), X(\tau))\right|^2\right] \\
\leq & 2\mathbb{E}\left[e^{2c_0(t+\alpha)} K_3^2(t+\alpha)(1 + \|X_n(\tau_n)\|^k)^2\right] + C_{t,x} \\
\leq & 2e^{2c_0(t+\alpha)} K_3^2(t+\alpha)\mathbb{E}\left[(1 + \sup_{0\leq r\leq t+\alpha} \|X_n(r)\|^k)^2\right] + C_{t,x} \qquad (2.34) \\
\leq & 4e^{2c_0(t+\alpha)} K_3^2(t+\alpha)\left(1 + \mathbb{E}\left[\sup_{0\leq r\leq t+\alpha} \|X_n(r)\|^{2k}\right]\right) + C_{t,x} \\
\leq & C(1 + K(1 + \|x_n\|^{2k})) + C_{t,x} \\
\leq & C(1 + K(1 + (\|x\| + \alpha)^{2k})) + C_{t,x} < \infty,
\end{aligned}$$

where we use equations (2.32), (2.33), (2.7) and polynomial growth of G in $\partial((0,\infty) \times D)$. As in Lemma 2.3.1, let $\epsilon > 0$, then there exists $\delta(\epsilon) > 0$ such that

$$\sup_{n\geq N_1} \int_E Y_n d\mathbb{P} < \frac{\epsilon}{2} \qquad (2.35)$$

for all $E \in \mathcal{F}$, with $\mathbb{P}[E] < \delta(\epsilon)$.
Let $E_{M,n,\eta,\beta}$ be defined as in equation (2.22), and chose $M > 0$ and $N_2 \in \mathbb{N}$ such that

$$\mathbb{P}[\Omega \setminus E_{M,n,\eta,\beta}] < \delta(\epsilon),$$

for all $n \geq N_2$.
For simplicity of notation, denote $E_{M,n,\eta,\beta}$ as E. Then

$$\begin{aligned}
|v_2(t_n, x_n) - v_2(t,x)| &\leq \int_E Y_n d\mathbb{P} + \int_{\Omega\setminus E} Y_n d\mathbb{P} \\
&\leq \int_E Y_n d\mathbb{P} + \frac{\epsilon}{2}.
\end{aligned}$$

Let A and D_t be defined as in Lemma 2.3.1 (equations (2.24) and (2.25)). Then on the set E we get that for all $n \geq N_1$ and $0 \leq s \leq t + \alpha$,

$$(\xi_n(s), X_n(s)), (\xi(s), X(s)) \in A.$$

So

$$\begin{aligned}\int_E Y_n d\mathbb{P} &\le \int_E e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} \\ &\quad \times \left|G(\xi_n(\tau_n), X_n(\tau_n)) - G(\xi(\tau), X(\tau))\right| d\mathbb{P} \qquad (2.36)\\ &+ \int_E \left|G(\xi(\tau), X(\tau))\right| \\ &\quad \left|e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} - e^{\int_0^{\tau} c(\xi(r),X(r))dr}\right| d\mathbb{P}. \qquad (2.37)\end{aligned}$$

We study each addend of the righthand side separately.

$$\begin{aligned}(2.36) &\le e^{c_0(t+\alpha)} \int_E |G(t_n - \tau_n, X_n(\tau_n)) - G(t_n - \tau_n, X(\tau_n))| d\mathbb{P} \qquad (2.38)\\ &+ e^{c_0(t+\alpha)} \int_E |G(t_n - \tau_n, X(\tau_n)) - G(t - \tau, X(\tau))| d\mathbb{P}. \qquad (2.39)\end{aligned}$$

First we get a bound for (2.38). Since G is continuous, then it is uniformly continuous on A. Then for $\epsilon > 0$, there exists $\gamma(c_0, t, \alpha, \epsilon, M)$ such that

$$|G((t_1, x_1)) - G(t_2, x_2))| < \frac{\epsilon}{8e^{c_0(t+\alpha)}}$$

for all $(t_1, x_1), (t_2, x_2) \in A$ with $\|(t_1, x_1) - (t_2, x_2)\| < \gamma(c_0, t, \alpha, \epsilon, M)$. On the set E, we have $(t_n - \tau_n, X_n(\tau_n)), (t_n - \tau_n, X(\tau_n)) \in A$ and

$$\|(t_n - \tau_n, X_n(\tau_n)) - (t_n - \tau_n, X(\tau_n))\| < \eta.$$

So if we choose $\eta < \gamma$, we get

$$(2.38) < \frac{\epsilon}{8e^{c_0(t+\alpha)}}.$$

Next we study (2.39). Thanks to Theorem A.1.2 we know that $\tau_n \xrightarrow[n\to\infty]{a.s} \tau$. This and the continuity of $X(\cdot)$ and G implies that

$$G(t_n - \tau_n, X(\tau_n)) \xrightarrow[n\to\infty]{a.s.} G(t - \tau, X(\tau)).$$

On the set E we have that $(t_n - \tau_n, X(\tau_n)), (t - \tau, X(\tau)) \in A$ and so

$$|G(t_n - \tau_n, X(\tau_n)) - G(t - \tau, X(\tau))| \mathbf{1}_E \le 2K_G(A).$$

By the Dominated Convergence Theorem, there exists, $N_3 \in \mathbb{N}$ such that

$$(2.39) < \frac{\epsilon}{8e^{c_0(t+\alpha)}}$$

for all $n \geq N_3$.
To give a bound for (2.37) we observe that on the set E

$$\begin{aligned}
&\left|\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr - \int_0^{\tau} c(\xi(r), X(r))dr\right| \\
&\quad \leq \int_0^{\tau_n\wedge\tau} |c(\xi_n(r), X_n(r)) - c(\xi(r), X(r))|dr \\
&\qquad + \int_{\tau_n\wedge\tau}^{\tau_n\vee\tau} \left(|c(\xi_n(r), X_n(r))|\mathbf{1}_{\tau_n\geq\tau} + |c(\xi(r), X(r))|\mathbf{1}_{\tau_n<\tau}\right) dr \\
&\quad \leq \int_0^{\tau_n\wedge\tau} L_c(A\cap D_t)(|t_n - t|^\lambda + \|X_n(r) - X(r)\|)dr + K_c(A\cap D_t)|\tau_n - \tau| \\
&\quad \leq L_c(A\cap D_t)(t+\alpha)(|t_n - t|^\lambda + \eta) + K_c(A\cap D_t)\beta.
\end{aligned}$$

Making a similar argument as the one made in equations (2.30) and (2.31) we get

$$\begin{aligned}
&\left|e^{\int_0^{\tau_n} c(\xi_n(r),X_n(r))dr} - e^{\int_0^{\tau} c(\xi(r),X(r))dr}\right| \\
&\qquad \leq e^{c_0(t+\alpha)}e\left[L_c(A\cap D_t)(t+\alpha)(|t_n - t|^\lambda + \eta) + K_c(A\cap D_t)\beta\right],
\end{aligned}$$

if $|t_n - t|^\lambda < \frac{1}{3L_c(A\cap D_t)(t+\alpha)}$, $\eta < \frac{1}{3L_c(A\cap D_t)(t+\alpha)}$ and $\beta < \frac{1}{3K_c(A\cap D_t)}$. Then

$$(2.37) \leq K_G(A)e^{c_0(t+\alpha)}e\left[L_c(A\cap D_t)(t+\alpha)(|t_n - t|^\lambda + \eta) + K_c(A\cap D_t)\beta\right].$$

Hence, to prove continuity we proceed as follows

- Let $\epsilon > 0$ and $0 < \alpha \ll 1$.
- Let $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$
$$\|(t_n, x_n) - (t, x)\| < \alpha.$$
- Let $\delta(\epsilon) > 0$ that fulfils the uniformly integrable condition (2.35).
- Take $M > 0$ such that $\mathbb{P}[\|X\|_{t+\alpha} > M] < \frac{\delta(\epsilon)}{3}$.

- Define $A := [0, t+\alpha] \times [-M-1, M+1]^d$ and $D_t := [0, t+\alpha] \times \overline{D}$.
- Let

$$\eta < \min\left\{1, \gamma(c_0, t, \alpha, \epsilon, M), \frac{1}{3L_c(A \cap D_t)(t+\alpha)}, \frac{\epsilon}{12K_G(A)e^{c_0(t+\alpha)}eL_c(A \cap D_t)(t+\alpha)}\right\}.$$

- Chose $N_2 \in \mathbb{N}$ such that $\mathbb{P}\left[\|X_n - X\|_{t+\alpha} > \eta\right] < \frac{\delta(\epsilon)}{3}$ for all $n \geq N_2$
- Let

$$\beta < \min\left\{\frac{1}{3K_c(A \cap D_t)}, \frac{\epsilon}{12K_G(A)e^{c_0(t+\alpha)}eK_c(A \cap D_t)}\right\}$$

- Chose $N_3 \in \mathbb{N}$ such that $\mathbb{P}\left[|\tau_n - \tau| > \beta\right] < \frac{\delta(\epsilon)}{3}$ for all $n \geq N_3$.
- Let $N_4 \in \mathbb{N}$ such that

$$|t_n - t|^\lambda < \min\left\{\frac{1}{3(L_c(A \cap D_t)(t+\alpha)}, \frac{\epsilon}{12K_G(A \cap D_t)e^{c_0(t+\alpha)}eL_c(A \cap D_t)(t+\alpha)}\right\}.$$

  for all $n \geq N_4$.
- Let $N_5 \in \mathbb{N}$ to get

$$\int_E |G(t_n - \tau_n, X(\tau_n)) - G(t - \tau, X(\tau))| d\mathbb{P} < \frac{\epsilon}{8e^{c_0(t+\alpha)}},$$

  for all $n \geq N_4$.

So for $N = N_1 \vee N_2 \vee N_3 \vee N_4 \vee N_5$, we have that if $n \geq N$ then

$$|v_2(t_n, x_n) - v_2(t, x)| < \epsilon,$$

and we conclude that v_2 is continuous over $(0, \infty) \times D$.
Next we prove the continuity in the boundary. Let $(t_n, x_n) \xrightarrow[n \to \infty]{} (t, x)$,

where $(t_n, x_n) \in (0,\infty) \times D$ and $(t,x) \in \partial((0,\infty) \times D)$, that is, either $t = 0$ or $x \in \partial D$. In both cases we get that $\tau = 0$ a.s.. We need to prove that

$$|v_2(t_n, x_n) - G(t,x)| \xrightarrow[n\to\infty]{} 0.$$

Let $0 < \alpha \ll 1$ and $N_1 \in \mathbb{N}$ such that

$$\|(t_n, x_n) - (t,x)\| < \alpha.$$

So, for all $n \geq N_1$,

$$\tau_n \leq t_n < t + \alpha.$$

We have that

$$\begin{aligned} |v_2(t_n, x_n) - G(t,x)| &\leq \mathbb{E}\Big[e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} \\ &\qquad \times |G(t_n - \tau_n, X_n(\tau_n)) - G(t,x)|\Big] \qquad (2.40) \\ &+ \mathbb{E}\Big[|G(t,x)| \Big|e^{\int_0^{\tau_n} c(\xi_n(r), X(r))dr} - 1\Big|\Big]. \qquad (2.41) \end{aligned}$$

Because $\left|e^{\int_0^{\tau_n} c(\xi_n, X_n)} - 1\right| \leq e^{c_0(t+\alpha)} + 1$ and $\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr \xrightarrow[n\to\infty]{a.s.} 0$ (due to Theorem A.1.2, that $\tau = 0$ a.s. and the continuity of c in $[0,\infty) \times \overline{D}$), by the Dominated Convergence Theorem we get for (2.41)

$$\mathbb{E}\Big[|G(t,x)| \Big|e^{\int_0^{\tau_n} c(\xi_n(r), X(r))dr} - 1\Big|\Big] \xrightarrow[n\to\infty]{} 0.$$

Next we work with (2.40). As in equation (2.34) we can prove that the sequence

$$\left\{e^{\int_0^{\tau_n} c(\xi_n(r), X_n(r))dr} |G(t_n - \tau_n, X_n(\tau_n)) - G(t,x)|\right\}_{n \geq N_1}$$

is uniformly integrable. We have that

$$\begin{aligned} (2.40) \;\leq\; & e^{c_0(t+\alpha)} \mathbb{E}\left[|G(t_n - \tau_n, X_n(\tau_n)) - G(t_n - \tau_n, X(\tau_n))|\right] \quad (2.42) \\ & + e^{c_0(t+\alpha)} \mathbb{E}\left[|G(t_n - \tau_n, X(\tau_n)) - G(t,x))|\right]. \quad (2.43) \end{aligned}$$

We repeat the same arguments made for the estimates to equations (2.38) and (2.39) with equation (2.42) and (2.43) respectively. Then we can prove that

$$\mathbb{E}\left[|G(t_n - \tau_n, X_n(\tau_n)) - G(t_n - \tau_n, X(\tau_n))|\right] \xrightarrow[n\to\infty]{} 0,$$

and

$$\mathbb{E}\left[|G(t_n - \tau_n, X(\tau_n)) - G(t,x))|\right] \xrightarrow[n\to\infty]{} 0.$$

So $v_2 \in C([0,\infty) \times \overline{D})$ and the proof is complete. □

2.3.2 Differentiability of v.

Let $0 \leq T_0 < T_1$ and $A \subset D$ be a bounded, open, connected set with C^2 boundary. Consider the following parabolic differential equation

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) = -f(t,x) \quad (t,x) \in (T_0,T_1] \times A, \\ u(T_0,x) = v(T_0,x) \quad \text{for } x \in A, \\ u(t,x) = v(t,x) \quad \text{for } (t,x) \in (T_0,T_1] \times \partial A. \end{aligned} \tag{2.44}$$

where the boundary data is v. If we assume **H0**, **H1** and **H2**, by the continuity of v (Theorem 2.3.1) and Theorem A.3.2 we can guarantee the existence of a unique classical solution to equation (2.44). To prove the regularity of v, we show that it coincides with the solution to equation (2.44) in the set $(T_0,T_1) \times A$ and so $v \in C^{1,2}((T_0,T_1) \times A)$. Since T_0, T_1 and A are arbitrary, we get the desired regularity. We are ready to prove the next Theorem.

Theorem 2.3.2. *Assume **H0**, **H1** and **H2**. Let v be defined as in equation* (2.16). *Then $v \in C^{1,2,\lambda}_{loc}((0,\infty) \times \mathbb{R}^d)$.*

Proof. Let w be the solution to equation (2.44). Define the following stopping times

$$\begin{aligned} \theta_T :=& \inf\{s > 0 | \xi(s) < T_0\} \\ \theta_A :=& \inf\{s > 0 | X(s) \notin \overline{A}\}, \\ \theta :=& \theta_T \wedge \theta_A. \end{aligned}$$

Following the same arguments of Section 5 in Chapter 6 of [27], we can prove that w has the following representation

$$\begin{aligned} w(t,x) =& \mathbb{E}_x \left[\int_0^\theta e^{\int_0^s c(t-r,X(r))dr} f(t-s,X(s))ds \right] \\ &+ \mathbb{E}_x \left[e^{\int_0^\theta c(t-r,X(r))ds} v(t-\theta, X(\theta)) \right]. \end{aligned} \tag{2.45}$$

Next we prove that v satisfies the following equality

$$\begin{aligned} v(t,x) =& \mathbb{E}_{t,x} \left[\int_0^\theta e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds \right] \\ &+ \mathbb{E}_{t,x} \left[e^{\int_0^\theta c(\xi(r),X(r))ds} v(\xi(\theta), X(\theta)) \right]. \end{aligned} \tag{2.46}$$

Let v_1 and v_2 be defined as in equations (2.17) and (2.18). We will use the following representation of v_1 and v_2,

$$v_1(t,x) = \mathbf{E}_{t,x}\left[\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right]$$

and

$$v_2(t,x) = \mathbf{E}_{t,x}\left[e^{\int_0^{\tau} c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\right].$$

First we work with v_1

$$\begin{aligned} v_1(t,x) &= \mathbf{E}_{t,x}\left[\mathbf{E}\left[\int_0^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\,\middle|\,\mathcal{F}_\theta\right]\right] \\ &= \mathbf{E}_{t,x}\left[\mathbf{E}\left[\int_0^{\theta} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\,\middle|\,\mathcal{F}_\theta\right]\right] \quad (2.47) \\ &\quad + \mathbf{E}_{t,x}\left[\mathbf{E}\left[\int_\theta^{\tau} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\,\middle|\,\mathcal{F}_\theta\right]\right] \quad (2.48) \end{aligned}$$

We study the addends of the righthand side separately

$$(2.47) = \mathbf{E}_{t,x}\left[\int_0^{\theta} e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))ds\right].$$

For (2.48) we make a couple of changes of variable to get

$$\begin{aligned} (2.48) =& \mathbf{E}_{t,x}\left[\mathbf{E}\left[\int_0^{\tau-\theta} e^{\int_0^{s+\theta} c(\xi(r),X(r))dr} f(\xi(s+\theta),X(s+\theta))ds\,\middle|\,\mathcal{F}_\theta\right]\right] \\ =& \mathbf{E}_{t,x}\left[e^{\int_0^{\theta} c(\xi(r),X(r))dr}\mathbf{E}\left[\int_0^{\tau-\theta} e^{\int_0^{s} c(\xi(r+\theta)),X(r+\theta))dr} f(\xi(s+\theta),X(s+\theta))ds\,\middle|\,\mathcal{F}_\theta\right]\right] \end{aligned}$$

Since $\theta < \tau_D$ (see Remark 2.1.1) and θ is bounded, we get that

$$\begin{aligned} \tau =& \inf\{s > 0 | (\xi(s), X(s)) \notin [0,\infty) \times \overline{D}\} \\ =& \theta + \inf\{s > 0 | (\xi(s+\theta), X(s+\theta)) \notin [0,\infty) \times \overline{D}\}, \end{aligned}$$

so

$$\tau - \theta = \Theta_\theta \circ \tau \qquad (2.49)$$

where $\Theta.$ denotes the shift operator. Since the process (ξ, X) is a homogeneous strong Markov process, we get that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\Theta_\theta\circ\tau} e^{\int_0^s c(\Theta_\theta\circ(\xi,X)(r))dr} f(\Theta_\theta\circ(\xi,X)(s))ds \middle| \mathcal{F}_\theta\right] \\
=&\mathbb{E}_{\xi(\theta),X(\theta)}\left[\int_0^\tau e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(r),X(s))ds\right] \\
=&v_1(\xi(\theta),X(\theta)).
\end{aligned} \tag{2.50}$$

So

$$\begin{aligned}
v_1(t,x) =&\mathbb{E}_{t,x}\left[\int_0^\theta e^{\int_0^s c(\xi(r),X(r))dr} f(\xi(s),X(s))\right] \\
&+\mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr} v_1(\xi(\theta),X(\theta))\right].
\end{aligned} \tag{2.51}$$

Next we study v_2. Again, for the integral we use a couple of changes of variables to get

$$\begin{aligned}
v_2(t,x) =&\mathbb{E}_{t,x}\left[\mathbb{E}\left[e^{\int_0^\tau c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\middle| \mathcal{F}_\theta\right]\right] \\
=&\mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr}\mathbb{E}\left[e^{\int_0^{\tau-\theta} c(\xi(r+\theta),X(r+\theta))dr} G(\xi(\tau),X(\tau))\middle| \mathcal{F}_\theta\right]\right].
\end{aligned}$$

We write

$$G(\xi(\tau),X(\tau)) = G(\xi(\tau-\theta+\theta),X(\tau-\theta+\theta)).$$

Then, the argument of the conditional expectation can be written as

$$e^{\int_0^{\Theta_\theta\circ\tau} c(\Theta_\theta\circ(\xi,X)(r))dr} G(\Theta_\theta\circ(\xi,X)(\Theta_\theta\circ\tau))$$

Using a similar argument as the one in equations (2.49) and (2.50) we get

$$\begin{aligned}
v_2(t,x) =&\mathbb{E}_{t,x}\Big[e^{\int_0^\theta c(\xi(r),X(r))dr} \\
&\times\mathbb{E}_{\xi(\theta),X(\theta)}\left[e^{\int_0^{\tau-\theta} c(\xi(r),X(r))dr} G(\xi(\tau),X(\tau))\right]\Big] \\
=&\mathbb{E}_{t,x}\left[e^{\int_0^\theta c(\xi(r),X(r))dr} v_2(\xi(\theta),X(\theta))\right].
\end{aligned} \tag{2.52}$$

Combining equations (2.51) and (2.52) we prove that (2.46) holds.
So due to equations (2.45) and (2.46) we have that $v = w$. Since $w \in C^{1,2,\lambda}((T_0,T_1)\times A)$ (see Theorem A.3.2 below) and T_0 T_1 and A are arbitrary we get that $v \in C^{1,2}((0,\infty)\times\mathbb{R}^d)\cap C^{1,2,\lambda}_{\text{loc}}((0,\infty)\times\mathbb{R}^d)$ and the proof is complete. □

We are ready to proof the Main Theorem

Proof of Theorem 2.2.1. The proof follows from Theorems 2.3.1 and 2.3.2 and Lemmas 2.2.1 and 2.2.2. □

Chapter 1

Semilinear parabolic differential equations.

In this chapter we consider the Cauchy problem for a semilinear parabolic equation that arises in some stochastic optimal control problems

$$\begin{aligned} -u_t(t,x) + \sum_{ij} a_{ij}(t,x)D_{ij}(t,x) + \sup_{\alpha\in\Lambda}\{\mathcal{L}_1^\alpha[u](t,x) + f(t,x,\alpha)\} =& 0, \quad \text{in } (0,\infty)\times\mathbb{R}^d \\ u(0,x) =& h(x), \qquad x\in\mathbb{R}^d. \end{aligned} \tag{1.1}$$

where $a = \sigma\sigma'$ and

$$\mathcal{L}_1^\alpha[u](t,x) := \sum_i b_i(t,x,\alpha)D_i u(t,x) + c(t,x,\alpha)u(t,x).$$

Additionally, we apply these results to an stochastic optimal consumption problem.

We recall that many papers consider semilinear equations of this form (see e.g. [?], [?], [?], [?], [?], [?] and [?]). In all these cases, the basis for the solution to the HJB equation is a result proved by Fleming (see [?] Theorem VI.6.2). It is assumed that the control set Λ is compact, $c \equiv 0$, the functions $b, \sigma \in C^{1,2}$ with σ, σ_x and b_x bounded. The data functions f and h are assumed to have a polynomial growth and C^2 regularity.

We study the existence and uniqueness of a solution to equation (1.1). The existence's theorem is based on a linearization technique and the results proved in Chapter ??. For the uniqueness part we prove that the solution to equation (1.1) has a probabilistic representation via a Verification Theorem.

1.1 Hypotheses and notation.

We assume the following hypotheses on the coefficients, we denote them by **H3**.

H3:

1. $\Lambda \subset \mathbb{R}^m$ is a compact set.

2. σ follows the same hypotheses made in **H1'**.

3. Let $h(x) : \mathbb{R}^d \to \mathbb{R}$ be a locally Hölder continuous function of order β such that for some $k > 0$ and $K_3 > 0$ we have

$$|h(x)| \le K_3(1 + \|x\|^k),$$

for all $x \in \mathbb{R}^d$.

4. Let

$$\begin{aligned} b :& [0,\infty) \times \mathbb{R}^d \times \Lambda \to \mathbb{R}^d \\ c :& [0,\infty) \times \mathbb{R}^d \times \Lambda \to \mathbb{R} \\ f :& [0,\infty) \times \mathbb{R}^d \times \Lambda \to \mathbb{R} \end{aligned}$$

be continuous functions such that

- (Continuity.) b, c and f have the following continuity. For all $T > 0$, $M \ge 1$, there exists $L_3(T, M)$ such that for all $t, s \in [0, T]$, $\|x\|, \|y\| \le M$ and $\alpha, \gamma \in \Lambda$,
  - (Locally Lipschitz.)

$$\|\psi(t, x, \alpha) - \psi(t, y, \gamma)\| \le L_3(T, M)(\|x - y\| + \|\alpha - \gamma\|).$$

  - (Locally Hölder.)

$$\|\psi(t, x, \alpha) - \psi(s, x, \alpha)\| \le L_3(T, M)|t - s|^\beta.$$

- (Growth.) There exists $c_0 \ge 0$ such that

$$c(t, x) \le c_0 \quad \text{for all } (t, x, \alpha) \in [0,\infty) \times \mathbb{R}^d \times \Lambda.$$

There exists k such that for $T > 0$, exists a constant $K_4(T)$

$$\begin{aligned}\|b(t,x,\alpha)\| \leq& K_4(T)(1+\|x\|)\\ |f(t,x,\alpha)| \leq& K_4(T)(1+\|x\|^k),\end{aligned}$$

for all $0 \leq t \leq T$, $\alpha \in \Lambda$ and $x \in \mathbb{R}^d$

5. For all $\psi \in HL^{0,1,\beta_0,\beta_1}((0,\infty) \times \mathbb{R}^d)$, $\beta_0, \beta_1 \in (0,1]$, let

$$A_\psi(t,x) := \operatorname{argmax}_{\alpha\in\Lambda} \left\{\mathcal{L}_1^\alpha[\psi](t,x) + f(t,x,\alpha)\right\}.$$

Then $A_\psi \in HL^{0,0,\beta_0,\beta_1}((0,\infty) \times \mathbb{R}^d)$.

1.1.1 Additional notation.

The space $C^{1,2,\beta}_{\text{loc}}((0,\infty) \times \mathbb{R}^d)$ is the space of all functions such that they and all their derivatives up to the second order in x and first order in t, are locally Hölder of order β.

We denote by $HL^{k,m,\beta_0,\beta_1}((0,\infty) \times D) \subset C^{k,m}((0,\infty) \times D)$, with $\beta_0, \beta_1 \in (0,1]$, the space of all continuous function such that all their derivatives up to order k in t and order m in x, are locally Hölder continuous of order β_0 in t and locally Hölder continuous of order β_1 in x. If $\beta_1 = 1$ we denote by $HL^{k,m,\beta_0}((0,\infty) \times D)$.

We use the following notation for the Sobolev and Hölder norms. Let $R \subset [0,\infty) \times \mathbb{R}^d$, $f : R \to \mathbb{R}$ be an arbitrary function, $\alpha \in (0,1]$ and

$1 < p < \infty$, then

$$\begin{aligned}
\|f\|_R &:= \sup_{(t,z)\in R} |f(t,z)|,\\
|f|_R^\alpha &:= \|f\|_R + \sup_{(t,z_1)\neq(t,z_2)\in R} \frac{|f(t,z_1)-f(t,z_2)|}{|z_1-z_2|^\alpha}\\
&\quad + \sup_{(t_1,z)\neq(t_2,z)\in R} \frac{|f(t,z_1)-f(t,z_2)|}{|t_1-t_2|^{\alpha/2}},\\
|f|_R^{1,\alpha} &:= |f|_R^\alpha + \sum_i |D_i f|_R^\alpha,\\
|f|_R^{2,\alpha} &:= |f|_R^{1,\alpha} + |f_t|_R^\alpha + \sum_{i,j} |D_{ij} f|_R^\alpha,\\
\|f\|_{p;R} &:= \left(\int_R |f|^p dz\right)^{\frac{1}{p}},\\
\|f\|_{2,p;R} &:= \|f\|_{p;R} + \|f_t\|_{p;R} + \sum_i \|D_i f\|_{p;R} + \sum_{i,j} \|D_{ij} f\|_{p;R}.
\end{aligned}$$

1.2 Main result.

We are ready to prove the main Theorem of this chapter. The proof is based in Theorem **??**, the ideas made in Appendix E for Theorem 6.1 of Chapter VI in [**?**] and some standard arguments for Verification Theorems.

Theorem 1.2.1. *Assume **H3**. Then there exists a unique classical solution* $u \in C([0,\infty)\times\mathbb{R}^d) \cap C_{loc}^{1,2,\lambda}((0,\infty)\times\mathbb{R}^d)$ *for some* $\lambda \in (0,1)$*, to equation* (1.1)*. The solution has the representation*

$$u(t,x) = \sup_{a\in\mathcal{A}} \{u(t,x;a)\},$$

where

$$\begin{aligned}
u(t,x;a) :=& \mathbb{E}_x\left[\int_0^t e^{\int_0^s c(t-r,X(r;a),a_r)dr} f(t-s,X(s;a),a_r)ds\right]\\
&+ \mathbb{E}_x\left[e^{\int_0^t c(t-r,X(r;a),a_r)dr} h(X(t;a))\right],
\end{aligned}$$

$\mathcal{A}$ *is the set of all predictable processes,* a_s*, defined in some probability space* $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_s\}_{s\geq 0})$*, such that*

$$\mathbb{P}\left[\textit{for all } s \geq 0; a_s \in \Lambda\right] = 1,$$

and $X(s;a)$ is the solution to the stochastic differential equation

$$dX(s;a) = b(t-s, X(s), a_s)ds + \sigma(t-s, X(s))dW(s), \quad X(0) = x. \quad (1.2)$$

Furthermore, for all $T > 0$,

$$\sup_{0\le t\le T} |u(t,x)| \le C(T, c_0, K_1, K_2, K_3, K_4)(1 + \|x\|^k), \qquad x \in \mathbb{R}^d,$$

where c_0, k, K_i, $i = 1, \dots, 4$ are the constants defined in ***H1***, ***H2*** *and* ***H3***.

The proof is divided in two Theorems: a Verification Theorem and an Existence one.

1.2.1 Verification Theorem.

Theorem 1.2.2 (Verification Theorem.). *Assume* ***H3***. *Assume also there exists $v \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2,\lambda}_{loc}((0,\infty) \times \mathbb{R}^d)$ for some $\lambda \in (0,1)$, solution to equation* (1.1) *such that for all $T > 0$*

$$\sup_{0\le t\le T} |v(t,x)| \le C(T)(1 + \|x\|^\mu), \qquad x \in \mathbb{R}^d, \quad (1.3)$$

for some $\mu > 0$. Let $a \in \mathcal{A}$, then

$$u(t,x;a) \le v(t,x) = u(t,x;a^*)$$

where $a^*_s := A_v(s, X(s)) \in \mathcal{A}$. In particular, the solution is unique and

$$v(t,x) = \sup_{a\in\mathcal{A}} \{u(t,x;a)\}.$$

Proof. Let $\mathcal{L}_2$ and $\mathcal{L}_1^\alpha$ be the differential operators

$$\mathcal{L}_2[u] := -u_t + \sum_{ij} a_{ij} D_{ij} u, \quad (1.4)$$

and

$$\mathcal{L}_1^\alpha[u] := \sum_i b_i(\cdot,\alpha) + c(\cdot,\alpha)u \quad (1.5)$$

and denote

$$f^\alpha = f(\cdot,\alpha). \quad (1.6)$$

Let $a \in \mathcal{A}$ be any admissible process and denote by $X(s) = X(s;a)$ the solution to equation (1.2). Since $v \in C^{1,2}$, applying Ito's rule we get for $s \leq t$

$$
\begin{aligned}
e^{\int_0^s c(t-r,X(r),a_r)dr} v(t-s,X(s)) = v(t,x) \\
&+ \int_0^s e^{\int_0^r c(t-y,X(y),a_y)dy} \left(\mathcal{L}_2 + \mathcal{L}_1^{a_s}\right)[v](t-r,X(r))dr \\
&+ \int_0^s e^{\int_0^r c(t-y,X(y),a_y)dy} Dv(t-r,X(r))\sigma(t-r,X(r))dW(r),
\end{aligned}
$$

Since $e^{\int_0^r c dy} Dv(\cdot)\sigma(\cdot)$ is locally bounded, we conclude that

$$
\left\{ e^{\int_0^s c(t-r,X(r),a_r)dr} v(t-s,X(s)) - \int_0^s e^{\int_0^r c(t-y,X(y),a_y)dy} \left(\mathcal{L}_2 + \mathcal{L}_1^{a_s}\right)[v](t-r,X(r))dr \right\}_{0\leq s\leq t}
$$

is a local martingale. Let $\{\tau_n\}_{n\geq 1}$ be a sequence of localization times for the local martingale. Hence, using equation (1.1)

$$
\begin{aligned}
v(t,x) =& \mathbb{E}_x\left[e^{\int_0^{s\wedge\tau_n} c(t-r,X(r),a_r)dr} v((t-\cdot,X)(s\wedge\tau_n)) \right] \\
&- \mathbb{E}_x\left[\int_0^{s\wedge\tau_n} e^{\int_0^r c(t-y,X(y),a_y)dy} \left(\mathcal{L}_2 + \mathcal{L}_1^{a_s}\right)[v](t-r,X(r))dr \right] \\
\geq& \mathbb{E}_x\left[e^{\int_0^{s\wedge\tau_n} c(t-r,X(r),a_r)dr} v((t-\cdot,X)(s\wedge\tau_n)) \right] \\
&+ \mathbb{E}_x\left[\int_0^{s\wedge\tau_n} e^{\int_0^r c(t-y,X(y),a_y)dy} f(t-r,X(r),a_r)dr \right].
\end{aligned}
$$

For all $n \in \mathbb{N}$ and $s \leq t$, using (1.3) we get

$$
|e^{\int_0^{s\wedge\tau_n} c(t-r,X(r),a_r)dr} v((t-\cdot,X)(s\wedge\tau_n))| \leq e^{c_0 t} C(t)(1 + \sup_{r\in[0,t]} \{\|X(r)\|^\mu\}),
$$

and

$$
\begin{aligned}
\int_0^{s\wedge\tau_n} & e^{\int_0^r c(t-y,X(y),a_y)dy} f(t-r,X(r),a_r)dr \\
&\leq \int_0^{s\wedge\tau_n} e^{rc_0} K_4(t)(1+\|X(s)\|)^k ds \\
&\leq t e^{tc_0} K_4(t)(1 + \sup_{r\in[0,t]} \{\|X(r)\|^k\}).
\end{aligned}
$$

Hence by the Dominated Convergence Theorem letting $n \to \infty$ and $s \uparrow t$

$$\begin{aligned} v(t,x) \geq & \mathbb{E}_x \left[\int_0^t e^{\int_0^s c(t-r,X(r),a_r)dr} f(t-s,X(s),a_r)ds \right] \\ & + \mathbb{E}_x \left[e^{\int_0^t c(t-r,X(r),a_r)dr} h(X(t)) \right]. \end{aligned}$$

And so $u(t,x;a) \leq v(t,x)$.

Since $v \in C^{1,2,\lambda}_{\text{loc}}((0,\infty) \times \mathbb{R}^d)$, it follows from **H3** that A_v is locally Hölder in t and locally Lipschitz in x and so equation (1.2) admits a strong solution with $a^*_s := A_v(s, X(s))$. This implies that the strategy $a^*_s \in \mathcal{A}$. Repeating the same arguments made with the arbitrary process we get that for a^*_s

$$u(t,x;a^*) = v(t,x)$$

and the proof is complete. □

1.2.2 Existence Theorem.

Theorem 1.2.3 (Existence Theorem.)**.** *Assume **H3**. Then there exists a unique classical solution* $u \in C([0,\infty) \times \mathbb{R}^d) \cap C^{1,2,\lambda}_{loc}((0,\infty) \times \mathbb{R}^d)$ *for some* $\lambda \in (0,1)$*, to equation* (1.1)*. Furthermore, for all* $T > 0$,

$$\sup_{0 \leq t \leq T} |u(t,x)| \leq C(T, c_0, K_1, K_2, K_3, K_4)(1 + \|x\|^k), \qquad x \in \mathbb{R}^d, \tag{1.7}$$

where c_0*,* k*,* K_i*,* $i = 1, \ldots, 4$ *are the constants defined in **H1**, **H2** and **H3**.*

Proof. The proof is divided in three main steps: First we construct a candidate solution to equation (1.1) by approximation with linear parabolic equations. Second we prove that this function is a weak solution to equation (1.1) and finally we prove that it is a classical solution.

Let $\mathcal{L}_2$, $\mathcal{L}_1^\alpha$ and f^α be defined as in equations (1.4), (1.5) and (1.6), respectively.

Let $\alpha_0 \in \Lambda$ and $u^{(0)}$ be the solution to

$$\begin{aligned} \mathcal{L}_2[u^{(0)}] + \mathcal{L}_1^{\alpha_0}[u^{(0)}] + f^{\alpha_0} =& 0, \quad (0,\infty) \times \mathbb{R}^d \\ u^{(0)}(0,x) =& h(x), \quad x \in \mathbb{R}^d. \end{aligned}$$

For $n \geq 1$, let

$$A^{(n-1)} := \text{argmax}_{\alpha \in \Lambda} \left\{ \mathcal{L}_1^\alpha[u^{(n-1)}] + f^\alpha \right\},$$

and $u^{(n)}$ be the solution to

$$\begin{aligned} \mathcal{L}_2[u^{(n)}] + \mathcal{L}_1^{A^{(n-1)}}[u^{(n)}] + f^{A^{(n-1)}} =&0, \quad (0,\infty)\times\mathbb{R}^d \\ u^{(n)}(0,x) =&h(x), \quad x\in\mathbb{R}^d. \end{aligned} \tag{1.8}$$

If $u^{(n-1)} \in C^{1,2,\beta}$ then $Du^{(n-1)} \in C^{0,1,\beta}$ and so $u^{(n-1)} \in HL^{0,1,\beta}$. Hence by hypothesis, $A^{(n-1)} \in HL^{0,0,\beta}$. This and **H3** implies that the coefficients of equation (1.8) satisfies the hypotheses of Theorem **??** and so the sequence $\{u^{(n)}\}$ is well defined and each $u^{(n)} \in C^{1,2,\beta}$.

Next, we prove that $u^{(n)} \le u^{(n+1)}$ for all $n \in \mathbb{N}$. Because

$$A^{(n)} \in \operatorname{argmax}\left\{\mathcal{L}_1^{\alpha}[u^{(n)}] + f^{\alpha}\right\}$$

we have that

$$\begin{aligned} 0 =&\mathcal{L}_2[u^{(n)}] + \mathcal{L}_1^{A^{(n-1)}}[u^{(n)}] + f^{A^{(n-1)}} \\ \le&\mathcal{L}_2[u^{(n)}] + \mathcal{L}_1^{A^{(n)}}[u^{(n)}] + f^{A^{(n)}}. \end{aligned}$$

Subtracting this to equation (1.8) for $n+1$ we get

$$0 \le \mathcal{L}_2[u^{(n+1)} - u^{(n)}] + \mathcal{L}_1^{A^{(n)}}[u^{(n+1)} - u^{(n)}] \tag{1.9}$$

in $(0,\infty)\times\mathbb{R}^d$. Thanks to the Maximum Principle Theorem 1.2 in [**?**] we prove that

$$u^{(n)} \le u^{(n+1)}.$$

Since K_4 does not depend on α, then using equation (**??**) in Theorem **??** we get that for all $n \in \mathbb{N}$

$$\sup_{0\le t\le T}\{|u^{(n)}(t,x)|\} \le C(T)(1+\|x\|^k), \qquad x\in\mathbb{R}^d \tag{1.10}$$

where the constant $C(T)$ is independent of n. Then the sequence $\{u^{(n)}(t,x)\}$ is bounded from above and increasing. For each $(t,x)\in[0,\infty)\times\mathbb{R}^d$, let

$$u^*(t,x) := \lim_{n\to\infty} u^{(n)}(t,x),$$

and let R be a bounded, open, connected subset of $(0,\infty)\times\mathbb{R}^d$ with smooth boundary. Since all the coefficients of equation (1.8) are locally bounded, with bound independent of n, it follows from Theorem 7.22 in [**?**] that for all $p > 1$

$$\|u^{(n)}\|_{2,p,R} \leq M_1.$$

where M_1 does not depend on n. Since the Sobolev space $W^{2,p}(R)$ is embedded in the Hölder space $C^{0,1,\lambda}(R)$, for some $0 < \lambda < 1$, there exists M_2 such that for all $n \in \mathbb{N}$

$$|u^{(n)}|_R^{1,\lambda} \leq M_2.$$

So we get that on R, $Du^{(n)}$ converges uniformly to Du^*, and $u_t^{(n)}$ and $D^2u^{(n)}$ converge weakly in $L^p(R)$ to u_t^* and D^2u^* respectively. To argue this, we use the fact that the space $W^{2,p}(R)$ is compactly embedded in $W^{2,q}(R)$ for some $q > p$. So there exists a subsequence of $\{u^{(n)}\}$ convergent in $W^{2,q}(R)$ to some function v. Since $u^{(n)}$ converges pointwisely to u^*, then $u^* = v$.
For $(t, x) \in R$, let

$$A^*(t,x) := \text{argmax}_{\alpha\in\Lambda}\{\mathcal{L}_2^\alpha[u^*](t,x) + f^\alpha(t,x)\}.$$

Since u^* and Du^* are Hölder continuous then by hypotheses, $A^* \in C^{0,0,\lambda}(R)$. We have the following inequalities

$$\begin{aligned} \mathcal{L}_2[u^*] + \mathcal{L}_1^{A^*}[u^*] + f^{A^*} \geq & \mathcal{L}_2[u^*] + \mathcal{L}_1^{A^{(n)}}[u^*] + f^{A^{(n)}} \\ = & \mathcal{L}_2[u^* - u^{(n)}] + \mathcal{L}_1^{A^{(n)}}[u^* - u^{(n)}]. \end{aligned} \tag{1.11}$$

The righthand side converges weakly to 0, this implies that that u^* satisfies weakly

$$\mathcal{L}_2[u^*] + \mathcal{L}_1^{A^*}[u^*] + f^{A^*} \geq 0. \tag{1.12}$$

On the other side,

$$\begin{aligned} \mathcal{L}_2[u^{(n)}] + \mathcal{L}_1^{A^*}[u^{(n)}] + f^{A^*} \leq & \mathcal{L}_2[u^{(n)}] + \mathcal{L}_1^{A^{(n)}}[u^{(n)}] + f^{A^{(n)}} \\ = & \mathcal{L}_2[u^{(n)} - u^{(n+1)}] + \mathcal{L}_1^{A^{(n)}}[u^{(n)} - u^{(n+1)}]. \end{aligned}$$

Again the righthand side converges weakly to 0 in R and so we prove that

$$\mathcal{L}_2[u^*] + \mathcal{L}_1^{A^*}[u^*] + f^{A^*} \leq 0. \tag{1.13}$$

Combining equations (1.12) and (1.13) we prove that u^* satisfies weakly in R

$$\mathcal{L}_2[u^*] + \mathcal{L}_1^{A^*}[u^*] + f^{A^*} = 0$$

It follows from Theorem 4.9 in [?] and the Hölder continuity of $\mathcal{L}_1^{A^*}[u^*] + f^{A^*}$ that

$$|u^*|_R^{2,\lambda} \leq C\left(\|u^*\|_R + \left|\mathcal{L}_1^{A^*}[u^*] + f^{A^*}\right|_R^{\lambda}\right),$$

where C depends on the Hölder norm of the coefficients of $\mathcal{L}_2$ and the local ellipticity in R. So $u^* \in C^{1,2,\lambda}(R)$. Since R is arbitrary, then $u^* \in C^{1,2,\lambda}_{\text{loc}}((0,\infty) \times \mathbb{R}^d)$ and satisfies (1.1). The boundary condition is fulfilled since $u^{(n)}(0,x) = h(x)$ and h is locally Hölder.

Finally, equation (1.10) proves that (1.7) is true and the proof is complete. □

1.3 Optimal consumption model.

In this section we present an stochastic optimal consumption model. We consider the wealth of an individual which is dynamically allocated in two investment instruments: a non-risk bonus and a risky asset, both depending on a external factor. The investment strategy is fixed. Our problem is to maximize a logarithmic utility function over all admissible consumption strategies. This kind of problems have been studied in [?], [?] and [?] where they consider a HARA utility function and the optimization is made over the investment and consumption strategies. Our approach is via a HJB equation. This problem shows situations in which the hypotheses of the present chapter appear naturally.

1.3.1 The model.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s\geq 0})$ be a complete filtered probability space and let $\{W_1(s), W_2(s)\}_{s\geq 0}$ be a two dimensional Brownian motion defined in it. We consider an incomplete market with an external factor

$$Y(s) = y + \int_0^s g(Y(r))dr + \beta(\rho W_1(s) + \epsilon W_2(s)),$$

with $0 \leq \rho \leq 1$, $\epsilon = \sqrt{1-\rho^2}$, and investment instruments

$$\begin{aligned} dZ_0(s) =& Z_0(s) r(Y(s))ds, \\ dZ(s) =& Z(s)\mu(Y(s))ds + Z(s)\nu(Y(s))dW_1(s), \end{aligned}$$

We assume that $g, \mu, \nu, r : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz with μ, ν and r bounded, ν strictly positive and g with at most linear growth.
Let $\mathcal{A}$ denote the set of all admissible investment strategies. We select them over all predictable process A_s with respect to $\mathcal{F}_s$, such that

$$\mathbb{P}\left[\text{for all } s \geq 0; A_s \in [0,1]\right] = 1$$

that is, A_s denotes the proportion of the wealth consumed at time s.
Let $m : \mathbb{R}^2 \to \mathbb{R}$ be a bounded, strictly positive, locally Lipschitz function that represents the proportion of the wealth invested in the risky asset.
For $A \in \mathcal{A}$, the wealth process $X(s)$ has the following dynamic

$$\begin{aligned} dX(s) =& - A_s X(s)ds + X(s)(1 - m(X(s), Y(s)))\frac{dZ_0(s)}{Z_0(s)} \\ &+ X(s)m(X(s), Y(s))\frac{dZ(s)}{Z(s)} \\ =& X(s)[-A_s + r(Y(s))(1 - m(X(s), Y(s))) + \mu(Y(s))m(X(s), Y(s))]ds \\ &+ X(s)m(X(s), Y(s))\nu(Y(s))dW_1(s), \end{aligned} \tag{1.14}$$

with $X(0) = x > 0$. This process is strictly positive and has the following representation

$$X(t) = x \exp\left\{\int_0^t \left(r(1-m) + \mu m - A_s - \frac{1}{2}m^2\nu^2\right) ds + \int_0^t m\nu dW_1(s)\right\}.$$

1.3.2 The value function and the HJB equation.

The objective is to maximize the expected consumption utility

$$V(t, x, y; A) := \mathbb{E}_{x,y}\left[\int_0^t \ln(A_s X(s) + 1)ds\right] \tag{1.15}$$

in a finite horizon, over the set of admissible strategies. Let V be the value function

$$V(t, x, y) := \sup_{A \in \mathcal{A}}\{V(t, x, y; A)\}.$$

To study the regularity of V and the existence of an optimal consumption strategy we consider the following HJB equation

$$\begin{aligned}
-u_t &+ \frac{1}{2}x^2 m(x,y)^2\nu(y)^2 u_{xx} + \beta\rho x m(x,y)\nu(y)u_{xy} + \frac{1}{2}\beta^2 u_{yy} \\
&+ x[r(y)(1-m(x,y)) + \mu(y)m(x,y)]u_x + g(y)u_y \\
&+ \sup_{\alpha\in[0,1]}\{-\alpha x u_x + \ln(\alpha x + 1)\} = 0 \qquad (t,x,y)\in(0,\infty)\times(0,\infty)\times\mathbb{R}, \\
&u(0,x,y) = 0 \quad (x,y)\in(0,\infty)\times\mathbb{R}^d.
\end{aligned} \tag{1.16}$$

For simplicity, we omit the (t,x,y) variables in the functions' notation. For any $(x,z)\in(0,\infty)\times\mathbb{R}^d$, the function $-xz\alpha+\ln(x\alpha+1)$ is strictly concave in α and has a unique maximum. Then for $\alpha\in[0,1]$ the supremum is attained at

$$\psi(x,z) := \begin{cases} 0, & \text{if } \frac{1-z}{xz}\le 0, \\ \frac{1-z}{xz}, & \text{if } 0<\frac{1-z}{xz}<1, \\ 1, & \text{if } 1\le\frac{1-z}{xz}. \end{cases} \tag{1.17}$$

Equation (1.16) is written as

$$\begin{aligned}
-u_t &+ \frac{1}{2}x^2 m^2\nu^2 u_{xx} + \beta\rho x m\nu u_{xy} + \frac{1}{2}\beta^2 u_{yy} \\
&+x[-\psi(x,u_x) + r(1-m) + \mu m]u_x + g u_y \\
&+ \ln(\psi(x,u_x)x+1) = 0 \quad (t,x,y)\in(0,\infty)\times(0,\infty)\times\mathbb{R}, \\
&u(0,x,y) = 0 \quad (x,y)\in(0,\infty)\times\mathbb{R}.
\end{aligned} \tag{1.18}$$

The coefficients of this equation do not fulfil the ellipticity condition at $x=0$. However, thanks to the kind of degeneracy, we will be able to prove the existence of a classical solution to equation (1.18).

1.3.3 Verification Theorem.

In this section we propose and prove a Verification Theorem. This Theorem asserts that, in case of existing a classical solution to (1.18), it has to be the value function and hence is unique. Also, this Theorem proves the existence of an optimal consumption strategy.

Theorem 1.3.1 (Verification Theorem.). *Let $g, \mu, \nu, r : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz functions such that, μ, ν and r are bounded, ν is strictly positive and g has a linear growth. Let $m : \mathbb{R}^2 \to \mathbb{R}$ be a bounded, strictly positive and locally Lipschitz function. Assume also there exists $v \in C([0,\infty) \times [0,\infty) \times \mathbb{R}) \cap C^{1,2,\beta}_{loc}((0,\infty) \times (0,\infty) \times \mathbb{R}^d)$ for some $\beta \in (0,1)$, solution to equation* (1.18) *such that for all $T > 0$*

$$\sup_{0 \le t \le T} |v(t,x,y)| \le C(T)(1 + \|(x,y)\|), \quad (x,y) \in [0,\infty) \times \mathbb{R}^d. \tag{1.19}$$

Let $A \in \mathcal{A}$ be any admissible strategy, then

$$V(t,x,y;A) \le v(t,x,y) = V(t,x,y;A^*),$$

where $A^*_s = \psi(X(s), v_x(t-s, X(s), Y(s)))$. In particular, the solution is unique and $v(t,x,y) = V(t,x,y)$.

Proof. The proof is similar as the one of Theorem 1.2.2 observing that the process $X(s;A)$ is strictly positive and hence we can repeat our analysis restricted to the set $(0,\infty) \times (0,\infty) \times \mathbb{R}^d$. □

1.3.4 Existence of a classical solution.

In this section we prove the existence of a solution to equation (1.18) with the properties required by the Verification Theorem.

Theorem 1.3.2. *Assume the hypotheses on the functions r, μ, ν, g and m made in the Verification Theorem. Then, there exists a unique solution $u \in C([0,\infty) \times [0,\infty) \times \mathbb{R}^d) \cap C^{1,2,\beta}_{loc}((0,\infty) \times (0,\infty) \times \mathbb{R}^d)$ for some $\beta \in (0,1)$, to equation* (1.18) *such that for all $T > 0$*

$$\sup_{0 \le t \le T} |u(t,x,y)| \le C(T)(1 + \|(x,y)\|), \quad (x,y) \in [0,\infty) \times \mathbb{R}^d. \tag{1.20}$$

Proof. Let $\mathcal{D}_2$ and $\mathcal{D}_1^\gamma$ be the differential operators

$$\mathcal{D}_2[u] := -u_t + \frac{1}{2}x^2 m^2 \nu^2 u_{xx} + \beta \rho x m \nu u_{xy} + \frac{1}{2}\beta^2 u_{yy}$$

and

$$\mathcal{D}_1^\gamma[u] := x[r(1-m) + \mu m - \gamma]u_x + g u_y.$$

We cannot apply Theorem 1.2.3 because of the degeneracy of the differential operator $\mathcal{D}_2$ at $x = 0$, however we can proceed as in its proof.
Equation (1.18) can be written as

$$\begin{aligned} \mathcal{D}_2[u] + \mathcal{D}_1^{\psi(x,u_x)}[u] + \ln(\psi(x,u_x)x+1)\} =& 0 \quad \text{in } (0,\infty)\times(0,\infty)\times\mathbb{R} \\ u(0,x,y) =& 0 \quad \text{in } (0,\infty)\times\mathbb{R}. \end{aligned}$$

Let $u^{(0)}$ be a solution to equation

$$\begin{aligned} \mathcal{D}_2[u^{(0)}] + \mathcal{D}_1^{1/2}[u^{(0)}] + \ln\left(\frac{1}{2}x+1\right) =& 0 \quad \text{in } (0,\infty)\times(0,\infty)\times\mathbb{R} \\ u^{(0)}(0,x,y) =& 0 \quad \text{in } (0,\infty)\times\mathbb{R}. \end{aligned}$$

For $n = 1, 2, \ldots$ let $u^{(n)}$ be a solution to

$$\begin{aligned} \mathcal{D}_2[u^{(n)}] + \mathcal{D}_1^{\psi(x,u_x^{(n-1)})}[u^{(n)}] + \ln(\psi(x,u_x^{(n-1)})x+1) =& 0 \quad \text{in } (0,\infty)\times(0,\infty)\times\mathbb{R} \\ u^{(n)}(0,x,y) =& 0 \quad \text{in } (0,\infty)\times\mathbb{R}. \end{aligned} \tag{1.21}$$

If $u^{(n-1)} \in C^{1,2,\beta}_{\text{loc}}$ then $u_x^{(n-1)} \in C^{0,1,\beta}_{\text{loc}}$ and so is locally Hölder in t and locally Lipschitz in (x, y). Since the function

$$H(x) = \begin{cases} 0, & \text{if } x \le 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } 1 \le x, \end{cases}$$

is Lipschitz, then $\psi(x, u_x^{(n-1)}) = H\left(\frac{1-u_x^{(n-1)}}{xu_x^{(n-1)}}\right)$ is locally Hölder in t and locally Lipschitz in (x, y) whenever $u^{(n-1)}$ is a classical solution.
The differential operator $\mathcal{D}_2$ is degenerated for $x = 0$, so we cannot apply Theorem **??** exactly as it is. However the kind of degeneracy $(a_{11}, a_{12}, b_1 = 0)$ implies that the stochastic process $X(s)$ associated to these equations is strictly positive and hence never reaches the set $x = 0$ in a finite time. This allows us to repeat the same arguments made in this paper to prove that on the set $(0,\infty) \times (0,\infty) \times \mathbb{R}$ we have a classical solution to equation (1.21) for all $n \in \mathbb{N}$. On this set, since m and ν are strictly positives, the matrix

$$\begin{pmatrix} x^2m^2\nu^2 & 2\beta\rho x m\nu \\ 2\beta\rho x m\nu & \beta^2 \end{pmatrix}$$

satisfies a local ellipticity condition. Let

$$u^{(n)}(t,x,y) = \mathbb{E}_{x,y}\left[\int_0^t \ln(\psi(X_n(s), u_x^{(n-1)})X_n(s)+1)ds\right]$$

where X_n is the solution to

$$dX_n(s) = X_n(s)[-\psi(X_n(s), u_x^{(n-1)}) + r(1-m) + \mu m]ds + X_n(s)m\nu dW_1(s),$$

with $X_n(0) = x$, that is,

$$\begin{aligned} X_n(t) =& x\exp\left\{\int_0^t (r(1-m)+\mu m - \psi ds\right\} \\ & \times \exp\left\{\int_0^t m\nu dW_1(s) - \int_0^t \frac{1}{2}m^2\nu^2 ds\right\}. \end{aligned} \tag{1.22}$$

As in Lemma **??**, we can prove that $u^{(n)}$ is continuous and repeating the same argument made in the proof to Theorem **??** we can prove that $u^{(n)} \in C^{1,2,\beta}_{\text{loc}}$ on the set $(0,\infty)\times(0,\infty)\times\mathbb{R}$. Using equation (1.22), a martingale argument and the boundedness of r, μ, ν, m and ψ, we can prove that $u^{(n)}$ is continuous at $x=0$,

$$\begin{aligned} 0 \le u^{(n)}(t,x,y) \le & \int_0^t \mathbb{E}_{x,y}\left[\psi(X_n(s), u_x^{(n-1)})X_n(s)\right] ds \\ \le & \int_0^t \mathbb{E}_{x,y}\left[xe^{Cs}\exp\left\{\int_0^s m\nu dW_1(r) - \int_0^s \frac{1}{2}m^2\nu^2 dr\right\}\right] ds \\ =& x\int_0^t e^{Cs}\mathbb{E}_{x,y}\left[\exp\left\{\int_0^s m\nu dW_1(r) - \int_0^s \frac{1}{2}m^2\nu^2 dr\right\}\right] ds \\ =& x\int_0^t e^{Cs} ds \xrightarrow[x\to 0]{} 0. \end{aligned}$$

Hence, for all $n \in \mathbb{N}$, $u^{(n)}$ is a classical solution to equation (1.21).
We repeat the argument in the proof of Theorem 1.2.3 to prove that $u := \lim_{n\to\infty} u^{(n)}$ is a classical solution to equation (1.18) and the proof is complete. □

Risk process.

In this chapter we focus in the problem when an insurance company puts its reserve capital in some investment instruments: a non-risk bonus and a risky asset. We are interested in the analysis of the probability of survival when the investment instruments depend in an external factor. Our goal is to maximize the survival probability over all admissible investment strategies. This problem was solved by Hipp and Plum in [?] and [?] when the non-risk rate is constant and the risky asset is a Geometric Brownian motion. Schmidli in [?] solved the same problem with both investment and proportional reinsurance.

Other problems such as investment, reinsurance, payment of dividends, severity of the ruin and combinations of them have been studied by many authors, e.g. [?], [?], [?], [?], [?], [?], [?] (see [?] for a very nice survey of this theory).

In this chapter we propose a HJB equation for the optimal survival probability and prove a Verification Theorem.

0.1 The model.

The model has the following parts: First the classical Cramér-Lundberg process

$$R(t) = x + ct - S(t).$$

where $x \geq 0$ is the initial capital, $c > 0$ stands for the premium income rate and $S(t) = \sum_{n=1}^{N(t)} \xi_n$, where $\{N(t)\}_{t\geq 0}$ is a Poisson process with intensity λ and jump times $\{\eta_n\}_{n=1}^{\infty}$; $\{\xi_n\}_{n=1}^{\infty}$ are i.i.d. positive random variables independent of the Poisson process, corresponding to the incoming claims, with common distribution Q and mean $\mu < \infty$.
Also let $\{W_1(t), W_2(t)\}_{t\geq 0}$ be a two dimensional standard Brownian motion independent of the process $R(t)$.

We work on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the smallest augmented right continuous filtration such that the process $\{R(t), W_1(t), W_2(t)\}_{t\geq 0}$ is measurable.
The external factor has the following dynamic

$$Y(t) = y + \int_0^t g(Y(s))ds + \beta(\rho W_1(t) + \epsilon W_2(t)), \tag{1}$$

with $0 \leq \rho \leq 1$, $\epsilon = \sqrt{1-\rho^2}$.
For investment we have a non-risk bonus and a risky asset, both depending on the external factor

$$\begin{aligned} dZ_0(t) =& Z_0(t)r(Y(t))dt, \\ dZ(t) =& Z(t)\mu(Y(t))dt + Z(t)\sigma(Y(t))dW_1(t), \end{aligned} \tag{2}$$

Let $\mathcal{A}$ denote the set of all admissible investment strategies. We select them over all predictable process A_t with respect to $\mathcal{F}_t$, such that

$$\mathbb{P}\left[\int_0^t A_s^2 ds < \infty\right] = 1, \quad \text{for all } t > 0.$$

We consider two cases: $r \equiv 0$ and r “arbitrary”. We assume either one of the following Hypothesis and denote them by $\mathbf{H}_0$ and $\mathbf{H}_1$ respectively.

1. $\mathbf{H}_0$
   - $r \equiv 0$.
   - g is Lipschitz continuous with linear growth and satisfies
   $$\int_0^x \exp\left\{-\int_0^z g(u)du\right\} dz \xrightarrow[x\to\pm\infty]{} \infty.$$
   - $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ are bounded Lipschitz continuous and satisfies $0 < \sigma_0 \leq \sigma(\cdot) \leq \sigma_1$, $0 < \mu_0 \leq \mu(\cdot)$ for some constants $\mu_0, \sigma_0, \sigma_1$.
2. $\mathbf{H}_1$
   - g is Lipschitz continuous with linear growth and satisfies
   $$\int_0^x \exp\left\{-\int_0^z g(u)du\right\} dz \xrightarrow[x\to\pm\infty]{} \infty.$$

- $r, \mu, \sigma : \mathbb{R} \to \mathbb{R}$ are bounded Lipschitz continuous and satisfies $0 < \sigma_0 \leq \sigma(\cdot)$ and $0 < r_0 \leq r(\cdot) < \mu(\cdot)$ for some constants r_0, σ_0.

Remark 0.1.1. *The condition made on g makes the process $Y(t)$ recurrent, that is, for all $a, b \in \mathbb{R}$*

$$\mathbb{P}_a[Y(t) = b \ i.o.] = 1.$$

Since its diffusion coefficient never vanishes, then the process satisfies

$$\mathbb{P}\left[\sup_{t>0} Y(t) = \infty\right] = \mathbb{P}\left[\inf_{t<0} Y(t) = -\infty\right] = 1.$$

(See [?] Proposition 5.22). Processes like the mean-reverting Ornstein-Ullenbeck fulfil this condition.

The reserve process with investment strategy $\{A_t\}_{t\geq 0}$ is

$$\begin{aligned} X(t; A) =& R(t) + \int_0^t A_s \frac{dZ(s)}{Z(s)} + \int_0^t (X(s-) - A_s)\frac{dZ_0(s)}{Z_0(s)} \\ =& x + \int_0^t [c + b(Y(s))A_s + r(Y(s))X(s-)]ds \\ &+ \int_0^t \sigma(Y(s))A_s dW_1(s) - S(t), \end{aligned} \tag{3}$$

where $b(\cdot) := (\mu - r)(\cdot)$.

We define the ruin time as

$$\tau(A) := \inf\{t > 0 | X(t; A) < 0\}$$

and the respective probability of ruin and survival

$$\psi(x, y; A) := \mathbb{P}\left[\tau(A) < \infty | X(0; A) = x, Y(0) = y\right],$$

$$\delta(x, y; A) := \mathbb{P}\left[\tau(A) = \infty | X(0; A) = x, Y(0) = y\right].$$

Also let

$$\delta(x, y) := \sup_{A \in \mathcal{A}}\{\delta(x, y; A)\}.$$

Our goal is to analyze the survival probability under an optimal investment strategy A_t^* that maximize it over all admissible strategies, that is,

$$\delta(x, y; A^*) = \delta(x, y)$$

0.2 HJB equation.

In this section we present an argument to propose a HJB equation which solution is the optimal probability of survival. Let $\alpha \in \mathbb{R}$ and define the process (X, Y) as

$$\begin{aligned} dX(t) =& (c + b(Y(t))\alpha) + r(Y(t))X(t-))dt \\ &+ \sigma(Y(t))\alpha dW_1(t) - dS(t) \\ dY(t) =& g(Y(t))dt + \beta d\tilde{W}(t), \end{aligned}$$

where $\tilde{W}(t) := \rho W_1(t) + \epsilon W_2(t)$. Since this is a Markov process, for $0 < h \ll 1$,

$$\begin{aligned} \delta(x, y; \alpha) =& \mathbb{E}_{x,y} \left[\mathbb{I}_{\{\tau(\alpha)=\infty\}} \right] \\ =& \mathbb{E}_{x,y} \left[\mathbb{E} \left[\mathbb{I}_{\{\tau(\alpha)=\infty\}} | \mathcal{F}_h \right] \right] \\ =& \mathbb{E}_{x,y} \left[\mathbb{E}_{X(h),Y(h)} \left[\mathbb{I}_{\{\tau(\alpha)=\infty\}} \right] \right] \\ =& \mathbb{E}_{x,y} \left[\delta(X(h), Y(h); \alpha) \right]. \end{aligned} \tag{4}$$

For simplicity of the notation we drop the index α and the arguments of the functions. If we suppose that $\delta \in C^2$, the integrals with respect to the Brownian motion and the Compensated Poisson process are martingales and all the interchanges between limits and integrals can be made, we get by Itô's rule

$$\begin{aligned} \delta(X(h), Y(h)) =& \delta(x, y) + \int_0^h g\delta_y ds + \int_0^h (c + b\alpha + rX)\delta_x ds \\ &+ \frac{1}{2} \left[\int_0^h \beta^2\rho^2\delta_{yy} ds + \int_0^h \beta^2\epsilon^2\delta_{yy} ds + \int_0^h \sigma^2\alpha^2\delta_{xx} ds + \int_0^h 2\beta\rho\sigma\alpha\delta_{xy} ds \right] \\ &+ \int_0^h \sigma\alpha\delta_x dW_1(s) + \int_0^h \beta\rho\delta_y dW_1(s) + \int_0^h \beta\epsilon\delta_y dW_2(s) \\ &+ \sum_{\{n\geq 1, \eta_n \leq h\}} [\delta(X(\eta_n-) - \xi_n, Y(\eta_n)) - \delta(X(\eta_n-), Y(\eta_n))]. \end{aligned} \tag{5}$$

Now let M be the Poisson Random Measure over $(0, \infty) \times (0, \infty)$ associated with the Lévy Process $S(t)$, with intensity measure $\nu(\cdot) := \lambda Q(\cdot)$, and let

$\tilde{M} = M - \nu$ be the compensated Poisson Random Measure. Hence

$$\sum_{\{n\geq 1, \eta_n \leq h\}} [\delta(X(\eta_n -) - \xi_n, Y(\eta_n)) - \delta(X(\eta_n -), Y(\eta_n))]$$
$$= \iint_{[0,h]\times(0,\infty)} [\delta(X(s-) - z, Y(s)) - \delta(X(s-), Y(s))] M(ds, dz)$$
$$= \iint_{[0,h]\times(0,\infty)} [\delta(X(s-) - z, Y(s)) - \delta(X(s-), Y(s))] \tilde{M}(ds, dz)$$
$$+ \iint_{[0,h]\times(0,\infty)} [\delta(X(s-) - z, Y(s)) - \delta(X(s-), Y(s))] \lambda ds dQ(z).$$

Now taking expectations in (5), substituting in (4), dividing by h, letting $h \to 0$ and taking supremum over all $\alpha \in \mathbb{R}$ we get

$$\sup_{\alpha\in\mathbb{R}} \{\mathcal{L}^\alpha[f](x, y)\} = 0 \tag{6}$$

where

$$\begin{aligned}\mathcal{L}^\alpha[f](x, y) &:= g(y) f_y(x, y) + (c + b(y)\alpha + xr(y)) f_x(x, y)\\ &+ \frac{1}{2}\beta^2 f_{yy}(x, y) + \frac{1}{2}\sigma(y)^2\alpha^2 f_{xx}(x, y) + \beta\rho\sigma(y) f_{xy}(x, y)\\ &+ \lambda \int_0^\infty (f(x - z, y) - f(x, y)) dQ(z).\end{aligned} \tag{7}$$

If $f_{xx} < 0$, the supremum is attained at

$$\alpha^*(x, y) = -\frac{b(y) f_x(x, y) + \beta\rho\sigma(y) f_{xy}(x, y)}{\sigma^2(y) f_{xx}(x, y)}. \tag{8}$$

0.3 Verification theorem.

In this section we propose and prove a Verification Theorem. This theorem asserts that, in case of existing a solution to (6), this solution has to be the optimal probability of survival. To be optimal, this solution has to be strictly increasing and concave in x. Because of the oscillatory of the paths of the Brownian motion, if we have that $\alpha^*(0, y) \neq 0$ then immediate ruin will occur, and so the strategy won't be optimal (because no investment would be better). Hence the boundary condition must be that $\alpha^*(0, y) = 0$ for all

$y \in \mathbb{R}$. If we assume this, we have that between jumps the "derivative" of the process $X(t)$ in 0 is given by $dX(t) = cdt$ and so the process will not ruin by investment. In fact the following Lemma asserts this observation (see [**?**] Theorem 9.4.1).

Lemma 0.3.1. *Consider a stochastic differential equation in* $\mathbb{R}^n$

$$d\xi(t) = b(\xi(t))dt + \sigma(\xi(t))dW(t),$$

with strong solution. Denote $a := \sigma * \sigma^t$
Let $G \subset \mathbb{R}^n$ *be an open set. Also let* $\nu(x) = (\nu_1, \ldots, \nu_n)$ *denote the inward normal at* x *to* ∂G, *and* $\rho(x) := dist(x, \partial G)$.
If for all $x \in \partial G$

$$\sum_{i,j=1}^{n} a_{ij}\nu_i\nu_j = 0,$$

and

$$\sum_{i=1}^{n} b_i\nu_i + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 \rho}{\partial x_i \partial x_j} \geq 0,$$

then

$$\mathbb{P}_y\left[\xi(t) \in G \text{ for all } t > 0\right] = 1$$

if $y \in G$.

We present the main result of this chapter.

Theorem 0.3.1 (Verification theorem.)**.** *Assume* $\mathbf{H}_0$ *or* $\mathbf{H}_1$. *Assume also there exists a solution* $f(x,y)$ *of Equation* (6) *with maximizing function* α^* *locally Lipschitz continuous, with the following properties*

1. $f(x,y) = 0$ *for* $(x,y) \in (-\infty, 0) \times \mathbb{R}$.
2. $f \in C^2[(0,\infty) \times \mathbb{R}] \cap C[[0,\infty) \times \mathbb{R}]$.
3. $f_x > 0$ *and* $f_{xx} < 0$.
4. $\alpha^*(0, \cdot) = 0$

Then f *is bounded, furthermore* $f(\infty, y)$ *is constant and for any admissible strategy* A *it's satisfied*

$$\delta(x,y;A) \leq \frac{f(x,y)}{f(\infty,y)} \leq \delta(x,y;A^*),$$

where $A^*_t = \alpha^*(X(t-), Y(t))$. *And hence we get the equality for this strategy.*

To prove the Theorem we need the following lemmas (for the proofs see Section **??** in Appendix **??**).

Lemma 0.3.2. *Consider stochastic processes* $\gamma(t)$, $\mu(t)$ *and* $\sigma(t)$ *such that* $0 < \mu_0 \leq \mu(t)$, $0 < \sigma_0 \leq \sigma(t) \leq \sigma_1$ *for some constants* $\mu_0, \sigma_0, \sigma_1$. *Also assume that* $\mu(t)$ *and* $\sigma(t)$ *are continuous processes. For* $a, b, c > 0$ *define the process*

$$\pi(t) := \gamma(t) + a + b\int_0^t \mu(s)ds + c\int_0^t \sigma(s)dW(s)$$

If $\gamma(t) \geq 0$ *for some* $A \in \mathcal{F}$ *then*

$$\pi(t) \xrightarrow[t\to\infty]{} \infty,$$

over the set A.

Lemma 0.3.3. *Let* $\pi(t) = x + \alpha\int_0^t \mu(s)ds + \beta\int_0^t \sigma(s)dW(s)$, *with* $0 < \sigma \leq \sigma_1$ *and* $0 < \mu_0 \leq \mu$, *then*

$$\mathbb{P}\left[\textit{for some } t; \pi(t) < 0\right] \leq \exp\left\{-\frac{2\alpha\mu_0}{\beta^2\sigma_1^2}x\right\}.$$

We proceed to the proof of the Verification Theorem

Proof of the Verification Theorem. We prove the theorem in two cases. In the first one we assume $\mathbf{H}_0$, that is $r \equiv 0$. In the second case we assume $\mathbf{H}_1$ and so we have an "arbitrary" r. Both cases are divided in two main steps. In the first step we consider an arbitrary admissible strategy, and we proof that its probability of survival is a lower bound for the solution of the HJB equation. In the second step we work with the optimal strategy and we prove that its probability of survival is an upper bound for the solution of the HJB equation.

Case 1. Assume $\mathbf{H}_0$.

Step 1. Lower bound.

Let $A = \{A_t\}_{t\geq 0}$ be any admissible strategy. Let $(X, Y)(t)$ be the risk process defined in equations (3) and (1), with investment strategy A, $(X, Y)(0) = (x, y)$ and τ its ruin time. To prove the boundness of f we need to prove that over the set $\{\tau = \infty\}$, the process $X(t) \xrightarrow[t\to\infty]{} \infty$. Instead of this we consider a family $\{X^\epsilon\}$ of risk processes asymptotically close to the process

X, with the property that $X^\epsilon(t) \xrightarrow[t\to\infty]{} \infty$ over the set $\{\tau = \infty\}$. This idea was proposed by Hipp and Plum in [?] and [?]. Let A^ϵ be defined as

$$A_t^\epsilon := A_t + \epsilon,$$

and $X^\epsilon(0) = x + \epsilon$ and τ^ϵ its ruin time.
We analyze the process (X^ϵ, Y).

$$\begin{aligned} X^\epsilon(t) =& x + \epsilon + \int_0^t (c + \mu(Y(s))(A_s + \epsilon^2))ds \\ &+ \int_0^t \sigma(Y(s))(A_s + \epsilon^2)dW_1(s) - S(t) \\ =& X(t) + \epsilon + \epsilon^2 \int_0^t \mu(Y(s))ds + \epsilon^2 \int_0^t \sigma(Y(s))W_1(s). \end{aligned} \tag{9}$$

Hence the following contain its true

$$\{\tau^\epsilon < \tau\} \subset \left\{\text{for some } t, \epsilon + \epsilon^2 \int_0^t \mu(Y(s))ds + \epsilon^2 \int_0^t \sigma(Y(s))dW_1(s) < 0\right\}$$

By Lemma 0.3.3, we see that

$$\begin{aligned} &\mathbb{P}[\tau^\epsilon < \tau] \\ &\quad \le \mathbb{P}\left[\text{for some } t > 0; \epsilon + \epsilon^2 \int_0^t \mu(Y(s))ds + \epsilon^2 \int_0^t \sigma(Y(s))W_1(t) < 0\right] \\ &\quad \le \exp\left(-\frac{2\mu_0}{\sigma_1^2 \epsilon}\right). \end{aligned} \tag{10}$$

Also, we have that over the set $\{\tau = \infty\}$, the process $X(t) \ge 0$. Then, thanks to Equation (9) and Lemma 0.3.2 we have

$$X^\epsilon(t) \xrightarrow[t\to\infty]{} \infty \tag{11}$$

over $\{\tau = \infty\}$. So (X^ϵ, Y) has the asserted properties.
Following the idea made by Schmidli in [?] we propose the following stopping times. This stopping times are used to prove the boundness of f.

Let $a \in \mathbb{R}$, $n, m, M \in \mathbb{N}$. For the process (X^ϵ, Y) let

$$\begin{aligned}\alpha_n^\epsilon &:= \inf\{t > 0 | X_t^\epsilon \notin [0, n]\}\\ \gamma_M &:= \inf\{t > 0 | Y(t) \notin [-M, M]\}\\ \nu_m^a &:= \inf\{t > m | Y(t) = a\}\end{aligned}$$

It follows from the non-explosion in finite time of the process Y and from Remark 0.1.1 that

$$\gamma_M \xrightarrow[M\to\infty]{a.s.} \infty,$$

$\nu_m^a < \infty$ a.s. and

$$\nu_m^a \xrightarrow[m\to\infty]{a.s.} \infty.$$

We also have that

$$\alpha_n^\epsilon \xrightarrow[n\to\infty]{a.s.} \tau^\epsilon.$$

Because $f \in C^2$, applying Ito's rule we get

$$\begin{aligned}&f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a)) - \int_0^{t\wedge\alpha_n^\epsilon\wedge\gamma_M\wedge\nu_m^a} \mathcal{L}^{A_s^\epsilon}(X^\epsilon(s-), Y(s))ds =\\ &f(x+\epsilon, y) + \int_0^{t\wedge\alpha_n^\epsilon\wedge\gamma_M\wedge\nu_m^a} \sigma(Y(s))A_s^\epsilon f_x(X^\epsilon(s-), Y(s))dW_1(s)\\ &+ \int_0^{t\wedge\alpha_n^\epsilon\wedge\gamma_M\wedge\nu_m^a} \beta f_y(X^\epsilon(s-), Y(s))d\tilde{W}(s)\\ &+ \iint_{(0,t\wedge\alpha_n^\epsilon\wedge\gamma_M\wedge\nu_m^a)\times(0,\infty)} [f(X^\epsilon(s-) - z, Y(s)) - f(X(s-), Y(s))]\tilde{M}(ds, dz).\end{aligned}$$

where $\mathcal{L}^\alpha$ is defined as in equation (7). For $0 < s < t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a$ we have that $(X^\epsilon, Y)(s) \in [0, n] \times [-M, M]$. Over this set f, f_x and f_y are bounded and since A^ϵ is admissible we have that the integrals with respect to the Brownian motions and the compensated Poisson process are local martingales. Hence we get that

$$\left\{ f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a)) - \int_0^{t\wedge\alpha_n^\epsilon\wedge\gamma_M\wedge\nu_m^a} \mathcal{L}^{A_s^\epsilon}(X^\epsilon(s-), Y(s))ds \right\}_{t\geq 0}$$

is a local martingale.

Let $\{\pi_k\}_{k=1}^{\infty}$ be a sequence of localization times such that $\lim_{k\to\infty}\pi_k = \infty$ a.s. For $0 < s < t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a \wedge \pi_k$, we get that $\mathcal{L}^{A_s^\epsilon}(X^\epsilon(s-), Y(s)) \leq 0$. Then for all $t \geq 0$, $n, m, M, k \in \mathbb{N}$

$$\mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a \wedge \pi_k))\right] \leq f(x+\epsilon, y).$$

Over the set $(-\infty, n] \times [-M, M]$, f is bounded, so by the Dominated Convergence Theorem we have

$$\begin{aligned} f(x+\epsilon, y) &\geq \lim_{k\to\infty} \mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a \wedge \pi_k))\right] \\ &= \mathbb{E}\left[\lim_{k\to\infty} f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a \wedge \pi_k))\right] \\ &= \mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a))\right]. \end{aligned}$$

Because $f \geq 0$ we apply Fatou's Lemma for $n \to \infty$ to get

$$\begin{aligned} f(x+\epsilon, y) &\geq \liminf_{n\to\infty} \mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a))\right] \\ &\geq \mathbb{E}\left[\liminf_{n\to\infty} f((X^\epsilon, Y)(t \wedge \alpha_n^\epsilon \wedge \gamma_M \wedge \nu_m^a))\right] \\ &= \mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \tau^\epsilon \wedge \gamma_M \wedge \nu_m^a))\right]. \end{aligned}$$

Repeating the same argument for $M \to \infty$ we get

$$f(x+\epsilon, y) \geq \mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \tau^\epsilon \wedge \nu_m^a))\right].$$

Letting $t \to \infty$, since $\nu_m^a < \infty$ a.s., and multiplying by $\mathbb{I}_{\{\tau^\epsilon = \infty\}}$

$$\begin{aligned} f(x+\epsilon, y) &\geq \liminf_{t\to\infty} \mathbb{E}\left[f((X^\epsilon, Y)(t \wedge \tau^\epsilon \wedge \nu_m^a))\mathbb{I}_{\{\tau^\epsilon=\infty\}}\right] \\ &\geq \mathbb{E}\left[\liminf_{t\to\infty} f((X^\epsilon, Y)(t \wedge \tau^\epsilon \wedge \nu_m^a))\mathbb{I}_{\{\tau^\epsilon=\infty\}}\right] \\ &= \mathbb{E}\left[f((X^\epsilon, Y)(\tau^\epsilon \wedge \nu_m^a))\mathbb{I}_{\{\tau^\epsilon=\infty\}}\right] \\ &= \mathbb{E}\left[f(X^\epsilon(\nu_m^a), a)\mathbb{I}_{\{\tau^\epsilon=\infty\}}\right], \end{aligned}$$

since we are over the set $\{\tau^\epsilon = \infty\}$.

Since $0 \leq \mathbb{I}_{\{\tau=\infty\}} \leq 1$

$$f(x+\epsilon, y) \geq \mathbb{E}\left[f(X^\epsilon(\nu_m^a), a)\mathbb{I}_{\{\tau^\epsilon=\infty, \tau=\infty\}}\right]$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$, thanks to equation (11), we get

$$\begin{aligned} f(x+\epsilon, y) &\geq \liminf_{m\to\infty} \mathbb{E}\left[f(X^\epsilon(\nu_m^a), a)\mathbb{I}_{\{\tau^\epsilon=\infty,\tau=\infty\}}\right] \\ &\geq \mathbb{E}\left[\liminf_{m\to\infty} f(X^\epsilon(\nu_m^a), a)\mathbb{I}_{\{\tau^\epsilon=\infty,\tau=\infty\}}\right] \\ &= f(\infty, a)\mathbb{P}\left[\tau^\epsilon = \infty, \tau = \infty\right] \\ &\geq f(\infty, a)\left(\mathbb{P}\left[\tau = \infty\right] - \exp\left\{-\frac{2\mu_0}{\sigma_1 \epsilon}\right\}\right). \end{aligned}$$

The last inequality follows from equation (10) and

$$\begin{aligned} \mathbb{P}\left[\tau = \infty\right] &= \mathbb{P}\left[\tau = \infty, \tau^\epsilon = \infty\right] + \mathbb{P}\left[\tau = \infty, \tau^\epsilon < \tau\right] \\ &\leq \mathbb{P}\left[\tau = \infty, \tau^\epsilon = \infty\right] + \mathbb{P}\left[\tau^\epsilon < \tau\right]. \end{aligned}$$

Letting $\epsilon \downarrow 0$

$$f(x, y) \geq f(\infty, a)\mathbb{P}\left[\tau = \infty\right].$$

For the strategy $A \equiv 0$ we have that $\mathbb{P}\left[\tau(0) = \infty\right] > 0$, and since $f(x, y)$ is finite, we have that for all $a \in \mathbb{R}$, $f(\infty, a)$ is finite. So

$$0 < \mathbb{P}\left[\tau(0) = \infty\right] \leq \frac{f(x, y)}{f(\infty, a)}$$

implies that $\limsup_{a\to\pm\infty} f(\infty, a) < \infty$. So $f(\infty, a)$ is a bounded function. Now since f is increasing in x we have that $f(x, y) \leq f(\infty, y)$ and hence f is a bounded function. Finally we get that for any admissible strategy A and $a \in \mathbb{R}$

$$\delta(x, y; A) \leq \frac{f(x, y)}{f(\infty, a)}.$$

Step 2. Upper bound.
Let (X^*, Y) be the risk process with investment strategy $A^*(t)$ (this strategy is admissible since α^* is locally Lipschitz) and
$(X^*, Y)(0) = (x, y)$. For this process we define the following stopping times

$$\begin{aligned} \alpha_n^* &:= \inf\{t > 0 | X_t^* \notin [0, n]\} \\ \gamma_M &:= \inf\{t > 0 | Y(t) \notin [-M, M]\} \\ \nu_m^a &:= \inf\{t > m | Y(t) = a\} \end{aligned}$$

For $f(X^*, Y)$, by Ito's rule we have

$$f((X^*,Y)(t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a)) - \int_0^{t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a} \mathcal{L}^{A_s^*}(X^*(s-),Y(s))ds =$$

$$f(x,y) + \int_0^{t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a} \sigma(Y(s))A_s^* f_x(X^*(s-),Y(s))dW_1(s)$$

$$+ \int_0^{t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a} \beta f_y(X^*(s-),Y(s))d\tilde{W}(s)$$

$$+ \iint_{(0,t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a)\times(0,\infty)} [f(X^*(s-)-z,Y(s)) - f(X(s-),Y(s))]\tilde{M}(ds,dz).$$

For $0 < s < t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a$ we have that $(X^*,Y)(s) \in [0,n]\times[-M,M]$. Over this set f, f_x, f_y and A^* are bounded, hence the integrals with respect to the Brownian motions and the compensated Poisson process are martingales. Also we have that $\mathcal{L}^{A_s^*}(X^*(s-),Y(s)) = 0$. Hence we get that

$$\{f((X^*,Y)(t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a))\}_{t\geq 0}$$

is a martingale. Then for all $t \geq 0$, $n,m,M \in \mathbb{N}$

$$f(x,y) = \mathbb{E}\left[f((X^*,Y)(t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a))\right].$$

Now since f is bounded, by the Dominated Convergence Theorem we get

$$\begin{aligned} f(x,y) &= \lim_{n\to\infty} \mathbb{E}\left[f((X^*,Y)(t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a))\right] \\ &= \mathbb{E}\left[\lim_{n\to\infty} f((X^*,Y)(t\wedge\alpha_n^*\wedge\gamma_M\wedge\nu_m^a))\right] \\ &= \mathbb{E}\left[f((X^*,Y)(t\wedge\tau^*\wedge\gamma_M\wedge\nu_m^a))\right]. \end{aligned}$$

By the same argument, since $\gamma_M \xrightarrow[M\to\infty]{a.s.} \infty$, we get

$$f(x,y) = \mathbb{E}\left[f((X^*,Y)(t\wedge\tau^*\wedge\nu_m^a))\right].$$

Letting $t\to\infty$, since $\nu_m^a < \infty$ a.s.,

$$f(x,y) = \mathbb{E}\left[f((X^*,Y)(\tau^*\wedge\nu_m^a))\right].$$

Now

$$\begin{aligned} f(x,y) =& \mathbb{E}\left[f((X^*,Y)(\tau^*\wedge\nu_m^a))(\mathbb{I}_{\{\tau^*=\infty\}} + \mathbb{I}_{\{\tau^*<\infty\}})\right] \\ =& \mathbb{E}\left[f((X^*,Y)(\nu_m^a))\mathbb{I}_{\{\tau^*=\infty\}}\right] + \mathbb{E}\left[f((X^*,Y)(\tau^*\wedge\nu_m^a))\mathbb{I}_{\{\tau^*<\infty\}}\right] \\ =& \mathbb{E}\left[f(X^*(\nu_m^a),a)\mathbb{I}_{\{\tau^*=\infty\}}\right] + \mathbb{E}\left[f((X^*,Y)(\tau^*\wedge\nu_m^a))\mathbb{I}_{\{\tau^*<\infty\}}\right] \\ \leq& f(\infty,a)\mathbb{P}\left[\tau^*=\infty\right] + \mathbb{E}\left[f((X^*,Y)(\tau^*\wedge\nu_m^a))\mathbb{I}_{\{\tau^*<\infty\}}\right] \end{aligned}$$

because f is increasing in x. Since $\nu_m^a \xrightarrow[m\to\infty]{a.s.} \infty$, letting $m \to \infty$ in the second integral we get

$$\begin{aligned} f(x,y) \leq & f(\infty,a)\mathbb{P}\left[\tau^* = \infty\right] + \lim_{m\to\infty} \mathbb{E}\left[f((X^*,Y)(\tau^* \wedge \nu_m^a))\mathbb{I}_{\{\tau^*<\infty\}}\right] \\ = & f(\infty,a)\mathbb{P}\left[\tau^* = \infty\right] + \mathbb{E}\left[f((X^*,Y)(\tau^*))\mathbb{I}_{\{\tau^*<\infty\}}\right] \\ = & f(\infty,a)\mathbb{P}\left[\tau^* = \infty\right] + \mathbb{E}\left[f(0,Y(\tau^*))\mathbb{I}_{\{\tau^*<\infty, X^*(\tau^*)=0\}}\right], \end{aligned}$$

because $f(x,y) = 0$ for $x < 0$.
We have that

$$\mathbb{P}\left[X^*(\tau^*) = 0, \tau^* < \infty\right] = 0. \tag{12}$$

This is true: since the process (X^*,Y) is a strong Markov process, then between any two jump times $[\eta_n, \eta_{n+1})$ the process has the following dynamic

$$\begin{aligned} dX^*(t) = & (c + b(Y(t))A^*(X^*(t),Y(t)) + r(Y(t))X^*(t))dt \\ & + \sigma(Y(t))A^*(X(t),Y(t))dW_1(t), \\ dY(t) = & g(Y(t))dt + \beta(\rho dW_1(t) + \epsilon dW_2(t)). \end{aligned}$$

Now since $A^*(0,\cdot) = 0$, the process degenerates on the boundary $\{0\} \times \mathbb{R}$. Following the notation in Lemma 0.3.1 we have that

$$\sum a_{ij}(0,y)\nu_i\nu_j = \sigma(y)^2 A^*(0,y)^2 = 0$$

and

$$\sum b_i\nu_i + \frac{1}{2}\sum a_{ij}\frac{\partial^2\rho}{\partial x_i \partial x_j} = c + b(y)A^*(0,y) = c > 0,$$

hence (X^*,Y) cannot cross the boundary continuously and so (12) follows.

So for all $a \in \mathbb{R}$.

$$\delta(x,y;A) \leq \frac{f(x,y)}{f(\infty,a)} \leq \delta(x,y;A^*).$$

Since A^* is an admissible strategy we get the equality for this strategy and so $f(\infty,a)$ has to be constant. Substituting y for a we finally conclude

$$\delta(x,y;A) \leq \frac{f(x,y)}{f(\infty,y)} = \delta(x,y;A^*).$$

Case 2. Assume $\mathbf{H}_1$.
We now proceed to the case with non risk bonus, that is, $0 < r_0 \leq r(\cdot)$. In this case we don't further have the assumption $\sigma(\cdot) \leq \sigma_1$. We follow again the idea proposed by Hipp and Plum [**?**] for the auxiliary process X^ϵ. In this case we work with the following processes.

Process	Strategy	Initial State	Ruin time
$(X,Y)(t)$	A_t	(x,y)	τ
$(X^\epsilon,Y)(t)$	A_t	$(x+\epsilon,y)$	τ^ϵ
$(X^*,Y)(t)$	A^*_t	(x,y)	τ^*

We have that

$$X^\epsilon(t) - X(t) = \epsilon + \int_0^t r(Y(s))(X^\epsilon(s) - X(s))ds$$

and solving the equation we get

$$X^\epsilon(t) = X(t) + \epsilon \exp\left\{\int_0^t r(Y(s))ds\right\}.$$

So over the set $\{\tau = \infty\}$ we have that $X(t) \geq 0$ and so

$$\begin{aligned} X^\epsilon(t) =& X(t) + \epsilon \exp\left\{\int_0^t r(Y(s))ds\right\} \\ \geq& \epsilon \exp\left\{\int_0^t r(Y(s))ds\right\} \\ \geq& \epsilon \exp\{r_0 t\} \xrightarrow[t\to\infty]{} \infty \end{aligned}$$

Also we get that for all $t > 0$, $X^\epsilon(t) \geq X(t)$ a.s., and so

$$\mathbb{P}\left[\tau^\epsilon < \tau\right] = 0.$$

From here the proof follows exactly the same as in the case without bonus. We only rest to prove that in this case, there exists an strategy A such that $\mathbb{P}\left[\tau(A) = \infty\right] > 0$, an argument used to prove the boundness of f. For this we consider again the case $A \equiv 0$. We work with the following processes

$$X_0(t) = x - \epsilon + ct - S(t)$$

and

$$X_1(t) = x + \int_0^t (c + r(Y(s))X(s-))ds - S(t).$$

Since the jumps of the processes are the same we have that the process $\{X_0(t) - X_1(t)\}_{t\geq 0}$ has continuous paths. Now over the set $\{\tau_0 = \infty\}$, we get

$$\begin{aligned} X_0(t) - X_1(t) =& -\epsilon + \int_0^t r(Y(s))(-X_1(s))ds \\ \leq& -\epsilon + \int_0^t r(Y(s))(X_0(s) - X_1(s))ds. \end{aligned}$$

By Gronwall's Lemma (Lemma **??**), over the set $\{\tau_0 = \infty\}$ we have

$$X_0(t) - X_1(t) \leq -x \exp\left\{-\int_0^t r(Y(s))ds\right\},$$

so

$$\begin{aligned} X_1(t) \geq & X_0(t) + \epsilon \exp\left\{-\int_0^t r(Y(s))ds\right\} \\ \geq & X_0(t) \end{aligned}$$

And so we conclude

$$\mathbb{P}\left[\tau_1 = \infty\right] \geq \mathbb{P}\left[\tau_0 = \infty\right] > 0,$$

which finish the proof. □

Auxiliary results.

.1 Continuity of the stopping times.

Theorem .1.1. *Let $\{Z(t)\}_{t\geq 0}$ be a stochastic process with continuous paths a.s., $A\subset\mathbb{R}^d$ an open, connected set with regular boundary. Let*

$$\tau := \inf\{t>0|Z(t)\notin\overline{A}\}.$$

Assume that $\mathbb{P}\left[\tau<\infty|Z(0)=z\right]=1$ and $\mathbb{P}\left[\tau=\tau'|Z(0)=z\right]=1$ for all $z\in A$, where

$$\tau' := \inf\{t>0|Z(t)\notin A\}.$$

For $a>0$, define

$$A_a := \{x\in\mathbb{R}^d|d(x,\partial A)<a\}$$

and

$$\begin{aligned}A_{a+} :=& A\cup A_a,\\ A_{a-} :=& A\setminus A_a\end{aligned}$$

and the corresponding exit times

$$\begin{aligned}\tau_{a+} &:= \inf\{t>0|Z(t)\notin\overline{A_{a+}}\},\\ \tau_{a-} &:= \inf\{t>0|Z(t)\notin\overline{A_{a-}}\}.\end{aligned}$$

Then, if $Z(0)=z\in A$,

$$\begin{aligned}\tau_{a+} &\xrightarrow[a\downarrow 0]{a.s.} \tau,\\ \tau_{a-} &\xrightarrow[a\downarrow 0]{a.s.} \tau.\end{aligned}$$

Proof. Let $Z(0) = z \in A$ and

$$B_z := \{Z(t) \text{ is continuous}\} \cap \{\tau = \tau'\} \cap \{\tau < \infty\}.$$

By the hypotheses we have that $\mathbb{P}[B_z] = 1$. Observe that $\tau_{a-} \leq \tau \leq \tau_{a+}$ for all $a > 0$, then we need to prove that for all $\omega \in B_z$ and $\alpha > 0$, there exists $\gamma(\omega, \alpha) > 0$ such that for all $0 < a < \gamma$

$$0 \leq \tau_{a+}(\omega) - \tau(\omega) < \alpha$$

and

$$0 \leq \tau(\omega) - \tau_{a-}(\omega) < \alpha.$$

Let $\omega \in B_z$ and $\alpha > 0$, then $Z(t, \omega)$ is continuous and $\tau(\omega) < \infty$. We first prove the continuity for τ_{a+}. Define

$$\gamma_+(\alpha, \omega) := \sup_{\{t \in [\tau(\omega), \tau(\omega)+\alpha), Z(t,\omega) \notin \overline{A}\}} \{d(Z(t, \omega), \partial A)\}.$$

Since $Z(t, \omega)$ is continuous, then $\gamma_+(\alpha, \omega) > 0$. So there exists $t_+ \in [\tau(\omega), \tau(\omega)+\alpha)$ such that $Z(t_+, \omega) \notin A_{a+}$ for all $0 < a \leq \frac{\gamma_+}{2}$. Let $\gamma := \frac{\gamma_+}{2}$, then for all $0 < a \leq \gamma$ we get that $\tau_{a+}(\omega) \in [\tau(\omega), \tau(\omega) + \alpha)$ and so

$$0 \leq \tau_{a+}(\omega) - \tau(\omega) < \alpha.$$

For τ_{a-} we proceed in a similar way. Let

$$\beta(\omega) := \inf_{t \leq \tau(\omega) - \alpha} \{d(Z(t, \omega), \partial A)\}.$$

Since $\tau(\omega) = \tau'(\omega)$ and $Z(t, \omega)$ is continuous, we get that $\beta(\omega) > 0$. Define

$$\gamma_-(\alpha, \omega) := \sup_{t \in (\tau(\omega)-\alpha, \tau(\omega)]} \{d(Z(t, \omega), \partial A)\}.$$

Again, it follows from the continuity of $Z(t, \omega)$ that $\gamma_-(\alpha, \omega) > 0$. So there exists $t_- \in (\tau(\omega) - \alpha, \tau(\omega)]$ such that $Z(t_-, \omega) \notin A_{a-}$ for all $0 < a \leq \frac{\beta \wedge \gamma_-}{2}$. Let $\gamma := \frac{\beta \wedge \gamma_-}{2}$, then for all $0 < a \leq \gamma$ we get that $\tau_{a-}(\omega) \in (\tau(\omega) - \alpha, \tau(\omega)]$ and so

$$0 \leq \tau(\omega) - \tau_{a-}(\omega) < \alpha$$

and the proof is complete. □

Theorem .1.2. *Let $\{Z(t)\}_{t\geq 0}$ be a stochastic process with $Z(0) = z \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$ an open, connected set with regular boundary. Let $\{z_n\}$ be a sequence such that*

$$z_n \xrightarrow[n\to\infty]{} z_0$$

with $z_n, z_0 \in A$. Denote by Z_0 and Z_n the stochastic processes with initial conditions z_0 and z_n, respectively. Define

$$\tau := \inf\{t > 0 | Z_0(t) \notin \overline{A}\},$$
$$\tau_n := \inf\{t > 0 | Z_n(t) \notin \overline{A}\}.$$

Assume that $\tau < \infty$ and $\tau = \tau'$ a.s., where

$$\tau' := \inf\{t > 0 | Z_0(t) \notin A\}.$$

Let $\{\Sigma(t, z)\}_{t\geq 0, z\in\mathbb{R}^d}$ denote the flow process of Z. If $\Sigma(t, z)$ is continuous a.s., then

$$\tau_n \xrightarrow[n\to\infty]{a.s.} \tau.$$

Proof. Let

$$B := \{\Sigma(t, z) \text{ is continuous}\} \cap \{\tau < \infty\} \cap \{\tau = \tau'\}.$$

As a consequence of the hypotheses we get that $\mathbb{P}[B] = 1$. We need to prove that for all $\omega \in B$ and $\epsilon > 0$, there exists $N(\epsilon, \omega) \in \mathbb{N}$ such that for all $n \geq N(\epsilon, \omega)$

$$|\tau(\omega) - \tau_n(\omega)| < \epsilon.$$

Let $\omega \in B$ and $\epsilon > 0$. For the process Z_0 define A_{a+}, A_{a-}, τ_{a+} and τ_{a-} as in Theorem .1.1. Since $\tau_{a+} \xrightarrow[a\downarrow 0]{a.s.} \tau$ and $\tau(\omega) < \infty$, then there exists $a_0(\omega) > 0$ such that $M(\omega) := \tau_{a_0}(\omega) < \infty$. So, for all $0 < a < a_0$ we get that

$$\tau_{a-}(\omega) \leq \tau(\omega) \leq \tau_{a+}(\omega) \leq M(\omega). \tag{1}$$

Let $a_1(\omega, \epsilon) > 0$ such that for all $0 < a \leq a_1$

$$\begin{aligned} \tau_{a+}(\omega) \leq& \tau(\omega) + \epsilon, \\ \tau_{a-}(\omega) \geq& \tau(\omega) - \epsilon. \end{aligned} \tag{2}$$

Let $r > 0$ and $N_1(r) \in \mathbb{N}$ such that for all $n \geq N_1$, $z_n \in [z_0 - r, z_0 + r]^d$. Since $\Sigma(t, x)(\omega)$ is continuous on $[0, \infty) \times \mathbb{R}^d$, then over the compact set

$C(r,\omega) := [0, M(\omega)] \times [z_0 - r, z_0 + r]^d$, $\Sigma(t,x)(\omega)$ is uniformly continuous. Define $a_2(\omega,\epsilon) := \frac{a_0(\omega)\wedge a_1(\omega,\epsilon)}{2}$. Then there exists $\gamma(C_r,\omega,a_2) > 0$ such that if $|z_n - z_0| < \gamma$, then

$$\sup_{0\le t\le M(\omega)} |\Sigma(t,z_n) - \Sigma(t,z_0)| = \sup_{0\le t\le M(\omega)} |Z_n(t)(\omega) - Z_0(t)(\omega)| < a_2(\omega,\epsilon). \quad (3)$$

Let $N_2(\omega) \in \mathbb{N}$ such that $|z_n - z_0| < \gamma$ for all $n \ge N_2$. Then for all $n \ge N_2$, thanks to equations (1) and (3)

$$\tau_{a_2-}(\omega) \le \tau_n(\omega) \le \tau_{a_2+}(\omega).$$

Let $N(\omega,\epsilon) := N_1 \vee N_2$, then for all $n \ge N(\omega,\epsilon)$, combining the last equation and equation (2) we get

$$\tau(\omega) - \epsilon \le \tau_n(\omega) \le \tau(\omega) + \epsilon.$$

So we conclude that for all $\omega \in B$, $\tau_n(\omega) \to \tau(\omega)$, and hence $\tau_n \xrightarrow{a.s.} \tau$. □

.2 Lemmas for the Risk Process.

In this section we present and prove the Lemmas needed in the proof of the Verification Theorem for the Risk Process.

Lemma .2.1. *Consider stochastic processes $\gamma(t)$, $\mu(t)$ and $\sigma(t)$ such that $0 < \mu_0 \le \mu(t)$, $0 < \sigma_0 \le \sigma(t) \le \sigma_1$ for some constants $\mu_0, \sigma_0, \sigma_1$. Also assume that $\mu(t)$ and $\sigma(t)$ are continuous processes. For $a, b, c > 0$ define the process*

$$\pi(t) := \gamma(t) + a + b\int_0^t \mu(s)ds + c\int_0^t \sigma(s)dW(s)$$

If $\gamma(t) \ge 0$ for some $A \in \mathcal{F}$ then

$$\pi(t) \xrightarrow[t\to\infty]{} \infty,$$

over the set A.

Proof. Notice that $\int_0^t \sigma(s)dW(s)$ is a martingale thanks to the boundness of σ. We use the fact that for any continuous local martingale M that vanishes at 0 we get

$$\frac{M(t)}{\langle M\rangle(t)} \xrightarrow[t\to\infty]{a.s.} 0$$

over the set $\{\langle M\rangle(\infty)=\infty\}$ (see [**?**] page 186). Since σ is bounded we have no explosions in finite time. Besides we know that

$$\left\langle \int_0^{\cdot} \sigma(s)dW(s) \right\rangle (t) = \int_0^t \sigma^2(s)ds.$$

Over the set A we have $\gamma(t)\geq 0$, so

$$\begin{aligned}
\pi(t) &= \gamma(t) + a + b\int_0^t \mu(s)ds + c\int_0^t \sigma(s)W(s) \\
&\geq a + b\int_0^t \mu(s)ds + c\int_0^t \sigma(s)dW(s) \\
&\geq a + b\mu_0 t + c\int_0^t \sigma(s)dW(s) \\
&= t\left(\frac{a}{t} + b\mu_0 + c\frac{\int_0^t \sigma(s)dW(s)}{\left\langle \int_0^{\cdot} \sigma(s)dW(s)\right\rangle(t)}\frac{\int_0^t \sigma^2(s)ds}{t}\right) \\
&\xrightarrow[t\to\infty]{} \infty
\end{aligned}$$

since for all $t>0$

$$\sigma_0^2 \leq \frac{\int_0^t \sigma^2(s)ds}{t} \leq \sigma_1^2.$$

□

Lemma .2.2. *Let* $\pi(t) = x + \alpha\int_0^t \mu(s)ds + \beta\int_0^t \sigma(s)dW(s)$*, with* $0<\sigma\leq\sigma_1$ *and* $0<\mu_0\leq\mu$*, then*

$$\mathbb{P}\left[\textit{for some } t; \pi(t)<0\right] \leq \exp\left\{-\frac{2\alpha\mu_0}{\beta^2\sigma_1^2}x\right\}.$$

Proof. Let

$$\kappa(t) = -\alpha\int_0^t \mu(s)ds - \beta\int_0^t \sigma(s)dW(s).$$

For $r>0$ define

$$\begin{aligned}
\gamma(t;r) &= \exp\left\{r\kappa(t) + r\alpha\int_0^t \mu(s)ds - \frac{r^2\beta^2}{2}\int_0^t \sigma^2(s)ds\right\} \\
&= \exp\left\{-r\beta\int_0^t \sigma^2(s)dW(s) - \frac{r^2\beta^2}{2}\int_0^t \sigma^2(s)ds\right\}
\end{aligned}$$

and

$$\theta(r) := \alpha\mu_0 r - \frac{\beta^2\sigma_1^2}{2}r^2$$

Hence

$$\begin{aligned}
&\mathbb{P}[\text{for some } t; \pi(t) < 0] = \mathbb{P}\left[\text{for some } t; \kappa(t) > x\right] \\
&=\mathbb{P}\left[\text{for some } t > 0; \gamma(t;r) > \exp\left\{rx + r\alpha\int_0^t \mu(s)ds - \frac{r^2\beta^2}{2}\int_0^t \sigma^2(s)ds\right\}\right] \\
&\leq\mathbb{P}\left[\text{for some } t > 0; \gamma(t;r) > \exp\left\{rx + r\alpha\mu_0 t - \frac{r^2\beta^2\sigma_1^2}{2}t\right\}\right] \\
&=\mathbb{P}\left[\text{for some } t > 0; \gamma(t;r) > \exp\left\{rx + \theta(r)t\right\}\right]
\end{aligned}$$

For $\hat{r} := \frac{2\alpha\mu_0}{\beta^2\sigma_1^2}$, we have that $\theta(\hat{r}) = 0$, so

$$\begin{aligned}
\mathbb{P}\left[\text{for some } t; \pi(t) < 0\right] \leq &\mathbb{P}\left[\text{for some } t > 0; \gamma(t;\hat{r}) > \exp\{\hat{r}x\}\right] \\
=&\mathbb{P}\left[\sup_{t>0}\{\gamma(t;\hat{r})\} > \exp\{\hat{r}x\}\right].
\end{aligned}$$

On the other hand we have by Novikov's criteria (see [**?**] page 351) that $\gamma(t;r)$ is a martingale since

$$\left\langle \int_0^\cdot (-r\beta\sigma(s))dW(s) \right\rangle(t) = \int_0^t r^2\beta^2\sigma^2(s)ds.$$

Using Doob's inequality for positive supermartingales we get

$$\begin{aligned}
\mathbb{P}\left[\text{for some } t; \pi(t) < 0\right] \leq &\frac{\mathbb{E}\left[\gamma(0;\hat{r})\right]}{\exp\{\hat{r}x\}} \\
= &\exp\{-\hat{r}x\}.
\end{aligned}$$

which finishes the proof. □

.3 Additional results.

Theorem .3.1. *Let* $\{(t_n, x_n)\}_{n\in\mathbb{N}} \subset [0,\infty) \times \mathbb{R}^d$ *be a sequence such that* $(t_n, x_n) \xrightarrow[n\to\infty]{} (t, x)$. *Denote by* X_n *and* X *the solutions of the following equations*

$$dX_n(s) = b(t_n - s, X_n(s))ds + \sigma(t_n - s, X_n(s))dW(s), \quad X_n(0) = x_n,$$

and

$$dX(s) = b(t-s, X(s))ds + \sigma(t-s, X(s))dW(s), \quad X(0) = x.$$

Then for all $T > 0$

$$\|X_n - X\|_T \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Proof. This Theorem is consequence of Theorem 1.5 in Chapter V of [**?**]. □

The following theorem is Theorem 9 of Chapter 3 in [**?**]. Let $0 \le T_0 < T_1$ and let $A \subset \mathbb{R}^d$ be a bounded open set with C^2 boundary. Since σ, b, c and f are locally Lipschitz, then they are locally Hölder of any order $\beta \in (0,1)$

Theorem .3.2. *Assume **H1** and **H2**. Consider the following differential equation*

$$\begin{aligned} -u_t(t,x) + \mathcal{L}[u](t,x) + c(t,x)u(t,x) = -f(t,x) \quad & (t,x) \in [T_0, T_1] \times A, \\ u(T_0, x) = g(T_0, x) \quad & \text{for } x \in A, \\ u(t,x) = g(t,x) \quad & \text{for } (t,x) \in (T_0, T_1] \times \partial A. \end{aligned} \tag{4}$$

If g *is continuous then there exists a classical solution* $w \in C([T_0, T_1) \times \overline{A}) \cap C^{1,2,\beta}((T_0, T_1) \times A)$ *of equation* (4).

Remark .3.1. *Let* w *be the solution of equation* (4) *and define* z *as* $w(t,x) = e^{c_0 t} z(t,x)$ *in* $[T_0, T_1] \times A$*. Then* z *fulfils equation* (4) *with* $c' = c - c_0$ *and* $f'(t,x) = e^{c_0 t} f(t,x)$*. And so the hypotheses of Theorem 9 of Chapter 3 in [**?**] are satisfied.*

Lemma .3.1 (Gronwall's Lemma.)**.** *Let* $\alpha \in \mathbb{R}$, $k(t) \ge 0$ *continuous and* $\gamma \in C(\mathbb{R})$*. If*

$$\gamma(t) \le \alpha + \int_0^t k(s)\gamma(s)ds$$

then

$$\gamma(t) \le \alpha \exp\left\{ \int_0^t k(s)ds \right\}.$$

Proof. Let

$$\eta(t) := \alpha + \int_0^t k(s)\gamma(s)ds.$$

Since $k(s)\gamma(s)$ is continuous, then $\eta(t)$ is differentiable and $\eta'(t) = k(t)\gamma(t)$ (see [**?**] page 349). Hence

$$\begin{aligned} \eta'(t) &= k(t)\gamma(t) \\ &\leq k(t)\left(\alpha + \int_0^t k(s)\gamma(s)ds\right) \\ &= k(t)\eta(t). \end{aligned}$$

Multiplying by $\exp\left\{-\int_0^t k(s)ds\right\}$ we get

$$\eta'(t)\exp\left\{-\int_0^t k(s)ds\right\} - k(t)\eta(t)\exp\left\{-\int_0^t k(s)ds\right\} \leq 0$$

from which

$$\left(\eta(t)\exp\left\{-\int_0^t k(s)ds\right\}\right)' \leq 0.$$

This implies that it is a decreasing function and so

$$\eta(t)\exp\left\{-\int_0^t k(s)ds\right\} \leq \eta(0) = \alpha,$$

which finish the proof. □