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"The Mexican peso was devalued 55 percent against the US dollar" 1987/11/18

For the past several years the Mexican economy has developed many problems. The main problems, the large debt, the continuous devaluation of the peso, the stagnation of the economy, etc., are these days present in all discussions by both economists and non-economists.

Among the economists, these discussions have taken different directions, one of which is related with the Keynesians' against the Neoclassicals' (IMF, Friedman) proposals, for the economy. This in turn has brought about the discussion of the original works of Keynes and Walras and their developments in order to better understand both to help analyze the actual situation of Mexican economy and highlight possible solutions. That is one of the reasons for studying the General Equilibrium Theory.

The theory of general equilibrium analysis is the determination of the market process as a device that allows the decentralized allocation of consumption and production resources. This decentralized allocation of an economy with multiple individual agents implies the solution of the possible conflict that could arise in a society. Ultimately, the social device called 'market' should reconcile individual interests without any mechanism that would impose direct duties to the individual agents. Walras' work is the departure point of the decentralized competitive market theory in which the individual agents define their activity plans (production and consumption) adjusting them by the social signals that the prices represent in their parametric function.

Prices are the only social signals that individual agents take into account to define their economic behavior. Outside these signals only strictly private elements exist (consumers' preference field and demand functions, producers' production sets and supply functions), and any communication among the agents before the transactions is excluded by definition. The possibility of communication is excluded when it is assumed that the agents passively adjust themselves to given prices, that is, act as price takers.

The competitive market has been traditionally conceived to be integrated by agents who cannot have an influence on the prices, controlling the amount of demanded and supplied commodities. However this implies the existence of an infinite number of agents because in such a case the influence of each one can be neglected. Later works conceive competitive market as one in which each agent passively adjusts himself to a given existing price system. His behavior is like a price taker, and this statement is totally independent of the capacity that some agents have to have an influence upon price formation.

The theory of General Equilibrium derived the individual agents taking into account only strictly private information elements, for example, initial endowments, preferences, etc. The market is the social device that brings the social information to the agents by means of price.

The central element of Walras' contribution can be summarized as follows. In the framework of a feasible situation for the whole economy, the market defines a price system for which the individual production and consumption plans are mutually compatible. Those are the most satisfactory individual plans for each agent at this price level in the sense that individual agents maximize utilities or profits. The supply and demand functions could be aggregated and all the transactions could be effectively carried out, if and only if, demand equals supply for all commodities. Therefore the tradition of the theory of general equilibrium starting with Walras implies the necessity of proving the existence of the competitive equilibrium. This is the tradition that guides the works of Wald, Arrow, Debreu, Mc Kenzie, Gale, Hahn, Nikaido and others.

This tradition determines the direction of research which by the 1950's resulted in a clear mathematical statement in which the theory rigorously conceives the individual agents as well as the characteristics that the competitive equilibrium has. The conditions in which a competitive equilibrium with those characteristics exist are stated in this framework. It is important to remark that there seems to exist a consensus between economists concerning the proof of existence of such equilibrium. Generally, economists consider that the proofs are rigorous and that they effectively solve the problems of existence of decentralized economies' equilibrium. This fact leads the research in two directions: one concerned with the problem of relaxing the restrictive assumptions; and the other concentrating on the dynamic process that effectively lead to the equilibrium already proved.

The second direction of the research mentioned above is what Walras called "the empirical solution on the market by the mechanism of free competiton". Now it is recognized that Walras' solution was achieved by a price formation process 'tatonnement' in which the non-equilibrium transactions are excluded and the Market Secretary both centralizes the individual agent's supply and demand informations for each vector price, and does the necessary adjustments to ensure the convergence toward equilibrium.

Negishi's works (1962) try to relax the restriction that does not accept non-equilibrium transactions. On the other hand a great number of disequilibrium works deal with the idea of avoiding the Market Secretary. Fisher (1983) points out that this last question still does not have a satisfactory answer. These last works are also in the Walras tradition, but we are going to limit our analysis up to the Arrow-Debreu model.

The three chapters of this work provide an analysis of general equilibrium theory. The first chapter states the Walras model as the departure point of general equilibrium theory. The second chapter follows the historical development of the Walras model ending with the Arrow-Debreu model. In the third chapter we discuss two problematic assumptions of general equilibrium existence proof: the boundness of individual possibilities production and consumption sets; and the divisibiliby of commodities. Finally in the appendix, we shall deal with some methodological problems that often appear in mathematical economy and furnish some general remarks on the concept of equilibrium and utility theory.

The principal idea of this work is to study in some detail the general equilibrium theory as one of the important lines of investigation of the economy since 1900. We will follow the historical process and point out both the improvements and the new problems that arise at each step. We are going to stress the procedure followed to bind the individual sets and the difficulties that arise when we remove the divisibility of commodities assumption. We will show that, even with all the modifications and new formulations, this theory still has some problems.

Chapter I Walrasian General Equilibrium Theory.

The theory of general equlibrium is concerned with the determination of the concept of market as a device that allows the decentralized allocation, of given consumption and production of an economy with multiple individual agents implies the solution of the possible conflict that could arise in a society without imposing any constraints of production or consumption upon individual agents.

Walras' work is the departure point in this area of theorizing. The original Walrasian system of general equilibrium has helped modern economic theory to expand in several directions. Discussions of the existence of solution of general equilibrium system began with Walras' formulation of the system.

In this chapter the general framework of Walras' work will be outlined. There will then be an attempt to portray Walras' exchange model as he himself did, which will be followed by Hansen's interpretation of the production model. First we use Walras notation, even though it is difficult to deal with, but later, in the production system we use the modern notation. It is important to note in the equations of the production system the different way that the equilibrium is reached in the market of products and in the market of productive service, that is, the difference between the equilibrium conditions. At the end of this chapter we shall furnish the simplified system. Its principal difference is that production process is not explicitly analyzed and the equilibrium conditions are stated in terms of which excess demand equals zero.

1. General Equilibrium Theory.

Auguste Walras was led to the concept of rareté or scarcity in the course of investigation into the philosophical foundations of the concept of property. In scarcity the senior Walras saw the foundation both of property and of economic value. Deep in the idea of scarcity lay the seed of neoclassical allocation theory, and Leon Walras's role was to draw it out and bring it to his theory of general equilibrium. As in all his works Leon Walras exhibits, his social ideals in this theory.

Walras was interested in the transformation of pure economics into a mathematical science and his prime task was to cast into mathematical form whatever part of the science that was feasible to handle in this way. He started with what he regarded as the simplest part; the pure and static theory of the exchange of two commodities. He knows and describes the complexities involved in attempting a static theory of the allocation of a given set of productive resources. In the first stage production is not considered.

Walras describes the problem as "Given certain quantities of commodities, to formulate a system of equations of which the prices of the commodities are the roots". Jaffe concisely describes Walras's origin of marginal utility: "From the very start, Leon Walras introduced his marginal utility theory immediately into his analysis of market price determination without considering it in any other context. His whole attention was focused on market phenomena and not on consumption". Thus, it is the marginal adjustment (not the utility as such) that matters.

Walras presented his first equation system in 1874 in Elements d'Economie Politique Pure; the final version appeared in the Edition definitive 1926.

First we are going to deal with logical structure of the conditions or relations in Walras's system used to determine the equilibrium values of all the economic variables in perfect equilibrium and pure competition, that is: the equilibrium consists of both the prices of all products and factors of production and the quantities of these products and factors that would be bought, by all the households and firms. Since the determination of these quantities implies the determination of individual as well as group and social incomes, this theory also determines such features as total income and employment in the society. Although the latter are macroeconomic aspects, they are considered to be the result of the microeconomic aspects in nature. In this sense it is therefore not correct to contrast Keynesian macroanalysis with Walrasian microanalysis.

Walras' theory of prices refers primarily to the prices of services of products and factors, and not to the prices of products and factors. This amounts to the same thing only in regard to products and factors that can be utilized only once; for more durable goods, the problem of the prices of products and factors themselves is a different problem that is solved in a second level of analysis. Walras conceived the equilibrium prices to be, the actual level around which prices oscillate in real life, but they are referred to here as the prices that would be paid in perfect equilibrium and pure competition. "Equilibrium..., is and ideal and not a real state. It never happens in the real world... Yet equilibrium is the normal state, in the sense that it is the state towards which things spontaneously tend under a régime of free competition..."(Walras1954) Walras grouped his productive services into services of land, labor, and 'capital proper' but this does not mean acceptance of the old triad of factors, in fact he admitted m indefinite but fixed number of means of production and services.

The Walrasian entrepreneur is the agent (person or corporation) who buys raw materials from other entrepreneurs, hires land, labor force or capital goods from landowners, worker or capitalists respectively, and sells the the products that result from the combination of their services for his account.

Though Walras constructed the model theoretically, at the same time, he identified practically, the various markets through which his economic mechanism works and the interaction of which constitutes his system. He found two fundamental markets, those of the products and the productive services, and in addition the market that determines the prices of the capitals, and the market of means of payment.

The strict association of every part of the argument with an identifiable market, is an essential feature of Walras' procedure which starts with a theoretical solution of an equilibrium problem and then investigates the manner in which this theoretical solution works out 'practically' in the corresponding market.

The equilibria in the two basic markets, the consumers goods and the service markets, and the way in which they interact, simultaneously determining one another, are of decisive importance in Walras' theory.

In Walras' solution, there is an emphasis on inventories of goods whose existence presupposes a certain past behavior of producers and consumers. Because here current reproduction presupposes certain expectations, the system still depicts a process in time, even if it is perfectly stationary. However Walras tried to build up an idealized equilibrium state in which the smooth and instantaneous adaptation to the conditions obtaining at the moment of all existing goods and processes are feasible. Households and firms merely declare what they would respectively buy and sell (produce) at hazard prices. These prices are announced experimentally by some agent in the market, and households and firms are free to change their minds if these prices do not turn out to be the equilibrium prices. Should equilibrium prices not occur, other prices are announced, and other declarations of willingness to buy or sell (produce) are written down on vouchers (pieces of paper that do not carry any obligation), until both the demand and supply are equal

The only mechanism of reaction to these variations of experimental prices that Walras recognizes is to raise the prices of commodities or services, when their demand is greater than the supply, and to reduce the prices of commodities or services, when their supply is greater than the demand.

a) Walras' Theory of Exchange.

Walras based his structure on an elaborate theory of exchange, which not only provides the theoretical description of the maximization behavior of consumers, but also displays the fundamental properties of economic action in general. This theory was based on a marginal utility explanation of economic value.

Walras fully recognized the possibilities that there may be no solution to the general equilibrium problem, or that the answer might lie in multiple equilibria. However, he thought unique equilibrium prices would almost always emerge if there are many commodities in the market.

To begin with, let's assume there are N persons with definite tastes. They appear on the market given any quantities of some kinds of well-defined M commodities. In order to take advantage of the possibilities that this market may offer to them, they want to maximize their satisfaction as far as their original possessions permit. If we accept the usual continuity and differentiablity assumption, the marginal utility functions of every participant, for every commodity, not only exist but are functions of the quantity of this commodity alone and they are monotonically decreasing. We then have: n(m-1) behavior equations expressing for all n participants the quantities they will give away or acquire at any given system of prices in terms of the numeraire (exchange relations), with the condition that they will go on exchanging until the marginal utilities to them of the quantities of all the quantity of all commodities that can be had for a unit of numeraire are equal; n equations such that all the quantities, the participants acquire (+) and give away (-), each quantity multiplied by its price in the standard commodity, must add up to zero; m equations such that, for every commodity, the total amount of quantity given away must equal the total amount of quantity acquired for the market as a whole. Thus we have m(n+1) equations. But then we shall show that one of them is dependent on the others. Finally we are left with m(n+1)-1 independent ones by which to determine the variables (unknowns), and the m equilibrium prices and the mn quantities exchanged by the households. Since the two first sets of equations considered by themselves, are homogenous of zero degree in the prices, it is only the exchange ratios and not the absolute prices which we can determine, though we can translate these ratios into absolute prices by means of the numeraire price identity.

A function $x_1=f(x_2,x_3,....x_r)$ is called homogeneous of zero degree if, h being any positive arbitrary constant, the dependent variable remains the same when the independent ones are multiplied by h, so that $x_1=f(hx_2,hx_3,....hx_r)$. Putting now $h=1/x_1$, we get $x_1=f(1,x_2/x_1,....x_r/x_1)$, that is to say a relation in which the former independent variables of which there are r, are replaced by ratios of which there are only r-1.

The question of the existence in the mathematical sense now arises; that is, whether there does exist a set of values that will satisfy the conditions of general equilibrium. In other words, are the equations capable of being simultaneously solved. Is this independent of whether there is any tendency in the market to establish these solutions or equilibrium values.

Walras could not answer this question satisfactorily even though he saw the possibility that the system of equations may not admit of any solution at all. He also saw, and even proved, that the solution, even if it exists, may not be unique. All he claimed was that solutions exist normally and that, if the commodities in the market are numerous, there will in general be a unique solution. But perhaps he was not fully aware that this solution need not be economically meaningful in the sense that an actual system might work with it.

Wald stated more rigorously within the Walrasian assumptions themselves the conditions on which the existence of solutions, and specially of a unique solution, depends. And Amoroso said that under 'tolerable' restrictions the existence theorem stands if, as we must, make total and marginal utility a function of all the commodities that enter a household's budget.

It is one of Walras' greatest merits to have distinguished between the existence and the stability problems. However, he treated the problem of stability in a peculiar way, because he posed it in connenction with what is logically an entirely different problem. He stated it in terms of the problem of the relation between the mathematical solution of his equations and the processes of any actual market. He wanted to show that the empirical method used in perfectly competitive markets and the theoretical method tend to produce the same equilibrium configuration, bringing forth the question of how the mechanism of competitive markets drives the system toward equilibrium and keeps it there.

So long as no other mechanism of reaction is admitted than the one exclusively considered by Walras' market mechanism, we can say that equibrium will be attained under these assumptions, that this equilibrium will be unique and stable, and that in this equilibrium will be those prices and quantities we can get from our theoretical solution. But it is clear that we are not dealing with markets of real life but highly abstract creations of the mind.

Equilibrium values in the perfect market are established by a game of trial and error (tatonnement) with prices being adjusted and quantities being readjusted in response. Suppose that all prices except one do equate the respective demands and offers. The one price that does not equate demand and offer must change according to the presupposed rule. But if we do adapt it, we thereby upset the equilibria in all other sections of the market, since they equate supply and demand only in the previous prices. Therefore we have in turn to adjust the others, and Walras gave a reason why the new configuration is nearer to equilibrium than was the original one. That is to say the effects of the adjustment of the price that was originally out of line upon the excess demand of the corresponding commodity, are direct, strong and in the same direction, wherewa the effect of the necessary readjustments of the other prices are indirect, weaker, and not all in the same direction; in part they compensate one another. But this evidently lacks rigor.

b)Walras' Theory of Production.

The theory of production is an attempt to explain the manner in which the mechanism of pure competition allocates the services of the different kinds and qualities of natural agents, labor power, and produced means of production.

The theory of production tells us which quantities of which products each firm will decide to produce, and which quantities of which productive services it will buy in view of the given tastes of prospective consumers of its products and the given propensities of these same consumers considered as 'owners' of productive services. The quantities of services potentially available during a given period of time are given, but they need not be completely absorbed by production, nor do they necessarily go to waste if they are not, because an essential feature of the Walrasian schema is that they are all of them capable of being consumed by their owners directly.

Walras' problem was to show how these data (total quantities and owners' propensities to consume) interlock with the consumers' tastes, so as to produce a consistent set of quantities and values. He looked for a solution symmetrical to the exchange one. For that reason he introduced into his mechanism an entrepreneur who role is to buy productive service and sell ionsumer's goods without any initiative or income of his own (not a capitalist at all).

In Walras' theory, the households were really the agents that determine the economic process. He was aware of the fact that the production and the adaptation to the production involves delays, but at first he simply neglected these delays.

Walras' schema sets several conditions: constant coefficients of production (for each product, only one technological possible way of producing it; there are not economies or diseconomies of scale): absence of any overhead, and; all firms in every industry produce exactly equal amounts of product. Under these conditions, there exists a unique set of solutions that covers both consumers' and producers' behavior. Walras' solution comes to this: the households that furnish the services have definite and single-valued schedules of willingness to part with these services. The prices of consumers' goods are determined simultaneously with the prices of the services and with reference to one another.

We express this by making everybody's offer of every service he owns a function of all prices (both of consumers' goods and services) and, for the same reason, everybody's demand for every commodity another function of all prices (both of the services and the consumer's goods). Everybody's demand for the numeraire commodity follows simply from everybody's balance equation, in which offers are offers of services and only the demands refer to commodities. From these individual demands and offers we get the aggregate offers of services and the aggregate demands for products in the market, as a function of all service and product prices. Then by the assumption of technologically fixed and constant coefficients of production we obtain the remaining restrictions that we need for the determination of prices. To determine prices we need the equations, equal in number to the number of services, which express that the quantities of the services employed in all industries must add up to the total offer of these services, and the equations, equal in number to the number of services used in each industry, each multiplied by the price of these services, must equal the unit price of the industry's product or that in all industries average cost. In the Walrasian case, marginal cost must equal price.

Walras did not present an answer that will satisfy the standards of a mathematician, although he saw all the hurdles that stand in the way of an affirmative answer. Also we have to consider that the existence of a set of solutions or even of non-negative solutions does not necessarily mean the existence of economically meaningful solutions.

In this model the stability and the presence in the economic process of a tendency to establish an equilibrium set of prices and quantities has serious problems related to the delays involved in the rearrangements that are used to achieve equilibrium.

2. Walras General Equilibrium System

a) Theory of Exchange.

Following Walras let A,B,C,... designate m commodities; d_{ab} designate the effective demand for A in exchange for B; P_{ab} the price of A in terms of B (v_a/v_b); Q_b quantity of commodity B.

And assuming that each party to the exchange is a holder of only one commodity then every holder of a quantity Q_b of commodity B, comes to the market prepared to exchange; a certain quantity o_{ba} of B for certain quantity d_{ab} of A; a certain quantity o_{bc} of B for certain quantity d_{cb} of C; a certain quantity o_{bd} of B for certain quantity d_{db} of D; etc. These follow respectively the equations of exchange.

$$\begin{aligned} d_{ab}v_a &= o_{ba}v_b \\ d_{cb}v_c &= o_{bc}v_b \\ d_{db}v_d &= o_{bd}v_b \\ &\vdots \\ &\vdots \end{aligned} \qquad (\text{I}.2.a.1)$$

Thus, we are going to take away from the market a quantity d_{ab} of A, d_{cb} of C, d_{db} of D, etc and a quantity y of B equal to

$$\begin{aligned} & y=Q_b-o_{ba}-o_{bc}-o_{bd}-\cdots \\ \text{ie. } & y=Q_b-d_{ab}P_{ab}-d_{cb}P_{cb}-\cdots \\ \text{ie. } & y=Q_b-d_{ab}P_{ab}-d_{cb}P_{cb}-\cdots \end{aligned} \qquad (\text{I}.2.a.2)$$

the determination of d_{ab}, d_{cb}, d_{db}, etc is impossible unless the prices are known, then we can express

$$d_{ab}=f_{ab}(P_{ab},P_{cb},P_{db},\ldots)$$

$$d_{cb}=f_{cb}(P_{ab},P_{cb},P_{db},\ldots) \qquad (\text{I}.2.a.3)$$

$$\vdots$$

In a similar manner, Walras obtained equations to express the several traders schedules of all other holders of B for A, C, D.... Then, adding these equations of individual demand, Walras obtained the (m-1) equations of total demand for A, C, D.... in exchange for B.

$$D_{ab}=F_{ab}(P_{ab},P_{cb},P_{db},\ldots)$$

$$D_{cb}=F_{cb}(P_{ab},P_{cb},P_{db},\ldots) \qquad (\text{I}.2.a.4)$$

$$D_{db}=F_{db}(P_{ab},P_{cb},P_{db},\ldots)$$

$$\ldots$$

And the (m-1) equations of total demand for B, C, D.... in exchange for A.

$$D_{ba}=F_{ba}(P_{ba},P_{ca},P_{da},\ldots)$$

$$D_{ca}=F_{ca}(P_{ba},P_{ca},P_{da},\ldots)$$

$$D_{da}=F_{da}(P_{ba},P_{ca},P_{da},\ldots)$$

$$\ldots$$

and so on. In all he has m (m-1) equations. Also he has the (m-1) equations of exchange of A for B, C, D.

$$D_{ab}=D_{ba}P_{ba} \qquad D_{ac}=D_{ca}P_{ca} \qquad D_{ad}=D_{da}P_{da} \quad \cdots \qquad (\text{I}.2.a.5)$$

the (m-1) equations of exchange of B for A, C, D....

$$D_{ba}=D_{ab}P_{ab} \qquad D_{bc}=D_{cb}P_{cb} \qquad D_{bd}=D_{db}P_{db} \qquad (\text{I}.2.a.5)$$

and so on. In all he has again m(m-1) equations, that are 2m(m-1) equations. And these equations connect precisely 2m(m-1) unknowns, for these one m(m-1) prices and m(m-1) total quantities exchanged when the m commodities are considered two at a time.

Walras remarks "We have in mind not to pose and solve the problem in question as if it were a real problem in a given concrete situation, but solely to formulate scientifically the nature of the problem which actually arises in the market where it is solved empirically."

Hence, if one wishes to leave arbitrage operations (indirect exchange) aside we have to state, "We do not have a perfect or general equilibrium unless the price of one of any two commodities in terms of the other is equal to the ratio of the prices of these two commodities in terms of any third commodity". Thus, the following equation would have to be satisfied

$$
\begin{array}{llll}
P_{ab}=1/P_{ba} & P_{cb}=P_{ca}/P_{ba} & P_{bd}=P_{ba}/P_{da} & \cdots \\
P_{ac}=1/P_{ca} & P_{bc}=P_{ba}/P_{ca} & P_{dc}=P_{da}/P_{ca} & \cdots \\
P_{ad}=1/P_{da} & P_{db}=P_{da}/P_{ba} & P_{cd}=P_{ca}/P_{da} & \cdots \\
\cdots & & &
\end{array}
\qquad (I.2.a.6)
$$

there are (m-1)(m-1) equations of general equilibrium, which contained implicitly m(m-1)/2 equations expressing the reciprocal relationship between prices. The commodity in terms of which the prices of all the others are expressed is the numeraire.

When a single general market is substituted for several special markets we have to express equality between the demand and supply of each commodity in terms of and in exchange for all the commodities taken together, and designate the prices in terms of A simply by P_a, P_b, P_c.... the equations (I.2.a.5) become:

$$D_{ab}+D_{ac}+D_{ab}+\cdots=D_{ba}P_b+D_{ca}P_c+D_{da}P_d+\cdots$$
$$D_{ba}+D_{bc}+D_{bd}+\cdots=D_{ab}1/P_b+D_{cb}P_c/P_b+D_{db}P_d/P_b+\cdots \qquad (I.2.a.7)$$
$$D_{ca}+D_{cb}+D_{cd}+\cdots=D_{ac}1/P_c+D_{bc}P_b/P_c+D_{dc}P_d/P_c+\cdots$$
$$\cdots$$

If we multiply the second equation of (I.2.a.7) by P_b we obtain

$$D_{ba}P_b+D_{bc}P_b+D_{ba}p_b+\cdots=D_{ab}+D_{cb}P_c+D_{db}P_d+\cdots$$

where $D_{bc}P_b=D_{cb}P_c$; $D_{bd}P_b=D_{db}P_d$; etc; cancelling we obtain

$$D_{ba}P_b=D_{ab} \qquad (I.2.a.8)$$

If we multiply the third equation of (I.2.a.7) by P_c we can get $D_{ca}P_c=D_{ac}$ and so on.

Finally if we add up all these last equations we obtain

$$D_{ab}+D_{ac}+\cdots=D_{ba}P_b+D_{ca}P_c+\cdots$$

it is the first equation of (I.2.a.7)

The meaning of this is that the first equation is linearly dependent on the remaining (m-1) equations system and we can omit it because it does not add new information.

Thus we finally have (m-1) equations of exchange, m(m-1) equations of demand and (m-1)(m-1) general equilibrium equations, making a total of 2m(m-1) equations the roots of which are the m(m-1) prices of the m commodities in terms of one another and the m(m-1) quantities of the m commodities which are exchanged for one another.

Exchange of several commodities for one another.

If each party is the holder of several commodities and if, in this case, the prices of m-1 of the m commodities are expressed in terms of the numeraire, maximum satisfaction will be achieved by each trader when the ratios of the scarcity of the commodities not used as the numeraire and the rareté of the commodity so used equal the prices created.

Now let party (1) be a holder of q_{a1} of A, q_{b1} of B.... let $r=\phi_{a1}(q)$, $r=\phi_{b1}(q)$.... be his equations of utility for commodities during a given period of time. Let P_b, P_c, P_d.... be the respective prices of commodities B, C, D.... in terms of A. And let x_1, y_1, z_1, w_1.... be the quantities of A, B, C.... respectively which the party will add to his original holds at these prices. If additions are positive it means demand, or if they are negative it means supply. The following relation must hold.

$$x_1+y_1P_b+z_1P_c+w_1P_d+\cdots=0 \qquad (I.2.a.9)$$

Walras expressed the attainment of maximum satisfaction by the following system.

$$\phi_{b1}(q_{b1}+y_1)=P_b\phi_{a1}(q_{a1}+x_1) \qquad (I.2.a.10)$$
$$\phi_{c1}(q_{c1}+z_1)=P_c\phi_{a1}(q_{a1}+y_1)$$
$$\cdots$$

constituting in all (m-1) equations, which together with the equation (I.2.a.9) give us a system of m equations. And he assumed that m-1 of the m unknowns x_1, y_1, z_1.... are eliminated one after another from these equations so that he is left with only one equation expressing the mth unknown as a function of the prices. But this is based upon the assumption that all systems of simultaneous equations have a solution where the number of equations equal the number of unknowns, and that does not always follow.

The equations of demand or supply of B, C, D.... by party (1) are the following.

$$y_1=f_{b1}(P_b,P_c,P_d,\ldots)$$
$$z_1=f_{c1}(P_b,P_c,P_d,\ldots) \qquad (I.2.a.11)$$
$$w_1=f_{d1}(P_b,P_c,P_d,\ldots)$$

while his demand or supply of A is given by the equation,

$$x_1=-(y_1P_b+z_1P_c+w_1P_d+\cdots) \qquad (I.2.a.11)$$

He did the same in the case of parties (2), (3),... ,obtaining in this way the trading schedule of the parties from the utility functions and the original stocks.

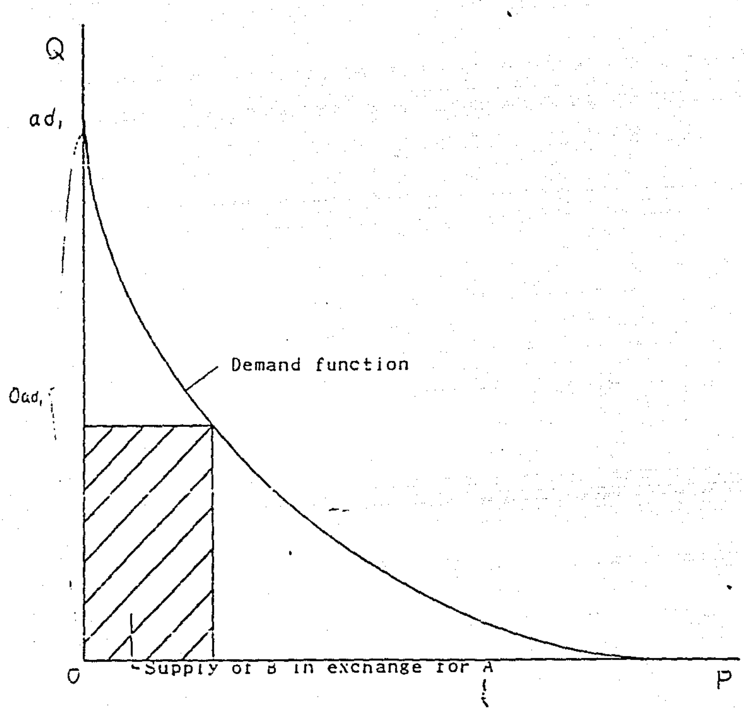
Walras stated that "(m-1) prices of (m-1) of the commodities are determined mathematically in terms of the mth commodity which serves as the numéraire, when the following three conditions are satisfied: first that each and every party to the exchange obtains the maximum satisfaction of his wants, the ratios of his raretés then being equal to the prices; second that each and every party give up quantities that stand in a definite ratio to the quantites received and vice versa, there being only one price in terms of the numeraire for each commodity, namely the price at which total effective demand equals total effective supply; and third that there be no occasion for arbitrage transactions, the equlibrium price of one of any two commodities in terms of the other being equal to the ratio of the prices of these two commodites in terms of any third commodity." And for the market to be in a state of equilibrium "it is necessary and sufficient that at these prices the effective demand for each commodity equal its effective supply. When this equality is absent, the attainment of equilibrium prices requires a rise in the prices of those commodities the effective demand for which is greater than the effective supply, and a fall in the prices of those commodites the effective supply of which is greater than effective demand".

Adding the equations of individual demand Walras obtains the total demand. There is nothing to indicate that these curves are continuous, more than that they are generally discontinuous. But Walras said that when we add the equations, we may suppose the total demand curve is continuous. In fact, whenever a very small increase in price takes place, at least one of the holders of B, out of a large number of them, decreases his demand and a very small decrease in the total demand of A will result.

Oad_1 represents the quantity of A effectively demanded at the price zero. (Graph 1) This quantity depends upon a certain kind of utility of the commodity which we shall call extensive utility, and it is found in the capacity of the particular kind of wealth under consideration to fill wants that are more or less extensive of numerous, depending upon the number of people that feel them and the degree to which they feel them. This first attribute of utility is simple and absolute, in the sense that the extensive utility of A does not influence anything but the demand curves of A, and the extensive utility of B does not influence anything but the demand curve of B. (This is true only if the induvidual demand for A and for B are independent of each other.)

There is another attribute which determines the slope of the curve. The slope is a ratio which depends upon what we shall call intensive utility, and affects the quantity of the commodity consumed in relation to the magnitude of the sacrifice which must be made to procure it, and also depends on the original stock of B.

Walras designated by the term effective utility the sum total of wants satisfied by any given quantity consumed of a commodity including both the intensive and extensive utility, and designated by the term rareté (scarcity) the intensity of the last want satisfied by any given quantity of a commodity consumed.



Graph I

Analytically Walras expressed "if we are given effective utilities as functions of the quantities consumed according to the equations $u=\Phi_{a,1}(q)$ and $v=\Phi_{b,1}(q)$ then the raretés are designated by the derivatives $\Phi_{a,1}'(q)$ and $\Phi_{a,1}'(q)$. If ,on the other hand, we are given the raretés as functions of the quantities consumed, according to the equations $r=\phi_{a,1}(q)$, $r=\phi_{b,1}(q)$ then the effective utilities are designated by the definite integrals

$$\int_0^q \phi_{a,1}(q)dq \quad \text{and} \quad \int_0^q \phi_{b,1}(q)dq$$

then

$$u=\Phi(q)=\int_0^q \phi(q)dq \qquad \text{(I .2.a.12)}$$

$$r=\Phi'(q)=\phi_u$$

b) Theory of Production.

Here Walras takes into account that commodities are products which result from the combination of productive factors. And he determines the price of productive services and formulates the law of the cost of production or cost price.

From now on we shall set up the production equations following Hansen; we shall introduce different notation than Walras used.

We assume that the number of commodities produced in the economy is n, with π_i and q_i being price and quantity respectively of the ith commodity. The number of productive services, labor, land and so forth, is m, with Π_j and Q_j being the price and quantity respectively of the jth productive service. Following Walras, we assume that produced goods are not used as inputs in the production process. All the n goods produces are thus (consumer) goods for final use.

<u>The demand for commodity equations</u> are as follows.

$$Q_i=Q_i(\Pi_1,\ldots,\Pi_n;\Pi_1,\ldots,\Pi_m) \qquad i=1,\ldots,n \qquad \text{(I .2.b.1)}$$

So we have one equation like this for each commodity. to obtain these equations we can, following Walras, derive them on the basis of consumers' utility maximization, assuming that all income from productive services accrues to the consumer, or we can postulate directly as Cassel did.

For the system to be in equilibrium, the price of each good must be equal to its costs. Let a_{ij} designate the technical coefficient, that is, the quantity of productive service i necessary for producing one unit of commodity j.

Then the cost equations are as follows.

$$\pi_i=\sum_{j=1}^{m} a_{ij}\Pi_j \qquad i=1,\ldots,n \qquad (I.2.b.2)$$

that is the price must be equal to total average costs.

Cassel and Walras also differ in their approaches to technical coefficients. Cassel considered the technical coefficients constants, but Walras said that they depend on the choice of technique, which, assuming profit maximization, must depend upon the prices of productive services. He obtained thus mn equations expressing the posibilities of technical substitution.

$$a_{ij}=a_{ij}(\Pi_1,\ldots,\Pi_m) \qquad i=1,\ldots,n \quad j=1,\ldots,m \qquad (I.2.b.3)$$

Since it is assumed that produced commodities are not used as inputs and, there are constant returns to scale, the price for produced commodities do not enter into these functions.

The supply equations for productive services have this form

$$Q_j=Q_j(\Pi_1,\ldots,\Pi_n;\Pi_1,\ldots,\Pi_m) \qquad J=i,\ldots,m \qquad (I.2.b.4)$$

Cassel postulated them and Walras derived them.

Equilibrium is assumed to require that all productive services be exhausted in production, which means that demand equals supply. Then the m equilibrium conditions stated as follows

$$Q_j = \sum_{i=1}^{n} a_{ij} q_i \qquad j=1,\ldots,m \qquad (1.2.b.5)$$

All the prices involved in all at these equations are in terms of a numeraire. We also have the equation $\Pi_n = 1$ (1.2.b.6) as the equation of the price of the numeraire commodity in terms of itself.

Walras formulated the law of the establishment of current on equilibrium prices in production as follows: "Given several services by means of which various products can be manufactured and assuming that these services are exchanged for their products through the medium of a numéraire, for the market to be in equilibrium, or for the prices of all the services and all the products in terms of the numeraire to be stationary, it is necessary and sufficient (1) that the effective demand for each service and each product be equal to its effective supply at these prices; and (2) that the selling prices of the products be equal to the cost of the services employed in making them. If this twofold equality does not exist, in order to achieve the first it is necessary to raise the prices of those services or products the effective demand for which is greater than the effective supply, and to lower the price of those services or products the effective supply of which is greater than the effective demand. In order to achieve the second, it is necessary to increase the output of those products the selling price of which is greater than the cost of production and to decrease the output of those products of which the cost of production is greater than the selling price."

Here we can note an interesting difference between the equilibrium conditions for products and those for productive service. For the former Walras used the condition of demand price equal supply price, where for the service markets the condition is that the demand quantity be equal to the supply quantity. For that reason the law of motion in both markets is different. In the product market Marshall's excess price hypothesis rules, and in the productive services market Walras' excess demand hypothesis rules.

Finally the general equilibrium system consists of equations (I .2.b.1) to (I .2.b.6). There are n prices Π_i and n quantities q_i of products, m prices Π_j and m quantities Q_j of productive services and nm technical coefficient a_{ij}. Altogether there are 2n+2m+nm unknowns. Furthermore there are n equations (I .2.b.1) n equations (I .2.b.2) nm equations (I .2.b.3) m equations (I .2.b.4) m equations (I .2.b.5) and 1 equation (I .2.b.6) that means 2n+2m+nm+1 equations. Walras showed that one of the equations is dependent on the system and to prove the dependency, it can be shown that for the system as a whole total planned expenditure for commodities must be equal to total income from planned sales of productive services.

$$\sum_{i=1}^{n} q_i \pi_i \equiv \sum_{j=1}^{m} Q_j \Pi_j \qquad (I .2.b.7)$$

This equation is arrived at by adding up all individual budget restrictions, assuming that the individual does not change his cash holdings.

From this equation we can derive any of the demand or supply functions as follows. For example

$$q_n \equiv \sum_{j=1}^{m} Q_j \Pi_j - \sum_{i=1}^{n-1} q_i \pi_i$$

Thus we can disregard one equation and yet solve the system with the 2n+2m +nm independent equations.

3. The Symplified System.

The original Walrasian system of general equlibrium has helped modern economic theory to expand in several directions. But also another simplified version in which production process is not explicitly analyzed has proved very useful. This system was first suggested by J. R. Hicks. We shallstate Hansen's model.

The symplified demand and supply system.

We assume that there are n goods in the economy (produced goods or nonproduced factor services). Prices are assumed to be expressed in terms of a numeraire (good number n). We let q_i denote the quantites of the goods, and π_i the prices, being $\pi_n=1$. We distinguish explicitly between quantity in demand q_i^d and quantity in supply q_i^s. Then the simplified Walrasian system consists of

n demand equations $q_i^d=q_i^d(\Pi_1, \Pi_2, \ldots, \Pi_{n-1})$ $i=1,\ldots,n$ (I .3.1)

n supply equations $q_i^s=q_i^s(\Pi_1, \Pi_2, \ldots, \Pi_{n-1})$ $i=1,\ldots,n$ (I .3.2)

and n equilibrium conditions $q_i^d=q_i^s, \cdots$ (I .3.3)

If we define the excess demand $q_i^x=q_i^d-q_i^s$ the excess demand system equilibrium conditions are stated as follows:

$$q_i^x(\Pi_1, \ldots, \Pi_{n-1})=0 \qquad i=1,\ldots,n \qquad \text{(I .3.4)}$$

It follows also that one equation depends upon the rest, and the system can be solved for the prices expressed in terms of the numeraire.

The increasing in generality must be paid for by increasing abstraction and deceasing concreteness. This last system states that in equilibrium all excess demands are zero. If this were the only feature of the system it would be rather uninteresting. However with the addition of further specification, important questions can be asked and answered. For example Samuelson has shown that specifications of the dynamic forces behind the state of equilibrium will enable us to say something about the effects on the equilibrium of a change in demand or supply.

It is important to remark also that even the characteristics of the model changed in the following years some essential features still remain unchanged. But also it is true that the core of general equilibrium took a long time to form and it was not until the early 1950s that it was really developed.

Chapter II Competitive Equilibrium.

The general equilibrium story begins with Walras, but the state of knowledge of general equilibrium analysis in 1930 was better defined by the Walras-Cassel model. In this chapter we are going to develop from the Walras-Cassel model to sketch Wald, Hicks, Von Neumann, Morgenstern, and Koopmans' advancement of Walras' work and point out some of the improvements and the new problems that each step involves. The Arrow-Debreu model of general equilibrium will then be reviewed because in this model Walras' work acquires a complete and sophisticated elaboration involving the use of topological results. Also in the Arrow-Debreu model one of the most important problems in the general equilibrium theory, the problem of existence of general equilibrium, has a different formulation. Finally some of the problems that the proof of existence presetnts in Arrow-Debreu's and McKenzie's works are highlighted.

1. Competitive Equilibrium (1930-1954).

The general equilibrium story begins with Walras, but the state of knowledge of general equilibrium analysis in 1930 can be defined by Gustav Cassel's (1932) The Theory of Social Economy. In this book Cassel set out the divisions between the consumers and producers, and integrated the market outcomes in product and factor markets. He argued against marginal utility and value, placed prices at the center of his allocation theory, and introduced his market principles to create a formal system. Cassel presented a Walrasian system without utility and organized its components in a way that would later suggest an approach to the existence question.

Cassel specifically restricted his argument to commodities (products or factors) that are scarce, and therefore he necessarily thought of factors as having positive prices. In addition, because the technical coefficients $A_{ij} > 0$, all product prices are nonnegative. Cassel extended the discussion of the system to a society which is progressing at a uniform rate.

Walras' solution to the problem of existence of equilibrium was unsatisfactory. For this reason, it was a constant concern of general equilibrium authors. However, in the Karl Menger's seminar in Vienna, one of the most important contribution, in this direction was made, which reformulated the Walras-Cassel system allowing inequalities.

Let A_{ij} be the amount of ith productive resource necessary to produce one unit of the jth commodity (product or service). Let X_j $j=1,\ldots,n$ be the output of the jth commodity in the economy. Let v_i $i=1,\ldots,m$, be the amount of the ith factor available in the economy. Let p be an n-vector which gives the prices of the commodities that prevail in the economy, and let w be an m-vector which gives the prices of the factors. Schlesinger's reformulation of the Walras-Cassel system can be described as follows:

$$\sum_{j=1}^{n} a_{ij} x_j \leqq v_i \qquad i=i,\ldots,m \tag{II.1.1}$$

$$\sum_{j=1}^{n} a_{ij} x_j < V_i \Rightarrow w_i=0 \tag{II.1.2}$$

If there is an excess supply of the ith factor, its price will be zero.

$$\sum_{i=1}^{m} w_i a_{ij} = p_j \qquad i=1,2,\ldots,n \tag{II.1.3}$$

$$p=f(x) \quad \text{ie.} \quad p_j=f_j(x_1,\ldots,x_n) \qquad j=1,\ldots,n \tag{II.1.4}$$

The inverse demand function expresses the price of each commodity as a function of the quantity demanded.

$$\begin{aligned} &v_i=\overline{v_i} && \text{constant} && \\ &p_j \geqq 0 && x_j \geqq 0 && j=1,\ldots,n \\ & && w_i \geqq 0 && i=i,\ldots,m \end{aligned} \tag{II.1.5}$$

Wald proved the existence of a unique solution (p,x,v) in the above system under the following assumptions:

i) $a_{ij} \geqq 0 \quad \forall\, i,j$

ii) $\overline{v} > 0$ a_{ij} constant $\forall i,j$

iii) $\forall j$ $\exists$ at least one i such that $a_{ij} > 0$

iv) The demand function f_j is single-valued, continuous, and defined $\forall x > 0$

v) If $\{x^q\}$ is a sequence of commodity vectors such that $x^q \to \overline{x}$ with $\overline{x}_j = 0 \Rightarrow f_j(x^q) \to \infty$. It means that the demand for each commodity will never be zero for any finite price, that is, every commodity is indispensable to the consumer. This assumption is unrealistic.

vi) Given $x, x' > 0$ and letting $p = f(x)$ and $p' = f(x')$ we have either $px < px'$ or $p'x' < p'x$. This assumption is needed in the proof of the uniqueness of the equilibrium solution.

For the proof see Wald, "Über die Eindeutige Positive Lösbarkeit der neuen Produktionsgl eichungen" The great achievement of Wald was the proof... of the unique solution...provided that the functions...connecting the prices of the products with the quantities produced satisfy certain conditions implied by the principle of marginal utility". (Menger 1973)

In 1936 Wald wrote "On some systems of equations of mathematical economics". In this paper he reviewed the theorems of the two published papers and discussed the ideas behind various assumptions, such as the assumption of revealed-preference. Dealing with the equations of exchange, Wald assumed very strong demand restrictions, also assumed that indifference curves are given by differential equations. That is why the first order conditions and budget equations appear. The assumptions that no individual holds negative stocks, that there are positive stocks of each good, that each individual has a positive endowment, and diminishing marginal utility prevails, are used to ensure a competitive exchange equilibrium as long as the marginal utility of a good is independent of the amount held of other goods. In other words substitutes and complements are excluded.

Kuhn and Dorfman-Samuelson-Solow (using duality theorem of linear programming and Kakutani's fixed point theorem) gave a proof of Wald's theorem.

Let have the following two types of problems

$$\underset{x\in R^n}{\text{Maximize}}\ p\cdot x \qquad \text{(Max)}$$

$$\text{Subject to } A\cdot x \leqq r \quad \text{and} \quad x \geqq 0$$

where p is a given vector in R^n, r is a given vector in R^m, and A is an m×n matrix.

$$\underset{w\in R^m}{\text{Minimize}}\ w\cdot r \qquad \text{(Min)}$$

$$\text{Subject to } A'\cdot X \geqq p \quad \text{and} \quad w \geqq 0$$

where A' is the transpose of A.

Each one of these problems is called the "dual problem" of the other.

Duality theorem

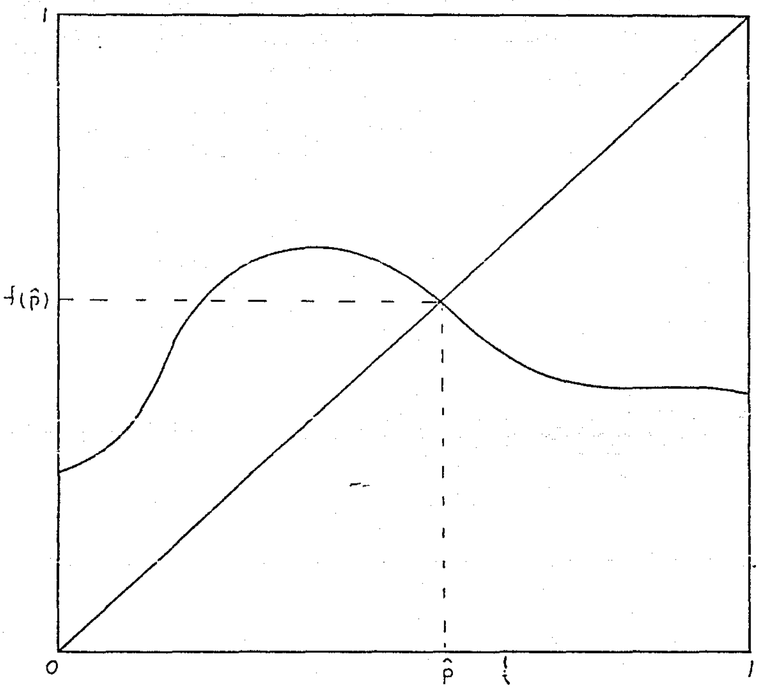
i) there exists an optimal solution $\hat{x}$ for the Max problem if and only if there exists an optimal solution $\hat{w}$ for the Min one.

ii) the inequalities $\hat{x} \geqq 0$ and $\hat{w} \geqq 0$ satisfy $A\cdot\hat{x} \leqq r$ $A'\hat{w} \geqq p$ and $\hat{w}\cdot(r-A\cdot\hat{x}) = \hat{x}\cdot(A'\cdot\hat{w}-p)=0$ if and only if $\hat{x}$ is optimal for (Max) and $\hat{w}$ is optimal for (Min), and $p\cdot\hat{x}=r\cdot\hat{w}$

Kakutani fixed point theorem: Let S be a non empty, compact convex subset of R^n. Let F be an upper semicontinuous function from S into itself such that, for all $p\in S$, the set F(p) is non empty and convex. Then there exists a $\hat{p}$ in S such that $\hat{p}\in F(\hat{p})$.

This theorem is a generalization of the following theorem, which is called Brower's fixed point theorem.

Brower fixed point theorem: Let S be a nonempty, compact, convex subset of R^n, and let F be a single-valued continuous function from S into itself. Then there exists a $\hat{p}$ in S such that $\hat{p}=F(\hat{p})$. (Graph 2)



Graph 2 Illustration of Brouwer's Fixed Point theorem.
S = [0,1]. A continuous function from S to S
must cross the diagonal line; thus f(p) = p

One sketch of the proof by Kuhn is as follows.

Let $X \equiv \{x \mid x \geqq 0, A \cdot x \leqq \bar{v}\}$ be the feasible set where $A=[a_{ij}]$. After proving that X is non empty, compact and convex, he considered the following problem:

$$\underset{x}{\text{Maximize}}: \ px$$

$$\text{Subject to } A \cdot x \leqq \bar{v} \text{ and } x \geqq 0$$

Define $p \equiv f(x) \ \forall x > 0$ such that $x \in X$.

Let x be fixed, obtain p and then solved the linear programming problem, then we have a mapping $x \xrightarrow{F(x)} p \rightarrow x^*$; $F(x): X \rightarrow X$. Extend this mapping to the whole X and denote $\phi(x)$. Show that $f(x)$ continuous $\Rightarrow \phi(x)$ upper semicontinuous and $\phi(x)$ is nonempty and convex $\forall x \in X$.

By Kakutani's theorem $\exists \hat{x} \in X$ such that $\hat{x} \in \phi(\hat{x})$

By assumption (v)(Wald) $\hat{x} > 0$.

Then we have $\hat{x} > 0$ and $\hat{p} = f(\hat{x})$ such that $\hat{x}$ solves the Max linear programming problem.

By duality theorem, $\exists \hat{w}$ for the dual problem

$$\text{Minimize}: \ w \cdot \bar{v}$$

$$\text{Subject to } A' \cdot w = \hat{p}, \ w \geqq 0$$

then $(\hat{p}, \hat{x}, \hat{w})$ constitutes a solution for Schlesinger's version of the Walras-Cassel model.

Note that if we allow $A' \cdot \hat{w} \geqq \hat{p}$ and not only $A' \cdot \hat{w} = \hat{p}$ then we admit zero production for some goods and we do not need the problematic (v) assumption. Instead we need an assumption to guarantee that the output of at least one commodity will be positive.

To prove uniqueness, suppose $(\hat{p}, \hat{x}, \hat{w})$ and (p^*, x^*, w^*) are two different equilibria.

Since both are solutions to the Max linear program we have,

$$\hat{p}\hat{x} \geqq \hat{p}x \quad \forall x \in X \quad \text{where } \hat{p}=f(\hat{x}) \quad \text{in particular} \quad \hat{p}\hat{x} \geqq \hat{p}x^{*}$$

Similarly, $p^{*}x^{*} \geqq p^{*}\hat{x}$ where $p^{*}=f(x^{*})$ which for assumption vi) implies that $\hat{p}\hat{x} < \hat{p}x^{*}$. This contradicts $\hat{p}\hat{x} \geqq \hat{p}x^{*}$.

But still there are some unsatisfactory points. What will guarantee the existence of the inverse demand function and its continuity? No behavioral rule for the consumers or for the producers is stated. The production set is for the economy as a whole and not for each producer, only one type of production set is considered, etc. As one can see from now on the problem is no longer one of finding solutions for simultaneous equations or inequalities.

In September 1932, the state of Colorado, Cowles' home base, chartered the Cowles Commission for Research in Economics, a group whose purpose was "to educate and benefit its members and mankind, and to advance the scientific study and development of economic theory in its relation to mathematics and statistics". The Econometric Society sponsored the Cowles Commission, and the first issue of Econometrica appeared in January 1933. Yet the times were hostile to mathematical economics. The central problems of economic science were focused on the depression and mass unemployment. Many young economists shaped the policy experiments of the New Deal. Theoretical work followed such events. In addition the theoretical explosion associated with Keynes' General Theory of Employment, Interest, and Money in 1936 consumed the interest of economists with a taste for theory.

However, the Cowles Commission maintained their focus on mathematical economics. "The (1947) reorientation which Marschak and his new staff wrought in the Cowles Commission's research program is sketched in the following passage from the Annual Report for 1943: The method of the studies ...is conditioned by the following four characteristics of economic data and economic theory: a) the theory is a system of simultaneous equations, not a single equation; b) some or all of these equations include 'random' terms; ...c) many data are given in the form of time series...d) many published data refer to aggregates...To develop and improve suitable methods seems, at the present state of our knowledge, at least as important as to obtain immediate results. Accordingly, the Commission has planned the publication of studies on the general theory of economic measurements...It is planned to continue these methodological studies systematically" (Christ 1952).

With such a program there was clear recognition of the centrality of general equilibrium analysis in the development of economic theories that provided a basis for empirical work. A primary reference in this direction was John R. Hicks' Value and Capital (1939).

Hicks stated "Wicksell cannot be blamed for a neglect of capital and interest, which problems were indeed his main preoccupation. But, writing before Pareto, he had not the advantage of being able to use Pareto's improvements in value theory; and (largely in consequence, I believe) his capital theory is limited to considering the artificial abstraction of a stationary state. Subject to this limitation, he did wonders; his theory of money and interest, ..., has been the foundation of modern monetary theory. Our present task may therefore be expressed in historical terms as follows. We have to reconsider the value theory of Pareto (and Walras), and then to apply this improved value theory to those dynamic problems of capital which Wicksell could not reach with the tools at his command" (Hicks, 1939).

Hicks developed the classical general equilibrium theory from the theory of the household and the theory of the firm in modern neoclassical language. He then provided an equilibrium and stability analysis. The most modern results in value theory were integrated in the analysis, and the properties of the enriched system were explored. Hicks distinguished two concepts of stability, imperfect stability and perfect stability. The Hicksian method of stability analysis is essentially that of comparative statics. The macroeconomic orientation of Hicks' argument linked the concerns of the Keynesian macroeconomics with a framework of general equilibrium theory, in which the microeconomics was articulated.

Samuelson contrasted his approach with that of Hicks and further explored the problem by analyzing the IS-LM Keynesian model as a system of simultaneous equations that deserved an explicit analysis of equilibrium. Samuelson assumed the existence of equilibrium. He writes the fundamental assumption of stability analysis as a system of differential equations, thus reducing the problem of stability analysis to the examination of the dynamic system generated by the differential equations. The dynamic approach by Samuelson handles the general equilibrium nature of the stability analysis, that is, the repercussion among various markets, and makes clear the dynamic character of the adjustment process toward an equilibrium. Samuelson then takes a linear approximation of the system of differential equations and he considers the relation between the Hicksian stability and the 'true dynamic stability' in his linear approximation system. He concludes that: 1) for the two commodity case, the two conditions are equivalent, 2) for the three commodity case, the Hicks condition for perfect stability is sufficient for the dynamic stability, and 3) for the n-commodity case ($n>3$), the Hicks condition for perfect stability is neither necessary nor sufficient for true dynamic stability. For the proof and a more detailed explanation see Samuelson (Foundation of Economics analysis 1947).

In 1928 von Neumann wrote "The theory of games". This paper contains an articulation of games with many finite strategies as well as the first proof of the min max theorem.

Min max theorem

For all games Γ $V_1'=V_2'$ ie. $\max_{\xi} \min_{\eta} K(\xi, \eta)=\min \max K(\xi, \eta)$ or equivalently a saddle point of $K(\xi, \eta)$ exist.

This theorem is valid for all functions $K(\xi, \eta)$ of the form

$$K(\xi, \eta)=\sum_{\tau_1=1}^{\sigma_1} \sum_{\tau_2=1}^{\sigma_2} H(\tau_1, \tau_2)\xi_{\tau_1}\eta_{\tau_2}$$

the coefficients $H(\tau_1, \tau_2)$ are absolutely unrestricted. The proof use a fixed point argument to establish the existence of a saddle point for a function $h(\xi, \eta)=\sum_{p=1}^{M+1} \sum_{q=1}^{N+1} \alpha_{pq}\xi_p\eta_q$ where the α_{pq} are constants and ξ_p and η_q are vertices of simplexes of appropriate dimension. A simplex in R^n is the set of points $\{x \mid x_i \in [0,1] \text{ and } \sum_{i=1}^{n} x_i=1\}$. The minmax theorem has, as a context, certain dual systems of inequalities with explicit non-negativity constraints on the ξ_p and η_q.

Weintraub said about "On an Economic Equation System and a Generalization of the Brower Fixed Point Theorem" (von Neumann 1936), "In my view, the single most important article in mathematical economics". The paper contains the first explicit statement of what has been subsequently called the activity analysis model of production, exhibits a model of competitive equilibrium, contains the first rigorous, formal, and fully explicit model in non aggregative capital theory, and also contains the first use in economics of certain now common tools: explicit duality arguments, explicit fixed point techniques for an existence proof, and convexity arguments.

Von Neumann solved the problem initially posed by Cassel and further defined by Wald, of establishing an equilibrium in a uniformly expanding economy. Wald had emphasized factor use and supply and the problem of allocating scarce resources. Von Neumann established that, for an economy in such an equilibrium, the rate of interest equals the rate of growth. Such a result, however, paid explicit attention to the price-quantity duality, the complementary slackness conditions induced by the non-negativity constraints, and the convexity of the production and price sets induced by returns to scale and homogeneity.

Von Neumann's model

Von Neumann assumes n goods $\delta_1, \dots, \delta_n$ and m processes $p_1, \dots, p_m$ so, if the processes are linear

$$P_i: \sum_{j=1}^{n} a_{ij}\delta_j \rightarrow \sum_{j=1}^{n} b_{ij}\delta_j$$

where a_{ij} is the amount of good δ_i used up in process p_i operating at unit intensity, while b_{ij} is the amount of good δ_j produced by the same process. If x_i is the intensity of the ith process, y_j is the price of good j, the economy expands at a rate α, and β is the interest factor, then the equations of the economy are

$$x_i \geqq 0$$

$$y_j \geqq 0$$

x_i and y_j are dual non-negative variables.

$$\sum_{i=1}^{m} x_i > 0 \qquad (\text{II}.1.6)$$

$$\sum_{j=1}^{n} y_j > 0$$

The above equations represent viability assumption. And the following are the complementary slackness conditions for the dual inequality systems

$$\alpha \sum_{i=1}^{m} a_{ij}x_i \leq \sum_{i=1}^{m} b_{ij}x_i \qquad (\text{II}.1.7)$$

$$\text{if } \alpha \sum_{i=1}^{m} a_{ij}x_i < \sum_{i=1}^{m} b_{ij}x_i \Rightarrow y_j=0$$

$$\beta \sum_{j=1}^{n} a_{ij}y_j \geqq \sum_{j=1}^{n} b_{ij}y_j \qquad (\text{II}.1.8)$$

$$\text{and if } \beta \sum_{j=1}^{n} a_{ij}y_j > \sum_{j=1}^{n} b_{ij}y_j \Rightarrow x_i=0$$

This model assumes that there are constant returns to scale, that the natural factors of production, including labor, can be expressed in unlimited quantities, that consumption of goods takes place only through the processes of production which include the necessities of life consumed by workers and employees, that is, all income in excess of necessities of life will be reinvested, and that $a_{ij}+b_{ij}>0$ to prevent subeconomies.

To prove that there exists x's and y's satisfaying the model conditions, von Neumann considers x', y' where

$$x'_i \geqq 0 \qquad y'_j \geqq 0$$
$$\sum_{i=1}^{m} x'_i > 0 \qquad \sum_{j=1}^{n} y'_j > 0 \tag{II.1.9}$$

then defines

$$\phi(x', y') = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x'_i y'_j}{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x'_i y'_j} \tag{II.1.10}$$

where x'_i, y'_j are variables. Then he shows that a solution to the original system exists, if and only if, for $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$

$$\min_{y'} \phi(x, y') = \phi(x, y) \quad \text{and} \quad \max_{x'} \phi(x', y) = \phi(x, y) \tag{II.1.11}$$

making the existence of an equilibrium equivalent to the existence of a saddle point of $\phi(x', y')$. And at this saddle point or equilibrium point $\alpha = \beta = \phi(x, y)$.

Morgenstern had had a continuing interest in the interaction of events and predictions, especially the interaction of agents when their predictions and the foresight link their behaviors in the marketplace. Von Neumann had a similar interest in strategic behavior, so they decided to "write a paper showing economists the essence and significance of game theory as it then existed". This collaboration resulted in the Theory of Games and Economic Behavior (von Neumann and Morgenstern 1947). This book has a definition and axiomatization of the mathematical concept of a game, and the fundamental existence theorem for two persons zero-sum games that states that any two-person zero-sum game has an equilibrium solution in mixed strategies. This theorem is related to the minmax problem which appears in a more general setting in certain economic models. "It seems worth remarking that two widely different problems related to mathematical economics -although discussed by entirely different methods- lead to the same mathematical problem and at that to one of a rather uncommon type 'the minmax type'. There may be some deeper formal connections here" (von Neumann and Morgenstern 1947).

Von Neumann's minmax theorem could be proved easily using Kakutani's point point theorem. However, Jean Ville gave a simpler way to prove it based on convexity arguments, and the supporting hyperplane theorem.

The supporting hyperplane theorem is now recognized as a corollary to the Hahn-Banach theorem of functional analysis. The version used by economists states that, given a convex set and a point outside that set, there is a plane through the point that does not intersect the interior of the convex set.

Section 16 of the Theory of Games and Economic Behavior contains the tools used in the proof of the fundamental minmax theorem of Section 17. Section 16 presaged an entirely new approach to the structure of economic optimization theory, an approach that led to a global characterization of objective functions and constraint sets through convexity arguments. Section 17 has the proof of the minmax theorem, originally due to Ville but much improved by von Neumann and Morgenstern. Their thinking about the possibility of a numerical representation of utility led to an axiomatization of choices in risk situations; this allowed the inference of the existence of a real continuous function as an order-preserving representation of utility. Now called the von Neumann-Morgenstern utility indicator, it is unique up to linear transformations and can be interpreted as 'expected utility'.

This line of analysis may be carried ahead to the early 1950s. In 1952 there appeared in Econometrica "An Axiomatic Approach to Measurable Utility" (Herstein and Milnor 1953) on the von Neumann-Morgenstern axiom system. Then Nash's work provides a direct bridge between the papers on the theory of games and those on the existence of equilibrium. "Equilibrium points in N-person games" (Nash 1950) generalized the von Neumann-Morgenstern equilibrium for two-person zero-sum games to n-person games. Nash defined an equilibrium of an n-person game as an n-tuple of strategies (one for each player) such that the strategy of any player is optimal (yields the highest payoff) against the equilibrium (n-1)-tuple defined by the remaining players. The proof used Kakutani's fixed point theorem to show that all n-person zero-sum games possess such an equilibrium, now called Nash equilibrium.

It was these works, and the development of activity analysis and programming that led to the articulation of the genaral equilibrium model and to the analysis of equilibrium by Arrow, Debreu and McKenzie.

Koopmans identified four lines of research that had jointly created the subject matter of linear programming or activity analysis: discussions on generalizations of the Walrasian equation systems of mathematical economics; the 'new' welfare economics, with the underlying idea that the comparison of the benefits from the alternative uses of each good, where not secured by competitive market situations, can be built into the administrative processes that decide the allocation of that good; the work on interindustry relationships, initiated, developed, and stimulated largely by Leontief; and the work by Dantzig and Wood on the organization of defense, the conduct of the war, and other specifically war-related allocation problems. In 1951 Koopmans axiomatized production through activity analysis, in a fashion analogous to the contemporaneous axiomatization of utility and consumer choice. The idea of his paper ("Analysis of production as an efficient combination of activities") was to go behind the given technique that economists used in their production function analysis. After he developed his model it was clear that production analysis can be formally developed from the properties of cones in R^n or alternatively, by special kinds of convex sets.

In 1949 the Cowles Commision hosted a conference on linear programming. In this conference economists developed a framework for the programming approach, one that emphasized convexity and the allied properties of general topology and algebra. Thus Weintraub said, "by mid 1949 and certainly by 1950, mathematical economists had (1) some knowledge of attempts, and successes, in establishing the existence of equilibrium in sensibly specified economic models; (2) a basic understanding of useful ways to model interrelated constrained-choice systems; and (3) fixed point theorem techniques for demonstrating the compatibility of strategies or independent choices. The problem of showing the existence of a competitive equilibrium was accesible; the work remained to be done" (Weintraub 1985).

2. Arrow-Debreu Model

Simultaneously in August 1950, Gerard Debreu and Kenneth J. Arrow presented models of a competitive economy. They proved that competitive equilibria for these models are Pareto efficient and that Pareto-efficient allocations can be realized by a price system such that the allocation is also a competitive equilibrium.

In "The coefficient of resource utilization" (Debreu 1951) Debreu gave a non-calculus proof of the intrinsic existence of price systems associated with the optimal complexes of physical resources, the new basic theorem of new welfare economics. This proof is based on convexity as he himself stated. Debreu defined individual consumption vectors, and an ordering on consumption vectors. Production is treated by total input vectors. Then he exploited the set theoretic structure of both the consumption and production spaces to obtain the definiton of competitive equilibrium and a characterization of that equilibrium as Pareto efficient.

In 1951 Arrow presented "An Extension of the Basic Theorems of Classical Welfare Economics." The paper's summary states that "The classical theorem of welfare economics on the relation between the price system and the achievement of optimal economic welfare is reviewed from the viewpoint of convex set theory. It is found that the theorem can be extended to cover the cases where the social optima are of the nature of corner maxima, and also where there are points of saturation in the preference fields of the members of society."

Arrow began by reviewing the marginal analysis treatment of equilibrium and Pareto efficiency, pointing out the difficulties in ensuring, in a calculus treatment, non-negative prices and the necessary production of every product by every firm. Using the separating hyperplane theorem for convex sets, Arrow demonstrated that equilibrium outcomes are optimal and that optimal distributions generate price vectors (which are defined by hyperplanes known to support the appropriate convex sets) that equate supply and demand, and are thus competitive equilibria. Debreu in his 1952 paper "A Social Equilibrium Existence Theorem" gave the existence theorem with general conditions under which a social system has an equilibrium, that is, a situation where the action of every agent belongs to his restricting subset and no agent has an incentive to choose another action. This theorem has been used by Arrow and Debreu to prove the existence of equilibrium for a classical competitive economic system in their paper "Existence of an Equilibrium for a Competitive Economy." These works are the support of the modern version of the model of a competitive economy, now called the Arrow-Debreu model.

We shall introduce some general concepts and meanings that Debreu stated in his "Theory of Value".

A commodity is defined by a specification of all its physical characteristics, of its availability date, and of its availability location.

As soon as one of these three factors changes, a different commodity results. Debreu makes a convention that for some of the economic agents inputs will be represented by non-negative numbers and outputs by non-positive numbers, and for others vice versa. Debreu assumes perfect divisibility of the commodities saying that it is imposed by the present stage of development of economics, and that it is quite acceptable for an economic agent producing or consuming a large number of the commodity. This assumption is stated when he says that the quantity of any commodity can be <u>any</u> real number.

It is assumed that there is only a finite number n of distinguishable commodities. The space R^n will be called the commodity space. For any economic agent an action is a specification for each commodity of the quantity that he will make available or that will be made available to him, and is represented by a point of R^n.

With each commodity is associated a real number, its price. This price is specified at the beginning, but shall be paid on the delivery date at the delivery place. Debreu is assuming implicitly that markets exist for all commodities. The price may be positive, null or negative if the respective commodity is scarce, free or noxious. The price system is a point in R^n, and the value of an action a is $\sum_{j=1}^{n} p_j a_j$.

Debreu considered two classes of economic agents, producers and consumers, who play different roles in the economy. However, an individual can play both roles, that of a producer and that of a consumer at the same time.

A producer is an economic agent whose role is to choose a production plan. It is assumed that there are m producers and each one has his production plan where output are represented by positive numbers, and inputs by negatives. This plan is represented by a point y^k of R^n, the commodity space. The set Y_k of all the productions possible for the kth producer is called his production set. $y=\sum_{k=1}^{m} y^k$ is called the total production or the total supply and the set $Y=\sum_{k=1}^{m} Y_k$ is the total production set.

Given a price system p and a production y^k the profit of the kth producer is $p \cdot y^k$. The total profit is $p \cdot y$. Given the price system p the kth producer chooses his production y^k in his production set Y_k so as to maximize his profit. Debreu stated that the resulting action is called an equilibrium production of the kth producer relative to p.

A consumer is an economic agent whose role is to choose a consumption plan. It is assumed that there are ℓ consumers for whom the inputs are represented by positive numbers and his outputs by negative numbers. The consumption plan of the ith consumer is represented by a point of the commodity space. The set X_i of all consumptions possible for the ith consumer is called his consumption set. $x=\sum_{i=1}^{\ell} x^i$ is called the total consumption on the total demand and the set $X=\sum_{i=1}^{\ell} X_i$ is called the total consumption set.

Consumers' preferences.

We call a preference relation on a consumption set X a binary relation $\gtrsim$ on $X \subset R^n$ satisfying the following condition.

i) Reflexive $x \gtrsim x \quad \forall x \in X$

ii) Transitive $x \gtrsim y \quad y \gtrsim z \Rightarrow x \gtrsim z \quad \forall x, y, z \in X$ (Ⅱ.2.1)

iii) For any $x,\ y \in X$ either $x \gtrsim y$ or $y \gtrsim x$.

Notation: The pair $(X, \gtrsim)$ is a preference field.

Given a preference field $(X, \gtrsim)$, a strong preference relation is denoted by $x > y$, that is, is preferred to y if $x \gtrsim y$ but not $y \gtrsim x$. This relation satisfies

i) $x > x$ for no $x \in X$ (Ⅱ.2.2)

ii) If $x \gtrsim y \ y \gtrsim z \ \forall x,\ y,\ z \in X$ and either $x > y$ or $y > z$ holds then $x > z$.

In particular $x > y,\ y > z \Rightarrow x > z$.

iii) For every x, $y \in X$ either $x > y$ or $y \geq x$.

An indifference relation denoted by $x \sim y$, that is x is indifferent to y if both $x \geq y$ and $y \geq x$ hold, this relation satisfies.

i) $x \sim x \quad \forall x \in X$

ii) For any x, $y \in X \quad x \sim y \Rightarrow y \sim x$ (Ⅱ.2.3)

iii) For any x, y, $z \in X$, $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

This relation induces a classification of the elements of X to indifference classes.

A point x in X is called a satiation consumption if no possible consumption is prefered to it by the consumer. It is assumed that no satiation consumption exists for any consumer.

If $(X, \geq)$ is a preference field that $\geq$ is continuous and X is a connected space, Debreu demonstrated that a continuous utility function exists such that

$$\forall x, y \in X \quad u(x) \geq u(y) \Leftrightarrow x \geq y \qquad (Ⅱ.2.4)$$

Now we shall state the Arrow-Debreu model. The model involves ℓ consumers, denoted by $i=i,\ldots,\ell$, m producers denoted by $k=1,\ldots,m$, and n commodities denoted by $j=1,\ldots,n$.

Consumers- Each consumer i has a preference field $(x_i, \geq_i)$ where x_i is his consumption set in R^n. He possesses an initial holding of some amount of goods denoted by a vector $a^i=(a^i_j) \in R^n$, where a^i_j is the amount of the jth commodity held by him.

Producers- Each producer k has a production technology set Y_k.

The principal working of the economy, whether it is achieved in a centralized mode or not, is to allocate production and the resulting products among consumers.

If each producer chooses a process y^k from his technology set Y_k, then $y=\sum_{k=1}^{m} y^k$ represents the total production. The sum $a=\sum_{i=1}^{\ell} a^i$ represents the aggregate initial holding. $\{a\}+\sum_{k=1}^{m} Y_k$ represents the aggregate initial holding adding the aggregate technology set, that is, the totality of all aggregate supply vectors availabe.

If each consumer chooses a consumption plan x^i from his consumption set X_i, then $x=\sum_{i=1}^{\ell} x^i$ represents the total demand.

An allocation of production and consumption means the choice of an $(\ell+m)$ tuple $(x^1,\ldots,x^\ell, y^1,\ldots,y^m)$ such that $x^i \in X_i \quad \forall i$ and $y^k \in Y_k \quad \forall k$ such that the aggregate demand equals the aggregate supply.

Let us assume the following conditions for consumers (c) and producers (p).

c.1) $\forall i$ the consumption set X_i is a closed convex set in R^n.

c.2) Each X_i has a lower bound c^i such that $x^i \geqq c^i \quad \forall x^i \in X_i$ (minimum of subsistence)

c.3) Each preference field $(X_i, \gtrsim_i)$ satisfies $x > y$ for $x, y \in X_i \Rightarrow \alpha x + \beta y > y$ $\alpha > 0,\ \beta > 0,\ \alpha + \beta = 1$.

c.4) Each preference relation $\gtrsim_i$ is continuous.

c.3 and c.4) $\Rightarrow$ convexity of each preference field $(X_i, \gtrsim_i)$

p.1) Each technology set Y_k is a subset of R^n containing the origin, the process of inaction.

p.2) Y_k is convex and closed in R^n.

p.3) The aggregate technology set $Y=\sum_{k=1}^{m} Y_k$ satisfies $Y \cap R^{n}_{+}=\{0\}$ (the impossibility of the land of Cockaigne) where R^{n}_{+} is the nonnegative orthant of R^n.

p.4) $Y \cap (-Y)=\{0\}$ (the irreversibility of aggregate prosesses)

c.p) there are lm constants $\alpha_{ik} \geqq 0 \quad i=1,\ldots,\ell \quad k=1,\ldots,m$ such that $\sum_{i=1}^{\ell} \alpha_{ik}=1 \quad k=1,\ldots,m$ where α_{ik} represents the relative share of consumer i in the profit π_k of producer k, as $\alpha_{ik}\pi_k$ is the share of consumer i in π_k. That means the exhaustion of the profit π_k by its distribution among consumers' as shareholders.

Now we can state the definition of a competitive equilibrium.

An $(\ell+m+1)$ tuple $(\hat{x}^1, \hat{x}^2, \hat{x}^3, \ldots, \hat{x}^\ell, \hat{y}^1, \ldots, \hat{y}^m, \hat{p})$ of menus of consumption $\hat{x}^i \in X_i$, production processes $\hat{y}^k \in Y_k$ and an n-dimension price vector $\hat{p} \geqq 0$ is called a competitive equilibrium of the model if the following conditions are fulfilled.

i) The maximum profit of each producer under $\hat{p}$, that is,

$$\pi_k(\hat{p}) = \langle \hat{p}, \hat{y}^k \rangle = \max \langle \hat{p}, y \rangle \quad \forall y \in Y_k \quad k=1, \ldots, m \qquad (\text{II}.2.5)$$

[The producers maximize their profits solely by controlling input-output configurations at their disposal under the equilibrium price vector $\hat{p}$ conceived as an uncontrllable datum.

ii) The optimum preference of each consumer subject to budget constraints under $\hat{p}$, that is, for each $i=1,\ldots,\ell$ $\hat{x}^i$ is a most preferable menu of consumption among all x in X_i fulfilling the budget constraint

$$\langle \hat{p}, x \rangle \leqq \langle \hat{p}, a^i \rangle + \Sigma\, \alpha_{ik}\pi_k(\hat{p}) \qquad (\text{II}.2.6)$$

initial holdings+dividends=Income of consumer i

The consumers achieve their most preferable menus of consumption solely by controlling their possible schemes of demand (and/or supply) subject to their budgets under $\hat{p}$.

iii) The balance of aggregate supply and demand (ie. market equilibrium) that is

$$\sum_{i=1}^{\ell} \hat{x}^i \leqq \sum_{i=1}^{\ell} a^i + \sum_{k=1}^{m} \hat{y}^k \qquad (\text{II}.2.7)$$

with equality holding in this equation for the jth component relation if the corresponding price $\hat{p}_j$ is positive. Then $\hat{p}$ is referred to as an equilibrium price vector.

Theorem Arrow and Debreu 1954. If each consumer i has a positive initial holding a^i in the sense that there is a commodity bundle b^i in X_i fulfilling $a^i > b^i$ and if he has no satiation point, then there exists a competitive equilibrium.

There exists other versions of this theorem which drop the unrealistic assumption of positive initial holdings but they have to impose further conditions on the preference fields and the aggregate technology set. Also some versions make a relaxation of the convexity and closedness of individual technology sets.

Further generalizations and elaborations of the Walrasian conjecture for the existence of a competitive equilibrium were made by Debreu (1959,1962), McKenzie(1959), Nikaido(1959b), Uzawa(1959-1960, 1962a,b), Morishima(1960b, 1964), Negishi(1961), etc.

As Takayama said "Let us sketch an outline of some of the important and difficult problems in the proof of existence...the first thorough recognition of these problems is due to Arrow-Debreu...
(i) The survival problem. This is the question of assuring that every consumer can survive, given the equilibrium conditions. If an equilibrium exists, the equilibrium prices of the resource held by some consumer may be so low that he may not be able to subsist on the income he obtains from his resources. The first requirement for this problem, of course, is that the aggregate supply set contains a point which is the sum of the minimal subsistence consumption requirements for each consumer (otherwise some consumer is bound to die). ... this means that there exist $x_i \in X_i$, for all i and $y \in Y$ such that $x=y+\bar{x}$, where $x \equiv \Sigma x_i$. The second requirement is that each consumer be able to subsist with the resources (including labor) he holds without engaging in exchange. This can be guaranteed if each consumer's consumption set, with his resources added, has an intersection with the aggregate production set of the economy. In fact, we need a little more. For example, we may require that not only must such an intersection be nonempty, it must also have an interior point. This corresponds to the cheaper-point assumption. ... Essentially, it guarantees the (upper semi-) continuity of each consumer's demand function.

(ii) Satiation. When an equilibrium price prevails, some consumer, because the prices of his resources are very high, may be able to purchase a consumption bundle such that he is satiated. As we said ··· the nonsatiation assumption is needed to establish the lower semicontinuity of the budget function (hence the upper semicontinuity of the demand function). Arrow and Debreu simply assumed that every consumer is nonsatiated in his (somewhat modified) consumption set. This is a strong assumption. The relaxation of this assumption is possible and is attempted in the literature (for example, McKenzie 1955, 1959).

(iii) Utility function and the production set. Arrow and Debreu assumed the existence of a continuous utility function for each consumer. McKenzie's formulation is in terms of a preference relation, although his assumptions imply the existence of a continuous utility function. The crucial assumption in this connection, which is common in all the existence proofs, is the convexity of individual preferences. Arrow and Debreu assumed the existence of a fixed number of firms, each of which has a convex production set. McKenzie 1959 assumes that the aggregate production set is a convex cone so that constant returns to scale prevails in the aggregate. McKenzie does not assume the irreversibility of the production processes, nor does he assume free disposability of commodities.

(iv) The number of producers. In Arrow and Debreu 1954 and subsequent works such as Debreu 1959, it is assumed that the total number of firms (producers) is fixed (at, say, k). It is well known and can easily be checked that diminishing returns to scale for an individual producer implies a positive profit, which in turn should imply that firms enter the market. Constant returns to scale for the aggregate production set can be justified on the basis of an adjustment in the number of firms, which are small in size compared to the industry. Diminishing returns to scale for an individual firm typically occur when there are certain limitational fixed factors, such as managerial ability or entrepreneurship, which are not explicitly introduced in the model (and are not marketed). Therefore, diminishing returns to scale (for each firm) plus a finite fixed set of firms imply the scarcity of certain commodities (factors) and freezing the assignment to various production processes of these commodities. (Such a model will not be useful for exploring possible effects of a redistribution of these resources.) Under diminishing returns to scale, firms may make profits, which are attributable to payments for the use of such resources as entrepreneurial skills or special talents of some kind. In McKenzie's model 1959, such resources are explicitly included in the list of commodities (and marketed) and the number of firms need not be fixed, so that we can safely assume constant returns to scale for the aggregate production set. McKenzie also shows the concordance of his model with the usual Hicksian model of a fixed number of firms, each of which has a closed and convex production set" (Takayama 1974)

3. Some Remarks.

In 1930 there was a general model of two classes of agents and their actions, with both a market of supply and demand for products, and a market of supply and demand for productive services. These two markets were at least partially integrated. Choices of agents were optimizing choices, and the notion of equilibrium was understood as a balancing of forces, even though it was not taken to be something that had to be proved.

Problems arose when negative prices could not be ruled out as solutions to the Cassel model. Schlesinger showed that the assumptions of the model could be rephrased to allow agent to be based on an optimization framework, precluding the possibility of negative prices and allowing free goods to result from market choices and equilibrium outcomes. He did not prove that equilibrium exists.

Wald's papers (1934, 1935, 1936) showed that the agents, markets, and optimization assumptions entailed the existence of competitive equilibrium.

During these years the use of linear programming became possible.

Von Neumann (1936) presented a growth model with assumptions about agent choice, and optimization markets, and, also, proved that there exists an equilibrium using game theory.

Yet the interpretations of the basic terms were very restrictive. Hicks' (1939) model expanded the set of permissible interpretations with more explicit concepts of agent, optimizing choice, commodities, utility and equilibrium.

Koopmans' production model was incorporated into the standard general equilibrium model.

Within a short period of time Arrow and Debreu presented a proof, as did McKenzie independently, of the existence of a competitive equilibrium in such canonical models although their work made some more restrictive assumptions. Debreu later worked in his axiomatic analysis of economic equilibrium. In Debreu's work this model acquired its more explicit form but its refinement and further development continues today.

Chapter Ⅲ Fundamental Problems on the Demonstration of General Competitive Equilibrium.

At the level of individual agents, general equilibrium theory ientifies the problem of unboundness of individual consumption and production possibility set. This problem may imply that individual consumption and production functions remains undefined. The solution to this problem is presented by general equlibrium authors in the chapter providing the proof of existence of equilibrium. They take the definition of production and assumptions on available resources as a starting point, and show that the set of feasible allocations for the entire economy is a compact set. However, this can not be considered a satisfactory solution to the problem of unboundness of individual possibility sets because the definition of individual agents in a decentralized economy implies that individual agents do not possess information at a global level. This in turn, has a negative impact on the meaning of the proof of existence of a general competitive equilibrium.

An other problem that has not been properly solved is related to the assumption that all commodities are completely divisible. We stated some of the alterations of the results of general equilibrium which are made when we avoid this assumption.

1. The sketch of the demonstration.

The definition of competitive equilibrium already stated in Chaper II is the one generally adopted by the later theoreticians of general equilibrium. This definition means that each agent, a consumer or producer, has an optimum behavior acting upon their own initiative as price takers. The Theory of General Equilibrium needs to state clearly the linkage between the subjective conditions (i,ii) and the objective conditions (iii).

The Theory states the consumer's behavior constructing a demand function $\phi^{i}(p)$ defined for all $p \geqq 0$ by

$$\phi^i(p)=\{x^i \mid x^i \in X_i, x^i \succsim_i x \text{ for all } x \in X_i \cap E \qquad (\text{Ⅲ}.1.1)$$

$$\text{subject to } \langle p,x\rangle \leqq \langle p,a^i\rangle+\sum_{k=1}^{m}\alpha_{ik}\pi_k \quad i=1,\ldots,n\}$$

That is derived from his preference optimization.

It also states the producer's behavior constructing a supply function $\psi_k(p)$ and his profit function $\pi_k(p)$

$$\psi_k(p)=\{y^k \mid \langle p,y^k\rangle=\max\langle p,y\rangle \text{ over all } y \in Y_k \cap E\} \qquad (\text{Ⅲ}.1.2)$$

$$\pi_k(p)=\max\langle p,y\rangle \text{ over all } y \in Y_k \cap E \quad (k=1,\ldots,m)$$

derived from his profit maximization.

These are only 'notions' of supply and demand functions. These functions are only describing the general agents' behavior. However it is not enough to state that the agents passively adjust to the prices, prices representing uncontrolled data, to obtain the supply and demand functions. That means that with the above elements it is not always possible to have supply and demand functions that have nonempty image sets for all semipositive price vectors.

This statement would allow the avoidance of the speculative behavior of the individual agents. For the agents the prices are 'parameters' in Lange's sense and escape the agents' control. This traditional way of studying the price formation leads to the 'endemic problem of the economic theory'. If the agents stay passive as price takers, who will change the prices in the adjustment process?

To shape their preference field $(X_i, \succsim_i)$ or their production sets Y_k, the individual agents do not need any information about other agents or about the whole economy. On the contrary, from the theory's standpoint it is necessary to avoid all communication between the individual agents, because if they communicate they may anticipate the price movement and behave as non competitive agents. It is in that context that the metaphor of the agent who is disabled to go to the market makes logical sense.

After the theory constructs the individual agents and defines the competitive equilibrium characteristics, it proceeds with the demonstration that in a decentralized economy exists an equilibrium with the required properties.

The demonstration is based upon the demonstration of a theorem that has different versions in Gale 1955, Debreu 1959, Nikaido 1968 and Arrow-Hahn 1971. We are going to follow Nikaido's one.

The Theorem states as follows:

Put $P_n=\{p|p\geqq 0,\ \sum_{j=1}^{n} p_j=1\}$ (the standard simplex of R^n) and let Γ be a compact convex subset of R^n. Suppose that there is given a set-valued mapping

$$\chi : P_n \rightarrow 2^{\Gamma}$$

which, for easy reference, will be called an excess supply function. It is assumed that the mapping satisfies the followin conditions:

i) $\chi : P_n \rightarrow 2^{\Gamma}$ is a closed mapping that carries each point of P^n to a non empty convex subset of Γ.

ii) The Walras law in the general sense holds, i.e.

$$\langle p,u\rangle \geqq 0 \quad \text{for } u\in \chi(p) \quad \text{where } u=y-x \qquad (\text{III}.1.3)$$

then there is some $\hat{p}$ in P_n such that $\chi(\hat{p}) \cap R^{n\ominus} \neq \phi$

This mapping $\chi(p)$ represents the behavior of the individual agents. It is assumed that the possibility of the aggregate actions exists and they construct the function in such a way that the excess suppy u is in $\chi(p)$.

The proof is worked out by constructing a suitable mapping and applying Kakutani's fixed point theorem. Let a single-valued mapping be defined by

$$\Theta = \Gamma \times P_n \to P_n$$

This mapping is constructed with the next two formulas that are possible to be considered as the expression of the rule of behavior of the 'auctioner' or Market Secretary

$$\Theta(u,p) = (\Theta_i(u,p)), \quad u = u_i \in \Gamma, \quad (p_i) \in p_n$$

$$\Theta_i(u,p) = \frac{P_i + \max(-u_i, 0)}{1 + \sum_{j=1}^{n} \max(-u_j, 0)} \qquad i = 1, \ldots, n \qquad (\text{III}.1.4)$$

Then by constructing the cartesian product of Θ with the mapping χ, which can be thought of as a mapping defined on $\Gamma \times P_n$. We obtain a new mapping, as follow

$$f = \chi \times \Theta : \Gamma \times P_n \to 2^{\Gamma \times P_n} \qquad (\text{III}.1.5)$$

$$f(u,p) = \chi(p) \times \{\Theta(u,p)\}$$

Nikaido shows that all the conditions of Kakutani's point fixed theorem are fulfilled by f, that is,

i) Both Γ and P_n are compact convex sets in R^n. Hence $\Gamma \times P_n$ is a compact convex set in $R^n \times R^n$;

ii) $\chi(p)$ is by assumption a convex subset of Γ for each p, while $\{\Theta(u,p)\}$ is a special convex subset, consisting of one element, of P_n for each $(u,p) \in \Gamma \times P_n$. Whence for each $(u,p) \in \Gamma \times P_n$ the image of (u,p) is a convex subset of $\Gamma \times P_n$,

iii) χ, Θ are closed mappings. Hence f is also a closed mapping.

Therefor, by virtue of Kakutani's fixed point theorem, f has a fixed point $(\hat{u}, \hat{p})$ so that $(\hat{u}, \hat{p}) \in f(\hat{u}, \hat{p})$. This last expression can be reduced to the relations for the component mappings $\hat{u} \in \chi(\hat{p})$, $\hat{p} = \Theta(\hat{u}, \hat{p})$. This theorem will be completely proven when it fulfills the condition $\hat{u} \geq 0$ which Nikaido already proved.

The demonstration of the existence of a competitive equilibrium, in which the consumers optimize their satisfaction and the producers their profit according to the definition of competitive equilibrium stated above, is worked out conceiving the mapping χ (p) as an aggregate function of excess supply. It should be remembered that the individual supply and demand functions are included in the aggregate excess supply function.

This function must fulfill all the conditions of Kakutani's fixed-point theorem. Essentially, it must be closed mapping, defined in a non-empty convex and compact set and which image must be a convex non-empty set. That means, that the aggregate excess supply function of the mapping: $\chi : P_* \rightarrow 2^r$ must be defined on an attainable convex and compact set for the economy as a whole. Arrow-Hahn 1971, Nikaido 1968 and Debreu 1959 called our attention to the fact that the individual sets X_i and Y_k may be unbounded sets, therefore, it is necessary to impose further conditions to ensure that they are bounded and then compact.

The problem is the same in all these works, even though the approaches are different. It is Nikaido who stated more explicitly the problem when he discusses the consumers choice: "It should be noted that the statement that merely formulates the pattern of a competitive consumer's choice: it by no means ensures even the existence of commoditiy bundles in X_i satisfying the budget constraint. Even when the budget constraint is consistent with the consumption set, the existence of a most preferable commodity bundle need not be automatically guaranteed" Nikaido 1968.

Arrow and Hahn stated in their chapter about the individual producers. (p63) "If Y_f is unbounded, then at certain p it may be that the firm would like to produce on an infinitely large scale. This possibility, as such, does not make it impossible to conduct an analysis of market equilibrium with positive prices; although the firm is taken to suppose that it can sell and but whatever quantities it likes at the going prices, the economy, in fact, may be incapable of producing outputs and using inputs in unlimited amounts. Indeed, if we are interested in a world of scarcity, we ought to exclude the possiblity."

The main source of problems arises from the possibility of the unbounded individual sets, in such a way that the existence of the most preferable commodity bundle or more rentable activity vector is not ensured.

"to overcome such difficulties, we will substitute certain virtual supply and demand functions for the true ones which are difficult to define. The former can be defined by narrowing the ranges of consumer's and producer's choice to suitable bounded subsets" Nikaido 1968.

The compactness of the set and the intelligibility of the individual supply and demand functions $\psi_k(p)$, $\phi_i(p)$ depends on the possibility of narrowing the ranges of the individual agents. These functions are defined only on compact sets. Furthermore as Nikaido remarks, the statement of consumers in competitive equilibrium definition merely formulates the pattern of a consumer's choice, however the definition of the functions is not guaranteed.

It is interesting to point out that other authors did not refer to this problem explicitly, and some of them who mentioned it did not explain a procedure to overcome it. Weintraub 1982 and Takayama 1974 merely stated that the compactness of the sets is guaranteed because they are lower-bounded (by the subsistence level) and because the budget constraint ensures the upper-bound. Indeed this second statement needs a proof. Some other authors assume from the beginning that the individual sets are closed and bounded, that is, compact sets. (Quirk y Saposnik 1968, Varian 1980, Malinvaud 1975, Lancaster 1971.)

Having stated the problem, now the difficulty is to make a rigorous definition of the functions. It is neccessary to bound in a suitable manner the sets of individual choice, in such a way that the competitive equilibrium subjective conditions have a coherent linkage with the objective ones. Nikaido stated this problem when he defined the properties of the actions of the individual agents in equilibrium conditions. First he recognized that the actions $(\hat{x},\hat{y})$ in competitive equilibrium conditions must satisfy the condition

$$(a+Y-X)\cap R^{n\oplus} \ni a+\Sigma\hat{Y}_k-\Sigma\hat{X}_i \qquad (\text{III}.1.6)$$

Then he defined the individual sets that fulfill that condition.

$$\begin{aligned}\forall i \quad \tilde{X}_i&=\{x^i \mid x^i\in X_i,\ (a+Y-\sum_{s\neq i}X_s-X^i)\}\cap R^{n\oplus}\neq\phi\\ \forall k \quad \tilde{Y}_k&=\{y^k \mid y^k\in Y_k,\ (a+Y_k-\sum_{t\neq k}Y_t-X)\}\cap R^{n\oplus}\neq\phi\end{aligned} \qquad (\text{III}.1.7)$$

That is, the individual sets are narrowed to that actions x^i or y^k for which the excess supply is zero or positive. Thus, the set $\tilde{X}_i$ consists of all the $x^i\in X_i$ that, when we add them to the total demand X, given the initial endowments and the production possibilities of the economy, they do not constitute a disequilibrium factor; that is, the $\tilde{X}_i$ consist of all the x^i that are compatible with the idea of balanced vector in a wide sense.

On the other hand, the sets $\tilde{Y}_k$ consist of all the y^k that when they are added to the total supply of the economy, given X and the initial endowments, they allow the economy to maintain a balanced vector. Nikaido shows that the sets $\tilde{X}_i$ and $\tilde{Y}_k$ are bounded for all i and k. The demonstration is based on the fact that, given the definition of the sets, it suffices to show that ℓ+m sequences $\{x^{iv}\}$ in X_i $\{Y^{kv}\}$ in Y_k satisfy the condition

$$a+\sum_{k=1}^{m}y^{kv}-\sum_{i=1}^{\ell}x^{iv}\geqq 0 \qquad (\text{III}.1.8)$$

Remember that also the sets $\hat{X}_i$ and $\hat{Y}_k$ are closed. Then define virtual suitable supply and demand functions. Thus the allocations associated with competitive equilibria are in the bounded sets $\tilde{X}_i$ and $\tilde{Y}_k$, although the existence of competitive equilibria themselves has not yet been established. On that basis next define virtual supply and demand functions. The first step is the construction of an hypercube

$$E=\{x \mid \xi_j \leqq x_j \leqq h_j,\ (j=1,\dots,n)\} \qquad (\text{III}.1.9)$$

such that $0, b^i \in E$, $\tilde{X}_i$, $\tilde{Y}_k \subset E^\circ$ $(i=1,\ell, k=1,\dots,m)$

where E° denotes the interior of E.

With this we can obtain $\ell+m$ nonempty compact sets $X_i \cap E$, $Y_k \cap E$. With this setup, we define individual supply and demand functions. These functions allow us to make a rigorous characterization of individual agents by means of individual functions defined on compact sets, and by means of the search of general equilibrium in the feasible set.

It should be noted that when the vector a is included in the definition of the sets $\tilde{X}_i$, $\tilde{Y}_k$, it implies that we are narrowing the individual sets as a function of the knowledge of the feasible set for the economy as a whole. This was also Arrow-Hahn and Debreu's procedure to secure that the individual sets are bounded. The idea of scarcity is the clue to bound the sets. This idea is included in two statements; one is the limitation of available resources of the economy, that is the intersection of the economy feasible production set and the positive orthant is only the origin.

The limitation of available resources plays an important role in Debreu and Arrow-Hahn's definition of the feasible set of the economy. They considered that the available resources and the global technology are given. Thus if $z(w)$ is the excess supply function, and if the possibilities of the economy are $W=X\times Y$, then the feasible set is $\hat{W}=W\cap\{w \mid z(w)\geqq 0\}$, and the competitive equilibrium, if it exists, must be in the interior of $\hat{W}$.

Let us remark that the dimension of the cube is given by the way the possible sets with the elements of the feasible set of the economy, cut the commodity space. Ultimately the dimension of the cube is determined by the available resources of the economy. Therefore the theory constructs the virtual supply and demand functions that express the behavior of the individual agents as a result of their action as price takers in a search of self-improvement. The individual agents make their choice in the narrow sets $X_i \cap E$ and $Y_k \cap E$. The functions are as follows:

Individual Supply functions

$$\psi^k(p)=\{y^k \mid \langle p,y^k\rangle=\max\langle p,y\rangle \quad \forall y \in Y_k \cap E\} \qquad (\text{III}.1.10)$$

and the corresponding profit function

$$\pi_k(p)=\max\langle p,y\rangle \quad \forall y \in Y_k \cap E \quad (k=1,\ldots,m) \qquad (\text{III}.1.11)$$

Individual demand functions

$$\phi^i(p)=\{x_i \mid x_i \in X_i,\ X^i \gtrsim_i X \quad \forall x \in X_i \cap E \qquad (\text{III}.1.12)$$
$$\text{such that} \quad \langle p,x\rangle \leqq \langle p,a_i\rangle+\Sigma\,\alpha_{ik}\pi_k(p)\} \quad (i=1,\ldots)$$

Now we have supply and profit functions defined on compact sets. These functions are continuous on a nonempty set $Y_k \cap E$ then reach the maximum and are clearly intelligible. On the other hand, the demand functions now redefined allow us to overcome the problem stated above. That is $\phi^i(p) \neq \phi$ because the budget constraint

$$\langle p,x\rangle \leqq \langle p,a_i\rangle+\sum_{k=1}^{m}\alpha_{ik}\pi_k(p) \qquad (\text{III}.1.13)$$

is consistent with $x \in X_i \in E$ for all $p \geqq 0$. Therefore it can be shown that

ϕ^{i} is a set-valued mapping that sends a semipositive p to a nonempty compact convex subset $X_i \cap E$.

Consequently the aggregate demand function $\phi(p)$, the aggregate supply function $\psi(p)$, and the aggregate excess supply function $\chi(p)$ are defined by

$$\phi^{i}(p)=\sum_{i=1}^{s}\phi^{i}(p)$$
$$\psi(p)=a+\sum_{k=1}^{m}\psi^{k}(p) \qquad (\text{III}.1.14)$$
$$\chi(p)=\psi(p)-\phi(p)$$

This aggregate excess supply function is used in the proof of the theorem stated before and the theorem of existence of competitive equilibrium.

2. Individual agents.

Dealing with individual agents, the General Equilibrium Theory did not solve the problem of the definition of the individual supply and demand functions. It means that the construction of the individual agents is incoherent. The theory follows a tortuous route revealing the existence of fundamental difficulties. One of the best examples is given by Debreu's work. In the chapter about the individual producer, he points out a problem. "Given an arbitrary p, there may be no maximum profit...Let therefore T_j' be the set of p in R^{a} for which the set of maximizers is not empty...Thus with each price system p in T_j' is associated the nonempty set $\eta_j(p)$ of possible productions maximizing profit for that p". Debreu 1959

Debreu introduces the assumption that immediately avoids the problem, but now the question arises in which conditions does the set T_j' exist? In the framework of the individual producer analysis Debreu merely states that "It will be shown...how, under certain rather weak assumptions, the production set Y_j can be replaced by a certain non empty compact subset of Y_j" (Debreu

1959). And in the same way the problem comes up in the individual consumer analysis "Given an arbitrary price-wealth pair (p,w) the set $\{x_i \in X_i | p \cdot x_i \leqq w_i\}$ in which the consumer must choose may be empty. Let therefore S_i be the set of (p,w) in $R^{\ell+m}$ for which this is not so..."

Later on he notices the problem again. "Given an arbitrary pair (p,w) in S_i, $\psi_i(p,w)$ may have no greatest element. Let therefore S_i' be the set of (p,w) in S_i for which the set of greatest elements of $Y_i(p,w)$ is not empty..." (Debreu 1959).

Again Debreu introduces the assumption that removes the problem in this analysis level. At the end of the chapter he state "...under certain weak assumptions, the consumption set X_i can be replaced by a certain non empty compact subset of X_i".

Debreu follows a procedure that consists in assuming those things that must be proven rigorously in the chapters which deal with the individual agents. In fact, without rigorous proofs of the coherence of supply and demand function definition, the individual agents simply are undetermined entities, that is, they do not exist.

It is important to remark that in the cases mentioned above Debreu postpones the analysis on the assumptions that guaranteed the existence of compact sets to the chapter about general equilibrium existence. This is not merely a methodological choice, but and imposition of a logical sequency on Debreu. In fact, it is in this chapter that the conditions to apply fixed point theorm demand an explanation- of the 'certain weak assumptions' that allow one to bound in a suitable way the individual sets to guarantee the definition of functions. The real motivation to follow this way is that in the level of the theory of the individual agent elements that allow the bounding of the action possible sets do not exist.

It is necessary to insist on this point. The only way to bound the individual sets is to connect the global initial resources of the economy as a whole with the individual production and consumption possibilities. In other words, this procedure implies the possibility of a linkage between objective data (the available resources of the economy) and some subjective elements (the production and consumption possiblity sets). Here is where the meaning of the linkage between the objective and subjective conditions of competitive equilibrium gets its real dimension and not in the nature of the parametric prices as Lange said. In Arrow-Hahn's terms, in the theory the individual producer must assume that given the prices he can sell and but any amount of commodity, but the economy, in fact, can not produce unlimited amounts of commodities.

From the angle of the necessity of the general equilibrium theory which needs a consistent construction of individual agents, this situation brings about negative consequences. The most clever authors of General Equilibrium Theory did not notice the problem. Nikaido's quote is revealing. "To overcome such difficulties (that consumption and technology sets need not be bounded), we will substitute certain virtual supply and demand functions for the true ones which are difficult to define" Nikaido 1968.

This is problematic in the sense that the true ones are not difficult to define, but impossible to define in an intelligible way. Consequently only the virtual functions are intelligibles, the true ones are meaningless.

The main cause of this difficulty resides in the fact that in a first instance, the device of building a virtual economy could be considered as a valid resource in the demonstration of the existence of general equilibrium. However, this device can not be carried to the chapters about individual agents and be taken into account only as 'certain weak assumptions that allow one to bound the individual sets'. The problem is not that the device is a weak or strong assumption, but the bounding derived form that is unintelligible at the level of individual agents (decentralized) whose functions we are trying to define. The solution invoked for general equilibrium theory about this problem is incompatible with the idea of a decentralized economy. This has negative repercussions on the meaning of the proof of existence of competitive equilibrium.

It is opportune to remark that the problem of the definition of the individual supply and demand functions is different and must be logically solved before than the problem of the behavior of the individual agents. Generally the theory deals with this problem in a different chapter than the one on the proof of existence but it is important to note that stability analysis or price formation assumes the definition of individual supply and demand functions as a solved problem. Similarly, the coherent definition of these functions must be guaranteed to ensure that the mappings used in the existence proof be intelligible in the framework of a decentralized economy.

3. Divisible commodities.

The usual demonstrations of the existence of a competitive equilibrium require, by assumption, the divisiblity of commodities. "This assumption of perfect divisibility is imposed... "Debreu 1959. This assumption is needed to ensure that the choice of economic agents is continuous as required by the fixed point theorem on which these demonstrations are based. Even this assumption is not realistic. Therefore it is important to analyze whether the perfect divisibility of commodities is only a convenient assumption for the existence of a competitive equilibrium or a determining one, that is, the presence of indivisible commodities alters the results.

This problem has been examined in the literature particularly Henry 1970, Dierker 1971, Broom 1972, Mas-Colell 1977, Montesano 1982. Henry has demonstrated that a competitive equilibrium exists if there are two commodities only one of which is indivisible, and $m \geqq 2$ agents. He also demonstrated that competitive equilibrium does not necessarily exist if there are more than two commodities of which at least one is indivisible.

Dierker, Broome and Mas-Colell have demonstrated with different assumptions, that when the number of agents is very large, an equilibrium can be defined with a price vector that determines agent's choices not feasible by an amount that is negligible with regard to the size of the economy. These demonstrations require assumptions that are not completeltely satisfactory "mainly because some fundamental features of economies with indivisible commodities are disregarded. The principal one, from a realistic point of view, is that the number of indivisible commodities is not constant (as the demonstrations assume) and that the amount of each indivisible commodity does not necessarily increase when the number of agents increase" Montesano 1982.

Broome and Mas-Colell assume the existence, along with indivisible commodities, of a divisible commodity that is strongly preferred to the indivisible ones. On the contrary, if fundamental indivisibilities prevail, not only may a competitive equilibrium not exist, but there may be no tendency toward equilibrium even when the size of the economy increases.

A methodological problem in the analysis of an economy with indivisible commodites is the definition of the homogeneity of commodities. This problem is also in the economy with divisible commodities, but here acquires more importance. The homogeneity of commodities cannot be merely a physical one like in Debreu's works "A commodity is ... defined by a specification of all its physical characteristics, of its available date, and of its availability location. As soon as one of these three factors changes, a different commodity results". Furthermore, a psychological test is not sufficient either. In addition, commodities considered perfect substitutes by all agents can have different prices. This last condition can be essential to the existence of a competitive equilibrium. For Montesano this problem has as a methodological consequence that "an economy with indivisible commodities must be generally analyzed by considering them all unique, i.e., commodities of which there is only one unit".

Montesano shows with different examples that a pure exchange economy with both unique commodities and with divisible and indivisible commodities does not necessarily allow a competitive equilibrium. In economies with indivisible commodities if the commodities that are physically and psychologically homogeneous have the same price they can exclude the existence of a competitive equilibrium which exists if the prices are different.

Another important point that differentiates economies with divisible commodities from those with indivisible commodities, is the link between competitive equilibria and Pareto optima. For economies with indivisible commodities competitive equilibrium can exist that is not a Pareto optimum (see Quirk and Saposnik 1968 and Montesano 1968). Therefore, it is not valid that all competitive equilibrium are Pareto optima as in the other economies. Furthermore, it is not true that all Pareto optima can be obtained through competitive equilibria (Montesano 1968).

Another important difference in the case of indivisible commodities economies, is that Walras' law does not hold in the narrow sense. That is we do not have the following relation

$$\langle p, u \rangle = 0 \quad \forall u \in \chi(p) \quad \text{where } u = y - x$$

but only, the Walras' law relation in the general sense.

$$\langle p, u \rangle \geqq 0 \quad \forall u \in \chi(p) \quad \text{where } u = y - x$$

This can be understood as follows: by varying the exchanged quantities of indivisible commodities, an agent may find it more convenient to exchange a commodity bundle, which is valued more at current prices, for another bundle, which is valued less but which he prefers, without the possibility of balancing its value, because of the indivisibility of the commodities.

Therefore we are again at a point that general equilibrium authors have not solved yet and which shows that the assumption of unrealistic assumptions introduces more problems. This opens a gap between the economic reality and the theoretical and mathematical interpretation.

The existence of general equilibrium depends on excessive assumptions about the commodities and the individual agents that are neither convincing nor intuitive. Therefore the work of the demonstration of the existence of general equilibrium without all of these unrealistic assumptions is still to be done.

4. Conclusion.

The use of mathematical formulations in economic theory was introduced in this century. It was developed mainly in connection with General Equilibrium Theory. In the second half of this century it has increasingly expanded and now covers a variety of branches of economics. But even if this gives the appearance of being scientific, many times the use of mathematics has on the contrary generated vagueness. As Walsh and Gram have said "Mathematical operations are performed upon entities that cannot be defined: calculations are made in terms of units that cannot be measured; accounting identites are mistaken for functional relationships; correlations are confused with causal laws; differences are identified with changes; and one-way movements in time are treated like movements to and fro in space. The complexity of models is elaborated merely for display, far and away beyond the possibility of application to reality". This is the reason for the concern to understand the true meaning of the relationships that appear in the general equilibrium model and to discuss the link between the theoretical statements of objective and subjective conditions and the reality that they attempt to describe. It is in these links that we find general equilibrium to have some of its fundamental problems.

The necessity of having compact sets is important not only to be able to demonstrate competitive general equilibrium existence, but also for the intelligibility of the individual agent (producer or consumer) theory. If the problem of having compact sets acquires its real dimension on the context of the demonstration of equilibrium existence, it is because, with the elements which are used to construct the individual agents, we cannot ensure the boundness of individual possibility sets.

The fixed point theorem requires the boundness of individual possiblity sets, because if the individual sets are unbounded the possibility set of the whole economy will be unbounded, too. Here the general equilibrium theory appeals to the introduction of an objective element, that is, the feasible resources for the whole economy, to guarantee the boundness of the total set. The general equilibrium theorists use the notion of feasible set that is completely intelligible in the whole economy level. But they make a big mistake when they consider that this procedure which ensures the boundness is also clear in the individual agents level.

If the individual sets are unbounded sets, the consequences for the theory are negative because the key point of the construction of individual agents will be undefined. In other words these agents do no exist in economic terms, they can not define their actions for a determinate price vector. In consequence, we cannot talk about the excess supply function and the demonstration of competitive general equilibrium becomes meaningless.

The $X_i \cap e$ and $Y_k \cap E$, represent an impossible intersection between the private and the social. The individual agents can not have the social information they need to make the intersection. To be possible, the individual agents must be able to restrict their individual sets X_i and Y_k to the subsets $\tilde{X}_i$ and $\tilde{Y}_k$ that are compatible with the general equilibrium idea. Each agent needs the following information: the availabe resources for the whole economy, and the possibiliby set of the whole economy. With this information the individual agents can determine the feasible set for the whole economy and therefore restrict their sets in a suitable way. Unfortunately, the required information can only be held by a central agent, and not by individual agents in a decentralized economy. By no means is this a satisfactory solution of the problem. As previously stated within the elements of individual agents theory, it is impossible to define compact possibility sets.

The solution that the theory brings implies a procedure that is incompatible with the theory's subject. In fact, this theory is in a line of thought that conceives the market as a social device which, even if there is no intervention political, allows the interests and passions of the isolated individual to harmonize in an autonomous way. In this manner the society is integrated by a process in the economical sphere. In this process economics is identified as the realm in which the individual agents can and must give free rein to their personal and egoistic interests. The invisible hand will be in charge and lead to a harmonious process ensuring prosperity of society. This has been and continuous to be the tradition in the framework of market theory in classical and neoclassical thought.

The elements initially provided by general equilibrium theory in defining an "individual agent" seem to be compatible with Smith's idea of natural harmony of the market. However, when the theory of general equilibrium conceives the individual agent as an entity who has the information of the whole society, and is capable of perceiving and restricting, their individual possibility set, it falls into a double contradiction with its own subject. On the one hand, in the name of feasiblily when the initial unbounded sets are substituted by suitable bounded sets, it implies that the interests or plans of the individual agent have been harmonized before the price formation process begins. In other words the theory is assuming the result that is looked for. On the other hand, the theory is not dealing with a decentralized economy any more because the isolated individuals of that economy have information that only could be given by a central agent. In other words, the Market Secretary is a problem in the general equilibrium theory. The invisible hand is not only visible but also is required before the process begins. The central agent is present in the construction of individual agents. Therefore the demonstration of existence of general equilibrium does not correspond to a decentralized economy.

When the theory of general equilibrium introduces social information elements to define individual agents, the door is opened to the non competitive behavior of individual agents and also shows the incapability to construct the idea of decentralized economy in a consistent way.

Even though we recognize that the problem already discussed is one of the most important ones within the general equilibrium framework, there exist some others. One which is already stated in this work is in relation to the assumption of the divisibility of commodities. This assumption is also required in the demonstration of the existence of general equilibrium, and when we avoid this assumption new problems arise like the problem of definition of homogenous commodities. The problem of indivisible commodities also has negative consequences for the theory because then the choice of individual agents can be discontinuous and the fixed point theorem could not be applied. Therefore we can not ensure the existence of equilibrium. Also the relation to Pareto optima point now becomes questionable. Here we have again in front of us a gap between the mathematical requirements of the theory and the objective conditions of a reality that the theory attempts to describe.

The contradictions which general equilibrium theory incurs could be interpreted as indications of deeper problems. In the general equilibrium scheme the social objectivity is built from the interaction of isolated individuals. The natural harmony of individual interests replace the artificial harmony and within economic thought becomes a result of the market social device.

The fundamental premise of market theory conceived in such a way, is obviously the result of conceiving the economic agents as isolated individuals, not harmonized a priori. In other words, the theory of natural harmonization of individual interests implies the possiblity of conceiving in an individual way each one of the interests that the market must harmonize.

It is for all of these reasons that we can conclude that the fundamental problem of general equilibrium theory comes from the impossibility of a coherent articulation between the subjective and objective conditions in the framework of a theory that need both to be acceptable. This problem is related to a deeper one that can be stated as follows; the conception of the society as an agglomeration (natural harmonization) of isolated individuals is logically problematic.

In spite of all these problems on the demonstration of General Copmetitive Equilibrium and on General Equilibrium Theory. Some economists consider that the proofs are rigorous and that they effectively solve the problem of existence of decentralized economies' equilibrium. This fact leads the research in two directions: one concerned with the problem of relaxing the restrictive assumptions; for example some works deal with the idea of avoiding the Market Secretary, and the other concentrating on the dyanamic process that leads to the equilibrium.

Appendix

One of the most important episodes in economics in this century, the development of General Equilibrium theory is linked with mathematics, and it is clear that the interaction of mathematics and economic reasoning has been fruitful. Nevertheless we must always be aware of problems that could arise.

When dealing with mathematical economics it is important to ask what the role played by mathematics in economics is.

"A training in mathematics is helpful by giving command over a marvellously terse and exact language for expressing clearly some general relations and some short processes of economic reasoning". (Marshall 1949)

Is it only that or something more.

One of the problems in mathematical economics is the gap between theory and the reality of economic life. Some economists express this by saying that the task of the economist is to discover the 'truth' about the real economic world. The real world provides the facts, and some of these facts can be put together to make a coherent theory. Here arises a new problem, what is economic fact; and what is a theory? Furthermore when we are using mathematical formulations for economic statements we have to be sure that we are not making the gap wider.

Relations between facts and theory, between mathematics and economics, need to be worked out with the discussion of some methodological questions.

In the history of economic analysis, there are two different traditions in the theory of general equilibrium, classical models of surplus and accumulation and neoclassical models of the allocation of given resources. We shall deal with the second ones. The founders of the neoclassical school Jevons, Menger and Walras, took as an expository point of departure the model of pure exchange which was the polar opposite of the classical. At the beginning this school had to deal with resistance. Mathematical methods of reasoning began to play an important decisive role in the pure theory of economics during this period.

In the Appendix Part I we shall point out methodological problems involved in the Positive and Neopositive approaches and then we shall deal with the relations between economical and physical systems.

In Appendix Part II we shall begin with the initial relations between mathematics and economics, and then we are going to furnish some general remarks on the concept of equilibrium before we make a simple sketch of different lines followed by the equilibrium authors. Afterwards we shall state in certain detail the utility theory that was a continuous reference point in this work.

Appendix Part I Mathematics, Economy and Methodology.

1. Some Methodological Problems.

In their everyday work, most economists do not seem to be aware of methodological problems. They make theoretical models (concerning hypothetical economics) or build pragmatic models; they think every model is valid provided it is of some practical use in being able to forecast some particular future event.

Almost always, the results of their analyses are used to make specific ideologies acceptable or, at least, more appealing. And more than this, as far as almost any model can be adjusted to fit the empirical data, wecan notice the inadequacy of the neopositivist approach to economics. But, as most economists still believe in it, it is important to say something about this.

This initial observation leads us to the necessity of dealing with epistemology, and that's why we shall therefor discuss some issues that economists rarely discuss.

Among the most argued epistemological problems, -which one not only in argued in economics-, are the following:

a) the ideology and its relations with scientific explanation.

b) casual explanation against teleological explanation.

c) the social role of science and the role of the society in the development of science.

d) how evolution and discontinuous change can be dealt with by a scientific theory.

e) Finally and related to d) the cognitive value of stochastic models.

These problems that appear in what is called the metatheory of the science, have different characteristics in economics. For example neither of the two concepts of scientific growth, as an accumulative process or as a discontinuous and disconnected process, seems to apply to economics. Meanwhile marginalism has been rescued by "planometrics" in socialist countries, classical economics has not been superseded.

Economic analysis was born with a normative aim. Not in vain the works of the classical economists oriented to "knowledge of the economic processes to be used to induce or at least facilitate behavior suited to increasing the nations wealth". And after the capitalist system became consolidated, research more and more clearly pursued apologetic aims, not usually explicitly stated.

2. Positivist and Neopositivist Approaches

A good example for such research is the school of economic thought generically called neoclassical, which has a positivist approach due to the conscious goal of having the same success as in physics.

According to this approach the main purpose of economics is believed to be the explanation of the economic system. To do this, we need to assume that the fundamental properties of the system are independent of its history, that is, it is assumed that all history is summarized by the initial conditions. The normal (rational) structure of the system can be represented by a system of simultaneous equations that can be arrived at by an axiomatic theory, and stability analysis will be not a search but an imposition of a property the model must have if it is a valid scientific interpretation of the real economy.

In the marginalist theory of general equilibrium, these relations are arrived at by assuming optimizing behavior. Assuming this behavior leads to the discovery of some properties that should be confirmed by observation.

The the positivist approach exalts the criterion of empirical observability. Only facts produced by individuals choices are scientifically relevant, but not collective choices.

Marx substantially adheres to the view that individual needs are the goals of the economic system; social needs are essentially the aggregation of individual needs, which depends on income distribution. (Capital, Chapter X Vol III)

In the neopositivist approach to economics the preconception that the theoretical models must reflect the properties of the real system has been abandoned. The interest of economists has concentrated on formalization of the theory to clarify the axioms and the deductive methods to derive meaningful theorems. Empirical research is used to produce forecasting models.

The gap between theory and empirical research was made easier by Popper's reformulation of the criteria of verification of scientific theories (disprobation or falsation). Since then analysis of the logical structures of economics theories has been able to progress faster. Neoclassical economists now say that the coherence of their viewpoint has been better assessed; the problem of the existence of a solution for the model of general equilibrium correctly posed; useful theorems of comparative statistics correctly derived; and theoretically fruitful generalizations of the model obtained.

Nevertheless, the classical positivist model has remained unchanged, and this fact has affected the choice and the solution of theoretical problems. On the other hand, the practical models are not only specifications of the theoretical ones, their kinship with theory is in fact very loose and sometimes rather ambiguous. In the neoclassical approach empirical models can hardly supply the indications that empirical research is usually asked for to confirm the theory. But this does not worry economists too much. The theoreticians promote their experiments as contributions to the development of scientific knowledge. The pragmatists think that economic research must pursue practical goals, and that economic models can be applied to forecasting future developments of the economy and determining the best forms of intervention by which we can change them in order to attain predetermined objectives.

But only when the essential structure is stable enough, can predictive models be effectively used for practical purposes. We have to admit that in general what the economists discover are trends.

With the separation of theory and empirical investigation, the neopositivist approach produced two consequences: on the one hand the birth of econometrics was made possible; on the other hand, with the acceptance of the possible existence of functions which had not been observed, such as the social welfare function, the path was opened to normative economics.

The invariance of the principal features of a market economy is reaffirmed, in terms weaker than those entitled by the classical theory and by the other versions of marginalism. The market for the neopositivists, is compatible with the pursuit of social aims. As a consequence, both the presence of changes in the income distribution and the use of resources for social purposes can be justified on the basis of the results of economical analysis (welfare economy).

The essential features of the system are supposed to be stable. And they must be determined before the system is analysed according to the methods borrowed from physics. In fact, it is because of these features that the market can guarantee the coexistence of optimal individual choices. Henceforth, this new conception of the economic system allows a of a variety of rational economical structures among which the society can choose (Pareto optima).

In the neopositivistic framework, one assumes that the essential relations of the system are predetermined, and moreover both the instrumental variables and their changes are thought to be structurally controllable.

From a theoretical point of view, the social goals cannot be explained by economic theory itself, since as they originate outside of it. On the other hand, the economic model guarantees that, given the goals the optimal social actions can be achieved. If the goals the system pursues, its initial conditions, and its structure are predetermined.

However, the model does not ensure the concrete achievement of optimal social actions; to obtain them the following conditions must be fulfilled:

the goals must be attainable and the hypotheses must hold true: This last condition involves the empirical verification of the hypotheses themselves.

In the neopositivistic model, the cognitive value of the normative models is the same as that of the descriptive ones; indeed it depends on its predictive capabilities, since only the controller can modify the economical process in a predetermined way. The relation between these nonnegative and descriptive models only depends on the logical process with which the first is obtained.

In the economic development of socialist countries, we can observe the same tendencies which appear in the neopositivist conception. As far as the mathematical economy is concerned, the theoretical tendency is to define the structural features to be considered as inalterable, and to determine from these the necessary mechanism of the rationality of the system; it also aims to distinguish the goals of the mechanism of efficiency from those determined by political decisions. In these countries, the practical tendency has had a development strictly related to mathematical programming, control theory, cybernetics and systems theory.

3. Physical and Economic Systems.

In reality, the economic system cannot be directly perceived, nor can it be "cut out" of the social world, in the same way in which the physical system is cut out of the natural world.

In a physical system, every law can be formulated and tested independently on other laws. As a consequence, even though a law is satisfactory for the solution of some particular concrete problems, the set of laws associated to a particular physical system (laws we can refer to as local ones) cannot provide a satisfactory explanation for the system, when the system itself is considered as a whole included in a larger one. In the physical system, it is also possible to isolate the system from its environment, and to take precise and repeated measurements of each input and each output.

It is not possible in the case of an economic system as in a physical one to isolate and formulate each one of its laws in an independent fashion.

In spite of this, according to Popper, even in the case of the economy some laws have been discovered which are similar to the natural laws which we have referred above as local. But even these local economic laws are logical connections between assessments. This fact is different when we are dealing with physical laws, as the latter can be empirically verified.

The economic system cannot be described if we disregard the interaction between the system as a whole and its parts. The approach followed by economists (from Quesnay to Walras) has always been a system approach (take the system as a whole like the sum of different parts related among themselves, and vies each part and the whole through laws).

We cannot assume that the system remains unchanged while we measure its inputs and outputs; nor can we always easily define even which are the inputs and outputs: moreover, it is possible that what is an input according to one theory may be an output according to another.

In spite of the peculiarities of the subject, economists have adopted models like those developed in physics. In neoclassical economcs, mathematical methods quite soon become equated with scientific method (Neoclassical theoricians think that the scientism criteria is formalization).

Popper's statement that "The success of mathematical economics shows that one social science at least has gone through its Newtonian revolution", contrasts with Hutchinson's remark that "the mathematical 'revolution' in economics has been one mainly of form, with very little or nonempirical; testable, predictive content involved". However, at times the mathematical made of economics has limited the critical development of economic theory.

The fundamental reason for the different role played by the mathematical method in economics as compared with its role in physical sciences is the protopostulate of the space time invariability of the phenomena. Such a protopostulate in economics is only a convenient -for the neopositivist school, indeed a necessary- working hypothesis.

In physics the results of research are cumulative. Every new theory, except in the case of parallel theories, supersedes the previous theory. And at the same time it allows us to define in which context the old theory can give an interprtation concrete process. Preanalytic knowledge, does not prevent the results of research from being generally accepted, at least for their practical usefulness.

To apply the scientific method to the economic system it must be conceived in a particular way that makes it possible to qualify the normal system as a rational structure; the operators of the economic system are assumed to be powerless and the market to behave as a physical system and the idea that stability of systems could not be assumed a priori. However in applying the scientific method to economics both the continuity assumption must be carefully considered.

In the 1940's the hypothesis of atomistic behavior, which allowed economists to apply the scheme of celestial mechanics used in physics to the economic system (by the theory of general equilibrium), was changed by the formulation and application to economics of the game theory. Edgeworth has already shown why the mechanical scheme cannot be used to explain market behavior in the case of oligopoly.

4. Economic System

According to Friedman "the relevant question to ask about 'assumptions' of a theory is not whether they are descriptively 'realistic', for they never are, but whether they are sufficiently good approximations for the purpose in hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions. The two supposedly independent tests thus reduce to one test".

Assumptions are arrived at by a cognitive process that can not escape critical evaluation. At the beginning of the process there are always some preliminary knowledge and a point of view that must be reconsidered after the results of the scientific research.

It is the positivist approach that caused the illusion that all cognitive processes could be resolved into scientific analysis, that is why the two pillars that sustained the positivist point of view have collapsed. These pillars are, firstly, the assumption that a definite logical system could be arrived at on which any other system could be based, and, secondly, the assumption that scientific development was resolvable into an accumulation of definitive results.

We cannot avoid uniting the results of all cognitive processes into a systematic whole, as coherently as possible, which is always to some extent inherited and which cannot be resolved in a well connected scientific theory. To admit that pure science does not exist does not mean deny that it is imposible to define scientific criteria or to submit all ideological premises to critical assessment.

What we now need to emphasize is that all scientific theories entail some preliminary knowledge that can be qualified as preanalytic, since it cannot be previously confirmed by the application of scientific criteria: It has in fact some ideological content associated with a particuiar *weltanschauung*.

Preanalytic knowledge is particularly relevant in economic analysis owing to the peculiar nature of the system economists propose to study. The underlying ideology may assume the form of ideological prejudices.

For classical economists the economic system is a subsystem of the social system (according to Marx the subsystem that will eventually determine the evolution of the whole social system). Marginalists, on the contrary, consider all social phenomena can be considered homogeneous from a specific standpoint; they all entail the use of scarce resources for the achievement of nonhomogeneous goals. In fact, since not all social phenomena are considered by economists but only those in which the economic aspect is dominant, the set of phenomena analyzed by marginalists roughly coincides with the classical economic system: a difference remains in the way the system is considered.

We should conceive a social system in a restricted sense, existing side by side with an economic system, both being inferred from the society as a whole. In this way the results of the two studies open the way to a dialectic cognitive process aimed at understanding the whole.

Appendix Part II Equilibrium Analysis.

1. Mathematics and Economy.

Schumpeter said that, the works of Jevons, Menger, Walras, Marshall, Wiccksell, Bohm-Bawerck, Clark, Pareto, Fisher,... proved that a mechanism of theoretical analysis whose basic features were the same everywhere existed. These authors works

existed. These authors works cover practically all of what we may call the primary poriod work in general theory, except that of the Marxists.

Schumpeter asked, why, then, do the structures of these leaders having the same basic features look so different? His answer was because there were plenty of differences in technique, in details, and in views on individual problems, and in addition because leaders and followers alike overemphasized them.

The most important differences in technique were seen when they dealt with the use or the refusal to use the calculus and systems of simultaneous equations. An example of the differences in detail is the dispute on real cost. As examples of differences in views on individual problems Schumprter pointed out the differences in the theory of capital and the different attitudes as regards partial analysis.

He also said that the underlying unit of that period's 'general economics' was not true in the first part of the period only in the Classical Situation that emerged roughly around 1900.

The new theory had to deal with resistance, because the changes looked very 'revolutionary'. Then as always, the majority of the economists were absorbed in the investigation of the facts and practical problems of the public policy. This majority reinforced by the historical and institutional group, had little use of the theory and did not welcome a variation on it.

They never accepted it as an instrument of research, and considered the 'marginalism' as speculative philosophy or as a new sectarian 'ism' which it was precisely their duty to eliminate by what they considered truly scientific and realistic research. Hence they made sweeping judgements upon it.

Mathematical methods of reasoning began to play an important and decisive role in the pure theory of economics during this period. The use of numerical examples or the restatement in an algebraic form of a non mathematical result does not constitute mathematical economics. Only when the reasoning itself that produces the result is explicitly mathematical is it mathematical economics.

At the beginning economists used mathematics only for statistical material then they began to use mathematics in quantitative analysis.

Before 1914 it was rare that publications required any knowledge of technical mathematics of their readers or even their writers, what was required was basic algebra, analytical geometry and general ideas of the logic of calculus.

Familiarity with the logic of calculus and concepts such as variables, functions, limits, continuity, derivatives and differentials, maxima and minima, systems of equations, determinateness, and stability changes one's whole attitude to the problems that arise from theoretical schemes of quantitive relations between things: problems obtain a new accuracy, and the points at which they lose become clear, new methods of proof emerge. Even when we know only a little about the relations between our variables we can get maximum results, and in some sense some controversial points vanish.

"Mathematical theory is more than a translation of non-mathematical theory in the language of symbols, but its results can, in general, be translated into non-mathematical language". But it is in the attempt to translate from non mathematical theory into a mathematical one or the reverse, that most economists fail.

The most important theories of marginal utility and the marginal productivity were worked out also by economists without any mathematical background. They think that except for a few otiose refinements, mathematical reasoning in economics has not added anything to what could be found out without it.

Cournot (1801-77) proposed to deal with a number of problems that lent themselves particularly well to treatment by calculus . Cournot also recognized, that for a complete and rigorous solution of the problems of parts of the economic system it is 'indispensable to take the entire system into consideration' (which is precisely what Walras was to do). But he believed that 'this would surpass the powers of mathematical analysis and of our practical methods of calculation' and therefore he envisaged instead the possibility of treating such problems in terms of a small set of aggregates in which social income and its variations are the most important.

Cournot was the first theorist to prove what mathematics can do for economics, and he was the master of partial analysis, but he had only a vague and nonoperational idea of general equilibrium.

The specifically econometric program -mathematical theory plus statistical figures- was struggling toward concious formulation all the time but, with some important exceptions, did not succeed. And the theory of the period did not lend itself to the insertion of such results. The majority of theorists, including some of the greatest, were completely unaware of the posibility of a theory that might eventually achieve numerical results. However, Cournot, Jevons, Pareto, Marshall were exceptions.

2. The Concept of Equilibrium.

When we are dealing with the concept of equilibrium we have to show clearly what we understand by words such as statics, dynamics, stationary state, evolution, because in this way we can avoid useless discussions. Following Schumpeter, by static analysis we mean a method of dealing with economic phenomena that tries to establish relations between elements of the economic system -prices and quantities of commodities- all of which refer to the same point of time.

The elements of the economic system that interact at a given point of time are evidently the result of preceding configurations, and the way in which they interact itself is not less evidently influenced by what people expect future configurations to be. Hence we are led to take into account past and expected future values of the variables, lags, sequences, rates of change, cumulative magnitudes, expectations, and so on. The methods that aim at doing this constitute economic dynamics.

In some sense static theory involves a higher level of abstraction and at the same time static theory may be said to constitute a special case of a more general dynamic theory. Always static theory has historically preceded dynamic theory and the reasons for this seem to be as obvious as they are sound -static theory is much simpler to work out, its propositions are easier to prove, and it seems closer to logical essentials.

Stationary state is an economic process that merely reproduces itself. Essentially it is a simplif ing device, but it is also something more. When we try to visualize how such a process might look and which of the phenomena of reality might be present in it we ipso facto discover which of them are lacking, and we thus acquire a tool of analysis. The term evolution may be used in a wider and in a narrower sense. In the wider sense it comprises all the phenomena that make an economic process non-stationary. In the narrower sense it comprises these phenomena minus those that may be described in terms of continuous variations of rates within an unchanging framework of institutions, tastes, or technological horizons, and will be included in the concept of growth.

At least in logical principle, statics and dynamics on the one hand, and stationary and evolutionary states, on the other, are independent of one another. We may describe a stationary process by a dynamic model. We may also describe an evolutionary process by a succession of static models (Comparative statics).

Improvements in the analytic apparatus of economics were worked out during these years but not quickly enough, or rigorously enough, to take full effect upon analysts practice before 1914. This slowed down the advance and explains some of the most serious shortcomings of the actual achievements.

The concept of the stationary state had been used to denote an actual state of the economy to be expected at some future time; eg. in Marx's model of simple reproduction.the simple reproduction.

Cassel, after Marshall, made an extension of the idea of the stationary state to balanced progress, that is, to the case of a society in which population and wealth grow at about the same rate and in which 'methods of production and the conditions of trade change but little. This conception has acquired the models not only of a stagnating but also of an expanding economy.

More clearly perceived than rigorously defined, the system of economics state did emerge these years, but the nature of statics economics did emerge, but the nature of economic dynamics was not even clearly understood. Some authors identified economic dynamics with a historical theory of change or with a theory that allows for trends; other authors with a theory of general interdependence as opposed to a partial analysis of sectional phenomena; still others with a theory of a modern as against the tradition bound economy of Middle Ages; and a few simply with the theory of small variations of economic quantites.

All this shows the importance, even for purely practical purposes, of logically rigorous definitions: for had the nature of the statics of the day been subjected to rigorous analysis, the problems of the dynamics would have emerged almost of themselves. But only suggestions that point toward the dynamics were found.

From the workshop of Walras the static theory of the economic universe emerged in the form of a large number of quantitative relations (equations) between economic elements of variables (prices and quantities of consumable and productive goods or services) that were conceived as simultaneously determining one another.

The Walrasian system of simultaneous equations, however, brought in a host of new problems of a specifically logical or mathematical nature that are much more delicate and go much deeper than Walras or anyone else had ever realized. In general they turn upon determinateness, equilibrium and stability.

The methodology of analyticalmethods is stated thus: since the economic system cannot be treated as a set of undefined things, we must first define what its elements are to mean before we can formulate the exact problem of their determination in terms of certain properties of the functions (relations) which this meaning involves. Then follows logically the proof that the problem can in fact be solved (proof of the existence of a solution) and, finally, the investigation into the 'laws' that the solution reveals (the properties of the solution). A set of quantities (variables) can be said to be determined if we can indicate relations to which they must conform and which will restrict the possible range of their values. If the relations determine just single value or sequence of values, we speak of unique determination, this is, of course, satisfactory case. The relations may yield, however, more than one possible value or sequence of values, which is not completely satisfactory. In particular the relations may determine only a range, and compel theorists to achieve a higher degree of determinateness.

If the relations, which are derived from a survey of the 'meaning' of a phenomenon, are such as to determine a set of values of the variables that will display no tendency to vary under the sole influence of the facts included in those relations per se, we call it equilibrium: we say that those relations define equilibrium conditions or an equilibrium position of the system and that there exists a set of values of the variables that satisfies equilibrium conditions.

Multiple equilibria are not necessarily useless but, the existence of a 'uniquely determined equilibrium set of values' is of the utmost importance, even if the proof has to be purchased at the price of very restrictive assumptions. Without any possibility of proving the existence of uniquely determined equilibrium of a small number of possible equilibria at however high a level of abstraction, a field of phenomena is really a chaos that is not under analytic control.

Relations that link variables with the same time subscript or different ones, may define a static or a dynamic equilibrium. Whether static or dynamic, equilibrium may be stable, neutral or unstable. Roughly speaking stable equilibrium value is an equilibrium value that, if changed by a small amount, calls into action forces that will tend to reproduce the old value; neutral equilibrium is and equilibrium value that does not know any such forces; and unstable equilibrium is an equilibrium value, change in which calls forth forces that tend to move the system farther and farther away from equilibrium values.

The economists of the period retaining the habit of their 'classic' predecessors, consider 'competition' as the normal case from which to build up their general analysis (Cournot built his analysis from the monopoly case). It is clear that the generalized description of economic behavior is greatly simplified by the assumption that the prices of all products and 'factors' cannot be perceptibly influenced by the individual household or the individual firm, and hence may be treated as given (as parameters) the theory of their behavior. These prices may then be determined, in general, by the mass effect of the actions of all households and all firms in the 'markets', and the households and firms have no choice but to adapt to the ruling prices the quantities of commodities and services they wish to buy or to sell. Jevons added his law of indifference, which defines the concept of the perfect market in which there cannot exist, at any moment, more than one price for each homogeneous commodity. These two features, competition and the law of indifference, were included in Walras' free concurrence.

The mechanism of pure competition is supposed to function through everybody's wish to maximize his net advantage (satisfaction or monetary gain) by means of attempts at optimal adaptation of quantities to be bought and sold. Walras was very much aware of the difficulties involved in the fact that this adaptation will produce different results according to the range of knowledge, promptness of decision, and 'rationality' of actors, and according to the expectations they entertain about the future course of prices, and the fact that their action is subject to additional restrictions that proceed from the situations they have made themselves by their past decisions. However, absorbed in the pioneer task of working out the essentials of the mathematical theory of the economic process, he said he had no choice but to simplify in a huge manner: that is he postulated that the quantities of productive services that enter into the unit of every product (coefficients of production) are constant technological data; that there is no such thing as fixed cost; that all the firms in an industry produce the same kind of product by the same method, in equal quantities; that the productive process takes no time; and that problems of location may be neglected.

Marshall, however, did not take this approach. As opposed to Walras, who was bent more on scraping off everything he did not consider essential to his theoretical schema, Marshall following the English tradition, was bent on salvaging every bit of real life he could possible leave in. This was more than a mere dislike of naked abstractions, it was an awareness of a set of problems that developed into the theory of monopolistic or imperfect competition later on.

Some economists accept pure monopoly and pure competition as the two fundamental patterns and next proceed by investigating how their hybrids work out. On the contrary others look upon the hybrids as fundamental and on pure monopoly and pure competition as limiting cases.

3. Utility theory.

The importance of utility theory extends far beyond of consumers' behavior, into those of production and income formation. From Aristotelian roots, this theory was developed by the scholastic doctors whose analyses of value and price in terms of 'utility and scarcity' lacked only the marginal apparatus. However in the first period it was not fruitful and the classical economists did not realize the possibilities of the utility approach to the phenomenon of the economic behavior.

Later Walras, Jevons and Menger rediscovered the theory for themselves and constructed upon it a theoretical structure. They all treated utility as a psychological fact gained from introspection, and as the 'cause' of value: They felt little or no compunction about its measurability. They all thought that the possessor of any commodity derived the utility from it and the magnitude of the utility was dependent on the quantity of that commodity alone.

The historical alliance of utility theory with utilitarian philosophy seems rather obvious, but the modern utilitarian theorists themselves claim it is not difficult to show that the utility theory of value is entirely independent of any hedonist postulates of philosophies. According to these theorists, it does not state or imply anything about the nature of the wants or desires from which it starts. The utility theory of value is better designated as a logic than a psychology of values. Early utility theorists talked about physical facts with the utmost confidence, thinking that it is preferable to derive a given set of propositions from externally or 'objectively' observable facts, if it can be done, than to derive the same set of propositions from premises established by introspection. Utility theory can be utilized to achieve this goal if it is used to furnish the assumptions or 'restrictions' that we need within the equilibrium theory of values and prices.

Many authors have also held that, by probing into the "psychology" of value in use, the utility theory contributed nothing to the understanding of economic processes. (Kautsky, Lexis, etc.)

If it is assumed that the consumer derives utility only from the goods he purchases the amount of utility is a function of the quantities of goods acquired, and that he will try to get the maximum possible amount of utility. Utility will be maximized when the marginal (last) unit of expenditure in each direction brings in the same increment of utility. In such a case, a transference of expenditure from one direction to another will bring in the diminishing of the total utility, for marginal utility of any commodity is assumed diminishing. Marshall in this way came to the conclusion that total utility is maximized when the marginal utilities of the commodities obtained are proportional to their prices.

Cardinal utility.

In the beginning, utility theoricians considered that marginal and total utility are directly measurable quantities. They associated utility sensations with a real number,unique except for the choice of a unit which is to be interpreted as a unit sensation. Bohm-Bawerk was the first who recognized practical difficulties. Marshall thought that a direct measure of utility or motive or pleasantness and unpleasantness of sensations could not be taken but that they could be measured inderectly by their observable effects. For example, the utility of pleasure may be indicated by the sum of money a man is prepared to give up in order to obtain it rather than to do without it. Both indirect and direct measures are conventionally called the theory of cardinal utility, and it is a uniquely determined real function of the quantities of the commodities (per stated period of time) at the disposal of individual or household.

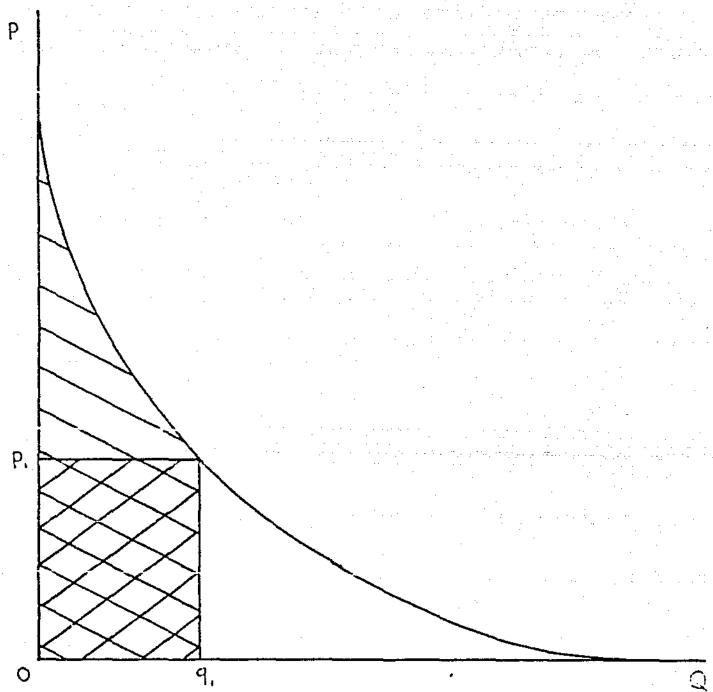
Antonelli in the "Sulla teoria matematica della economia politica"(1886) had become concerned with fundamental ideas that came to be developed more clearly. Edgeworth did away with the assumption that the utility of every commodity is a function of the quantity of this commodity alone, and made the utility enjoyed by an individual a function of all the commodities that enter his budget. Then Marshall attempted (upon Dupuit's idea) to make the measurement of utility operational by means of the concept of Consumer's Rent. His idea was to measure the total utility accruing to an individual from the consumption of a given quantity of a commodity by the sum of money represented by the definite integral of the individual's demand function taken from zero to the quantity. The consumers' surplus is the difference between this integral and the price actually paid times the quantity bought. (Graph A1) This conjecture is open to a number of objections and was badly received at first, later, however, Hicks revived it because of its usefulness in welfare economics.

Ordinal utility.

The objection to measurability was the most serious of those that were raised against the marginal utility theory. Among utility theoreticians there is in fact no compelling necessity of insisting upon measurability so long as they are interested only in a maximum problem. There was no question of people's ability to compare satisfactions expected from the possession of different sets of goods without measuring them, that is to say, the ability of people to order such sets in a given preference system, referred to as ordinal utility.

It is possible to describe ordinal utility by means of any monotonically increasing function though not uniquely determined. This function has in fact no economic meaning because what it is devised to tell us is whether there is increase, decrease, or equality of utility. Pareto call such a function an Index function.

However, it was not the index function as such, but another set of constructs that became characteristic of this stage of value theory. It was the indifference surfaces or, in the case of two commodities, the



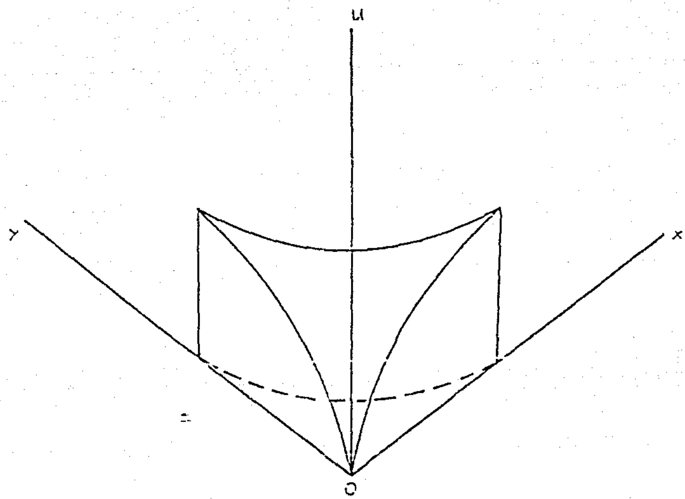
Graph A1 Consumers Rent.

indifference curves, that were independently discovered by Edgeworth for other purposes.

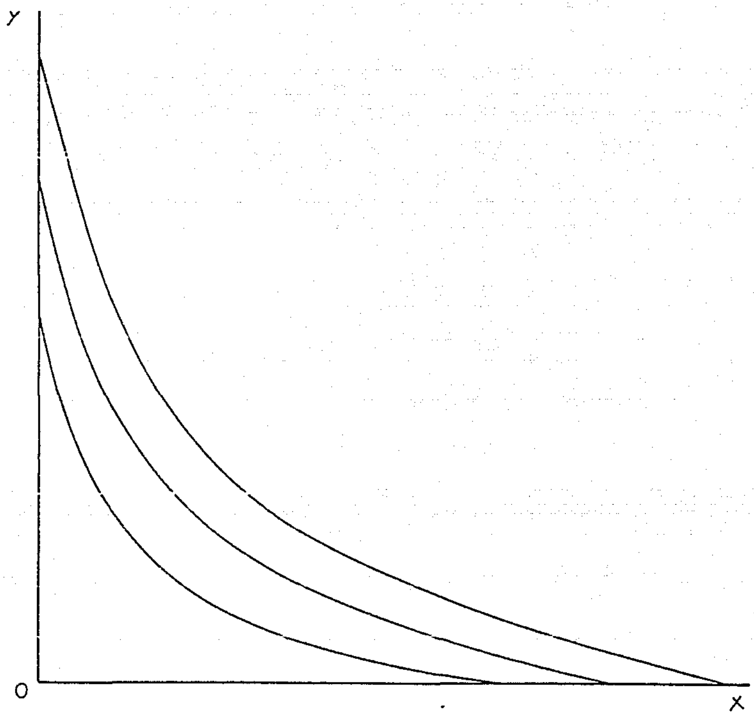
In the two commodity case, we can portray the quantities of these commodities on two of the co-ordinates of a three dimensional diagram and represent by the third co-ordinate the varying amounts of total utility which corresponds to all possible combinations of the two commodities. The result is utility surface that rises from the origin as the quantities of the two commodities increase, and possibly flattens out later on, Pareto called it the 'Hill of pleasure'. If we cut out this surface by means of a succession of planes parallel to the plane of the two commodity co-ordinates the intersection is a set which has a constant total utility. (Graph A2) The quantities of the two commodities will vary in such a way that the increase of one just compensates the individual for the corresponding decrease of the other. These are what Edgeworth called indifference curves and if we project them on the commodity plane, we get the 'indifference map'. (Graph A3) He used it in his barter exchange theory.

For if X has a positive marginal utility, an increase in the quantity of Y must increase total utility, and bring us on to a higher indifference curve, similarly if we only increase the quantity of X. It is possible to stay on the same indifference curve if these movements are compensated. So Pareto stated that these indifference curves must slope downwards to the right because each commodity has a positive marginal utility. The slope of the curve passing through any point P(x,y) is the amount of y which needed to compensate the loss of a small unit of x, that is the ratio of the marginal of x to the marginal utility of y. Pareto also said that these curves will be convex to the origin.

As soon as we project the indifference lines on the commodity plane, the utility dimension vanishes from the picture so that their meaning is no longer dependent on any hypothesis of measurability. The indifference lines tell us no more than that the individual may consider certain combinations



Graph A2 Utility Surface intersected with a plane
and its projection.



Graph A3 Indifference map.

of the two commodities as equally desirable, and that he prefers combinations represented by any 'higher' indifference curve to combinations represented by any 'lower' one.

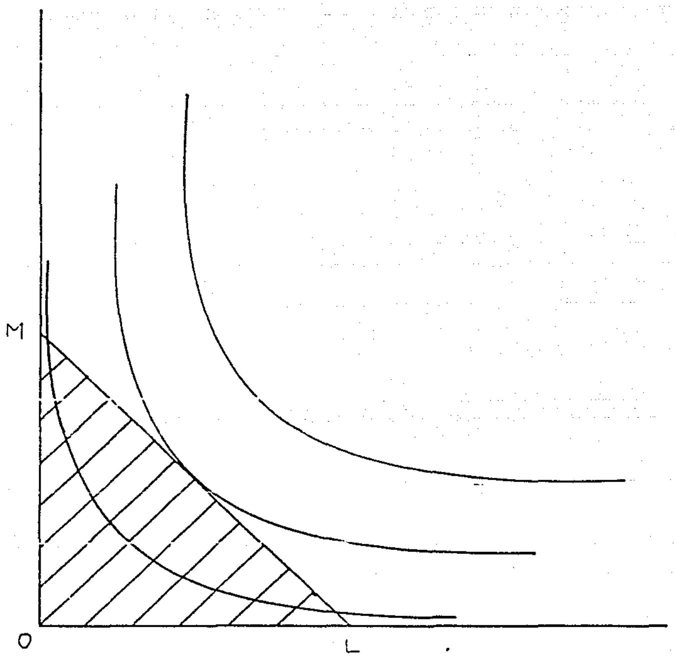
If a length OL is marked off along the X-axis, representing the amount of X which the consumer could get if he expands all his income to this commodity, and an amount OM on the Y-axis, representing similarly the amount of Y, then any point on the line LM represents a pair of quantities of the two commodities which he could obtain, and the slope of the line LM is the price ratio. Through any point of the line there will pass an indiference curve. As the utility will be maximized on the point where the live LM is the tangent of the indifference curve, (Graph A4) that point also shows the proportionality between marginal utilities and prices.

After Pareto proceeded to develop the idea of ordinal utility and eventually worked out the fundamentals of the modern theory of value, a further advance was made by Johnson and Slutsky, although it was not until 1934 that the job was completely done by Allen and Hicks.

Fisher presented an analysis completely free from utility assumptions that worked only with indifference maps in the modern sense. He, and later Allen and Hicks, considered indifference curves as the starting points of the analysis. This approach differed from that taken by Edgeworth, who derived indifference curves from a utility surface.

The theory of Allen and Hicks was the first completely independent of the existence of an index function and completely free from the lingering shadows of even marginal utility. Marginal utility is replaced in their system by the marginal rate of substitution. Indifference curves are satisfactorily defined for individual households but the question of what meaning is to be attached to collective indifference curves remains.

If the indifference curves assume less than does utility analysis, they still assume more than is necessary for the conjectures of equilibrium theory. From a practical point of view, drawing purely imaginary



Graph A4 Maximum utility point.

indifference curves is not better than speaking of purely imaginary utility functions because they both use nothing observable in principle, and they only use potential observations which so far nobody has been able to make in fact. The writing of the equations of equilibrium theory requires no other postulate than this: faced with a given set of prices and a given income, the individual chooses to buy or sell in a uniquely determinate way. This was later formulated by Samuelson as the consistency postulate.

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